

Hall's Marriage Theorem

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Abstract

A proof of Hall's Marriage Theorem due to Halmos and Vaughan [1].

```
theory Marriage
imports Main
begin
```

```
theorem marriage-necessary:
```

```
  fixes A :: 'a ⇒ 'b set and I :: 'a set
```

```
  assumes finite I and ∀ i∈I. finite (A i)
```

```
  and ∃ R. (∀ i∈I. R i ∈ A i) ∧ inj-on R I (is ∃ R. ?R R A & ?inj R A)
```

```
  shows ∀ J⊆I. card J ≤ card (⋃ (A ` J))
```

```
proof clarify
```

```
  fix J
```

```
  assume J ⊆ I
```

```
  show card J ≤ card (⋃ (A ` J))
```

```
proof -
```

```
  from assms(3) obtain R where ?R R A and ?inj R A by auto
```

```
  have inj-on R J by (rule subset-inj-on[OF ‹?inj R A› ‹J⊆I›])
```

```
  moreover have (R ` J) ⊆ (⋃ (A ` J)) using ‹J⊆I› ‹?R R A› by auto
```

```
  moreover have finite (⋃ (A ` J)) using ‹J⊆I› assms
```

```
    by (metis finite-UN-I finite-subset subsetD)
```

```
  ultimately show ?thesis by (rule card-inj-on-le)
```

```
qed
```

```
qed
```

The proof by Halmos and Vaughan:

```
theorem marriage-HV:
```

```
  fixes A :: 'a ⇒ 'b set and I :: 'a set
```

```
  assumes finite I and ∀ i∈I. finite (A i)
```

```
  and ∀ J⊆I. card J ≤ card (⋃ (A ` J)) (is ?M A I)
```

```
  shows ∃ R. (∀ i∈I. R i ∈ A i) ∧ inj-on R I
```

```
    (is ?SDR A I is ∃ R. ?R R A I & ?inj R A I)
```

```
proof -
```

```
  { fix I
```

```
    have finite I ⇒ ∀ i∈I. finite (A i) ⇒ ?M A I ⇒ ?SDR A I
```

```

proof(induct arbitrary: A rule: finite-psubset-induct)
  case (psubset I)
  show ?case
  proof (cases)
    assume  $I = \{\}$  then show ?thesis by simp
  next
    assume  $I \neq \{\}$ 
    have  $\forall i \in I. A\ i \neq \{\}$ 
    proof (rule ccontr)
      assume  $\neg (\forall i \in I. A\ i \neq \{\})$ 
      then obtain  $i$  where  $i \in I$   $A\ i = \{\}$  by blast
      hence  $\{i\} \subseteq I$  by auto
      from  $mp[OF\ spec[OF\ psubset.prem1(2)]\ this]$   $\langle A\ i = \{\} \rangle$ 
      show False by simp
    qed
  show ?thesis
  proof cases
    assume case1:  $\forall K \subseteq I. K \neq \{\} \longrightarrow \text{card}(\bigcup(A \text{ ` } K)) \geq \text{card } K + 1$ 
    show ?thesis
    proof -
      from  $\langle I \neq \{\} \rangle$  obtain  $n$  where  $n \in I$  by auto
      with  $\langle \forall i \in I. A\ i \neq \{\} \rangle$  have  $A\ n \neq \{\}$  by auto
      then obtain  $x$  where  $x \in A\ n$  by auto
      let  $?A' = \lambda i. A\ i - \{x\}$  let  $?I' = I - \{n\}$ 
      from  $\langle n \in I \rangle$  have  $?I' \subseteq I$ 
      by (metis DiffD2 Diff-subset insertI1 psubset-eq)
      have  $fin'$ :  $\forall i \in ?I'. \text{finite} (?A'\ i)$  using psubset.prem1 by auto
      have  $?M\ ?A'\ ?I'$ 
    proof clarify
      fix  $J$ 
      assume  $J \subseteq ?I'$ 
      hence  $J \subseteq I$  by (metis  $\langle I - \{n\} \subseteq I \rangle$  subset-psubset-trans)
      show  $\text{card } J \leq \text{card}(\bigcup_{i \in J} A\ i - \{x\})$ 
    proof cases
      assume  $J = \{\}$  thus ?thesis by auto
    next
      assume  $J \neq \{\}$ 
      hence  $\text{card } J + 1 \leq \text{card}(\bigcup(A \text{ ` } J))$  using case1  $\langle J \subseteq I \rangle$  by blast
      moreover
      have  $\text{card}(\bigcup(A \text{ ` } J)) - 1 \leq \text{card}(\bigcup_{i \in J} A\ i - \{x\})$  (is  $?l \leq ?r$ )
    proof -
      have finite J using  $\langle J \subseteq I \rangle$  psubset(1)
      by (metis psubset-imp-subset finite-subset)
      hence  $1: \text{finite}(\bigcup(A \text{ ` } J))$ 
      using  $\langle \forall i \in I. \text{finite}(A\ i) \rangle$   $\langle J \subseteq I \rangle$  by force
      have  $?l = \text{card}(\bigcup(A \text{ ` } J)) - \text{card}\{x\}$  by simp
      also have  $\dots \leq \text{card}(\bigcup(A \text{ ` } J) - \{x\})$  using  $1$ 
      by (metis diff-card-le-card-Diff finite.intros)
      also have  $\bigcup(A \text{ ` } J) - \{x\} = (\bigcup_{i \in J} A\ i - \{x\})$  by blast
  
```

finally show *?thesis* .
qed
ultimately show *?thesis* **by** *arith*
qed
qed
from *psubset(2)*[*OF* $\langle ?I' \subset I \rangle$ *fin'* $\langle ?M ?A' ?I' \rangle$]
obtain *R'* **where** $?R R' ?A' ?I' ?inj R' ?A' ?I'$ **by** *auto*
let $?Rx = R'(n := x)$
have $?R ?Rx A I$ **using** $\langle x \in A n \rangle \langle ?R R' ?A' ?I' \rangle$ **by** *force*
have $\forall i \in ?I'. ?Rx i \neq x$ **using** $\langle ?R R' ?A' ?I' \rangle$ **by** *auto*
hence $?inj ?Rx A I$ **using** $\langle ?inj R' ?A' ?I' \rangle$
by(*auto simp: inj-on-def*)
with $\langle ?R ?Rx A I \rangle$ **show** *?thesis* **by** *auto*
qed
next
assume $\neg (\forall K \subset I. K \neq \{\}) \longrightarrow \text{card} (\bigcup (A \text{ ' } K)) \geq \text{card } K + 1$
then obtain *K* **where**
 $K \subset I K \neq \{\}$ **and** *c1*: $\neg (\text{card} (\bigcup (A \text{ ' } K)) \geq \text{card } K + 1)$ **by** *auto*
with *psubset.prem(2)* **have** $\text{card} (\bigcup (A \text{ ' } K)) \geq \text{card } K$ **by** *auto*
with *c1* **have** *case2*: $\text{card} (\bigcup (A \text{ ' } K)) = \text{card } K$ **by** *auto*
from $\langle K \subset I \rangle \langle \text{finite } I \rangle$ **have** *finite K* **by** (*auto intro: finite-subset*)
from *psubset.prem(1)* $\langle K \subset I \rangle$
have $\forall i \in K. \text{finite } (A i) \forall J \subseteq K. \text{card } J \leq \text{card} (\bigcup (A \text{ ' } J))$ **by** *auto*
from *psubset(2)*[*OF* $\langle K \subset I \rangle$ *this*]
obtain *R1* **where** $?R R1 A K ?inj R1 A K$ **by** *auto*
let $?AK = \lambda i. A i - \bigcup (A \text{ ' } K)$ **let** $?IK = I - K$
from $\langle K \neq \{\} \rangle \langle K \subset I \rangle$ **have** $?IK \subset I$ **by** *auto*
have $\forall i \in ?IK. \text{finite } (?AK i)$ **using** *psubset.prem(1)* **by** *auto*
have $?M ?AK ?IK$
proof *clarify*
fix *J* **assume** $J \subseteq ?IK$
with $\langle \text{finite } I \rangle$ **have** *finite J* **by**(*auto intro: finite-subset*)
show $\text{card } J \leq \text{card} (\bigcup (?AK \text{ ' } J))$
proof–
from $\langle J \subseteq ?IK \rangle$ **have** $J \cap K = \{\}$ **by** *auto*
have $\text{card } J = \text{card}(J \cup K) - \text{card } K$
using $\langle \text{finite } J \rangle \langle \text{finite } K \rangle \langle J \cap K = \{\} \rangle$
by (*auto simp: card-Un-disjoint*)
also have $\text{card}(J \cup K) \leq \text{card} (\bigcup (A \text{ ' } (J \cup K)))$
proof –
from $\langle J \subseteq ?IK \rangle \langle K \subset I \rangle$ **have** $J \cup K \subseteq I$ **by** *auto*
with *psubset.prem(2)* **show** *?thesis* **by** *blast*
qed
also have $\dots - \text{card } K = \text{card} (\bigcup (?AK \text{ ' } J) \cup \bigcup (A \text{ ' } K)) - \text{card } K$
proof–
have $\bigcup (A \text{ ' } (J \cup K)) = \bigcup (?AK \text{ ' } J) \cup \bigcup (A \text{ ' } K)$
using $\langle J \subseteq ?IK \rangle$ **by** *auto*
thus *?thesis* **by** *simp*
qed

```

also have ... = card (⋃ (?AK ' J)) + card(⋃(A ' K)) - card K
proof-
  have finite (⋃ (?AK ' J)) using ⟨finite J⟩ ⟨J⊆?IK⟩ psubset(3)
  by(blast intro: finite-UN-I finite-Diff)
  moreover have finite (⋃(A ' K))
  using ⟨finite K⟩ ⟨∀i∈K. finite (A i)⟩ by auto
  moreover have ⋃ (?AK ' J) ∩ ⋃(A ' K) = {} by auto
  ultimately show ?thesis
  by (simp add: card-Un-disjoint del:Un-Diff-cancel2)
qed
also have ... = card (⋃ (?AK ' J)) using case2 by simp
finally show ?thesis by simp
qed
qed
from psubset(2)[OF ⟨?IK⊂I⟩ ⟨∀i∈?IK. finite (?AK i)⟩ ⟨∀J⊆?IK. card
J ≤ card (⋃i∈J. A i - ⋃(A ' K))⟩]
obtain R2 where ?R R2 ?AK ?IK ?inj R2 ?AK ?IK by auto
let ?R12 = λi. if i∈K then R1 i else R2 i
have ∀i∈I. ?R12 i ∈ A i using ⟨?R R1 A K⟩⟨?R R2 ?AK ?IK⟩ by auto
moreover have ∀i∈I. ∀j∈I. i≠j⟶?R12 i ≠ ?R12 j
proof clarify
  fix i j assume i∈I j∈I i≠j ?R12 i = ?R12 j
  show False
  proof-
    { assume i∈K ∧ j∈K ∨ i∉K ∧ j∉K
      with ⟨?inj R1 A K⟩ ⟨?inj R2 ?AK ?IK⟩ ⟨?R12 i=?R12 j⟩ ⟨i≠j⟩ ⟨i∈I⟩
    }
    { assume i∈K ∧ j∉K ∨ i∉K ∧ j∈K
      with ⟨?R R1 A K⟩ ⟨?R R2 ?AK ?IK⟩ ⟨?R12 i=?R12 j⟩ ⟨j∈I⟩ ⟨i∈I⟩
    }
    have ?thesis by (fastforce simp: inj-on-def)
  } moreover
  { assume i∈K ∧ j∉K ∨ i∉K ∧ j∈K
    with ⟨?R R1 A K⟩ ⟨?R R2 ?AK ?IK⟩ ⟨?R12 i=?R12 j⟩ ⟨j∈I⟩ ⟨i∈I⟩
  }
  have ?thesis by auto (metis Diff-iff)
  } ultimately show ?thesis by blast
qed
qed
ultimately show ?thesis unfolding inj-on-def by fast
qed
qed
qed
}
with assms ⟨?M A I⟩ show ?thesis by auto
qed

```

The proof by Rado:

```

theorem marriage-Rado:
  fixes A :: 'a ⇒ 'b set and I :: 'a set
  assumes finite I and ∀ i∈I. finite (A i)
  and ∀ J⊆I. card J ≤ card (⋃(A ' J)) (is ?M A)
  shows ∃R. (∀i∈I. R i ∈ A i) ∧ inj-on R I

```

```

(is ?SDR A is  $\exists R. ?R R A \ \& \ ?inj R A$ )
proof -
{ have  $\forall i \in I. finite (A i) \implies ?M A \implies ?SDR A$ 
  proof(induct n ==  $\sum i \in I. card(A i) - 1$  arbitrary: A)
    case 0
    have  $\forall i \in I. \exists a. A(i) = \{a\}$ 
    proof (rule ccontr)
      assume  $\neg (\forall i \in I. \exists a. A i = \{a\})$ 
      then obtain i where  $i: i:I \ \forall a. A i \neq \{a\}$  by blast
      hence  $\{i\} \subseteq I$  by auto
      from  $0(1-2)$  mp[OF spec[OF 0.prem(2)]]  $\langle \{i\} \subseteq I \rangle$   $\langle finite I \rangle$  i
      show False by (auto simp: card-le-Suc-iff)
    qed
    then obtain R where  $R: \forall i \in I. A i = \{R i\}$  by metis
    then have  $\forall i \in I. R i \in A i$  by blast
    moreover have inj-on R I
    proof (auto simp: inj-on-def)
      fix x y assume  $x \in I \ y \in I \ R x = R y$ 
      with R spec[OF 0.prem(2), of  $\{x,y\}$ ] show  $x=y$ 
      by (simp add: le-Suc-eq card-insert-if-split: if-splits)
    qed
    ultimately show ?case by blast
  next
  case (Suc n)
  from Suc.hyps(2)[symmetric, THEN sum-SucD]
  obtain i where  $i: i:I \ 2 \leq card(A i)$  by auto
  then obtain  $x1 \ x2$  where  $x1 : A i \ x2 : A i \ x1 \neq x2$ 
    using Suc(3) by (fastforce simp: card-le-Suc-iff eval-nat-numeral)
  let ?Ai x =  $A i - \{x\}$  let ?A x =  $A(i:=?Ai x)$ 
  let ?U J =  $\bigcup (A ' J)$  let ?Ui J x =  $?U J \cup ?Ai x$ 
  have  $n1: n = (\sum j \in I. card (?A x1 j) - 1)$ 
    using Suc.hyps(2) Suc.prem(1) i  $\langle finite I \rangle$   $\langle x1:A i \rangle$ 
    by (auto simp: sum.remove card-Diff-singleton)
  have  $n2: n = (\sum j \in I. card (?A x2 j) - 1)$ 
    using Suc.hyps(2) Suc.prem(1) i  $\langle finite I \rangle$   $\langle x2:A i \rangle$ 
    by (auto simp: sum.remove card-Diff-singleton)
  have  $finx1: \forall j \in I. finite (?A x1 j)$  by (simp add: Suc(3))
  have  $finx2: \forall j \in I. finite (?A x2 j)$  by (simp add: Suc(3))
  { fix x assume  $\neg ?M (A(i:= ?Ai x))$ 
    with Suc.prem(2) obtain J
      where  $J: J \subseteq I \ card J > card(\bigcup ((A(i:= ?Ai x) ' J)))$ 
      by (auto simp add: not-less-eq-eq Suc-le-eq)
    note fJi = finite-Diff[OF finite-subset[OF  $J \subseteq I$ ]  $\langle finite I \rangle$ ], of  $\{i\}$ 
    have  $fU: finite(?U (J-\{i\}))$  using  $\langle J \subseteq I \rangle$ 
      by (metis Diff-iff Suc(3) finite-UN[OF fJi] subsetD)
    have  $i \in J$  using J Suc.prem(2)
      by (simp-all add: UNION-fun-upd not-le[symmetric] del: fun-upd-apply
        split: if-splits)
    hence  $card(J-\{i\}) \geq card(?Ui (J-\{i\}) x)$ 
  }
}

```

```

    using fJi J by(simp add: UNION-fun-upd del: fun-upd-apply)
  hence  $\exists J \subseteq I. i \notin J \wedge \text{card}(J) \geq \text{card}(\text{?Ui } J \ x) \wedge \text{finite}(\text{?U } J)$ 
    by (metis DiffD2 J(1) fU  $\langle i \in J \rangle$  insertI1 subset-insertI2 subset-insert-iff)
} note lem = this
have ?M (?A x1)  $\vee$  ?M (?A x2) — Rado's Lemma
proof(rule ccontr)
  assume  $\neg$  (?M (?A x1)  $\vee$  ?M (?A x2))
  with lem obtain J1 J2 where
    J1:  $J1 \subseteq I \ i \notin J1 \ \text{card } J1 \geq \text{card}(\text{?Ui } J1 \ x1) \ \text{finite}(\text{?U } J1)$  and
    J2:  $J2 \subseteq I \ i \notin J2 \ \text{card } J2 \geq \text{card}(\text{?Ui } J2 \ x2) \ \text{finite}(\text{?U } J2)$ 
  by metis
  note fin1 = finite-subset[OF  $\langle J1 \subseteq I \rangle$  assms(1)]
  note fin2 = finite-subset[OF  $\langle J2 \subseteq I \rangle$  assms(1)]
  have finUi1: finite(?Ui J1 x1) using Suc(3) by(blast intro: J1(4) i(1))
  have finUi2: finite(?Ui J2 x2) using Suc(3) by(blast intro: J2(4) i(1))
  have card J1 + card J2 + 1 = card(J1  $\cup$  J2) + 1 + card(J1  $\cap$  J2)
    by simp (metis card-Un-Int fin1 fin2)
  also have card(J1  $\cup$  J2) + 1 = card(insert i (J1  $\cup$  J2))
    using  $\langle i \notin J1 \rangle \langle i \notin J2 \rangle$  fin1 fin2 by simp
  also have ...  $\leq$  card ( $\bigcup$  (A ' insert i (J1  $\cup$  J2))) (is -  $\leq$  card ?M)
    by (metis J1(1) J2(1) Suc(4) Un-least i(1) insert-subset)
  also have ?M = ?Ui J1 x1  $\cup$  ?Ui J2 x2 using  $\langle x1 \neq x2 \rangle$  by auto
  also have card(J1  $\cap$  J2)  $\leq$  card( $\bigcup$  (A ' (J1  $\cap$  J2)))
    by (metis J2(1) Suc(4) le-infI2)
  also have ...  $\leq$  card(?U J1  $\cap$  ?U J2) by(blast intro: card-mono J1(4))
  also have ...  $\leq$  card(?Ui J1 x1  $\cap$  ?Ui J2 x2)
    using Suc(3)  $\langle i \in I \rangle$  by(blast intro: card-mono J1(4))
  finally show False using J1(3) J2(3)
    by(auto simp add: card-Un-Int[symmetric, OF finUi1 finUi2])
qed
thus ?case using Suc.hyps(1)[OF n1 finx1] Suc.hyps(1)[OF n2 finx2]
  by (metis DiffD1 fun-upd-def)
qed
} with assms  $\langle ?M \ A \rangle$  show ?thesis by auto
qed
end

```

References

- [1] P. R. Halmos and H. E. Vaughan. The marriage problem. *American Journal of Mathematics*, 72:214–215, 1950.