Lower Semicontinuous Functions

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June 17, 2024

Abstract

We define the notions of lower and upper semicontinuity for functions from a metric space to the extended real line. We prove that a function is both lower and upper semicontinuous if and only if it is continuous. We also give several equivalent characterizations of lower semicontinuity. In particular, we prove that a function is lower semicontinuous if and only if its epigraph is a closed set. Also, we introduce the notion of the lower semicontinuous hull of an arbitrary function and prove its basic properties.

Contents

1 Lower semicontinuous functions

theory *Lower-Semicontinuous*
imports *HOL−Analysis.Multivariate-Analysis*
begin

1.1 Relative interior in one dimension

lemma *rel-interior-ereal-semiline*:
  fixes $a :: ereal$
  shows *rel-interior* $\{y.\ a \leq ereal\ y\} = \{y.\ a < ereal\ y\}$
⟨*proof*⟩

lemma *closed-ereal-semiline*:
  fixes $a :: ereal$
  shows *closed* $\{y.\ a \leq ereal\ y\}$
⟨*proof*⟩

lemma *ereal-semiline-unique*:
fixes $a\ b$ :: *ereal*
shows $\{y.\ a \le ereal\ y\} = \{y.\ b \le ereal\ y\} \longleftrightarrow a = b$
⟨*proof*⟩

1.2 Lower and upper semicontinuity

definition
lsc-at :: $'a \Rightarrow ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
$lsc\text{-}at\ x0\ f \longleftrightarrow (\forall X\ l.\ X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow f\ x0 \le l)$

definition
usc-at :: $'a \Rightarrow ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
$usc\text{-}at\ x0\ f \longleftrightarrow (\forall X\ l.\ X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow l \le f\ x0)$

lemma *lsc-at-mem*:
assumes *lsc-at x0 f*
assumes $x \longrightarrow x0$
assumes $(f \circ x) \longrightarrow A$
shows $f\ x0 \le A$
⟨*proof*⟩

lemma *usc-at-mem*:
assumes *usc-at x0 f*
assumes $x \longrightarrow x0$
assumes $(f \circ x) \longrightarrow A$
shows $f\ x0 \ge A$
⟨*proof*⟩

lemma *lsc-at-open*:
fixes f :: $'a::first\text{-}countable\text{-}topology \Rightarrow 'b::\{complete\text{-}linorder,\ linorder\text{-}topology\}$
shows $lsc\text{-}at\ x0\ f \longleftrightarrow$
$(\forall S.\ open\ S \wedge f\ x0 \in S \longrightarrow (\exists T.\ open\ T \wedge x0 \in T \wedge (\forall x'{\in}T.\ f\ x' \le f\ x0 \longrightarrow f\ x' \in S)))$
(**is** $?lhs \longleftrightarrow\ ?rhs$)
⟨*proof*⟩

lemma *lsc-at-open-mem*:
fixes f :: $'a::first\text{-}countable\text{-}topology \Rightarrow 'b::\{complete\text{-}linorder,\ linorder\text{-}topology\}$
assumes *lsc-at x0 f*
assumes $open\ S \wedge f\ x0 : S$
obtains T **where** $open\ T \wedge x0 \in T \wedge (\forall x'{\in}T.\ (f\ x' \le f\ x0 \longrightarrow f\ x' \in S))$
⟨*proof*⟩

lemma *lsc-at-MInfty*:
fixes f :: $'a::topological\text{-}space \Rightarrow ereal$

assumes $f\ x0 = -\infty$
shows *lsc-at x0 f*
⟨*proof*⟩

lemma *lsc-at-PInfty*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
assumes $f\ x0 = \infty$
shows *lsc-at x0 f* $\longleftrightarrow$ *continuous* (*at x0*) *f*
⟨*proof*⟩

lemma *lsc-at-real*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
assumes $|f\ x0| \neq \infty$
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall e.\ e>0 \longrightarrow (\exists T.\ open\ T \wedge x0 : T \wedge (\forall y \in T.\ f\ y > f\ x0 - e)))$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-at-ereal*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall C<f(x0).\ \exists T.\ open\ T \wedge x0 \in T \wedge (\forall y \in T.\ f\ y > C))$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lst-at-ball*:
fixes f :: $'a$::*metric-space* => *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall C<f(x0).\ \exists d>0.\ \forall y \in (ball\ x0\ d).\ C<f(y))$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lst-at-delta*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $(\forall C<f(x0).\ \exists d>0.\ \forall y.\ dist\ x0\ y < d \longrightarrow C < f\ y)$
(**is** *?lhs* $\longleftrightarrow$ *?rhs*)
⟨*proof*⟩

lemma *lsc-liminf-at*:
fixes f :: $'a$::*metric-space* $\Rightarrow$ *ereal*
shows *lsc-at x0 f* $\longleftrightarrow$ $f\ x0 \leq Liminf\ (at\ x0)\ f$
⟨*proof*⟩

lemma *lsc-liminf-at-eq*:

fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ (*f x0* = *min* (*f x0*) (*Liminf* (*at x0*) *f*))
⟨*proof*⟩

lemma *lsc-imp-liminf*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes *lsc-at x0 f*
assumes x ⟶ *x0*
shows *f x0* ≤ *liminf* (*f* ∘ *x*)
⟨*proof*⟩

lemma *lsc-liminf*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ (∀ *x*. *x* ⟶ *x0* ⟶ *f x0* ≤ *liminf* (*f* ∘ *x*))
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ (∀ *x c*. *x* ⟶ *x0* ∧ (∀ *n*. *f*(*x n*)≤*c*) ⟶ *f*(*x0*)≤*c*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially-gen*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0 f* ⟷ (∀ *x c c0*. *x* ⟶ *x0* ∧ *c* ⟶ *c0* ∧ (∀ *n*. *f*(*x n*)≤*c n*) ⟶ *f*(*x0*)≤*c0*)
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-sequentially-mem*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
assumes *lsc-at x0 f*
assumes *x* ⟶ *x0 c* ⟶ *c0*
assumes ∀ *n*. *f*(*x n*)≤*c n*
shows *f*(*x0*)≤*c0*
⟨*proof*⟩

lemma *lsc-uminus*:
fixes f :: $'a$::*metric-space* ⇒ *ereal*
shows *lsc-at x0* (λ*x*. −*f x*) ⟷ *usc-at x0 f*
⟨*proof*⟩

lemma *usc-limsup*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $usc\text{-}at\ x0\ f \longleftrightarrow (\forall x.\ x \longlonglongrightarrow x0 \longrightarrow f\ x0 \geq limsup\ (f \circ x))$
(**is** $?lhs \longleftrightarrow ?rhs$)
⟨*proof*⟩

lemma *usc-imp-limsup*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
assumes $usc\text{-}at\ x0\ f$
assumes $x \longlonglongrightarrow x0$
shows $f\ x0 \geq limsup\ (f \circ x)$
⟨*proof*⟩

lemma *usc-limsup-at*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $usc\text{-}at\ x0\ f \longleftrightarrow f\ x0 \geq Limsup\ (at\ x0)\ f$
⟨*proof*⟩

lemma *continuous-iff-lsc-usc*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $continuous\ (at\ x0)\ f \longleftrightarrow (lsc\text{-}at\ x0\ f) \wedge (usc\text{-}at\ x0\ f)$
⟨*proof*⟩

lemma *continuous-lsc-compose*:
assumes $lsc\text{-}at\ (g\ x0)\ f\ continuous\ (at\ x0)\ g$
shows $lsc\text{-}at\ x0\ (f \circ g)$
⟨*proof*⟩

lemma *continuous-isCont*:
$continuous\ (at\ x0)\ f \longleftrightarrow isCont\ f\ x0$
⟨*proof*⟩

lemma *isCont-iff-lsc-usc*:
fixes $f :: 'a::metric\text{-}space \Rightarrow ereal$
shows $isCont\ f\ x0 \longleftrightarrow (lsc\text{-}at\ x0\ f) \wedge (usc\text{-}at\ x0\ f)$
⟨*proof*⟩

definition
$lsc :: ('a::topological\text{-}space \Rightarrow 'b::order\text{-}topology) \Rightarrow bool$ **where**
$lsc\ f \longleftrightarrow (\forall x.\ lsc\text{-}at\ x\ f)$

definition

usc :: (*'a*::*topological-space* ⇒ *'b*::*order-topology*) ⇒ *bool* **where**
usc f ⟷ (∀ *x*. *usc-at x f*)

lemma *continuous-UNIV-iff-lsc-usc*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (∀ *x*. *continuous* (*at x*) *f*) ⟷ (*lsc f*) ∧ (*usc f*)
⟨*proof*⟩

1.3 Epigraphs

definition *Epigraph S* (*f*::- ⇒ *ereal*) = {*xy*. *fst xy* : *S* ∧ *f* (*fst xy*) ≤ *ereal*(*snd xy*)}

lemma *mem-Epigraph*: (*x*, *y*) ∈ *Epigraph S f* ⟷ *x* ∈ *S* ∧ *f x* ≤ *ereal y* ⟨*proof*⟩

lemma *ereal-closed-levels*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (∀ *y*. *closed* {*x*. *f*(*x*)≤*y*}) ⟷ (∀ *r*. *closed* {*x*. *f*(*x*)≤*ereal r*})
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-iff*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (*lsc f* ⟷ (∀ *y*. *closed* {*x*. *f*(*x*)≤*y*})) ∧ (*lsc f* ⟷ *closed* (*Epigraph UNIV f*))
⟨*proof*⟩

definition *lsc-hull* :: (*'a*::*metric-space* ⇒ *ereal*) ⇒ (*'a*::*metric-space* ⇒ *ereal*) **where**
lsc-hull f = (*SOME g*. *Epigraph UNIV g* = *closure*(*Epigraph UNIV f*))

lemma *epigraph-mono*:
fixes *f* :: *'a*::*metric-space* ⇒ *ereal*
shows (*x*,*y*):*Epigraph UNIV f* ∧ *y*≤*z* ⟶ (*x*,*z*):*Epigraph UNIV f*
⟨*proof*⟩

lemma *closed-epigraph-lines*:
fixes *S* :: (*'a*::*metric-space* ∗ *'b*::*metric-space*) *set*
assumes *closed S*
shows *closed* {*z*. (*x*, *z*) : *S*}
⟨*proof*⟩

lemma *mono-epigraph*:
  fixes S :: ('a::*metric-space* $*$ *real*) *set*
  assumes *mono*: $\forall$ x y z. (x,y):S $\wedge$ $y \le z$ $\longrightarrow$ (x,z):S
  assumes *closed* S
  shows $\exists$ g. ((*Epigraph UNIV* g) = S)
⟨*proof*⟩

lemma *lsc-hull-exists*:
  fixes f :: 'a::*metric-space* $\Rightarrow$ *ereal*
  shows $\exists$ g. *Epigraph UNIV* g = *closure* (*Epigraph UNIV* f)
⟨*proof*⟩

lemma *epigraph-invertible*:
  assumes *Epigraph UNIV* f = *Epigraph UNIV* g
  shows $f=g$
⟨*proof*⟩

lemma *lsc-hull-ex-unique*:
  fixes f :: 'a::*metric-space* $\Rightarrow$ *ereal*
  shows $\exists!g$. *Epigraph UNIV* g = *closure* (*Epigraph UNIV* f)
⟨*proof*⟩

lemma *epigraph-lsc-hull*:
  fixes f :: 'a::*metric-space* $\Rightarrow$ *ereal*
  shows *Epigraph UNIV* (*lsc-hull* f) = *closure*(*Epigraph UNIV* f)
⟨*proof*⟩

lemma *lsc-hull-expl*:
  (g = *lsc-hull* f) $\longleftrightarrow$ (*Epigraph UNIV* g = *closure*(*Epigraph UNIV* f))
⟨*proof*⟩

lemma *lsc-lsc-hull*: *lsc* (*lsc-hull* f)
  ⟨*proof*⟩

lemma *epigraph-subset-iff*:
  fixes f g :: 'a::*metric-space* $\Rightarrow$ *ereal*
  shows *Epigraph UNIV* f $\le$ *Epigraph UNIV* g $\longleftrightarrow$ ($\forall$ x. g x $\le$ f x)
⟨*proof*⟩

lemma *lsc-hull-le*: (*lsc-hull* f) x $\le$ f x

⟨*proof*⟩

lemma *lsc-hull-greatest*:
fixes $f\ g :: {}'a::metric\text{-}space \Rightarrow ereal$
assumes $lsc\ g\ \forall x.\ g\ x \leq f\ x$
shows $\forall x.\ g\ x \leq (lsc\text{-}hull\ f)\ x$
⟨*proof*⟩

lemma *lsc-hull-iff-greatest*:
fixes $f\ g :: {}'a::metric\text{-}space \Rightarrow ereal$
shows $(g = lsc\text{-}hull\ f) \longleftrightarrow$
$lsc\ g \wedge (\forall x.\ g\ x \leq f\ x) \wedge (\forall h.\ lsc\ h \wedge (\forall x.\ h\ x \leq f\ x) \longrightarrow (\forall x.\ h\ x \leq g\ x))$
(**is** *?lhs* ⟷ *?rhs*)
⟨*proof*⟩

lemma *lsc-hull-mono*:
fixes $f\ g :: {}'a::metric\text{-}space \Rightarrow ereal$
assumes $\forall x.\ g\ x \leq f\ x$
shows $\forall x.\ (lsc\text{-}hull\ g)\ x \leq (lsc\text{-}hull\ f)\ x$
⟨*proof*⟩

lemma *lsc-hull-lsc*:
$lsc\ f \longleftrightarrow (f = lsc\text{-}hull\ f)$
⟨*proof*⟩

lemma *lsc-hull-liminf-at*:
fixes $f :: {}'a::metric\text{-}space \Rightarrow ereal$
shows $\forall x.\ (lsc\text{-}hull\ f)\ x = min\ (f\ x)\ (Liminf\ (at\ x)\ f)$
⟨*proof*⟩

lemma *lsc-hull-same-inf*:
fixes $f :: {}'a::metric\text{-}space \Rightarrow ereal$
shows $(INF\ x.\ lsc\text{-}hull\ f\ x) = (INF\ x.\ f\ x)$
⟨*proof*⟩

1.4 Convex Functions

definition
$convex\text{-}on :: {}'a::real\text{-}vector\ set \Rightarrow ('a \Rightarrow ereal) \Rightarrow bool$ **where**
$convex\text{-}on\ s\ f \longleftrightarrow$
$(\forall x \in s.\ \forall y \in s.\ \forall u \geq 0.\ \forall v \geq 0.\ u + v = 1$
$\longrightarrow f\ (u *_R x + v *_R y) \leq ereal\ u * f\ x + ereal\ v * f\ y)$

lemma *convex-on-ereal-mem*:
assumes *convex-on s f*
assumes *x:s y:s*
assumes *u≥0 v≥0 u+v=1*
shows $f\ (u *_R x + v *_R y) \leq$ *ereal u* $*$ *f x* $+$ *ereal v* $*$ *f y*
⟨*proof*⟩

lemma *convex-on-ereal-subset*: *convex-on t f* $\Longrightarrow s \leq t \Longrightarrow$ *convex-on s f*
⟨*proof*⟩

lemma *convex-on-ereal-univ*: *convex-on UNIV f* $\longleftrightarrow$ ($\forall$ *S*. *convex-on S f*)
⟨*proof*⟩

lemma *ereal-pos-sum-distrib-left*:
fixes f :: *'a* $\Rightarrow$ *ereal*
assumes $r \geq 0$ $r \neq \infty$
shows $r *$ *sum f A* $=$ *sum* ($\lambda n.\ r * f\ n$) *A*
⟨*proof*⟩

lemma *convex-ereal-add*:
fixes *f g* :: *'a::real-vector* $\Rightarrow$ *ereal*
assumes *convex-on s f convex-on s g*
shows *convex-on s* ($\lambda x.\ f\ x + g\ x$)
⟨*proof*⟩

lemma *convex-ereal-cmul*:
assumes $0 \leq$ (*c::ereal*) *convex-on s f*
shows *convex-on s* ($\lambda x.\ c * f\ x$)
⟨*proof*⟩

lemma *convex-ereal-max*:
fixes *f g* :: *'a::real-vector* $\Rightarrow$ *ereal*
assumes *convex-on s f convex-on s g*
shows *convex-on s* ($\lambda x.$ *max* ($f\ x$) ($g\ x$))
⟨*proof*⟩

lemma *convex-on-ereal-alt*:
fixes *C* :: *'a::real-vector set*
assumes *convex C*
shows *convex-on C f* $=$
($\forall x \in C.\ \forall y \in C.\ \forall m$ :: *real*. $m \geq 0 \wedge m \leq 1$
$\longrightarrow f\ (m *_R x + (1 - m) *_R y) \leq$ (*ereal m*) $* f\ x + (1 -$ (*ereal m*)) $* f\ y$)

⟨*proof*⟩

lemma *convex-on-ereal-alt-mem*:
fixes C :: *'a::real-vector set*
assumes *convex C*
assumes *convex-on C f*
assumes $x : C\ y : C$
assumes $(m{::}real) \geq 0\ m \leq 1$
shows $f\ (m *_R x + (1 - m) *_R y) \leq (ereal\ m) * f\ x + (1 - (ereal\ m)) * f\ y$
⟨*proof*⟩

lemma *ereal-add-right-mono*: $(a{::}ereal) \leq b \Longrightarrow a + c \leq b + c$
⟨*proof*⟩

lemma *convex-on-ereal-sum-aux*:
assumes $1{-}a{>}0$
shows $(1 - ereal\ a) * (ereal\ (c\ /\ (1 - a)) * b) = (ereal\ c) * b$
⟨*proof*⟩

lemma *convex-on-ereal-sum*:
fixes a :: *'a ⇒ real*
fixes y :: *'a ⇒ 'b::real-vector*
fixes f :: *'b ⇒ ereal*
assumes *finite s* $s \neq \{\}$
assumes *convex-on C f*
assumes *convex C*
assumes $(SUM\ i : s.\ a\ i) = 1$
assumes $\forall i.\ i \in s \longrightarrow a\ i \geq 0$
assumes $\forall i.\ i \in s \longrightarrow y\ i \in C$
shows $f\ (SUM\ i : s.\ a\ i *_R y\ i) \leq (SUM\ i : s.\ ereal\ (a\ i) * f\ (y\ i))$
⟨*proof*⟩

lemma *sum-2*: $sum\ u\ \{1{::}nat..2\} = (u\ 1){+}(u\ 2)$
⟨*proof*⟩

lemma *convex-on-ereal-iff*:
assumes *convex s*
shows *convex-on s f* $\longleftrightarrow (\forall k\ u\ x.\ (\forall i{\in}\{1..k{::}nat\}.\ 0 \leq u\ i \wedge x\ i : s) \wedge sum\ u$
$\{1..k\} = 1 \longrightarrow$
$f\ (sum\ (\lambda i.\ u\ i *_R x\ i)\ \{1..k\}\ ) \leq sum\ (\lambda i.\ (ereal\ (u\ i)) * f(x\ i))\ \{1..k\}\ )$
(**is** *?rhs* ⟷ *?lhs*)
⟨*proof*⟩

lemma *convex-Epigraph*:
assumes *convex S*
shows *convex*(*Epigraph S f*) ⟷ *convex-on S f*
⟨*proof*⟩

lemma *convex-EpigraphI*:
convex-on s f ⟹ *convex s* ⟹ *convex*(*Epigraph s f*)
⟨*proof*⟩

definition
concave-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
concave-on S f ⟷ *convex-on S* (λ*x*. − *f x*)

definition
finite-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
finite-on S f ⟷ (∀ *x*∈*S*. (*f x* ≠ ∞ ∧ *f x* ≠ −∞))

definition
affine-on :: *'a::real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
affine-on S f ⟷ (*convex-on S f* ∧ *concave-on S f* ∧ *finite-on S f*)

definition
domain (*f*::- ⇒ *ereal*) = {*x*. *f x* < ∞}

lemma *domain-Epigraph-aux*:
assumes *x* ≠ ∞
shows ∃ *r*. *x*≤*ereal r*
⟨*proof*⟩

lemma *domain-Epigraph*:
domain f = {*x*. ∃ *y*. (*x*,*y*) ∈ *Epigraph UNIV f*}
⟨*proof*⟩

lemma *domain-Epigraph-fst*:
domain f = *fst* ‘ (*Epigraph UNIV f*)
⟨*proof*⟩

lemma *convex-on-domain*:
convex-on (*domain f*) *f* ⟷ *convex-on UNIV f*
⟨*proof*⟩

lemma *convex-on-domain2*:

convex-on (*domain f*) *f* ⟷ (∀ *S*. *convex-on S f*)
⟨*proof*⟩

lemma *convex-domain*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows *convex* (*domain f*)
⟨*proof*⟩

lemma *infinite-convex-domain-iff*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *ereal*
assumes ∀ *x*. (*f x* = ∞ | *f x* = −∞)
shows *convex-on UNIV f* ⟷ *convex* (*domain f*)
⟨*proof*⟩

lemma *convex-PInfty-outside*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f convex S*
shows *convex-on UNIV* (λ*x*. *if x*:*S then* (*f x*) *else* ∞)
⟨*proof*⟩

definition
proper-on :: *'a*::*real-vector set* ⇒ (*'a* ⇒ *ereal*) ⇒ *bool* **where**
proper-on S f ⟷ ((∀ *x*∈*S*. *f x* ≠ −∞) ∧ (∃ *x*∈*S*. *f x* ≠ ∞))

definition
proper :: (*'a*::*real-vector* ⇒ *ereal*) ⇒ *bool* **where**
proper f ⟷ *proper-on UNIV f*

lemma *proper-iff*:
proper f ⟷ ((∀ *x*. *f x* ≠ −∞) ∧ (∃ *x*. *f x* ≠ ∞))
⟨*proof*⟩

lemma *improper-iff*:
~(*proper f*) ⟷ ((∃ *x*. *f x* = −∞) | (∀ *x*. *f x* = ∞))
⟨*proof*⟩

lemma *ereal-MInf-plus*[*simp*]: −∞ + *x* = (*if x* = ∞ *then* ∞ *else* −∞::*ereal*)
⟨*proof*⟩

lemma *convex-improper*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*

assumes ~(*proper f*)
shows ∀ *x*∈*rel-interior*(*domain f*). *f x* = −∞
⟨*proof*⟩

lemma *convex-improper2*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes ~(*proper f*)
shows *f x* = ∞ | *f x* = −∞ | *x* : *rel-frontier* (*domain f*)
⟨*proof*⟩

lemma *convex-lsc-improper*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes ~(*proper f*)
assumes *lsc f*
shows *f x* = ∞ | *f x* = −∞
⟨*proof*⟩

lemma *convex-lsc-hull*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows *convex-on UNIV* (*lsc-hull f*)
⟨*proof*⟩

lemma *improper-lsc-hull*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes ~(*proper f*)
shows ~(*proper* (*lsc-hull f*))
⟨*proof*⟩

lemma *lsc-hull-convex-improper*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes ~(*proper f*)
shows ∀ *x*∈*rel-interior*(*domain f*). (*lsc-hull f*) *x* = *f x*
⟨*proof*⟩

lemma *convex-with-rel-open-domain*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *rel-open* (*domain f*)
shows (∀ *x*. *f x* > −∞) | (∀ *x*. (*f x* = ∞ | *f x* = −∞))

⟨*proof*⟩

lemma *convex-with-UNIV-domain*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  assumes *domain f* = *UNIV*
  shows (∀ x. $f\ x > -\infty$) ∨ (∀ x. $f\ x = -\infty$)
⟨*proof*⟩

lemma *rel-interior-Epigraph*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  shows (x,z) : *rel-interior* (*Epigraph UNIV f*) ⟷
    (x : *rel-interior* (*domain f*) ∧ $f\ x$ < *ereal z*)
⟨*proof*⟩

lemma *rel-interior-EpigraphI*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  shows *rel-interior* (*Epigraph UNIV f*) =
    {(x,z) |x z. x : *rel-interior* (*domain f*) ∧ $f\ x$ < *ereal z*}
⟨*proof*⟩

lemma *convex-less-ri-domain*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  assumes ∃ x. $f\ x < a$
  shows ∃ x∈*rel-interior* (*domain f*). $f\ x < a$
⟨*proof*⟩

lemma *rel-interior-eq-between*:
  fixes S T :: ($'m$::*euclidean-space*) *set*
  assumes *convex S convex T*
  shows (*rel-interior S* = *rel-interior T*) ⟷ (*rel-interior S* ≤ T ∧ T ≤ *closure S*)
⟨*proof*⟩

lemma *convex-less-in-riS*:
  fixes f :: $'a$::*euclidean-space* ⇒ *ereal*
  assumes *convex-on UNIV f*
  assumes *convex S rel-interior S* ≤ *domain f*

assumes $\exists\, x \in closure\ S.\ f\ x < a$
shows $\exists\, x \in rel\text{-}interior\ S.\ f\ x < a$
⟨*proof*⟩

lemma *convex-less-inS*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f*
assumes *convex S* $S \leq$ *domain f*
assumes $\exists\, x \in closure\ S.\ f\ x < a$
shows $\exists\, x \in S.\ f\ x < a$
⟨*proof*⟩

lemma *convex-finite-geq-on-closure*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f*
assumes *convex S finite-on S f*
assumes $\forall\, x \in S.\ f\ x \geq a$
shows $\forall\, x \in closure\ S.\ f\ x \geq a$
⟨*proof*⟩

lemma *lsc-hull-of-convex-determined*:
fixes $f\ g$ :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f convex-on UNIV g*
assumes *rel-interior* (*domain f*) = *rel-interior* (*domain g*)
assumes $\forall\, x \in$ *rel-interior* (*domain f*). $f\ x = g\ x$
shows *lsc-hull f* = *lsc-hull g*
⟨*proof*⟩

lemma *domain-lsc-hull-between*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
shows *domain f* $\leq$ *domain* (*lsc-hull f*)
$\wedge$ *domain* (*lsc-hull f*) $\leq$ *closure* (*domain f*)
⟨*proof*⟩

lemma *domain-vs-domain-lsc-hull*:
fixes f :: $'a::euclidean\text{-}space \Rightarrow ereal$
assumes *convex-on UNIV f*
shows *rel-interior*(*domain* (*lsc-hull f*)) = *rel-interior*(*domain f*)
$\wedge$ *closure*(*domain* (*lsc-hull f*)) = *closure*(*domain f*)
$\wedge$ *aff-dim*(*domain* (*lsc-hull f*)) = *aff-dim*(*domain f*)
⟨*proof*⟩

lemma *vertical-line-affine*:

fixes *x* :: *'a::euclidean-space*
shows *affine* {(*x*,*m*::*real*)|*m*. *m*:*UNIV*}
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees-onRI*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows ∀ *x*∈*rel-interior* (*domain f*). (*f x* = (*lsc-hull f*) *x*)
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees-outside*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows ∀ *x*. *x* ∉ *closure* (*domain f*) ⟶ (*f x* = (*lsc-hull f*) *x*)
⟨*proof*⟩

lemma *lsc-hull-of-convex-agrees*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
shows ∀ *x*. (*f x* = (*lsc-hull f*) *x*) | *x* : *rel-frontier* (*domain f*)
⟨*proof*⟩

lemma *lsc-hull-of-proper-convex-proper*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f proper f*
shows *proper* (*lsc-hull f*)
⟨*proof*⟩

lemma *lsc-hull-of-proper-convex*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f proper f*
shows *lsc* (*lsc-hull f*) ∧ *proper* (*lsc-hull f*) ∧ *convex-on UNIV* (*lsc-hull f*) ∧
(∀ *x*. (*f x* = (*lsc-hull f*) *x*) | *x* : *rel-frontier* (*domain f*))
⟨*proof*⟩

lemma *affine-no-rel-frontier*:
fixes *S* :: (*'n::euclidean-space*) *set*
assumes *affine S*
shows *rel-frontier S* = {}
⟨*proof*⟩

lemma *convex-with-affine-domain-is-lsc*:

fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *affine* (*domain f*)
shows *lsc f*
⟨*proof*⟩

lemma *convex-finite-is-lsc*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *finite-on UNIV f*
shows *lsc f*
⟨*proof*⟩

lemma *always-eventually-within*:
(∀ *x*∈*S*. *P x*) ⟹ *eventually P* (*at x within S*)
⟨*proof*⟩

lemma *ereal-divide-pos*:
assumes (*a::ereal*)>*0* *b*>*0*
shows *a*/(*ereal b*)>*0*
⟨*proof*⟩

lemma *real-interval-limpt*:
assumes *a*<*b*
shows (*b::real*) *islimpt* {*a*..<*b*}
⟨*proof*⟩

lemma *lsc-hull-of-convex-aux*:
Limsup (*at 1 within* {*0*..<*1*}) (λ*m*. *ereal* ((*1*−*m*)∗*a*+*m*∗*b*)) ≤ *ereal b*
⟨*proof*⟩

lemma *lsc-hull-of-convex*:
fixes *f* :: *'a::euclidean-space* ⇒ *ereal*
assumes *convex-on UNIV f*
assumes *x* : *rel-interior* (*domain f*)
shows ((λ*m*. *f*((*1*−*m*)$*_R$ *x* + *m* $*_R$ *y*)) ⟶ (*lsc-hull f*) *y*) (*at 1 within* {*0*..<*1*})
(**is** (*?g* ⟶ - *y*) -)
⟨*proof*⟩

end