

# Lower Semicontinuous Functions

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## Abstract

We define the notions of lower and upper semicontinuity for functions from a metric space to the extended real line. We prove that a function is both lower and upper semicontinuous if and only if it is continuous. We also give several equivalent characterizations of lower semicontinuity. In particular, we prove that a function is lower semicontinuous if and only if its epigraph is a closed set. Also, we introduce the notion of the lower semicontinuous hull of an arbitrary function and prove its basic properties.

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## 1 Lower semicontinuous functions

```
theory Lower-Semicontinuous
imports HOL-Analysis.Multivariate-Analysis
begin
```

### 1.1 Relative interior in one dimension

```
lemma rel-interior-ereal-semiline:
  fixes a :: ereal
  shows rel-interior {y. a ≤ ereal y} = {y. a < ereal y}
proof (cases a)
  case (real r) then show ?thesis
    using rel-interior-real-semiline[of r]
    by (simp add: atLeast-def greaterThan-def)
next case PInf thus ?thesis using rel-interior-empty by auto
next case MInf thus ?thesis using rel-interior-UNIV by auto
qed
```

**lemma** *closed-ereal-semiline*:  
**fixes**  $a :: \text{ereal}$   
**shows**  $\text{closed } \{y. a \leq \text{ereal } y\}$   
**proof** (*cases a*)  
**case** (*real r*) **then show** *?thesis*  
**using** *closed-real-atLeast unfolding atLeast-def by simp*  
**qed** *auto*

**lemma** *ereal-semiline-unique*:  
**fixes**  $a b :: \text{ereal}$   
**shows**  $\{y. a \leq \text{ereal } y\} = \{y. b \leq \text{ereal } y\} \longleftrightarrow a = b$   
**by** (*metis mem-Collect-eq ereal-le-real order-antisym*)

## 1.2 Lower and upper semicontinuity

**definition**  
 $\text{lsc-at} :: 'a \Rightarrow ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{lsc-at } x0 f \longleftrightarrow (\forall X l. X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow f x0 \leq l)$

**definition**  
 $\text{usc-at} :: 'a \Rightarrow ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{usc-at } x0 f \longleftrightarrow (\forall X l. X \longrightarrow x0 \wedge (f \circ X) \longrightarrow l \longrightarrow l \leq f x0)$

**lemma** *lsc-at-mem*:  
**assumes** *lsc-at x0 f*  
**assumes**  $x \longrightarrow x0$   
**assumes**  $(f \circ x) \longrightarrow A$   
**shows**  $f x0 \leq A$   
**using** *assms lsc-at-def[of x0 f] by blast*

**lemma** *usc-at-mem*:  
**assumes** *usc-at x0 f*  
**assumes**  $x \longrightarrow x0$   
**assumes**  $(f \circ x) \longrightarrow A$   
**shows**  $f x0 \geq A$   
**using** *assms usc-at-def[of x0 f] by blast*

**lemma** *lsc-at-open*:  
**fixes**  $f :: 'a::\text{first-countable-topology} \Rightarrow 'b::\{\text{complete-linorder, linorder-topology}\}$   
**shows**  $\text{lsc-at } x0 f \longleftrightarrow$   
 $(\forall S. \text{open } S \wedge f x0 \in S \longrightarrow (\exists T. \text{open } T \wedge x0 \in T \wedge (\forall x' \in T. f x' \leq f x0 \longrightarrow f x' \in S)))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
**proof** –  
**{ assume  $\sim ?rhs$**   
**from this obtain S where S-def:**  
 $\text{open } S \wedge f x0 : S \wedge (\forall T. (\text{open } T \wedge x0 \in T) \longrightarrow (\exists x' \in T. f x' \leq f x0 \wedge f x' \in S))$

$\notin S$ ) by *metis*  
**define**  $X$  **where**  $X = \{x'. f x' \leq f x0 \wedge f x' \notin S\}$   
**hence**  $x0$  *islimpt*  $X$  **unfolding** *islimpt-def* **using**  $S$ -def **by** *auto*  
**from this obtain**  $x$  **where**  $x$ -def:  $(\forall n. x n \in X) \wedge x \longrightarrow x0$   
**using** *islimpt-sequential*[of  $x0$   $X$ ] **by** *auto*  
**hence not:**  $\sim(f \circ x) \longrightarrow (f x0)$  **unfolding** *lim-explicit* **using**  $X$ -def  $S$ -def **by**  
*auto*  
**from** *compact-complete-linorder*[of  $f \circ x$ ] **obtain**  $l$   $r$  **where**  $r$ -def: *strict-mono*  $r$   
 $\wedge ((f \circ x) \circ r) \longrightarrow l$  **by** *auto*  
**{ assume**  $l : S$  **hence**  $\exists N. \forall n \geq N. f(x(r n)) \in S$   
**using**  $r$ -def *lim-explicit*[of  $f \circ x \circ r$   $l$ ]  $S$ -def **by** *auto*  
**hence** *False* **using**  $x$ -def  $X$ -def **by** *auto*  
**}** **hence**  $l$ -prop:  $l \notin S \wedge l \leq f x0$   
**using**  $r$ -def  $x$ -def  $X$ -def *Lim-bounded*[of  $f \circ x \circ r$ ]  
**by** *auto*  
**{ assume**  $f x0 \leq l$  **hence**  $f x0 = l$  **using**  $l$ -prop **by** *auto*  
**hence** *False* **using**  $l$ -prop  $S$ -def **by** *auto*  
**}**  
**hence**  $\exists x l. x \longrightarrow x0 \wedge (f \circ x) \longrightarrow l \wedge \sim(f x0 \leq l)$   
**apply**(*rule-tac*  $x=x \circ r$  **in**  $exI$ ) **apply**(*rule-tac*  $x=l$  **in**  $exI$ )  
**using**  $r$ -def  $x$ -def **by** (*auto simp add: o-assoc LIMSEQ-subseq-LIMSEQ*)  
**hence**  $\sim ?lhs$  **unfolding** *lsc-at-def* **by** *blast*  
**}**  
**moreover**  
**{ assume**  $?rhs$   
**{ fix**  $x A$  **assume**  $x$ -def:  $x \longrightarrow x0$   $(f \circ x) \longrightarrow A$   
**{ assume**  $A \neq f x0$   
**from this obtain**  $S V$  **where**  $SV$ -def:  $open S \wedge open V \wedge f x0 : S \wedge A : V$   
 $\wedge S$  *Int*  $V = \{\}$   
**using** *hausdorff*[of  $f x0$   $A$ ] **by** *auto*  
**from this obtain**  $T$  **where**  $T$ -def:  $open T \wedge x0 : T \wedge (\forall x' \in T. (f x' \leq f x0$   
 $\longrightarrow f x' \in S))$   
**using**  $\langle ?rhs \rangle$  **by** *metis*  
**from this obtain**  $N1$  **where**  $\forall n \geq N1. x n \in T$  **using**  $x$ -def *lim-explicit*[of  $x$   
 $x0$ ] **by** *auto*  
**hence**  $*$ :  $\forall n \geq N1. (f(x n) \leq f x0 \longrightarrow f(x n) \in S)$  **using**  $T$ -def **by** *auto*  
**from**  $SV$ -def **obtain**  $N2$  **where**  $\forall n \geq N2. f(x n) \in V$   
**using** *lim-explicit*[of  $f \circ x$   $A$ ]  $x$ -def **by** *auto*  
**hence**  $\forall n \geq (\max N1 N2). \neg(f(x n) \leq f x0)$  **using**  $SV$ -def  $*$  **by** *auto*  
**hence**  $\forall n \geq (\max N1 N2). f(x n) \geq f x0$  **by** *auto*  
**hence**  $f x0 \leq A$  **using** *Lim-bounded2*[of  $f \circ x$   $A$   $\max N1 N2$   $f x0$ ]  $x$ -def **by**  
*auto*  
**}** **hence**  $f x0 \leq A$  **by** *auto*  
**}** **hence**  $?lhs$  **unfolding** *lsc-at-def* **by** *blast*  
**}** **ultimately show**  $?thesis$  **by** *blast*  
**qed**

lemma *lsc-at-open-mem*:

**fixes**  $f :: 'a::\text{first-countable-topology} \Rightarrow 'b::\{\text{complete-linorder}, \text{linorder-topology}\}$   
**assumes**  $\text{lsc-at } x0\ f$   
**assumes**  $\text{open } S \wedge f\ x0 : S$   
**obtains**  $T$  **where**  $\text{open } T \wedge x0 \in T \wedge (\forall x' \in T. (f\ x' \leq f\ x0 \longrightarrow f\ x' \in S))$   
**using**  $\text{assms lsc-at-open[of } x0\ f]$  **by**  $\text{blast}$

**lemma**  $\text{lsc-at-MInfty}$ :  
**fixes**  $f :: 'a::\text{topological-space} \Rightarrow \text{ereal}$   
**assumes**  $f\ x0 = -\infty$   
**shows**  $\text{lsc-at } x0\ f$   
**unfolding**  $\text{lsc-at-def}$  **using**  $\text{assms}$  **by**  $\text{auto}$

**lemma**  $\text{lsc-at-PInfty}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $f\ x0 = \infty$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow \text{continuous (at } x0) f$   
**unfolding**  $\text{lsc-at-open continuous-at-open}$  **using**  $\text{assms}$  **by**  $\text{auto}$

**lemma**  $\text{lsc-at-real}$ :  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**assumes**  $|f\ x0| \neq \infty$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall e. e > 0 \longrightarrow (\exists T. \text{open } T \wedge x0 : T \wedge (\forall y \in T. f\ y > f\ x0 - e)))$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
**proof** –  
**obtain**  $m$  **where**  $m\text{-def}: f\ x0 = \text{ereal } m$  **using**  $\text{assms}$  **by**  $(\text{cases } f\ x0)$   $\text{auto}$   
**{ assume**  $\text{lsc}: \text{lsc-at } x0\ f$   
**{ fix**  $e$  **assume**  $e\text{-def}: (e :: \text{ereal}) > 0$   
**hence**  $*$ :  $f\ x0 : \{f\ x0 - e <..< f\ x0 + e\}$  **using**  $\text{assms}$   $\text{ereal-between}$  **by**  $\text{auto}$   
**from this obtain**  $T$  **where**  $T\text{-def}: \text{open } T \wedge x0 : T \wedge (\forall x' \in T. f\ x' \leq f\ x0 \longrightarrow f\ x' \in \{f\ x0 - e <..< f\ x0 + e\})$   
**apply**  $(\text{subst lsc-at-open-mem[of } x0\ f\ \{f\ x0 - e <..< f\ x0 + e\}])$  **using**  $\text{lsc}$   
 $e\text{-def}$  **by**  $\text{auto}$   
**{ fix**  $y$  **assume**  $y:T$   
**{ assume**  $f\ y \leq f\ x0$  **hence**  $f\ y > f\ x0 - e$  **using**  $T\text{-def } \langle y:T \rangle$  **by**  $\text{auto}$  }  
**moreover**  
**{ assume**  $f\ y > f\ x0$  **hence**  $\dots > f\ x0 - e$  **using**  $*$  **by**  $\text{auto}$  }  
**ultimately have**  $f\ y > f\ x0 - e$  **using**  $\text{not-le}$  **by**  $\text{blast}$   
**} hence**  $\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. f\ y > f\ x0 - e)$  **using**  $T\text{-def}$  **by**  
 $\text{auto}$   
**} hence ?rhs by auto**  
**}**  
**moreover**  
**{ assume**  $?rhs$   
**{ fix**  $S$  **assume**  $S\text{-def}: \text{open } S \wedge f\ x0 : S$   
**from this obtain**  $e$  **where**  $e\text{-def}: e > 0 \wedge \{f\ x0 - e <..< f\ x0 + e\} \leq S$

**apply** (*subst ereal-open-cont-interval*[of  $S f x0$ ]) **using** *assms* **by** *auto*  
**from this obtain**  $T$  **where**  $T$ -def:  $\text{open } T \wedge x0 : T \wedge (\forall y \in T. f y > f x0 - e)$   
**using**  $\langle ?rhs \rangle$ [*rule-format, of e*] **by** *auto*  
**{ fix**  $y$  **assume**  $y : T f y \leq f x0$  **hence**  $f y < f x0 + e$   
**using** *assms e-def ereal-between*[of  $f x0 e$ ] **by** *auto*  
**hence**  $f y \in S$  **using** *e-def T-def*  $\langle y \in T \rangle$  **by** *auto*  
**} hence**  $\exists T. \text{open } T \wedge x0 : T \wedge (\forall y \in T. (f y \leq f x0 \longrightarrow f y \in S))$  **using**  
 $T$ -def **by** *auto*  
**} hence** *lsc-at*  $x0 f$  **using** *lsc-at-open* **by** *auto*  
**} ultimately show** *?thesis* **by** *auto*  
**qed**

**lemma** *lsc-at-ereal*:

**fixes**  $f :: 'a :: \text{metric-space} \Rightarrow \text{ereal}$

**shows**  $\text{lsc-at } x0 f \longleftrightarrow (\forall C < f(x0)). \exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. f y > C)$

(**is** *?lhs*  $\longleftrightarrow$  *?rhs*)

**proof** –

**{ assume**  $f x0 = -\infty$  **hence** *?thesis* **using** *lsc-at-MInfty* **by** *auto* **}**

**moreover**

**{ assume** *pinf*:  $f x0 = \infty$

**{ assume** *lsc*: *lsc-at*  $x0 f$

**{ fix**  $C$  **assume**  $C < f x0$

**hence**  $\text{open } \{C <..\} \wedge f x0 : \{C <..\}$  **by** *auto*

**from this obtain**  $T$  **where**  $T$ -def:  $\text{open } T \wedge x0 \in T \wedge (\forall y \in T. f y \in \{C <..\})$

**using** *pinf lsc lsc-at-PInfty*[of  $f x0$ ] **unfolding** *continuous-at-open* **by** *metis*

**hence**  $\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. C < f y)$  **by** *auto*

**} hence** *?rhs* **by** *auto*

**}**

**moreover**

**{ assume** *?rhs*

**{ fix**  $S$  **assume**  $S$ -def:  $\text{open } S \wedge f x0 : S$

**then obtain**  $C$  **where**  $C$ -def:  $\text{ereal } C < f x0 \wedge \{\text{ereal } C <..\} \leq S$  **using** *pinf*

*open-PInfty* **by** *auto*

**then obtain**  $T$  **where**  $T$ -def:  $\text{open } T \wedge x0 : T \wedge (\forall y \in T. f y \in S)$

**using**  $\langle ?rhs \rangle$ [*rule-format, of ereal C*] **by** *auto*

**hence**  $\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. (f y \leq f x0 \longrightarrow f y \in S))$  **using**

$T$ -def **by** *auto*

**} hence** *lsc-at*  $x0 f$  **using** *lsc-at-open* **by** *auto*

**} ultimately have** *?thesis* **by** *blast*

**}**

**moreover**

**{ assume** *fin*:  $f x0 \neq -\infty f x0 \neq \infty$

**{ assume** *lsc*: *lsc-at*  $x0 f$

**{ fix**  $C$  **assume**  $C < f x0$

**hence**  $f x0 - C > 0$  **using** *fin ereal-less-minus-iff* **by** *auto*

**from this obtain**  $T$  **where**  $T$ -def:  $\text{open } T \wedge x0 \in T \wedge (\forall y \in T. f x0 - (f x0 - C) < f y)$

```

    using lsc-at-real[of f x0] lsc fin by auto
    moreover have f x0 - (f x0 - C) = C using fin apply (cases f x0, cases
C) by auto
    ultimately have  $\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. C < f y)$  by auto
  } hence ?rhs by auto
}
moreover
{ assume ?rhs
  { fix e :: ereal assume e > 0
    hence f x0 - e < f x0 using fin apply (cases f x0, cases e) by auto
    hence  $\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. f x0 - e < f y)$  using fin <?rhs> by
auto
  } hence lsc-at x0 f using lsc-at-real[of f x0] fin by auto
} ultimately have ?thesis by blast
} ultimately show ?thesis by blast
qed

```

lemma *lsc-at-ball*:

fixes  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

shows  $\text{lsc-at } x0 f \iff (\forall C < f(x0). \exists d > 0. \forall y \in (\text{ball } x0 d). C < f(y))$

(is ?lhs  $\iff$  ?rhs)

proof

assume lsc: *lsc-at* x0 f

show ?rhs

proof (intro strip)

fix C :: ereal assume C < f x0

then obtain T where open T  $\wedge$  x0 : T  $\wedge$  ( $\forall y \in T. C < f y$ )

using lsc *lsc-at-ereal*[of x0 f] by auto

then show  $\exists d. d > 0 \wedge (\forall y \in (\text{ball } x0 d). C < f y)$

by (force simp add: *open-contains-ball*)

qed

next

assume ?rhs

{ fix C :: ereal assume C < f x0

then obtain d where  $d > 0 \wedge (\forall y \in (\text{ball } x0 d). C < f y)$  using <?rhs> by

auto

hence  $\exists T. \text{open } T \wedge x0 \in T \wedge (\forall y \in T. C < f y)$

by (meson *Elementary-Metric-Spaces.open-ball centre-in-ball*)

} then show ?lhs using *lsc-at-ereal*[of x0 f] by auto

qed

lemma *lsc-at-delta*:

fixes  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

shows  $\text{lsc-at } x0 f \iff (\forall C < f(x0). \exists d > 0. \forall y. \text{dist } x0 y < d \longrightarrow C < f y)$

(is ?lhs  $\iff$  ?rhs)

proof -

**have**  $?rhs \longleftrightarrow (\forall C < f(x0). \exists d > 0. \forall y \in (ball\ x0\ d). C < f\ y)$  **unfolding** *ball-def*  
**by** *auto*  
**thus**  $?thesis$  **using** *lst-at-ball*[of  $x0\ f$ ] **by** *auto*  
**qed**

**lemma** *lsc-liminf-at*:  
**fixes**  $f :: 'a::metric-space \Rightarrow ereal$   
**shows**  $lsc-at\ x0\ f \longleftrightarrow f\ x0 \leq Liminf\ (at\ x0)\ f$   
**unfolding** *lst-at-ball le-Liminf-iff eventually-at*  
**by** (*intro arg-cong*[where  $f=All$ ] *imp-cong refl ext ex-cong*)  
*(auto simp: dist-commute zero-less-dist-iff)*

**lemma** *lsc-liminf-at-eq*:  
**fixes**  $f :: 'a::metric-space \Rightarrow ereal$   
**shows**  $lsc-at\ x0\ f \longleftrightarrow (f\ x0 = min\ (f\ x0)\ (Liminf\ (at\ x0)\ f))$   
**by** (*metis inf-ereal-def le-iff-inf lsc-liminf-at*)

**lemma** *lsc-imp-liminf*:  
**fixes**  $f :: 'a::metric-space \Rightarrow ereal$   
**assumes** *lsc-at*  $x0\ f$   
**assumes**  $x \longrightarrow x0$   
**shows**  $f\ x0 \leq liminf\ (f \circ x)$   
**proof** (*cases*  $f\ x0$ )  
**case** *PInf* **then show**  $?thesis$  **using** *assms lsc-at-PInfty*[of  $f\ x0$ ] *lim-imp-Liminf*[of  
 $- f \circ x$ ]  
*continuous-at-sequentially*[of  $x0\ f$ ] **by** *auto*  
**next**  
**case** (*real*  $r$ )  
**{** **fix**  $e$  **assume** *e-def*: ( $e :: ereal$ ) $>0$   
**from** *this* **obtain**  $T$  **where** *T-def*:  $open\ T \wedge x0 : T \wedge (\forall y \in T. f\ y > f\ x0 -$   
 $e)$   
**using** *lsc-at-real*[of  $f\ x0$ ] *real assms* **by** *auto*  
**from** *this* **obtain**  $N$  **where** *N-def*:  $\forall n \geq N. x\ n \in T$   
**apply** (*subst tendsto-obtains-N*[of  $x\ x0\ T$ ]) **using** *assms* **by** *auto*  
**hence**  $\forall n \geq N. f\ x0 - e < (f \circ x)\ n$  **using** *T-def* **by** *auto*  
**hence**  $liminf\ (f \circ x) \geq f\ x0 - e$  **by** (*intro Liminf-bounded*) (*auto simp:*  
*eventually-sequentially intro!: exI*[of  $- N$ ])  
**hence**  $f\ x0 \leq liminf\ (f \circ x) + e$  **apply** (*cases*  $e$ ) **unfolding** *ereal-minus-le-iff*  
**by** *auto*  
**}**  
**then show**  $?thesis$   
**using** *ereal-le-epsilon* **by** *blast*  
**qed** *auto*

**lemma** *lsc-liminf*:  
**fixes**  $f :: 'a::metric-space \Rightarrow ereal$

shows  $\text{lsc-at } x0 \ f \longleftrightarrow (\forall x. x \longrightarrow x0 \longrightarrow f \ x0 \leq \text{liminf } (f \circ x))$   
(is  $?lhs \longleftrightarrow ?rhs$ )  
**proof**  
assume  $?rhs$   
{ **fix**  $x \ A$  **assume**  $x\text{-def}: x \longrightarrow x0 \ (f \circ x) \longrightarrow A$   
**hence**  $f \ x0 \leq A$  **using**  $\langle ?rhs \rangle \text{lim-imp-Liminf[of sequentially]}$  **by** *auto*  
} **thus**  $?lhs$  **unfolding**  $\text{lsc-at-def}$  **by** *blast*  
**qed** (use  $\text{lsc-imp-liminf}$  in *auto*)

**lemma**  $\text{lsc-sequentially}$ :

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

shows  $\text{lsc-at } x0 \ f \longleftrightarrow (\forall x \ c. x \longrightarrow x0 \wedge (\forall n. f(x \ n) \leq c) \longrightarrow f(x0) \leq c)$   
(is  $?lhs \longleftrightarrow ?rhs$ )

**proof**

assume  $?rhs$

{ **fix**  $x \ l$  **assume**  $x \longrightarrow x0 \ (f \circ x) \longrightarrow l$   
{ **assume**  $l = \infty$  **hence**  $f \ x0 \leq l$  **by** *auto* }

**moreover**

{ **assume**  $l = -\infty$   
{ **fix**  $B :: \text{real}$  **obtain**  $N$  **where**  $N\text{-def}: \forall n \geq N. f(x \ n) \leq \text{ereal } B$   
**using**  $\text{Lim-MInfty[of } f \circ x]$   $\langle (f \circ x) \longrightarrow l \rangle \langle l = -\infty \rangle$  **by** *auto*  
**define**  $g$  **where**  $g \ n = (\text{if } n \geq N \text{ then } x \ n \ \text{else } x \ N)$  **for**  $n$   
**hence**  $g \longrightarrow x0$   
**by** (*intro filterlim-cong[THEN iffD1, OF refl refl - \langle x \longrightarrow x0 \rangle]*)  
(*auto simp: eventually-sequentially*)  
**moreover** **have**  $\forall n. f(g \ n) \leq \text{ereal } B$  **using**  $g\text{-def } N\text{-def}$  **by** *auto*  
**ultimately** **have**  $f \ x0 \leq \text{ereal } B$  **using**  $\langle ?rhs \rangle$  **by** *auto*  
} **hence**  $f \ x0 = -\infty$  **using**  $\text{ereal-bot}$  **by** *auto*  
**hence**  $f \ x0 \leq l$  **by** *auto* }

**moreover**

{ **assume**  $\text{fin}: |l| \neq \infty$   
{ **fix**  $e$  **assume**  $e\text{-def}: (e :: \text{ereal}) > 0$   
**from** *this* **obtain**  $N$  **where**  $N\text{-def}: \forall n \geq N. f(x \ n) \in \{l - e <..< l + e\}$   
**apply** (*subst tendsto-obtains-N[of } f \circ x \ l \ \{l - e <..< l + e\}]*)  
**using**  $\text{fin } e\text{-def } \text{ereal-between}$   $\langle (f \circ x) \longrightarrow l \rangle$  **by** *auto*  
**define**  $g$  **where**  $g \ n = (\text{if } n \geq N \text{ then } x \ n \ \text{else } x \ N)$  **for**  $n$   
**hence**  $g \longrightarrow x0$   
**by** (*intro filterlim-cong[THEN iffD1, OF refl refl - \langle x \longrightarrow x0 \rangle]*)  
(*auto simp: eventually-sequentially*)  
**moreover** **have**  $\forall n. f(g \ n) \leq l + e$  **using**  $g\text{-def } N\text{-def}$  **by** *auto*  
**ultimately** **have**  $f \ x0 \leq l + e$  **using**  $\langle ?rhs \rangle$  **by** *auto*  
} **hence**  $f \ x0 \leq l$  **using**  $\text{ereal-le-epsilon}$  **by** *auto*  
} **ultimately** **have**  $f \ x0 \leq l$  **by** *blast*  
} **then** **show**  $?lhs$  **unfolding**  $\text{lsc-at-def}$  **by** *auto*

**next**

assume  $\text{lsc}: ?lhs$

{ **fix**  $x \ c$  **assume**  $xc\text{-def}: x \longrightarrow x0 \wedge (\forall n. f(x \ n) \leq c)$   
**hence**  $\text{liminf } (f \circ x) \leq c$



**using** *Limsup-bounded*[**where**  $F = \text{sequentially}$  **and**  $X = f \circ x$  **and**  $C = c$ ]  
*Liminf-le-Limsup*[*of sequentially f o x*]  
**by auto**  
**hence**  $f\ x0 \leq c$  **using** *lsc xc-def lsc-imp-liminf*[*of x0 f x*] **by auto**  
**}** **thus** *?rhs* **by auto**  
**qed**

**lemma** *lsc-sequentially-gen*:

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{lsc-at } x0\ f \longleftrightarrow (\forall x\ c\ c0. x \longrightarrow x0 \wedge c \longrightarrow c0 \wedge (\forall n. f(x\ n) \leq c\ n) \longrightarrow f(x0) \leq c0)$   
**(is** *?lhs*  $\longleftrightarrow$  *?rhs***)**

**proof**

**assume** *?rhs*

**{** **fix**  $x\ c0$  **assume**  $a: x \longrightarrow x0 \wedge (\forall n. f(x\ n) \leq c0)$

**define**  $c$  **where**  $c = (\lambda n::\text{nat}. c0)$

**hence**  $c \longrightarrow c0$  **by auto**

**hence**  $f(x0) \leq c0$  **using**  $\langle ?rhs \rangle$ [*rule-format, of x c c0*] **using** *a c-def* **by auto**

**}** **then show** *?lhs* **using** *lsc-sequentially*[*of x0 f*] **by auto**

**next**

**assume** *lsc: lsc-at x0 f*

**{** **fix**  $x\ c\ c0$  **assume**  $xc\text{-def}: x \longrightarrow x0 \wedge c \longrightarrow c0 \wedge (\forall n. f(x\ n) \leq c\ n)$

**hence**  $\text{liminf}(f \circ x) \leq c0$

**using** *Liminf-mono*[*of f o x c sequentially*] *lim-imp-Liminf*[*of sequentially*] **by auto**

**hence**  $f\ x0 \leq c0$  **using** *lsc xc-def lsc-imp-liminf*[*of x0 f x*] **by auto**

**}** **then show** *?rhs* **by auto**

**qed**

**lemma** *lsc-sequentially-mem*:

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

**assumes** *lsc-at x0 f*

**assumes**  $x \longrightarrow x0\ c \longrightarrow c0$

**assumes**  $\forall n. f(x\ n) \leq c\ n$

**shows**  $f(x0) \leq c0$

**using** *lsc-sequentially-gen*[*of x0 f*] *assms* **by auto**

**lemma** *lsc-uminus*:

**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$

**shows**  $\text{lsc-at } x0\ (\lambda x. -f\ x) \longleftrightarrow \text{usc-at } x0\ f$

**proof**

**assume** *lsc: lsc-at x0 (λx. -f x)*

**{** **fix**  $x\ A$  **assume**  $x\text{-def}: x \longrightarrow x0\ (f \circ x) \longrightarrow A$

**hence**  $(\lambda i. -f(x\ i)) \longrightarrow -A$  **using** *tendsto-uminus-ereal*[*of f o x A*] **by auto**

**auto**

**hence**  $((\lambda x. -f\ x) \circ x) \longrightarrow -A$  **unfolding** *o-def* **by auto**

hence  $-f x0 \leq -A$  **apply** (*subst lsc-at-mem*[of  $x0$   $(\lambda x. -f x)$   $x$ ]) **using** *lsc*  
*x-def* **by** *auto*

hence  $f x0 \geq A$  **by** *auto*

} **then show** *usc-at x0 f* **unfolding** *usc-at-def* **by** *auto*

**next**

**assume** *usc: usc-at x0 f*

{ **fix**  $x A$  **assume** *x-def*:  $x \longrightarrow x0$   $((\lambda x. -f x) \circ x) \longrightarrow A$

hence  $(\lambda i. -f (x i)) \longrightarrow A$  **unfolding** *o-def* **by** *auto*

hence  $(\lambda i. f (x i)) \longrightarrow -A$  **using** *tendsto-uminus-ereal*[of  $(\lambda i. -f (x i))$   
*A*] **by** *auto*

hence  $(f \circ x) \longrightarrow -A$  **unfolding** *o-def* **by** *auto*

hence  $f x0 \geq -A$  **apply** (*subst usc-at-mem*[of  $x0$   $f$   $x$ ]) **using** *usc x-def* **by**  
*auto*

hence  $-f x0 \leq A$  **by** (*auto simp: ereal-uminus-le-reorder*)

} **then show** *lsc-at x0*  $(\lambda x. -f x)$  **unfolding** *lsc-at-def* **by** *auto*

**qed**

**lemma** *usc-limsup*:

**fixes**  $f :: 'a::metric-space \Rightarrow ereal$

**shows** *usc-at x0 f*  $\longleftrightarrow (\forall x. x \longrightarrow x0 \longrightarrow f x0 \geq \text{limsup } (f \circ x))$

(**is** *?lhs*  $\longleftrightarrow$  *?rhs*)

**proof** –

**have** *usc-at x0 f*  $\longleftrightarrow (\forall x. x \longrightarrow x0 \longrightarrow -f x0 \leq \text{liminf } ((\lambda x. -f x) \circ x))$

**using** *lsc-uminus*[of  $x0$   $f$ ] *lsc-liminf*[of  $x0$   $(\lambda x. -f x)$ ] **by** *auto*

**moreover**

{ **fix**  $x$  **assume**  $x \longrightarrow x0$

**have**  $(-f x0 \leq -\text{limsup } (f \circ x)) \longleftrightarrow (-f x0 \leq \text{liminf } ((\lambda x. -f x) \circ x))$

**using** *ereal-Liminf-uminus*[of  $-f \circ x$ ] **unfolding** *o-def* **by** *auto*

hence  $(f x0 \geq \text{limsup } (f \circ x)) \longleftrightarrow (-f x0 \leq \text{liminf } ((\lambda x. -f x) \circ x))$

**by** *auto*

} **ultimately show** *?thesis* **by** *auto*

**qed**

**lemma** *usc-imp-limsup*:

**fixes**  $f :: 'a::metric-space \Rightarrow ereal$

**assumes** *usc-at x0 f*

**assumes**  $x \longrightarrow x0$

**shows**  $f x0 \geq \text{limsup } (f \circ x)$

**using** *assms usc-limsup*[of  $x0$   $f$ ] **by** *auto*

**lemma** *usc-limsup-at*:

**fixes**  $f :: 'a::metric-space \Rightarrow ereal$

**shows** *usc-at x0 f*  $\longleftrightarrow f x0 \geq \text{Limsup } (at x0) f$

**proof** –

**have** *usc-at x0 f*  $\longleftrightarrow$  *lsc-at x0*  $(\lambda x. -(f x))$  **by** (*metis lsc-uminus*)

also have ...  $\longleftrightarrow -(f\ x0) \leq \text{Liminf (at } x0) (\lambda x. -(f\ x))$  **by** (*metis lsc-liminf-at*)  
also have ...  $\longleftrightarrow -(f\ x0) \leq -(\text{Limsup (at } x0) f)$  **by** (*metis ereal-Liminf-uminus*)  
**finally show** *?thesis* **by** *auto*  
**qed**

**lemma** *continuous-iff-lsc-usc*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{continuous (at } x0) f \longleftrightarrow (\text{lsc-at } x0 f) \wedge (\text{usc-at } x0 f)$   
**proof** –  
{ **assume**  $a: \text{continuous (at } x0) f$   
  { **fix**  $x$  **assume**  $x \longrightarrow x0$   
    **hence**  $(f \circ x) \longrightarrow f\ x0$  **using**  $a$  **continuous-imp-tendsto**[of  $x0\ f\ x$ ] **by** *auto*  
    **hence**  $\text{liminf (f } \circ x) = f\ x0 \wedge \text{limsup (f } \circ x) = f\ x0$   
      **using**  $\text{lim-imp-Liminf}$ [of *sequentially*]  $\text{lim-imp-Limsup}$ [of *sequentially*] **by**  
*auto*  
  } **hence**  $\text{lsc-at } x0 f \wedge \text{usc-at } x0 f$  **unfolding**  $\text{lsc-liminf usc-limsup}$  **by** *auto*  
} }  
**moreover**  
{ **assume**  $a: (\text{lsc-at } x0 f) \wedge (\text{usc-at } x0 f)$   
  { **fix**  $x$  **assume**  $x \longrightarrow x0$   
    **hence**  $\text{limsup (f } \circ x) \leq f\ x0$  **using**  $a$  **unfolding**  $\text{usc-limsup}$  **by** *auto*  
    **moreover have** ...  $\leq \text{liminf (f } \circ x)$  **using**  $a$   $\langle x \longrightarrow x0 \rangle$  **unfolding**  
*lsc-liminf* **by** *auto*  
    **ultimately have**  $\text{limsup (f } \circ x) = f\ x0 \wedge \text{liminf (f } \circ x) = f\ x0$   
      **using**  $\text{Liminf-le-Limsup}$ [of *sequentially*  $f \circ x$ ] **by** *auto*  
    **hence**  $(f \circ x) \longrightarrow f\ x0$  **using**  $\text{Liminf-eq-Limsup}$ [of *sequentially*]  
      **by** (*simp add: tendsto-iff-Liminf-eq-Limsup*)  
  } **hence**  $\text{continuous (at } x0) f$   
    **using**  $\text{continuous-at-sequentially}$ [of  $x0\ f$ ] **by** *auto*  
} **ultimately show** *?thesis* **by** *blast*  
**qed**

**lemma** *continuous-lsc-compose*:  
**assumes**  $\text{lsc-at (g } x0) f$   $\text{continuous (at } x0) g$   
**shows**  $\text{lsc-at } x0 (f \circ g)$   
**proof** –  
{ **fix**  $x\ L$  **assume**  $x \longrightarrow x0$   $(f \circ g \circ x) \longrightarrow L$   
  **hence**  $f(g\ x0) \leq L$  **apply** (*subst lsc-at-mem*[of  $g\ x0\ f\ g \circ x\ L$ ])  
    **using**  $\text{assms continuous-imp-tendsto}$ [of  $x0\ g\ x$ ] **unfolding** *o-def* **by** *auto*  
} **from this show** *?thesis* **unfolding**  $\text{lsc-at-def}$  **by** *auto*  
**qed**

**lemma** *continuous-isCont*:  
 $\text{continuous (at } x0) f \longleftrightarrow \text{isCont } f\ x0$   
**by** (*metis continuous-at isCont-def*)

**lemma** *isCont-iff-lsc-usc*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{isCont } f \ x0 \longleftrightarrow (\text{lsc-at } x0 \ f) \wedge (\text{usc-at } x0 \ f)$   
**by** (*metis continuous-iff-lsc-usc continuous-isCont*)

**definition**  
 $\text{lsc} :: ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{lsc } f \longleftrightarrow (\forall x. \text{lsc-at } x \ f)$

**definition**  
 $\text{usc} :: ('a::\text{topological-space} \Rightarrow 'b::\text{order-topology}) \Rightarrow \text{bool}$  **where**  
 $\text{usc } f \longleftrightarrow (\forall x. \text{usc-at } x \ f)$

**lemma** *continuous-UNIV-iff-lsc-usc*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $(\forall x. \text{continuous } (\text{at } x) \ f) \longleftrightarrow (\text{lsc } f) \wedge (\text{usc } f)$   
**by** (*metis continuous-iff-lsc-usc lsc-def usc-def*)

### 1.3 Epigraphs

**definition** *Epigraph*  $S \ f ::- \Rightarrow \text{ereal} = \{xy. \text{fst } xy : S \wedge f \ (\text{fst } xy) \leq \text{ereal}(\text{snd } xy)\}$

**lemma** *mem-Epigraph*:  $(x, y) \in \text{Epigraph } S \ f \longleftrightarrow x \in S \wedge f \ x \leq \text{ereal } y$  **unfolding**  
*Epigraph-def* **by** *auto*

**lemma** *ereal-closed-levels*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $(\forall y. \text{closed } \{x. f(x) \leq y\}) \longleftrightarrow (\forall r. \text{closed } \{x. f(x) \leq \text{ereal } r\})$   
**(is ?lhs  $\longleftrightarrow$  ?rhs)**  
**proof** –  
**{ assume ?rhs**  
**{ fix } y :: eréal**  
**{ assume }  $y \neq \infty \wedge y \neq -\infty$  hence  $\text{closed } \{x. f(x) \leq y\}$  using  $\langle ?rhs \rangle$  by (cases**  
*y) auto }*  
**moreover**  
**{ assume }  $y = \infty$  hence  $\text{closed } \{x. f(x) \leq y\}$  by *auto* }**  
**moreover**  
**{ assume }  $y = -\infty$**   
**hence }  $\{x. f(x) \leq y\} = \text{Inter } \{\{x. f(x) \leq \text{ereal } r\} \mid r. r : \text{UNIV}\}$  using *ereal-bot***  
**by } *auto***  
**hence }  $\text{closed } \{x. f(x) \leq y\}$  using *closed-Inter*  $\langle ?rhs \rangle$  by *auto***  
**} ultimately have  $\text{closed } \{x. f(x) \leq y\}$  by *blast***  
**} hence ?lhs by *auto***

} from this show ?thesis by auto  
qed

lemma lsc-iff:

fixes f :: 'a::metric-space  $\Rightarrow$  ereal

shows (lsc f  $\longleftrightarrow$  ( $\forall y. \text{closed } \{x. f(x) \leq y\}$ ))  $\wedge$  (lsc f  $\longleftrightarrow$  closed (Epigraph UNIV f))

proof –

{ assume lsc f

{ fix z z0 assume a:  $\forall n. z n \in (\text{Epigraph UNIV } f) \wedge z \longrightarrow z0$

{ fix n have z n : (Epigraph UNIV f) using a by auto

hence f (fst (z n))  $\leq$  ereal(snd (z n)) using a unfolding Epigraph-def by

auto

hence  $\exists xn cn. z n = (xn, cn) \wedge f(xn) \leq$  ereal cn

apply (rule-tac x=fst (z n) in exI) apply (rule-tac x=snd (z n) in exI)

by auto

} from this obtain x c where xc-def:  $\forall n. z n = (x n, c n) \wedge f(x n) \leq$  ereal (c

n) by metis

from this a have  $\exists x0 c0. z0 = (x0, c0) \wedge x \longrightarrow x0 \wedge c \longrightarrow c0$

apply (rule-tac x=fst z0 in exI) apply (rule-tac x=snd z0 in exI)

using tendsto-fst[of z z0] tendsto-snd[of z z0] by auto

from this obtain x0 c0 where xc0-def:  $z0 = (x0, c0) \wedge x \longrightarrow x0 \wedge c \longrightarrow c0$  by auto

hence f(x0)  $\leq$  ereal c0 apply (subst lsc-sequentially-mem[of x0 f x ereal  $\circ$  c ereal c0])

using <lsc f> xc-def unfolding lsc-def unfolding o-def by auto

hence z0 : (Epigraph UNIV f) unfolding Epigraph-def using xc0-def by auto

} hence closed (Epigraph UNIV f) by (simp add: closed-sequential-limits)

}

moreover

{ assume closed (Epigraph UNIV f)

hence \*:  $\forall x l. (\forall n. f (\text{fst } (x n)) \leq \text{ereal}(\text{snd } (x n))) \wedge x \longrightarrow l \longrightarrow$

f (fst l)  $\leq$  ereal(snd l) unfolding Epigraph-def closed-sequential-limits by auto

{ fix r :: real

{ fix z z0 assume a:  $\forall n. f(z n) \leq$  ereal r  $\wedge z \longrightarrow z0$

hence f(z0)  $\leq$  ereal r using \*[rule-format, of ( $\lambda n. (z n, r)$ ) (z0, r)]

tendsto-Pair[of z z0] by auto

} hence closed {x. f(x)  $\leq$  ereal r} by (simp add: closed-sequential-limits)

} hence  $\forall y. \text{closed } \{x. f(x) \leq y\}$  using ereal-closed-levels by auto

}

moreover

{ assume a:  $\forall y. \text{closed } \{x. f(x) \leq y\}$

{ fix x0

{ fix x l assume x  $\longrightarrow$  x0 (f  $\circ$  x)  $\longrightarrow$  l

{ assume l =  $\infty$  hence f x0  $\leq$  l by auto }

moreover

{ assume mi: l =  $-\infty$

{ fix B :: real

```

obtain  $N$  where  $N$ -def:  $\forall n \geq N. f(x\ n) \leq_{ereal} B$ 
  using  $mi \langle f \circ x \rangle \longrightarrow l$   $Lim$ -MInfty[ $of\ f \circ x$ ] by auto
  { fix  $d$  assume  $(d :: real) > 0$ 
    from this obtain  $N1$  where  $N1$ -def:  $\forall n \geq N1. dist\ (x\ n)\ x0 < d$ 
      using  $\langle x \longrightarrow x0 \rangle$  unfolding lim-sequentially by auto
      hence  $\exists y. dist\ y\ x0 < d \wedge y : \{x. f(x) \leq_{ereal} B\}$ 
      apply (rule-tac  $x=x$  (max  $N\ N1$ ) in  $exI$ ) using  $N$ -def by auto
    }
  hence  $x0 : closure\ \{x. f(x) \leq_{ereal} B\}$  apply (subst closure-approachable)
by auto
  hence  $f\ x0 \leq_{ereal} B$  using  $a$  by auto
} hence  $f\ x0 \leq l$  using ereal-bot[ $of\ f\ x0$ ] by auto
}
moreover
{ assume  $fin: |l| \neq \infty$ 
  { fix  $e$  assume  $e$ -def:  $(e :: ereal) > 0$ 
    from this obtain  $N$  where  $N$ -def:  $\forall n \geq N. f(x\ n) : \{l - e <.. < l + e\}$ 
      apply (subst tendsto-obtains-N[ $of\ f \circ x\ l\ \{l - e <.. < l + e\}$ ])
      using  $fin$   $e$ -def ereal-between  $\langle f \circ x \rangle \longrightarrow l$  by auto
      hence  $*$ :  $\forall n \geq N. x\ n : \{x. f(x) \leq l + e\}$  using  $N$ -def by auto
      { fix  $d$  assume  $(d :: real) > 0$ 
        from this obtain  $N1$  where  $N1$ -def:  $\forall n \geq N1. dist\ (x\ n)\ x0 < d$ 
          using  $\langle x \longrightarrow x0 \rangle$  unfolding lim-sequentially by auto
          hence  $\exists y. dist\ y\ x0 < d \wedge y : \{x. f(x) \leq l + e\}$ 
          apply (rule-tac  $x=x$  (max  $N\ N1$ ) in  $exI$ ) using  $*$  by auto
        }
      hence  $x0 : closure\ \{x. f(x) \leq l + e\}$  apply (subst closure-approachable) by
auto
      hence  $f\ x0 \leq l + e$  using  $a$  by auto
    } hence  $f\ x0 \leq l$  using ereal-le-epsilon by auto
  } ultimately have  $f\ x0 \leq l$  by blast
} hence lsc-at  $x0\ f$  unfolding lsc-at-def by auto
} hence lsc  $f$  unfolding lsc-def by auto
}
ultimately show ?thesis by auto
qed

```

**definition** *lsc-hull* ::  $(a :: metric-space \Rightarrow ereal) \Rightarrow (a :: metric-space \Rightarrow ereal)$  **where**  
*lsc-hull*  $f = (SOME\ g. Epigraph\ UNIV\ g = closure(Epigraph\ UNIV\ f))$

**lemma** *epigraph-mono*:

```

fixes  $f :: a :: metric-space \Rightarrow ereal$ 
shows  $(x, y) : Epigraph\ UNIV\ f \wedge y \leq z \longrightarrow (x, z) : Epigraph\ UNIV\ f$ 
unfolding Epigraph-def apply auto
using ereal-less-eq(3)[ $of\ y\ z$ ] order-trans-rules(23) by blast

```

**lemma** *closed-epigraph-lines*:  
**fixes**  $S :: ('a::\text{metric-space} * 'b::\text{metric-space}) \text{ set}$   
**assumes**  $\text{closed } S$   
**shows**  $\text{closed } \{z. (x, z) : S\}$   
**proof** –  
{ **fix**  $z$  **assume**  $z\text{-def}: z : \text{closure } \{z. (x, z) : S\}$   
{ **fix**  $e :: \text{real}$  **assume**  $e > 0$   
**from**  $\text{this}$  **obtain**  $y$  **where**  $y\text{-def}: (x, y) : S \wedge \text{dist } y \ z < e$   
**using**  $\text{closure-approachable}[\text{of } z \ \{z. (x, z) : S\}] \ z\text{-def}$  **by**  $\text{auto}$   
**moreover** **have**  $\text{dist } (x, y) \ (x, z) = \text{dist } y \ z$  **unfolding**  $\text{dist-prod-def}$  **by**  $\text{auto}$   
**ultimately** **have**  $\exists s. s \in S \wedge \text{dist } s \ (x, z) < e$  **apply**( $\text{rule-tac } x=(x, y)$  **in**  $\text{exI}$ )  
**by**  $\text{auto}$   
} **hence**  $(x, z) : S$  **using**  $\text{closed-approachable}[\text{of } S \ (x, z)] \ \text{assms}$  **by**  $\text{auto}$   
} **hence**  $\text{closure } \{z. (x, z) : S\} \leq \{z. (x, z) : S\}$  **by**  $\text{blast}$   
**from**  $\text{this}$  **show**  $?thesis$  **using**  $\text{closure-subset-eq}$  **by**  $\text{auto}$   
**qed**

**lemma** *mono-epigraph*:  
**fixes**  $S :: ('a::\text{metric-space} * \text{real}) \text{ set}$   
**assumes**  $\text{mono}: \forall x \ y \ z. (x, y) : S \wedge y \leq z \longrightarrow (x, z) : S$   
**assumes**  $\text{closed } S$   
**shows**  $\exists g. ((\text{Epigraph UNIV } g) = S)$   
**proof** –  
{ **fix**  $x$   
**have**  $\text{closed } \{z. (x, z) : S\}$  **using**  $\langle \text{closed } S \rangle \ \text{closed-epigraph-lines}$  **by**  $\text{auto}$   
**hence**  $\exists a. \{z. (x, z) : S\} = \{z. a \leq \text{ereal } z\}$  **apply** ( $\text{subst mono-closed-ereal}$ )  
**using**  $\text{mono}$  **by**  $\text{auto}$   
} **from**  $\text{this}$  **obtain**  $g$  **where**  $g\text{-def}: \forall x. \{z. (x, z) : S\} = \{z. g \ x \leq \text{ereal } z\}$  **by**  $\text{metis}$   
{ **fix**  $s$   
**have**  $s : S \longleftrightarrow (\text{fst } s, \text{snd } s) : S$  **by**  $\text{auto}$   
**also** **have**  $\dots \longleftrightarrow g(\text{fst } s) \leq \text{ereal } (\text{snd } s)$  **using**  $g\text{-def}[\text{rule-format, of } \text{fst } s]$  **by**  $\text{blast}$   
**finally** **have**  $s : S \longleftrightarrow g(\text{fst } s) \leq \text{ereal } (\text{snd } s)$  **by**  $\text{auto}$   
}  
**hence**  $(\text{Epigraph UNIV } g) = S$  **unfolding**  $\text{Epigraph-def}$  **by**  $\text{auto}$   
**from**  $\text{this}$  **show**  $?thesis$  **by**  $\text{auto}$   
**qed**

**lemma** *lsc-hull-exists*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\exists g. \text{Epigraph UNIV } g = \text{closure } (\text{Epigraph UNIV } f)$   
**proof** –  
{ **fix**  $x \ y \ z$  **assume**  $xy: (x, y) : \text{closure } (\text{Epigraph UNIV } f) \wedge y \leq z$   
{ **fix**  $e :: \text{real}$  **assume**  $e > 0$   
**hence**  $\exists ya \in \text{Epigraph UNIV } f. \text{dist } ya \ (x, y) < e$

**using** *xy closure-approachable*[of  $(x,y)$  *Epigraph UNIV f*] **by** *auto*  
**from this obtain**  $a\ b$  **where**  $ab: (a,b):\text{Epigraph UNIV } f \wedge \text{dist } (a,b) (x,y) < e$   
**by** *auto*  
**moreover have**  $\text{dist } (a,b) (x,y) = \text{dist } (a,b+(z-y)) (x,z)$   
**unfolding** *dist-prod-def dist-norm* **by** (*simp add: algebra-simps*)  
**moreover have**  $(a,b+(z-y)):\text{Epigraph UNIV } f$  **apply** (*subst epigraph-mono*[of  
-  $b$ ]) **using**  $ab\ xy$  **by** *auto*  
**ultimately have**  $\exists w \in \text{Epigraph UNIV } f. \text{dist } w (x, z) < e$  **by** *auto*  
**} hence**  $(x,z):\text{closure } (\text{Epigraph UNIV } f)$  **using** *closure-approachable* **by** *auto*  
**}**  
**hence**  $\forall x\ y\ z. (x,y) \in \text{closure } (\text{Epigraph UNIV } f) \wedge y \leq z \longrightarrow (x,z) \in \text{closure } (\text{Epigraph UNIV } f)$  **by** *auto*  
**from this show** *?thesis* **using** *mono-epigraph*[of  $\text{closure } (\text{Epigraph UNIV } f)$ ] **by**  
*auto*  
**qed**

**lemma** *epigraph-invertible*:  
**assumes**  $\text{Epigraph UNIV } f = \text{Epigraph UNIV } g$   
**shows**  $f=g$   
**proof** –  
**{ fix**  $x$   
**{ fix**  $C$  **have**  $f\ x \leq \text{ereal } C \longleftrightarrow (x,C) : \text{Epigraph UNIV } f$  **unfolding** *Epigraph-def*  
**by** *auto*  
**also have**  $\dots \longleftrightarrow (x,C) : \text{Epigraph UNIV } g$  **using** *assms* **by** *auto*  
**also have**  $\dots \longleftrightarrow g\ x \leq \text{ereal } C$  **unfolding** *Epigraph-def* **by** *auto*  
**finally have**  $f\ x \leq \text{ereal } C \longleftrightarrow g\ x \leq \text{ereal } C$  **by** *auto*  
**} hence**  $g\ x = f\ x$  **using** *ereal-le-real*[of  $g\ x\ f\ x$ ] *ereal-le-real*[of  $f\ x\ g\ x$ ] **by** *auto*  
**} from this show** *?thesis* **by** (*simp add: ext*)  
**qed**

**lemma** *lsc-hull-ex-unique*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\exists! g. \text{Epigraph UNIV } g = \text{closure } (\text{Epigraph UNIV } f)$   
**using** *lsc-hull-exists epigraph-invertible* **by** *metis*

**lemma** *epigraph-lsc-hull*:  
**fixes**  $f :: 'a::\text{metric-space} \Rightarrow \text{ereal}$   
**shows**  $\text{Epigraph UNIV } (\text{lsc-hull } f) = \text{closure}(\text{Epigraph UNIV } f)$   
**proof** –  
**have**  $\exists g. \text{Epigraph UNIV } g = \text{closure } (\text{Epigraph UNIV } f)$  **using** *lsc-hull-exists* **by**  
*auto*  
**thus** *?thesis* **unfolding** *lsc-hull-def*  
**using** *some-eq-ex*[of  $(\lambda g. \text{Epigraph UNIV } g = \text{closure}(\text{Epigraph UNIV } f))$ ] **by**  
*auto*  
**qed**



```

lemma lsc-hull-expl:
  ( $g = \text{lsc-hull } f$ )  $\longleftrightarrow$  ( $\text{Epigraph UNIV } g = \text{closure}(\text{Epigraph UNIV } f)$ )
proof –
{ assume  $\text{Epigraph UNIV } g = \text{closure}(\text{Epigraph UNIV } f)$ 
  hence  $\text{lsc-hull } f = g$  unfolding lsc-hull-def apply ( $\text{subst some1-equality}[of - g]$ )
  using lsc-hull-ex-unique by metis+
}
from this show ?thesis using epigraph-lsc-hull by auto
qed

```

```

lemma lsc-lsc-hull:  $\text{lsc } (\text{lsc-hull } f)$ 
  using epigraph-lsc-hull[ $of f$ ] lsc-iff[ $of \text{lsc-hull } f$ ] by auto

```

```

lemma epigraph-subset-iff:
  fixes  $f g :: 'a::\text{metric-space} \Rightarrow \text{ereal}$ 
  shows  $\text{Epigraph UNIV } f \leq \text{Epigraph UNIV } g \longleftrightarrow (\forall x. g x \leq f x)$ 
proof –
{ assume  $\text{epi}: \text{Epigraph UNIV } f \leq \text{Epigraph UNIV } g$ 
  { fix  $x$ 
    { fix  $z$  assume  $f x \leq \text{ereal } z$ 
      hence  $(x,z) \in \text{Epigraph UNIV } f$  unfolding Epigraph-def by auto
      hence  $(x,z) \in \text{Epigraph UNIV } g$  using epi by auto
      hence  $g x \leq \text{ereal } z$  unfolding Epigraph-def by auto
    } hence  $g x \leq f x$  apply ( $\text{subst } \text{ereal-le-real}$ ) by auto
  }
}
moreover
{ assume  $\text{le}: \forall x. g x \leq f x$ 
  { fix  $x y$  assume  $(x,y) \in \text{Epigraph UNIV } f$ 
    hence  $f x \leq \text{ereal } y$  unfolding Epigraph-def by auto
    moreover have  $g x \leq f x$  using le by auto
    ultimately have  $g x \leq \text{ereal } y$  by auto
    hence  $(x,y) \in \text{Epigraph UNIV } g$  unfolding Epigraph-def by auto
  }
}
ultimately show ?thesis by auto
qed

```

```

lemma lsc-hull-le:  $(\text{lsc-hull } f) x \leq f x$ 
  using epigraph-lsc-hull[ $of f$ ] closure-subset epigraph-subset-iff[ $of f \text{lsc-hull } f$ ] by
auto

```

```

lemma lsc-hull-greatest:
fixes  $f g :: 'a::\text{metric-space} \Rightarrow \text{ereal}$ 

```

**assumes**  $lsc\ g\ \forall x. g\ x \leq f\ x$   
**shows**  $\forall x. g\ x \leq (lsc\text{-}hull\ f)\ x$   
**proof** –  
**have**  $closure(Epigraph\ UNIV\ f) \leq Epigraph\ UNIV\ g$   
**using**  $lsc\text{-}iff\ epigraph\text{-}subset\text{-}iff\ assms$  **by**  $(metis\ closure\text{-}minimal)$   
**from this show**  $?thesis$  **using**  $epigraph\text{-}subset\text{-}iff\ lsc\text{-}hull\text{-}expl$  **by**  $metis$   
**qed**

**lemma**  $lsc\text{-}hull\text{-}iff\text{-}greatest$ :  
**fixes**  $f\ g :: 'a::metric\text{-}space \Rightarrow ereal$   
**shows**  $(g = lsc\text{-}hull\ f) \longleftrightarrow$   
 $lsc\ g \wedge (\forall x. g\ x \leq f\ x) \wedge (\forall h. lsc\ h \wedge (\forall x. h\ x \leq f\ x) \longrightarrow (\forall x. h\ x \leq g\ x))$   
**(is**  $?lhs \longleftrightarrow ?rhs$ **)**  
**proof** –  
**{ assume**  $?lhs$  **hence**  $?rhs$  **using**  $lsc\text{-}lsc\text{-}hull\ lsc\text{-}hull\text{-}le\ lsc\text{-}hull\text{-}greatest$  **by**  $metis$   
**}**  
**moreover**  
**{ assume**  $?rhs$   
**{ fix**  $x$  **have**  $(lsc\text{-}hull\ f)\ x \leq g\ x$  **using**  $\langle ?rhs \rangle\ lsc\text{-}lsc\text{-}hull\ lsc\text{-}hull\text{-}le$  **by**  $metis$   
**moreover have**  $(lsc\text{-}hull\ f)\ x \geq g\ x$  **using**  $\langle ?rhs \rangle\ lsc\text{-}hull\text{-}greatest$  **by**  $metis$   
**ultimately have**  $(lsc\text{-}hull\ f)\ x = g\ x$  **by**  $auto$   
**} hence**  $?lhs$  **by**  $(simp\ add:\ ext)$   
**} ultimately show**  $?thesis$  **by**  $blast$   
**qed**

**lemma**  $lsc\text{-}hull\text{-}mono$ :  
**fixes**  $f\ g :: 'a::metric\text{-}space \Rightarrow ereal$   
**assumes**  $\forall x. g\ x \leq f\ x$   
**shows**  $\forall x. (lsc\text{-}hull\ g)\ x \leq (lsc\text{-}hull\ f)\ x$   
**proof** –  
**{ fix**  $x$  **have**  $(lsc\text{-}hull\ g)\ x \leq g\ x$  **using**  $lsc\text{-}hull\text{-}le[of\ g\ x]$  **by**  $auto$   
**also have**  $\dots \leq f\ x$  **using**  $assms$  **by**  $auto$   
**finally have**  $(lsc\text{-}hull\ g)\ x \leq f\ x$  **by**  $auto$   
**} note**  $* = this$   
**show**  $?thesis$  **apply**  $(subst\ lsc\text{-}hull\text{-}greatest)$  **using**  $lsc\text{-}lsc\text{-}hull[of\ g]\ *$  **by**  $auto$   
**qed**

**lemma**  $lsc\text{-}hull\text{-}lsc$ :  
 $lsc\ f \longleftrightarrow (f = lsc\text{-}hull\ f)$   
**using**  $lsc\text{-}hull\text{-}iff\text{-}greatest[of\ f\ f]$  **by**  $auto$

**lemma**  $lsc\text{-}hull\text{-}liminf\text{-}at$ :  
**fixes**  $f :: 'a::metric\text{-}space \Rightarrow ereal$   
**shows**  $\forall x. (lsc\text{-}hull\ f)\ x = \min(f\ x)\ (Liminf\ (at\ x)\ f)$   
**proof** –

```

{ fix x z assume (x,z):Epigraph UNIV (λx. min (f x) (Liminf (at x) f))
  hence xz-def: ereal z ≥ (SUP e∈{0<..}. INF y∈ball x e. f y)
    unfolding Epigraph-def min-Liminf-at by auto
  { fix e::real assume e>0
    hence e/sqrt 2>0 using ⟨e>0⟩ by simp
    from this obtain e1 where e1-def: e1<e/sqrt 2 ∧ e1>0 using dense by auto
    hence (SUP e∈{0<..}. INF y∈ball x e. f y) ≥ (INF y∈ball x e1. f y)
      by (auto intro: SUP-upper)
    hence ereal z ≥ (INF y∈ball x e1. f y) using xz-def by auto
    hence *: ∀ y>ereal z. ∃ t. t ∈ ball x e1 ∧ f t ≤ y
      by (simp add: Bex-def Inf-le-iff-less)
    obtain t where t-def: t : ball x e1 ∧ f t ≤ ereal(z+e1)
      using e1-def *[rule-format, of ereal(z+e1)] by auto
    hence epi: (t,z+e1):Epigraph UNIV f unfolding Epigraph-def by auto
    have dist x t < e1 using t-def unfolding ball-def dist-norm by auto
    hence sqrt (e1 ^ 2 + dist x t ^ 2) < e
      using e1-def apply (subst real-sqrt-sum-squares-less) by auto
    moreover have dist (x,z) (t,z+e1) = sqrt (e1 ^ 2 + dist x t ^ 2)
      unfolding dist-prod-def dist-norm by (simp add: algebra-simps)
    ultimately have dist (x,z) (t,z+e1) < e by auto
    hence ∃ y∈Epigraph UNIV f. dist y (x, z) < e
      apply (rule-tac x=(t,z+e1) in bexI) apply (simp add: dist-commute) using
    epi by auto
  } hence (x,z) : closure (Epigraph UNIV f)
    using closure-approachable[of (x,z) Epigraph UNIV f] by auto
}
}
moreover
{ fix x z assume xz-def: (x,z):closure (Epigraph UNIV f)
  { fix e::real assume e>0
    from this obtain y where y-def: y:(Epigraph UNIV f) ∧ dist y (x,z) < e
      using closure-approachable[of (x,z) Epigraph UNIV f] xz-def by metis
    have dist (fst y) x ≤ sqrt ((dist (fst y) x) ^ 2 + (dist (snd y) z) ^ 2)
      by (auto intro: real-sqrt-sum-squares-ge1)
    also have ... < e using y-def unfolding dist-prod-def by (simp add: alge-
    bra-simps)
    finally have dist (fst y) x < e by auto
    hence h1: fst y:ball x e unfolding ball-def by (simp add: dist-commute)
    have dist (snd y) z ≤ sqrt ((dist (fst y) x) ^ 2 + (dist (snd y) z) ^ 2)
      by (auto intro: real-sqrt-sum-squares-ge2)
    also have ... < e using y-def unfolding dist-prod-def by (simp add: alge-
    bra-simps)
    finally have h2: dist (snd y) z < e by auto
    have (INF y∈ball x e. f y) ≤ f(fst y) using h1 by (simp add: INF-lower)
    also have ... ≤ ereal(snd y) using y-def unfolding Epigraph-def by auto
    also have ... < ereal(z+e) using h2 unfolding dist-norm by auto
    finally have (INF y∈ball x e. f y) < ereal(z+e) by auto
  } hence *: ∀ e>0. (INF y∈ball x e. f y) < ereal(z+e) by auto
}
{ fix e assume (e::real)>0

```

```

{ fix d assume (d::real)>0
  { assume d<e
    have (INF y∈ball x d. f y) < ereal(z+d) using * ⟨d>0⟩ by auto
    also have ... < ereal(z+e) using ⟨d<e⟩ by auto
    finally have (INF y∈ball x d. f y) < ereal(z+e) by auto
  }
  moreover
  { assume ~⟨d<e⟩
    hence ball x e ≤ ball x d by auto
    hence (INF y∈ball x d. f y) ≤ (INF y∈ball x e. f y) apply (subst INF-mono)
  }
  by auto
  also have ... < ereal(z+e) using * ⟨e>0⟩ by auto
  finally have (INF y∈ball x d. f y) < ereal(z+e) by auto
  } ultimately have (INF y∈ball x d. f y) < ereal(z+e) by blast
  hence (INF y∈ball x d. f y) ≤ ereal(z+e) by auto
  } hence min (f x) (Liminf (at x) f) ≤ ereal (z+e) unfolding min-Liminf-at
  by (auto intro: SUP-least)
  } hence min (f x) (Liminf (at x) f) ≤ ereal z using ereal-le-epsilon2 by auto
  hence (x,z):Epigraph UNIV (λx. min (f x) (Liminf (at x) f)) unfolding Epi-
  graph-def by auto
  }
  ultimately have Epigraph UNIV (λx. min (f x) (Liminf (at x) f))= closure
  (Epigraph UNIV f) by auto
  hence (λx. min (f x) (Liminf (at x) f)) = lsc-hull f
    using epigraph-invertible epigraph-lsc-hull[of f] by blast
  from this show ?thesis by metis
  qed

```

**lemma** *lsc-hull-same-inf*:

```

  fixes f :: 'a::metric-space ⇒ ereal
  shows (INF x. lsc-hull f x) = (INF x. f x)
  proof-
  { fix x
    have (INF x. f x) ≤ (INF y∈ball x 1. f y) apply (subst INF-mono) by auto
    also have ... ≤ min (f x) (Liminf (at x) f) unfolding min-Liminf-at by (auto
    intro: SUP-upper)
    also have ...=(lsc-hull f) x using lsc-hull-liminf-at[of f] by auto
    finally have (INF x. f x) ≤ (lsc-hull f) x by auto
  } hence (INF x. f x) ≤ (INF x. lsc-hull f x) apply (subst INF-greatest) by auto
  moreover have (INF x. lsc-hull f x) ≤ (INF x. f x)
    apply (subst INF-mono) using lsc-hull-le by auto
  ultimately show ?thesis by auto
  qed

```

## 1.4 Convex Functions

**definition**

*convex-on* :: 'a::real-vector set ⇒ ('a ⇒ ereal) ⇒ bool **where**

$convex-on\ s\ f \iff$   
 $(\forall x \in s. \forall y \in s. \forall u \geq 0. \forall v \geq 0. u + v = 1$   
 $\implies f(u *_R x + v *_R y) \leq ereal\ u * f\ x + ereal\ v * f\ y)$

**lemma** *convex-on-ereal-mem*:  
**assumes** *convex-on s f*  
**assumes** *x:s y:s*  
**assumes**  $u \geq 0\ v \geq 0\ u+v=1$   
**shows**  $f(u *_R x + v *_R y) \leq ereal\ u * f\ x + ereal\ v * f\ y$   
**using** *assms unfolding convex-on-def* **by** *auto*

**lemma** *convex-on-ereal-subset*:  $convex-on\ t\ f \implies s \leq t \implies convex-on\ s\ f$   
**unfolding** *convex-on-def* **by** *auto*

**lemma** *convex-on-ereal-univ*:  $convex-on\ UNIV\ f \iff (\forall S. convex-on\ S\ f)$   
**using** *convex-on-ereal-subset* **by** *auto*

**lemma** *ereal-pos-sum-distrib-left*:  
**fixes**  $f :: 'a \Rightarrow ereal$   
**assumes**  $r \geq 0\ r \neq \infty$   
**shows**  $r * sum\ f\ A = sum\ (\lambda n. r * f\ n)\ A$   
**proof** (*cases finite A*)  
**case** *True*  
**thus** *?thesis*  
**proof** *induct*  
**case** *empty* **thus** *?case* **by** *simp*  
**next**  
**case** (*insert x A*) **thus** *?case* **using** *assms* **by** (*simp add: ereal-pos-distrib*)  
**qed**  
**next**  
**case** *False* **thus** *?thesis* **by** *simp*  
**qed**

**lemma** *convex-ereal-add*:  
**fixes**  $f\ g :: 'a::real-vector \Rightarrow ereal$   
**assumes** *convex-on s f convex-on s g*  
**shows**  $convex-on\ s\ (\lambda x. f\ x + g\ x)$   
**proof** –  
**{** **fix**  $x\ y$  **assume**  $x:s\ y:s$  **moreover**  
**fix**  $u\ v :: real$  **assume**  $uv: 0 \leq u\ 0 \leq v\ u + v = 1$   
**ultimately** **have**  $f(u *_R x + v *_R y) + g(u *_R x + v *_R y)$   
 $\leq (ereal\ u * f\ x + ereal\ v * f\ y) + (ereal\ u * g\ x + ereal\ v * g\ y)$   
**using** *assms unfolding convex-on-def* **by** (*auto simp add: add-mono*)  
**also** **have**  $\dots = (ereal\ u * f\ x + ereal\ u * g\ x) + (ereal\ v * f\ y + ereal\ v * g\ y)$   
**by** (*simp add: algebra-simps*)  
**}**

**also have**  $\dots = \text{ereal } u * (f x + g x) + \text{ereal } v * (f y + g y)$   
**using**  $uv$  **by** (*simp add: ereal-pos-distrib*)  
**finally have**  $f (u *_R x + v *_R y) + g (u *_R x + v *_R y)$   
 $\leq \text{ereal } u * (f x + g x) + \text{ereal } v * (f y + g y)$  **by auto** }  
**thus** *?thesis* **unfolding** *convex-on-def* **by auto**  
**qed**

**lemma** *convex-ereal-cmul*:

**assumes**  $0 \leq (c::\text{ereal})$  *convex-on*  $s$   $f$   
**shows** *convex-on*  $s$   $(\lambda x. c * f x)$   
**proof** –  
{ **fix**  $x y$  **assume**  $x:s y:s$  **moreover**  
**fix**  $u v ::\text{real}$  **assume**  $uv: 0 \leq u \ 0 \leq v \ u + v = 1$   
**ultimately have**  $f (u *_R x + v *_R y) \leq (\text{ereal } u * f x + \text{ereal } v * f y)$   
**using** *assms* **unfolding** *convex-on-def* **by auto**  
**hence**  $c * f (u *_R x + v *_R y) \leq c * (\text{ereal } u * f x + \text{ereal } v * f y)$   
**using** *assms* **by** (*intro ereal-mult-left-mono*) **auto**  
**also have**  $\dots \leq c * (\text{ereal } u * f x) + c * (\text{ereal } v * f y)$   
**using** *assms* **by** (*simp add: ereal-le-distrib*)  
**also have**  $\dots = \text{ereal } u *(c * f x) + \text{ereal } v *(c * f y)$  **by** (*simp add: algebra-simps*)  
**finally have**  $c * f (u *_R x + v *_R y)$   
 $\leq \text{ereal } u * (c * f x) + \text{ereal } v * (c * f y)$  **by auto** }  
**thus** *?thesis* **unfolding** *convex-on-def* **by auto**  
**qed**

**lemma** *convex-ereal-max*:

**fixes**  $f g :: 'a::\text{real-vector} \Rightarrow \text{ereal}$   
**assumes** *convex-on*  $s$   $f$  *convex-on*  $s$   $g$   
**shows** *convex-on*  $s$   $(\lambda x. \max (f x) (g x))$   
**proof** –  
{ **fix**  $x y$  **assume**  $x:s y:s$  **moreover**  
**fix**  $u v ::\text{real}$  **assume**  $uv: 0 \leq u \ 0 \leq v \ u + v = 1$   
**ultimately have**  $\max (f (u *_R x + v *_R y)) (g (u *_R x + v *_R y))$   
 $\leq \max (\text{ereal } u * f x + \text{ereal } v * f y) (\text{ereal } u * g x + \text{ereal } v * g y)$   
**apply** (*subst max.mono*) **using** *assms* **unfolding** *convex-on-def* **by auto**  
**also have**  $\dots \leq \text{ereal } u * \max (f x) (g x) + \text{ereal } v * \max (f y) (g y)$   
**apply** (*subst max.boundedI*)  
**apply** (*subst add-mono*) **prefer** 4 **apply** (*subst add-mono*)  
**by** (*subst ereal-mult-left-mono, auto simp add: uv*)+  
**finally have**  $\max (f (u *_R x + v *_R y)) (g (u *_R x + v *_R y))$   
 $\leq \text{ereal } u * \max (f x) (g x) + \text{ereal } v * \max (f y) (g y)$  **by auto** }  
**thus** *?thesis* **unfolding** *convex-on-def* **by auto**  
**qed**

**lemma** *convex-on-ereal-alt*:

**fixes**  $C :: 'a::\text{real-vector set}$

```

assumes convex C
shows convex-on C f =
  ( $\forall x \in C. \forall y \in C. \forall m :: \text{real}. m \geq 0 \wedge m \leq 1$ 
     $\longrightarrow f (m *_R x + (1 - m) *_R y) \leq (\text{ereal } m) * f x + (1 - (\text{ereal } m)) * f y$ )
proof safe
  fix x y fix m :: real
  have[simp]:  $\text{ereal } (1 - m) = (1 - \text{ereal } m)$ 
    using ereal-minus(1)[of 1 m] by (auto simp: one-ereal-def)
  assume asms: convex-on C f x : C y : C 0 ≤ m m ≤ 1
  from this[unfolded convex-on-def, rule-format]
  have  $\forall u v. ((0 \leq u \wedge 0 \leq v \wedge u + v = 1) \longrightarrow$ 
     $f (u *_R x + v *_R y) \leq (\text{ereal } u) * f x + (\text{ereal } v) * f y)$  by auto
  from this[rule-format, of m 1 - m, simplified] asms
  show  $f (m *_R x + (1 - m) *_R y)$ 
     $\leq (\text{ereal } m) * f x + (1 - \text{ereal } m) * f y$  by auto
next
  assume asm:  $\forall x \in C. \forall y \in C. \forall m. 0 \leq m \wedge m \leq 1$ 
     $\longrightarrow f (m *_R x + (1 - m) *_R y) \leq (\text{ereal } m) * f x + (1 - \text{ereal } m) * f y$ 
  { fix x y fix u v :: real
    assume lasm:  $x \in C y \in C u \geq 0 v \geq 0 u + v = 1$ 
    hence[simp]:  $1 - u = v \ 1 - \text{ereal } u = \text{ereal } v$ 
    using ereal-minus(1)[of 1 m] by (auto simp: one-ereal-def)
    from asm[rule-format, of x y u]
    have  $f (u *_R x + v *_R y) \leq (\text{ereal } u) * f x + (\text{ereal } v) * f y$ 
    using lasm by auto }
  thus convex-on C f unfolding convex-on-def by auto
qed

```

**lemma** *convex-on-ereal-alt-mem:*

```

fixes C :: 'a::real-vector set
assumes convex C
assumes convex-on C f
assumes x : C y : C
assumes (m::real) ≥ 0 m ≤ 1
shows  $f (m *_R x + (1 - m) *_R y) \leq (\text{ereal } m) * f x + (1 - (\text{ereal } m)) * f y$ 
by (metis asms convex-on-ereal-alt)

```

**lemma** *ereal-add-right-mono:*  $(a::\text{ereal}) \leq b \implies a + c \leq b + c$

**by** (*metis add-mono order-refl*)

**lemma** *convex-on-ereal-sum-aux:*

```

assumes  $1 - a > 0$ 
shows  $(1 - \text{ereal } a) * (\text{ereal } (c / (1 - a)) * b) = (\text{ereal } c) * b$ 
by (metis mult.assoc mult.commute eq-divide-eq ereal-minus(1) asms
  one-ereal-def less-le times-ereal.simps(1))

```

```

lemma convex-on-ereal-sum:
  fixes a :: 'a ⇒ real
  fixes y :: 'a ⇒ 'b::real-vector
  fixes f :: 'b ⇒ ereal
  assumes finite s s ≠ {}
  assumes convex-on C f
  assumes convex C
  assumes (SUM i : s. a i) = 1
  assumes ∀i. i ∈ s → a i ≥ 0
  assumes ∀i. i ∈ s → y i ∈ C
  shows f (SUM i : s. a i *R y i) ≤ (SUM i : s. ereal (a i) * f (y i))
using assms(1,2,5-)
proof (induct s arbitrary:a rule:finite-ne-induct)
  case (singleton i)
  hence ai: a i = 1 by auto
  thus ?case by (auto simp: one-ereal-def[symmetric])
next
  case (insert i s)
  from ‹convex-on C f›
  have conv: ∀x y m. ((x ∈ C ∧ y ∈ C ∧ 0 ≤ m ∧ m ≤ 1)
    → f (m *R x + (1 - m) *R y) ≤ (ereal m) * f x + (1 - ereal m) * f y)
    using convex-on-ereal-alt[of C f] ‹convex C› by auto
  { assume a i = 1
    hence (SUM j : s. a j) = 0
      using insert by auto
    hence ∀j. (j ∈ s → a j = 0)
      using insert by (simp add: sum-nonneg-eq-0-iff)
    hence ?case using insert.hyps(1-3) ‹a i = 1›
      by (simp add: zero-ereal-def[symmetric] one-ereal-def[symmetric]) }
  moreover
  { assume asm: a i ≠ 1
    from insert have yai: y i : C a i ≥ 0 by auto
    have fis: finite (insert i s) using insert by auto
    hence ai1: a i ≤ 1 using sum-nonneg-leq-bound[of insert i s a] insert by simp
    hence a i < 1 using asm by auto
    hence i0: 1 - a i > 0 by auto
    hence i1: 1 - ereal (a i) > 0 using ereal-minus(1)[of 1 a i]
      by (simp add: zero-ereal-def[symmetric] one-ereal-def[symmetric])
    have i2: 1 - ereal (a i) ≠ ∞ using ereal-minus(1)[of 1]
      by (simp add: zero-ereal-def[symmetric] one-ereal-def[symmetric])
    let ?a j = a j / (1 - a i)
    have a-nonneg: ∧j. j ∈ s ⇒ 0 ≤ a j / (1 - a i)
      using i0 insert
      by (metis insert-iff divide-nonneg-pos)
    have (SUM j : insert i s. a j) = 1 using insert by auto
    hence (SUM j : s. a j) = 1 - a i using sum.insert insert by fastforce
    hence (SUM j : s. a j) / (1 - a i) = 1 using i0 by auto
    hence a1: (SUM j : s. ?a j) = 1 unfolding sum-divide-distrib by simp
    have asum: (SUM j : s. ?a j *R y j) : C
  }

```



**using** *insert convex-sum*[*OF*  $\langle$ *finite s* $\rangle$   
 $\langle$ *convex C* $\rangle$  *a1 a-nonneg*] **by** *auto*  
**have** *asum-le*:  $f (SUM j : s. ?a j *_{R} y j) \leq (SUM j : s. ereal (?a j) * f (y j))$   
**using** *a-nonneg a1 insert* **by** *blast*  
**have**  $f (SUM j : insert i s. a j *_{R} y j) = f ((SUM j : s. a j *_{R} y j) + a i *_{R} y i)$   
**using** *sum.insert*[*of s i*  $\lambda j. a j *_{R} y j$ , *OF*  $\langle$ *finite s* $\rangle$   $\langle$ *i*  $\notin$  *s* $\rangle$ ]  
**by** (*auto simp only: add.commute*)  
**also have**  $\dots = f (((1 - a i) * inverse (1 - a i)) *_{R} (SUM j : s. a j *_{R} y j) + a i *_{R} y i)$   
**using** *i0* **by** *auto*  
**also have**  $\dots = f ((1 - a i) *_{R} (SUM j : s. (a j * inverse (1 - a i)) *_{R} y j) + a i *_{R} y i)$   
**using** *scaleR-right.sum*[*of inverse (1 - a i)*  $\lambda j. a j *_{R} y j$  *s, symmetric*] **by**  
(*auto simp: algebra-simps*)  
**also have**  $\dots = f ((1 - a i) *_{R} (SUM j : s. ?a j *_{R} y j) + a i *_{R} y i)$   
**by** (*auto simp: divide-inverse*)  
**also have**  $\dots \leq (1 - ereal (a i)) * f ((SUM j : s. ?a j *_{R} y j)) + (ereal (a i)) * f (y i)$   
**using** *conv[rule-format, of y i (SUM j : s. ?a j \*\_{R} y j) a i]*  
**using** *yai(1) asum yai(2) ai1* **by** (*auto simp add: add.commute*)  
**also have**  $\dots \leq (1 - ereal (a i)) * (SUM j : s. ereal (?a j) * f (y j)) + (ereal (a i)) * f (y i)$   
**using** *ereal-add-right-mono*[*OF* *ereal-mult-left-mono*[*of - - 1 - ereal (a i), OF asum-le less-imp-le*[*OF i1*]], *of (ereal (a i)) \* f (y i)*] **by** *simp*  
**also have**  $\dots = (SUM j : s. (1 - ereal (a i)) * (ereal (?a j) * f (y j))) + (ereal (a i)) * f (y i)$   
**unfolding** *ereal-pos-sum-distrib-left*[*of 1 - ereal (a i)*  $\lambda j. (ereal (?a j)) * f (y j)$ , *OF less-imp-le*[*OF i1*] *i2*] **by** *auto*  
**also have**  $\dots = (SUM j : s. (ereal (a j)) * f (y j)) + (ereal (a i)) * f (y i)$   
**using** *i0 convex-on-ereal-sum-aux* **by** *auto*  
**also have**  $\dots = (ereal (a i)) * f (y i) + (SUM j : s. (ereal (a j)) * f (y j))$  **by**  
(*simp add: add.commute*)  
**also have**  $\dots = (SUM j : insert i s. (ereal (a j)) * f (y j))$  **using** *insert* **by**  
*auto*  
**finally have**  $f (SUM j : insert i s. a j *_{R} y j) \leq (SUM j : insert i s. (ereal (a j)) * f (y j))$  **by** *simp* }  
**ultimately show** *?case* **by** *auto*  
**qed**

**lemma** *sum-2*:  $sum u \{1::nat..2\} = (u 1) + (u 2)$

**proof** –

**have**  $\{1::nat..2\} = \{1::nat, 2\}$  **by** *auto*

**thus** *?thesis* **by** *auto*

**qed**

**lemma** *convex-on-ereal-iff*:

```

assumes convex s
shows convex-on s f  $\longleftrightarrow$  ( $\forall k u x. (\forall i \in \{1..k::nat\}. 0 \leq u i \wedge x i : s) \wedge \text{sum } u$ 
 $\{1..k\} = 1 \longrightarrow$ 
 $f (\text{sum } (\lambda i. u i *_R x i) \{1..k\}) \leq \text{sum } (\lambda i. (\text{ereal } (u i)) * f(x i)) \{1..k\}$ 
 $(\text{is } ?rhs \longleftrightarrow ?lhs)$ )
proof -
{ assume ?rhs
  { fix k u x
    have zero:  $\sim(\text{sum } u \{1..0::nat\} = (1::real))$  by auto
    assume ( $\forall i \in \{1..k::nat\}. (0::real) \leq u i \wedge x i \in s$ )
    moreover assume *:  $\text{sum } u \{1..k\} = 1$ 
    moreover from * have  $k \neq 0$  using zero by metis
    ultimately have  $f (\text{sum } (\lambda i. u i *_R x i) \{1..k\})$ 
       $\leq \text{sum } (\lambda i. (\text{ereal } (u i)) * f(x i)) \{1..k\}$ 
      using convex-on-ereal-sum[of  $\{1..k\}$  s f u x] using assms <?rhs> by auto
    } hence ?lhs by auto
  }
}
moreover
{ assume ?lhs
  { fix x y u v
    assume xys:  $x:s y:s$ 
    assume uv1:  $u \geq 0 v \geq 0 u + v = (1::real)$ 
    define xy where  $xy = (\lambda i::nat. \text{if } i=1 \text{ then } x \text{ else } y)$ 
    define uv where  $uv = (\lambda i::nat. \text{if } i=1 \text{ then } u \text{ else } v)$ 
    have  $\forall i \in \{1..2::nat\}. (0 \leq uv i) \wedge (xy i : s)$  unfolding xy-def uv-def using
xys uv1 by auto
    moreover have  $\text{sum } uv \{1..2\} = 1$  using sum-2[of uv] unfolding uv-def
using uv1 by auto
    moreover have  $(\text{SUM } i = 1..2. uv i *_R xy i) = u *_R x + v *_R y$ 
      using sum-2[of  $(\lambda i. uv i *_R xy i)$ ] unfolding xy-def uv-def using xys uv1
by auto
    moreover have  $(\text{SUM } i = 1..2. \text{ereal } (uv i) * f(xy i)) = \text{ereal } u * f x + \text{ereal } v * f y$ 
      using sum-2[of  $(\lambda i. \text{ereal } (uv i) * f(xy i))$ ] unfolding xy-def uv-def using
xys uv1 by auto
    ultimately have  $f (u *_R x + v *_R y) \leq \text{ereal } u * f x + \text{ereal } v * f y$ 
      using <?lhs>[rule-format, of 2 uv xy] by auto
    } hence ?rhs unfolding convex-on-def by auto
  } ultimately show ?thesis by blast
}
qed

```

**lemma** convex-Epigraph:

```

assumes convex S
shows convex(Epigraph S f)  $\longleftrightarrow$  convex-on S f
proof -
{ assume rhs: convex(Epigraph S f)
  { fix x y assume xy:  $x:S y:S$ 
    fix u v ::real assume uv:  $0 \leq u 0 \leq v u + v = 1$ 

```

**have**  $f (u *_R x + v *_R y) \leq \text{ereal } u * f x + \text{ereal } v * f y$   
**proof**–  
{ **assume**  $u=0 \mid v=0$  **hence** ?thesis **using**  $uv$  **by** (auto simp: zero-ereal-def[symmetric]) }  
one-ereal-def[symmetric] }  
**moreover**  
{ **assume**  $f x = \infty \mid f y = \infty$  **hence** ?thesis **using**  $uv$  **by** (auto simp:  
zero-ereal-def[symmetric] one-ereal-def[symmetric]) }  
**moreover**  
{ **assume**  $a: f x = -\infty \wedge (f y \neq \infty) \wedge \sim(u=0)$   
**from this obtain**  $z$  **where**  $f y \leq \text{ereal } z$  **apply** (cases  $f y$ ) **by** auto  
**hence**  $yz: (y,z) : \text{Epigraph } S f$  **unfolding** Epigraph-def **using**  $xy$  **by** auto  
{ **fix**  $C::\text{real}$   
**have**  $(x, (1/u)*(C - v * z)) : \text{Epigraph } S f$  **unfolding** Epigraph-def **using**  
 $a xy$  **by** auto  
**hence**  $(u *_R x + v *_R y, C) : \text{Epigraph } S f$   
**using** rhs convexD[of Epigraph  $S f$   $(x, (1/u)*(C - v * z)) (y,z) u v]$   $uv$   
 $yz a$  **by** auto  
**hence**  $(f (u *_R x + v *_R y) \leq \text{ereal } C)$  **unfolding** Epigraph-def **by** auto  
} **hence**  $f (u *_R x + v *_R y) = -\infty$  **using** eréal-bot **by** auto  
**hence** ?thesis **by** auto }  
**moreover**  
{ **assume**  $a: (f x \neq \infty) \wedge f y = -\infty \wedge \sim(v=0)$   
**from this obtain**  $z$  **where**  $f x \leq \text{ereal } z$  **apply** (cases  $f x$ ) **by** auto  
**hence**  $xz: (x,z) : \text{Epigraph } S f$  **unfolding** Epigraph-def **using**  $xy$  **by** auto  
{ **fix**  $C::\text{real}$   
**have**  $(y, (1/v)*(C - u * z)) : \text{Epigraph } S f$  **unfolding** Epigraph-def **using**  
 $a xy$  **by** auto  
**hence**  $(u *_R x + v *_R y, C) : \text{Epigraph } S f$   
**using** rhs convexD[of Epigraph  $S f$   $(x,z) (y, (1/v)*(C - u * z)) u v]$   $uv$   
 $xz a$  **by** auto  
**hence**  $(f (u *_R x + v *_R y) \leq \text{ereal } C)$  **unfolding** Epigraph-def **by** auto  
} **hence**  $f (u *_R x + v *_R y) = -\infty$  **using** eréal-bot **by** auto  
**hence** ?thesis **by** auto }  
**moreover**  
{ **assume**  $a: f x \neq \infty \wedge f x \neq -\infty \wedge f y \neq \infty \wedge f y \neq -\infty$   
**from this obtain**  $fx fy$  **where**  $fx: f x = \text{ereal } fx \wedge f y = \text{ereal } fy$   
**apply** (cases  $f x$ , cases  $f y$ ) **by** auto  
**hence**  $(x, fx) : \text{Epigraph } S f \wedge (y, fy) : \text{Epigraph } S f$  **unfolding** Epigraph-def  
**using**  $xy$  **by** auto  
**hence**  $(u *_R x + v *_R y, u * fx + v * fy) : \text{Epigraph } S f$   
**using** rhs convexD[of Epigraph  $S f$   $(x,fx) (y,fy) u v]$   $uv$  **by** auto  
**hence** ?thesis **unfolding** Epigraph-def **using**  $fx fy$  **by** auto  
} **ultimately show** ?thesis **by** blast  
**qed**  
} **hence** convex-on  $S f$  **unfolding** convex-on-def **by** auto  
}  
**moreover**  
{ **assume**  $lhs: \text{convex-on } S f$   
{ **fix**  $x y fx fy$  **assume**  $xy: (x,fx):\text{Epigraph } S f (y,fy):\text{Epigraph } S f$

```

fix  $u\ v :: \text{real}$  assume  $uv: 0 \leq u\ 0 \leq v\ u + v = 1$ 
hence  $le: f\ x \leq \text{ereal}\ f\ x \wedge f\ y \leq \text{ereal}\ f\ y$  using  $xy$  unfolding  $\text{Epigraph-def}$  by
 $auto$ 
have  $x:S \wedge y:S$  using  $xy$  unfolding  $\text{Epigraph-def}$  by  $auto$ 
moreover hence  $inS: u *_R\ x + v *_R\ y : S$  using  $assms\ uv\ \text{convexD}[of\ S]$  by
 $auto$ 
ultimately have  $f(u *_R\ x + v *_R\ y) \leq (\text{ereal}\ u) * f\ x + (\text{ereal}\ v) * f\ y$ 
using  $lhs\ \text{convex-on-ereal-mem}[of\ S\ f\ x\ y\ u\ v]$   $uv$  by  $auto$ 
also have  $\dots \leq (\text{ereal}\ u) * (\text{ereal}\ f\ x) + (\text{ereal}\ v) * (\text{ereal}\ f\ y)$ 
apply  $(subst\ \text{add-mono})$  apply  $(subst\ \text{ereal-mult-left-mono})$ 
prefer  $4$  apply  $(subst\ \text{ereal-mult-left-mono})$  using  $le\ uv$  by  $auto$ 
also have  $\dots = \text{ereal}\ (u * f\ x + v * f\ y)$  by  $auto$ 
finally have  $(u *_R\ x + v *_R\ y, u * f\ x + v * f\ y) : \text{Epigraph}\ S\ f$ 
unfolding  $\text{Epigraph-def}$  using  $inS$  by  $auto$ 
} hence  $\text{convex}(\text{Epigraph}\ S\ f)$  unfolding  $\text{convex-def}$  by  $auto$ 
}
ultimately show  $?thesis$  by  $auto$ 
qed

```

**lemma**  $\text{convex-EpigraphI}$ :  
 $\text{convex-on}\ s\ f \implies \text{convex}\ s \implies \text{convex}(\text{Epigraph}\ s\ f)$   
**unfolding**  $\text{convex-Epigraph}$  **by**  $auto$

**definition**  
 $\text{concave-on} :: 'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
 $\text{concave-on}\ S\ f \iff \text{convex-on}\ S\ (\lambda x. -f\ x)$

**definition**  
 $\text{finite-on} :: 'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
 $\text{finite-on}\ S\ f \iff (\forall x \in S. (f\ x \neq \infty \wedge f\ x \neq -\infty))$

**definition**  
 $\text{affine-on} :: 'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
 $\text{affine-on}\ S\ f \iff (\text{convex-on}\ S\ f \wedge \text{concave-on}\ S\ f \wedge \text{finite-on}\ S\ f)$

**definition**  
 $\text{domain}\ (f::- \Rightarrow \text{ereal}) = \{x. f\ x < \infty\}$

**lemma**  $\text{domain-Epigraph-aux}$ :  
**assumes**  $x \neq \infty$   
**shows**  $\exists r. x \leq \text{ereal}\ r$   
**using**  $assms$  **by**  $(cases\ x)\ auto$

**lemma**  $\text{domain-Epigraph}$ :  
 $\text{domain}\ f = \{x. \exists y. (x, y) \in \text{Epigraph}\ UNIV\ f\}$

**unfolding** *domain-def Epigraph-def* **using** *domain-Epigraph-aux* **by** *auto*

**lemma** *domain-Epigraph-fst*:

*domain f = fst ' (Epigraph UNIV f)*

**proof** –

{ **fix** *x* **assume** *x:domain f*

**from** *this* **obtain** *y* **where** *(x,y):Epigraph UNIV f* **using** *domain-Epigraph[of f]* **by** *auto*

**moreover** *have x = fst (x,y)* **by** *auto*

**ultimately** *have x:fst ' (Epigraph UNIV f)* **unfolding** *image-def* **by** *blast*

} **from** *this* **show** *?thesis* **using** *domain-Epigraph[of f]* **by** *auto*

**qed**

**lemma** *convex-on-domain*:

*convex-on (domain f) f  $\longleftrightarrow$  convex-on UNIV f*

**proof** –

{ **assume** *lhs: convex-on (domain f) f*

{ **fix** *x y*

**fix** *u v ::real* **assume** *uv: 0 ≤ u 0 ≤ v u + v = 1*

**have** *f (u \*<sub>R</sub> x + v \*<sub>R</sub> y) ≤ ereal u \* f x + ereal v \* f y*

**proof** –

{ **assume** *f x = ∞ | f y = ∞* **hence** *?thesis* **using** *uv* **by** (*auto simp: zero-ereal-def[symmetric] one-ereal-def[symmetric]*) }

**moreover**

{ **assume**  $\sim$  (*f x = ∞ | f y = ∞*)

**hence** *x : domain f  $\wedge$  y : domain f* **unfolding** *domain-def* **by** *auto*

**hence** *?thesis* **using** *lhs* **unfolding** *convex-on-def* **using** *uv* **by** *auto*

} **ultimately** **show** *?thesis* **by** *blast*

**qed** }

**hence** *convex-on UNIV f* **unfolding** *convex-on-def* **by** *auto*

} **from** *this* **show** *?thesis* **using** *convex-on-ereal-subset* **by** *auto*

**qed**

**lemma** *convex-on-domain2*:

*convex-on (domain f) f  $\longleftrightarrow$  ( $\forall S.$  *convex-on S f*)*

**by** (*metis convex-on-domain convex-on-ereal-univ*)

**lemma** *convex-domain*:

**fixes** *f :: 'a::euclidean-space  $\Rightarrow$  ereal*

**assumes** *convex-on UNIV f*

**shows** *convex (domain f)*

**proof** –

**have** *convex (Epigraph UNIV f)* **using** *assms convex-Epigraph* **by** *auto*

**thus** *?thesis* **unfolding** *domain-Epigraph-fst*

**apply** (*subst convex-linear-image*) **using** *linear-fst linear-conv-bounded-linear* **by**

*auto*  
**qed**

**lemma** *infinite-convex-domain-iff*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\forall x. (f\ x = \infty \mid f\ x = -\infty)$   
**shows**  $\text{convex-on UNIV } f \longleftrightarrow \text{convex } (\text{domain } f)$   
**proof** –  
{ **assume**  $\text{dom}: \text{convex } (\text{domain } f)$   
{ **fix**  $x\ y$  **assume**  $x:\text{domain } f\ y:\text{domain } f$  **moreover**  
**fix**  $u\ v :: \text{real}$  **assume**  $uv: 0 \leq u\ 0 \leq v\ u + v = 1$   
**ultimately have**  $u *_R x + v *_R y : \text{domain } f$   
**using**  $\text{dom unfolding convex-def by auto}$   
**hence**  $f(u *_R x + v *_R y) = -\infty$   
**using**  $\text{assms unfolding domain-def by auto}$   
} **hence**  $\text{convex-on } (\text{domain } f)\ f$  **unfolding convex-on-def by auto**  
**hence**  $\text{convex-on UNIV } f$  **by**  $(\text{metis convex-on-domain2})$   
} **thus** *?thesis* **by**  $(\text{metis convex-domain})$   
**qed**

**lemma** *convex-PInfty-outside*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes**  $\text{convex-on UNIV } f\ \text{convex } S$   
**shows**  $\text{convex-on UNIV } (\lambda x. \text{if } x:S \text{ then } (f\ x) \text{ else } \infty)$   
**proof** –  
**define**  $g$  **where**  $g\ x = (\text{if } x:S \text{ then } -\infty \text{ else } \infty::\text{ereal})$  **for**  $x$   
**hence**  $\text{convex-on UNIV } g$  **apply**  $(\text{subst infinite-convex-domain-iff})$   
**using**  $\text{assms unfolding domain-def by auto}$   
**moreover have**  $(\lambda x. \text{if } x:S \text{ then } (f\ x) \text{ else } \infty) = (\lambda x. \text{max } (f\ x)\ (g\ x))$   
**apply**  $(\text{subst fun-eq-iff})$  **unfolding g-def by auto**  
**ultimately show** *?thesis* **using**  $\text{convex-ereal-max assms by auto}$   
**qed**

**definition**  
 $\text{proper-on} :: 'a::\text{real-vector set} \Rightarrow ('a \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
 $\text{proper-on } S\ f \longleftrightarrow ((\forall x \in S. f\ x \neq -\infty) \wedge (\exists x \in S. f\ x \neq \infty))$

**definition**  
 $\text{proper} :: ('a::\text{real-vector} \Rightarrow \text{ereal}) \Rightarrow \text{bool}$  **where**  
 $\text{proper } f \longleftrightarrow \text{proper-on UNIV } f$

**lemma** *proper-iff*:  
 $\text{proper } f \longleftrightarrow ((\forall x. f\ x \neq -\infty) \wedge (\exists x. f\ x \neq \infty))$   
**unfolding**  $\text{proper-def proper-on-def by auto}$

**lemma improper-iff:**  
 $\sim(\text{proper } f) \longleftrightarrow ((\exists x. f x = -\infty) \mid (\forall x. f x = \infty))$   
**by** (*simp add: proper-iff*)

**lemma ereal-MInf-plus[*simp*]:**  $-\infty + x = (\text{if } x = \infty \text{ then } \infty \text{ else } -\infty::\text{ereal})$   
**by** *simp*

**lemma convex-improper:**  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes**  $\sim(\text{proper } f)$   
**shows**  $\forall x \in \text{rel-interior}(\text{domain } f). f x = -\infty$   
**proof** –  
**{ assume**  $\text{domain } f = \{\}$  **hence** *?thesis using rel-interior-empty by auto* **}**  
**moreover**  
**{ assume**  $\text{nemp: domain } f \neq \{\}$   
**then obtain**  $u$  **where**  $u\text{-def: } f u = -\infty$  **using** *assms improper-iff[of f] unfolding domain-def by auto*  
**hence**  $u\text{dom: } u:\text{domain } f$  **unfolding** *domain-def by auto*  
**{ fix**  $x$  **assume**  $x:\text{rel-interior}(\text{domain } f)$   
**then obtain**  $m$  **where**  $m\text{-def: } m > 1 \wedge (1 - m) *_R u + m *_R x : \text{domain } f$   
**using** *convex-rel-interior-iff[of domain f x] nemp convex-domain[of f] assms udom by auto*  
**define**  $v$  **where**  $v = 1/m$   
**hence**  $v01: 0 < v \wedge v < 1$  **using** *m-def by auto*  
**define**  $y$  **where**  $y = (1 - m) *_R u + m *_R x$   
**have**  $x = (1 - v) *_R u + v *_R y$  **unfolding** *v-def y-def using m-def by (simp add: algebra-simps)*  
**hence**  $f x \leq (1 - \text{ereal } v) * f u + \text{ereal } v * f y$   
**using** *convex-on-ereal-alt-mem[of UNIV f y u v] assms convex-UNIV v01 by (simp add: add commute)*  
**moreover** **have**  $f y < \infty$  **using** *m-def y-def unfolding domain-def by auto*  
**moreover** **have**  $*$ :  $0 < 1 - \text{ereal } v$  **using** *v01 by (metis diff-gt-0-iff-gt ereal-less(2) ereal-minus(1) one-ereal-def)*  
**moreover** **from**  $*$  **have**  $(1 - \text{ereal } v) * f u = -\infty$  **using** *u-def by auto*  
**ultimately** **have**  $f x = -\infty$  **using** *v01 by simp*  
**}** **hence** *?thesis by auto*  
**}** **ultimately** **show** *?thesis by blast*  
**qed**

**lemma convex-improper2:**  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**assumes**  $\sim(\text{proper } f)$   
**shows**  $f x = \infty \mid f x = -\infty \mid x : \text{rel-frontier}(\text{domain } f)$   
**proof** –  
**{ assume**  $a: \sim(f x = \infty \mid f x = -\infty)$

**hence**  $x$ : *domain f* **unfolding** *domain-def* **by** *auto*  
**hence**  $x$ : *closure (domain f)* **using** *closure-subset* **by** *auto*  
**moreover** **have**  $x \notin$  *rel-interior (domain f)* **using** *assms convex-improper a* **by**  
*auto*  
**ultimately** **have**  $x$ : *rel-frontier (domain f)* **unfolding** *rel-frontier-def* **by** *auto*  
**} thus** *?thesis* **by** *auto*  
**qed**

**lemma** *convex-lsc-improper*:

**fixes**  $f$  :: ' $a$ ::*euclidean-space*  $\Rightarrow$  *ereal*  
**assumes** *convex-on UNIV f*  
**assumes**  $\sim$ (*proper f*)  
**assumes** *lsc f*  
**shows**  $f x = \infty \mid f x = -\infty$

**proof** –

**{ fix**  $x$  **assume**  $f x \neq \infty$   
**hence** *lsc-at x f* **using** *assms unfolding lsc-def* **by** *auto*  
**have**  $x$ : *domain f* **unfolding** *domain-def* **using**  $\langle f x \neq \infty \rangle$  **by** *auto*  
**hence**  $x$ : *closure (domain f)* **using** *closure-subset* **by** *auto*  
**hence**  $x$ -*def*:  $x$ : *closure (rel-interior (domain f))*  
**by** (*metis assms(1) convex-closure-rel-interior convex-domain*)  
**{ fix**  $C$  **assume**  $C < f x$   
**from** *this* **obtain**  $d$  **where**  $d$ -*def*:  $d > 0 \wedge (\forall y. \text{dist } x \ y < d \longrightarrow C < f y)$   
**using** *lst-at-delta[of x f] lsc-at x f* **by** *auto*  
**from** *this* **obtain**  $y$  **where**  $y$ -*def*:  $y$ :*rel-interior (domain f)*  $\wedge$  *dist y x < d*  
**using**  $x$ -*def* *closure-approachable[of x rel-interior (domain f)]* **by** *auto*  
**hence**  $f y = -\infty$  **by** (*metis assms(1) assms(2) convex-improper*)  
**moreover** **have**  $C < f y$  **using**  $y$ -*def*  $d$ -*def* **by** (*simp add: dist-commute*)  
**ultimately** **have** *False* **by** *auto*  
**} hence**  $f x = -\infty$  **by** *auto*  
**} from** *this* **show** *?thesis* **by** *auto*  
**qed**

**lemma** *convex-lsc-hull*:

**fixes**  $f$  :: ' $a$ ::*euclidean-space*  $\Rightarrow$  *ereal*  
**assumes** *convex-on UNIV f*  
**shows** *convex-on UNIV (lsc-hull f)*

**proof** –

**have** *convex(Epigraph UNIV f)* **by** (*metis assms convex-EpigraphI convex-UNIV*)  
**hence** *convex (Epigraph UNIV (lsc-hull f))* **by** (*metis convex-closure epigraph-lsc-hull*)  
**thus** *?thesis* **by** (*metis convex-Epigraph convex-UNIV*)

**qed**

**lemma** *improper-lsc-hull*:

**fixes**  $f$  :: ' $a$ ::*euclidean-space*  $\Rightarrow$  *ereal*  
**assumes**  $\sim$ (*proper f*)



**shows**  $\sim(\text{proper } (\text{lsc-hull } f))$   
**proof** –  
{  
  **assume** \*:  $\forall x. f x = \infty$   
  **then have**  $\text{lsc } f$   
  **by** (*metis* (*full-types*) *UNIV-I* *lsc-at-open* *lsc-def* *open-UNIV*)  
  **with** \* **have**  $\forall x. (\text{lsc-hull } f) x = \infty$  **by** (*metis* *lsc-hull-lsc*)  
}  
**hence**  $(\forall x. f x = \infty) \longleftrightarrow (\forall x. (\text{lsc-hull } f) x = \infty)$   
  **by** (*metis* *ereal-infty-less-eq(1)* *lsc-hull-le*)  
**moreover have**  $(\exists x. f x = -\infty) \longrightarrow (\exists x. (\text{lsc-hull } f) x = -\infty)$   
  **by** (*metis* *ereal-infty-less-eq2(2)* *lsc-hull-le*)  
**ultimately show** *?thesis* **using** *assms* **unfolding** *improper-iff* **by** *auto*  
**qed**

**lemma** *lsc-hull-convex-improper*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV*  $f$   
**assumes**  $\sim(\text{proper } f)$   
**shows**  $\forall x \in \text{rel-interior}(\text{domain } f). (\text{lsc-hull } f) x = f x$   
**by** (*metis* *assms* *convex-improper* *ereal-infty-less-eq(2)* *lsc-hull-le*)

**lemma** *convex-with-rel-open-domain*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV*  $f$   
**assumes** *rel-open* ( $\text{domain } f$ )  
**shows**  $(\forall x. f x > -\infty) \mid (\forall x. (f x = \infty \mid f x = -\infty))$   
**proof** –  
{ **assume**  $\neg(\forall x. f x > -\infty)$   
  **hence**  $\neg(\text{proper } f)$  **using** *proper-iff*  
  **by** (*simp* *add: proper-iff*)  
  **hence**  $\forall x \in \text{rel-interior}(\text{domain } f). f x = -\infty$  **by** (*metis* *assms(1)* *convex-improper*)  
  **hence**  $\forall x \in \text{domain } f. f x = -\infty$  **by** (*metis* *assms(2)* *rel-open-def*)  
  **hence**  $\forall x. (f x = \infty \mid f x = -\infty)$  **unfolding** *domain-def* **by** *auto*  
} **thus** *?thesis* **by** *auto*  
**qed**

**lemma** *convex-with-UNIV-domain*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV*  $f$   
**assumes**  $\text{domain } f = \text{UNIV}$   
**shows**  $(\forall x. f x > -\infty) \vee (\forall x. f x = -\infty)$   
**by** (*metis* *assms* *convex-improper* *ereal-MInfty-lessI* *proper-iff* *rel-interior-UNIV* *UNIV-I*)

```

lemma rel-interior-Epigraph:
  fixes  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$ 
  assumes convex-on UNIV f
  shows  $(x,z) : \text{rel-interior} (\text{Epigraph UNIV } f) \longleftrightarrow$ 
     $(x : \text{rel-interior} (\text{domain } f) \wedge f x < \text{ereal } z)$ 
apply (subst rel-interior-projection[of - ( $\lambda y. \{z. (y, z) : \text{Epigraph UNIV } f\}$ )])
apply (metis assms convex-EpigraphI convex-UNIV convex-on-ereal-univ)
unfolding domain-Epigraph Epigraph-def using rel-interior-ereal-semiline by auto

```

```

lemma rel-interior-EpigraphI:
  fixes  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$ 
  assumes convex-on UNIV f
  shows  $\text{rel-interior} (\text{Epigraph UNIV } f) =$ 
     $\{(x,z) \mid x z. x : \text{rel-interior} (\text{domain } f) \wedge f x < \text{ereal } z\}$ 
proof -
  { fix  $x z$ 
    have  $(x,z) : \text{rel-interior} (\text{Epigraph UNIV } f) \longleftrightarrow (x : \text{rel-interior} (\text{domain } f) \wedge f x < \text{ereal } z)$ 
      using rel-interior-Epigraph[of f x z] assms by auto
    } thus ?thesis by auto
qed

```

```

lemma convex-less-ri-domain:
  fixes  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$ 
  assumes convex-on UNIV f
  assumes  $\exists x. f x < a$ 
  shows  $\exists x \in \text{rel-interior} (\text{domain } f). f x < a$ 
proof -
  define  $A$  where  $A = \{(x::'a::\text{euclidean-space}, m) \mid x m. \text{ereal } m < a\}$ 
  obtain  $x$  where  $f x < a$  using assms by auto
  then obtain  $z$  where  $z\text{-def}: f x < \text{ereal } z \wedge \text{ereal } z < a$  using ereal-dense2 by
auto
  hence  $(x,z) : A \wedge (x,z) : \text{Epigraph UNIV } f$  unfolding A-def Epigraph-def by auto
  hence  $A \text{ Int } (\text{Epigraph UNIV } f) \neq \{\}$  unfolding A-def Epigraph-def using
assms by auto
  moreover have open A proof(cases a)
    case real hence  $A = \{y. \text{inner } (0::'a, 1) y < \text{real-of-ereal } a\}$  using A-def by
auto
    thus ?thesis using open-halfspace-lt by auto
    next case PInf thus ?thesis using A-def by auto
    next case MInf thus ?thesis using A-def by auto
  qed
  ultimately have  $A \text{ Int } \text{rel-interior}(\text{Epigraph UNIV } f) \neq \{\}$ 
  by (metis assms(1) convex-Epigraph convex-UNIV
    open-Int-closure-eq-empty open-inter-closure-rel-interior)
  then obtain  $x_0 z_0$  where  $(x_0, z_0) : A \wedge x_0 : \text{rel-interior} (\text{domain } f) \wedge f x_0 <$ 

```

*ereal*  $z0$   
**using** *rel-interior-Epigraph*[of  $f$ ] *assms* **by** *auto*  
**thus** *?thesis* **apply**(*rule-tac*  $x=x0$  **in** *bexF*) **using** *A-def* **by** *auto*  
**qed**

**lemma** *rel-interior-eq-between*:  
**fixes**  $S T :: ('m::euclidean-space)$  *set*  
**assumes** *convex*  $S$  *convex*  $T$   
**shows**  $(rel-interior\ S = rel-interior\ T) \longleftrightarrow (rel-interior\ S \leq T \wedge T \leq closure\ S)$   
**by** (*metis* *assms* *closure-eq-between* *convex-closure-rel-interior* *convex-rel-interior-closure*)

**lemma** *convex-less-in-riS*:  
**fixes**  $f :: 'a::euclidean-space \Rightarrow ereal$   
**assumes** *convex-on* *UNIV*  $f$   
**assumes** *convex*  $S$  *rel-interior*  $S \leq domain\ f$   
**assumes**  $\exists x \in closure\ S. f\ x < a$   
**shows**  $\exists x \in rel-interior\ S. f\ x < a$   
**proof** –  
**define**  $g$  **where**  $g\ x = (if\ x \in closure\ S\ then\ (f\ x)\ else\ \infty)$  **for**  $x$   
**hence**  $\exists x. g\ x < a$  **using** *assms* **by** *auto*  
**have** *convg*: *convex-on* *UNIV*  $g$  **unfolding** *g-def* **apply** (*subst* *convex-PInfty-outside*)  
**using** *assms* *convex-closure* **by** *auto*  
**hence**  $*$ :  $\exists x \in rel-interior\ (domain\ g). g\ x < a$  **apply** (*subst* *convex-less-ri-domain*)  
**using** *assms* *g-def* **by** *auto*  
**have** *convex*  $(domain\ g)$  **by** (*metis* *convg* *convex-domain*)  
**moreover** **have** *rel-interior*  $S \leq domain\ g \wedge domain\ g \leq closure\ S$   
**using** *g-def* *assms* *rel-interior-subset-closure* **unfolding** *domain-def* **by** *auto*  
**ultimately** **have** *rel-interior*  $(domain\ g) = rel-interior\ S$   
**by** (*metis* *assms*(2) *rel-interior-eq-between*)  
**thus** *?thesis*  
**by** (*metis*  $*$  *g-def* *less-ereal.simps*(2))  
**qed**

**lemma** *convex-less-in-S*:  
**fixes**  $f :: 'a::euclidean-space \Rightarrow ereal$   
**assumes** *convex-on* *UNIV*  $f$   
**assumes** *convex*  $S$   $S \leq domain\ f$   
**assumes**  $\exists x \in closure\ S. f\ x < a$   
**shows**  $\exists x \in S. f\ x < a$   
**using** *convex-less-in-riS*[of  $f\ S\ a$ ] *rel-interior-subset*[of  $S$ ] *assms* **by** *auto*

**lemma** *convex-finite-geq-on-closure*:  
**fixes**  $f :: 'a::euclidean-space \Rightarrow ereal$

```

assumes convex-on UNIV f
assumes convex S finite-on S f
assumes  $\forall x \in S. f x \geq a$ 
shows  $\forall x \in \text{closure } S. f x \geq a$ 
proof –
have  $S \leq \text{domain } f$  using assms unfolding finite-on-def domain-def by auto
{ assume  $\neg(\forall x \in \text{closure } S. f x \geq a)$ 
  hence  $\exists x \in \text{closure } S. f x < a$  by (simp add: not-le)
  hence False using convex-less-inS[of f S a] assms ⟨S ≤ domain f⟩ by auto
} thus ?thesis by auto
qed

```

```

lemma lsc-hull-of-convex-determined:
  fixes  $f g :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$ 
  assumes convex-on UNIV f convex-on UNIV g
  assumes  $\text{rel-interior } (\text{domain } f) = \text{rel-interior } (\text{domain } g)$ 
  assumes  $\forall x \in \text{rel-interior } (\text{domain } f). f x = g x$ 
  shows  $\text{lsc-hull } f = \text{lsc-hull } g$ 
proof –
  have  $\text{rel-interior } (\text{Epigraph UNIV } f) = \text{rel-interior } (\text{Epigraph UNIV } g)$ 
  apply (subst rel-interior-EpigraphI, metis assms)+ using assms by auto
  hence  $\text{closure } (\text{Epigraph UNIV } f) = \text{closure } (\text{Epigraph UNIV } g)$ 
  by (metis assms convex-EpigraphI convex-UNIV convex-closure-rel-interior)
  thus ?thesis by (metis lsc-hull-expl)
qed

```

```

lemma domain-lsc-hull-between:
  fixes  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$ 
  shows  $\text{domain } f \leq \text{domain } (\text{lsc-hull } f)$ 
   $\wedge \text{domain } (\text{lsc-hull } f) \leq \text{closure } (\text{domain } f)$ 
proof –
{ fix  $x$  assume  $x \in \text{domain } f$ 
  hence  $x \in \text{domain } (\text{lsc-hull } f)$  unfolding domain-def using lsc-hull-le[of f x] by auto
} moreover
{ fix  $x$  assume  $x \in \text{domain } (\text{lsc-hull } f)$ 
  hence  $\min (f x) (\text{Liminf } (at x) f) < \infty$  unfolding domain-def using lsc-hull-liminf-at[of f] by auto
  then obtain  $z$  where  $z$ -def:  $\min (f x) (\text{Liminf } (at x) f) < z \wedge z < \infty$  by (metis dense)
  { fix  $e :: \text{real}$  assume  $e > 0$ 
    hence  $\text{Inf } (f ` \text{ball } x e) \leq \min (f x) (\text{Liminf } (at x) f)$ 
    unfolding min-Liminf-at apply (subst SUP-upper) by auto
    hence  $\exists y. y \in \text{ball } x e \wedge f y \leq z$ 
    using Inf-le-iff-less [of f ball x e min (f x) (Liminf (at x) f)]  $z$ -def by (auto simp add: Bex-def)
    hence  $\exists y. \text{dist } x y < e \wedge y \in \text{domain } f$  unfolding domain-def ball-def using

```

*z-def* **by** *auto*  
**}** **hence**  $x \in \text{closure}(\text{domain } f)$  **unfolding** *closure-approachable* **by** (*auto simp add: dist-commute*)  
**}** **ultimately show** *?thesis* **by** *auto*  
**qed**

**lemma** *domain-vs-domain-lsc-hull:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows**  $\text{rel-interior}(\text{domain } (\text{lsc-hull } f)) = \text{rel-interior}(\text{domain } f)$   
 $\wedge \text{closure}(\text{domain } (\text{lsc-hull } f)) = \text{closure}(\text{domain } f)$   
 $\wedge \text{aff-dim}(\text{domain } (\text{lsc-hull } f)) = \text{aff-dim}(\text{domain } f)$   
**proof** –  
**have** *convex (domain f)* **by** (*metis assms convex-domain*)  
**moreover have** *convex (domain (lsc-hull f))* **by** (*metis assms convex-domain convex-lsc-hull*)  
**moreover have**  $\text{rel-interior } (\text{domain } f) \leq \text{domain } (\text{lsc-hull } f)$   
 $\wedge \text{domain } (\text{lsc-hull } f) \leq \text{closure } (\text{domain } f)$   
**by** (*metis domain-lsc-hull-between rel-interior-subset subset-trans*)  
**ultimately show** *?thesis* **by** (*metis closure-eq-between rel-interior-aff-dim rel-interior-eq-between*)  
**qed**

**lemma** *vertical-line-affine:*

**fixes**  $x :: 'a::\text{euclidean-space}$   
**shows** *affine  $\{(x, m::\text{real}) \mid m. m:UNIV\}$*   
**unfolding** *affine-def* **by** (*auto simp add: pth-8*)

**lemma** *lsc-hull-of-convex-agrees-onRI:*

**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows**  $\forall x \in \text{rel-interior } (\text{domain } f). (f \ x = (\text{lsc-hull } f) \ x)$   
**proof** –  
**have** *cEpi: convex (Epigraph UNIV f)* **by** (*metis assms convex-EpigraphI convex-UNIV*)  
**{** **fix**  $x$  **assume** *x-def:  $x : \text{rel-interior } (\text{domain } f)$*   
**hence**  $f \ x < \infty$  **unfolding** *domain-def* **using** *rel-interior-subset* **by** *auto*  
**then obtain**  $r$  **where** *r-def:  $(x, r) : \text{rel-interior } (\text{Epigraph } UNIV \ f)$*   
**using** *assms x-def rel-interior-Epigraph[of f x]* **by** (*metis ereal-dense2*)  
**define**  $M$  **where**  $M = \{(x, m::\text{real}) \mid m. m:UNIV\}$   
**hence** *affine M* **using** *vertical-line-affine* **by** *auto*  
**moreover have**  $\text{rel-interior } (\text{Epigraph } UNIV \ f) \ \text{Int } M \neq \{\}$  **using** *r-def M-def*  
**by** *auto*  
**ultimately have**  $\ast: \text{closure } (\text{Epigraph } UNIV \ f) \ \text{Int } M = \text{closure } (\text{Epigraph } UNIV \ f \ \text{Int } M)$   
**using** *convex-affine-closure-Int[of Epigraph UNIV f M] cEpi* **by** *auto*  
**have**  $\text{Epigraph } UNIV \ f \ \text{Int } M = \{x\} \times \{m. f \ x \leq \text{ereal } m\}$

**unfolding** *Epigraph-def M-def* **by** *auto*  
**moreover have**  $\text{closed}(\{x\} \times \{m. f x \leq \text{ereal } m\})$  **apply** (*subst closed-Times*)  
**using** *closed-ereal-semiline* **by** *auto*  
**ultimately have**  $\{x\} \times \{m. f x \leq \text{ereal } m\} = \text{closure}(\text{Epigraph UNIV } f) \text{ Int } M$   
**by** (*metis \* Int-commute closure-closed*)  
**also have**  $\dots = \text{Epigraph UNIV } (\text{lsc-hull } f) \text{ Int } M$  **by** (*metis Int-commute epi-graph-lsc-hull*)  
**also have**  $\dots = \{x\} \times \{m. ((\text{lsc-hull } f) x) \leq \text{ereal } m\}$   
**unfolding** *Epigraph-def M-def* **by** *auto*  
**finally have**  $\{m. f x \leq \text{ereal } m\} = \{m. \text{lsc-hull } f x \leq \text{ereal } m\}$  **by** *auto*  
**hence**  $f x = (\text{lsc-hull } f) x$  **using** *ereal-semiline-unique* **by** *auto*  
**} thus** *?thesis* **by** *auto*  
**qed**

**lemma** *lsc-hull-of-convex-agrees-outside*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows**  $\forall x. x \notin \text{closure}(\text{domain } f) \longrightarrow (f x = (\text{lsc-hull } f) x)$   
**proof** –  
**{ fix**  $x$  **assume**  $x \notin \text{closure}(\text{domain } f)$   
**hence**  $x \notin \text{domain}(\text{lsc-hull } f)$  **using** *domain-lsc-hull-between* **by** *auto*  
**hence**  $(\text{lsc-hull } f) x = \infty$  **unfolding** *domain-def* **by** *auto*  
**hence**  $f x = (\text{lsc-hull } f) x$  **using** *lsc-hull-le[of f x]* **by** *auto*  
**} thus** *?thesis* **by** *auto*  
**qed**

**lemma** *lsc-hull-of-convex-agrees*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f*  
**shows**  $\forall x. (f x = (\text{lsc-hull } f) x) \mid x : \text{rel-frontier}(\text{domain } f)$   
**by** (*metis DiffI assms lsc-hull-of-convex-agrees-onRI lsc-hull-of-convex-agrees-outside rel-frontier-def*)

**lemma** *lsc-hull-of-proper-convex-proper*:  
**fixes**  $f :: 'a::\text{euclidean-space} \Rightarrow \text{ereal}$   
**assumes** *convex-on UNIV f proper f*  
**shows** *proper (lsc-hull f)*  
**proof** –  
**obtain**  $x$  **where**  $x\text{-def}: x:\text{rel-interior}(\text{domain } f) \wedge f x < \infty$   
**by** (*metis assms convex-less-ri-domain ereal-less-PInfty proper-iff*)  
**hence**  $f x = (\text{lsc-hull } f) x$  **using** *lsc-hull-of-convex-agrees[of f]* **assms**  
**unfolding** *rel-frontier-def* **by** *auto*  
**moreover have**  $f x > -\infty$   
**using** *assms proper-iff* **by** *blast*  
**ultimately have**  $(\text{lsc-hull } f) x < \infty \wedge (\text{lsc-hull } f) x > -\infty$  **using**  $x\text{-def}$  **by** *auto*  
**thus** *?thesis* **using** *convex-lsc-improper[of lsc-hull f x]*

*lsc-lsc-hull*[of *f*] *assms convex-lsc-hull*[of *f*] **by auto**  
**qed**

**lemma** *lsc-hull-of-proper-convex*:

**fixes** *f* :: 'a::euclidean-space  $\Rightarrow$  ereal  
**assumes** *convex-on UNIV f proper f*  
**shows** *lsc (lsc-hull f)  $\wedge$  proper (lsc-hull f)  $\wedge$  convex-on UNIV (lsc-hull f)  $\wedge$*   
*( $\forall x. (f x = (lsc-hull f) x) \mid x : \text{rel-frontier (domain f)}$ )*  
**by** (*metis assms convex-lsc-hull lsc-hull-of-convex-agrees lsc-hull-of-proper-convex-proper lsc-lsc-hull*)

**lemma** *affine-no-rel-frontier*:

**fixes** *S* :: ('n::euclidean-space) set  
**assumes** *affine S*  
**shows** *rel-frontier S = {}*  
**unfolding** *rel-frontier-def* **using** *assms affine-closed*[of *S*]  
*closure-closed*[of *S*] *affine-rel-open*[of *S*] *rel-open-def*[of *S*] **by auto**

**lemma** *convex-with-affine-domain-is-lsc*:

**fixes** *f* :: 'a::euclidean-space  $\Rightarrow$  ereal  
**assumes** *convex-on UNIV f*  
**assumes** *affine (domain f)*  
**shows** *lsc f*  
**by** (*metis assms affine-no-rel-frontier emptyE lsc-def lsc-hull-liminf-at lsc-hull-of-convex-agrees lsc-liminf-at-eq*)

**lemma** *convex-finite-is-lsc*:

**fixes** *f* :: 'a::euclidean-space  $\Rightarrow$  ereal  
**assumes** *convex-on UNIV f*  
**assumes** *finite-on UNIV f*  
**shows** *lsc f*  
**proof** –  
**have** *affine (domain f)*  
**using** *assms affine-UNIV* **unfolding** *finite-on-def domain-def* **by auto**  
**thus** *?thesis* **by** (*metis assms(1) convex-with-affine-domain-is-lsc*)  
**qed**

**lemma** *always-eventually-within*:

*( $\forall x \in S. P x$ )  $\implies$  eventually *P* (at *x* within *S*)*  
**unfolding** *eventually-at-filter* **by auto**

**lemma** *ereal-divide-pos*:

**assumes** *(a::ereal) > 0 b > 0*

**shows**  $a/(ereal\ b) > 0$   
**by** (*metis PInfty-eq-infinity assms ereal.simps(2) ereal-less(2) ereal-less-divide-pos ereal-mult-zero*)

**lemma** *real-interval-limpt*:  
**assumes**  $a < b$   
**shows**  $(b::real)\ islimpt\ \{a..<b\}$   
**proof** –  
**{ fix**  $T$  **assume**  $b:T$  **open**  $T$   
**then obtain**  $e$  **where**  $e-def: e > 0 \wedge cball\ b\ e \leq T$  **using** *open-contains-cball[of T]* **by** *auto*  
**hence**  $(b-e):cball\ b\ e$  **unfolding** *cball-def dist-norm* **by** *auto*  
**moreover**  
**{ assume**  $a \geq b-e$  **hence**  $a:cball\ b\ e$  **unfolding** *cball-def dist-norm* **using**  $\langle a < b \rangle$   
**by** *auto* **}**  
**ultimately have**  $max\ a\ (b-e):cball\ b\ e$   
**by** (*metis max.absorb1 max.absorb2 linear*)  
**hence**  $max\ a\ (b-e):T$  **using**  $e-def$  **by** *auto*  
**moreover have**  $max\ a\ (b-e):\{a..<b\}$  **using**  $e-def\ \langle a < b \rangle$  **by** *auto*  
**ultimately have**  $\exists y \in \{a..<b\}. y : T \wedge y \neq b$  **by** *auto*  
**}** **thus** *?thesis* **unfolding** *islimpt-def* **by** *auto*  
**qed**

**lemma** *lsc-hull-of-convex-aux*:  
 $Limsup\ (at\ 1\ within\ \{0..<1\})\ (\lambda m. ereal\ ((1-m)*a+m*b)) \leq ereal\ b$   
**proof** –  
**have** *nontr*:  $\sim\ trivial-limit\ (at\ 1\ within\ \{0..<1::real\})$   
**apply** (*subst trivial-limit-within*) **using** *real-interval-limpt* **by** *auto*  
**have**  $((\lambda m. ereal\ ((1-m)*a+m*b)) \longrightarrow (1 - 1) * a + 1 * b)$   $(at\ 1\ within\ \{0..<1\})$   
**unfolding** *lim-ereal* **by** (*intro tendsto-intros*)  
**from** *lim-imp-Limsup[OF nontr this]* **show** *?thesis* **by** *simp*  
**qed**

**lemma** *lsc-hull-of-convex*:  
**fixes**  $f :: 'a::euclidean-space \Rightarrow ereal$   
**assumes** *convex-on UNIV f*  
**assumes**  $x : rel-interior\ (domain\ f)$   
**shows**  $((\lambda m. f((1-m)*_R\ x + m*_R\ y)) \longrightarrow (lsc-hull\ f)\ y)$   $(at\ 1\ within\ \{0..<1\})$   
**(is**  $(?g \longrightarrow -\ y)$  **-)**  
**proof** (*cases y=x*)  
**case** *True*  
**hence**  $?g = (\lambda m. f\ y)$  **by** (*simp add: algebra-simps*)  
**hence**  $(?g \longrightarrow f\ y)$   $(at\ 1\ within\ \{0..<1\})$  **by** *simp*  
**moreover have**  $(lsc-hull\ f)\ y = f\ y$  **by** (*metis \langle y=x \rangle assms lsc-hull-of-convex-agrees-onRI*)  
**ultimately show** *?thesis* **by** *auto*



```

next
  case False
  have aux:  $\forall m. y - ((1 - m) *_R x + m *_R y) = (1 - m) *_R (y - x)$  by (simp add:
algebra-simps)
  have (lsc-hull f)  $y = \min (f y) (Liminf (at y) f)$  by (metis lsc-hull-liminf-at)
  also have  $\dots \leq Liminf (at 1 \text{ within } \{0..<1\}) ?g$  unfolding min-Liminf-at
unfolding Liminf-within
  apply (subst SUP-mono) apply (rule-tac x=n/norm(y-x) in bezI)
  apply (subst INF-mono) apply (rule-tac x=(1 - m) *_R x + m *_R y in bezI)
prefer 2
  unfolding ball-def dist-norm by (auto simp add: aux <y≠x> less-divide-eq)
  finally have  $*$ : (lsc-hull f)  $y \leq Liminf (at 1 \text{ within } \{0..<1\}) ?g$  by auto
  { fix b assume ereal b  $\geq (lsc-hull f) y$ 
  hence yb:  $(y, b) : \text{closure}(\text{Epigraph UNIV } f)$  by (metis epigraph-lsc-hull mem-Epigraph
  UNIV-I)
  have  $x : \text{domain } f$  by (metis assms(2) rel-interior-subset rev-subsetD)
  hence  $f x < \infty$  unfolding domain-def by auto
  then obtain a where ereal a  $> f x$  by (metis ereal-dense2)
  hence xa:  $(x, a) : \text{rel-interior}(\text{Epigraph UNIV } f)$  by (metis assms rel-interior-Epigraph)
  { fix m :: real assume  $0 \leq m \wedge m < 1$ 
  hence  $(y, b) - (1 - m) *_R ((y, b) - (x, a)) : \text{rel-interior}(\text{Epigraph UNIV } f)$ 
  apply (subst rel-interior-closure-convex-shrink)
  apply (metis assms(1) convex-Epigraph convex-UNIV convex-on-ereal-univ)
  using yb xa by auto
  hence  $f (y - (1 - m) *_R (y - x)) < \text{ereal } (b - (1 - m) * (b - a))$ 
  using assms(1) rel-interior-Epigraph by auto
  hence  $?g m \leq \text{ereal } ((1 - m) * a + m * b)$  by (simp add: algebra-simps)
  }
  hence eventually  $(\lambda m. ?g m \leq \text{ereal } ((1 - m) * a + m * b))$ 
  (at 1 within \{0..<1\}) apply (subst always-eventually-within) by auto
  hence  $Limsup (at 1 \text{ within } \{0..<1\}) ?g \leq Limsup (at 1 \text{ within } \{0..<1\}) (\lambda m. \text{ereal } ((1 - m) * a + m * b))$ 
  apply (subst Limsup-mono) by auto
  also have  $\dots \leq \text{ereal } b$  using lsc-hull-of-convex-aux by auto
  finally have  $Limsup (at 1 \text{ within } \{0..<1\}) ?g \leq \text{ereal } b$  by auto
  }
  hence  $Limsup (at 1 \text{ within } \{0..<1\}) ?g \leq (lsc-hull f) y$ 
  using ereal-le-real[of (lsc-hull f) y] by auto
  moreover have nontr:  $\sim \text{trivial-limit} (at (1 :: \text{real}) \text{ within } \{0..<1\})$ 
  apply (subst trivial-limit-within) using real-interval-limpt by auto
  moreover hence  $Liminf (at 1 \text{ within } \{0..<1\}) ?g \leq Limsup (at 1 \text{ within } \{0..<1\}) ?g$ 
  apply (subst Liminf-le-Limsup) by auto
  ultimately have  $Limsup (at 1 \text{ within } \{0..<1\}) ?g = (lsc-hull f) y$ 
   $\wedge Liminf (at 1 \text{ within } \{0..<1\}) ?g = (lsc-hull f) y$ 
  using  $*$  by auto
  thus thesis apply (subst Liminf-eq-Limsup) using nontr by auto
qed

```

**end**