

The Localization of a Commutative Ring

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Abstract

We formalize the localization [1, II, §4] of a commutative ring R with respect to a multiplicative subset (i.e. a submonoid of R seen as a multiplicative monoid).

This localization is itself a commutative ring and we build the natural homomorphism of rings from R to its localization.

Contents

1 The Localization of a Commutative Ring	1
1.1 Localization	1
1.2 The Natural Homomorphism from a Ring to Its Localization	7

2 Acknowledgements	7
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theory *Localization*

imports *Main HOL-Algebra.Group HOL-Algebra.Ring HOL-Algebra.AbelCoset*
begin

Contents:

- We define the localization of a commutative ring R with respect to a multiplicative subset, i.e. with respect to a submonoid of R (seen as a multiplicative monoid), cf. [*rec-rng-of-frac*].
- We prove that this localization is a commutative ring (cf. [*crng-rng-of-frac*]) equipped with a homomorphism of rings from R (cf. [*rng-to-rng-of-frac-is-ring-hom*]).

1 The Localization of a Commutative Ring

1.1 Localization

locale *submonoid = monoid M for M (structure) +*

fixes *S*

assumes *subset : S ⊆ carrier M*

and *m-closed* [*intro*, *simp*] : $\llbracket x \in S; y \in S \rrbracket \implies x \otimes y \in S$
and *one-closed* [*simp*] : $\mathbf{1} \in S$

lemma (**in** *submonoid*) *is-submonoid: submonoid M S*
 ⟨*proof*⟩

locale *mult-submonoid-of-rng = ring R + submonoid R S for R and S*

locale *mult-submonoid-of-crng = cring R + mult-submonoid-of-rng R S for R and S*

locale *eq-obj-rng-of-frac = cring R + mult-submonoid-of-crng R S for R (structure) and S +*
fixes *rel*
defines $rel \equiv (\text{carrier} = \text{carrier } R \times S, eq = \lambda(r,s) (r',s'). \exists t \in S. t \otimes ((s' \otimes r) \ominus (s \otimes r')) = \mathbf{0})$

lemma (**in** *abelian-group*) *minus-to-eq* :
assumes *abelian-group G and x ∈ carrier G and y ∈ carrier G and x ⊖ y = 0*
shows $x = y$
 ⟨*proof*⟩

lemma (**in** *eq-obj-rng-of-frac*) *equiv-obj-rng-of-frac*:
shows *equivalence rel*
 ⟨*proof*⟩

definition *eq-class-of-rng-of-frac*:: $- \Rightarrow 'a \Rightarrow 'b \Rightarrow \text{-set} (\text{infix } \langle | \rangle 10)$
where $r \text{ |}_{rel} s \equiv \{(r', s') \in \text{carrier } rel. (r, s) \text{ .}=\text{rel} (r', s')\}$

lemma *class-of-to-rel*:
shows $\text{class-of}_{rel} (r, s) = (r \text{ |}_{rel} s)$
 ⟨*proof*⟩

lemma (**in** *eq-obj-rng-of-frac*) *zero-in-mult-submonoid*:
assumes $\mathbf{0} \in S$ **and** $(r, s) \in \text{carrier } rel$ **and** $(r', s') \in \text{carrier } rel$
shows $(r \text{ |}_{rel} s) = (r' \text{ |}_{rel} s')$
 ⟨*proof*⟩

definition *set-eq-class-of-rng-of-frac*:: $- \Rightarrow \text{-set} (\langle \text{set}'\text{-class}'\text{-of}\rangle)$
where $\text{set-class-of}_{rel} \equiv \{(r \text{ |}_{rel} s) \mid r \text{ s}. (r, s) \in \text{carrier } rel\}$

lemma *elem-eq-class*:
assumes *equivalence S and x ∈ carrier S and y ∈ carrier S and x .=_S y*
shows $\text{class-of}_S x = \text{class-of}_S y$
 ⟨*proof*⟩

lemma (**in** *abelian-group*) *four-elem-comm*:
assumes $a \in \text{carrier } G$ **and** $b \in \text{carrier } G$ **and** $c \in \text{carrier } G$ **and** $d \in \text{carrier } G$

G
shows $a \oplus c \oplus b \oplus d = a \oplus b \oplus c \oplus d$
 $\langle proof \rangle$

lemma (in *abelian-monoid*) *right-add-eq*:
assumes $a = b$
shows $c \oplus a = c \oplus b$
 $\langle proof \rangle$

lemma (in *abelian-monoid*) *right-minus-eq*:
assumes $a = b$
shows $c \ominus a = c \ominus b$
 $\langle proof \rangle$

lemma (in *abelian-group*) *inv-add*:
assumes $a \in carrier\ G$ **and** $b \in carrier\ G$
shows $\ominus (a \oplus b) = \ominus a \oplus b$
 $\langle proof \rangle$

lemma (in *abelian-group*) *right-inv-add*:
assumes $a \in carrier\ G$ **and** $b \in carrier\ G$ **and** $c \in carrier\ G$
shows $c \oplus a \oplus b = c \oplus (a \oplus b)$
 $\langle proof \rangle$

context *eq-obj-rng-of-frac*
begin

definition *carrier-rng-of-frac*:: - *partial-object*
where *carrier-rng-of-frac* \equiv $(\langle carrier = set-class-of_{rel} \rangle)$

definition *mult-rng-of-frac*:: $[-set, -set] \Rightarrow -set$
where *mult-rng-of-frac* $X\ Y \equiv$
 $let\ x' = (SOME\ x.\ x \in X)\ in$
 $let\ y' = (SOME\ y.\ y \in Y)\ in$
 $(fst\ x' \otimes fst\ y')|_{rel}\ (snd\ x' \otimes snd\ y')$

definition *rec-monoid-rng-of-frac*:: - *monoid*
where *rec-monoid-rng-of-frac* \equiv $(\langle carrier = set-class-of_{rel},\ mult = mult-rng-of-frac,$
 $one = (\mathbf{1}|_{rel}\ \mathbf{1}) \rangle)$

lemma *member-class-to-carrier*:
assumes $x \in (r\ |_{rel}\ s)$ **and** $y \in (r'\ |_{rel}\ s')$
shows $(fst\ x \otimes fst\ y,\ snd\ x \otimes snd\ y) \in carrier\ rel$
 $\langle proof \rangle$

lemma *member-class-to-member-class*:
assumes $x \in (r\ |_{rel}\ s)$ **and** $y \in (r'\ |_{rel}\ s')$
shows $(fst\ x \otimes fst\ y\ |_{rel}\ snd\ x \otimes snd\ y) \in set-class-of_{rel}$
 $\langle proof \rangle$

lemma *closed-mult-rng-of-frac* :

assumes $(r, s) \in \text{carrier rel}$ **and** $(t, u) \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \otimes_{\text{rec-monoid-rng-of-frac}} (t \mid_{\text{rel}} u) \in \text{set-class-of}_{\text{rel}}$

<proof>

lemma *non-empty-class*:

assumes $(r, s) \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \neq \{\}$

<proof>

lemma *mult-rng-of-frac-fundamental-lemma*:

assumes $(r, s) \in \text{carrier rel}$ **and** $(r', s') \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \otimes_{\text{rec-monoid-rng-of-frac}} (r' \mid_{\text{rel}} s') = (r \otimes r' \mid_{\text{rel}} s \otimes s')$

<proof>

lemma *member-class-to-assoc*:

assumes $x \in (r \mid_{\text{rel}} s)$ **and** $y \in (t \mid_{\text{rel}} u)$ **and** $z \in (v \mid_{\text{rel}} w)$

shows $((fst\ x \otimes fst\ y) \otimes fst\ z \mid_{\text{rel}} (snd\ x \otimes snd\ y) \otimes snd\ z) = (fst\ x \otimes (fst\ y \otimes fst\ z) \mid_{\text{rel}} snd\ x \otimes (snd\ y \otimes snd\ z))$

<proof>

lemma *assoc-mult-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$ **and** $(t, u) \in \text{carrier rel}$ **and** $(v, w) \in \text{carrier rel}$

shows $((r \mid_{\text{rel}} s) \otimes_{\text{rec-monoid-rng-of-frac}} (t \mid_{\text{rel}} u)) \otimes_{\text{rec-monoid-rng-of-frac}} (v \mid_{\text{rel}} w) =$

$(r \mid_{\text{rel}} s) \otimes_{\text{rec-monoid-rng-of-frac}} ((t \mid_{\text{rel}} u) \otimes_{\text{rec-monoid-rng-of-frac}} (v \mid_{\text{rel}} w))$

<proof>

lemma *left-unit-mult-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$

shows $\mathbf{1}_{\text{rec-monoid-rng-of-frac}} \otimes_{\text{rec-monoid-rng-of-frac}} (r \mid_{\text{rel}} s) = (r \mid_{\text{rel}} s)$

<proof>

lemma *right-unit-mult-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \otimes_{\text{rec-monoid-rng-of-frac}} \mathbf{1}_{\text{rec-monoid-rng-of-frac}} = (r \mid_{\text{rel}} s)$

<proof>

lemma *monoid-rng-of-frac*:

shows *monoid* (*rec-monoid-rng-of-frac*)

<proof>

lemma *comm-mult-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$ **and** $(r', s') \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \otimes_{\text{rec-monoid-rng-of-frac}} (r' \mid_{\text{rel}} s') = (r' \mid_{\text{rel}} s') \otimes_{\text{rec-monoid-rng-of-frac}} (r \mid_{\text{rel}} s)$

<proof>

lemma *comm-monoid-rng-of-frac*:
shows *comm-monoid* (*rec-monoid-rng-of-frac*)
 ⟨*proof*⟩

definition *add-rng-of-frac*:: [-set, -set] ⇒ -set
where *add-rng-of-frac* $X Y \equiv$
 let $x' = (\text{SOME } x. x \in X)$ in
 let $y' = (\text{SOME } y. y \in Y)$ in
 ($\text{snd } y' \otimes \text{fst } x' \oplus \text{snd } x' \otimes \text{fst } y'$) $|_{\text{rel}}$ ($\text{snd } x' \otimes \text{snd } y'$)

definition *rec-rng-of-frac*:: - ring
where *rec-rng-of-frac* \equiv
 (| *carrier* = *set-class-of* $_{\text{rel}}$, *mult* = *mult-rng-of-frac*, *one* = ($\mathbf{1} |_{\text{rel}} \mathbf{1}$), *zero* = ($\mathbf{0} |_{\text{rel}}$
 $\mathbf{1}$), *add* = *add-rng-of-frac* |)

lemma *add-rng-of-frac-fundamental-lemma*:
assumes $(r, s) \in \text{carrier } \text{rel}$ **and** $(r', s') \in \text{carrier } \text{rel}$
shows $(r |_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r' |_{\text{rel}} s') = (s' \otimes r \oplus s \otimes r' |_{\text{rel}} s \otimes s')$
 ⟨*proof*⟩

lemma *closed-add-rng-of-frac*:
assumes $(r, s) \in \text{carrier } \text{rel}$ **and** $(r', s') \in \text{carrier } \text{rel}$
shows $(r |_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r' |_{\text{rel}} s') \in \text{set-class-of } \text{rel}$
 ⟨*proof*⟩

lemma *closed-rel-add*:
assumes $(r, s) \in \text{carrier } \text{rel}$ **and** $(r', s') \in \text{carrier } \text{rel}$
shows $(s' \otimes r \oplus s \otimes r', s \otimes s')$ $\in \text{carrier } \text{rel}$
 ⟨*proof*⟩

lemma *assoc-add-rng-of-frac*:
assumes $(r, s) \in \text{carrier } \text{rel}$ **and** $(r', s') \in \text{carrier } \text{rel}$ **and** $(r'', s'') \in \text{carrier } \text{rel}$
shows $(r |_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r' |_{\text{rel}} s') \oplus_{\text{rec-rng-of-frac}} (r'' |_{\text{rel}} s'') =$
 $(r |_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} ((r' |_{\text{rel}} s') \oplus_{\text{rec-rng-of-frac}} (r'' |_{\text{rel}} s''))$
 ⟨*proof*⟩

lemma *add-rng-of-frac-zero*:
shows $(\mathbf{0} |_{\text{rel}} \mathbf{1}) \in \text{set-class-of } \text{rel}$
 ⟨*proof*⟩

lemma *l-unit-add-rng-of-frac*:
assumes $(r, s) \in \text{carrier } \text{rel}$
shows $\mathbf{0}_{\text{rec-rng-of-frac}} \oplus_{\text{rec-rng-of-frac}} (r |_{\text{rel}} s) = (r |_{\text{rel}} s)$
 ⟨*proof*⟩

lemma *r-unit-add-rng-of-frac*:
assumes $(r, s) \in \text{carrier } \text{rel}$
shows $(r |_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} \mathbf{0}_{\text{rec-rng-of-frac}} = (r |_{\text{rel}} s)$

<proof>

lemma *comm-add-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$ **and** $(r', s') \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r' \mid_{\text{rel}} s') = (r' \mid_{\text{rel}} s') \oplus_{\text{rec-rng-of-frac}} (r \mid_{\text{rel}} s)$

<proof>

lemma *class-of-zero-rng-of-frac*:

assumes $s \in S$

shows $(\mathbf{0} \mid_{\text{rel}} s) = \mathbf{0}_{\text{rec-rng-of-frac}}$

<proof>

lemma *r-inv-add-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$

shows $(r \mid_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (\ominus r \mid_{\text{rel}} s) = \mathbf{0}_{\text{rec-rng-of-frac}}$

<proof>

lemma *l-inv-add-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$

shows $(\ominus r \mid_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r \mid_{\text{rel}} s) = \mathbf{0}_{\text{rec-rng-of-frac}}$

<proof>

lemma *abelian-group-rng-of-frac*:

shows *abelian-group* (*rec-rng-of-frac*)

<proof>

lemma *r-distr-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$ **and** $(r', s') \in \text{carrier rel}$ **and** $(r'', s'') \in \text{carrier rel}$

shows $((r \mid_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r' \mid_{\text{rel}} s')) \otimes_{\text{rec-rng-of-frac}} (r'' \mid_{\text{rel}} s'') =$

$(r \mid_{\text{rel}} s) \otimes_{\text{rec-rng-of-frac}} (r'' \mid_{\text{rel}} s'') \oplus_{\text{rec-rng-of-frac}} (r' \mid_{\text{rel}} s') \otimes_{\text{rec-rng-of-frac}} (r'' \mid_{\text{rel}} s'')$

<proof>

lemma *l-distr-rng-of-frac*:

assumes $(r, s) \in \text{carrier rel}$ **and** $(r', s') \in \text{carrier rel}$ **and** $(r'', s'') \in \text{carrier rel}$

shows $(r'' \mid_{\text{rel}} s'') \otimes_{\text{rec-rng-of-frac}} ((r \mid_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r' \mid_{\text{rel}} s')) =$

$(r'' \mid_{\text{rel}} s'') \otimes_{\text{rec-rng-of-frac}} (r \mid_{\text{rel}} s) \oplus_{\text{rec-rng-of-frac}} (r'' \mid_{\text{rel}} s'') \otimes_{\text{rec-rng-of-frac}} (r' \mid_{\text{rel}} s')$

<proof>

lemma *rng-rng-of-frac*:

shows *ring* (*rec-rng-of-frac*)

<proof>

lemma *crng-rng-of-frac*:

shows *cring* (*rec-rng-of-frac*)

<proof>

lemma *simp-in-frac*:

assumes $(r, s) \in \text{carrier } \text{rel}$ **and** $s' \in S$
shows $(r \mid_{\text{rel}} s) = (s' \otimes r \mid_{\text{rel}} s' \otimes s)$
 ⟨proof⟩

1.2 The Natural Homomorphism from a Ring to Its Localization

definition *rng-to-rng-of-frac* :: $'a \Rightarrow ('a \times 'a)$ set where
rng-to-rng-of-frac $r \equiv (r \mid_{\text{rel}} \mathbf{1})$

lemma *rng-to-rng-of-frac-is-ring-hom* :
shows *rng-to-rng-of-frac* \in *ring-hom* R *rec-rng-of-frac*
 ⟨proof⟩

lemma *Im-rng-to-rng-of-frac-unit*:
assumes $x \in \text{rng-to-rng-of-frac } 'S$
shows $x \in \text{Units } \text{rec-rng-of-frac}$
 ⟨proof⟩

lemma *eq-class-to-rel*:
assumes $(r, s) \in \text{carrier } R \times S$ **and** $(r', s') \in \text{carrier } R \times S$ **and** $(r \mid_{\text{rel}} s) = (r' \mid_{\text{rel}} s')$
shows $(r, s) \text{ .}=\text{rel } (r', s')$
 ⟨proof⟩

lemma *rng-to-rng-of-frac-without-zero-div-is-inj*:
assumes $\mathbf{0} \notin S$ **and** $\forall a \in \text{carrier } R. \forall b \in \text{carrier } R. a \otimes b = \mathbf{0} \longrightarrow a = \mathbf{0} \vee b = \mathbf{0}$
shows *a-kernel* R *rec-rng-of-frac* *rng-to-rng-of-frac* = $\{\mathbf{0}\}$
 ⟨proof⟩

end

end

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References

- [1] S. Lang. *Algebra*. Springer, revised third edition edition, 2002.