

Reasoning about Lists via List Interleaving

Pasquale Noce

Security Certification Specialist at Arjo Systems - Gep S.p.A.

pasquale dot noce dot lavoro at gmail dot com

pasquale dot noce at arjowiggins-it dot com

May 21, 2019

Abstract

Among the various mathematical tools introduced in his outstanding work on Communicating Sequential Processes, Hoare has defined "interleaves" as the predicate satisfied by any three lists such that the first list may be split into sublists alternately extracted from the other two ones, whatever is the criterion for extracting an item from either one list or the other in each step.

This paper enriches Hoare's definition by identifying such criterion with the truth value of a predicate taking as inputs the head and the tail of the first list. This enhanced "interleaves" predicate turns out to permit the proof of equalities between lists without the need of an induction. Some rules that allow to infer "interleaves" statements without induction, particularly applying to the addition or removal of a prefix to the input lists, are also proven. Finally, a stronger version of the predicate, named "Interleaves", is shown to fulfil further rules applying to the addition or removal of a suffix to the input lists.

Contents

1 List interleaving	1
1.1 A first version of interleaving	2
1.2 A second, stronger version of interleaving	11

1 List interleaving

```
theory ListInterleaving
imports Main
begin
```

Among the various mathematical tools introduced in his outstanding work on Communicating Sequential Processes [1], Hoare has defined *interleaves*

as the predicate satisfied by any three lists s , t , u such that s may be split into sublists alternately extracted from t and u , whatever is the criterion for extracting an item from either t or u in each step.

This paper enriches Hoare's definition by identifying such criterion with the truth value of a predicate taking as inputs the head and the tail of s . This enhanced *interleaves* predicate turns out to permit the proof of equalities between lists without the need of an induction. Some rules that allow to infer *interleaves* statements without induction, particularly applying to the addition of a prefix to the input lists, are also proven. Finally, a stronger version of the predicate, named *Interleaves*, is shown to fulfil further rules applying to the addition of a suffix to the input lists.

Throughout this paper, the salient points of definitions and proofs are commented; for additional information, cf. Isabelle documentation, particularly [5], [4], [3], and [2]. For a sample nontrivial application of the mathematical machinery developed in this paper, cf. [6].

1.1 A first version of interleaving

Here below is the definition of predicate *interleaves*, as well as of a convenient symbolic notation for it. As in the definition of predicate *interleaves* formulated in [1], the recursive decomposition of the input lists is performed by item prepending. In the case of a list ws constructed recursively by item appending rather than prepending, the statement that it satisfies predicate *interleaves* with two further lists can nevertheless be proven by induction using as input *rev ws*, rather than ws itself.

With respect to Hoare's homonymous predicate, *interleaves* takes as an additional input a predicate P , which is a function of a single item and a list. Then, for statement *interleaves* P $(x \# xs)$ $(y \# ys)$ $(z \# zs)$ to hold, the item picked for being x must be y if P x xs , z otherwise. On the contrary, in case either the second or the third list is empty, the truth value of P x xs does not matter and list $x \# xs$ just has to match the other nonempty one, if any.

```

fun interleaves ::
  ('a  $\Rightarrow$  'a list  $\Rightarrow$  bool)  $\Rightarrow$  'a list  $\Rightarrow$  'a list  $\Rightarrow$  'a list  $\Rightarrow$  bool where
  interleaves P (x # xs) (y # ys) (z # zs) = (if P x xs
    then x = y  $\wedge$  interleaves P xs ys (z # zs)
    else x = z  $\wedge$  interleaves P xs (y # ys) zs) |
  interleaves P (x # xs) (y # ys) [] =
    (x = y  $\wedge$  interleaves P xs ys []) |
  interleaves P (x # xs) [] (z # zs) =
    (x = z  $\wedge$  interleaves P xs [] zs) |
  interleaves - (- # -) [] [] = False |
  interleaves - [] (- # -) - = False |

```

interleaves - [] - (- # -) = *False* |
interleaves - [] [] = *True*

abbreviation *interleaves-syntax* ::

'a list \Rightarrow 'a list \Rightarrow 'a list \Rightarrow ('a \Rightarrow 'a list \Rightarrow bool) \Rightarrow bool

((- \simeq {-, -, -}) [60, 60, 60] 51)

where $xs \simeq \{ys, zs, P\} \equiv \text{interleaves } P \text{ } xs \text{ } ys \text{ } zs$

The advantage provided by this enhanced *interleaves* predicate is that in case $xs \simeq \{ys, zs, P\}$, fixing the values of xs and either ys or zs has the effect of fixing the value of the remaining list, too. Namely, if $xs \simeq \{ys', zs, P\}$ also holds, then $ys = ys'$, and likewise, if $xs \simeq \{ys, zs', P\}$ also holds, then $zs = zs'$. Therefore, once two *interleaves* statements $xs \simeq \{ys, zs, P\}$, $xs' \simeq \{ys', zs', P\}$ have been proven along with equalities $xs = xs'$, $P = P'$, and either $zs = zs'$ or $ys = ys'$, possibly by induction, the remaining equality, i.e. respectively $ys = ys'$ and $zs = zs'$, can be inferred without the need of a further induction.

Here below is the proof of this property as well as of other ones. Particularly, it is also proven that in case $xs \simeq \{ys, zs, P\}$, lists ys and zs can be swapped by replacing predicate P with its negation.

It is worth noting that fixing the values of ys and zs does not fix the value of xs instead. A counterexample is $ys = [y]$, $zs = [z]$ with $y \neq z$, $P \ y \ [z] = \text{True}$, $P \ z \ [y] = \text{False}$, in which case both of the *interleaves* statements $[y, z] \simeq \{ys, zs, P\}$ and $[z, y] \simeq \{ys, zs, P\}$ hold.

lemma *interleaves-length* [rule-format]:

$xs \simeq \{ys, zs, P\} \longrightarrow \text{length } xs = \text{length } ys + \text{length } zs$

proof (*induction* $P \ xs \ ys \ zs$ rule: *interleaves.induct*, *simp-all*)

qed (*rule conjI*, (*rule-tac* [!] *impI*)+, *simp-all*)

lemma *interleaves-nil*:

$[] \simeq \{ys, zs, P\} \implies ys = [] \wedge zs = []$

by (*rule interleaves.cases* [*of* (P , [], ys , zs)], *simp-all*)

lemma *interleaves-swap*:

$xs \simeq \{ys, zs, P\} = xs \simeq \{zs, ys, \lambda w \ ws. \neg P \ w \ ws\}$

proof (*induction* $P \ xs \ ys \ zs$ rule: *interleaves.induct*, *simp-all*)

fix $y' :: 'a$ **and** $ys' \ zs' \ P'$

show $\neg [] \simeq \{zs', y' \# ys', \lambda w \ ws. \neg P' \ w \ ws\}$ **by** (*cases* zs' , *simp-all*)

qed

lemma *interleaves-equal-fst* [rule-format]:

$xs \simeq \{ys, zs, P\} \longrightarrow xs \simeq \{ys', zs, P\} \longrightarrow ys = ys'$

proof (*induction* xs arbitrary: $ys \ ys' \ zs$, (*rule-tac* [!] *impI*)+)

fix $ys \ ys' \ zs$

assume $[] \simeq \{ys, zs, P\}$

hence $ys = [] \wedge zs = []$ by (rule interleaves-nil)
 moreover assume $[] \simeq \{ys', zs, P\}$
 hence $ys' = [] \wedge zs = []$ by (rule interleaves-nil)
 ultimately show $ys = ys'$ by simp

next

fix $x xs ys ys' zs$

assume

$A: \bigwedge ys ys' zs. xs \simeq \{ys, zs, P\} \longrightarrow xs \simeq \{ys', zs, P\} \longrightarrow ys = ys'$ and

$B: x \# xs \simeq \{ys, zs, P\}$ and

$B': x \# xs \simeq \{ys', zs, P\}$

show $ys = ys'$

proof (cases zs , case-tac [2] ys , case-tac [2-3] ys' , simp-all)

assume $C: zs = []$

hence $\exists w ws. ys = w \# ws$ using B by (cases ys , simp-all)

then obtain $w ws$ where $Y: ys = w \# ws$ by blast

hence $D: w = x$ using B and C by simp

have $xs \simeq \{ws, [], P\}$ using B and C and Y by simp

moreover have $\exists w' ws'. ys' = w' \# ws'$

using B' and C by (cases ys' , simp-all)

then obtain $w' ws'$ where $Y': ys' = w' \# ws'$ by blast

hence $D': w' = x$ using B' and C by simp

have $xs \simeq \{ws', [], P\}$ using B' and C and Y' by simp

moreover have $xs \simeq \{ws, [], P\} \longrightarrow xs \simeq \{ws', [], P\} \longrightarrow ws = ws'$

using A .

ultimately have $ws = ws'$ by simp

with Y and Y' and D and D' show ?thesis by simp

next

fix $v vs w' ws'$

assume $C: zs = v \# vs$ and $ys = []$

hence $D: xs \simeq \{[], vs, P\}$ using B by simp

assume $E: ys' = w' \# ws'$

have $P x xs \vee \neg P x xs$ by simp

moreover {

assume $P x xs$

hence $xs \simeq \{ws', v \# vs, P\}$ using B' and C and E by simp

hence $\text{length } xs = \text{Suc } (\text{length } vs) + \text{length } ws'$

by (simp add: interleaves-length)

moreover have $\text{length } xs = \text{length } vs$

using D by (simp add: interleaves-length)

ultimately have False by simp

}

moreover {

assume $\neg P x xs$

hence $xs \simeq \{w' \# ws', vs, P\}$ using B' and C and E by simp

moreover have $xs \simeq \{[], vs, P\} \longrightarrow xs \simeq \{w' \# ws', vs, P\} \longrightarrow$

$[] = w' \# ws'$

using A .

ultimately have $[] = w' \# ws'$ using D by simp

hence False by simp

```

}
ultimately show False ..
next
fix v vs w ws
assume C:  $zs = v \# vs$  and  $ys' = []$ 
hence D:  $xs \simeq \{[], vs, P\}$  using B' by simp
assume E:  $ys = w \# ws$ 
have  $P \ x \ xs \vee \neg P \ x \ xs$  by simp
moreover {
  assume  $P \ x \ xs$ 
  hence  $xs \simeq \{ws, v \# vs, P\}$  using B and C and E by simp
  hence  $length \ xs = Suc \ (length \ vs) + length \ ws$ 
  by (simp add: interleaves-length)
  moreover have  $length \ xs = length \ vs$ 
  using D by (simp add: interleaves-length)
  ultimately have False by simp
}
moreover {
  assume  $\neg P \ x \ xs$ 
  hence  $xs \simeq \{w \# ws, vs, P\}$  using B and C and E by simp
  moreover have  $xs \simeq \{[], vs, P\} \longrightarrow xs \simeq \{w \# ws, vs, P\} \longrightarrow [] = w \# ws$ 
  using A .
  ultimately have  $[] = w \# ws$  using D by simp
  hence False by simp
}
ultimately show False ..
next
fix v vs w ws w' ws'
assume C:  $zs = v \# vs$  and D:  $ys = w \# ws$  and D':  $ys' = w' \# ws'$ 
have  $P \ x \ xs \vee \neg P \ x \ xs$  by simp
moreover {
  assume E:  $P \ x \ xs$ 
  hence F:  $w = x$  using B and C and D by simp
  have  $xs \simeq \{ws, v \# vs, P\}$  using B and C and D and E by simp
  moreover have F':  $w' = x$  using B' and C and D' and E by simp
  have  $xs \simeq \{ws', v \# vs, P\}$  using B' and C and D' and E by simp
  moreover have  $xs \simeq \{ws, v \# vs, P\} \longrightarrow xs \simeq \{ws', v \# vs, P\} \longrightarrow$ 
     $ws = ws'$ 
  using A .
  ultimately have  $ws = ws'$  by simp
  hence  $w = w' \wedge ws = ws'$  using F and F' by simp
}
moreover {
  assume E:  $\neg P \ x \ xs$ 
  hence  $xs \simeq \{w \# ws, vs, P\}$  using B and C and D by simp
  moreover have  $xs \simeq \{w' \# ws', vs, P\}$ 
  using B' and C and D' and E by simp
  moreover have  $xs \simeq \{w \# ws, vs, P\} \longrightarrow xs \simeq \{w' \# ws', vs, P\} \longrightarrow$ 
     $w \# ws = w' \# ws'$ 
}

```

using A .
 ultimately have $w \# ws = w' \# ws'$ by *simp*
 hence $w = w' \wedge ws = ws'$ by *simp*
 }
 ultimately show $w = w' \wedge ws = ws' ..$
 qed
 qed

lemma *interleaves-equal-snd*:
 $xs \simeq \{ys, zs, P\} \implies xs \simeq \{ys, zs', P\} \implies zs = zs'$
by (*subst (asm) (1 2) interleaves-swap, rule interleaves-equal-fst*)

Since *interleaves* statements permit to prove equalities between lists without the need of an induction, it is useful to search for rules that allow to infer such statements themselves without induction, which is precisely what is done here below. Particularly, it is proven that under proper assumptions, predicate *interleaves* keeps being satisfied by applying a filter, a mapping, or the addition or removal of a prefix to the input lists.

lemma *interleaves-all-nil*:
 $xs \simeq \{xs, [], P\}$
by (*induction xs, simp-all*)

lemma *interleaves-nil-all*:
 $xs \simeq \{[], xs, P\}$
by (*induction xs, simp-all*)

lemma *interleaves-equal-all-nil*:
 $xs \simeq \{ys, [], P\} \implies xs = ys$
by (*insert interleaves-all-nil, rule interleaves-equal-fst*)

lemma *interleaves-equal-nil-all*:
 $xs \simeq \{[], zs, P\} \implies xs = zs$
by (*insert interleaves-nil-all, rule interleaves-equal-snd*)

lemma *interleaves-filter* [*rule-format*]:
 assumes $A: \forall x xs. P x (\text{filter } Q xs) = P x xs$
 shows $xs \simeq \{ys, zs, P\} \longrightarrow \text{filter } Q xs \simeq \{\text{filter } Q ys, \text{filter } Q zs, P\}$
proof (*induction xs arbitrary: ys zs, rule-tac [!] impI, simp*)
 fix $ys zs$
 assume $\square \simeq \{ys, zs, P\}$
 hence $ys = \square \wedge zs = \square$ **by** (*rule interleaves-nil*)
 thus $\square \simeq \{\text{filter } Q ys, \text{filter } Q zs, P\}$ **by** *simp*
next
 fix $x xs ys zs$
 assume
 $B: \bigwedge ys' zs'. xs \simeq \{ys', zs', P\} \longrightarrow$

$\text{filter } Q \text{ } xs \simeq \{\text{filter } Q \text{ } ys', \text{filter } Q \text{ } zs', P\}$ **and**
 $C: x \# xs \simeq \{ys, zs, P\}$
show $\text{filter } Q (x \# xs) \simeq \{\text{filter } Q \text{ } ys, \text{filter } Q \text{ } zs, P\}$
proof (*cases* ys , *case-tac* $!$ zs , *simp-all del*: filter.simps , *rule ccontr*)
 assume $ys = []$ **and** $zs = []$
 thus *False* **using** C **by** *simp*
next
 fix $z \text{ } zs'$
 assume $ys = []$ **and** $zs = z \# zs'$
 hence $D: x = z \wedge xs \simeq \{[], zs', P\}$ **using** C **by** *simp*
 moreover **have** $xs \simeq \{[], zs', P\} \longrightarrow$
 $\text{filter } Q \text{ } xs \simeq \{\text{filter } Q \text{ } [], \text{filter } Q \text{ } zs', P\}$
 using B .
 ultimately **have** $\text{filter } Q \text{ } xs \simeq \{[], \text{filter } Q \text{ } zs', P\}$ **by** *simp*
 thus $\text{filter } Q (x \# xs) \simeq \{[], \text{filter } Q (z \# zs'), P\}$ **using** D **by** *simp*
next
 fix $y \text{ } ys'$
 assume $ys = y \# ys'$ **and** $zs = []$
 hence $D: x = y \wedge xs \simeq \{ys', [], P\}$ **using** C **by** *simp*
 moreover **have** $xs \simeq \{ys', [], P\} \longrightarrow$
 $\text{filter } Q \text{ } xs \simeq \{\text{filter } Q \text{ } ys', \text{filter } Q \text{ } [], P\}$
 using B .
 ultimately **have** $\text{filter } Q \text{ } xs \simeq \{\text{filter } Q \text{ } ys', [], P\}$ **by** *simp*
 thus $\text{filter } Q (x \# xs) \simeq \{\text{filter } Q (y \# ys'), [], P\}$ **using** D **by** *simp*
next
 fix $y \text{ } ys' \text{ } z \text{ } zs'$
 assume $ys = y \# ys'$ **and** $zs = z \# zs'$
 hence $D: x \# xs \simeq \{y \# ys', z \# zs', P\}$ **using** C **by** *simp*
 show $\text{filter } Q (x \# xs) \simeq \{\text{filter } Q (y \# ys'), \text{filter } Q (z \# zs'), P\}$
 proof (*cases* $P \text{ } x \text{ } xs$)
 case *True*
 hence $E: P \text{ } x$ ($\text{filter } Q \text{ } xs$) **using** A **by** *simp*
 have $F: x = y \wedge xs \simeq \{ys', z \# zs', P\}$ **using** D **and** *True* **by** *simp*
 moreover **have** $xs \simeq \{ys', z \# zs', P\} \longrightarrow$
 $\text{filter } Q \text{ } xs \simeq \{\text{filter } Q \text{ } ys', \text{filter } Q (z \# zs'), P\}$
 using B .
 ultimately **have** $G: \text{filter } Q \text{ } xs \simeq \{\text{filter } Q \text{ } ys', \text{filter } Q (z \# zs'), P\}$
 by *simp*
 show *?thesis*
 proof (*cases* $Q \text{ } x$)
 assume $Q \text{ } x$
 hence $\text{filter } Q (x \# xs) = x \# \text{filter } Q \text{ } xs$ **by** *simp*
 moreover **have** $\text{filter } Q (y \# ys') = x \# \text{filter } Q \text{ } ys'$
 using $\langle Q \text{ } x \rangle$ **and** F **by** *simp*
 ultimately **show** *?thesis* **using** E **and** G
 by (*cases* $\text{filter } Q (z \# zs')$, *simp-all*)
next
 assume $\neg Q \text{ } x$
 hence $\text{filter } Q (x \# xs) = \text{filter } Q \text{ } xs$ **by** *simp*

```

    moreover have filter Q (y # ys') = filter Q ys'
    using (¬ Q x) and F by simp
    ultimately show ?thesis using E and G
    by (cases filter Q (z # zs'), simp-all)
  qed
next
case False
hence E: ¬ P x (filter Q xs) using A by simp
have F: x = z ∧ xs ≃ {y # ys', zs', P} using D and False by simp
moreover have xs ≃ {y # ys', zs', P} ⟶
  filter Q xs ≃ {filter Q (y # ys'), filter Q zs', P}
using B .
ultimately have G: filter Q xs ≃ {filter Q (y # ys'), filter Q zs', P}
by simp
show ?thesis
proof (cases Q x)
  assume Q x
  hence filter Q (x # xs) = x # filter Q xs by simp
  moreover have filter Q (z # zs') = x # filter Q zs'
  using (Q x) and F by simp
  ultimately show ?thesis using E and G
  by (cases filter Q (y # ys'), simp-all)
next
  assume ¬ Q x
  hence filter Q (x # xs) = filter Q xs by simp
  moreover have filter Q (z # zs') = filter Q zs'
  using (¬ Q x) and F by simp
  ultimately show ?thesis using E and G
  by (cases filter Q (z # zs'), simp-all)
qed
qed
qed
qed

lemma interleaves-map [rule-format]:
  assumes A: inj f
  shows xs ≃ {ys, zs, P} ⟶
    map f xs ≃ {map f ys, map f zs, λw ws. P (inv f w) (map (inv f) ws)}
    (is - ⟶ - ≃ {-, -, ?P'})
proof (induction xs arbitrary: ys zs, rule-tac [!] impI, simp-all)
  fix ys zs
  assume [] ≃ {ys, zs, P}
  hence ys = [] ∧ zs = [] by (rule interleaves-nil)
  thus [] ≃ {map f ys, map f zs, ?P'} by simp
next
  fix x xs ys zs
  assume
    B: ∧ys zs. xs ≃ {ys, zs, P} ⟶ map f xs ≃ {map f ys, map f zs, ?P'} and
    C: x # xs ≃ {ys, zs, P}

```


show $f x \# \text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } ys, \text{map } f \text{ } zs, ?P' \}$
proof (*cases* ys , *case-tac* $[\] \text{ } zs$, *simp-all del: interleaves.simps(1)*)
 assume $ys = [\]$ **and** $zs = [\]$
 thus *False* **using** C **by** *simp*
next
 fix $z \text{ } zs'$
 assume $ys = [\]$ **and** $zs = z \# zs'$
 hence $x = z \wedge xs \simeq \{ [\], zs', P \}$ **using** C **by** *simp*
 moreover **have** $xs \simeq \{ [\], zs', P \} \longrightarrow \text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } [\], \text{map } f \text{ } zs', ?P' \}$
 using B .
 ultimately **show** $f x = f z \wedge \text{map } f \text{ } xs \simeq \{ [\], \text{map } f \text{ } zs', ?P' \}$ **by** *simp*
next
 fix $y \text{ } ys'$
 assume $ys = y \# ys'$ **and** $zs = [\]$
 hence $x = y \wedge xs \simeq \{ ys', [\], P \}$ **using** C **by** *simp*
 moreover **have** $xs \simeq \{ ys', [\], P \} \longrightarrow \text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } ys', \text{map } f \text{ } [\], ?P' \}$
 using B .
 ultimately **show** $f x = f y \wedge \text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } ys', [\], ?P' \}$ **by** *simp*
next
 fix $y \text{ } ys' \text{ } z \text{ } zs'$
 assume $ys = y \# ys'$ **and** $zs = z \# zs'$
 hence $D: x \# xs \simeq \{ y \# ys', z \# zs', P \}$ **using** C **by** *simp*
 show $f x \# \text{map } f \text{ } xs \simeq \{ f y \# \text{map } f \text{ } ys', f z \# \text{map } f \text{ } zs', ?P' \}$
 proof (*cases* $P \text{ } x \text{ } xs$)
 case *True*
 hence $E: ?P' (f x) (\text{map } f \text{ } xs)$ **using** A **by** *simp*
 have $x = y \wedge xs \simeq \{ ys', z \# zs', P \}$ **using** D **and** *True* **by** *simp*
 moreover **have** $xs \simeq \{ ys', z \# zs', P \} \longrightarrow$
 $\text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } ys', \text{map } f \text{ } (z \# zs'), ?P' \}$
 using B .
 ultimately **have** $f x = f y \wedge \text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } ys', \text{map } f \text{ } (z \# zs'), ?P' \}$
 by *simp*
 thus *?thesis* **using** E **by** *simp*
next
 case *False*
 hence $E: \neg ?P' (f x) (\text{map } f \text{ } xs)$ **using** A **by** *simp*
 have $x = z \wedge xs \simeq \{ y \# ys', zs', P \}$ **using** D **and** *False* **by** *simp*
 moreover **have** $xs \simeq \{ y \# ys', zs', P \} \longrightarrow$
 $\text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } (y \# ys'), \text{map } f \text{ } zs', ?P' \}$
 using B .
 ultimately **have** $f x = f z \wedge \text{map } f \text{ } xs \simeq \{ \text{map } f \text{ } (y \# ys'), \text{map } f \text{ } zs', ?P' \}$
 by *simp*
 thus *?thesis* **using** E **by** *simp*
qed
qed
qed

lemma *interleaves-prefix-fst-1* [*rule-format*]:
 assumes $A: xs \simeq \{ ys, zs, P \}$

shows $(\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)) \longrightarrow$
 $ws @ xs \simeq \{ws @ ys, zs, P\}$
proof (*induction ws, simp-all add: A, rule impI*)
fix $w\ ws$
assume $B: \forall n < Suc (\text{length } ws). P ((w \# ws) ! n) (\text{drop } n ws @ xs)$
assume $(\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)) \longrightarrow$
 $ws @ xs \simeq \{ws @ ys, zs, P\}$
moreover have $\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)$
proof (*rule allI, rule impI*)
fix n
assume $n < \text{length } ws$
moreover have $Suc\ n < Suc (\text{length } ws) \longrightarrow$
 $P ((w \# ws) ! (Suc\ n)) (\text{drop } (Suc\ n) ws @ xs)$
using $B ..$
ultimately show $P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)$ **by** *simp*
qed
ultimately have $ws @ xs \simeq \{ws @ ys, zs, P\} ..$
moreover have $0 < Suc (\text{length } ws) \longrightarrow P ((w \# ws) ! 0) (\text{drop } 0 ws @ xs)$
using $B ..$
hence $P w (ws @ xs)$ **by** *simp*
ultimately show $w \# ws @ xs \simeq \{w \# ws @ ys, zs, P\}$ **by** (*cases zs, simp-all*)
qed

lemma *interleaves-prefix-fst-2* [*rule-format*]:

$ws @ xs \simeq \{ws @ ys, zs, P\} \longrightarrow$
 $(\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)) \longrightarrow$
 $xs \simeq \{ys, zs, P\}$
proof (*induction ws, simp-all, (rule impI)+*)
fix $w\ ws$
assume $A: \forall n < Suc (\text{length } ws). P ((w \# ws) ! n) (\text{drop } n ws @ xs)$
hence $0 < Suc (\text{length } ws) \longrightarrow P ((w \# ws) ! 0) (\text{drop } 0 ws @ xs) ..$
hence $P w (ws @ xs)$ **by** *simp*
moreover assume $w \# ws @ xs \simeq \{w \# ws @ ys, zs, P\}$
ultimately have $ws @ xs \simeq \{ws @ ys, zs, P\}$ **by** (*cases zs, simp-all*)
moreover assume $ws @ xs \simeq \{ws @ ys, zs, P\} \longrightarrow$
 $(\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)) \longrightarrow$
 $xs \simeq \{ys, zs, P\}$
ultimately have $(\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)) \longrightarrow$
 $xs \simeq \{ys, zs, P\}$
by *simp*
moreover have $\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)$
proof (*rule allI, rule impI*)
fix n
assume $n < \text{length } ws$
moreover have $Suc\ n < Suc (\text{length } ws) \longrightarrow$
 $P ((w \# ws) ! (Suc\ n)) (\text{drop } (Suc\ n) ws @ xs)$
using $A ..$
ultimately show $P (ws ! n) (\text{drop } (Suc\ n) ws @ xs)$ **by** *simp*
qed

ultimately show $xs \simeq \{ys, zs, P\}$..
qed

lemma *interleaves-prefix-fst* [rule-format]:
 $\forall n < \text{length } ws. P (ws ! n) (\text{drop } (Suc\ n) \text{ } ws @ xs) \implies$
 $xs \simeq \{ys, zs, P\} = ws @ xs \simeq \{ws @ ys, zs, P\}$
proof (*rule iffI, erule interleaves-prefix-fst-1, simp*)
qed (*erule interleaves-prefix-fst-2, simp*)

lemma *interleaves-prefix-snd* [rule-format]:
 $\forall n < \text{length } ws. \neg P (ws ! n) (\text{drop } (Suc\ n) \text{ } ws @ xs) \implies$
 $xs \simeq \{ys, zs, P\} = ws @ xs \simeq \{ys, ws @ zs, P\}$
proof (*subst (1 2) interleaves-swap*)
qed (*rule interleaves-prefix-fst, simp*)

1.2 A second, stronger version of interleaving

Simple counterexamples show that unlike prefixes, the addition or removal of suffixes to the input lists does not generally preserve the validity of predicate *interleaves*. In fact, if $P\ y\ [x] = True$ with $x \neq y$, then $[y, x] \simeq \{[x], [y], P\}$ does not hold although $[y] \simeq \{\ [], [y], \lambda w\ ws. P\ w\ (ws @ [x])\}$ does, by virtue of lemma $?xs \simeq \{\ [], ?xs, ?P\}$. Similarly, $[x, y] \simeq \{\ [], [y, x], \lambda w\ ws. P\ w\ (ws @ [x])\}$ does not hold for $x \neq y$ even though $[x, y, x] \simeq \{[x], [y, x], P\}$ does.

Both counterexamples would not work any longer if the truth value of the input predicate were significant even if either the second or the third list is empty. In fact, in the former case, condition $P\ y\ [x] = True$ would entail the falseness of statement $[y] \simeq \{\ [], [y], \lambda w\ ws. P\ w\ (ws @ [x])\}$, so that the validity of rule $[y] \simeq \{\ [], [y], \lambda w\ ws. P\ w\ (ws @ [x])\} \implies [y, x] \simeq \{[x], [y], P\}$ would be preserved. In the latter case, statement $[x, y, x] \simeq \{[x], [y, x], P\}$ may only hold provided the last item x of the first list is extracted from the third one, which would require that $\neg P\ x\ []$; thus, subordinating rule $[x, y, x] \simeq \{[x], [y, x], P\} \implies [x, y] \simeq \{\ [], [y, x], \lambda w\ ws. P\ w\ (ws @ [x])\}$ to condition $P\ x\ []$ would preserve its validity.

This argument suggests that in order to obtain an *interleaves* predicate whose validity is also preserved upon the addition or removal of a suffix to the input lists, the truth value of the input predicate must matter until both the second and the third list are empty. In what follows, such a stronger version of the predicate, named *Interleaves*, is defined along with a convenient symbolic notation for it.

fun *Interleaves* ::
 $('a \Rightarrow 'a\ list \Rightarrow bool) \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow 'a\ list \Rightarrow bool$ **where**
Interleaves $P\ (x \# xs)\ (y \# ys)\ (z \# zs) = (if\ P\ x\ xs$
 $then\ x = y \wedge \text{Interleaves } P\ xs\ ys\ (z \# zs)$

```

else x = z ∧ Interleaves P xs (y # ys) zs |
Interleaves P (x # xs) (y # ys) [] =
(P x xs ∧ x = y ∧ Interleaves P xs ys []) |
Interleaves P (x # xs) [] (z # zs) =
(¬ P x xs ∧ x = z ∧ Interleaves P xs [] zs) |
Interleaves - (- # -) [] [] = False |
Interleaves - [] (- # -) - = False |
Interleaves - [] - (- # -) = False |
Interleaves - [] [] [] = True

```

abbreviation *Interleaves-syntax* ::

```

'a list ⇒ 'a list ⇒ 'a list ⇒ ('a ⇒ 'a list ⇒ bool) ⇒ bool
((¬ ≅ {-, -, -}) [60, 60, 60] 51)
where xs ≅ {ys, zs, P} ≡ Interleaves P xs ys zs

```

In what follows, it is proven that predicate *Interleaves* is actually not weaker than, viz. is a sufficient condition for, predicate *interleaves*. Moreover, the former predicate is shown to fulfil the same rules as the latter, although sometimes under more stringent assumptions (cf. lemmas *Interleaves-all-nil*, *Interleaves-nil-all* with lemmas $?xs \simeq \{?xs, [], ?P\}$, $?xs \simeq \{[], ?xs, ?P\}$), and to have the further property that under proper assumptions, its validity is preserved upon the addition or removal of a suffix to the input lists.

lemma *Interleaves-interleaves* [rule-format]:

```

xs ≅ {ys, zs, P} ⟶ xs ≅ {ys, zs, P}
proof (induction P xs ys zs rule: interleaves.induct, simp-all)
qed (rule conjI, (rule-tac [!] impI)+, simp-all)

```

lemma *Interleaves-length*:

```

xs ≅ {ys, zs, P} ⟹ length xs = length ys + length zs
by (drule Interleaves-interleaves, rule interleaves-length)

```

lemma *Interleaves-nil*:

```

[] ≅ {ys, zs, P} ⟹ ys = [] ∧ zs = []
by (drule Interleaves-interleaves, rule interleaves-nil)

```

lemma *Interleaves-swap*:

```

xs ≅ {ys, zs, P} = xs ≅ {zs, ys, λw ws. ¬ P w ws}
proof (induction P xs ys zs rule: Interleaves.induct, simp-all)
  fix y' :: 'a and ys' zs' P'
  show ¬ [] ≅ {zs', y' # ys', λw ws. ¬ P' w ws} by (cases zs', simp-all)
qed

```

lemma *Interleaves-equal-fst*:

```

xs ≅ {ys, zs, P} ⟹ xs ≅ {ys', zs, P} ⟹ ys = ys'
by ((drule Interleaves-interleaves)+, rule interleaves-equal-fst)

```

lemma *Interleaves-equal-snd*:

$xs \cong \{ys, zs, P\} \implies xs \cong \{ys, zs', P\} \implies zs = zs'$
by ((*drule Interleaves-interleaves*)+, *rule interleaves-equal-snd*)

lemma *Interleaves-equal-all-nil*:

$xs \cong \{\square, P\} \implies xs = ys$
by (*drule Interleaves-interleaves*, *rule interleaves-equal-all-nil*)

lemma *Interleaves-equal-nil-all*:

$xs \cong \{\square, zs, P\} \implies xs = zs$
by (*drule Interleaves-interleaves*, *rule interleaves-equal-nil-all*)

lemma *Interleaves-filter* [*rule-format*]:

assumes $A: \forall x xs. P x (\text{filter } Q xs) = P x xs$
shows $xs \cong \{ys, zs, P\} \longrightarrow \text{filter } Q xs \cong \{\text{filter } Q ys, \text{filter } Q zs, P\}$
proof (*induction xs arbitrary: ys zs, rule-tac* [!] *impI, simp*)
fix $ys zs$
assume $\square \cong \{ys, zs, P\}$
hence $ys = \square \wedge zs = \square$ **by** (*rule Interleaves-nil*)
thus $\square \cong \{\text{filter } Q ys, \text{filter } Q zs, P\}$ **by** *simp*

next

fix $x xs ys zs$

assume

$B: \bigwedge ys' zs'. xs \cong \{ys', zs', P\} \longrightarrow$
 $\text{filter } Q xs \cong \{\text{filter } Q ys', \text{filter } Q zs', P\}$ **and**
 $C: x \# xs \cong \{ys, zs, P\}$

show $\text{filter } Q (x \# xs) \cong \{\text{filter } Q ys, \text{filter } Q zs, P\}$

proof (*cases ys, case-tac* [!] *zs, simp-all del: filter.simps, rule ccontr*)

assume $ys = \square$ **and** $zs = \square$
thus *False* **using** C **by** *simp*

next

fix $z zs'$

assume $ys = \square$ **and** $zs = z \# zs'$

hence $D: \neg P x xs \wedge x = z \wedge xs \cong \{\square, zs', P\}$ **using** C **by** *simp+*

moreover **have** $xs \cong \{\square, zs', P\} \longrightarrow$

$\text{filter } Q xs \cong \{\text{filter } Q \square, \text{filter } Q zs', P\}$

using B .

ultimately **have** $\text{filter } Q xs \cong \{\square, \text{filter } Q zs', P\}$ **by** *simp*

moreover **have** $\neg P x (\text{filter } Q xs)$ **using** A **and** D **by** *simp+*

ultimately **show** $\text{filter } Q (x \# xs) \cong \{\square, \text{filter } Q (z \# zs'), P\}$

using D **by** *simp*

next

fix $y ys'$

assume $ys = y \# ys'$ **and** $zs = \square$

hence $D: P x xs \wedge x = y \wedge xs \cong \{ys', \square, P\}$ **using** C **by** *simp+*

moreover **have** $xs \cong \{ys', \square, P\} \longrightarrow$

$\text{filter } Q xs \cong \{\text{filter } Q ys', \text{filter } Q \square, P\}$

using B .

ultimately **have** $\text{filter } Q xs \cong \{\text{filter } Q ys', \square, P\}$ **by** *simp*

moreover have $P x$ (*filter Q xs*) **using** A **and** D **by** *simp+*
ultimately show $\text{filter } Q (x \# xs) \cong \{\text{filter } Q (y \# ys'), [], P\}$
using D **by** *simp*

next
fix $y \ ys' \ z \ zs'$
assume $ys = y \# ys'$ **and** $zs = z \# zs'$
hence $D: x \# xs \cong \{y \# ys', z \# zs', P\}$ **using** C **by** *simp*
show $\text{filter } Q (x \# xs) \cong \{\text{filter } Q (y \# ys'), \text{filter } Q (z \# zs'), P\}$
proof (*cases P x xs*)
case $True$
hence $E: P x$ (*filter Q xs*) **using** A **by** *simp*
have $F: x = y \wedge xs \cong \{ys', z \# zs', P\}$ **using** D **and** $True$ **by** *simp*
moreover have $xs \cong \{ys', z \# zs', P\} \longrightarrow$
 $\text{filter } Q xs \cong \{\text{filter } Q ys', \text{filter } Q (z \# zs'), P\}$
using B .
ultimately have $G: \text{filter } Q xs \cong \{\text{filter } Q ys', \text{filter } Q (z \# zs'), P\}$
by *simp*
show *?thesis*
proof (*cases Q x*)
assume $Q x$
hence $\text{filter } Q (x \# xs) = x \# \text{filter } Q xs$ **by** *simp*
moreover have $\text{filter } Q (y \# ys') = x \# \text{filter } Q ys'$
using $\langle Q x \rangle$ **and** F **by** *simp*
ultimately show *?thesis* **using** E **and** G
by (*cases filter Q (z # zs'), simp-all*)

next
assume $\neg Q x$
hence $\text{filter } Q (x \# xs) = \text{filter } Q xs$ **by** *simp*
moreover have $\text{filter } Q (y \# ys') = \text{filter } Q ys'$
using $\langle \neg Q x \rangle$ **and** F **by** *simp*
ultimately show *?thesis* **using** E **and** G
by (*cases filter Q (z # zs'), simp-all*)

qed

next
case $False$
hence $E: \neg P x$ (*filter Q xs*) **using** A **by** *simp*
have $F: x = z \wedge xs \cong \{y \# ys', zs', P\}$ **using** D **and** $False$ **by** *simp*
moreover have $xs \cong \{y \# ys', zs', P\} \longrightarrow$
 $\text{filter } Q xs \cong \{\text{filter } Q (y \# ys'), \text{filter } Q zs', P\}$
using B .
ultimately have $G: \text{filter } Q xs \cong \{\text{filter } Q (y \# ys'), \text{filter } Q zs', P\}$
by *simp*
show *?thesis*
proof (*cases Q x*)
assume $Q x$
hence $\text{filter } Q (x \# xs) = x \# \text{filter } Q xs$ **by** *simp*
moreover have $\text{filter } Q (z \# zs') = x \# \text{filter } Q zs'$
using $\langle Q x \rangle$ **and** F **by** *simp*
ultimately show *?thesis* **using** E **and** G

by (cases filter Q ($y \# ys'$), simp-all)
 next
 assume $\neg Q x$
 hence filter Q ($x \# xs$) = filter Q xs by simp
 moreover have filter Q ($z \# zs'$) = filter Q zs'
 using $\langle \neg Q x \rangle$ and F by simp
 ultimately show ?thesis using E and G
 by (cases filter Q ($z \# zs'$), simp-all)
 qed
 qed
 qed
 qed

lemma *Interleaves-map* [rule-format]:

assumes A : inj f
 shows $xs \cong \{ys, zs, P\} \longrightarrow$
 $map\ f\ xs \cong \{map\ f\ ys, map\ f\ zs, \lambda w\ ws. P\ (inv\ f\ w)\ (map\ (inv\ f)\ ws)\}$
 (is $- \longrightarrow - \cong \{-, -, ?P'\}$)
 proof (induction xs arbitrary: $ys\ zs$, rule-tac [!] impI, simp-all)
 fix $ys\ zs$
 assume $\square \cong \{ys, zs, P\}$
 hence $ys = \square \wedge zs = \square$ by (rule *Interleaves-nil*)
 thus $\square \cong \{map\ f\ ys, map\ f\ zs, ?P'\}$ by simp
 next
 fix $x\ xs\ ys\ zs$
 assume
 B : $\bigwedge ys\ zs. xs \cong \{ys, zs, P\} \longrightarrow map\ f\ xs \cong \{map\ f\ ys, map\ f\ zs, ?P'\}$ and
 C : $x \# xs \cong \{ys, zs, P\}$
 show $f\ x \# map\ f\ xs \cong \{map\ f\ ys, map\ f\ zs, ?P'\}$
 proof (cases ys , case-tac [!] zs , simp-all del: *Interleaves.simps(1-3)*)
 assume $ys = \square$ and $zs = \square$
 thus *False* using C by simp
 next
 fix $z\ zs'$
 assume $ys = \square$ and $zs = z \# zs'$
 hence D : $\neg P\ x\ xs \wedge x = z \wedge xs \cong \{\square, zs', P\}$ using C by simp+
 moreover have $xs \cong \{\square, zs', P\} \longrightarrow map\ f\ xs \cong \{map\ f\ \square, map\ f\ zs', ?P'\}$
 using B .
 ultimately have $map\ f\ xs \cong \{\square, map\ f\ zs', ?P'\}$ by simp
 moreover have $\neg ?P'\ (f\ x)\ (map\ f\ xs)$ using A and D by simp+
 ultimately show $f\ x \# map\ f\ xs \cong \{\square, f\ z \# map\ f\ zs', ?P'\}$
 using D by simp
 next
 fix $y\ ys'$
 assume $ys = y \# ys'$ and $zs = \square$
 hence D : $P\ x\ xs \wedge x = y \wedge xs \cong \{ys', \square, P\}$ using C by simp+
 moreover have $xs \cong \{ys', \square, P\} \longrightarrow map\ f\ xs \cong \{map\ f\ ys', map\ f\ \square, ?P'\}$
 using B .
 ultimately have $map\ f\ xs \cong \{map\ f\ ys', \square, ?P'\}$ by simp

moreover have $?P' (f x) (map f xs)$ **using** A **and** D **by** *simp+*
ultimately show $f x \# map f xs \cong \{f y \# map f ys', [], ?P'\}$
using D **by** *simp*
next
fix $y \ ys' \ z \ zs'$
assume $ys = y \# ys'$ **and** $zs = z \# zs'$
hence $D: x \# xs \cong \{y \# ys', z \# zs', P\}$ **using** C **by** *simp*
show $f x \# map f xs \cong \{f y \# map f ys', f z \# map f zs', ?P'\}$
proof (*cases P x xs*)
case *True*
hence $E: ?P' (f x) (map f xs)$ **using** A **by** *simp*
have $x = y \wedge xs \cong \{ys', z \# zs', P\}$ **using** D **and** *True* **by** *simp*
moreover have $xs \cong \{ys', z \# zs', P\} \longrightarrow$
 $map f xs \cong \{map f ys', map f (z \# zs'), ?P'\}$
using B .
ultimately have $f x = f y \wedge map f xs \cong \{map f ys', map f (z \# zs'), ?P'\}$
by *simp*
thus *?thesis* **using** E **by** *simp*
next
case *False*
hence $E: \neg ?P' (f x) (map f xs)$ **using** A **by** *simp*
have $x = z \wedge xs \cong \{y \# ys', zs', P\}$ **using** D **and** *False* **by** *simp*
moreover have $xs \cong \{y \# ys', zs', P\} \longrightarrow$
 $map f xs \cong \{map f (y \# ys'), map f zs', ?P'\}$
using B .
ultimately have $f x = f z \wedge map f xs \cong \{map f (y \# ys'), map f zs', ?P'\}$
by *simp*
thus *?thesis* **using** E **by** *simp*
qed
qed
qed

lemma *Interleaves-prefix-fst-1* [*rule-format*]:
assumes $A: xs \cong \{ys, zs, P\}$
shows $(\forall n < length \ ws. P (ws ! n) (drop (Suc n) ws @ xs)) \longrightarrow$
 $ws @ xs \cong \{ws @ ys, zs, P\}$
proof (*induction ws, simp-all add: A, rule impI*)
fix $w \ ws$
assume $B: \forall n < Suc (length \ ws). P ((w \# ws) ! n) (drop n \ ws @ xs)$
assume $(\forall n < length \ ws. P (ws ! n) (drop (Suc n) ws @ xs)) \longrightarrow$
 $ws @ xs \cong \{ws @ ys, zs, P\}$
moreover have $\forall n < length \ ws. P (ws ! n) (drop (Suc n) ws @ xs)$
proof (*rule allI, rule impI*)
fix n
assume $n < length \ ws$
moreover have $Suc \ n < Suc (length \ ws) \longrightarrow$
 $P ((w \# ws) ! (Suc \ n)) (drop (Suc \ n) \ ws @ xs)$
using B ..
ultimately show $P (ws ! n) (drop (Suc \ n) \ ws @ xs)$ **by** *simp*

qed
ultimately have $ws @ xs \cong \{ws @ ys, zs, P\} ..$
moreover have $0 < \text{Suc} (\text{length } ws) \longrightarrow P ((w \# ws) ! 0) (\text{drop } 0 \text{ } ws @ xs)$
using $B ..$
hence $P w (ws @ xs)$ **by** *simp*
ultimately show $w \# ws @ xs \cong \{w \# ws @ ys, zs, P\}$ **by** (*cases zs, simp-all*)
qed

lemma *Interleaves-prefix-fst-2* [*rule-format*]:

$ws @ xs \cong \{ws @ ys, zs, P\} \longrightarrow$
 $(\forall n < \text{length } ws. P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs)) \longrightarrow$
 $xs \cong \{ys, zs, P\}$
proof (*induction ws, simp-all, (rule impI)+*)
fix $w\ ws$
assume $A: \forall n < \text{Suc} (\text{length } ws). P ((w \# ws) ! n) (\text{drop } n \text{ } ws @ xs)$
hence $0 < \text{Suc} (\text{length } ws) \longrightarrow P ((w \# ws) ! 0) (\text{drop } 0 \text{ } ws @ xs) ..$
hence $P w (ws @ xs)$ **by** *simp*
moreover assume $w \# ws @ xs \cong \{w \# ws @ ys, zs, P\}$
ultimately have $ws @ xs \cong \{ws @ ys, zs, P\}$ **by** (*cases zs, simp-all*)
moreover assume $ws @ xs \cong \{ws @ ys, zs, P\} \longrightarrow$
 $(\forall n < \text{length } ws. P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs)) \longrightarrow$
 $xs \cong \{ys, zs, P\}$
ultimately have $(\forall n < \text{length } ws. P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs)) \longrightarrow$
 $xs \cong \{ys, zs, P\}$
by *simp*
moreover have $\forall n < \text{length } ws. P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs)$
proof (*rule allI, rule impI*)
fix n
assume $n < \text{length } ws$
moreover have $\text{Suc } n < \text{Suc} (\text{length } ws) \longrightarrow$
 $P ((w \# ws) ! (\text{Suc } n)) (\text{drop} (\text{Suc } n) \text{ } ws @ xs)$
using $A ..$
ultimately show $P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs)$ **by** *simp*
qed
ultimately show $xs \cong \{ys, zs, P\} ..$
qed

lemma *Interleaves-prefix-fst* [*rule-format*]:

$\forall n < \text{length } ws. P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs) \implies$
 $xs \cong \{ys, zs, P\} = ws @ xs \cong \{ws @ ys, zs, P\}$
proof (*rule iffI, erule Interleaves-prefix-fst-1, simp*)
qed (*erule Interleaves-prefix-fst-2, simp*)

lemma *Interleaves-prefix-snd* [*rule-format*]:

$\forall n < \text{length } ws. \neg P (ws ! n) (\text{drop} (Suc\ n) \text{ } ws @ xs) \implies$
 $xs \cong \{ys, zs, P\} = ws @ xs \cong \{ys, ws @ zs, P\}$
proof (*subst (1 2) Interleaves-swap*)
qed (*rule Interleaves-prefix-fst, simp*)

lemma *Interleaves-all-nil-1* [rule-format]:
 $xs \cong \{xs, [], P\} \longrightarrow (\forall n < \text{length } xs. P (xs ! n) (\text{drop } (Suc\ n) xs))$
proof (*induction xs, simp-all, rule impI, erule conjE, rule allI, rule impI*)
fix $x\ xs\ n$
assume
 $xs \cong \{xs, [], P\} \longrightarrow (\forall n < \text{length } xs. P (xs ! n) (\text{drop } (Suc\ n) xs))$ **and**
 $xs \cong \{xs, [], P\}$
hence $A: \forall n < \text{length } xs. P (xs ! n) (\text{drop } (Suc\ n) xs) ..$
assume
 $B: P\ x\ xs$ **and**
 $C: n < Suc\ (\text{length } xs)$
show $P ((x \# xs) ! n) (\text{drop } n\ xs)$
proof (*cases n, simp-all add: B*)
case ($Suc\ m$)
have $m < \text{length } xs \longrightarrow P (xs ! m) (\text{drop } (Suc\ m) xs)$ **using** $A ..$
moreover have $m < \text{length } xs$ **using** C **and** Suc **by** *simp*
ultimately show $P (xs ! m) (\text{drop } (Suc\ m) xs) ..$
qed
qed

lemma *Interleaves-all-nil-2* [rule-format]:
 $\forall n < \text{length } xs. P (xs ! n) (\text{drop } (Suc\ n) xs) \implies xs \cong \{xs, [], P\}$
by (*insert Interleaves-prefix-fst [of xs P [] []], simp*)

lemma *Interleaves-all-nil*:
 $xs \cong \{xs, [], P\} = (\forall n < \text{length } xs. P (xs ! n) (\text{drop } (Suc\ n) xs))$
proof (*rule iffI, rule allI, rule impI, rule Interleaves-all-nil-1, assumption+*)
qed (*rule Interleaves-all-nil-2, simp*)

lemma *Interleaves-nil-all*:
 $xs \cong \{[], xs, P\} = (\forall n < \text{length } xs. \neg P (xs ! n) (\text{drop } (Suc\ n) xs))$
by (*subst Interleaves-swap, simp add: Interleaves-all-nil*)

lemma *Interleaves-suffix-one-aux*:
assumes $A: P\ x\ []$
shows $\neg xs @ [x] \cong \{[], zs, P\}$
proof (*induction xs arbitrary: zs, simp-all, rule-tac [!] notI*)
fix zs
assume $[x] \cong \{[], zs, P\}$
thus *False* **by** (*cases zs, simp-all add: A*)
next
fix $w\ xs\ zs$
assume $B: \bigwedge zs. \neg xs @ [x] \cong \{[], zs, P\}$
assume $w \# xs @ [x] \cong \{[], zs, P\}$
thus *False* **proof** (*cases zs, simp-all, (erule-tac conjE)+*)
fix zs'
assume $xs @ [x] \cong \{[], zs', P\}$
moreover have $\neg xs @ [x] \cong \{[], zs', P\}$ **using** $B .$
ultimately show *False* **by** *contradiction*

qed
qed

lemma *Interleaves-suffix-one-fst-2* [rule-format]:

assumes $A: P\ x\ []$

shows $xs\ @\ [x] \cong \{ys\ @\ [x],\ zs,\ P\} \longrightarrow xs \cong \{ys,\ zs,\ \lambda w\ ws.\ P\ w\ (ws\ @\ [x])\}$
(is \longrightarrow \cong $\{-,\ -, \ ?P'\}$)

proof (*induction* xs *arbitrary*: $ys\ zs$, *rule-tac* $[\!|]$ *impI*, *simp-all*)

fix $ys\ zs$

assume $[x] \cong \{ys\ @\ [x],\ zs,\ P\}$

hence $B: length\ [x] = length\ (ys\ @\ [x]) + length\ zs$

by (*rule* *Interleaves-length*)

have $ys: ys = []$ **by** (*cases* ys , *simp*, *insert* B , *simp*)

then have $zs = []$ **by** (*cases* zs , *simp*, *insert* B , *simp*)

with ys **show** $[] \cong \{ys,\ zs,\ ?P'\}$ **by** *simp*

next

fix $w\ xs\ ys\ zs$

assume $B: \bigwedge ys\ zs.\ xs\ @\ [x] \cong \{ys\ @\ [x],\ zs,\ P\} \longrightarrow xs \cong \{ys,\ zs,\ ?P'\}$

assume $w\ \#\ xs\ @\ [x] \cong \{ys\ @\ [x],\ zs,\ P\}$

thus $w\ \#\ xs \cong \{ys,\ zs,\ ?P'\}$

proof (*cases* zs , *case-tac* $[\!|]$ ys , *simp-all* *del*: *Interleaves.simps*(1,3),
(erule-tac $[1-2]$ *conjE*) $+$)

assume $xs\ @\ [x] \cong \{[],\ [],\ P\}$

thus *False* **by** (*cases* xs , *simp-all*)

next

fix ys'

have $xs\ @\ [x] \cong \{ys'\ @\ [x],\ [],\ P\} \longrightarrow xs \cong \{ys',\ [],\ ?P'\}$ **using** B .

moreover assume $xs\ @\ [x] \cong \{ys'\ @\ [x],\ [],\ P\}$

ultimately show $xs \cong \{ys',\ [],\ ?P'\}$..

next

fix $z'\ zs'$

assume $w\ \#\ xs\ @\ [x] \cong \{[x],\ z'\ \#\ zs',\ P\}$

thus $w\ \#\ xs \cong \{[],\ z'\ \#\ zs',\ ?P'\}$

proof (*cases* $P\ w\ (xs\ @\ [x])$, *simp-all*, *erule-tac* $[\!|]$ *conjE*)

assume $xs\ @\ [x] \cong \{[],\ z'\ \#\ zs',\ P\}$

moreover have $\neg xs\ @\ [x] \cong \{[],\ z'\ \#\ zs',\ P\}$

using A **by** (*rule* *Interleaves-suffix-one-aux*)

ultimately show *False* **by** *contradiction*

next

have $xs\ @\ [x] \cong \{[x],\ zs',\ P\} \longrightarrow xs \cong \{[],\ zs',\ ?P'\}$ **using** B **by** *simp*

moreover assume $xs\ @\ [x] \cong \{[x],\ zs',\ P\}$

ultimately show $xs \cong \{[],\ zs',\ ?P'\}$..

qed

next

fix $y'\ ys'\ z'\ zs'$

assume $w\ \#\ xs\ @\ [x] \cong \{y'\ \#\ ys'\ @\ [x],\ z'\ \#\ zs',\ P\}$

thus $w\ \#\ xs \cong \{y'\ \#\ ys',\ z'\ \#\ zs',\ ?P'\}$

proof (*cases* $P\ w\ (xs\ @\ [x])$, *simp-all*, *erule-tac* $[\!|]$ *conjE*)

have $xs\ @\ [x] \cong \{ys'\ @\ [x],\ z'\ \#\ zs',\ P\} \longrightarrow xs \cong \{ys',\ z'\ \#\ zs',\ ?P'\}$

```

    using B .
    moreover assume  $xs @ [x] \cong \{ys' @ [x], z' \# zs', P\}$ 
    ultimately show  $xs \cong \{ys', z' \# zs', ?P'\}$  ..
  next
  have  $xs @ [x] \cong \{y' \# ys' @ [x], zs', P\} \longrightarrow xs \cong \{y' \# ys', zs', ?P'\}$ 
    using B by simp
  moreover assume  $xs @ [x] \cong \{y' \# ys' @ [x], zs', P\}$ 
  ultimately show  $xs \cong \{y' \# ys', zs', ?P'\}$  ..
qed
qed
qed

lemma Interleaves-suffix-fst-1 [rule-format]:
  assumes A:  $\forall n < \text{length } ws. P (ws ! n) (\text{drop } (\text{Suc } n) ws)$ 
  shows  $xs \cong \{ys, zs, \lambda v vs. P v (vs @ ws)\} \longrightarrow xs @ ws \cong \{ys @ ws, zs, P\}$ 
    (is  $- \cong \{-, -, ?P'\} \longrightarrow -$ )
proof (induction xs arbitrary: ys zs, rule-tac [!] impI, simp-all)
  fix ys zs
  assume []  $\cong \{ys, zs, ?P'\}$ 
  hence  $ys = [] \wedge zs = []$  by (rule Interleaves-nil)
  thus  $ws \cong \{ys @ ws, zs, P\}$  using A by (simp add: Interleaves-all-nil)
next
  fix x xs ys zs
  assume A:  $\bigwedge ys zs. xs \cong \{ys, zs, ?P'\} \longrightarrow xs @ ws \cong \{ys @ ws, zs, P\}$ 
  assume  $x \# xs \cong \{ys, zs, ?P'\}$ 
  thus  $x \# xs @ ws \cong \{ys @ ws, zs, P\}$ 
  proof (rule-tac Interleaves.cases [of  $(?P', x \# xs, ys, zs)$ ],
    simp-all del: Interleaves.simps(1),
    (erule-tac conjE)+, (erule-tac [2] conjE)+, (erule-tac [3] conjE)+)
    fix  $P' x' xs' y' ys' z' zs'$ 
    assume
      B:  $x' \# xs' \cong \{y' \# ys', z' \# zs', P'\}$  and
      C:  $?P' = P'$  and
      D:  $xs = xs'$ 
    show  $x' \# xs' @ ws \cong \{y' \# ys' @ ws, z' \# zs', P\}$ 
    proof (cases  $P' x' xs'$ )
      have  $xs \cong \{ys', z' \# zs', ?P'\} \longrightarrow xs @ ws \cong \{ys' @ ws, z' \# zs', P\}$ 
        using A .
      moreover case True
      hence  $xs \cong \{ys', z' \# zs', ?P'\}$  using B and C and D by simp
      ultimately have  $xs @ ws \cong \{ys' @ ws, z' \# zs', P\}$  ..
      moreover have  $P x' (xs' @ ws)$  using C [symmetric] and True by simp
      moreover have  $x' = y'$  using B and True by simp
      ultimately show ?thesis using D by simp
    next
    have  $xs \cong \{y' \# ys', zs', ?P'\} \longrightarrow xs @ ws \cong \{(y' \# ys') @ ws, zs', P\}$ 
      using A .
    moreover case False
    hence  $xs \cong \{y' \# ys', zs', ?P'\}$  using B and C and D by simp

```

ultimately have $xs @ ws \cong \{(y' \# ys') @ ws, zs', P\} ..$
 moreover have $\neg P x' (xs' @ ws)$ using C [symmetric] and *False by simp*
 moreover have $x' = z'$ using B and *False by simp*
 ultimately show *?thesis* using D by *simp*

qed

next

fix $P' x' xs' y' ys'$

have $xs \cong \{ys', [], ?P'\} \longrightarrow xs @ ws \cong \{ys' @ ws, [], P\}$ using A .

moreover assume

$xs' \cong \{ys', [], P'\}$ and

$B: ?P' = P'$ and

$C: xs = xs'$

hence $xs \cong \{ys', [], ?P'\}$ by *simp*

ultimately have $xs' @ ws \cong \{ys' @ ws, [], P\}$ using C by *simp*

moreover assume

$P' x' xs'$ and

$x' = y'$

hence $P y' (xs' @ ws)$ using B [symmetric] by *simp*

ultimately show $P y' (xs' @ ws) \wedge xs' @ ws \cong \{ys' @ ws, [], P\}$ by *simp*

next

fix $P' x' xs' z' zs'$

have $xs \cong \{[], zs', ?P'\} \longrightarrow xs @ ws \cong \{[] @ ws, zs', P\}$ using A .

moreover assume

$xs' \cong \{[], zs', P'\}$ and

$B: ?P' = P'$ and

$C: xs = xs'$

hence $xs \cong \{[], zs', ?P'\}$ by *simp*

ultimately have $xs' @ ws \cong \{ws, zs', P\}$ using C by *simp*

moreover assume

$\neg P' x' xs'$ and

$x' = z'$

hence $\neg P z' (xs' @ ws)$ using B [symmetric] by *simp*

ultimately show $z' \# xs' @ ws \cong \{ws, z' \# zs', P\}$ by (*cases ws, simp-all*)

qed

qed

lemma *Interleaves-suffix-one-fst-1* [rule-format]:

$P x [] \Longrightarrow$

$xs \cong \{ys, zs, \lambda w ws. P w (ws @ [x])\} \Longrightarrow xs @ [x] \cong \{ys @ [x], zs, P\}$

by (*rule Interleaves-suffix-fst-1, simp*)

lemma *Interleaves-suffix-one-fst*:

$P x [] \Longrightarrow$

$xs \cong \{ys, zs, \lambda w ws. P w (ws @ [x])\} = xs @ [x] \cong \{ys @ [x], zs, P\}$

proof (*rule iffI, rule Interleaves-suffix-one-fst-1, assumption+*)

qed (*rule Interleaves-suffix-one-fst-2*)

lemma *Interleaves-suffix-one-snd*:

$\neg P x [] \Longrightarrow$

$xs \cong \{ys, zs, \lambda w ws. P w (ws @ [x])\} = xs @ [x] \cong \{ys, zs @ [x], P\}$
by (*subst* (1 2) *Interleaves-swap*, *rule Interleaves-suffix-one-fst*)

lemma *Interleaves-suffix-aux* [*rule-format*]:

$(\forall n < \text{length } ws. P (ws ! n) (\text{drop } (\text{Suc } n) ws)) \longrightarrow$
 $x \# xs @ ws \cong \{ws, zs, P\} \longrightarrow$
 $\neg P x (xs @ ws)$

proof (*induction ws arbitrary: P rule: rev-induct, simp-all,*
rule impI, (rule-tac [2] impI)+)

fix P

assume $x \# xs \cong \{\ [], zs, P\}$

thus $\neg P x xs$ **by** (*cases zs, simp-all*)

next

fix $w ws P$

assume

$A: \bigwedge P'. (\forall n < \text{length } ws. P' (ws ! n) (\text{drop } (\text{Suc } n) ws)) \longrightarrow$
 $x \# xs @ ws \cong \{ws, zs, P'\} \longrightarrow \neg P' x (xs @ ws)$ **and**

$B: \forall n < \text{Suc } (\text{length } ws). P ((ws @ [w]) ! n)$
 $(\text{drop } (\text{Suc } n) ws @ \text{drop } (\text{Suc } n - \text{length } ws) [w])$

assume $x \# xs @ ws @ [w] \cong \{ws @ [w], zs, P\}$

hence $C: (x \# xs @ ws) @ [w] \cong \{ws @ [w], zs, P\}$ **by** *simp*

let $?P' = \lambda v vs. P v (vs @ [w])$

have $(\forall n < \text{length } ws. ?P' (ws ! n) (\text{drop } (\text{Suc } n) ws)) \longrightarrow$
 $x \# xs @ ws \cong \{ws, zs, ?P'\} \longrightarrow \neg ?P' x (xs @ ws)$

using A .

moreover have $\forall n < \text{length } ws. ?P' (ws ! n) (\text{drop } (\text{Suc } n) ws)$

proof (*rule allI, rule impI*)

fix n

assume $D: n < \text{length } ws$

moreover have $n < \text{Suc } (\text{length } ws) \longrightarrow P ((ws @ [w]) ! n)$
 $(\text{drop } (\text{Suc } n) ws @ \text{drop } (\text{Suc } n - \text{length } ws) [w])$

using B ..

ultimately have $P ((ws @ [w]) ! n) (\text{drop } (\text{Suc } n) ws @ [w])$ **by** *simp*

moreover have $n < \text{length } (\text{butlast } (ws @ [w]))$ **using** D **by** *simp*

hence $\text{butlast } (ws @ [w]) ! n = (ws @ [w]) ! n$ **by** (*rule nth-butlast*)

ultimately show $P (ws ! n) (\text{drop } (\text{Suc } n) ws @ [w])$ **by** *simp*

qed

ultimately have $x \# xs @ ws \cong \{ws, zs, ?P'\} \longrightarrow \neg ?P' x (xs @ ws)$..

moreover have $\text{length } ws < \text{Suc } (\text{length } ws) \longrightarrow P ((ws @ [w]) ! \text{length } ws)$
 $(\text{drop } (\text{Suc } (\text{length } ws)) ws @ \text{drop } (\text{Suc } (\text{length } ws) - \text{length } ws) [w])$

using B ..

hence $P w []$ **by** *simp*

hence $x \# xs @ ws \cong \{ws, zs, ?P'\}$

using C **by** (*rule Interleaves-suffix-one-fst-2*)

ultimately have $\neg ?P' x (xs @ ws)$..

thus $\neg P x (xs @ ws @ [w])$ **by** *simp*

qed

lemma *Interleaves-suffix-fst-2* [*rule-format*]:

```

assumes  $A: \forall n < \text{length } ws. P (ws ! n) (\text{drop } (\text{Suc } n) ws)$ 
shows  $xs @ ws \cong \{ys @ ws, zs, P\} \longrightarrow xs \cong \{ys, zs, \lambda v vs. P v (vs @ ws)\}$ 
  (is  $- \longrightarrow - \cong \{-, -, ?P'\}$ )
proof (induction  $xs$  arbitrary:  $ys$   $zs$ , rule-tac  $[\ ]$  impI, simp-all)
  fix  $ys$   $zs$ 
  assume  $ws \cong \{ys @ ws, zs, P\}$ 
  hence  $B: \text{length } ws = \text{length } (ys @ ws) + \text{length } zs$ 
  by (rule Interleaves-length)
  have  $ys: ys = [\ ]$  by (cases  $ys$ , simp, insert  $B$ , simp)
  then have  $zs = [\ ]$  by (cases  $zs$ , simp, insert  $B$ , simp)
  with  $ys$  show  $[\ ] \cong \{ys, zs, ?P'\}$  by simp
next
  fix  $x$   $xs$   $ys$   $zs$ 
  assume  $B: \bigwedge ys\ zs. xs @ ws \cong \{ys @ ws, zs, P\} \longrightarrow xs \cong \{ys, zs, ?P'\}$ 
  assume  $x \# xs @ ws \cong \{ys @ ws, zs, P\}$ 
  thus  $x \# xs \cong \{ys, zs, ?P'\}$ 
  proof (cases  $zs$ , case-tac  $[\ ]$   $ys$ , simp-all del: Interleaves.simps(1,3),
    (erule-tac  $[\ ]$  conjE)+
    assume  $C: x \# xs @ ws \cong \{ws, [\ ], P\}$ 
    have  $\text{length } (x \# xs @ ws) = \text{length } ws + \text{length } [\ ]$ 
    by (insert Interleaves-length  $[OF\ C]$ , simp)
    thus False by simp
  next
  fix  $ys'$ 
  have  $xs @ ws \cong \{ys' @ ws, [\ ], P\} \longrightarrow xs \cong \{ys', [\ ], ?P'\}$  using  $B$  .
  moreover assume  $xs @ ws \cong \{ys' @ ws, [\ ], P\}$ 
  ultimately show  $xs \cong \{ys', [\ ], ?P'\}$  ..
next
  fix  $z'$   $zs'$ 
  assume  $x \# xs @ ws \cong \{ws, z' \# zs', P\}$ 
  thus  $x \# xs \cong \{[\ ], z' \# zs', ?P'\}$ 
  proof (cases  $P\ x\ (xs @ ws)$ , simp-all)
    case True
    moreover assume  $x \# xs @ ws \cong \{ws, z' \# zs', P\}$ 
    with  $A$  [rule-format] have  $\neg P\ x\ (xs @ ws)$ 
    by (rule Interleaves-suffix-aux)
    ultimately show False by contradiction
  next
  case False
  moreover assume  $x \# xs @ ws \cong \{ws, z' \# zs', P\}$ 
  ultimately have  $x = z' \wedge xs @ ws \cong \{ws, zs', P\}$  by (cases  $ws$ , simp-all)
  moreover have  $xs @ ws \cong \{[\ ] @ ws, zs', P\} \longrightarrow xs \cong \{[\ ], zs', ?P'\}$ 
  using  $B$  .
  ultimately show  $x = z' \wedge xs \cong \{[\ ], zs', ?P'\}$  by simp
  qed
next
  fix  $y'$   $ys'$   $z'$   $zs'$ 
  assume  $x \# xs @ ws \cong \{y' \# ys' @ ws, z' \# zs', P\}$ 
  thus  $x \# xs \cong \{y' \# ys', z' \# zs', ?P'\}$ 

```

```

proof (cases P x (xs @ ws), simp-all, erule-tac [!] conjE)
  have xs @ ws  $\cong$  {ys' @ ws, z' # zs', P}  $\longrightarrow$  xs  $\cong$  {ys', z' # zs', ?P'}
    using B .
  moreover assume xs @ ws  $\cong$  {ys' @ ws, z' # zs', P}
  ultimately show xs  $\cong$  {ys', z' # zs', ?P'} ..
next
  have xs @ ws  $\cong$  {y' # ys' @ ws, zs', P}  $\longrightarrow$  xs  $\cong$  {y' # ys', zs', ?P'}
    using B by simp
  moreover assume xs @ ws  $\cong$  {y' # ys' @ ws, zs', P}
  ultimately show xs  $\cong$  {y' # ys', zs', ?P'} ..
qed
qed
qed

```

```

lemma Interleaves-suffix-fst [rule-format]:
   $\forall n < \text{length } ws. P (ws ! n) (\text{drop } (\text{Suc } n) ws) \implies$ 
   $xs \cong \{ys, zs, \lambda v \text{ vs. } P v (vs @ ws)\} = xs @ ws \cong \{ys @ ws, zs, P\}$ 
proof (rule iffI, rule Interleaves-suffix-fst-1, simp-all)
qed (rule Interleaves-suffix-fst-2, simp)

```

```

lemma Interleaves-suffix-snd [rule-format]:
   $\forall n < \text{length } ws. \neg P (ws ! n) (\text{drop } (\text{Suc } n) ws) \implies$ 
   $xs \cong \{ys, zs, \lambda v \text{ vs. } P v (vs @ ws)\} = xs @ ws \cong \{ys, zs @ ws, P\}$ 
by (subst (1 2) Interleaves-swap, rule Interleaves-suffix-fst, simp)

```

end

References

- [1] C. A. R. Hoare. *Communicating Sequential Processes*. Prentice-Hall, Inc., 1985.
- [2] A. Krauss. *Defining Recursive Functions in Isabelle/HOL*. <http://isabelle.in.tum.de/website-Isabelle2015/dist/Isabelle2015/doc/functions.pdf>.
- [3] T. Nipkow. *A Tutorial Introduction to Structured Isar Proofs*. <http://isabelle.in.tum.de/website-Isabelle2011/dist/Isabelle2011/doc/isar-overview.pdf>.
- [4] T. Nipkow. *Programming and Proving in Isabelle/HOL*, May 2015. <http://isabelle.in.tum.de/website-Isabelle2015/dist/Isabelle2015/doc/prog-prove.pdf>.
- [5] T. Nipkow, L. Paulson, and M. Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, May 2015. <http://isabelle.in.tum.de/website-Isabelle2015/dist/Isabelle2015/doc/tutorial.pdf>.

- [6] P. Noce. The ipurge unwinding theorem for csp noninterference security. *Archive of Formal Proofs*, June 2015. http://isa-afp.org/entries/Noninterference_Ipurge_Unwinding.shtml, Formal proof development.