

Liouville Numbers

Manuel Eberl

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Abstract

In this work, we define the concept of Liouville numbers as well as the standard construction to obtain Liouville numbers and we prove their most important properties: irrationality and transcendence.

This is historically interesting since Liouville numbers constructed in the standard way were the first numbers that were proven to be transcendental. The proof is very elementary and requires only standard arithmetic and the Mean Value Theorem for polynomials and the boundedness of polynomials on compact intervals.

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1 Liouville Numbers

1.1 Preliminary lemmas

theory *Liouville-Numbers-Misc*

imports

Complex-Main

HOL-Computational-Algebra.Polynomial

begin

We will require these inequalities on factorials to show properties of the standard construction later.

lemma *fact-ineq*: $n \geq 1 \implies \text{fact } n + k \leq \text{fact } (n + k)$

proof (*induction k*)

case (*Suc k*)

from *Suc* **have** $\text{fact } n + \text{Suc } k \leq \text{fact } (n + k) + 1$ **by** *simp*

also from *Suc* **have** $\dots \leq \text{fact } (n + \text{Suc } k)$ **by** *simp*

finally show *?case* .

qed *simp-all*

lemma *Ints-sum*:

assumes $\bigwedge x. x \in A \implies f x \in \mathbb{Z}$

shows $\text{sum } f A \in \mathbb{Z}$

by (*cases finite A, insert assms, induction A rule: finite-induct*)
(*auto intro!: Ints-add*)

lemma *suminf-split-initial-segment'*:

summable ($f :: \text{nat} \Rightarrow 'a :: \text{real-normed-vector}$) \implies

$\text{suminf } f = (\sum n. f (n + k + 1)) + \text{sum } f \{..k\}$

by (*subst suminf-split-initial-segment[of - Suc k], assumption, subst lessThan-Suc-atMost*)

simp-all

lemma *Rats-eq-int-div-int'*: $(\mathbb{Q} :: \text{real set}) = \{\text{of-int } p / \text{of-int } q \mid p \ q. \ q > 0\}$

proof *safe*

fix $x :: \text{real}$ **assume** $x \in \mathbb{Q}$

then obtain $p \ q$ **where** $pq: x = \text{of-int } p / \text{of-int } q \ q \neq 0$

by (*subst (asm) Rats-eq-int-div-int*) *auto*

show $\exists p \ q. x = \text{real-of-int } p / \text{real-of-int } q \wedge 0 < q$

proof (*cases q > 0*)

case *False*

show *?thesis* **by** (*rule exI[of - -p], rule exI[of - -q]*) (*insert False pq, auto*)

qed (*insert pq, force*)

qed *auto*

lemma *Rats-cases'*:

assumes $(x :: \text{real}) \in \mathbb{Q}$

obtains $p \ q$ **where** $q > 0 \ x = \text{of-int } p / \text{of-int } q$

using *assms* **by** (*subst (asm) Rats-eq-int-div-int'*) *auto*

The following inequality gives a lower bound for the absolute value of an integer polynomial at a rational point that is not a root.

lemma *int-poly-rat-no-root-ge*:

fixes $p :: \text{real poly}$ **and** $a \ b :: \text{int}$

assumes $\bigwedge n. \text{coeff } p \ n \in \mathbb{Z}$

assumes $b > 0 \ \text{poly } p \ (a / b) \neq 0$

defines $n \equiv \text{degree } p$

shows $\text{abs } (\text{poly } p \ (a / b)) \geq 1 / \text{of-int } b ^ n$

proof –

let $?S = (\sum i \leq n. \text{coeff } p \ i * \text{of-int } a ^ i * (\text{of-int } b ^ (n - i)))$

from $\langle b > 0 \rangle$ **have** $\text{eq}: ?S = \text{of-int } b ^ n * \text{poly } p \ (a / b)$

by (*simp add: poly-altdef power-divide mult-ac n-def sum-distrib-left power-diff*)

have $?S \in \mathbb{Z}$ **by** (*intro Ints-sum Ints-mult assms Ints-power*) *simp-all*

moreover from *assms* **have** $?S \neq 0$ **by** (*subst eq*) *auto*

ultimately have $\text{abs } ?S \geq 1$ **by** (*elim Ints-cases*) *simp*

with $\text{eq } \langle b > 0 \rangle$ **show** *?thesis* **by** (*simp add: field-simps abs-mult*)

qed

end

theory *Liouville-Numbers*

imports

Complex-Main

HOL-Computational-Algebra.Polynomial

Liouville-Numbers-Misc

begin

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. “Well“, in this context, means that the error of the n -th rational in the sequence is bounded by the n -th power of its denominator.

Our approach will be the following:

- Liouville numbers cannot be rational.
- Any irrational algebraic number cannot be approximated in the Liouville sense
- Therefore, all Liouville numbers are transcendental.
- The standard construction fulfils all the properties of Liouville numbers.

1.2 Definition of Liouville numbers

The following definitions and proofs are largely adapted from those in the Wikipedia article on Liouville numbers. [1]

A Liouville number is a real number that can be approximated well – but not perfectly – by a sequence of rational numbers. The error of the n -th term $\frac{p_n}{q_n}$ is at most q_n^{-n} , where $p_n \in \mathbb{Z}$ and $q_n \in \mathbb{Z}_{\geq 2}$.

We will say that such a number can be approximated in the Liouville sense.

locale *liouville* =

fixes $x :: \text{real}$ **and** $p\ q :: \text{nat} \Rightarrow \text{int}$

assumes *approx-int-pos*: $\text{abs}(x - p\ n / q\ n) > 0$

and *denom-gt-1*: $q\ n > 1$

and *approx-int*: $\text{abs}(x - p\ n / q\ n) < 1 / \text{of-int}(q\ n) ^ n$

First, we show that any Liouville number is irrational.

lemma (*in liouville*) *irrational*: $x \notin \mathbb{Q}$

proof

assume $x \in \mathbb{Q}$

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then obtain  $c d :: \text{int}$  where  $d: x = \text{of-int } c / \text{of-int } d \ d > 0$ 
  by (elim Rats-cases') simp
define  $n$  where  $n = \text{Suc } (\text{nat } \lceil \log 2 \ d \rceil)$ 
note  $q\text{-gt-1} = \text{denom-gt-1}[\text{of } n]$ 

have  $\text{neg}: c * q \ n \neq d * p \ n$ 
proof
  assume  $c * q \ n = d * p \ n$ 
  hence  $\text{of-int } (c * q \ n) = \text{of-int } (d * p \ n)$  by (simp only: )
  with approx-int-pos[ $\text{of } n$ ]  $d \ q\text{-gt-1}$  show False by (auto simp: field-simps)
qed
hence  $\text{abs-pos}: 0 < \text{abs } (c * q \ n - d * p \ n)$  by simp

from  $q\text{-gt-1} \ d$  have  $\text{of-int } d = 2^{\text{powr } \log 2 \ d}$  by (subst powr-log-cancel) simp-all
also from  $d$  have  $\log 2 \ (\text{of-int } d) \geq \log 2 \ 1$  by (subst log-le-cancel-iff) simp-all
hence  $2^{\text{powr } \log 2 \ d} \leq 2^{\text{powr } \text{of-nat } (\text{nat } \lceil \log 2 \ d \rceil)}$ 
  by (intro powr-mono) simp-all
also have  $\dots = \text{of-int } (2^{\text{nat } \lceil \log 2 \ d \rceil})$ 
  by (subst powr-realpow) simp-all
finally have  $d \leq 2^{\text{nat } \lceil \log 2 \ (\text{of-int } d) \rceil}$ 
  by (subst (asm) of-int-le-iff)
also have  $\dots * q \ n \leq q \ n^{\text{Suc } (\text{nat } \lceil \log 2 \ (\text{of-int } d) \rceil)}$ 
  by (subst power-Suc, subst mult.commute, intro mult-left-mono power-mono,
insert q-gt-1) simp-all
also from  $q\text{-gt-1}$  have  $\dots = q \ n^{\text{Suc } (\text{nat } \lceil \log 2 \ (\text{of-int } d) \rceil)}$  by (simp add: n-def)
finally have  $1 / \text{of-int } (q \ n^{\text{Suc } (\text{nat } \lceil \log 2 \ (\text{of-int } d) \rceil)}) \leq 1 / \text{real-of-int } (d * q \ n)$  using  $q\text{-gt-1} \ d$ 
  by (intro divide-left-mono Rings.mult-pos-pos of-int-pos, subst of-int-le-iff)
simp-all
also have  $\dots \leq \text{of-int } (\text{abs } (c * q \ n - d * p \ n)) / \text{real-of-int } (d * q \ n)$  using
 $q\text{-gt-1} \ d \ \text{abs-pos}$ 
  by (intro divide-right-mono) (linarith, simp)
also have  $\dots = \text{abs } (x - \text{of-int } (p \ n) / \text{of-int } (q \ n))$  using  $q\text{-gt-1} \ d(2)$ 
  by (simp-all add: divide-simps d(1) mult-ac)
finally show False using approx-int[ $\text{of } n$ ] by simp
qed

```

Next, any irrational algebraic number cannot be approximated with rational numbers in the Liouville sense.

lemma *liouville-irrational-algebraic*:

```

fixes  $x :: \text{real}$ 
assumes irrationsl:  $x \notin \mathbb{Q}$  and algebraic  $x$ 
obtains  $c :: \text{real}$  and  $n :: \text{nat}$ 
  where  $c > 0$  and  $\bigwedge (p::\text{int}) (q::\text{int}). q > 0 \implies \text{abs } (x - p / q) > c / \text{of-int } q^{\text{Suc } n}$ 
proof –
from  $\langle \text{algebraic } x \rangle$  obtain  $p$  where  $p: \bigwedge i. \text{coeff } p \ i \in \mathbb{Z} \ p \neq 0 \ \text{poly } p \ x = 0$ 
  by (elim algebraicE) blast
define  $n$  where  $n = \text{degree } p$ 

```

— The derivative of p is bounded within $\{x - 1..x + 1\}$.
let $?f = \lambda t. |poly (pderiv p) t|$
define M **where** $M = (SUP\ t \in \{x-1..x+1\}. ?f\ t)$
define $roots$ **where** $roots = \{x. poly\ p\ x = 0\} - \{x\}$

define A -set **where** A -set = $\{1, 1/M\} \cup \{abs\ (x' - x) \mid x'. x' \in roots\}$
define A' **where** $A' = Min\ A$ -set
define A **where** $A = A' / 2$

— We define A to be a positive real number that is less than $1, 1 / M$ and any distance from x to another root of p .

— Properties of M , our upper bound for the derivative of p :

have $\exists s \in \{x-1..x+1\}. \forall y \in \{x-1..x+1\}. ?f\ s \geq ?f\ y$
by (*intro continuous-attains-sup*) (*auto intro!: continuous-intros*)
hence bdd : bdd -above ($?f\ \{x-1..x+1\}$) **by** *auto*

have M -pos: $M > 0$

proof —

from p **have** $pderiv\ p \neq 0$ **by** (*auto dest!: pderiv-iszero*)
hence $finite\ \{x. poly\ (pderiv\ p)\ x = 0\}$ **using** *poly-roots-finite* **by** *blast*
moreover **have** $\neg finite\ \{x-1..x+1\}$ **by** (*simp add: infinite-Icc*)
ultimately **have** $\neg finite\ (\{x-1..x+1\} - \{x. poly\ (pderiv\ p)\ x = 0\})$ **by** *simp*
hence $\{x-1..x+1\} - \{x. poly\ (pderiv\ p)\ x = 0\} \neq \{\}$ **by** (*intro notI*) *simp*
then **obtain** t **where** $t: t \in \{x-1..x+1\}$ **and** $poly\ (pderiv\ p)\ t \neq 0$ **by** *blast*
hence $0 < ?f\ t$ **by** *simp*
also **from** t **and** bdd **have** $\dots \leq M$ **unfolding** M -def **by** (*rule cSUP-upper*)
finally **show** $M > 0$.

qed

have M -sup: $?f\ t \leq M$ **if** $t \in \{x-1..x+1\}$ **for** t

proof —

from *that* **and** bdd **show** $?f\ t \leq M$
unfolding M -def **by** (*rule cSUP-upper*)

qed

— Properties of A :

from p *poly-roots-finite*[*of p*] **have** $finite\ A$ -set
unfolding A -set-def $roots$ -def **by** *simp*
have $x \notin roots$ **unfolding** $roots$ -def **by** *auto*
hence $A' > 0$ **using** *Min-gr-iff*[*OF* $\langle finite\ A$ -set \rangle , *folded* A' -def, *of 0*]
by (*auto simp: A-set-def M-pos*)
hence A -pos: $A > 0$ **by** (*simp add: A-def*)

from $\langle A' > 0 \rangle$ **have** $A < A'$ **by** (*simp add: A-def*)

moreover **from** *Min-le*[*OF* $\langle finite\ A$ -set \rangle , *folded* A' -def]

have $A' \leq 1\ A' \leq 1/M \wedge x'. x' \neq x \implies poly\ p\ x' = 0 \implies A' \leq abs\ (x' - x)$

unfolding A -set-def $roots$ -def **by** *auto*

ultimately **have** A -less: $A < 1\ A < 1/M \wedge x'. x' \neq x \implies poly\ p\ x' = 0 \implies A$

< abs (x' - x)
 by fastforce+

— Finally: no rational number can approximate x “well enough”.

have $\forall (a::int) (b::int). b > 0 \longrightarrow |x - \text{of-int } a / \text{of-int } b| > A / \text{of-int } b \wedge n$
 proof (intro allI impI, rule ccontr)

fix a b :: int

assume b: b > 0 and $\neg(A / \text{of-int } b \wedge n < |x - \text{of-int } a / \text{of-int } b|)$

hence ab: abs (x - of-int a / of-int b) $\leq A / \text{of-int } b \wedge n$ by simp

also from A-pos b have $A / \text{of-int } b \wedge n \leq A / 1$

by (intro divide-left-mono) simp-all

finally have ab': abs (x - a / b) $\leq A$ by simp

also have ... ≤ 1 using A-less by simp

finally have ab'': a / b $\in \{x-1..x+1\}$ by auto

have no-root: poly p (a / b) $\neq 0$

proof

assume poly p (a / b) = 0

moreover from $\langle x \notin \mathbb{Q} \rangle$ have $x \neq a / b$ by auto

ultimately have $A < \text{abs } (a / b - x)$ using A-less(3) by simp

with ab' show False by simp

qed

have $\exists x0 \in \{x-1..x+1\}. \text{poly } p (a / b) - \text{poly } p x = (a / b - x) * \text{poly } (pderiv p) x0$

using ab'' by (intro poly-MVT') (auto simp: min-def max-def)

with p obtain x0 :: real where x0:

$x0 \in \{x-1..x+1\} \text{poly } p (a / b) = (a / b - x) * \text{poly } (pderiv p) x0$ by auto

from x0(2) no-root have deriv-pos: poly (pderiv p) x0 $\neq 0$ by auto

from b p no-root have p-ab: abs (poly p (a / b)) $\geq 1 / \text{of-int } b \wedge n$

unfolding n-def by (intro int-poly-rat-no-root-ge)

note ab

also from A-less b have $A / \text{of-int } b \wedge n < (1/M) / \text{of-int } b \wedge n$

by (intro divide-strict-right-mono) simp-all

also have ... = $(1 / b \wedge n) / M$ by simp

also from M-pos ab p-ab have ... $\leq \text{abs } (\text{poly } p (a / b)) / M$

by (intro divide-right-mono) simp-all

also from deriv-pos M-pos x0(1)

have ... $\leq \text{abs } (\text{poly } p (a / b)) / \text{abs } (\text{poly } (pderiv p) x0)$

by (intro divide-left-mono M-sup) simp-all

also have $|\text{poly } p (a / b)| = |a / b - x| * |\text{poly } (pderiv p) x0|$

by (subst x0(2)) (simp add: abs-mult)

with deriv-pos have $|\text{poly } p (a / b)| / |\text{poly } (pderiv p) x0| = |x - a / b|$

by (subst nonzero-divide-eq-eq) simp-all

finally show False by simp

qed
 with *A-pos* show *?thesis* using *that[of A n]* by *blast*
 qed

Since Liouville numbers are irrational, but can be approximated well by rational numbers in the Liouville sense, they must be transcendental.

lemma (in *liouville*) *transcendental*: \neg *algebraic* *x*
proof

assume *algebraic* *x*
 from *liouville-irrational-algebraic[OF irrational this]*
 obtain *c n* where *cn*:
 $c > 0 \wedge p \ q. \ q > 0 \implies c / \text{real-of-int } q \wedge n < |x - \text{real-of-int } p / \text{real-of-int } q|$
 by *auto*

define *r* where $r = \text{nat } \lceil \log 2 (1 / c) \rceil$
 define *m* where $m = n + r$
 from *cn(1)* have $(1/c) = 2 \text{ powr } \log 2 (1/c)$ by (*subst powr-log-cancel*) *simp-all*
 also have $\log 2 (1/c) \leq \text{of-nat } (\text{nat } \lceil \log 2 (1/c) \rceil)$ by *linarith*
 hence $2 \text{ powr } \log 2 (1/c) \leq 2 \text{ powr } \dots$ by (*intro powr-mono*) *simp-all*
 also have $\dots = 2 \wedge r$ unfolding *r-def* by (*simp add: powr-realpow*)
 finally have $1 / (2 \wedge r) \leq c$ using *cn(1)* by (*simp add: field-simps*)

have $\text{abs } (x - p \ m / q \ m) < 1 / \text{of-int } (q \ m) \wedge m$ by (*rule approx-int*)
 also have $\dots = (1 / \text{of-int } (q \ m) \wedge r) * (1 / \text{real-of-int } (q \ m) \wedge n)$
 by (*simp add: m-def power-add*)
 also from *denom-gt-1[of m]* have $1 / \text{real-of-int } (q \ m) \wedge r \leq 1 / 2 \wedge r$
 by (*intro divide-left-mono power-mono*) *simp-all*
 also have $\dots \leq c$ by *fact*
 finally have $\text{abs } (x - p \ m / q \ m) < c / \text{of-int } (q \ m) \wedge n$
 using *denom-gt-1[of m]* by \neg (*simp-all add: divide-right-mono*)
 with *cn(2)[of q m p m]* *denom-gt-1[of m]* show *False* by *simp*
 qed

1.3 Standard construction for Liouville numbers

We now define the standard construction for Liouville numbers.

definition *standard-liouville* :: $(\text{nat} \Rightarrow \text{int}) \Rightarrow \text{int} \Rightarrow \text{real}$ **where**
standard-liouville *p q* = $(\sum k. p \ k / \text{of-int } q \wedge \text{fact } (\text{Suc } k))$

lemma *standard-liouville-summable*:
 fixes *p* :: $\text{nat} \Rightarrow \text{int}$ and *q* :: *int*
 assumes $q > 1$ range $p \subseteq \{0..<q\}$
 shows *summable* $(\lambda k. p \ k / \text{of-int } q \wedge \text{fact } (\text{Suc } k))$
proof (*rule summable-comparison-test'*)
 from *assms(1)* show *summable* $(\lambda n. (1 / q) \wedge n)$
 by (*intro summable-geometric*) *simp-all*
next
 fix *n* :: *nat*

from *assms* **have** $p \ n \in \{0..<q\}$ **by** *blast*
with *assms*(1) **have** $\text{norm } (p \ n \ / \ \text{of-int } q \ ^{\wedge} \ \text{fact } (\text{Suc } n)) \leq$
 $q \ / \ \text{of-int } q \ ^{\wedge} \ (\text{fact } (\text{Suc } n))$ **by** (*auto simp: field-simps*)
also from *assms*(1) **have** $\dots = 1 \ / \ \text{of-int } q \ ^{\wedge} \ (\text{fact } (\text{Suc } n) - 1)$
by (*subst power-diff*) (*auto simp del: fact-Suc*)
also have $\text{Suc } n \leq \text{fact } (\text{Suc } n)$ **by** (*rule fact-ge-self*)
with *assms*(1) **have** $1 \ / \ \text{real-of-int } q \ ^{\wedge} \ (\text{fact } (\text{Suc } n) - 1) \leq 1 \ / \ \text{of-int } q \ ^{\wedge} \ n$
by (*intro divide-left-mono power-increasing*)
(*auto simp del: fact-Suc simp add: algebra-simps*)
finally show $\text{norm } (p \ n \ / \ \text{of-int } q \ ^{\wedge} \ \text{fact } (\text{Suc } n)) \leq (1 \ / \ q) \ ^{\wedge} \ n$
by (*simp add: power-divide*)
qed

lemma *standard-liouville-sums*:

assumes $q > 1$ *range* $p \subseteq \{0..<q\}$
shows $(\lambda k. p \ k \ / \ \text{of-int } q \ ^{\wedge} \ \text{fact } (\text{Suc } k))$ *sums standard-liouville* $p \ q$
using *standard-liouville-summable*[*OF assms*] **unfolding** *standard-liouville-def*
by (*simp add: summable-sums*)

Now we prove that the standard construction indeed yields Liouville numbers.

lemma *standard-liouville-is-liouville*:

assumes $q > 1$ *range* $p \subseteq \{0..<q\}$ *frequently* $(\lambda n. p \ n \neq 0)$ *sequentially*
defines $b \equiv \lambda n. q \ ^{\wedge} \ \text{fact } (\text{Suc } n)$
defines $a \equiv \lambda n. (\sum k \leq n. p \ k * q \ ^{\wedge} \ (\text{fact } (\text{Suc } n) - \text{fact } (\text{Suc } k)))$
shows *liouville* (*standard-liouville* $p \ q$) $a \ b$

proof

fix $n :: \text{nat}$
from *assms*(1) **have** $1 < q \ ^{\wedge} \ 1$ **by** *simp*
also from *assms*(1) **have** $\dots \leq b \ n$ **unfolding** *b-def*
by (*intro power-increasing*) (*simp-all del: fact-Suc*)
finally show $b \ n > 1$.

note *summable* = *standard-liouville-summable*[*OF assms*(1,2)]

let $?S = \sum k. p \ (k + n + 1) \ / \ \text{of-int } q \ ^{\wedge} \ (\text{fact } (\text{Suc } (k + n + 1)))$

let $?C = (q - 1) \ / \ \text{of-int } q \ ^{\wedge} \ (\text{fact } (n+2))$

have $a \ n \ / \ b \ n = (\sum k \leq n. p \ k * (\text{of-int } q \ ^{\wedge} \ (\text{fact } (n+1) - \text{fact } (k+1))) \ / \ \text{of-int } q \ ^{\wedge} \ (\text{fact } (n+1)))$

by (*simp add: a-def b-def sum-divide-distrib of-int-sum*)

also have $\dots = (\sum k \leq n. p \ k \ / \ \text{of-int } q \ ^{\wedge} \ (\text{fact } (\text{Suc } k)))$

by (*intro sum.cong refl, subst inverse-divide [symmetric], subst power-diff [symmetric]*)

(*insert assms*(1), *simp-all add: divide-simps fact-mono-nat del: fact-Suc*)

also have *standard-liouville* $p \ q - \dots = ?S$ **unfolding** *standard-liouville-def*

by (*subst diff-eq-eq*) (*intro suminf-split-initial-segment' summable*)

finally have *abs* (*standard-liouville* $p \ q - a \ n \ / \ b \ n$) = *abs* $?S$ **by** (*simp only:*)

moreover from *assms* **have** $?S \geq 0$

by (*intro suminf-nonneg allI divide-nonneg-pos summable-ignore-initial-segment*)

summable) *force+*
ultimately have $eq: abs (standard-liouville\ p\ q - a\ n / b\ n) = ?S$ **by** *simp*

also have $?S \leq (\sum k. ?C * (1 / q) ^ k)$
proof (*intro suminf-le allI summable-ignore-initial-segment summable*)
from *assms* **show** *summable* $(\lambda k. ?C * (1 / q) ^ k)$
by (*intro summable-mult summable-geometric*) *simp-all*
next
fix $k :: nat$
from *assms* **have** $p (k + n + 1) \leq q - 1$ **by** *force*
with $\langle q > 1 \rangle$ **have** $p (k + n + 1) / of-int\ q ^ fact (Suc (k + n + 1)) \leq$
 $(q - 1) / of-int\ q ^ (fact (Suc (k + n + 1)))$
by (*intro divide-right-mono*) (*simp-all del: fact-Suc*)
also from $\langle q > 1 \rangle$ **have** $\dots \leq (q - 1) / of-int\ q ^ (fact (n + 2) + k)$
using *fact-ineq*[*of n+2 k*]
by (*intro divide-left-mono power-increasing*) (*simp-all add: add-ac*)
also have $\dots = ?C * (1 / q) ^ k$
by (*simp add: field-simps power-add del: fact-Suc*)
finally show $p (k + n + 1) / of-int\ q ^ fact (Suc (k + n + 1)) \leq \dots$
qed
also from *assms* **have** $\dots = ?C * (\sum k. (1 / q) ^ k)$
by (*intro suminf-mult summable-geometric*) *simp-all*
also from *assms* **have** $(\sum k. (1 / q) ^ k) = 1 / (1 - 1 / q)$
by (*intro suminf-geometric*) *simp-all*
also from *assms*(1) **have** $?C * \dots = of-int\ q ^ 1 / of-int\ q ^ fact (n + 2)$
by (*simp add: field-simps*)
also from *assms*(1) **have** $\dots \leq of-int\ q ^ fact (n + 1) / of-int\ q ^ fact (n + 2)$
by (*intro divide-right-mono power-increasing*) (*simp-all add: field-simps del: fact-Suc*)
also from *assms*(1) **have** $\dots = 1 / (of-int\ q ^ (fact (n + 2) - fact (n + 1)))$
by (*subst power-diff*) *simp-all*
also have $fact (n + 2) - fact (n + 1) = (n + 1) * fact (n + 1)$ **by** (*simp add: algebra-simps*)
also from *assms*(1) **have** $1 / (of-int\ q ^ \dots) < 1 / (real-of-int\ q ^ (fact (n + 1) * n))$
by (*intro divide-strict-left-mono power-increasing mult-right-mono*) *simp-all*
also have $\dots = 1 / of-int (b\ n) ^ n$
by (*simp add: power-mult b-def power-divide del: fact-Suc*)
finally show $|standard-liouville\ p\ q - a\ n / b\ n| < 1 / of-int (b\ n) ^ n$.

from *assms*(3) **obtain** k **where** $k: k \geq n + 1\ p\ k \neq 0$
by (*auto simp: frequently-def eventually-at-top-linorder*)
define k' **where** $k' = k - n - 1$
from *assms* k **have** $p\ k \geq 0$ **by** *force*
with k *assms* **have** $k': p (k' + n + 1) > 0$ **unfolding** k' -*def* **by** *force*
with *assms*(1,2) **have** $?S > 0$
by (*intro suminf-pos2*[*of - k'*] *summable-ignore-initial-segment summable*)
(*force intro!: divide-pos-pos divide-nonneg-pos*)
with *eq* **show** $|standard-liouville\ p\ q - a\ n / b\ n| > 0$ **by** (*simp only:*)

qed

We can now show our main result: any standard Liouville number is transcendental.

theorem *transcendental-standard-liouville:*

assumes $q > 1$ range $p \subseteq \{0..<q\}$ frequently $(\lambda k. p k \neq 0)$ sequentially

shows \neg algebraic (standard-liouville $p q$)

proof –

from *assms* **interpret**

liouville standard-liouville p q

$\lambda n. (\sum k \leq n. p k * q ^ \wedge (fact (Suc n) - fact (Suc k)))$

$\lambda n. q ^ \wedge fact (Suc n)$

by (*rule standard-liouville-is-liouville*)

from *transcendental* **show** *?thesis* .

qed

In particular: The the standard construction for constant sequences, such as the “classic” Liouville constant $\sum_{n=1}^{\infty} 10^{-n!} = 0.11000100\dots$, are transcendental.

This shows that Liouville numbers exists and therefore gives a concrete and elementary proof that transcendental numbers exist.

corollary *transcendental-standard-standard-liouville:*

$a \in \{0 < .. < b\} \implies \neg$ algebraic (standard-liouville $(\lambda-. int a) (int b)$)

by (*intro transcendental-standard-liouville*) *auto*

corollary *transcendental-liouville-constant:*

\neg algebraic (standard-liouville $(\lambda-. 1) 10$)

by (*intro transcendental-standard-liouville*) *auto*

end

References

- [1] Wikipedia. Liouville number — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Liouville_number&oldid=696910651, 2015. [Online; accessed 22-July-2004].