Lebesgue-Stieltjes Integral

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May 8, 2026

Abstract

This entry formalizes some basic facts and lemmas relating to the integration with respect to the Lebesgue-Stieltjes measure (interval measure). It includes the well-known formula to calculate the Lebesgue-Stieltjes integral:

$$\int g(x)\, dF(x) = \int g(x) F'(x)\, dx.$$

Contents

theory *Preliminaries-LSI*
  imports *HOL−Library.Rewrite HOL−Analysis.Analysis*
begin

context *order-topology*
begin

lemma
  assumes $a < b$
  shows *at-within-Ioo-at-right*: *at* a *within* $\{a<..<b\}$ = *at-right* a **and**
    at-within-Ioo-at-left: *at* b *within* $\{a<..<b\}$ = *at-left* b
    ⟨*proof*⟩

end

lemma *Int-atLeastAtMost-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..b\}$ = $\{a..b\}$
  ⟨*proof*⟩

lemma *Int-greaterThanAtMost-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..b\}$ = $\{a<..b\}$
  ⟨*proof*⟩

lemma *Int-atLeastLessThan-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..<b\} = \{a..<b\}$
⟨*proof*⟩

lemma *Int-greaterThanLessThan-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..<b\} = \{a<..<b\}$
⟨*proof*⟩

lemma *constant-on-empty*[*simp*]: *f constant-on* $\{\}$
⟨*proof*⟩

lemma *constant-on-Un*:
assumes *f constant-on A f constant-on B* $A \cap B \neq \{\}$
shows *f constant-on* $A \cup B$
⟨*proof*⟩

lemma *differentiable-transform-open*:
assumes *f differentiable* (*at x*)
and $x \in s$
and *open s*
and $\bigwedge x'.\ x' \in s \Longrightarrow f\ x' = g\ x'$
shows *g differentiable* (*at x*)
⟨*proof*⟩

lemma *differentiable-eq-field-differentiable-real*:
fixes *f* :: *real* ⇒ *real*
shows *f differentiable F* ⟷ *f field-differentiable F*
⟨*proof*⟩

lemma *differentiable-on-eq-field-differentiable-real*:
fixes *f* :: *real* ⇒ *real*
shows *f differentiable-on s* ⟷ (∀ *x*∈*s*. *f field-differentiable* (*at x within s*))
⟨*proof*⟩

lemma *set-borel-measurable-UNIV*[*simp*]:
fixes *f* :: *'a* :: *real-vector* ⇒ *real*
shows *set-borel-measurable M UNIV f* ⟷ *f* ∈ *borel-measurable M*
⟨*proof*⟩

lemma *deriv-measurable-real*:
fixes *f* :: *real* ⇒ *real*
assumes *f differentiable-on S open S f* ∈ *borel-measurable borel*
shows *set-borel-measurable borel S* (*deriv f*)
⟨*proof*⟩

corollary *deriv-measurable-real-UNIV*:
fixes *f* :: *real* ⇒ *real*

assumes *f differentiable-on UNIV f* ∈ *borel-measurable borel*
shows *deriv f* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *piecewise-differentiable-on-deriv-measurable-real*:
fixes *f* :: *real* ⇒ *real*
assumes *f piecewise-differentiable-on S open S f* ∈ *borel-measurable borel*
shows *set-borel-measurable borel S* (*deriv f*)
⟨*proof*⟩

corollary *piecewise-differentiable-on-deriv-measurable-real-UNIV*:
fixes *f* :: *real* ⇒ *real*
assumes *f piecewise-differentiable-on UNIV f* ∈ *borel-measurable borel*
shows (*deriv f*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *einterval-empty*:
fixes *a b* :: *ereal*
assumes $a \geq b$
shows *einterval a b* = {}
⟨*proof*⟩

lemma *einterval-split*:
fixes *a b* :: *ereal* **and** *s*::*real*
assumes *s* ∈ *einterval a b*
shows *einterval a b* − {*s*} = *einterval a s* ∪ *einterval s b*
⟨*proof*⟩

lemma *einterval-Ioc-approximation*:
fixes *a b* :: *ereal*
assumes $a < b$
obtains *u l* :: *nat* ⇒ *real* **where**
einterval a b = (⋃*i*. {*l i* <.. *u i*})
incseq u decseq l ⋀*i*. *l i* < *u i* ⋀*i*. *a* < *l i* ⋀*i*. *u i* < *b*
l ⟶ *a u* ⟶ *b*
⟨*proof*⟩

lemma *measure-eqI-Ioc*:
fixes *M N* :: *real measure*
assumes *sets*: *sets M* = *sets borel sets N* = *sets borel*
assumes *fin*: ⋀*a b*. *a* ≤ *b* ⟹ *emeasure M* {*a*<..*b*} < ∞
assumes *eq*: ⋀*a b*. *a* ≤ *b* ⟹ *emeasure M* {*a*<..*b*} = *emeasure N* {*a*<..*b*}
shows *M* = *N*
⟨*proof*⟩

lemma *measure-einterval-eqI-Ioc*:
fixes *M N* :: *real measure* **and** *a b* :: *ereal*

assumes *Mborel*: $sets\ M = sets\ borel$ **and** *Nborel*: $sets\ N = sets\ borel$ **and**
$\bigwedge s\ t.\ a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \Longrightarrow emeasure\ M\ \{s{<}..t\} \neq \infty$ **and**
$\bigwedge s\ t.\ a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \Longrightarrow emeasure\ M\ \{s{<}..t\} = emeasure\ N\ \{s{<}..t\}$
shows $restrict\text{-}space\ M\ (einterval\ a\ b) = restrict\text{-}space\ N\ (einterval\ a\ b)$
⟨*proof*⟩

lemma *nn-integral-disjoint-pair2*:
assumes $B \in sets\ M\ C \in sets\ M\ B \cap C = \{\}$ **and**
[*measurable*]: $(\lambda x.\ f\ x * indicator\ B\ x) \in borel\text{-}measurable\ M$ **and**
[*measurable*]: $(\lambda x.\ f\ x * indicator\ C\ x) \in borel\text{-}measurable\ M$
shows $(\int^+ x \in B \cup C.\ f\ x\ \partial M) = (\int^+ x \in B.\ f\ x\ \partial M) + (\int^+ x \in C.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *set-nn-integral-interval-measure-bounded-finite*:
fixes $F :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $A :: real\ set$ **and** $M{::}real$
assumes $bounded\ A\ \bigwedge x.\ x \in A \Longrightarrow h\ x \leq M\ A \in sets\ borel$ **and**
$mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$
shows $(\int^+ x{\in}A.\ h\ x\ \partial(interval\text{-}measure\ F)) < \infty$
⟨*proof*⟩

end
theory *Lebesgue-Stieltjes-Integral*
imports *Wlog.Wlog Preliminaries-LSI*
begin

1 Interval Measure Integral

1.1 Basic Calculations

lemma *interval-measure-const-null*:
fixes $c{::}real$
shows $interval\text{-}measure\ (\lambda\text{-}.\ c) = null\text{-}measure\ lborel$
⟨*proof*⟩

lemma *interval-measure-singleton*:
fixes $F :: real \Rightarrow real$ **and** $s{::}real$
assumes $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F$
shows $(interval\text{-}measure\ F)\ \{s\} = F\ s - Lim\ (at\text{-}left\ s)\ F$
⟨*proof*⟩

lemma *interval-measure-singleton-continuous*:
fixes $F :: real \Rightarrow real$ **and** $s{::}real$
assumes $mono\ F\ \bigwedge x.\ continuous\ (at\text{-}right\ x)\ F\ isCont\ F\ s$
shows $(interval\text{-}measure\ F)\ \{s\} = 0$
⟨*proof*⟩

1.2 Changing the Underlying Function

lemma *einterval-nn-integral-interval-measure-cong*:
fixes F G :: *real* ⇒ *real* **and** h :: *real* ⇒ *ennreal* **and** a b :: *ereal*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F **and**
mono G $\bigwedge x$. *continuous* (*at-right* x) G **and**
(F − G) *constant-on* (*einterval* a b) **and**
h ∈ *borel-measurable borel*
shows ($\int^+ x$∈(*einterval* a b). h x ∂(*interval-measure* F)) =
($\int^+ x$∈(*einterval* a b). h x ∂(*interval-measure* G))
⟨*proof*⟩

corollary *Ioo-nn-integral-interval-measure-cong*:
fixes F G :: *real* ⇒ *real* **and** h :: *real* ⇒ *ennreal* **and** r s :: *real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F **and**
mono G $\bigwedge x$. *continuous* (*at-right* x) G **and**
(F − G) *constant-on* {r<..<s} **and**
h ∈ *borel-measurable borel*
shows ($\int^+ x$∈{r<..<s}. h x ∂(*interval-measure* F)) = ($\int^+ x$∈{r<..<s}. h x ∂(*interval-measure* G))
⟨*proof*⟩

corollary *Ioi-nn-integral-interval-measure-cong*:
fixes F G :: *real* ⇒ *real* **and** h :: *real* ⇒ *ennreal* **and** r::*real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F **and**
mono G $\bigwedge x$. *continuous* (*at-right* x) G **and**
(F − G) *constant-on* {r<..} **and**
h ∈ *borel-measurable borel*
shows ($\int^+ x$∈{r<..}. h x ∂(*interval-measure* F)) = ($\int^+ x$∈{r<..}. h x ∂(*interval-measure* G))
⟨*proof*⟩

corollary *Iio-nn-integral-interval-measure-cong*:
fixes F G :: *real* ⇒ *real* **and** h :: *real* ⇒ *ennreal* **and** s::*real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F **and**
mono G $\bigwedge x$. *continuous* (*at-right* x) G **and**
(F − G) *constant-on* {..<s} **and**
h ∈ *borel-measurable borel*
shows ($\int^+ x$∈{..<s}. h x ∂(*interval-measure* F)) = ($\int^+ x$∈{..<s}. h x ∂(*interval-measure* G))
⟨*proof*⟩

corollary *nn-integral-interval-measure-cong*:
fixes F G :: *real* ⇒ *real* **and** h :: *real* ⇒ *ennreal*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F **and**
mono G $\bigwedge x$. *continuous* (*at-right* x) G **and**
(F − G) *constant-on* *UNIV* **and**
h ∈ *borel-measurable borel*
shows ($\int^+ x$. h x ∂(*interval-measure* F)) = ($\int^+ x$. h x ∂(*interval-measure* G))
⟨*proof*⟩

lemma *singleton-nn-integral-interval-measure-cong*:
fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *s*::*real*
assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
F s − *Lim* (*at-left s*) *F* = *G s* − *Lim* (*at-left s*) *G* **and**
h ∈ *borel-measurable borel*
shows (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

lemma *singleton-const-nn-integral-interval-measure-cong*:
fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..*s*} **and** *r* < *s* **and**
h ∈ *borel-measurable borel*
shows (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

lemma *Ioc-nn-integral-interval-measure-cong*:
fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..*s*} **and**
h ∈ *borel-measurable borel*
shows (∫ ⁺*x*∈{*r*<..*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*r*<..*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

lemma *Iic-nn-integral-interval-measure-cong*:
fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *s*::*real*
assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {..*s*} **and**
h ∈ *borel-measurable borel*
shows (∫ ⁺*x*∈{..*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{..*s*}. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

lemma *Ico-nn-integral-interval-measure-cong*:
fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..<*s*} **and**
F r − *Lim* (*at-left r*) *F* = *G r* − *Lim* (*at-left r*) *G* **and**
h ∈ *borel-measurable borel*
shows (∫ ⁺*x*∈{*r*..<*s*}. *h x* ∂(*interval-measure F*)) = (∫ ⁺*x*∈{*r*..<*s*}. *h x* ∂(*interval-measure*

G))
⟨*proof*⟩

corollary *Ico-Cont-nn-integral-interval-measure-cong*:
 fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
 assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
  mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
  (*F* − *G*) *constant-on* {*r*<..<*s*} **and**
  isCont F r isCont G r **and**
  h ∈ *borel-measurable borel*
 shows (∫⁺*x*∈{*r*..<*s*}. *h x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*..<*s*}. *h x* ∂(*interval-measure*
G))
⟨*proof*⟩

lemma *Ici-nn-integral-interval-measure-cong*:
 fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r*::*real*
 assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
  mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
  (*F* − *G*) *constant-on* {*r*<..} **and**
  F r − *Lim* (*at-left r*) *F* = *G r* − *Lim* (*at-left r*) *G* **and**
  h ∈ *borel-measurable borel*
 shows (∫⁺*x*∈{*r*..}. *h x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*..}. *h x* ∂(*interval-measure*
G))
⟨*proof*⟩

corollary *Ici-Cont-nn-integral-interval-measure-cong*:
 fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r*::*real*
 assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
  mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
  (*F* − *G*) *constant-on* {*r*<..} **and**
  isCont F r isCont G r **and**
  h ∈ *borel-measurable borel*
 shows (∫⁺*x*∈{*r*..}. *h x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*..}. *h x* ∂(*interval-measure*
G))
⟨*proof*⟩

lemma *Icc-nn-integral-interval-measure-cong*:
 fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
 assumes *mono F* ⋀*x*. *continuous* (*at-right x*) *F* **and**
  mono G ⋀*x*. *continuous* (*at-right x*) *G* **and**
  (*F* − *G*) *constant-on* {*r*<..*s*} **and**
  F r − *Lim* (*at-left r*) *F* = *G r* − *Lim* (*at-left r*) *G* **and**
  h ∈ *borel-measurable borel*
 shows (∫⁺*x*∈{*r*..*s*}. *h x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*..*s*}. *h x* ∂(*interval-measure*
G))
⟨*proof*⟩

corollary *Icc-Cont-nn-integral-interval-measure-cong*:
 fixes *F G* :: *real* ⇒ *real* **and** *h* :: *real* ⇒ *ennreal* **and** *r s* :: *real*

assumes *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F* **and**
mono G $\bigwedge x$. *continuous* (*at-right x*) *G* **and**
(*F* − *G*) *constant-on* {*r*<..*s*} **and**
isCont F r isCont G r **and**
h ∈ *borel-measurable borel*
shows ($\int^+ x \in \{r..s\}$. *h x* ∂(*interval-measure F*)) = ($\int^+ x \in \{r..s\}$. *h x* ∂(*interval-measure G*))
⟨*proof*⟩

1.3 Restricting the Integral

lemma *nn-integral-interval-measure-Ici*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r*::*real*
assumes *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F g* ∈ *borel-measurable borel* **and**
F constant-on {..<*r*}
shows ($\int^+ x$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{r..\}$. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

lemma *nn-integral-interval-measure-Ioi*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r*::*real*
assumes *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F g* ∈ *borel-measurable borel* **and**
F constant-on {..<*r*} *isCont F r*
shows ($\int^+ x$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{r<..\}$. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

lemma *nn-integral-interval-measure-Iic*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *s*::*real*
assumes *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F g* ∈ *borel-measurable borel* **and**
F constant-on {*s*<..}
shows ($\int^+ x$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{..s\}$. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

lemma *nn-integral-interval-measure-Iio*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *s*::*real*
assumes *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F g* ∈ *borel-measurable borel* **and**
F constant-on {*s*<..} *isCont F s*
shows ($\int^+ x$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{..<s\}$. *g x* ∂(*interval-measure F*))
⟨*proof*⟩

lemma *nn-integral-interval-measure-Icc*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
assumes *mono F* $\bigwedge x$. *continuous* (*at-right x*) *F g* ∈ *borel-measurable borel* **and**
F constant-on {..<*r*} *F constant-on* {*s*<..}
shows ($\int^+ x$. *g x* ∂(*interval-measure F*)) = ($\int^+ x \in \{r..s\}$. *g x* ∂(*interval-measure F*))

⟨*proof*⟩

lemma *nn-integral-interval-measure-Ioc*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** r s :: *real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F $g \in$ *borel-measurable borel* **and**
F *constant-on* {..<r} F *constant-on* {s<..} *isCont* F *rf*
shows ($\int^+ x$. g x ∂(*interval-measure* F)) = ($\int^+ x\in\{r<..s\}$. g x ∂(*interval-measure* F))
⟨*proof*⟩

lemma *nn-integral-interval-measure-Ico*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** r s :: *real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F $g \in$ *borel-measurable borel*
F *constant-on* {..<r} F *constant-on* {s<..} *isCont* F s
shows ($\int^+ x$. g x ∂(*interval-measure* F)) = ($\int^+ x\in\{r..<s\}$. g x ∂(*interval-measure* F))
⟨*proof*⟩

lemma *nn-integral-interval-measure-Ioo*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** r s :: *real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F $g \in$ *borel-measurable borel* **and**
F *constant-on* {..<r} F *constant-on* {s<..} *isCont* F r *isCont* F s
shows ($\int^+ x$. g x ∂(*interval-measure* F)) = ($\int^+ x\in\{r<..<s\}$. g x ∂(*interval-measure* F))
⟨*proof*⟩

1.4 Calculation by the Derivative

proposition *set-nn-integral-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** a b :: *ereal* **and** A :: *real set*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *differentiable-on* (*einterval* a b) **and**
g-msr: $g \in$ *borel-measurable lborel* **and**
$A \in$ *sets borel* $A \subseteq$ *einterval* a b
shows ($\int^+ x\in A$. g x ∂(*interval-measure* F)) = ($\int^+ x\in A$. g x $*$ *deriv* F x ∂*lborel*)
⟨*proof*⟩

corollary *nn-integral-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *differentiable-on UNIV* **and**
$g \in$ *borel-measurable lborel*
shows ($\int^+ x$. g x ∂(*interval-measure* F)) = ($\int^+ x$. g x $*$ *deriv* F x ∂*lborel*)
⟨*proof*⟩

corollary *Ioi-nn-integral-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** r::*real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *differentiable-on* {r<..} **and**
$g \in$ *borel-measurable lborel*
shows ($\int^+ x\in\{r<..\}$. g x ∂(*interval-measure* F)) = ($\int^+ x\in\{r<..\}$. g x $*$ *deriv* F

x $\partial lborel$)
⟨*proof*⟩

corollary *Iio-nn-integral-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** s::*real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *differentiable-on* $\{..<s\}$ **and**
$g \in$ *borel-measurable lborel*
shows $(\int^+ x \in \{..<s\}.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{..<s\}.\ g\ x * \textit{deriv}\ F\ x\ \partial \textit{lborel})$
⟨*proof*⟩

corollary *Ioo-nn-integral-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** r s :: *real*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *differentiable-on* $\{r<..<s\}$ **and**
$g \in$ *borel-measurable lborel*
shows $(\int^+ x \in \{r<..<s\}.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x \in \{r<..<s\}.\ g\ x * \textit{deriv}\ F\ x\ \partial \textit{lborel})$
⟨*proof*⟩

lemma *set-nn-integral-finite-nondifferentiable-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** a b :: *ereal* **and** S :: *real set*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F $g \in$ *borel-measurable lborel* **and**
cont: *continuous-on* (*einterval* a b) F **and**
diff: F *differentiable-on einterval* a b $- S$ **and**
fin: *finite* S
shows $(\int^+ x \in \textit{einterval}\ a\ b.\ g\ x\ \partial(\textit{interval-measure}\ F)) =$
$(\int^+ x \in \textit{einterval}\ a\ b.\ g\ x * \textit{deriv}\ F\ x\ \partial \textit{lborel})$
⟨*proof*⟩

proposition *set-nn-integral-piecewise-differentiable-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** a b :: *ereal*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *piecewise-differentiable-on* (*einterval* a b)
$g \in$ *borel-measurable lborel*
shows $(\int^+ x \in \textit{einterval}\ a\ b.\ g\ x\ \partial(\textit{interval-measure}\ F)) =$
$(\int^+ x \in \textit{einterval}\ a\ b.\ g\ x * \textit{deriv}\ F\ x\ \partial \textit{lborel})$
⟨*proof*⟩

corollary *nn-integral-piecewise-differentiable-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal*
assumes *mono* F $\bigwedge x$. *continuous* (*at-right* x) F F *piecewise-differentiable-on* *UNIV*
$g \in$ *borel-measurable lborel*
shows $(\int^+ x.\ g\ x\ \partial(\textit{interval-measure}\ F)) = (\int^+ x.\ g\ x * \textit{deriv}\ F\ x\ \partial \textit{lborel})$
⟨*proof*⟩

corollary *Ioi-nn-integral-piecewise-differentiable-interval-measure-deriv*:
fixes F :: *real* ⇒ *real* **and** g :: *real* ⇒ *ennreal* **and** r::*real*

assumes *mono F* ⋀*x. continuous* (*at-right x*) *F F piecewise-differentiable-on* {*r*<..}
g ∈ *borel-measurable lborel*
shows (∫⁺*x*∈{*r*<..}. *g x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*<..}. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

corollary *Iio-nn-integral-piecewise-differentiable-interval-measure-deriv*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *s*::*real*
assumes *mono F* ⋀*x. continuous* (*at-right x*) *F F piecewise-differentiable-on* {..<*s*}
g ∈ *borel-measurable lborel*
shows (∫⁺*x*∈{..<*s*}. *g x* ∂(*interval-measure F*)) = (∫⁺*x*∈{..<*s*}. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

corollary *Ioo-nn-integral-piecewise-differentiable-interval-measure-deriv*:
fixes *F* :: *real* ⇒ *real* **and** *g* :: *real* ⇒ *ennreal* **and** *r s* :: *real*
assumes *mono F* ⋀*x. continuous* (*at-right x*) *F F piecewise-differentiable-on* {*r*<..<*s*}
g ∈ *borel-measurable lborel*
shows (∫⁺*x*∈{*r*<..<*s*}. *g x* ∂(*interval-measure F*)) = (∫⁺*x*∈{*r*<..<*s*}. *g x* ∗ *deriv F x* ∂*lborel*)
⟨*proof*⟩

end