

Lebesgue-Stieltjes Integral

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Abstract

This entry formalizes some basic facts and lemmas relating to the integration with respect to the Lebesgue-Stieltjes measure (interval measure). It includes the well-known formula to calculate the Lebesgue-Stieltjes integral:

$$\int g(x) dF(x) = \int g(x)F'(x) dx.$$

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theory *Preliminaries-LSI*

imports *HOL-Library.Rewrite HOL-Analysis.Analysis*

begin

context *order-topology*

begin

lemma

assumes $a < b$

shows *at-within-Ioo-at-right: at a within {a<..**and***

*at-within-Ioo-at-left: at b within {a<..**at-left***

using *assms at-within-nhd*

apply (*metis inf.right-idem open-lessThan greaterThanLessThan-eq lessThan-iff*)

using *assms at-within-nhd*

by (*smt (verit, ccfv-SIG) inf.idem inf-aci*

greaterThanLessThan-def greaterThan-iff open-greaterThan)

end

lemma *Int-atLeastAtMost-Unbounded[simp]: {a..} Int {..b} = {a..b}*

by *auto*

lemma *Int-greaterThanAtMost-Unbounded[simp]*: $\{a<..\} \text{ Int } \{..b\} = \{a<..b\}$
by *auto*

lemma *Int-atLeastLessThan-Unbounded[simp]*: $\{a..\} \text{ Int } \{..<b\} = \{a..<b\}$
by *auto*

lemma *Int-greaterThanLessThan-Unbounded[simp]*: $\{a<..\} \text{ Int } \{..<b\} = \{a<..<b\}$
by *auto*

lemma *constant-on-empty[simp]*: $f \text{ constant-on } \{\}$
by (*simp add: constant-on-def*)

lemma *constant-on-Un*:
assumes $f \text{ constant-on } A \ f \text{ constant-on } B \ A \cap B \neq \{\}$
shows $f \text{ constant-on } A \cup B$
proof –
from *assms* **obtain** c **where** $c \in A \cap B$ **by** *auto*
with *assms* **have** $\bigwedge x. x \in A \cup B \implies f x = f c$ **unfolding** *constant-on-def* **by**
auto
thus *?thesis* **unfolding** *constant-on-def* **by** *simp*
qed

lemma *differentiable-transform-open*:
assumes $f \text{ differentiable (at } x)$
and $x \in s$
and *open* s
and $\bigwedge x'. x' \in s \implies f x' = g x'$
shows $g \text{ differentiable (at } x)$
using *assms* *has-derivative-transform-within-open at-within-open* **unfolding** *differentiable-def*
by *blast*

lemma *differentiable-eq-field-differentiable-real*:
fixes $f :: \text{real} \Rightarrow \text{real}$
shows $f \text{ differentiable } F \iff f \text{ field-differentiable } F$
unfolding *field-differentiable-def differentiable-def has-real-derivative*
using *has-real-derivative-iff* **by** *presburger*

lemma *differentiable-on-eq-field-differentiable-real*:
fixes $f :: \text{real} \Rightarrow \text{real}$
shows $f \text{ differentiable-on } s \iff (\forall x \in s. f \text{ field-differentiable (at } x \text{ within } s))$
unfolding *differentiable-on-def* **using** *differentiable-eq-field-differentiable-real* **by**
simp

lemma *set-borel-measurable-UNIV*[simp]:
fixes $f :: 'a :: \text{real-vector} \Rightarrow \text{real}$
shows *set-borel-measurable* $M \text{ UNIV } f \iff f \in \text{borel-measurable } M$
unfolding *set-borel-measurable-def* **by** *simp*

lemma *deriv-measurable-real*:
fixes $f :: \text{real} \Rightarrow \text{real}$
assumes f *differentiable-on* S *open* S $f \in \text{borel-measurable borel}$
shows *set-borel-measurable* $\text{borel } S$ $(\text{deriv } f)$
proof –
have $\bigwedge x. x \in S \implies \text{deriv } f x = \lim (\lambda i. (f (x + 1 / \text{Suc } i) - f x) / (1 / \text{Suc } i))$
proof –
fix x **assume** $x \in S$: $x \in S$
hence f *field-differentiable* (at x within S)
using *differentiable-on-eq-field-differentiable-real* **assms** **by** *simp*
hence f *has-field-derivative* $\text{deriv } f x$ (at x)
using *assms DERIV-deriv-iff-field-differentiable* $x \in S$ *at-within-open* **by** *force*
hence $(\lambda h. (f (x+h) - f x) / h) \rightarrow 0 \implies \text{deriv } f x$ **using** *DERIV-def* **by** *auto*
hence $\forall d. (\forall i. d i \in \text{UNIV} - \{0::\text{real}\}) \implies d \longrightarrow 0 \implies$
 $((\lambda h. (f (x+h) - f x) / h) \circ d) \longrightarrow \text{deriv } f x$
using *tendsto-at-iff-sequentially* **by** *blast*
moreover **have** $\forall i. 1 / \text{Suc } i \in \text{UNIV} - \{0::\text{real}\}$ **by** *simp*
moreover **have** $(\lambda i. 1 / \text{Suc } i) \longrightarrow 0$ **using** *LIMSEQ-Suc* *lim-const-over-n*
by *blast*
ultimately **have** $((\lambda h. (f (x+h) - f x) / h) \circ (\lambda i. 1 / \text{Suc } i)) \longrightarrow \text{deriv } f x$ **by** *auto*
thus $\text{deriv } f x = \lim (\lambda i. (f (x + 1 / \text{Suc } i) - f x) / (1 / \text{Suc } i))$
unfolding *comp-def* **by** *(simp add: limI)*
qed
moreover **have** $(\lambda x. \text{indicator } S x * \lim (\lambda i. (f (x + 1 / \text{Suc } i) - f x) / (1 / \text{Suc } i)))$
 $\in \text{borel-measurable borel}$
using *assms* **by** *(measurable, simp, measurable)*
ultimately **show** *?thesis*
unfolding *set-borel-measurable-def* *real-scaleR-def*
by *(smt (verit) indicator-simps measurable-cong mult-eq-0-iff)*
qed

corollary *deriv-measurable-real-UNIV*:
fixes $f :: \text{real} \Rightarrow \text{real}$
assumes f *differentiable-on* UNIV $f \in \text{borel-measurable borel}$
shows $\text{deriv } f \in \text{borel-measurable borel}$
using *deriv-measurable-real*[**where** $S = \text{UNIV}$] *assms* **by** *simp*

lemma *piecewise-differentiable-on-deriv-measurable-real*:
fixes $f :: \text{real} \Rightarrow \text{real}$
assumes f *piecewise-differentiable-on* S *open* S $f \in \text{borel-measurable borel}$
shows *set-borel-measurable* $\text{borel } S$ $(\text{deriv } f)$

proof –

from *assms* **obtain** T **where** *fin*: *finite* T **and**
 diff: $\bigwedge x. x \in S - T \implies f$ *differentiable at* x *within* S
 unfolding *piecewise-differentiable-on-def* **by** *blast*
with *assms* **have** *open* $(S - T)$ **using** *finite-imp-closed* **by** *blast*
moreover *hence* f *differentiable-on* $(S - T)$
 unfolding *differentiable-on-def* **using** *assms* **by** (*metis* *Diff-iff at-within-open*
diff)
ultimately *have* *set-borel-measurable* *borel* $(S - T)$ (*deriv* f)
 by (*intro* *deriv-measurable-real*; *simp* *add*: *assms*)
thus *?thesis*
 unfolding *set-borel-measurable-def* *real-scaleR-def*
 by (*rule* *measurable-discrete-difference*[**where** $X=T$])
 (*simp-all* *add*: *countable-finite* *fin* *indicator-diff*)

qed

corollary *piecewise-differentiable-on-deriv-measurable-real-UNIV*:

fixes $f :: \text{real} \Rightarrow \text{real}$
 assumes f *piecewise-differentiable-on* $UNIV$ $f \in \text{borel-measurable borel}$
 shows $(\text{deriv } f) \in \text{borel-measurable borel}$
 using *piecewise-differentiable-on-deriv-measurable-real*[**where** $S=UNIV$] *assms*
by *simp*

lemma *einterval-empty*:

fixes $a b :: \text{ereal}$
 assumes $a \geq b$
 shows *einterval* $a b = \{\}$
 using *assms* *einterval-iff* **by** *auto*

lemma *einterval-split*:

fixes $a b :: \text{ereal}$ **and** $s :: \text{real}$
 assumes $s \in \text{einterval } a b$
 shows *einterval* $a b - \{s\} = \text{einterval } a s \cup \text{einterval } s b$

proof

show *einterval* $a b - \{s\} \subseteq \text{einterval } a (\text{ereal } s) \cup \text{einterval } (\text{ereal } s) b$
 using *assms* *einterval-iff* **by** *auto*
 show *einterval* $a (\text{ereal } s) \cup \text{einterval } (\text{ereal } s) b \subseteq \text{einterval } a b - \{s\}$
 using *assms* *einterval-iff*
 by (*smt* (*verit*, *ccfv-threshold*) *Diff-iff* *Diff-insert-absorb* *Un-iff*
 order.strict-trans *less-ereal.simps* *subsetI*)

qed

lemma *einterval-Ioc-approximation*:

fixes $a b :: \text{ereal}$
 assumes $a < b$
 obtains $u l :: \text{nat} \Rightarrow \text{real}$ **where**
 einterval $a b = (\bigcup i. \{l\ i <.. u\ i\})$

$incseq\ u\ decseq\ l \wedge i. l\ i < u\ i \wedge i. a < l\ i \wedge i. u\ i < b$
 $l \longrightarrow a\ u \longrightarrow b$

proof –

from *assms* **obtain** $u\ l :: nat \Rightarrow real$ **where**

einterval $a\ b = (\bigcup i. \{l\ i .. u\ i\})$

$incseq\ u\ decseq\ l \wedge i. l\ i < u\ i \wedge i. a < ereal\ (l\ i) \wedge i. ereal\ (u\ i) < b$
 $(\lambda i. ereal\ (l\ i)) \longrightarrow a\ (\lambda i. ereal\ (u\ i)) \longrightarrow b$

using *einterval-Icc-approximation* **by** (*smt* (*verit*) *Sup.SUP-cong*)

moreover **have** $(\bigcup i. \{l\ i .. u\ i\}) = (\bigcup i. \{l\ i <.. u\ i\})$

proof

show $(\bigcup i. \{l\ i .. u\ i\}) \subseteq (\bigcup i. \{l\ i <.. u\ i\})$

proof

fix x **assume** $x \in (\bigcup i. \{l\ i .. u\ i\})$

from *this* **obtain** i **where** $x_i: x \in \{l\ i .. u\ i\}$ **by** *blast*

obtain j **where** $j \geq i\ l\ j < l\ i$

proof (*cases* $\langle a = -\infty \rangle$)

case *True*

with $\langle (\lambda i. ereal\ (l\ i)) \longrightarrow a \rangle$ **obtain** k **where** $l\ k \leq ereal\ (l\ i - 1)$

using *Lim-MInfty* **by** (*meson nle-le*)

hence $l\ k < l\ i$ **by** *simp*

hence $l\ (max\ i\ k + 1) < l\ i$

using $\langle decseq\ l \rangle$ **by** (*smt* (*verit*, *del-insts*) *decseq-defle-add1 max.cobounded2*)

moreover **have** $max\ i\ k + 1 \geq i$ **by** *simp*

ultimately **show** *?thesis* **using** *that* **by** *simp*

next

case *False*

with *assms* **obtain** ar **where** $aar: a = ereal\ ar$ **using** *less-ereal.simps* **by**
(*metis ereal-cases*)

with $\langle (\lambda i. ereal\ (l\ i)) \longrightarrow a \rangle$ **have** $l \longrightarrow ar$ **by** *auto*

moreover **have** $0 < l\ i - ar$ **using** $\langle \lambda i. a < ereal\ (l\ i) \rangle$ *aar* **by** *simp*

ultimately **obtain** k **where** $\bigwedge h. h \geq k \implies norm\ (l\ h - ar) < l\ i - ar$

using *LIMSEQ-D* **by** *blast*

hence $\bigwedge h. h \geq k \implies l\ h < l\ i$ **by** *force*

hence $l\ (max\ i\ k + 1) < l\ i$ **by** *simp*

moreover **have** $max\ i\ k + 1 \geq i$ **by** *simp*

ultimately **show** *?thesis* **using** *that* **by** *simp*

qed

hence $x \in \{l\ j <.. u\ j\}$ **using** *xi assms* $\langle incseq\ u \rangle$ *monoD* **by** *fastforce*

thus $x \in (\bigcup i. \{l\ i <.. u\ i\})$ **by** *blast*

qed

next

have $\bigwedge i. \{l\ i <.. u\ i\} \subseteq \{l\ i .. u\ i\}$ **by** *force*

thus $(\bigcup i. \{l\ i <.. u\ i\}) \subseteq (\bigcup i. \{l\ i .. u\ i\})$ **by** *blast*

qed

ultimately **show** *?thesis* **using** *that* **by** *simp*

qed

lemma *measure-eqI-Ioc*:

```

fixes  $M N$  :: real measure
assumes sets:  $sets\ M = sets\ borel\ sets\ N = sets\ borel$ 
assumes fin:  $\bigwedge a\ b. a \leq b \implies emeasure\ M\ \{a<..b\} < \infty$ 
assumes eq:  $\bigwedge a\ b. a \leq b \implies emeasure\ M\ \{a<..b\} = emeasure\ N\ \{a<..b\}$ 
shows  $M = N$ 
proof (rule measure-eqI-generator-eq-countable)
  let  $?Ioc = \lambda(a::real,b::real). \{a<..b\}$  let  $?E = range\ ?Ioc$ 
  show Int-stable  $?E$  using Int-stable-def Int-greaterThanAtMost by force
  show  $?E \subseteq Pow\ UNIV$  sets  $M = sigma\text{-sets}\ UNIV\ ?E$  sets  $N = sigma\text{-sets}\ UNIV\ ?E$ 
  unfolding sets by (auto simp add: borel-sigma-sets-Ioc)
  show  $\bigwedge I. I \in ?E \implies emeasure\ M\ I = emeasure\ N\ I$ 
  proof -
    fix  $I$  assume  $I \in ?E$ 
    then obtain  $a\ b$  where  $I = \{a<..b\}$  by auto
    thus  $emeasure\ M\ I = emeasure\ N\ I$  by (smt (verit, best) eq greaterThanAtMost-empty)
  qed
  show  $?Ioc\ ' (Rats \times Rats) \subseteq ?E$  ( $\bigcup_{i \in (Rats \times Rats)}. ?Ioc\ i = UNIV$ )
    using Rats-no-bot-less Rats-no-top-le by auto
  show countable ( $?Ioc\ ' (Rats \times Rats)$ ) using countable-rat by blast
  show  $\bigwedge I. I \in ?Ioc\ ' (Rats \times Rats) \implies emeasure\ M\ I \neq \infty$ 
  proof -
    fix  $I$  assume  $I \in ?Ioc\ ' (Rats \times Rats)$ 
    then obtain  $a\ b$  where  $(a,b) \in (Rats \times Rats)$   $I = \{a<..b\}$  by blast
    thus  $emeasure\ M\ I \neq \infty$  by (smt (verit, best) Ioc-inj fin order.strict-implies-not-eq)
  qed
qed

```

lemma *measure-einterval-eqI-Ioc*:

```

fixes  $M N$  :: real measure and  $a\ b$  :: ereal
assumes Mborel:  $sets\ M = sets\ borel$  and Nborel:  $sets\ N = sets\ borel$  and
   $\bigwedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies emeasure\ M\ \{s<..t\} \neq \infty$  and
   $\bigwedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies emeasure\ M\ \{s<..t\} = emeasure\ N\ \{s<..t\}$ 
shows restrict-space  $M\ (einterval\ a\ b) = restrict\text{-space}\ N\ (einterval\ a\ b)$ 
proof (cases  $\langle a < b \rangle$ )

```

case ab : *True*

let $?Iab = einterval\ a\ b$

let $?Iocs = \{\{s<..t\} \mid s\ t :: real. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b\}$

have $Iocs\text{-}Iab$: $?Iocs \subseteq Pow\ ?Iab$

by *safe (metis einterval-iff greaterThanAtMost-iff ereal-less-le le-ereal-less less-eq-ereal-def)*

obtain $u\ l$:: *nat* \Rightarrow *real* **where**

$Iab\text{-}Un$: $?Iab = (\bigcup i. \{l\ i <.. u\ i\})$ **and**

lu : $\bigwedge i. l\ i < u\ i$ **and** al : $\bigwedge i. a < l\ i$ **and** ub : $\bigwedge i. u\ i < b$

```

using einterval-Ioc-approximation ab by (smt (verit) Inf.INF-cong)

let ?Iocn = λn::nat. {l n <.. u n}

have Iabsigma: ∧L :: real measure. sets L = sets borel ⇒
  sets (restrict-space L ?Iab) = sigma-sets ?Iab ?Iocs
proof –
  fix L :: real measure assume asm: sets L = sets borel
  have sets (restrict-space L ?Iab) = (inter ?Iab) ‘ sigma-sets UNIV (range
(λ(s,t) ⇒ {s<..t}))
  by (rewrite sets-restrict-space, rewrite asm, rewrite borel-sigma-sets-Ioc) simp
  also have ... = sigma-sets ?Iab ((inter ?Iab) ‘ {{s<..t} | s t. True})
  apply (rewrite restricted-sigma)
  apply (metis Pow-UNIV borel-einterval borel-sigma-sets-Ioc sets-measure-of
top-greatest)
  unfolding image-def by simp-all
  also have ... = sigma-sets ?Iab ?Iocs
proof
  have (inter ?Iab) ‘ {{s<..t} | s t. True} ⊆ sigma-sets ?Iab ?Iocs
proof
  fix A assume A ∈ (inter ?Iab) ‘ {{s<..t} | s t. True}
  from this obtain s t where Aabst: A = ?Iab ∩ {s<..t} by blast
  also have ... = (∪ n. {l n <.. u n} ∩ {s<..t}) using Aabst Iab-Un UN-simps
by force
  also have ... = (∪ n. {max (l n) s <.. min (u n) t}) by simp
  moreover have
     $\bigwedge n. \{ \max (l n) s <.. \min (u n) t \} \in \text{sigma-sets } ?Iab ?Iocs$ 
proof –
  fix n
  show  $\{ \max (l n) s <.. \min (u n) t \} \in \text{sigma-sets } ?Iab ?Iocs$ 
proof (cases ‹max (l n) s ≤ min (u n) t›)
    case True
    moreover have a < ereal (max (l n) s) using al by (metis less-ereal-le
max-def)
    moreover have ereal (min (u n) t) < b using ub by (smt (verit)
ereal-less-le)
    ultimately have  $\{ \max (l n) s <.. \min (u n) t \} \in ?Iocs$  by blast
    thus ?thesis ..
  next
  case False
  hence  $\{ \max (l n) s <.. \min (u n) t \} = \{ \}$  by auto
  thus ?thesis by (metis sigma-sets.Empty)
  qed
qed
ultimately show A ∈ sigma-sets ?Iab ?Iocs by (simp add: sigma-sets.Union)
qed
thus sigma-sets ?Iab ((inter ?Iab) ‘ {{s<..t} | s t. True}) ⊆ sigma-sets ?Iab
?Iocs
by (rule sigma-sets-mono)

```

next
have $?Iocs \subseteq \text{sigma-sets } ?Iab ((\text{inter } ?Iab) \text{ ' } \{\{s<..t\} \mid s t. \text{True}\})$
proof
fix A **assume** $A \in ?Iocs$
from this obtain $s t$ **where** $a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b$ **and** $A = \{s<..t\}$ **by** *blast*
hence $A = ?Iab \cap \{s<..t\}$ **using** *Iocs-Iab* **by** *blast*
hence $A \in (\text{inter } ?Iab) \text{ ' } \{\{s<..t\} \mid s t. \text{True}\}$ **by** *auto*
thus $A \in \text{sigma-sets } ?Iab ((\text{inter } ?Iab) \text{ ' } \{\{s<..t\} \mid s t. \text{True}\})$ **by** *blast*
qed
thus $\text{sigma-sets } ?Iab ?Iocs \subseteq \text{sigma-sets } ?Iab ((\text{inter } ?Iab) \text{ ' } \{\{s<..t\} \mid s t. \text{True}\})$
True)
by (*rule sigma-sets-mono*)
qed
finally show $\text{sets } (\text{restrict-space } L ?Iab) = \text{sigma-sets } ?Iab ?Iocs$.
qed

have *Int-stable ?Iocs*
proof (*rule Int-stableI*)
fix $A B$ **assume** $A \in ?Iocs B \in ?Iocs$
from this obtain $sA tA sB tB$ **where**
 $asA: a < \text{ereal } sA$ **and** $sAtA: sA \leq tA$ **and** $tAb: \text{ereal } tA < b$ **and** $A = \{sA <.. tA\}$ **and**
 $asB: a < \text{ereal } sB$ **and** $sBtB: sB \leq tB$ **and** $eBb: \text{ereal } tB < b$ **and** $B = \{sB <.. tB\}$
by *blast*
hence $AB: A \cap B = \{\max sA sB <.. \min tA tB\}$ **using** *Int-greaterThanAtMost*
by *blast*
thus $A \cap B \in ?Iocs$
proof (*cases <max sA sB ≤ min tA tB>*)
case *True*
moreover have $a < \text{ereal } (\max sA sB) \text{ereal } (\min tA tB) < b$
using $asA tAb asB eBb$ **by** *linarith+*
ultimately show *?thesis* **using** AB **by** *blast*
next
case *False*
hence $A \cap B = \{\}$ **using** AB **by** *force*
thus $A \cap B \in ?Iocs$ **using** asA *less-ereal-le sAtA tAb* **by** *fastforce*
qed
qed
moreover note *Iocs-Iab*
moreover have $\bigwedge A. A \in ?Iocs \implies$
 $\text{emeasure } (\text{restrict-space } M ?Iab) A = \text{emeasure } (\text{restrict-space } N ?Iab) A$
proof –
fix A **assume** $asm: A \in ?Iocs$
hence $[simp]: A \subseteq ?Iab$ **using** *Iocs-Iab* **by** *blast*
have $\text{emeasure } (\text{restrict-space } M ?Iab) A = \text{emeasure } M A$
by (*rewrite emeasure-restrict-space; simp add: assms*)
also have $\dots = \text{emeasure } N A$ **using** $assms$ asm **by** *force*

also have $\dots = \text{emeasure } (\text{restrict-space } N \text{ ?Iab}) A$
by (*rewrite emeasure-restrict-space; simp add: assms*)
finally show $\text{emeasure } (\text{restrict-space } M \text{ ?Iab}) A = \text{emeasure } (\text{restrict-space } N \text{ ?Iab}) A$.
qed
moreover have $\text{sets } (\text{restrict-space } M \text{ ?Iab}) = \text{sigma-sets } ?Iab \text{ ?Iocs}$ **using** *Iab-sigma assms* **by** *simp*
moreover have $\text{sets } (\text{restrict-space } N \text{ ?Iab}) = \text{sigma-sets } ?Iab \text{ ?Iocs}$ **using** *Iab-sigma assms* **by** *simp*
moreover have $\text{rngIocn: range } ?Iocn \subseteq ?Iocs$
using *al lu ub* **by** (*smt (verit) image-subset-iff mem-Collect-eq*)
moreover have $(\bigcup n. ?Iocn \ n) = ?Iab$ **using** *Iab-Un* **by** *simp*
moreover have $\bigwedge n. \text{emeasure } (\text{restrict-space } M \text{ ?Iab}) (?Iocn \ n) \neq \infty$
proof –
fix $n::\text{nat}$
have $\{l \ n <.. \ u \ n\} \subseteq ?Iab$ **using** *Iab-Un* **by** *blast*
moreover have $\text{emeasure } M \ \{l \ n <.. \ u \ n\} \neq \infty$ **using** *assms al lu ub less-imp-le*
by *metis*
ultimately show $\text{emeasure } (\text{restrict-space } M \text{ ?Iab}) (?Iocn \ n) \neq \infty$
by (*rewrite emeasure-restrict-space; simp add: assms*)
qed
ultimately show $\text{restrict-space } M \text{ ?Iab} = \text{restrict-space } N \text{ ?Iab}$
by (*rule measure-eqI-generator-eq[where A=?Iocn]; simp*)

next

case *False*
hence $\text{einterval } a \ b = \{\}$ **using** *einterval-empty* **by** *simp*
thus *?thesis* **by** (*metis sets.empty-sets space-empty-eq-bot space-restrict-space2*)

qed

lemma *nn-integral-disjoint-pair2*:

assumes $B \in \text{sets } M \ C \in \text{sets } M \ B \cap C = \{\}$ **and**
 $[\text{measurable}]: (\lambda x. f \ x * \text{indicator } B \ x) \in \text{borel-measurable } M$ **and**
 $[\text{measurable}]: (\lambda x. f \ x * \text{indicator } C \ x) \in \text{borel-measurable } M$
shows $(\int^{+x \in B \cup C}. f \ x \ \partial M) = (\int^{+x \in B}. f \ x \ \partial M) + (\int^{+x \in C}. f \ x \ \partial M)$
proof –
have $[\text{measurable}]: (\lambda x. f \ x * \text{indicator } (B \cup C) \ x) \in \text{borel-measurable } M$
using *assms* **apply** (*rewrite indicator-add[THEN sym], simp*)
apply (*rewrite distrib-left*) **by** *simp*
have $(\int^{+x \in B \cup C}. f \ x \ \partial M) = (\int^{+x \in B \cup C}. f \ x * \text{indicator } (B \cup C) \ x \ \partial M)$
by (*metis indicator-simps mult-1-right mult-eq-0-iff*)
also have $\dots = (\int^{+x \in B}. f \ x * \text{indicator } (B \cup C) \ x \ \partial M) + (\int^{+x \in C}. f \ x * \text{indicator } (B \cup C) \ x \ \partial M)$
using *assms* **by** (*rewrite nn-integral-disjoint-pair; simp*)
moreover have $(\int^{+x \in B}. f \ x * \text{indicator } (B \cup C) \ x \ \partial M) = (\int^{+x \in B}. f \ x \ \partial M)$
by (*metis (mono-tags, lifting) indicator-inter-arith inf-sup-absorb mult.assoc mult commute*)

moreover have $(\int^{+x \in C}. f x * \text{indicator } (B \cup C) x \partial M) = (\int^{+x \in C}. f x \partial M)$
by (*metis (no-types, lifting) indicator-inter-arith inf-sup-absorb mult.commute mult.left-commute sup-commute*)
ultimately show *?thesis* **by** *simp*
qed

lemma *set-nn-integral-interval-measure-bounded-finite:*

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $A :: \text{real set}$ **and** $M :: \text{real}$
assumes *bounded A* $\bigwedge x. x \in A \implies h x \leq M$ $A \in \text{sets borel}$ **and**
mono F $\bigwedge x. \text{continuous (at-right } x) F$
shows $(\int^{+x \in A}. h x \partial(\text{interval-measure } F)) < \infty$

proof –

let $?IMF = \text{interval-measure } F$
obtain a **where** $a\text{-ub}: \bigwedge x. x \in A \implies |x| \leq a$
using *bounded-iff assms* **by** *force*
define c **where** $c \equiv |a| + 1$
have [*simp*]: $c > 0$
unfolding *c-def* **by** *auto*
have $Ac: A \subseteq \{-c <..c\}$
using *a-ub c-def* **by** *force*
have $(\int^{+x \in A}. h x \partial ?IMF) \leq (\int^{+x \in A}. M \partial ?IMF)$
using *nn-set-integral-mono assms* **by** (*simp add: indicator-def nn-integral-mono*)
also have $\dots \leq (\int^{+x \in \{-c <..c\}}. M \partial ?IMF)$
using *nn-set-integral-set-mono Ac* **by** *force*
also have $\dots = M * \text{emeasure } ?IMF \{-c <..c\}$
by (*rewrite nn-integral-cmult-indicator; simp*)
also have $\dots \leq |M| * \text{emeasure } ?IMF \{-c <..c\}$
using *abs-ge-self* **by** (*simp add: mult-right-mono*)
also have $\text{emeasure } ?IMF \{-c <..c\} = F c - F (-c)$
by (*simp add: emeasure-interval-measure-Ioc-eq ennreal-eq-0-iff monoD assms*)
finally have $(\int^{+x \in A}. h x \partial ?IMF) \leq |M| * (F c - F (-c))$
by (*simp add: ennreal-mult*)
then show *?thesis*
by (*simp add: le-less-trans*)

qed

end

theory *Lebesgue-Stieltjes-Integral*

imports *Wlog.Wlog Preliminaries-LSI*

begin

1 Interval Measure Integral

1.1 Basic Calculations

lemma *interval-measure-const-null:*

fixes $c :: \text{real}$
shows $\text{interval-measure } (\lambda-. c) = \text{null-measure lborel}$
proof –

have $\bigwedge s t. s \leq t \implies \text{emeasure } (\text{interval-measure } (\lambda-. c)) \{s <..t\} = 0$
using *emeasure-interval-measure-Ioc* **by** *force*
then show *?thesis* **by** (*intro measure-eqI-Ioc; simp*)
qed

lemma *interval-measure-singleton*:
fixes $F :: \text{real} \implies \text{real}$ **and** $s :: \text{real}$
assumes *mono F* $\bigwedge x. \text{continuous } (\text{at-right } x) F$
shows $(\text{interval-measure } F) \{s\} = F s - \text{Lim } (\text{at-left } s) F$
proof –
let $?dF = \text{interval-measure } F$
let $?I = \lambda n :: \text{nat}. \{s - 1/(n+1) <..s\}$
have *limsn*: $(\lambda n. s - 1 / (\text{real } n + 1)) \longrightarrow s$ **by** *real-asymp*
have $\{s\} = (\bigcap (\text{range } ?I))$
proof
have $\bigwedge n. s \in ?I n$ **by** *simp*
thus $\{s\} \subseteq (\bigcap (\text{range } ?I))$ **by** *simp*
show $(\bigcap (\text{range } ?I)) \subseteq \{s\}$
proof
fix x **assume** $x \in (\bigcap (\text{range } ?I))$
hence $x-I: \bigwedge n. x \in ?I n$ **by** *simp*
hence $\bigwedge n. s - 1 / (\text{real } n + 1) < x$ **by** *simp*
hence $s \leq x$
apply (*intro LIMSEQ-le-const2[OF limsn]*)
using *less-eq-real-def* **by** (*intro exI[of - 0]*) *simp*
moreover **have** $x \leq s$ **using** $x-I$ **by** *simp*
ultimately show $x \in \{s\}$ **by** *simp*
qed
qed
moreover **have** $(\lambda n. \text{emeasure } ?dF (?I n)) \longrightarrow \text{emeasure } ?dF (\bigcap (\text{range } ?I))$
proof (*rule Lim-emeasure-decseq*)
show $\text{range } ?I \subseteq \text{sets } ?dF$ **by** (*rewrite sets-interval-measure*) *force*
show *monotone* $(\leq) (\lambda J K. K \subseteq J) ?I$
proof (*rule monotoneI*)
fix $m n :: \text{nat}$ **assume** $m \leq n$
hence $s - 1 / (\text{real } m + 1) \leq s - 1 / (\text{real } n + 1)$
by (*metis add.commute of-nat-Suc Suc-le-mono diff-left-mono inverse-of-nat-le Suc-not-Zero*)
thus $?I n \subseteq ?I m$ **by** *force*
qed
show $\bigwedge n. \text{emeasure } ?dF (?I n) \neq \infty$
using *emeasure-interval-measure-Ioc* **by** (*simp add: assms monoD*)
qed
moreover **have** $(\lambda n. ?dF (?I n)) \longrightarrow F s - \text{Lim } (\text{at-left } s) F$
proof –
have $(F \longrightarrow \text{Sup } (F \text{ ‘ } \{..<s\})) (\text{at-left } s)$
using *Lim-left-bound*[*of UNIV s F F s*] *monoD*[*OF ‹mono F›*] **by** *simp*
hence $(F \longrightarrow \text{Lim } (\text{at-left } s) F) (\text{at-left } s)$ **by** (*simp add: tendsto-Lim*)

hence $(\lambda n::nat. F (s - 1 / (real\ n + 1))) \longrightarrow Lim\ (at\text{-}left\ s)\ F$
apply *(rewrite in asm tendsto-at-iff-sequentially)*
apply *(drule spec[of - $\lambda n. s - 1 / (real\ n + 1)$])*
unfolding *comp-def using limsn by simp*
hence $(\lambda n::nat. ennreal\ (F\ s - F\ (s - 1 / (real\ n + 1))))$
 $\longrightarrow\ ennreal\ (F\ s - Lim\ (at\text{-}left\ s)\ F)$
by *(intro tendsto-intros)*
thus *?thesis by (rewrite emeasure-interval-measure-Ioc; simp add: assms monoD)*
qed
ultimately show *?thesis using LIMSEQ-unique by simp*
qed

lemma *interval-measure-singleton-continuous:*
fixes $F :: real \Rightarrow real$ **and** $s::real$
assumes $mono\ F \wedge x. continuous\ (at\text{-}right\ x)\ F$ **isCont** $F\ s$
shows $(interval\text{-}measure\ F)\ \{s\} = 0$
proof –
have $Lim\ (at\text{-}left\ s)\ F = F\ s$
using *assms continuous-at-imp-continuous-within continuous-within*
tendsto-Lim trivial-limit-at-left-real by blast
with *interval-measure-singleton assms show ?thesis by simp*
qed

1.2 Changing the Underlying Function

lemma *einterval-nn-integral-interval-measure-cong:*
fixes $F\ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $a\ b :: ereal$
assumes $mono\ F \wedge x. continuous\ (at\text{-}right\ x)\ F$ **and**
 $mono\ G \wedge x. continuous\ (at\text{-}right\ x)\ G$ **and**
 $(F - G)\ constant\text{-}on\ (einterval\ a\ b)$ **and**
 $h \in borel\text{-}measurable\ borel$
shows $(\int^{+x \in (einterval\ a\ b). h\ x}\ \partial(interval\text{-}measure\ F)) =$
 $(\int^{+x \in (einterval\ a\ b). h\ x}\ \partial(interval\text{-}measure\ G))$
proof –
let $?IM = interval\text{-}measure$
let $?Iab = einterval\ a\ b$
from *assms obtain c where FG-c: $\wedge x. x \in ?Iab \implies F\ x - G\ x = c$*
unfolding *constant-on-def by force*
have $\wedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies emeasure\ (?IM\ F)\ \{s<..t\} \neq$
 ∞
using *assms unfolding mono-def by (rewrite emeasure-interval-measure-Ioc; simp)*
moreover have $\wedge s\ t. a < ereal\ s \wedge s \leq t \wedge ereal\ t < b \implies$
 $emeasure\ (?IM\ F)\ \{s<..t\} = emeasure\ (?IM\ G)\ \{s<..t\}$
proof –
fix $s\ t$ **assume** *astb: $a < ereal\ s \wedge s \leq t \wedge ereal\ t < b$*
hence $F\ t - F\ s = G\ t - G\ s$
using *FG-c by (smt (verit) einterval-iff ereal-less-le less-ereal-le)*
thus $emeasure\ (?IM\ F)\ \{s<..t\} = emeasure\ (?IM\ G)\ \{s<..t\}$

using *assms astb unfolding mono-def* **by** (*rewrite emeasure-interval-measure-Ioc;*
simp)⁺
qed
ultimately have *restrict-space (?IM F) ?Iab = restrict-space (?IM G) ?Iab*
by (*intro measure-einterval-eqI-Ioc; simp*)
thus *?thesis using nn-integral-restrict-space[THEN sym]*
by (*rewrite nn-integral-restrict-space[THEN sym], simp*)⁺ *simp*
qed

corollary *Ioo-nn-integral-interval-measure-cong:*

fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r s :: \text{real}$
assumes *mono F \wedge x. continuous (at-right x) F* **and**
mono G \wedge x. continuous (at-right x) G **and**
*(F - G) constant-on {r<.. **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r < .. < s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r < .. < s\}. h x$
 $\partial(\text{interval-measure } G))$
using *einterval-eq-Icc assms einterval-nn-integral-interval-measure-cong*
by (*metis (mono-tags, lifting) nn-integral-cong*)*

corollary *Ioi-nn-integral-interval-measure-cong:*

fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r :: \text{real}$
assumes *mono F \wedge x. continuous (at-right x) F* **and**
mono G \wedge x. continuous (at-right x) G **and**
(F - G) constant-on {r<..} **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r < ..\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r < ..\}. h x \partial(\text{interval-measure}$
 $G))$
using *einterval-eq-Ici assms einterval-nn-integral-interval-measure-cong*
by (*metis (mono-tags, lifting) nn-integral-cong*)

corollary *Iio-nn-integral-interval-measure-cong:*

fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $s :: \text{real}$
assumes *mono F \wedge x. continuous (at-right x) F* **and**
mono G \wedge x. continuous (at-right x) G **and**
*(F - G) constant-on {.. **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{..
 $G))$
using *einterval-eq-Iic assms einterval-nn-integral-interval-measure-cong*
by (*metis (mono-tags, lifting) nn-integral-cong*)$*

corollary *nn-integral-interval-measure-cong:*

fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$
assumes *mono F \wedge x. continuous (at-right x) F* **and**
mono G \wedge x. continuous (at-right x) G **and**
(F - G) constant-on UNIV **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x. h x \partial(\text{interval-measure } F)) = (\int^+ x. h x \partial(\text{interval-measure } G))$

using *einterval-nn-integral-interval-measure-cong einterval-eq-UNIV assms*
by (*metis (mono-tags, lifting) mult-1-right*
indicator-eq-1-iff nn-integral-cong space-interval-measure)

lemma *singleton-nn-integral-interval-measure-cong*:

fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ **and**

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$ **and**

$F s - \text{Lim (at-left } s) F = G s - \text{Lim (at-left } s) G$ **and**

$h \in \text{borel-measurable borel}$

shows $(\int^+ x \in \{s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{s\}. h x \partial(\text{interval-measure } G))$

using *interval-measure-singleton assms* **by** (*rewrite nn-integral-indicator-singleton; simp*)

lemma *singleton-const-nn-integral-interval-measure-cong*:

fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ **and**

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$ **and**

$(F - G) \text{ constant-on } \{r <..s\}$ **and** $r < s$ **and**

$h \in \text{borel-measurable borel}$

shows $(\int^+ x \in \{s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{s\}. h x \partial(\text{interval-measure } G))$

proof –

have [*simp*]: $\{..<s\} \cap \{r<..\} = \{r<.. **using** *assms* **by** *force*$

have [*simp*]: *at s within* $\{r<..$

using *at-within-Ioo-at-left assms trivial-limit-at-left-real* **by** *metis*

from *assms* **obtain** $c :: \text{real}$ **where** $FGc: \wedge x. x \in \{r<..$

unfolding *constant-on-def* **by** *force*

have $\text{Lim (at } s \text{ within } \{r<..$

proof –

have $(F \longrightarrow \text{Sup } (F \text{ ' } (\{..<s\} \cap \{r<..\})))$ (*at s within* $\{..<s\} \cap \{r<..\}$)

apply (*rule Lim-left-bound*[**where** $I=\{r<..\}$ **and** $K=F s$])

using *assms monoD* **by** *force+*

thus *?thesis* **by** (*intro tendsto-Lim; simp*)

qed

moreover **have** $\text{Lim (at } s \text{ within } \{r<..$

proof –

have $(G \longrightarrow \text{Sup } (G \text{ ' } (\{..<s\} \cap \{r<..\})))$ (*at s within* $\{..<s\} \cap \{r<..\}$)

apply (*rule Lim-left-bound*[**where** $I=\{r<..\}$ **and** $K=G s$])

using *assms monoD* **by** *force+*

thus *?thesis* **by** (*intro tendsto-Lim; simp*)

qed

moreover **have** $\text{Sup } (F \text{ ' } \{r<..$

proof –

have *bdd-above* $(G \text{ ' } \{r<.. **using** *assms* **by** (*metis bdd-above-Ioo bdd-above-image-mono*)$

moreover **have** $\wedge x. x \in \{r<.. **using** *FGc* **by** *force*$

ultimately **show** *?thesis* **using** *FGc assms* **by** (*rewrite Sup-add-eq*[*THEN sym*]; *simp*)

qed
ultimately have $\text{Lim (at-left } s) F - \text{Lim (at-left } s) G = c$
using *at-within-Ioo-at-left assms* **by force**
hence $F s - \text{Lim (at-left } s) F = G s - \text{Lim (at-left } s) G$ **using** *FGc assms* **by**
fastforce
thus *?thesis* **by** (*intro singleton-nn-integral-interval-measure-cong; simp add:*
assms)
qed

lemma *Ioc-nn-integral-interval-measure-cong*:
fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r s :: \text{real}$
assumes *mono F* $\wedge x. \text{continuous (at-right } x) F$ **and**
mono G $\wedge x. \text{continuous (at-right } x) G$ **and**
 $(F - G) \text{ constant-on } \{r <..s\}$ **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^{+x \in \{r <..s\}}. h x \partial(\text{interval-measure } F)) = (\int^{+x \in \{r <..s\}}. h x \partial(\text{interval-measure } G))$
proof –
let $?IM = \text{interval-measure}$
show *?thesis*
proof (*cases* $\langle r < s \rangle$)
case *True*
hence [*simp*]: *insert s* $\{r <..<s\} = \{r <..s\}$ **by force**
have $(\int^{+x \in \{r <..<s\}}. h x \partial(?IM F)) = (\int^{+x \in \{r <..<s\}}. h x \partial(?IM G))$
proof –
have $\{r <..<s\} = \text{einterval (ereal } r) \text{ (ereal } s)$ **by simp**
moreover have $(F - G) \text{ constant-on (einterval (ereal } r) \text{ (ereal } s))$
proof –
have $(\text{einterval (ereal } r) \text{ (ereal } s)) \subseteq \{r <..s\}$ **by force**
with *assms constant-on-subset* **show** *?thesis* **by simp**
qed
ultimately show *?thesis* **using** *assms einterval-nn-integral-interval-measure-cong*
by *presburger*
qed
moreover have $(\int^{+x \in \{s\}}. h x \partial(?IM F)) = (\int^{+x \in \{s\}}. h x \partial(?IM G))$
by (*rule singleton-const-nn-integral-interval-measure-cong*[**where** $r=r$]; *simp*
add: assms True)
ultimately have $(\int^{+x \in \{r <..<s\}}. h x \partial(?IM F)) + (\int^{+x \in \{s\}}. h x \partial(?IM F)) =$
 $(\int^{+x \in \{r <..<s\}}. h x \partial(?IM G)) + (\int^{+x \in \{s\}}. h x \partial(?IM G))$
by *simp*
moreover have
 $\wedge H. (\int^{+x \in \{r <..<s\}}. h x \partial(?IM H)) + (\int^{+x \in \{s\}}. h x \partial(?IM H)) =$
 $(\int^{+x \in \{r <..s\}}. h x \partial(?IM H))$
by (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable; simp add: assms*)
ultimately show *?thesis* **by simp**
next
case *False*
hence $\{r <..s\} = \{\}$ **using** *greaterThanAtMost-empty-iff* **by simp**

thus ?thesis by simp
qed
qed

lemma *Iic-nn-integral-interval-measure-cong*:

fixes $F G :: \text{real} \Rightarrow \text{real}$ and $h :: \text{real} \Rightarrow \text{ennreal}$ and $s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ and

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$ and

$(F - G) \text{ constant-on } \{..s\}$ and

$h \in \text{borel-measurable borel}$

shows $(\int^{+x \in \{..s\}}. h x \partial(\text{interval-measure } F)) = (\int^{+x \in \{..s\}}. h x \partial(\text{interval-measure } G))$

proof –

let $?IM = \text{interval-measure}$

have $[simp]: \text{insert } s \{..<s\} = \{..s\}$ by force

have $(\int^{+x \in \{..<s\}}. h x \partial(?IM F)) = (\int^{+x \in \{..<s\}}. h x \partial(?IM G))$

proof –

have $\{..<s\} = \text{einterval } (-\infty) (\text{ereal } s)$ by simp

moreover have $(F - G) \text{ constant-on } (\text{einterval } (-\infty) (\text{ereal } s))$

proof –

have $(\text{einterval } (-\infty) (\text{ereal } s)) \subseteq \{..s\}$ by force

with *assms constant-on-subset* show ?thesis by simp

qed

ultimately show ?thesis using *assms einterval-nn-integral-interval-measure-cong*

by *presburger*

qed

moreover have $(\int^{+x \in \{s\}}. h x \partial(?IM F)) = (\int^{+x \in \{s\}}. h x \partial(?IM G))$

proof –

have $(F - G) \text{ constant-on } \{s-1 <..s\}$

using *assms constant-on-subset* by (*metis atMost-iff greaterThanAtMost-iff subset-eq*)

thus ?thesis

by (*intro singleton-const-nn-integral-interval-measure-cong*[**where** $r=s-1$];

simp add: assms)

qed

ultimately have $(\int^{+x \in \{..<s\}}. h x \partial(?IM F)) + (\int^{+x \in \{s\}}. h x \partial(?IM F)) =$
 $(\int^{+x \in \{..<s\}}. h x \partial(?IM G)) + (\int^{+x \in \{s\}}. h x \partial(?IM G))$

by *simp*

moreover have

$\wedge H. (\int^{+x \in \{..<s\}}. h x \partial(?IM H)) + (\int^{+x \in \{s\}}. h x \partial(?IM H)) = (\int^{+x \in \{..s\}}. h x \partial(?IM H))$

by (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable; simp add: assms*)

ultimately show ?thesis by *simp*

qed

lemma *Ico-nn-integral-interval-measure-cong*:

fixes $F G :: \text{real} \Rightarrow \text{real}$ and $h :: \text{real} \Rightarrow \text{ennreal}$ and $r s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ and

$\text{mono } G \wedge x. \text{continuous (at-right } x) G$ and

($F - G$ constant-on $\{r <..<s\}$ and
 $F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$ and
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r..<s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..<s\}. h x \partial(\text{interval-measure } G))$
proof –
let $?IM = \text{interval-measure}$
show $?thesis$
proof (*cases* $\langle r < s \rangle$)
case *True*
hence [*simp*]: *insert* r $\{r <..<s\} = \{r..<s\}$ **by** *force*
have $\bigwedge H. (\int^+ x \in \{r..<s\}. h x \partial(?IM H)) = (\int^+ x \in \{r\}. h x \partial(?IM H)) + (\int^+ x \in \{r <..<s\}. h x \partial(?IM H))$
by (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable*; *simp add: assms*)
moreover have $(\int^+ x \in \{r\}. h x \partial(?IM F)) = (\int^+ x \in \{r\}. h x \partial(?IM G))$
by (*rewrite singleton-nn-integral-interval-measure-cong*; *measurable*; *simp add: assms*)
moreover have $(\int^+ x \in \{r <..<s\}. h x \partial(?IM F)) = (\int^+ x \in \{r <..<s\}. h x \partial(?IM G))$
apply (*rewrite einterval-eq-Icc*[*THEN sym*])+
by (*rewrite einterval-nn-integral-interval-measure-cong*; *measurable*; *simp add: assms*)
ultimately show $?thesis$ **by** *simp*
next
case *False*
thus $?thesis$ **by** *simp*
qed
qed

corollary *Ico-Cont-nn-integral-interval-measure-cong*:
fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r s :: \text{real}$
assumes *mono* $F \bigwedge x. \text{continuous (at-right } x) F$ **and**
mono $G \bigwedge x. \text{continuous (at-right } x) G$ **and**
($F - G$ constant-on $\{r <..<s\}$ and
***isCont* $F r$ *isCont* $G r$ and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r..<s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..<s\}. h x \partial(\text{interval-measure } G))$
proof –
from *assms* **have** $F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$
by (*metis add-0 continuous-at-imp-continuous-at-within continuous-within eq-diff-eq tendsto-Lim trivial-limit-at-left-real*)
thus $?thesis$
using *assms* **by** (*intro Ico-nn-integral-interval-measure-cong*; *simp*)
qed

lemma *Ici-nn-integral-interval-measure-cong*:
fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r :: \text{real}$
assumes *mono* $F \bigwedge x. \text{continuous (at-right } x) F$ **and**

mono $G \wedge x$. *continuous (at-right x)* G **and**
 $(F - G)$ *constant-on* $\{r<..\}$ **and**
 $F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$ **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } G))$
proof –
let $?IM = \text{interval-measure}$
have $[\text{simp}]$: *insert* $r \{r<..\} = \{r..\}$ **by force**
have $\wedge H$. $(\int^+ x \in \{r..\}. h x \partial(?IM H)) = (\int^+ x \in \{r\}. h x \partial(?IM H)) + (\int^+ x \in \{r<..\}. h x \partial(?IM H))$
by (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable*; *simp add*: *assms*)
moreover have $(\int^+ x \in \{r\}. h x \partial(?IM F)) = (\int^+ x \in \{r\}. h x \partial(?IM G))$
by (*rewrite singleton-nn-integral-interval-measure-cong*; *measurable*; *simp add*: *assms*)
moreover have $(\int^+ x \in \{r<..\}. h x \partial(?IM F)) = (\int^+ x \in \{r<..\}. h x \partial(?IM G))$
apply (*rewrite einterval-eq-Ici*[*THEN sym*])+
by (*rewrite einterval-nn-integral-interval-measure-cong*; *measurable*; *simp add*: *assms*)
ultimately show *?thesis by simp*
qed

corollary *Ici-Cont-nn-integral-interval-measure-cong*:
fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r :: \text{real}$
assumes **mono** $F \wedge x$. *continuous (at-right x)* F **and**
mono $G \wedge x$. *continuous (at-right x)* G **and**
 $(F - G)$ *constant-on* $\{r<..\}$ **and**
 $\text{isCont } F r \text{ isCont } G r$ **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..\}. h x \partial(\text{interval-measure } G))$
proof –
from *assms* **have** $F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$
by (*metis continuous-at-imp-continuous-within continuous-within tendsto-Lim trivial-limit-at-left-real diff-self*)
thus *?thesis*
using *assms* **by** (*intro Ici-nn-integral-interval-measure-cong*; *simp*)
qed

lemma *Icc-nn-integral-interval-measure-cong*:
fixes $F G :: \text{real} \Rightarrow \text{real}$ **and** $h :: \text{real} \Rightarrow \text{ennreal}$ **and** $r s :: \text{real}$
assumes **mono** $F \wedge x$. *continuous (at-right x)* F **and**
mono $G \wedge x$. *continuous (at-right x)* G **and**
 $(F - G)$ *constant-on* $\{r<..s\}$ **and**
 $F r - \text{Lim (at-left } r) F = G r - \text{Lim (at-left } r) G$ **and**
 $h \in \text{borel-measurable borel}$
shows $(\int^+ x \in \{r..s\}. h x \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..s\}. h x \partial(\text{interval-measure } G))$
proof –

let $?IM = interval\text{-}measure$
show $?thesis$
proof (*cases* $\langle r \leq s \rangle$)
 case *True*
 hence [*simp*]: *insert* r $\{r <..s\} = \{r..s\}$ **by** *force*
 have $\bigwedge H. (\int^{+x \in \{r..s\}}. h \ x \ \partial(?IM \ H)) = (\int^{+x \in \{r\}}. h \ x \ \partial(?IM \ H)) +$
 $(\int^{+x \in \{r <..s\}}. h \ x \ \partial(?IM \ H))$
 by (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *measurable*; *simp add: assms*)
 moreover have $(\int^{+x \in \{r\}}. h \ x \ \partial(?IM \ F)) = (\int^{+x \in \{r\}}. h \ x \ \partial(?IM \ G))$
 by (*rewrite singleton-nn-integral-interval-measure-cong*; *measurable*; *simp add: assms*)
 moreover have $(\int^{+x \in \{r <..s\}}. h \ x \ \partial(?IM \ F)) = (\int^{+x \in \{r <..s\}}. h \ x \ \partial(?IM \ G))$
 by (*rewrite Ioc-nn-integral-interval-measure-cong*; *measurable*; *simp add: assms*)
 ultimately show $?thesis$ **by** *simp*
 next
 case *False*
 thus $?thesis$ **by** *simp*
 qed
qed

corollary *Icc-Cont-nn-integral-interval-measure-cong*:
 fixes $F \ G :: real \Rightarrow real$ **and** $h :: real \Rightarrow ennreal$ **and** $r \ s :: real$
 assumes *mono* $F \bigwedge x. continuous \ (at\text{-}right \ x) \ F$ **and**
 mono $G \bigwedge x. continuous \ (at\text{-}right \ x) \ G$ **and**
 $(F - G)$ *constant-on* $\{r <..s\}$ **and**
 isCont $F \ r$ *isCont* $G \ r$ **and**
 $h \in borel\text{-}measurable \ borel$
 shows $(\int^{+x \in \{r..s\}}. h \ x \ \partial(interval\text{-}measure \ F)) = (\int^{+x \in \{r..s\}}. h \ x \ \partial(interval\text{-}measure \ G))$
proof –
 from *assms* **have** $F \ r - Lim \ (at\text{-}left \ r) \ F = G \ r - Lim \ (at\text{-}left \ r) \ G$
 by (*metis add-0 continuous-at-imp-continuous-at-within continuous-within eq-diff-eq tendsto-Lim trivial-limit-at-left-real*)
 thus $?thesis$
 using *assms* **by** (*intro Icc-nn-integral-interval-measure-cong*; *simp*)
qed

1.3 Restricting the Integral

lemma *nn-integral-interval-measure-Ici*:
 fixes $F :: real \Rightarrow real$ **and** $g :: real \Rightarrow ennreal$ **and** $r :: real$
 assumes *mono* $F \bigwedge x. continuous \ (at\text{-}right \ x) \ F$ $g \in borel\text{-}measurable \ borel$ **and**
 F *constant-on* $\{..<r\}$
 shows $(\int^{+x}. g \ x \ \partial(interval\text{-}measure \ F)) = (\int^{+x \in \{r..s\}}. g \ x \ \partial(interval\text{-}measure \ F))$
proof –
 let $?IMF = interval\text{-}measure \ F$

have $\{..<r\} \cup \{r..\} = UNIV$ **by** *auto*
moreover have $\{..<r\} \cap \{r..\} = \{\}$ **by** *auto*
ultimately have $(\int^{+x}. g x \partial?IMF) = (\int^{+x \in \{..<r\}}. g x \partial?IMF) + (\int^{+x \in \{r..\}}. g x \partial?IMF)$
using *assms* **by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *simp*)
also have $(\int^{+x \in \{..<r\}}. g x \partial?IMF) = 0$
using *assms interval-measure-const-null mono-on-const*
by (*rewrite Iio-nn-integral-interval-measure-cong*[**where** $G=\lambda-.0$]; *simp add: fun-diff-def*)
finally show *?thesis* **by** *simp*
qed

lemma *nn-integral-interval-measure-Ioi*:
fixes $F :: real \Rightarrow real$ **and** $g :: real \Rightarrow ennreal$ **and** $r::real$
assumes *mono* $F \wedge x. continuous (at-right x) F$ $g \in borel-measurable borel$ **and**
 F *constant-on* $\{..<r\}$ *isCont* $F r$
shows $(\int^{+x}. g x \partial(interval-measure F)) = (\int^{+x \in \{r<..\}}. g x \partial(interval-measure F))$
proof –
let $?IMF = interval-measure F$
have [*simp*]: *insert* $r \{r<..\} = \{r..\}$ **by** *force*
have $(\int^{+x}. g x \partial?IMF) = (\int^{+x \in \{r..\}}. g x \partial?IMF)$
by (*rule nn-integral-interval-measure-Ici*; *simp add: assms*)
also have $\dots = (\int^{+x \in \{r\}}. g x \partial?IMF) + (\int^{+x \in \{r<..\}}. g x \partial?IMF)$
using *assms* **by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *simp*)
also have $(\int^{+x \in \{r\}}. g x \partial?IMF) = 0$ **using** *assms interval-measure-singleton-continuous*
by *simp*
finally show *?thesis* **by** *simp*
qed

lemma *nn-integral-interval-measure-Ici*:
fixes $F :: real \Rightarrow real$ **and** $g :: real \Rightarrow ennreal$ **and** $s::real$
assumes *mono* $F \wedge x. continuous (at-right x) F$ $g \in borel-measurable borel$ **and**
 F *constant-on* $\{s<..\}$
shows $(\int^{+x}. g x \partial(interval-measure F)) = (\int^{+x \in \{..s\}}. g x \partial(interval-measure F))$
proof –
let $?IMF = interval-measure F$
have $\{..s\} \cup \{s<..\} = UNIV$ **by** *auto*
moreover have $\{..s\} \cap \{s<..\} = \{\}$ **by** *auto*
ultimately have $(\int^{+x}. g x \partial?IMF) = (\int^{+x \in \{..s\}}. g x \partial?IMF) + (\int^{+x \in \{s<..\}}. g x \partial?IMF)$
using *assms* **by** (*rewrite nn-integral-disjoint-pair*[*THEN sym*]; *simp*)
also have $(\int^{+x \in \{s<..\}}. g x \partial?IMF) = 0$
using *assms interval-measure-const-null mono-on-const*
by (*rewrite Ioi-nn-integral-interval-measure-cong*[**where** $G=\lambda-.0$]; *simp add: fun-diff-def*)
finally show *?thesis* **by** *simp*
qed

lemma *nn-integral-interval-measure-Iio*:
fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $s :: \text{real}$
assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ $g \in \text{borel-measurable borel}$ **and**
 $F \text{ constant-on } \{s < ..\}$ $\text{isCont } F \ s$
shows $(\int^+ x. g \ x \ \partial(\text{interval-measure } F)) = (\int^+ x \in \{.. < s\}. g \ x \ \partial(\text{interval-measure } F))$

proof –
let $?IMF = \text{interval-measure } F$
have $[simp]: \text{insert } s \ \{.. < s\} = \{..s\}$ **by force**
have $(\int^+ x. g \ x \ \partial?IMF) = (\int^+ x \in \{..s\}. g \ x \ \partial?IMF)$
by $(\text{rule nn-integral-interval-measure-Iic; simp add: assms})$
also have $\dots = (\int^+ x \in \{s\}. g \ x \ \partial?IMF) + (\int^+ x \in \{.. < s\}. g \ x \ \partial?IMF)$
using $\text{assms by (rewrite nn-integral-disjoint-pair[THEN sym]; simp)}$
also have $(\int^+ x \in \{s\}. g \ x \ \partial?IMF) = 0$
using $\text{assms interval-measure-singleton-continuous by simp}$
finally show $?thesis$ **by simp**

qed

lemma *nn-integral-interval-measure-Icc*:
fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r \ s :: \text{real}$
assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ $g \in \text{borel-measurable borel}$ **and**
 $F \text{ constant-on } \{.. < r\}$ $F \text{ constant-on } \{s < ..\}$
shows $(\int^+ x. g \ x \ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..s\}. g \ x \ \partial(\text{interval-measure } F))$

proof –
let $?IMF = \text{interval-measure } F$
show $?thesis$
proof $(\text{cases } \langle r \leq s \rangle)$
case True
have $(\int^+ x. g \ x \ \partial?IMF) = (\int^+ x \in \{r..s\}. g \ x \ \partial?IMF)$
by $(\text{rule nn-integral-interval-measure-Ici; simp add: assms})$
also have $\dots = (\int^+ x \in \{r..s\}. g \ x \ \partial?IMF)$
proof –
have $\{r..s\} = \{r..s\} \cup \{s < ..\}$ **using** $\text{assms using True by force}$
moreover have $\{r..s\} \cap \{s < ..\} = \{\}$ **by auto**
ultimately have $(\int^+ x \in \{r..s\}. g \ x \ \partial?IMF) = (\int^+ x \in \{r..s\}. g \ x \ \partial?IMF) +$
 $(\int^+ x \in \{s < ..\}. g \ x \ \partial?IMF)$
using $\text{assms by (rewrite nn-integral-disjoint-pair[THEN sym]; simp)}$
also have $(\int^+ x \in \{s < ..\}. g \ x \ \partial?IMF) = 0$
using $\text{assms interval-measure-const-null mono-on-const}$
by $(\text{rewrite Ioi-nn-integral-interval-measure-cong[where } G = \lambda. 0.]); \text{simp add: fun-diff-def})$
finally show $?thesis$ **by simp**

qed

finally show $?thesis$.

next
case False
hence $[simp]: \{s < ..\} \cup \{.. < r\} = \text{UNIV}$ **and** $[simp]: \{r < ..s\} = \{\}$ **using False**

by *force+*
have F *constant-on* $(\{s<..\} \cup \{..<r\})$ **using** *assms constant-on-Un False* **by**
force
hence [*simp*]: F *constant-on UNIV* **by** *simp*
hence [*simp*]: *isCont* F r
by (*metis (no-types, lifting) ext UNIV-I constant-on-def continuous-const*)
have $(\int^{+x}. g \ x \ \partial ?IMF) = 0$
by (*rewrite nn-integral-interval-measure-cong[where G= λ -.0]*;
simp add: fun-diff-def interval-measure-const-null mono-on-const assms)
also have $\dots = (\int^{+x \in \{r..s\}}. g \ x \ \partial ?IMF)$
by (*rewrite Icc-Cont-nn-integral-interval-measure-cong[where G= λ -.0]*;
simp add: assms fun-diff-def interval-measure-const-null mono-on-const)
finally show *?thesis* .
qed
qed

lemma *nn-integral-interval-measure-Ioc*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r \ s :: \text{real}$
assumes *mono* $F \ \wedge x.$ *continuous (at-right x)* $F \ g \in$ *borel-measurable borel* **and**
 F *constant-on* $\{..<r\}$ F *constant-on* $\{s<..\}$ *isCont* F r
shows $(\int^{+x}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{r<..s\}}. g \ x \ \partial(\text{interval-measure } F))$
proof –
let $?IMF = \text{interval-measure } F$
show *?thesis*
proof (*cases* $\langle r \leq s \rangle$)
case *True*
hence [*simp*]: *insert* $r \ \{r<..s\} = \{r..s\}$ **by** *force*
have $(\int^{+x}. g \ x \ \partial ?IMF) = (\int^{+x \in \{r..s\}}. g \ x \ \partial ?IMF)$
by (*rule nn-integral-interval-measure-Icc; simp add: assms*)
also have $\dots = (\int^{+x \in \{r\}}. g \ x \ \partial ?IMF) + (\int^{+x \in \{r<..s\}}. g \ x \ \partial ?IMF)$
using *assms* **by** (*rewrite nn-integral-disjoint-pair[THEN sym]; simp*)
also have $\dots = (\int^{+x \in \{r<..s\}}. g \ x \ \partial ?IMF)$
using *assms interval-measure-singleton-continuous* **by** *simp*
finally show *?thesis* .
next
case *False*
hence F *constant-on* $(\{s<..\} \cup \{..<r\})$ **by** (*intro constant-on-Un; simp add: assms*)
moreover have $\{s<..\} \cup \{..<r\} = \text{UNIV}$ **using** *False* **by** *force*
ultimately have [*simp*]: F *constant-on UNIV* **by** *simp*
hence [*simp*]: F *constant-on* $\{r<..s\}$ **using** *constant-on-subset* **by** *blast*
have $(\int^{+x}. g \ x \ \partial ?IMF) = 0$
by (*rewrite nn-integral-interval-measure-cong[where G= λ -.0]*;
simp add: fun-diff-def interval-measure-const-null mono-on-const assms)
also have $\dots = (\int^{+x \in \{r<..s\}}. g \ x \ \partial ?IMF)$
by (*rewrite Ioc-nn-integral-interval-measure-cong[where G= λ -.0]*;
simp add: assms fun-diff-def interval-measure-const-null mono-on-const)
finally show *?thesis* .

qed
qed

lemma *nn-integral-interval-measure-Ico:*

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r\ s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ $g \in \text{borel-measurable borel}$

$F \text{ constant-on } \{..<r\}$ $F \text{ constant-on } \{s<..\}$ $\text{isCont } F\ s$

shows $(\int^+ x. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r..<s\}. g\ x\ \partial(\text{interval-measure } F))$

proof –

let $?IMF = \text{interval-measure } F$

show $?thesis$

proof (*cases* $\langle r \leq s \rangle$)

case *True*

hence $[\text{simp}]: \text{insert } s\ \{r..<s\} = \{r..s\}$ **by** *force*

have $(\int^+ x. g\ x\ \partial?IMF) = (\int^+ x \in \{r..s\}. g\ x\ \partial?IMF)$

by (*rule nn-integral-interval-measure-Icc; simp add: assms*)

also have $\dots = (\int^+ x \in \{r..<s\}. g\ x\ \partial?IMF) + (\int^+ x \in \{s\}. g\ x\ \partial?IMF)$

using *assms* **by** (*rewrite nn-integral-disjoint-pair[THEN sym]; simp*)

also have $\dots = (\int^+ x \in \{r..<s\}. g\ x\ \partial?IMF)$

using *interval-measure-singleton-continuous assms* **by** *simp*

finally show $?thesis$.

next

case *False*

hence $F \text{ constant-on } (\{s<..\} \cup \{..<r\})$ **by** (*intro constant-on-Un; simp add: assms*)

moreover have $\{s<..\} \cup \{..<r\} = \text{UNIV}$ **using** *False* **by** *force*

ultimately have $[\text{simp}]: F \text{ constant-on UNIV}$ **by** *simp*

hence $[\text{simp}]: \text{isCont } F\ r$

by (*metis (no-types, lifting) ext UNIV-I constant-on-def continuous-const*)

have $[\text{simp}]: F \text{ constant-on } \{r<..\<s\}$ **using** *constant-on-subset[of F UNIV]* **by** *simp*

have $(\int^+ x. g\ x\ \partial?IMF) = 0$

by (*rewrite nn-integral-interval-measure-cong[where G= λ -.0];*

simp add: fun-diff-def interval-measure-const-null mono-on-const assms)

also have $\dots = (\int^+ x \in \{r..<s\}. g\ x\ \partial?IMF)$

by (*rewrite Ico-Cont-nn-integral-interval-measure-cong[where G= λ -.0];*

simp add: assms fun-diff-def interval-measure-const-null mono-on-const)

finally show $?thesis$.

qed

qed

lemma *nn-integral-interval-measure-Ioo:*

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r\ s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F$ $g \in \text{borel-measurable borel}$ **and**

$F \text{ constant-on } \{..<r\}$ $F \text{ constant-on } \{s<..\}$ $\text{isCont } F\ r$ $\text{isCont } F\ s$

shows $(\int^+ x. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r<..\<s\}. g\ x\ \partial(\text{interval-measure } F))$

proof –

let $?IMF = \text{interval-measure } F$
show $?thesis$
proof (*cases* $\langle r < s \rangle$)
 case *True*
 hence $[simp]: \text{insert } r \{r <.. < s\} = \{r.. < s\}$ **using** *assms* **by** *force*
 have $(\int^{+x}. g \ x \ \partial ?IMF) = (\int^{+x \in \{r.. < s\}}. g \ x \ \partial ?IMF)$
 using *assms* **by** (*intro nn-integral-interval-measure-Ico; simp*)
 also have $\dots = (\int^{+x \in \{r\}}. g \ x \ \partial ?IMF) + (\int^{+x \in \{r <.. < s\}}. g \ x \ \partial ?IMF)$
 using *assms* **by** (*rewrite nn-integral-disjoint-pair[THEN sym]; simp*)
 also have $\dots = (\int^{+x \in \{r <.. < s\}}. g \ x \ \partial ?IMF)$
 using *assms interval-measure-singleton-continuous* **by** *simp*
 finally show $?thesis$.
 next
 case *False*
 hence $\{..r\} \cap \{s..\} \neq \{\}$ **by** *auto*
 moreover have F *constant-on* $\{..r\}$
 proof -
 from *assms* **obtain** y **where** $Fy: \bigwedge x. x \in \{.. < r\} \implies F \ x = y$ **unfolding**
 constant-on-def **by** *blast*
 have $(F \longrightarrow F \ r)$ (*at-left* r) **using** *assms continuous-within filterlim-at-split*
 by *blast*
 hence $((\lambda-. y) \longrightarrow F \ r)$ (*at-left* r)
 by (*rewrite tendsto-cong; simp add: eventually-at-left-field lt-ex Fy*)
 moreover have $((\lambda-. y) \longrightarrow y)$ (*at-left* r) **by** (*rule tendsto-const*)
 ultimately have $F \ r = y$ **by** (*intro tendsto-unique; auto*)
 with *assms* **show** $?thesis$
 unfolding *constant-on-def* **using** Fy *less-eq-real-def* **by** (*intro exI[of - y]*)
 auto
 qed
 moreover have F *constant-on* $\{s..\}$
 proof -
 from *assms* **obtain** y **where** $Fy: \bigwedge x. x \in \{s <..\} \implies F \ x = y$ **unfolding**
 constant-on-def **by** *blast*
 have $(F \longrightarrow F \ s)$ (*at-right* s) **using** *assms continuous-within filterlim-at-split*
 by *blast*
 hence $((\lambda-. y) \longrightarrow F \ s)$ (*at-right* s)
 by (*rewrite tendsto-cong; simp add: eventually-at-right-field gt-ex Fy*)
 moreover have $((\lambda-. y) \longrightarrow y)$ (*at-right* r) **by** (*rule tendsto-const*)
 ultimately have $F \ s = y$ **by** (*intro tendsto-unique; auto*)
 with *assms* **show** $?thesis$
 unfolding *constant-on-def* **using** Fy *less-eq-real-def* **by** (*intro exI[of - y]*)
 auto
 qed
 ultimately have F *constant-on* $\{..r\} \cup \{s..\}$ **by** (*intro constant-on-Un; simp*)
 moreover have $\{..r\} \cup \{s..\} = UNIV$ **using** *False* **by** *auto*
 ultimately have $[simp]: F$ *constant-on* $UNIV$ **by** *simp*
 hence $[simp]: F$ *constant-on* $\{r <.. < s\}$ **using** *constant-on-subset* **by** *blast*
 hence $(\int^{+t}. g \ t \ \partial ?IMF) = 0$
 by (*rewrite nn-integral-interval-measure-cong[where G= $\lambda-.0$]*);

simp add: fun-diff-def interval-measure-const-null mono-on-const assms)
also have ... = $(\int^{+t \in \{r < .. < s\}}. g \ t \ \partial ?IMF)$
by (*rewrite Ioo-nn-integral-interval-measure-cong*[**where** $G = \lambda -. 0$]);
simp add: assms fun-diff-def interval-measure-const-null mono-on-const)
finally show *?thesis* .
qed
qed

1.4 Calculation by the Derivative

proposition *set-nn-integral-interval-measure-deriv:*

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $a \ b :: \text{ereal}$ **and** $A :: \text{real set}$
assumes *mono F* \wedge *continuous (at-right x) F* *F differentiable-on (einterval a b)* **and**

g-msr: g \in *borel-measurable lborel* **and**

$A \in$ *sets borel* $A \subseteq$ *einterval a b*

shows $(\int^{+x \in A}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in A}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$

proof –

let *?Iab* = *einterval a b*

let *?IM* = *interval-measure*

wlog $A = ?Iab$ **generalizing** $A \ g$ **keeping** *g-msr*

proof –

from *assms* **have** *indicator A* \in *borel-measurable lborel* **by** *measurable*

with *assms* **have** *gA-msr: ($\lambda x. g \ x * \text{indicator } A \ x$)* \in *borel-measurable lborel*

by *measurable*

have $(\int^{+x \in A}. g \ x \ \partial(?IM \ F)) = (\int^{+x \in ?Iab}. g \ x * \text{indicator } A \ x \ \partial(?IM \ F))$

using *assms* **by** (*simp add: mult-ac mult-indicator-subset*)

also have ... = $(\int^{+x \in ?Iab}. g \ x * \text{indicator } A \ x * \text{deriv } F \ x \ \partial \text{lborel})$

using *hypothesis gA-msr* **by** *simp*

also have ... = $(\int^{+x \in ?Iab}. g \ x * \text{deriv } F \ x * \text{indicator } A \ x \ \partial \text{lborel})$

using *mult.commute mult.assoc* **by** (*metis (no-types, lifting)*)

also have ... = $(\int^{+x \in A}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$

using *assms* **by** (*simp add: mult.assoc mult-indicator-subset*)

finally show *?thesis* .

qed

hence *A-I: A = ?Iab* **by** *simp*

thus *?thesis*

proof (*cases* $\langle a < b \rangle$)

case *True*

let *?dFIab* = $\lambda x. \text{deriv } F \ x * \text{indicator } ?Iab \ x$

have *dFsmr: set-borel-measurable borel (?Iab) (deriv F)*

by (*rule deriv-measurable-real; simp add: assms borel-measurable-mono*)

hence [*measurable*]: $(\lambda x. \text{ennreal } (?dFIab \ x)) \in$ *borel-measurable borel*

using *assms* **unfolding** *set-borel-measurable-def* **by** (*rewrite mult.commute*)

simp

have *IMFst: $\bigwedge s \ t. a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b \implies \text{emeasure } (?IM \ F)$*

$\{s <..t\} = F t - F s$
using *assms monoD* **by** (*rewrite emeasure-interval-measure-Ioc; force*)
moreover have $\bigwedge s t. a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b \implies$
 $\text{emeasure } (?IM F) \{s <..t\} = \text{emeasure } (\text{density lborel } ?dFIab) \{s <..t\}$
proof –
fix $s t :: \text{real}$
assume *asm*: $a < \text{ereal } s \wedge s \leq t \wedge \text{ereal } t < b$
hence $\text{emeasure } (?IM F) \{s <..t\} = F t - F s$ **by** (*rule IMFst*)
also have $\dots = \int^+ x. \text{ennreal } (?dFIab x) * \text{indicator } \{s..t\} x \partial \text{lborel}$
proof –
have [*simp*]: $\bigwedge x. x \in \{s..t\} \implies x \in ?Iab$
using *asm einterval-iff ereal-less-le less-ereal-le* **by** *force*
hence [*simp*]: $\bigwedge x. s \leq x \wedge x \leq t \implies F$ *differentiable at x within ?Iab*
by (*intro differentiable-onD; simp add: assms*)
have $\bigwedge x. x \in \{s..t\} \implies \text{DERIV } F x :> \text{deriv } F x$
apply (*rewrite DERIV-deriv-iff-real-differentiable*)
by (*rewrite at-within-open[where S=?Iab, THEN sym]; simp*)
moreover hence $\bigwedge x. x \in \{s..t\} \implies 0 \leq \text{deriv } F x$
using *assms mono-imp-mono-on mono-on-imp-deriv-nonneg* **by** (*metis UNIV-I interior-UNIV*)
moreover have $(\lambda x. \text{deriv } F x * \text{indicator } ?Iab x) \in \text{borel-measurable borel}$
using *dFmsr unfolding set-borel-measurable-def* **by** (*simp add: mult.commute*)
ultimately show *?thesis* **by** (*rewrite nn-integral-FTC-Icc; simp add: asm*)
qed
also have $\dots = \int^+ x. \text{ennreal } (?dFIab x) * \text{indicator } \{s <..t\} x \partial \text{lborel}$
proof –
have *sym-diff* $\{s..t\} \{s <..t\} = \{s\}$ **using** *asm* **by** *force*
thus *?thesis* **by** (*intro nn-integral-null-delta*) *auto*
qed
also have $\dots = \text{emeasure } (\text{density lborel } ?dFIab) \{s <..t\}$
by (*rewrite emeasure-density; simp add: assms True*)
finally show $\text{emeasure } (?IM F) \{s <..t\} = \text{emeasure } (\text{density lborel } ?dFIab)$
 $\{s <..t\} .$
qed
ultimately have *IMdsty*:
 $\text{restrict-space } (?IM F) ?Iab = \text{restrict-space } (\text{density lborel } ?dFIab) ?Iab$
by (*intro measure-einterval-eqI-Ioc; simp*)
show *?thesis*
proof –
have $(\int^+_{x \in ?Iab} g x \partial (?IM F)) = (\int^+ x. g x \partial (\text{restrict-space } (?IM F) ?Iab))$
by (*rewrite nn-integral-restrict-space[THEN sym]; simp*)
also have $\dots = (\int^+ x. g x \partial (\text{restrict-space } (\text{density lborel } ?dFIab) ?Iab))$
using *IMdsty* **by** *simp*
also have $\dots = (\int^+ x. g x * \text{indicator } ?Iab x \partial (\text{density lborel } ?dFIab))$
by (*rewrite nn-integral-restrict-space; simp*)
also have $\dots =$
 $(\int^+ x. \text{ennreal } (\text{deriv } F x * \text{indicator } ?Iab x) * (g x * \text{indicator } ?Iab x)$
 $\partial \text{lborel})$
using *g-msr* **by** (*rewrite nn-integral-density; simp*)

also have $\dots = (\int^{+x \in ?Iab}. g \ x * \text{ennreal} (\text{deriv } F \ x) \ \partial \text{lborel})$
by (*rule nn-integral-cong*) (*simp add: mult.left-commute indicator-times-eq-if*)
finally show *?thesis* **using** *A-I* **by** *simp*
qed
next
case *False*
hence $einterval \ a \ b = \{\}$ **using** *einterval-empty* **by** *simp*
hence $A = \{\}$ **using** *assms A-I* **by** *simp*
thus *?thesis* **by** *simp*
qed

qed

corollary *nn-integral-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$
assumes *mono F* $\wedge x.$ *continuous (at-right x) F F differentiable-on UNIV* **and**
 $g \in \text{borel-measurable lborel}$
shows $(\int^{+x}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$
using *set-nn-integral-interval-measure-deriv einterval-eq-UNIV indicator-UNIV*
assms
by (*metis (mono-tags, lifting) mult.right-neutral nn-integral-cong*
space-in-borel verit-eq-simplify(6))

corollary *Ioi-nn-integral-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r :: \text{real}$
assumes *mono F* $\wedge x.$ *continuous (at-right x) F F differentiable-on {r<..}* **and**
 $g \in \text{borel-measurable lborel}$
shows $(\int^{+x \in \{r<..\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{r<..\}}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$
proof –
let $?I = einterval$
let $?IMF = interval-measure \ F$
have $(\int^{+x \in \{r<..\}}. g \ x \ \partial ?IMF) = (\int^{+x \in (?I \ (ereal \ r) \ \infty)}. g \ x \ \partial ?IMF)$ **by** *simp*
also have $\dots = (\int^{+x \in (?I \ (ereal \ r) \ \infty)}. g \ x * \text{ennreal} (\text{deriv } F \ x) \ \partial \text{lborel})$
using *assms* **by** (*rewrite set-nn-integral-interval-measure-deriv[of - ereal r \infty]*;
simp)
also have $\dots = (\int^{+x \in \{r<..\}}. g \ x * \text{ennreal} (\text{deriv } F \ x) \ \partial \text{lborel})$ **by** *simp*
finally show *?thesis* .
qed

corollary *Iio-nn-integral-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $s :: \text{real}$
assumes *mono F* $\wedge x.$ *continuous (at-right x) F F differentiable-on {..<s}* **and**
 $g \in \text{borel-measurable lborel}$
shows $(\int^{+x \in \{..<s\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{..<s\}}. g \ x * \text{deriv } F \ x \ \partial \text{lborel})$
proof –
let $?I = einterval$
let $?IMF = interval-measure \ F$

have $(\int^{+x \in \{..<s\}}. g x \partial ?IMF) = (\int^{+x \in (?I (-\infty) (ereal s))}. g x \partial ?IMF)$ **by**
simp
also have $\dots = (\int^{+x \in (?I (-\infty) (ereal s))}. g x * ennreal (deriv F x) \partial lborel)$
using *assms* **by** *(rewrite set-nn-integral-interval-measure-deriv[of - -\infty ereal s]; simp)*
also have $\dots = (\int^{+x \in \{..<s\}}. g x * deriv F x \partial lborel)$ **by** *simp*
finally show *?thesis* .
qed

corollary *Ioo-nn-integral-interval-measure-deriv:*

fixes $F :: real \Rightarrow real$ **and** $g :: real \Rightarrow ennreal$ **and** $r s :: real$
assumes *mono F \wedge x. continuous (at-right x) F F differentiable-on {r<..~~}~~*
and

$g \in \text{borel-measurable } lborel$
shows $(\int^{+x \in \{r<..$

proof –

let $?I = \text{einterval}$
let $?IMF = \text{interval-measure } F$
have $(\int^{+x \in \{r<..
by *simp*
also have $\dots = (\int^{+x \in (?I (ereal r) (ereal s))}. g x * ennreal (deriv F x) \partial lborel)$
using *assms* **by** *(rewrite set-nn-integral-interval-measure-deriv[of - ereal r ereal s]; simp)*
also have $\dots = (\int^{+x \in \{r<.. **by** *simp*
finally show *?thesis* .
qed$$

lemma *set-nn-integral-finite-nondifferentiable-interval-measure-deriv:*

fixes $F :: real \Rightarrow real$ **and** $g :: real \Rightarrow ennreal$ **and** $a b :: ereal$ **and** $S :: real \text{ set}$
assumes *mono F \wedge x. continuous (at-right x) F g \in borel-measurable lborel and*
cont: continuous-on (einterval a b) F and
diff: F differentiable-on einterval a b - S and
fin: finite S
shows $(\int^{+x \in \text{einterval } a b}. g x \partial (\text{interval-measure } F)) =$
 $(\int^{+x \in \text{einterval } a b}. g x * deriv F x \partial lborel)$
using *fin cont diff*
proof *(induction S arbitrary: a b rule: finite-psubset-induct)*

let $?I = \text{einterval}$
let $?IMF = \text{interval-measure } F$

have *[measurable]: F \in borel-measurable borel*
using *assms borel-measurable-mono* **by** *blast*

case *IH: (psubset S)*

thus *?case*

proof *(cases S)*

case *emptyI*

thus *?thesis* **using** *assms set-nn-integral-interval-measure-deriv IH* **by** *simp*
next
case (*insertI S' s*)
hence *Ss: S - {s} ⊂ S ∧ finite (S - {s})* **by** *force+*
have [*simp*]: *?I a (ereal s) ∩ ?I (ereal s) b = {}* **using** *einterval-iff* **by** *force*
show *?thesis*
proof (*cases ‹s ∈ ?I a b›*)

case *True*

hence [*simp*]: *a < b* **using** *einterval-iff* **by** *force*
from *True* **have** *Iabs: ?I a b - {s} = ?I a (ereal s) ∪ ?I (ereal s) b*
by (*rule einterval-split*)
hence [*simp*]: *?I a (ereal s) ⊆ ?I a b* **and** [*simp*]: *?I (ereal s) b ⊆ ?I a b* **by**
auto

have *F* *piecewise-differentiable-on ?I a b*
using *IH unfolding differentiable-on-def piecewise-differentiable-on-def*
by (*metis DiffE at-within-open finite-imp-closed open-Diff open-einterval*)
hence [*measurable*]: *set-borel-measurable borel (?I a b) (deriv F)*
by (*rule piecewise-differentiable-on-deriv-measurable-real; simp add: assms*)
hence [*measurable*]: $\bigwedge A. A \subseteq ?I a b \implies A \in \text{sets borel} \implies$
 $(\lambda x. g x * \text{ennreal (deriv F x)} * \text{indicator A x}) \in \text{borel-measurable borel}$
proof –
fix *A* **assume** *A ⊆ ?I a b* **and** *A ∈ sets borel*
hence *set-borel-measurable borel A (deriv F)*
using *set-borel-measurable-subset[of - ?I a b - A]* **by** *measurable*
hence $(\lambda x. \text{ennreal (deriv F x)} * \text{indicator A x}) \in \text{borel-measurable borel}$
apply (*rewrite mult.commute, rewrite indicator-mult-ennreal*)
unfolding *set-borel-measurable-def* **by** *simp*
thus $(\lambda x. g x * \text{ennreal (deriv F x)} * \text{indicator A x}) \in \text{borel-measurable borel}$
using *assms measurable-lborel1* **by** (*rewrite mult.assoc; simp*)
qed

let *?Sl = S ∩ ?I a (ereal s)*
have *?Sl ⊂ S ∧ finite ?Sl* **using** *Ss einterval-iff* **by** *force+*
moreover **have** *continuous-on (?I a (ereal s)) F*
using *IH True*
by (*smt (verit) at-within-open order.strict-trans*
continuous-on-eq-continuous-within einterval-iff open-einterval)
moreover **have** *F* *differentiable-on ?I a (ereal s) - ?Sl*
apply (*rule differentiable-on-subset[of - ?I a b - S], simp add: IH*)
using *True einterval-iff eereal-less-le* **by** *fastforce*
ultimately **have**
 $II: (\int^{+x} \in ?I a (ereal s). g x \partial ?IMF) =$
 $(\int^{+x} \in ?I a (ereal s). g x * \text{ennreal (deriv F x)} \partial \text{lborel})$
using *IH assms* **by** *simp*

let *?Sr = S ∩ ?I (ereal s) b*

have $?Sr \subset S \wedge \text{finite } ?Sr$ **using** $Ss \text{ einterval-iff}$ **by** force+
moreover have $\text{continuous-on } (?I \text{ (ereal } s) \ b) \ F$ **using** $IH \ \text{True}$
by $(\text{smt } (\text{verit}) \ \text{at-within-open } \text{order.strict-trans}$
 $\text{continuous-on-eq-continuous-within } \text{einterval-iff } \text{open-einterval})$
moreover have $F \text{ differentiable-on } ?I \text{ (ereal } s) \ b - ?Sr$
apply $(\text{rule } \text{differentiable-on-subset}[\text{of } - ?I \ a \ b - S], \ \text{simp add: } IH)$
using $\text{True einterval-iff } \text{order.strict-trans}$ **by** fastforce
ultimately have
 $Ir: (\int^+ x \in ?I \text{ (ereal } s) \ b. \ g \ x \ \partial ?IMF) =$
 $(\int^+ x \in ?I \text{ (ereal } s) \ b. \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel})$
using $IH \ \text{assms}$ **by** simp

have $\text{isCont } F \ s$
using $\text{True } IH$ **by** $(\text{metis } \text{at-within-open } \text{continuous-on-eq-continuous-within}$
 $\text{open-einterval})$
hence $\text{emeasure } ?IMF \ \{s\} = \text{ennreal } (\text{deriv } F \ s) * \text{emeasure } \text{lborel } \{s\}$
using assms **by** $(\text{rewrite } \text{interval-measure-singleton-continuous}; \ \text{simp})$
hence $Is: (\int^+ x \in \{s\}. \ g \ x \ \partial ?IMF) = (\int^+ x \in \{s\}. \ g \ x * \text{deriv } F \ x \ \partial \text{lborel})$
by $(\text{rewrite } \text{nn-integral-indicator-singleton}; \ \text{simp})$

show $?thesis$
proof $-$
have $[\text{simp}]: \text{insert } s \ (?I \ a \ b) = ?I \ a \ b$ **using** $\text{insert-absorb } \text{True}$ **by** force
have $(\int^+ x \in ?I \ a \ b. \ g \ x \ \partial ?IMF) = (\int^+ x \in ?I \ a \ b - \{s\}. \ g \ x \ \partial ?IMF) +$
 $(\int^+ x \in \{s\}. \ g \ x \ \partial ?IMF)$
using True assms **by** $(\text{rewrite } \text{nn-integral-disjoint-pair}[\text{THEN } \text{sym}]; \ \text{simp})$
also have $\dots = (\int^+ x \in ?I \ a \ \text{(ereal } s). \ g \ x \ \partial ?IMF) +$
 $(\int^+ x \in ?I \ \text{(ereal } s) \ b. \ g \ x \ \partial ?IMF) + (\int^+ x \in \{s\}. \ g \ x \ \partial ?IMF)$
using $\text{True assms } Iabs$ **by** $(\text{rewrite in } \sqcup + - \text{nn-integral-disjoint-pair}[\text{THEN}$
 $\text{sym}]; \ \text{simp})$
also have $\dots = (\int^+ x \in ?I \ a \ \text{(ereal } s). \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel}) +$
 $(\int^+ x \in ?I \ \text{(ereal } s) \ b. \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel}) +$
 $(\int^+ x \in \{s\}. \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel})$
using $I \ Ir \ Is$ **by** simp
also have $\dots = (\int^+ x \in ?I \ a \ b - \{s\}. \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel}) +$
 $(\int^+ x \in \{s\}. \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel})$
by $(\text{rewrite } \text{nn-integral-disjoint-pair2}[\text{THEN } \text{sym}]; \ \text{simp add: } Iabs)$
also have $\dots = (\int^+ x \in ?I \ a \ b. \ g \ x * \text{ennreal } (\text{deriv } F \ x) \ \partial \text{lborel})$
by $(\text{rewrite } \text{nn-integral-disjoint-pair2}[\text{THEN } \text{sym}]; \ \text{simp add: } \text{True})$
finally show $?thesis .$

qed

next

case False
hence $?I \ a \ b - S = ?I \ a \ b - (S - \{s\})$ **by** blast
with IH **have** $F \text{ differentiable-on } ?I \ a \ b - (S - \{s\})$ **by** simp
then show $?thesis$ **using** $\text{assms } IH \ Ss$ **by** simp

qed
 qed
 qed

proposition *set-nn-integral-piecewise-differentiable-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $a\ b :: \text{ereal}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on (einterval } a\ b)$

$g \in \text{borel-measurable lborel}$

shows $(\int^+ x \in \text{einterval } a\ b. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \text{einterval } a\ b. g\ x * \text{deriv } F\ x\ \partial\text{lborel})$

proof –

let $?Iab = \text{einterval } a\ b$

from *assms* **have** $\text{continuous-on } ?Iab\ F$ **by** (*intro piecewise-differentiable-on-imp-continuous-on*)

moreover from *assms* **obtain** S **where** $\text{finite } S$ **and** $\wedge x. x \in ?Iab - S \implies F$

differentiable (at } x)

unfolding *piecewise-differentiable-on-def* **by** (*metis DiffE at-within-open open-einterval*)

moreover hence F *differentiable-on* $?Iab - S$

using *differentiable-at-imp-differentiable-on* **by** *blast*

ultimately show *?thesis*

using *assms* **by** (*intro set-nn-integral-finite-nondifferentiable-interval-measure-deriv*)

simp

qed

corollary *nn-integral-piecewise-differentiable-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on UNIV}$

$g \in \text{borel-measurable lborel}$

shows $(\int^+ x. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x. g\ x * \text{deriv } F\ x\ \partial\text{lborel})$

using *set-nn-integral-piecewise-differentiable-interval-measure-deriv*

einterval-eq-UNIV indicator-UNIV assms

by (*metis (mono-tags, lifting) mult.right-neutral nn-integral-cong*)

corollary *Ioi-nn-integral-piecewise-differentiable-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on } \{r < ..\}$

$g \in \text{borel-measurable lborel}$

shows $(\int^+ x \in \{r < ..\}. g\ x\ \partial(\text{interval-measure } F)) = (\int^+ x \in \{r < ..\}. g\ x * \text{deriv } F\ x\ \partial\text{lborel})$

using *set-nn-integral-piecewise-differentiable-interval-measure-deriv assms*

by (*metis (mono-tags, lifting) einterval-eq-Ici nn-integral-cong*)

corollary *Iio-nn-integral-piecewise-differentiable-interval-measure-deriv*:

fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $s :: \text{real}$

assumes $\text{mono } F \wedge x. \text{continuous (at-right } x) F F \text{ piecewise-differentiable-on } \{.. < s\}$

$g \in \text{borel-measurable lborel}$

shows $(\int^{+x \in \{..<s\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{..<s\}}. g \ x \ * \ \text{deriv } F \ x \ \partial\text{lborel})$
using *set-nn-integral-piecewise-differentiable-interval-measure-deriv assms*
by (*metis (mono-tags, lifting) einterval-eq-Iic nn-integral-cong*)

corollary *Ioo-nn-integral-piecewise-differentiable-interval-measure-deriv*:
fixes $F :: \text{real} \Rightarrow \text{real}$ **and** $g :: \text{real} \Rightarrow \text{ennreal}$ **and** $r \ s :: \text{real}$
assumes $\text{mono } F \ \wedge \ x. \text{continuous (at-right } x) \ F \ F \ \text{piecewise-differentiable-on } \{r < .. < s\}$
 $g \in \text{borel-measurable lborel}$
shows $(\int^{+x \in \{r < .. < s\}}. g \ x \ \partial(\text{interval-measure } F)) = (\int^{+x \in \{r < .. < s\}}. g \ x \ * \ \text{deriv } F \ x \ \partial\text{lborel})$
using *set-nn-integral-piecewise-differentiable-interval-measure-deriv assms*
by (*metis (mono-tags, lifting) einterval-eq-Icc nn-integral-cong*)

end