

# Lambert Series

Manuel Eberl

May 14, 2024

## Abstract

This entry provides a formalisation of *Lambert series*, i.e. series of the form  $L(a_n, q) = \sum_{n=1}^{\infty} a_n q^n / (1 - q^n)$  where  $a_n$  is a sequence of real or complex numbers. Proofs for all the basic properties are provided, such as

- the precise region in which  $L(a_n, q)$  converges
- the functional equation  $L(a_n, \frac{1}{q}) = -(\sum_{n=1}^{\infty} a_n) - L(a_n, q)$
- the power series expansion of  $L(a_n, q)$  at  $q = 0$
- the connection  $L(a_n, q) = \sum_{k=1}^{\infty} f(q^k)$  for  $f(z) = \sum_{n=1}^{\infty} a_n z^n$  that links a Lambert series to its “corresponding” power series
- connections to various number-theoretic functions, e.g. the divisor  $\sigma$  function via  $\sum_{n=1}^{\infty} \sigma_{\alpha}(n) q^n = L(n^{\alpha}, q)$

The formalisation mainly follows the chapter on Lambert series in Konrad Knopp’s classic textbook *Theory and Application of Infinite Series* [1] and includes all results presented therein.

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# 1 Missing Library Material

```
theory Lambert_Series_Library
```

```
imports
```

```
  "HOL-Complex_Analysis.Complex_Analysis"
```

```
  "HOL-Library.Landau_Symbols"
```

```
  "HOL-Real_Asymp.Real_Asymp"
```

```
begin
```

## 1.1 Miscellaneous

```
lemma power_less_1_iff: "x ≥ 0 ⇒ (x :: real) ^ n < 1 ↔ x < 1 ∧ n > 0"
```

```
  by (metis not_gr_zero not_less_iff_gr_or_eq power_0 real_root_lt_1_iff real_root_pos2)
```

```
lemma fls_nth_sum: "fls_nth (∑ x∈A. f x) n = (∑ x∈A. fls_nth (f x) n)"
```

```
  by (induction A rule: infinite_finite_induct) auto
```

```
lemma two_times_choose_two: "2 * (n choose 2) = n * (n - 1)"
```

```
  unfolding choose_two by (simp add: algebra_simps)
```

```
lemma Nats_not_empty [simp]: "N ≠ {}"
```

```
  using Nats_1 by blast
```

## 1.2 Infinite sums

```
lemma has_sum_iff: "(f has_sum S) A ↔ f summable_on A ∧ infsum f A = S"
```

```
  using infsumI summable_iff_has_sum_infsum by blast
```

```
lemma summable_on_reindex_bij_witness:
```

```
  assumes "∧ a. a ∈ S ⇒ i (j a) = a"
```

```
  assumes "∧ a. a ∈ S ⇒ j a ∈ T"
```

```
  assumes "∧ b. b ∈ T ⇒ j (i b) = b"
```

```
  assumes "∧ b. b ∈ T ⇒ i b ∈ S"
```

```
  assumes "∧ a. a ∈ S ⇒ h (j a) = g a"
```

```
  shows "g summable_on S ↔ h summable_on T"
```

```
  using has_sum_reindex_bij_witness[of S i j T h g, OF assms refl]
```

```
  by (simp add: summable_on_def)
```

```
lemma has_sum_diff:
```

```
  fixes f g :: "'a ⇒ 'b::{topological_ab_group_add}"
```

```
  assumes <(f has_sum a) A>
```

```
  assumes <(g has_sum b) A>
```

```
  shows <((λx. f x - g x) has_sum (a - b)) A>
```

```
  using has_sum_add[of f A a "λx. -g x" "-b"] assms by (simp add: has_sum_uminus)
```

```
lemma summable_on_diff:
```

```

fixes f g :: "'a ⇒ 'b::{topological_ab_group_add}"
assumes <f summable_on A>
assumes <g summable_on A>
shows <(λx. f x - g x) summable_on A>
by (metis (full_types) assms summable_on_def has_sum_diff)

lemma infsum_diff:
  fixes f g :: "'a ⇒ 'b::{topological_ab_group_add, t2_space}"
  assumes <f summable_on A>
  assumes <g summable_on A>
  shows <infsum (λx. f x - g x) A = infsum f A - infsum g A>
proof -
  have <((λx. f x - g x) has_sum (infsum f A - infsum g A)) A>
    by (simp add: assms has_sum_diff)
  then show ?thesis
    using infsumI by blast
qed

lemma summable_norm_add:
  assumes "summable (λn. norm (f n))" "summable (λn. norm (g n))"
  shows "summable (λn. norm (f n + g n))"
proof (rule summable_comparison_test)
  show "summable (λn. norm (f n) + norm (g n))"
    by (intro summable_add assms)
  show "∃N. ∀n≥N. norm (norm (f n + g n)) ≤ norm (f n) + norm (g n)"
    by (intro exI[of _ 0] allI impI) (auto simp: norm_triangle_ineq)
qed

lemma summable_norm_diff:
  assumes "summable (λn. norm (f n))" "summable (λn. norm (g n))"
  shows "summable (λn. norm (f n - g n))"
  using summable_norm_add[of f "λn. -g n"] assms by simp

lemma sums_imp_has_prod_exp:
  fixes f :: "'_ ⇒ 'a::{real_normed_field,banach}"
  assumes "f sums F"
  shows "(λn. exp (f n)) has_prod exp F"
proof -
  have "(λn. exp (∑ i≤n. f i)) ⟶ exp F"
    by (intro tendsto_intros) (use assms in <auto simp: sums_def' atLeast0AtMost>)
  also have "(λn. exp (∑ i≤n. f i)) = (λn. ∏ i≤n. exp (f i))"
    by (simp add: exp_sum)
  finally have "raw_has_prod (λn. exp (f n)) 0 (exp F)"
    unfolding raw_has_prod_def by auto
  thus ?thesis
    unfolding has_prod_def by blast
qed

lemma telescope_summable_iff:

```

```

fixes f :: "nat ⇒ 'a::{real_normed_vector}"
shows "summable (λn. f (Suc n) - f n) ⟷ convergent f"
proof
  assume "convergent f"
  thus "summable (λn. f (Suc n) - f n)"
    using telescope_summable[of f] by (auto simp: convergent_def)
next
  assume "summable (λn. f (Suc n) - f n)"
  hence "convergent (λn. ∑ i<n. f (Suc i) - f i)"
    by (simp add: summable_iff_convergent)
  also have "(λn. ∑ i<n. f (Suc i) - f i) = (λn. f n - f 0)"
    by (subst sum_lessThan_telescope) auto
  also have "convergent ... ⟷ convergent f"
    by (rule convergent_diff_const_right_iff)
  finally show "convergent f" .
qed

lemma telescope_summable_iff':
  fixes f :: "nat ⇒ 'a::{real_normed_vector}"
  shows "summable (λn. f n - f (Suc n)) ⟷ convergent f"
  using telescope_summable_iff[of "λn. -f n"] by (simp flip: convergent_minus_iff)

lemma norm_summable_mult_bounded:
  assumes "summable (λn. norm (f n))"
  assumes "g ∈ O(λ_. 1)"
  shows "summable (λn. norm (f n * g n))"
proof -
  from assms(2) obtain C where C: "C > 0" "eventually (λn. norm (g n)
≤ C) at_top"
  by (auto elim!: landau_o.bigE)
  show ?thesis
  proof (rule summable_comparison_test_ev)
    show "summable (λn. norm (f n) * C)"
      by (subst mult.commute) (intro summable_mult assms)
    show "eventually (λn. norm (norm (f n * g n)) ≤ norm (f n) * C) at_top"
      using C(2) by eventually_elim (use C(1) in <auto intro!: mult_mono
simp: norm_mult>)
  qed
qed

lemma summable_powser_comparison_test_bigo:
  fixes f g :: "nat ⇒ 'a :: {real_normed_field, banach}"
  assumes "summable f" "g ∈ O(λn. f n * c ^ n)" "norm c < 1"
  shows "summable (λn. norm (g n))"
proof (rule summable_comparison_test_bigo)
  have "summable (λn. norm (f n * c ^ n))"
    by (rule powser_insidea[of _ 1]) (use assms in auto)
  thus "summable (λn. norm (norm (f n * c ^ n)))"
    by simp

```

```

    show "(λn. norm (g n)) ∈ O(λn. norm (f n * c ^ n))"
      using assms(2) by simp
qed

lemma geometric_sums_gen:
  assumes "norm (x :: 'a :: real_normed_field) < 1"
  shows "(λn. x ^ (n + k)) sums (x ^ k / (1 - x))"
proof -
  have "(λn. x ^ k * x ^ n) sums (x ^ k * (1 / (1 - x)))"
    by (intro sums_mult geometric_sums assms)
  thus ?thesis
    by (simp add: power_add mult_ac)
qed

lemma has_sum_geometric:
  fixes x :: "'a :: {real_normed_field, banach}"
  assumes "norm x < 1"
  shows "((λn. x ^ n) has_sum (x ^ m / (1 - x))) {m..}"
proof -
  have "((λn. x ^ n) has_sum (1 / (1 - x))) UNIV"
    using assms
    by (intro norm_summable_imp_has_sum)
      (auto intro: geometric_sums summable_geometric simp: norm_power)
  hence "((λn. x ^ m * x ^ n) has_sum (x ^ m * (1 / (1 - x)))) UNIV"
    by (rule has_sum_cmult_right)
  also have "?this ↔ ?thesis"
    by (rule has_sum_reindex_bij_witness[of _ "λn. n - m" "λn. n + m"])
  (auto simp: power_add)
  finally show ?thesis .
qed

lemma n_powser_sums:
  fixes q :: "'a :: {real_normed_field, banach}"
  assumes q: "norm q < 1"
  shows "(λn. of_nat n * q ^ n) sums (q / (1 - q) ^ 2)"
proof -
  have "(λn. q * (of_nat (Suc n) * q ^ n)) sums (q * (1 / (1 - q)^2))"
    using q by (intro sums_mult geometric_deriv_sums)
  also have "(λn. q * (of_nat (Suc n) * q ^ n)) = (λn. of_nat (Suc n)
    * q ^ Suc n)"
    by (simp add: algebra_simps)
  finally have "(λn. of_nat n * q ^ n) sums (q * (1 / (1 - q)^2) + of_nat
    0 * q ^ 0)"
    by (rule sums_Suc)
  thus "(λn. of_nat n * q ^ n) sums (q / (1 - q) ^ 2)"
    by simp
qed

```

### 1.3 Convergence radius

```

lemma tendsto_imp_conv_radius_eq:
  assumes "(λn. ereal (norm (f n) powr (1 / real n))) → c'" "c =
  inverse c'"
  shows "conv_radius f = c"
proof -
  have "(λn. ereal (root n (norm (f n)))) → c'"
  proof (rule Lim_transform_eventually)
    show "(λn. ereal (norm (f n) powr (1 / real n))) → c'"
      using assms by simp
    show "∀F x in sequentially. ereal (norm (f x) powr (1 / real x))
  =
      ereal (root x (norm (f x)))"
      using eventually_gt_at_top[of 0]
  proof eventually_elim
    case (elim n)
    show ?case using elim
      by (cases "f n = 0") (simp_all add: root_powr_inverse)
  qed
  qed
  thus ?thesis
    unfolding conv_radius_def using assms by (simp add: limsup_root_limit)
qed

lemma conv_radius_powr_real: "conv_radius (λn. real n powr a) = 1"
proof (rule tendsto_imp_conv_radius_eq)
  have "(λn. ereal ((real n powr a) powr (1 / real n))) → ereal 1"
    by (rule tendsto_ereal) real_asymp
  thus "(λn. ereal (norm (real n powr a) powr (1 / real n))) → ereal
  1"
    by simp
qed (simp_all add: one_ereal_def)

lemma conv_radius_one_over: "conv_radius (λn. 1 / of_nat n :: 'a :: {real_normed_field,
banach}) = 1"
proof (rule tendsto_imp_conv_radius_eq)
  have "(λn. ereal ((1 / n) powr (1 / real n))) → ereal 1"
    by (rule tendsto_ereal) real_asymp
  thus "(λn. ereal (norm (1 / of_nat n :: 'a) powr (1 / real n))) →
  ereal 1"
    by (simp add: norm_divide)
qed (simp_all add: one_ereal_def)

lemma conv_radius_mono:
  assumes "eventually (λn. norm (f n) ≥ norm (g n)) sequentially"
  shows "conv_radius f ≤ conv_radius g"
  unfolding conv_radius_def
proof (rule ereal_inverse_antimono[OF _ Limsup_mono])
  have "limsup (λn. 0) ≤ limsup (λn. ereal (root n (norm (g n))))"

```

```

    by (rule Limsup_mono) (auto intro!: eventually_mono[OF eventually_gt_at_top[of
0]])
    thus "limsup ( $\lambda n. \text{ereal} (\text{root } n (\text{norm } (g \ n)))) \geq 0"$ 
    by (simp add: Limsup_const)
next
    show " $\forall_F x$  in sequentially.  $\text{ereal} (\text{root } x (\text{norm } (g \ x))) \leq \text{ereal} (\text{root }
x (\text{norm } (f \ x)))"$ 
    using assms eventually_gt_at_top[of 0] by eventually_elim auto
qed

```

```

lemma conv_radius_const [simp]:
  assumes "c  $\neq$  0"
  shows "conv_radius ( $\lambda_. c$ ) = 1"
proof (rule tendsto_imp_conv_radius_eq)
  show " $(\lambda n. \text{ereal} (\text{norm } c \text{ powr } (1 / \text{real } n))) \longrightarrow \text{ereal } 1"$ 
  by (rule tendsto_ereal) (use assms in real_asymp)
qed auto

```

```

lemma conv_radius_bigo_polynomial:
  assumes "f  $\in O(\lambda n. \text{of\_nat } n \wedge k)"$ 
  shows "conv_radius f  $\geq$  1"
proof -
  from assms obtain C where ev: "C > 0" "eventually ( $\lambda n. \text{norm } (f \ n) \leq
C * \text{real } n \wedge k$ ) at_top"
  by (elim landau_o.bigE) (auto simp: norm_power)
  have " $(\lambda x. (C * \text{real } x \wedge k) \text{ powr } (1 / \text{real } x)) \longrightarrow 1"$ 
  using ev(1) by real_asymp
  hence "conv_radius ( $\lambda n. C * \text{real } n \wedge k$ ) = inverse (ereal 1)" using ev(1)
  by (intro tendsto_imp_conv_radius_eq[OF _ refl] tendsto_ereal)
  (simp add: norm_mult norm_power abs_mult)
  moreover have "conv_radius ( $\lambda n. C * \text{real } n \wedge k$ )  $\leq$  conv_radius f"
  by (intro conv_radius_mono eventually_mono[OF ev(2)]) auto
  ultimately show ?thesis
  by (simp add: one_ereal_def)
qed

```

## 1.4 Limits

```

lemma oscillation_imp_not_tendsto:
  assumes "eventually ( $\lambda n. f (g \ n) \in A$ ) sequentially" "filterlim g F
sequentially"
  assumes "eventually ( $\lambda n. f (h \ n) \in B$ ) sequentially" "filterlim h F
sequentially"
  assumes "closed A" "closed B" "A  $\cap$  B = {}"
  shows " $\neg$ filterlim f (nhds c) F"
proof
  assume *: "filterlim f (nhds c) F"
  have "filterlim ( $\lambda n. f (g \ n)$ ) (nhds c) sequentially"
  using * assms(2) by (rule filterlim_compose)

```



```

with assms(1,5) have "c ∈ A"
  by (metis Lim_in_closed_set sequentially_bot)
have "filterlim (λn. f (h n)) (nhds c) sequentially"
  using * assms(4) by (rule filterlim_compose)
with assms(3,6) have "c ∈ B"
  by (metis Lim_in_closed_set sequentially_bot)
with <c ∈ A> and <A ∩ B = {}> show False
  by blast
qed

lemma oscillation_imp_not_convergent:
  assumes "frequently (λn. f n ∈ A) sequentially"
  assumes "frequently (λn. f n ∈ B) sequentially"
  assumes "closed A" "closed B" "A ∩ B = {}"
  shows "¬convergent f"
proof -
  obtain g :: "nat ⇒ nat" where g: "strict_mono g" "∧n. f (g n) ∈ A"
    using assms(1) infinite_enumerate
    unfolding cofinite_eq_sequentially [symmetric] INFM_iff_infinite
    by blast
  obtain h :: "nat ⇒ nat" where h: "strict_mono h" "∧n. f (h n) ∈ B"
    using assms(2) infinite_enumerate
    unfolding cofinite_eq_sequentially [symmetric] INFM_iff_infinite
    by blast
  have "¬f ⟶ L" for L
  proof (rule oscillation_imp_not_tendsto)
    show "∀F n in sequentially. f (g n) ∈ A" "∀F n in sequentially.
f (h n) ∈ B"
    using g h by auto
  qed (use g h assms in <auto intro: filterlim_subseq>)
  thus ?thesis
    unfolding convergent_def by blast
qed

lemma seq_bigo_1_iff:
  "g ∈ O(λ_. 1) ⟷ bounded (range g)"
proof
  assume "g ∈ O(λ_. 1)"
  then obtain C where "eventually (λn. norm (g n) ≤ C) at_top"
    by (elim landau_o.bigE) auto
  then obtain N where "∧n. n ≥ N ⟹ norm (g n) ≤ C"
    by (auto simp: eventually_at_top_linorder)
  hence "norm (g n) ≤ Max (insert C (norm ` g ` {...<N}))" for n
    by (cases "n < N") (auto simp: Max_ge_iff)
  thus "bounded (range g)"
    by (auto simp: bounded_iff)
next
  assume "bounded (range g)"
  then obtain C where "norm (g n) ≤ C" for n

```

```

    by (auto simp: bounded_iff)
  thus "g ∈ O(λ_. 1)"
    by (intro bigoI[where c = C]) auto
qed

```

```

lemma incseq_convergent':
  assumes "incseq (g :: nat ⇒ real)" "g ∈ O(λ_. 1)"
  shows "convergent g"
proof -
  from assms(2) have "bounded (range g)"
    by (simp add: seq_bigo_1_iff)
  then obtain C where C: "|g n| ≤ C" for n
    unfolding bounded_iff by auto
  show ?thesis
  proof (rule incseq_convergent)
    show "incseq g"
      by fact
    next
    have "g i ≤ C" for i :: nat
      using C[of i] by auto
    thus "∀i. g i ≤ C"
      by blast
  qed (auto simp: convergent_def)
qed

```

```

lemma decseq_convergent':
  assumes "decseq (g :: nat ⇒ real)" "g ∈ O(λ_. 1)"
  shows "convergent g"
  using incseq_convergent'[of "λn. -g n "] assms
  by (simp flip: convergent_minus_iff add: decseq_eq_incseq)

```

```

lemma filterlim_of_int_iff:
  fixes c :: "'a :: real_normed_algebra_1"
  assumes "F ≠ bot"
  shows "filterlim (λx. of_int (f x)) (nhds c) F ↔
    (∃c'. c = of_int c' ∧ eventually (λx. f x = c') F)"

```

```

proof
  assume "∃c'. c = of_int c' ∧ eventually (λx. f x = c') F"
  then obtain c' where c': "c = of_int c'" "eventually (λx. f x = c')
F"
    by blast
  from c'(2) have "eventually (λx. of_int (f x) = c) F"
    by eventually_elim (auto simp: c'(1))
  thus "filterlim (λx. of_int (f x)) (nhds c) F"
    by (rule tendsto_eventually)
next
  assume *: "filterlim (λx. of_int (f x)) (nhds c) F"
  show "(∃c'. c = of_int c' ∧ eventually (λx. f x = c') F)"
  proof (cases "c ∈ ℤ")

```

```

    case False
    hence "setdist {c}  $\mathbb{Z}$  > 0"
      by (subst setdist_gt_0_compact_closed) auto
    with * have "eventually ( $\lambda x. \text{dist} (\text{of\_int} (f x)) c < \text{setdist} \{c\} \mathbb{Z}$ ) F"
      unfolding tendsto_iff by blast
    then obtain x where "dist (of_int (f x)) c < setdist {c}  $\mathbb{Z}$ "
      using <F  $\neq$  bot> eventually_happens by blast
    moreover have "dist c (of_int (f x))  $\geq$  setdist {c}  $\mathbb{Z}$ "
      by (rule setdist_le_dist) auto
    ultimately show ?thesis
      by (simp add: dist_commute)
  next
  case True
  then obtain c' where c: "c = of_int c'"
    by (elim Ints_cases)
  have "eventually ( $\lambda x. \text{dist} (\text{of\_int} (f x)) c < 1$ ) F"
    using * unfolding tendsto_iff by auto
  hence "eventually ( $\lambda x. f x = c'$ ) F"
    by eventually_elim (auto simp: c dist_of_int)
  with c show ?thesis
    by auto
qed
qed

lemma filterlim_of_nat_iff:
  fixes c :: "'a :: real_normed_algebra_1"
  assumes "F  $\neq$  bot"
  shows "filterlim ( $\lambda x. \text{of\_nat} (f x)$ ) (nhds c) F  $\longleftrightarrow$ 
    ( $\exists c'. c = \text{of\_nat} c' \wedge \text{eventually} (\lambda x. f x = c') F$ )"
proof
  assume " $\exists c'. c = \text{of\_nat} c' \wedge \text{eventually} (\lambda x. f x = c') F$ "
  then obtain c' where c': "c = of_nat c'" "eventually ( $\lambda x. f x = c'$ ) F"
  F"
    by blast
  from c'(2) have "eventually ( $\lambda x. \text{of\_nat} (f x) = c$ ) F"
    by eventually_elim (auto simp: c'(1))
  thus "filterlim ( $\lambda x. \text{of\_nat} (f x)$ ) (nhds c) F"
    by (rule tendsto_eventually)
next
  assume *: "filterlim ( $\lambda x. \text{of\_nat} (f x)$ ) (nhds c) F"
  show " $(\exists c'. c = \text{of\_nat} c' \wedge \text{eventually} (\lambda x. f x = c') F)$ "
  proof (cases "c  $\in$   $\mathbb{N}$ ")
    case False
    hence "setdist {c}  $\mathbb{N}$  > 0"
      by (subst setdist_gt_0_compact_closed) auto
    with * have "eventually ( $\lambda x. \text{dist} (\text{of\_nat} (f x)) c < \text{setdist} \{c\} \mathbb{N}$ ) F"
      unfolding tendsto_iff by blast

```

```

then obtain x where "dist (of_nat (f x)) c < setdist {c} ℕ"
  using <F ≠ bot> eventually_happens by blast
moreover have "dist c (of_nat (f x)) ≥ setdist {c} ℕ"
  by (rule setdist_le_dist) auto
ultimately show ?thesis
  by (simp add: dist_commute)
next
case True
then obtain c' where c: "c = of_nat c'"
  by (elim Nats_cases)
have "eventually (λx. dist (of_nat (f x)) c < 1) F"
  using * unfolding tendsto_iff by auto
hence "eventually (λx. f x = c') F"
  by eventually_elim (auto simp: c dist_of_nat)
with c show ?thesis
  by auto
qed
qed

lemma uniform_limit_compose:
  assumes "uniform_limit B (λx y. f x y) (λy. f' y) F" "∧y. y ∈ A ⇒
g y ∈ B"
  shows "uniform_limit A (λx y. f x (g y)) (λy. f' (g y)) F"
proof -
  have "uniform_limit (g ` A) (λx y. f x y) (λy. f' y) F"
    using assms(1) by (rule uniform_limit_on_subset) (use assms(2) in
blast)
  thus "uniform_limit A (λx y. f x (g y)) (λy. f' (g y)) F"
    unfolding uniform_limit_iff by auto
qed

lemma uniform_limit_const':
  assumes "filterlim f (nhds c) F"
  shows "uniform_limit A (λx y. f x) (λy. c) F"
proof -
  have "∀F n in F. ∀x∈A. dist (f n) c < ε" if ε: "ε > 0" for ε :: real
  proof -
    from assms and ε have "∀F n in F. dist (f n) c < ε"
      unfolding tendsto_iff by blast
    thus ?thesis
      by eventually_elim auto
  qed
  thus ?thesis
    unfolding uniform_limit_iff by blast
qed

lemma uniform_limit_singleton_iff [simp]:
  "uniform_limit {x} f g F ↔ filterlim (λy. f y x) (nhds (g x)) F"
  by (simp add: uniform_limit_iff tendsto_iff)

```

end

## 2 Some Facts About Number-Theoretic Functions

theory *Number\_Theoretic\_Functions\_Extras*

imports

"Dirichlet\_Series.Dirichlet\_Series\_Analysis"

"Dirichlet\_Series.Divisor\_Count"

Lambert\_Series\_Library

begin

lemma (in *nat\_power\_field*) *nat\_power\_minus*:

" $a \neq 0 \vee n \neq 0 \implies \text{nat\_power } n \ (-a) = \text{inverse } (\text{nat\_power } n \ a)$ "

using *nat\_power\_diff*[of *n 0 a*] by (cases "*n = 0*") (*simp\_all add: field\_simps*)

lemma *divisor\_sigma\_minus*:

fixes *a* :: "'a :: {nat\_power\_field, field\_char\_0}"

shows "*divisor\_sigma* (-*a*) *n* = *divisor\_sigma* *a* *n* / *nat\_power* *n* *a*"

proof (cases "*n = 0*")

case *n*: *False*

have "*divisor\_sigma* (-*a* :: 'a) *n* = ( $\sum d \mid d \text{ dvd } n. \text{nat\_power } d \ (-a)$ )"

by (*simp add: divisor\_sigma\_def*)

also have "... = ( $\sum d \mid d \text{ dvd } n. \text{nat\_power } d \ a / \text{nat\_power } n \ a$ )"

using *n* by (intro *sum.reindex\_bij\_witness*[of \_ " $\lambda d. n \text{ div } d$ " " $\lambda d. n \text{ div } d$ "])

(*auto elim!: dvdE simp: field\_simps nat\_power\_minus nat\_power\_mult\_distrib*)

also have "... = *divisor\_sigma* *a* *n* / *nat\_power* *n* *a*"

by (*simp add: sum\_divide\_distrib divisor\_sigma\_def*)

finally show ?thesis .

qed auto

lemma *norm\_moebius\_mu*:

"*norm* (*moebius\_mu* *n* :: 'a :: {real\_normed\_algebra\_1, comm\_ring\_1}) =  
*ind\_squarefree* *n*"

by (*subst of\_int\_moebius\_mu [symmetric]*, *subst norm\_of\_int*) (*auto simp: abs\_moebius\_mu*)

lemma *conv\_radius\_nat\_power*: "*conv\_radius* ( $\lambda n. \text{nat\_power } n \ a$  :: 'a ::  
{*nat\_power\_normed\_field*, *banach*}) = 1"

proof (rule *tendsto\_imp\_conv\_radius\_eq*)

show " $(\lambda n. \text{ereal } (\text{norm } (\text{nat\_power } n \ a) \text{ powr } (1 / \text{real } n))) \longrightarrow \text{ereal } 1$ "

proof (rule *Lim\_transform\_eventually*)

show " $(\lambda n. \text{ereal } ((\text{real } n \text{ powr } (a \cdot 1)) \text{ powr } (1 / \text{real } n))) \longrightarrow \text{ereal } 1$ "

by (rule *tendsto\_ereal*) *real\_asymp*

```

    show "eventually (λn. (real n powr (a · 1)) powr (1 / real n) =
      ereal (norm (nat_power n a :: 'a) powr (1 / real n))) at_top"
      using eventually_gt_at_top[of 0] by eventually_elim (simp add: norm_nat_power)
  qed
qed (simp_all add: one_ereal_def)

```

```

lemma not_convergent_liouville_lambda:
  "¬convergent (liouville_lambda :: nat ⇒ 'a :: {real_normed_algebra,
comm_ring_1, semiring_char_0})"
proof -
  have "¬(liouville_lambda :: nat ⇒ 'a) → c" for c
  proof (rule oscillation_imp_not_tendsto)
    show "eventually (λn. liouville_lambda (2 ^ (2 * n)) ∈ {1 :: 'a})
sequentially"
      by auto
    show "filterlim (λn. 2 ^ (2 * n) :: nat) at_top sequentially"
      by real_asymp
    show "eventually (λn. liouville_lambda (2 ^ (2 * n + 1)) ∈ {-1 ::
'a}) sequentially"
      by (subst liouville_lambda.power) auto
    show "filterlim (λn. 2 ^ (2 * n + 1) :: nat) at_top sequentially"
      by real_asymp
  qed auto
  thus ?thesis
    by (auto simp: convergent_def)
qed

```

```

lemma conv_radius_liouville_lambda:
  "conv_radius (liouville_lambda :: nat ⇒ 'a :: {real_normed_field, banach})
= 1"
proof -
  have "¬summable (liouville_lambda :: nat ⇒ 'a)"
    using not_convergent_liouville_lambda[where ?'a = 'a] summable_LIMSEQ_zero
    by (auto simp: convergent_def)
  hence "conv_radius (liouville_lambda :: nat ⇒ 'a) ≤ norm (1 :: 'a)"
    by (intro conv_radius_leI') auto
  moreover have "conv_radius (liouville_lambda :: nat ⇒ 'a) ≥ 1"
  proof (rule conv_radius_bigo_polynomial)
    show "(liouville_lambda :: nat ⇒ 'a) ∈ O(λn. of_nat n ^ 0)"
      by (intro bigoI[of _ 1] eventually_mono[OF eventually_gt_at_top[of
0]])
      (auto simp: liouville_lambda_def norm_power)
  qed
  ultimately show ?thesis
    by (intro antisym) (auto simp: one_ereal_def)
qed

```

```

lemma not_convergent_mangoldt: "¬convergent (mangoldt :: nat ⇒ 'a ::
{real_normed_algebra_1})"

```

```

proof -
  have *: "¬primepow (6 * n :: nat)" for n
    by (rule not_primepowI[of 2 3]) auto
  have "¬(mangoldt :: nat ⇒ 'a) ⟶ c" for c
  proof (rule oscillation_imp_not_tendsto)
    show "eventually (λn. mangoldt (2 ^ n) ∈ {of_real (ln 2) :: 'a})
  sequentially"
    by (auto simp: mangoldt_primepow)
    show "filterlim (λn. 2 ^ n :: nat) at_top sequentially"
    by real_asymp
    show "eventually (λn. mangoldt (6 * n) ∈ {0 :: 'a}) sequentially"
    using * by (auto simp: mangoldt_def)
    show "filterlim (λn. 6 * n :: nat) at_top sequentially"
    by real_asymp
  qed auto
  thus ?thesis
    by (auto simp: convergent_def)
qed

lemma conv_radius_mangoldt:
  "conv_radius (mangoldt :: nat ⇒ 'a :: {real_normed_field, banach})
= 1"
proof -
  have "¬summable (mangoldt :: nat ⇒ 'a)"
    using not_convergent_mangoldt[where ?'a = 'a] summable_LIMSEQ_zero
    by (auto simp: convergent_def)
  hence "conv_radius (mangoldt :: nat ⇒ 'a) ≤ norm (1 :: 'a)"
    by (intro conv_radius_leI') auto
  moreover have "conv_radius (mangoldt :: nat ⇒ 'a) ≥ 1"
  proof (rule conv_radius_bigo_polynomial)
    have "(mangoldt :: nat ⇒ 'a) ∈ O(λn. of_real (ln (real n)))"
    by (intro bigoI[of _ 1] eventually_mono[OF eventually_gt_at_top[of
0]])
    (auto simp: mangoldt_le)
    also have "(λn. of_real (ln (real n))) ∈ O(λn. of_nat n :: 'a)"
    by (subst landau_o.big.norm_iff [symmetric], unfold norm_of_real
norm_of_nat) real_asymp
    finally show "(mangoldt :: nat ⇒ 'a) ∈ O(λn. of_nat n ^ 1)"
    by simp
  qed
  ultimately show ?thesis
    by (intro antisym) (auto simp: one_ereal_def)
qed

lemma not_convergent_moebius_mu: "¬convergent (moebius_mu :: nat ⇒ 'a
:: real_normed_field)"
proof (rule oscillation_imp_not_convergent)
  have "infinite {p. prime (p :: nat)}"
  by (rule primes_infinite)

```

```

hence "frequently (prime :: nat ⇒ bool) cofinite"
  by (simp add: Inf_many_def)
hence "frequently (λn. moebius_mu n = (-1 :: 'a)) cofinite"
  by (rule frequently_elim1) (simp add: moebius_mu.prime)
thus "frequently (λn. moebius_mu n ∈ {(-1 :: 'a)}) sequentially"
  using cofinite_eq_sequentially by fastforce
next
have "infinite (range (λn. 4 * n :: nat))"
  by (subst finite_image_iff) (auto simp: inj_on_def)
moreover {
  have "¬squarefree (2 ^ 2 :: nat)"
    by (subst squarefree_power_iff) auto
  also have "2 ^ 2 = (4 :: nat)"
    by simp
  finally have "range (λn. 4 * n :: nat) ⊆ {n::nat. ¬squarefree n}"
    by (auto dest: squarefree_multD)
}
ultimately have "frequently (λn::nat. ¬squarefree n) cofinite"
  unfolding INFM_iff_infinite using finite_subset by blast
thus "∃F n in sequentially. moebius_mu n ∈ {0}"
  unfolding cofinite_eq_sequentially by (rule frequently_elim1) auto
qed auto

lemma conv_radius_moebius_mu:
  "conv_radius (moebius_mu :: nat ⇒ 'a :: {real_normed_field, banach})
  = 1"
proof -
  have "¬summable (moebius_mu :: nat ⇒ 'a)"
    using not_convergent_moebius_mu[where ?'a = 'a] summable_LIMSEQ_zero
    by (auto simp: convergent_def)
  hence "conv_radius (moebius_mu :: nat ⇒ 'a) ≤ norm (1 :: 'a)"
    by (intro conv_radius_leI') auto
  moreover have "conv_radius (moebius_mu :: nat ⇒ 'a) ≥ conv_radius
  (λ_. 1 :: 'a)"
    by (intro conv_radius_mono always_eventually) (auto simp: norm_moebius_mu
ind_def)
  ultimately show ?thesis
    by (intro antisym) (auto simp: one_ereal_def)
qed

lemma not_convergent_totient:
  "¬convergent (λn. of_nat (totient n) :: 'a :: {real_normed_field, banach})"
proof
  assume "convergent (λn. of_nat (totient n) :: 'a)"
  then obtain L where L: "eventually (λn. totient n = L) at_top"
    by (auto simp: convergent_def filterlim_of_nat_iff)
  then obtain N where N: "∧n. n ≥ N ⇒ totient n = L"
    unfolding eventually_at_top_linorder by blast
  obtain p q where "prime p" "p > N" "prime q" "q > p"

```



```

    using bigger_prime by blast
  with N[of p] N[of q] show False
  by (auto simp: totient_prime)
qed

lemma conv_radius_totient:
  "conv_radius (λn. of_nat (totient n) :: 'a :: {real_normed_field, banach})
  = 1"
proof -
  have "¬summable (λn. of_nat (totient n) :: 'a)"
    using not_convergent_totient[where ?'a = 'a] summable_LIMSEQ_zero
    by (auto simp: convergent_def)
  hence "conv_radius (λn. of_nat (totient n) :: 'a) ≤ norm (1 :: 'a)"
    by (intro conv_radius_leI') auto
  moreover have "conv_radius (λn. of_nat (totient n) :: 'a) ≥ 1"
  proof (rule conv_radius_bigo_polynomial)
    show "(λn. of_nat (totient n)) ∈ O(λn. of_nat n ^ 1)"
      by (intro bigoI[of _ 1] always_eventually) (auto simp: totient_le)
  qed
  ultimately show ?thesis
    by (intro antisym) (auto simp: one_ereal_def)
qed

end

```

### 3 Some Abel-Style Summation Tests

```

theory Summation_Tests_More
  imports "HOL-Analysis.Analysis" "HOL-Library.Landau_Symbols" Lambert_Series_Library
begin

```

The following five summation tests are taken from Chapter 10 of Knopp's textbook [?]. He introduces a strong variant of Abel's summation test and then deduces from it four summation tests named after Abel, Dirichlet, du Bois-Reymond, and Dedekind.

```

lemma abel_partial_summation:
  fixes f g :: "nat ⇒ 'a :: comm_ring_1"
  defines "F ≡ (λn. ∑ k≤n. f k)"
  shows "(∑ r=n+1..n+k. f r * g r) =
    (∑ r=n+1..n+k. F r * (g r - g (Suc r))) -
    F n * g (Suc n) + F (n + k) * g (n + k + 1)"
  by (induction k) (auto simp: F_def algebra_simps)

```

```

theorem abel_summation_test_strong:
  fixes f g :: "nat ⇒ 'a :: {real_normed_field, banach}"
  defines "F ≡ (λn. ∑ k≤n. f k)"
  assumes "summable (λr. F r * (g r - g (Suc r)))"
  assumes "convergent (λr. F r * g (Suc r))"

```

```

shows "summable ( $\lambda r. f r * g r$ )"
unfolding summable_iff_convergent'
proof (rule Cauchy_convergent)
show "Cauchy ( $\lambda n. \sum_{r \leq n}. f r * g r$ )"
  unfolding Cauchy_def
proof safe
  fix  $\varepsilon :: \text{real}$  assume  $\varepsilon: " \varepsilon > 0 "$ 
  let ?A = " $\lambda n. (\sum_{r \leq n}. F r * (g r - g (Suc r)))$ "
  from assms(2) have "Cauchy ( $\lambda n. \sum_{r \leq n}. F r * (g r - g (Suc r))$ )"
    unfolding summable_iff_convergent' using convergent_Cauchy by blast
  moreover have " $\varepsilon / 2 > 0$ "
    using  $\varepsilon$  by auto
  ultimately obtain M1 where M1: " $\text{dist } (?A m) (?A n) < \varepsilon / 2$ " if " $m \geq M1$ " " $n \geq M1$ " for  $m n$ 
    unfolding Cauchy_def by fast
  have M1': " $\text{norm } (\sum_{r=m..n}. F r * (g r - g (Suc r))) < \varepsilon / 2$ " if " $M1 < m$ " " $m \leq Suc n$ " for  $m n$ 
    proof -
      have " $\text{dist } (?A n) (?A (m - 1)) < \varepsilon / 2$ "
        by (rule M1) (use that in auto)
      also have " $\text{dist } (?A n) (?A (m - 1)) = \text{norm } (\sum_{r \in \{..n\} - \{..m - 1\}}. F r * (g r - g (Suc r)))$ "
        unfolding dist_norm using that by (subst Groups_Big.sum_diff)
      auto
      also have " $\{..n\} - \{..m - 1\} = \{m..n\}$ "
        using that by auto
      finally show ?thesis .
    qed

  from  $\varepsilon$  have " $\varepsilon / 4 > 0$ "
    by auto
  from assms(3) obtain c where " $(\lambda r. F r * g (Suc r)) \longrightarrow c$ "
    by (auto simp: convergent_def)
  with " $< \varepsilon / 4 > 0$ " have "eventually ( $\lambda r. \text{dist } (F r * g (Suc r)) c < \varepsilon / 4$ ) sequentially"
    using tendstoD by blast
  then obtain M2 where M2: " $\text{dist } (F r * g (Suc r)) c < \varepsilon / 4$ " if " $r \geq M2$ " for  $r$ 
    unfolding eventually_at_top_linorder by blast

  show " $\exists M. \forall m \geq M. \forall n \geq M. \text{dist } (\sum_{r \leq m}. f r * g r) (\sum_{r \leq n}. f r * g r) < \varepsilon$ "
    proof (rule exI[of _ "max M1 M2"], safe)
      fix  $m n :: \text{nat}$  assume " $m \geq \text{max } M1 M2$ " " $n \geq \text{max } M1 M2$ "
      thus " $\text{dist } (\sum_{r \leq m}. f r * g r) (\sum_{r \leq n}. f r * g r) < \varepsilon$ "
        proof (induction  $m n$  rule: linorder_wlog)
          case (le  $m n$ )
          define  $k$  where " $k = n - m$ "
          from le have  $n\_eq: "n = m + k"$ 

```

```

    by (auto simp: k_def)
  have "dist  $(\sum_{r \leq m}. f r * g r)$   $(\sum_{r \leq n}. f r * g r) =$ 
    norm  $(\sum_{r \leq n}. f r * g r) - (\sum_{r \leq m}. f r * g r)$ "
    by (simp add: dist_norm norm_minus_commute)
  also have " $(\sum_{r \leq n}. f r * g r) - (\sum_{r \leq m}. f r * g r) = (\sum_{r \in \{..n\} - \{..m\}}.$ 
 $f r * g r)$ "
    using le by (subst Groups_Big.sum_diff) auto
  also have " $\{..n\} - \{..m\} = \{m+1..m+k\}$ "
    by (auto simp: n_eq)
  also have " $(\sum_{r=m+1..m+k}. f r * g r) =$ 
     $(\sum_{r=m+1..m+k}. F r * (g r - g (Suc r))) -$ 
     $F m * g (Suc m) + F (m + k) * g (Suc (m + k))$ "
    unfolding F_def by (subst abel_partial_summation) simp_all
  also have "norm ...  $\leq$  norm  $(\sum_{r=m+1..m+k}. F r * (g r - g (Suc$ 
 $r))) +$ 
    dist  $(F m * g (Suc m)) c +$  dist  $(F (m + k) * g (Suc$ 
 $(m + k))) c$ "
    by norm
  also have "...  $< \varepsilon / 2 + \varepsilon / 4 + \varepsilon / 4$ "
    using le by (intro add_strict_mono M1' M2) auto
  also have "... =  $\varepsilon$ "
    by simp
  finally show "dist  $(\sum_{r \leq m}. f r * g r)$   $(\sum_{r \leq n}. f r * g r) < \varepsilon$ "
.
qed (simp add: dist_commute max_commute)
qed
qed
qed

```

```

corollary abel_summation_test:
  fixes f g :: "nat  $\Rightarrow$  real"
  assumes "summable f"
  assumes "incseq g" "g  $\in O(\lambda_. 1)$ "
  shows "summable  $(\lambda r. f r * g r)$ "
proof (rule abel_summation_test_strong)
  have "convergent g"
    by (rule incseq_convergent') fact+
  thus "convergent  $(\lambda n. (\sum_{k \leq n}. f k) * g (Suc n))$ " using assms(1)
    by (intro convergent_mult) (simp_all add: convergent_Suc_iff summable_iff_convergent')
  show "summable  $(\lambda n. (\sum_{k \leq n}. f k) * (g n - g (Suc n)))$ "
  proof (subst mult_commute, rule summable_norm_cancel, rule norm_summable_mult_bounded)
    have "summable  $(\lambda n. g (Suc n) - g n)$ "
      using <convergent g> by (simp add: telescope_summable_iff)
    also have " $(\lambda n. g (Suc n) - g n) = (\lambda n. norm (g n - g (Suc n)))$ "
      using <incseq g> by (auto simp: incseq_def fun_eq_iff)
    finally show "summable  $(\lambda n. norm (g n - g (Suc n)))$ " .
  next
    have "bounded  $(\text{range } (\lambda n. \sum_{k < n}. f k))$ "
      by (rule summable_imp_sums_bounded) fact
  end

```

```

    hence "(λn. ∑ k<n. f k) ∈ O(λ_. 1)"
      by (simp add: seq_bigo_1_iff)
    hence "(λn. ∑ k<Suc n. f k) ∈ O(λ_. 1)"
      by (rule landau_o.big.compose) (rule filterlim_Suc)
    also have "(λn. {..<Suc n}) = (λn. {..n})"
      by auto
    finally show "(λn. ∑ k≤n. f k) ∈ O(λ_. 1)" .
  qed
qed

corollary dirichlet_summation_test:
  fixes f g :: "nat ⇒ real"
  assumes "(λn. ∑ r≤n. f r) ∈ O(λ_. 1)"
  assumes "decseq g" "g ∈ o(λ_. 1)"
  shows "summable (λr. f r * g r)"
proof (rule abel_summation_test_strong)
  have "(λx. g (Suc x)) ∈ o(λx. 1)"
    using assms(3) by (rule landau_o.small.compose) (rule filterlim_Suc)
  have "(λn. (∑ r≤n. f r) * g (Suc n)) ∈ o(λ_. 1 * 1)"
    by (rule landau_o.big_small_mult) fact+
  thus "convergent (λr. sum f {..r} * g (Suc r))"
    by (auto dest!: smalloD_tendsto simp: convergent_def)
next
  have "g ⟶ 0"
    using assms(3) by (auto dest!: smalloD_tendsto simp: convergent_def)
  hence "convergent g"
    by (auto simp: convergent_def)
  show "summable (λn. (∑ k≤n. f k) * (g n - g (Suc n)))"
  proof (subst mult.commute, rule summable_norm_cancel, rule norm_summable_mult_bounded)
    have "summable (λn. g n - g (Suc n))"
      using <convergent g> by (simp add: telescope_summable_iff')
    also have "(λn. g n - g (Suc n)) = (λn. norm (g n - g (Suc n)))"
      using <decseq g> by (auto simp: decseq_Suc_iff fun_eq_iff)
    finally show "summable (λn. norm (g n - g (Suc n)))" .
  qed fact
qed

```

```

corollary dubois_reymond_summation_test:
  fixes f g :: "nat ⇒ 'a :: {real_normed_field, banach}"
  assumes "summable f"
  assumes "summable (λr. norm (g r - g (Suc r)))"
  shows "summable (λr. f r * g r)"
proof (rule abel_summation_test_strong)
  have "summable (λr. g r - g (Suc r))"
    using assms(2) by (rule summable_norm_cancel)
  hence "convergent g"
    by (subst (asm) telescope_summable_iff')
  show "convergent (λr. sum f {..r} * g (Suc r))"
    using assms(1) <convergent g>

```

```

    by (intro convergent_mult) (auto simp: convergent_Suc_iff summable_iff_convergent')

show "summable (λn. (∑ k≤n. f k) * (g n - g (Suc n)))"
proof (subst mult.commute, rule summable_norm_cancel, rule norm_summable_mult_bounded)
  have "bounded (range (λn. ∑ k<n. f k))"
    by (rule summable_imp_sums_bounded) fact
  hence "(λn. ∑ k<n. f k) ∈ O(λ_. 1)"
    by (simp add: seq_bigo_1_iff)
  hence "(λn. ∑ k<Suc n. f k) ∈ O(λ_. 1)"
    by (rule landau_o.big.compose) (rule filterlim_Suc)
  also have "(λn. {..<Suc n}) = (λn. {..n})"
    by auto
  finally show "(λn. ∑ k≤n. f k) ∈ O(λ_. 1)" .
qed fact
qed

corollary dedekind_summation_test:
  fixes f g :: "nat ⇒ 'a :: {real_normed_field, banach}"
  assumes "(λn. ∑ k≤n. f k) ∈ O(λ_. 1)"
  assumes "summable (λr. norm (g r - g (Suc r)))"
  assumes "g ∈ o(λ_. 1)"
  shows "summable (λr. f r * g r)"
proof (rule abel_summation_test_strong)
  have "(λx. g (Suc x)) ∈ o(λx. 1)"
    using assms(3) by (rule landau_o.small.compose) (rule filterlim_Suc)
  have "(λn. (∑ r≤n. f r) * g (Suc n)) ∈ o(λ_. 1 * 1)"
    by (rule landau_o.big_small_mult) fact+
  thus "convergent (λr. sum f {..r} * g (Suc r))"
    by (auto dest!: smalloD_tendsto simp: convergent_def)
  show "summable (λn. (∑ k≤n. f k) * (g n - g (Suc n)))"
    by (subst mult.commute, rule summable_norm_cancel, rule norm_summable_mult_bounded)
fact+
qed

end

```

## 4 Lambert Series

```

theory Lambert_Series
  imports
    "HOL-Complex_Analysis.Complex_Analysis"
    "HOL-Real_Asymp.Real_Asymp"
    "Dirichlet_Series.Dirichlet_Series_Analysis"
    "Dirichlet_Series.Divisor_Count"
    Polylog.Polylog
    Lambert_Series_Library
    Number_Theoretic_Functions_Extras
    Summation_Tests_More
begin

```

## 4.1 Definition

Given any sequence  $a(n)$  for  $n \geq 1$ , the corresponding *Lambert series* is defined as

$$L(a, q) = \sum_{n=1}^{\infty} a(n) \frac{q^n}{1 - q^n} .$$

**definition** `lambert` :: "(nat  $\Rightarrow$  'a :: {real\_normed\_field, banach})  $\Rightarrow$  'a  $\Rightarrow$  'a" where

```
"lambert a q =
  (let f = ( $\lambda$ n. a (Suc n) * q ^ (Suc n) / (1 - q ^ (Suc n))) in
   if summable f then  $\sum$ n. f n else 0)"
```

**lemma** `lambert_eqI`:

```
assumes "( $\lambda$ n. a (Suc n) * q ^ (Suc n) / (1 - q ^ (Suc n))) sums x"
shows "lambert a q = x"
using assms unfolding lambert_def Let_def sums_iff by simp
```

**lemma** `lambert_cong` [`cong`]:

```
"( $\bigwedge$ n. n > 0  $\implies$  a n = a' n)  $\implies$  q = q'  $\implies$  lambert a q = lambert a' q'"
by (simp add: lambert_def)
```

**lemma** `lambert_0` [`simp`]: "lambert a 0 = 0"

```
by (simp add: lambert_def)
```

**lemma** `lambert_0'` [`simp`]: "lambert ( $\lambda$ \_. 0) q = 0"

```
by (simp add: lambert_def)
```

**lemma** `lambert_cmult`: "lambert ( $\lambda$ n. c \* a n) q = c \* lambert a q"

**proof** (cases "c = 0")

```
case False
```

```
define f where "f = ( $\lambda$ n. a (Suc n) * q ^ (Suc n) / (1 - q ^ (Suc n)))"
show ?thesis
```

```
proof (cases "summable f")
```

```
case True
```

```
hence "( $\lambda$ n. c * (a (Suc n) * q ^ (Suc n) / (1 - q ^ (Suc n)))) sums
(c * ( $\sum$ n. f n))"
```

```
unfolding mult.assoc by (intro sums_mult) (auto simp: f_def)
```

```
thus ?thesis using True
```

```
by (intro lambert_eqI) (auto simp: lambert_def f_def algebra_simps)
```

```
next
```

```
case False
```

```
hence " $\neg$ summable ( $\lambda$ n. c * f n)"
```

```
using <c  $\neq$  0> by simp
```

```
with False show ?thesis
```

```
by (simp add: lambert_def f_def algebra_simps)
```

```
qed
```

```
qed auto
```

```
lemma lambert_cmult': "lambert (λn. a n * c) q = lambert a q * c"
  using lambert_cmult[of c a q] by (simp add: mult_ac)
```

```
lemma lambert_uminus: "lambert (λn. -a n) q = -lambert a q"
  using lambert_cmult[of "-1" a q] by simp
```

We will later see that if  $\sum_{n=1}^{\infty} a(n)$  exists then the Lambert series converges everywhere except on the unit circle; otherwise it has the same convergence radius as  $a$  (and that radius then has to be  $< 1$ ).

```
definition lambert_conv_radius :: "(nat ⇒ 'a :: {banach, real_normed_field})
⇒ ereal"
  where "lambert_conv_radius a = (if summable a then ∞ else conv_radius
a)"
```

```
lemma lambert_conv_radius_gt_1_iff: "lambert_conv_radius a > 1 ⟷ summable
a"
```

**proof**

```
  assume *: "lambert_conv_radius a > 1"
  {
    assume "¬summable a"
    hence "conv_radius a > 1"
      using * by (auto simp: lambert_conv_radius_def)
    hence "summable (λn. a n * 1 ^ n)"
      by (intro summable_in_conv_radius) (auto simp: one_ereal_def)
    with <¬summable a> have False
      by simp
  }
  thus "summable a"
    by blast
```

```
qed (auto simp: lambert_conv_radius_def)
```

## 4.2 Uniform convergence, continuity, holomorphicity

We will now show some (uniform) convergence results for  $L(a, q)$ , which will then give us the holomorphicity and continuity of  $L(a, q)$ . We will also show some absolute summability results.

**context**

```
  fixes a :: "nat ⇒ 'a :: {real_normed_field, banach}"
  fixes f :: "nat ⇒ 'a ⇒ 'a" and A :: "'a"
  defines "f ≡ λk q. a k * q ^ k / (1 - q ^ k)"
  defines "A ≡ (∑ n. a (Suc n))"
```

**begin**

Let  $a(n)$  have convergence radius  $r$ . In discs of radius  $\min(1, r)$ , the Lambert series for  $a(n)$  converges uniformly. This is a simple application of Weierstraß's  $M$  test.

```
lemma uniform_limit_lambert1_aux:
```

```

fixes r :: real
assumes "0 < r" "r < min 1 (conv_radius a)"
shows "uniform_limit (ball 0 r) (λn q. (∑ k<n. f (Suc k) q)) (λq.
∑ k. f (Suc k) q) sequentially"
proof -
  from assms have r: "r > 0" "r < 1" "r < conv_radius a"
  by auto
  show "uniform_limit (ball 0 r) (λn q. (∑ k<n. f (Suc k) q)) (λq. ∑ k.
f (Suc k) q) sequentially"
  proof (rule Weierstrass_m_test_ev)
    have "eventually (λk. 1 - r ^ k ≥ 1 / 2) at_top"
    using r by real_asymp
    hence "eventually (λk. ∀q∈ball 0 r. norm (f k q) ≤ 2 * norm (a k)
* r ^ k) at_top"
    using eventually_gt_at_top[of 0]
  proof eventually_elim
    case k: (elim k)
    show "∀q∈ball 0 r. norm (f k q) ≤ 2 * norm (a k) * r ^ k"
    proof
      fix q :: 'a assume q: "q ∈ ball 0 r"
      have "norm (f k q) = norm (a k) * norm q ^ k / norm (1 - q ^ k)"
      by (simp add: f_def norm_mult norm_divide norm_power)
      also {
        have "1 / 2 ≤ 1 - r ^ k"
        using k by simp
        also have "... ≤ norm (1 :: 'a) - norm (q ^ k)"
        using q by (auto simp: norm_power intro!: power_mono)
        also have "... ≤ norm (1 - q ^ k)"
        by norm
        finally have "norm (1 - q ^ k) ≥ 1 / 2" .
      }
      hence "norm (a k) * norm q ^ k / norm (1 - q ^ k) ≤
norm (a k) * r ^ k / (1 / 2)"
      using q r k
      by (intro mult_mono power_mono frac_le)
      (auto intro!: mult_pos_pos simp: power_less_1_iff norm_power
dest!: power_eq_1_iff)
      finally show "norm (f k q) ≤ 2 * norm (a k) * r ^ k"
      by simp
    qed
  qed
  thus "∀F k in sequentially. ∀q∈ball 0 r. norm (f (Suc k) q) ≤ 2
* norm (a (Suc k)) * r ^ Suc k"
  by (rule eventually_compose_filterlim[OF _ filterlim_Suc])
next
  have "summable (λk. 2 * (norm (a (Suc k)) * of_real r ^ Suc k))"
  by (subst summable_Suc_iff, intro summable_mult abs_summable_in_conv_radius)
(use r in auto)
  thus "summable (λk. 2 * norm (a (Suc k)) * r ^ Suc k)"

```



```

    using <r > 0> by (simp add: norm_mult norm_power mult_ac)
  qed
qed

lemma uniform_limit_lambert1:
  fixes r :: real
  assumes "0 < r" "r < min 1 (conv_radius a)"
  shows "uniform_limit (ball 0 r) (λn q. (∑ k<n. f (Suc k) q)) (lambert
a) sequentially"
proof -
  have lim: "uniform_limit (ball 0 r) (λn q. (∑ k<n. f (Suc k) q)) (λq.
∑ k. f (Suc k) q) sequentially"
    using assms by (rule uniform_limit_lambert1_aux)
  also have "?thesis ↔ ?thesis"
  proof (intro uniform_limit_cong ballI allI refl always_eventually)
    fix q :: 'a assume q: "q ∈ ball 0 r"
    have *: "(λn. a (Suc n) * q ^ Suc n / (1 - q ^ Suc n)) sums (∑ k.
f (Suc k) q)"
      using tendsto_uniform_limitI[OF lim q] unfolding f_def by (simp
add: sums_def)
    show "(∑ k. f (Suc k) q) = lambert a q"
      by (rule sym, rule lambert_eqI) (fact *)
  qed
  finally show ?thesis .
qed

```

Since  $a_n \frac{q^n}{1-q^n} = -a_n - a_n \frac{(\frac{1}{q})^n}{1-(\frac{1}{q})^n}$ , we can substitute  $q \mapsto \frac{1}{q}$  in the above uniform convergence result to deduce that uniform convergence also holds on any annulus  $r \leq |q| \leq R$  with  $1 < r < R$ .

```

lemma uniform_limit_lambert2:
  fixes r R :: real
  assumes r: "1 < r" "r < R"
  assumes "summable a"
  defines "D ≡ cball 0 R - ball 0 r"
  shows "uniform_limit D (λn q. (∑ k<n. f (Suc k) q)) (λq. -A - lambert
a (1 / q)) sequentially"
proof -
  define g where "g = (λn q. a n * (1 / q) ^ n / (1 - (1 / q) ^ n))"
  from r(1) obtain r' where r': "1 < r'" "r' < r"
  using dense by blast
  have "uniform_limit D (λn q. ∑ k<n. f (Suc k) (1 / q)) (λq. lambert
a (1 / q)) sequentially"
    using uniform_limit_lambert1
  proof (rule uniform_limit_compose[where g = "λq. 1 / q"])
    have "conv_radius a ≥ norm (of_real 1 :: 'a)"
      by (rule conv_radius_geI) (use <summable a> in auto)
    hence "min 1 (conv_radius a) = 1"
      by (simp add: min_def one_ereal_def)
  qed

```

```

with r' show "1 / r' > 0" "ereal (1 / r') < min 1 (conv_radius a)"
  by auto
next
show "1 / q ∈ ball 0 (1 / r'" if "q ∈ D" for q
  using that r r' by (auto simp: D_def norm_divide divide_simps not_less)
qed
moreover have "summable (λn. a (Suc n))"
  using <summable a> by (simp add: summable_Suc_iff)
hence "(λn. ∑ k<n. a (Suc k)) → A"
  unfolding A_def summable_sums_iff sums_def .
ultimately have "uniform_limit D (λn q. -(∑ k<n. a (Suc k)) - (∑ k<n.
f (Suc k) (1/q)))
  (λq. -A - lambert a (1 / q)) sequentially"
  by (intro uniform_limit_intros uniform_limit_const')
also have "?this ↔ ?thesis"
proof (intro uniform_limit_cong refl always_eventually allI ballI)
  fix n :: nat and q :: 'a
  assume q: "q ∈ D"
  hence q': "q ≠ 0"
    using r by (auto simp: D_def)
  have q'': "q ^ k ≠ 1" if "k > 0" for k
    using q r that by (auto dest!: power_eq_1_iff simp: D_def)
  have "-(∑ k<n. a (Suc k)) - (∑ k<n. f (Suc k) (1 / q)) =
    (∑ k<n. -a (Suc k) - f (Suc k) (1 / q))"
    by (simp add: sum_negf sum_subtractf)
  also have *: "-a k - f k (1 / q) = f k q" if "k > 0" for k
    using that q' q''[of k] by (auto simp: f_def field_simps)
  have "(∑ k<n. -a (Suc k) - f (Suc k) (1 / q)) = (∑ k<n. f (Suc k)
q)"
    by (intro sum.cong refl *) auto
  finally show "-(∑ k<n. a (Suc k)) - (∑ k<n. f (Suc k) (1 / q)) = (∑ k<n.
f (Suc k) q)" .
qed
finally show ?thesis .
qed

```

With some more book-keeping, we show that the series converges uniformly in all compact sets that do not touch the unit circle and, if  $\sum_{n=1}^{\infty} a(n)$  does not exist, lie fully within the convergence radius of  $a(n)$ . This is mentioned in Knopp's Theorem 259.

**theorem uniform\_limit\_lambert:**

assumes "compact X" "X ⊆ eball 0 (lambert\_conv\_radius a) - sphere 0 1"

shows "uniform\_limit X (λn q. (∑ k<n. f (Suc k) q)) (lambert a) sequentially"

**proof -**

from assms have norm\_neq\_1: "norm x ≠ 1" if "x ∈ X" for x

using that by auto

have 1: "uniform\_limit (X ∩ cball 0 1) (λn q. (∑ k<n. f (Suc k) q)) (lambert a) sequentially"

```

proof (cases "X ∩ cball 0 1 ⊆ {0}")
  case True
  hence "X ∩ cball 0 1 = {} ∨ X ∩ cball 0 1 = {0}"
    by auto
  thus ?thesis
    by (elim disjE) (simp_all add: f_def lambert_def)
next
  case False
  obtain r :: real
    where r: "r > 0" "r < 1" "r < conv_radius a" "∧x. x ∈ X ∩ cball
0 1 ⇒ norm x < r"
  proof -
    define c where "c = (if lambert_conv_radius a > 1 then 1
      else real_of_ereal (lambert_conv_radius a))"
    have "compact ((ereal ∘ norm) ` (X ∩ cball 0 1))"
      by (intro compact_continuous_image continuous_intros compact_insert)
    (use assms in auto)
    hence "Sup ((ereal ∘ norm) ` (X ∩ cball 0 1)) ∈ (ereal ∘ norm)
` (X ∩ cball 0 1)"
      using False by (intro closed_contains_Sup_cl) (auto intro: compact_imp_closed)
    then obtain q where q: "q ∈ (X ∩ cball 0 1)"
      "ereal (norm q) = Sup ((ereal ∘ norm) ` (X ∩ cball 0 1))"
    unfolding o_def by force
    have q': "norm q' ≤ norm q" if "q' ∈ X ∩ cball 0 1" for q'
      using Sup_upper q that unfolding o_def by (metis ereal_less_eq(3)
imageI insertCI)
    have "q ≠ 0"
      using q' False by force
    have "conv_radius a ≥ norm (1 :: 'a)" if "summable a"
      by (rule conv_radius_geI) (use that in auto)
    hence "norm q < conv_radius a" "norm q < 1"
      using q(1) assms <q ≠ 0>
      by (auto simp: lambert_conv_radius_def eball_def ereal_le_less
split: if_splits)
    then obtain r where r: "norm q < r" "r < 1" "r < conv_radius a"
      by (smt (verit, ccfv_SIG) ereal_dense2 ereal_less(3) less_ereal.simps(1)
linorder_not_le order_less_le_trans)
    show ?thesis
      proof (rule that[of r])
        show "r > 0"
          using r q <q ≠ 0> by (smt (verit) norm_ge_zero)
        show "r < 1" "ereal r < conv_radius a"
          using r by auto
        show "norm q' < r" if "q' ∈ X ∩ cball 0 1" for q'
          using assms q q' that <q ≠ 0> r
          by (force simp: lambert_conv_radius_def eball_def split: if_splits)
      qed
    qed
  have "uniform_limit (ball 0 r) (λn q. (∑ k<n. f (Suc k) q)) (lambert

```

```

a) sequentially"
  by (rule uniform_limit_lambert1) (use r in auto)
  thus "uniform_limit (X ∩ cball 0 1) (λn q. (∑ k<n. f (Suc k) q))
(lambert a) sequentially"
  by (rule uniform_limit_on_subset) (use r in auto)
qed

have 2: "uniform_limit (X - ball 0 1) (λn q. (∑ k<n. f (Suc k) q))
(lambert a) sequentially"
proof (cases "∃q∈X. norm q > 1")
  case False
  hence *: "X - ball 0 1 = {}"
  using norm_neq_1 by fastforce
  show ?thesis
  unfolding * by simp
next
  case True
  then obtain q where q: "q ∈ X" "norm q > 1"
  by auto
  with assms have "lambert_conv_radius a > 1"
  by (smt (verit, ccfv_SIG) Diff_subset dist_0_norm ereal_less(3)
in_eball_iff linorder_not_le order_less_le_trans subsetD)
  hence "summable a"
  by (simp add: lambert_conv_radius_gt_1_iff)
  obtain r where r: "r > 1" "∧x. x ∈ X - cball 0 1 ⇒ norm x > r"
  proof -
    have compact: "compact (norm ` (X - ball 0 1))"
    by (intro compact_continuous_image compact_diff continuous_intros)
    (use assms in auto)
    hence "Inf (norm ` (X - ball 0 1)) ∈ norm ` (X - ball 0 1)"
    using q by (intro closed_contains_Inf)
    (auto intro: compact_imp_closed bounded_imp_bdd_below
compact_imp_bounded)
    then obtain q' where q': "q' ∈ X - ball 0 1" "norm q' = Inf (norm
` (X - ball 0 1))"
    by force
    have "norm q' > 1"
    using q' assms by auto
    then obtain r where r: "1 < r" "r < norm q'"
    using dense by blast

  show ?thesis
  proof (rule that[of r])
    show "r > 1"
    using q' assms r by auto
    show "norm q'' > r" if "q'' ∈ X - cball 0 1" for q''
    proof -
      have "r < Inf (norm ` (X - ball 0 1))"
      using r q' by simp

```

```

    also have "... ≤ norm q'"
      by (rule cInf_lower)
        (use that assms compact in <auto intro!: bounded_imp_bdd_below
compact_imp_bounded>)
    finally show ?thesis .
  qed
qed
qed

obtain R where R: "R > r" "∧x. x ∈ X ⇒ norm x < R"
proof -
  have "bounded X"
    using assms compact_imp_bounded by blast
  then obtain R where R: "norm x ≤ R" if "x ∈ X" for x
    unfolding bounded_iff by blast
  show ?thesis
  proof (rule that[of "R + 1"])
    show "r < R + 1"
      using r(2)[of q] q R[of q] by auto
    show "norm x < R + 1" if "x ∈ X" for x
      using R[of x] that by auto
  qed
qed

  have lim: "uniform_limit (cball 0 R - ball 0 r) (λn q. (∑ k<n. f
(Suc k) q))
    (λq. -A - Lambert a (1 / q)) sequentially"
    by (rule uniform_limit_Lambert2) (use r R <summable a> in auto)
  also have "?this ↔ uniform_limit (cball 0 R - ball 0 r) (λn q. (∑ k<n.
f (Suc k) q))
    (Lambert a) sequentially"
  proof (intro uniform_limit_cong refl always_eventually_allI ballI)
    fix q :: 'a assume q: "q ∈ cball 0 R - ball 0 r"
    with lim have "(λn. (∑ k<n. f (Suc k) q)) → -A - Lambert a
(1 / q)"
      by (rule tendsto_uniform_limitI)
    hence "(λk. f (Suc k) q) sums (-A - Lambert a (1 / q))"
      by (simp add: sums_def)
    thus "-A - Lambert a (1 / q) = Lambert a q"
      unfolding Lambert_def[of a q] by (simp add: sums_iff f_def)
  qed
  finally show ?thesis
    by (rule uniform_limit_on_subset) (use r R norm_neq_1 in force)+
qed

  have "uniform_limit ((X ∩ cball 0 1) ∪ (X - ball 0 1))
    (λn q. (∑ k<n. f (Suc k) q)) (Lambert a) sequentially"
    using 1 2 by (rule uniform_limit_on_Un)
  also have "(X ∩ cball 0 1) ∪ (X - ball 0 1) = X"

```

```

    using norm_neq_1 by auto
  finally show ?thesis .
qed

```

```

lemma sums_lambert:
  assumes "norm q < lambert_conv_radius a" "norm q ≠ 1"
  shows "(λk. f (Suc k) q) sums lambert a q"
proof -
  have "(λn. (∑ k<n. f (Suc k) q)) → lambert a q"
  proof (rule tendsto_uniform_limitI[OF uniform_limit_lambert])
    show "compact {q}" "{q} ⊆ eball 0 (lambert_conv_radius a) - sphere
0 1"
      using assms by auto
  qed auto
  thus ?thesis
    by (simp add: sums_def)
qed

```

A side effect of this: the functional equation

$$L(a, \frac{1}{q}) = -(\sum_{n=1}^{\infty} a(n)) - L(a, q) ,$$

which is valid for all  $q$  with  $q \neq 0$  and  $|q| \neq 1$  if  $\sum_{n=1}^{\infty} a(n)$  exists.

```

theorem lambert_reciprocal:
  assumes "summable a" and "q ≠ 0" and "norm q ≠ 1"
  shows "lambert a (1 / q) = -A - lambert a q"
proof -
  have *: "lambert a (1 / q) = -A - lambert a q" if q: "norm q > 1" for
q
  proof -
    obtain r where r: "1 < r" "r < norm q"
      using q dense by blast
    have "uniform_limit (cball 0 (norm q + 1) - ball 0 r)
      (λn q. ∑ k<n. f (Suc k) q)
      (λq. - A - lambert a (1 / q)) sequentially"
      by (rule uniform_limit_lambert2) (use assms r in auto)
    hence "(λk. f (Suc k) q) sums (-A - lambert a (1 / q))"
      unfolding sums_def by (rule tendsto_uniform_limitI) (use r in auto)
    moreover have "uniform_limit {q} (λn q. ∑ k<n. f (Suc k) q) (lambert
a) sequentially"
      using assms r
      by (intro uniform_limit_lambert compact_diff compact_cball)
      (auto simp: lambert_conv_radius_def)
    hence "(λk. f (Suc k) q) sums lambert a q"
      unfolding sums_def by (rule tendsto_uniform_limitI) (use r in auto)
    ultimately show ?thesis
      by (simp add: sums_iff algebra_simps)
  qed

```

```

show ?thesis
proof (cases "norm q > 1")
  case False
  thus ?thesis using assms
    using *[of "1 / q"] by (auto simp: norm_divide)
qed (use *[of q] in auto)
qed

```

```

lemma summable_lambert:
  assumes "norm q < lambert_conv_radius a" "norm q ≠ 1"
  shows "summable (λk. f k q)"
  using sums_lambert[OF assms] unfolding sums_iff by (subst (asm) summable_Suc_iff)
auto

```

We have shown that the Lambert series for  $a(n)$  converges everywhere except on the unit circle if  $\sum_{n=1}^{\infty} a(n)$  exists, and it converges within the convergence radius of  $R$  of  $a(n)$  otherwise.

We will now show that within  $\min(1, R)$ , this convergence is absolute.

```

lemma norm_summable_lambert:
  assumes "norm q < min 1 (conv_radius a)"
  shows "summable (λk. norm (f k q))"
proof (cases "q = 0")
  case [simp]: True
  have "eventually (λk. norm (f k q) = 0) at_top"
    using eventually_gt_at_top[of 0] by eventually_elim (auto simp: f_def)
  thus ?thesis
    using summable_cong by fastforce
next
  case False
  define R where "R = (if conv_radius a > 1 then 1 else real_of_ereal
(conv_radius a))"
  have R: "ereal R = min 1 (conv_radius a)"
    using conv_radius_nonneg[of a] by (cases "conv_radius a") (auto simp:
R_def)
  with assms have "norm q < R"
    by (metis less_ereal.simps(1))
  then obtain r where r: "norm q < r" "r < R"
    using dense by blast
  hence "r > 0"
    using norm_ge_zero le_less_trans by blast
  have "r < 1"
    using r R by (metis ereal_less(3) less_ereal.simps(1) min_less_iff_conj)
  have "r < conv_radius a"
    using r R by (metis less_ereal.simps(1) min_less_iff_conj)
  have "norm q < 1"
    using assms by auto
  note r' = ⟨r > 0⟩ ⟨r < 1⟩ ⟨r < conv_radius a⟩
  show ?thesis
  proof (rule summable_powser_comparison_test_bigo)

```

```

    show "summable (λn. a n * of_real r ^ n)"
      by (rule summable_in_conv_radius) (use r r' R in auto)
  next
    have "(λn. norm (f n q)) = (λn. norm (a n) * norm q ^ n / norm (1
- q ^ n))"
      by (simp add: f_def norm_mult norm_divide norm_power)
    also have "1 - norm q ^ n ≤ norm (1 - q ^ n)" for n
      by (metis norm_one norm_power norm_triangle_ineq2)
    hence "(λn. norm (a n) * norm q ^ n / norm (1 - q ^ n)) ∈
      O(λn. norm (a n) * (norm q ^ n / (1 - norm q ^ n)))"
      using <q ≠ 0> <norm q < 1>
      by (intro bigO[of _ 1] eventually_mono[OF eventually_gt_at_top[of
0]])
      (auto intro!: mult_left_mono divide_left_mono mult_pos_pos add_pos_pos

      dest!: power_eq_1_iff simp: power_le_one power_less_1_iff)
    also have "(λn. norm (a n) * (norm q ^ n / (1 - norm q ^ n))) ∈ O(λn.
norm (a n) * norm q ^ n)"
      by (intro landau_o.big.mult_left) (use <q ≠ 0> <norm q < 1> in
real_asymp)
    also have "(λn. norm (a n) * norm q ^ n) =
      (λn. norm (a n * of_real r ^ n * (q / of_real r) ^ n))"
      using <r > 0> by (intro ext) (auto simp: norm_mult norm_divide norm_power
power_divide)
    finally show "(λn. f n q) ∈ O(λn. a n * of_real r ^ n * (q / of_real
r) ^ n)"
      by (subst (asm) landau_o.big.norm_iff)
  next
    show "norm (q / of_real r) < 1"
      using r r' by (simp add: norm_divide field_simps)
  qed
qed

```

If additionally  $\sum_{k=1}^{\infty} a(k)$  converges absolutely, the absolute convergence of the Lambert series also holds everywhere.

**lemma norm\_summable\_lambert':**

assumes "summable (λk. norm (a k))" and "norm q ≠ 1"

shows "summable (λk. norm (f k q))"

**proof -**

have \*: "summable (λk. norm (f k q))" if q: "norm q < 1" for q

**proof -**

have "conv\_radius a ≥ norm (1 :: 'a)"

using assms(1) by (intro conv\_radius\_geI) (auto dest: summable\_norm\_cancel)

with q have "ereal (norm q) < conv\_radius a"

by (simp add: ereal\_le\_less)

thus ?thesis

using assms(2) q by (intro norm\_summable\_lambert) auto

**qed**



```

show ?thesis
proof (cases "norm q < 1")
  case True
  thus ?thesis using *[of q] by simp
next
  case False
  hence [simp]: "q ≠ 0"
  by auto
  have [simp]: "q ^ k = 1 ↔ k = 0" for k
  using assms(2) by (auto dest: power_eq_1_iff)
  have "summable (λk. norm (a k + f k (inverse q)))"
  using False assms(2) by (intro summable_norm_add assms *) (auto
simp: norm_divide field_simps)
  moreover have "eventually (λk. f k q = -a k - f k (inverse q)) at_top"
  using eventually_gt_at_top[of 0]
  by eventually_elim (use False assms(2) in <auto simp: fun_eq_iff
field_simps f_def>)
  hence "eventually (λk. norm (f k q) = norm (a k + f k (inverse q)))
at_top"
  by eventually_elim (auto simp: norm_uminus_minus)
  ultimately show ?thesis
  using summable_cong by fast
qed
qed

lemma abs_summable_on_lambert:
  assumes "norm q < min 1 (conv_radius a)"
  shows "(λk. f k q) abs_summable_on {1..}"
proof -
  have "summable (λk. norm (f (Suc k) q))"
  by (subst summable_Suc_iff, rule norm_summable_lambert) (use assms
in auto)
  hence "(λk. f (Suc k) q) abs_summable_on UNIV"
  by (subst summable_on_UNIV_nonneg_real_iff) auto
  also have "?this ↔ ?thesis"
  by (intro summable_on_reindex_bij_witness[of _ "λk. k - 1" Suc]) auto
  finally show ?thesis .
qed

lemma abs_summable_on_lambert':
  assumes "summable (λk. norm (a k))" and "norm q ≠ 1"
  shows "(λk. f k q) abs_summable_on {1..}"
proof -
  have "summable (λk. norm (f (Suc k) q))"
  by (subst summable_Suc_iff, rule norm_summable_lambert') (use assms
in auto)
  hence "(λk. f (Suc k) q) abs_summable_on UNIV"
  by (subst summable_on_UNIV_nonneg_real_iff) auto
  also have "?this ↔ ?thesis"

```

```

    by (intro summable_on_reindex_bij_witness[of _ "λk. k - 1" Suc]) auto
    finally show ?thesis .
qed

```

```

lemma summable_on_lambert:
  assumes "norm q < min 1 (conv_radius a)"
  shows "(λk. f k q) summable_on {1..}"
  using abs_summable_summable[OF abs_summable_on_lambert[OF assms]] .

```

```

lemma has_sum_lambert:
  assumes "norm q < min 1 (conv_radius a)"
  shows "((λk. f k q) has_sum lambert a q) {1..}"
proof -
  have "(λk. f (Suc k) q) has_sum lambert a q) UNIV"
  proof (rule norm_summable_imp_has_sum)
    show "summable (λn. norm (f (Suc n) q))"
      using norm_summable_lambert[OF assms] by (subst summable_Suc_iff)
    show "(λk. f (Suc k) q) sums lambert a q"
      by (rule sums_lambert) (use assms in <auto simp: lambert_conv_radius_def>)
  qed
  also have "?this ⟷ ?thesis"
    by (intro has_sum_reindex_bij_witness[of _ "λk. k - 1" Suc]) auto
  finally show ?thesis .
qed

```

We can also show a more precise convergence result that essentially fully reduces the question of convergence of a Lambert series to that of its “corresponding” power series:  $\sum_{k=1}^{\infty} a(k) \frac{q^k}{1-q^k}$  converges if and only if the “corresponding” power series  $\sum_{k=1}^{\infty} a(k)q^k$  converges or if  $\sum_{k=1}^{\infty} a(k)$  converges. This is Theorem 259 in Knopp’s book. A key ingredient, aside from the results we have amassed so far, is the du-Bois Reymond summation test.

```

theorem summable_lambert_iff:
  assumes "norm q ≠ 1"
  shows "summable (λk. f k q) ⟷ summable a ∨ summable (λk. a k * q ^ k)"
proof (cases "summable a")
  case True
  hence "summable (λk. f k q)" using assms
    by (intro summable_lambert) (auto simp: lambert_conv_radius_def)
  with True show ?thesis
    by auto
next
  case not_summable: False
  have [simp]: "q ^ k ≠ 1" if "k > 0" for k
    using that assms by (auto dest: power_eq_1_iff)
  have "summable (λk. f k q) ⟷ summable (λk. a k * q ^ k)"
  proof (cases "norm q < 1")
    case False

```

```

with assms have q: "norm q > 1"
  by simp
have "¬summable (λk. a k * q ^ k)"
  using q not_summable_conv_radius_geI[of a q] summable_in_conv_radius[of
1 a]
  by (auto simp: ereal_le_less one_ereal_def)
moreover have "¬summable (λk. f k q)"
proof
  assume "summable (λk. f k q)"
  hence *: "summable (λk. a k / (1 - q ^ k) * q ^ k)"
    by (auto simp: f_def)
  hence "summable (λk. a k / (1 - q ^ k) * 1 ^ k)"
    by (rule powser_inside) (use q in auto)
  hence "summable (λk. a k / (1 - q ^ k) - a k / (1 - q ^ k) * q ^
k)"
    by (intro summable_diff *) auto
  moreover have "eventually (λk. a k / (1 - q ^ k) - a k / (1 - q
^ k) * q ^ k = a k) at_top"
    using eventually_gt_at_top[of 0]
    by eventually_elim (simp add: divide_simps, simp add: algebra_simps)
  ultimately have "summable a"
    using summable_cong by fast
  with not_summable show False
    by contradiction
qed
ultimately show ?thesis
  by simp
next
case q: True
show ?thesis
proof
  assume "summable (λk. f k q)"
  hence "summable (λk. f k q * (1 - q ^ k))"
  proof (rule dubois_reymond_summation_test)
    have "summable (λk. norm (q - 1) * norm q ^ k)"
      using q by (intro summable_mult summable_geometric) auto
    also have "(λk. norm (q - 1) * norm q ^ k) = (λk. norm ((q - 1)
* q ^ k))"
      by (simp add: norm_mult norm_power)
    also have "... = (λk. norm (1 - q ^ k - (1 - q ^ Suc k)))"
      by (simp add: algebra_simps)
    finally show "summable (λk. norm (1 - q ^ k - (1 - q ^ Suc k)))"
    .
  qed
  moreover have "eventually (λk. f k q * (1 - q ^ k) = a k * q ^
k) at_top"
    using eventually_gt_at_top[of 0] by eventually_elim (auto simp:
divide_simps f_def)
  ultimately show "summable (λk. a k * q ^ k)"

```

```

    using summable_cong by fast
  next
    assume "summable (λk. a k * q ^ k)"
    hence "summable (λk. a k * q ^ k * (1 / (1 - q ^ k)))"
    proof (rule dubois_reymond_summation_test)
      show "summable (λk. norm (1 / (1 - q ^ k) - 1 / (1 - q ^ Suc
k)))"
      proof (rule summable_comparison_test_ev)
        show "summable (λk. 2 / (1 - norm q) ^ 2 * norm q ^ k)"
          using q by (intro summable_mult summable_geometric) auto
        next
          show "eventually (λk. norm (norm (1 / (1 - q ^ k) - 1 / (1
- q ^ Suc k))) ≤
                                2 / (1 - norm q) ^ 2 * norm q ^ k) at_top"
            using eventually_gt_at_top[of 0]
          proof eventually_elim
            case k: (elim k)
            have "norm (1 - q) ≤ norm (1 :: 'a) + 1"
              using q by norm
            hence 1: "norm (1 - q) ≤ 2"
              by simp
            have 2: "norm (1 - q ^ 1) ≥ 1 - norm q" if 1: "1 > 0" for
1
              proof -
                have "norm (1 - q ^ 1) ≥ norm (1 :: 'a) - norm (q ^ 1)"
                  by norm
                moreover have "norm (q ^ 1) ≤ norm q ^ 1"
                  using q 1 unfolding norm_power by (intro power_decreasing)
                auto
                ultimately show ?thesis
                  by simp
              qed
            have "1 / (1 - q ^ k) - 1 / (1 - q ^ Suc k) =
              (q ^ k - q ^ Suc k) / ((1 - q ^ k) * (1 - q ^ Suc
k))"
              using k by (simp add: divide_simps del: power_Suc)
            also have "... = (1 - q) * q ^ k / ((1 - q ^ k) * (1 - q ^
Suc k))"
              by (simp add: algebra_simps)
            also have "norm ... = norm (1 - q) * norm q ^ k / (norm (1
- q ^ k) * norm (1 - q ^ Suc k))"
              by (simp add: norm_mult norm_divide norm_power)
            also have "... ≤ 2 * norm q ^ k / ((1 - norm q) * (1 - norm
q))" using q k
              by (intro frac_le mult_mono mult_pos_pos zero_le_power norm_ge_zero
1 2)
              (auto simp del: power_Suc)
            finally show ?case

```

```

      by (simp add: power2_eq_square)
    qed
  qed
  qed
  thus "summable ( $\lambda k. f k q$ )"
    by (simp add: f_def)
  qed
  qed
  with not_summable show ?thesis
    by blast
  qed
end

lemma holomorphic_lambert [holomorphic_intros]:
  assumes "X  $\subseteq$  eball 0 (lambert_conv_radius a) - sphere 0 1"
  shows "lambert a holomorphic_on X"
  proof -
    have "lambert a holomorphic_on eball 0 (lambert_conv_radius a) - sphere
    0 1"
    proof (rule holomorphic_uniform_sequence)
      show "open (eball 0 (lambert_conv_radius a) - sphere (0 :: complex)
    1)"
        by (intro open_Diff) auto
    next
      fix q :: complex
      assume q: "q  $\in$  eball 0 (lambert_conv_radius a) - sphere 0 1"
      have "open (eball 0 (lambert_conv_radius a) - sphere 0 1 :: complex
    set)"
        by auto
      then obtain r where r: "r > 0" "cball q r  $\subseteq$  eball 0 (lambert_conv_radius
    a) - sphere 0 1"
        unfolding open_contains_cball using q by blast
      define f where "f  $\equiv$   $\lambda k q. a k * q ^ k / (1 - q ^ k)$ "
      show " $\exists d > 0. cball q d \subseteq eball 0 (lambert_conv_radius a) - sphere
    0 1 \wedge$ 
        uniform_limit (cball q d) ( $\lambda n q. \sum_{k < n. f (Suc k) q}$ ) (lambert
    a) sequentially"
        proof (intro exI[of _ r] conjI)
          show "uniform_limit (cball q r) ( $\lambda n q. \sum_{k < n. f (Suc k) q}$ ) (lambert
    a) sequentially"
            unfolding f_def by (rule uniform_limit_lambert) (use r in auto)
          qed (use r in auto)
        next
          show " $(\lambda q. \sum_{k < n. a (Suc k) * q ^ Suc k / (1 - q ^ Suc k))$  holomorphic_on
        eball 0 (lambert_conv_radius a) - sphere 0 1" for n :: nat
            by (intro holomorphic_intros) (auto simp del: power_Suc dest!: power_eq_1_iff)
          qed
    qed
  end

```

```

    thus ?thesis
      by (rule holomorphic_on_subset) fact
qed

lemma holomorphic_lambert' [holomorphic_intros]:
  assumes "f holomorphic_on A" "\z. z ∈ A ⇒ f z ∈ eball 0 (lambert_conv_radius
a) - sphere 0 1"
  shows "(λz. lambert a (f z)) holomorphic_on A"
  by (rule holomorphic_on_compose_gen[OF assms(1) holomorphic_lambert[OF
order.refl],
      unfolded o_def])
      (use assms(2) in auto)

lemma analytic_lambert [analytic_intros]:
  fixes a :: "nat ⇒ complex"
  assumes "A ⊆ eball 0 (lambert_conv_radius a) - sphere 0 1"
  shows "lambert a analytic_on A"
proof -
  have "open (eball 0 (lambert_conv_radius a) - sphere 0 1 :: complex
set)"
    by auto
  hence "lambert a analytic_on eball 0 (lambert_conv_radius a) - sphere
0 1"
    using holomorphic_lambert[OF order.refl, of a]
    by (auto simp: analytic_on_open)
  thus ?thesis
    by (rule analytic_on_subset) fact
qed

lemma analytic_lambert' [analytic_intros]:
  assumes "f analytic_on A" "\z. z ∈ A ⇒ f z ∈ eball 0 (lambert_conv_radius
a) - sphere 0 1"
  shows "(λz. lambert a (f z)) analytic_on A"
  by (rule analytic_on_compose_gen[OF assms(1) analytic_lambert[OF order.refl],
unfolded o_def])
      (use assms(2) in auto)

lemma continuous_on_lambert [continuous_intros]:
  fixes a :: "nat ⇒ 'a :: {real_normed_field, banach, heine_borel}"
  assumes "A ⊆ eball 0 (lambert_conv_radius a) - sphere 0 1"
  shows "continuous_on A (lambert a)"
proof -
  have "isCont (lambert a) q" if q: "q ∈ eball 0 (lambert_conv_radius
a) - sphere 0 1" for q
  proof -
    have "open (eball 0 (lambert_conv_radius a) - sphere 0 1 :: 'a set)"
      by auto
    with q obtain r where r: "r > 0" "cball q r ⊆ eball 0 (lambert_conv_radius
a) - sphere 0 1"

```

```

      unfolding open_contains_cball by blast
      have "continuous_on (cball q r) (lambert a)"
      proof (rule uniform_limit_theorem)
        show "uniform_limit (cball q r)
              ( $\lambda n q. \sum k < n. a (Suc k) * q ^ Suc k / (1 - q ^ Suc k)$ ) (lambert
a) at_top"
          by (intro uniform_limit_lambert) (use r in auto)
        show " $\forall_F n$  in sequentially. continuous_on (cball q r)
              ( $\lambda q. \sum k < n. a (Suc k) * q ^ Suc k / (1 - q ^ Suc k)$ )" using
ing q r
          by (auto intro!: always_eventually_continuous_intros simp del:
power_Suc dest!: power_eq_1_iff)
        qed auto
        hence "continuous_on (ball q r) (lambert a)"
          by (rule continuous_on_subset) auto
        thus ?thesis using <r > 0>
          by (subst (asm) continuous_on_eq_continuous_at) auto
      qed
      with assms show ?thesis
      by (intro continuous_at_imp_continuous_on) auto
    qed

lemma continuous_on_lambert' [continuous_intros]:
  fixes a :: "nat  $\Rightarrow$  'a :: {real_normed_field, banach, heine_borel}"
  assumes "continuous_on A f" " $\bigwedge z. z \in A \implies f z \in \text{eball } 0 (\text{lambert\_conv\_radius } a) - \text{sphere } 0 \ 1$ "
  shows "continuous_on A ( $\lambda z. \text{lambert } a (f z)$ )"
  by (rule continuous_on_compose2[OF continuous_on_lambert[OF order.refl]
assms(1)])
  (use assms(2) in auto)

lemma tendsto_lambert [tendsto_intros]:
  fixes a :: "nat  $\Rightarrow$  'a :: {real_normed_field, banach, heine_borel}"
  assumes "(f  $\longrightarrow$  c) F" "c  $\in$  eball 0 (lambert_conv_radius a) - sphere
0 1"
  shows "(( $\lambda x. \text{lambert } a (f x)$ )  $\longrightarrow$  lambert a c) F"
proof -
  have "open (eball 0 (lambert_conv_radius a) - sphere 0 1 :: 'a set)"
    by (intro open_Diff closed_sphere) auto
  hence "isCont (lambert a) c"
    using continuous_on_lambert[OF order.refl]
    by (intro continuous_on_interior) (use assms in <auto simp: interior_open>)
  thus ?thesis
    using assms(1) by (simp add: isCont_tendsto_compose)
qed

If  $\sum_{n=1}^{\infty} a(n)$  exists, the Lambert series of  $a(n)$  tends to it for  $q \rightarrow \infty$ .
lemma tendsto_lambert_at_infinity:
  assumes "summable (a :: nat  $\Rightarrow$  'a :: {real_normed_field, banach, heine_borel})"

```

```

shows "(lambert a ⟶ - (∑ n. a (Suc n))) at_infinity"
proof (rule Lim_transform_eventually)
  have "((λq. - (∑ n. a (Suc n)) - lambert a (1 / q)) ⟶ - (∑ n. a (Suc n)) - lambert a 0) at_infinity"
    by (rule tendsto_diff tendsto_lambert tendsto_intros tendsto_divide_0 filterlim_ident)+
    (use assms in <auto simp: lambert_conv_radius_def>)
  thus "((λq. - (∑ n. a (Suc n)) - lambert a (1 / q)) ⟶ - (∑ n. a (Suc n))) at_infinity"
    by simp
next
have "eventually (λq :: 'a. norm q > 1) at_infinity"
  by (metis dual_order.strict_trans1 eventually_at_infinity gt_ex)
thus "∀F x in at_infinity. - (∑ n. a (Suc n)) - lambert a (1 / x) = lambert a x"
  by eventually_elim (subst lambert_reciprocal, use assms in auto)
qed

```

### 4.3 Power series expansion

By exchanging the order of summation, we can prove the power series expansion of  $L(a, q)$  as

$$L(a, q) = \sum_{n=1}^{\infty} (a * 1)(n) q^n$$

where  $*$  denotes the Dirichlet product, i.e.  $(a * 1)(n) = \sum_{d|n} a(d)$ .

This gives particularly nice results when  $a(n)$  is a number-theoretic function.

```

theorem has_sum_lambert_powser:
  assumes "norm q < min 1 (conv_radius a)"
  assumes "dirichlet_prod a (λ_. 1) = b"
  shows "(λn. b n * q ^ n) has_sum lambert a q {1..}"
proof -
  have q: "norm q < 1" "norm q < conv_radius a"
    using assms by auto
  have q': "norm q < lambert_conv_radius a"
    using q by (auto simp: lambert_conv_radius_def)
  have "((λ(n, k). a n * q ^ (k * n)) has_sum lambert a q) ({1..} × {1..})"
  proof (rule has_sum_SigmaI; (unfold prod.case)?)
    show "((λn. a n * q ^ n / (1 - q ^ n)) has_sum lambert a q) {1..}"
      by (intro has_sum_lambert) (use assms in <auto simp: lambert_conv_radius_def>)
  next
    show "(λ(n, k). a n * q ^ (k * n)) summable_on {1..} × {1..}"
  proof (rule abs_summable_summable)
    show "(λ(n, k). a n * q ^ (k * n)) abs_summable_on {1..} × {1..}"
  proof (rule summable_on_SigmaI; (unfold prod.case)?)
    fix n :: nat
    assume n: "n ∈ {1..}"
    have "((λk. norm (a n) * (norm q ^ n) ^ k) has_sum

```



```

      (norm (a n) * (norm q ^ n / (1 - norm q ^ n))) {1..}"
    using has_sum_geometric[of "norm q ^ n" 1] q n
    by (intro has_sum_cmult_right) (auto simp: norm_power power_less_1_iff)
  thus "(( $\lambda k$ . norm (a n * q ^ (k * n))) has_sum
    (norm (a n) * norm q ^ n / (1 - norm q ^ n))) {1..}"
    by (simp add: norm_mult norm_power norm_divide mult_ac flip:
power_mult)
  next
    show " $(\lambda n$ . norm (a n) * norm q ^ n / (1 - norm q ^ n)) summable_on
{1..}"
    proof (rule abs_summable_summable)
      show " $(\lambda x$ . norm (a x) * norm q ^ x / (1 - norm q ^ x)) abs_summable_on
{1..}"
        using abs_summable_on_lambert[of "of_real (norm q)" " $\lambda n$ . norm
(a n)"] q' q
        by (cases "summable ( $\lambda n$ . norm (a n))")
            (auto simp: lambert_conv_radius_def split: if_splits)
      qed
    qed auto
  qed
next
  fix n :: nat
  assume n: "n  $\in$  {1..}"
  have "(( $\lambda k$ . a n * (q ^ n) ^ k) has_sum a n * (q ^ n / (1 - q ^ n)))
{1..}"
    using has_sum_geometric[of "q ^ n" 1] q n
    by (intro has_sum_cmult_right) (auto simp: power_less_1_iff norm_power)
  thus "(( $\lambda k$ . a n * q ^ (k * n)) has_sum a n * q ^ n / (1 - q ^ n))
{1..}"
    by (simp add: mult_ac flip: power_mult)
  qed
  also have "?this  $\longleftrightarrow$  (( $\lambda(n, d)$ . a d * q ^ n) has_sum lambert a q) (SIGMA
n:{1..}. {d. d dvd n})"
    by (rule has_sum_reindex_bij_witness[of _ " $\lambda(m, d)$ . (d, m div d)"
" $\lambda(n, k)$ . (n * k, n)"])
      (auto simp: mult_ac)
  finally have 1: "(( $\lambda(n, d)$ . a d * q ^ n) has_sum lambert a q) (SIGMA
n:{1..}. {d. d dvd n})" .

  have 2: "(( $\lambda d$ . a d * q ^ n) has_sum (b n * q ^ n)) {d. d dvd n}" if
n: "n > 0" for n
  proof -
    have "(( $\lambda d$ . a d * q ^ n) has_sum (( $\sum d \mid d \text{ dvd } n$ . a d) * q ^ n))
{d. d dvd n}"
      by (intro has_sum_cmult_left has_sum_finite finite_divisors_nat
n)
    also have "(( $\sum d \mid d \text{ dvd } n$ . a d) = dirichlet_prod a ( $\lambda$ _. 1) n"
      by (simp add: dirichlet_prod_def)
    also have "... = b n"

```

```

    using assms by simp
    finally show ?thesis .
qed

show "(( $\lambda n. b n * q ^ n$ ) has_sum_lambert a q) {1..}"
  using has_sum_SigmaD[OF 1, of " $\lambda n. b n * q ^ n$ "] 2 by simp
qed

lemma sums_lambert_powser:
  assumes "norm q < min 1 (conv_radius a)"
  assumes "dirichlet_prod a ( $\lambda_. 1$ ) = b"
  shows " $(\lambda n. b n * q ^ n)$  sums_lambert a q"
proof -
  from assms(2) have [simp]: "b 0 = 0"
  using dirichlet_prod_0 by blast
  have "(( $\lambda n. b n * q ^ n$ ) has_sum_lambert a q) {1..}"
  by (rule has_sum_lambert_powser) fact+
  also have "?this  $\longleftrightarrow$  (( $\lambda n. b n * q ^ n$ ) has_sum_lambert a q) UNIV"
  by (intro has_sum_cong_neutral) (auto simp: not_le)
  finally show ?thesis
  by (rule has_sum_imp_sums)
qed

lemma conv_radius_dirichlet_prod_1_ge:
  fixes a b :: "nat  $\Rightarrow$  'a :: {real_normed_field, banach}"
  defines "b  $\equiv$  dirichlet_prod a ( $\lambda_. 1$ )"
  shows "conv_radius b  $\geq$  min 1 (conv_radius a)"
proof (rule conv_radius_geI_ex)
  fix r :: real
  assume r: "0 < r" "r < min 1 (conv_radius a)"
  have " $(\lambda n. b n * \text{of\_real } r ^ n)$  sums_lambert a (of_real r)"
  using r by (intro sums_lambert_powser) (auto simp: b_def)
  thus " $\exists z. \text{norm } z = r \wedge \text{summable } (\lambda n. b n * z ^ n)$ "
  using r(1) by (intro exI[of _ "of_real r"]) (auto simp: sums_iff)
qed

lemma sums_lambert_powser':
  assumes "norm q < min 1 (conv_radius a)"
  assumes "fds b = fds a * fds_zeta" "b 0 = 0"
  shows " $(\lambda n. b n * q ^ n)$  sums_lambert a q"
  using assms(1)
proof (rule sums_lambert_powser)
  have "fds_nth (fds a * fds_zeta) = dirichlet_prod a ( $\lambda_. 1$ )"
  by (auto simp: fds_nth_mult)
  also have "fds a * fds_zeta = fds b"
  using assms(2) by (simp only: )
  also have "fds_nth (fds b) = b"
  using assms(3) by (auto simp: fun_eq_iff fds_nth_fds)
  finally show "dirichlet_prod a ( $\lambda_. 1$ ) = b" ..

```

qed

### 4.3.1 Divisor $\sigma$ function

For any  $q$  with  $|q| < 1$  and any  $\alpha \in \mathbb{C}$ , we have

$$\sum_{n=1}^{\infty} \sigma_{\alpha}(n) q^n = \sum_{n=1}^{\infty} n^{\alpha} \frac{q^n}{1 - q^n}$$

where  $\sigma_{\alpha}(n)$  is the divisor  $\sigma$  function, i.e.  $\sigma_{\alpha}(n) = \sum_{d|n} d^{\alpha}$ .

```

lemma divisor_sigma_powser_conv_lambert:
  fixes  $\alpha$   $q$  :: "'a :: {nat_power_normed_field, banach}"
  assumes  $q$ : "norm  $q$  < 1"
  shows "( $\lambda n$ . divisor_sigma  $\alpha$   $n$  *  $q$  ^  $n$ ) sums_lambert ( $\lambda n$ . nat_power
 $n$   $\alpha$ )  $q$ "
proof (rule sums_lambert_powser')
  have "conv_radius ( $\lambda n$ . nat_power  $n$   $\alpha$ ) = 1"
  proof (rule tendsto_imp_conv_radius_eq)
    have "( $\lambda n$ . ereal ((real  $n$  powr ( $\alpha$  * 1)) powr (1 / real  $n$ )))  $\longrightarrow$ 
1"
      unfolding one_ereal_def by (rule tendsto_ereal) real_asymp
    also have "?this  $\longleftrightarrow$  ( $\lambda n$ . ereal (norm (nat_power  $n$   $\alpha$ ) powr (1 / real
 $n$ )))  $\longrightarrow$  1"
      by (intro filterlim_cong_refl eventually_mono[OF eventually_gt_at_top[of
0]])
      (simp add: norm_nat_power)
    finally show "( $\lambda n$ . ereal (norm (nat_power  $n$   $\alpha$ ) powr (1 / real  $n$ )))
 $\longrightarrow$  1"
      by (simp add: norm_nat_power)
  qed auto
  thus "ereal (norm  $q$ ) < min 1 (conv_radius ( $\lambda n$ . nat_power  $n$   $\alpha$ ))"
  using  $q$  by simp
next
  have "fds (divisor_sigma  $\alpha$ ) = fds_zeta * fds_shift  $\alpha$  fds_zeta"
  by (rule fds_divisor_sigma)
  also have "fds_shift  $\alpha$  fds_zeta = fds ( $\lambda n$ . nat_power  $n$   $\alpha$ )"
  by (simp add: fds_eq_iff)
  finally show "fds (divisor_sigma  $\alpha$ ) = fds ( $\lambda n$ . nat_power  $n$   $\alpha$ ) * fds_zeta"
  by (simp add: mult commute)
qed auto

```

```

lemma divisor_count_powser_conv_lambert:
  fixes  $q$  :: "'a :: {nat_power_normed_field, banach}"
  assumes  $q$ : "norm  $q$  < 1"
  shows "( $\lambda n$ . of_nat (divisor_count  $n$ ) *  $q$  ^  $n$ ) sums_lambert ( $\lambda$ . 1)
 $q$ "
  using divisor_sigma_powser_conv_lambert[OF assms, of 0]
  by (simp add: divisor_sigma_0_left)

```

### 4.3.2 Möbius $\mu$ function

For any  $q$  with  $|q| < 1$ , we have

$$\sum_{n=1}^{\infty} \mu(n) \frac{q^n}{1 - q^n} = q$$

where  $\mu(n)$  is Möbius'  $\mu$  function, which is 0 if  $n$  is not squarefree (i.e. contains the same prime factor more than once) and otherwise equal to  $(-1)^k$ , where  $k$  is the number of prime factors of  $n$ .

```

lemma lambert_moebius_mu:
  fixes q :: "'a :: {real_normed_field, banach}"
  assumes q: "norm q < 1"
  shows "lambert moebius_mu q = q"
proof -
  have "(λn. indicator {1} n * q ^ n) sums lambert moebius_mu q"
  proof (rule sums_lambert_powser')
    have "fds moebius_mu * fds_zeta = (1 :: 'a fds)"
      using fds_zeta_times_moebius_mu[where ?'a = 'a] by (simp only:
mult.commute)
    also have "(1 :: 'a fds) = fds (indicator {1})"
      by (auto simp: fds_eq_iff fds_nth_one)
    finally show "fds (indicator {1} :: nat ⇒ 'a) = fds moebius_mu * fds_zeta"
  ..
  qed (use q in <auto simp: conv_radius_moebius_mu>)
  also have "(λn. indicator {1} n * q ^ n) = (λn. (if n = 1 then 1 else
0) * q ^ n)"
    by auto
  finally have "... sums lambert moebius_mu q" .
  moreover have "(λn. (if n = 1 then 1 else 0) * q ^ n) sums (q ^ 1)"
    by (rule powser_sums_if)
  ultimately show "lambert moebius_mu q = q"
    by (simp add: sums_iff)
qed

```

```

lemma lambert_conv_radius_moebius_mu:
  "lambert_conv_radius (moebius_mu :: nat ⇒ 'a :: {real_normed_field,
banach}) = 1"
proof -
  have "¬summable (moebius_mu :: nat ⇒ 'a)"
    using not_convergent_moebius_mu[where ?'a = 'a] summable_LIMSEQ_zero
    by (auto simp: convergent_def)
  thus ?thesis
    by (simp add: lambert_conv_radius_def conv_radius_moebius_mu)
qed

```

### 4.3.3 Euler's totient function $\varphi$

For any  $q$  with  $|q| < 1$ , we have

$$\frac{q}{(1-q)^2} = \sum_{n=1}^{\infty} nq^n = \sum_{n=1}^{\infty} \varphi(n) \frac{q^n}{1-q^n}$$

where  $\varphi(n)$  is Euler's totient function, i.e. the number of positive integers not greater than  $n$  that are coprime to  $n$ .

```

lemma lambert_totient:
  fixes q :: "'a :: {real_normed_field, banach}"
  assumes q: "norm q < 1"
  shows "lambert ( $\lambda n. \text{of\_nat } (\text{totient } n) :: 'a$ ) q = q / (1 - q) ^ 2"
proof -
  have "( $\lambda n. \text{of\_nat } n * q ^ n$ ) sums lambert ( $\lambda n. \text{of\_nat } (\text{totient } n) :: 'a$ ) q"
  proof (rule sums_lambert_powser')
    show "fds (of_nat :: nat  $\Rightarrow$  'a) = fds ( $\lambda n. \text{of\_nat } (\text{totient } n)$ ) * fds_zeta"
    by (rule fds_totient_times_zeta [symmetric])
  qed (use q in <auto simp: conv_radius_totient>)
  from sums_unique2[OF this n_powser_sums[OF q]] show ?thesis .
qed

```

```

lemma lambert_conv_radius_totient:
  "lambert_conv_radius ( $\lambda n. \text{of\_nat } (\text{totient } n) :: 'a :: \{real\_normed\_field, banach\}$ ) = 1"
proof -
  have "\summable ( $\lambda n. \text{of\_nat } (\text{totient } n) :: 'a :: \{real\_normed\_field, banach\}$ )"
  using not_convergent_totient[where ?'a = 'a] summable_LIMSEQ_zero
  by (auto simp: convergent_def)
  thus ?thesis
  by (simp add: lambert_conv_radius_def conv_radius_totient)
qed

```

### 4.3.4 Mangoldt's $\Lambda$ function

For any  $q$  with  $|q| < 1$ , we have

$$\sum_{n=1}^{\infty} \ln n q^n = \sum_{n=1}^{\infty} \Lambda(n) \frac{q^n}{1-q^n}$$

where  $\Lambda(n)$  is Mangoldt's function, which is defined to be equal to  $\log n$  if  $n$  is prime and 0 otherwise.

```

lemma lambert_mangoldt:
  fixes q :: "'a :: {real_normed_field, banach}"

```

```

    assumes q: "norm q < 1"
    shows "(λn. of_real (ln (Suc n)) * q ^ (Suc n)) sums lambert mangoldt
q"
proof -
  have "(λn. (if n = 0 then 0 else of_real (ln n)) * q ^ n) sums lambert
mangoldt q"
    by (rule sums_lambert_powser')
    (use q fds_mangoldt_times_zeta[where ?'a = 'a] in <auto simp: conv_radius_mangoldt>)
  also have "?this ↔ (λn. (if Suc n = 0 then 0 else of_real (ln (Suc
n))) * q ^ Suc n) sums lambert mangoldt q"
    by (subst sums_Suc_iff) auto
  also have "... ↔ ?thesis"
    by simp
  finally show ?thesis .
qed

```

```

lemma lambert_conv_radius_mangoldt:
  "lambert_conv_radius (mangoldt :: nat ⇒ 'a :: {real_normed_field, banach})
= 1"
proof -
  have "¬summable (mangoldt :: nat ⇒ 'a :: {real_normed_field, banach})"
    using not_convergent_mangoldt[where ?'a = 'a] summable_LIMSEQ_zero
    by (auto simp: convergent_def)
  thus ?thesis
    by (simp add: lambert_conv_radius_def conv_radius_mangoldt)
qed

```

#### 4.3.5 Liouville's $\lambda$ function

For any  $q$  with  $|q| < 1$ , we have

$$\sum_{n=1}^{\infty} q^{n^2} = \sum_{n=1}^{\infty} \lambda(n) \frac{q^n}{1 - q^n}$$

where  $\lambda(n)$  is Liouville's function, which is defined as the number of prime factors of  $n$  (taking multiplicity into account).

```

lemma lambert_liouville_lambda:
  fixes q :: "'a :: {real_normed_field, banach}"
  assumes q: "norm q < 1"
  shows "(λn. ind is_square n * q ^ n) sums lambert liouville_lambda
q"
  by (rule sums_lambert_powser')
  (use q fds_liouville_lambda_times_zeta[where ?'a = 'a]
  in <auto simp: conv_radius_liouville_lambda mult_ac>)

```

```

lemma lambert_liouville_lambda':
  fixes q :: "'a :: {real_normed_field, banach}"
  assumes q: "norm q < 1"

```

```

shows "(λn. q ^ ((n+1) ^ 2)) sums lambert liouville_lambda q"
proof -
  have "(λn. ind is_square ((n+1)^2) * q ^ ((n+1) ^ 2)) sums lambert liouville_lambda
  q ↔
    (λn. ind is_square n * q ^ n) sums lambert liouville_lambda q"
  proof (rule sums_mono_reindex)
    show "strict_mono (λn::nat. (n + 1)^2)"
      by (intro strict_monoI power_strict_mono) auto
    show "ind is_square n * q ^ n = 0" if "n ∉ range (λn. (n + 1)^2)" for
  n
  proof (rule ccontr)
    assume "ind is_square n * q ^ n ≠ 0"
    hence "is_square n" "n > 0"
      by (auto simp: ind_def split: if_splits)
    then obtain m where "n = m ^ 2" "m > 0"
      by (elim is_nth_powerE) auto
    hence "n = ((m - 1) + 1) ^ 2"
      by auto
    with that show False by blast
  qed
  qed
  thus ?thesis
    using lambert_liouville_lambda[OF assms] by simp
  qed

lemma lambert_conv_radius_liouville_lambda:
  "lambert_conv_radius (liouville_lambda :: nat ⇒ 'a :: {real_normed_field,
  banach}) = 1"
  proof -
    have "¬sumnable (liouville_lambda :: nat ⇒ 'a)"
      using not_convergent_liouville_lambda[where ?'a = 'a] sumnable_LIMSEQ_zero
      by (auto simp: convergent_def)
    thus ?thesis
      by (simp add: lambert_conv_radius_def conv_radius_liouville_lambda)
  qed

```

#### 4.4 Expressing a Lambert series in terms of a power series

Let  $a(n)$  be a sequence of numbers. Then we can express the value of the Lambert series as an infinite sum in terms of the “normal” power series  $f(q) = \sum_{k=1}^{\infty} a(k)q^k$ :

$$L(a, q) = \sum_{n=1}^{\infty} f(q^n)$$

The proof is quite obvious, by expanding  $f(q^n)$  into its power series and then switching the order of summation.

This gives us a number of interesting relationships, including a connection between  $L(n^a, q)$  and the polylogarithm function  $\text{Li}_{-a}$ .

```

theorem lambert_conv_powser_has_sum:
  assumes q: "norm q < min 1 (conv_radius a)" and [simp]: "a 0 = 0"
  defines "f ≡ (λq. ∑ n. a n * q ^ n)"
  shows "((λn. f (q ^ n)) has_sum lambert a q) {1..}"
proof -
  have "((λ(k, n). a k * (q ^ k) ^ n) has_sum lambert a q) ({1..} × {1..})"
  proof (rule has_sum_SigmaI; (unfold prod.case?))
    fix k :: nat
    assume k: "k ∈ {1..}"
    show "((λy. a k * (q ^ k) ^ y) has_sum (a k * (q ^ k / (1 - q ^ k))))
{1..}"
      by (intro has_sum_cmult_right has_sum_geometric_from_1)
        (use q k in <auto simp: power_less_1_iff norm_power>)
  next
    have "((λk. a k * q ^ k / (1 - q ^ k)) has_sum lambert a q) {1..}"
      using q by (intro has_sum_lambert) (auto simp: lambert_conv_radius_def
split: if_splits)
    thus "((λk. a k * (q ^ k / (1 - q ^ k))) has_sum lambert a q) {1..}"
      by (simp add: field_simps)
  next
    show "(λ(k, n). a k * (q ^ k) ^ n) summable_on {1..} × {1..}"
    proof (rule abs_summable_summable, rule summable_on_SigmaI;
      (unfold prod.case norm_divide norm_power norm_mult norm_one
norm_of_nat?))
      fix k :: nat assume k: "k ∈ {1..}"
      show "((λn. norm (a k) * (norm q ^ k) ^ n) has_sum
      (norm (a k) * (norm q ^ k / (1 - norm q ^ k)))) {1..}"
        by (intro has_sum_cmult_right has_sum_geometric_from_1)
          (use k q in <auto simp: power_less_1_iff>)
    next
      show "(λn. norm (a n) * (norm q ^ n / (1 - norm q ^ n))) summable_on
{1..}"
        using summable_on_lambert[of "norm q" "λn. norm (a n)"] q
        by (auto simp: lambert_conv_radius_def split: if_splits)
      qed auto
    qed
  hence *: "((λ(n, k). a k * q ^ (k * n)) has_sum lambert a q) ({1..}
× {1..})"
    by (subst (asm) has_sum_swap) (simp_all flip: power_mult add: mult.commute)

show ?thesis
proof (rule has_sum_SigmaD [OF *]; unfold prod.case)
  fix n :: nat assume n: "n ∈ {1..}"
  have "ereal (norm q ^ n) ≤ ereal (norm q ^ 1)"
    unfolding ereal_less_eq using n q by (intro power_decreasing) auto
  also have "... < conv_radius a"
    using q by (simp add: less_eq_ereal_def)
  finally have norm_q_n: "norm q ^ n < conv_radius a" .

```



```

    have "((λk. a k * (q ^ n) ^ k) has_sum f (q ^ n)) UNIV"
    proof (rule norm_summable_imp_has_sum)
      from norm_q_n show "((λk. a k * (q ^ n) ^ k) sums f (q ^ n))"
        unfolding f_def by (intro summable_sums summable_in_conv_radius)
    (auto simp: norm_power)
    next
      show "summable (λk. norm (a k * (q ^ n) ^ k))"
        by (rule abs_summable_in_conv_radius) (use norm_q_n in <auto simp:
norm_power>)
    qed
    also have "?this  $\longleftrightarrow$  ((λk. a k * q ^ (k * n)) has_sum f (q ^ n)) {1..}"

      by (intro has_sum_cong_neutral) (auto simp: mult.commute not_le
simp flip: power_mult)
    finally show ... .
  qed
qed

lemma lambert_conv_powser_has_sum':
  assumes "norm q < r" and "r ≤ 1"
  assumes "∧q. norm q < r  $\implies$  (λn. a (Suc n) * q ^ Suc n) sums f q"
  shows "((λn. f (q ^ n)) has_sum lambert a q) {1..}"
proof -
  define a' where "a' = (λk. if k = 0 then 0 else a k)"
  have "norm q < r"
  by fact
  also have "r ≤ conv_radius a"
proof (rule conv_radius_geI_ex)
  fix r' assume r': "0 < r'" "ereal r' < ereal r"
  show "∃x. norm x = r' ∧ summable (λn. a n * x ^ n)"
    by (rule exI[of _ "of_real r'"], subst summable_Suc_iff [symmetric])

      (use assms(3)[of "of_real r'"] r' in <simp add: sums_iff>)
  qed
  also have "conv_radius a = conv_radius a'"
    by (intro conv_radius_cong eventually_mono[OF eventually_gt_at_top[of
0]]) (auto simp: a'_def)
  finally have "((λn. (∑ k. a' k * (q ^ n) ^ k)) has_sum lambert a' q)
{1..}"
    using assms(1,2) by (intro lambert_conv_powser_has_sum) (auto simp:
a'_def)
  also have "?this  $\longleftrightarrow$  ((λn. f (q ^ n)) has_sum lambert a' q) {1..}"
proof (rule has_sum_cong)
  fix n :: nat assume n: "n ∈ {1..}"
  have "norm q ^ n ≤ norm q ^ 1"
    using assms(1,2) n by (intro power_decreasing) auto
  also have "... < r"
    using assms(1,2) by simp
  finally have "(λk. a (Suc k) * (q ^ n) ^ Suc k) sums f (q ^ n)"

```

```

    by (intro assms) (auto simp: norm_power)
  hence "( $\lambda k. a' (Suc k) * (q ^ n) ^ Suc k$ ) sums f (q ^ n)"
    by (simp add: a'_def)
  hence "( $\lambda k. a' k * (q ^ n) ^ k$ ) sums f (q ^ n)"
    by (subst (asm) sums_Suc_iff) (simp add: a'_def)
  thus "( $\sum k. a' k * (q ^ n) ^ k$ ) = f (q ^ n)"
    by (simp add: sums_iff)
qed
also have "lambert a' q = lambert a q"
  by (simp add: a'_def)
finally show ?thesis .
qed

lemma lambert_conv_powser_sums:
  assumes q: "norm q < min 1 (conv_radius a)" and [simp]: "a 0 = 0"
  defines "f  $\equiv$  ( $\lambda q. \sum n. a n * q ^ n$ )"
  shows "( $\lambda n. f (q ^ Suc n)$ ) sums lambert a q"
proof -
  have "( $\lambda n. f (q ^ n)$ ) has_sum lambert a q {1..}"
    unfolding f_def by (rule lambert_conv_powser_has_sum) fact+
  also have "?this  $\longleftrightarrow$  (( $\lambda n. f (q ^ Suc n)$ ) has_sum lambert a q) UNIV"
    by (rule has_sum_reindex_bij_witness[of _ Suc " $\lambda n. n - 1$ "]) auto
  finally show ?thesis
    by (rule has_sum_imp_sums)
qed

lemma lambert_conv_powser_sums':
  assumes "norm q < r" and "r  $\leq$  1"
  assumes " $\bigwedge q. \text{norm } q < r \implies (\lambda n. a (Suc n) * q ^ Suc n) \text{ sums } f \text{ } q$ "
  shows "( $\lambda n. f (q ^ Suc n)$ ) sums lambert a q"
proof -
  have "( $\lambda n. f (q ^ n)$ ) has_sum lambert a q {1..}"
    by (rule lambert_conv_powser_has_sum') fact+
  also have "?this  $\longleftrightarrow$  (( $\lambda n. f (q ^ Suc n)$ ) has_sum lambert a q) UNIV"
    by (rule has_sum_reindex_bij_witness[of _ Suc " $\lambda n. n - 1$ "]) auto
  finally show ?thesis
    by (rule has_sum_imp_sums)
qed

lemma lambert_mult_exp_conv_powser_has_sum:
  assumes "norm q < r" and "r  $\leq$  1" and c: "norm c  $\leq$  1"
  assumes " $\bigwedge q. \text{norm } q < r \implies (\lambda n. a (Suc n) * q ^ Suc n) \text{ sums } f \text{ } q$ "
  shows "( $\lambda n. f (c * q ^ n)$ ) has_sum lambert ( $\lambda n. c ^ n * a n$ ) q"
  {1..}"
proof (rule lambert_conv_powser_has_sum')
  fix q :: 'a assume q: "norm q < r"
  have "norm c * norm q  $\leq$  1 * norm q"
    using q c by (intro mult_right_mono) auto
  hence cq: "norm c * norm q < r"

```

```

    using q by simp
  have "(λn. a (Suc n) * (c * q) ^ Suc n) sums f (c * q)"
    by (rule assms) (use cq in <simp add: norm_mult>)
  thus "(λn. c ^ Suc n * a (Suc n) * q ^ Suc n) sums f (c * q)"
    by (simp add: power_mult_distrib algebra_simps)
qed (use assms in auto)

lemma lambert_mult_exp_conv_powser_sums:
  assumes "norm q < r" and "r ≤ 1" and c: "norm c ≤ 1"
  assumes "∧q. norm q < r ⇒ (λn. a (Suc n) * q ^ Suc n) sums f q"
  shows "(λn. f (c * q ^ Suc n)) sums lambert (λn. c ^ n * a n) q"
proof (rule lambert_conv_powser_sums')
  fix q :: 'a assume q: "norm q < r"
  have "norm c * norm q ≤ 1 * norm q"
    using q c by (intro mult_right_mono) auto
  hence cq: "norm c * norm q < r"
    using q by simp
  have "(λn. a (Suc n) * (c * q) ^ Suc n) sums f (c * q)"
    by (rule assms) (use cq in <simp add: norm_mult>)
  thus "(λn. c ^ Suc n * a (Suc n) * q ^ Suc n) sums f (c * q)"
    by (simp add: power_mult_distrib algebra_simps)
qed (use assms in auto)

lemma lambert_power_int_has_sum_polylog_gen:
  fixes q :: complex
  assumes q: "norm q < 1" and c: "norm c ≤ 1"
  shows "(λn. polylog (-a) (c * q ^ n)) has_sum lambert (λn. c ^ n *
of_nat n powi a) q {1..}"
  using q
proof (rule lambert_mult_exp_conv_powser_has_sum)
  show "(λn. of_nat (Suc n) powi a * q ^ Suc n) sums polylog (-a) q"
    if "norm q < 1" for q :: complex
    using sums_polylog[of q "-a"] that by simp
qed (use assms in auto)

lemma has_sum_lambert_recip_complex_gen:
  fixes q :: complex
  assumes q: "norm q < 1" and c: "norm c ≤ 1"
  shows "(λk. -ln (1 - c * q ^ k)) has_sum lambert (λn. c ^ n / of_nat
n) q {1..}"
proof -
  have "(λn. polylog 1 (c * q ^ n)) has_sum lambert (λn. c ^ n * of_nat
n powi - 1) q {1..}"
    using lambert_power_int_has_sum_polylog_gen[OF q, of c "-1"] q c by
simp
  also have "?this ⟷ ((λn::nat. -ln (1 - c * q ^ n)) has_sum
lambert (λn. c ^ n * of_nat n powi -1) q) {1..}"
proof (intro has_sum_cong)
  fix n :: nat assume n: "n ∈ {1..}"

```

```

    have "norm c * norm (q ^ n) ≤ 1 * norm (q ^ n)"
      using q c by (intro mult_right_mono) auto
    also have "norm (q ^ n) ≤ norm q ^ 1"
      using n q unfolding norm_power by (intro power_decreasing) auto
    also have "1 * norm q ^ 1 < 1"
      using q by simp
    finally have cq: "norm (c * q ^ n) < 1"
      by (simp_all add: norm_mult)
    thus "polylog 1 (c * q ^ n) = -ln (1 - c * q ^ n)"
      by (subst polylog_1) auto
  qed
  also have "(λn. c ^ n * of_nat n powi -1) = (λn. c ^ n / of_nat n ::
complex)"
    by (auto simp: field_simps)
  finally show ?thesis .
qed

lemma has_sum_lambert_recip_complex:
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "((λk. -ln (1 - q ^ k)) has_sum lambert (λn. 1 / of_nat n)
q) {1..}"
  using has_sum_lambert_recip_complex_gen[OF assms, of 1] by simp

lemma has_sum_lambert_recip_complex':
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "((λk. -ln (1 + q ^ k)) has_sum lambert (λn. (-1) ^ n / of_nat
n) q) {1..}"
  using has_sum_lambert_recip_complex_gen[OF assms, of "-1"] by simp

lemma has_sum_lambert_poly_complex:
  fixes q :: complex and a :: nat
  assumes q: "norm q < 1" and a: "a > 0"
  defines "E ≡ poly (eulerian_poly a)"
  shows "((λn. E (q ^ n) * q ^ n / (1 - q ^ n) ^ (a + 1)) has_sum
lambert (λn. complex_of_nat n ^ a) q) {1..}"

proof -
  have "((λn. polylog (-a) (q ^ n)) has_sum lambert (λn. of_nat n ^ a)
q) {1..}"
    using lambert_power_int_has_sum_polylog_gen[OF q, of 1 a] q by simp
  also have "?this ↔ ((λn. E (q ^ n) * q ^ n / (1 - q ^ n) ^ (a+1))
has_sum lambert (λn. of_nat n ^ a) q) {1..}"
    proof (rule has_sum_cong)
      fix n :: nat assume n: "n ∈ {1..}"
      have "norm (q ^ n) ≤ norm q ^ 1"
        using n q unfolding norm_power by (intro power_decreasing) auto
      also have "... < 1"
        using q by simp
    end

```

```

    finally show "polylog (-a) (q ^ n) = E (q ^ n) * q ^ n / (1 - q ^ n)
    ^ (a+1)"
      using a by (subst polylog_neg_int_left)
                (auto simp: E_def power_int_diff power_int_minus divide_simps)
  qed
  finally show ?thesis .
qed

```

```

lemma lambert_minus1_power_has_sum:
  assumes q: "norm q < 1"
  shows "((λn. q ^ n / (1 + q ^ n)) has_sum lambert (λn. (-1) ^ Suc
n) q) {1..}"
  using q
proof (rule lambert_conv_powser_has_sum')
  show "(λn. (-1) ^ Suc (Suc n) * q ^ Suc n) sums (q / (1 + q))" if "norm
q < 1" for q :: 'a
  proof -
    have "(λn. (-1) ^ Suc (Suc n) * q ^ Suc n) sums (1 - 1 / (1 + q))"
      using sums_minus[OF geometric_sums[of "-q"]] that
      by (subst sums_Suc_iff) (auto simp: field_simps power_minus')
    also have "1 - 1 / (1 + q) = q / (1 + q)"
      using that by (auto simp: divide_simps add_eq_0_iff)
    finally show ?thesis .
  qed
qed auto

```

```

lemma lambert_exp_has_sum:
  fixes q :: "'a :: {real_normed_field, banach}"
  assumes q: "norm q < 1" and a: "norm a ≤ 1"
  shows "((λn. a * q ^ n / (1 - a * q ^ n)) has_sum lambert (λn. a ^ n
^ n) q) {1..}"
proof -
  have "((λn. a * q ^ n / (1 - a * q ^ n)) has_sum lambert (λn. a ^ n
* 1) q) {1..}"
  using q
  proof (rule lambert_mult_exp_conv_powser_has_sum)
    show "(λn. 1 * q ^ Suc n) sums (q / (1 - q))" if "norm q < 1" for
q :: 'a
    using geometric_sums_gen[of q 1] that by simp
  qed (use a in auto)
  thus ?thesis
  by simp
qed

```

## 4.5 Connection to Euler's function

In this section, we show a connection between Lambert series and Euler's function:

$$\varphi(q) = \prod_{k=1}^{\infty} (1 - q^k)$$

(not to be confused with Euler's totient function, commonly denoted with  $\varphi(n)$ )

For this, we apply the results from the previous section to  $a(n) = \frac{1}{n}$  to obtain:

$$\sum_{k=1}^{\infty} \ln(1 - q^k) = -L\left(\frac{1}{n}, q\right)$$

```

lemma sums_lambert_recip_complex:
  fixes q :: complex
  assumes q: "norm q < 1"
  shows  "((λk. -ln (1 - q ^ Suc k)) sums_lambert (λn. 1 / of_nat n)
q)"
  using q
proof (rule lambert_conv_powser_sums')
  show "(λk. 1 / of_nat (Suc k) * q ^ Suc k) sums - Ln (1 - q)" if "norm
q < 1" for q
    using sums_minus[OF Ln_series[of "-q"]] that
    by (subst sums_Suc_iff) (simp_all add: power_minus')
qed auto

```

```

lemma sums_lambert_recip_complex':
  fixes q :: complex
  assumes q: "norm q < 1"
  shows  "((λk. -ln (1 + q ^ Suc k)) sums_lambert (λn. (-1)^n / of_nat
n) q)"
  using q
proof (rule lambert_conv_powser_sums')
  show "(λk. (-1)^Suc k / of_nat (Suc k) * q ^ Suc k) sums - Ln (1 +
q)" if "norm q < 1" for q
    using sums_minus[OF Ln_series[of "q"]] that
    by (subst sums_Suc_iff) (simp_all add: power_minus')
qed auto

```

By exponentiating this, we get:

$$\varphi(q) \stackrel{\text{def}}{=} \prod_{n=1}^{\infty} (1 - q^n) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1 - q^n}\right)$$

In other words, the Lambert sum  $\sum \frac{1}{n} \frac{q^n}{1 - q^n}$  is a logarithm of Euler's function  $\varphi(q)$ .

Note that this does not show that this is *the* logarithm of  $\varphi(q)$ , but merely that it is *one* of the branches of the multi-valued logarithm of  $\varphi(q)$ . Nevertheless, we will – just like is typically in textbooks – ignore this in our informal explanations and write  $\ln \varphi(q)$ .

```

theorem euler_phi_conv_lambert:
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "(λn. 1 - q ^ Suc n) has_prod exp (-lambert (λn. 1 / of_nat n)
q)"
proof -
  have not_1: "q ^ n ≠ 1" if "n > 0" for n
    using that q by (auto dest: power_eq_1_iff)
  have "(λn. exp (-ln (1 - q ^ Suc n))) has_prod exp (lambert (λn. 1 /
of_nat n) q)"
    by (intro sums_imp_has_prod_exp sums_lambert_recip_complex q)
  also have "(λn. exp (-ln (1 - q ^ Suc n))) = (λn. inverse (1 - q ^ Suc
n))"
    using q unfolding exp_minus by (subst exp_Ln) (auto simp del: power_Suc
simp: not_1)
  finally have "(λn. inverse (inverse (1 - q ^ Suc n))) has_prod
inverse (exp (lambert (λn. 1 / complex_of_nat n) q))"
    by (intro has_prod_inverse)
  thus ?thesis
    using q by (simp del: power_Suc add: exp_minus)
qed

```

With our general results on Lambert series, we also know that  $\ln \varphi(q)$  has the power series expansion

$$\ln \varphi(q) = - \sum_{n=1}^{\infty} \sigma_{-1}(n) q^n = - \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} q^n .$$

```

lemma ln_euler_phi_powser:
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "(λn. divisor_sigma (-1) n * q ^ n) sums lambert (λn. 1 / of_nat
n) q"
  using divisor_sigma_powser_conv_lambert[OF q, of "-1"]
  by (simp add: powr_minus divide_inverse)

```

```

lemma ln_euler_phi_powser':
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "(λn. divisor_sum n / n * q ^ n) sums lambert (λn. 1 / of_nat
n) q"
  using ln_euler_phi_powser[OF q]
  by (simp add: divisor_sigma_minus divisor_sigma_1_left mult_ac)

```

We also show the following variant of the above, also mentioned by Knopp:

```

theorem euler_phi_variant_conv_lambert:
  fixes q :: complex
  assumes q: "norm q < 1"
  shows "(λn. 1 + q ^ Suc n) has_prod exp (-lambert (λn. (-1) ^ n / of_nat
n) q)"
proof -
  have not_1: "q ^ n ≠ -1" if "n > 0" for n
  proof
    assume "q ^ n = -1"
    hence "norm q ^ n = 1"
      by (simp flip: norm_power)
    thus False
      using that q by (auto dest: power_eq_1_iff)
  qed
  have "(λn. exp (-ln (1 + q ^ Suc n))) has_prod exp (lambert (λn. (-1)^n
/ of_nat n) q)"
    by (intro sums_imp_has_prod_exp sums_lambert_recip_complex' q)
  also have "(λn. exp (-ln (1 + q ^ Suc n))) = (λn. inverse (1 + q ^ Suc
n))"
    using q unfolding exp_minus
    by (subst exp_Ln) (auto simp del: power_Suc simp: not_1 add_eq_0_iff)
  finally have "(λn. inverse (inverse (1 + q ^ Suc n))) has_prod
inverse (exp (lambert (λn. (-1)^n / complex_of_nat
n) q))"
    by (intro has_prod_inverse)
  thus ?thesis
    using q by (simp del: power_Suc add: exp_minus)
qed

```

## 4.6 Application: Fibonacci numbers

Lastly, we show a connection between the Fibonacci numbers and Lambert series, namely that:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sqrt{5} \left[ L\left(1, \frac{1}{2}(3 - \sqrt{5})\right) - L\left(1, \frac{1}{2}(7 - 3\sqrt{5})\right) \right]$$

```

lemma fib_closed_form_alt:
  defines "φ ≡ (1 + sqrt 5) / 2"
  shows "real (fib n) = (φ ^ n - (-1 / φ) ^ n) / sqrt 5"
proof -
  have "real (fib n) = (φ ^ n - ((1 - sqrt 5) / 2) ^ n) / sqrt 5"
    unfolding φ_def by (rule fib_closed_form)
  also have "1 + sqrt 5 > 0"
    by (intro add_pos_pos) auto
  hence "(1 - sqrt 5) / 2 = -(1 / φ)"
    by (simp add: φ_def field_simps)
  finally show ?thesis

```



```

    by simp
qed

theorem sum_inv_even_fib_conv_lambert:
  defines "L ≡ lambert (λ_. 1)"
  shows "(λn. 1 / real (fib (2*n))) has_sum
    (sqrt 5 * (L ((3 - sqrt 5) / 2) - L ((7 - 3 * sqrt 5) / 2)))"
{1..}"
proof -
  define φ :: real where "φ = (1 + sqrt 5) / 2"
  have "1 + sqrt 5 > 0"
    by (intro add_pos_pos) auto
  hence [simp]: "1 + sqrt 5 ≠ 0"
    by auto
  have pos: "φ > 1"
    by (auto simp: φ_def intro: add_pos_pos)
  have [simp]: "φ ^ k = 1 ↔ k = 0" for k
    by (auto simp: φ_def dest: power_eq_1_iff)
  have "(λn. 1 * (1/φ^2)^n / (1 - (1/φ^2)^n) - 1 * (1/φ^4)^n / (1 -
    (1/φ^4)^n))
    has_sum (L (1/φ^2) - L(1/φ^4)) {1..}"
    unfolding L_def using pos
    by (intro has_sum_diff has_sum_lambert) (auto simp: field_simps intro!:
one_less_power)
  also have "(λn. 1 * (1/φ^2)^n / (1 - (1/φ^2)^n) - 1 * (1/φ^4)^n /
    (1 - (1/φ^4)^n)) =
    (λn. 1 / (φ ^ (2 * n) - (1 / φ) ^ (2 * n)))" using pos
    by (simp add: divide_simps fun_eq_iff flip: power_mult)
    (simp add: algebra_simps flip: power_add)?
  also have "... has_sum (L (1/φ^2) - L(1/φ^4)) {1..} ↔
    ((λn. 1 / sqrt 5 * (1 / real (fib (2 * n)))) has_sum (L (1/φ^2)
- L(1/φ^4)) {1..}"
    proof (intro has_sum_cong)
      fix n :: nat assume n: "n ∈ {1..}"
      have "φ ^ (2 * n) - (1 / φ) ^ (2 * n) = (φ ^ (2 * n) - (-1 / φ) ^
    (2 * n))"
        by simp
      also have "... = real (fib (2 * n)) * sqrt 5"
        by (subst fib_closed_form_alt) (simp add: φ_def)
      finally show "1 / (φ ^ (2 * n) - (1 / φ) ^ (2 * n)) = 1 / sqrt 5 *
    (1 / real (fib (2 * n)))"
        by simp
    qed
  also have "1 / φ ^ 2 = (3 - sqrt 5) / 2"
    by (simp add: φ_def power_divide power2_eq_square divide_simps) (auto
simp: algebra_simps)?
  also have "1 / φ ^ 4 = (7 - 3 * sqrt 5) / 2"
    by (simp add: φ_def power_divide eval_nat_numeral divide_simps) (auto
simp: algebra_simps)?

```

```

    finally have "((λn. sqrt 5 * (1 / sqrt 5 * (1 / real (fib (2 * n))))))
has_sum
    sqrt 5 * (L ((3 - sqrt 5) / 2) - L ((7 - 3 * sqrt 5)
/ 2)) {1..}"
    by (intro has_sum_cmult_right)
    thus ?thesis
    by simp
qed
end

```

## References

- [1] K. Knopp. *Theorie und Anwendung der Unendlichen Reihen*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013.