

Converting Linear-Time Temporal Logic to Generalized Büchi Automata

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Abstract

We formalize linear-time temporal logic (LTL) and the algorithm by Gerth et al. to convert LTL formulas to generalized Büchi automata. We also formalize some syntactic rewrite rules that can be applied to optimize the LTL formula before conversion. Moreover, we integrate the Stuttering Equivalence AFP-Entry by Stefan Merz, adapting the lemma that next-free LTL formula cannot distinguish between stuttering equivalent runs to our setting.

We use the Isabelle Refinement and Collection framework, as well as the Autoref tool, to obtain a refined version of our algorithm, from which efficiently executable code can be extracted.

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 3 |
| 2 | LTL to GBA translation | 3 |
| 2.1 | Statistics | 3 |
| 2.2 | Preliminaries | 4 |
| 2.3 | Creation of States | 5 |
| 2.4 | Creation of GBA | 14 |
| 3 | Refinement to Efficient Code | 22 |
| 3.1 | Parametricity Setup Boilerplate | 22 |
| 3.1.1 | LTL Formulas | 22 |
| 3.1.2 | Nodes | 26 |
| 3.2 | Massaging the Abstract Algorithm | 28 |
| 3.2.1 | Creation of the Nodes | 28 |
| 3.2.2 | Creation of GBA from Nodes | 30 |
| 3.3 | Refinement to Efficient Data Structures | 33 |
| 3.3.1 | Creation of GBA from Nodes | 33 |
| 3.3.2 | Creation of Graph | 35 |

1 Introduction

In LTL model checking obtaining an equivalent automaton from a linear temporal logic (LTL) formula makes up an important nontrivial part of the whole process. Gerth et al. [2] present a simple tableau-based construction, which takes an LTL formula and decomposes it according to its structure gaining the desired automaton step-by-step.

In this entry, we formalize Linear Temporal Logic (LTL), some optimizing syntactic rewrite rules on LTL formulas, and Gerth's algorithm. Using the Isabelle Refinement Framework, we extract efficient code from our formalization.

Moreover, we connect our LTL formalization to the one of Stefan Merz [3], adapting the lemma that next-free LTL formula cannot distinguish between stuttering equivalent runs to our setting.

This work is part of the CAVA project [1] to implement an executable fully verified LTL model checker.

2 LTL to GBA translation

```
theory LTL-to-GBA
imports
  CAVA-Base.CAVA-Base
  LTL.LTL
  CAVA-Automata.Automata
begin
```

2.1 Statistics

```
code-printing
code-module Gerth-Statistics  $\rightarrow$  (SML)  $\langle$ 
  structure Gerth-Statistics = struct
    val active = Unsynchronized.ref false
    val data = Unsynchronized.ref (0,0,0)

    fun is-active () = !active
    fun set-data num-states num-init num-acc = (
      active := true;
      data := (num-states, num-init, num-acc)
    )

    fun to-string () = let
      val (num-states, num-init, num-acc) = !data
    in
      Num states: ^ IntInf.toString (num-states) ^\n
      ^ Num initial: ^ IntInf.toString num-init ^\n
      ^ Num acc-classes: ^ IntInf.toString num-acc ^\n
    end
  end
end
```

```

    end

    val - = Statistics.register-stat (Gerth LTL-to-GBA,is-active,to-string)
  end
>
code-reserved SML Gerth-Statistics

consts
  stat-set-data-int :: integer ⇒ integer ⇒ integer ⇒ unit

code-printing
  constant stat-set-data-int ↦ (SML) Gerth'-Statistics.set'-data

definition stat-set-data ns ni na
  ≡ stat-set-data-int (integer-of-nat ns) (integer-of-nat ni) (integer-of-nat na)

lemma [autoref-rules]:
  (stat-set-data,stat-set-data) ∈ nat-rel → nat-rel → nat-rel → unit-rel
  ⟨proof⟩

abbreviation stat-set-data-nres ns ni na ≡ RETURN (stat-set-data ns ni na)

lemma discard-stat-refine[refine]:
  m1 ≤ m2 ⇒ stat-set-data-nres ns ni na ≫ m1 ≤ m2 ⟨proof⟩

```

2.2 Preliminaries

Some very special lemmas for reasoning about the nres-monad

lemma SPEC-rule-nested2:

$$\llbracket m \leq \text{SPEC } P; \bigwedge r1\ r2. P (r1, r2) \rrbracket \Longrightarrow g (r1, r2) \leq \text{SPEC } P$$

$$\Longrightarrow m \leq \text{SPEC } (\lambda r'. g\ r' \leq \text{SPEC } P)$$
 ⟨proof⟩

lemma SPEC-rule-param2:
assumes $f\ x \leq \text{SPEC } (P\ x)$
and $\bigwedge r1\ r2. (P\ x) (r1, r2) \Longrightarrow (P\ x') (r1, r2)$
shows $f\ x \leq \text{SPEC } (P\ x')$
 ⟨proof⟩

lemma SPEC-rule-weak:
assumes $f\ x \leq \text{SPEC } (Q\ x)$ **and** $f\ x \leq \text{SPEC } (P\ x)$
and $\bigwedge r1\ r2. \llbracket (Q\ x) (r1, r2); (P\ x) (r1, r2) \rrbracket \Longrightarrow (P\ x') (r1, r2)$
shows $f\ x \leq \text{SPEC } (P\ x')$
 ⟨proof⟩

lemma SPEC-rule-weak-nested2: $\llbracket f \leq \text{SPEC } Q; f \leq \text{SPEC } P; \bigwedge r1\ r2. \llbracket Q (r1, r2); P (r1, r2) \rrbracket \Longrightarrow g (r1, r2) \leq \text{SPEC } P$
 $\Longrightarrow f \leq \text{SPEC } (\lambda r'. g\ r' \leq \text{SPEC } P)$
 ⟨proof⟩

2.3 Creation of States

In this section, the first part of the algorithm, which creates the states of the automaton, is formalized.

type-synonym *node-name* = *nat*

type-synonym 'a *frml* = 'a *ltrl*

type-synonym 'a *interp* = 'a *set word*

record 'a *node* =
 name :: *node-name*
 incoming :: *node-name set*
 new :: 'a *frml set*
 old :: 'a *frml set*
 next :: 'a *frml set*

context
begin

fun *new1* **where**
 new1 (μ *and_r* ψ) = $\{\mu, \psi\}$
| *new1* (μ *U_r* ψ) = $\{\mu\}$
| *new1* (μ *R_r* ψ) = $\{\psi\}$
| *new1* (μ *or_r* ψ) = $\{\mu\}$
| *new1* - = $\{\}$

fun *next1* **where**
 next1 (*X_r* ψ) = $\{\psi\}$
| *next1* (μ *U_r* ψ) = $\{\mu$ *U_r* $\psi\}$
| *next1* (μ *R_r* ψ) = $\{\mu$ *R_r* $\psi\}$
| *next1* - = $\{\}$

fun *new2* **where**
 new2 (μ *U_r* ψ) = $\{\psi\}$
| *new2* (μ *R_r* ψ) = $\{\mu, \psi\}$
| *new2* (μ *or_r* ψ) = $\{\psi\}$
| *new2* - = $\{\}$

definition *expand-init* $\equiv 0$

definition *expand-new-name* $\equiv \text{Suc}$

lemma *expand-new-name-expand-init*: *expand-init* < *expand-new-name nm*
 <*proof*>

lemma *expand-new-name-step*[*intro*]:

fixes $n :: 'a \text{ node}$
shows $\text{name } n < \text{expand-new-name } (\text{name } n)$
 $\langle \text{proof} \rangle$

lemma $\text{expand-new-name--less-zero}[\text{intro}]$: $0 < \text{expand-new-name } nm$
 $\langle \text{proof} \rangle$

abbreviation

$\text{upd-incoming-f } n \equiv (\lambda n'.$
 $\text{if } (\text{old } n' = \text{old } n \wedge \text{next } n' = \text{next } n) \text{ then}$
 $\text{ } n' \langle \text{incoming} := \text{incoming } n \cup \text{incoming } n' \rangle$
 $\text{else } n'$
 \rangle

definition

$\text{upd-incoming } n \text{ ns} \equiv ((\text{upd-incoming-f } n) \text{ ` } ns)$

lemma $\text{upd-incoming--elem}$:

assumes $nd \in \text{upd-incoming } n \text{ ns}$

shows $nd \in ns$

$\vee (\exists nd' \in ns. nd = nd' \langle \text{incoming} := \text{incoming } n \cup \text{incoming } nd' \rangle) \wedge$
 $\text{old } nd' = \text{old } n \wedge$
 $\text{next } nd' = \text{next } n$

$\langle \text{proof} \rangle$

lemma $\text{upd-incoming--ident-node}$:

assumes $nd \in \text{upd-incoming } n \text{ ns}$ **and** $nd \in ns$

shows $\text{incoming } n \subseteq \text{incoming } nd \vee \neg (\text{old } nd = \text{old } n \wedge \text{next } nd = \text{next } n)$

$\langle \text{proof} \rangle$

lemma $\text{upd-incoming--ident}$:

assumes $\forall n \in ns. P \ n$

and $\bigwedge n. \llbracket n \in ns; P \ n \rrbracket \implies (\bigwedge v. P \ (n \langle \text{incoming} := v \rangle))$

shows $\forall n \in \text{upd-incoming } n \text{ ns}. P \ n$

$\langle \text{proof} \rangle$

lemma $\text{name-upd-incoming-f}[\text{simp}]$: $\text{name } (\text{upd-incoming-f } n \ x) = \text{name } x$

$\langle \text{proof} \rangle$

lemma $\text{name-upd-incoming}[\text{simp}]$:

$\text{name } \text{ ` } (\text{upd-incoming } n \ ns) = \text{name } \text{ ` } ns$ (**is** $?lhs = ?rhs$)

$\langle \text{proof} \rangle$

abbreviation expand-body

where

$\text{expand-body} \equiv (\lambda \text{expand } (n, ns).$

```

if new n = {} then (
  if ( $\exists n' \in ns. old\ n' = old\ n \wedge next\ n' = next\ n$ ) then
    RETURN (name n, upd-incoming n ns)
  else
    expand (
      (
        name=expand-new-name (name n),
        incoming={name n},
        new=next n,
        old={},
        next={}
      ),
      {n}  $\cup$  ns)
    ) else do {
       $\varphi \leftarrow SPEC (\lambda x. x \in (new\ n))$ ;
      let n = n( $new := new\ n - \{\varphi\}$ );
      if ( $\exists q. \varphi = prop_r(q) \vee \varphi = nprop_r(q)$ ) then
        (if ( $not_r\ \varphi \in old\ n$ ) then RETURN (name n, ns)
          else expand (n( $old := \{\varphi\} \cup old\ n$ ), ns))
        )
      else if  $\varphi = true_r$  then expand (n( $old := \{\varphi\} \cup old\ n$ ), ns)
      else if  $\varphi = false_r$  then RETURN (name n, ns)
      else if ( $\exists \nu\ \mu. (\varphi = \nu\ and_r\ \mu) \vee (\varphi = X_r\ \nu)$ ) then
        expand (
          n(
            new := new1  $\varphi \cup new\ n$ ,
            old :=  $\{\varphi\} \cup old\ n$ ,
            next := next1  $\varphi \cup next\ n$ 
          ),
          ns)
        )
      else do {
        (nm, nds)  $\leftarrow$  expand (
          n(
            new := new1  $\varphi \cup new\ n$ ,
            old :=  $\{\varphi\} \cup old\ n$ ,
            next := next1  $\varphi \cup next\ n$ 
          ),
          ns);
        expand (n( $name := nm, new := new2\ \varphi \cup new\ n, old := \{\varphi\} \cup old\ n$ ),
          nds)
        )
      }
    }
  )
)

```

lemma *expand-body-mono*: trimono *expand-body* \langle proof \rangle

definition *expand* :: ('a node \times ('a node set)) \Rightarrow (node-name \times 'a node set) nres
where *expand* \equiv REC *expand-body*

lemma *REC-rule-old*:

fixes $x::'x$
assumes M : *trimono body*
assumes $I0$: Φx
assumes IS : $\bigwedge f x. [\bigwedge x. \Phi x \implies f x \leq M x; \Phi x; f \leq REC \text{ body}]$
 $\implies \text{body } f x \leq M x$
shows $REC \text{ body } x \leq M x$
 $\langle \text{proof} \rangle$

lemma *expand-rec-rule*:
assumes $I0$: Φx
assumes IS : $\bigwedge f x. [\bigwedge x. f x \leq \text{expand } x; \bigwedge x. \Phi x \implies f x \leq M x; \Phi x]$
 $\implies \text{expand-body } f x \leq M x$
shows $\text{expand } x \leq M x$
 $\langle \text{proof} \rangle$

abbreviation

expand-assm-incoming n-ns
 $\equiv (\forall nm \in \text{incoming } (fst \ n\text{-ns}). \text{ name } (fst \ n\text{-ns}) > nm)$
 $\wedge 0 < \text{ name } (fst \ n\text{-ns})$
 $\wedge (\forall q \in \text{snd } n\text{-ns}.$
 $\text{ name } (fst \ n\text{-ns}) > \text{ name } q$
 $\wedge (\forall nm \in \text{incoming } q. \text{ name } (fst \ n\text{-ns}) > nm))$

abbreviation

expand-rslt-incoming nm-nds
 $\equiv (\forall q \in \text{snd } nm\text{-nds}. (fst \ nm\text{-nds} > \text{ name } q \wedge (\forall nm' \in \text{incoming } q. fst \ nm\text{-nds} > nm')))$

abbreviation

expand-rslt-name n-ns nm-nds
 $\equiv (\text{ name } (fst \ n\text{-ns}) \leq fst \ nm\text{-nds} \wedge \text{ name } ' (snd \ n\text{-ns}) \subseteq \text{ name } ' (snd \ nm\text{-nds}))$
 $\wedge \text{ name } ' (snd \ nm\text{-nds})$
 $= \text{ name } ' (snd \ n\text{-ns}) \cup \text{ name } ' \{nd \in \text{snd } nm\text{-nds}. \text{ name } nd \geq \text{ name } (fst \ n\text{-ns})\}$

abbreviation

expand-name-ident nds
 $\equiv (\forall q \in nds. \exists ! q' \in nds. \text{ name } q = \text{ name } q')$

abbreviation

expand-assm-exist ξ n-ns
 $\equiv \{\eta. \exists \mu. \mu \ U_r \ \eta \in \text{old } (fst \ n\text{-ns}) \wedge \xi \models_r \eta\} \subseteq \text{new } (fst \ n\text{-ns}) \cup \text{old } (fst \ n\text{-ns})$
 $\wedge (\forall \psi \in \text{new } (fst \ n\text{-ns}). \xi \models_r \psi)$
 $\wedge (\forall \psi \in \text{old } (fst \ n\text{-ns}). \xi \models_r \psi)$
 $\wedge (\forall \psi \in \text{next } (fst \ n\text{-ns}). \xi \models_r X_r \ \psi)$

abbreviation

expand-rslt-exist--node-prop ξ n nd
 $\equiv \text{incoming } n \subseteq \text{incoming } nd$
 $\wedge (\forall \psi \in \text{old } nd. \xi \models_r \psi) \wedge (\forall \psi \in \text{next } nd. \xi \models_r X_r \ \psi)$

$$\wedge \{ \eta. \exists \mu. \mu U_r \eta \in \text{old } nd \wedge \xi \models_r \eta \} \subseteq \text{old } nd$$

abbreviation

$$\begin{aligned} & \text{expand-rslt-exist } \xi \text{ } n\text{-ns } nm\text{-nds} \\ & \equiv (\exists nd \in \text{snd } nm\text{-nds}. \text{expand-rslt-exist--node-prop } \xi \text{ (fst } n\text{-ns) } nd) \end{aligned}$$

abbreviation

$$\begin{aligned} & \text{expand-rslt-exist-eq--node } n \text{ } nd \\ & \equiv \text{name } n = \text{name } nd \\ & \wedge \text{old } n = \text{old } nd \\ & \wedge \text{next } n = \text{next } nd \\ & \wedge \text{incoming } n \subseteq \text{incoming } nd \end{aligned}$$

abbreviation

$$\begin{aligned} & \text{expand-rslt-exist-eq } n\text{-ns } nm\text{-nds} \equiv \\ & (\forall n \in \text{snd } n\text{-ns}. \exists nd \in \text{snd } nm\text{-nds}. \text{expand-rslt-exist-eq--node } n \text{ } nd) \end{aligned}$$

lemma *expand-name-propag*:

assumes *expand-asm-incoming* $n\text{-ns} \wedge \text{expand-name-ident} (\text{snd } n\text{-ns})$ (**is** $?Q$ $n\text{-ns}$)

$$\begin{aligned} \text{shows } \text{expand } n\text{-ns} \leq \text{SPEC } (\lambda r. & \text{expand-rslt-incoming } r \\ & \wedge \text{expand-rslt-name } n\text{-ns } r \\ & \wedge \text{expand-name-ident} (\text{snd } r)) \end{aligned}$$

(**is** $\text{expand} - \leq \text{SPEC} (?P \text{ } n\text{-ns})$)

<proof>

lemmas *expand-name-propag--incoming* = *SPEC-rule-conjunct1*[*OF expand-name-propag*]

lemmas *expand-name-propag--name* =

$$\text{SPEC-rule-conjunct1}[\text{OF SPEC-rule-conjunct2}[\text{OF } \text{expand-name-propag}]]$$

lemmas *expand-name-propag--name-ident* =

$$\text{SPEC-rule-conjunct2}[\text{OF SPEC-rule-conjunct2}[\text{OF } \text{expand-name-propag}]]$$

lemma *expand-rslt-exist-eq*:

$$\begin{aligned} \text{shows } \text{expand } n\text{-ns} \leq \text{SPEC } (\text{expand-rslt-exist-eq } n\text{-ns}) \\ (\text{is } - \leq \text{SPEC} (?P \text{ } n\text{-ns})) \end{aligned}$$

<proof>

lemma *expand-prop-exist*:

$$\text{expand } n\text{-ns} \leq \text{SPEC } (\lambda r. \text{expand-asm-exist } \xi \text{ } n\text{-ns} \longrightarrow \text{expand-rslt-exist } \xi \text{ } n\text{-ns} \text{ } r)$$

(**is** $- \leq \text{SPEC} (?P \text{ } n\text{-ns})$)

<proof>

Termination proof

definition $\text{expand}_T :: ('a \text{ node} \times ('a \text{ node set})) \Rightarrow (\text{node-name} \times 'a \text{ node set}) \text{ nres}$

where $\text{expand}_T \text{ } n\text{-ns} \equiv \text{REC}_T \text{ } \text{expand-body } n\text{-ns}$

abbreviation $\text{old-next-pair } n \equiv (\text{old } n, \text{next } n)$

abbreviation *old-next-limit* $\varphi \equiv \text{Pow}(\text{subfrmlsr } \varphi) \times \text{Pow}(\text{subfrmlsr } \varphi)$

lemma *old-next-limit-finite*: *finite* (*old-next-limit* φ)
(*proof*)

definition

expand-ord $\varphi \equiv$
inv-image (*finite-psupset* (*old-next-limit* φ) *<*lex*>* *less-than*)
($\lambda(n, ns). (\text{old-next-pair } ' ns, \text{size-set } (new\ n))$))

lemma *expand-ord-wf[simp]*: *wf* (*expand-ord* φ)
(*proof*)

abbreviation

expand-inv-node $\varphi\ n$
 $\equiv \text{finite } (new\ n) \wedge \text{finite } (old\ n) \wedge \text{finite } (next\ n)$
 $\wedge (new\ n) \cup (old\ n) \cup (next\ n) \subseteq \text{subfrmlsr } \varphi$

abbreviation

expand-inv-result $\varphi\ ns \equiv \text{finite } ns \wedge (\forall n' \in ns. (new\ n') \cup (old\ n') \cup (next\ n') \subseteq \text{subfrmlsr } \varphi)$

definition

expand-inv $\varphi\ n\ ns \equiv (\text{case } n\ ns \text{ of } (n, ns) \Rightarrow \text{expand-inv-node } \varphi\ n \wedge \text{expand-inv-result } \varphi\ ns)$

lemma *new1-less-sum*:

size-set (*new1* φ) $<$ *size-set* $\{\varphi\}$
(*proof*)

lemma *new2-less-sum*:

size-set (*new2* φ) $<$ *size-set* $\{\varphi\}$
(*proof*)

lemma *new1-finite[intro]*: *finite* (*new1* ψ)
(*proof*)

lemma *new1-subset-frmls*: $\varphi \in \text{new1 } \psi \implies \varphi \in \text{subfrmlsr } \psi$
(*proof*)

lemma *new2-finite[intro]*: *finite* (*new2* ψ)
(*proof*)

lemma *new2-subset-frmls*: $\varphi \in \text{new2 } \psi \implies \varphi \in \text{subfrmlsr } \psi$
(*proof*)

lemma *next1-finite[intro]*: *finite* (*next1* ψ)
(*proof*)

lemma *next1-subset-frmls*: $\varphi \in \text{next1 } \psi \implies \varphi \in \text{subfrmlsr } \psi$
(*proof*)

lemma *expand-inv-impl[intro!]*:
assumes *expand-inv* φ (n , ns)
and *newn*: $\psi \in \text{new } n$
and *old-next-pair* ‘ $ns \subseteq \text{old-next-pair}$ ‘ ns'
and *expand-inv-result* φ ns'
and ($n' = n \langle \text{new} := \text{new } n - \{\psi\},$
 $\text{old} := \{\psi\} \cup \text{old } n \rangle \vee$
 $(n' = n \langle \text{new} := \text{new1 } \psi \cup (\text{new } n - \{\psi\}),$
 $\text{old} := \{\psi\} \cup \text{old } n,$
 $\text{next} := \text{next1 } \psi \cup \text{next } n \rangle \vee$
 $(n' = n \langle \text{name} := nm,$
 $\text{new} := \text{new2 } \psi \cup (\text{new } n - \{\psi\}),$
 $\text{old} := \{\psi\} \cup \text{old } n \rangle)$
(is $?case1 \vee ?case2 \vee ?case3)$
shows $((n', ns'), (n, ns)) \in \text{expand-ord } \varphi \wedge \text{expand-inv } \varphi (n', ns')$
(is $?concl1 \wedge ?concl2)$
 $\langle \text{proof} \rangle$

lemma *expand-term-prop-help*:
assumes $((n', ns'), (n, ns)) \in \text{expand-ord } \varphi \wedge \text{expand-inv } \varphi (n', ns')$
and *assm-rule*: $\llbracket \text{expand-inv } \varphi (n', ns'); ((n', ns'), (n, ns)) \in \text{expand-ord } \varphi \rrbracket$
 $\implies f (n', ns') \leq \text{SPEC } P$
shows $f (n', ns') \leq \text{SPEC } P$
 $\langle \text{proof} \rangle$

lemma *expand-inv-upd-incoming*:
assumes *expand-inv* φ (n , ns)
shows *expand-inv-result* φ (*upd-incoming* n ns)
 $\langle \text{proof} \rangle$

lemma *upd-incoming-eq-old-next-pair*: *old-next-pair* ‘ $ns = \text{old-next-pair}$ ‘ (*upd-incoming* n ns)
(is $?A = ?B)$
 $\langle \text{proof} \rangle$

lemma *expand-term-prop*:
 $\text{expand-inv } \varphi \text{ } n\text{-}ns \implies$
 $\text{expand}_T \text{ } n\text{-}ns \leq \text{SPEC } (\lambda(-, nds). \text{old-next-pair}$ ‘ $\text{snd } n\text{-}ns \subseteq \text{old-next-pair}$ ‘ nds
 $\wedge \text{expand-inv-result } \varphi \text{ } nds)$
(is $- \implies - \leq \text{SPEC } (?P \text{ } n\text{-}ns)$
 $\langle \text{proof} \rangle$

lemma *expand-eq-expand_T*:
assumes *inv*: *expand-inv* φ $n\text{-}ns$
shows $\text{expand}_T \text{ } n\text{-}ns = \text{expand } n\text{-}ns$
 $\langle \text{proof} \rangle$

lemma *expand-nofail*:
assumes *inv*: *expand-inv* φ *n-ns*
shows *nofail* (*expand_T* *n-ns*)
<proof>

lemma [*intro!*]: *expand-inv* φ (
 (
 name = *expand-new-name* *expand-init*,
 incoming = {*expand-init*},
 new = { φ },
 old = {},
 next = {})
 {})
<proof>

definition *create-graph* :: '*a* *frml* \Rightarrow '*a* *node set nres*

where

```

create-graph  $\varphi \equiv$ 
do {
  (-, nds)  $\leftarrow$  expand (
    (
      name = expand-new-name expand-init,
      incoming = {expand-init},
      new = { $\varphi$ },
      old = {},
      next = {}
    )::'a node,
    {}::'a node set);
  RETURN nds
}

```

definition *create-graph_T* :: '*a* *frml* \Rightarrow '*a* *node set nres*

where

```

create-graphT  $\varphi \equiv$  do {
  (-, nds)  $\leftarrow$  expandT (
    (
      name = expand-new-name expand-init,
      incoming = {expand-init},
      new = { $\varphi$ },
      old = {},
      next = {}
    )::'a node,
    {}::'a node set);
  RETURN nds
}

```

lemma *create-graph-eq-create-graph_T*: *create-graph* $\varphi =$ *create-graph_T* φ

<proof>

lemma *create-graph-finite*: *create-graph* $\varphi \leq$ *SPEC finite*
<proof>

lemma *create-graph-nofail*: *nofail* (*create-graph* φ)
<proof>

abbreviation

create-graph-rslt-exist ξ *nds*
 $\equiv \exists nd \in nds.$
 expand-init \in *incoming* *nd*
 $\wedge (\forall \psi \in old\ nd. \xi \models_r \psi) \wedge (\forall \psi \in next\ nd. \xi \models_r X_r \psi)$
 $\wedge \{\eta. \exists \mu. \mu U_r \eta \in old\ nd \wedge \xi \models_r \eta\} \subseteq old\ nd$

lemma *L4-7*:
assumes $\xi \models_r \varphi$
shows *create-graph* $\varphi \leq$ *SPEC* (*create-graph-rslt-exist* ξ)
<proof>

lemma *expand-incoming-name-exist*:
assumes *name* (*fst* *n-ns*) $>$ *expand-init*
 $\wedge (\forall nm \in incoming\ (fst\ n-ns). nm \neq expand-init \longrightarrow nm \in name\ ' (snd\ n-ns))$
 $\wedge expand-assm-incoming\ n-ns \wedge expand-name-ident\ (snd\ n-ns)$ (**is** $?Q\ n-ns$)
and $\forall nd \in snd\ n-ns.$
 name *nd* $>$ *expand-init*
 $\wedge (\forall nm \in incoming\ nd. nm \neq expand-init \longrightarrow nm \in name\ ' (snd\ n-ns))$
 (**is** $?P\ (snd\ n-ns)$)
shows *expand* *n-ns* \leq *SPEC* ($\lambda nm-nds. ?P\ (snd\ nm-nds)$)
<proof>

lemma *create-graph--incoming-name-exist*:
create-graph $\varphi \leq$ *SPEC* ($\lambda nds. \forall nd \in nds. expand-init < name\ nd \wedge (\forall nm \in incoming\ nd. nm \neq expand-init \longrightarrow nm \in name\ ' nds)$)
<proof>

abbreviation

expand-rslt-all--ex-equiv ξ *nd* *nds* \equiv
($\exists nd' \in nds.$
 name *nd* \in *incoming* *nd'*
 $\wedge (\forall \psi \in old\ nd'. suffix\ 1\ \xi \models_r \psi) \wedge (\forall \psi \in next\ nd'. suffix\ 1\ \xi \models_r X_r \psi)$
 $\wedge \{\eta. \exists \mu. \mu U_r \eta \in old\ nd' \wedge suffix\ 1\ \xi \models_r \eta\} \subseteq old\ nd'$)

abbreviation

expand-rslt-all ξ *n-ns* *nm-nds* \equiv
($\forall nd \in snd\ nm-nds. name\ nd \notin name\ ' (snd\ n-ns) \wedge$

$$\begin{aligned}
& (\forall \psi \in \text{old } nd. \xi \models_r \psi) \wedge (\forall \psi \in \text{next } nd. \xi \models_r X_r \psi) \\
& \longrightarrow \text{expand-rslt-all-ex-equiv } \xi \text{ nd } (\text{snd } nm\text{-nds})
\end{aligned}$$

lemma *expand-prop-all*:

assumes *expand-assm-incoming* $n\text{-ns} \wedge \text{expand-name-ident } (\text{snd } n\text{-ns})$ (**is** $?Q$ $n\text{-ns}$)

shows $\text{expand } n\text{-ns} \leq \text{SPEC } (\text{expand-rslt-all } \xi \text{ } n\text{-ns})$

(**is** $\leq \text{SPEC } (?P \text{ } n\text{-ns})$)

<proof>

abbreviation

create-graph-rslt-all $\xi \text{ nds}$

$\equiv \forall q \in \text{nds}. (\forall \psi \in \text{old } q. \xi \models_r \psi) \wedge (\forall \psi \in \text{next } q. \xi \models_r X_r \psi)$

$\longrightarrow (\exists q' \in \text{nds}. \text{name } q \in \text{incoming } q')$

$\wedge (\forall \psi \in \text{old } q'. \text{suffix } 1 \xi \models_r \psi)$

$\wedge (\forall \psi \in \text{next } q'. \text{suffix } 1 \xi \models_r X_r \psi)$

$\wedge \{\eta. \exists \mu. \mu U_r \eta \in \text{old } q' \wedge \text{suffix } 1 \xi \models_r \eta\} \subseteq \text{old } q'$

lemma *L4-5: create-graph* $\varphi \leq \text{SPEC } (\text{create-graph-rslt-all } \xi)$

<proof>

2.4 Creation of GBA

This section formalizes the second part of the algorithm, that creates the actual generalized Büchi automata from the set of nodes.

definition *create-gba-from-nodes* :: $'a \text{ frml} \Rightarrow 'a \text{ node set} \Rightarrow ('a \text{ node}, 'a \text{ set}) \text{ gba-rec}$

where *create-gba-from-nodes* $\varphi \text{ qs} \equiv \langle$

$g\text{-}V = \text{qs},$

$g\text{-}E = \{(q, q'). q \in \text{qs} \wedge q' \in \text{qs} \wedge \text{name } q \in \text{incoming } q'\},$

$g\text{-}V0 = \{q \in \text{qs}. \text{expand-init} \in \text{incoming } q\},$

$gbg\text{-}F = \{\{q \in \text{qs}. \mu U_r \eta \in \text{old } q \longrightarrow \eta \in \text{old } q\} \mid \mu \eta. \mu U_r \eta \in \text{subfrmlsr } \varphi\},$

$gba\text{-}L = \lambda q \text{ l}. q \in \text{qs} \wedge \{p. \text{prop}_r(p) \in \text{old } q\} \subseteq l \wedge \{p. \text{nprop}_r(p) \in \text{old } q\} \cap l = \{\}$

\rangle

end

locale *create-gba-from-nodes-precond* =

fixes $\varphi :: 'a \text{ ltlr}$

fixes $\text{qs} :: 'a \text{ node set}$

assumes $\text{res}: \text{inres } (\text{create-graph } \varphi) \text{ qs}$

begin

lemma *finite-qs[simp, intro!]: finite* qs

<proof>

lemma *create-gba-from-nodes--invar: gba* $(\text{create-gba-from-nodes } \varphi \text{ qs})$

<proof>

sublocale *gba: gba* $\text{create-gba-from-nodes } \varphi \text{ qs}$

<proof>

lemma *create-gba-from-nodes--fin*: *finite* ($g\text{-}V$ (*create-gba-from-nodes* φ qs))
<proof>

lemma *create-gba-from-nodes--ipath*:
ipath $gba.E$ $r \longleftrightarrow (\forall i. r\ i \in qs \wedge name\ (r\ i) \in incoming\ (r\ (Suc\ i)))$
<proof>

lemma *create-gba-from-nodes--is-run*:
 $gba.is\text{-}run\ r \longleftrightarrow expand\text{-}init \in incoming\ (r\ 0)$
 $\wedge (\forall i. r\ i \in qs \wedge name\ (r\ i) \in incoming\ (r\ (Suc\ i)))$
<proof>

context
begin

abbreviation
 $auto\text{-}run\text{-}j\ j\ \xi\ q \equiv$
 $(\forall \psi \in old\ q. suffix\ j\ \xi \models_r \psi) \wedge (\forall \psi \in next\ q. suffix\ j\ \xi \models_r X_r\ \psi) \wedge$
 $\{\eta. \exists \mu. \mu\ U_r\ \eta \in old\ q \wedge suffix\ j\ \xi \models_r \eta\} \subseteq old\ q$

fun *auto-run* :: [*'a* *interpret*, *'a* *node set*] \Rightarrow *'a* *node word*
where

$auto\text{-}run\ \xi\ nds\ 0$
 $= (SOME\ q. q \in nds \wedge expand\text{-}init \in incoming\ q \wedge auto\text{-}run\text{-}j\ 0\ \xi\ q)$
 $| auto\text{-}run\ \xi\ nds\ (Suc\ k)$
 $= (SOME\ q'. q' \in nds \wedge name\ (auto\text{-}run\ \xi\ nds\ k) \in incoming\ q'$
 $\wedge auto\text{-}run\text{-}j\ (Suc\ k)\ \xi\ q')$

lemma *run-propag-on-create-graph*:
assumes *ipath* $gba.E$ σ
shows $\sigma\ k \in qs \wedge name\ (\sigma\ k) \in incoming\ (\sigma\ (k+1))$
<proof>

lemma *expand-false-propag*:
assumes $false_r \notin old\ (fst\ n\text{-}ns) \wedge (\forall nd \in snd\ n\text{-}ns. false_r \notin old\ nd)$
(is *?Q* *n-ns*)
shows $expand\ n\text{-}ns \leq SPEC\ (\lambda nm\text{-}nds. \forall nd \in snd\ nm\text{-}nds. false_r \notin old\ nd)$
<proof>

lemma *false-propag-on-create-graph*: *create-graph* $\varphi \leq SPEC\ (\lambda nds. \forall nd \in nds. false_r \notin old\ nd)$
<proof>

lemma *expand-and-propag*:

assumes μ and_r $\eta \in \text{old } (fst \ n\text{-}ns)$

$\longrightarrow \{\mu, \eta\} \subseteq \text{old } (fst \ n\text{-}ns) \cup \text{new } (fst \ n\text{-}ns)$ (**is** ?Q $n\text{-}ns$)

and $\forall nd \in \text{snd } n\text{-}ns. \mu$ and_r $\eta \in \text{old } nd \longrightarrow \{\mu, \eta\} \subseteq \text{old } nd$ (**is** ?P ($\text{snd } n\text{-}ns$))

shows $\text{expand } n\text{-}ns \leq \text{SPEC } (\lambda nm\text{-}nds. ?P (\text{snd } nm\text{-}nds))$

<proof>

lemma *and-propag-on-create-graph*:

create-graph $\varphi \leq \text{SPEC } (\lambda nds. \forall nd \in nds. \mu$ and_r $\eta \in \text{old } nd \longrightarrow \{\mu, \eta\} \subseteq \text{old } nd)$

<proof>

lemma *expand-or-propag*:

assumes μ or_r $\eta \in \text{old } (fst \ n\text{-}ns)$

$\longrightarrow \{\mu, \eta\} \cap (\text{old } (fst \ n\text{-}ns) \cup \text{new } (fst \ n\text{-}ns)) \neq \{\}$ (**is** ?Q $n\text{-}ns$)

and $\forall nd \in \text{snd } n\text{-}ns. \mu$ or_r $\eta \in \text{old } nd \longrightarrow \{\mu, \eta\} \cap \text{old } nd \neq \{\}$

(**is** ?P ($\text{snd } n\text{-}ns$))

shows $\text{expand } n\text{-}ns \leq \text{SPEC } (\lambda nm\text{-}nds. ?P (\text{snd } nm\text{-}nds))$

<proof>

lemma *or-propag-on-create-graph*:

create-graph $\varphi \leq \text{SPEC } (\lambda nds. \forall nd \in nds. \mu$ or_r $\eta \in \text{old } nd \longrightarrow \{\mu, \eta\} \cap \text{old } nd \neq \{\})$

<proof>

abbreviation

next-propag--assm $\mu \ n\text{-}ns \equiv$

$(X_r \ \mu \in \text{old } (fst \ n\text{-}ns) \longrightarrow \mu \in \text{next } (fst \ n\text{-}ns))$

$\wedge (\forall nd \in \text{snd } n\text{-}ns. X_r \ \mu \in \text{old } nd \wedge \text{name } nd \in \text{incoming } (fst \ n\text{-}ns))$

$\longrightarrow \mu \in \text{old } (fst \ n\text{-}ns) \cup \text{new } (fst \ n\text{-}ns)$

abbreviation

next-propag--rslt $\mu \ ns \equiv$

$\forall nd \in ns. \forall nd' \in ns. X_r \ \mu \in \text{old } nd \wedge \text{name } nd \in \text{incoming } nd' \longrightarrow \mu \in \text{old } nd'$

lemma *expand-next-propag*:

fixes $n\text{-}ns :: - \times 'a$ node set

assumes *next-propag--assm* $\mu \ n\text{-}ns$

\wedge *next-propag--rslt* $\mu \ (\text{snd } n\text{-}ns)$

\wedge *expand--assm--incoming* $n\text{-}ns$

\wedge *expand--name--ident* ($\text{snd } n\text{-}ns$) (**is** ?Q $n\text{-}ns$)

shows $\text{expand } n\text{-}ns \leq \text{SPEC } (\lambda r. \text{next-propag--rslt } \mu \ (\text{snd } r))$

(**is** $- \leq \text{SPEC } ?P$)

<proof>

lemma *next-propag-on-create-graph*:

create-graph $\varphi \leq \text{SPEC } (\lambda nds. \forall n \in nds. \forall n' \in nds. X_r \ \mu \in \text{old } n \wedge \text{name } n \in \text{incoming } n')$

$n' \longrightarrow \mu \in \text{old } n'$
 ⟨proof⟩

abbreviation

$\text{release-propag--assm } \mu \eta \text{ } n\text{-ns} \equiv$
 $(\mu R_r \eta \in \text{old } (fst \text{ } n\text{-ns}))$
 $\longrightarrow \{\mu, \eta\} \subseteq \text{old } (fst \text{ } n\text{-ns}) \cup \text{new } (fst \text{ } n\text{-ns}) \vee$
 $(\eta \in \text{old } (fst \text{ } n\text{-ns}) \cup \text{new } (fst \text{ } n\text{-ns})) \wedge \mu R_r \eta \in \text{next } (fst \text{ } n\text{-ns}))$
 $\wedge (\forall nd \in \text{snd } n\text{-ns.}$
 $\mu R_r \eta \in \text{old } nd \wedge \text{name } nd \in \text{incoming } (fst \text{ } n\text{-ns}))$
 $\longrightarrow \{\mu, \eta\} \subseteq \text{old } nd \vee$
 $(\eta \in \text{old } nd \wedge \mu R_r \eta \in \text{old } (fst \text{ } n\text{-ns}) \cup \text{new } (fst \text{ } n\text{-ns})))$

abbreviation

$\text{release-propag--rslt } \mu \eta \text{ } ns \equiv$
 $\forall nd \in ns.$
 $\forall nd' \in ns.$
 $\mu R_r \eta \in \text{old } nd \wedge \text{name } nd \in \text{incoming } nd'$
 $\longrightarrow \{\mu, \eta\} \subseteq \text{old } nd \vee$
 $(\eta \in \text{old } nd \wedge \mu R_r \eta \in \text{old } nd')$

lemma *expand-release-propag:*

fixes $n\text{-ns} :: - \times 'a \text{ node set}$

assumes $\text{release-propag--assm } \mu \eta \text{ } n\text{-ns}$

$\wedge \text{release-propag--rslt } \mu \eta \text{ } (\text{snd } n\text{-ns})$

$\wedge \text{expand-assm-incoming } n\text{-ns}$

$\wedge \text{expand-name-ident } (\text{snd } n\text{-ns})$ (**is** $?Q \text{ } n\text{-ns}$)

shows $\text{expand } n\text{-ns} \leq \text{SPEC } (\lambda r. \text{release-propag--rslt } \mu \eta \text{ } (\text{snd } r))$

(**is** $- \leq \text{SPEC } ?P$)

⟨proof⟩

lemma *release-propag-on-create-graph:*

create-graph φ

$\leq \text{SPEC } (\lambda nds. \forall n \in nds. \forall n' \in nds. \mu R_r \eta \in \text{old } n \wedge \text{name } n \in \text{incoming } n'$
 $\longrightarrow (\{\mu, \eta\} \subseteq \text{old } n \vee \eta \in \text{old } n \wedge \mu R_r \eta \in \text{old } n'))$

⟨proof⟩

abbreviation

$\text{until-propag--assm } f g \text{ } n\text{-ns} \equiv$
 $(f U_r g \in \text{old } (fst \text{ } n\text{-ns}))$
 $\longrightarrow (g \in \text{old } (fst \text{ } n\text{-ns}) \cup \text{new } (fst \text{ } n\text{-ns}))$
 $\vee (f \in \text{old } (fst \text{ } n\text{-ns}) \cup \text{new } (fst \text{ } n\text{-ns}) \wedge f U_r g \in \text{next } (fst \text{ } n\text{-ns})))$
 $\wedge (\forall nd \in \text{snd } n\text{-ns. } f U_r g \in \text{old } nd \wedge \text{name } nd \in \text{incoming } (fst \text{ } n\text{-ns}))$
 $\longrightarrow (g \in \text{old } nd \vee (f \in \text{old } nd \wedge f U_r g \in \text{old } (fst \text{ } n\text{-ns}) \cup \text{new } (fst \text{ } n\text{-ns})))$

abbreviation

$\text{until-propag--rslt } f g \text{ } ns \equiv$

$$\forall n \in ns. \forall nd \in ns. f U_r g \in old\ n \wedge name\ n \in incoming\ nd \\ \longrightarrow (g \in old\ n \vee (f \in old\ n \wedge f U_r g \in old\ nd))$$

lemma *expand-until-propag*:

fixes $n\text{-}ns :: - \times 'a\ node\ set$

assumes *until-propag--assm* $\mu\ \eta\ n\text{-}ns$

\wedge *until-propag--rslt* $\mu\ \eta\ (snd\ n\text{-}ns)$

\wedge *expand-assm-incoming* $n\text{-}ns$

\wedge *expand-name-ident* $(snd\ n\text{-}ns)$ (**is** $?Q\ n\text{-}ns$)

shows *expand* $n\text{-}ns \leq SPEC\ (\lambda r. \text{until-propag--rslt}\ \mu\ \eta\ (snd\ r))$

(**is** $- \leq SPEC\ ?P$)

<proof>

lemma *until-propag-on-create-graph*:

create-graph $\varphi \leq SPEC\ (\lambda nds. \forall n \in nds. \forall n' \in nds. \mu\ U_r\ \eta \in old\ n \wedge name\ n \in incoming\ n')$

$\longrightarrow (\eta \in old\ n \vee \mu \in old\ n \wedge \mu\ U_r\ \eta \in old\ n')$

<proof>

definition *all-subfrmls* $:: 'a\ node \Rightarrow 'a\ frml\ set$

where *all-subfrmls* $n \equiv \bigcup (subfrmlsr\ ' (new\ n \cup old\ n \cup next\ n))$

lemma *all-subfrmls--UnionD*:

assumes $(\bigcup x \in A. subfrmlsr\ x) \subseteq B$

and $x \in A$

and $y \in subfrmlsr\ x$

shows $y \in B$

<proof>

lemma *expand-all-subfrmls-propag*:

assumes *all-subfrmls* $(fst\ n\text{-}ns) \subseteq B \wedge (\forall nd \in snd\ n\text{-}ns. all\text{-}subfrmls\ nd \subseteq B)$ (**is** $?Q\ n\text{-}ns$)

shows *expand* $n\text{-}ns \leq SPEC\ (\lambda r. \forall nd \in snd\ r. all\text{-}subfrmls\ nd \subseteq B)$

(**is** $- \leq SPEC\ ?P$)

<proof>

lemma *old-propag-on-create-graph*: *create-graph* $\varphi \leq SPEC\ (\lambda nds. \forall n \in nds. old\ n \subseteq subfrmlsr\ \varphi)$

<proof>

lemma *L4-2--aux*:

assumes *run*: *ipath* $gba.E\ \sigma$

and $\mu\ U_r\ \eta \in old\ (\sigma\ 0)$

and $\forall j. (\forall i < j. \{\mu, \mu\ U_r\ \eta\} \subseteq old\ (\sigma\ i)) \longrightarrow \eta \notin old\ (\sigma\ j)$

shows $\forall i. \{\mu, \mu\ U_r\ \eta\} \subseteq old\ (\sigma\ i) \wedge \eta \notin old\ (\sigma\ i)$

<proof>

lemma *L4-2a*:

assumes $ipath\ gba.E\ \sigma$
and $\mu\ U_r\ \eta \in old\ (\sigma\ 0)$
shows $(\forall i. \{\mu, \mu\ U_r\ \eta\} \subseteq old\ (\sigma\ i) \wedge \eta \notin old\ (\sigma\ i))$
 $\vee (\exists j. (\forall i < j. \{\mu, \mu\ U_r\ \eta\} \subseteq old\ (\sigma\ i) \wedge \eta \in old\ (\sigma\ j)))$
(is $?A \vee ?B$
 $\langle proof \rangle$

lemma $L4-2b$:
assumes $run: ipath\ gba.E\ \sigma$
and $\mu\ U_r\ \eta \in old\ (\sigma\ 0)$
and $ACC: gba.is-acc\ \sigma$
shows $\exists j. (\forall i < j. \{\mu, \mu\ U_r\ \eta\} \subseteq old\ (\sigma\ i) \wedge \eta \in old\ (\sigma\ j))$
 $\langle proof \rangle$

lemma $L4-2c$:
assumes $run: ipath\ gba.E\ \sigma$
and $\mu\ R_r\ \eta \in old\ (\sigma\ 0)$
shows $\forall i. \eta \in old\ (\sigma\ i) \vee (\exists j < i. \mu \in old\ (\sigma\ j))$
 $\langle proof \rangle$

lemma $L4-8'$:
assumes $ipath\ gba.E\ \sigma$ **(is** $?inf-run\ \sigma$)
and $gba.is-acc\ \sigma$ **(is** $?gbarel-accept\ \sigma$)
and $\forall i. gba.L\ (\sigma\ i)\ (\xi\ i)$ **(is** $?lgbarel-accept\ \xi\ \sigma$)
and $\psi \in old\ (\sigma\ 0)$
shows $\xi \models_r \psi$
 $\langle proof \rangle$

lemma $L4-8$:
assumes $gba.is-acc-run\ \sigma$
and $\forall i. gba.L\ (\sigma\ i)\ (\xi\ i)$
and $\psi \in old\ (\sigma\ 0)$
shows $\xi \models_r \psi$
 $\langle proof \rangle$

lemma $expand-expand-init-propag$:
assumes $(\forall nm \in incoming\ n'. nm < name\ n')$
 $\wedge name\ n' \leq name\ (fst\ n-ns)$
 $\wedge (incoming\ n' \cap incoming\ (fst\ n-ns) \neq \{\})$
 $\longrightarrow new\ n' \subseteq old\ (fst\ n-ns) \cup new\ (fst\ n-ns)$
(is $?Q\ n-ns$)
and $\forall nd \in snd\ n-ns. \forall nm \in incoming\ n'. nm \in incoming\ nd \longrightarrow new\ n' \subseteq old\ nd$
(is $?P\ (snd\ n-ns)$)
shows $expand\ n-ns \leq SPEC\ (\lambda r. name\ n' \leq fst\ r \wedge ?P\ (snd\ r))$
 $\langle proof \rangle$

lemma $expand-init-propag-on-create-graph$:
 $create-graph\ \varphi \leq SPEC(\lambda nds. \forall nd \in nds. expand-init \in incoming\ nd \longrightarrow \varphi \in old$

nd)
⟨*proof*⟩

lemma *L4-6*:
assumes $q \in gba.V0$
shows $\varphi \in old\ q$
⟨*proof*⟩

lemma *L4-9*:
assumes $acc: gba.accept\ \xi$
shows $\xi \models_r \varphi$
⟨*proof*⟩

lemma *L4-10*:
assumes $\xi \models_r \varphi$
shows $gba.accept\ \xi$
⟨*proof*⟩

end
end

lemma *create-graph--name-ident*: $create_graph\ \varphi \leq SPEC\ (\lambda nds. \forall q \in nds. \exists! q' \in nds. name\ q = name\ q')$
⟨*proof*⟩

corollary *create-graph--name-inj*: $create_graph\ \varphi \leq SPEC\ (\lambda nds. inj_on\ name\ nds)$
⟨*proof*⟩

definition
create-gba φ
 $\equiv do\ \{ nds \leftarrow create_graph_T\ \varphi;$
 $RETURN\ (create_gba_from_nodes\ \varphi\ nds)\ \}$

lemma *create-graph-precond*: $create_graph\ \varphi \leq SPEC\ (create_gba_from_nodes_precond\ \varphi)$
⟨*proof*⟩

lemma *create-gba--invar*: $create_gba\ \varphi \leq SPEC\ gba$
⟨*proof*⟩

lemma *create-gba-acc*:
shows $create_gba\ \varphi \leq SPEC(\lambda \mathcal{A}. \forall \xi. gba.accept\ \mathcal{A}\ \xi \longleftrightarrow \xi \models_r \varphi)$
⟨*proof*⟩

lemma *create-gba--name-inj*:
shows $create_gba\ \varphi \leq SPEC(\lambda \mathcal{A}. (inj_on\ name\ (g-V\ \mathcal{A})))$
⟨*proof*⟩

lemma *create-gba--fin:*

shows $\text{create-gba } \varphi \leq \text{SPEC}(\lambda \mathcal{A}. (\text{finite } (g\text{-}V \ \mathcal{A})))$
<proof>

lemma *create-graph-old-finite:*

$\text{create-graph } \varphi \leq \text{SPEC} (\lambda \text{nds}. \forall \text{nd} \in \text{nds}. \text{finite } (\text{old } \text{nd}))$
<proof>

lemma *create-gba--old-fin:*

shows $\text{create-gba } \varphi \leq \text{SPEC}(\lambda \mathcal{A}. \forall \text{nd} \in g\text{-}V \ \mathcal{A}. \text{finite } (\text{old } \text{nd}))$
<proof>

lemma *create-gba--incoming-exists:*

shows $\text{create-gba } \varphi$
 $\leq \text{SPEC}(\lambda \mathcal{A}. \forall \text{nd} \in g\text{-}V \ \mathcal{A}. \text{incoming } \text{nd} \subseteq \text{insert } \text{expand-init } (\text{name } ' (g\text{-}V \ \mathcal{A})))$
<proof>

lemma *create-gba--no-init:*

shows $\text{create-gba } \varphi \leq \text{SPEC}(\lambda \mathcal{A}. \text{expand-init } \notin \text{name } ' (g\text{-}V \ \mathcal{A}))$
<proof>

definition *nds-invars* $\text{nds} \equiv$

$\text{inj-on name } \text{nds}$
 $\wedge \text{finite } \text{nds}$
 $\wedge \text{expand-init } \notin \text{name } ' \text{nds}$
 $\wedge (\forall \text{nd} \in \text{nds}.$
 $\quad \text{finite } (\text{old } \text{nd})$
 $\quad \wedge \text{incoming } \text{nd} \subseteq \text{insert } \text{expand-init } (\text{name } ' \text{nds}))$

lemma *create-gba-nds-invars:* $\text{create-gba } \varphi \leq \text{SPEC} (\lambda \mathcal{A}. \text{nds-invars } (g\text{-}V \ \mathcal{A}))$

<proof>

theorem *T4-1:*

$\text{create-gba } \varphi \leq \text{SPEC}(\lambda \mathcal{A}. \text{gba } \mathcal{A}$
 $\quad \wedge \text{finite } (g\text{-}V \ \mathcal{A})$
 $\quad \wedge (\forall \xi. \text{gba.accept } \mathcal{A} \ \xi \longleftrightarrow \xi \models_r \varphi)$
 $\quad \wedge (\text{nds-invars } (g\text{-}V \ \mathcal{A})))$
<proof>

definition *create-name-gba* $\varphi \equiv \text{do } \{$

$\quad G \leftarrow \text{create-gba } \varphi;$
 $\quad \text{ASSERT } (\text{nds-invars } (g\text{-}V \ G));$
 $\quad \text{RETURN } (\text{gba-rename name } G)$
 $\}$

theorem *create-name-gba-correct:*

create-name-gba $\varphi \leq \text{SPEC}(\lambda A. \text{gba } A \wedge \text{finite } (g-V A) \wedge (\forall \xi. \text{gba.accept } A \xi \longleftrightarrow \xi \models_r \varphi))$
 <proof>

definition *create-name-igba* :: 'a::linorder ltlr \Rightarrow - **where**
create-name-igba $\varphi \equiv$ do {
 A \leftarrow *create-name-gba* φ ;
 A' \leftarrow *gba-to-idx* A;
 stat-set-data-nres (card (g-V A)) (card (g-V0 A')) (igbg-num-acc A');
 RETURN A'
 }

lemma *create-name-igba-correct*: *create-name-igba* $\varphi \leq \text{SPEC } (\lambda G. \text{igba } G \wedge \text{finite } (g-V G) \wedge (\forall \xi. \text{igba.accept } G \xi \longleftrightarrow \xi \models_r \varphi))$
 <proof>

context

notes [*refine-vcg*] = *order-trans*[*OF create-name-gba-correct*]
begin

lemma *create-name-igba* $\varphi \leq \text{SPEC } (\lambda G. \text{igba } G \wedge (\forall \xi. \text{igba.accept } G \xi \longleftrightarrow \xi \models_r \varphi))$
 <proof>

end

end

3 Refinement to Efficient Code

theory *LTL-to-GBA-impl*

imports

LTL-to-GBA
Deriving.Compare-Instances
CAVA-Automata.Automata-Impl
CAVA-Base.CAVA-Code-Target

begin

3.1 Parametricity Setup Boilerplate

3.1.1 LTL Formulas

derive *linorder ltlr*

inductive-set *ltlr-rel* for *R* where

(*True-ltlr*, *True-ltlr*) \in *ltlr-rel* *R*
 | (*False-ltlr*, *False-ltlr*) \in *ltlr-rel* *R*
 | (*a*, *a'*) \in *R* \implies (*Prop-ltlr a*, *Prop-ltlr a'*) \in *ltlr-rel* *R*

$| (a, a') \in R \implies (Nprop\text{-}ltrl\ a, Nprop\text{-}ltrl\ a') \in ltlr\text{-}rel\ R$
 $| \llbracket (a, a') \in ltlr\text{-}rel\ R; (b, b') \in ltlr\text{-}rel\ R \rrbracket$
 $\implies (And\text{-}ltrl\ a\ b, And\text{-}ltrl\ a'\ b') \in ltlr\text{-}rel\ R$
 $| \llbracket (a, a') \in ltlr\text{-}rel\ R; (b, b') \in ltlr\text{-}rel\ R \rrbracket$
 $\implies (Or\text{-}ltrl\ a\ b, Or\text{-}ltrl\ a'\ b') \in ltlr\text{-}rel\ R$
 $| \llbracket (a, a') \in ltlr\text{-}rel\ R \rrbracket \implies (Next\text{-}ltrl\ a, Next\text{-}ltrl\ a') \in ltlr\text{-}rel\ R$
 $| \llbracket (a, a') \in ltlr\text{-}rel\ R; (b, b') \in ltlr\text{-}rel\ R \rrbracket$
 $\implies (Until\text{-}ltrl\ a\ b, Until\text{-}ltrl\ a'\ b') \in ltlr\text{-}rel\ R$
 $| \llbracket (a, a') \in ltlr\text{-}rel\ R; (b, b') \in ltlr\text{-}rel\ R \rrbracket$
 $\implies (Release\text{-}ltrl\ a\ b, Release\text{-}ltrl\ a'\ b') \in ltlr\text{-}rel\ R$

lemmas *ltrl-rel-induct*[*induct set*]
 $= ltlr\text{-}rel.\text{induct}[simplified\ relAPP\text{-}def[of\ ltlr\text{-}rel,\ symmetric]]$
lemmas *ltrl-rel-cases*[*cases set*]
 $= ltlr\text{-}rel.\text{cases}[simplified\ relAPP\text{-}def[of\ ltlr\text{-}rel,\ symmetric]]$
lemmas *ltrl-rel-intros*
 $= ltlr\text{-}rel.\text{intros}[simplified\ relAPP\text{-}def[of\ ltlr\text{-}rel,\ symmetric]]$

inductive-simps *ltrl-rel-left-simps*[*simplified relAPP-def[of ltrl-rel, symmetric]*]:

$(True\text{-}ltrl\ z) \in ltlr\text{-}rel\ R$
 $(False\text{-}ltrl\ z) \in ltlr\text{-}rel\ R$
 $(Prop\text{-}ltrl\ p, z) \in ltlr\text{-}rel\ R$
 $(Nprop\text{-}ltrl\ p, z) \in ltlr\text{-}rel\ R$
 $(And\text{-}ltrl\ a\ b, z) \in ltlr\text{-}rel\ R$
 $(Or\text{-}ltrl\ a\ b, z) \in ltlr\text{-}rel\ R$
 $(Next\text{-}ltrl\ a, z) \in ltlr\text{-}rel\ R$
 $(Until\text{-}ltrl\ a\ b, z) \in ltlr\text{-}rel\ R$
 $(Release\text{-}ltrl\ a\ b, z) \in ltlr\text{-}rel\ R$

lemma *ltrl-rel-sv*[*relator-props*]:
assumes *SV*: *single-valued R*
shows *single-valued* $(\langle R \rangle ltlr\text{-}rel)$
 $\langle proof \rangle$

lemma *ltrl-rel-id*[*relator-props*]: $\llbracket R = Id \rrbracket \implies \langle R \rangle ltlr\text{-}rel = Id$
 $\langle proof \rangle$

lemma *ltrl-rel-id-simp*[*simp*]: $\langle Id \rangle ltlr\text{-}rel = Id$ $\langle proof \rangle$

consts *i-ltrl* :: *interface* \Rightarrow *interface*

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of ltrl-rel i-ltrl*]

thm *ltrl-rel-intros*[*no-vars*]

lemma *ltrl-con-param*[*param, autoref-rules*]:

$(True\text{-}ltrl, True\text{-}ltrl) \in \langle R \rangle ltlr\text{-}rel$
 $(False\text{-}ltrl, False\text{-}ltrl) \in \langle R \rangle ltlr\text{-}rel$
 $(Prop\text{-}ltrl, Prop\text{-}ltrl) \in R \rightarrow \langle R \rangle ltlr\text{-}rel$
 $(Nprop\text{-}ltrl, Nprop\text{-}ltrl) \in R \rightarrow \langle R \rangle ltlr\text{-}rel$

$(And\text{-ltrl}, And\text{-ltrl}) \in \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel}$
 $(Or\text{-ltrl}, Or\text{-ltrl}) \in \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel}$
 $(Next\text{-ltrl}, Next\text{-ltrl}) \in \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel}$
 $(Until\text{-ltrl}, Until\text{-ltrl}) \in \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel}$
 $(Release\text{-ltrl}, Release\text{-ltrl}) \in \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel}$
 $\langle proof \rangle$

lemma *case-ltrl-param*[*param, autoref-rules*]:

$(case\text{-ltrl}, case\text{-ltrl}) \in Rv \rightarrow Rv \rightarrow (R \rightarrow Rv)$
 $\rightarrow (R \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv) \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv$
 $\langle proof \rangle$

lemma *rec-ltrl-param*[*param, autoref-rules*]:

$(rec\text{-ltrl}, rec\text{-ltrl}) \in Rv \rightarrow Rv \rightarrow (R \rightarrow Rv)$
 $\rightarrow (R \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv \rightarrow Rv \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv \rightarrow Rv \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow Rv \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv \rightarrow Rv \rightarrow Rv)$
 $\rightarrow (\langle R \rangle \text{ltrl-rel} \rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv \rightarrow Rv \rightarrow Rv)$
 $\rightarrow \langle R \rangle \text{ltrl-rel} \rightarrow Rv$

$\langle proof \rangle$

lemma *case-ltrl-mono*[*refine-mono*]:

assumes $\varphi = True\text{-ltrl} \implies a \leq a'$
assumes $\varphi = False\text{-ltrl} \implies b \leq b'$
assumes $\bigwedge p. \varphi = Prop\text{-ltrl } p \implies c p \leq c' p$
assumes $\bigwedge p. \varphi = Nprop\text{-ltrl } p \implies d p \leq d' p$
assumes $\bigwedge \mu \nu. \varphi = And\text{-ltrl } \mu \nu \implies e \mu \nu \leq e' \mu \nu$
assumes $\bigwedge \mu \nu. \varphi = Or\text{-ltrl } \mu \nu \implies f \mu \nu \leq f' \mu \nu$
assumes $\bigwedge \mu. \varphi = Next\text{-ltrl } \mu \implies g \mu \leq g' \mu$
assumes $\bigwedge \mu \nu. \varphi = Until\text{-ltrl } \mu \nu \implies h \mu \nu \leq h' \mu \nu$
assumes $\bigwedge \mu \nu. \varphi = Release\text{-ltrl } \mu \nu \implies i \mu \nu \leq i' \mu \nu$
shows $case\text{-ltrl } a b c d e f g h i \varphi \leq case\text{-ltrl } a' b' c' d' e' f' g' h' i' \varphi$
 $\langle proof \rangle$

primrec *ltrl-eq* **where**

$ltrl\text{-eq } eq \ True\text{-ltrl } f \iff (case\ f\ of\ True\text{-ltrl} \Rightarrow True \mid - \Rightarrow False)$
 $\mid ltrl\text{-eq } eq \ False\text{-ltrl } f \iff (case\ f\ of\ False\text{-ltrl} \Rightarrow True \mid - \Rightarrow False)$
 $\mid ltrl\text{-eq } eq \ (Prop\text{-ltrl } p) \ f \iff (case\ f\ of\ Prop\text{-ltrl } p' \Rightarrow eq\ p\ p' \mid - \Rightarrow False)$
 $\mid ltrl\text{-eq } eq \ (Nprop\text{-ltrl } p) \ f \iff (case\ f\ of\ Nprop\text{-ltrl } p' \Rightarrow eq\ p\ p' \mid - \Rightarrow False)$
 $\mid ltrl\text{-eq } eq \ (And\text{-ltrl } \mu \nu) \ f$
 $\iff (case\ f\ of\ And\text{-ltrl } \mu' \nu' \Rightarrow ltrl\text{-eq } eq\ \mu\ \mu' \wedge ltrl\text{-eq } eq\ \nu\ \nu' \mid - \Rightarrow False)$

$| \text{ltlr-eq eq } (Or\text{-ltlr } \mu \nu) f$
 $\longleftrightarrow (case\ f\ of\ Or\text{-ltlr } \mu' \nu' \Rightarrow \text{ltlr-eq eq } \mu \mu' \wedge \text{ltlr-eq eq } \nu \nu' \mid - \Rightarrow False)$
 $| \text{ltlr-eq eq } (Next\text{-ltlr } \varphi) f$
 $\longleftrightarrow (case\ f\ of\ Next\text{-ltlr } \varphi' \Rightarrow \text{ltlr-eq eq } \varphi \varphi' \mid - \Rightarrow False)$
 $| \text{ltlr-eq eq } (Until\text{-ltlr } \mu \nu) f$
 $\longleftrightarrow (case\ f\ of\ Until\text{-ltlr } \mu' \nu' \Rightarrow \text{ltlr-eq eq } \mu \mu' \wedge \text{ltlr-eq eq } \nu \nu' \mid - \Rightarrow False)$
 $| \text{ltlr-eq eq } (Release\text{-ltlr } \mu \nu) f$
 $\longleftrightarrow (case\ f\ of$
 $\quad Release\text{-ltlr } \mu' \nu' \Rightarrow \text{ltlr-eq eq } \mu \mu' \wedge \text{ltlr-eq eq } \nu \nu'$
 $\mid - \Rightarrow False)$

lemma *ltlr-eq-autoref*[*autoref-rules*]:
assumes *EQP*: $(eq, (=)) \in R \rightarrow R \rightarrow bool\text{-rel}$
shows $(\text{ltlr-eq eq}, (=)) \in \langle R \rangle \text{ltlr-rel} \rightarrow \langle R \rangle \text{ltlr-rel} \rightarrow bool\text{-rel}$
 $\langle proof \rangle$

lemma *ltlr-dflt-cmp*[*autoref-rules-raw*]:
assumes *PREFER-id* *R*
shows
 $(dflt\text{-cmp } (\leq) (<), dflt\text{-cmp } (\leq) (<))$
 $\in \langle R \rangle \text{ltlr-rel} \rightarrow \langle R \rangle \text{ltlr-rel} \rightarrow comp\text{-res-rel}$
 $\langle proof \rangle$

type-synonym
 $node\text{-name-impl} = node\text{-name}$

abbreviation $(input)\ node\text{-name-rel} \equiv Id :: (node\text{-name-impl} \times node\text{-name})\ set$

lemma *case-ltlr-gtransfer*:
assumes
 $\gamma\ ai \leq a$
 $\gamma\ bi \leq b$
 $\bigwedge a. \gamma\ (ci\ a) \leq c\ a$
 $\bigwedge a. \gamma\ (di\ a) \leq d\ a$
 $\bigwedge \text{ltlr1}\ \text{ltlr2}. \gamma\ (ei\ \text{ltlr1}\ \text{ltlr2}) \leq e\ \text{ltlr1}\ \text{ltlr2}$
 $\bigwedge \text{ltlr1}\ \text{ltlr2}. \gamma\ (fi\ \text{ltlr1}\ \text{ltlr2}) \leq f\ \text{ltlr1}\ \text{ltlr2}$
 $\bigwedge \text{ltlr}. \gamma\ (gi\ \text{ltlr}) \leq g\ \text{ltlr}$
 $\bigwedge \text{ltlr1}\ \text{ltlr2}. \gamma\ (hi\ \text{ltlr1}\ \text{ltlr2}) \leq h\ \text{ltlr1}\ \text{ltlr2}$
 $\bigwedge \text{ltlr1}\ \text{ltlr2}. \gamma\ (iiv\ \text{ltlr1}\ \text{ltlr2}) \leq i\ \text{ltlr1}\ \text{ltlr2}$
shows $\gamma\ (case\text{-ltlr}\ ai\ bi\ ci\ di\ ei\ fi\ gi\ hi\ iiv\ \varphi)$
 $\leq (case\text{-ltlr}\ a\ b\ c\ d\ e\ f\ g\ h\ i\ \varphi)$
 $\langle proof \rangle$

lemmas [*refine-transfer*]
 $= case\text{-ltlr-gtransfer}[\mathbf{where}\ \gamma = nres\text{-of}]\ case\text{-ltlr-gtransfer}[\mathbf{where}\ \gamma = RETURN]$

lemma [*refine-transfer*]:
assumes
 $ai \neq dSUCCEED$

$bi \neq dSUCCEED$
 $\bigwedge a. ci \ a \neq dSUCCEED$
 $\bigwedge a. di \ a \neq dSUCCEED$
 $\bigwedge ltr1 \ ltr2. ei \ ltr1 \ ltr2 \neq dSUCCEED$
 $\bigwedge ltr1 \ ltr2. fi \ ltr1 \ ltr2 \neq dSUCCEED$
 $\bigwedge ltr. gi \ ltr \neq dSUCCEED$
 $\bigwedge ltr1 \ ltr2. hi \ ltr1 \ ltr2 \neq dSUCCEED$
 $\bigwedge ltr1 \ ltr2. iiv \ ltr1 \ ltr2 \neq dSUCCEED$
shows $case-ltr \ ai \ bi \ ci \ di \ ei \ fi \ gi \ hi \ iiv \ \varphi \neq dSUCCEED$
 $\langle proof \rangle$

3.1.2 Nodes

record $'a \ node-impl =$
 $name-impl \ :: \ node-name-impl$
 $incoming-impl \ :: \ (node-name-impl,unit) \ RBT-Impl.rbt$
 $new-impl \ :: \ 'a \ frml \ list$
 $old-impl \ :: \ 'a \ frml \ list$
 $next-impl \ :: \ 'a \ frml \ list$

definition $node-rel \ where \ node-rel-def-internal: \ node-rel \ Re \ R \equiv \{(\$
 \langle $name-impl = namei,$
 $incoming-impl = inci,$
 $new-impl = newi,$
 $old-impl = oldi,$
 $next-impl = nexti,$
 $\dots = morei$
 $\rangle,$
 \langle $name = name,$
 $incoming = inc,$
 $new=new,$
 $old=old,$
 $next = next,$
 $\dots = more$
 $\rangle \mid namei \ name \ inci \ inc \ newi \ new \ oldi \ old \ nexti \ next \ morei \ more.$
 $(namei, name) \in node-name-rel$
 $\wedge (inci, inc) \in (node-name-rel) \ dflt-rs-rel$
 $\wedge (newi, new) \in (\langle R \rangle ltr-rel) \ lss.rel$
 $\wedge (oldi, old) \in (\langle R \rangle ltr-rel) \ lss.rel$
 $\wedge (nexti, next) \in (\langle R \rangle ltr-rel) \ lss.rel$
 $\wedge (morei, more) \in Re$
 $\}$

lemma $node-rel-def: \ \langle Re, R \rangle node-rel = \{(\$
 \langle $name-impl = namei,$
 $incoming-impl = inci,$
 $new-impl = newi,$
 $old-impl = oldi,$
 $next-impl = nexti,$

```

... = morei
),
( name = name,
  incoming = inc,
  new=new,
  old=old,
  next = next,
  ... = more
) ) | namei name inci inc newi new oldi old nexti next morei more.
  (namei,name)∈node-name-rel
  ∧ (inci,inc)∈⟨node-name-rel⟩dflt-rs-rel
  ∧ (newi,new)∈⟨⟨R⟩ltrl-rel⟩lss.rel
  ∧ (oldi,old)∈⟨⟨R⟩ltrl-rel⟩lss.rel
  ∧ (nexti,next)∈⟨⟨R⟩ltrl-rel⟩lss.rel
  ∧ (morei,more)∈Re
} ⟨proof⟩

```

lemma *node-rel-sv*[*relator-props*]:
single-valued Re \implies *single-valued R* \implies *single-valued* ($\langle Re,R \rangle$ node-rel)
 ⟨proof⟩

consts *i-node* :: *interface* \Rightarrow *interface* \Rightarrow *interface*

lemmas [*autoref-rel-intf*] = *REL-INTFI*[of *node-rel i-node*]

lemma [*autoref-rules*]: (*node-impl-ext*, *node-ext*) \in
node-name-rel
 \rightarrow \langle node-name-rel \rangle dflt-rs-rel
 \rightarrow $\langle\langle R \rangle$ ltrl-rel \rangle lss.rel
 \rightarrow $\langle\langle R \rangle$ ltrl-rel \rangle lss.rel
 \rightarrow $\langle\langle R \rangle$ ltrl-rel \rangle lss.rel
 \rightarrow *Re*
 \rightarrow $\langle Re,R \rangle$ node-rel
 ⟨proof⟩

lemma [*autoref-rules*]:
 (*node-impl.name-impl-update*,*node.name-update*)
 \in (*node-name-rel* \rightarrow *node-name-rel*) \rightarrow $\langle Re,R \rangle$ node-rel \rightarrow $\langle Re,R \rangle$ node-rel
 (*node-impl.incoming-impl-update*,*node.incoming-update*)
 \in (\langle node-name-rel \rangle dflt-rs-rel \rightarrow \langle node-name-rel \rangle dflt-rs-rel)
 \rightarrow $\langle Re,R \rangle$ node-rel
 \rightarrow $\langle Re,R \rangle$ node-rel
 (*node-impl.new-impl-update*,*node.new-update*)
 \in ($\langle\langle R \rangle$ ltrl-rel \rangle lss.rel \rightarrow $\langle\langle R \rangle$ ltrl-rel \rangle lss.rel) \rightarrow $\langle Re,R \rangle$ node-rel \rightarrow $\langle Re,R \rangle$ node-rel
 (*node-impl.old-impl-update*,*node.old-update*)
 \in ($\langle\langle R \rangle$ ltrl-rel \rangle lss.rel \rightarrow $\langle\langle R \rangle$ ltrl-rel \rangle lss.rel) \rightarrow $\langle Re,R \rangle$ node-rel \rightarrow $\langle Re,R \rangle$ node-rel
 (*node-impl.next-impl-update*,*node.next-update*)
 \in ($\langle\langle R \rangle$ ltrl-rel \rangle lss.rel \rightarrow $\langle\langle R \rangle$ ltrl-rel \rangle lss.rel) \rightarrow $\langle Re,R \rangle$ node-rel \rightarrow $\langle Re,R \rangle$ node-rel

$(node-impl.more-update, node.more-update)$
 $\in (Re \rightarrow Re) \rightarrow \langle Re, R \rangle node-rel \rightarrow \langle Re, R \rangle node-rel$
 $\langle proof \rangle$

term *name*

lemma [*autoref-rules*]:

$(node-impl.name-impl, node.name) \in \langle Re, R \rangle node-rel \rightarrow node-name-rel$
 $(node-impl.incoming-impl, node.incoming)$
 $\in \langle Re, R \rangle node-rel \rightarrow \langle node-name-rel \rangle dflt-rs-rel$
 $(node-impl.new-impl, node.new) \in \langle Re, R \rangle node-rel \rightarrow \langle \langle R \rangle ltlr-rel \rangle lss.rel$
 $(node-impl.old-impl, node.old) \in \langle Re, R \rangle node-rel \rightarrow \langle \langle R \rangle ltlr-rel \rangle lss.rel$
 $(node-impl.next-impl, node.next) \in \langle Re, R \rangle node-rel \rightarrow \langle \langle R \rangle ltlr-rel \rangle lss.rel$
 $(node-impl.more, node.more) \in \langle Re, R \rangle node-rel \rightarrow Re$
 $\langle proof \rangle$

3.2 Messaging the Abstract Algorithm

In a first step, we do some refinement steps on the abstract data structures, with the goal to make the algorithm more efficient.

3.2.1 Creation of the Nodes

In the expand-algorithm, we replace nested conditionals by case-distinctions, and slightly stratify the code.

abbreviation (*input*) *expand2* *exp* *n* *ns* φ *n1* *nx1* *n2* $\equiv do \{$
 $(nm, nds) \leftarrow exp ($
 $n(|$
 $new := insert\ n1\ (new\ n),$
 $old := insert\ \varphi\ (old\ n),$
 $next := nx1 \cup next\ n\ |),$
 $ns);$
 $exp\ (n(|\ name := nm,\ new := n2 \cup new\ n,\ old := \{\varphi\} \cup old\ n\ |),\ nds)$
 $\}$

definition *expand-aimpl* $\equiv REC_T (\lambda expand\ (n, ns).$

$if\ new\ n = \{\}$ then (
 $if\ (\exists n' \in ns.\ old\ n' = old\ n \wedge next\ n' = next\ n)$ then
 $RETURN\ (name\ n,\ upd-incoming\ n\ ns)$
 $else\ do \{$
 $ASSERT\ (n \notin ns);$
 $ASSERT\ (name\ n \notin name'ns);$
 $expand\ (|$
 $name=expand-new-name\ (name\ n),$
 $incoming=\{name\ n\},$
 $new=next\ n,$
 $old=\{\},$

```

        next={ } ],
        insert n ns)
    }
) else do {
   $\varphi \leftarrow SPEC (\lambda x. x \in (new\ n));$ 
  let n = n(| new := new n - { $\varphi$ } |);
  case  $\varphi$  of
    propr(q)  $\Rightarrow$ 
      if npropr(q)  $\in$  old n then RETURN (name n, ns)
      else expand (n(| old := { $\varphi$ }  $\cup$  old n |), ns)
    | npropr(q)  $\Rightarrow$ 
      if propr(q)  $\in$  old n then RETURN (name n, ns)
      else expand (n(| old := { $\varphi$ }  $\cup$  old n |), ns)
    | truer  $\Rightarrow$  expand (n(| old := { $\varphi$ }  $\cup$  old n |), ns)
    | falser  $\Rightarrow$  RETURN (name n, ns)
    |  $\nu$  andr  $\mu \Rightarrow$  expand (n(|
      new := insert  $\nu$  (insert  $\mu$  (new n)),
      old := { $\varphi$ }  $\cup$  old n,
      next := next n |), ns)
    | Xr  $\nu \Rightarrow$  expand
      (n(| new := new n, old := { $\varphi$ }  $\cup$  old n, next := insert  $\nu$  (next n) |), ns)
    |  $\mu$  orr  $\nu \Rightarrow$  expand2 expand n ns  $\varphi$   $\mu$  { } {  $\nu$  }
    |  $\mu$  Ur  $\nu \Rightarrow$  expand2 expand n ns  $\varphi$   $\mu$  { $\varphi$ } {  $\nu$  }
    |  $\mu$  Rr  $\nu \Rightarrow$  expand2 expand n ns  $\varphi$   $\nu$  { $\varphi$ } {  $\mu, \nu$  }
  }
)

```

lemma *expand-aimpl-refine*:

fixes $n\text{-}ns :: ('a\ node \times -)$
defines $R \equiv Id \cap \{(-, (n, ns)). \forall n' \in ns. n > name\ n'\}$
defines $R' \equiv Id \cap \{(-, (n, ns)). \forall n' \in ns. name\ n > name\ n'\}$
assumes [*refine*]: $(n\text{-}ns', n\text{-}ns) \in R'$
shows $expand\text{-}aimpl\ n\text{-}ns' \leq \Downarrow R (expand_T\ n\text{-}ns)$
<proof>

thm *create-graph-def*

definition *create-graph-aimpl* $\varphi = do$ {
 (-, nds) \leftarrow
 expand-aimpl
 (|name = expand-new-name expand-init, incoming = {expand-init},
 new = { φ }, old = { }, next = { }|),
 { }|);
 RETURN nds

}

lemma *create-graph-aimpl-refine*: $create-graph-aimpl \varphi \leq \Downarrow Id (create-graph_T \varphi)$
 ⟨proof⟩

3.2.2 Creation of GBA from Nodes

We summarize creation of the GBA and renaming of the nodes into one step

lemma *create-name-gba-alt*: $create-name-gba \varphi = do \{$
 $nds \leftarrow create-graph_T \varphi;$
 $ASSERT (nds-invars nds);$
 $RETURN (gba-rename-ext (\lambda. ()) name (create-gba-from-nodes \varphi nds))$
 $\}$
 ⟨proof⟩

In the following, we implement the components of the renamed GBA separately.

definition *build-succ nds* =
 FOREACH
 $nds (\lambda q' s.$
 FOREACH
 $(incoming\ q') (\lambda qn\ s.$
 if $qn = expand-init$ then
 RETURN s
 else
 RETURN $(s(qn \mapsto insert (name\ q') (the-default\ \{\}) (s\ qn))))$
) s
) Map.empty

lemma *build-succ-aux1*:
 assumes [simp]: $finite\ nds$
 assumes [simp]: $\bigwedge q. q \in nds \implies finite (incoming\ q)$
 shows $build-succ\ nds \leq SPEC (\lambda r. r = (\lambda qn.$
 $dflt-None-set \{qn'. \exists q'.$
 $q' \in nds \wedge qn' = name\ q' \wedge qn \in incoming\ q' \wedge qn \neq expand-init$
 $\}))$
 ⟨proof⟩

lemma *build-succ-aux2*:
 assumes NINIT: $expand-init \notin name'nds$
 assumes CL: $\bigwedge nd. nd \in nds \implies incoming\ nd \subseteq insert\ expand-init (name'nds)$
 shows
 $(\lambda qn. dflt-None-set$
 $\{qn'. \exists q'. q' \in nds \wedge qn' = name\ q' \wedge qn \in incoming\ q' \wedge qn \neq expand-init\})$
 = $(\lambda qn. dflt-None-set (succ-of-E$
 $(rename-E\ name\ \{(q, q'). q \in nds \wedge q' \in nds \wedge name\ q \in incoming\ q'\})\ qn))$

(is ($\lambda qn. \text{dflt-None-set } (?L \text{ } qn)$) = ($\lambda qn. \text{dflt-None-set } (?R \text{ } qn)$))
 ⟨proof⟩

lemma *build-succ-correct*:

assumes *NINIT*: $\text{expand-init} \notin \text{name}'\text{nds}$

assumes *FIN*: *finite nds*

assumes *CL*: $\bigwedge nd. nd \in \text{nds} \implies \text{incoming } nd \subseteq \text{insert } \text{expand-init } (\text{name}'\text{nds})$

shows $\text{build-succ } \text{nds} \leq \text{SPEC } (\lambda r.$

$\text{E-of-succ } (\lambda qn. \text{the-default } \{\}) (r \text{ } qn))$

$= \text{rename-E } (\lambda u. \text{name } u) \{(q, q'). q \in \text{nds} \wedge q' \in \text{nds} \wedge \text{name } q \in \text{incoming } q'\}$

⟨proof⟩

primrec *until-frmlsr* :: $'a \text{ frml} \Rightarrow ('a \text{ frml} \times 'a \text{ frml}) \text{ set}$ **where**

$\text{until-frmlsr } (\mu \text{ and}_r \psi) = (\text{until-frmlsr } \mu) \cup (\text{until-frmlsr } \psi)$

| $\text{until-frmlsr } (X_r \mu) = \text{until-frmlsr } \mu$

| $\text{until-frmlsr } (\mu \text{ U}_r \psi) = \text{insert } (\mu, \psi) ((\text{until-frmlsr } \mu) \cup (\text{until-frmlsr } \psi))$

| $\text{until-frmlsr } (\mu \text{ R}_r \psi) = (\text{until-frmlsr } \mu) \cup (\text{until-frmlsr } \psi)$

| $\text{until-frmlsr } (\mu \text{ or}_r \psi) = (\text{until-frmlsr } \mu) \cup (\text{until-frmlsr } \psi)$

| $\text{until-frmlsr } (\text{true}_r) = \{\}$

| $\text{until-frmlsr } (\text{false}_r) = \{\}$

| $\text{until-frmlsr } (\text{prop}_r(-)) = \{\}$

| $\text{until-frmlsr } (\text{nprop}_r(-)) = \{\}$

lemma *until-frmlsr-correct*:

$\text{until-frmlsr } \varphi = \{(\mu, \eta). \text{Until-ltlr } \mu \ \eta \in \text{subfrmlsr } \varphi\}$

⟨proof⟩

definition *build-F nds* φ

$\equiv (\lambda(\mu, \eta). \text{name } ' \{q \in \text{nds}. (\text{Until-ltlr } \mu \ \eta \in \text{old } q \longrightarrow \eta \in \text{old } q)\}) '$

$\text{until-frmlsr } \varphi$

lemma *build-F-correct*: $\text{build-F nds } \varphi =$

$\{\text{name } ' A \mid A. \exists \mu \ \eta. A = \{q \in \text{nds}. \text{Until-ltlr } \mu \ \eta \in \text{old } q \longrightarrow \eta \in \text{old } q\} \wedge$

$\text{Until-ltlr } \mu \ \eta \in \text{subfrmlsr } \varphi\}$

⟨proof⟩

definition *pn-props* $ps \equiv \text{FOREACH}i$

$(\lambda it (P, N). P = \{p. \text{Prop-ltlr } p \in ps - it\} \wedge N = \{p. \text{Nprop-ltlr } p \in ps - it\})$

$ps (\lambda p (P, N).$

$\text{case } p \text{ of } \text{Prop-ltlr } p \Rightarrow \text{RETURN } (\text{insert } p \ P, N)$

| $\text{Nprop-ltlr } p \Rightarrow \text{RETURN } (P, \text{insert } p \ N)$

| $- \Rightarrow \text{RETURN } (P, N)$

) ({} , {})

lemma *pn-props-correct*:

assumes [*simp*]: *finite ps*
shows *pn-props ps* ≤ *SPEC*($\lambda r. r =$
 $\{\{p. \text{Prop-ltlr } p \in ps\}, \{p. \text{Nprop-ltlr } p \in ps\}\}$)
 $\langle \text{proof} \rangle$

definition *pn-map nds* ≡ *FOREACH nds*

($\lambda nd m. \text{do } \{$
 $\text{PN} \leftarrow \text{pn-props } (\text{old } nd);$
 $\text{RETURN } (m(\text{name } nd \mapsto \text{PN}))$
 $\}$) *Map.empty*

lemma *pn-map-correct*:

assumes [*simp*]: *finite nds*
assumes *FIN'*: $\bigwedge nd. nd \in nds \implies \text{finite } (\text{old } nd)$
assumes *INJ*: *inj-on name nds*
shows *pn-map nds* ≤ *SPEC* ($\lambda r. \forall qn.$
 $\text{case } r \text{ } qn \text{ of}$
 $\text{None} \Rightarrow qn \notin \text{name}'nds$
 $| \text{Some } (P, N) \Rightarrow qn \in \text{name}'nds$
 $\wedge P = \{p. \text{Prop-ltlr } p \in \text{old } (\text{the-inv-into } nds \text{ name } qn)\}$
 $\wedge N = \{p. \text{Nprop-ltlr } p \in \text{old } (\text{the-inv-into } nds \text{ name } qn)\}$
 $\}$)
 $\langle \text{proof} \rangle$

definition *cr-rename-gba nds* φ ≡ *do* {

let $V = \text{name}' nds;$
 $V0 = \text{name}' \{q \in nds. \text{expand-init} \in \text{incoming } q\};$
 $\text{succmap} \leftarrow \text{build-succ } nds;$
 $E = E\text{-of-succ } (\text{the-default } \{\} \text{ o succmap});$
 $F = \text{build-F } nds \ \varphi;$
 $\text{pnm} \leftarrow \text{pn-map } nds;$
 $\text{let } L = (\lambda qn l. \text{case } \text{pnm } qn \text{ of}$
 $\text{None} \Rightarrow \text{False}$
 $| \text{Some } (P, N) \Rightarrow (\forall p \in P. p \in (l ::_r \langle Id \rangle \text{fun-set-rel})) \wedge (\forall p \in N. p \notin l)$
 $\});$
 $\text{RETURN } ((\downarrow g\text{-}V = V, g\text{-}E = E, g\text{-}V0 = V0, gbg\text{-}F = F, gba\text{-}L = L))$
 $\}$

lemma *cr-rename-gba-refine*:

assumes *INV*: *nds-invars nds*
assumes *REL[simplified]*: $(nds', nds) \in Id \ (\varphi', \varphi) \in Id$
shows *cr-rename-gba nds'* φ'
 $\leq \downarrow Id \ (\text{RETURN } (gba\text{-rename-ext } (\lambda \cdot. ()) \text{ name } (\text{create-gba-from-nodes } \varphi \text{ nds})))$
 $\langle \text{proof} \rangle$

definition *create-name-gba-aimpl* $\varphi \equiv \text{do } \{$
 $\text{nds} \leftarrow \text{create-graph-aimpl } \varphi;$
 $\text{ASSERT } (\text{nds-invars } \text{nds});$
 $\text{cr-rename-gba } \text{nds } \varphi$
 $\}$

lemma *create-name-gba-aimpl-refine*:
 $\text{create-name-gba-aimpl } \varphi \leq \Downarrow \text{Id } (\text{create-name-gba } \varphi)$
 $\langle \text{proof} \rangle$

3.3 Refinement to Efficient Data Structures

3.3.1 Creation of GBA from Nodes

schematic-goal *until-frmlsr-impl-aux*:
assumes [*relator-props*, *simp*]: $R = \text{Id}$
shows ($?c, \text{until-frmlsr}$)
 $\in \langle \langle R :: (- \times - :: \text{linorder}) \text{ set} \rangle \rangle \text{ltlr-rel} \rightarrow \langle \langle R \rangle \rangle \text{ltlr-rel} \times_r \langle \langle R \rangle \rangle \text{ltlr-rel} \rangle \text{dflt-rs-rel}$
 $\langle \text{proof} \rangle$

concrete-definition *until-frmlsr-impl* **uses** *until-frmlsr-impl-aux*
lemmas [*autoref-rules*] = *until-frmlsr-impl.refine*[*OF PREFER-id-D*]

schematic-goal *build-succ-impl-aux*:
shows ($?c, \text{build-succ}$) \in
 $\langle \langle Rm, R \rangle \text{node-rel} \rangle \text{list-set-rel}$
 $\rightarrow \langle \langle \text{nat-rel}, \langle \text{nat-rel} \rangle \text{list-set-rel} \rangle \text{iam-map-rel} \rangle \text{nres-rel}$
 $\langle \text{proof} \rangle$

concrete-definition *build-succ-impl* **uses** *build-succ-impl-aux*
lemmas [*autoref-rules*] = *build-succ-impl.refine*

schematic-goal *build-succ-code-aux*: $\text{RETURN } ?c \leq \text{build-succ-impl } x$
 $\langle \text{proof} \rangle$

concrete-definition *build-succ-code* **uses** *build-succ-code-aux*
lemmas [*refine-transfer*] = *build-succ-code.refine*

schematic-goal *build-F-impl-aux*:
assumes [*relator-props*]: $R = \text{Id}$
shows ($?c, \text{build-F}$) \in

$\langle\langle Rm, R \rangle \text{node-rel}\rangle \text{list-set-rel} \rightarrow \langle R \rangle \text{ltlr-rel} \rightarrow \langle\langle \text{nat-rel} \rangle \text{list-set-rel}\rangle \text{list-set-rel}$
 $\langle \text{proof} \rangle$

concrete-definition *build-F-impl uses build-F-impl-aux*
lemmas [*autoref-rules*] = *build-F-impl.refine[OF PREFER-id-D]*

schematic-goal *pn-map-impl-aux*:
shows ($?c, \text{pn-map}$) \in
 $\langle\langle Rm, Id \rangle \text{node-rel}\rangle \text{list-set-rel}$
 $\rightarrow \langle\langle \text{nat-rel}, \langle Id \rangle \text{list-set-rel} \times_r \langle Id \rangle \text{list-set-rel} \rangle \text{iam-map-rel}\rangle \text{nres-rel}$
 $\langle \text{proof} \rangle$

concrete-definition *pn-map-impl uses pn-map-impl-aux*
lemma *pn-map-impl-autoref[autoref-rules]*:
assumes *PREFER-id R*
shows (*pn-map-impl, pn-map*) \in
 $\langle\langle Rm, R \rangle \text{node-rel}\rangle \text{list-set-rel}$
 $\rightarrow \langle\langle \text{nat-rel}, \langle R \rangle \text{list-set-rel} \times_r \langle R \rangle \text{list-set-rel} \rangle \text{iam-map-rel}\rangle \text{nres-rel}$
 $\langle \text{proof} \rangle$

schematic-goal *pn-map-code-aux: RETURN ?c ≤ pn-map-impl x*
 $\langle \text{proof} \rangle$

concrete-definition *pn-map-code uses pn-map-code-aux*
lemmas [*refine-transfer*] = *pn-map-code.refine*

schematic-goal *cr-rename-gba-impl-aux*:
assumes *ID[relator-props]: R=Id*
notes [*autoref-tyrel del*] = *tyrel-dflt-linorder-set*
notes [*autoref-tyrel*] = *ty-REL[of ⟨nat-rel⟩list-set-rel]*
shows ($?c, \text{cr-rename-gba}$) \in
 $\langle\langle Rm, R \rangle \text{node-rel}\rangle \text{list-set-rel} \rightarrow \langle R \rangle \text{ltlr-rel} \rightarrow (\text{?R}::(\text{?'c} \times -) \text{set})$
 $\langle \text{proof} \rangle$

concrete-definition *cr-rename-gba-impl uses cr-rename-gba-impl-aux*

thm *cr-rename-gba-impl.refine*

lemma *cr-rename-gba-autoref[autoref-rules]*:
assumes *PREFER-id R*
shows (*cr-rename-gba-impl, cr-rename-gba*) \in
 $\langle\langle Rm, R \rangle \text{node-rel}\rangle \text{list-set-rel} \rightarrow \langle R \rangle \text{ltlr-rel} \rightarrow$
 $\langle \text{gbav-impl-rel-ext unit-rel nat-rel} \langle\langle R \rangle \text{fun-set-rel} \rangle \rangle \text{nres-rel}$
 $\langle \text{proof} \rangle$

schematic-goal *cr-rename-gba-code-aux: RETURN ?c ≤ cr-rename-gba-impl x y*
 $\langle \text{proof} \rangle$

concrete-definition *cr-rename-gba-code* **uses** *cr-rename-gba-code-aux*
lemmas [*refine-transfer*] = *cr-rename-gba-code.refine*

3.3.2 Creation of Graph

The implementation of the node-set. The relation enforces that there are no different nodes with the same name. This effectively establishes an additional invariant, made explicit by an assertion in the refined program. This invariant allows for a more efficient implementation.

definition *ls-nds-rel-def-internal*:

ls-nds-rel $R \equiv \langle R \rangle \text{list-set-rel} \cap \{(-,s). \text{inj-on name } s\}$

lemma *ls-nds-rel-def*: $\langle R \rangle \text{ls-nds-rel} = \langle R \rangle \text{list-set-rel} \cap \{(-,s). \text{inj-on name } s\}$
 $\langle \text{proof} \rangle$

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of ls-nds-rel i-set*]

lemma *ls-nds-rel-sv*[*relator-props*]:

assumes *single-valued* R

shows *single-valued* $(\langle R \rangle \text{ls-nds-rel})$

$\langle \text{proof} \rangle$

context begin interpretation *autoref-syn* $\langle \text{proof} \rangle$

lemma *lsnds-empty-autoref*[*autoref-rules*]:

assumes *PREFER-id* R

shows $([],\{\}) \in \langle R \rangle \text{ls-nds-rel}$

$\langle \text{proof} \rangle$

lemma *lsnds-insert-autoref*[*autoref-rules*]:

assumes *SIDE-PRECOND* (*name* $n \notin \text{name}'ns$)

assumes $(n',n) \in R$

assumes $(ns',ns) \in \langle R \rangle \text{ls-nds-rel}$

shows $(n' \# ns', (OP \text{ insert} :: R \rightarrow \langle R \rangle \text{ls-nds-rel} \rightarrow \langle R \rangle \text{ls-nds-rel}) \$ n \$ ns)$
 $\in \langle R \rangle \text{ls-nds-rel}$

$\langle \text{proof} \rangle$

lemma *ls-nds-image-autoref-aux*:

assumes [*autoref-rules*]: $(fi,f) \in Ra \rightarrow Rb$

assumes $(l,s) \in \langle Ra \rangle \text{ls-nds-rel}$

assumes [*simp*]: $\forall x. \text{name } (f x) = \text{name } x$

shows $(\text{map } fi \ l, f's) \in \langle Rb \rangle \text{ls-nds-rel}$

$\langle \text{proof} \rangle$

lemma *ls-nds-image-autoref*[*autoref-rules*]:

assumes $(fi,f) \in Ra \rightarrow Rb$

assumes *SIDE-PRECOND* ($\forall x. \text{name } (f x) = \text{name } x$)

shows $(\text{map } fi, (OP \text{ '}) :: (Ra \rightarrow Rb) \rightarrow \langle Ra \rangle \text{ls-nds-rel} \rightarrow \langle Rb \rangle \text{ls-nds-rel}) \$ f)$
 $\in \langle Ra \rangle \text{ls-nds-rel} \rightarrow \langle Rb \rangle \text{ls-nds-rel}$

$\langle \text{proof} \rangle$

lemma *list-set-autoref-to-list*[*autoref-ga-rules*]:
shows *is-set-to-sorted-list* (λ - . True) *R ls-nds-rel id*
<proof>

end

context begin interpretation *autoref-syn* *<proof>*

lemma [*autoref-itype*]:
upd-incoming
 $\vdash_i \langle Im, I \rangle_i i\text{-node} \rightarrow_i \langle \langle Im', I \rangle_i i\text{-node} \rangle_i i\text{-set} \rightarrow_i \langle \langle Im', I \rangle_i i\text{-node} \rangle_i i\text{-set}$
<proof>

end

term *upd-incoming*

schematic-goal *upd-incoming-impl-aux*:
assumes *REL-IS-ID R*
shows ($?c$, *upd-incoming*) $\in \langle Rm1, R \rangle \text{node-rel}$
 $\rightarrow \langle \langle Rm2, R \rangle \text{node-rel} \rangle \text{ls-nds-rel}$
 $\rightarrow \langle \langle Rm2, R \rangle \text{node-rel} \rangle \text{ls-nds-rel}$
<proof>

concrete-definition *upd-incoming-impl* **uses** *upd-incoming-impl-aux*

lemmas [*autoref-rules*] = *upd-incoming-impl.refine[OF PREFER-D[of REL-IS-ID]]*

schematic-goal *expand-impl-aux*: ($?c$, *expand-aimpl*) \in
 $\langle \text{unit-rel}, \text{Id} \rangle \text{node-rel} \times_r \langle \langle \text{unit-rel}, \text{Id} \rangle \text{node-rel} \rangle \text{ls-nds-rel}$
 $\rightarrow \langle \text{nat-rel} \times_r \langle \langle \text{unit-rel}, \text{Id} \rangle \text{node-rel} \rangle \text{ls-nds-rel} \rangle \text{nres-rel}$
<proof>

concrete-definition *expand-impl* **uses** *expand-impl-aux*

context begin interpretation *autoref-syn* *<proof>*

lemma [*autoref-itype*]: *expand_T*
 $\vdash_i \langle \langle i\text{-unit}, I \rangle_i i\text{-node}, \langle \langle i\text{-unit}, I \rangle_i i\text{-node} \rangle_i i\text{-set} \rangle_i i\text{-prod}$
 $\rightarrow_i \langle \langle i\text{-nat}, \langle \langle i\text{-unit}, I \rangle_i i\text{-node} \rangle_i i\text{-set} \rangle_i i\text{-prod} \rangle_i i\text{-nres}$ *<proof>*

lemma *expand-autoref*[*autoref-rules*]:
assumes *ID: PREFER-id R*
assumes *A: (n-ns', n-ns)*
 $\in \langle \text{unit-rel}, R \rangle \text{node-rel} \times_r \langle \langle \text{unit-rel}, R \rangle \text{node-rel} \rangle \text{list-set-rel}$
assumes *B: SIDE-PRECOND* (
 $\text{let } (n, ns) = n\text{-ns in inj-on name ns} \wedge (\forall n' \in ns. \text{name } n > \text{name } n')$
)
shows (*expand-impl n-ns'*,
(*OP expand-aimpl*
 $\vdash \langle \text{unit-rel}, R \rangle \text{node-rel} \times_r \langle \langle \text{unit-rel}, R \rangle \text{node-rel} \rangle \text{list-set-rel}$

$\rightarrow \langle \text{nat-rel} \times_r \langle \langle \text{unit-rel}, R \rangle \text{node-rel} \rangle \text{list-set-rel} \rangle \text{nres-rel} \rangle \$n\text{-ns}$
 $\in \langle \text{nat-rel} \times_r \langle \langle \text{unit-rel}, R \rangle \text{node-rel} \rangle \text{list-set-rel} \rangle \text{nres-rel}$
 $\langle \text{proof} \rangle$

end

schematic-goal *expand-code-aux*: RETURN ?c ≤ expand-impl n-ns
 $\langle \text{proof} \rangle$

concrete-definition *expand-code* uses *expand-code-aux*

prepare-code-thms *expand-code-def*

lemmas [*refine-transfer*] = *expand-code.refine*

schematic-goal *create-graph-impl-aux*:

assumes *ID*: $R = \text{Id}$

shows (?c, *create-graph-aimpl*)

$\in \langle R \rangle \text{ltlr-rel} \rightarrow \langle \langle \langle \text{unit-rel}, R \rangle \text{node-rel} \rangle \text{list-set-rel} \rangle \text{nres-rel}$

$\langle \text{proof} \rangle$

concrete-definition *create-graph-impl* uses *create-graph-impl-aux*

lemmas [*autoref-rules*] = *create-graph-impl.refine*[OF *PREFER-id-D*]

schematic-goal *create-graph-code-aux*: RETURN ?c ≤ create-graph-impl φ
 $\langle \text{proof} \rangle$

concrete-definition *create-graph-code* uses *create-graph-code-aux*

lemmas [*refine-transfer*] = *create-graph-code.refine*

schematic-goal *create-name-gba-impl-aux*:

(?c, (*create-name-gba-aimpl*:: 'a::linorder ltlr \Rightarrow -))

$\in \langle \text{Id} \rangle \text{ltlr-rel} \rightarrow (\text{?R}::(\text{?'c} \times -) \text{ set})$

$\langle \text{proof} \rangle$

concrete-definition *create-name-gba-impl* uses *create-name-gba-impl-aux*

lemma *create-name-gba-autoref*[*autoref-rules*]:

assumes *PREFER-id R*

shows

(*create-name-gba-impl*, *create-name-gba*)

$\in \langle R \rangle \text{ltlr-rel} \rightarrow \langle \text{gbav-impl-rel-ext unit-rel nat-rel} (\langle R \rangle \text{fun-set-rel}) \rangle \text{nres-rel}$

(**is** $\text{-:} \rightarrow \langle \text{?R} \rangle \text{nres-rel}$)

$\langle \text{proof} \rangle$

schematic-goal *create-name-gba-code-aux*: RETURN ?c ≤ create-name-gba-impl
 φ
 $\langle \text{proof} \rangle$

concrete-definition *create-name-gba-code* uses *create-name-gba-code-aux*

lemmas [*refine-transfer*] = *create-name-gba-code.refine*

```

schematic-goal create-name-igba-impl-aux:
  assumes RID: R=Id
  shows (?c,create-name-igba)∈
  ⟨R⟩ltlr-rel → ⟨igbav-impl-rel-ext unit-rel nat-rel ((R)fun-set-rel)⟩nres-rel
  ⟨proof⟩
concrete-definition create-name-igba-impl uses create-name-igba-impl-aux
lemmas [autoref-rules] = create-name-igba-impl.refine[OF PREFER-id-D]

schematic-goal create-name-igba-code-aux: RETURN ?c ≤ create-name-igba-impl
  φ
  ⟨proof⟩
concrete-definition create-name-igba-code uses create-name-igba-code-aux
lemmas [refine-transfer] = create-name-igba-code.refine

export-code create-name-igba-code checking SML

end

```

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