The Kolmogorov-Chentsov Theorem

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Abstract

Continuous-time stochastic processes often carry the condition of having almost-surely continuous paths. If some process X satisfies certain bounds on its expectation, then the Kolmogorov-Chentsov theorem lets us construct a modification of X, i.e. a process X' such that $\forall t. X_t = X'_t$ almost surely, that has Hölder continuous paths.

In this work, we mechanise the Kolmogorov-Chentsov theorem. To get there, we develop a theory of stochastic processes, together with Hölder continuity, convergence in measure, and arbitrary intervals of dyadic rationals.

With this, we pave the way towards a construction of Brownian motion. The work is based on the exposition in Achim Klenke's probability theory text [1].

Contents

1 Supporting lemmas

theory *Kolmogorov-Chentsov-Extras*
  imports *HOL−Probability.Probability*
begin

lemma *atLeastAtMost-induct*[*consumes 1*, *case-names base Suc*]:
  assumes $x \in \{n..m\}$
    and $P\ n$
    and $\bigwedge k.\ [\![k \geq n;\ k < m;\ P\ k]\!] \Longrightarrow P\ (Suc\ k)$
  shows $P\ x$
  ⟨*proof*⟩

lemma *eventually-prodI′*:
  assumes $eventually\ P\ F\ eventually\ Q\ G\ \forall x\ y.\ P\ x \longrightarrow Q\ y \longrightarrow R\ (x,y)$
  shows $eventually\ R\ (F \times_F\ G)$
  ⟨*proof*⟩

Analogous to $[\![almost\text{-}everywhere\ ?M\ ?P;\ \bigwedge N.\ [\![\bigwedge x.\ x \in space\ ?M - N \Longrightarrow ?P\ x;\ N \in null\text{-}sets\ ?M]\!] \Longrightarrow ?thesis]\!] \Longrightarrow ?thesis$

lemma *AE-I3*:
  assumes $\bigwedge x.\ x \in space\ M - N \Longrightarrow P\ x\ N \in null\text{-}sets\ M$
  shows $AE\ x\ in\ M.\ P\ x$
  ⟨*proof*⟩

Extends $[\![(?f \longrightarrow ?l)\ ?F;\ (?g \longrightarrow ?m)\ ?F]\!] \Longrightarrow ((\lambda x.\ dist\ (?f\ x)\ (?g\ x)) \longrightarrow dist\ ?l\ ?m)\ ?F$

lemma *tendsto-dist-prod*:
  fixes $l\ m :: 'a :: metric\text{-}space$
  assumes f: $(f \longrightarrow l)\ F$
    and g: $(g \longrightarrow m)\ G$
  shows $((\lambda x.\ dist\ (f\ (fst\ x))\ (g\ (snd\ x))) \longrightarrow dist\ l\ m)\ (F \times_F\ G)$
⟨*proof*⟩

lemma *borel-measurable-at-within-metric* [*measurable*]:
  fixes $f :: 'a :: first\text{-}countable\text{-}topology \Rightarrow 'b \Rightarrow 'c :: metric\text{-}space$
  assumes [*measurable*]: $\bigwedge t.\ t \in S \Longrightarrow f\ t \in borel\text{-}measurable\ M$
    and *convergent*: $\bigwedge \omega.\ \omega \in space\ M \Longrightarrow \exists\, l.\ ((\lambda t.\ f\ t\ \omega) \longrightarrow l)\ (at\ x\ within\ S)$
    and *nontrivial*: $at\ x\ within\ S \neq \bot$
  shows $(\lambda \omega.\ Lim\ (at\ x\ within\ S)\ (\lambda t.\ f\ t\ \omega)) \in borel\text{-}measurable\ M$
⟨*proof*⟩

lemma *Max-finite-image-ex*:
  assumes $finite\ S\ S \neq \{\}\ P\ (MAX\ k{\in}S.\ f\ k)$
  shows $\exists\, k \in S.\ P\ (f\ k)$
  ⟨*proof*⟩

lemma *measure-leq-emeasure-ennreal*: $0 \leq x \Longrightarrow emeasure\ M\ A \leq ennreal\ x \Longrightarrow measure\ M\ A \leq x$

⟨*proof*⟩

lemma *Union-add-subset*: $(m :: nat) \leq n \Longrightarrow (\bigcup k.\ A\ (k + n)) \subseteq (\bigcup k.\ A\ (k + m))$
⟨*proof*⟩

lemma *floor-in-Nats* [*simp*]: $x \geq 0 \Longrightarrow \lfloor x \rfloor \in \mathbb{N}$
⟨*proof*⟩

lemma *triangle-ineq-list*:
fixes $l :: ('a :: metric\text{-}space)\ list$
assumes $l \neq []$
shows $dist\ (hd\ l)\ (last\ l) \leq (\sum\ i{=}1..length\ l - 1.\ dist\ (l!i)\ (l!(i{-}1)))$
⟨*proof*⟩

lemma *triangle-ineq-sum*:
fixes $f :: nat \Rightarrow 'a :: metric\text{-}space$
assumes $n \leq m$
shows $dist\ (f\ n)\ (f\ m) \leq (\sum\ i{=}Suc\ n..m.\ dist\ (f\ i)\ (f\ (i{-}1)))$
⟨*proof*⟩

lemma (**in** *product-prob-space*) *indep-vars-PiM-coordinate*:
assumes $I \neq \{\}$
shows *prob-space.indep-vars* $(\Pi_M\ i{\in}I.\ M\ i)\ M\ (\lambda x\ f.\ f\ x)\ I$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-sets-indep-set*:
assumes *indep-sets* $F\ I\ i \in I\ j \in I\ i \neq j$
shows *indep-set* $(F\ i)\ (F\ j)$
⟨*proof*⟩

lemma (**in** *prob-space*) *indep-vars-indep-var*:
assumes *indep-vars* $M'\ X\ I\ i \in I\ j \in I\ i \neq j$
shows *indep-var* $(M'\ i)\ (X\ i)\ (M'\ j)\ (X\ j)$
⟨*proof*⟩

end

2 Intervals of dyadic rationals

theory *Dyadic-Interval*
imports *HOL−Analysis.Analysis*
begin

In this file we describe intervals of dyadic numbers $S..T$ for reals S T. We use the floor and ceiling functions to approximate the numbers with increasing accuracy.

lemma *frac-floor*: $\lfloor x \rfloor = x - frac\ x$

⟨*proof*⟩

lemma *frac-ceil*: $\lceil x\rceil = x + frac\ (-\ x)$
⟨*proof*⟩

lemma *floor-pow2-lim*: $(\lambda n.\ \lfloor 2\hat{}n * T\rfloor\ /\ 2\ \hat{}\ n) \longrightarrow T$
⟨*proof*⟩

lemma *floor-pow2-leq*: $\lfloor 2\hat{}n * T\rfloor\ /\ 2\ \hat{}\ n \leq T$
⟨*proof*⟩

lemma *ceil-pow2-lim*: $(\lambda n.\ \lceil 2\hat{}n * T\rceil\ /\ 2\ \hat{}\ n) \longrightarrow T$
⟨*proof*⟩

lemma *ceil-pow2-geq*: $\lceil 2\hat{}n * T\rceil\ /\ 2\ \hat{}\ n \geq T$
⟨*proof*⟩

`dyadic_interval_step` $n\ S\ T$ is the collection of dyadic numbers in $\{S..T\}$ with denominator 2^n. As $n \to \infty$ this collection approximates $\{S..T\}$. Compare with $dyadics \equiv \bigcup_{k\ m}\ \{of\text{-}nat\ m\ /\ (2::?'a)^k\}$

definition *dyadic-interval-step* :: *nat* ⇒ *real* ⇒ *real* ⇒ *real set*
where $dyadic\text{-}interval\text{-}step\ n\ S\ T \equiv (\lambda k.\ k\ /\ (2\ \hat{}\ n))\ `\ \{\lceil 2\hat{}n * S\rceil..\lfloor 2\hat{}n * T\rfloor\}$

definition *dyadic-interval* :: *real* ⇒ *real* ⇒ *real set*
where $dyadic\text{-}interval\ S\ T \equiv (\bigcup n.\ dyadic\text{-}interval\text{-}step\ n\ S\ T)$

lemma *dyadic-interval-step-empty*[*simp*]: $T < S \Longrightarrow dyadic\text{-}interval\text{-}step\ n\ S\ T = \{\}$
⟨*proof*⟩

lemma *dyadic-interval-step-singleton*[*simp*]: $X \in \mathbb{Z} \Longrightarrow dyadic\text{-}interval\text{-}step\ n\ X\ X = \{X\}$
⟨*proof*⟩

lemma *dyadic-interval-step-zero* [*simp*]: $dyadic\text{-}interval\text{-}step\ 0\ S\ T = real\text{-}of\text{-}int\ `\ \{\lceil S\rceil\ ..\ \lfloor T\rfloor\}$
⟨*proof*⟩

lemma *dyadic-interval-step-mem* [*intro*]:
assumes$x \geq 0\ T \geq 0\ x \leq T$
shows $\lfloor 2\hat{}n * x\rfloor\ /\ 2\ \hat{}\ n \in dyadic\text{-}interval\text{-}step\ n\ 0\ T$
⟨*proof*⟩

lemma *dyadic-interval-step-iff*:
$x \in dyadic\text{-}interval\text{-}step\ n\ S\ T \longleftrightarrow$
$(\exists k.\ k \geq \lceil 2\hat{}n * S\rceil \wedge k \leq \lfloor 2\hat{}n * T\rfloor \wedge x = k/2\hat{}n)$
⟨*proof*⟩

lemma *dyadic-interval-step-memI* [*intro*]:

assumes $\exists k::int.\ x = k/2\hat{\ }n\ x \geq S\ x \leq T$
shows $x \in$ *dyadic-interval-step n S T*
⟨*proof*⟩

lemma *mem-dyadic-interval*: $x \in$ *dyadic-interval S T* $\longleftrightarrow$ $(\exists n.\ x \in$ *dyadic-interval-step n S T*$)$
⟨*proof*⟩

lemma *mem-dyadic-intervalI*: $\exists n.\ x \in$ *dyadic-interval-step n S T* $\Longrightarrow x \in$ *dyadic-interval S T*
⟨*proof*⟩

lemma *dyadic-step-leq*: $x \in$ *dyadic-interval-step n S T* $\Longrightarrow x \leq T$
⟨*proof*⟩

lemma *dyadics-leq*: $x \in$ *dyadic-interval S T* $\Longrightarrow x \leq T$
⟨*proof*⟩

lemma *dyadic-step-geq*: $x \in$ *dyadic-interval-step n S T* $\Longrightarrow x \geq S$
⟨*proof*⟩

lemma *dyadics-geq*: $x \in$ *dyadic-interval S T* $\Longrightarrow x \geq S$
⟨*proof*⟩

corollary *dyadic-interval-subset-interval* [*simp*]: (*dyadic-interval 0 T*) $\subseteq \{0..T\}$
⟨*proof*⟩

lemma *zero-in-dyadics*: $T \geq 0 \Longrightarrow 0 \in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

The following theorem is useful for reasoning with at_within

lemma *dyadic-interval-converging-sequence*:
assumes $t \in \{0..T\}$ $T \neq 0$
shows $\exists s.\ \forall n.\ s\ n \in$ *dyadic-interval 0 T* $- \{t\} \wedge s \longrightarrow t$
⟨*proof*⟩

lemma *dyadic-interval-dense*: *closure* (*dyadic-interval 0 T*) $= \{0..T\}$
⟨*proof*⟩

corollary *dyadic-interval-islimpt*:
assumes $T > 0$ $t \in \{0..T\}$
shows *t islimpt dyadic-interval 0 T*
⟨*proof*⟩

corollary *at-within-dyadic-interval-nontrivial*[*simp*]:
assumes $T > 0$ $t \in \{0..T\}$
shows (*at t within dyadic-interval 0 T*) $\neq$ *bot*
⟨*proof*⟩

lemma *dyadic-interval-step-finite*[*simp*]: *finite* (*dyadic-interval-step n S T*)
⟨*proof*⟩

lemma *dyadic-interval-countable*[*simp*]: *countable* (*dyadic-interval S T*)
⟨*proof*⟩

lemma *floor-pow2-add-leq*:
fixes T :: *real*
shows $\lfloor 2\hat{\ }n * T\rfloor$ / 2 ^ $n \leq \lfloor 2$ ^ $(n+k) * T\rfloor$ / 2 ^ $(n+k)$
⟨*proof*⟩

corollary *floor-pow2-mono*: *mono* ($\lambda n.$ $\lfloor 2\hat{\ }n * (T$ :: *real*$)\rfloor$ / 2 ^ n)
⟨*proof*⟩

lemma *dyadic-interval-step-Max*: $T \geq 0 \Longrightarrow$ *Max* (*dyadic-interval-step n 0 T*) = $\lfloor 2\hat{\ }n * T\rfloor$ / $2\hat{\ }n$
⟨*proof*⟩

lemma *dyadic-interval-step-subset*:
$n \leq m \Longrightarrow$ *dyadic-interval-step n 0 T* $\subseteq$ *dyadic-interval-step m 0 T*
⟨*proof*⟩

corollary *dyadic-interval-step-mono*:
assumes $x \in$ *dyadic-interval-step n 0 T* $n \leq m$
shows $x \in$ *dyadic-interval-step m 0 T*
⟨*proof*⟩

lemma *dyadic-as-natural*:
assumes $x \in$ *dyadic-interval-step n 0 T*
shows $\exists!k.$ x = *real k* / 2 ^ n
⟨*proof*⟩

lemma *dyadic-of-natural*:
assumes *real k* / 2 ^ $n \leq T$
shows *real k* / 2 ^ n $\in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-interval-minus*:
assumes $x \in$ *dyadic-interval-step n 0 T* $y \in$ *dyadic-interval-step n 0 T* $x \leq y$
shows $y - x \in$ *dyadic-interval-step n 0 T*
⟨*proof*⟩

lemma *dyadic-times-nat*: $x \in$ *dyadic-interval-step n 0 T* $\Longrightarrow$ ($x * 2$ ^ n) $\in \mathbb{N}$
⟨*proof*⟩

definition *dyadic-expansion x n b k* $\equiv$ *set b* $\subseteq \{0,1\}$
$\wedge$ *length b* = n $\wedge$ x = *real-of-int k* + ($\sum m{\in}\{1..n\}$. *real* (b ! $(m-1)$) / 2 ^ m)

lemma *dyadic-expansionI*:

assumes *set* $b \subseteq \{0,1\}$ *length* $b = n$ $x = k + (\sum m \in \{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m)$
shows *dyadic-expansion* $x\ n\ b\ k$
⟨*proof*⟩

lemma *dyadic-expansionD*:
assumes *dyadic-expansion* $x\ n\ b\ k$
shows *set* $b \subseteq \{0,1\}$
and *length* $b = n$
and $x = k + (\sum m \in \{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m)$
⟨*proof*⟩

lemma *dyadic-expansion-ex*:
assumes $x \in$ *dyadic-interval-step* $n\ 0\ T$
shows $\exists\, b\ k.$ *dyadic-expansion* $x\ n\ b\ k$
⟨*proof*⟩

lemma *dyadic-expansion-frac-le-1*:
assumes *dyadic-expansion* $x\ n\ b\ k$
shows $(\sum m \in \{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m) < 1$
⟨*proof*⟩

lemma *dyadic-expansion-frac-range*:
assumes *dyadic-expansion* $x\ n\ b\ k$ $m \in \{1..n\}$
shows $b\ !\ (m-1) \in \{0,1\}$
⟨*proof*⟩

lemma *dyadic-expansion-interval*:
assumes *dyadic-expansion* $x\ n\ b\ k$ $x \in \{S..T\}$
shows $x \in$ *dyadic-interval-step* $n\ S\ T$
⟨*proof*⟩

lemma *dyadic-expansion-nth-geq*:
assumes *dyadic-expansion* $x\ n\ b\ k$ $m \in \{1..n\}$ $b\ !\ (m-1) = 1$
shows $x \geq k + 1/2\hat{}m$
⟨*proof*⟩

lemma *dyadic-expansion-frac-geq-0*:
assumes *dyadic-expansion* $x\ n\ b\ k$
shows $(\sum m \in \{1..n\}.\ (b\ !\ (m-1))\ /\ 2\ \hat{}\ m) \geq 0$
⟨*proof*⟩

lemma *dyadic-expansion-frac*:
assumes *dyadic-expansion* $x\ n\ b\ k$
shows *frac* $x = (\sum m \in \{1..n\}.\ (b\ !\ (m-1))/\ 2\ \hat{}\ m)$
⟨*proof*⟩

lemma *dyadic-expansion-floor*:
assumes *dyadic-expansion* $x\ n\ b\ k$

shows $k = \lfloor x \rfloor$
$\langle proof \rangle$

lemma *sum-interval-pow2-inv*: $(\sum m \in \{Suc\ l..n\}.\ (1 :: real) \ /\ 2\ \hat{}\ m) = 1\ /\ 2\hat{}l - 1/2\hat{}n$ **if** $l < n$
$\langle proof \rangle$

lemma *dyadic-expansion-unique*:
assumes *dyadic-expansion x n b k*
and *dyadic-expansion x n c j*
shows $b = c \wedge j = k$
$\langle proof \rangle$

end

3 Hölder continuity

theory *Holder-Continuous*
imports *HOL−Analysis.Analysis*
begin

Hölder continuity is a weaker version of Lipschitz continuity.

definition *holder-at-within* :: $real \Rightarrow 'a\ set \Rightarrow 'a \Rightarrow ('a :: metric\text{-}space \Rightarrow 'b :: metric\text{-}space) \Rightarrow bool$ **where**
holder-at-within $\gamma\ D\ r\ \varphi \equiv \gamma \in \{0<..1\} \wedge$
$(\exists \varepsilon > 0.\ \exists C \geq 0.\ \forall s \in D.\ dist\ r\ s < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma)$

definition *local-holder-on* :: $real \Rightarrow 'a :: metric\text{-}space\ set \Rightarrow ('a \Rightarrow 'b :: metric\text{-}space) \Rightarrow bool$ **where**
local-holder-on $\gamma\ D\ \varphi \equiv \gamma \in \{0<..1\} \wedge$
$(\forall t \in D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r \in D.\ \forall s \in D.\ dist\ s\ t < \varepsilon \wedge dist\ r\ t < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$

definition *holder-on* :: $real \Rightarrow 'a :: metric\text{-}space\ set \Rightarrow ('a \Rightarrow 'b :: metric\text{-}space) \Rightarrow bool$ (*-−holder′-on 1000*) **where**
$\gamma{-}holder\text{-}on\ D\ \varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge (\exists C \geq 0.\ (\forall r \in D.\ \forall s \in D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$

lemma *holder-onI*:
assumes $\gamma \in \{0<..1\}\ \exists C \geq 0.\ (\forall r \in D.\ \forall s \in D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma)$
shows $\gamma{-}holder\text{-}on\ D\ \varphi$
$\langle proof \rangle$

We prove various equivalent formulations of local holder continuity, using open and closed balls and inequalities.

lemma *local-holder-on-cball*:

local-holder-on γ D $\varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge$
$(\forall t\in D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r\in cball\ t\ \varepsilon \cap D.\ \forall s\in cball\ t\ \varepsilon \cap D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
(**is** $?L \longleftrightarrow ?R$)
⟨*proof*⟩

corollary *local-holder-on-leq-def*: *local-holder-on* γ D $\varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge$
$(\forall t\in D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r\in D.\ \forall s\in D.\ dist\ s\ t \leq \varepsilon \wedge dist\ r\ t \leq \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
⟨*proof*⟩

corollary *local-holder-on-ball*: *local-holder-on* γ D $\varphi \longleftrightarrow \gamma \in \{0<..1\} \wedge$
$(\forall t\in D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r\in ball\ t\ \varepsilon \cap D.\ \forall s\in ball\ t\ \varepsilon \cap D.\ dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
⟨*proof*⟩

lemma *local-holder-on-altdef*:
assumes $D \neq \{\}$
shows *local-holder-on* γ D $\varphi = (\forall t \in D.\ (\exists \varepsilon > 0.\ (\gamma{-}holder{-}on\ ((cball\ t\ \varepsilon) \cap D)\ \varphi)))$
⟨*proof*⟩

lemma *local-holder-on-cong*[*cong*]:
assumes $\gamma = \varepsilon$ $C = D$ $\bigwedge x.\ x \in C \Longrightarrow \varphi\ x = \psi\ x$
shows *local-holder-on* γ C $\varphi \longleftrightarrow$ *local-holder-on* ε D ψ
⟨*proof*⟩

lemma *local-holder-onI*:
assumes $\gamma \in \{0<..1\}$ $(\forall t\in D.\ \exists \varepsilon > 0.\ \exists C \geq 0.\ (\forall r\in D.\ \forall s\in D.\ dist\ s\ t < \varepsilon \wedge dist\ r\ t < \varepsilon \longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma))$
shows *local-holder-on* γ D φ
⟨*proof*⟩

lemma *local-holder-ballI*:
assumes $\gamma \in \{0<..1\}$
and $\bigwedge t.\ t \in D \Longrightarrow \exists \varepsilon > 0.\ \exists C \geq 0.\ \forall r\in ball\ t\ \varepsilon \cap D.\ \forall s\in ball\ t\ \varepsilon \cap D.$
$dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma$
shows *local-holder-on* γ D φ
⟨*proof*⟩

lemma *local-holder-onE*:
assumes *local-holder*: *local-holder-on* γ D φ
and *gamma*: $\gamma \in \{0<..1\}$
and $t \in D$
obtains ε C **where** $\varepsilon > 0$ $C \geq 0$
$\bigwedge r\ s.\ r \in ball\ t\ \varepsilon \cap D \Longrightarrow s \in ball\ t\ \varepsilon \cap D \Longrightarrow dist\ (\varphi\ r)\ (\varphi\ s) \leq C * dist\ r\ s\ powr\ \gamma$
⟨*proof*⟩

Holder continuity matches up with the existing definitions in *HOL−Analysis.Lipschitz*

lemma *holder-1-eq-lipschitz*: $1{-}holder\text{-}on\ D\ \varphi = (\exists\, C.\ lipschitz\text{-}on\ C\ D\ \varphi)$
⟨*proof*⟩

lemma *local-holder-1-eq-local-lipschitz*:
assumes $T \neq \{\}$
shows $local\text{-}holder\text{-}on\ 1\ D\ \varphi = local\text{-}lipschitz\ T\ D\ (\lambda\text{-}.\ \varphi)$
⟨*proof*⟩

lemma *local-holder-refine*:
assumes g: $local\text{-}holder\text{-}on\ g\ D\ \varphi\ g \leq 1$
and h: $h \leq g\ h > 0$
shows $local\text{-}holder\text{-}on\ h\ D\ \varphi$
⟨*proof*⟩

lemma *holder-uniform-continuous*:
assumes $\gamma{-}holder\text{-}on\ X\ \varphi$
shows $uniformly\text{-}continuous\text{-}on\ X\ \varphi$
⟨*proof*⟩

corollary *holder-on-continuous-on*: $\gamma{-}holder\text{-}on\ X\ \varphi \Longrightarrow continuous\text{-}on\ X\ \varphi$
⟨*proof*⟩

lemma *holder-implies-local-holder*: $\gamma{-}holder\text{-}on\ D\ \varphi \Longrightarrow local\text{-}holder\text{-}on\ \gamma\ D\ \varphi$
⟨*proof*⟩

lemma *local-holder-imp-continuous*:
assumes *local-holder*: $local\text{-}holder\text{-}on\ \gamma\ X\ \varphi$
shows $continuous\text{-}on\ X\ \varphi$
⟨*proof*⟩

lemma *local-holder-compact-imp-holder*:
assumes $compact\ I\ local\text{-}holder\text{-}on\ \gamma\ I\ \varphi$
shows $\gamma{-}holder\text{-}on\ I\ \varphi$
⟨*proof*⟩

lemma *holder-const*: $\gamma{-}holder\text{-}on\ C\ (\lambda\text{-}.\ c) \longleftrightarrow \gamma \in \{0{<}..1\}$
⟨*proof*⟩

lemma *local-holder-const*: $local\text{-}holder\text{-}on\ \gamma\ C\ (\lambda\text{-}.\ c) \longleftrightarrow \gamma \in \{0{<}..1\}$
⟨*proof*⟩

end

4 Convergence in measure

theory *Measure-Convergence*
imports *HOL−Probability.Probability*
begin

We use measure rather than emeasure because ennreal is not a metric space, which we need to reason about convergence. By intersecting with the set of finite measure A, we don't run into issues where infinity is collapsed to 0.

For finite measures this definition is equal to the definition without set A – see below.

definition *tendsto-measure* :: $'b$ *measure* $\Rightarrow$ $('a \Rightarrow 'b \Rightarrow ('c :: \{$*second-countable-topology*,*metric-space*$\}))$ $\Rightarrow ('b \Rightarrow 'c) \Rightarrow 'a$ *filter* $\Rightarrow$ *bool*
where *tendsto-measure M X l F* $\equiv$ $(\forall n.\ X\ n \in$ *borel-measurable M*$)$ $\wedge$ $l \in$ *borel-measurable M* $\wedge$
$(\forall \varepsilon > 0.\ \forall A \in$ *fmeasurable M*.
$((\lambda n.\ measure\ M\ (\{\omega \in space\ M.\ dist\ (X\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0)\ F)$

abbreviation (**in** *prob-space*) *tendsto-prob* (**infixr** $\longrightarrow_P$ *55*) **where**
$(f \longrightarrow_P l)\ F \equiv$ *tendsto-measure M f l F*

lemma *tendsto-measure-measurable*[*measurable-dest*]:
tendsto-measure M X l F $\Longrightarrow$ $X\ n \in$ *borel-measurable M*
⟨*proof*⟩

lemma *tendsto-measure-measurable-lim*[*measurable-dest*]:
tendsto-measure M X l F $\Longrightarrow$ $l \in$ *borel-measurable M*
⟨*proof*⟩

lemma *tendsto-measure-mono*: $F \le F' \Longrightarrow$ *tendsto-measure M f l F'* $\Longrightarrow$ *tendsto-measure M f l F*
⟨*proof*⟩

lemma *tendsto-measureI*:
assumes [*measurable*]: $\bigwedge n.\ X\ n \in$ *borel-measurable M* $l \in$ *borel-measurable M*
and $\bigwedge \varepsilon\ A.\ \varepsilon > 0 \Longrightarrow A \in$ *fmeasurable M* $\Longrightarrow$
$((\lambda n.\ measure\ M\ (\{\omega \in space\ M.\ dist\ (X\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0)\ F$
shows *tendsto-measure M X l F*
⟨*proof*⟩

lemma (**in** *finite-measure*) *finite-tendsto-measureI*:
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M* $f \in$ *borel-measurable M*
and $\bigwedge \varepsilon.\ \varepsilon > 0 \Longrightarrow ((\lambda n.\ measure\ M\ \{\omega \in space\ M.\ dist\ (f'\ n\ \omega)\ (f\ \omega) > \varepsilon\}) \longrightarrow 0)\ F$
shows *tendsto-measure M f' f F*
⟨*proof*⟩

lemma (**in** *finite-measure*) *finite-tendsto-measureD*:
assumes [*measurable*]: *tendsto-measure M f' f F*
shows $(\forall \varepsilon > 0.\ ((\lambda n.\ measure\ M\ \{\omega \in space\ M.\ dist\ (f'\ n\ \omega)\ (f\ \omega) > \varepsilon\}) \longrightarrow 0)\ F)$
⟨*proof*⟩

lemma (**in** *finite-measure*) *tendsto-measure-leq*:

assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable* $M\ f \in$ *borel-measurable* M
shows *tendsto-measure* $M\ f'\ f\ F \longleftrightarrow$
$(\forall \varepsilon > 0.\ ((\lambda n.\ \mathit{measure}\ M\ \{\omega \in \mathit{space}\ M.\ \mathit{dist}\ (f'\ n\ \omega)\ (f\ \omega) \geq \varepsilon\}) \longrightarrow 0)$
$F)$ (**is** $?L \longleftrightarrow ?R$)
⟨*proof*⟩

abbreviation *LIMSEQ-measure* $M\ f\ l \equiv$ *tendsto-measure* $M\ f\ l$ *sequentially*

lemma *LIMSEQ-measure-def*: *LIMSEQ-measure* $M\ f\ l \longleftrightarrow$
$(\forall n.\ f\ n \in$ *borel-measurable* $M) \wedge (l \in$ *borel-measurable* $M) \wedge$
$(\forall \varepsilon > 0.\ \forall A \in$ *fmeasurable* $M.$
$(\lambda n.\ \mathit{measure}\ M\ (\{\omega \in \mathit{space}\ M.\ \mathit{dist}\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0)$
⟨*proof*⟩

lemma *LIMSEQ-measureD*:
assumes *LIMSEQ-measure* $M\ f\ l\ \varepsilon > 0\ A \in$ *fmeasurable* M
shows $(\lambda n.\ \mathit{measure}\ M\ (\{\omega \in \mathit{space}\ M.\ \mathit{dist}\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0$
⟨*proof*⟩

lemma *fmeasurable-inter*: $[\![A \in \mathit{sets}\ M;\ B \in \mathit{fmeasurable}\ M]\!] \Longrightarrow A \cap B \in$ *fmeasurable* M
⟨*proof*⟩

lemma *LIMSEQ-measure-emeasure*:
assumes *LIMSEQ-measure* $M\ f\ l\ \varepsilon > 0\ A \in$ *fmeasurable* M
and [*measurable*]: $\bigwedge i.\ f\ i \in$ *borel-measurable* $M\ l \in$ *borel-measurable* M
shows $(\lambda n.\ \mathit{emeasure}\ M\ (\{\omega \in \mathit{space}\ M.\ \mathit{dist}\ (f\ n\ \omega)\ (l\ \omega) > \varepsilon\} \cap A)) \longrightarrow 0$
⟨*proof*⟩

lemma *measure-Lim-within-LIMSEQ*:
fixes $a :: 'a ::$ *first-countable-topology*
assumes $\bigwedge t.\ X\ t \in$ *borel-measurable* $M\ L \in$ *borel-measurable* M
assumes $\bigwedge S.\ [\![(\forall n.\ S\ n \neq a \wedge S\ n \in T);\ S \longrightarrow a]\!] \Longrightarrow$ *LIMSEQ-measure* M $(\lambda n.\ X\ (S\ n))\ L$
shows *tendsto-measure* $M\ X\ L$ (*at a within* T)
⟨*proof*⟩

definition *tendsto-AE* $:: 'b$ *measure* $\Rightarrow ('a \Rightarrow 'b \Rightarrow 'c ::$ *topological-space*$) \Rightarrow ('b \Rightarrow 'c) \Rightarrow 'a$ *filter* $\Rightarrow$ *bool* **where**
tendsto-AE $M\ f'\ l\ F \longleftrightarrow (AE\ \omega\ in\ M.\ ((\lambda n.\ f'\ n\ \omega) \longrightarrow l\ \omega)\ F)$

lemma *LIMSEQ-ae-pointwise*: $(\bigwedge x.\ (\lambda n.\ f\ n\ x) \longrightarrow l\ x) \Longrightarrow$ *tendsto-AE* $M\ f\ l$ *sequentially*
⟨*proof*⟩

lemma *tendsto-AE-within-LIMSEQ*:
fixes $a :: 'a ::$ *first-countable-topology*
assumes $\bigwedge S.\ [\![(\forall n.\ S\ n \neq a \wedge S\ n \in T);\ S \longrightarrow a]\!] \Longrightarrow$ *tendsto-AE* $M\ (\lambda n.\ X\ (S\ n))\ L$ *sequentially*

shows *tendsto-AE M X L* (*at a within T*)
⟨*proof*⟩

lemma *LIMSEQ-dominated-convergence*:
fixes $X :: nat \Rightarrow real$
assumes $X \longrightarrow L$ $(\bigwedge n.\ Y\ n \leq X\ n)$ $(\bigwedge n.\ Y\ n \geq L)$
shows $Y \longrightarrow L$
⟨*proof*⟩

Klenke remark 6.4

lemma *measure-conv-imp-AE-sequentially*:
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* $\in$ *borel-measurable M*
and *tendsto-AE M f′ f sequentially*
shows *LIMSEQ-measure M f′ f*
⟨*proof*⟩

corollary *LIMSEQ-measure-pointwise*:
assumes $\bigwedge x.\ (\lambda n.\ f\ n\ x) \longrightarrow f'\ x$ $\bigwedge n.\ f\ n \in$ *borel-measurable M f′* $\in$ *borel-measurable M*
shows *LIMSEQ-measure M f f′*
⟨*proof*⟩

lemma *Lim-measure-pointwise*:
fixes *a* :: *′a* :: *first-countable-topology*
assumes $\bigwedge x.\ ((\lambda n.\ f\ n\ x) \longrightarrow f'\ x)$(*at a within T*) $\bigwedge n.\ f\ n \in$ *borel-measurable M f′* $\in$ *borel-measurable M*
shows *tendsto-measure M f f′* (*at a within T*)
⟨*proof*⟩

corollary *measure-conv-imp-AE-at-within*:
fixes *x* :: *′a* :: *first-countable-topology*
assumes [*measurable*]: $\bigwedge n.\ f'\ n \in$ *borel-measurable M f* $\in$ *borel-measurable M*
and *tendsto-AE M f′ f* (*at x within S*)
shows *tendsto-measure M f′ f* (*at x within S*)
⟨*proof*⟩

Klenke remark 6.5

lemma (**in** *sigma-finite-measure*) *LIMSEQ-measure-unique-AE*:
fixes *f* :: *nat* ⇒ *′a* ⇒ *′b* :: {*second-countable-topology*,*metric-space*}
assumes [*measurable*]: $\bigwedge n.\ f\ n \in$ *borel-measurable M l* $\in$ *borel-measurable M l′* $\in$ *borel-measurable M*
and *LIMSEQ-measure M f l LIMSEQ-measure M f l′*
shows *AE x in M*. $l\ x = l'\ x$
⟨*proof*⟩

corollary (**in** *sigma-finite-measure*) *LIMSEQ-ae-unique-AE*:
fixes *f* :: *nat* ⇒ *′a* ⇒ *′b* :: {*second-countable-topology*,*metric-space*}
assumes $\bigwedge n.\ f\ n \in$ *borel-measurable M l* $\in$ *borel-measurable M l′* $\in$ *borel-measurable M*

and *tendsto-AE M f l sequentially tendsto-AE M f l′ sequentially*
shows *AE x in M. l x = l′ x*
⟨*proof*⟩

lemma (**in** *sigma-finite-measure*) *tendsto-measure-at-within-eq-AE*:
fixes *f* :: *′b* :: *first-countable-topology* ⇒ *′a* ⇒ *′c* :: {*second-countable-topology*,*metric-space*}
assumes *f-measurable*: $\bigwedge x.\ x \in S \Longrightarrow f\ x \in$ *borel-measurable M*
and *l-measurable*: *l* ∈ *borel-measurable M l′* ∈ *borel-measurable M*
and *tendsto*: *tendsto-measure M f l* (*at t within S*) *tendsto-measure M f l′* (*at t within S*)
and *not-bot*: (*at t within S*) ≠ ⊥
shows *AE x in M. l x = l′ x*
⟨*proof*⟩
end

5 Stochastic processes

theory *Stochastic-Processes*
imports *Kolmogorov-Chentsov-Extras Dyadic-Interval*
begin

A stochastic process is an indexed collection of random variables. For compatibility with `product_prob_space` we don't enforce conditions on the index set I in the assumptions.

locale *stochastic-process* = *prob-space* +
fixes *M′* :: *′b measure*
and *I* :: *′t set*
and *X* :: *′t* ⇒ *′a* ⇒ *′b*
assumes *random-process*[*measurable*]: ⋀*i. random-variable M′* (*X i*)

sublocale *stochastic-process* ⊆ *product*: *product-prob-space* (λ*t. distr M M′* (*X t*))
⟨*proof*⟩

lemma (**in** *prob-space*) *stochastic-processI*:
assumes ⋀*i. random-variable M′* (*X i*)
shows *stochastic-process M M′ X*
⟨*proof*⟩

typedef (*′t*, *′a*, *′b*) *stochastic-process* =
{(*M* :: *′a measure*, *M′* :: *′b measure*, *I* :: *′t set*, *X* :: *′t* ⇒ *′a* ⇒ *′b*).
stochastic-process M M′ X}
⟨*proof*⟩

setup-lifting *type-definition-stochastic-process*

lift-definition *proc-source* :: (*′t*,*′a*,*′b*) *stochastic-process* ⇒ *′a measure*
is *fst* ⟨*proof*⟩

interpretation *proc-source*: *prob-space proc-source X*
  ⟨*proof*⟩

lift-definition *proc-target* :: $('t,'a,'b)$ *stochastic-process* ⇒ *'b measure*
  is *fst* ∘ *snd* ⟨*proof*⟩

lift-definition *proc-index* :: $('t,'a,'b)$ *stochastic-process* ⇒ *'t set*
  is *fst* ∘ *snd* ∘ *snd* ⟨*proof*⟩

lift-definition *process* :: $('t,'a,'b)$ *stochastic-process* ⇒ *'t* ⇒ *'a* ⇒ *'b*
  is *snd* ∘ *snd* ∘ *snd* ⟨*proof*⟩

declare [[*coercion process*]]

lemma *stochastic-process-construct* [*simp*]: *stochastic-process* (*proc-source X*) (*proc-target X*) (*process X*)
  ⟨*proof*⟩

interpretation *stochastic-process proc-source X proc-target X proc-index X process X*
  ⟨*proof*⟩

lemma *stochastic-process-measurable* [*measurable*]: *process X t* ∈ (*proc-source X*) $\rightarrow_M$ (*proc-target X*)
  ⟨*proof*⟩

Here we construct a process on a given index set. For this we need to produce measurable functions for indices outside the index set; we use the constant function, but it needs to point at an element of the target set to be measurable.

context *prob-space*
begin

lift-definition *process-of* :: *'b measure* ⇒ *'t set* ⇒ (*'t* ⇒ *'a* ⇒ *'b*) ⇒ *'b* ⇒ $('t,'a,'b)$ *stochastic-process*
  is λ *M' I X ω*. *if* (∀ *t* ∈ *I*. *X t* ∈ *M* $\rightarrow_M$ *M'*) ∧ *ω* ∈ *space M'*
  then (*M*, *M'*, *I*, (λ*t*. *if t* ∈ *I then X t else* (λ-. *ω*)))
  else (*return* (*sigma UNIV* {{}, *UNIV*}) (*SOME x. True*), *sigma UNIV UNIV*, *I*, λ- -. *ω*)
  ⟨*proof*⟩

lemma *index-process-of*[*simp*]: *proc-index* (*process-of M' I X ω*) = *I*
  ⟨*proof*⟩

lemma
  assumes ∀ *t* ∈ *I*. *X t* ∈ *M* $\rightarrow_M$ *M' ω* ∈ *space M'*
  shows
    source-process-of[*simp*]: *proc-source* (*process-of M' I X ω*) = *M* **and**
    target-process-of[*simp*]: *proc-target* (*process-of M' I X ω*) = *M'* **and**

process-process-of[*simp*]: *process* (*process-of* M' I X ω) = (λt. *if* $t \in I$ *then* X t *else* (λ-. ω))
⟨*proof*⟩

lemma *process-of-apply*:
assumes $\forall t \in I$. X $t \in M \rightarrow_M M'$ $\omega \in$ *space* M' $t \in I$
shows *process* (*process-of* M' I X ω) $t = X$ t
⟨*proof*⟩
end

We define the finite-dimensional distributions of our process.

lift-definition *distributions* :: ('t, 'a, 'b) *stochastic-process* ⇒ 't *set* ⇒ ('t ⇒ 'b) *measure*
is $\lambda(M, M', -, X)$ T. (Π_M $t{\in}T$. *distr* M M' (X t)) ⟨*proof*⟩

lemma *distributions-altdef*: *distributions* X T = (Π_M $t{\in}T$. *distr* (*proc-source* X) (*proc-target* X) (X t))
⟨*proof*⟩

lemma *prob-space-distributions*: *prob-space* (*distributions* X J)
⟨*proof*⟩

lemma *sets-distributions*: *sets* (*distributions* X J) = *sets* (*PiM* J (λ-. (*proc-target* X)))
⟨*proof*⟩

lemma *space-distributions*: *space* (*distributions* X J) = (Π_E $i{\in}J$. *space* (*proc-target* X))
⟨*proof*⟩

lemma *emeasure-distributions*:
assumes *finite* J $\bigwedge j.\ j{\in}J \Longrightarrow A\ j \in$ *sets* (*proc-target* X)
shows *emeasure* (*distributions* X J) (Pi_E J A) = ($\prod j{\in}J$. *emeasure* (*distr* (*proc-source* X) (*proc-target* X) (X j)) (A j))
⟨*proof*⟩

interpretation *projective-family* (*proc-index* X) *distributions* X (λ-. *proc-target* X)
⟨*proof*⟩

locale *polish-stochastic* = *stochastic-process* M *borel* :: 'b::*polish-space measure* I X
for M **and** I **and** X

lemma *distributed-cong-random-variable*:
assumes $M = K$ $N = L$ *AE* x *in* M. X $x = Y$ x $X \in M \rightarrow_M N$ $Y \in K \rightarrow_M L$ $f \in$ *borel-measurable* N
shows *distributed* M N X $f \longleftrightarrow$ *distributed* K L Y f

$\langle proof \rangle$

For all sorted lists of indices, the increments specified by this list are independent

lift-definition *indep-increments* :: $('t :: linorder, 'a, 'b :: minus)$ *stochastic-process* $\Rightarrow$ *bool* **is**
$\lambda(M, M', I, X).$
$(\forall l.\ set\ l \subseteq I \wedge sorted\ l \wedge length\ l \geq 2 \longrightarrow$
$prob\text{-}space.indep\text{-}vars\ M\ (\lambda\text{-}.\ M')\ (\lambda k\ v.\ X\ (l!k)\ v - X\ (l!(k-1))\ v)\ \{1..<length\ l\})$ $\langle proof \rangle$

lemma *indep-incrementsE*:
assumes *indep-increments X*
and $set\ l \subseteq proc\text{-}index\ X \wedge sorted\ l \wedge length\ l \geq 2$
shows $prob\text{-}space.indep\text{-}vars\ (proc\text{-}source\ X)\ (\lambda\text{-}.\ proc\text{-}target\ X)$
$(\lambda k\ v.\ X\ (l!k)\ v - X\ (l!(k-1))\ v)\ \{1..<length\ l\}$
$\langle proof \rangle$

lemma *indep-incrementsI*:
assumes $\bigwedge l.\ set\ l \subseteq proc\text{-}index\ X \Longrightarrow sorted\ l \Longrightarrow length\ l \geq 2 \Longrightarrow$
$prob\text{-}space.indep\text{-}vars\ (proc\text{-}source\ X)\ (\lambda\text{-}.\ proc\text{-}target\ X)\ (\lambda k\ v.\ X\ (l!k)\ v - X\ (l!(k-1))\ v)\ \{1..<length\ l\}$
shows *indep-increments X*
$\langle proof \rangle$

lemma *indep-increments-indep-var*:
assumes $indep\text{-}increments\ X\ h \in proc\text{-}index\ X\ j \in proc\text{-}index\ X\ k \in proc\text{-}index\ X\ h \leq j\ j \leq k$
shows $prob\text{-}space.indep\text{-}var\ (proc\text{-}source\ X)\ (proc\text{-}target\ X)\ (\lambda v.\ X\ j\ v - X\ h\ v)\ (proc\text{-}target\ X)\ (\lambda v.\ X\ k\ v - X\ j\ v)$
$\langle proof \rangle$

definition $stationary\text{-}increments\ X \longleftrightarrow (\forall t1\ t2\ k.\ t1 > 0 \wedge t2 > 0 \wedge k > 0 \longrightarrow$

$distr\ (proc\text{-}source\ X)\ (proc\text{-}target\ X)\ (\lambda v.\ X\ (t1 + k)\ v - X\ t1\ v) =$
$distr\ (proc\text{-}source\ X)\ (proc\text{-}target\ X)\ (\lambda v.\ X\ (t2 + k)\ v - X\ t2\ v))$

Processes on the same source measure space, with the same index space, but not necessarily the same target measure since we only care about the measurable target space, not the measure

lift-definition *compatible* :: $('t,'a,'b)$ *stochastic-process* $\Rightarrow$ $('t,'a,'b)$ *stochastic-process* $\Rightarrow$ *bool*
is $\lambda(Mx, M'x, Ix, X)\ (My, M'y, Iy, \text{-}).\ Mx = My \wedge sets\ M'x = sets\ M'y \wedge Ix = Iy$ $\langle proof \rangle$

lemma *compatibleI*:
assumes $proc\text{-}source\ X = proc\text{-}source\ Y\ sets\ (proc\text{-}target\ X) = sets\ (proc\text{-}target\ Y)$
$proc\text{-}index\ X = proc\text{-}index\ Y$

shows *compatible X Y*
⟨*proof*⟩

lemma
assumes *compatible X Y*
shows
compatible-source [*dest*]: *proc-source X* = *proc-source Y* **and**
compatible-target [*dest*]: *sets* (*proc-target X*) = *sets* (*proc-target Y*) **and**
compatible-index [*dest*]: *proc-index X* = *proc-index Y*
⟨*proof*⟩

lemma *compatible-refl* [*simp*]: *compatible X X*
⟨*proof*⟩

lemma *compatible-sym*: *compatible X Y* $\Longrightarrow$ *compatible Y X*
⟨*proof*⟩

lemma *compatible-trans*:
assumes *compatible X Y compatible Y Z*
shows *compatible X Z*
⟨*proof*⟩

lemma (**in** *prob-space*) *compatible-process-of*:
assumes *measurable*: $\forall t \in I.\ X\ t \in M \rightarrow_M M'$ $\forall t \in I.\ Y\ t \in M \rightarrow_M M'$
and $a \in$ *space* M' $b \in$ *space* M'
shows *compatible* (*process-of* M' *I X a*) (*process-of* M' *I Y b*)
⟨*proof*⟩

definition *modification* :: ('t,'a,'b) *stochastic-process* ⇒ ('t,'a,'b) *stochastic-process* ⇒ *bool* **where**
modification X Y $\longleftrightarrow$ *compatible X Y* ∧ (∀ *t* ∈ *proc-index X*. *AE x in proc-source X*. *X t x* = *Y t x*)

lemma *modificationI* [*intro*]:
assumes *compatible X Y* $\bigwedge t.\ t \in$ *proc-index X* $\Longrightarrow$ *AE x in proc-source X*. *X t x* = *Y t x*
shows *modification X Y*
⟨*proof*⟩

lemma *modificationD* [*dest*]:
assumes *modification X Y*
shows *compatible X Y*
and $\bigwedge t.\ t \in$ *proc-index X* $\Longrightarrow$ *AE x in proc-source X*. *X t x* = *Y t x*
⟨*proof*⟩

lemma *modification-null-set*:
assumes *modification X Y t* ∈ *proc-index X*
obtains *N* **where** {*x* ∈ *space* (*proc-source X*). *X t x* ≠ *Y t x*} ⊆ *N N* ∈ *null-sets* (*proc-source X*)

$\langle proof \rangle$

lemma *modification-refl* [*simp*]: *modification X X*
$\langle proof \rangle$

lemma *modification-sym*: *modification X Y* $\Longrightarrow$ *modification Y X*
$\langle proof \rangle$

lemma *modification-trans*:
assumes *modification X Y modification Y Z*
shows *modification X Z*
$\langle proof \rangle$

lemma *modification-imp-identical-distributions*:
assumes *modification*: *modification X Y*
and *index*: $T \subseteq$ *proc-index X*
shows *distributions X T* = *distributions Y T*
$\langle proof \rangle$

definition *indistinguishable* :: $('t,'a,'b)$ *stochastic-process* $\Rightarrow$ $('t,'a,'b)$ *stochastic-process* $\Rightarrow$ *bool* **where**
indistinguishable X Y $\longleftrightarrow$ *compatible X Y* $\wedge$
($\exists\, N \in$ *null-sets* (*proc-source X*). $\forall\, t \in$ *proc-index X*. $\{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$)

lemma *indistinguishableI*:
assumes *compatible X Y*
and $\exists\, N \in$ *null-sets* (*proc-source X*). ($\forall\, t \in$ *proc-index X*. $\{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$)
shows *indistinguishable X Y*
$\langle proof \rangle$

lemma *indistinguishable-null-set*:
assumes *indistinguishable X Y*
obtains N **where**
$N \in$ *null-sets* (*proc-source X*)
$\bigwedge t.\ t \in$ *proc-index X* $\Longrightarrow \{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$
$\langle proof \rangle$

lemma *indistinguishableD*:
assumes *indistinguishable X Y*
shows *compatible X Y*
and $\exists\, N \in$ *null-sets* (*proc-source X*). ($\forall\, t \in$ *proc-index X*. $\{x \in$ *space* (*proc-source X*). $X\ t\ x \neq Y\ t\ x\} \subseteq N$)
$\langle proof \rangle$

lemma *indistinguishable-eq-AE*:
assumes *indistinguishable X Y*
shows *AE x in proc-source X*. $\forall\, t \in$ *proc-index X*. $X\ t\ x = Y\ t\ x$

⟨*proof*⟩

lemma *indistinguishable-null-ex*:
assumes *indistinguishable X Y*
shows $\exists\, N \in$ *null-sets* (*proc-source X*). $\{x \in$ *space* (*proc-source X*).$\exists\, t \in$ *proc-index* *X*. $X\ t\ x \neq Y\ t\ x\} \subseteq N$
⟨*proof*⟩

lemma *indistinguishable-refl* [*simp*]: *indistinguishable X X*
⟨*proof*⟩

lemma *indistinguishable-sym*: *indistinguishable X Y* $\Longrightarrow$ *indistinguishable Y X*
⟨*proof*⟩

lemma *indistinguishable-trans*:
assumes *indistinguishable X Y indistinguishable Y Z*
shows *indistinguishable X Z*
⟨*proof*⟩

lemma *indistinguishable-modification*: *indistinguishable X Y* $\Longrightarrow$ *modification X Y*
⟨*proof*⟩

Klenke 21.5(i)

lemma *modification-countable*:
assumes *modification X Y countable* (*proc-index X*)
shows *indistinguishable X Y*
⟨*proof*⟩

Klenke 21.5(ii). The textbook statement is more general - we reduce right continuity to regular continuity

lemma *modification-continuous-indistinguishable*:
fixes *X* :: (*real*, *'a*, *'b* :: *metric-space*) *stochastic-process*
assumes *modification*: *modification X Y*
and *interval*: $\exists\, T > 0$. *proc-index X* = $\{0..T\}$
and *rc*: *AE* ω *in proc-source X*. *continuous-on* (*proc-index X*) (λt. $X\ t\ \omega$)
(**is** *AE* ω *in proc-source X*. *?cont-X* ω)
AE ω *in proc-source Y*. *continuous-on* (*proc-index Y*) (λt. $Y\ t\ \omega$)
(**is** *AE* ω *in proc-source Y*. *?cont-Y* ω)
shows *indistinguishable X Y*
⟨*proof*⟩

lift-definition *restrict-index* :: (*'t*, *'a*, *'b*) *stochastic-process* ⇒ *'t set* ⇒ (*'t*, *'a*, *'b*) *stochastic-process*
is $\lambda(M, M', I, X)\ T.\ (M, M', T, X)$ ⟨*proof*⟩

lemma
shows

restrict-index-source[*simp*]: *proc-source* (*restrict-index X T*) = *proc-source X*
and
restrict-index-target[*simp*]: *proc-target* (*restrict-index X T*) = *proc-target X* **and**
restrict-index-index[*simp*]: *proc-index* (*restrict-index X T*) = *T* **and**
restrict-index-process[*simp*]: *process* (*restrict-index X T*) = *process X*
⟨*proof*⟩

lemma *restrict-index-override*[*simp*]: *restrict-index* (*restrict-index X T*) *S* = *restrict-index X S*
⟨*proof*⟩

lemma *compatible-restrict-index*:
assumes *compatible X Y*
shows *compatible* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *modification-restrict-index*:
assumes *modification X Y S* ⊆ *proc-index X*
shows *modification* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *indistinguishable-restrict-index*:
assumes *indistinguishable X Y S* ⊆ *proc-index X*
shows *indistinguishable* (*restrict-index X S*) (*restrict-index Y S*)
⟨*proof*⟩

lemma *AE-eq-minus* [*intro*]:
fixes *a* :: *'a* ⇒ (*'b* :: *real-normed-vector*)
assumes *AE x in M*. *a x* = *b x* *AE x in M*. *c x* = *d x*
shows *AE x in M*. *a x* − *c x* = *b x* − *d x*
⟨*proof*⟩

lemma *modification-indep-increments*:
fixes *X Y* :: (*'a* :: *linorder*, *'b*, *'c* :: {*second-countable-topology*, *real-normed-vector*}) *stochastic-process*
assumes *modification X Y sets* (*proc-target Y*) = *sets borel*
shows *indep-increments X* ⟹ *indep-increments Y*
⟨*proof*⟩

end

6 The Kolomgorov-Chentsov theorem

theory *Kolmogorov-Chentsov*
imports *Stochastic-Processes Holder-Continuous Dyadic-Interval Measure-Convergence*
begin

6.1 Supporting lemmas

The main contribution of this file is the Kolmogorov-Chentsov theorem: given a stochastic process that satisfies some continuity properties, we can construct a Hölder continuous modification. We first prove some auxiliary lemmas before moving on to the main construction.

Klenke 5.11: Markov inequality. Compare with $\llbracket(\lambda x.\ ?u\ x * indicator\ ?A\ x) \in borel\text{-}measurable\ ?M;\ ?A \in sets\ ?M\rrbracket \Longrightarrow emeasure\ ?M\ \{x \in ?A.\ 1 \leq ?c * ?u\ x\} \leq ?c * set\text{-}nn\text{-}integral\ ?M\ ?A\ ?u$

lemma *nn-integral-Markov-inequality-extended*:
  fixes $f :: real \Rightarrow ennreal$ **and** $\varepsilon :: real$ **and** $X :: 'a \Rightarrow real$
  assumes *mono*: $mono\text{-}on\ (range\ X \cup \{0<..\})\ f$
    and *finite*: $\bigwedge x.\ f\ x < \infty$
    and *e*: $\varepsilon > 0\ f\ \varepsilon > 0$
    and [*measurable*]: $X \in borel\text{-}measurable\ M$
  shows $emeasure\ M\ \{p \in space\ M.\ (X\ p) \geq \varepsilon\} \leq (\int^+ x.\ f\ (X\ x)\ \partial M)\ /\ f\ \varepsilon$
⟨*proof*⟩

lemma *nn-integral-Markov-inequality-extended-rnv*:
  fixes $f :: real \Rightarrow real$ **and** $\varepsilon :: real$ **and** $X :: 'a \Rightarrow 'b :: real\text{-}normed\text{-}vector$
  assumes [*measurable*]: $X \in borel\text{-}measurable\ M$
    and *mono*: $mono\text{-}on\ \{0..\}\ f$
    and *e*: $\varepsilon > 0\ f\ \varepsilon > 0$
  shows $emeasure\ M\ \{p \in space\ M.\ norm\ (X\ p) \geq \varepsilon\} \leq (\int^+ x.\ f\ (norm\ (X\ x))\ \partial M)\ /\ f\ \varepsilon$
  ⟨*proof*⟩

6.2 Kolmogorov-Chentsov

Klenke theorem 21.6 - Kolmorogov-Chentsov

locale *Kolmogorov-Chentsov* =
  fixes $X :: (real, 'a, 'b :: polish\text{-}space)\ stochastic\text{-}process$
    and $a\ b\ C\ \gamma :: real$
  assumes *index*[*simp*]: $proc\text{-}index\ X = \{0..\}$
    and *target-borel*[*simp*]: $proc\text{-}target\ X = borel$
    and *gt-0*: $a > 0\ b > 0\ C > 0$
    and *b-leq-a*: $b \leq a$
    and *gamma*: $\gamma \in \{0<..<b/a\}$
    and *expectation*: $\bigwedge s\ t.\ \llbracket s \geq 0;\ t \geq 0\rrbracket \Longrightarrow$
        $(\int^+ x.\ dist\ (X\ t\ x)\ (X\ s\ x)\ powr\ a\ \partial proc\text{-}source\ X) \leq C * dist\ t\ s\ powr\ (1+b)$
begin

lemma *gamma-0-1*[*simp*]: $\gamma \in \{0<..1\}$
  ⟨*proof*⟩

lemma *gamma-gt-0*[*simp*]: $\gamma > 0$

⟨*proof*⟩

lemma *gamma-le-1*[*simp*]: $\gamma \leq 1$
⟨*proof*⟩

abbreviation *source* ≡ *proc-source X*

lemma *X-borel-measurable*[*measurable*]: *X t* ∈ *borel-measurable source* **for** *t*
⟨*proof*⟩

lemma *markov*: $\mathcal{P}(x\ in\ source.\ \varepsilon \leq dist\ (X\ t\ x)\ (X\ s\ x)) \leq (C * dist\ t\ s\ powr\ (1 + b))\ /\ \varepsilon\ powr\ a$
if $s \geq 0$ $t \geq 0$ $\varepsilon > 0$ **for** $s\ t\ \varepsilon$
⟨*proof*⟩

lemma *conv-in-prob*:
assumes $t \geq 0$
shows *tendsto-measure* (*proc-source X*) *X* (*X t*) (*at t within* {*0..*})
⟨*proof*⟩

lemma *conv-in-prob-finite*:
assumes $t \geq 0$
shows *tendsto-measure* (*proc-source X*) *X* (*X t*) (*at t within* {*0..T*})
⟨*proof*⟩

lemma *incr*: $\mathcal{P}(x\ in\ source.\ 2\ powr\ (-\ \gamma * n) \leq dist\ (X\ ((k - 1) * 2\ powr - n)\ x)\ (X\ (k * 2\ powr - n)\ x))$
$\leq C * 2\ powr\ (-n * (1{+}b{-}a{*}\gamma))$
if $k \geq 1$ $n \geq 0$ **for** $k\ n$
⟨*proof*⟩

end

In order to construct the modification of X, it suffices to construct a modification of X on $\{0..T\}$ for all finite T, from which we construct the modification on $\{0..\}$ via a countable union.

locale *Kolmogorov-Chentsov-finite* = *Kolmogorov-Chentsov* +
fixes *T* :: *real*
assumes *zero-le-T*: $0 < T$
begin

A_n will characterise the set of states with increments that exceed the bounds required for Hölder continuity. As $n \longrightarrow \infty$, this approaches the set of states for which X is not Hölder continuous. We define N as this limit, and show that N is a null set. On $\omega \in \Omega - N$, we show that $X(\omega)$ is Hölder continuous (and therefore uniformly continuious) on the dyadic rationals, and construct a modification by taking the continuous extension on the reals.

definition $A \equiv \lambda n.$ *if* $2\ \hat{}\ n * T < 1$ *then space source else*

$\{x \in \mathit{space\ source}.$
$\mathit{Max}\ \{\mathit{dist}\ (X\ (\mathit{real\text{-}of\text{-}int}\ (k - 1) * 2\ \mathit{powr} - \mathit{real}\ n)\ x)\ (X\ (\mathit{real\text{-}of\text{-}int}\ k * 2\ \mathit{powr} - \mathit{real}\ n)\ x)$
$\mid k.\ k \in \{1..\lfloor 2\hat{\ }n * T\rfloor\}\} \geq 2\ \mathit{powr}\ (-\gamma * \mathit{real}\ n)\}$

abbreviation $B \equiv \lambda n.\ (\bigcup m.\ A\ (m + n))$

abbreviation $N \equiv \bigcap(\mathit{range}\ B)$

lemma *A-geq*: $2\ \hat{}\ n * T \geq 1 \Longrightarrow A\ n = \{x \in \mathit{space\ source}.$
$\mathit{Max}\ \{\mathit{dist}\ (X\ (\mathit{real\text{-}of\text{-}int}\ (k - 1) * 2\ \mathit{powr} - \mathit{real}\ n)\ x)\ (X\ (\mathit{real\text{-}of\text{-}int}\ k * 2\ \mathit{powr} - \mathit{real}\ n)\ x)$
$\mid k.\ k \in \{1..\lfloor 2\hat{\ }n * T\rfloor\}\} \geq 2\ \mathit{powr}\ (-\gamma * \mathit{real}\ n)\}$ **for** n
⟨*proof*⟩

lemma *A-measurable*[*measurable*]: $A\ n \in \mathit{sets\ source}$
⟨*proof*⟩

lemma *emeasure-A-leq*:
fixes $n :: \mathit{nat}$
assumes [*simp*]: $2\hat{\ }n * T \geq 1$
shows $\mathit{emeasure\ source}\ (A\ n) \leq C * T * 2\ \mathit{powr}\ (-\ n * (b - a * \gamma))$
⟨*proof*⟩

lemma *measure-A-leq*:
assumes $2\hat{\ }n * T \geq 1$
shows $\mathit{measure\ source}\ (A\ n) \leq C * T * 2\ \mathit{powr}\ (-\ n * (b - a * \gamma))$
⟨*proof*⟩

lemma *summable-A*: $\mathit{summable}\ (\lambda m.\ \mathit{measure\ source}\ (A\ m))$
⟨*proof*⟩

lemma *lim-B*: $(\lambda n.\ \mathit{measure\ source}\ (B\ n)) \longrightarrow 0$
⟨*proof*⟩

lemma *N-null*: $N \in \mathit{null\text{-}sets\ source}$
⟨*proof*⟩

lemma *notin-N-index*:
assumes $\omega \in \mathit{space\ source} - N$
obtains n_0 **where** $\omega \notin (\bigcup n.\ A\ (n + n_0))$
⟨*proof*⟩

context
fixes ω
assumes ω: $\omega \in \mathit{space\ source} - N$
begin

definition $n_0 \equiv \mathit{SOME}\ m.\ \omega \notin (\bigcup n.\ A\ (n + m)) \wedge m > 0$

lemma
  shows *n-zero*: $\omega \notin (\bigcup n.\ A\ (n + n_0))$
    and *n-zero-nonzero*: $n_0 > 0$
$\langle proof \rangle$

lemma *nzero-ge*: $\bigwedge n.\ n \geq n_0 \Longrightarrow 2\hat{}n * T \geq 1$
$\langle proof \rangle$

lemma *omega-notin*: $\bigwedge n.\ n \geq n_0 \Longrightarrow \omega \notin A\ n$
  $\langle proof \rangle$

Klenke 21.7

lemma *X-dyadic-incr*:
  assumes $k \in \{1..\lfloor 2\hat{}n * T\rfloor\}$ $n \geq n_0$
  shows *dist* $(X\ ((\mathit{real\text{-}of\text{-}int}\ k{-}1)/2\hat{}n)\ \omega)\ (X\ (\mathit{real\text{-}of\text{-}int}\ k/2\hat{}n)\ \omega) < 2\ \mathit{powr}\ (-\ \gamma * n)$
$\langle proof \rangle$

Klenke (21.8)

lemma *dist-dyadic-mn*:
  assumes *mn*: $n_0 \leq n$ $n \leq m$
    and *t-dyadic*: $t \in \mathit{dyadic\text{-}interval\text{-}step}\ m\ 0\ T$
    and *u-dyadic-n*: $u \in \mathit{dyadic\text{-}interval\text{-}step}\ n\ 0\ T$
    and *ut*: $u \leq t$ $t - u < 2/2\hat{}n$
  shows *dist* $(X\ u\ \omega)\ (X\ t\ \omega) \leq 2\ \mathit{powr}\ (-\ \gamma * n)\ /\ (1 - 2\ \mathit{powr}\ -\ \gamma)$
$\langle proof \rangle$

lemma *dist-dyadic-fixed*:
  assumes *mn*: $n_0 \leq n$ $n \leq m$
    and *s-dyadic*: $s \in \mathit{dyadic\text{-}interval\text{-}step}\ m\ 0\ T$
    and *t-dyadic*: $t \in \mathit{dyadic\text{-}interval\text{-}step}\ m\ 0\ T$
    and *st*: $s \leq t$ $t - s \leq 1/2\hat{}n$
  shows *dist* $(X\ t\ \omega)\ (X\ s\ \omega) \leq 2 * 2\ \mathit{powr}\ (-\ \gamma * n)\ /\ (1 - 2\ \mathit{powr}\ -\ \gamma)$
$\langle proof \rangle$

definition $C_0 \equiv 2 * 2\ \mathit{powr}\ \gamma\ /\ (1 - 2\ \mathit{powr}\ -\ \gamma)$

lemma *C-zero-ge*[*simp*]: $C_0 > 0$
  $\langle proof \rangle$

Klenke (21.9)

Let $s, t \in D$ with $|s - t| \leq \frac{1}{2^n_0}$. By choosing the minimal $n \geq n_0$ such that $|t - s| \geq 2^{-n}$, we obtain by $[\![n_0 \leq ?n;\ ?n \leq ?m;\ ?s \in \mathit{dyadic\text{-}interval\text{-}step}\ ?m\ 0\ T;\ ?t \in \mathit{dyadic\text{-}interval\text{-}step}\ ?m\ 0\ T;\ ?s \leq ?t;\ ?t - ?s \leq 1\ /\ 2^{?n}]\!] \Longrightarrow \mathit{dist}\ (\mathit{process}\ X\ ?t\ \omega)\ (\mathit{process}\ X\ ?s\ \omega) \leq 2 * 2\ \mathit{powr}\ (-\ \gamma * \mathit{real}\ ?n)\ /\ (1 - 2\ \mathit{powr}\ -\ \gamma)$:

$$|X_t(\omega) - X_s(\omega)| \leq C_0|t - s|^\gamma$$

lemma *dist-dyadic*:
  assumes *t*: $t \in$ *dyadic-interval 0 T*
    and *s*: $s \in$ *dyadic-interval 0 T*
    and *st-dist*: $dist\ t\ s \leq 1\ /\ 2\ \hat{}\ n_0$
  shows $dist\ (X\ t\ \omega)\ (X\ s\ \omega) \leq C_0 * (dist\ t\ s)\ powr\ \gamma$
⟨*proof*⟩

definition $K \equiv C_0 * (2\hat{}nat\ \lceil 2\hat{}n_0 * T\rceil)\ powr\ (1 - \gamma)$

lemma C_0*-le-K*: $C_0 \leq K$
  ⟨*proof*⟩

lemma *K-pos*: $0 < K$
  ⟨*proof*⟩

Klenke (21.10)

lemma *X-dyadic-le-K'*:
  assumes *dyadic*: $s \in$ *dyadic-interval 0 T* $t \in$ *dyadic-interval 0 T*
    and *st*: $s \leq t$
  shows $dist\ (X\ s\ \omega)\ (X\ t\ \omega) \leq K * dist\ s\ t\ powr\ \gamma$
⟨*proof*⟩

lemma *X-dyadic-le-K*:
  assumes $s \in$ *dyadic-interval 0 T*
    and $t \in$ *dyadic-interval 0 T*
  shows $dist\ (X\ s\ \omega)\ (X\ t\ \omega) \leq K * dist\ s\ t\ powr\ \gamma$
  ⟨*proof*⟩

corollary *holder-dyadic*: γ*−holder-on* (*dyadic-interval 0 T*) $(\lambda t.\ X\ t\ \omega)$
  ⟨*proof*⟩

lemma *uniformly-continuous-dyadic*: *uniformly-continuous-on* (*dyadic-interval 0 T*) $(\lambda t.\ X\ t\ \omega)$
  ⟨*proof*⟩

lemma *Lim-exists*: $\exists L.\ ((\lambda s.\ X\ s\ \omega) \longrightarrow L)$ (*at t within* (*dyadic-interval 0 T*))
  if $t \in \{0..T\}$
  ⟨*proof*⟩

lemma *Lim-unique*: $\exists! L.\ ((\lambda s.\ X\ s\ \omega) \longrightarrow L)$ (*at t within* (*dyadic-interval 0 T*))

  if $t \in \{0..T\}$
  ⟨*proof*⟩

definition $L \equiv (\lambda t.$ (*Lim* (*at t within dyadic-interval 0 T*) $(\lambda s.\ X\ s\ \omega)))$

lemma *X-tendsto-L*:
  assumes $t \in \{0..T\}$

shows $((\lambda s.\ X\ s\ \omega) \longrightarrow L\ t)$ (*at t within* (*dyadic-interval 0 T*))
⟨*proof*⟩

lemma *L-dist-K*:
assumes *s*: $s \in \{0..T\}$
and *t*: $t \in \{0..T\}$
shows *dist* $(L\ s)\ (L\ t) \leq K *$ *dist s t powr* γ
⟨*proof*⟩

corollary *L-holder*: γ*−holder-on* $\{0..T\}$ *L*
⟨*proof*⟩

corollary *L-local-holder*: *local-holder-on* γ $\{0..T\}$ *L*
⟨*proof*⟩

lemma *X-dyadic-eq-L*:
assumes $t \in$ *dyadic-interval 0 T*
shows $X\ t\ \omega = L\ t$
⟨*proof*⟩
end

definition *default* :: *'b* **where** *default* = (*SOME x. True*)

definition *X-tilde* :: *real* ⇒ *'a* ⇒ *'b* **where**
X-tilde ≡ ($\lambda t\ \omega$. *if* $\omega \in N$ *then default else* (*Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.\ X\ s\ \omega$)))

lemma *X-tilde-not-N-Lim*:
assumes $\omega \in$ *space source* − *N*
shows *X-tilde t* ω = *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.\ X\ s\ \omega$)
⟨*proof*⟩

lemma *X-tilde-not-N-L*:
assumes $\omega \in$ *space source* − *N*
shows *X-tilde t* $\omega = L\ \omega\ t$
⟨*proof*⟩

lemma *local-holder-X-tilde*: *local-holder-on* γ $\{0..T\}$ (λt. *X-tilde t* ω)
if $\omega \in$ *space source* **for** ω
⟨*proof*⟩

corollary *X-tilde-eq-L-AE*: *AE* ω *in source. X-tilde t* $\omega = L\ \omega\ t$
⟨*proof*⟩

corollary *X-tilde-eq-Lim-AE*:
AE ω *in source. X-tilde t* ω = *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.\ X\ s\ \omega$)
⟨*proof*⟩

lemma *X-tilde-tendsto-AE*: $t \in \{0..T\} \Longrightarrow$ *tendsto-AE source X* (*X-tilde t*) (*at t*

within dyadic-interval 0 T)
⟨*proof*⟩

end

context *Kolmogorov-Chentsov-finite*
begin

By (21.5) $0 \leq ?t \Longrightarrow$ *tendsto-measure source* (*process X*) (*process X ?t*) (*at ?t within* $\{0..?T\}$) and (21.11) $?\omega \in$ *space source* $- (\bigcap_n \bigcup_m A\ (m + n)) \Longrightarrow L\ ?\omega \equiv \lambda t.$ *Lim* (*at t within dyadic-interval 0 T*) ($\lambda s.$ *process X s* $?\omega$), $P[X \neq \tilde{X}] = 0$

lemma *X-tilde-measurable*[*measurable*]:
assumes $t \in \{0..T\}$
shows *X-tilde t* $\in$ *borel-measurable source*
⟨*proof*⟩

lemma *X-eq-X-tilde-AE*: *AE* ω *in source. X t* ω = *X-tilde t* ω **if** $t \in \{0..T\}$ **for** *t*
⟨*proof*⟩

lemma *X-tilde-modification*: *modification* (*restrict-index X* $\{0..T\}$)
(*prob-space.process-of source* (*proc-target X*) $\{0..T\}$ *X-tilde default*)
⟨*proof*⟩
end

We have now shown that we can construct a modification of X for any interval $\{0..T\}$. We want to extend this result to construct a modification on the interval $\{0..\}$ - this can be constructed by gluing together all modifications with natural-valued T which results in a countable union of modifications, which itself is a modification.

context *Kolmogorov-Chentsov*
begin

lemma *Kolmogorov-Chentsov-finite*: $T > 0 \Longrightarrow$ *Kolmogorov-Chentsov-finite X a b C* γ *T*
⟨*proof*⟩

definition *Mod* $\equiv \lambda T.$ *SOME Y. modification* (*restrict-index X* $\{0..T\}$) $Y \wedge$
($\forall x \in$*space source. local-holder-on* γ $\{0..T\}$ ($\lambda t.$ *Y t x*))

lemma *Mod*: *modification* (*restrict-index X* $\{0..T\}$) (*Mod T*)
($\forall x \in$*space source. local-holder-on* γ $\{0..T\}$ ($\lambda t.$ (*Mod T*) *t x*)) **if** $0 < T$ **for** *T*
⟨*proof*⟩

lemma *compatible-Mod*: *compatible* (*restrict-index X* $\{0..T\}$) (*Mod T*) **if** $0 < T$ **for** *T*
⟨*proof*⟩

lemma *Mod-source*[*simp*]: *proc-source* (*Mod T*) = *source* **if** $0 < T$ **for** *T*
⟨*proof*⟩

lemma *Mod-target*: *sets* (*proc-target* (*Mod T*)) = *sets* (*proc-target X*) **if** $0 < T$ **for** *T*
⟨*proof*⟩

lemma *Mod-index*[*simp*]: $0 < T \Longrightarrow$ *proc-index* (*Mod T*) = $\{0..T\}$
⟨*proof*⟩

lemma *indistinguishable-mod*:
indistinguishable (*restrict-index* (*Mod S*) $\{0..min\ S\ T\}$) (*restrict-index* (*Mod T*) $\{0..min\ S\ T\}$)
if $S > 0$ $T > 0$ **for** *S T*
⟨*proof*⟩

definition *N S T* ≡ *SOME N*. *N* ∈ *null-sets source* ∧ $\{\omega \in$ *space source*. $\exists t \in \{0..min\ S\ T\}$. (*Mod S*) $t\ \omega \neq$ (*Mod T*) $t\ \omega\} \subseteq N$

lemma *N*:
assumes $S > 0$ $T > 0$
shows *N S T* ∈ *null-sets source* ∧ $\{\omega \in$ *space source*. $\exists t \in \{0..min\ S\ T\}$. (*Mod S*) $t\ \omega \neq$ (*Mod T*) $t\ \omega\} \subseteq$ *N S T*
⟨*proof*⟩

definition *N-inf* **where** *N-inf* ≡ $(\bigcup S \in \mathbb{N} - \{0\}.\ (\bigcup\ T \in \mathbb{N} - \{0\}.\ N\ S\ T))$

lemma *N-inf-null*: *N-inf* ∈ *null-sets source*
⟨*proof*⟩

lemma *Mod-eq-N-inf*: (*Mod S*) $t\ \omega$ = (*Mod T*) $t\ \omega$
if $\omega \in$ *space source* − *N-inf* $t \in \{0..min\ S\ T\}$ $S \in \mathbb{N} - \{0\}$ $T \in \mathbb{N} - \{0\}$ **for** $\omega\ t\ S\ T$
⟨*proof*⟩

definition *default* :: *'b* **where** *default* = (*SOME x. True*)

definition *X-mod* ≡ $\lambda t\ \omega$. *if* $\omega \in$ *space source* − *N-inf then* (*Mod* $\lfloor t+1 \rfloor$) $t\ \omega$ *else default*

definition *X-mod-process* ≡ *prob-space.process-of source* (*proc-target X*) $\{0..\}$ *X-mod default*

lemma *Mod-measurable*[*measurable*]: $\forall t \in \{0..\}$. *X-mod t* ∈ *source* $\rightarrow_M$ *proc-target X*
⟨*proof*⟩

lemma *modification-X-mod-process*: *modification X X-mod-process*
⟨*proof*⟩

lemma *local-holder-X-mod*: *local-holder-on* γ $\{0..\}$ $(\lambda t.\ X\text{-}mod\ t\ \omega)$ **for** ω
$\langle proof \rangle$

lemma *local-holder-X-mod-process*: *local-holder-on* γ $\{0..\}$ $(\lambda t.\ X\text{-}mod\text{-}process\ t\ \omega)$
for ω
$\langle proof \rangle$

theorem *continuous-modification*:
$\exists X'.$ *modification* $X\ X' \wedge (\forall \omega.$ *local-holder-on* γ $\{0..\}$ $(\lambda t.\ X'\ t\ \omega))$
$\langle proof \rangle$

end

theorem *Kolmogorov-Chentsov*:
fixes X :: (*real*, $'a$, $'b$ :: *polish-space*) *stochastic-process*
and $a\ b\ C\ \gamma$ :: *real*
assumes *index*[*simp*]: *proc-index* $X = \{0..\}$
and *target-borel*[*simp*]: *proc-target* X = *borel*
and *gt-0*: $a > 0\ b > 0\ C > 0$
and *b-leq-a*: $b \leq a$
and *gamma*: $\gamma \in \{0{<}..{<}b/a\}$
and *expectation*: $\bigwedge s\ t.\ [\![s \geq 0;\ t \geq 0]\!] \Longrightarrow$
$(\int^{+} x.\ dist\ (X\ t\ x)\ (X\ s\ x)\ powr\ a\ \partial proc\text{-}source\ X) \leq C * dist\ t\ s\ powr$
$(1+b)$
shows $\exists X'.$ *modification* $X\ X' \wedge (\forall \omega.$ *local-holder-on* γ $\{0..\}$ $(\lambda t.\ X'\ t\ \omega))$
$\langle proof \rangle$
end

References

[1] A. Klenke. *Probability theory: a comprehensive course.* Springer Science & Business Media, 2020.