

# The Königsberg Bridge Problem and the Friendship Theorem

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## Abstract

This development provides a formalization of undirected graphs and simple graphs, which are based on Benedikt Nordhoff and Peter Lammich's simple formalization of labelled directed graphs [4] in the archive. Then, with our formalization of graphs, we have shown both necessary and sufficient conditions for Eulerian trails and circuits [2] as well as the fact that the Königsberg Bridge problem does not have a solution. In addition, we have also shown the Friendship Theorem in simple graphs[1, 3].

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**theory** *MoreGraph* **imports** *Complex-Main Dijkstra-Shortest-Path.Graph*  
**begin**

## 1 Undirected Multigraph and undirected trails

**locale** *valid-unMultigraph=valid-graph* *G* **for**  $G::('v,'w)$  *graph+*  
**assumes** *corres[simp]*:  $(v,w,u') \in \text{edges } G \longleftrightarrow (u',w,v) \in \text{edges } G$   
**and** *no-id[simp]*:  $(v,w,v) \notin \text{edges } G$

**fun** (**in** *valid-unMultigraph*) *is-trail* ::  $'v \Rightarrow ('v,'w)$  *path*  $\Rightarrow 'v \Rightarrow \text{bool}$  **where**  
*is-trail*  $v \ [] \ v' \longleftrightarrow v=v' \wedge v' \in V \ |$   
*is-trail*  $v \ ((v1,w,v2)\#ps) \ v' \longleftrightarrow v=v1 \wedge (v1,w,v2) \in E \wedge$   
 $(v1,w,v2) \notin \text{set } ps \wedge (v2,w,v1) \notin \text{set } ps \wedge \text{is-trail } v2 \ ps \ v'$

## 2 Degrees and related properties

**definition** *degree* ::  $'v \Rightarrow ('v,'w)$  *graph*  $\Rightarrow \text{nat}$  **where**  
*degree*  $v \ g \equiv \text{card}(\{e. e \in \text{edges } g \wedge \text{fst } e = v\})$

**definition** *odd-nodes-set* ::  $('v,'w)$  *graph*  $\Rightarrow 'v$  *set* **where**  
*odd-nodes-set*  $g \equiv \{v. v \in \text{nodes } g \wedge \text{odd}(\text{degree } v \ g)\}$

**definition** *num-of-odd-nodes* ::  $('v, 'w)$  *graph*  $\Rightarrow \text{nat}$  **where**  
*num-of-odd-nodes*  $g \equiv \text{card}(\text{odd-nodes-set } g)$

**definition** *num-of-even-nodes* ::  $('v, 'w)$  *graph*  $\Rightarrow \text{nat}$  **where**  
*num-of-even-nodes*  $g \equiv \text{card}(\{v. v \in \text{nodes } g \wedge \text{even}(\text{degree } v \ g)\})$

**definition** *del-unEdge* **where** *del-unEdge*  $v \ e \ v' \ g \equiv ()$   
 $\text{nodes} = \text{nodes } g, \text{edges} = \text{edges } g - \{(v,e,v'),(v',e,v)\} \ |$

**definition** *rev-path* ::  $('v,'w)$  *path*  $\Rightarrow ('v,'w)$  *path* **where**  
*rev-path*  $ps \equiv \text{map } (\lambda(a,b,c).(c,b,a)) (\text{rev } ps)$

**fun** *rem-unPath*::  $('v,'w)$  *path*  $\Rightarrow ('v,'w)$  *graph*  $\Rightarrow ('v,'w)$  *graph* **where**  
*rem-unPath*  $[] \ g = g$   
*rem-unPath*  $((v,w,v')\#ps) \ g =$   
*rem-unPath*  $ps \ (\text{del-unEdge } v \ w \ v' \ g)$

**lemma** *del-undirected*: *del-unEdge*  $v \ e \ v' \ g = \text{delete-edge } v' \ e \ v \ (\text{delete-edge } v \ e \ v' \ g)$   
 $\langle \text{proof} \rangle$

**lemma** *delete-edge-sym*: *del-unEdge*  $v \ e \ v' \ g = \text{del-unEdge } v' \ e \ v \ g$

*<proof>*

**lemma** *del-unEdge-valid[simp]*: **assumes** *valid-unMultigraph g*  
**shows** *valid-unMultigraph (del-unEdge v e v' g)*  
*<proof>*

**lemma** *set-compre-diff*:  $\{x \in A - B. P x\} = \{x \in A. P x\} - \{x \in B. P x\}$  *<proof>*  
**lemma** *set-compre-subset*:  $B \subseteq A \implies \{x \in B. P x\} \subseteq \{x \in A. P x\}$  *<proof>*

**lemma** *del-edge-undirected-degree-plus*: *finite (edges g)  $\implies (v,e,v') \in edges g$*   
 *$\implies (v',e,v) \in edges g \implies degree v (del-unEdge v e v' g) + 1 = degree v g$*   
*<proof>*

**lemma** *del-edge-undirected-degree-plus'*: *finite (edges g)  $\implies (v,e,v') \in edges g$*   
 *$\implies (v',e,v) \in edges g \implies degree v' (del-unEdge v e v' g) + 1 = degree v' g$*   
*<proof>*

**lemma** *del-edge-undirected-degree-minus[simp]*: *finite (edges g)  $\implies (v,e,v') \in edges g$*   
 *$\implies (v',e,v) \in edges g \implies degree v (del-unEdge v e v' g) = degree v g - (1::nat)$*   
*<proof>*

**lemma** *del-edge-undirected-degree-minus'[simp]*: *finite (edges g)  $\implies (v,e,v') \in edges g$*   
 *$\implies (v',e,v) \in edges g \implies degree v' (del-unEdge v e v' g) = degree v' g - (1::nat)$*   
*<proof>*

**lemma** *del-unEdge-com*: *del-unEdge v w v' (del-unEdge n e n' g)*  
*= del-unEdge n e n' (del-unEdge v w v' g)*  
*<proof>*

**lemma** *rem-unPath-com*: *rem-unPath ps (del-unEdge v w v' g)*  
*= del-unEdge v w v' (rem-unPath ps g)*  
*<proof>*

**lemma** *rem-unPath-valid[intro]*:  
*valid-unMultigraph g  $\implies valid-unMultigraph (rem-unPath ps g)$*   
*<proof>*

**lemma** (**in** *valid-unMultigraph*) *degree-frame*:  
**assumes** *finite (edges G) x  $\notin \{v, v'\}$*   
**shows** *degree x (del-unEdge v w v' G) = degree x G (is ?L=?R)*  
*<proof>*

**lemma** *[simp]*: *rev-path [] = []* *<proof>*

**lemma** *rev-path-append[simp]*:  $rev\text{-}path\ (xs@ys) = (rev\text{-}path\ ys) @ (rev\text{-}path\ xs)$   
*<proof>*

**lemma** *rev-path-double[simp]*:  $rev\text{-}path(rev\text{-}path\ xs)=xs$   
*<proof>*

**lemma** *del-UnEdge-node[simp]*:  $v \in nodes\ (del\text{-}unEdge\ u\ e\ u'\ G) \longleftrightarrow v \in nodes\ G$   
*<proof>*

**lemma** *[intro!]*:  $finite\ (edges\ G) \implies finite\ (edges\ (del\text{-}unEdge\ u\ e\ u'\ G))$   
*<proof>*

**lemma** *[intro!]*:  $finite\ (nodes\ G) \implies finite\ (nodes\ (del\text{-}unEdge\ u\ e\ u'\ G))$   
*<proof>*

**lemma** *[intro!]*:  $finite\ (edges\ G) \implies finite\ (edges\ (rem\text{-}unPath\ ps\ G))$   
*<proof>*

**lemma** *del-UnEdge-frame[intro]*:  
 $x \in edges\ g \implies x \neq (v,e,v') \implies x \neq (v',e,v) \implies x \in edges\ (del\text{-}unEdge\ v\ e\ v'\ g)$   
*<proof>*

**lemma** *[intro!]*:  $finite\ (nodes\ G) \implies finite\ (odd\text{-}nodes\text{-}set\ G)$   
*<proof>*

**lemma** *[simp]*:  $nodes\ (del\text{-}unEdge\ u\ e\ u'\ G) = nodes\ G$   
*<proof>*

**lemma** *[simp]*:  $nodes\ (rem\text{-}unPath\ ps\ G) = nodes\ G$   
*<proof>*

**lemma** *[intro!]*:  $finite\ (nodes\ G) \implies finite\ (nodes\ (rem\text{-}unPath\ ps\ G))$  *<proof>*

**lemma** *in-set-rev-path[simp]*:  $(v',w,v) \in set\ (rev\text{-}path\ ps) \longleftrightarrow (v,w,v') \in set\ ps$   
*<proof>*

**lemma** *rem-unPath-edges*:  
 $edges\ (rem\text{-}unPath\ ps\ G) = edges\ G - (set\ ps \cup set\ (rev\text{-}path\ ps))$   
*<proof>*

**lemma** *rem-unPath-graph [simp]*:  
 $rem\text{-}unPath\ (rev\text{-}path\ ps)\ G = rem\text{-}unPath\ ps\ G$   
*<proof>*

**lemma** *distinct-rev-path[simp]*:  $distinct\ (rev\text{-}path\ ps) \longleftrightarrow distinct\ ps$   
*<proof>*

**lemma** *(in valid-unMultigraph) is-path-rev*:  $is\text{-}path\ v'\ (rev\text{-}path\ ps)\ v \longleftrightarrow is\text{-}path\ v\ ps\ v'$

*<proof>*

**lemma** (in *valid-unMultigraph*) *singleton-distinct-path* [intro]:

$(v, w, v') \in E \implies \text{is-trail } v [(v, w, v')] v'$

*<proof>*

**lemma** (in *valid-unMultigraph*) *is-trail-path*:

$\text{is-trail } v \text{ ps } v' \iff \text{is-path } v \text{ ps } v' \wedge \text{distinct ps} \wedge (\text{set ps} \cap \text{set (rev-path ps)} = \{\})$

*<proof>*

**lemma** (in *valid-unMultigraph*) *is-trail-rev*:

$\text{is-trail } v' (\text{rev-path ps}) v \iff \text{is-trail } v \text{ ps } v'$

*<proof>*

**lemma** (in *valid-unMultigraph*) *is-trail-intro*[intro]:

$\text{is-trail } v' \text{ ps } v \implies \text{is-path } v' \text{ ps } v$  *<proof>*

**lemma** (in *valid-unMultigraph*) *is-trail-split*:

$\text{is-trail } v (p1 @ p2) v' \implies (\exists u. \text{is-trail } v \text{ p1 } u \wedge \text{is-trail } u \text{ p2 } v')$

*<proof>*

**lemma** (in *valid-unMultigraph*) *is-trail-split'*:  $\text{is-trail } v (p1 @ (u, w, u') \# p2) v'$

$\implies \text{is-trail } v \text{ p1 } u \wedge (u, w, u') \in E \wedge \text{is-trail } u' \text{ p2 } v'$

*<proof>*

**lemma** (in *valid-unMultigraph*) *distinct-elim*[simp]:

**assumes**  $\text{is-trail } v ((v1, w, v2) \# \text{ps}) v'$

**shows**  $(v1, w, v2) \in \text{edges}(\text{rem-unPath ps } G) \iff (v1, w, v2) \in E$

*<proof>*

**lemma** *distinct-path-subset*:

**assumes** *valid-unMultigraph* *G1* *valid-unMultigraph* *G2*  $\text{edges } G1 \subseteq \text{edges } G2$

$\text{nodes } G1 \subseteq \text{nodes } G2$

**assumes** *distinct-G1*: *valid-unMultigraph.is-trail* *G1* *v* *ps* *v'*

**shows** *valid-unMultigraph.is-trail* *G2* *v* *ps* *v'* *<proof>*

**lemma** (in *valid-unMultigraph*) *distinct-path-intro'*:

**assumes** *valid-unMultigraph.is-trail* (*rem-unPath* *p* *G*) *v* *ps* *v'*

**shows** *is-trail* *v* *ps* *v'*

*<proof>*

**lemma** (in *valid-unMultigraph*) *distinct-path-intro*:

**assumes** *valid-unMultigraph.is-trail* (*del-unEdge* *x1* *x2* *x3* *G*) *v* *ps* *v'*

**shows** *is-trail* *v* *ps* *v'*

*<proof>*

**lemma** (in *valid-unMultigraph*) *distinct-elim-rev*[simp]:

**assumes** *is-trail*  $v ((v1,w,v2)\#ps)$   $v'$   
**shows**  $(v2,w,v1)\in\text{edges}(\text{rem-unPath } ps \ G) \longleftrightarrow (v2,w,v1)\in E$   
 ⟨*proof*⟩

**lemma** (in *valid-unMultigraph*) *del-UnEdge-even*:  
**assumes**  $(v,w,v') \in E$  *finite*  $E$   
**shows**  $v\in\text{odd-nodes-set}(\text{del-unEdge } v \ w \ v' \ G) \longleftrightarrow \text{even}(\text{degree } v \ G)$   
 ⟨*proof*⟩

**lemma** (in *valid-unMultigraph*) *del-UnEdge-even'*:  
**assumes**  $(v,w,v') \in E$  *finite*  $E$   
**shows**  $v'\in\text{odd-nodes-set}(\text{del-unEdge } v \ w \ v' \ G) \longleftrightarrow \text{even}(\text{degree } v' \ G)$   
 ⟨*proof*⟩

**lemma** *del-UnEdge-even-even*:  
**assumes** *valid-unMultigraph*  $G$  *finite*(*edges*  $G$ ) *finite*(*nodes*  $G$ )  $(v, w, v')\in\text{edges } G$   
**assumes** *parity-assms*: *even* (*degree*  $v \ G$ ) *even* (*degree*  $v' \ G$ )  
**shows**  $\text{num-of-odd-nodes}(\text{del-unEdge } v \ w \ v' \ G)=\text{num-of-odd-nodes } G + 2$   
 ⟨*proof*⟩

**lemma** *del-UnEdge-even-odd*:  
**assumes** *valid-unMultigraph*  $G$  *finite*(*edges*  $G$ ) *finite*(*nodes*  $G$ )  $(v, w, v')\in\text{edges } G$   
**assumes** *parity-assms*: *even* (*degree*  $v \ G$ ) *odd* (*degree*  $v' \ G$ )  
**shows**  $\text{num-of-odd-nodes}(\text{del-unEdge } v \ w \ v' \ G)=\text{num-of-odd-nodes } G$   
 ⟨*proof*⟩

**lemma** *del-UnEdge-odd-even*:  
**assumes** *valid-unMultigraph*  $G$  *finite*(*edges*  $G$ ) *finite*(*nodes*  $G$ )  $(v, w, v')\in\text{edges } G$   
**assumes** *parity-assms*: *odd* (*degree*  $v \ G$ ) *even* (*degree*  $v' \ G$ )  
**shows**  $\text{num-of-odd-nodes}(\text{del-unEdge } v \ w \ v' \ G)=\text{num-of-odd-nodes } G$   
 ⟨*proof*⟩

**lemma** *del-UnEdge-odd-odd*:  
**assumes** *valid-unMultigraph*  $G$  *finite*(*edges*  $G$ ) *finite*(*nodes*  $G$ )  $(v, w, v')\in\text{edges } G$   
**assumes** *parity-assms*: *odd* (*degree*  $v \ G$ ) *odd* (*degree*  $v' \ G$ )  
**shows**  $\text{num-of-odd-nodes } G=\text{num-of-odd-nodes}(\text{del-unEdge } v \ w \ v' \ G)+2$   
 ⟨*proof*⟩

**lemma** (in *valid-unMultigraph*) *rem-UnPath-parity-v'*:  
**assumes** *finite*  $E$  *is-trail*  $v \ ps \ v'$   
**shows**  $v\neq v' \longleftrightarrow (\text{odd}(\text{degree } v'(\text{rem-unPath } ps \ G)) = \text{even}(\text{degree } v' \ G))$  ⟨*proof*⟩

**lemma** (in *valid-unMultigraph*) *rem-UnPath-parity-v*:  
**assumes** *finite*  $E$  *is-trail*  $v \ ps \ v'$   
**shows**  $v\neq v' \longleftrightarrow (\text{odd}(\text{degree } v(\text{rem-unPath } ps \ G)) = \text{even}(\text{degree } v \ G))$

*<proof>*

**lemma** (in *valid-unMultigraph*) *rem-UnPath-parity-others*:  
assumes *finite E is-trail v ps v' n*  $v, v' \notin \{v, v'\}$   
shows  $\text{even}(\text{degree } n(\text{rem-unPath } ps \ G)) = \text{even}(\text{degree } n \ G)$  *<proof>*

**lemma** (in *valid-unMultigraph*) *rem-UnPath-even*:  
assumes *finite E finite V is-trail v ps v'*  
assumes *parity-assms: even (degree v' G)*  
shows  $\text{num-of-odd-nodes}(\text{rem-unPath } ps \ G) = \text{num-of-odd-nodes } G$   
 $+ (\text{if even}(\text{degree } v \ G) \wedge v \neq v' \text{ then } 2 \text{ else } 0)$  *<proof>*

**lemma** (in *valid-unMultigraph*) *rem-UnPath-odd*:  
assumes *finite E finite V is-trail v ps v'*  
assumes *parity-assms: odd (degree v' G)*  
shows  $\text{num-of-odd-nodes}(\text{rem-unPath } ps \ G) = \text{num-of-odd-nodes } G$   
 $+ (\text{if odd}(\text{degree } v \ G) \wedge v \neq v' \text{ then } -2 \text{ else } 0)$  *<proof>*

**lemma** (in *valid-unMultigraph*) *rem-UnPath-cycle*:  
assumes *finite E finite V is-trail v ps v' v=v'*  
shows  $\text{num-of-odd-nodes}(\text{rem-unPath } ps \ G) = \text{num-of-odd-nodes } G$  (is ?L=?R)  
*<proof>*

### 3 Connectivity

**definition** (in *valid-unMultigraph*) *connected::bool where*  
 $\text{connected} \equiv \forall v \in V. \forall v' \in V. v \neq v' \longrightarrow (\exists ps. \text{is-path } v \ ps \ v')$

**lemma** (in *valid-unMultigraph*) *connected*  $\implies \forall v \in V. \forall v' \in V. v \neq v' \longrightarrow (\exists ps. \text{is-trail } v \ ps \ v')$   
*<proof>*

**lemma** (in *valid-unMultigraph*) *no-rep-length: is-trail v ps v'  $\implies \text{length } ps = \text{card}(\text{set } ps)$*   
*<proof>*

**lemma** (in *valid-unMultigraph*) *path-in-edges: is-trail v ps v'  $\implies \text{set } ps \subseteq E$*   
*<proof>*

**lemma** (in *valid-unMultigraph*) *trail-bound*:  
assumes *finite E is-trail v ps v'*  
shows  $\text{length } ps \leq \text{card } E$   
*<proof>*

**definition** (in *valid-unMultigraph*) *exist-path-length:: 'v  $\Rightarrow \text{nat} \Rightarrow \text{bool}$  where*  
 $\text{exist-path-length } v \ l \equiv \exists v' ps. \text{is-trail } v' \ ps \ v \wedge \text{length } ps = l$

**lemma** (in *valid-unMultigraph*) *longest-path*:

**assumes** *finite E n*  $n \in V$   
**shows**  $\exists v. \exists \text{max-path. is-trail } v \text{ max-path } n \wedge$   
 $(\forall v'. \forall e \in E. \neg \text{is-trail } v' (e \# \text{max-path}) n)$   
 $\langle \text{proof} \rangle$

**lemma** *even-card'*:  
**assumes** *even(card A)*  $x \in A$   
**shows**  $\exists y \in A. y \neq x$   
 $\langle \text{proof} \rangle$

**lemma** *odd-card*:  
**assumes** *finite A odd(card A)*  
**shows**  $\exists x. x \in A$   
 $\langle \text{proof} \rangle$

**lemma** (in *valid-unMultigraph*) *extend-distinct-path*:  
**assumes** *finite E is-trail v' ps v*  
**assumes** *parity-assms*:  $(\text{even } (\text{degree } v' G) \wedge v' \neq v) \vee (\text{odd } (\text{degree } v' G) \wedge v' = v)$   
**shows**  $\exists e v1. \text{is-trail } v1 (e \# ps) v$   
 $\langle \text{proof} \rangle$

replace an edge (or its reverse in a path) by another path (in an undirected graph)

**fun** *replace-by-UnPath*::  $('v, 'w) \text{ path} \Rightarrow 'v \times 'w \times 'v \Rightarrow ('v, 'w) \text{ path} \Rightarrow ('v, 'w) \text{ path}$   
**where**

*replace-by-UnPath* [] - - = [] |  
*replace-by-UnPath* (x#xs) (v,e,v') ps =  
 (if x=(v,e,v') then ps@*replace-by-UnPath* xs (v,e,v') ps  
 else if x=(v',e,v) then (rev-path ps)@*replace-by-UnPath* xs (v,e,v') ps  
 else x#*replace-by-UnPath* xs (v,e,v') ps)

**lemma** (in *valid-unMultigraph*) *del-unEdge-connectivity*:  
**assumes** *connected*  $\exists ps. \text{valid-graph.is-path } (\text{del-unEdge } v e v' G) v ps v'$   
**shows** *valid-unMultigraph.connected*  $(\text{del-unEdge } v e v' G)$   
 $\langle \text{proof} \rangle$

**lemma** (in *valid-unMultigraph*) *path-between-odds*:  
**assumes** *odd(degrees v G) odd(degrees v' G) finite E v ≠ v' num-of-odd-nodes G=2*  
**shows**  $\exists ps. \text{is-trail } v ps v'$   
 $\langle \text{proof} \rangle$

**lemma** (in *valid-unMultigraph*) *del-unEdge-even-connectivity*:  
**assumes** *finite E finite V connected*  $\forall n \in V. \text{even}(\text{degree } n G) (v, e, v') \in E$   
**shows** *valid-unMultigraph.connected*  $(\text{del-unEdge } v e v' G)$   
 $\langle \text{proof} \rangle$

**lemma** (in *valid-graph*) *path-end:ps ≠ []*  $\implies \text{is-path } v ps v' \implies v' = \text{snd}(\text{snd}(\text{last}$

ps))  
 ⟨proof⟩

**lemma** (in *valid-unMultigraph*) *connectivity-split*:

**assumes** *connected*  $\neg$ *valid-unMultigraph.connected* (*del-unEdge*  $v w v' G$ )  
 ( $v, w, v' \in E$ )

**obtains**  $G1 G2$  **where**

*nodes*  $G1 = \{n. \exists ps. \text{valid-graph.is-path } (\text{del-unEdge } v w v' G) n ps v\}$

**and** *edges*  $G1 = \{(n, e, n'). (n, e, n') \in \text{edges } (\text{del-unEdge } v w v' G)\}$

$\wedge n \in \text{nodes } G1 \wedge n' \in \text{nodes } G1\}$

**and** *nodes*  $G2 = \{n. \exists ps. \text{valid-graph.is-path } (\text{del-unEdge } v w v' G) n ps v'\}$

**and** *edges*  $G2 = \{(n, e, n'). (n, e, n') \in \text{edges } (\text{del-unEdge } v w v' G)\}$

$\wedge n \in \text{nodes } G2 \wedge n' \in \text{nodes } G2\}$

**and** *edges*  $G1 \cup \text{edges } G2 = \text{edges } (\text{del-unEdge } v w v' G)$

**and** *edges*  $G1 \cap \text{edges } G2 = \{\}$

**and** *nodes*  $G1 \cup \text{nodes } G2 = \text{nodes } (\text{del-unEdge } v w v' G)$

**and** *nodes*  $G1 \cap \text{nodes } G2 = \{\}$

**and** *valid-unMultigraph*  $G1$

**and** *valid-unMultigraph*  $G2$

**and** *valid-unMultigraph.connected*  $G1$

**and** *valid-unMultigraph.connected*  $G2$

⟨proof⟩

**lemma** *sub-graph-degree-frame*:

**assumes** *valid-graph*  $G2$  *edges*  $G1 \cup \text{edges } G2 = \text{edges } G$  *nodes*  $G1 \cap \text{nodes } G2 = \{\}$   $n \in \text{nodes } G1$

**shows** *degree*  $n G = \text{degree } n G1$

⟨proof⟩

**lemma** *odd-nodes-no-edge[simp]*: *finite* (*nodes*  $g$ )  $\implies \text{num-of-odd-nodes } (g \setminus \text{edges} := \{\}) = 0$

⟨proof⟩

## 4 Adjacent nodes

**definition** (in *valid-unMultigraph*) *adjacent*::  $'v \Rightarrow 'v \Rightarrow \text{bool}$  **where**

*adjacent*  $v v' \equiv \exists w. (v, w, v') \in E$

**lemma** (in *valid-unMultigraph*) *adjacent-sym*: *adjacent*  $v v' \longleftrightarrow \text{adjacent } v' v$

⟨proof⟩

**lemma** (in *valid-unMultigraph*) *adjacent-no-loop[simp]*: *adjacent*  $v v' \implies v \neq v'$

⟨proof⟩

**lemma** (in *valid-unMultigraph*) *adjacent-V[simp]*:

**assumes** *adjacent*  $v v'$

**shows**  $v \in V v' \in V$

⟨proof⟩

**lemma** (in *valid-unMultigraph*) *adjacent-finite*:  
 $finite\ E \implies finite\ \{n.\ adjacent\ v\ n\}$   
 <proof>

## 5 Undirected simple graph

**locale** *valid-unSimpGraph=valid-unMultigraph G* **for**  $G::('v,'w)\ graph+$   
**assumes** *no-multi[simp]*:  $(v,w,u) \in edges\ G \implies (v,w',u) \in edges\ G \implies w = w'$

**lemma** (in *valid-unSimpGraph*) *finV-to-finE[simp]*:  
**assumes** *finite V*  
**shows** *finite E*  
 <proof>

**lemma** *del-unEdge-valid'[simp]:valid-unSimpGraph G*  $\implies$   
 $valid-unSimpGraph\ (del-unEdge\ v\ w\ u\ G)$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *del-UnEdge-non-adj*:  
 $(v,w,u) \in E \implies \neg valid-unMultigraph.adjacent\ (del-unEdge\ v\ w\ u\ G)\ v\ u$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *degree-adjacent*:  $finite\ E \implies degree\ v\ G = card\ \{n.\ adjacent\ v\ n\}$   
 <proof>

**end**

**theory** *KoenigsbergBridge* **imports** *MoreGraph*  
**begin**

## 6 Definition of Eulerian trails and circuits

**definition** (in *valid-unMultigraph*) *is-Eulerian-trail*::  $'v \Rightarrow ('v,'w)\ path \Rightarrow 'v \Rightarrow bool$   
**where**  
 $is-Eulerian-trail\ v\ ps\ v' \equiv is-trail\ v\ ps\ v' \wedge edges\ (rem-unPath\ ps\ G) = \{\}$

**definition** (in *valid-unMultigraph*) *is-Eulerian-circuit*::  $'v \Rightarrow ('v,'w)\ path \Rightarrow 'v \Rightarrow bool$   
**where**  
 $is-Eulerian-circuit\ v\ ps\ v' \equiv (v=v') \wedge (is-Eulerian-trail\ v\ ps\ v')$

## 7 Necessary conditions for Eulerian trails and circuits

**lemma** (in *valid-unMultigraph*) *euclerian-rev*:  
*is-Eulerian-trail v' (rev-path ps) v=is-Eulerian-trail v ps v'*  
 ⟨*proof*⟩

**theorem** (in *valid-unMultigraph*) *euclerian-cycle-ex*:  
**assumes** *is-Eulerian-circuit v ps v' finite V finite E*  
**shows**  $\forall v \in V. \text{even } (\text{degree } v \ G)$   
 ⟨*proof*⟩

**theorem** (in *valid-unMultigraph*) *euclerian-path-ex*:  
**assumes** *is-Eulerian-trail v ps v' finite V finite E*  
**shows**  $(\forall v \in V. \text{even } (\text{degree } v \ G)) \vee (\text{num-of-odd-nodes } G = 2)$   
 ⟨*proof*⟩

## 8 Specific case of the Konigsberg Bridge Problem

**datatype** *kon-node* = *a* | *b* | *c* | *d*

**datatype** *kon-bridge* = *ab1* | *ab2* | *ac1* | *ac2* | *ad1* | *bd1* | *cd1*

**definition** *kon-graph* :: (*kon-node*,*kon-bridge*) *graph* **where**  
*kon-graph* ≡ (nodes={*a*,*b*,*c*,*d*},  
 edges={(*a*,*ab1*,*b*), (*b*,*ab1*,*a*),  
 (*a*,*ab2*,*b*), (*b*,*ab2*,*a*),  
 (*a*,*ac1*,*c*), (*c*,*ac1*,*a*),  
 (*a*,*ac2*,*c*), (*c*,*ac2*,*a*),  
 (*a*,*ad1*,*d*), (*d*,*ad1*,*a*),  
 (*b*,*bd1*,*d*), (*d*,*bd1*,*b*),  
 (*c*,*cd1*,*d*), (*d*,*cd1*,*c*)} )

**instantiation** *kon-node* :: *enum*

**begin**

**definition** [*simp*]: *enum-class.enum* = [*a*,*b*,*c*,*d*]

**definition** [*simp*]: *enum-class.enum-all* *P*  $\longleftrightarrow P \ a \ \wedge \ P \ b \ \wedge \ P \ c \ \wedge \ P \ d$

**definition** [*simp*]: *enum-class.enum-ex* *P*  $\longleftrightarrow P \ a \ \vee \ P \ b \ \vee \ P \ c \ \vee \ P \ d$

**instance** ⟨*proof*⟩

**end**

**instantiation** *kon-bridge* :: *enum*

**begin**

**definition** [*simp*]: *enum-class.enum* = [*ab1*,*ab2*,*ac1*,*ac2*,*ad1*,*cd1*,*bd1*]

**definition** [*simp*]: *enum-class.enum-all* *P*  $\longleftrightarrow P \ ab1 \ \wedge \ P \ ab2 \ \wedge \ P \ ac1 \ \wedge \ P \ ac2$

$\wedge P\ ad1 \ \wedge P\ bd1$   
 $\wedge P\ cd1$

**definition** *[simp]:enum-class.enum-ex*  $P \longleftrightarrow P\ ab1 \vee P\ ab2 \vee P\ ac1 \vee P\ ac2$   
 $\vee P\ ad1 \vee P\ bd1$   
 $\vee P\ cd1$

**instance**  $\langle proof \rangle$   
**end**

**interpretation** *kon-graph: valid-unMultigraph kon-graph*  
 $\langle proof \rangle$

**theorem**  $\neg kon\text{-graph.is-Eulerian-trail}\ v1\ p\ v2$   
 $\langle proof \rangle$

## 9 Sufficient conditions for Eulerian trails and circuits

**lemma** (in *valid-unMultigraph*) *eulerian-cons*:

**assumes**

*valid-unMultigraph.is-Eulerian-trail*  $(del\text{-unEdge}\ v0\ w\ v1\ G)\ v1\ ps\ v2$   
 $(v0,w,v1) \in E$

**shows** *is-Eulerian-trail*  $v0\ ((v0,w,v1)\#ps)\ v2$   
 $\langle proof \rangle$

**lemma** (in *valid-unMultigraph*) *eulerian-cons'*:

**assumes**

*valid-unMultigraph.is-Eulerian-trail*  $(del\text{-unEdge}\ v2\ w\ v3\ G)\ v1\ ps\ v2$   
 $(v2,w,v3) \in E$

**shows** *is-Eulerian-trail*  $v1\ (ps@[v2,w,v3])\ v3$   
 $\langle proof \rangle$

**lemma** *eulerian-split*:

**assumes**  $nodes\ G1 \cap nodes\ G2 = \{\}$   $edges\ G1 \cap edges\ G2 = \{\}$

*valid-unMultigraph*  $G1$  *valid-unMultigraph*  $G2$

*valid-unMultigraph.is-Eulerian-trail*  $G1\ v1\ ps1\ v1'$

*valid-unMultigraph.is-Eulerian-trail*  $G2\ v2\ ps2\ v2'$

**shows** *valid-unMultigraph.is-Eulerian-trail*  $(nodes=nodes\ G1 \cup nodes\ G2,$   
 $edges=edges\ G1 \cup edges\ G2 \cup \{(v1',w,v2),(v2,w,v1')\})\ v1\ (ps1@[v1',w,v2])\#ps2)$   
 $v2'$   
 $\langle proof \rangle$

**lemma** (in *valid-unMultigraph*) *eulerian-sufficient*:

**assumes** *finite*  $V$  *finite*  $E$  *connected*  $V \neq \{\}$

**shows** *num-of-odd-nodes*  $G = 2 \implies$

$(\exists v \in V. \exists v' \in V. \exists ps. odd(degrees\ v\ G) \wedge odd(degrees\ v'\ G) \wedge (v \neq v') \wedge is\text{-Eulerian-trail}\ v\ ps\ v')$

**and** *num-of-odd-nodes*  $G = 0 \implies (\forall v \in V. \exists ps. is\text{-Eulerian-circuit}\ v\ ps\ v)$

<proof>  
 end

**theory** *FriendshipTheory*  
**imports** *MoreGraph HOL-Number-Theory.Number-Theory*  
**begin**

## 10 Common steps

**definition** (in *valid-unSimpGraph*) *non-adj* :: '*v* ⇒ '*v* ⇒ bool **where**  
*non-adj* *v v'* ≡  $v \in V \wedge v' \in V \wedge v \neq v' \wedge \neg \text{adjacent } v \ v'$

**lemma** (in *valid-unSimpGraph*) *no-quad*:  
**assumes**  $\bigwedge v \ u. \ v \in V \implies u \in V \implies v \neq u \implies \exists! n. \ \text{adjacent } v \ n \wedge \text{adjacent } u \ n$   
**shows**  $\neg (\exists v1 \ v2 \ v3 \ v4. \ v2 \neq v4 \wedge v1 \neq v3 \wedge \text{adjacent } v1 \ v2 \wedge \text{adjacent } v2 \ v3 \wedge$   
 $\text{adjacent } v3 \ v4$   
 $\wedge \text{adjacent } v4 \ v1)$   
 <proof>

**lemma** *even-card-set*:  
**assumes** *finite* *A* **and**  $\forall x \in A. \ f \ x \in A \wedge f \ x \neq x \wedge f \ (f \ x) = x$   
**shows** *even*(*card* *A*) <proof>

**lemma** (in *valid-unSimpGraph*) *even-degree*:  
**assumes** *friend-assm*:  $\bigwedge v \ u. \ v \in V \implies u \in V \implies v \neq u \implies \exists! n. \ \text{adjacent } v \ n \wedge$   
 $\text{adjacent } u \ n$   
**and** *finite* *E*  
**shows**  $\forall v \in V. \ \text{even}(\text{degree } v \ G)$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *degree-two-windmill*:  
**assumes** *friend-assm*:  $\bigwedge v \ u. \ v \in V \implies u \in V \implies v \neq u \implies \exists! n. \ \text{adjacent } v \ n \wedge$   
 $\text{adjacent } u \ n$   
**and** *finite* *E* **and** *card* *V* ≥ 2  
**shows**  $(\exists v \in V. \ \text{degree } v \ G = 2) \longleftrightarrow (\exists v. \ \forall n \in V. \ n \neq v \longrightarrow \text{adjacent } v \ n)$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *regular*:  
**assumes** *friend-assm*:  $\bigwedge v \ u. \ v \in V \implies u \in V \implies v \neq u \implies \exists! n. \ \text{adjacent } v \ n \wedge$   
 $\text{adjacent } u \ n$   
**and** *finite* *E* **and** *finite* *V* **and**  $\neg (\exists v \in V. \ \text{degree } v \ G = 2)$   
**shows**  $\exists k. \ \forall v \in V. \ \text{degree } v \ G = k$   
 <proof>

## 11 Exclusive steps for combinatorial proofs

**fun** (in *valid-unSimpGraph*) *adj-path*:: '*v* ⇒ '*v* list ⇒ bool **where**

$adj\text{-path } v [] = (v \in V)$   
 $| adj\text{-path } v (u \# us) = (adjacent\ v\ u \wedge adj\text{-path } u\ us)$

**lemma** (in *valid-unSimpGraph*) *adj-path-butlast*:  
 $adj\text{-path } v\ ps \implies adj\text{-path } v\ (butlast\ ps)$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *adj-path-V*:  
 $adj\text{-path } v\ ps \implies set\ ps \subseteq V$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *adj-path-V'*:  
 $adj\text{-path } v\ ps \implies v \in V$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *adj-path-app*:  
 $adj\text{-path } v\ ps \implies ps \neq [] \implies adjacent\ (last\ ps)\ u \implies adj\text{-path } v\ (ps @ [u])$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *adj-path-app'*:  
 $adj\text{-path } v\ (ps @ [q]) \implies ps \neq [] \implies adjacent\ (last\ ps)\ q$   
 <proof>

**lemma** *card-partition'*:  
**assumes**  $\forall v \in A. card\ \{n. R\ v\ n\} = k\ k > 0$  *finite A*  
 $\forall v1\ v2. v1 \neq v2 \longrightarrow \{n. R\ v1\ n\} \cap \{n. R\ v2\ n\} = \{\}$   
**shows**  $card\ (\bigcup v \in A. \{n. R\ v\ n\}) = k * card\ A$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *path-count*:  
**assumes**  $k\text{-adj}: \bigwedge v. v \in V \implies card\ \{n. adjacent\ v\ n\} = k$  **and**  $v \in V$  **and** *finite V* **and**  $k > 0$   
**shows**  $card\ \{ps. length\ ps = l \wedge adj\text{-path } v\ ps\} = k^l$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *total-v-num*:  
**assumes** *friend-assm*:  $\bigwedge v\ u. v \in V \implies u \in V \implies v \neq u \implies \exists! n. adjacent\ v\ n \wedge adjacent\ u\ n$   
**and** *finite E* **and** *finite V* **and**  $V \neq \{\}$  **and**  $\forall v \in V. degree\ v\ G = k$  **and**  $k > 0$   
**shows**  $card\ V = k * k - k + 1$   
 <proof>

**lemma** *rotate-eq*:  $rotate1\ xs = rotate1\ ys \implies xs = ys$   
 <proof>

**lemma** *rotate-diff*:  $rotate\ m\ xs = rotate\ n\ xs \implies rotate\ (m - n)\ xs = xs$   
 <proof>

**lemma** (in *valid-unSimpGraph*) *exist-degree-two*:  
**assumes** *friend-asm*:  $\bigwedge v u. v \in V \implies u \in V \implies v \neq u \implies \exists! n. \text{adjacent } v \ n \wedge \text{adjacent } u \ n$   
**and** *finite E* **and** *finite V* **and** *card V*  $\geq 2$   
**shows**  $\exists v \in V. \text{degree } v \ G = 2$   
*<proof>*

**theorem** (in *valid-unSimpGraph*) *friendship-thm*:  
**assumes** *friend-asm*:  $\bigwedge v u. v \in V \implies u \in V \implies v \neq u \implies \exists! n. \text{adjacent } v \ n \wedge \text{adjacent } u \ n$   
**and** *finite V*  
**shows**  $\exists v. \forall n \in V. n \neq v \longrightarrow \text{adjacent } v \ n$   
*<proof>*

**end**

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