

# A Formalization of Knuth–Bendix Orders\*

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## Abstract

We define a generalized version of Knuth–Bendix orders, including subterm coefficient functions. For these orders we formalize several properties such as strong normalization, the subterm property, closure properties under substitutions and contexts, as well as ground totality.

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# 1 Introduction

In their seminal paper [2], Knuth and Bendix introduced two important concepts: a procedure that allows us to solve certain instances of the word problem – (Knuth–Bendix) completion – as well as a specific order on terms that is useful to orient equations in the aforementioned procedure – the Knuth–Bendix order (or KBO, for short).

This AFP-entry is about the formalization of KBO. Note that there are several variants of KBO [2, 1, 3, 7, 4], e.g., incorporating quasi-precedences, infinite signatures, subterm coefficient functions, and generalized weight functions. In fact, not for all of these variants well-foundedness has been proven. We give the first well-foundedness proof for a variant of KBO that combines infinite signatures, quasi-precedences, and subterm coefficient functions. Our proof is direct, i.e., it does not depend on Kruskal’s tree theorem.

This formalization is used in the `IsaFoR/CeTAproject` [6] for certifying untrusted termination and confluence proofs. For more details we refer to our RTA paper [5].

# 2 Order Pairs

An order pair consists of two relations  $S$  and  $NS$ , where  $S$  is a strict order and  $NS$  a compatible non-strict order, such that the combination of  $S$  and  $NS$  always results in strict decrease.

**theory** *Order-Pair*

**imports** *Abstract–Rewriting.Relative-Rewriting*

**begin**

**named-theorems** *order-simps*

**declare** *O-assoc*[*order-simps*]

**locale** *pre-order-pair* =

**fixes**  $S :: 'a\ rel$

**and**  $NS :: 'a\ rel$

**assumes** *refl-NS*: *refl NS*

**and** *trans-S*: *trans S*

**and** *trans-NS*: *trans NS*

**begin**

**lemma** *refl-NS-point*:  $(s, s) \in NS$  *<proof>*

**lemma** *NS-O-NS*[*order-simps*]:  $NS\ O\ NS = NS\ NS\ O\ NS\ O\ T = NS\ O\ T$  *<proof>*

**lemma** *trancl-NS*[*order-simps*]:  $NS^+ = NS$  *<proof>*

**lemma** *rtrancl-NS*[*order-simps*]:  $NS^* = NS$   
*<proof>*

**lemma** *trancl-S*[*order-simps*]:  $S^+ = S$  *<proof>*

**lemma** *S-O-S*:  $S O S \subseteq S S O S O T \subseteq S O T$   
*<proof>*

**lemma** *trans-S-point*:  $\bigwedge x y z. (x, y) \in S \implies (y, z) \in S \implies (x, z) \in S$   
*<proof>*

**lemma** *trans-NS-point*:  $\bigwedge x y z. (x, y) \in NS \implies (y, z) \in NS \implies (x, z) \in NS$   
*<proof>*

**end**

**locale** *compat-pair* =

**fixes**  $S NS :: 'a \text{ rel}$

**assumes** *compat-NS-S*:  $NS O S \subseteq S$

**and** *compat-S-NS*:  $S O NS \subseteq S$

**begin**

**lemma** *compat-NS-S-point*:  $\bigwedge x y z. (x, y) \in NS \implies (y, z) \in S \implies (x, z) \in S$   
*<proof>*

**lemma** *compat-S-NS-point*:  $\bigwedge x y z. (x, y) \in S \implies (y, z) \in NS \implies (x, z) \in S$   
*<proof>*

**lemma** *S-O-rtrancl-NS*[*order-simps*]:  $S O NS^* = S S O NS^* O T = S O T$   
*<proof>*

**lemma** *rtrancl-NS-O-S*[*order-simps*]:  $NS^* O S = S NS^* O S O T = S O T$   
*<proof>*

**end**

**locale** *order-pair* = *pre-order-pair*  $S NS$  + *compat-pair*  $S NS$

**for**  $S NS :: 'a \text{ rel}$

**begin**

**lemma** *S-O-NS*[*order-simps*]:  $S O NS = S S O NS O T = S O T$  *<proof>*

**lemma** *NS-O-S*[*order-simps*]:  $NS O S = S NS O S O T = S O T$  *<proof>*

**lemma** *compat-rtrancl*:

**assumes** *ab*:  $(a, b) \in S$

**and** *bc*:  $(b, c) \in (NS \cup S)^*$

**shows**  $(a, c) \in S$

*<proof>*

**end**

```

locale SN-ars =
  fixes S :: 'a rel
  assumes SN: SN S

locale SN-pair = compat-pair S NS + SN-ars S for S NS :: 'a rel

locale SN-order-pair = order-pair S NS + SN-ars S for S NS :: 'a rel

sublocale SN-order-pair  $\subseteq$  SN-pair  $\langle$ proof $\rangle$ 

end

```

### 3 Lexicographic Extension

**theory** *Lexicographic-Extension*

**imports**

*Matrix.Utility*

*Order-Pair*

**begin**

In this theory we define the lexicographic extension of an order pair, so that it generalizes the existing notion ( $\langle *lex* \rangle$ ) which is based on a single order only.

Our main result is that this extension yields again an order pair.

**fun** *lex-two* :: 'a rel  $\Rightarrow$  'a rel  $\Rightarrow$  'b rel  $\Rightarrow$  ('a  $\times$  'b) rel

**where**

*lex-two* s ns s2 =  $\{((a1, b1), (a2, b2)) . (a1, a2) \in s \vee (a1, a2) \in ns \wedge (b1, b2) \in s2\}$

**lemma** *lex-two*:

**assumes** *compat*: ns O s  $\subseteq$  s

**and** *SN-s*: SN s

**and** *SN-s2*: SN s2

**shows** SN (*lex-two* s ns s2) (**is** SN ?r)

$\langle$ proof $\rangle$

**lemma** *lex-two-compat*:

**assumes** *compat1*: ns1 O s1  $\subseteq$  s1

**and** *compat1'*: s1 O ns1  $\subseteq$  s1

**and** *trans1*: s1 O s1  $\subseteq$  s1

**and** *trans1'*: ns1 O ns1  $\subseteq$  ns1

**and** *compat2*: ns2 O s2  $\subseteq$  s2

**and** ns: (ab1, ab2)  $\in$  *lex-two* s1 ns1 ns2

**and** s: (ab2, ab3)  $\in$  *lex-two* s1 ns1 s2

**shows** (ab1, ab3)  $\in$  *lex-two* s1 ns1 s2

$\langle$ proof $\rangle$

**lemma** *lex-two-compat'*:

```

assumes compat1: ns1 O s1  $\subseteq$  s1
and compat1': s1 O ns1  $\subseteq$  s1
and trans1: s1 O s1  $\subseteq$  s1
and trans1': ns1 O ns1  $\subseteq$  ns1
and compat2': s2 O ns2  $\subseteq$  s2
and s: (ab1, ab2)  $\in$  lex-two s1 ns1 s2
and ns: (ab2, ab3)  $\in$  lex-two s1 ns1 ns2
shows (ab1, ab3)  $\in$  lex-two s1 ns1 s2
<proof>

```

```

lemma lex-two-compat2:
assumes ns1 O s1  $\subseteq$  s1 s1 O ns1  $\subseteq$  s1 s1 O s1  $\subseteq$  s1 ns1 O ns1  $\subseteq$  ns1 ns2 O
s2  $\subseteq$  s2
shows lex-two s1 ns1 ns2 O lex-two s1 ns1 s2  $\subseteq$  lex-two s1 ns1 s2
<proof>

```

```

lemma lex-two-compat'2:
assumes ns1 O s1  $\subseteq$  s1 s1 O ns1  $\subseteq$  s1 s1 O s1  $\subseteq$  s1 ns1 O ns1  $\subseteq$  ns1 s2 O ns2
 $\subseteq$  s2
shows lex-two s1 ns1 s2 O lex-two s1 ns1 ns2  $\subseteq$  lex-two s1 ns1 s2
<proof>

```

```

lemma lex-two-refl:
assumes r1: refl ns1 and r2: refl ns2
shows refl (lex-two s1 ns1 ns2)
<proof>

```

```

lemma lex-two-order-pair:
assumes o1: order-pair s1 ns1 and o2: order-pair s2 ns2
shows order-pair (lex-two s1 ns1 s2) (lex-two s1 ns1 ns2)
<proof>

```

```

lemma lex-two-SN-order-pair:
assumes o1: SN-order-pair s1 ns1 and o2: SN-order-pair s2 ns2
shows SN-order-pair (lex-two s1 ns1 s2) (lex-two s1 ns1 ns2)
<proof>

```

In the unbounded lexicographic extension, there is no restriction on the lengths of the lists. Therefore it is possible to compare lists of different lengths. This usually results a non-terminating relation, e.g.,  $[1] > [0, 1] > [0, 0, 1] > \dots$

```

fun lex-ext-unbounded :: ('a  $\Rightarrow$  'a  $\Rightarrow$  bool  $\times$  bool)  $\Rightarrow$  'a list  $\Rightarrow$  'a list  $\Rightarrow$  bool  $\times$  bool
where lex-ext-unbounded f [] [] = (False, True) |
lex-ext-unbounded f (- # -) [] = (True, True) |
lex-ext-unbounded f [] (- # -) = (False, False) |
lex-ext-unbounded f (a # as) (b # bs) =
  (let (stri, nstri) = f a b in
   if stri then (True, True)
   else if nstri then lex-ext-unbounded f as bs

```

else (False, False))

**lemma** *lex-ext-unbounded-iff*: (lex-ext-unbounded f xs ys) = (  
 (( $\exists i < \text{length } xs. i < \text{length } ys \wedge (\forall j < i. \text{snd } (f (xs ! j) (ys ! j))) \wedge \text{fst } (f (xs ! i) (ys ! i))$ ))  $\vee$   
 ( $\forall i < \text{length } ys. \text{snd } (f (xs ! i) (ys ! i)) \wedge \text{length } xs > \text{length } ys$ ),  
 (( $\exists i < \text{length } xs. i < \text{length } ys \wedge (\forall j < i. \text{snd } (f (xs ! j) (ys ! j))) \wedge \text{fst } (f (xs ! i) (ys ! i))$ ))  $\vee$   
 ( $\forall i < \text{length } ys. \text{snd } (f (xs ! i) (ys ! i)) \wedge \text{length } xs \geq \text{length } ys$ )  
 (is ?lex xs ys = (?stri xs ys, ?nstri xs ys))  
 <proof>

**declare** *lex-ext-unbounded.simps*[simp del]

The lexicographic extension of an order pair takes a natural number as maximum bound. A decrease with lists of unequal lengths will never be successful if the length of the second list exceeds this bound. The bound is essential to preserve strong normalization.

**definition** *lex-ext* :: ('a  $\Rightarrow$  'a  $\Rightarrow$  bool  $\times$  bool)  $\Rightarrow$  nat  $\Rightarrow$  'a list  $\Rightarrow$  'a list  $\Rightarrow$  bool  $\times$  bool

**where**

*lex-ext* f n ss ts =  
 (let lts = length ts in  
 if (length ss = lts  $\vee$  lts  $\leq$  n) then lex-ext-unbounded f ss ts  
 else (False, False))

**lemma** *lex-ext-iff*: (lex-ext f m xs ys) = (  
 (length xs = length ys  $\vee$  length ys  $\leq$  m)  $\wedge$  (( $\exists i < \text{length } xs. i < \text{length } ys \wedge (\forall j < i. \text{snd } (f (xs ! j) (ys ! j))) \wedge \text{fst } (f (xs ! i) (ys ! i))$ ))  $\vee$   
 ( $\forall i < \text{length } ys. \text{snd } (f (xs ! i) (ys ! i)) \wedge \text{length } xs > \text{length } ys$ ),  
 (length xs = length ys  $\vee$  length ys  $\leq$  m)  $\wedge$   
 (( $\exists i < \text{length } xs. i < \text{length } ys \wedge (\forall j < i. \text{snd } (f (xs ! j) (ys ! j))) \wedge \text{fst } (f (xs ! i) (ys ! i))$ ))  $\vee$   
 ( $\forall i < \text{length } ys. \text{snd } (f (xs ! i) (ys ! i)) \wedge \text{length } xs \geq \text{length } ys$ )  
 <proof>

**lemma** *lex-ext-to-lex-ext-unbounded*:

**assumes** length xs  $\leq$  n **and** length ys  $\leq$  n  
**shows** lex-ext f n xs ys = lex-ext-unbounded f xs ys  
 <proof>

**lemma** *lex-ext-stri-imp-nstri*:

**assumes** fst (lex-ext f m xs ys)  
**shows** snd (lex-ext f m xs ys)  
 <proof>

**lemma** *nstri-lex-ext-map*:

**assumes**  $\bigwedge s t. s \in \text{set } ss \implies t \in \text{set } ts \implies \text{fst } (\text{order } s t) \implies \text{fst } (\text{order}' (f s) (f t))$   
**and**  $\bigwedge s t. s \in \text{set } ss \implies t \in \text{set } ts \implies \text{snd } (\text{order } s t) \implies \text{snd } (\text{order}' (f s) (f t))$   
**and**  $\text{snd } (\text{lex-ext } \text{order } n ss ts)$   
**shows**  $\text{snd } (\text{lex-ext } \text{order}' n (\text{map } f ss) (\text{map } f ts))$   
 $\langle \text{proof} \rangle$

**lemma** *stri-lex-ext-map*:

**assumes**  $\bigwedge s t. s \in \text{set } ss \implies t \in \text{set } ts \implies \text{fst } (\text{order } s t) \implies \text{fst } (\text{order}' (f s) (f t))$   
**and**  $\bigwedge s t. s \in \text{set } ss \implies t \in \text{set } ts \implies \text{snd } (\text{order } s t) \implies \text{snd } (\text{order}' (f s) (f t))$   
**and**  $\text{fst } (\text{lex-ext } \text{order } n ss ts)$   
**shows**  $\text{fst } (\text{lex-ext } \text{order}' n (\text{map } f ss) (\text{map } f ts))$   
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-arg-empty*:  $\text{snd } (\text{lex-ext } f n [] xs) \implies xs = []$   
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-co-compat*:

**assumes**  $\bigwedge s t. s \in \text{set } ss \implies t \in \text{set } ts \implies \text{fst } (\text{order } s t) \implies \text{snd } (\text{order}' t s)$   
 $\implies \text{False}$   
**and**  $\bigwedge s t. s \in \text{set } ss \implies t \in \text{set } ts \implies \text{snd } (\text{order } s t) \implies \text{fst } (\text{order}' t s)$   
 $\implies \text{False}$   
**and**  $\bigwedge s t. \text{fst } (\text{order } s t) \implies \text{snd } (\text{order } s t)$   
**and**  $\text{fst } (\text{lex-ext } \text{order } n ss ts)$   
**and**  $\text{snd } (\text{lex-ext } \text{order}' n ts ss)$   
**shows** *False*  
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-irrefl*: **assumes**  $\bigwedge x. x \in \text{set } xs \implies \neg \text{fst } (\text{rel } x x)$   
**shows**  $\neg \text{fst } (\text{lex-ext } \text{rel } n xs xs)$   
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-unbounded-stri-imp-nstri*:

**assumes**  $\text{fst } (\text{lex-ext-unbounded } f xs ys)$   
**shows**  $\text{snd } (\text{lex-ext-unbounded } f xs ys)$   
 $\langle \text{proof} \rangle$

**lemma** *all-nstri-imp-lex-nstri*: **assumes**  $\forall i < \text{length } ys. \text{snd } (f (xs ! i) (ys ! i))$   
**and**  $\text{length } xs \geq \text{length } ys$  **and**  $\text{length } xs = \text{length } ys \vee \text{length } ys \leq m$   
**shows**  $\text{snd } (\text{lex-ext } f m xs ys)$   
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-cong[fundef-cong]*: **fixes**  $f g m1 m2 xs1 xs2 ys1 ys2$

**assumes**  $\text{length } xs1 = \text{length } ys1$  **and**  $m1 = m2$  **and**  $\text{length } xs2 = \text{length } ys2$   
**and**  $\bigwedge i. \llbracket i < \text{length } ys1; i < \text{length } ys2 \rrbracket \implies f (xs1 ! i) (xs2 ! i) = g (ys1 ! i)$

(*ys2 ! i*)  
**shows**  $\text{lex-ext } f \ m1 \ xs1 \ xs2 = \text{lex-ext } g \ m2 \ ys1 \ ys2$   
 ⟨*proof*⟩

**lemma** *lex-ext-unbounded-cong*[*fundef-cong*]: **assumes**  $as = as'$  **and**  $bs = bs'$   
**and**  $\bigwedge i. i < \text{length } as' \implies i < \text{length } bs' \implies f (as' ! i) (bs' ! i) = g (as' ! i) (bs' ! i)$  **shows**  $\text{lex-ext-unbounded } f \ as \ bs = \text{lex-ext-unbounded } g \ as' \ bs'$   
 ⟨*proof*⟩

Compatibility is the key property to ensure transitivity of the order.

We prove compatibility locally, i.e., it only has to hold for elements of the argument lists. Locality is essential for being applicable in recursively defined term orders such as KBO.

**lemma** *lex-ext-compat*:

**assumes** *compat*:  $\bigwedge s \ t \ u. \llbracket s \in \text{set } ss; t \in \text{set } ts; u \in \text{set } us \rrbracket \implies$   
 $(\text{snd } (f \ s \ t) \wedge \text{fst } (f \ t \ u) \longrightarrow \text{fst } (f \ s \ u)) \wedge$   
 $(\text{fst } (f \ s \ t) \wedge \text{snd } (f \ t \ u) \longrightarrow \text{fst } (f \ s \ u)) \wedge$   
 $(\text{snd } (f \ s \ t) \wedge \text{snd } (f \ t \ u) \longrightarrow \text{snd } (f \ s \ u)) \wedge$   
 $(\text{fst } (f \ s \ t) \wedge \text{fst } (f \ t \ u) \longrightarrow \text{fst } (f \ s \ u))$

**shows**

$(\text{snd } (\text{lex-ext } f \ n \ ss \ ts) \wedge \text{fst } (\text{lex-ext } f \ n \ ts \ us) \longrightarrow \text{fst } (\text{lex-ext } f \ n \ ss \ us)) \wedge$   
 $(\text{fst } (\text{lex-ext } f \ n \ ss \ ts) \wedge \text{snd } (\text{lex-ext } f \ n \ ts \ us) \longrightarrow \text{fst } (\text{lex-ext } f \ n \ ss \ us)) \wedge$   
 $(\text{snd } (\text{lex-ext } f \ n \ ss \ ts) \wedge \text{snd } (\text{lex-ext } f \ n \ ts \ us) \longrightarrow \text{snd } (\text{lex-ext } f \ n \ ss \ us)) \wedge$   
 $(\text{fst } (\text{lex-ext } f \ n \ ss \ ts) \wedge \text{fst } (\text{lex-ext } f \ n \ ts \ us) \longrightarrow \text{fst } (\text{lex-ext } f \ n \ ss \ us))$

⟨*proof*⟩

**lemma** *lex-ext-unbounded-map*:

**assumes** *S*:  $\bigwedge i. i < \text{length } ss \implies i < \text{length } ts \implies \text{fst } (r \ (ss ! i) \ (ts ! i)) \implies$   
 $\text{fst } (r \ (\text{map } f \ ss ! i) \ (\text{map } f \ ts ! i))$   
**and** *NS*:  $\bigwedge i. i < \text{length } ss \implies i < \text{length } ts \implies \text{snd } (r \ (ss ! i) \ (ts ! i)) \implies$   
 $\text{snd } (r \ (\text{map } f \ ss ! i) \ (\text{map } f \ ts ! i))$   
**shows**  $(\text{fst } (\text{lex-ext-unbounded } r \ ss \ ts) \longrightarrow \text{fst } (\text{lex-ext-unbounded } r \ (\text{map } f \ ss) \ (\text{map } f \ ts))) \wedge$   
 $(\text{snd } (\text{lex-ext-unbounded } r \ ss \ ts) \longrightarrow \text{snd } (\text{lex-ext-unbounded } r \ (\text{map } f \ ss) \ (\text{map } f \ ts)))$   
 ⟨*proof*⟩

**lemma** *lex-ext-unbounded-map-S*:

**assumes** *S*:  $\bigwedge i. i < \text{length } ss \implies i < \text{length } ts \implies \text{fst } (r \ (ss ! i) \ (ts ! i)) \implies$   
 $\text{fst } (r \ (\text{map } f \ ss ! i) \ (\text{map } f \ ts ! i))$   
**and** *NS*:  $\bigwedge i. i < \text{length } ss \implies i < \text{length } ts \implies \text{snd } (r \ (ss ! i) \ (ts ! i)) \implies$   
 $\text{snd } (r \ (\text{map } f \ ss ! i) \ (\text{map } f \ ts ! i))$   
**and** *stri*:  $\text{fst } (\text{lex-ext-unbounded } r \ ss \ ts)$   
**shows**  $\text{fst } (\text{lex-ext-unbounded } r \ (\text{map } f \ ss) \ (\text{map } f \ ts))$   
 ⟨*proof*⟩

**lemma** *lex-ext-unbounded-map-NS*:

**assumes**  $S: \bigwedge i. i < \text{length } ss \implies i < \text{length } ts \implies \text{fst } (r (ss ! i) (ts ! i)) \implies \text{fst } (r (\text{map } f \text{ } ss ! i) (\text{map } f \text{ } ts ! i))$   
**and**  $NS: \bigwedge i. i < \text{length } ss \implies i < \text{length } ts \implies \text{snd } (r (ss ! i) (ts ! i)) \implies \text{snd } (r (\text{map } f \text{ } ss ! i) (\text{map } f \text{ } ts ! i))$   
**and**  $nstri: \text{snd } (\text{lex-ext-unbounded } r \text{ } ss \text{ } ts)$   
**shows**  $\text{snd } (\text{lex-ext-unbounded } r (\text{map } f \text{ } ss) (\text{map } f \text{ } ts))$   
 $\langle \text{proof} \rangle$

Strong normalization with local SN assumption

**lemma** *lex-ext-SN*:

**assumes**  $\text{compat}: \bigwedge x y z. \llbracket \text{snd } (g x y); \text{fst } (g y z) \rrbracket \implies \text{fst } (g x z)$   
**shows**  $SN \{ (ys, xs). (\forall y \in \text{set } ys. SN\text{-on } \{ (s, t). \text{fst } (g s t) \} \{y\}) \wedge \text{fst } (\text{lex-ext } g \text{ } m \text{ } ys \text{ } xs) \}$   
**(is**  $SN \{ (ys, xs). ?\text{cond } ys \text{ } xs \}$   
 $\langle \text{proof} \rangle$

Strong normalization with global SN assumption is immediate consequence.

**lemma** *lex-ext-SN-2*:

**assumes**  $\text{compat}: \bigwedge x y z. \llbracket \text{snd } (g x y); \text{fst } (g y z) \rrbracket \implies \text{fst } (g x z)$   
**and**  $SN: SN \{ (s, t). \text{fst } (g s t) \}$   
**shows**  $SN \{ (ys, xs). \text{fst } (\text{lex-ext } g \text{ } m \text{ } ys \text{ } xs) \}$   
 $\langle \text{proof} \rangle$

The empty list is the least element in the lexicographic extension.

**lemma** *lex-ext-least-1*:  $\text{snd } (\text{lex-ext } f \text{ } m \text{ } xs \text{ } [])$   
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-least-2*:  $\neg \text{fst } (\text{lex-ext } f \text{ } m \text{ } [] \text{ } ys)$   
 $\langle \text{proof} \rangle$

Preservation of totality on lists of same length.

**lemma** *lex-ext-unbounded-total*:

**assumes**  $\forall (s, t) \in \text{set } (\text{zip } ss \text{ } ts). s = t \vee \text{fst } (f s t) \vee \text{fst } (f t s)$   
**and**  $\text{refl}: \bigwedge t. \text{snd } (f t t)$   
**and**  $\text{length } ss = \text{length } ts$   
**shows**  $ss = ts \vee \text{fst } (\text{lex-ext-unbounded } f \text{ } ss \text{ } ts) \vee \text{fst } (\text{lex-ext-unbounded } f \text{ } ts \text{ } ss)$   
 $\langle \text{proof} \rangle$

**lemma** *lex-ext-total*:

**assumes**  $\forall (s, t) \in \text{set } (\text{zip } ss \text{ } ts). s = t \vee \text{fst } (f s t) \vee \text{fst } (f t s)$   
**and**  $\bigwedge t. \text{snd } (f t t)$   
**and**  $\text{len}: \text{length } ss = \text{length } ts$   
**shows**  $ss = ts \vee \text{fst } (\text{lex-ext } f \text{ } n \text{ } ss \text{ } ts) \vee \text{fst } (\text{lex-ext } f \text{ } n \text{ } ts \text{ } ss)$   
 $\langle \text{proof} \rangle$

Monotonicity of the lexicographic extension.

**lemma** *lex-ext-unbounded-mono*:

```

assumes  $\bigwedge i. \llbracket i < \text{length } xs; i < \text{length } ys; \text{fst } (P \ (xs \ ! \ i) \ (ys \ ! \ i)) \rrbracket \implies \text{fst } (P' \ (xs \ ! \ i) \ (ys \ ! \ i))$ 
and  $\bigwedge i. \llbracket i < \text{length } xs; i < \text{length } ys; \text{snd } (P \ (xs \ ! \ i) \ (ys \ ! \ i)) \rrbracket \implies \text{snd } (P' \ (xs \ ! \ i) \ (ys \ ! \ i))$ 
shows
  ( $\text{fst } (\text{lex-ext-unbounded } P \ xs \ ys) \longrightarrow \text{fst } (\text{lex-ext-unbounded } P' \ xs \ ys)$ )  $\wedge$ 
  ( $\text{snd } (\text{lex-ext-unbounded } P \ xs \ ys) \longrightarrow \text{snd } (\text{lex-ext-unbounded } P' \ xs \ ys)$ )
  (is ( $?l1 \ xs \ ys \longrightarrow ?r1 \ xs \ ys$ )  $\wedge$  ( $?l2 \ xs \ ys \longrightarrow ?r2 \ xs \ ys$ ))
   $\langle \text{proof} \rangle$ 

```

**lemma** *lex-ext-local-mono* [*mono*]:

```

assumes  $\bigwedge s \ t. s \in \text{set } ts \implies t \in \text{set } ss \implies \text{ord } s \ t \implies \text{ord}' \ s \ t$ 
shows  $\text{fst } (\text{lex-ext } (\lambda x \ y. (\text{ord } x \ y, (x, y) \in \text{ns-rel})) \ (\text{length } ts) \ ts \ ss) \longrightarrow$ 
   $\text{fst } (\text{lex-ext } (\lambda x \ y. (\text{ord}' \ x \ y, (x, y) \in \text{ns-rel})) \ (\text{length } ts) \ ts \ ss)$ 
   $\langle \text{proof} \rangle$ 

```

**lemma** *lex-ext-mono* [*mono*]:

```

assumes  $\bigwedge s \ t. \text{ord } s \ t \longrightarrow \text{ord}' \ s \ t$ 
shows  $\text{fst } (\text{lex-ext } (\lambda x \ y. (\text{ord } x \ y, (x, y) \in \text{ns})) \ (\text{length } ts) \ ts \ ss) \longrightarrow$ 
   $\text{fst } (\text{lex-ext } (\lambda x \ y. (\text{ord}' \ x \ y, (x, y) \in \text{ns})) \ (\text{length } ts) \ ts \ ss)$ 
   $\langle \text{proof} \rangle$ 

```

**end**

## 4 KBO

Below, we formalize a variant of KBO that includes subterm coefficient functions.

A more standard definition is obtained by setting all subterm coefficients to 1. For this special case it would be possible to define more efficient code-equations that do not have to evaluate subterm coefficients at all.

**theory** *KBO*

**imports**

*Lexicographic-Extension*  
*First-Order-Terms.Subterm-and-Context*  
*Polynomial-Factorization.Missing-List*

**begin**

### 4.1 Subterm Coefficient Functions

Given a function *scf*, associating positions with subterm coefficients, and a list *xs*, the function *scf-list* yields an expanded list where each element of *xs* is replicated a number of times according to its subterm coefficient.

**definition** *scf-list* ::  $(\text{nat} \Rightarrow \text{nat}) \Rightarrow 'a \text{ list} \Rightarrow 'a \text{ list}$

**where**

$\text{scf-list } scf \ xs = \text{concat } (\text{map } (\lambda(x, i). \text{replicate } (scf \ i) \ x) \ (\text{zip } xs \ [0 \ .. < \ \text{length } xs]))$

**lemma** *set-scf-list* [*simp*]:  
**assumes**  $\forall i < \text{length } xs. \text{scf } i > 0$   
**shows**  $\text{set } (\text{scf-list } \text{scf } xs) = \text{set } xs$   
 $\langle \text{proof} \rangle$

**lemma** *scf-list-subset*:  $\text{set } (\text{scf-list } \text{scf } xs) \subseteq \text{set } xs$   
 $\langle \text{proof} \rangle$

**lemma** *scf-list-empty* [*simp*]:  
 $\text{scf-list } \text{scf } [] = []$   $\langle \text{proof} \rangle$

**lemma** *scf-list-bef-i-aft* [*simp*]:  
 $\text{scf-list } \text{scf } (\text{bef } @ i \# \text{aft}) =$   
 $\text{scf-list } \text{scf } \text{bef } @ \text{replicate } (\text{scf } (\text{length } \text{bef})) i @$   
 $\text{scf-list } (\lambda i. \text{scf } (\text{Suc } (\text{length } \text{bef } + i))) \text{aft}$   
 $\langle \text{proof} \rangle$

**lemma** *scf-list-map* [*simp*]:  
 $\text{scf-list } \text{scf } (\text{map } f \text{ } xs) = \text{map } f (\text{scf-list } \text{scf } xs)$   
 $\langle \text{proof} \rangle$

The function *scf-term* replicates each argument a number of times according to its subterm coefficient function.

**fun** *scf-term* ::  $('f \times \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}) \Rightarrow ('f, 'v) \text{ term} \Rightarrow ('f, 'v) \text{ term}$   
**where**  
 $\text{scf-term } \text{scf } (\text{Var } x) = (\text{Var } x) |$   
 $\text{scf-term } \text{scf } (\text{Fun } f \text{ } ts) = \text{Fun } f (\text{scf-list } (\text{scf } (f, \text{length } ts)) (\text{map } (\text{scf-term } \text{scf})$   
 $ts))$

**lemma** *vars-term-scf-subset*:  
 $\text{vars-term } (\text{scf-term } \text{scf } s) \subseteq \text{vars-term } s$   
 $\langle \text{proof} \rangle$

**lemma** *scf-term-subst*:  
 $\text{scf-term } \text{scf } (t \cdot \sigma) = \text{scf-term } \text{scf } t \cdot (\lambda x. \text{scf-term } \text{scf } (\sigma x))$   
 $\langle \text{proof} \rangle$

## 4.2 Weight Functions

**locale** *weight-fun* =  
**fixes**  $w :: 'f \times \text{nat} \Rightarrow \text{nat}$   
**and**  $w0 :: \text{nat}$   
**and**  $\text{scf} :: 'f \times \text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$   
**begin**

The *weight* of a term is computed recursively, where variables have weight  $w0$  and the weight of a compound term is computed by adding the weight of its root symbol  $w(f, n)$  to the weighted sum where weights of arguments are multiplied according to their subterm coefficients.

```

fun weight :: ('f, 'v) term  $\Rightarrow$  nat
  where
    weight (Var x) = w0 |
    weight (Fun f ts) =
      (let n = length ts; scff = scf (f, n) in
       w (f, n) + sum-list (map ( $\lambda$  (ti, i). weight ti * scff i) (zip ts [0 ..< n])))

```

Alternatively, we can replicate arguments via *scf-list*. The advantage is that then both *weight* and *scf-term* are defined via *scf-list*.

```

lemma weight-simp [simp]:
  weight (Fun f ts) = w (f, length ts) + sum-list (map weight (scf-list (scf (f,
length ts)) ts))
<proof>

```

```

declare weight.simps(2)[simp del]

```

```

abbreviation SCF  $\equiv$  scf-term scf

```

```

lemma sum-list-scf-list:
  assumes  $\bigwedge i. i < \text{length } ts \implies f\ i > 0$ 
  shows sum-list (map weight ts)  $\leq$  sum-list (map weight (scf-list f ts))
  <proof>

```

```

end

```

### 4.3 Definition of KBO

The precedence is given by three parameters:

- a predicate *pr-strict* for strict decrease between two function symbols,
- a predicate *pr-weak* for weak decrease between two function symbols, and
- a function indicating whether a symbol is least in the precedence.

```

locale kbo = weight-fun w w0 scf
  for w w0 and scf :: 'f  $\times$  nat  $\Rightarrow$  nat  $\Rightarrow$  nat +
  fixes least :: 'f  $\Rightarrow$  bool
  and pr-strict :: 'f  $\times$  nat  $\Rightarrow$  'f  $\times$  nat  $\Rightarrow$  bool
  and pr-weak :: 'f  $\times$  nat  $\Rightarrow$  'f  $\times$  nat  $\Rightarrow$  bool
begin

```

The result of *kbo* is a pair of Booleans encoding strict/weak decrease.

Interestingly, the bound on the lengths of the lists in the lexicographic extension is not required for KBO.

```

fun kbo :: ('f, 'v) term  $\Rightarrow$  ('f, 'v) term  $\Rightarrow$  bool  $\times$  bool
  where

```

```

kbo s t = (if (vars-term-ms (SCF t)  $\subseteq\#$  vars-term-ms (SCF s)  $\wedge$  weight t  $\leq$ 
weight s)
  then if (weight t < weight s)
    then (True, True)
  else (case s of
    Var y  $\Rightarrow$  (False, (case t of Var x  $\Rightarrow$  x = y | Fun g ts  $\Rightarrow$  ts = []  $\wedge$  least g))
  | Fun f ss  $\Rightarrow$  (case t of
    Var x  $\Rightarrow$  (True, True)
  | Fun g ts  $\Rightarrow$  if pr-strict (f, length ss) (g, length ts)
    then (True, True)
    else if pr-weak (f, length ss) (g, length ts)
    then lex-ext-unbounded kbo ss ts
    else (False, False)))
  else (False, False))

```

Abbreviations for strict (S) and nonstrict (NS) KBO.

**abbreviation**  $S \equiv \lambda s t. \text{fst} (kbo s t)$

**abbreviation**  $NS \equiv \lambda s t. \text{snd} (kbo s t)$

For code-generation we do not compute the weights of  $s$  and  $t$  repeatedly.

**lemma** *kbo-code*:  $kbo s t = (\text{let } wt = \text{weight } t; ws = \text{weight } s \text{ in}$   
 $\text{if } (\text{vars-term-ms } (SCF t) \subseteq\# \text{ vars-term-ms } (SCF s) \wedge wt \leq ws)$   
 $\text{then if } wt < ws$   
 $\text{then } (True, True)$   
 $\text{else } (\text{case } s \text{ of}$   
 $\text{Var } y \Rightarrow (False, (\text{case } t \text{ of Var } x \Rightarrow True \mid \text{Fun } g \text{ ts} \Rightarrow \text{ts} = [] \wedge \text{least } g))$   
 $\mid \text{Fun } f \text{ ss} \Rightarrow (\text{case } t \text{ of}$   
 $\text{Var } x \Rightarrow (True, True)$   
 $\mid \text{Fun } g \text{ ts} \Rightarrow \text{let } ff = (f, \text{length } ss); gg = (g, \text{length } ts) \text{ in}$   
 $\text{if pr-strict } ff \text{ gg}$   
 $\text{then } (True, True)$   
 $\text{else if pr-weak } ff \text{ gg}$   
 $\text{then lex-ext-unbounded kbo ss ts}$   
 $\text{else } (False, False)))$   
 $\text{else } (False, False))$   
 $\langle \text{proof} \rangle$

**end**

**declare**  $kbo.kbo\text{-code}[code]$

**declare**  $\text{weight-fun.weight.simps}[code]$

**lemma** *mset-replicate-mono*:

**assumes**  $m1 \subseteq\# m2$

**shows**  $\sum\# (mset (\text{replicate } n \text{ } m1)) \subseteq\# \sum\# (mset (\text{replicate } n \text{ } m2))$

$\langle \text{proof} \rangle$

While the locale *kbo* only fixes its parameters, we now demand that these parameters are sensible, e.g., encoding a well-founded precedence, etc.

**locale** *admissible-kbo* =

*kbo w w0 scf least pr-strict pr-weak*  
**for** *w w0 pr-strict pr-weak* **and** *least :: 'f ⇒ bool and scf +*  
**assumes** *w0: w (f, 0) ≥ w0 w0 > 0*  
**and** *adm: w (f, 1) = 0 ⇒ pr-weak (f, 1) (g, n)*  
**and** *least: least f = (w (f, 0) = w0 ∧ (∀ g. w (g, 0) = w0 → pr-weak (g, 0)*  
*(f, 0)))*  
**and** *scf: i < n ⇒ scf (f, n) i > 0*  
**and** *pr-weak-refl [simp]: pr-weak fn fn*  
**and** *pr-weak-trans: pr-weak fn gm ⇒ pr-weak gm hk ⇒ pr-weak fn hk*  
**and** *pr-strict: pr-strict fn gm ⇔ pr-weak fn gm ∧ ¬ pr-weak gm fn*  
**and** *pr-SN: SN {(fn, gm). pr-strict fn gm}*  
**begin**

**lemma** *weight-w0: weight t ≥ w0*  
*<proof>*

**lemma** *weight-gt-0: weight t > 0*  
*<proof>*

**lemma** *weight-0 [iff]: weight t = 0 ⇔ False*  
*<proof>*

**lemma** *not-S-Var: ¬ S (Var x) t*  
*<proof>*

**lemma** *S-imp-NS: S s t ⇒ NS s t*  
*<proof>*

#### 4.4 Reflexivity and Irreflexivity

**lemma** *NS-refl: NS s s*  
*<proof>*

**lemma** *pr-strict-irrefl: ¬ pr-strict fn fn*  
*<proof>*

**lemma** *S-irrefl: ¬ S t t*  
*<proof>*

#### 4.5 Monotonicity (a.k.a. Closure under Contexts)

**lemma** *S-mono-one:*  
**assumes** *S: S s t*  
**shows** *S (Fun f (ss1 @ s # ss2)) (Fun f (ss1 @ t # ss2))*  
*<proof>*

**lemma** *S-ctxt: S s t ⇒ S (C⟨s⟩) (C⟨t⟩)*  
*<proof>*

**lemma** *NS-mono-one:*

**assumes**  $NS$ :  $NS\ s\ t$  **shows**  $NS\ (Fun\ f\ (ss1\ @\ s\ \# \ ss2))\ (Fun\ f\ (ss1\ @\ t\ \# \ ss2))$   
 $\langle proof \rangle$

**lemma**  $NS\text{-ctxt}$ :  $NS\ s\ t \implies NS\ (C\langle s \rangle)\ (C\langle t \rangle)$   
 $\langle proof \rangle$

## 4.6 The Subterm Property

**lemma**  $NS\text{-Var-imp-eq-least}$ :  $NS\ (Var\ x)\ t \implies t = Var\ x \vee (\exists\ f.\ t = Fun\ f\ [] \wedge\ least\ f)$   
 $\langle proof \rangle$

**lemma**  $kbo\text{-supt-one}$ :  $NS\ s\ (t :: ('f, 'v)\ term) \implies S\ (Fun\ f\ (bef\ @\ s\ \# \ aft))\ t$   
 $\langle proof \rangle$

**lemma**  $S\text{-supt}$ :  
**assumes**  $supt$ :  $s \triangleright t$   
**shows**  $S\ s\ t$   
 $\langle proof \rangle$

**lemma**  $NS\text{-supteq}$ :  
**assumes**  $s \trianglerighteq t$   
**shows**  $NS\ s\ t$   
 $\langle proof \rangle$

## 4.7 Least Elements

**lemma**  $NS\text{-all-least}$ :  
**assumes**  $l$ :  $least\ f$   
**shows**  $NS\ t\ (Fun\ f\ [])$   
 $\langle proof \rangle$

**lemma**  $not\text{-S-least}$ :  
**assumes**  $l$ :  $least\ f$   
**shows**  $\neg S\ (Fun\ f\ [])\ t$   
 $\langle proof \rangle$

**lemma**  $NS\text{-least-least}$ :  
**assumes**  $l$ :  $least\ f$   
**and**  $NS$ :  $NS\ (Fun\ f\ [])\ t$   
**shows**  $\exists\ g.\ t = Fun\ g\ [] \wedge least\ g$   
 $\langle proof \rangle$

## 4.8 Stability (a.k.a. Closure under Substitutions)

**lemma**  $weight\text{-subst}$ :  $weight\ (t \cdot \sigma) =$   
 $weight\ t + sum\ mset\ (image\ mset\ (\lambda\ x.\ weight\ (\sigma\ x) - w0)\ (vars\ term\ ms\ (SCF\ t)))$   
 $\langle proof \rangle$

**lemma** *weight-stable-le*:  
**assumes** *ws*:  $\text{weight } s \leq \text{weight } t$   
**and** *vs*:  $\text{vars-term-ms } (SCF\ s) \subseteq\# \text{vars-term-ms } (SCF\ t)$   
**shows**  $\text{weight } (s \cdot \sigma) \leq \text{weight } (t \cdot \sigma)$   
 $\langle \text{proof} \rangle$

**lemma** *weight-stable-lt*:  
**assumes** *ws*:  $\text{weight } s < \text{weight } t$   
**and** *vs*:  $\text{vars-term-ms } (SCF\ s) \subseteq\# \text{vars-term-ms } (SCF\ t)$   
**shows**  $\text{weight } (s \cdot \sigma) < \text{weight } (t \cdot \sigma)$   
 $\langle \text{proof} \rangle$

KBO is stable, i.e., closed under substitutions.

**lemma** *kbo-stable*:  
**fixes**  $\sigma :: ('f, 'v)\ \text{subst}$   
**assumes**  $NS\ s\ t$   
**shows**  $(S\ s\ t \longrightarrow S\ (s \cdot \sigma)\ (t \cdot \sigma)) \wedge NS\ (s \cdot \sigma)\ (t \cdot \sigma)$  (**is**  $?P\ s\ t$ )  
 $\langle \text{proof} \rangle$

**lemma** *S-subst*:  
 $S\ s\ t \Longrightarrow S\ (s \cdot (\sigma :: ('f, 'v)\ \text{subst}))\ (t \cdot \sigma)$   
 $\langle \text{proof} \rangle$

**lemma** *NS-subst*:  $NS\ s\ t \Longrightarrow NS\ (s \cdot (\sigma :: ('f, 'v)\ \text{subst}))\ (t \cdot \sigma)$   $\langle \text{proof} \rangle$

## 4.9 Transitivity and Compatibility

**lemma** *kbo-trans*:  $(S\ s\ t \longrightarrow NS\ t\ u \longrightarrow S\ s\ u) \wedge$   
 $(NS\ s\ t \longrightarrow S\ t\ u \longrightarrow S\ s\ u) \wedge$   
 $(NS\ s\ t \longrightarrow NS\ t\ u \longrightarrow NS\ s\ u)$   
**(is**  $?P\ s\ t\ u$ )  
 $\langle \text{proof} \rangle$

**lemma** *S-trans*:  $S\ s\ t \Longrightarrow S\ t\ u \Longrightarrow S\ s\ u$   $\langle \text{proof} \rangle$

**lemma** *NS-trans*:  $NS\ s\ t \Longrightarrow NS\ t\ u \Longrightarrow NS\ s\ u$   $\langle \text{proof} \rangle$

**lemma** *NS-S-compat*:  $NS\ s\ t \Longrightarrow S\ t\ u \Longrightarrow S\ s\ u$   $\langle \text{proof} \rangle$

**lemma** *S-NS-compat*:  $S\ s\ t \Longrightarrow NS\ t\ u \Longrightarrow S\ s\ u$   $\langle \text{proof} \rangle$

## 4.10 Strong Normalization (a.k.a. Well-Foundedness)

**lemma** *kbo-strongly-normalizing*:  
**fixes**  $s :: ('f, 'v)\ \text{term}$   
**shows**  $SN\text{-on } \{(s, t). S\ s\ t\} \{s\}$   
 $\langle \text{proof} \rangle$

**lemma** *S-SN*:  $SN\ \{(x, y). S\ x\ y\}$   
 $\langle \text{proof} \rangle$

## 4.11 Ground Totality

**lemma** *ground-SCF* [*simp*]:  
   $ground (SCF t) = ground t$   
  ⟨*proof*⟩

**declare** *kbo.simps*[*simp del*]

**lemma** *ground-vars-term-ms*:  $ground t \implies vars-term-ms t = \{\#\}$   
  ⟨*proof*⟩

**context**

**fixes**  $F :: ('f \times nat) set$

**assumes** *pr-weak*:  $pr-weak = pr-strict^{==}$

**and** *pr-gtotal*:  $\bigwedge g. f \in F \implies g \in F \implies f = g \vee pr-strict f g \vee pr-strict g f$

**begin**

**lemma** *S-ground-total*:

**assumes** *funas-term*  $s \subseteq F$  **and** *ground*  $s$  **and** *funas-term*  $t \subseteq F$  **and** *ground*  $t$

**shows**  $s = t \vee S s t \vee S t s$

  ⟨*proof*⟩

**end**

## 4.12 Summary

At this point we have shown well-foundedness *S-SN*, transitivity and compatibility *S-trans NS-trans NS-S-compat S-NS-compat*, closure under substitutions *S-subst NS-subst*, closure under contexts *S-ctxt NS-ctxt*, the subterm property *S-supt NS-supteq*, reflexivity of the weak *NS-refl* and irreflexivity of the strict part *S-irrefl*, and ground-totality *S-ground-total*.

In particular, this allows us to show that KBO is an instance of strongly normalizing order pairs (*SN-order-pair*).

**sublocale** *SN-order-pair*  $\{(x, y). S x y\} \{(x, y). NS x y\}$

  ⟨*proof*⟩

**end**

**end**

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