

# Knight’s Tour Revisited Revisited

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## Abstract

This is a formalization of the article “Knight’s Tour Revisited” by Cull and De Curtins where they prove the existence of a Knight’s path for arbitrary  $n \times m$ -boards with  $\min(n, m) \geq 5$ . If  $n \cdot m$  is even, then there exists a Knight’s circuit.

A Knight’s Path is a sequence of moves of a Knight on a chessboard s.t. the Knight visits every square of a chessboard exactly once. Finding a Knight’s path is a an instance of the Hamiltonian path problem.

During the formalization two mistakes in the original proof were discovered. These mistakes are corrected in this formalization.

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```
theory KnightsTour
  imports Main
begin
```

## 1 Introduction and Definitions

A Knight's path is a sequence of moves on a chessboard s.t. every step in sequence is a valid move for a Knight and that the Knight visits every square on the boards exactly once. A Knight is a chess figure that is only able to move two squares vertically and one square horizontally or two squares horizontally and one square vertically. Finding a Knight's path is an instance of the Hamiltonian Path Problem. A Knight's circuit is a Knight's path, where additionally the Knight can move from the last square to the first square of the path, forming a loop.

Cull and De Curtins [1] prove the existence of a Knight's path on a  $n \times m$ -board for sufficiently large  $n$  and  $m$ . The main idea for the proof is to inductively construct a Knight's path for the  $n \times m$ -board from a few pre-computed Knight's paths for small boards, i.e.  $5 \times 5$ ,  $8 \times 6$ , ...,  $8 \times 9$ . The paths for small boards are transformed (i.e. transpose, mirror, translate) and concatenated to create paths for larger boards.

While formalizing the proofs I discovered two mistakes in the original proof in [1]: (i) the pre-computed path for the  $6 \times 6$ -board that ends in the upper-left (in Figure 2) and (ii) the pre-computed path for the  $8 \times 8$ -board that ends in the upper-left (in Figure 5) are incorrect: on the  $6 \times 6$ -board the Knight cannot step from square 26 to square 27; in the  $8 \times 8$ -board the Knight cannot step from square 27 to square 28. In this formalization I have replaced the two incorrect paths with correct paths.

A square on a board is identified by its coordinates.

**type-synonym**  $square = int \times int$

A board is represented as a set of squares. Note, that this allows boards to have an arbitrary shape and do not necessarily need to be rectangular.

**type-synonym**  $board = square\ set$

A (rectangular)  $(n \times m)$ -board is the set of all squares  $(i, j)$  where  $1 \leq i \leq n$  and  $1 \leq j \leq m$ .  $(1, 1)$  is the lower-left corner, and  $(n, m)$  is the upper-right corner.

**definition**  $board :: nat \Rightarrow nat \Rightarrow board$  **where**

$$board\ n\ m = \{(i, j) \mid i\ j.\ 1 \leq i \wedge i \leq int\ n \wedge 1 \leq j \wedge j \leq int\ m\}$$

A path is a sequence of steps on a board. A path is represented by the list of visited squares on the board. Each square on the  $(n \times m)$ -board is identified by its coordinates  $(i, j)$ .

**type-synonym**  $path = square\ list$

A Knight can only move two squares vertically and one square horizontally or two squares horizontally and one square vertically. Thus, a knight at position  $(i, j)$  can only move to  $(i \pm 1, j \pm 2)$  or  $(i \pm 2, j \pm 1)$ .

**definition**  $valid\ step :: square \Rightarrow square \Rightarrow bool$  **where**

$$valid\ step\ s_i\ s_j \equiv (case\ s_i\ of\ (i, j) \Rightarrow s_j \in \{(i+1, j+2), (i-1, j+2), (i+1, j-2), (i-1, j-2), (i+2, j+1), (i-2, j+1), (i+2, j-1), (i-2, j-1)\})$$

Now we define an inductive predicate that characterizes a Knight's path. A square  $s_i$  can be pre-pended to a current Knight's path  $s_j \# ps$  if (i) there is a valid step from the square  $s_i$  to the first square  $s_j$  of the current path and (ii) the square  $s_i$  has not been visited yet.

**inductive**  $knights\ path :: board \Rightarrow path \Rightarrow bool$  **where**

$$\begin{aligned} & knights\ path\ \{s_i\}\ [s_i] \\ | & s_i \notin b \Longrightarrow valid\ step\ s_i\ s_j \Longrightarrow knights\ path\ b\ (s_j \# ps) \Longrightarrow knights\ path\ (b \cup \{s_i\}) \\ & (s_i \# s_j \# ps) \end{aligned}$$

**code-pred**  $knights\ path$  .

A sequence is a Knight's circuit iff the sequence is a Knight's path and there is a valid step from the last square to the first square.

**definition**  $knights\ circuit\ b\ ps \equiv (knights\ path\ b\ ps \wedge valid\ step\ (last\ ps)\ (hd\ ps))$

## 2 Executable Checker for a Knight's Path

This section gives the implementation and correctness-proof for an executable checker for a knights-path w.r.t. the definition  $knights\ path$ .

## 2.1 Implementation of an Executable Checker

**fun** *row-exec* :: *nat*  $\Rightarrow$  *int set* **where**

*row-exec* 0 = {}  
| *row-exec* m = *insert* (*int* m) (*row-exec* (m-1))

**fun** *board-exec-aux* :: *nat*  $\Rightarrow$  *int set*  $\Rightarrow$  *board* **where**

*board-exec-aux* 0 M = {}  
| *board-exec-aux* k M = {(*int* k, j) | j. j  $\in$  M}  $\cup$  *board-exec-aux* (k-1) M

Compute a board.

**fun** *board-exec* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *board* **where**

*board-exec* n m = *board-exec-aux* n (*row-exec* m)

**fun** *step-checker* :: *square*  $\Rightarrow$  *square*  $\Rightarrow$  *bool* **where**

*step-checker* (i, j) (i', j') =  
((i+1, j+2) = (i', j')  $\vee$  (i-1, j+2) = (i', j')  $\vee$  (i+1, j-2) = (i', j')  $\vee$  (i-1, j-2) = (i', j'))  
 $\vee$  (i+2, j+1) = (i', j')  $\vee$  (i-2, j+1) = (i', j')  $\vee$  (i+2, j-1) = (i', j')  $\vee$  (i-2, j-1) = (i', j'))

**fun** *path-checker* :: *board*  $\Rightarrow$  *path*  $\Rightarrow$  *bool* **where**

*path-checker* b [] = *False*  
| *path-checker* b [s<sub>i</sub>] = (s<sub>i</sub> = b)  
| *path-checker* b (s<sub>i</sub>#s<sub>j</sub>#ps) = (s<sub>i</sub>  $\in$  b  $\wedge$  *step-checker* s<sub>i</sub> s<sub>j</sub>  $\wedge$  *path-checker* (b - {s<sub>i</sub>}) (s<sub>j</sub>#ps))

**fun** *circuit-checker* :: *board*  $\Rightarrow$  *path*  $\Rightarrow$  *bool* **where**

*circuit-checker* b ps = (*path-checker* b ps  $\wedge$  *step-checker* (last ps) (hd ps))

## 2.2 Correctness Proof of the Executable Checker

**lemma** *row-exec-leq*: j  $\in$  *row-exec* m  $\longleftrightarrow$  1  $\leq$  j  $\wedge$  j  $\leq$  *int* m

**by** (*induction* m) *auto*

**lemma** *board-exec-aux-leq-mem*: (i, j)  $\in$  *board-exec-aux* k M  $\longleftrightarrow$  1  $\leq$  i  $\wedge$  i  $\leq$  *int* k  $\wedge$  j  $\in$  M

**by** (*induction* k M *rule*: *board-exec-aux.induct*) *auto*

**lemma** *board-exec-leq*: (i, j)  $\in$  *board-exec* n m  $\longleftrightarrow$  1  $\leq$  i  $\wedge$  i  $\leq$  *int* n  $\wedge$  1  $\leq$  j  $\wedge$  j  $\leq$  *int* m

**using** *board-exec-aux-leq-mem* *row-exec-leq* **by** *auto*

**lemma** *board-exec-correct*: *board* n m = *board-exec* n m

**unfolding** *board-def* **using** *board-exec-leq* **by** *auto*

**lemma** *step-checker-correct*: *step-checker* s<sub>i</sub> s<sub>j</sub>  $\longleftrightarrow$  *valid-step* s<sub>i</sub> s<sub>j</sub>

**proof**

**assume** *step-checker* s<sub>i</sub> s<sub>j</sub>

**then show** *valid-step* s<sub>i</sub> s<sub>j</sub>

```

    unfolding valid-step-def
    apply (cases si)
    apply (cases sj)
    apply auto
    done
next
  assume assms: valid-step si sj
  then show step-checker si sj
    unfolding valid-step-def by auto
qed

```

```

lemma step-checker-rev: step-checker (i,j) (i',j')  $\implies$  step-checker (i',j') (i,j)
  apply (simp only: step-checker.simps)
  by (elim disjE) auto

```

```

lemma knights-path-intro-rev:
  assumes si  $\in$  b valid-step si sj knights-path (b - {si}) (sj#ps)
  shows knights-path b (si#sj#ps)
  using assms
proof -
  assume assms: si  $\in$  b valid-step si sj knights-path (b - {si}) (sj#ps)
  then have si  $\notin$  (b - {si}) b - {si}  $\cup$  {si} = b
    by auto
  then show ?thesis
    using assms knights-path.intros(2)[of si b - {si}] by auto
qed

```

Final correctness corollary for the executable checker *path-checker*.

```

lemma path-checker-correct: path-checker b ps  $\longleftrightarrow$  knights-path b ps
proof
  assume path-checker b ps
  then show knights-path b ps
  proof (induction rule: path-checker.induct)
    case (3 si sj xs b)
    then show ?case using step-checker-correct knights-path-intro-rev by auto
  qed (auto intro: knights-path.intros)
next
  assume knights-path b ps
  then show path-checker b ps
    using step-checker-correct
    by (induction rule: knights-path.induct) (auto elim: knights-path.cases)
qed

```

```

corollary knights-path-exec-simp: knights-path (board n m) ps  $\longleftrightarrow$  path-checker
(board-exec n m) ps
  using board-exec-correct path-checker-correct[symmetric] by simp

```

```

lemma circuit-checker-correct: circuit-checker b ps  $\longleftrightarrow$  knights-circuit b ps
  unfolding knights-circuit-def using path-checker-correct step-checker-correct by

```

*auto*

**corollary** *knights-circuit-exec-simp*:

*knights-circuit* (board  $n$   $m$ )  $ps \longleftrightarrow$  *circuit-checker* (board-exec  $n$   $m$ )  $ps$   
using *board-exec-correct* *circuit-checker-correct*[*symmetric*] **by** *simp*

### 3 Basic Properties of *knights-path* and *knights-circuit*

**lemma** *board-leq-subset*:  $n_1 \leq n_2 \wedge m_1 \leq m_2 \implies$  board  $n_1$   $m_1 \subseteq$  board  $n_2$   $m_2$   
unfolding *board-def* **by** *auto*

**lemma** *finite-row-exec*: *finite* (row-exec  $m$ )  
**by** (*induction*  $m$ ) *auto*

**lemma** *finite-board-exec-aux*: *finite*  $M \implies$  *finite* (board-exec-aux  $n$   $M$ )  
**by** (*induction*  $n$ ) *auto*

**lemma** *board-finite*: *finite* (board  $n$   $m$ )  
using *finite-board-exec-aux* *finite-row-exec* **by** (*simp only*: *board-exec-correct*) *auto*

**lemma** *card-row-exec*: *card* (row-exec  $m$ ) =  $m$

**proof** (*induction*  $m$ )

**case** (*Suc*  $m$ )

**have** *int* (*Suc*  $m$ )  $\notin$  row-exec  $m$

using *row-exec-leq* **by** *auto*

**then have** *card* (*insert* (*int* (*Suc*  $m$ )) (row-exec  $m$ )) =  $1 +$  *card* (row-exec  $m$ )

using *card-Suc-eq* **by** (*metis* *Suc plus-1-eq-Suc* row-exec.simps(1))

**then have** *card* (row-exec (*Suc*  $m$ )) =  $1 +$  *card* (row-exec  $m$ )

**by** *auto*

**then show** *?case* **using** *Suc.IH* **by** *auto*

**qed** *auto*

**lemma** *set-comp-ins*:

$\{(k,j) \mid j. j \in \text{insert } x M\} = \text{insert } (k,x) \{(k,j) \mid j. j \in M\}$  (**is**  $?Mi = ?iM$ )

**proof**

**show**  $?Mi \subseteq ?iM$

**proof**

**fix**  $y$  **assume**  $y \in ?Mi$

**then obtain**  $j$  **where** [*simp*]:  $y = (k,j)$  **and**  $j \in \text{insert } x M$  **by** *blast*

**then have**  $j = x \vee j \in M$  **by** *auto*

**then show**  $y \in ?iM$  **by** (*elim disjE*) *auto*

**qed**

**next**

**show**  $?iM \subseteq ?Mi$

**proof**

**fix**  $y$  **assume**  $y \in ?iM$

**then obtain**  $j$  **where** [*simp*]:  $y = (k,j)$  **and**  $j \in \text{insert } x M$  **by** *blast*

**then have**  $j = x \vee j \in M$  **by** *auto*

**then show**  $y \in ?Mi$  **by** (*elim disjE*) *auto*

**qed**  
**qed**

**lemma** *finite-card-set-comp*:  $\text{finite } M \implies \text{card } \{(k,j) \mid j. j \in M\} = \text{card } M$   
**proof** (*induction M rule: finite-induct*)  
  **case** (*insert x M*)  
  **then show** *?case* **using** *set-comp-ins[of k x M]* **by auto**  
**qed auto**

**lemma** *card-board-exec-aux*:  $\text{finite } M \implies \text{card } (\text{board-exec-aux } k \ M) = k * \text{card } M$   
**proof** (*induction k*)  
  **case** (*Suc k*)  
  **let**  $?M' = \{(int \ (Suc \ k), j) \mid j. j \in M\}$   
  **let**  $?rec-k = \text{board-exec-aux } k \ M$

**have** *finite*:  $\text{finite } ?M' \ \text{finite } ?rec-k$   
  **using** *Suc finite-board-exec-aux* **by auto**  
  **then have** *card-Un-simp*:  $\text{card } (?M' \cup ?rec-k) = \text{card } ?M' + \text{card } ?rec-k$   
  **using** *board-exec-aux-leg-mem card-Un-Int[of ?M' ?rec-k]* **by auto**

**have** *card-M*:  $\text{card } ?M' = \text{card } M$   
  **using** *Suc finite-card-set-comp* **by auto**

**have**  $\text{card } (\text{board-exec-aux } (Suc \ k) \ M) = \text{card } ?M' + \text{card } ?rec-k$   
  **using** *card-Un-simp* **by auto**  
  **also have**  $\dots = \text{card } M + k * \text{card } M$   
  **using** *Suc card-M* **by auto**  
  **also have**  $\dots = (Suc \ k) * \text{card } M$   
  **by auto**  
  **finally show** *?case* .

**qed auto**

**lemma** *card-board*:  $\text{card } (\text{board } n \ m) = n * m$   
**proof** –  
  **have**  $\text{card } (\text{board } n \ m) = \text{card } (\text{board-exec-aux } n \ (\text{row-exec } m))$   
  **using** *board-exec-correct* **by auto**  
  **also have**  $\dots = n * m$   
  **using** *card-row-exec card-board-exec-aux finite-row-exec* **by auto**  
  **finally show** *?thesis* .

**qed**

**lemma** *knights-path-board-non-empty*:  $\text{knights-path } b \ ps \implies b \neq \{\}$   
**by** (*induction arbitrary: ps rule: knights-path.induct*) **auto**

**lemma** *knights-path-board-m-n-geq-1*:  $\text{knights-path } (\text{board } n \ m) \ ps \implies \min \ n \ m \geq 1$   
**unfolding** *board-def* **using** *knights-path-board-non-empty* **by fastforce**

**lemma** *knights-path-non-nil*:  $\text{knights-path } b \ ps \implies ps \neq []$

by (induction arbitrary: b rule: knights-path.induct) auto

**lemma** *knights-path-set-eq*: knights-path b ps  $\implies$  set ps = b  
 by (induction rule: knights-path.induct) auto

**lemma** *knights-path-subset*:  
 knights-path b<sub>1</sub> ps<sub>1</sub>  $\implies$  knights-path b<sub>2</sub> ps<sub>2</sub>  $\implies$  set ps<sub>1</sub>  $\subseteq$  set ps<sub>2</sub>  $\iff$  b<sub>1</sub>  $\subseteq$  b<sub>2</sub>  
 using knights-path-set-eq by auto

**lemma** *knights-path-board-unique*: knights-path b<sub>1</sub> ps  $\implies$  knights-path b<sub>2</sub> ps  $\implies$   
 b<sub>1</sub> = b<sub>2</sub>  
 using knights-path-set-eq by auto

**lemma** *valid-step-neq*: valid-step s<sub>i</sub> s<sub>j</sub>  $\implies$  s<sub>i</sub>  $\neq$  s<sub>j</sub>  
 unfolding valid-step-def by auto

**lemma** *valid-step-non-transitive*: valid-step s<sub>i</sub> s<sub>j</sub>  $\implies$  valid-step s<sub>j</sub> s<sub>k</sub>  $\implies$   $\neg$ valid-step  
 s<sub>i</sub> s<sub>k</sub>  
**proof** –  
 assume *assms*: valid-step s<sub>i</sub> s<sub>j</sub> valid-step s<sub>j</sub> s<sub>k</sub>  
 obtain i<sub>i</sub> j<sub>i</sub> i<sub>j</sub> j<sub>j</sub> i<sub>k</sub> j<sub>k</sub> **where** [simp]: s<sub>i</sub> = (i<sub>i</sub>,j<sub>i</sub>) s<sub>j</sub> = (i<sub>j</sub>,j<sub>j</sub>) s<sub>k</sub> = (i<sub>k</sub>,j<sub>k</sub>) **by**  
 force  
 then have step-checker (i<sub>i</sub>,j<sub>i</sub>) (i<sub>j</sub>,j<sub>j</sub>) step-checker (i<sub>j</sub>,j<sub>j</sub>) (i<sub>k</sub>,j<sub>k</sub>)  
 using *assms* step-checker-correct by auto  
 then show  $\neg$ valid-step s<sub>i</sub> s<sub>k</sub>  
 apply (simp add: step-checker-correct[symmetric])  
 apply (elim disjE)  
 apply auto  
 done  
**qed**

**lemma** *knights-path-distinct*: knights-path b ps  $\implies$  distinct ps  
**proof** (induction rule: knights-path.induct)  
 case (2 s<sub>i</sub> b s<sub>j</sub> ps)  
 then have s<sub>i</sub>  $\notin$  set (s<sub>j</sub> # ps)  
 using knights-path-set-eq valid-step-neq by blast  
 then show ?case using 2 by auto  
**qed** auto

**lemma** *knights-path-length*: knights-path b ps  $\implies$  length ps = card b  
 using knights-path-set-eq knights-path-distinct by (metis distinct-card)

**lemma** *knights-path-take*:  
 assumes *knights-path* b ps 0 < k k < length ps  
 shows knights-path (set (take k ps)) (take k ps)  
 using *assms*  
**proof** (induction arbitrary: k rule: knights-path.induct)  
 case (2 s<sub>i</sub> b s<sub>j</sub> ps)  
 then have k = 1  $\vee$  k = 2  $\vee$  2 < k by force



```

then show ?case
  using 2
proof (elim disjE)
  assume k = 2
  then have take k (s_i#s_j#ps) = [s_i,s_j] s_i ∉ {s_j} using 2 valid-step-neq by
auto
  then show ?thesis using 2 knights-path.intros by auto
next
  assume 2 < k
  then have k-simps: k-2 = k-1-1 0 < k-2 k-2 < length ps and
    take-simp1: take k (s_i#s_j#ps) = s_i#take (k-1) (s_j#ps) and
    take-simp2: take k (s_i#s_j#ps) = s_i#s_j#take (k-1-1) ps
  using assms 2 take-Cons'[of k s_i s_j#ps] take-Cons'[of k-1 s_j ps] by auto
  then have knights-path (set (take (k-1) (s_j#ps))) (take (k-1) (s_j#ps))
  using 2 k-simps by auto
  then have kp: knights-path (set (take (k-1) (s_j#ps))) (s_j#take (k-2) ps)
  using take-Cons'[of k-1 s_j ps] by (auto simp: k-simps elim: knights-path.cases)

  have no-mem: s_i ∉ set (take (k-1) (s_j#ps))
  using 2 set-take-subset[of k-1 s_j#ps] knights-path-set-eq by blast
  have knights-path (set (take (k-1) (s_j#ps)) ∪ {s_i}) (s_i#s_j#take (k-2) ps)
  using knights-path.intros(2)[OF no-mem ⟨valid-step s_i s_j⟩ kp] by auto
  then show ?thesis using k-simps take-simp2 knights-path-set-eq by metis
qed (auto intro: knights-path.intros)
qed auto

lemma knights-path-drop:
  assumes knights-path b ps 0 < k k < length ps
  shows knights-path (set (drop k ps)) (drop k ps)
  using assms
proof (induction arbitrary: k rule: knights-path.induct)
  case (2 s_i b s_j ps)
  then have (k = 1 ∧ ps = []) ∨ (k = 1 ∧ ps ≠ []) ∨ 1 < k by force
  then show ?case
  using 2
proof (elim disjE)
  assume k = 1 ∧ ps ≠ []
  then show ?thesis using 2 knights-path-set-eq by force
next
  assume 1 < k
  then have 0 < k-1 k-1 < length (s_j#ps) drop k (s_i#s_j#ps) = drop (k-1)
(s_j#ps)
  using assms 2 drop-Cons'[of k s_i s_j#ps] by auto
  then show ?thesis
  using 2 by auto
qed (auto intro: knights-path.intros)
qed auto

```

A Knight's path can be split to form two new disjoint Knight's paths.

**corollary** *knights-path-split*:

**assumes** *knights-path*  $b$   $ps$   $0 < k$   $k < \text{length } ps$   
**shows**  
 $\exists b_1 b_2. \text{knights-path } b_1 (\text{take } k \text{ } ps) \wedge \text{knights-path } b_2 (\text{drop } k \text{ } ps) \wedge b_1 \cup b_2 = b$   
 $\wedge b_1 \cap b_2 = \{\}$   
**using** *assms*  
**proof** –  
**let**  $?b_1 = \text{set } (\text{take } k \text{ } ps)$   
**let**  $?b_2 = \text{set } (\text{drop } k \text{ } ps)$   
**have**  $kp1: \text{knights-path } ?b_1 (\text{take } k \text{ } ps)$  **and**  $kp2: \text{knights-path } ?b_2 (\text{drop } k \text{ } ps)$   
**using** *assms* *knights-path-take* *knights-path-drop* **by** *auto*  
**have**  $\text{union}: ?b_1 \cup ?b_2 = b$   
**using** *assms* *knights-path-set-eq* **by** (*metis* *append-take-drop-id* *set-append*)  
**have**  $\text{inter}: ?b_1 \cap ?b_2 = \{\}$   
**using** *assms* *knights-path-distinct* **by** (*metis* *append-take-drop-id* *distinct-append*)  
**show** *thesis* **using**  $kp1$   $kp2$   $\text{union}$   $\text{inter}$  **by** *auto*  
**qed**

Append two disjoint Knight's paths.

**corollary** *knights-path-append*:

**assumes** *knights-path*  $b_1$   $ps_1$  *knights-path*  $b_2$   $ps_2$   $b_1 \cap b_2 = \{\}$  *valid-step* (*last*  $ps_1$ ) (*hd*  $ps_2$ )  
**shows** *knights-path*  $(b_1 \cup b_2)$   $(ps_1 @ ps_2)$   
**using** *assms*  
**proof** (*induction* *arbitrary*:  $ps_2$   $b_2$  *rule*: *knights-path.induct*)  
**case** ( $1$   $s_i$ )  
**then** **have**  $s_i \notin b_2$   $ps_2 \neq []$  *valid-step*  $s_i$  (*hd*  $ps_2$ ) *knights-path*  $b_2$  (*hd*  $ps_2 \# tl$   $ps_2$ )  
  
**using** *knights-path-non-nil* **by** *auto*  
**then** **have** *knights-path*  $(b_2 \cup \{s_i\})$   $(s_i \# hd$   $ps_2 \# tl$   $ps_2)$   
**using** *knights-path.intros* **by** *blast*  
**then** **show** *case* **using**  $\langle ps_2 \neq [] \rangle$  **by** *auto*  
**next**  
**case** ( $2$   $s_i$   $b_1$   $s_j$   $ps_1$ )  
**then** **have**  $s_i \notin b_1 \cup b_2$  *valid-step*  $s_i$   $s_j$  *knights-path*  $(b_1 \cup b_2)$   $(s_j \# ps_1 @ ps_2)$  **by** *auto*  
**then** **have** *knights-path*  $(b_1 \cup b_2 \cup \{s_i\})$   $(s_i \# s_j \# ps_1 @ ps_2)$   
**using** *knights-path.intros* **by** *auto*  
**then** **show** *case* **by** *auto*  
**qed**

**lemma** *valid-step-rev*: *valid-step*  $s_i$   $s_j \implies \text{valid-step } s_j$   $s_i$

**using** *step-checker-correct* *step-checker-rev* **by** (*metis* *prod.exhaust-sel*)

Reverse a Knight's path.

**corollary** *knights-path-rev*:

**assumes** *knights-path*  $b$   $ps$   
**shows** *knights-path*  $b$  (*rev*  $ps$ )  
**using** *assms*

```

proof (induction rule: knights-path.induct)
  case (2 si b sj ps)
  then have knights-path {si} [si] b ∩ {si} = {} valid-step (last (rev (sj # ps)))
  (hd [si])
  using valid-step-rev by (auto intro: knights-path.intros)
  then have knights-path (b ∪ {si}) ((rev (sj # ps))@[si])
  using 2 knights-path-append by blast
  then show ?case by auto
qed (auto intro: knights-path.intros)

```

Reverse a Knight's circuit.

```

corollary knights-circuit-rev:
  assumes knights-circuit b ps
  shows knights-circuit b (rev ps)
  using assms knights-path-rev valid-step-rev
  unfolding knights-circuit-def by (auto simp: hd-rev last-rev)

```

```

lemma knights-circuit-rotate1:
  assumes knights-circuit b (si # ps)
  shows knights-circuit b (ps@[si])
proof (cases ps = [])
  case True
  then show ?thesis using assms by auto
next
  case False
  have kp1: knights-path b (si # ps) valid-step (last (si # ps)) (hd (si # ps))
  using assms unfolding knights-circuit-def by auto
  then have kp-elim: si ∉ (b - {si}) valid-step si (hd ps) knights-path (b - {si})
  ps
  using ⟨ps ≠ []⟩ by (auto elim: knights-path.cases)
  then have vs': valid-step (last (ps@[si])) (hd (ps@[si]))
  using ⟨ps ≠ []⟩ valid-step-rev by auto

  have kp2: knights-path {si} [si] (b - {si}) ∩ {si} = {}
  by (auto intro: knights-path.intros)

  have vs: valid-step (last ps) (hd [si])
  using ⟨ps ≠ []⟩ ⟨valid-step (last (si # ps)) (hd (si # ps))⟩ by auto

  have (b - {si}) ∪ {si} = b
  using kp1 kp-elim knights-path-set-eq by force
  then show ?thesis
  unfolding knights-circuit-def
  using vs knights-path-append[OF ⟨knights-path (b - {si}) ps⟩ kp2] vs' by auto
qed

```

A Knight's circuit can be rotated to start at any square on the board.

```

lemma knights-circuit-rotate-to:
  assumes knights-circuit b ps hd (drop k ps) = si k < length ps
  shows  $\exists ps'. \text{knights-circuit } b \text{ } ps' \wedge \text{hd } ps' = s_i$ 
  using assms
proof (induction k arbitrary: b ps)
  case (Suc k)
  let ?sj=hd ps
  let ?ps'=tl ps
  show ?case
  proof (cases si = ?sj)
    case True
    then show ?thesis using Suc by auto
  next
  case False
  then have ?ps' ≠ []
  using Suc by (metis drop-Nil drop-Suc drop-eq-Nil2 le-antisym nat-less-le)
  then have knights-circuit b (?sj#?ps')
  using Suc by (metis list.exhaust-sel tl-Nil)
  then have knights-circuit b (?ps'@[?sj]) hd (drop k (?ps'@[?sj])) = si
  using Suc knights-circuit-rotate1 by (auto simp: drop-Suc)
  then show ?thesis using Suc by auto
qed
qed auto

```

For positive boards (1,1) can only have (2,3) and (3,2) as a neighbour.

```

lemma valid-step-1-1:
  assumes valid-step (1,1) (i,j) i > 0 j > 0
  shows  $(i,j) = (2,3) \vee (i,j) = (3,2)$ 
  using assms unfolding valid-step-def by auto

```

```

lemma list-len-g-1-split:  $\text{length } xs > 1 \implies \exists x_1 x_2 xs'. xs = x_1 \# x_2 \# xs'$ 
proof (induction xs)
  case (Cons x xs)
  then have length xs > 0 by auto
  then have length xs ≥ 1 by presburger
  then have length xs = 1 ∨ length xs > 1 by auto
  then show ?case
  proof (elim disjE)
    assume length xs = 1
    then obtain x1 where [simp]: xs = [x1]
    using length-Suc-conv[of xs 0] by auto
    then show ?thesis by auto
  next
  assume  $1 < \text{length } xs$ 
  then show ?thesis using Cons by auto
qed
qed auto

```

```

lemma list-len-g-3-split:  $\text{length } xs > 3 \implies \exists x_1 x_2 xs' x_3. xs = x_1 \# x_2 \# xs' \@[x_3]$ 

```

```

proof (induction xs)
  case (Cons x xs)
  then have length xs = 3  $\vee$  length xs > 3 by auto
  then show ?case
  proof (elim disjE)
    assume length xs = 3
    then obtain x1 xs1 where [simp]: xs = x1#xs1 length xs1 = 2
      using length-Suc-conv[of xs 2] by auto
    then obtain x2 xs2 where [simp]: xs1 = x2#xs2 length xs2 = 1
      using length-Suc-conv[of xs1 1] by auto
    then obtain x3 where [simp]: xs2 = [x3]
      using length-Suc-conv[of xs2 0] by auto
    then show ?thesis by auto
  next
    assume length xs > 3
    then show ?thesis using Cons by auto
  qed
qed auto

```

Any Knight's circuit on a positive board can be rotated to start with (1,1) and end with (3,2).

**corollary** rotate-knights-circuit:

```

assumes knights-circuit (board n m) ps min n m  $\geq$  5
shows  $\exists$  ps. knights-circuit (board n m) ps  $\wedge$  hd ps = (1,1)  $\wedge$  last ps = (3,2)
using assms
proof –
  let ?b=board n m
  have knights-path ?b ps
    using assms unfolding knights-circuit-def by auto
  then have (1,1)  $\in$  set ps
    using assms knights-path-set-eq by (auto simp: board-def)
  then obtain k where hd (drop k ps) = (1,1) k < length ps
    by (metis hd-drop-conv-nth in-set-conv-nth)
  then obtain psr where psr-prems: knights-circuit ?b psr hd psr = (1,1)
    using assms knights-circuit-rotate-to by blast
  then have kp: knights-path ?b psr and valid-step (last psr) (1,1)
    unfolding knights-circuit-def by auto

  have (1,1)  $\in$  ?b (1,2)  $\in$  ?b (1,3)  $\in$  ?b
    using assms unfolding board-def by auto
  then have (1,1)  $\in$  set psr (1,2)  $\in$  set psr (1,3)  $\in$  set psr
    using kp knights-path-set-eq by auto

  have 3 < card ?b
    using assms board-leq-subset card-board[of 5 5]
      card-mono[OF board-finite[of n m], of board 5 5] by auto
  then have 3 < length psr
    using knights-path-length kp by auto
  then obtain sj ps' sk where [simp]: psr = (1,1)#sj#ps'@[sk]

```

```

    using ⟨hd psr = (1,1)⟩ list-len-g-3-split[of psr] by auto
  have sj ≠ sk
    using kp knights-path-distinct by force

  have vs-sk: valid-step sk (1,1)
    using ⟨valid-step (last psr) (1,1)⟩ by simp

  have vs-sj: valid-step (1,1) sj and kp': knights-path (?b - {(1,1)}) (sj#ps'@[sk])
    using kp by (auto elim: knights-path.cases)

  have sj ∈ set psr sk ∈ set psr by auto
  then have sj ∈ ?b sk ∈ ?b
    using kp knights-path-set-eq by blast+
  then have 0 < fst sj ∧ 0 < snd sj 0 < fst sk ∧ 0 < snd sk
    unfolding board-def by auto
  then have sk = (2,3) ∨ sk = (3,2) sj = (2,3) ∨ sj = (3,2)
    using vs-sk vs-sj valid-step-1-1 valid-step-rev by (metis prod.collapse)+
  then have sk = (3,2) ∨ sj = (3,2)
    using ⟨sj ≠ sk⟩ by auto
  then show ?thesis
  proof (elim disjE)
    assume sk = (3,2)
    then have last psr = (3,2) by auto
    then show ?thesis using psr-prems by auto
  next
    assume sj = (3,2)
    then have vs: valid-step (last ((1,1)#rev (sj#ps'@[sk]))) (hd ((1,1)#rev
(sj#ps'@[sk])))
      unfolding valid-step-def by auto

    have rev-simp: rev (sj#ps'@[sk]) = sk#(rev ps'@[sj]) by auto

    have knights-path (?b - {(1,1)}) (rev (sj#ps'@[sk]))
      using knights-path-rev[OF kp'] by auto
    then have (1,1) ∉ (?b - {(1,1)}) valid-step (1,1) sk
      knights-path (?b - {(1,1)}) (sk#(rev ps'@[sj]))
      using assms vs-sk valid-step-rev by (auto simp: rev-simp)
    then have knights-path (?b - {(1,1)} ∪ {(1,1)}) ((1,1)#sk#(rev ps'@[sj]))
      using knights-path.intros(2)[of (1,1) ?b - {(1,1)} sk (rev ps'@[sj])] by auto
    then have knights-path ?b ((1,1)#rev (sj#ps'@[sk]))
      using assms by (simp add: board-def insert-absorb rev-simp)
    then have knights-circuit ?b ((1,1)#rev (sj#ps'@[sk]))
      unfolding knights-circuit-def using vs by auto
    then show ?thesis
      using ⟨sj = (3,2)⟩ by auto
  qed
qed

```

## 4 Transposing Paths and Boards

### 4.1 Implementation of Path and Board Transposition

**definition** *transpose-square*  $s_i = (\text{case } s_i \text{ of } (i,j) \Rightarrow (j,i))$

**fun** *transpose* :: *path*  $\Rightarrow$  *path* **where**  
  *transpose* [] = []  
| *transpose* ( $s_i \# ps$ ) = (*transpose-square*  $s_i$ ) # *transpose*  $ps$

**definition** *transpose-board* :: *board*  $\Rightarrow$  *board* **where**  
  *transpose-board*  $b \equiv \{(j,i) \mid i\ j. (i,j) \in b\}$

### 4.2 Correctness of Path and Board Transposition

**lemma** *transpose2*: *transpose-square* (*transpose-square*  $s_i$ ) =  $s_i$   
  **unfolding** *transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-nil*:  $ps = [] \iff \text{transpose } ps = []$   
  **using** *transpose.elims* **by** *blast*

**lemma** *transpose-length*: *length*  $ps = \text{length } (\text{transpose } ps)$   
  **by** (*induction ps*) *auto*

**lemma** *hd-transpose*:  $ps \neq [] \implies \text{hd } (\text{transpose } ps) = \text{transpose-square } (\text{hd } ps)$   
  **by** (*induction ps*) (*auto simp: transpose-square-def*)

**lemma** *last-transpose*:  $ps \neq [] \implies \text{last } (\text{transpose } ps) = \text{transpose-square } (\text{last } ps)$

**proof** (*induction ps*)  
  **case** (*Cons*  $s_i$   $ps$ )  
  **then show** ?*case*  
  **proof** (*cases*  $ps = []$ )  
    **case** *True*  
    **then show** ?*thesis* **using** *Cons* **by** (*auto simp: transpose-square-def*)  
  **next**  
  **case** *False*  
  **then show** ?*thesis* **using** *Cons transpose-nil* **by** *auto*  
**qed**  
**qed** *auto*

**lemma** *take-transpose*:  
  **shows** *take*  $k$  (*transpose*  $ps$ ) = *transpose* (*take*  $k$   $ps$ )  
**proof** (*induction ps arbitrary: k*)  
  **case** *Nil*  
  **then show** ?*case* **by** *auto*  
**next**  
  **case** (*Cons*  $s_i$   $ps$ )  
  **then obtain**  $i\ j$  **where**  $s_i = (i,j)$  **by** *force*  
  **then have**  $k = 0 \vee k > 0$  **by** *auto*  
  **then show** ?*case*

**proof** (*elim disjE*)  
**assume**  $k > 0$   
**then show** *?thesis* **using** *Cons.IH* **by** (*auto simp: <math>\langle s\_i = (i,j) \rangle take-Cons'</math>*)  
**qed** *auto*  
**qed**

**lemma** *drop-transpose*:  
**shows**  $drop\ k\ (transpose\ ps) = transpose\ (drop\ k\ ps)$   
**proof** (*induction ps arbitrary: k*)  
**case** *Nil*  
**then show** *?case* **by** *auto*  
**next**  
**case** (*Cons s<sub>i</sub> ps*)  
**then obtain** *i j* **where**  $s_i = (i,j)$  **by** *force*  
**then have**  $k = 0 \vee k > 0$  **by** *auto*  
**then show** *?case*  
**proof** (*elim disjE*)  
**assume**  $k > 0$   
**then show** *?thesis* **using** *Cons.IH* **by** (*auto simp: <math>\langle s\_i = (i,j) \rangle drop-Cons'</math>*)  
**qed** *auto*  
**qed**

**lemma** *transpose-board-correct*:  $s_i \in b \iff (transpose-square\ s_i) \in transpose-board\ b$   
**unfolding** *transpose-board-def transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-board*:  $transpose-board\ (board\ n\ m) = board\ m\ n$   
**unfolding** *board-def* **using** *transpose-board-correct* **by** (*auto simp: transpose-square-def*)

**lemma** *insert-transpose-board*:  
 $insert\ (transpose-square\ s_i)\ (transpose-board\ b) = transpose-board\ (insert\ s_i\ b)$   
**unfolding** *transpose-board-def transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-board2*:  $transpose-board\ (transpose-board\ b) = b$   
**unfolding** *transpose-board-def* **by** *auto*

**lemma** *transpose-union*:  $transpose-board\ (b_1 \cup b_2) = transpose-board\ b_1 \cup transpose-board\ b_2$   
**unfolding** *transpose-board-def* **by** *auto*

**lemma** *transpose-valid-step*:  
 $valid-step\ s_i\ s_j \iff valid-step\ (transpose-square\ s_i)\ (transpose-square\ s_j)$   
**unfolding** *valid-step-def transpose-square-def* **by** (*auto split: prod.splits*)

**lemma** *transpose-knights-path'*:  
**assumes** *knights-path b ps*  
**shows** *knights-path (transpose-board b) (transpose ps)*  
**using** *assms*  
**proof** (*induction rule: knights-path.induct*)



```

case (1  $s_i$ )
then have  $\text{transpose-board } \{s_i\} = \{\text{transpose-square } s_i\}$   $\text{transpose } [s_i] = [\text{transpose-square } s_i]$ 
using  $\text{transpose-board-correct}$  by (auto simp:  $\text{transpose-square-def}$   $\text{split: prod.splits}$ )
then show ?case by (auto intro:  $\text{knight-path.intros}$ )
next
case (2  $s_i$   $b$   $s_j$   $ps$ )
then have  $\text{prems: transpose-square } s_i \notin \text{transpose-board } b$ 
 $\text{valid-step } (\text{transpose-square } s_i) (\text{transpose-square } s_j)$ 
and  $\text{transpose } (s_j \# ps) = \text{transpose-square } s_j \# \text{transpose } ps$ 
using 2  $\text{transpose-board-correct}$   $\text{transpose-valid-step}$  by auto
then show ?case
using 2  $\text{knight-path.intros}(2)[\text{OF } \text{prems}]$   $\text{insert-transpose-board}$  by auto
qed

```

**corollary**  $\text{transpose-knight-path}$ :

```

assumes  $\text{knight-path } (\text{board } n \ m) \ ps$ 
shows  $\text{knight-path } (\text{board } m \ n) \ (\text{transpose } ps)$ 
using  $\text{assms } \text{transpose-knight-path}'[\text{of } \text{board } n \ m \ ps]$  by (auto simp:  $\text{transpose-board}$ )

```

**corollary**  $\text{transpose-knight-circuit}$ :

```

assumes  $\text{knight-circuit } (\text{board } n \ m) \ ps$ 
shows  $\text{knight-circuit } (\text{board } m \ n) \ (\text{transpose } ps)$ 
using  $\text{assms}$ 

```

**proof** –

```

have  $\text{knight-path } (\text{board } n \ m) \ ps$  and  $vs: \text{valid-step } (\text{last } ps) \ (\text{hd } ps)$ 
using  $\text{assms}$   $\text{unfolding } \text{knight-circuit-def}$  by auto
then have  $kp-t: \text{knight-path } (\text{board } m \ n) \ (\text{transpose } ps)$  and  $ps \neq []$ 
using  $\text{transpose-knight-path}$   $\text{knight-path-non-nil}$  by auto
then have  $\text{valid-step } (\text{last } (\text{transpose } ps)) \ (\text{hd } (\text{transpose } ps))$ 
using  $vs$   $\text{hd-transpose}$   $\text{last-transpose}$   $\text{transpose-valid-step}$  by auto
then show ?thesis using  $kp-t$  by (auto simp:  $\text{knight-circuit-def}$ )
qed

```

## 5 Mirroring Paths and Boards

### 5.1 Implementation of Path and Board Mirroring

```

abbreviation  $\text{min1 } ps \equiv \text{Min } ((fst) \ ' \ \text{set } ps)$ 
abbreviation  $\text{max1 } ps \equiv \text{Max } ((fst) \ ' \ \text{set } ps)$ 
abbreviation  $\text{min2 } ps \equiv \text{Min } ((snd) \ ' \ \text{set } ps)$ 
abbreviation  $\text{max2 } ps \equiv \text{Max } ((snd) \ ' \ \text{set } ps)$ 

```

**definition**  $\text{mirror1-square} :: \text{int} \Rightarrow \text{square} \Rightarrow \text{square}$  **where**

```

 $\text{mirror1-square } n \ s_i = (\text{case } s_i \ \text{of } (i,j) \Rightarrow (n-i,j))$ 

```

**fun**  $\text{mirror1-aux} :: \text{int} \Rightarrow \text{path} \Rightarrow \text{path}$  **where**

```

 $\text{mirror1-aux } n \ [] = []$ 

```

|  $mirror1\text{-aux } n (s_i\#ps) = (mirror1\text{-square } n s_i)\#mirror1\text{-aux } n ps$

**definition**  $mirror1 ps = mirror1\text{-aux } (max1 ps + min1 ps) ps$

**definition**  $mirror1\text{-board} :: int \Rightarrow board \Rightarrow board$  **where**  
 $mirror1\text{-board } n b \equiv \{mirror1\text{-square } n s_i \mid s_i. s_i \in b\}$

**definition**  $mirror2\text{-square} :: int \Rightarrow square \Rightarrow square$  **where**  
 $mirror2\text{-square } m s_i = (case s_i of (i,j) \Rightarrow (i,m-j))$

**fun**  $mirror2\text{-aux} :: int \Rightarrow path \Rightarrow path$  **where**  
 $mirror2\text{-aux } m [] = []$   
|  $mirror2\text{-aux } m (s_i\#ps) = (mirror2\text{-square } m s_i)\#mirror2\text{-aux } m ps$

**definition**  $mirror2 ps = mirror2\text{-aux } (max2 ps + min2 ps) ps$

**definition**  $mirror2\text{-board} :: int \Rightarrow board \Rightarrow board$  **where**  
 $mirror2\text{-board } m b \equiv \{mirror2\text{-square } m s_i \mid s_i. s_i \in b\}$

## 5.2 Correctness of Path and Board Mirroring

**lemma**  $mirror1\text{-board-id}: mirror1\text{-board } (int\ n+1) (board\ n\ m) = board\ n\ m$  (**is -**  
 $= ?b$ )

**proof**

**show**  $mirror1\text{-board } (int\ n+1) ?b \subseteq ?b$

**proof**

**fix**  $s_i'$

**assume**  $assms: s_i' \in mirror1\text{-board } (int\ n+1) ?b$

**then obtain**  $i' j'$  **where**  $[simp]: s_i' = (i',j')$  **by force**

**then have**  $(i',j') \in mirror1\text{-board } (int\ n+1) ?b$

**using**  $assms$  **by auto**

**then obtain**  $i j$  **where**  $(i,j) \in ?b$   $mirror1\text{-square } (int\ n+1) (i,j) = (i',j')$

**unfolding**  $mirror1\text{-board-def}$  **by auto**

**then have**  $1 \leq i \wedge i \leq int\ n\ 1 \leq j \wedge j \leq int\ m$   $i'=(int\ n+1)-i\ j'=j$

**unfolding**  $board-def\ mirror1\text{-square-def}$  **by auto**

**then have**  $1 \leq i' \wedge i' \leq int\ n\ 1 \leq j' \wedge j' \leq int\ m$

**by auto**

**then show**  $s_i' \in ?b$

**unfolding**  $board-def$  **by auto**

**qed**

**next**

**show**  $?b \subseteq mirror1\text{-board } (int\ n+1) ?b$

**proof**

**fix**  $s_i$

**assume**  $assms: s_i \in ?b$

**then obtain**  $i j$  **where**  $[simp]: s_i = (i,j)$  **by force**

**then have**  $(i,j) \in ?b$

**using**  $assms$  **by auto**

**then have**  $1 \leq i \wedge i \leq int\ n\ 1 \leq j \wedge j \leq int\ m$

**unfolding** *board-def* **by** *auto*  
**then obtain**  $i' j'$  **where**  $i'=(int\ n+1)-i\ j'=j$  **by** *auto*  
**then have**  $(i',j') \in ?b\ mirror1-square\ (int\ n+1)\ (i',j') = (i,j)$   
**using**  $\langle 1 \leq i \wedge i \leq int\ n \rangle\ \langle 1 \leq j \wedge j \leq int\ m \rangle$   
**unfolding** *mirror1-square-def* **by** (*auto simp: board-def*)  
**then show**  $s_i \in mirror1-board\ (int\ n+1)\ ?b$   
**unfolding** *mirror1-board-def* **by** *force*  
**qed**  
**qed**

**lemma** *mirror2-board-id*:  $mirror2-board\ (int\ m+1)\ (board\ n\ m) = board\ n\ m$  (*is - = ?b*)

**proof**

**show**  $mirror2-board\ (int\ m+1)\ ?b \subseteq ?b$

**proof**

**fix**  $s_i'$

**assume** *assms*:  $s_i' \in mirror2-board\ (int\ m+1)\ ?b$

**then obtain**  $i' j'$  **where** [*simp*]:  $s_i' = (i',j')$  **by** *force*

**then have**  $(i',j') \in mirror2-board\ (int\ m+1)\ ?b$

**using** *assms* **by** *auto*

**then obtain**  $i\ j$  **where**  $(i,j) \in ?b\ mirror2-square\ (int\ m+1)\ (i,j) = (i',j')$

**unfolding** *mirror2-board-def* **by** *auto*

**then have**  $1 \leq i \wedge i \leq int\ n\ 1 \leq j \wedge j \leq int\ m\ i'=i\ j'=(int\ m+1)-j$

**unfolding** *board-def mirror2-square-def* **by** *auto*

**then have**  $1 \leq i' \wedge i' \leq int\ n\ 1 \leq j' \wedge j' \leq int\ m$

**by** *auto*

**then show**  $s_i' \in ?b$

**unfolding** *board-def* **by** *auto*

**qed**

**next**

**show**  $?b \subseteq mirror2-board\ (int\ m+1)\ ?b$

**proof**

**fix**  $s_i$

**assume** *assms*:  $s_i \in ?b$

**then obtain**  $i\ j$  **where** [*simp*]:  $s_i = (i,j)$  **by** *force*

**then have**  $(i,j) \in ?b$

**using** *assms* **by** *auto*

**then have**  $1 \leq i \wedge i \leq int\ n\ 1 \leq j \wedge j \leq int\ m$

**unfolding** *board-def* **by** *auto*

**then obtain**  $i' j'$  **where**  $i'=i\ j'=(int\ m+1)-j$  **by** *auto*

**then have**  $(i',j') \in ?b\ mirror2-square\ (int\ m+1)\ (i',j') = (i,j)$

**using**  $\langle 1 \leq i \wedge i \leq int\ n \rangle\ \langle 1 \leq j \wedge j \leq int\ m \rangle$

**unfolding** *mirror2-square-def* **by** (*auto simp: board-def*)

**then show**  $s_i \in mirror2-board\ (int\ m+1)\ ?b$

**unfolding** *mirror2-board-def* **by** *force*

**qed**

**qed**

**lemma** *knights-path-min1*:  $knights-path\ (board\ n\ m)\ ps \implies min1\ ps = 1$

**proof** –  
**assume** *assms*: *knights-path* (board *n m*) *ps*  
**then have**  $\min n m \geq 1$   
**using** *knights-path-board-m-n-geq-1* **by** *auto*  
**then have**  $(1,1) \in \text{board } n m$  **and**  $ge-1: \forall (i,j) \in \text{board } n m. i \geq 1$   
**unfolding** *board-def* **by** *auto*  
**then have** *finite*: *finite* ((*fst*) ‘ board *n m*) **and**  
*non-empty*: (*fst*) ‘ board *n m*  $\neq \{\}$  **and**  
*mem-1*:  $1 \in (\text{fst}) \text{ ‘ board } n m$   
**using** *board-finite* **by** *auto* (*metis fstI image-eqI*)  
**then have** *Min* ((*fst*) ‘ board *n m*) = 1  
**using** *ge-1* **by** (*auto simp: Min-eq-iff*)  
**then show** *?thesis*  
**using** *assms knights-path-set-eq* **by** *auto*  
**qed**

**lemma** *knights-path-min2*: *knights-path* (board *n m*) *ps*  $\implies \min2 ps = 1$   
**proof** –  
**assume** *assms*: *knights-path* (board *n m*) *ps*  
**then have**  $\min n m \geq 1$   
**using** *knights-path-board-m-n-geq-1* **by** *auto*  
**then have**  $(1,1) \in \text{board } n m$  **and**  $ge-1: \forall (i,j) \in \text{board } n m. j \geq 1$   
**unfolding** *board-def* **by** *auto*  
**then have** *finite*: *finite* ((*snd*) ‘ board *n m*) **and**  
*non-empty*: (*snd*) ‘ board *n m*  $\neq \{\}$  **and**  
*mem-1*:  $1 \in (\text{snd}) \text{ ‘ board } n m$   
**using** *board-finite* **by** *auto* (*metis sndI image-eqI*)  
**then have** *Min* ((*snd*) ‘ board *n m*) = 1  
**using** *ge-1* **by** (*auto simp: Min-eq-iff*)  
**then show** *?thesis*  
**using** *assms knights-path-set-eq* **by** *auto*  
**qed**

**lemma** *knights-path-max1*: *knights-path* (board *n m*) *ps*  $\implies \max1 ps = \text{int } n$   
**proof** –  
**assume** *assms*: *knights-path* (board *n m*) *ps*  
**then have**  $\min n m \geq 1$   
**using** *knights-path-board-m-n-geq-1* **by** *auto*  
**then have**  $(\text{int } n, 1) \in \text{board } n m$  **and**  $leq-n: \forall (i,j) \in \text{board } n m. i \leq \text{int } n$   
**unfolding** *board-def* **by** *auto*  
**then have** *finite*: *finite* ((*fst*) ‘ board *n m*) **and**  
*non-empty*: (*fst*) ‘ board *n m*  $\neq \{\}$  **and**  
*mem-1*:  $\text{int } n \in (\text{fst}) \text{ ‘ board } n m$   
**using** *board-finite* **by** *auto* (*metis fstI image-eqI*)  
**then have** *Max* ((*fst*) ‘ board *n m*) = *int n*  
**using** *leq-n* **by** (*auto simp: Max-eq-iff*)  
**then show** *?thesis*  
**using** *assms knights-path-set-eq* **by** *auto*  
**qed**

**lemma** *knights-path-max2*: *knights-path* (board *n m*) *ps*  $\implies$  *max2 ps = int m*  
**proof** –  
**assume** *assms*: *knights-path* (board *n m*) *ps*  
**then have** *min n m  $\geq$  1*  
**using** *knights-path-board-m-n-geq-1* **by** *auto*  
**then have**  $(1, \text{int } m) \in \text{board } n \ m$  **and** *leq-m*:  $\forall (i,j) \in \text{board } n \ m. j \leq \text{int } m$   
**unfolding** *board-def* **by** *auto*  
**then have** *finite*: *finite* ((*snd*) ‘board *n m*) **and**  
*non-empty*: (*snd*) ‘board *n m*  $\neq$  {} **and**  
*mem-1*: *int m*  $\in$  (*snd*) ‘board *n m*  
**using** *board-finite* **by** *auto* (*metis sndI image-eqI*)  
**then have** *Max* ((*snd*) ‘board *n m*) = *int m*  
**using** *leq-m* **by** (*auto simp: Max-eq-iff*)  
**then show** *?thesis*  
**using** *assms knights-path-set-eq* **by** *auto*  
**qed**

**lemma** *mirror1-aux-nil*: *ps = []  $\longleftrightarrow$  mirror1-aux m ps = []*  
**using** *mirror1-aux.elims* **by** *blast*

**lemma** *mirror1-nil*: *ps = []  $\longleftrightarrow$  mirror1 ps = []*  
**unfolding** *mirror1-def* **using** *mirror1-aux-nil* **by** *blast*

**lemma** *mirror2-aux-nil*: *ps = []  $\longleftrightarrow$  mirror2-aux m ps = []*  
**using** *mirror2-aux.elims* **by** *blast*

**lemma** *mirror2-nil*: *ps = []  $\longleftrightarrow$  mirror2 ps = []*  
**unfolding** *mirror2-def* **using** *mirror2-aux-nil* **by** *blast*

**lemma** *length-mirror1-aux*: *length ps = length (mirror1-aux n ps)*  
**by** (*induction ps*) *auto*

**lemma** *length-mirror1*: *length ps = length (mirror1 ps)*  
**unfolding** *mirror1-def* **using** *length-mirror1-aux* **by** *auto*

**lemma** *length-mirror2-aux*: *length ps = length (mirror2-aux n ps)*  
**by** (*induction ps*) *auto*

**lemma** *length-mirror2*: *length ps = length (mirror2 ps)*  
**unfolding** *mirror2-def* **using** *length-mirror2-aux* **by** *auto*

**lemma** *mirror1-board-iff*:  $s_i \notin b \longleftrightarrow \text{mirror1-square } n \ s_i \notin \text{mirror1-board } n \ b$   
**unfolding** *mirror1-board-def mirror1-square-def* **by** (*auto split: prod.splits*)

**lemma** *mirror2-board-iff*:  $s_i \notin b \longleftrightarrow \text{mirror2-square } n \ s_i \notin \text{mirror2-board } n \ b$   
**unfolding** *mirror2-board-def mirror2-square-def* **by** (*auto split: prod.splits*)

**lemma** *insert-mirror1-board*:

*insert* (*mirror1-square*  $n$   $s_i$ ) (*mirror1-board*  $n$   $b$ ) = *mirror1-board*  $n$  (*insert*  $s_i$   $b$ )  
**unfolding** *mirror1-board-def* *mirror1-square-def* **by** (*auto split: prod.splits*)

**lemma** *insert-mirror2-board*:

*insert* (*mirror2-square*  $n$   $s_i$ ) (*mirror2-board*  $n$   $b$ ) = *mirror2-board*  $n$  (*insert*  $s_i$   $b$ )  
**unfolding** *mirror2-board-def* *mirror2-square-def* **by** (*auto split: prod.splits*)

**lemma** *valid-step-mirror1*:

*valid-step*  $s_i$   $s_j$   $\longleftrightarrow$  *valid-step* (*mirror1-square*  $n$   $s_i$ ) (*mirror1-square*  $n$   $s_j$ )

**proof**

**assume** *assms*: *valid-step*  $s_i$   $s_j$

**obtain**  $i$   $j$   $i'$   $j'$  **where** [*simp*]:  $s_i = (i,j)$   $s_j = (i',j')$  **by** *force*

**then have** *valid-step* ( $n-i,j$ ) ( $n-i',j'$ )

**using** *assms* **unfolding** *valid-step-def*

**apply** *simp*

**apply** (*elim disjE*)

**apply** *auto*

**done**

**then show** *valid-step* (*mirror1-square*  $n$   $s_i$ ) (*mirror1-square*  $n$   $s_j$ )

**unfolding** *mirror1-square-def* **by** *auto*

**next**

**assume** *assms*: *valid-step* (*mirror1-square*  $n$   $s_i$ ) (*mirror1-square*  $n$   $s_j$ )

**obtain**  $i$   $j$   $i'$   $j'$  **where** [*simp*]:  $s_i = (i,j)$   $s_j = (i',j')$  **by** *force*

**then have** *valid-step* ( $i,j$ ) ( $i',j'$ )

**using** *assms* **unfolding** *valid-step-def* *mirror1-square-def*

**apply** *simp*

**apply** (*elim disjE*)

**apply** *auto*

**done**

**then show** *valid-step*  $s_i$   $s_j$

**unfolding** *mirror1-square-def* **by** *auto*

**qed**

**lemma** *valid-step-mirror2*:

*valid-step*  $s_i$   $s_j$   $\longleftrightarrow$  *valid-step* (*mirror2-square*  $m$   $s_i$ ) (*mirror2-square*  $m$   $s_j$ )

**proof**

**assume** *assms*: *valid-step*  $s_i$   $s_j$

**obtain**  $i$   $j$   $i'$   $j'$  **where** [*simp*]:  $s_i = (i,j)$   $s_j = (i',j')$  **by** *force*

**then have** *valid-step* ( $i,m-j$ ) ( $i',m-j'$ )

**using** *assms* **unfolding** *valid-step-def*

**apply** *simp*

**apply** (*elim disjE*)

**apply** *auto*

**done**

**then show** *valid-step* (*mirror2-square*  $m$   $s_i$ ) (*mirror2-square*  $m$   $s_j$ )

**unfolding** *mirror2-square-def* **by** *auto*

**next**

**assume** *assms*: *valid-step* (*mirror2-square*  $m$   $s_i$ ) (*mirror2-square*  $m$   $s_j$ )

**obtain**  $i$   $j$   $i'$   $j'$  **where** [*simp*]:  $s_i = (i,j)$   $s_j = (i',j')$  **by** *force*

```

then have valid-step (i,j) (i',j')
  using assms unfolding valid-step-def mirror2-square-def
  apply simp
  apply (elim disjE)
  apply auto
  done
then show valid-step si sj
  unfolding mirror1-square-def by auto
qed

```

```

lemma hd-mirror1:
  assumes knights-path (board n m) ps hd ps = (i,j)
  shows hd (mirror1 ps) = (int n+1-i,j)
  using assms
proof -
  have hd (mirror1 ps) = hd (mirror1-aux (int n+1) ps)
    unfolding mirror1-def using assms knights-path-min1 knights-path-max1 by
auto
  also have ... = hd (mirror1-aux (int n+1) ((hd ps)#(tl ps)))
    using assms knights-path-non-nil by (metis list.collapse)
  also have ... = (int n+1-i,j)
    using assms by (auto simp: mirror1-square-def)
  finally show ?thesis .
qed

```

```

lemma last-mirror1-aux:
  assumes ps ≠ [] last ps = (i,j)
  shows last (mirror1-aux n ps) = (n-i,j)
  using assms
proof (induction ps)
  case (Cons si ps)
  then show ?case
    using mirror1-aux-nil Cons by (cases ps = []) (auto simp: mirror1-square-def)
qed auto

```

```

lemma last-mirror1:
  assumes knights-path (board n m) ps last ps = (i,j)
  shows last (mirror1 ps) = (int n+1-i,j)
  unfolding mirror1-def using assms last-mirror1-aux knights-path-non-nil
  by (simp add: knights-path-max1 knights-path-min1)

```

```

lemma hd-mirror2:
  assumes knights-path (board n m) ps hd ps = (i,j)
  shows hd (mirror2 ps) = (i,int m+1-j)
  using assms
proof -
  have hd (mirror2 ps) = hd (mirror2-aux (int m+1) ps)
    unfolding mirror2-def using assms knights-path-min2 knights-path-max2 by
auto

```

**also have** ... =  $hd (mirror2\text{-}aux (int\ m+1) ((hd\ ps)\#(tl\ ps)))$   
**using** *assms knights-path-non-nil* **by** (*metis list.collapse*)  
**also have** ... =  $(i, int\ m+1-j)$   
**using** *assms* **by** (*auto simp: mirror2-square-def*)  
**finally show** ?thesis .  
**qed**

**lemma** *last-mirror2-aux*:  
**assumes**  $ps \neq []$   $last\ ps = (i,j)$   
**shows**  $last (mirror2\text{-}aux\ m\ ps) = (i, m-j)$   
**using** *assms*  
**proof** (*induction ps*)  
**case** (*Cons s<sub>i</sub> ps*)  
**then show** ?case  
**using** *mirror2-aux-nil Cons* **by** (*cases ps = []*) (*auto simp: mirror2-square-def*)  
**qed** *auto*

**lemma** *last-mirror2*:  
**assumes** *knights-path* (*board n m*)  $ps$   $last\ ps = (i,j)$   
**shows**  $last (mirror2\ ps) = (i, int\ m+1-j)$   
**unfolding** *mirror2-def* **using** *assms last-mirror2-aux knights-path-non-nil*  
**by** (*simp add: knights-path-max2 knights-path-min2*)

**lemma** *mirror1-aux-knights-path*:  
**assumes** *knights-path*  $b\ ps$   
**shows** *knights-path* (*mirror1-board n b*) (*mirror1-aux n ps*)  
**using** *assms*  
**proof** (*induction rule: knights-path.induct*)  
**case** (*1 s<sub>i</sub>*)  
**then have** *mirror1-board n {s<sub>i</sub>} = {mirror1-square n s<sub>i</sub>}*  
**unfolding** *mirror1-board-def* **by** *blast*  
**then show** ?case **by** (*auto intro: knights-path.intros*)  
**next**  
**case** (*2 s<sub>i</sub> b s<sub>j</sub> ps*)  
**then have** *prems: mirror1-square n s<sub>i</sub> ∉ mirror1-board n b*  
*valid-step (mirror1-square n s<sub>i</sub>) (mirror1-square n s<sub>j</sub>)*  
**and** *mirror1-aux n (s<sub>j</sub>#ps) = mirror1-square n s<sub>j</sub>#mirror1-aux n ps*  
**using** *2 mirror1-board-iff valid-step-mirror1* **by** *auto*  
**then show** ?case  
**using** *2 knights-path.intros(2)[OF prems] insert-mirror1-board* **by** *auto*  
**qed**

**corollary** *mirror1-knights-path*:  
**assumes** *knights-path* (*board n m*)  $ps$   
**shows** *knights-path* (*board n m*) (*mirror1 ps*)  
**using** *assms*  
**proof** –  
**have** [*simp*]:  $min1\ ps = 1\ max1\ ps = int\ n$   
**using** *assms knights-path-min1 knights-path-max1* **by** *auto*



```

then have mirror1-board (int n+1) (board n m) = (board n m)
using mirror1-board-id by auto
then have knights-path (board n m) (mirror1-aux (int n+1) ps)
using assms mirror1-aux-knights-path[of board n m ps int n+1] by auto
then show ?thesis unfolding mirror1-def by auto
qed

```

```

lemma mirror2-aux-knights-path:
assumes knights-path b ps
shows knights-path (mirror2-board n b) (mirror2-aux n ps)
using assms
proof (induction rule: knights-path.induct)
case (1 si)
then have mirror2-board n {si} = {mirror2-square n si}
unfolding mirror2-board-def by blast
then show ?case by (auto intro: knights-path.intros)
next
case (2 si b sj ps)
then have prems: mirror2-square n si ∉ mirror2-board n b
valid-step (mirror2-square n si) (mirror2-square n sj)
and mirror2-aux n (sj#ps) = mirror2-square n sj#mirror2-aux n ps
using 2 mirror2-board-iff valid-step-mirror2 by auto
then show ?case
using 2 knights-path.intros(2)[OF prems] insert-mirror2-board by auto
qed

```

```

corollary mirror2-knights-path:
assumes knights-path (board n m) ps
shows knights-path (board n m) (mirror2 ps)
proof –
have [simp]: min2 ps = 1 max2 ps = int m
using assms knights-path-min2 knights-path-max2 by auto
then have mirror2-board (int m+1) (board n m) = (board n m)
using mirror2-board-id by auto
then have knights-path (board n m) (mirror2-aux (int m+1) ps)
using assms mirror2-aux-knights-path[of board n m ps int m+1] by auto
then show ?thesis unfolding mirror2-def by auto
qed

```

### 5.3 Rotate Knight's Paths

Transposing (*KnightsTour.transpose*) and mirroring (along first axis *mirror1*) a Knight's path preserves the Knight's path's property. Tranpose+Mirror1 equals a 90deg-clockwise turn.

```

corollary rot90-knights-path:
assumes knights-path (board n m) ps
shows knights-path (board m n) (mirror1 (transpose ps))
using assms transpose-knights-path mirror1-knights-path by auto

```

**lemma** *hd-rot90-knights-path*:  
**assumes** *knights-path* (board  $n$   $m$ )  $ps$   $hd\ ps = (i,j)$   
**shows**  $hd\ (mirror1\ (transpose\ ps)) = (int\ m+1-j,i)$   
**using** *assms*  
**proof** –  
**have**  $hd\ (transpose\ ps) = (j,i)$  *knights-path* (board  $m$   $n$ ) (transpose  $ps$ )  
**using** *assms knights-path-non-nil hd-transpose transpose-knights-path*  
**by** (auto simp: transpose-square-def)  
**then show** *?thesis* **using** *hd-mirror1* **by** auto  
**qed**

**lemma** *last-rot90-knights-path*:  
**assumes** *knights-path* (board  $n$   $m$ )  $ps$   $last\ ps = (i,j)$   
**shows**  $last\ (mirror1\ (transpose\ ps)) = (int\ m+1-j,i)$   
**using** *assms*  
**proof** –  
**have**  $last\ (transpose\ ps) = (j,i)$  *knights-path* (board  $m$   $n$ ) (transpose  $ps$ )  
**using** *assms knights-path-non-nil last-transpose transpose-knights-path*  
**by** (auto simp: transpose-square-def)  
**then show** *?thesis* **using** *last-mirror1* **by** auto  
**qed**

## 6 Translating Paths and Boards

When constructing knight's paths for larger boards multiple knight's paths for smaller boards are concatenated. To concatenate paths the the coordinates in the path need to be translated. Therefore, simple auxiliary functions are provided.

### 6.1 Implementation of Path and Board Translation

Translate the coordinates for a path by  $(k_1,k_2)$ .

**fun** *trans-path* ::  $int \times int \Rightarrow path \Rightarrow path$  **where**  
*trans-path*  $(k_1,k_2)$  [] = []  
| *trans-path*  $(k_1,k_2)$   $((i,j)\#xs) = (i+k_1,j+k_2)\#(trans-path\ (k_1,k_2)\ xs)$

Translate the coordinates of a board by  $(k_1,k_2)$ .

**definition** *trans-board* ::  $int \times int \Rightarrow board \Rightarrow board$  **where**  
*trans-board*  $t\ b \equiv (case\ t\ of\ (k_1,k_2) \Rightarrow \{(i+k_1,j+k_2)|i\ j.\ (i,j) \in b\})$

### 6.2 Correctness of Path and Board Translation

**lemma** *trans-path-length*:  $length\ ps = length\ (trans-path\ (k_1,k_2)\ ps)$   
**by** (induction  $ps$ ) auto

**lemma** *trans-path-non-nil*:  $ps \neq [] \implies trans-path\ (k_1,k_2)\ ps \neq []$   
**by** (induction  $ps$ ) auto

**lemma** *trans-path-correct*:  $(i,j) \in \text{set } ps \iff (i+k_1,j+k_2) \in \text{set } (\text{trans-path } (k_1,k_2) ps)$   
**proof** (*induction ps*)  
  **case** (*Cons s<sub>i</sub> ps*)  
  **then show** ?*case* **by** (*cases s<sub>i</sub>*) *auto*  
**qed** *auto*

**lemma** *trans-path-non-nil-last*:  
 $ps \neq [] \implies \text{last } (\text{trans-path } (k_1,k_2) ps) = \text{last } (\text{trans-path } (k_1,k_2) ((i,j)\#ps))$   
**using** *trans-path-non-nil* **by** (*induction ps*) *auto*

**lemma** *hd-trans-path*:  
**assumes**  $ps \neq []$   $\text{hd } ps = (i,j)$   
**shows**  $\text{hd } (\text{trans-path } (k_1,k_2) ps) = (i+k_1,j+k_2)$   
**using** *assms* **by** (*induction ps*) *auto*

**lemma** *last-trans-path*:  
**assumes**  $ps \neq []$   $\text{last } ps = (i,j)$   
**shows**  $\text{last } (\text{trans-path } (k_1,k_2) ps) = (i+k_1,j+k_2)$   
**using** *assms*  
**proof** (*induction ps*)  
  **case** (*Cons s<sub>i</sub> ps*)  
  **then show** ?*case*  
    **using** *trans-path-non-nil-last[symmetric]*  
    **apply** (*cases s<sub>i</sub>*)  
    **apply** (*cases ps = []*)  
    **apply** *auto*  
  **done**  
**qed** (*auto*)

**lemma** *take-trans*:  
**shows**  $\text{take } k (\text{trans-path } (k_1,k_2) ps) = \text{trans-path } (k_1,k_2) (\text{take } k ps)$   
**proof** (*induction ps arbitrary: k*)  
  **case** *Nil*  
  **then show** ?*case* **by** *auto*  
**next**  
  **case** (*Cons s<sub>i</sub> ps*)  
  **then obtain** *i j* **where**  $s_i = (i,j)$  **by** *force*  
  **then have**  $k = 0 \vee k > 0$  **by** *auto*  
  **then show** ?*case*  
  **proof** (*elim disjE*)  
    **assume**  $k > 0$   
    **then show** ?*thesis* **using** *Cons.IH* **by** (*auto simp: <s<sub>i</sub> = (i,j)> take-Cons'*)  
  **qed** *auto*  
**qed**

**lemma** *drop-trans*:  
**shows**  $\text{drop } k (\text{trans-path } (k_1,k_2) ps) = \text{trans-path } (k_1,k_2) (\text{drop } k ps)$

```

proof (induction ps arbitrary: k)
  case Nil
  then show ?case by auto
next
  case (Cons si ps)
  then obtain i j where si = (i,j) by force
  then have k = 0 ∨ k > 0 by auto
  then show ?case
  proof (elim disjE)
    assume k > 0
    then show ?thesis using Cons.IH by (auto simp: ⟨si = (i,j)⟩ drop-Cons')
  qed auto
qed

lemma trans-board-correct: (i,j) ∈ b ⟷ (i+k1,j+k2) ∈ trans-board (k1,k2) b
  unfolding trans-board-def by auto

lemma board-subset: n1 ≤ n2 ⟹ m1 ≤ m2 ⟹ board n1 m1 ⊆ board n2 m2
  unfolding board-def by auto

Board concatenation

corollary board-concat:
  shows board n m1 ∪ trans-board (0,int m1) (board n m2) = board n (m1+m2)
  (is ?b1 ∪ ?b2 = ?b)
proof
  show ?b1 ∪ ?b2 ⊆ ?b unfolding board-def trans-board-def by auto
next
  show ?b ⊆ ?b1 ∪ ?b2
  proof
    fix x
    assume x ∈ ?b
    then obtain i j where x-split: x = (i,j) 1 ≤ i ∧ i ≤ int n 1 ≤ j ∧ j ≤ int
      (m1+m2)
    unfolding board-def by auto
    then have j ≤ int m1 ∨ (int m1 < j ∧ j ≤ int (m1+m2)) by auto
    then show x ∈ ?b1 ∪ ?b2
  proof
    assume j ≤ int m1
    then show x ∈ ?b1 ∪ ?b2 using x-split unfolding board-def by auto
  next
    assume asm: int m1 < j ∧ j ≤ int (m1+m2)
    then have (i,j-int m1) ∈ board n m2 using x-split unfolding board-def by
      auto
    then show x ∈ ?b1 ∪ ?b2
    using x-split asm trans-board-correct[of i j-int m1 board n m2 0 int m1] by
      auto
  qed
qed
qed

```

**lemma** *transpose-trans-board*:

*transpose-board (trans-board (k<sub>1</sub>,k<sub>2</sub>) b) = trans-board (k<sub>2</sub>,k<sub>1</sub>) (transpose-board b)*  
**unfolding** *transpose-board-def trans-board-def* **by** *blast*

**corollary** *board-concatT*:

**shows** *board n<sub>1</sub> m ∪ trans-board (int n<sub>1</sub>,0) (board n<sub>2</sub> m) = board (n<sub>1</sub>+n<sub>2</sub>) m* **(is**  
*?b<sub>1</sub> ∪ ?b<sub>2</sub> = ?b)*

**proof** –

**let** *?b<sub>1</sub>T=board m n<sub>1</sub>*

**let** *?b<sub>2</sub>T=trans-board (0,int n<sub>1</sub>) (board m n<sub>2</sub>)*

**have** *?b<sub>1</sub> ∪ ?b<sub>2</sub> = transpose-board (?b<sub>1</sub>T ∪ ?b<sub>2</sub>T)*

**using** *transpose-board2 transpose-union transpose-board transpose-trans-board*

**by** *auto*

**also have** *... = transpose-board (board m (n<sub>1</sub>+n<sub>2</sub>))*

**using** *board-concat* **by** *auto*

**also have** *... = board (n<sub>1</sub>+n<sub>2</sub>) m*

**using** *transpose-board* **by** *auto*

**finally show** *?thesis* .

**qed**

**lemma** *trans-valid-step*:

*valid-step (i,j) (i',j') ⇒ valid-step (i+k<sub>1</sub>,j+k<sub>2</sub>) (i'+k<sub>1</sub>,j'+k<sub>2</sub>)*

**unfolding** *valid-step-def* **by** *auto*

Translating a path and a boards preserves the validity.

**lemma** *trans-knights-path*:

**assumes** *knights-path b ps*

**shows** *knights-path (trans-board (k<sub>1</sub>,k<sub>2</sub>) b) (trans-path (k<sub>1</sub>,k<sub>2</sub>) ps)*

**using** *assms*

**proof** (*induction rule: knights-path.induct*)

**case** (*ℓ s<sub>i</sub> b s<sub>j</sub> xs*)

**then obtain** *i j i' j'* **where** *split: s<sub>i</sub> = (i,j) s<sub>j</sub> = (i',j')* **by** *force*

**let** *?s<sub>i</sub>=(i+k<sub>1</sub>,j+k<sub>2</sub>)*

**let** *?s<sub>j</sub>=(i'+k<sub>1</sub>,j'+k<sub>2</sub>)*

**let** *?xs=trans-path (k<sub>1</sub>,k<sub>2</sub>) xs*

**let** *?b=trans-board (k<sub>1</sub>,k<sub>2</sub>) b*

**have** *simps: trans-path (k<sub>1</sub>,k<sub>2</sub>) (s<sub>i</sub>#s<sub>j</sub>#xs) = ?s<sub>i</sub>#?s<sub>j</sub>#?xs*

*?b ∪ {?s<sub>i</sub>} = trans-board (k<sub>1</sub>,k<sub>2</sub>) (b ∪ {s<sub>i</sub>})*

**unfolding** *trans-board-def* **using** *split* **by** *auto*

**have** *?s<sub>i</sub> ∉ ?b valid-step ?s<sub>i</sub> ?s<sub>j</sub> knights-path ?b (?s<sub>j</sub>#?xs)*

**using** *ℓ split trans-valid-step* **by** (*auto simp: trans-board-def*)

**then have** *knights-path (?b ∪ {?s<sub>i</sub>}) (?s<sub>i</sub>#?s<sub>j</sub>#?xs)*

**using** *knights-path.intros* **by** *auto*

**then show** *?case* **using** *simps* **by** *auto*

**qed** (*auto simp: trans-board-def intro: knights-path.intros*)

Predicate that indicates if two squares *s<sub>i</sub>* and *s<sub>j</sub>* are adjacent in *ps*.

**definition** *step-in* :: *path ⇒ square ⇒ square ⇒ bool* **where**

$step\text{-}in\ ps\ s_i\ s_j \equiv (\exists k. 0 < k \wedge k < length\ ps \wedge last\ (take\ k\ ps) = s_i \wedge hd\ (drop\ k\ ps) = s_j)$

**lemma** *step-in-Cons*:  $step\text{-}in\ ps\ s_i\ s_j \implies step\text{-}in\ (s_k\#ps)\ s_i\ s_j$

**proof** –

**assume** *step-in ps s<sub>i</sub> s<sub>j</sub>*

**then obtain** *k* **where**  $0 < k \wedge k < length\ ps \wedge last\ (take\ k\ ps) = s_i \wedge hd\ (drop\ k\ ps) = s_j$

**unfolding** *step-in-def* **by** *auto*

**then have**  $0 < k+1 \wedge k+1 < length\ (s_k\#ps)$

$last\ (take\ (k+1)\ (s_k\#ps)) = s_i \wedge hd\ (drop\ (k+1)\ (s_k\#ps)) = s_j$

**by** *auto*

**then show** *?thesis*

**by** (*auto simp: step-in-def*)

**qed**

**lemma** *step-in-append*:  $step\text{-}in\ ps\ s_i\ s_j \implies step\text{-}in\ (ps@ps')\ s_i\ s_j$

**proof** –

**assume** *step-in ps s<sub>i</sub> s<sub>j</sub>*

**then obtain** *k* **where**  $0 < k \wedge k < length\ ps \wedge last\ (take\ k\ ps) = s_i \wedge hd\ (drop\ k\ ps) = s_j$

**unfolding** *step-in-def* **by** *auto*

**then have**  $0 < k \wedge k < length\ (ps@ps')$

$last\ (take\ k\ (ps@ps')) = s_i \wedge hd\ (drop\ k\ (ps@ps')) = s_j$

**by** *auto*

**then show** *?thesis*

**by** (*auto simp: step-in-def*)

**qed**

**lemma** *step-in-prepend*:  $step\text{-}in\ ps\ s_i\ s_j \implies step\text{-}in\ (ps'\@ps)\ s_i\ s_j$

**using** *step-in-Cons* **by** (*induction ps' arbitrary: ps*) *auto*

**lemma** *step-in-valid-step*:  $knights\text{-}path\ b\ ps \implies step\text{-}in\ ps\ s_i\ s_j \implies valid\text{-}step\ s_i\ s_j$

**proof** –

**assume** *assms: knights-path b ps step-in ps s<sub>i</sub> s<sub>j</sub>*

**then obtain** *k* **where** *k-prems*:  $0 < k \wedge k < length\ ps \wedge last\ (take\ k\ ps) = s_i \wedge hd\ (drop\ k\ ps) = s_j$

**unfolding** *step-in-def* **by** *auto*

**then have**  $k = 1 \vee k > 1$  **by** *auto*

**then show** *?thesis*

**proof** (*elim disjE*)

**assume**  $k = 1$

**then obtain** *ps'* **where**  $ps = s_i\#s_j\#ps'$

**using** *k-prems list-len-g-1-split* **by** *fastforce*

**then show** *?thesis*

**using** *assms* **by** (*auto elim: knights-path.cases*)

**next**

**assume**  $k > 1$

**then have**  $0 < k-1 \wedge k-1 < \text{length } ps$   
**using**  $k\text{-prems}$  **by** *auto*  
**then obtain**  $b$  **where**  $\text{knights-path } b \text{ (drop (k-1) ps)}$   
**using**  $\text{assms knights-path-split}$  **by** *blast*  
  
**obtain**  $ps'$  **where**  $\text{drop (k-1) ps} = s_i \# s_j \# ps'$   
**using**  $k\text{-prems } \langle 0 < k - 1 \wedge k - 1 < \text{length } ps \rangle$   
**by** (*metis Cons-nth-drop-Suc Suc-diff-1 hd-drop-conv-nth last-snoc take-hd-drop*)  
**then show**  $?thesis$   
**using**  $\langle \text{knights-path } b \text{ (drop (k-1) ps)} \rangle$  **by** (*auto elim: knights-path.cases*)  
**qed**  
**qed**

**lemma** *trans-step-in*:  
 $\text{step-in } ps \text{ (i,j) (i',j')} \implies \text{step-in (trans-path (k_1,k_2) ps) (i+k_1,j+k_2) (i'+k_1,j'+k_2)}$   
**proof** –  
**let**  $?ps' = \text{trans-path (k_1,k_2) ps}$   
**assume**  $\text{step-in } ps \text{ (i,j) (i',j')}$   
**then obtain**  $k$  **where**  $0 < k \wedge k < \text{length } ps$   $\text{last (take k ps) = (i,j)}$   $\text{hd (drop k ps) = (i',j')}$   
**unfolding** *step-in-def* **by** *auto*  
**then have**  $\text{take k ps} \neq []$   $\text{drop k ps} \neq []$  **by** *fastforce+*  
**then have**  $0 < k \wedge k < \text{length } ?ps'$   
 $\text{last (take k ?ps')} = (i+k_1,j+k_2)$   $\text{hd (drop k ?ps')} = (i'+k_1,j'+k_2)$   
**using** *trans-path-length*  
 $\text{last-trans-path[OF } \langle \text{take k ps} \neq [] \rangle \langle \text{last (take k ps) = (i,j)} \rangle \text{ take-trans}$   
 $\text{hd-trans-path[OF } \langle \text{drop k ps} \neq [] \rangle \langle \text{hd (drop k ps) = (i',j')} \rangle \text{ drop-trans}$   
**by** *auto*  
**then show**  $?thesis$   
**by** (*auto simp: step-in-def*)  
**qed**

**lemma** *transpose-step-in*:  
 $\text{step-in } ps \text{ s}_i \text{ s}_j \implies \text{step-in (transpose ps) (transpose-square s}_i\text{) (transpose-square s}_j\text{)}$   
 $(\text{is } - \implies \text{step-in } ?psT \text{ ?s}_iT \text{ ?s}_jT)$   
**proof** –  
**assume**  $\text{step-in } ps \text{ s}_i \text{ s}_j$   
**then obtain**  $k$  **where**  
 $k\text{-prems: } 0 < k \wedge k < \text{length } ps$   $\text{last (take k ps) = s}_i$   $\text{hd (drop k ps) = s}_j$   
**unfolding** *step-in-def* **by** *auto*  
**then have**  $\text{non-nil: take k ps} \neq []$   $\text{drop k ps} \neq []$  **by** *fastforce+*  
**have**  $\text{take k } ?psT = \text{transpose (take k ps)}$   $\text{drop k } ?psT = \text{transpose (drop k ps)}$   
**using** *take-transpose drop-transpose* **by** *auto*  
**then have**  $\text{last (take k } ?psT) = ?s}_iT$   $\text{hd (drop k } ?psT) = ?s}_jT$   
**using**  $\text{non-nil k-prems hd-transpose last-transpose}$  **by** *auto*  
**then show**  $\text{step-in } ?psT \text{ ?s}_iT \text{ ?s}_jT$   
**unfolding** *step-in-def* **using**  $k\text{-prems transpose-length}$  **by** *auto*  
**qed**

**lemma** *hd-take*:  $0 < k \implies \text{hd } xs = \text{hd } (\text{take } k \text{ } xs)$

**by** (*induction xs*) *auto*

**lemma** *last-drop*:  $k < \text{length } xs \implies \text{last } xs = \text{last } (\text{drop } k \text{ } xs)$

**by** (*induction xs*) *auto*

### 6.3 Concatenate Knight's Paths and Circuits

Concatenate two knight's path on a  $n \times m$ -board along the 2nd axis if the first path contains the step  $s_i \rightarrow s_j$  and there are valid steps  $s_i \rightarrow \text{hd } ps_2'$  and  $s_j \rightarrow \text{last } ps_2'$ , where  $ps_2'$  is  $ps_2$  is translated by  $m_1$ . An arbitrary step in  $ps_2$  is preserved.

**corollary** *knights-path-split-concat-si-prev*:

**assumes** *knights-path* (*board n m<sub>1</sub>*)  $ps_1$  *knights-path* (*board n m<sub>2</sub>*)  $ps_2$   
*step-in*  $ps_1$   $s_i$   $s_j$  *hd*  $ps_2 = (i_h, j_h)$  *last*  $ps_2 = (i_l, j_l)$  *step-in*  $ps_2$   $(i', j')$   
*valid-step*  $s_i$   $(i_h, \text{int } m_1 + j_h)$  *valid-step*  $(i_l, \text{int } m_1 + j_l)$   $s_j$

**shows**  $\exists ps. \text{knights-path } (\text{board } n \text{ } (m_1 + m_2)) \text{ } ps \wedge \text{hd } ps = \text{hd } ps_1$   
 $\wedge \text{last } ps = \text{last } ps_1 \wedge \text{step-in } ps \text{ } (i, \text{int } m_1 + j) \text{ } (i', \text{int } m_1 + j')$

**using** *assms*

**proof** –

**let**  $?b_1 = \text{board } n \text{ } m_1$

**let**  $?b_2 = \text{board } n \text{ } m_2$

**let**  $?ps_2' = \text{trans-path } (0, \text{int } m_1) \text{ } ps_2$

**let**  $?b' = \text{trans-board } (0, \text{int } m_1) \text{ } ?b_2$

**have**  $kp2'$ : *knights-path*  $?b' \text{ } ?ps_2'$  **using** *assms trans-knights-path* **by** *auto*

**then have**  $?ps_2' \neq []$  **using** *knights-path-non-nil* **by** *auto*

**obtain**  $k$  **where** *k-prems*:

$0 < k < \text{length } ps_1$  *last*  $(\text{take } k \text{ } ps_1) = s_i$  *hd*  $(\text{drop } k \text{ } ps_1) = s_j$

**using** *assms unfolding step-in-def* **by** *auto*

**let**  $?ps = (\text{take } k \text{ } ps_1) @ ?ps_2' @ (\text{drop } k \text{ } ps_1)$

**obtain**  $b_1$   $b_2$  **where** *b-prems*: *knights-path*  $b_1$   $(\text{take } k \text{ } ps_1)$  *knights-path*  $b_2$   $(\text{drop } k \text{ } ps_1)$

$b_1 \cup b_2 = ?b_1$   $b_1 \cap b_2 = \{\}$

**using** *assms*  $\langle 0 < k \rangle \langle k < \text{length } ps_1 \rangle$  *knights-path-split* **by** *blast*

**have** *hd*  $?ps_2' = (i_h, \text{int } m_1 + j_h)$  *last*  $?ps_2' = (i_l, \text{int } m_1 + j_l)$

**using** *assms knights-path-non-nil hd-trans-path last-trans-path* **by** *auto*

**then have** *hd*  $?ps_2' = (i_h, \text{int } m_1 + j_h)$  *last*  $((\text{take } k \text{ } ps_1) @ ?ps_2') = (i_l, \text{int } m_1 + j_l)$

**using**  $\langle ?ps_2' \neq [] \rangle$  **by** *auto*

**then have** *vs*: *valid-step*  $(\text{last } (\text{take } k \text{ } ps_1))$   $(\text{hd } ?ps_2')$

*valid-step*  $(\text{last } ((\text{take } k \text{ } ps_1) @ ?ps_2'))$   $(\text{hd } (\text{drop } k \text{ } ps_1))$

**using** *assms k-prems* **by** *auto*

**have**  $?b_1 \cap ?b' = \{\}$  **unfolding** *board-def trans-board-def* **by** *auto*

**then have**  $b_1 \cap ?b' = \{\} \wedge (b_1 \cup ?b') \cap b_2 = \{\}$  **using** *b-prems* **by** *blast*

**then have** *inter-empty*:  $b_1 \cap ?b' = \{\} \wedge (b_1 \cup ?b') \cap b_2 = \{\}$  **by** *auto*



**have** *knights-path* ( $b_1 \cup ?b'$ ) ((*take k ps<sub>1</sub>*) @ *?ps<sub>2</sub>'*)  
**using** *kp2'* *b-prems inter-empty vs knights-path-append* **by** *auto*  
**then have** *knights-path* ( $b_1 \cup ?b' \cup b_2$ ) *?ps*  
**using** *b-prems inter-empty vs knights-path-append*[**where**  $ps_1 = (\text{take } k \text{ } ps_1) @$   
*?ps<sub>2</sub>'*] **by** *auto*  
**then have** *knights-path* ( $?b_1 \cup ?b'$ ) *?ps*  
**using** *b-prems Un-commute Un-assoc* **by** *metis*  
**then have** *kp*: *knights-path* (*board n (m<sub>1</sub>+m<sub>2</sub>)*) *?ps*  
**using** *board-concat*[*of n m<sub>1</sub> m<sub>2</sub>*] **by** *auto*

**have** *hd*:  $hd \text{ } ?ps = hd \text{ } ps_1$   
**using** *assms*  $\langle 0 < k \rangle$  *knights-path-non-nil hd-take* **by** *auto*

**have** *last*:  $last \text{ } ?ps = last \text{ } ps_1$   
**using** *assms*  $\langle k < length \text{ } ps_1 \rangle$  *knights-path-non-nil last-drop* **by** *auto*

**have** *m-simps*:  $j + int \text{ } m_1 = int \text{ } m_1 + j \text{ } j' + int \text{ } m_1 = int \text{ } m_1 + j'$  **by** *auto*  
**have** *si*: *step-in* *?ps* ( $i, int \text{ } m_1 + j$ ) ( $i', int \text{ } m_1 + j'$ )  
**using** *assms* *step-in-append*[*OF step-in-prepend*[*OF trans-step-in*],  
*of ps<sub>2</sub> i j i' j' take k ps<sub>1</sub> 0 int m<sub>1</sub> drop k ps<sub>1</sub>*]  
**by** (*auto simp: m-simps*)

**show** *?thesis* **using** *kp hd last si* **by** *auto*  
**qed**

**lemma** *len1-hd-last*:  $length \text{ } xs = 1 \implies hd \text{ } xs = last \text{ } xs$   
**by** (*induction xs*) *auto*

Weaker version of  $\llbracket \text{knights-path} (\text{board } ?n \text{ } ?m_1) \text{ } ?ps_1; \text{knights-path} (\text{board } ?n \text{ } ?m_2) \text{ } ?ps_2; \text{step-in } ?ps_1 \text{ } ?s_i \text{ } ?s_j; hd \text{ } ?ps_2 = (?i_h, ?j_h); last \text{ } ?ps_2 = (?i_l, ?j_l); \text{step-in } ?ps_2 \text{ } (?i, ?j) (?i', ?j'); \text{valid-step } ?s_i \text{ } (?i_h, int \text{ } ?m_1 + ?j_h); \text{valid-step} (?i_l, int \text{ } ?m_1 + ?j_l) \text{ } ?s_j \rrbracket \implies \exists ps. \text{knights-path} (\text{board } ?n \text{ } (?m_1 + ?m_2)) ps \wedge hd \text{ } ps = hd \text{ } ?ps_1 \wedge last \text{ } ps = last \text{ } ?ps_1 \wedge \text{step-in } ps \text{ } (?i, int \text{ } ?m_1 + ?j) (?i', int \text{ } ?m_1 + ?j')$ .

**corollary** *knights-path-split-concat*:

**assumes** *knights-path* (*board n m<sub>1</sub>*) *ps<sub>1</sub>* *knights-path* (*board n m<sub>2</sub>*) *ps<sub>2</sub>*  
 $step-in \text{ } ps_1 \text{ } s_i \text{ } s_j \text{ } hd \text{ } ps_2 = (i_h, j_h) \text{ } last \text{ } ps_2 = (i_l, j_l)$   
 $valid-step \text{ } s_i \text{ } (i_h, int \text{ } m_1 + j_h) \text{ } valid-step \text{ } (i_l, int \text{ } m_1 + j_l) \text{ } s_j$   
**shows**  $\exists ps. \text{knights-path} (\text{board } n \text{ } (m_1 + m_2)) ps \wedge hd \text{ } ps = hd \text{ } ps_1 \wedge last \text{ } ps = last \text{ } ps_1$

**proof** –

**have**  $length \text{ } ps_2 = 1 \vee length \text{ } ps_2 > 1$   
**using** *assms* *knights-path-non-nil* **by** (*meson length-0-conv less-one linorder-neqE-nat*)  
**then show** *?thesis*  
**proof** (*elim disjE*)  
**let**  $?s_k = (i_h, int \text{ } m_1 + j_h)$   
**assume**  $length \text{ } ps_2 = 1$

```

then have  $(i_h, j_h) = (i, j_l)$ 
  using assms len1-hd-last by metis
then have valid-step  $s_i$   $?s_k$  valid-step  $?s_k$   $s_j$  valid-step  $s_i$   $s_j$ 
  using assms step-in-valid-step by auto
then show ?thesis
  using valid-step-non-transitive by blast
next
assume length  $ps_2 > 1$ 
then obtain  $i_1 j_1 i_2 j_2 ps_2'$  where  $ps_2 = (i_1, j_1) \# (i_2, j_2) \# ps_2'$ 
  using list-len-g-1-split by fastforce
then have last  $(take\ 1\ ps_2) = (i_1, j_1)$  hd  $(drop\ 1\ ps_2) = (i_2, j_2)$  by auto
then have step-in  $ps_2$   $(i_1, j_1)$   $(i_2, j_2)$  using  $\langle length\ ps_2 > 1 \rangle$  by (auto simp: step-in-def)
  then show ?thesis
    using assms knights-path-split-concat-si-prev by blast
qed
qed

```

Concatenate two knight's path on a  $n \times m$ -board along the 1st axis.

**corollary** *knights-path-split-concatT*:

```

assumes knights-path (board  $n_1$   $m$ )  $ps_1$  knights-path (board  $n_2$   $m$ )  $ps_2$ 
  step-in  $ps_1$   $s_i$   $s_j$  hd  $ps_2 = (i_h, j_h)$  last  $ps_2 = (i_l, j_l)$ 
  valid-step  $s_i$   $(int\ n_1 + i_h, j_h)$  valid-step  $(int\ n_1 + i_l, j_l)$   $s_j$ 
shows  $\exists ps. \textit{knights-path}$  (board  $(n_1 + n_2)$   $m$ )  $ps \wedge \textit{hd}\ ps = \textit{hd}\ ps_1 \wedge \textit{last}\ ps = \textit{last}\ ps_1$ 
  using assms
proof –
  let  $?ps_1 T = \textit{transpose}\ ps_1$ 
  let  $?ps_2 T = \textit{transpose}\ ps_2$ 
  have  $kps: \textit{knights-path}$  (board  $m$   $n_1$ )  $?ps_1 T$  knights-path (board  $m$   $n_2$ )  $?ps_2 T$ 
    using assms transpose-knights-path by auto

  let  $?s_i T = \textit{transpose-square}\ s_i$ 
  let  $?s_j T = \textit{transpose-square}\ s_j$ 
  have  $si: \textit{step-in}\ ?ps_1 T\ ?s_i T\ ?s_j T$ 
    using assms transpose-step-in by auto

  have  $ps_1 \neq []$   $ps_2 \neq []$ 
    using assms knights-path-non-nil by auto
  then have hd-last2: hd  $?ps_2 T = (j_h, i_h)$  last  $?ps_2 T = (j_l, i_l)$ 
    using assms hd-transpose last-transpose by (auto simp: transpose-square-def)

  have  $vs: \textit{valid-step}\ ?s_i T\ (j_h, int\ n_1 + i_h)$  valid-step  $(j_l, int\ n_1 + i_l)$   $?s_j T$ 
    using assms transpose-valid-step by (auto simp: transpose-square-def split: prod.splits)

  then obtain  $ps$  where
     $ps\ \textit{prems}: \textit{knights-path}$  (board  $m$   $(n_1 + n_2)$ )  $ps$  hd  $ps = \textit{hd}\ ?ps_1 T$  last  $ps = \textit{last}\ ?ps_1 T$ 

```

```

    using knights-path-split-concat[OF kps si hd-last2 vs] by auto
  then have  $ps \neq []$  using knights-path-non-nil by auto
  let  $?psT=transpose\ ps$ 
  have knights-path (board  $(n_1+n_2)\ m$ )  $?psT\ hd\ ?psT = hd\ ps_1\ last\ ?psT = last$ 
 $ps_1$ 
  using  $\langle ps_1 \neq [] \rangle \langle ps \neq [] \rangle ps-prems\ transpose-knights-path\ hd-transpose\ last-transpose$ 

  by (auto simp: transpose2)
  then show ?thesis by auto
qed

```

Concatenate two Knight's path along the 2nd axis. There is a valid step from the last square in the first Knight's path  $ps_1$  to the first square in the second Knight's path  $ps_2$ .

**corollary** *knights-path-concat*:

```

  assumes knights-path (board  $n\ m_1$ )  $ps_1$  knights-path (board  $n\ m_2$ )  $ps_2$ 
     $hd\ ps_2 = (i_h, j_h)\ valid-step\ (last\ ps_1)\ (i_h, int\ m_1 + j_h)$ 
  shows knights-path (board  $n\ (m_1+m_2)$ )  $(ps_1 @ (trans-path\ (0, int\ m_1)\ ps_2))$ 
proof –
  let  $?ps_2'=trans-path\ (0, int\ m_1)\ ps_2$ 
  let  $?b=trans-board\ (0, int\ m_1)\ (board\ n\ m_2)$ 
  have inter-empty:  $board\ n\ m_1 \cap ?b = \{\}$ 
    unfolding board-def trans-board-def by auto
  have  $hd\ ?ps_2' = (i_h, int\ m_1 + j_h)$ 
    using assms knights-path-non-nil hd-trans-path by auto
  then have kp: knights-path (board  $n\ m_1$ )  $ps_1$  knights-path  $?b\ ?ps_2'$  and
    vs: valid-step  $(last\ ps_1)\ (hd\ ?ps_2')$ 
    using assms trans-knights-path by auto
  then show knights-path (board  $n\ (m_1+m_2)$ )  $(ps_1 @ ?ps_2')$ 
    using knights-path-append[OF kp inter-empty vs] board-concat by auto
qed

```

Concatenate two Knight's path along the 2nd axis. The first Knight's path end in  $(2, m_1-1)$  (lower-right) and the second Knight's paths start in  $(1, 1)$  (lower-left).

**corollary** *knights-path-lr-concat*:

```

  assumes knights-path (board  $n\ m_1$ )  $ps_1$  knights-path (board  $n\ m_2$ )  $ps_2$ 
     $last\ ps_1 = (2, int\ m_1 - 1)\ hd\ ps_2 = (1, 1)$ 
  shows knights-path (board  $n\ (m_1+m_2)$ )  $(ps_1 @ (trans-path\ (0, int\ m_1)\ ps_2))$ 
proof –
  have valid-step  $(last\ ps_1)\ (1, int\ m_1 + 1)$ 
    using assms unfolding valid-step-def by auto
  then show ?thesis
    using assms knights-path-concat by auto
qed

```

Concatenate two Knight's circuits along the 2nd axis. In the first Knight's path the squares  $(2, m_1-1)$  and  $(4, m_1)$  are adjacent and the second Knight's circuit starts in  $(1, 1)$  (lower-left) and end in  $(3, 2)$ .

**corollary** *knights-circuit-lr-concat*:

```

assumes knights-circuit (board n m1) ps1 knights-circuit (board n m2) ps2
          step-in ps1 (2,int m1-1) (4,int m1)
          hd ps2 = (1,1) last ps2 = (3,2) step-in ps2 (2,int m2-1) (4,int m2)
shows ∃ ps. knights-circuit (board n (m1+m2)) ps ∧ step-in ps (2,int (m1+m2)-1)
          (4,int (m1+m2))
proof -
  have kp1: knights-path (board n m1) ps1 and kp2: knights-path (board n m2) ps2

  and vs: valid-step (last ps1) (hd ps1)
  using assms unfolding knights-circuit-def by auto

  have m-simps: int m1 + (int m2-1) = int (m1+m2)-1 int m1 + int m2 = int
  (m1+m2) by auto

  have valid-step (2,int m1-1) (1,int m1+1) valid-step (3,int m1+2) (4,int m1)
  unfolding valid-step-def by auto
  then obtain ps where knights-path (board n (m1+m2)) ps hd ps = hd ps1 last
  ps = last ps1 and
    si: step-in ps (2,int (m1+m2)-1) (4,int (m1+m2))
  using assms kp1 kp2
    knights-path-split-concat-si-prev[of n m1 ps1 m2 ps2 (2,int m1-1)
    (4,int m1) 1 1 3 2 2 int m2-1 4 int m2]

  by (auto simp only: m-simps)
  then have knights-circuit (board n (m1+m2)) ps
  using vs by (auto simp: knights-circuit-def)
  then show ?thesis
  using si by auto
qed

```

## 7 Parsing Paths

In this section functions are implemented to parse and construct paths. The parser converts the matrix representation (*(nat list) list*) used in [1] to a path (*path*).

for debugging

```

fun test-path :: path ⇒ bool where
  test-path (si#sj#xs) = (step-checker si sj ∧ test-path (sj#xs))
| test-path - = True

```

```

fun f-opt :: ('a ⇒ 'a) ⇒ 'a option ⇒ 'a option where
  f-opt - None = None
| f-opt f (Some a) = Some (f a)

```

```

fun add-opt-fst-sq :: int ⇒ square option ⇒ square option where
  add-opt-fst-sq - None = None
| add-opt-fst-sq k (Some (i,j)) = Some (k+i,j)

```

```

fun find-k-in-col :: nat ⇒ nat list ⇒ int option where
  find-k-in-col k [] = None
| find-k-in-col k (c#cs) = (if c = k then Some 1 else f-opt ((+) 1) (find-k-in-col k
cs))

```

```

fun find-k-sqr :: nat ⇒ (nat list) list ⇒ square option where
  find-k-sqr k [] = None
| find-k-sqr k (r#rs) = (case find-k-in-col k r of
  None ⇒ f-opt (λ(i,j). (i+1,j)) (find-k-sqr k rs)
| Some j ⇒ Some (1,j))

```

Auxiliary function to easily parse pre-computed boards from paper.

```

fun to-sqrs :: nat ⇒ (nat list) list ⇒ path option where
  to-sqrs 0 rs = Some []
| to-sqrs k rs = (case find-k-sqr k rs of
  None ⇒ None
| Some si ⇒ f-opt (λps. ps@[si]) (to-sqrs (k-1) rs))

```

```

fun num-elems :: (nat list) list ⇒ nat where
  num-elems (r#rs) = length r * length (r#rs)

```

Convert a matrix (*nat list list*) to a path (*path*). With this function we implicitly define the lower-left corner to be  $(1,1)$  and the upper-right corner to be  $(n,m)$ .

**definition**  $to\text{-}path\ rs \equiv to\text{-}sqrs\ (num\text{-}elems\ rs)\ (rev\ rs)$

Example

```

value to-path
  [[3,22,13,16,5],
  [12,17,4,21,14],
  [23,2,15,6,9],
  [18,11,8,25,20],
  [1,24,19,10,7::nat]]

```

## 8 Knight's Paths for $5 \times m$ -Boards

Given here are knight's paths,  $kp5xmlr$  and  $kp5xmur$ , for the  $(5 \times m)$ -board that start in the lower-left corner for  $m \in \{5,6,7,8,9\}$ . The path  $kp5xmlr$  ends in the lower-right corner, whereas the path  $kp5xmur$  ends in the upper-right corner. The tables show the visited squares numbered in ascending order.

**abbreviation**  $b5x5 \equiv board\ 5\ 5$

A Knight's path for the  $(5 \times 5)$ -board that starts in the lower-left and ends in the lower-right.

3	22	13	16	5
12	17	4	21	14
23	2	15	6	9
18	11	8	25	20
1	24	19	10	7

**abbreviation**  $kp5x5lr \equiv$  the (to-path

[[3,22,13,16,5],  
 [12,17,4,21,14],  
 [23,2,15,6,9],  
 [18,11,8,25,20],  
 [1,24,19,10,7]])

**lemma**  $kp-5x5-lr$ : knights-path  $b5x5$   $kp5x5lr$   
 by (simp only: knights-path-exec-simp) eval

**lemma**  $kp-5x5-lr-hd$ : hd  $kp5x5lr = (1,1)$  by eval

**lemma**  $kp-5x5-lr-last$ : last  $kp5x5lr = (2,4)$  by eval

**lemma**  $kp-5x5-lr-non-nil$ :  $kp5x5lr \neq []$  by eval

A Knight's path for the  $(5 \times 5)$ -board that starts in the lower-left and ends in the upper-right.

7	12	15	20	5
16	21	6	25	14
11	8	13	4	19
22	17	2	9	24
1	10	23	18	3

**abbreviation**  $kp5x5ur \equiv$  the (to-path

[[7,12,15,20,5],  
 [16,21,6,25,14],  
 [11,8,13,4,19],  
 [22,17,2,9,24],  
 [1,10,23,18,3]])

**lemma**  $kp-5x5-ur$ : knights-path  $b5x5$   $kp5x5ur$   
 by (simp only: knights-path-exec-simp) eval

**lemma**  $kp-5x5-ur-hd$ : hd  $kp5x5ur = (1,1)$  by eval

**lemma**  $kp-5x5-ur-last$ : last  $kp5x5ur = (4,4)$  by eval

**lemma**  $kp-5x5-ur-non-nil$ :  $kp5x5ur \neq []$  by eval

**abbreviation**  $b5x6 \equiv$  board 5 6

A Knight's path for the  $(5 \times 6)$ -board that starts in the lower-left and ends in the lower-right.

7	14	21	28	5	12
22	27	6	13	20	29
15	8	17	24	11	4
26	23	2	9	30	19
1	16	25	18	3	10

**abbreviation**  $kp5x6lr \equiv$  the (to-path

[[7,14,21,28,5,12],  
 [22,27,6,13,20,29],  
 [15,8,17,24,11,4],  
 [26,23,2,9,30,19],  
 [1,16,25,18,3,10]])

**lemma**  $kp-5x6-lr$ : *knights-path b5x6 kp5x6lr*

**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma**  $kp-5x6-lr-hd$ :  $hd\ kp5x6lr = (1,1)$  **by** *eval*

**lemma**  $kp-5x6-lr-last$ :  $last\ kp5x6lr = (2,5)$  **by** *eval*

**lemma**  $kp-5x6-lr-non-nil$ :  $kp5x6lr \neq []$  **by** *eval*

A Knight's path for the  $(5 \times 6)$ -board that starts in the lower-left and ends in the upper-right.

3	10	29	20	5	12
28	19	4	11	30	21
9	2	17	24	13	6
18	27	8	15	22	25
1	16	23	26	7	14

**abbreviation**  $kp5x6ur \equiv$  the (to-path

[[3,10,29,20,5,12],  
 [28,19,4,11,30,21],  
 [9,2,17,24,13,6],  
 [18,27,8,15,22,25],  
 [1,16,23,26,7,14]])

**lemma**  $kp-5x6-ur$ : *knights-path b5x6 kp5x6ur*

**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma**  $kp-5x6-ur-hd$ :  $hd\ kp5x6ur = (1,1)$  **by** *eval*

**lemma**  $kp-5x6-ur-last$ :  $last\ kp5x6ur = (4,5)$  **by** *eval*

**lemma**  $kp-5x6-ur-non-nil$ :  $kp5x6ur \neq []$  **by** *eval*

**abbreviation**  $b5x7 \equiv \text{board } 5 \ 7$

A Knight's path for the  $(5 \times 7)$ -board that starts in the lower-left and ends in the lower-right.

3	12	21	30	5	14	23
20	29	4	13	22	31	6
11	2	19	32	7	24	15
28	33	10	17	26	35	8
1	18	27	34	9	16	25

**abbreviation**  $kp5x7lr \equiv \text{the (to-path}$

$[[3,12,21,30,5,14,23],$   
 $[20,29,4,13,22,31,6],$   
 $[11,2,19,32,7,24,15],$   
 $[28,33,10,17,26,35,8],$   
 $[1,18,27,34,9,16,25]])$

**lemma**  $kp\text{-}5x7\text{-}lr$ : *knight's-path*  $b5x7$   $kp5x7lr$

**by** (*simp only: knight's-path-exec-simp*) *eval*

**lemma**  $kp\text{-}5x7\text{-}lr\text{-}hd$ :  $hd$   $kp5x7lr = (1,1)$  **by** *eval*

**lemma**  $kp\text{-}5x7\text{-}lr\text{-}last$ :  $last$   $kp5x7lr = (2,6)$  **by** *eval*

**lemma**  $kp\text{-}5x7\text{-}lr\text{-}non\text{-}nil$ :  $kp5x7lr \neq []$  **by** *eval*

A Knight's path for the  $(5 \times 7)$ -board that starts in the lower-left and ends in the upper-right.

3	32	11	34	5	26	13
10	19	4	25	12	35	6
31	2	33	20	23	14	27
18	9	24	29	16	7	22
1	30	17	8	21	28	15

**abbreviation**  $kp5x7ur \equiv \text{the (to-path}$

$[[3,32,11,34,5,26,13],$   
 $[10,19,4,25,12,35,6],$   
 $[31,2,33,20,23,14,27],$   
 $[18,9,24,29,16,7,22],$   
 $[1,30,17,8,21,28,15]])$

**lemma**  $kp\text{-}5x7\text{-}ur$ : *knight's-path*  $b5x7$   $kp5x7ur$

**by** (*simp only: knight's-path-exec-simp*) *eval*

**lemma**  $kp\text{-}5x7\text{-}ur\text{-}hd$ :  $hd$   $kp5x7ur = (1,1)$  **by** *eval*



**lemma** *kp-5x7-ur-last*: *last kp5x7ur = (4,6)* **by** *eval*

**lemma** *kp-5x7-ur-non-nil*: *kp5x7ur ≠ []* **by** *eval*

**abbreviation** *b5x8*  $\equiv$  *board 5 8*

A Knight's path for the  $(5 \times 8)$ -board that starts in the lower-left and ends in the lower-right.

3	12	37	26	5	14	17	28
34	23	4	13	36	27	6	15
11	2	35	38	25	16	29	18
22	33	24	9	20	31	40	7
1	10	21	32	39	8	19	30

**abbreviation** *kp5x8lr*  $\equiv$  *the (to-path*

*[[3,12,37,26,5,14,17,28],*  
*[34,23,4,13,36,27,6,15],*  
*[11,2,35,38,25,16,29,18],*  
*[22,33,24,9,20,31,40,7],*  
*[1,10,21,32,39,8,19,30]])*

**lemma** *kp-5x8-lr*: *knight's-path b5x8 kp5x8lr*

**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x8-lr-hd*: *hd kp5x8lr = (1,1)* **by** *eval*

**lemma** *kp-5x8-lr-last*: *last kp5x8lr = (2,7)* **by** *eval*

**lemma** *kp-5x8-lr-non-nil*: *kp5x8lr ≠ []* **by** *eval*

A Knight's path for the  $(5 \times 8)$ -board that starts in the lower-left and ends in the upper-right.

33	8	17	38	35	6	15	24
18	37	34	7	16	25	40	5
9	32	29	36	39	14	23	26
30	19	2	11	28	21	4	13
1	10	31	20	3	12	27	22

**abbreviation** *kp5x8ur*  $\equiv$  *the (to-path*

*[[33,8,17,38,35,6,15,24],*  
*[18,37,34,7,16,25,40,5],*  
*[9,32,29,36,39,14,23,26],*  
*[30,19,2,11,28,21,4,13],*  
*[1,10,31,20,3,12,27,22]])*

**lemma** *kp-5x8-ur*: *knight's-path b5x8 kp5x8ur*

**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x8-ur-hd*:  $hd\ kp5x8ur = (1,1)$  **by** *eval*

**lemma** *kp-5x8-ur-last*:  $last\ kp5x8ur = (4,7)$  **by** *eval*

**lemma** *kp-5x8-ur-non-nil*:  $kp5x8ur \neq []$  **by** *eval*

**abbreviation** *b5x9*  $\equiv board\ 5\ 9$

A Knight's path for the  $(5 \times 9)$ -board that starts in the lower-left and ends in the lower-right.

9	4	11	16	23	42	33	36	25
12	17	8	3	32	37	24	41	34
5	10	15	20	43	22	35	26	29
18	13	2	7	38	31	28	45	40
1	6	19	14	21	44	39	30	27

**abbreviation** *kp5x9lr*  $\equiv the\ (to-path$

$[[9,4,11,16,23,42,33,36,25],$   
 $[12,17,8,3,32,37,24,41,34],$   
 $[5,10,15,20,43,22,35,26,29],$   
 $[18,13,2,7,38,31,28,45,40],$   
 $[1,6,19,14,21,44,39,30,27]])$

**lemma** *kp-5x9-lr*: *knight's-path b5x9 kp5x9lr*  
**by** (*simp only: knight's-path-exec-simp*) *eval*

**lemma** *kp-5x9-lr-hd*:  $hd\ kp5x9lr = (1,1)$  **by** *eval*

**lemma** *kp-5x9-lr-last*:  $last\ kp5x9lr = (2,8)$  **by** *eval*

**lemma** *kp-5x9-lr-non-nil*:  $kp5x9lr \neq []$  **by** *eval*

A Knight's path for the  $(5 \times 9)$ -board that starts in the lower-left and ends in the upper-right.

9	4	11	16	27	32	35	40	25
12	17	8	3	36	41	26	45	34
5	10	15	20	31	28	33	24	39
18	13	2	7	42	37	22	29	44
1	6	19	14	21	30	43	38	23

**abbreviation** *kp5x9ur*  $\equiv the\ (to-path$

$[[9,4,11,16,27,32,35,40,25],$   
 $[12,17,8,3,36,41,26,45,34],$   
 $[5,10,15,20,31,28,33,24,39],$   
 $[18,13,2,7,42,37,22,29,44],$

[1,6,19,14,21,30,43,38,23]])  
**lemma** *kp-5x9-ur: knights-path b5x9 kp5x9ur*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-5x9-ur-hd: hd kp5x9ur = (1,1)* **by** *eval*

**lemma** *kp-5x9-ur-last: last kp5x9ur = (4,8)* **by** *eval*

**lemma** *kp-5x9-ur-non-nil: kp5x9ur ≠ []* **by** *eval*

**lemmas** *kp-5xm-lr =*

*kp-5x5-lr kp-5x5-lr-hd kp-5x5-lr-last kp-5x5-lr-non-nil*  
*kp-5x6-lr kp-5x6-lr-hd kp-5x6-lr-last kp-5x6-lr-non-nil*  
*kp-5x7-lr kp-5x7-lr-hd kp-5x7-lr-last kp-5x7-lr-non-nil*  
*kp-5x8-lr kp-5x8-lr-hd kp-5x8-lr-last kp-5x8-lr-non-nil*  
*kp-5x9-lr kp-5x9-lr-hd kp-5x9-lr-last kp-5x9-lr-non-nil*

**lemmas** *kp-5xm-ur =*

*kp-5x5-ur kp-5x5-ur-hd kp-5x5-ur-last kp-5x5-ur-non-nil*  
*kp-5x6-ur kp-5x6-ur-hd kp-5x6-ur-last kp-5x6-ur-non-nil*  
*kp-5x7-ur kp-5x7-ur-hd kp-5x7-ur-last kp-5x7-ur-non-nil*  
*kp-5x8-ur kp-5x8-ur-hd kp-5x8-ur-last kp-5x8-ur-non-nil*  
*kp-5x9-ur kp-5x9-ur-hd kp-5x9-ur-last kp-5x9-ur-non-nil*

For every  $5 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(2,m-1)$  (bottom-right).

**lemma** *knights-path-5xm-lr-exists:*

**assumes**  $m \geq 5$

**shows**  $\exists ps. \text{knights-path (board 5 } m) ps \wedge \text{hd } ps = (1,1) \wedge \text{last } ps = (2, \text{int } m-1)$

**using** *assms*

**proof** (*induction m rule: less-induct*)

**case** (*less m*)

**then have**  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  **by** *auto*

**then show** *?case*

**proof** (*elim disjE*)

**assume**  $m \in \{5,6,7,8,9\}$

**then show** *?thesis* **using** *kp-5xm-lr* **by** *fastforce*

**next**

**assume** *m-ge: 5 ≤ m-5*

**then obtain**  $ps_1$  **where**  $ps_1\text{-IH: knights-path (board 5 (m-5)) } ps_1 \text{ hd } ps_1 = (1,1)$

$\text{last } ps_1 = (2, \text{int } (m-5)-1) \text{ } ps_1 \neq []$

**using** *less.IH[of m-5] knights-path-non-nil* **by** *auto*

**let**  $?ps_2 = \text{kp5x5lr}$

**let**  $?ps_2' = ps_1 @ \text{trans-path (0, int (m-5)) } ?ps_2$

**have** *knights-path b5x5 ?ps\_2 hd ?ps\_2 = (1, 1) ?ps\_2 ≠ [] last ?ps\_2 = (2,4)*

**using** *kp-5xm-lr* **by** *auto*

**then have** *1: knights-path (board 5 m) ?ps\_2'*

```

using m-ge ps1-IH knights-path-lr-concat[of 5 m-5 ps1 5 ?ps2] by auto

have 2: hd ?ps2' = (1,1) using ps1-IH by auto

have last (trans-path (0,int (m-5)) ?ps2) = (2,int m-1)
using m-ge last-trans-path[OF  $\langle ?ps_2 \neq [] \rangle \langle last ?ps_2 = (2,4) \rangle$ ] by auto
then have 3: last ?ps2' = (2,int m-1)
using last-appendR[OF trans-path-non-nil[OF  $\langle ?ps_2 \neq [] \rangle$ ],symmetric] by
metis

show ?thesis using 1 2 3 by auto
qed
qed

```

For every  $5 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(4,m-1)$  (top-right).

**lemma** *knights-path-5xm-ur-exists*:

```

assumes  $m \geq 5$ 
shows  $\exists ps. \text{knights-path (board } 5 \text{ } m) ps \wedge hd ps = (1,1) \wedge last ps = (4, \text{int } m-1)$ 
using assms
proof –
have  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  using assms by auto
then show ?thesis
proof (elim disjE)
assume  $m \in \{5,6,7,8,9\}$ 
then show ?thesis using kp-5xm-ur by fastforce
next
assume m-ge:  $5 \leq m-5$ 
then obtain ps1 where ps-prems: knights-path (board 5 (m-5)) ps1 hd ps1 =
(1,1)

```

*last* *ps<sub>1</sub>* = (2,*int* (m-5)-1) *ps<sub>1</sub>*  $\neq []$

```

using knights-path-5xm-lr-exists[of (m-5)] knights-path-non-nil by auto
let ?ps2=kp5x5ur
let ?ps'=ps1 @ trans-path (0,int (m-5)) ?ps2
have knights-path b5x5 ?ps2 hd ?ps2 = (1, 1) ?ps2  $\neq []$ 
last ?ps2 = (4,4)
using kp-5xm-ur by auto
then have 1: knights-path (board 5 m) ?ps'
using m-ge ps-prems knights-path-lr-concat[of 5 m-5 ps1 5 ?ps2] by auto

have 2: hd ?ps' = (1,1) using ps-prems by auto

have last (trans-path (0,int (m-5)) ?ps2) = (4,int m-1)
using m-ge last-trans-path[OF  $\langle ?ps_2 \neq [] \rangle \langle last ?ps_2 = (4,4) \rangle$ ] by auto
then have 3: last ?ps' = (4,int m-1)
using last-appendR[OF trans-path-non-nil[OF  $\langle ?ps_2 \neq [] \rangle$ ],symmetric] by
metis

```

**show** *?thesis* **using** 1 2 3 **by** *auto*

**qed**  
**qed**

$5 \leq ?m \implies \exists ps. \text{knights-path (board 5 ?m) } ps \wedge \text{hd } ps = (1, 1) \wedge \text{last } ps = (2, \text{int } ?m - 1)$  and  $5 \leq ?m \implies \exists ps. \text{knights-path (board 5 ?m) } ps \wedge \text{hd } ps = (1, 1) \wedge \text{last } ps = (2, \text{int } ?m - 1)$  formalize Lemma 1 from [1].

**lemmas** *knights-path-5xm-exists = knights-path-5xm-lr-exists knights-path-5xm-ur-exists*

## 9 Knight's Paths and Circuits for $6 \times m$ -Boards

**abbreviation** *b6x5*  $\equiv$  *board 6 5*

A Knight's path for the  $(6 \times 5)$ -board that starts in the lower-left and ends in the upper-left.

10	19	4	29	12
3	30	11	20	5
18	9	24	13	28
25	2	17	6	21
16	23	8	27	14
1	26	15	22	7

**abbreviation** *kp6x5ul*  $\equiv$  *the (to-path*

*[[10,19,4,29,12],*  
*[3,30,11,20,5],*  
*[18,9,24,13,28],*  
*[25,2,17,6,21],*  
*[16,23,8,27,14],*  
*[1,26,15,22,7]])*

**lemma** *kp-6x5-ul: knights-path b6x5 kp6x5ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-6x5-ul-hd: hd kp6x5ul = (1,1)* **by** *eval*

**lemma** *kp-6x5-ul-last: last kp6x5ul = (5,2)* **by** *eval*

**lemma** *kp-6x5-ul-non-nil: kp6x5ul  $\neq$  []* **by** *eval*

A Knight's circuit for the  $(6 \times 5)$ -board.

16	9	6	27	18
7	26	17	14	5
10	15	8	19	28
25	30	23	4	13
22	11	2	29	20
1	24	21	12	3

**abbreviation**  $kc6x5 \equiv$  the (to-path

[[16,9,6,27,18],  
 [7,26,17,14,5],  
 [10,15,8,19,28],  
 [25,30,23,4,13],  
 [22,11,2,29,20],  
 [1,24,21,12,3]])

**lemma**  $kc-6x5$ : knights-circuit  $b6x5$   $kc6x5$   
 by (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc-6x5-hd$ : hd  $kc6x5 = (1,1)$  by eval

**lemma**  $kc-6x5-non-nil$ :  $kc6x5 \neq []$  by eval

**abbreviation**  $b6x6 \equiv$  board 6 6

The path given for the  $6 \times 6$ -board that ends in the upper-left is wrong. The Knight cannot move from square 26 to square 27.

14	23	6	28	12	21
7	36	13	22	5	27
24	15	29	35	20	11
30	8	17	26	34	4
16	25	2	32	10	19
1	31	9	18	3	33

**abbreviation**  $kp6x6ul-false \equiv$  the (to-path

[[14,23,6,28,12,21],  
 [7,36,13,22,5,27],  
 [24,15,29,35,20,11],  
 [30,8,17,26,34,4],  
 [16,25,2,32,10,19],  
 [1,31,9,18,3,33]])

**lemma**  $\neg$ knights-path  $b6x6$   $kp6x6ul-false$   
 by (simp only: knights-path-exec-simp) eval

I have computed a correct Knight's path for the  $6 \times 6$ -board that ends in the upper-left. A Knight's path for the  $(6 \times 6)$ -board that starts in the lower-left and ends in the upper-left.

8	25	10	21	6	23
11	36	7	24	33	20
26	9	34	3	22	5
35	12	15	30	19	32
14	27	2	17	4	29
1	16	13	28	31	18

**abbreviation** *kp6x6ul*  $\equiv$  the (to-path

[[8,25,10,21,6,23],  
[11,36,7,24,33,20],  
[26,9,34,3,22,5],  
[35,12,15,30,19,32],  
[14,27,2,17,4,29],  
[1,16,13,28,31,18]])

**lemma** *kp-6x6-ul*: knights-path b6x6 kp6x6ul

by (simp only: knights-path-exec-simp) eval

**lemma** *kp-6x6-ul-hd*: hd kp6x6ul = (1,1) by eval

**lemma** *kp-6x6-ul-last*: last kp6x6ul = (5,2) by eval

**lemma** *kp-6x6-ul-non-nil*: kp6x6ul  $\neq$  [] by eval

A Knight's circuit for the  $(6 \times 6)$ -board.

4	25	34	15	18	7
35	14	5	8	33	16
24	3	26	17	6	19
13	36	23	30	9	32
22	27	2	11	20	29
1	12	21	28	31	10

**abbreviation** *kc6x6*  $\equiv$  the (to-path

[[4,25,34,15,18,7],  
[35,14,5,8,33,16],  
[24,3,26,17,6,19],  
[13,36,23,30,9,32],  
[22,27,2,11,20,29],  
[1,12,21,28,31,10]])

**lemma** *kc-6x6*: knights-circuit b6x6 kc6x6

by (simp only: knights-circuit-exec-simp) eval

**lemma** *kc-6x6-hd*: hd kc6x6 = (1,1) by eval

**lemma** *kc-6x6-non-nil*: kc6x6  $\neq$  [] by eval

**abbreviation** *b6x7*  $\equiv$  board 6 7

A Knight's path for the  $(6 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

18	23	8	39	16	25	6
9	42	17	24	7	40	15
22	19	32	41	38	5	26
33	10	21	28	31	14	37
20	29	2	35	12	27	4
1	34	11	30	3	36	13

**abbreviation**  $kp6x7ul \equiv$  the (to-path

[[18,23,8,39,16,25,6],  
 [9,42,17,24,7,40,15],  
 [22,19,32,41,38,5,26],  
 [33,10,21,28,31,14,37],  
 [20,29,2,35,12,27,4],  
 [1,34,11,30,3,36,13]])

**lemma**  $kp-6x7-ul$ : knights-path  $b6x7$   $kp6x7ul$

by (simp only: knights-path-exec-simp) eval

**lemma**  $kp-6x7-ul-hd$ : hd  $kp6x7ul = (1,1)$  by eval

**lemma**  $kp-6x7-ul-last$ : last  $kp6x7ul = (5,2)$  by eval

**lemma**  $kp-6x7-ul-non-nil$ :  $kp6x7ul \neq []$  by eval

A Knight's circuit for the  $(6 \times 7)$ -board.

26	37	8	17	28	31	6
9	18	27	36	7	16	29
38	25	10	19	30	5	32
11	42	23	40	35	20	15
24	39	2	13	22	33	4
1	12	41	34	3	14	21

**abbreviation**  $kc6x7 \equiv$  the (to-path

[[26,37,8,17,28,31,6],  
 [9,18,27,36,7,16,29],  
 [38,25,10,19,30,5,32],  
 [11,42,23,40,35,20,15],  
 [24,39,2,13,22,33,4],  
 [1,12,41,34,3,14,21]])

**lemma**  $kc-6x7$ : knights-circuit  $b6x7$   $kc6x7$

by (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc-6x7-hd$ : hd  $kc6x7 = (1,1)$  by eval

**lemma**  $kc-6x7-non-nil$ :  $kc6x7 \neq []$  by eval

**abbreviation**  $b6x8 \equiv$  board 6 8



A Knight's path for the  $(6 \times 8)$ -board that starts in the lower-left and ends in the upper-left.

18	31	8	35	16	33	6	45
9	48	17	32	7	46	15	26
30	19	36	47	34	27	44	5
37	10	21	28	43	40	25	14
20	29	2	39	12	23	4	41
1	38	11	22	3	42	13	24

**abbreviation**  $kp6x8ul \equiv$  the (to-path

[[18,31,8,35,16,33,6,45],  
 [9,48,17,32,7,46,15,26],  
 [30,19,36,47,34,27,44,5],  
 [37,10,21,28,43,40,25,14],  
 [20,29,2,39,12,23,4,41],  
 [1,38,11,22,3,42,13,24]])

**lemma**  $kp\text{-}6x8\text{-}ul$ : knights-path  $b6x8$   $kp6x8ul$

by (simp only: knights-path-exec-simp) eval

**lemma**  $kp\text{-}6x8\text{-}ul\text{-}hd$ : hd  $kp6x8ul = (1,1)$  by eval

**lemma**  $kp\text{-}6x8\text{-}ul\text{-}last$ : last  $kp6x8ul = (5,2)$  by eval

**lemma**  $kp\text{-}6x8\text{-}ul\text{-}non\text{-}nil$ :  $kp6x8ul \neq []$  by eval

A Knight's circuit for the  $(6 \times 8)$ -board.

30	35	8	15	28	39	6	13
9	16	29	36	7	14	27	38
34	31	10	23	40	37	12	5
17	48	33	46	11	22	41	26
32	45	2	19	24	43	4	21
1	18	47	44	3	20	25	42

**abbreviation**  $kc6x8 \equiv$  the (to-path

[[30,35,8,15,28,39,6,13],  
 [9,16,29,36,7,14,27,38],  
 [34,31,10,23,40,37,12,5],  
 [17,48,33,46,11,22,41,26],  
 [32,45,2,19,24,43,4,21],  
 [1,18,47,44,3,20,25,42]])

**lemma**  $kc\text{-}6x8$ : knights-circuit  $b6x8$   $kc6x8$

by (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc\text{-}6x8\text{-}hd$ : hd  $kc6x8 = (1,1)$  by eval

**lemma** *kc-6x8-non-nil*:  $kc6x8 \neq []$  **by** *eval*

**abbreviation** *b6x9*  $\equiv$  *board 6 9*

A Knight's path for the  $(6 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

22	45	10	53	20	47	8	35	18
11	54	21	46	9	36	19	48	7
44	23	42	37	52	49	32	17	34
41	12	25	50	27	38	29	6	31
24	43	2	39	14	51	4	33	16
1	40	13	26	3	28	15	30	5

**abbreviation** *kp6x9ul*  $\equiv$  *the (to-path*

*[[22,45,10,53,20,47,8,35,18],*  
*[11,54,21,46,9,36,19,48,7],*  
*[44,23,42,37,52,49,32,17,34],*  
*[41,12,25,50,27,38,29,6,31],*  
*[24,43,2,39,14,51,4,33,16],*  
*[1,40,13,26,3,28,15,30,5]])*

**lemma** *kp-6x9-ul*: *knight's-path b6x9 kp6x9ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-6x9-ul-hd*: *hd kp6x9ul = (1,1)* **by** *eval*

**lemma** *kp-6x9-ul-last*: *last kp6x9ul = (5,2)* **by** *eval*

**lemma** *kp-6x9-ul-non-nil*:  $kp6x9ul \neq []$  **by** *eval*

A Knight's circuit for the  $(6 \times 9)$ -board.

14	49	4	51	24	39	6	29	22
3	52	13	40	5	32	23	42	7
48	15	50	25	38	41	28	21	30
53	2	37	12	33	26	31	8	43
16	47	54	35	18	45	10	27	20
1	36	17	46	11	34	19	44	9

**abbreviation** *kc6x9*  $\equiv$  *the (to-path*

*[[14,49,4,51,24,39,6,29,22],*  
*[3,52,13,40,5,32,23,42,7],*  
*[48,15,50,25,38,41,28,21,30],*  
*[53,2,37,12,33,26,31,8,43],*  
*[16,47,54,35,18,45,10,27,20],*  
*[1,36,17,46,11,34,19,44,9]])*

**lemma** *kc-6x9*: *knight's-circuit b6x9 kc6x9*

by (simp only: knights-circuit-exec-simp) eval

**lemma** *kc-6x9-hd*:  $hd\ kc6x9 = (1,1)$  by eval

**lemma** *kc-6x9-non-nil*:  $kc6x9 \neq []$  by eval

**lemmas** *kp-6xm-ul* =

*kp-6x5-ul kp-6x5-ul-hd kp-6x5-ul-last kp-6x5-ul-non-nil*  
*kp-6x6-ul kp-6x6-ul-hd kp-6x6-ul-last kp-6x6-ul-non-nil*  
*kp-6x7-ul kp-6x7-ul-hd kp-6x7-ul-last kp-6x7-ul-non-nil*  
*kp-6x8-ul kp-6x8-ul-hd kp-6x8-ul-last kp-6x8-ul-non-nil*  
*kp-6x9-ul kp-6x9-ul-hd kp-6x9-ul-last kp-6x9-ul-non-nil*

**lemmas** *kc-6xm* =

*kc-6x5 kc-6x5-hd kc-6x5-non-nil*  
*kc-6x6 kc-6x6-hd kc-6x6-non-nil*  
*kc-6x7 kc-6x7-hd kc-6x7-non-nil*  
*kc-6x8 kc-6x8-hd kc-6x8-non-nil*  
*kc-6x9 kc-6x9-hd kc-6x9-non-nil*

For every  $6 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(5,2)$  (top-left).

**lemma** *knights-path-6xm-ul-exists*:

**assumes**  $m \geq 5$

**shows**  $\exists ps. knights-path\ (board\ 6\ m)\ ps \wedge hd\ ps = (1,1) \wedge last\ ps = (5,2)$

**using** *assms*

**proof** (*induction m rule: less-induct*)

**case** (*less m*)

**then have**  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  by *auto*

**then show** *?case*

**proof** (*elim disjE*)

**assume**  $m \in \{5,6,7,8,9\}$

**then show** *?thesis* using *kp-6xm-ul* by *fastforce*

**next**

**let**  $?ps_1 = kp6x5ul$

**let**  $?b_1 = board\ 6\ 5$

**have**  $ps_1\text{-prems}: knights-path\ ?b_1\ ?ps_1\ hd\ ?ps_1 = (1,1)\ last\ ?ps_1 = (5,2)$

**using** *kp-6xm-ul* by *auto*

**assume**  $m\text{-ge}: 5 \leq m-5$

**then obtain**  $ps_2$  **where**  $ps_2\text{-IH}: knights-path\ (board\ 6\ (m-5))\ ps_2\ hd\ ps_2 = (1,1)$

$last\ ps_2 = (5,2)$

**using** *less.IH[of m-5]* *knights-path-non-nil* by *auto*

**have**  $27 < length\ ?ps_1\ last\ (take\ 27\ ?ps_1) = (2,4)\ hd\ (drop\ 27\ ?ps_1) = (4,5)$

**by** *eval+*

**then have** *step-in*  $?ps_1\ (2,4)\ (4,5)$

**unfolding** *step-in-def* using *zero-less-numeral* by *blast*

**then have** *step-in*  $?ps_1\ (2,4)\ (4,5)$

```

      valid-step (2,4) (1,int 5+1)
      valid-step (5,int 5+2) (4,5)
    unfolding valid-step-def by auto
  then show ?thesis
    using ⟨5 ≤ m-5⟩ ps1-prems ps2-IH knights-path-split-concat[of 6 5 ?ps1 m-5
ps2] by auto
  qed
qed

```

For every  $6 \times m$ -board with  $m \geq 5$  there exists a knight's circuit.

**lemma** *knights-circuit-6xm-exists*:

```

  assumes m ≥ 5
  shows ∃ ps. knights-circuit (board 6 m) ps
  using assms
proof -
  have m ∈ {5,6,7,8,9} ∨ 5 ≤ m-5 using assms by auto
  then show ?thesis
  proof (elim disjE)
    assume m ∈ {5,6,7,8,9}
    then show ?thesis using kc-6xm by fastforce
  next
    let ?ps1=rev kc6x5
    have knights-circuit b6x5 ?ps1 last ?ps1 = (1,1)
      using kc-6xm knights-circuit-rev by (auto simp: last-rev)
    then have ps1-prems: knights-path b6x5 ?ps1 valid-step (last ?ps1) (hd ?ps1)
      unfolding knights-circuit-def using valid-step-rev by auto
    assume m-ge: 5 ≤ m-5
    then obtain ps2 where ps2-prems: knights-path (board 6 (m-5)) ps2 hd ps2
      = (1,1)
      last ps2 = (5,2)
      using knights-path-6xm-ul-exists[of (m-5)] knights-path-non-nil by auto

    have 2 < length ?ps1 last (take 2 ?ps1) = (2,4) hd (drop 2 ?ps1) = (4,5) by
eval+
    then have step-in ?ps1 (2,4) (4,5)
      unfolding step-in-def using zero-less-numeral by blast
    then have step-in ?ps1 (2,4) (4,5)
      valid-step (2,4) (1,int 5+1)
      valid-step (5,int 5+2) (4,5)
      unfolding valid-step-def by auto
    then have ∃ ps. knights-path (board 6 m) ps ∧ hd ps = hd ?ps1 ∧ last ps = last
?ps1
      using m-ge ps1-prems ps2-prems knights-path-split-concat[of 6 5 ?ps1 m-5
ps2] by auto
    then show ?thesis using ps1-prems by (auto simp: knights-circuit-def)
  qed
qed

```

$5 \leq ?m \implies \exists ps. \text{knights-path (board 6 ?m) ps} \wedge \text{hd ps} = (1, 1) \wedge \text{last ps}$

= (5, 2) and  $5 \leq ?m \implies \exists ps. \text{knights-circuit (board } 6 \text{ } ?m) ps$  formalize  
 Lemma 2 from [1].

lemmas *knights-path-6xm-exists = knights-path-6xm-ul-exists knights-circuit-6xm-exists*

## 10 Knight's Paths and Circuits for $8 \times m$ -Boards

abbreviation *b8x5*  $\equiv$  board 8 5

A Knight's path for the  $(8 \times 5)$ -board that starts in the lower-left and ends in the upper-left.

28	7	22	39	26
23	40	27	6	21
8	29	38	25	14
37	24	15	20	5
16	9	30	13	34
31	36	33	4	19
10	17	2	35	12
1	32	11	18	3

abbreviation *kp8x5ul*  $\equiv$  the (to-path

[[28,7,22,39,26],  
 [23,40,27,6,21],  
 [8,29,38,25,14],  
 [37,24,15,20,5],  
 [16,9,30,13,34],  
 [31,36,33,4,19],  
 [10,17,2,35,12],  
 [1,32,11,18,3]])

**lemma** *kp-8x5-ul*: *knights-path b8x5 kp8x5ul*

**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-8x5-ul-hd*: *hd kp8x5ul = (1,1)* **by** *eval*

**lemma** *kp-8x5-ul-last*: *last kp8x5ul = (7,2)* **by** *eval*

**lemma** *kp-8x5-ul-non-nil*: *kp8x5ul  $\neq$  []* **by** *eval*

A Knight's circuit for the  $(8 \times 5)$ -board.

26	7	28	15	24
31	16	25	6	29
8	27	30	23	14
17	32	39	34	5
38	9	18	13	22
19	40	33	4	35
10	37	2	21	12
1	20	11	36	3

**abbreviation**  $kc8x5 \equiv$  the (to-path

[[26,7,28,15,24],  
 [31,16,25,6,29],  
 [8,27,30,23,14],  
 [17,32,39,34,5],  
 [38,9,18,13,22],  
 [19,40,33,4,35],  
 [10,37,2,21,12],  
 [1,20,11,36,3]])

**lemma**  $kc-8x5$ : knights-circuit  $b8x5$   $kc8x5$   
 by (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc-8x5-hd$ : hd  $kc8x5 = (1,1)$  by eval

**lemma**  $kc-8x5-last$ : last  $kc8x5 = (3,2)$  by eval

**lemma**  $kc-8x5-non-nil$ :  $kc8x5 \neq []$  by eval

**lemma**  $kc-8x5-si$ : step-in  $kc8x5$  (2,4) (4,5) (is step-in ?ps - -)

**proof** -

have  $0 < (21::nat) 21 < \text{length } ?ps$  last (take 21 ?ps) = (2,4) hd (drop 21 ?ps)  
 = (4,5)

by eval+

then show ?thesis unfolding step-in-def by blast

qed

**abbreviation**  $b8x6 \equiv$  board 8 6

A Knight's path for the  $(8 \times 6)$ -board that starts in the lower-left and ends in the upper-left.

42	11	26	9	34	13
25	48	43	12	27	8
44	41	10	33	14	35
47	24	45	20	7	28
40	19	32	3	36	15
23	46	21	6	29	4
18	39	2	31	16	37
1	22	17	38	5	30

**abbreviation**  $kp8x6ul \equiv$  the (to-path

[[42,11,26,9,34,13],  
 [25,48,43,12,27,8],  
 [44,41,10,33,14,35],  
 [47,24,45,20,7,28],  
 [40,19,32,3,36,15],  
 [23,46,21,6,29,4],  
 [18,39,2,31,16,37],  
 [1,22,17,38,5,30]])

**lemma**  $kp\text{-}8x6\text{-}ul$ : knights-path  $b8x6$   $kp8x6ul$   
 by (simp only: knights-path-exec-simp) eval

**lemma**  $kp\text{-}8x6\text{-}ul\text{-}hd$ :  $hd$   $kp8x6ul = (1,1)$  by eval

**lemma**  $kp\text{-}8x6\text{-}ul\text{-}last$ :  $last$   $kp8x6ul = (7,2)$  by eval

**lemma**  $kp\text{-}8x6\text{-}ul\text{-}non\text{-}nil$ :  $kp8x6ul \neq []$  by eval

A Knight's circuit for the  $(8 \times 6)$ -board. I have reversed circuit s.t. the circuit steps from  $(2,5)$  to  $(4,6)$  and not the other way around. This makes the proofs easier.

8	29	24	45	12	37
25	46	9	38	23	44
30	7	28	13	36	11
47	26	39	10	43	22
6	31	4	27	14	35
3	48	17	40	21	42
32	5	2	19	34	15
1	18	33	16	41	20

**abbreviation**  $kc8x6 \equiv$  the (to-path

[[8,29,24,45,12,37],  
 [25,46,9,38,23,44],  
 [30,7,28,13,36,11],  
 [47,26,39,10,43,22],  
 [6,31,4,27,14,35],

```
[3,48,17,40,21,42],
[32,5,2,19,34,15],
[1,18,33,16,41,20]]
```

**lemma** *kc-8x6*: *knights-circuit b8x6 kc8x6*  
**by** (*simp only: knights-circuit-exec-simp*) *eval*

**lemma** *kc-8x6-hd*: *hd kc8x6 = (1,1)* **by** *eval*

**lemma** *kc-8x6-non-nil*: *kc8x6 ≠ []* **by** *eval*

**lemma** *kc-8x6-si*: *step-in kc8x6 (2,5) (4,6) (is step-in ?ps -)*  
**proof** –

**have**  $0 < (34::nat) \ 34 < \text{length } ?ps$

**last** (*take 34 ?ps*) = (2,5) **hd** (*drop 34 ?ps*) = (4,6) **by** *eval+*

**then show** *?thesis unfolding step-in-def* **by** *blast*  
**qed**

**abbreviation** *b8x7*  $\equiv$  *board 8 7*

A Knight's path for the  $(8 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

```
38 19 6 55 46 21 8
5 56 39 20 7 54 45
18 37 4 47 34 9 22
3 48 35 40 53 44 33
36 17 52 49 32 23 10
51 2 29 14 41 26 43
16 13 50 31 28 11 24
1 30 15 12 25 42 27
```

**abbreviation** *kp8x7ul*  $\equiv$  *the (to-path*

```
[[38,19,6,55,46,21,8],
[5,56,39,20,7,54,45],
[18,37,4,47,34,9,22],
[3,48,35,40,53,44,33],
[36,17,52,49,32,23,10],
[51,2,29,14,41,26,43],
[16,13,50,31,28,11,24],
[1,30,15,12,25,42,27]])
```

**lemma** *kp-8x7-ul*: *knights-path b8x7 kp8x7ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**lemma** *kp-8x7-ul-hd*: *hd kp8x7ul = (1,1)* **by** *eval*

**lemma** *kp-8x7-ul-last*: *last kp8x7ul = (7,2)* **by** *eval*

**lemma** *kp-8x7-ul-non-nil*: *kp8x7ul ≠ []* **by** *eval*



A Knight's circuit for the  $(8 \times 7)$ -board. I have reversed circuit s.t. the circuit steps from  $(2,6)$  to  $(4,7)$  and not the other way around. This makes the proofs easier.

36	31	18	53	20	29	44
17	54	35	30	45	52	21
32	37	46	19	8	43	28
55	16	7	34	27	22	51
38	33	26	47	6	9	42
3	56	15	12	25	50	23
14	39	2	5	48	41	10
1	4	13	40	11	24	49

**abbreviation**  $kc8x7 \equiv$  the (to-path

```

[[36,31,18,53,20,29,44],
 [17,54,35,30,45,52,21],
 [32,37,46,19,8,43,28],
 [55,16,7,34,27,22,51],
 [38,33,26,47,6,9,42],
 [3,56,15,12,25,50,23],
 [14,39,2,5,48,41,10],
 [1,4,13,40,11,24,49]]

```

**lemma**  $kc-8x7$ : knights-circuit  $b8x7$   $kc8x7$   
**by** (simp only: knights-circuit-exec-simp) eval

**lemma**  $kc-8x7-hd$ : hd  $kc8x7 = (1,1)$  **by** eval

**lemma**  $kc-8x7-non-nil$ :  $kc8x7 \neq []$  **by** eval

**lemma**  $kc-8x7-si$ : step-in  $kc8x7$   $(2,6)$   $(4,7)$  (is step-in ?ps - -)

**proof** -

**have**  $0 < (41::nat)$   $41 < \text{length } ?ps$

$\text{last } (\text{take } 41 ?ps) = (2,6)$   $\text{hd } (\text{drop } 41 ?ps) = (4,7)$  **by** eval+

**then show** ?thesis **unfolding** step-in-def **by** blast

**qed**

**abbreviation**  $b8x8 \equiv$  board 8 8

The path given for the  $8 \times 8$ -board that ends in the upper-left is wrong. The Knight cannot move from square 27 to square 28.

24	11	37	9	26	21	39	7
36	64	24	22	38	8	27	20
12	23	10	53	58	49	6	28
63	35	61	50	55	52	19	40
46	13	54	57	48	59	29	5
34	62	47	60	51	56	41	18
14	45	2	32	16	43	4	30
1	33	15	44	3	31	17	42

**abbreviation** *kp8x8ul-false*  $\equiv$  the (to-path

[[24,11,37,9,26,21,39,7],  
 [36,64,25,22,38,8,27,20],  
 [12,23,10,53,58,49,6,28],  
 [63,35,61,50,55,52,19,40],  
 [46,13,54,57,48,59,29,5],  
 [34,62,47,60,51,56,41,18],  
 [14,45,2,32,16,43,4,30],  
 [1,33,15,44,3,31,17,42]])

**lemma**  $\neg$ knights-path b8x8 kp8x8ul-false  
 by (simp only: knights-path-exec-simp) eval

I have computed a correct Knight's path for the  $8 \times 8$ -board that ends in the upper-left.

38	41	36	27	32	43	20	25
35	64	39	42	21	26	29	44
40	37	6	33	28	31	24	19
5	34	63	14	7	22	45	30
62	13	4	9	58	49	18	23
3	10	61	52	15	8	57	46
12	53	2	59	48	55	50	17
1	60	11	54	51	16	47	56

**abbreviation** *kp8x8ul*  $\equiv$  the (to-path

[[38,41,36,27,32,43,20,25],  
 [35,64,39,42,21,26,29,44],  
 [40,37,6,33,28,31,24,19],  
 [5,34,63,14,7,22,45,30],  
 [62,13,4,9,58,49,18,23],  
 [3,10,61,52,15,8,57,46],  
 [12,53,2,59,48,55,50,17],  
 [1,60,11,54,51,16,47,56]])

**lemma** *kp-8x8-ul: knights-path b8x8 kp8x8ul*  
 by (simp only: knights-path-exec-simp) eval

**lemma** *kp-8x8-ul-hd*: *hd kp8x8ul = (1,1) by eval*

**lemma** *kp-8x8-ul-last*: *last kp8x8ul = (7,2) by eval*

**lemma** *kp-8x8-ul-non-nil*: *kp8x8ul ≠ [] by eval*

A Knight's circuit for the  $(8 \times 8)$ -board.

48	13	30	9	56	45	28	7
31	10	47	50	29	8	57	44
14	49	12	55	46	59	6	27
11	32	37	60	51	54	43	58
36	15	52	63	38	61	26	5
33	64	35	18	53	40	23	42
16	19	2	39	62	21	4	25
1	34	17	20	3	24	41	22

**abbreviation** *kc8x8*  $\equiv$  *the (to-path*

*[[48,13,30,9,56,45,28,7],*  
*[31,10,47,50,29,8,57,44],*  
*[14,49,12,55,46,59,6,27],*  
*[11,32,37,60,51,54,43,58],*  
*[36,15,52,63,38,61,26,5],*  
*[33,64,35,18,53,40,23,42],*  
*[16,19,2,39,62,21,4,25],*  
*[1,34,17,20,3,24,41,22]]*

**lemma** *kc-8x8*: *knights-circuit b8x8 kc8x8*  
*by (simp only: knights-circuit-exec-simp) eval*

**lemma** *kc-8x8-hd*: *hd kc8x8 = (1,1) by eval*

**lemma** *kc-8x8-non-nil*: *kc8x8 ≠ [] by eval*

**lemma** *kc-8x8-si*: *step-in kc8x8 (2,7) (4,8) (is step-in ?ps - -)*

**proof** –

**have**  $0 < (4::nat) < \text{length } ?ps$

$\text{last } (\text{take } 4 ?ps) = (2,7) \text{ hd } (\text{drop } 4 ?ps) = (4,8)$  **by eval+**

**then show** *?thesis unfolding step-in-def by blast*

**qed**

**abbreviation** *b8x9*  $\equiv$  *board 8 9*

A Knight's path for the  $(8 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

32	47	6	71	30	45	8	43	26
5	72	31	46	7	70	27	22	9
48	33	4	29	64	23	44	25	42
3	60	35	62	69	28	41	10	21
34	49	68	65	36	63	24	55	40
59	2	61	16	67	56	37	20	11
50	15	66	57	52	13	18	39	54
1	58	51	14	17	38	53	12	19

**abbreviation**  $kp8x9ul \equiv$  the (to-path

[[32,47,6,71,30,45,8,43,26],  
[5,72,31,46,7,70,27,22,9],  
[48,33,4,29,64,23,44,25,42],  
[3,60,35,62,69,28,41,10,21],  
[34,49,68,65,36,63,24,55,40],  
[59,2,61,16,67,56,37,20,11],  
[50,15,66,57,52,13,18,39,54],  
[1,58,51,14,17,38,53,12,19]])

**lemma**  $kp-8x9-ul$ : knights-path  $b8x9$   $kp8x9ul$

by (simp only: knights-path-exec-simp) eval

**lemma**  $kp-8x9-ul-hd$ :  $hd$   $kp8x9ul = (1,1)$  by eval

**lemma**  $kp-8x9-ul-last$ :  $last$   $kp8x9ul = (7,2)$  by eval

**lemma**  $kp-8x9-ul-non-nil$ :  $kp8x9ul \neq []$  by eval

A Knight's circuit for the  $(8 \times 9)$ -board.

42	19	38	5	36	21	34	7	60
39	4	41	20	63	6	59	22	33
18	43	70	37	58	35	68	61	8
3	40	49	64	69	62	57	32	23
50	17	44	71	48	67	54	9	56
45	2	65	14	27	12	29	24	31
16	51	72	47	66	53	26	55	10
1	46	15	52	13	28	11	30	25

**abbreviation**  $kc8x9 \equiv$  the (to-path

[[42,19,38,5,36,21,34,7,60],  
[39,4,41,20,63,6,59,22,33],  
[18,43,70,37,58,35,68,61,8],  
[3,40,49,64,69,62,57,32,23],  
[50,17,44,71,48,67,54,9,56],  
[45,2,65,14,27,12,29,24,31],  
[16,51,72,47,66,53,26,55,10],

```

[1,46,15,52,13,28,11,30,25]])
lemma kc-8x9: knights-circuit b8x9 kc8x9
  by (simp only: knights-circuit-exec-simp) eval

lemma kc-8x9-hd: hd kc8x9 = (1,1) by eval

lemma kc-8x9-non-nil: kc8x9 ≠ [] by eval

lemma kc-8x9-si: step-in kc8x9 (2,8) (4,9) (is step-in ?ps - -)
proof -
  have  $0 < (55::nat) 55 < length\ ?ps$ 
    last (take 55 ?ps) = (2,8) hd (drop 55 ?ps) = (4,9) by eval+
  then show ?thesis unfolding step-in-def by blast
qed

```

```

lemmas kp-8xm-ul =
  kp-8x5-ul kp-8x5-ul-hd kp-8x5-ul-last kp-8x5-ul-non-nil
  kp-8x6-ul kp-8x6-ul-hd kp-8x6-ul-last kp-8x6-ul-non-nil
  kp-8x7-ul kp-8x7-ul-hd kp-8x7-ul-last kp-8x7-ul-non-nil
  kp-8x8-ul kp-8x8-ul-hd kp-8x8-ul-last kp-8x8-ul-non-nil
  kp-8x9-ul kp-8x9-ul-hd kp-8x9-ul-last kp-8x9-ul-non-nil

```

```

lemmas kc-8xm =
  kc-8x5 kc-8x5-hd kc-8x5-last kc-8x5-non-nil kc-8x5-si
  kc-8x6 kc-8x6-hd kc-8x6-non-nil kc-8x6-si
  kc-8x7 kc-8x7-hd kc-8x7-non-nil kc-8x7-si
  kc-8x8 kc-8x8-hd kc-8x8-non-nil kc-8x8-si
  kc-8x9 kc-8x9-hd kc-8x9-non-nil kc-8x9-si

```

For every  $8 \times m$ -board with  $m \geq 5$  there exists a knight's circuit.

```

lemma knights-circuit-8xm-exists:
  assumes  $m \geq 5$ 
  shows  $\exists ps. knights-circuit\ (board\ 8\ m)\ ps \wedge step-in\ ps\ (2, int\ m-1)\ (4, int\ m)$ 
  using assms
proof (induction m rule: less-induct)
  case (less m)
  then have  $m \in \{5,6,7,8,9\} \vee 5 \leq m-5$  by auto
  then show ?case
  proof (elim disjE)
    assume  $m \in \{5,6,7,8,9\}$ 
    then show ?thesis using kc-8xm by fastforce
  next
  let  $?ps_2 = kc8x5$ 
  let  $?b_2 = board\ 8\ 5$ 
  have  $ps_2-prems: knights-circuit\ ?b_2\ ?ps_2\ hd\ ?ps_2 = (1,1)\ last\ ?ps_2 = (3,2)$ 
  using kc-8xm by auto
  have  $21 < length\ ?ps_2\ last\ (take\ 21\ ?ps_2) = (2, int\ 5-1)\ hd\ (drop\ 21\ ?ps_2) =$ 
 $(4, int\ 5)$ 
  by eval+

```

```

then have si: step-in ?ps2 (2,int 5-1) (4,int 5)
  unfolding step-in-def using zero-less-numeral by blast
assume m-ge: 5 ≤ m-5
then obtain ps1 where ps1-IH: knights-circuit (board 8 (m-5)) ps1
  step-in ps1 (2,int (m-5)-1) (4,int (m-5))
  using less.IH[of m-5] knights-path-non-nil by auto
then show ?thesis
  using m-ge ps2-prems si knights-circuit-lr-concat[of 8 m-5 ps1 5 ?ps2] by
auto
qed
qed

```

For every  $8 \times m$ -board with  $m \geq 5$  there exists a knight's path that starts in  $(1,1)$  (bottom-left) and ends in  $(7,2)$  (top-left).

**lemma** *knights-path-8xm-ul-exists*:

```

assumes m ≥ 5
shows ∃ ps. knights-path (board 8 m) ps ∧ hd ps = (1,1) ∧ last ps = (7,2)
using assms
proof -
  have m ∈ {5,6,7,8,9} ∨ 5 ≤ m-5 using assms by auto
  then show ?thesis
  proof (elim disjE)
    assume m ∈ {5,6,7,8,9}
    then show ?thesis using kp-8xm-ul by fastforce
  next
  let ?ps1=kp8x5ul
  have ps1-prems: knights-path b8x5 ?ps1 hd ?ps1 = (1,1) last ?ps1 = (7,2)
    using kp-8xm-ul by auto
  assume m-ge: 5 ≤ m-5
  then have b-prems: 5 ≤ min 8 (m-5)
    unfolding board-def by auto

  obtain ps2 where knights-circuit (board 8 (m-5)) ps2
    using m-ge knights-circuit-8xm-exists[of (m-5)] knights-path-non-nil by auto
  then obtain ps2' where ps2'-prems': knights-circuit (board 8 (m-5)) ps2'
    hd ps2' = (1,1) last ps2' = (3,2)
    using b-prems ⟨5 ≤ min 8 (m-5)⟩ rotate-knights-circuit by blast
  then have ps2'-path: knights-path (board 8 (m-5)) (rev ps2')
    valid-step (last ps2') (hd ps2') hd (rev ps2') = (3,2) last (rev ps2') = (1,1)
    unfolding knights-circuit-def using knights-path-rev by (auto simp: hd-rev
last-rev)

  have 34 < length ?ps1 last (take 34 ?ps1) = (4,5) hd (drop 34 ?ps1) = (2,4)
by eval+
  then have step-in ?ps1 (4,5) (2,4)
    unfolding step-in-def using zero-less-numeral by blast
  then have step-in ?ps1 (4,5) (2,4)
    valid-step (4,5) (3,int 5+2)
    valid-step (1,int 5+1) (2,4)

```

**unfolding** *valid-step-def* **by** *auto*  
**then have**  $\exists ps. \text{knights-path (board } 8 \ m) \ ps \wedge \text{hd } ps = \text{hd } ?ps_1 \wedge \text{last } ps = \text{last } ?ps_1$   
**using** *m-ge ps<sub>1</sub>-prems ps<sub>2</sub>'-prems' ps<sub>2</sub>'-path*  
*knights-path-split-concat[of 8 5 ?ps<sub>1</sub> m-5 rev ps<sub>2</sub>]* **by** *auto*  
**then show** *?thesis* **using** *ps<sub>1</sub>-prems* **by** *auto*  
**qed**  
**qed**

$5 \leq ?m \implies \exists ps. \text{knights-circuit (board } 8 \ ?m) \ ps \wedge \text{step-in } ps \ (2, \text{int } ?m - 1) \ (4, \text{int } ?m)$  and  $5 \leq ?m \implies \exists ps. \text{knights-path (board } 8 \ ?m) \ ps \wedge \text{hd } ps = (1, 1) \wedge \text{last } ps = (7, 2)$  formalize Lemma 3 from [1].

**lemmas** *knights-path-8xm-exists = knights-circuit-8xm-exists knights-path-8xm-ul-exists*

## 11 Knight's Paths and Circuits for $n \times m$ -Boards

In this section the desired theorems are proved. The proof uses the previous lemmas to construct paths and circuits for arbitrary  $n \times m$ -boards.

A Knight's path for the  $(5 \times 5)$ -board that starts in the lower-left and ends in the upper-left.

7	20	9	14	5
10	25	6	21	16
19	8	15	4	13
24	11	2	17	22
1	18	23	12	3

**abbreviation** *kp5x5ul*  $\equiv$  *the (to-path*

$[[7,20,9,14,5],$   
 $[10,25,6,21,16],$   
 $[19,8,15,4,13],$   
 $[24,11,2,17,22],$   
 $[1,18,23,12,3]]$

**lemma** *kp-5x5-ul: knights-path b5x5 kp5x5ul*

**by** (*simp only: knights-path-exec-simp*) *eval*

A Knight's path for the  $(5 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

17	14	25	6	19	8	29
26	35	18	15	28	5	20
13	16	27	24	7	30	9
34	23	2	11	32	21	4
1	12	33	22	3	10	31

**abbreviation**  $kp5x7ul \equiv$  the (to-path

[[17,14,25,6,19,8,29],  
 [26,35,18,15,28,5,20],  
 [13,16,27,24,7,30,9],  
 [34,23,2,11,32,21,4],  
 [1,12,33,22,3,10,31]])

**lemma**  $kp-5x7-ul$ : knights-path  $b5x7$   $kp5x7ul$

by (simp only: knights-path-exec-simp) eval

A Knight's path for the  $(5 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

7	12	37	42	5	18	23	32	27
38	45	6	11	36	31	26	19	24
13	8	43	4	41	22	17	28	33
44	39	2	15	10	35	30	25	20
1	14	9	40	3	16	21	34	29

**abbreviation**  $kp5x9ul \equiv$  the (to-path

[[7,12,37,42,5,18,23,32,27],  
 [38,45,6,11,36,31,26,19,24],  
 [13,8,43,4,41,22,17,28,33],  
 [44,39,2,15,10,35,30,25,20],  
 [1,14,9,40,3,16,21,34,29]])

**lemma**  $kp-5x9-ul$ : knights-path  $b5x9$   $kp5x9ul$

by (simp only: knights-path-exec-simp) eval

**abbreviation**  $b7x7 \equiv$  board  $7\ 7$

A Knight's path for the  $(7 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

9	30	19	42	7	32	17
20	49	8	31	18	43	6
29	10	41	36	39	16	33
48	21	38	27	34	5	44
11	28	35	40	37	26	15
22	47	2	13	24	45	4
1	12	23	46	3	14	25

**abbreviation**  $kp7x7ul \equiv$  the (to-path

[[9,30,19,42,7,32,17],  
 [20,49,8,31,18,43,6],  
 [29,10,41,36,39,16,33],  
 [48,21,38,27,34,5,44],  
 [11,28,35,40,37,26,15],  
 [22,47,2,13,24,45,4],



[1,12,23,46,3,14,25]])

**lemma** *kp-7x7-ul: knights-path b7x7 kp7x7ul*

**by** (*simp only: knights-path-exec-simp*) *eval*

**abbreviation** *b7x9*  $\equiv$  *board 7 9*

A Knight's path for the  $(7 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

59	4	17	50	37	6	19	30	39
16	63	58	5	18	51	38	7	20
3	60	49	36	57	42	29	40	31
48	15	62	43	52	35	56	21	8
61	2	13	26	45	28	41	32	55
14	47	44	11	24	53	34	9	22
1	12	25	46	27	10	23	54	33

**abbreviation** *kp7x9ul*  $\equiv$  *the (to-path*

*[[59,4,17,50,37,6,19,30,39],*

*[16,63,58,5,18,51,38,7,20],*

*[3,60,49,36,57,42,29,40,31],*

*[48,15,62,43,52,35,56,21,8],*

*[61,2,13,26,45,28,41,32,55],*

*[14,47,44,11,24,53,34,9,22],*

*[1,12,25,46,27,10,23,54,33]])*

**lemma** *kp-7x9-ul: knights-path b7x9 kp7x9ul*

**by** (*simp only: knights-path-exec-simp*) *eval*

**abbreviation** *b9x7*  $\equiv$  *board 9 7*

A Knight's path for the  $(9 \times 7)$ -board that starts in the lower-left and ends in the upper-left.

5	20	53	48	7	22	31
52	63	6	21	32	55	8
19	4	49	54	47	30	23
62	51	46	33	56	9	58
3	18	61	50	59	24	29
14	43	34	45	28	57	10
17	2	15	60	35	38	25
42	13	44	27	40	11	36
1	16	41	12	37	26	39

**abbreviation** *kp9x7ul*  $\equiv$  *the (to-path*

*[[5,20,53,48,7,22,31],*

*[52,63,6,21,32,55,8],*

```

[19,4,49,54,47,30,23],
[62,51,46,33,56,9,58],
[3,18,61,50,59,24,29],
[14,43,34,45,28,57,10],
[17,2,15,60,35,38,25],
[42,13,44,27,40,11,36],
[1,16,41,12,37,26,39]])

```

**lemma** *kp-9x7-ul: knights-path b9x7 kp9x7ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*

**abbreviation** *b9x9*  $\equiv$  *board 9 9*

A Knight's path for the  $(9 \times 9)$ -board that starts in the lower-left and ends in the upper-left.

```

13 26 39 52 11 24 37 50 9
40 81 12 25 38 51 10 23 36
27 14 53 58 63 68 73 8 49
80 41 64 67 72 57 62 35 22
15 28 59 54 65 74 69 48 7
42 79 66 71 76 61 56 21 34
29 16 77 60 55 70 75 6 47
78 43 2 31 18 45 4 33 20
1 30 17 44 3 32 19 46 5

```

**abbreviation** *kp9x9ul*  $\equiv$  *the (to-path*

```

[[13,26,39,52,11,24,37,50,9],
[40,81,12,25,38,51,10,23,36],
[27,14,53,58,63,68,73,8,49],
[80,41,64,67,72,57,62,35,22],
[15,28,59,54,65,74,69,48,7],
[42,79,66,71,76,61,56,21,34],
[29,16,77,60,55,70,75,6,47],
[78,43,2,31,18,45,4,33,20],
[1,30,17,44,3,32,19,46,5]])

```

**lemma** *kp-9x9-ul: knights-path b9x9 kp9x9ul*  
**by** (*simp only: knights-path-exec-simp*) *eval*

The following lemma is a sub-proof used in Lemma 4 in [1]. I moved the sub-proof out to a separate lemma.

**lemma** *knights-circuit-exists-even-n-gr10:*

**assumes** *even n n*  $\geq 10$  *m*  $\geq 5$

$\exists ps.$  *knights-path (board (n-5) m) ps*  $\wedge$  *hd ps* = (*int (n-5)*,1)  
 $\wedge$  *last ps* = (*int (n-5)*-1,*int m*-1)

**shows**  $\exists ps.$  *knights-circuit (board m n) ps*

**using** *assms*

**proof** –

```

let ?b2=board (n-5) m
assume n ≥ 10
then obtain ps2 where ps2-prems: knights-path ?b2 ps2 hd ps2 = (int (n-5),1)

  last ps2 = (int (n-5)-1,int m-1)
  using assms by auto
let ?ps2-m2=mirror2 ps2
have ps2-m2-prems: knights-path ?b2 ?ps2-m2 hd ?ps2-m2 = (int (n-5),int m)
  last ?ps2-m2 = (int (n-5)-1,2)
  using ps2-prems mirror2-knights-path hd-mirror2 last-mirror2 by auto

obtain ps1 where ps1-prems: knights-path (board 5 m) ps1 hd ps1 = (1,1)last
ps1 = (2,int m-1)
  using assms knights-path-5xm-exists by auto
let ?ps1'=trans-path (int (n-5),0) ps1
let ?b1'=trans-board (int (n-5),0) (board 5 m)
have ps1'-prems: knights-path ?b1' ?ps1' hd ?ps1' = (int (n-5)+1,1)
  last ?ps1' = (int (n-5)+2,int m-1)
  using ps1-prems trans-knights-path knights-path-non-nil hd-trans-path last-trans-path
by auto

let ?ps=?ps1'@?ps2-m2
let ?psT=transpose ?ps

have n-5 ≥ 5 using ⟨n ≥ 10⟩ by auto
have inter: ?b1' ∩ ?b2 = {}
  unfolding trans-board-def board-def using ⟨n-5 ≥ 5⟩ by auto
have union: ?b1' ∪ ?b2 = board n m
  using ⟨n-5 ≥ 5⟩ board-concatT[of n-5 m 5] by auto

have vs: valid-step (last ?ps1') (hd ?ps2-m2) and valid-step (last ?ps2-m2) (hd
?ps1')
  unfolding valid-step-def using ps1'-prems ps2-m2-prems by auto
then have vs-c: valid-step (last ?ps) (hd ?ps)
  using ps1'-prems ps2-m2-prems knights-path-non-nil by auto

have knights-path (board n m) ?ps
  using ps1'-prems ps2-m2-prems inter vs union knights-path-append[of ?b1' ?ps1'
?b2 ?ps2-m2]
  by auto
then have knights-circuit (board n m) ?ps
  unfolding knights-circuit-def using vs-c by auto
then show ?thesis using transpose-knights-circuit by auto
qed

```

For every  $n \times m$ -board with  $\min n m \geq 5$  and odd  $n$  there exists a Knight's path that starts in  $(n, 1)$  (top-left) and ends in  $(n-1, m-1)$  (top-right).

This lemma formalizes Lemma 4 from [1]. Formalizing the proof of this lemma was quite challenging as a lot of details on how to exactly combine

the boards are left out in the original proof in [1].

**lemma** *knights-path-odd-n-exists*:

**assumes**  $odd\ n\ \wedge\ min\ n\ m \geq 5$

**shows**  $\exists ps. knights\text{-}path\ (board\ n\ m)\ ps \wedge hd\ ps = (int\ n,1) \wedge last\ ps = (int\ n-1, int\ m-1)$

**using** *assms*

**proof** –

**obtain**  $x$  **where**  $x = n + m$  **by** *auto*

**then show** *?thesis*

**using** *assms*

**proof** (*induction x arbitrary: n m rule: less-induct*)

**case** (*less x*)

**then have**  $m = 5 \vee m = 6 \vee m = 7 \vee m = 8 \vee m = 9 \vee m \geq 10$  **by** *auto*

**then show** *?case*

**proof** (*elim disjE*)

**assume** [*simp*]:  $m = 5$

**have**  $odd\ n\ n \geq 5$  **using** *less* **by** *auto*

**then have**  $n = 5 \vee n = 7 \vee n = 9 \vee n - 5 \geq 5$  **by** *presburger*

**then show** *?thesis*

**proof** (*elim disjE*)

**assume** [*simp*]:  $n = 5$

**let**  $?ps=mirror1\ (transpose\ kp5x5ul)$

**have**  $kp: knights\text{-}path\ (board\ n\ m)\ ?ps$

**using** *kp-5x5-ul rot90-knights-path* **by** *auto*

**have**  $hd\ ?ps = (int\ n,1)\ last\ ?ps = (int\ n-1, int\ m-1)$

**by** (*simp only: <m = 5> <n = 5> | eval*)**+**

**then show** *?thesis* **using**  $kp$  **by** *auto*

**next**

**assume** [*simp*]:  $n = 7$

**let**  $?ps=mirror1\ (transpose\ kp5x7ul)$

**have**  $kp: knights\text{-}path\ (board\ n\ m)\ ?ps$

**using** *kp-5x7-ul rot90-knights-path* **by** *auto*

**have**  $hd\ ?ps = (int\ n,1)\ last\ ?ps = (int\ n-1, int\ m-1)$

**by** (*simp only: <m = 5> <n = 7> | eval*)**+**

**then show** *?thesis* **using**  $kp$  **by** *auto*

**next**

**assume** [*simp*]:  $n = 9$

**let**  $?ps=mirror1\ (transpose\ kp5x9ul)$

**have**  $kp: knights\text{-}path\ (board\ n\ m)\ ?ps$

**using** *kp-5x9-ul rot90-knights-path* **by** *auto*

**have**  $hd\ ?ps = (int\ n,1)\ last\ ?ps = (int\ n-1, int\ m-1)$

**by** (*simp only: <m = 5> <n = 9> | eval*)**+**

**then show** *?thesis* **using**  $kp$  **by** *auto*

**next**

**let**  $?b_2=board\ m\ (n-5)$

**assume**  $n-5 \geq 5$

**then have**  $\exists ps. knights\text{-}circuit\ ?b_2\ ps$

**proof** –

**have**  $n-5 = 6 \vee n-5 = 8 \vee n-5 \geq 10$

```

    using ⟨ $n-5 \geq 5$ ⟩ less by presburger
  then show ?thesis
  proof (elim disjE)
    assume  $n-5 = 6$ 
    then obtain ps where knights-circuit (board (n-5) m) ps
      using knights-path-6xm-exists[of m] by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume  $n-5 = 8$ 
    then obtain ps where knights-circuit (board (n-5) m) ps
      using knights-path-8xm-exists[of m] by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume  $n-5 \geq 10$ 
    then show ?thesis
      using less less.IH[of n-10+m n-10 m]
        knights-circuit-exists-even-n-gr10[of n-5 m] by auto
  qed
  then obtain ps2 where knights-circuit ?b2 ps2 hd ps2 = (1,1) last ps2 =
(3,2)
    using ⟨ $n-5 \geq 5$ ⟩ rotate-knights-circuit[of m n-5] by auto
  then have rev-ps2-prems: knights-path ?b2 (rev ps2) valid-step (last ps2) (hd
ps2)
    hd (rev ps2) = (3,2) last (rev ps2) = (1,1)
  unfolding knights-circuit-def using knights-path-rev by (auto simp: hd-rev
last-rev)

  let ?ps1=kp5x5ul
  have ps1-prems: knights-path (board 5 5) ?ps1 hd ?ps1 = (1,1) last ?ps1 =
(4,2)
    using kp-5x5-ul by simp eval+

  have 16 < length ?ps1 last (take 16 ?ps1) = (4,5) hd (drop 16 ?ps1) =
(2,4) by eval+
  then have si: step-in ?ps1 (4,5) (2,4)
    unfolding step-in-def using zero-less-numeral by blast

  have vs: valid-step (4,5) (3,int 5+2) valid-step (1,int 5+1) (2,4)
    unfolding valid-step-def by auto

  obtain ps where knights-path (board m n) ps hd ps = (1,1) last ps = (4,2)
    using ⟨ $n-5 \geq 5$ ⟩ ps1-prems rev-ps2-prems si vs
      knights-path-split-concat[of 5 5 ?ps1 n-5 rev ps2 (4,5) (2,4)] by auto
  then show ?thesis
    using rot90-knights-path hd-rot90-knights-path last-rot90-knights-path by
fastforce

```

```

qed
next
  assume [simp]: m = 6
  then obtain ps where
    ps-prems: knights-path (board m n) ps hd ps = (1,1) last ps = (int m-1,2)
  using less knights-path-6xm-exists[of n] by auto
  let ?ps'=mirror1 (transpose ps)
  have knights-path (board n m) ?ps' hd ?ps' = (int n,1) last ?ps' = (int n-1,int
m-1)
  using ps-prems rot90-knights-path hd-rot90-knights-path last-rot90-knights-path
by auto
  then show ?thesis by auto
next
  assume [simp]: m = 7
  have odd n n ≥ 5 using less by auto
  then have n = 5 ∨ n = 7 ∨ n = 9 ∨ n-5 ≥ 5 by presburger
  then show ?thesis
  proof (elim disjE)
    assume [simp]: n = 5
    let ?ps=mirror1 kp5x7lr
    have kp: knights-path (board n m) ?ps
      using kp-5x7-lr mirror1-knights-path by auto
    have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
      by (simp only: ⟨m = 7⟩ ⟨n = 5⟩ | eval)+
    then show ?thesis using kp by auto
  next
    assume [simp]: n = 7
    let ?ps=mirror1 (transpose kp7x7ul)
    have kp: knights-path (board n m) ?ps
      using kp-7x7-ul rot90-knights-path by auto
    have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
      by (simp only: ⟨m = 7⟩ ⟨n = 7⟩ | eval)+
    then show ?thesis using kp by auto
  next
    assume [simp]: n = 9
    let ?ps=mirror1 (transpose kp7x9ul)
    have kp: knights-path (board n m) ?ps
      using kp-7x9-ul rot90-knights-path by auto
    have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
      by (simp only: ⟨m = 7⟩ ⟨n = 9⟩ | eval)+
    then show ?thesis using kp by auto
  next
    let ?b2=board m (n-5)
    let ?b2T=board (n-5) m
    assume n-5 ≥ 5
    then have ∃ ps. knights-circuit ?b2 ps
    proof -
      have n-5 = 6 ∨ n-5 = 8 ∨ n-5 ≥ 10
        using ⟨n-5 ≥ 5⟩ less by presburger

```

```

then show ?thesis
proof (elim disjE)
  assume  $n-5 = 6$ 
  then obtain  $ps$  where knights-circuit (board (n-5) m)  $ps$ 
    using knights-path-6xm-exists[of m] by auto
  then show ?thesis
    using transpose-knights-circuit by auto
next
  assume  $n-5 = 8$ 
  then obtain  $ps$  where knights-circuit (board (n-5) m)  $ps$ 
    using knights-path-8xm-exists[of m] by auto
  then show ?thesis
    using transpose-knights-circuit by auto
next
  assume  $n-5 \geq 10$ 
  then show ?thesis
    using less less.IH[of  $n-10+m$   $n-10$  m]
      knights-circuit-exists-even-n-gr10[of  $n-5$  m] by auto
  qed
qed
then obtain  $ps_2$  where  $ps_2$ -prems: knights-circuit ? $b_2$   $ps_2$  hd  $ps_2 = (1,1)$ 
  last  $ps_2 = (3,2)$ 
  using  $\langle n-5 \geq 5 \rangle$  rotate-knights-circuit[of m  $n-5$ ] by auto
let ? $ps_2T$  = transpose  $ps_2$ 
have  $ps_2T$ -prems: knights-path ? $b_2T$  ? $ps_2T$  hd ? $ps_2T = (1,1)$  last ? $ps_2T =$ 
(2,3)
  using  $ps_2$ -prems transpose-knights-path knights-path-non-nil hd-transpose
last-transpose
  unfolding knights-circuit-def transpose-square-def by auto

let ? $ps_1$  = kp5x7lr
have  $ps_1$ -prems: knights-path  $b5x7$  ? $ps_1$  hd ? $ps_1 = (1,1)$  last ? $ps_1 = (2,6)$ 
  using kp-5x7-lr by simp eval+

  have  $29 < \text{length } ?ps_1$  last (take 29 ? $ps_1$ ) = (4,2) hd (drop 29 ? $ps_1$ ) =
(5,4) by eval+
  then have  $si$ : step-in ? $ps_1$  (4,2) (5,4)
    unfolding step-in-def using zero-less-numeral by blast

have  $vs$ : valid-step (4,2) (int 5+1,1) valid-step (int 5+2,3) (5,4)
  unfolding valid-step-def by auto

obtain  $ps$  where knights-path (board n m)  $ps$  hd  $ps = (1,1)$  last  $ps = (2,6)$ 
  using  $\langle n-5 \geq 5 \rangle$   $ps_1$ -prems  $ps_2T$ -prems  $si$   $vs$ 
    knights-path-split-concatT[of 5 m ? $ps_1$   $n-5$  ? $ps_2T$  (4,2) (5,4)] by auto
then show ?thesis
  using mirror1-knights-path hd-mirror1 last-mirror1 by fastforce
qed
next

```

```

assume [simp]: m = 8
then obtain ps where ps-prems: knights-path (board m n) ps hd ps = (1,1)
  last ps = (int m-1,2)
  using less knights-path-8xm-exists[of n] by auto
let ?ps'=mirror1 (transpose ps)
have knights-path (board n m) ?ps' hd ?ps' = (int n,1) last ?ps' = (int n-1,int
m-1)
using ps-prems rot90-knights-path hd-rot90-knights-path last-rot90-knights-path
by auto
then show ?thesis by auto
next
assume [simp]: m = 9
have odd n n ≥ 5 using less by auto
then have n = 5 ∨ n = 7 ∨ n = 9 ∨ n-5 ≥ 5 by presburger
then show ?thesis
proof (elim disjE)
  assume [simp]: n = 5
  let ?ps=mirror1 kp5x9lr
  have kp: knights-path (board n m) ?ps
    using kp-5x9-lr mirror1-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 9⟩ ⟨n = 5⟩ | eval)+
  then show ?thesis using kp by auto
next
  assume [simp]: n = 7
  let ?ps=mirror1 (transpose kp9x7ul)
  have kp: knights-path (board n m) ?ps
    using kp-9x7-ul rot90-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 9⟩ ⟨n = 7⟩ | eval)+
  then show ?thesis using kp by auto
next
  assume [simp]: n = 9
  let ?ps=mirror1 (transpose kp9x9ul)
  have kp: knights-path (board n m) ?ps
    using kp-9x9-ul rot90-knights-path by auto
  have hd ?ps = (int n,1) last ?ps = (int n-1,int m-1)
    by (simp only: ⟨m = 9⟩ ⟨n = 9⟩ | eval)+
  then show ?thesis using kp by auto
next
  let ?b2=board m (n-5)
  let ?b2T=board (n-5) m
  assume n-5 ≥ 5
  then have ∃ ps. knights-circuit ?b2 ps
  proof -
    have n-5 = 6 ∨ n-5 = 8 ∨ n-5 ≥ 10
      using ⟨n-5 ≥ 5⟩ less by presburger
    then show ?thesis
  proof (elim disjE)

```



```

    assume  $n-5 = 6$ 
    then obtain  $ps$  where knights-circuit (board  $(n-5) m$ )  $ps$ 
      using knights-path-6xm-exists[of  $m$ ] by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume  $n-5 = 8$ 
    then obtain  $ps$  where knights-circuit (board  $(n-5) m$ )  $ps$ 
      using knights-path-8xm-exists[of  $m$ ] by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume  $n-5 \geq 10$ 
    then show ?thesis
      using less less.IH[of  $n-10+m$   $n-10$   $m$ ]
        knights-circuit-exists-even-n-gr10[of  $n-5$   $m$ ] by auto
  qed
  obtain  $ps_2$  where  $ps_2$ -prems: knights-circuit  $?b_2$   $ps_2$  hd  $ps_2 = (1,1)$ 
    last  $ps_2 = (3,2)$ 
    using  $\langle n-5 \geq 5 \rangle$  rotate-knights-circuit[of  $m$   $n-5$ ] by auto
  let  $?ps_2T = \text{transpose } (\text{rev } ps_2)$ 
  have  $ps_2T$ -prems: knights-path  $?b_2T$   $?ps_2T$  hd  $?ps_2T = (2,3)$  last  $?ps_2T =$ 
(1,1)
  using  $ps_2$ -prems knights-path-rev transpose-knights-path knights-path-non-nil

    hd-transpose last-transpose
  unfolding knights-circuit-def transpose-square-def by (auto simp: hd-rev
last-rev)

  let  $?ps_1 = kp5x9lr$ 
  have  $ps_1$ -prems: knights-path  $b5x9$   $?ps_1$  hd  $?ps_1 = (1,1)$  last  $?ps_1 = (2,8)$ 
    using kp-5x9-lr by simp eval+

  have  $16 < \text{length } ?ps_1$  last (take 16  $?ps_1$ ) =  $(5,4)$  hd (drop 16  $?ps_1$ ) =
(4,2) by eval+
  then have si: step-in  $?ps_1$   $(5,4)$   $(4,2)$ 
    unfolding step-in-def using zero-less-numeral by blast

  have vs: valid-step  $(5,4)$  (int 5+2,3) valid-step (int 5+1,1)  $(4,2)$ 
    unfolding valid-step-def by auto

  obtain  $ps$  where knights-path (board  $n$   $m$ )  $ps$  hd  $ps = (1,1)$  last  $ps = (2,8)$ 
    using  $\langle n-5 \geq 5 \rangle$   $ps_1$ -prems  $ps_2T$ -prems si vs
      knights-path-split-concatT[of 5  $m$   $?ps_1$   $n-5$   $?ps_2T$   $(5,4)$   $(4,2)$ ] by auto
  then show ?thesis
    using mirror1-knights-path hd-mirror1 last-mirror1 by fastforce
  qed
next

```

```

let ?b1=board n 5
let ?b2=board n (m-5)
assume m ≥ 10
then have n+5 < x 5 ≤ min n 5 n+(m-5) < x 5 ≤ min n (m-5)
  using less by auto
then obtain ps1 ps2 where kp-prems:
  knights-path ?b1 ps1 hd ps1 = (int n,1) last ps1 = (int n-1,4)
  knights-path (board n (m-5)) ps2 hd ps2 = (int n,1) last ps2 = (int n-1,int
(m-5)-1)
  using less.premss less.IH[of n+5 n 5] less.IH[of n+(m-5) n m-5] by auto
let ?ps=ps1@trans-path (0,int 5) ps2
have valid-step (last ps1) (int n,int 5+1)
  unfolding valid-step-def using kp-prems by auto
then have knights-path (board n m) ?ps hd ?ps = (int n,1) last ?ps = (int
n-1,int m-1)
  using <m ≥ 10> kp-prems knights-path-concat[of n 5 ps1 m-5 ps2]
  knights-path-non-nil trans-path-non-nil last-trans-path by auto
then show ?thesis by auto
qed
qed
qed

```

Auxiliary lemma that constructs a Knight's circuit if  $m \geq 5$  and  $n \geq 10 \wedge$  even  $n$ .

```

lemma knights-circuit-exists-n-even-gr-10:
  assumes n ≥ 10 ∧ even n m ≥ 5
  shows ∃ps. knights-circuit (board n m) ps
  using assms
proof -
  obtain ps1 where ps1-prems: knights-path (board 5 m) ps1 hd ps1 = (1,1)
  last ps1 = (2,int m-1)
  using assms knights-path-5xm-exists by auto
  let ?ps1'=trans-path (int (n-5),0) ps1
  let ?b5xm'=trans-board (int (n-5),0) (board 5 m)
  have ps1'-prems: knights-path ?b5xm' ?ps1' hd ?ps1' = (int (n-5)+1,1)
  last ?ps1' = (int (n-5)+2,int m-1)
  using ps1-prems trans-knights-path knights-path-non-nil hd-trans-path last-trans-path
by auto

  assume n ≥ 10 ∧ even n
  then have odd (n-5) min (n-5) m ≥ 5 using assms by auto
  then obtain ps2 where ps2-prems: knights-path (board (n-5) m) ps2 hd ps2 =
(int (n-5),1)
  last ps2 = (int (n-5)-1,int m-1)
  using knights-path-odd-n-exists[of n-5 m] by auto
  let ?ps2'=mirror2 ps2
  have ps2'-prems: knights-path (board (n-5) m) ?ps2' hd ?ps2' = (int (n-5),int
m)
  last ?ps2' = (int (n-5)-1,2)

```

```

using ps2-prems mirror2-knights-path hd-mirror2 last-mirror2 by auto

have inter: ?b5xm' ∩ board (n-5) m = {}
  unfolding trans-board-def board-def by auto

have union: board n m = ?b5xm' ∪ board (n-5) m
  using  $\langle n \geq 10 \wedge \text{even } n \rangle$  board-concatT[of n-5 m 5] by auto

have vs: valid-step (last ?ps1') (hd ?ps2') valid-step (last ?ps2') (hd ?ps1')
  using ps1'-prems ps2'-prems unfolding valid-step-def by auto

let ?ps=?ps1' @ ?ps2'
have last ?ps = last ?ps2' hd ?ps = hd ?ps1'
  using ps1'-prems ps2'-prems knights-path-non-nil by auto
then have vs-c: valid-step (last ?ps) (hd ?ps)
  using vs by auto

have knights-path (board n m) ?ps
  using ps1'-prems ps2'-prems inter union vs knights-path-append by auto
then show ?thesis
  using vs-c unfolding knights-circuit-def by blast
qed

```

Final Theorem 1: For every  $n \times m$ -board with  $\min n m \geq 5$  and  $n*m$  even there exists a Knight's circuit.

```

theorem knights-circuit-exists:
  assumes min n m ≥ 5 even (n*m)
  shows  $\exists ps. \text{knights-circuit (board n m) ps}$ 
  using assms
proof –
  have  $n = 6 \vee m = 6 \vee n = 8 \vee m = 8 \vee (n \geq 10 \wedge \text{even } n) \vee (m \geq 10 \wedge \text{even } m)$ 
    using assms by auto
  then show ?thesis
  proof (elim disjE)
    assume  $n = 6$ 
    then show ?thesis
      using assms knights-path-6xm-exists by auto
  next
    assume  $m = 6$ 
    then obtain ps where knights-circuit (board m n) ps
      using assms knights-path-6xm-exists by auto
    then show ?thesis
      using transpose-knights-circuit by auto
  next
    assume  $n = 8$ 
    then show ?thesis
      using assms knights-path-8xm-exists by auto
  next

```

```

assume  $m = 8$ 
then obtain  $ps$  where  $knights-circuit$  (board  $m$   $n$ )  $ps$ 
  using  $assms$   $knights-path-8xm-exists$  by  $auto$ 
then show  $?thesis$ 
  using  $transpose-knights-circuit$  by  $auto$ 
next
assume  $n \geq 10 \wedge even$   $n$ 
then show  $?thesis$ 
  using  $assms$   $knights-circuit-exists-n-even-gr-10$  by  $auto$ 
next
assume  $m \geq 10 \wedge even$   $m$ 
then obtain  $ps$  where  $knights-circuit$  (board  $m$   $n$ )  $ps$ 
  using  $assms$   $knights-circuit-exists-n-even-gr-10$  by  $auto$ 
then show  $?thesis$ 
  using  $transpose-knights-circuit$  by  $auto$ 
qed
qed

```

Final Theorem 2: for every  $n \times m$ -board with  $\min n m \geq 5$  there exists a Knight's path.

```

theorem  $knights-path-exists$ :
  assumes  $\min n m \geq 5$ 
  shows  $\exists ps. knights-path$  (board  $n$   $m$ )  $ps$ 
  using  $assms$ 
proof –
  have  $odd$   $n \vee odd$   $m \vee even$  ( $n*m$ ) by  $simp$ 
  then show  $?thesis$ 
  proof ( $elim$   $disjE$ )
    assume  $odd$   $n$ 
    then show  $?thesis$ 
      using  $assms$   $knights-path-odd-n-exists$  by  $auto$ 
  next
  assume  $odd$   $m$ 
  then obtain  $ps$  where  $knights-path$  (board  $m$   $n$ )  $ps$ 
    using  $assms$   $knights-path-odd-n-exists$  by  $auto$ 
  then show  $?thesis$ 
    using  $transpose-knights-path$  by  $auto$ 
  next
  assume  $even$  ( $n*m$ )
  then show  $?thesis$ 
    using  $assms$   $knights-circuit-exists$  by ( $auto$   $simp$ :  $knights-circuit-def$ )
qed
qed

```

THE END

end

## References

- [1] P. Cull and J. De Curtins. Knight's tour revisited. *Fibonacci Quarterly*, 16:276–285, 1978.