

Kleene Algebra with Tests and Demonic Refinement Algebras

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Abstract

We formalise Kleene algebra with tests (KAT) and demonic refinement algebra (DRA) with tests in Isabelle/HOL. KAT is relevant for program verification and correctness proofs in the partial correctness setting. DRA targets similar applications in the context of total correctness. Our formalisation contains the two most important models of these algebras: binary relations in the case of KAT and predicate transformers in the case of DRA. In addition, we derive the inference rules for Hoare logic in KAT and its relational model.

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1 Test Dioids

```

theory Test-Dioid
  imports Kleene-Algebra.Dioid
begin

```

Tests are embedded in a weak dioid, a dioid without the right annihilation and left distributivity axioms, using an operator t defined by a complementation operator. This allows us to use tests in weak settings, such as Probabilistic Kleene Algebra and Demonic Refinement Algebra.

1.1 Test Monoids

```

class n-op =
  fixes n-op :: 'a  $\Rightarrow$  'a ( $\langle n \rightarrow [90] \ 91$ )

class test-monoid = monoid-mult + n-op +
  assumes tm1 [simp]: n n 1 = 1
  and tm2 [simp]: n x  $\cdot$  n n x = n 1
  and tm3: n x  $\cdot$  n (n n z  $\cdot$  n n y) = n (n (n x  $\cdot$  n y)  $\cdot$  n (n x  $\cdot$  n z))

begin

```

```

definition a-zero :: 'a ( $\langle o \rangle$ ) where
  o  $\equiv$  n 1

```

```

abbreviation t x  $\equiv$  n n x

```

```

definition n-add-op :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a (infixl  $\langle \oplus \rangle$  65) where
  x  $\oplus$  y  $\equiv$  n (n x  $\cdot$  n y)

```

```

lemma n 1  $\cdot$  x = n 1
   $\langle proof \rangle$ 

```

```

lemma x  $\cdot$  n 1 = n 1
   $\langle proof \rangle$ 

```

```

lemma n 1  $\cdot$  x = n 1  $\implies$  n x  $\cdot$  y  $\cdot$  t z = n 1  $\implies$  n x  $\cdot$  y = n x  $\cdot$  y  $\cdot$  n z
   $\langle proof \rangle$ 

```

```

lemma n-t-closed [simp]: t (n x) = n x
   $\langle proof \rangle$ 

```

lemma *mult-t-closed* [simp]: $t (n x \cdot n y) = n x \cdot n y$
 $\langle proof \rangle$

lemma *n-comm-var*: $n (n x \cdot n y) = n (n y \cdot n x)$
 $\langle proof \rangle$

lemma *n-comm*: $n x \cdot n y = n y \cdot n x$
 $\langle proof \rangle$

lemma *huntington1* [simp]: $n (n (n n x \cdot n y) \cdot n (n n x \cdot n n y)) = n n x$
 $\langle proof \rangle$

lemma *huntington2* [simp]: $n (n x \oplus n n y) \oplus n (n x \oplus n y) = n n x$
 $\langle proof \rangle$

lemma *add-assoc*: $n x \oplus (n y \oplus n z) = (n x \oplus n y) \oplus n z$
 $\langle proof \rangle$

lemma *t-mult-closure*: $t x = x \implies t y = y \implies t (x \cdot y) = x \cdot y$
 $\langle proof \rangle$

lemma *n-t-compl* [simp]: $n x \oplus t x = 1$
 $\langle proof \rangle$

lemma *zero-least1* [simp]: $o \oplus n x = n x$
 $\langle proof \rangle$

lemma *zero-least2* [simp]: $o \cdot n x = o$
 $\langle proof \rangle$

lemma *zero-least3* [simp]: $n x \cdot o = o$
 $\langle proof \rangle$

lemma *one-greatest1* [simp]: $1 \oplus n x = 1$
 $\langle proof \rangle$

lemma *one-greatest2* [simp]: $n x \oplus 1 = 1$
 $\langle proof \rangle$

lemma *n-add-idem* [simp]: $n x \oplus n x = n x$
 $\langle proof \rangle$

lemma *n-mult-idem* [simp]: $n x \cdot n x = n x$
 $\langle proof \rangle$

lemma *n-preserve* [simp]: $n x \cdot n y \cdot n x = n y \cdot n x$
 $\langle proof \rangle$

lemma *n-preserve2* [*simp*]: $n\ x \cdot n\ y \cdot t\ x = o$
 $\langle proof \rangle$

lemma *de-morgan1* [*simp*]: $n\ (n\ x \cdot n\ y) = t\ x \oplus t\ y$
 $\langle proof \rangle$

lemma *de-morgan4* [*simp*]: $n\ (t\ x \oplus t\ y) = n\ x \cdot n\ y$
 $\langle proof \rangle$

lemma *n-absorb1* [*simp*]: $n\ x \oplus n\ x \cdot n\ y = n\ x$
 $\langle proof \rangle$

lemma *n-absorb2* [*simp*]: $n\ x \cdot (n\ x \oplus n\ y) = n\ x$
 $\langle proof \rangle$

lemma *n-distrib1*: $n\ x \cdot (n\ y \oplus n\ z) = (n\ x \cdot n\ y) \oplus (n\ x \cdot n\ z)$
 $\langle proof \rangle$

lemma *n-distrib1-opp*: $(n\ x \oplus n\ y) \cdot n\ z = (n\ x \cdot n\ z) \oplus (n\ y \cdot n\ z)$
 $\langle proof \rangle$

lemma *n-distrib2*: $n\ x \oplus n\ y \cdot n\ z = (n\ x \oplus n\ y) \cdot (n\ x \oplus n\ z)$
 $\langle proof \rangle$

lemma *n-distrib2-opp*: $n\ x \cdot n\ y \oplus n\ z = (n\ x \oplus n\ z) \cdot (n\ y \oplus n\ z)$
 $\langle proof \rangle$

definition *ts-ord* :: ' $a \Rightarrow 'a \Rightarrow bool$ (**infix** $\sqsubseteq$ 50) **where**
 $x \sqsubseteq y = (n\ x \cdot n\ y = n\ y)$

lemma *ts-ord-alt*: $n\ x \sqsubseteq n\ y \longleftrightarrow n\ x \oplus n\ y = n\ y$
 $\langle proof \rangle$

lemma *ts-ord-refl*: $n\ x \sqsubseteq n\ x$
 $\langle proof \rangle$

lemma *ts-ord-trans*: $n\ x \sqsubseteq n\ y \Longrightarrow n\ y \sqsubseteq n\ z \Longrightarrow n\ x \sqsubseteq n\ z$
 $\langle proof \rangle$

lemma *ts-ord-antisym*: $n\ x \sqsubseteq n\ y \Longrightarrow n\ y \sqsubseteq n\ x \Longrightarrow n\ x = n\ y$
 $\langle proof \rangle$

lemma *ts-ord-mult-isol*: $n\ x \sqsubseteq n\ y \Longrightarrow n\ z \cdot n\ x \sqsubseteq n\ z \cdot n\ y$
 $\langle proof \rangle$

lemma *ts-ord-mult-isor*: $n\ x \sqsubseteq n\ y \Longrightarrow n\ x \cdot n\ z \sqsubseteq n\ y \cdot n\ z$
 $\langle proof \rangle$

lemma *ts-ord-add-isol*: $n\ x \sqsubseteq n\ y \Longrightarrow n\ z \oplus n\ x \sqsubseteq n\ z \oplus n\ y$

$\langle proof \rangle$

lemma *ts-ord-add-isor*: $n\ x \sqsubseteq n\ y \implies n\ x \oplus n\ z \sqsubseteq n\ y \oplus n\ z$
 $\langle proof \rangle$

lemma *ts-ord-anti*: $n\ x \sqsubseteq n\ y \implies t\ y \sqsubseteq t\ x$
 $\langle proof \rangle$

lemma *ts-ord-anti-iff*: $n\ x \sqsubseteq n\ y \longleftrightarrow t\ y \sqsubseteq t\ x$
 $\langle proof \rangle$

lemma *zero-ts-ord*: $o \sqsubseteq n\ x$
 $\langle proof \rangle$

lemma *n-subid*: $n\ x \sqsubseteq 1$
 $\langle proof \rangle$

lemma *n-mult-lb1*: $n\ x \cdot n\ y \sqsubseteq n\ x$
 $\langle proof \rangle$

lemma *n-mult-lb2*: $n\ x \cdot n\ y \sqsubseteq n\ y$
 $\langle proof \rangle$

lemma *n-mult-glbI*: $n\ z \sqsubseteq n\ x \implies n\ z \sqsubseteq n\ y \implies n\ z \sqsubseteq n\ x \cdot n\ y$
 $\langle proof \rangle$

lemma *n-mult-glb*: $n\ z \sqsubseteq n\ x \wedge n\ z \sqsubseteq n\ y \longleftrightarrow n\ z \sqsubseteq n\ x \cdot n\ y$
 $\langle proof \rangle$

lemma *n-add-ub1*: $n\ x \sqsubseteq n\ x \oplus n\ y$
 $\langle proof \rangle$

lemma *n-add-ub2*: $n\ y \sqsubseteq n\ x \oplus n\ y$
 $\langle proof \rangle$

lemma *n-add-lubI*: $n\ x \sqsubseteq n\ z \implies n\ y \sqsubseteq n\ z \implies n\ x \oplus n\ y \sqsubseteq n\ z$
 $\langle proof \rangle$

lemma *n-add-lub*: $n\ x \sqsubseteq n\ z \wedge n\ y \sqsubseteq n\ z \longleftrightarrow n\ x \oplus n\ y \sqsubseteq n\ z$
 $\langle proof \rangle$

lemma *n-galois1*: $n\ x \sqsubseteq n\ y \oplus n\ z \longleftrightarrow n\ x \cdot t\ y \sqsubseteq n\ z$
 $\langle proof \rangle$

lemma *n-galois2*: $n\ x \sqsubseteq t\ y \oplus n\ z \longleftrightarrow n\ x \cdot n\ y \sqsubseteq n\ z$
 $\langle proof \rangle$

lemma *n-distrib-alt*: $n\ x \cdot n\ z = n\ y \cdot n\ z \implies n\ x \oplus n\ z = n\ y \oplus n\ z \implies n\ x = n\ y$

$\langle proof \rangle$

lemma *n-dist-var1*: $(n\ x \oplus n\ y) \cdot (t\ x \oplus n\ z) = t\ x \cdot n\ y \oplus n\ x \cdot n\ z$
 $\langle proof \rangle$

lemma *n-dist-var2*: $n\ (n\ x \cdot n\ y \oplus t\ x \cdot n\ z) = n\ x \cdot t\ y \oplus t\ x \cdot t\ z$
 $\langle proof \rangle$

end

1.2 Test Near-Semirings

class *test-near-semiring-zero1* = *ab-near-semiring-one-zero1* + *n-op* + *plus-ord* +
 assumes *test-one* [*simp*]: $n\ n\ 1 = 1$
 and *test-mult* [*simp*]: $n\ n\ (n\ x \cdot n\ y) = n\ x \cdot n\ y$
 and *test-mult-comp* [*simp*]: $n\ x \cdot n\ n\ x = 0$
 and *test-de-morgan* [*simp*]: $n\ (n\ n\ x \cdot n\ n\ y) = n\ x + n\ y$

begin

lemma *n-zero* [*simp*]: $n\ 0 = 1$
 $\langle proof \rangle$

lemma *n-one* [*simp*]: $n\ 1 = 0$
 $\langle proof \rangle$

lemma *one-idem* [*simp*]: $1 + 1 = 1$
 $\langle proof \rangle$

subclass *near-dioid-one-zero1*
 $\langle proof \rangle$

lemma *t-n-closed* [*simp*]: $n\ n\ (n\ x) = n\ x$
 $\langle proof \rangle$

lemma *t-de-morgan-var1* [*simp*]: $n\ (n\ x \cdot n\ y) = n\ n\ x + n\ n\ y$
 $\langle proof \rangle$

lemma *n-mult-comm*: $n\ x \cdot n\ y = n\ y \cdot n\ x$
 $\langle proof \rangle$

lemma *tm3'*: $n\ x \cdot n\ (n\ n\ z \cdot n\ n\ y) = n\ (n\ (n\ x \cdot n\ y) \cdot n\ (n\ x \cdot n\ z))$
 $\langle proof \rangle$

subclass *test-monoid*
 $\langle proof \rangle$

lemma *ord-transl* [*simp*]: $n\ x \leq n\ y \longleftrightarrow n\ x \sqsubseteq n\ y$
 $\langle proof \rangle$

lemma *add-transl* [*simp*]: $n\ x + n\ y = n\ x \oplus n\ y$
 $\langle proof \rangle$

lemma *zero-trans*: $0 = o$
 $\langle proof \rangle$

definition *test* :: 'a $\Rightarrow$ bool **where**
 $test\ p \equiv t\ p = p$

notation *n-op* ($\langle ! \rightarrow [101]\ 100$)

lemma *test-prop*: $(\forall x. test\ x \longrightarrow P\ x) \longleftrightarrow (\forall x. P\ (t\ x))$
 $\langle proof \rangle$

lemma *test-propI*: $test\ x \Longrightarrow P\ x \Longrightarrow P\ (t\ x)$
 $\langle proof \rangle$

lemma *test-propE* [*elim!*]: $test\ x \Longrightarrow P\ (t\ x) \Longrightarrow P\ x$
 $\langle proof \rangle$

lemma *test-comp-closed* [*simp*]: $test\ p \Longrightarrow test\ (!p)$
 $\langle proof \rangle$

lemma *test-double-comp-var*: $test\ p \Longrightarrow p = !(!p)$
 $\langle proof \rangle$

lemma *test-mult-closed*: $test\ p \Longrightarrow test\ q \Longrightarrow test\ (p \cdot q)$
 $\langle proof \rangle$

lemma *t-add-closed* [*simp*]: $t\ (n\ x + n\ y) = n\ x + n\ y$
 $\langle proof \rangle$

lemma *test-add-closed*: $test\ p \Longrightarrow test\ q \Longrightarrow test\ (p + q)$
 $\langle proof \rangle$

lemma *test-mult-comm-var*: $test\ p \Longrightarrow test\ q \Longrightarrow p \cdot q = q \cdot p$
 $\langle proof \rangle$

lemma *t-zero* [*simp*]: $t\ 0 = 0$
 $\langle proof \rangle$

lemma *test-zero-var*: $test\ 0$
 $\langle proof \rangle$

lemma *test-one-var*: $test\ 1$
 $\langle proof \rangle$

lemma *test-preserve*: $test\ p \Longrightarrow test\ q \Longrightarrow p \cdot q \cdot p = q \cdot p$

$\langle proof \rangle$

lemma *test-preserve2*: $test\ p \implies test\ q \implies p \cdot q \cdot !p = 0$
 $\langle proof \rangle$

lemma *n-subid'*: $n\ x \leq 1$
 $\langle proof \rangle$

lemma *test-subid*: $test\ p \implies p \leq 1$
 $\langle proof \rangle$

lemma *test-mult-idem-var* [*simp*]: $test\ p \implies p \cdot p = p$
 $\langle proof \rangle$

lemma *n-add-comp* [*simp*]: $n\ x + t\ x = 1$
 $\langle proof \rangle$

lemma *n-add-comp-var* [*simp*]: $t\ x + n\ x = 1$
 $\langle proof \rangle$

lemma *test-add-comp* [*simp*]: $test\ p \implies p + !p = 1$
 $\langle proof \rangle$

lemma *test-comp-mult1* [*simp*]: $test\ p \implies !p \cdot p = 0$
 $\langle proof \rangle$

lemma *test-comp-mult2* [*simp*]: $test\ p \implies p \cdot !p = 0$
 $\langle proof \rangle$

lemma *test-comp*: $test\ p \implies \exists q. test\ q \wedge p + q = 1 \wedge p \cdot q = 0$
 $\langle proof \rangle$

lemma *n-absorb1'* [*simp*]: $n\ x + n\ x \cdot n\ y = n\ x$
 $\langle proof \rangle$

lemma *test-absorb1* [*simp*]: $test\ p \implies test\ q \implies p + p \cdot q = p$
 $\langle proof \rangle$

lemma *n-absorb2'* [*simp*]: $n\ x \cdot (n\ x + n\ y) = n\ x$
 $\langle proof \rangle$

lemma *test-absorb2*: $test\ p \implies test\ q \implies p \cdot (p + q) = p$
 $\langle proof \rangle$

lemma *n-distrib-left*: $n\ x \cdot (n\ y + n\ z) = (n\ x \cdot n\ y) + (n\ x \cdot n\ z)$
 $\langle proof \rangle$

lemma *test-distrib-left*: $test\ p \implies test\ q \implies test\ r \implies p \cdot (q + r) = p \cdot q + p \cdot r$

r
 $\langle proof \rangle$

lemma *de-morgan1'*: $test\ p \implies test\ q \implies !p + !q = !(p \cdot q)$
 $\langle proof \rangle$

lemma *n-de-morgan-var2* [*simp*]: $n\ (n\ x + n\ y) = t\ x \cdot t\ y$
 $\langle proof \rangle$

lemma *n-de-morgan-var3* [*simp*]: $n\ (t\ x + t\ y) = n\ x \cdot n\ y$
 $\langle proof \rangle$

lemma *de-morgan2*: $test\ p \implies test\ q \implies !p \cdot !q = !(p + q)$
 $\langle proof \rangle$

lemma *de-morgan3*: $test\ p \implies test\ q \implies !(!p + !q) = p \cdot q$
 $\langle proof \rangle$

lemma *de-morgan4'*: $test\ p \implies test\ q \implies !(!p \cdot !q) = p + q$
 $\langle proof \rangle$

lemma *n-add-distr*: $n\ x + (n\ y \cdot n\ z) = (n\ x + n\ y) \cdot (n\ x + n\ z)$
 $\langle proof \rangle$

lemma *test-add-distr*: $test\ p \implies test\ q \implies test\ r \implies p + q \cdot r = (p + q) \cdot (p + r)$
 $\langle proof \rangle$

lemma *n-add-distl*: $(n\ x \cdot n\ y) + n\ z = (n\ x + n\ z) \cdot (n\ y + n\ z)$
 $\langle proof \rangle$

lemma *test-add-distl-var*: $test\ p \implies test\ q \implies test\ r \implies p \cdot q + r = (p + r) \cdot (q + r)$
 $\langle proof \rangle$

lemma *n-ord-def-alt*: $n\ x \leq n\ y \longleftrightarrow n\ x \cdot n\ y = n\ x$
 $\langle proof \rangle$

lemma *test-leq-mult-def-var*: $test\ p \implies test\ q \implies p \leq q \longleftrightarrow p \cdot q = p$
 $\langle proof \rangle$

lemma *n-anti*: $n\ x \leq n\ y \implies t\ y \leq t\ x$
 $\langle proof \rangle$

lemma *n-anti-iff*: $n\ x \leq n\ y \longleftrightarrow t\ y \leq t\ x$
 $\langle proof \rangle$

lemma *test-comp-anti-iff*: $test\ p \implies test\ q \implies p \leq q \longleftrightarrow !q \leq !p$
 $\langle proof \rangle$

lemma *n-restrictl*: $n\ x \cdot y \leq y$

<proof>

lemma *test-restrictl*: $\text{test } p \implies p \cdot x \leq x$

<proof>

lemma *n-mult-lb1'*: $n\ x \cdot n\ y \leq n\ x$

<proof>

lemma *test-mult-lb1*: $\text{test } p \implies \text{test } q \implies p \cdot q \leq p$

<proof>

lemma *n-mult-lb2'*: $n\ x \cdot n\ y \leq n\ y$

<proof>

lemma *test-mult-lb2*: $\text{test } p \implies \text{test } q \implies p \cdot q \leq q$

<proof>

lemma *n-mult-glbI'*: $n\ z \leq n\ x \implies n\ z \leq n\ y \implies n\ z \leq n\ x \cdot n\ y$

<proof>

lemma *test-mult-glbI*: $\text{test } p \implies \text{test } q \implies \text{test } r \implies p \leq q \implies p \leq r \implies p \leq q \cdot r$

<proof>

lemma *n-mult-glb'*: $n\ z \leq n\ x \wedge n\ z \leq n\ y \longleftrightarrow n\ z \leq n\ x \cdot n\ y$

<proof>

lemma *test-mult-glb*: $\text{test } p \implies \text{test } q \implies \text{test } r \implies p \leq q \wedge p \leq r \longleftrightarrow p \leq q \cdot r$

<proof>

lemma *n-galois1'*: $n\ x \leq n\ y + n\ z \longleftrightarrow n\ x \cdot t\ y \leq n\ z$

<proof>

lemma *test-galois1*: $\text{test } p \implies \text{test } q \implies \text{test } r \implies p \leq q + r \longleftrightarrow p \cdot !q \leq r$

<proof>

lemma *n-galois2'*: $n\ x \leq t\ y + n\ z \longleftrightarrow n\ x \cdot n\ y \leq n\ z$

<proof>

lemma *test-galois2*: $\text{test } p \implies \text{test } q \implies \text{test } r \implies p \leq !q + r \longleftrightarrow p \cdot q \leq r$

<proof>

lemma *n-huntington2*: $n\ (n\ x + t\ y) + n\ (n\ x + n\ y) = n\ n\ x$

<proof>

lemma *test-huntington2*: $\text{test } p \implies \text{test } q \implies !(p + q) + !(p + !q) = !p$

$\langle proof \rangle$

lemma *n-kat-1-opp*: $n\ x \cdot y \cdot n\ z = y \cdot n\ z \longleftrightarrow t\ x \cdot y \cdot n\ z = 0$
 $\langle proof \rangle$

lemma *test-eq4*: $test\ p \implies test\ q \implies !p \cdot x \cdot !q = x \cdot !q \longleftrightarrow p \cdot x \cdot !q = 0$
 $\langle proof \rangle$

lemma *n-kat-1-var*: $t\ x \cdot y \cdot t\ z = y \cdot t\ z \longleftrightarrow n\ x \cdot y \cdot t\ z = 0$
 $\langle proof \rangle$

lemma *test-kat-1*: $test\ p \implies test\ q \implies p \cdot x \cdot q = x \cdot q \longleftrightarrow !p \cdot x \cdot q = 0$
 $\langle proof \rangle$

lemma *n-kat-21-opp*: $y \cdot n\ z \leq n\ x \cdot y \implies n\ x \cdot y \cdot n\ z = y \cdot n\ z$
 $\langle proof \rangle$

lemma *test-kat-21-opp*: $test\ p \implies test\ q \implies x \cdot q \leq p \cdot x \longrightarrow p \cdot x \cdot q = x \cdot q$
 $\langle proof \rangle$

lemma $n\ x \cdot y \cdot n\ z = y \cdot n\ z \implies y \cdot n\ z \leq n\ x \cdot y$
 $\langle proof \rangle$

lemma *n-distrib-alt'*: $n\ x \cdot n\ z = n\ y \cdot n\ z \implies n\ x + n\ z = n\ y + n\ z \implies n\ x = n\ y$
 $\langle proof \rangle$

lemma *test-distrib-alt*: $test\ p \implies test\ q \implies test\ r \implies p \cdot r = q \cdot r \wedge p + r = q + r \longrightarrow p = q$
 $\langle proof \rangle$

lemma *n-eq1*: $n\ x \cdot y \leq z \wedge t\ x \cdot y \leq z \longleftrightarrow y \leq z$
 $\langle proof \rangle$

lemma *test-eq1*: $test\ p \implies y \leq x \longleftrightarrow p \cdot y \leq x \wedge !p \cdot y \leq x$
 $\langle proof \rangle$

lemma *n-dist-var1'*: $(n\ x + n\ y) \cdot (t\ x + n\ z) = t\ x \cdot n\ y + n\ x \cdot n\ z$
 $\langle proof \rangle$

lemma *test-dist-var1*: $test\ p \implies test\ q \implies test\ r \implies (p + q) \cdot (!p + r) = !p \cdot q + p \cdot r$
 $\langle proof \rangle$

lemma *n-dist-var2'*: $n\ (n\ x \cdot n\ y + t\ x \cdot n\ z) = n\ x \cdot t\ y + t\ x \cdot t\ z$
 $\langle proof \rangle$

lemma *test-dist-var2*: $test\ p \implies test\ q \implies test\ r \implies !(p \cdot q + !p \cdot r) = (p \cdot !q + !p \cdot !r)$

$\langle \text{proof} \rangle$

lemma *test-restrict*: $\text{test } p \implies x \cdot p \leq x$
 $\langle \text{proof} \rangle$

lemma *test-eq2*: $\text{test } p \implies z \leq p \cdot x + !p \cdot y \iff p \cdot z \leq p \cdot x \wedge !p \cdot z \leq !p \cdot y$
 $\langle \text{proof} \rangle$

lemma *test-eq3*: $\llbracket \text{test } p; \text{test } q \rrbracket \implies p \cdot x = p \cdot x \cdot q \iff p \cdot x \leq x \cdot q$
 $\langle \text{proof} \rangle$

lemma *test1*: $\llbracket \text{test } p; \text{test } q; p \cdot x \cdot !q = 0 \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{test } q; x \cdot !q = !p \cdot x \cdot !q \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle \text{proof} \rangle$

lemma *comm-add*: $\llbracket \text{test } p; p \cdot x = x \cdot p; p \cdot y = y \cdot p \rrbracket \implies p \cdot (x + y) = (x + y) \cdot p$
 $\langle \text{proof} \rangle$

lemma *comm-add-var*: $\llbracket \text{test } p; \text{test } q; \text{test } r; p \cdot x = x \cdot p; p \cdot y = y \cdot p \rrbracket \implies p \cdot (q \cdot x + r \cdot y) = (q \cdot x + r \cdot y) \cdot p$
 $\langle \text{proof} \rangle$

lemma *test* $p \implies p \cdot x = x \cdot p \implies p \cdot x = p \cdot x \cdot p \wedge !p \cdot x = !p \cdot x \cdot !p$
 $\langle \text{proof} \rangle$

lemma *test-distrib*: $\llbracket \text{test } p; \text{test } q \rrbracket \implies (p + q) \cdot (q \cdot y + !q \cdot x) = q \cdot y + !q \cdot p \cdot x$
 $\langle \text{proof} \rangle$

end

1.3 Test Near Semirings with Distributive Tests

We now make the assumption that tests distribute over finite sums of arbitrary elements from the left. This can be justified in models such as multirelations and probabilistic predicate transformers.

class *test-near-semiring-zero-distrib* = *test-near-semiring-zero* +
assumes *n-left-distrib*: $n \cdot x \cdot (y + z) = n \cdot x \cdot y + n \cdot x \cdot z$

begin

lemma *n-left-distrib-var*: $\text{test } p \implies p \cdot (x + y) = p \cdot x + p \cdot y$
 $\langle \text{proof} \rangle$

lemma *n-mult-left-iso*: $x \leq y \implies n \cdot z \cdot x \leq n \cdot z \cdot y$
 $\langle \text{proof} \rangle$

lemma *test-mult-iso*: $\text{test } p \implies x \leq y \implies p \cdot x \leq p \cdot y$

$\langle \text{proof} \rangle$

lemma $\text{test } p \implies x \cdot p \leq x$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{test } q \rrbracket \implies p \cdot x = p \cdot x \cdot q \longleftrightarrow p \cdot x \leq x \cdot q$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{test } q; p \cdot x \cdot !q = 0 \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{test } q; x \cdot !q = !p \cdot x \cdot !q \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle \text{proof} \rangle$

Next, we study tests with commutativity conditions.

lemma $\text{comm-add: test } p \implies p \cdot x = x \cdot p \implies p \cdot y = y \cdot p \implies p \cdot (x + y) = (x + y) \cdot p$
 $\langle \text{proof} \rangle$

lemma $\text{comm-add-var: test } p \implies \text{test } q \implies \text{test } r \implies p \cdot x = x \cdot p \implies p \cdot y = y \cdot p \implies p \cdot (q \cdot x + r \cdot y) = (q \cdot x + r \cdot y) \cdot p$
 $\langle \text{proof} \rangle$

lemma $\text{test-distrib: test } p \implies \text{test } q \implies (p + q) \cdot (q \cdot y + !q \cdot x) = q \cdot y + !q \cdot p \cdot x$
 $\langle \text{proof} \rangle$

end

1.4 Test Predioids

The following class is relevant for probabilistic Kleene algebras.

class $\text{test-pre-dioid-zero} = \text{test-near-semiring-zero-distrib} + \text{pre-dioid}$

begin

lemma $n\text{-restrictr: } x \cdot n \ y \leq x$
 $\langle \text{proof} \rangle$

lemma $\text{test-restrictr: test } p \implies x \cdot p \leq x$
 $\langle \text{proof} \rangle$

lemma $n\text{-kat-2: } n \ x \cdot y = n \ x \cdot y \cdot n \ z \longleftrightarrow n \ x \cdot y \leq y \cdot n \ z$
 $\langle \text{proof} \rangle$

lemma $\text{test-kat-2: test } p \implies \text{test } q \implies p \cdot x = p \cdot x \cdot q \longleftrightarrow p \cdot x \leq x \cdot q$
 $\langle \text{proof} \rangle$

lemma $n\text{-kat-2-opp: } y \cdot n \ z = n \ x \cdot y \cdot n \ z \longleftrightarrow y \cdot n \ z \leq n \ x \cdot y$

$\langle proof \rangle$

lemma *test-kat-2-opp*: $test\ p \implies test\ q \implies x \cdot q = p \cdot x \cdot q \iff x \cdot q \leq p \cdot x$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q; p \cdot x \cdot !q = 0 \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q; x \cdot !q = !p \cdot x \cdot !q \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q \rrbracket \implies x \cdot (p + q) \leq x \cdot p + x \cdot q$
 $\langle proof \rangle$

end

The following class is relevant for Demonic Refinement Algebras.

class *test-semiring-zerol* = *test-near-semiring-zerol* + *semiring-one-zerol*

begin

subclass *diod-one-zerol* $\langle proof \rangle$

subclass *test-pre-diod-zerol*
 $\langle proof \rangle$

lemma *n-kat-31*: $n\ x \cdot y \cdot t\ z = 0 \implies n\ x \cdot y \cdot n\ z = n\ x \cdot y$
 $\langle proof \rangle$

lemma *test-kat-31*: $test\ p \implies test\ q \implies p \cdot x \cdot !q = 0 \implies p \cdot x = p \cdot x \cdot q$
 $\langle proof \rangle$

lemma *n-kat-var*: $t\ x \cdot y \cdot t\ z = y \cdot t\ z \implies n\ x \cdot y \cdot n\ z = n\ x \cdot y$
 $\langle proof \rangle$

lemma *test1-var*: $test\ p \implies test\ q \implies x \cdot !q = !p \cdot x \cdot !q \implies p \cdot x = p \cdot x \cdot q$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q; p \cdot x \cdot !q = 0 \rrbracket \implies !p \cdot x \cdot q = 0$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q; p \cdot x = p \cdot x \cdot q \rrbracket \implies x \cdot !q = !p \cdot x \cdot !q$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q; p \cdot x = p \cdot x \cdot q \rrbracket \implies p \cdot x \cdot !q = 0$
 $\langle proof \rangle$

lemma $\llbracket test\ p; test\ q; p \cdot x = p \cdot x \cdot q \rrbracket \implies !p \cdot x \cdot q = 0$
 $\langle proof \rangle$

lemma $\llbracket \text{test } p; \text{ test } q; x \cdot !q = !p \cdot x \cdot !q \rrbracket \implies !p \cdot x \cdot q = 0$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{ test } q; !p \cdot x \cdot q = 0 \rrbracket \implies p \cdot x = p \cdot x \cdot q$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{ test } q; !p \cdot x \cdot q = 0 \rrbracket \implies x \cdot !q = !p \cdot x \cdot !q$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{ test } q; !p \cdot x \cdot q = 0 \rrbracket \implies p \cdot x \cdot !q = 0$
 $\langle \text{proof} \rangle$

lemma $\text{test } p \implies p \cdot x = p \cdot x \cdot p \wedge !p \cdot x = !p \cdot x \cdot !p \implies p \cdot x = x \cdot p$
 $\langle \text{proof} \rangle$

end

1.5 Test Semirings

The following class is relevant for Kleene Algebra with Tests.

class *test-semiring* = *test-semiring-zero1* + *semiring-one-zero*

begin

lemma *n-kat-1*: $n \cdot x \cdot y \cdot t \cdot z = 0 \iff n \cdot x \cdot y \cdot n \cdot z = n \cdot x \cdot y$
 $\langle \text{proof} \rangle$

lemma *test-kat-1'*: $\text{test } p \implies \text{test } q \implies p \cdot x \cdot !q = 0 \iff p \cdot x = p \cdot x \cdot q$
 $\langle \text{proof} \rangle$

lemma *n-kat-3*: $n \cdot x \cdot y \cdot t \cdot z = 0 \iff n \cdot x \cdot y \leq y \cdot n \cdot z$
 $\langle \text{proof} \rangle$

lemma *test-kat-3*: $\text{test } p \implies \text{test } q \implies p \cdot x \cdot !q = 0 \iff p \cdot x \leq x \cdot q$
 $\langle \text{proof} \rangle$

lemma *n-kat-prop*: $n \cdot x \cdot y \cdot n \cdot z = n \cdot x \cdot y \iff t \cdot x \cdot y \cdot t \cdot z = y \cdot t \cdot z$
 $\langle \text{proof} \rangle$

lemma *test-kat-prop*: $\text{test } p \implies \text{test } q \implies p \cdot x = p \cdot x \cdot q \iff x \cdot !q = !p \cdot x \cdot !q$
 $\langle \text{proof} \rangle$

lemma *n-kat-3-opp*: $t \cdot x \cdot y \cdot n \cdot z = 0 \iff y \cdot n \cdot z \leq n \cdot x \cdot y$
 $\langle \text{proof} \rangle$

lemma *kat-1-var*: $n \cdot x \cdot y \cdot n \cdot z = y \cdot n \cdot z \iff y \cdot n \cdot z \leq n \cdot x \cdot y$
 $\langle \text{proof} \rangle$

lemma $\llbracket \text{test } p; \text{ test } q \rrbracket \implies (p \cdot x \cdot !q = 0) \implies (!p \cdot x \cdot q = 0)$
 $\langle \text{proof} \rangle$

lemma $n \cdot x \cdot y + t \cdot x \cdot z = n \cdot x \cdot y \vee n \cdot x \cdot y + t \cdot x \cdot z = t \cdot x \cdot z$

$\langle \text{proof} \rangle$

end

end

2 Pre-Conway Algebra with Tests

theory *Conway-Tests*

imports *Kleene-Algebra.Conway Test-Dioid*

begin

class *near-conway-zero-tests* = *near-conway-zero* + *test-near-semiring-zero-distrib*

begin

lemma *n-preserve1-var*: $n \cdot x \cdot y \leq n \cdot x \cdot y \cdot n \cdot x \implies n \cdot x \cdot (n \cdot x \cdot y + t \cdot x \cdot z)^\dagger \leq (n \cdot x \cdot y)^\dagger \cdot n \cdot x$
 $\langle \text{proof} \rangle$

lemma *test-preserve1-var*: $\text{test } p \implies p \cdot x \leq p \cdot x \cdot p \implies p \cdot (p \cdot x + !p \cdot y)^\dagger \leq (p \cdot x)^\dagger \cdot p$
 $\langle \text{proof} \rangle$

end

class *test-pre-conway* = *pre-conway* + *test-pre-dioid-zero*

begin

subclass *near-conway-zero-tests*
 $\langle \text{proof} \rangle$

lemma *test-preserve*: $\text{test } p \implies p \cdot x \leq p \cdot x \cdot p \implies p \cdot x^\dagger = (p \cdot x)^\dagger \cdot p$
 $\langle \text{proof} \rangle$

lemma *test-preserve1*: $\text{test } p \implies p \cdot x \leq p \cdot x \cdot p \implies p \cdot (p \cdot x + !p \cdot y)^\dagger = (p \cdot x)^\dagger \cdot p$
 $\langle \text{proof} \rangle$

lemma *test-preserve2*: $\text{test } p \implies p \cdot x \cdot p = p \cdot x \implies p \cdot (p \cdot x + !p \cdot y)^\dagger \leq x^\dagger$
 $\langle \text{proof} \rangle$

end

end

3 Kleene Algebra with Tests

theory *KAT*

imports *Kleene-Algebra.Kleene-Algebra Conway-Tests*

begin

First, we study left Kleene algebras with tests which also have only a left zero. These structures can be expanded to demonic refinement algebras.

class *left-kat-zero1* = *left-kleene-algebra-zero1* + *test-semiring-zero1*

begin

lemma *star-n-export1*: $(n \cdot x \cdot y)^* \cdot n \cdot x \leq n \cdot x \cdot y^*$
 $\langle \text{proof} \rangle$

lemma *star-test-export1*: $\text{test } p \implies (p \cdot x)^* \cdot p \leq p \cdot x^*$
 $\langle \text{proof} \rangle$

lemma *star-n-export2*: $(n \cdot x \cdot y)^* \cdot n \cdot x \leq y^* \cdot n \cdot x$
 $\langle \text{proof} \rangle$

lemma *star-test-export2*: $\text{test } p \implies (p \cdot x)^* \cdot p \leq x^* \cdot p$
 $\langle \text{proof} \rangle$

lemma *star-n-export-left*: $x \cdot n \cdot y \leq n \cdot y \cdot x \implies x^* \cdot n \cdot y = n \cdot y \cdot (x \cdot n \cdot y)^*$
 $\langle \text{proof} \rangle$

lemma *star-test-export-left*: $\text{test } p \implies x \cdot p \leq p \cdot x \longrightarrow x^* \cdot p = p \cdot (x \cdot p)^*$
 $\langle \text{proof} \rangle$

lemma *star-n-export-right*: $x \cdot n \cdot y \leq n \cdot y \cdot x \implies x^* \cdot n \cdot y = (n \cdot y \cdot x)^* \cdot n \cdot y$
 $\langle \text{proof} \rangle$

lemma *star-test-export-right*: $\text{test } p \implies x \cdot p \leq p \cdot x \longrightarrow x^* \cdot p = (p \cdot x)^* \cdot p$
 $\langle \text{proof} \rangle$

lemma *star-n-folk*: $n \cdot z \cdot x = x \cdot n \cdot z \implies n \cdot z \cdot y = y \cdot n \cdot z \implies (n \cdot z \cdot x + t \cdot z \cdot y)^* \cdot n \cdot z = n \cdot z \cdot (n \cdot z \cdot x)^*$
 $\langle \text{proof} \rangle$

lemma *star-test-folk*: $\text{test } p \implies p \cdot x = x \cdot p \longrightarrow p \cdot y = y \cdot p \longrightarrow (p \cdot x + !p \cdot y)^* \cdot p = p \cdot (p \cdot x)^*$
 $\langle \text{proof} \rangle$

end

```

class kat-zero1 = kleene-algebra-zero1 + test-semiring-zero1
begin

sublocale conway: near-conway-zero1-tests star ⟨proof⟩

lemma n-star-sim-right:  $n\ y \cdot x = x \cdot n\ y \implies n\ y \cdot x^* = (n\ y \cdot x)^* \cdot n\ y$ 
  ⟨proof⟩

lemma star-sim-right:  $\text{test } p \implies p \cdot x = x \cdot p \longrightarrow p \cdot x^* = (p \cdot x)^* \cdot p$ 
  ⟨proof⟩

lemma n-star-sim-left:  $n\ y \cdot x = x \cdot n\ y \implies n\ y \cdot x^* = n\ y \cdot (x \cdot n\ y)^*$ 
  ⟨proof⟩

lemma star-sim-left:  $\text{test } p \implies p \cdot x = x \cdot p \longrightarrow p \cdot x^* = p \cdot (x \cdot p)^*$ 
  ⟨proof⟩

lemma n-comm-star:  $n\ z \cdot x = x \cdot n\ z \implies n\ z \cdot y = y \cdot n\ z \implies n\ z \cdot x \cdot (n\ z \cdot y)^* = n\ z \cdot x \cdot y^*$ 
  ⟨proof⟩

lemma comm-star:  $\text{test } p \implies p \cdot x = x \cdot p \longrightarrow p \cdot y = y \cdot p \longrightarrow p \cdot x \cdot (p \cdot y)^* = p \cdot x \cdot y^*$ 
  ⟨proof⟩

lemma n-star-sim-right-var:  $n\ y \cdot x = x \cdot n\ y \implies x^* \cdot n\ y = n\ y \cdot (x \cdot n\ y)^*$ 
  ⟨proof⟩

lemma star-sim-right-var:  $\text{test } p \implies p \cdot x = x \cdot p \longrightarrow x^* \cdot p = p \cdot (x \cdot p)^*$ 
  ⟨proof⟩

end

  Finally, we define Kleene algebra with tests.

class kat = kleene-algebra + test-semiring

begin

sublocale conway: test-pre-conway star ⟨proof⟩

end

end

```

4 Demonic Refinement Algebra with Tests

```

theory DRAT
  imports KAT Kleene-Algebra.DRA

```

begin

In this section, we define demonic refinement algebras with tests and prove the most important theorems from the literature. In this context, tests are also known as guards.

class *drat* = *dra* + *test-semiring-zero*
begin

subclass *kat-zero* *<proof>*

An assertion is a mapping from a guard to a subset similar to tests, but it aborts if the predicate does not hold.

definition *assertion* :: '*a* ⇒ '*a* (*⋅*^{*o*}) [101] 100) **where**
test p ⇒ *p*^{*o*} = !*p* · *⊤* + 1

lemma *asg*: $\llbracket \text{test } p; \text{test } q \rrbracket \implies q \leq 1 \wedge 1 \leq p^o$
<proof>

lemma *assertion-isol*: $\text{test } p \implies y \leq p^o \cdot x \longleftrightarrow p \cdot y \leq x$
<proof>

lemma *assertion-isor*: $\text{test } p \implies y \leq x \cdot p \longleftrightarrow y \cdot p^o \leq x$
<proof>

lemma *assertion-iso*: $\llbracket \text{test } p; \text{test } q \rrbracket \implies x \cdot q^o \leq p^o \cdot x \longleftrightarrow p \cdot x \leq x \cdot q$
<proof>

lemma *total-correctness*: $\llbracket \text{test } p; \text{test } q \rrbracket \implies p \cdot x \cdot !q = 0 \longleftrightarrow x \cdot !q \leq !p \cdot \top$
<proof>

lemma *test-iteration-sim*: $\llbracket \text{test } p; p \cdot x \leq x \cdot p \rrbracket \implies p \cdot x^\infty \leq x^\infty \cdot p$
<proof>

lemma *test-iteration-annir*: $\text{test } p \implies !p \cdot (p \cdot x)^\infty = !p$
<proof>

Next we give an example of a program transformation from von Wright [5].

lemma *loop-refinement*: $\llbracket \text{test } p; \text{test } q \rrbracket \implies (p \cdot x)^\infty \cdot !p \leq (p \cdot q \cdot x)^\infty \cdot !(p \cdot q) \cdot (p \cdot x)^\infty \cdot !p$
<proof>

Finally, we prove different versions of Back's atomicity refinement theorem for action systems.

lemma *atom-step1*: $r \cdot b \leq b \cdot r \implies (a + b + r)^\infty = b^\infty \cdot r^\infty \cdot (a \cdot b^\infty \cdot r^\infty)^\infty$
<proof>

lemma *atom-step2*:

assumes $s = s \cdot q \cdot q \cdot b = 0 \ r \cdot q \leq q \cdot r \ q \cdot l \leq l \cdot q \ r^\infty = r^\star \ \text{test } q$
shows $s \cdot l^\infty \cdot b^\infty \cdot r^\infty \cdot q \cdot (a \cdot b^\infty \cdot r^\infty \cdot q)^\infty \leq s \cdot l^\infty \cdot r^\infty \cdot (a \cdot b^\infty \cdot q \cdot r^\infty)^\infty$

$\langle proof \rangle$

lemma *atom-step3*:

assumes $r \cdot l \leq l \cdot r \ a \cdot l \leq l \cdot a \ b \cdot l \leq l \cdot b \ q \cdot l \leq l \cdot q \ b^\infty = b^\star$

shows $l^\infty \cdot r^\infty \cdot (a \cdot b^\infty \cdot q \cdot r^\infty)^\infty = (a \cdot b^\infty \cdot q + l + r)^\infty$

$\langle proof \rangle$

This is Back's atomicity refinement theorem, as specified by von Wright [5].

theorem *atom-ref-back*:

assumes $s = s \cdot q \ a = q \cdot a \ q \cdot b = 0$

$r \cdot b \leq b \cdot r \ r \cdot l \leq l \cdot r \ r \cdot q \leq q \cdot r$

$a \cdot l \leq l \cdot a \ b \cdot l \leq l \cdot b \ q \cdot l \leq l \cdot q$

$r^\infty = r^\star \ b^\infty = b^\star \ test \ q$

shows $s \cdot (a + b + r + l)^\infty \cdot q \leq s \cdot (a \cdot b^\infty \cdot q + r + l)^\infty$

$\langle proof \rangle$

This variant is due to Höfner, Struth and Sutcliffe [2].

theorem *atom-ref-back-struth*:

assumes $s \leq s \cdot q \ a \leq q \cdot a \ q \cdot b = 0$

$r \cdot b \leq b \cdot r \ r \cdot q \leq q \cdot r$

$(a + r + b) \cdot l \leq l \cdot (a + r + b) \ q \cdot l \leq l \cdot q$

$r^\infty = r^\star \ q \leq 1$

shows $s \cdot (a + b + r + l)^\infty \cdot q \leq s \cdot (a \cdot b^\infty \cdot q + r + l)^\infty$

$\langle proof \rangle$

Finally, we prove Cohen's [1] variation of the atomicity refinement theorem.

lemma *atom-ref-cohen*:

assumes $r \cdot p \cdot y \leq y \cdot r \ y \cdot p \cdot l \leq l \cdot y \ r \cdot p \cdot l \leq l \cdot r$

$p \cdot r \cdot !p = 0 \ p \cdot l \cdot !p = 0 \ !p \cdot l \cdot p = 0$

$y \cdot 0 = 0 \ r \cdot 0 = 0 \ test \ p$

shows $(y + r + l)^\infty = (p \cdot l)^\infty \cdot (y + !p \cdot l + r \cdot !p)^\infty \cdot (r \cdot p)^\infty$

$\langle proof \rangle$

end

end

5 Models for Demonic Refinement Algebra with Tests

theory *DRA-Models*

imports *DRAT*

begin

We formalise the predicate transformer model of demonic refinement algebra. Predicate transformers are formalised as strict and additive functions over a field of sets, or alternatively as costrict and multiplicative functions.

In the future, this should be merged with Preoteasa's more abstract formalisation [4].

no-notation

plus (**infixl** $\langle + \rangle$ 65) **and**
less-eq ($\langle'(\leq)'\rangle$) **and**
less-eq ($\langle(\text{notation}=\text{infix } \leq)\rangle - / \leq - \rangle$ [51, 51] 50)

notation *comp* (**infixl** $\langle \cdot \rangle$ 55)

type-synonym $'a \text{ bfun} = 'a \text{ set} \Rightarrow 'a \text{ set}$

Definitions of signature:

definition *top* :: $'a \text{ bfun}$ **where** $\text{top} \equiv \lambda x. \text{UNIV}$

definition *bot* :: $'a \text{ bfun}$ **where** $\text{bot} \equiv \lambda x. \{\}$

definition *adjoint* :: $'a \text{ bfun} \Rightarrow 'a \text{ bfun}$ **where** $\text{adjoint } f \equiv (\lambda p. - f (-p))$

definition *fun-inter* :: $'a \text{ bfun} \Rightarrow 'a \text{ bfun} \Rightarrow 'a \text{ bfun}$ (**infix** $\langle \sqcap \rangle$ 51) **where**
 $f \sqcap g \equiv \lambda p. f p \sqcap g p$

definition *fun-union* :: $'a \text{ bfun} \Rightarrow 'a \text{ bfun} \Rightarrow 'a \text{ bfun}$ (**infix** $\langle + \rangle$ 52) **where**
 $f + g \equiv \lambda p. f p \cup g p$

definition *fun-order* :: $'a \text{ bfun} \Rightarrow 'a \text{ bfun} \Rightarrow \text{bool}$ (**infix** $\langle \leq \rangle$ 50) **where**
 $f \leq g \equiv \forall p. f p \subseteq g p$

definition *fun-strict-order* :: $'a \text{ bfun} \Rightarrow 'a \text{ bfun} \Rightarrow \text{bool}$ (**infix** $\langle < \cdot \rangle$ 50) **where**
 $f < \cdot g \equiv f \leq g \wedge f \neq g$

definition *N* :: $'a \text{ bfun} \Rightarrow 'a \text{ bfun}$ **where**
 $N f \equiv ((\text{adjoint } f \circ \text{bot}) \sqcap \text{id})$

lemma *top-max*: $f \leq \text{top}$
 $\langle \text{proof} \rangle$

lemma *bot-min*: $\text{bot} \leq f$
 $\langle \text{proof} \rangle$

lemma *oder-def*: $f \sqcap g = f \implies f \leq g$
 $\langle \text{proof} \rangle$

lemma *order-def-var*: $f \leq g \implies f \sqcap g = f$
 $\langle \text{proof} \rangle$

lemma *adjoint-idem* [simp]: $\text{adjoint } (\text{adjoint } f) = f$
 $\langle \text{proof} \rangle$

lemma *adjoint-prop1* [simp]: $(f \circ \text{top}) \sqcap (\text{adjoint } f \circ \text{bot}) = \text{bot}$
 $\langle \text{proof} \rangle$

lemma *adjoint-prop2*[simp]: $(f \circ top) + (adjoint\ f \circ bot) = top$
 $\langle proof \rangle$

lemma *adjoint-mult*: $adjoint\ (f \circ g) = adjoint\ f \circ adjoint\ g$
 $\langle proof \rangle$

lemma *adjoint-top*[simp]: $adjoint\ top = bot$
 $\langle proof \rangle$

lemma *N-comp1*: $(N\ (N\ f)) + N\ f = id$
 $\langle proof \rangle$

lemma *N-comp2*: $(N\ (N\ f)) \circ N\ f = bot$
 $\langle proof \rangle$

lemma *N-comp3*: $N\ f \circ (N\ (N\ f)) = bot$
 $\langle proof \rangle$

lemma *N-de-morgan*: $N\ (N\ f) \circ N\ (N\ g) = N\ (N\ f) \sqcap N\ (N\ g)$
 $\langle proof \rangle$

lemma *conj-pred-aux* [simp]: $(\lambda p. x\ p \cup y\ p) = y \implies \forall p. x\ p \subseteq y\ p$
 $\langle proof \rangle$

Next, we define a type for conjunctive or multiplicative predicate transformers.

typedef *'a bool-op* = $\{f :: 'a\ bfun. (\forall g\ h. mono\ f \wedge f \circ g + f \circ h = f \circ (g + h) \wedge bot \circ f = bot)\}$
 $\langle proof \rangle$

setup-lifting *type-definition-bool-op*

Conjunctive predicate transformers form a dioid with tests without right annihilator.

instantiation *bool-op* :: (*type*) *dioid-one-zero*
begin

lift-definition *less-eq-bool-op* :: *'a bool-op* $\Rightarrow$ *'a bool-op* $\Rightarrow$ *bool* **is** *fun-order*
 $\langle proof \rangle$

lift-definition *less-bool-op* :: *'a bool-op* $\Rightarrow$ *'a bool-op* $\Rightarrow$ *bool* **is** $(<.)$ $\langle proof \rangle$

lift-definition *zero-bool-op* :: *'a bool-op* **is** *bot*
 $\langle proof \rangle$

lift-definition *one-bool-op* :: *'a bool-op* **is** *id*
 $\langle proof \rangle$

lift-definition *times-bool-op* :: *'a bool-op* $\Rightarrow$ *'a bool-op* $\Rightarrow$ *'a bool-op* **is** $(\circ)$
 $\langle proof \rangle$

```

lift-definition plus-bool-op :: 'a bool-op  $\Rightarrow$  'a bool-op  $\Rightarrow$  'a bool-op is (+)
   $\langle proof \rangle$ 

instance
   $\langle proof \rangle$ 

end

instantiation bool-op :: (type) test-semiring-zero1
begin
  lift-definition n-op-bool-op :: 'a bool-op  $\Rightarrow$  'a bool-op is N
     $\langle proof \rangle$ 

  instance
     $\langle proof \rangle$ 

end

definition fun-star :: 'a bfun  $\Rightarrow$  'a bfun where
  fun-star f = lfp ( $\lambda r. f \circ r + id$ )

definition fun-iteration :: 'a bfun  $\Rightarrow$  'a bfun where
  fun-iteration f = gfp ( $\lambda g. f \circ g + id$ )

  Verifying the iteration laws is left for future work. This could be obtained
  by integrating Preoteasa's approach [4].

end

```

6 Models for Kleene Algebra with Tests

```

theory KAT-Models
  imports Kleene-Algebra.Kleene-Algebra-Models KAT
begin

```

We show that binary relations under the obvious definitions form a Kleene algebra with tests.

```

interpretation rel-dioid-tests: test-semiring ( $\cup$ ) ( $O$ )  $Id$   $\{\}$  ( $\subseteq$ ) ( $\subset$ )  $\lambda x. Id \cap (- x)$ 
   $\langle proof \rangle$ 

```

```

interpretation rel-kat: kat ( $\cup$ ) ( $O$ )  $Id$   $\{\}$  ( $\subseteq$ ) ( $\subset$ ) rtrancl  $\lambda x. Id \cap (- x)$ 
   $\langle proof \rangle$ 

```

```

typedef 'a relation = UNIV::'a rel set  $\langle proof \rangle$ 

```

```

setup-lifting type-definition-relation

```

```

instantiation relation :: (type) kat
begin

  lift-definition n-op-relation :: 'a relation  $\Rightarrow$  'a relation is  $\lambda x. Id \cap ( - x)$   $\langle proof \rangle$ 
  lift-definition zero-relation :: 'a relation is  $\{\}$   $\langle proof \rangle$ 
  lift-definition star-relation :: 'a relation  $\Rightarrow$  'a relation is rtranc1  $\langle proof \rangle$ 
  lift-definition less-eq-relation :: 'a relation  $\Rightarrow$  'a relation  $\Rightarrow$  bool is  $(\subseteq)$   $\langle proof \rangle$ 
  lift-definition less-relation :: 'a relation  $\Rightarrow$  'a relation  $\Rightarrow$  bool is  $(\subset)$   $\langle proof \rangle$ 
  lift-definition one-relation :: 'a relation is Id  $\langle proof \rangle$ 
  lift-definition times-relation :: 'a relation  $\Rightarrow$  'a relation  $\Rightarrow$  'a relation is  $(O)$ 
 $\langle proof \rangle$ 
  lift-definition plus-relation :: 'a relation  $\Rightarrow$  'a relation  $\Rightarrow$  'a relation is  $(\cup)$ 
 $\langle proof \rangle$ 

  instance
     $\langle proof \rangle$ 

end

end

```

7 Transformation Theorem for while Loops

```

theory FolkTheorem
  imports Conway-Tests KAT DRAT
begin

```

We prove Kozen's transformation theorem for while loops [3] in a weak setting that unifies previous proofs in Kleene algebra with tests, demonic refinement algebras and a variant of probabilistic Kleene algebra.

```

context test-pre-conway
begin

```

```

abbreviation pres :: 'a  $\Rightarrow$  'a  $\Rightarrow$  bool where
  pres x y  $\equiv y \cdot x = y \cdot x \cdot y$ 

```

```

lemma pres-comp: pres y z  $\Longrightarrow$  pres x z  $\Longrightarrow$  pres (x  $\cdot$  y) z
 $\langle proof \rangle$ 

```

```

lemma test-pres1: test p  $\Longrightarrow$  test q  $\Longrightarrow$  pres p q
 $\langle proof \rangle$ 

```

```

lemma test-pres2: test p  $\Longrightarrow$  test q  $\Longrightarrow$  pres x q  $\Longrightarrow$  pres (p  $\cdot$  x) q
 $\langle proof \rangle$ 

```

```

lemma cond-helper1:
assumes test p and pres x p
shows  $p \cdot (p \cdot x + !p \cdot y)^\dagger \cdot (p \cdot z + !p \cdot w) = p \cdot x^\dagger \cdot z$ 

```


$\langle proof \rangle$

lemma *cond-helper2*:

assumes *test p and pres y (!p)*

shows $!p \cdot (p \cdot x + !p \cdot y)^\dagger \cdot (p \cdot z + !p \cdot w) = !p \cdot y^\dagger \cdot w$

$\langle proof \rangle$

lemma *cond-distr-var*:

assumes *test p and test q and test r*

shows $(q \cdot p + r \cdot !p) \cdot (p \cdot x + !p \cdot y) = q \cdot p \cdot x + r \cdot !p \cdot y$

$\langle proof \rangle$

lemma *cond-distr*:

assumes *test p and test q and test r*

shows $(p \cdot q + !p \cdot r) \cdot (p \cdot x + !p \cdot y) = p \cdot q \cdot x + !p \cdot r \cdot y$

$\langle proof \rangle$

theorem *conditional*:

assumes *test p and test q and test r1 and test r2*

and *pres x1 q and pres y1 q and pres x2 (!q) and pres y2 (!q)*

shows $(p \cdot q + !p \cdot !q) \cdot (q \cdot x1 + !q \cdot x2) \cdot ((q \cdot r1 + !q \cdot r2) \cdot (q \cdot y1 + !q \cdot y2))^\dagger \cdot !(q \cdot r1 + !q \cdot r2) =$

$(p \cdot q + !p \cdot !q) \cdot (p \cdot x1 \cdot (r1 \cdot y1)^\dagger \cdot !r1 + !p \cdot x2 \cdot (r2 \cdot y2)^\dagger \cdot !r2)$

$\langle proof \rangle$

theorem *nested-loops*:

assumes *test p and test q*

shows $p \cdot x \cdot ((p + q) \cdot (q \cdot y + !q \cdot x))^\dagger \cdot !(p + q) + !p = (p \cdot x \cdot (q \cdot y)^\dagger \cdot !q)^\dagger \cdot !p$

$\langle proof \rangle$

lemma *postcomputation*:

assumes *test p and pres y (!p)*

shows $!p \cdot y + p \cdot (p \cdot x \cdot (!p \cdot y + p))^\dagger \cdot !p = (p \cdot x)^\dagger \cdot !p \cdot y$

$\langle proof \rangle$

lemma *composition-helper*:

assumes *test p and test q*

and *pres x p*

shows $p \cdot (q \cdot x)^\dagger \cdot !q \cdot p = p \cdot (q \cdot x)^\dagger \cdot !q$

$\langle proof \rangle$

theorem *composition*:

assumes *test p and test q*

and *pres y p and pres y (!p)*

shows $(p \cdot x)^\dagger \cdot !p \cdot (q \cdot y)^\dagger \cdot !q = !p \cdot (q \cdot y)^\dagger \cdot !q + p \cdot (p \cdot x \cdot (!p \cdot (q \cdot y)^\dagger \cdot !q + p))^\dagger \cdot !p$

$\langle proof \rangle$

end

Kleene algebras with tests form pre-Conway algebras, therefore the transformation theorem is valid for KAT as well.

sublocale *kat* $\subseteq$ *pre-conway star* $\langle$ *proof* $\rangle$

Demonic refinement algebras form pre-Conway algebras, therefore the transformation theorem is valid for DRA as well.

sublocale *drat* $\subseteq$ *pre-conway strong-iteration*
 $\langle$ *proof* $\rangle$

We do not currently consider an expansion of probabilistic Kleene algebra.

end

8 Propositional Hoare Logic

theory *PHL-KAT*

imports *KAT Kleene-Algebra.PHL-KA*

begin

We define a class of pre-dioids with notions of assertions, tests and iteration. The above rules of PHL are derivable in that class.

class *at-pre-dioid* = *pre-dioid-one* +
 fixes *alpha* :: 'a $\Rightarrow$ 'a ($\langle\alpha\rangle$)
 and *tau* :: 'a $\Rightarrow$ 'a ($\langle\tau\rangle$)
 assumes *at-pres*: $\alpha \ x \cdot \tau \ y \leq \tau \ y \cdot \alpha \ x \cdot \tau \ y$
 and *as-left-supdistl*: $\alpha \ x \cdot (y + z) \leq \alpha \ x \cdot y + \alpha \ x \cdot z$

begin

Only the conditional and the iteration rule need to be considered (in addition to the export laws. In this context, they no longer depend on external assumptions. The other ones do not depend on any assumptions anyway.

lemma *at-phl-cond*:

assumes $\alpha \ u \cdot \tau \ v \cdot x \leq x \cdot z$ **and** $\alpha \ u \cdot \tau \ w \cdot y \leq y \cdot z$
shows $\alpha \ u \cdot (\tau \ v \cdot x + \tau \ w \cdot y) \leq (\tau \ v \cdot x + \tau \ w \cdot y) \cdot z$
 $\langle$ *proof* $\rangle$

lemma *ht-at-phl-cond*:

assumes $\{\alpha \ u \cdot \tau \ v\} \ x \ \{\!|z|\!\}$ **and** $\{\alpha \ u \cdot \tau \ w\} \ y \ \{\!|z|\!\}$
shows $\{\alpha \ u\} \ (\tau \ v \cdot x + \tau \ w \cdot y) \ \{\!|z|\!\}$
 $\langle$ *proof* $\rangle$

lemma *ht-at-phl-export1*:

assumes $\{\alpha \ x \cdot \tau \ y\} \ z \ \{\!|w|\!\}$
shows $\{\alpha \ x\} \ \tau \ y \cdot z \ \{\!|w|\!\}$
 $\langle$ *proof* $\rangle$

```

lemma ht-at-phl-export2:
assumes  $\llbracket x \rrbracket y \llbracket \alpha z \rrbracket$ 
shows  $\llbracket x \rrbracket y \cdot \tau w \llbracket \alpha z \cdot \tau w \rrbracket$ 
   $\langle proof \rangle$ 

end

```

```

class at-it-pre-diod = at-pre-diod + it-pre-diod
begin

```

```

lemma at-phl-while:
assumes  $\alpha p \cdot \tau s \cdot x \leq x \cdot \alpha p$ 
shows  $\alpha p \cdot (it (\tau s \cdot x) \cdot \tau w) \leq it (\tau s \cdot x) \cdot \tau w \cdot (\alpha p \cdot \tau w)$ 
   $\langle proof \rangle$ 

```

```

lemma ht-at-phl-while:
assumes  $\llbracket \alpha p \cdot \tau s \rrbracket x \llbracket \alpha p \rrbracket$ 
shows  $\llbracket \alpha p \rrbracket it (\tau s \cdot x) \cdot \tau w \llbracket \alpha p \cdot \tau w \rrbracket$ 
   $\langle proof \rangle$ 

```

```

end

```

The following statements show that pre-Conway algebras, Kleene algebras with tests and demonic refinement algebras form pre-diods with test and assertions. This automatically generates propositional Hoare logics for these structures.

```

sublocale test-pre-diod-zero1 < phl: at-pre-diod where  $\alpha = t$  and  $\tau = t$ 
   $\langle proof \rangle$ 

```

```

sublocale test-pre-conway < phl: at-it-pre-diod where  $\alpha = t$  and  $\tau = t$ 
and  $it = dagger$   $\langle proof \rangle$ 

```

```

sublocale kat-zero1 < phl: at-it-pre-diod where  $\alpha = t$  and  $\tau = t$  and  $it = star$ 
   $\langle proof \rangle$ 

```

```

context test-pre-diod-zero1 begin

```

```

abbreviation if-then-else :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a ( $\langle if - then - else - fi \rangle$  [64,64,64]
63) where
   $if\ p\ then\ x\ else\ y\ fi \equiv (p \cdot x + !p \cdot y)$ 

```

```

lemma phl-n-cond:
 $\llbracket n v \cdot n w \rrbracket x \llbracket z \rrbracket \Longrightarrow \llbracket n v \cdot t w \rrbracket y \llbracket z \rrbracket \Longrightarrow \llbracket n v \rrbracket n w \cdot x + t w \cdot y \llbracket z \rrbracket$ 
   $\langle proof \rangle$ 

```

```

lemma phl-test-cond:
assumes  $test\ p$  and  $test\ b$ 
and  $\llbracket p \cdot b \rrbracket x \llbracket q \rrbracket$  and  $\llbracket p \cdot !b \rrbracket y \llbracket q \rrbracket$ 

```

shows $\{p\} \ b \cdot x + !b \cdot y \ \{q\}$
 $\langle proof \rangle$

lemma *phl-cond-syntax*:
assumes *test p and test b*
and $\{p \cdot b\} \ x \ \{q\}$ **and** $\{p \cdot !b\} \ y \ \{q\}$
shows $\{p\} \ \text{if } b \text{ then } x \text{ else } y \text{ fi} \ \{q\}$
 $\langle proof \rangle$

lemma *phl-cond-syntax-iff*:
assumes *test p and test b*
shows $\{p \cdot b\} \ x \ \{q\} \wedge \{p \cdot !b\} \ y \ \{q\} \longleftrightarrow \{p\} \ \text{if } b \text{ then } x \text{ else } y \text{ fi} \ \{q\}$
 $\langle proof \rangle$

end

context *test-pre-conway*
begin

lemma *phl-n-while*:
assumes $\{n \ x \cdot \ n \ y\} \ z \ \{n \ x\}$
shows $\{n \ x\} \ (n \ y \cdot z)^{\dagger} \cdot t \ y \ \{n \ x \cdot t \ y\}$
 $\langle proof \rangle$

end

context *kat-zerol*
begin

abbreviation *while* :: 'a $\Rightarrow$ 'a $\Rightarrow$ 'a (*while - do - od* [64,64] 63) **where**
while *b do x od* $\equiv (b \cdot x)^{\star} \cdot !b$

lemma *phl-n-while*:
assumes $\{n \ x \cdot \ n \ y\} \ z \ \{n \ x\}$
shows $\{n \ x\} \ (n \ y \cdot z)^{\star} \cdot t \ y \ \{n \ x \cdot t \ y\}$
 $\langle proof \rangle$

lemma *phl-test-while*:
assumes *test p and test b*
and $\{p \cdot b\} \ x \ \{p\}$
shows $\{p\} \ (b \cdot x)^{\star} \cdot !b \ \{p \cdot !b\}$
 $\langle proof \rangle$

lemma *phl-while-syntax*:
assumes *test p and test b and* $\{p \cdot b\} \ x \ \{p\}$
shows $\{p\} \ \text{while } b \text{ do } x \text{ od} \ \{p \cdot !b\}$
 $\langle proof \rangle$

end

lemma (in kat) $test\ p \implies p \cdot x^* \leq x^* \cdot p \implies p \cdot x \leq x \cdot p$
⟨proof⟩

lemma (in kat) $test\ p \implies test\ b \implies p \cdot (b \cdot x)^* \cdot !b \leq (b \cdot x)^* \cdot !b \cdot p \cdot !b \implies p \cdot b \cdot x \leq x \cdot p$
(proof)

lemma (in kat) $test\ p \implies test\ q \implies p \cdot x \cdot y \leq x \cdot y \cdot q \implies (\exists r. test\ r \wedge p \cdot x \leq x \cdot r \wedge r \cdot y \leq y \cdot q)$
 $\langle proof \rangle$

$$\text{lemma (in } \textit{kat}) \text{ test } p \implies \text{test } q \implies p \cdot x \cdot y \cdot !q = 0 \implies (\exists r. \text{test } r \wedge p \cdot x \cdot !r = 0 \wedge r \cdot y \cdot !q = 0)$$

$$\langle \textit{proof} \rangle$$

The following facts should be moved. They show that the rules of Hoare logic based on Tarlecki triples are invertible.

abbreviation (in *near-diodid*) $tt :: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ ($\langle \langle \cdot \rangle \rangle - \langle \cdot \rangle \rangle$) **where**
 $\langle x \rangle \ y \ \langle z \rangle \equiv x \cdot y \leq z$

lemma (in *near-dioid-one*) *tt-skip*: $\llbracket p \rrbracket \leq \llbracket p \rrbracket$
 $\langle proof \rangle$

lemma (in *near-dioid*) *tt-cons1*: $(\exists q'. \langle p \rangle \ x \ \langle q' \rangle \wedge q' \leq q) \longleftrightarrow \langle p \rangle \ x \ \langle q \rangle$
<proof>

lemma (in *near-diod*) *tt-cons2*: $(\exists p'. \langle p' \rangle \ x \ \langle q \rangle \wedge p \leq p') \longleftrightarrow \langle p \rangle \ x \ \langle q \rangle$
<proof>

lemma (in *near-diod*) *tt-seq*: $(\exists r. \langle p \rangle \ x \ \langle r \rangle \wedge \langle r \rangle \ y \ \langle q \rangle) \longleftrightarrow \langle p \rangle \ x \cdot y \ \langle q \rangle$
<proof>

lemma (in dioid) tt-cond: $\langle p \cdot v \rangle x \langle q \rangle \wedge \langle p \cdot w \rangle y \langle q \rangle \longleftrightarrow \langle p \rangle (v \cdot x + w \cdot y) \langle q \rangle$
 $\langle proof \rangle$

lemma (in *kleene-algebra*) *tt-while*: $w \leq 1 \implies \langle p \cdot v \rangle x \langle p \rangle \implies \langle p \rangle (v \cdot x)^* \cdot w$
 $\langle p \rangle$
 $\langle proof \rangle$

The converse implication can be refuted. The situation is similar to the ht case.

lemma (in *kat*) *tt-while*: $test\ v \implies \langle p \rangle\ (v \cdot x)^{\star} \cdot !v\ \langle p \rangle \implies \langle p \cdot v \rangle\ x\ \langle p \rangle$
<proof>

lemma (in kat) *tt-while: test* $v \Longrightarrow \llbracket p \rrbracket (v \cdot x)^* \llbracket p \rrbracket \Longrightarrow \llbracket p \cdot v \rrbracket x \llbracket p \rrbracket$
 $\langle_{proof} \rangle$

Perhaps this holds with possibly infinite loops in DRA...

wlp in KAT

lemma (in *kat*) $\text{test } y \implies (\exists z. \text{test } z \wedge z \cdot x \cdot !y = 0)$
 ⟨*proof*⟩

end

9 Propositional Hoare Logic

theory *PHL-DRAT*
 imports *DRAT Kleene-Algebra.PHL-DRA PHL-KAT*
 begin

sublocale *drat* < *phl*: *at-it-pre-diod* **where** $\alpha = t$ **and** $\tau = t$ **and** *it* = *strong-iteration* ⟨*proof*⟩

context *drat*
 begin

no-notation *while* (⟨*while* - *do* - *od*⟩ [64,64] 63)

abbreviation *while* :: '*a* $\Rightarrow$ '*a* $\Rightarrow$ '*a* (⟨*while* - *do* - *od*⟩ [64,64] 63) **where**
while *b* *do* *x* *od* $\equiv (b \cdot x)^\infty \cdot !b$

lemma *phl-n-while*:
 assumes $\{n\ x \cdot n\ y\} z \{n\ x\}$
 shows $\{n\ x\} (n\ y \cdot z)^\infty \cdot t\ y \{n\ x \cdot t\ y\}$
 ⟨*proof*⟩

lemma *phl-test-while*:
 assumes *test* *p* **and** *test* *b*
 and $\{p \cdot b\} x \{p\}$
 shows $\{p\} (b \cdot x)^\infty \cdot !b \{p \cdot !b\}$
 ⟨*proof*⟩

lemma *phl-while-syntax*:
 assumes *test* *p* **and** *test* *b* **and** $\{p \cdot b\} x \{p\}$
 shows $\{p\} \text{while } b \text{ do } x \text{ od } \{p \cdot !b\}$
 ⟨*proof*⟩

end

end

10 Two sorted Kleene Algebra with Tests

theory *KAT2*
 imports *Kleene-Algebra.Kleene-Algebra*
 begin

As an alternative to the one-sorted implementation of tests, we provide a two-sorted, more conventional one. In this setting, Isabelle's Boolean algebra theory can be used. This alternative can be developed further along the lines of the one-sorted implementation.

syntax *-kat* :: 'a $\Rightarrow$ 'a ($\langle \cdot \rangle$)

$\langle ML \rangle$

locale *diod-tests* =

fixes *test* :: 'a::boolean-algebra $\Rightarrow$ 'b::diod-one-zero
and *not* :: 'b::diod-one-zero $\Rightarrow$ 'b::diod-one-zero ($\langle - \rangle$)
assumes *test-sup* [*simp*,*kat-hom*]: *test* (*sup* *p* *q*) = '*p* + *q*'
and *test-inf* [*simp*,*kat-hom*]: *test* (*inf* *p* *q*) = '*p* · *q*'
and *test-top* [*simp*,*kat-hom*]: *test* *top* = 1
and *test-bot* [*simp*,*kat-hom*]: *test* *bot* = 0
and *test-not* [*simp*,*kat-hom*]: *test* ($-$ *p*) = ' $-$ *p*'
and *test-iso-eq* [*kat-hom*]: *p* $\leq$ *q* $\longleftrightarrow$ '*p* $\leq$ *q*'

begin

notation *test* ($\langle \iota \rangle$)

lemma *test-eq* [*kat-hom*]: *p* = *q* $\longleftrightarrow$ '*p* = *q*'
 $\langle proof \rangle$

$\langle ML \rangle$

lemma *test-iso*: *p* $\leq$ *q* $\Longrightarrow$ '*p* $\leq$ *q*'
 $\langle proof \rangle$

lemma *test-meet-comm*: '*p* · *q* = *q* · *p*'
 $\langle proof \rangle$

lemmas *test-one-top*[*simp*] = *test-iso*[*OF top-greatest, simplified*]

lemma [*simp*]: ' $-$ *p* + *p* = 1'
 $\langle proof \rangle$

lemma [*simp*]: '*p* + ($-$ *p*) = 1'
 $\langle proof \rangle$

lemma [*simp*]: ' $(-)$ *p* · *p* = 0'
 $\langle proof \rangle$

lemma [*simp*]: '*p* · ($-$ *p*) = 0'
 $\langle proof \rangle$

end

```

locale kat =
  fixes test :: 'a::boolean-algebra  $\Rightarrow$  'b::kleene-algebra
  and not :: 'b::kleene-algebra  $\Rightarrow$  'b::kleene-algebra ( $\langle ! \rangle$ )
  assumes is-diod-tests: diod-tests test not

sublocale kat  $\subseteq$  diod-tests  $\langle$ proof $\rangle$ 

context kat
begin

notation test ( $\langle \iota \rangle$ )

lemma test-eq [kat-hom]:  $p = q \longleftrightarrow 'p = q'$ 
   $\langle$ proof $\rangle$ 

 $\langle$ ML $\rangle$ 

lemma test-iso:  $p \leq q \Longrightarrow 'p \leq q'$ 
   $\langle$ proof $\rangle$ 

lemma test-meet-comm:  $'p \cdot q = q \cdot p'$ 
   $\langle$ proof $\rangle$ 

lemmas test-one-top[simp] = test-iso[OF top-greatest, simplified]

lemma test-star [simp]:  $'p^\star = 1'$ 
   $\langle$ proof $\rangle$ 

lemmas [kat-hom] = test-star[symmetric]

lemma [simp]:  $'!p + p = 1'$ 
   $\langle$ proof $\rangle$ 

lemma [simp]:  $'p + !p = 1'$ 
   $\langle$ proof $\rangle$ 

lemma [simp]:  $'!p \cdot p = 0'$ 
   $\langle$ proof $\rangle$ 

lemma [simp]:  $'p \cdot !p = 0'$ 
   $\langle$ proof $\rangle$ 

definition hoare-triple :: 'b  $\Rightarrow$  'b  $\Rightarrow$  'b  $\Rightarrow$  bool ( $\langle \langle ! \rangle - \langle ! \rangle \rangle$ ) where
   $\langle !p \rangle \text{ c } \langle !q \rangle \equiv p \cdot c \leq c \cdot q$ 

declare hoare-triple-def[iff]

```


lemma *hoare-triple-def-var*: $'p \cdot c \leq c \cdot q \longleftrightarrow p \cdot c \cdot !q = 0'$
 $\langle proof \rangle$

lemmas $[intro!] = star-sim2[rule-format]$

lemma *hoare-weakening*: $p \leq p' \implies q' \leq q \implies '\llbracket p' \rrbracket c \llbracket q' \rrbracket' \implies '\llbracket p \rrbracket c \llbracket q \rrbracket'$
 $\langle proof \rangle$

lemma *hoare-star*: $'\llbracket p \rrbracket c \llbracket p \rrbracket' \implies '\llbracket p \rrbracket c^* \llbracket p \rrbracket'$
 $\langle proof \rangle$

lemmas $[vcg] = hoare-weakening[OF order-refl - hoare-star]$

lemma *hoare-test* $[vcg]$: $'p \cdot t \leq q' \implies '\llbracket p \rrbracket t \llbracket q \rrbracket'$
 $\langle proof \rangle$

lemma *hoare-mult* $[vcg]$: $'\llbracket p \rrbracket x \llbracket r \rrbracket' \implies '\llbracket r \rrbracket y \llbracket q \rrbracket' \implies '\llbracket p \rrbracket x \cdot y \llbracket q \rrbracket'$
 $\langle proof \rangle$

lemma $[simp]$: $'!p \cdot !p = !p'$
 $\langle proof \rangle$

lemma *hoare-plus* $[vcg]$: $'\llbracket p \rrbracket x \llbracket q \rrbracket' \implies '\llbracket p \rrbracket y \llbracket q \rrbracket' \implies '\llbracket p \rrbracket x + y \llbracket q \rrbracket'$
 $\langle proof \rangle$

definition *While* :: $'b \Rightarrow 'b \Rightarrow 'b \ (\langle While - Do - End \rangle [50,50] 51)$ **where**
 $While\ t\ Do\ c\ End = (t \cdot c)^* \cdot !t$

lemma *hoare-while*: $'\llbracket p \cdot t \rrbracket c \llbracket p \rrbracket' \implies '\llbracket p \rrbracket\ While\ t\ Do\ c\ End \llbracket !t \cdot p \rrbracket'$
 $\langle proof \rangle$

lemma $[vcg]$: $'\llbracket p \cdot t \rrbracket c \llbracket p \rrbracket' \implies '!t \cdot p \leq q' \implies '\llbracket p \rrbracket\ While\ t\ Do\ c\ End \llbracket q \rrbracket'$
 $\langle proof \rangle$

definition *If* :: $'b \Rightarrow 'b \Rightarrow 'b \Rightarrow 'b \ (\langle If - Then - Else \rightarrow [50,50,50] 51)$ **where**
 $If\ p\ Then\ c1\ Else\ c2 \equiv p \cdot c1 + !p \cdot c2$

lemma *hoare-if* $[vcg]$: $'\llbracket p \cdot t \rrbracket c1 \llbracket q \rrbracket' \implies '\llbracket p \cdot !t \rrbracket c2 \llbracket q \rrbracket' \implies '\llbracket p \rrbracket\ If\ t\ Then\ c1\ Else\ c2 \llbracket q \rrbracket'$
 $\langle proof \rangle$

end

end

11 Two sorted Demonic Refinement Algebras

theory *DRAT2*

imports *Kleene-Algebra.DRA*
begin

As an alternative to the one-sorted implementation of demonic refinement algebra with tests, we provide a two-sorted, more conventional one. This alternative can be developed further along the lines of the one-sorted implementation.

syntax $-dra :: 'a \Rightarrow 'a \ (\langle \cdot \rangle)$

$\langle ML \rangle$

locale *drat* =
fixes $test :: 'a::boolean-algebra \Rightarrow 'b::dra$
and $not :: 'b::dra \Rightarrow 'b::dra \ (\langle ! \rangle)$
assumes $test-sup \ [simp, kat-hom]: test \ (sup \ p \ q) = 'p + q'$
and $test-inf \ [simp, kat-hom]: test \ (inf \ p \ q) = 'p \cdot q'$
and $test-top \ [simp, kat-hom]: test \ top = 1$
and $test-bot \ [simp, kat-hom]: test \ bot = 0$
and $test-not \ [simp, kat-hom]: test \ (- \ p) = '!p'$
and $test-iso-eq \ [kat-hom]: p \leq q \longleftrightarrow 'p \leq q'$

begin

notation $test \ (\langle \cdot \rangle)$

lemma $test-eq \ [kat-hom]: p = q \longleftrightarrow 'p = q'$
 $\langle proof \rangle$

$\langle ML \rangle$

lemma $test-iso: p \leq q \Longrightarrow 'p \leq q'$
 $\langle proof \rangle$

lemma $test-meet-comm: 'p \cdot q = q \cdot p'$
 $\langle proof \rangle$

lemmas $test-one-top[simp] = test-iso[OF \ top-greatest, simplified]$

lemma $test-star \ [simp]: 'p^* = 1'$
 $\langle proof \rangle$

lemmas $[kat-hom] = test-star[symmetric]$

lemma $test-comp-add1 \ [simp]: !p + p = 1'$
 $\langle proof \rangle$

lemma $test-comp-add2 \ [simp]: 'p + !p = 1'$
 $\langle proof \rangle$

lemma *test-comp-mult1* [*simp*]: $!p \cdot p = 0$
 $\langle proof \rangle$

lemma *test-comp-mult2* [*simp*]: $p \cdot !p = 0$
 $\langle proof \rangle$

lemma *test-eq1*: $y \leq x \iff p \cdot y \leq x \wedge !p \cdot y \leq x$
 $\langle proof \rangle$

lemma $p \cdot x = p \cdot x \cdot q \implies p \cdot x \cdot !q = 0$
nitpick $\langle proof \rangle$

lemma *test1*: $p \cdot x \cdot !q = 0 \implies p \cdot x = p \cdot x \cdot q$
 $\langle proof \rangle$

lemma *test2*: $p \cdot q \cdot p = p \cdot q$
 $\langle proof \rangle$

lemma *test3*: $p \cdot q \cdot !p = 0$
 $\langle proof \rangle$

lemma *test4*: $!p \cdot q \cdot p = 0$
 $\langle proof \rangle$

lemma *total-correctness*: $p \cdot x \cdot !q = 0 \iff x \cdot !q \leq !p \cdot \top$
 $\langle proof \rangle$

lemma *test-iteration-sim*: $p \cdot x \leq x \cdot p \implies p \cdot x^\infty \leq x^\infty \cdot p$
 $\langle proof \rangle$

lemma *test-iteration-annir*: $!p \cdot (p \cdot x)^\infty = !p$
 $\langle proof \rangle$

end

end

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