

Kleene Algebras with Domain

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Abstract

Kleene algebras with domain are Kleene algebras endowed with an operation that maps each element of the algebra to its domain of definition (or its complement) in abstract fashion. They form a simple algebraic basis for Hoare logics, dynamic logics or predicate transformer semantics. We formalise a modular hierarchy of algebras with domain and antidomain (domain complement) operations in Isabelle/HOL that ranges from domain and antidomain semigroups to modal Kleene algebras and divergence Kleene algebras. We link these algebras with models of binary relations and program traces. We include some examples from modal logics, termination and program analysis.

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1 Introductory Remarks

These theory files are intended as a reference formalisation for variants of Kleene algebras with domain. The algebraic hierarchy is developed in a modular way from domain and antidomain semigroups to modal Kleene algebras in which forward and backward box and diamond operators interact via conjugations and Galois connections. Throughout the development we have aimed at readable proofs so that these theories can be seen as a machine-checked introduction to reasoning in this setting. Apart from that, the Isabelle code is only sparsely annotated, and we refer to a series of articles for further information.

Our formalisation follows the approaches of Desharnais, Jipsen and Struth to domain semigroups [3] and Desharnais and Struth to families of domain semirings and Kleene algebras with domain [7, 6]. The link with modal Kleene algebras, Hoare logics and predicate transformers has been elaborated by Möller and Struth [13]; a notion of divergence has been added by Desharnais, Möller and Struth [5]. A previous stage of this formalisation has been documented in a companion article [11].

The target model of these axiomatisations are binary relations, where the domain operation represents the set of those elements that are related to some other element. There is a vast amount of literature on axiomatising the domain of functions, especially in semigroup theory. The deterministic nature of functions, however, leads to different axiom sets. An integration of these approaches is left for future work.

Our Isabelle/HOL formalisation itself is based on a formalisation of variants of Kleene algebras [1]. An adaptation of Kleene algebras with domain to the setting of concurrent dynamic algebra [10] can also be found in the Archive of Formal Proofs [9]. A formalisation of the original two-sorted approach to Kleene algebra with domain [4] is left for future work as well.

2 Domain Semirings

```
theory Domain-Semiring
imports Kleene-Algebra.Kleene-Algebra
```

```
begin
```

2.1 Domain Semigroups and Domain Monoids

```
class domain-op =
  fixes domain-op :: 'a ⇒ 'a (d)
```

First we define the class of domain semigroups. Axioms are taken from [3].

```
class domain-semigroup = semigroup-mult + domain-op +
  assumes dsg1 [simp]: d x · x = x
  and dsg2 [simp]: d (x · d y) = d (x · y)
  and dsg3 [simp]: d (d x · y) = d x · d y
  and dsg4: d x · d y = d y · d x
```

```
begin
```

```
lemma domain-invol [simp]: d (d x) = d x
proof -
  have d (d x) = d (d (d x · x))
  by simp
  also have ... = d (d x · d x)
  using dsg3 by presburger
  also have ... = d (d x · x)
  by simp
  finally show ?thesis
  by simp
qed
```

The next lemmas show that domain elements form semilattices.

```
lemma dom-el-idem [simp]: d x · d x = d x
proof -
  have d x · d x = d (d x · x)
  using dsg3 by presburger
  thus ?thesis
  by simp
qed
```

lemma *dom-mult-closed* [*simp*]: $d (d x \cdot d y) = d x \cdot d y$
by *simp*

lemma *dom-lc3* [*simp*]: $d x \cdot d (x \cdot y) = d (x \cdot y)$

proof –

have $d x \cdot d (x \cdot y) = d (d x \cdot x \cdot y)$
using *dsg3 mult-assoc* **by** *presburger*
thus *?thesis*
by *simp*

qed

lemma *d-fixpoint*: $(\exists y. x = d y) \longleftrightarrow x = d x$
by *auto*

lemma *d-type*: $\forall P. (\forall x. x = d x \longrightarrow P x) \longleftrightarrow (\forall x. P (d x))$
by (*metis domain-invol*)

We define the semilattice ordering on domain semigroups and explore the semilattice of domain elements from the order point of view.

definition *ds-ord* :: $'a \Rightarrow 'a \Rightarrow \text{bool}$ (**infix** \sqsubseteq 50) **where**
 $x \sqsubseteq y \longleftrightarrow x = d x \cdot y$

lemma *ds-ord-refl*: $x \sqsubseteq x$
by (*simp add: ds-ord-def*)

lemma *ds-ord-trans*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq z \Longrightarrow x \sqsubseteq z$
proof –

assume $x \sqsubseteq y$ **and** $a: y \sqsubseteq z$
hence $b: x = d x \cdot y$
using *ds-ord-def* **by** *blast*
hence $x = d x \cdot d y \cdot z$
using *a ds-ord-def mult-assoc* **by** *force*
also have $\dots = d (d x \cdot y) \cdot z$
by *simp*
also have $\dots = d x \cdot z$
using *b* **by** *auto*
finally show *?thesis*
using *ds-ord-def* **by** *blast*

qed

lemma *ds-ord-antisym*: $x \sqsubseteq y \Longrightarrow y \sqsubseteq x \Longrightarrow x = y$

proof –

assume $a: x \sqsubseteq y$ **and** $y \sqsubseteq x$
hence $b: y = d y \cdot x$
using *ds-ord-def* **by** *auto*
have $x = d x \cdot d y \cdot x$
using *a b ds-ord-def mult-assoc* **by** *force*
also have $\dots = d y \cdot x$
by (*metis (full-types) b dsg3 dsg4*)

thus *?thesis*
using *b calculation by presburger*
qed

This relation is indeed an order.

sublocale *ds: ordering* $\langle(\sqsubseteq)\rangle$ $\langle\lambda x y. x \sqsubseteq y \wedge x \neq y\rangle$
proof

show $\langle x \sqsubseteq y \wedge x \neq y \longleftrightarrow x \sqsubseteq y \wedge x \neq y \rangle$ **for** $x y$
by (*rule refl*)
show $x \sqsubseteq x$ **for** x
by (*rule ds-ord-refl*)
show $x \sqsubseteq y \implies y \sqsubseteq z \implies x \sqsubseteq z$ **for** $x y z$
by (*rule ds-ord-trans*)
show $x \sqsubseteq y \implies y \sqsubseteq x \implies x = y$ **for** $x y$
by (*rule ds-ord-antisym*)
qed

declare *ds.refl* [*simp*]

lemma *ds-ord-eq*: $x \sqsubseteq d x \longleftrightarrow x = d x$
by (*simp add: ds-ord-def*)

lemma $x \sqsubseteq y \implies z \cdot x \sqsubseteq z \cdot y$

oops

lemma *ds-ord-iso-right*: $x \sqsubseteq y \implies x \cdot z \sqsubseteq y \cdot z$

proof –
assume $x \sqsubseteq y$
hence $a: x = d x \cdot y$
by (*simp add: ds-ord-def*)
hence $x \cdot z = d x \cdot y \cdot z$
by *auto*
also have $\dots = d (d x \cdot y \cdot z) \cdot d x \cdot y \cdot z$
using *dsg1 mult-assoc* **by** *presburger*
also have $\dots = d (x \cdot z) \cdot d x \cdot y \cdot z$
using *a* **by** *presburger*
finally show *?thesis*
using *ds-ord-def dsg4 mult-assoc* **by** *auto*
qed

The order on domain elements could as well be defined based on multiplication/meet.

lemma *ds-ord-sl-ord*: $d x \sqsubseteq d y \longleftrightarrow d x \cdot d y = d x$
using *ds-ord-def* **by** *auto*

lemma *ds-ord-1*: $d (x \cdot y) \sqsubseteq d x$
by (*simp add: ds-ord-sl-ord dsg4*)

lemma *ds-subid-aux*: $d x \cdot y \sqsubseteq y$
by (*simp add: ds-ord-def mult-assoc*)

lemma $y \cdot d x \sqsubseteq y$

oops

lemma *ds-dom-iso*: $x \sqsubseteq y \implies d x \sqsubseteq d y$

proof –

assume $x \sqsubseteq y$

hence $x = d x \cdot y$

by (*simp add: ds-ord-def*)

hence $d x = d (d x \cdot y)$

by *presburger*

also have $\dots = d x \cdot d y$

by *simp*

finally show *?thesis*

using *ds-ord-sl-ord* **by** *auto*

qed

lemma *ds-dom-llp*: $x \sqsubseteq d y \cdot x \longleftrightarrow d x \sqsubseteq d y$

proof

assume $x \sqsubseteq d y \cdot x$

hence $x = d y \cdot x$

by (*simp add: ds-subid-aux ds.antisym*)

hence $d x = d (d y \cdot x)$

by *presburger*

thus $d x \sqsubseteq d y$

using *ds-ord-sl-ord dsg4* **by** *force*

next

assume $d x \sqsubseteq d y$

thus $x \sqsubseteq d y \cdot x$

by (*metis (no-types) ds-ord-iso-right dsg1*)

qed

lemma *ds-dom-llp-strong*: $x = d y \cdot x \longleftrightarrow d x \sqsubseteq d y$

using *ds.eq-iff*

by (*simp add: ds-dom-llp ds.eq-iff ds-subid-aux*)

definition *refines* :: $'a \Rightarrow 'a \Rightarrow \text{bool}$

where $\text{refines } x y \equiv d y \sqsubseteq d x \wedge (d y) \cdot x \sqsubseteq y$

lemma *refines-refl*: $\text{refines } x x$

using *refines-def* **by** *simp*

lemma *refines-trans*: $\text{refines } x y \implies \text{refines } y z \implies \text{refines } x z$

unfolding *refines-def*

by (*metis domain-invol ds.trans dsg1 dsg3 ds-ord-def*)

```

lemma refines-antisym: refines  $x\ y \implies \text{refines } y\ x \implies x = y$ 
  apply (rule ds.antisym)
  apply (simp-all add: refines-def)
  apply (metis ds-dom-llp-strong)
  apply (metis ds-dom-llp-strong)
done

```

```

sublocale ref: ordering refines  $\lambda x\ y. (\text{refines } x\ y \wedge x \neq y)$ 
proof
  show  $\bigwedge x\ y. \text{refines } x\ y \wedge x \neq y \longleftrightarrow \text{refines } x\ y \wedge x \neq y$ 
  ..
  show  $\bigwedge x. \text{refines } x\ x$ 
    by (rule refines-refl)
  show  $\bigwedge x\ y\ z. \text{refines } x\ y \implies \text{refines } y\ z \implies \text{refines } x\ z$ 
    by (rule refines-trans)
  show  $\bigwedge x\ y. \text{refines } x\ y \implies \text{refines } y\ x \implies x = y$ 
    by (rule refines-antisym)
qed

end

```

We expand domain semigroups to domain monoids.

```

class domain-monoid = monoid-mult + domain-semigroup
begin

```

```

lemma dom-one [simp]:  $d\ 1 = 1$ 
proof -
  have  $1 = d\ 1 \cdot 1$ 
    using dsg1 by presburger
  thus ?thesis
    by simp
qed

```

```

lemma ds-subid-eq:  $x \sqsubseteq 1 \longleftrightarrow x = d\ x$ 
  by (simp add: ds-ord-def)

```

```

end

```

2.2 Domain Near-Semirings

The axioms for domain near-semirings are taken from [6].

```

class domain-near-semiring = ab-near-semiring + plus-ord + domain-op +
  assumes dns1 [simp]:  $d\ x \cdot x = x$ 
  and dns2 [simp]:  $d\ (x \cdot d\ y) = d\ (x \cdot y)$ 
  and dns3 [simp]:  $d\ (x + y) = d\ x + d\ y$ 
  and dns4:  $d\ x \cdot d\ y = d\ y \cdot d\ x$ 
  and dns5 [simp]:  $d\ x \cdot (d\ x + d\ y) = d\ x$ 

begin

```

Domain near-semirings are automatically dioids; addition is idempotent.

```

subclass near-dioid
proof
  show  $\bigwedge x. x + x = x$ 
  proof –
    fix x
    have a:  $d\ x = d\ x \cdot d\ (x + x)$ 
      using dns3 dns5 by presburger
    have  $d\ (x + x) = d\ (x + x + (x + x)) \cdot d\ (x + x)$ 
      by (metis (no-types) dns3 dns4 dns5)
    hence  $d\ (x + x) = d\ (x + x) + d\ (x + x)$ 
      by simp
    thus  $x + x = x$ 
      by (metis a dns1 dns4 distrib-right')
  qed
qed

```

Next we prepare to show that domain near-semirings are domain semigroups.

```

lemma dom-iso:  $x \leq y \implies d\ x \leq d\ y$ 
  using order-prop by auto

```

```

lemma dom-add-closed [simp]:  $d\ (d\ x + d\ y) = d\ x + d\ y$ 
proof –
  have  $d\ (d\ x + d\ y) = d\ (d\ x) + d\ (d\ y)$ 
    by simp
  thus ?thesis
    by (metis dns1 dns2 dns3 dns4)
qed

```

```

lemma dom-absorp-2 [simp]:  $d\ x + d\ x \cdot d\ y = d\ x$ 
proof –
  have  $d\ x + d\ x \cdot d\ y = d\ x \cdot d\ x + d\ x \cdot d\ y$ 
    by (metis add-idem' dns5)
  also have  $\dots = (d\ x + d\ y) \cdot d\ x$ 
    by (simp add: dns4)
  also have  $\dots = d\ x \cdot (d\ x + d\ y)$ 
    by (metis dom-add-closed dns4)
  finally show ?thesis
    by simp
qed

```

```

lemma dom-1:  $d\ (x \cdot y) \leq d\ x$ 
proof –
  have  $d\ (x \cdot y) = d\ (d\ x \cdot d\ (x \cdot y))$ 
    by (metis dns1 dns2 mult-assoc)
  also have  $\dots \leq d\ (d\ x) + d\ (d\ x \cdot d\ (x \cdot y))$ 
    by simp
  also have  $\dots = d\ (d\ x + d\ x \cdot d\ (x \cdot y))$ 
    using dns3 by presburger

```


also have $\dots = d (d x)$
by *simp*
finally show *?thesis*
by (*metis dom-add-closed add-idem'*)
qed

lemma *dom-subid-aux2*: $d x \cdot y \leq y$
proof –
have $d x \cdot y \leq d (x + d y) \cdot y$
by (*simp add: mult-isor*)
also have $\dots = (d x + d (d y)) \cdot d y \cdot y$
using *dns1 dns3 mult-assoc* **by** *presburger*
also have $\dots = (d y + d y \cdot d x) \cdot y$
by (*simp add: dns4 add-commute*)
finally show *?thesis*
by *simp*
qed

lemma *dom-glb*: $d x \leq d y \implies d x \leq d z \implies d x \leq d y \cdot d z$
by (*metis dns5 less-eq-def mult-isor*)

lemma *dom-glb-eq*: $d x \leq d y \cdot d z \iff d x \leq d y \wedge d x \leq d z$
proof –
have $d x \leq d z \longrightarrow d x \leq d z$
by *meson*
then show *?thesis*
by (*metis (no-types) dom-absorp-2 dom-glb dom-subid-aux2 local.dual-order.trans local.join.sup.coboundedI2*)
qed

lemma *dom-ord*: $d x \leq d y \iff d x \cdot d y = d x$
proof
assume $d x \leq d y$
hence $d x + d y = d y$
by (*simp add: less-eq-def*)
thus $d x \cdot d y = d x$
by (*metis dns5*)
next
assume $d x \cdot d y = d x$
thus $d x \leq d y$
by (*metis dom-subid-aux2*)
qed

lemma *dom-export* [*simp*]: $d (d x \cdot y) = d x \cdot d y$
proof (*rule order.antisym*)
have $d (d x \cdot y) = d (d (d x \cdot y)) \cdot d (d x \cdot y)$
using *dns1* **by** *presburger*
also have $\dots = d (d x \cdot d y) \cdot d (d x \cdot y)$
by (*metis dns1 dns2 mult-assoc*)

finally show $a: d (d x \cdot y) \leq d x \cdot d y$
 by (*metis (no-types) dom-add-closed dom-glb dom-1 add-idem' dns2 dns4*)
have $d (d x \cdot y) = d (d x \cdot y) \cdot d x$
 using *a dom-glb-eq dom-ord* by *force*
hence $d x \cdot d y = d (d x \cdot y) \cdot d y$
 by (*metis dns1 dns2 mult-assoc*)
thus $d x \cdot d y \leq d (d x \cdot y)$
 using *a dom-glb-eq dom-ord* by *auto*
qed

subclass *domain-semigroup*
 by (*unfold-locales, auto simp: dns4*)

We compare the domain semigroup ordering with that of the dioid.

lemma *d-two-orders*: $d x \sqsubseteq d y \iff d x \leq d y$
 by (*simp add: dom-ord ds-ord-sl-ord*)

lemma *two-orders*: $x \sqsubseteq y \implies x \leq y$
 by (*metis dom-subid-aux2 ds-ord-def*)

lemma $x \leq y \implies x \sqsubseteq y$

oops

Next we prove additional properties.

lemma *dom-subdist*: $d x \leq d (x + y)$
 by *simp*

lemma *dom-distrib*: $d x + d y \cdot d z = (d x + d y) \cdot (d x + d z)$

proof –

have $(d x + d y) \cdot (d x + d z) = d x \cdot (d x + d z) + d y \cdot (d x + d z)$

using *distrib-right'* by *blast*

also have $\dots = d x + (d x + d z) \cdot d y$

by (*metis (no-types) dns3 dns5 dsg4*)

also have $\dots = d x + d x \cdot d y + d z \cdot d y$

using *add-assoc' distrib-right'* by *presburger*

finally show *?thesis*

by (*simp add: dsg4*)

qed

lemma *dom-llp1*: $x \leq d y \cdot x \implies d x \leq d y$

proof –

assume $x \leq d y \cdot x$

hence $d x \leq d (d y \cdot x)$

using *dom-iso* by *blast*

also have $\dots = d y \cdot d x$

by *simp*

finally show $d x \leq d y$

by (*simp add: dom-glb-eq*)

qed

lemma *dom-llp2*: $d x \leq d y \implies x \leq d y \cdot x$
using *d-two-orders local.ds-dom-llp two-orders* **by** *blast*

lemma *dom-llp*: $x \leq d y \cdot x \longleftrightarrow d x \leq d y$
using *dom-llp1 dom-llp2* **by** *blast*

end

We expand domain near-semirings by an additive unit, using slightly different axioms.

class *domain-near-semiring-one* = *ab-near-semiring-one* + *plus-ord* + *domain-op*
+
assumes *dnso1* [*simp*]: $x + d x \cdot x = d x \cdot x$
and *dnso2* [*simp*]: $d (x \cdot d y) = d (x \cdot y)$
and *dnso3* [*simp*]: $d x + 1 = 1$
and *dnso4* [*simp*]: $d (x + y) = d x + d y$
and *dnso5*: $d x \cdot d y = d y \cdot d x$

begin

The previous axioms are derivable.

subclass *domain-near-semiring*

proof

show $\bigwedge x. d x \cdot x = x$

by (*metis add-commute local.dnso3 local.distrib-right' local.dnso1 local.mult-one1*)

show $\bigwedge x y. d (x \cdot d y) = d (x \cdot y)$

by *simp*

show $\bigwedge x y. d (x + y) = d x + d y$

by *simp*

show $\bigwedge x y. d x \cdot d y = d y \cdot d x$

by (*simp add: dnso5*)

show $\bigwedge x y. d x \cdot (d x + d y) = d x$

proof -

fix $x y$

have $\bigwedge x. 1 + d x = 1$

using *add-commute dnso3* **by** *presburger*

thus $d x \cdot (d x + d y) = d x$

by (*metis (no-types) a dnso2 dnso4 dnso5 distrib-right' mult-one1*)

qed

qed

subclass *domain-monoid* ..

lemma *dom-subid*: $d x \leq 1$

by (*simp add: less-eq-def*)

end

We add a left unit of multiplication.

```
class domain-near-semiring-one-zero1 = ab-near-semiring-one-zero1 + domain-near-semiring-one
+
assumes dns06 [simp]: d 0 = 0
```

begin

```
lemma domain-very-strict: d x = 0  $\longleftrightarrow$  x = 0
by (metis annihil dns1 dns06)
```

```
lemma dom-weakly-local: x · y = 0  $\longleftrightarrow$  x · d y = 0
```

proof –

```
have x · y = 0  $\longleftrightarrow$  d (x · y) = 0
by (simp add: domain-very-strict)
also have ...  $\longleftrightarrow$  d (x · d y) = 0
by simp
```

```
finally show ?thesis
using domain-very-strict by blast
```

qed

end

2.3 Domain Pre-Dioids

Pre-semirings with one and a left zero are automatically dioids. Hence there is no point defining domain pre-semirings separately from domain dioids. The axioms are once again from [6].

```
class domain-pre-doid-one = pre-doid-one + domain-op +
assumes dpd1 : x ≤ d x · x
and dpd2 [simp]: d (x · d y) = d (x · y)
and dpd3 [simp]: d x ≤ 1
and dpd4 [simp]: d (x + y) = d x + d y
```

begin

We prepare to show that every domain pre-doid with one is a domain near-doid with one.

```
lemma dns1'' [simp]: d x · x = x
```

proof (rule order.antisym)

```
show d x · x ≤ x
using dpd3 mult-isor by fastforce
show x ≤ d x · x
by (simp add: dpd1)
```

qed

```
lemma d-iso: x ≤ y  $\implies$  d x ≤ d y
by (metis dpd4 less-eq-def)
```

```

lemma domain-1'':  $d(x \cdot y) \leq d x$ 
proof –
  have  $d(x \cdot y) = d(x \cdot d y)$ 
    by simp
  also have  $\dots \leq d(x \cdot 1)$ 
    by (meson d-iso dpd3 mult-isol)
  finally show ?thesis
    by simp
qed

lemma domain-export'' [simp]:  $d(d x \cdot y) = d x \cdot d y$ 
proof (rule order.antisym)
  have one:  $d(d x \cdot y) \leq d x$ 
    by (metis dpd2 domain-1'' mult-onel)
  have two:  $d(d x \cdot y) \leq d y$ 
    using d-iso dpd3 mult-isol by fastforce
  have  $d(d x \cdot y) = d(d(d x \cdot y)) \cdot d(d x \cdot y)$ 
    by simp
  also have  $\dots = d(d x \cdot y) \cdot d(d x \cdot y)$ 
    by (metis dns1'' dpd2 mult-assoc)
  thus  $d(d x \cdot y) \leq d x \cdot d y$ 
    using mult-isol-var one two by force
next
  have  $d x \cdot d y \leq 1$ 
    by (metis dpd3 mult-1-right mult-isol order.trans)
  thus  $d x \cdot d y \leq d(d x \cdot y)$ 
    by (metis dns1'' dpd2 mult-isol mult-oner)
qed

lemma dom-subid-aux1'':  $d x \cdot y \leq y$ 
proof –
  have  $d x \cdot y \leq 1 \cdot y$ 
    using dpd3 mult-isol by blast
  thus ?thesis
    by simp
qed

lemma dom-subid-aux2'':  $x \cdot d y \leq x$ 
  using dpd3 mult-isol by fastforce

lemma d-comm:  $d x \cdot d y = d y \cdot d x$ 
proof (rule order.antisym)
  have  $d x \cdot d y = (d x \cdot d y) \cdot (d x \cdot d y)$ 
    by (metis dns1'' domain-export'')
  thus  $d x \cdot d y \leq d y \cdot d x$ 
    by (metis dom-subid-aux1'' dom-subid-aux2'' mult-isol-var)
next
  have  $d y \cdot d x = (d y \cdot d x) \cdot (d y \cdot d x)$ 
    by (metis dns1'' domain-export'')

```

```

thus  $d y \cdot d x \leq d x \cdot d y$ 
  by (metis dom-subid-aux1'' dom-subid-aux2'' mult-isol-var)
qed

```

```

subclass domain-near-semiring-one
  by (unfold-locales, auto simp: d-comm local.join.sup.absorb2)

```

```

lemma domain-subid:  $x \leq 1 \implies x \leq d x$ 
  by (metis dns1 mult-isol mult-oner)

```

```

lemma d-preserves-equation:  $d y \cdot x \leq x \cdot d z \iff d y \cdot x = d y \cdot x \cdot d z$ 
  by (metis dom-subid-aux2'' order.antisym local.dom-el-idem local.dom-subid-aux2
  local.order-prop local.subdistl mult-assoc)

```

```

lemma d-restrict-iff:  $(x \leq y) \iff (x \leq d x \cdot y)$ 
  by (metis dom-subid-aux2 dsg1 less-eq-def order-trans subdistl)

```

```

lemma d-restrict-iff-1:  $(d x \cdot y \leq z) \iff (d x \cdot y \leq d x \cdot z)$ 
  by (metis dom-subid-aux2 domain-1'' domain-invol dsg1 mult-isol-var order-trans)

```

```

end

```

We add once more a left unit of multiplication.

```

class domain-pre-dioid-one-zero1 = domain-pre-dioid-one + pre-dioid-one-zero1 +
  assumes dps5 [simp]:  $d 0 = 0$ 

```

```

begin

```

```

subclass domain-near-semiring-one-zero1
  by (unfold-locales, simp)

```

```

end

```

2.4 Domain Semirings

We do not consider domain semirings without units separately at the moment. The axioms are taken from from [7]

```

class domain-semiring1 = semiring-one-zero1 + plus-ord + domain-op +
  assumes dsr1 [simp]:  $x + d x \cdot x = d x \cdot x$ 
  and dsr2 [simp]:  $d (x \cdot d y) = d (x \cdot y)$ 
  and dsr3 [simp]:  $d x + 1 = 1$ 
  and dsr4 [simp]:  $d 0 = 0$ 
  and dsr5 [simp]:  $d (x + y) = d x + d y$ 

```

```

begin

```

Every domain semiring is automatically a domain pre-dioid with one and left zero.

```

subclass dioid-one-zero
  by (standard, metis add-commute dsr1 dsr3 distrib-left mult-oner)

subclass domain-pre-dioid-one-zero
  by (standard, auto simp: less-eq-def)

end

class domain-semiring = domain-semiringl + semiring-one-zero

```

2.5 The Algebra of Domain Elements

We show that the domain elements of a domain semiring form a distributive lattice. Unfortunately we cannot prove this within the type class of domain semirings.

```

typedef (overloaded) 'a d-element = {x :: 'a :: domain-semiring. x = d x}
  by (rule-tac x = 1 in exI, simp add: domain-subid ds.eq-iff)

```

```

setup-lifting type-definition-d-element

```

```

instantiation d-element :: (domain-semiring) bounded-lattice

```

```

begin

```

```

lift-definition less-eq-d-element :: 'a d-element  $\Rightarrow$  'a d-element  $\Rightarrow$  bool is ( $\leq$ ) .

```

```

lift-definition less-d-element :: 'a d-element  $\Rightarrow$  'a d-element  $\Rightarrow$  bool is ( $<$ ) .

```

```

lift-definition bot-d-element :: 'a d-element is 0
  by simp

```

```

lift-definition top-d-element :: 'a d-element is 1
  by simp

```

```

lift-definition inf-d-element :: 'a d-element  $\Rightarrow$  'a d-element  $\Rightarrow$  'a d-element is ( $\cdot$ )
  by (metis dsq3)

```

```

lift-definition sup-d-element :: 'a d-element  $\Rightarrow$  'a d-element  $\Rightarrow$  'a d-element is
  (+)
  by simp

```

```

instance

```

```

  apply (standard; transfer)
  apply (simp add: less-le-not-le)+
  apply (metis dom-subid-aux2'')
  apply (metis dom-subid-aux2)
  apply (metis dom-glb)
  apply simp+
  by (metis dom-subid)

```

end

instance *d-element* :: (*domain-semiring*) *distrib-lattice*
by (*standard, transfer, metis dom-distrib*)

2.6 Domain Semirings with a Greatest Element

If there is a greatest element in the semiring, then we have another equality.

class *domain-semiring-top* = *domain-semiring* + *order-top*

begin

notation *top* (\top)

lemma *kat-equivalence-greatest*: $d\ x \leq d\ y \longleftrightarrow x \leq d\ y \cdot \top$

proof

assume $d\ x \leq d\ y$

thus $x \leq d\ y \cdot \top$

by (*metis dsq1 mult-isol-var top-greatest*)

next

assume $x \leq d\ y \cdot \top$

thus $d\ x \leq d\ y$

using *dom-glb-eq dom-iso* **by** *fastforce*

qed

end

2.7 Forward Diamond Operators

context *domain-semiringl*

begin

We define a forward diamond operator over a domain semiring. A more modular consideration is not given at the moment.

definition *fd* :: $'a \Rightarrow 'a \Rightarrow 'a$ ($(\ |-\)$ -) [61,81] 82) **where**

$|x\rangle\ y = d\ (x \cdot y)$

lemma *fdia-d-simp* [*simp*]: $|x\rangle\ d\ y = |x\rangle\ y$

by (*simp add: fd-def*)

lemma *fdia-dom* [*simp*]: $|x\rangle\ 1 = d\ x$

by (*simp add: fd-def*)

lemma *fdia-add1*: $|x\rangle\ (y + z) = |x\rangle\ y + |x\rangle\ z$

by (*simp add: fd-def distrib-left*)

lemma *fdia-add2*: $|x + y\rangle\ z = |x\rangle\ z + |y\rangle\ z$

by (*simp add: fd-def distrib-right*)

lemma *fdia-mult*: $|x \cdot y\rangle z = |x\rangle |y\rangle z$
by (*simp add: fd-def mult-assoc*)

lemma *fdia-one* [*simp*]: $|1\rangle x = d x$
by (*simp add: fd-def*)

lemma *fdemodalisation1*: $d z \cdot |x\rangle y = 0 \iff d z \cdot x \cdot d y = 0$
proof –
have $d z \cdot |x\rangle y = 0 \iff d z \cdot d (x \cdot y) = 0$
by (*simp add: fd-def*)
also have $\dots \iff d z \cdot x \cdot d y = 0$
by (*metis annil dnsob dsg1 dsg3 mult-assoc*)
finally show *?thesis*
using *dom-weakly-local* **by** *auto*
qed

lemma *fdemodalisation2*: $|x\rangle y \leq d z \iff x \cdot d y \leq d z \cdot x$
proof
assume $|x\rangle y \leq d z$
hence $a: d (x \cdot d y) \leq d z$
by (*simp add: fd-def*)
have $x \cdot d y = d (x \cdot d y) \cdot x \cdot d y$
using *dsg1 mult-assoc* **by** *presburger*
also have $\dots \leq d z \cdot x \cdot d y$
using *a calculation dom-llp2 mult-assoc* **by** *auto*
finally show $x \cdot d y \leq d z \cdot x$
using *dom-subid-aux2'' order-trans* **by** *blast*
next
assume $x \cdot d y \leq d z \cdot x$
hence $d (x \cdot d y) \leq d (d z \cdot d x)$
using *dom-iso* **by** *fastforce*
also have $\dots \leq d (d z)$
using *domain-1''* **by** *blast*
finally show $|x\rangle y \leq d z$
by (*simp add: fd-def*)
qed

lemma *fd-iso1*: $d x \leq d y \implies |z\rangle x \leq |z\rangle y$
using *fd-def local.dom-iso local.mult-isol* **by** *fastforce*

lemma *fd-iso2*: $x \leq y \implies |x\rangle z \leq |y\rangle z$
by (*simp add: fd-def dom-iso mult-isol*)

lemma *fd-zero-var* [*simp*]: $|0\rangle x = 0$
by (*simp add: fd-def*)

lemma *fd-subdist-1*: $|x\rangle y \leq |x\rangle (y + z)$

```

    by (simp add: fd-iso1)

lemma fd-subdist-2:  $|x\rangle (d\ y \cdot d\ z) \leq |x\rangle\ y$ 
  by (simp add: fd-iso1 dom-subid-aux2'')

lemma fd-subdist:  $|x\rangle (d\ y \cdot d\ z) \leq |x\rangle\ y \cdot |x\rangle\ z$ 
  using fd-def fd-iso1 fd-subdist-2 dom-glb dom-subid-aux2 by auto

lemma fdia-export-1:  $d\ y \cdot |x\rangle\ z = |d\ y \cdot x\rangle\ z$ 
  by (simp add: fd-def mult-assoc)

end

context domain-semiring

begin

lemma fdia-zero [simp]:  $|x\rangle\ 0 = 0$ 
  by (simp add: fd-def)

end

```

2.8 Domain Kleene Algebras

We add the Kleene star to our considerations. Special domain axioms are not needed.

```
class domain-left-kleene-algebra = left-kleene-algebra-zero1 + domain-semiring1
```

```
begin
```

```

lemma dom-star [simp]:  $d\ (x^*) = 1$ 
proof -
  have  $d\ (x^*) = d\ (1 + x \cdot x^*)$ 
    by simp
  also have  $\dots = d\ 1 + d\ (x \cdot x^*)$ 
    using dns3 by blast
  finally show ?thesis
    using add-commute local.dsr3 by auto
qed

```

```

lemma fdia-star-unfold [simp]:  $|1\rangle\ y + |x\rangle\ |x^*\rangle\ y = |x^*\rangle\ y$ 
proof -
  have  $|1\rangle\ y + |x\rangle\ |x^*\rangle\ y = |1 + x \cdot x^*\rangle\ y$ 
    using local.fdia-add2 local.fdia-mult by presburger
  thus ?thesis
    by simp
qed

```

```
lemma fdia-star-unfoldr [simp]:  $|1\rangle\ y + |x^*\rangle\ |x\rangle\ y = |x^*\rangle\ y$ 
```

proof –
have $|1\rangle y + |x^*\rangle |x\rangle y = |1 + x^* \cdot x\rangle y$
using *fdia-add2 fdia-mult by presburger*
thus *?thesis*
by *simp*
qed

lemma *fdia-star-unfold-var [simp]: $d y + |x\rangle |x^*\rangle y = |x^*\rangle y$*
proof –
have $d y + |x\rangle |x^*\rangle y = |1\rangle y + |x\rangle |x^*\rangle y$
by *simp*
also have $\dots = |1 + x \cdot x^*\rangle y$
using *fdia-add2 fdia-mult by presburger*
finally show *?thesis*
by *simp*
qed

lemma *fdia-star-unfoldr-var [simp]: $d y + |x^*\rangle |x\rangle y = |x^*\rangle y$*
proof –
have $d y + |x^*\rangle |x\rangle y = |1\rangle y + |x^*\rangle |x\rangle y$
by *simp*
also have $\dots = |1 + x^* \cdot x\rangle y$
using *fdia-add2 fdia-mult by presburger*
finally show *?thesis*
by *simp*
qed

lemma *fdia-star-induct-var: $|x\rangle y \leq d y \implies |x^*\rangle y \leq d y$*
proof –
assume *a1: $|x\rangle y \leq d y$*
hence $x \cdot d y \leq d y \cdot x$
by (*simp add: fdemodalisation2*)
hence $x^* \cdot d y \leq d y \cdot x^*$
by (*simp add: star-sim1*)
thus *?thesis*
by (*simp add: fdemodalisation2*)
qed

lemma *fdia-star-induct: $d z + |x\rangle y \leq d y \implies |x^*\rangle z \leq d y$*
proof –
assume *a: $d z + |x\rangle y \leq d y$*
hence *b: $d z \leq d y$ and c: $|x\rangle y \leq d y$*
apply (*simp add: local.join.le-supE*)
using *a* **by** *auto*
hence *d: $|x^*\rangle z \leq |x^*\rangle y$*
using *fd-def fd-iso1* **by** *auto*
have $|x^*\rangle y \leq d y$
using *c fdia-star-induct-var* **by** *blast*
thus *?thesis*

```

    using d by fastforce
qed

lemma fdia-star-induct-eq: d z + |x⟩ y = d y  $\implies$  |x*⟩ z  $\leq$  d y
  by (simp add: fdia-star-induct)

end

class domain-kleene-algebra = kleene-algebra + domain-semiring

begin

subclass domain-left-kleene-algebra ..

end

end

```

3 Antidomain Semirings

```

theory Antidomain-Semiring
imports Domain-Semiring
begin

```

3.1 Antidomain Monoids

We axiomatise antidomain monoids, using the axioms of [3].

```

class antidomain-op =
  fixes antidomain-op :: 'a  $\Rightarrow$  'a (ad)

class antidomain-left-monoid = monoid-mult + antidomain-op +
  assumes am1 [simp]: ad x  $\cdot$  x = ad 1
  and am2: ad x  $\cdot$  ad y = ad y  $\cdot$  ad x
  and am3 [simp]: ad (ad x)  $\cdot$  x = x
  and am4 [simp]: ad (x  $\cdot$  y)  $\cdot$  ad (x  $\cdot$  ad y) = ad x
  and am5 [simp]: ad (x  $\cdot$  y)  $\cdot$  x  $\cdot$  ad y = ad (x  $\cdot$  y)  $\cdot$  x

begin

```

```

no-notation domain-op (d)
no-notation zero-class.zero (0)

```

We define a zero element and operations of domain and addition.

```

definition a-zero :: 'a (0) where
  0 = ad 1

```

```

definition am-d :: 'a  $\Rightarrow$  'a (d) where
  d x = ad (ad x)

```

definition *am-add-op* :: 'a ⇒ 'a ⇒ 'a (**infixl** ⊕ 65) **where**

$x \oplus y \equiv ad (ad x \cdot ad y)$

lemma *a-d-zero* [*simp*]: $ad x \cdot d x = 0$

by (*metis am1 am2 a-zero-def am-d-def*)

lemma *a-d-one* [*simp*]: $d x \oplus ad x = 1$

by (*metis am1 am3 mult-1-right am-d-def am-add-op-def*)

lemma *n-annil* [*simp*]: $0 \cdot x = 0$

proof –

have $0 \cdot x = d x \cdot ad x \cdot x$

by (*simp add: a-zero-def am-d-def*)

also have $\dots = d x \cdot 0$

by (*metis am1 mult-assoc a-zero-def*)

thus *?thesis*

by (*metis am1 am2 am3 mult-assoc a-zero-def*)

qed

lemma *a-mult-idem* [*simp*]: $ad x \cdot ad x = ad x$

proof –

have $ad x \cdot ad x = ad (1 \cdot x) \cdot 1 \cdot ad x$

by *simp*

also have $\dots = ad (1 \cdot x) \cdot 1$

using *am5* **by** *blast*

finally show *?thesis*

by *simp*

qed

lemma *a-add-idem* [*simp*]: $ad x \oplus ad x = ad x$

by (*metis am1 am3 am4 mult-1-right am-add-op-def*)

The next three axioms suffice to show that the domain elements form a Boolean algebra.

lemma *a-add-comm*: $x \oplus y = y \oplus x$

using *am2 am-add-op-def* **by** *auto*

lemma *a-add-assoc*: $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

proof –

have $\bigwedge x y. ad x \cdot ad (x \cdot y) = ad x$

by (*metis a-mult-idem am2 am4 mult-assoc*)

thus *?thesis*

by (*metis a-add-comm am-add-op-def local.am3 local.am4 mult-assoc*)

qed

lemma *huntington* [*simp*]: $ad (x \oplus y) \oplus ad (x \oplus ad y) = ad x$

using *a-add-idem am-add-op-def* **by** *auto*

lemma *a-absorb1* [*simp*]: $(ad\ x \oplus ad\ y) \cdot ad\ x = ad\ x$
by (*metis a-add-idem a-mult-idem am4 mult-assoc am-add-op-def*)

lemma *a-absorb2* [*simp*]: $ad\ x \oplus ad\ x \cdot ad\ y = ad\ x$

proof –

have $ad\ (ad\ x) \cdot ad\ (ad\ x \cdot ad\ y) = ad\ (ad\ x)$

by (*metis (no-types) a-mult-idem local.am4 local.mult.semigroup-axioms semigroup.assoc*)

then show *?thesis*

using *a-add-idem am-add-op-def* **by** *auto*

qed

The distributivity laws remain to be proved; our proofs follow those of Mad-
dux [12].

lemma *prod-split* [*simp*]: $ad\ x \cdot ad\ y \oplus ad\ x \cdot d\ y = ad\ x$

using *a-add-idem am-d-def am-add-op-def* **by** *auto*

lemma *sum-split* [*simp*]: $(ad\ x \oplus ad\ y) \cdot (ad\ x \oplus d\ y) = ad\ x$

using *a-add-idem am-d-def am-add-op-def* **by** *fastforce*

lemma *a-comp-simp* [*simp*]: $(ad\ x \oplus ad\ y) \cdot d\ x = ad\ y \cdot d\ x$

proof –

have $f1: (ad\ x \oplus ad\ y) \cdot d\ x = ad\ (ad\ (ad\ x) \cdot ad\ (ad\ y)) \cdot ad\ (ad\ x) \cdot ad\ (ad\ (ad\ y))$

by (*simp add: am-add-op-def am-d-def*)

have $f2: ad\ y = ad\ (ad\ (ad\ y))$

using *a-add-idem am-add-op-def* **by** *auto*

have $ad\ y = ad\ (ad\ (ad\ x) \cdot ad\ (ad\ y)) \cdot ad\ y$

by (*metis (no-types) a-absorb1 a-add-comm am-add-op-def*)

then show *?thesis*

using $f2\ f1$ **by** (*simp add: am-d-def local.am2 local.mult.semigroup-axioms semigroup.assoc*)

qed

lemma *a-distrib1*: $ad\ x \cdot (ad\ y \oplus ad\ z) = ad\ x \cdot ad\ y \oplus ad\ x \cdot ad\ z$

proof –

have $f1: \bigwedge a. ad\ (ad\ (ad\ (a::'a)) \cdot ad\ (ad\ a)) = ad\ a$

using *a-add-idem am-add-op-def* **by** *auto*

have $f2: \bigwedge a\ aa. ad\ ((a::'a) \cdot aa) \cdot (a \cdot ad\ aa) = ad\ (a \cdot aa) \cdot a$

using *local.am5 mult-assoc* **by** *auto*

have $f3: \bigwedge a. ad\ (ad\ (ad\ (a::'a))) = ad\ a$

using $f1$ **by** *simp*

have $\bigwedge a. ad\ (a::'a) \cdot ad\ a = ad\ a$

by *simp*

then have $\bigwedge a\ aa. ad\ (ad\ (ad\ (a::'a) \cdot ad\ aa)) = ad\ aa \cdot ad\ a$

using $f3\ f2$ **by** (*metis (no-types) local.am2 local.am4 mult-assoc*)

then have $ad\ x \cdot (ad\ y \oplus ad\ z) = ad\ x \cdot (ad\ y \oplus ad\ z) \cdot ad\ y \oplus ad\ x \cdot (ad\ y \oplus ad\ z) \cdot d\ y$

using *am-add-op-def am-d-def local.am2 local.am4* **by** *presburger*

also have ... = $ad\ x \cdot ad\ y \oplus ad\ x \cdot (ad\ y \oplus ad\ z) \cdot d\ y$
by (*simp add: mult-assoc*)
also have ... = $ad\ x \cdot ad\ y \oplus ad\ x \cdot ad\ z \cdot d\ y$
by (*simp add: mult-assoc*)
also have ... = $ad\ x \cdot ad\ y \oplus ad\ x \cdot ad\ y \cdot ad\ z \oplus ad\ x \cdot ad\ z \cdot d\ y$
by (*metis a-add-idem a-mult-idem local.am4 mult-assoc am-add-op-def*)
also have ... = $ad\ x \cdot ad\ y \oplus (ad\ x \cdot ad\ z \cdot ad\ y \oplus ad\ x \cdot ad\ z \cdot d\ y)$
by (*metis am2 mult-assoc a-add-assoc*)
finally show *?thesis*
by (*metis a-add-idem a-mult-idem am4 am-d-def am-add-op-def*)
qed

lemma *a-distrib2*: $ad\ x \oplus ad\ y \cdot ad\ z = (ad\ x \oplus ad\ y) \cdot (ad\ x \oplus ad\ z)$
proof –
have *f1*: $\bigwedge a\ aa\ ab.\ ad\ (ad\ (ad\ (a::'a) \cdot ad\ aa) \cdot ad\ (ad\ a \cdot ad\ ab)) = ad\ a \cdot ad\ (ad\ (ad\ aa) \cdot ad\ (ad\ ab))$
using *a-distrib1 am-add-op-def* **by** *auto*
have $\bigwedge a.\ ad\ (ad\ (ad\ (a::'a))) = ad\ a$
by (*metis a-absorb2 a-mult-idem am-add-op-def*)
then have $ad\ (ad\ (ad\ x) \cdot ad\ (ad\ y)) \cdot ad\ (ad\ (ad\ x) \cdot ad\ (ad\ z)) = ad\ (ad\ (ad\ x) \cdot ad\ (ad\ y \cdot ad\ z))$
using *f1* **by** (*metis (full-types)*)
then show *?thesis*
by (*simp add: am-add-op-def*)
qed

lemma *aa-loc [simp]*: $d\ (x \cdot d\ y) = d\ (x \cdot y)$
proof –
have *f1*: $x \cdot d\ y \cdot y = x \cdot y$
by (*metis am3 mult-assoc am-d-def*)
have *f2*: $\bigwedge w\ z.\ ad\ (w \cdot z) \cdot (w \cdot ad\ z) = ad\ (w \cdot z) \cdot w$
by (*metis am5 mult-assoc*)
hence *f3*: $\bigwedge z.\ ad\ (x \cdot y) \cdot (x \cdot z) = ad\ (x \cdot y) \cdot (x \cdot (ad\ (ad\ (ad\ y) \cdot y) \cdot z))$
using *f1* **by** (*metis (no-types) mult-assoc am-d-def*)
have $ad\ (x \cdot ad\ (ad\ y)) \cdot (x \cdot y) = 0$ **using** *f1*
by (*metis am1 mult-assoc n-annil a-zero-def am-d-def*)
thus *?thesis*
by (*metis a-d-zero am-d-def f3 local.am1 local.am2 local.am3 local.am4*)
qed

lemma *a-loc [simp]*: $ad\ (x \cdot d\ y) = ad\ (x \cdot y)$
proof –
have $\bigwedge a.\ ad\ (ad\ (ad\ (a::'a))) = ad\ a$
using *am-add-op-def am-d-def prod-split* **by** *auto*
then show *?thesis*
by (*metis (full-types) aa-loc am-d-def*)
qed

lemma *d-a-export [simp]*: $d\ (ad\ x \cdot y) = ad\ x \cdot d\ y$

proof –
have $f1: \bigwedge a \ aa. \ ad \ ((a::'a) \cdot \ ad \ (ad \ aa)) = \ ad \ (a \cdot \ aa)$
using $a\text{-loc}$ $am\text{-d-def}$ **by** $auto$
have $\bigwedge a. \ ad \ (ad \ (a::'a) \cdot \ a) = \ 1$
using $a\text{-d-one}$ $am\text{-add-op-def}$ $am\text{-d-def}$ **by** $auto$
then have $\bigwedge a \ aa. \ ad \ (ad \ (ad \ (a::'a) \cdot \ ad \ aa)) = \ ad \ a \cdot \ ad \ aa$
using $f1$ **by** $(metis \ a\text{-distrib2} \ am\text{-add-op-def} \ local.\text{mult-1-left})$
then show $?thesis$
using $f1$ **by** $(metis \ (no\text{-types}) \ am\text{-d-def})$
qed

Every antidomain monoid is a domain monoid.

sublocale $dm: \ domain\text{-monoid} \ am\text{-d} \ (\cdot) \ 1$
apply $(unfold\text{-locales})$
apply $(simp \ add: \ am\text{-d-def})$
apply $simp$
using $am\text{-d-def}$ $d\text{-a-export}$ **apply** $auto[1]$
by $(simp \ add: \ am\text{-d-def} \ local.\text{am2})$

lemma $ds\text{-ord-iso1}: \ x \sqsubseteq y \implies z \cdot x \sqsubseteq z \cdot y$

oops

lemma $a\text{-very-costrict}: \ ad \ x = 1 \longleftrightarrow x = 0$

proof
assume $a: \ ad \ x = 1$
hence $0 = \ ad \ x \cdot x$
using $a\text{-zero-def}$ **by** $force$
thus $x = 0$
by $(simp \ add: \ a)$
next
assume $x = 0$
thus $ad \ x = 1$
using $a\text{-zero-def}$ $am\text{-d-def}$ $dm.\text{dom-one}$ **by** $auto$
qed

lemma $a\text{-weak-loc}: \ x \cdot y = 0 \longleftrightarrow x \cdot d \ y = 0$

proof –
have $x \cdot y = 0 \longleftrightarrow ad \ (x \cdot y) = 1$
by $(simp \ add: \ a\text{-very-costrict})$
also have $\dots \longleftrightarrow ad \ (x \cdot d \ y) = 1$
by $simp$
finally show $?thesis$
using $a\text{-very-costrict}$ **by** $blast$
qed

lemma $a\text{-closure} \ [simp]: \ d \ (ad \ x) = \ ad \ x$
using $a\text{-add-idem}$ $am\text{-add-op-def}$ $am\text{-d-def}$ **by** $auto$

lemma *a-d-mult-closure* [*simp*]: $d (ad\ x \cdot ad\ y) = ad\ x \cdot ad\ y$
by *simp*

lemma *kat-3'*: $d\ x \cdot y \cdot ad\ z = 0 \implies d\ x \cdot y = d\ x \cdot y \cdot d\ z$
by (*metis dm.dom-one local.am5 local.mult-1-left a-zero-def am-d-def*)

lemma *s4* [*simp*]: $ad\ x \cdot ad (ad\ x \cdot y) = ad\ x \cdot ad\ y$

proof –

have $\bigwedge a\ aa. ad (a::'a) \cdot ad (ad\ aa) = ad (ad (ad\ a \cdot aa))$

using *am-d-def d-a-export* **by** *presburger*

then have $\bigwedge a\ aa. ad (ad (a::'a)) \cdot ad\ aa = ad (ad (ad\ aa \cdot a))$

using *local.am2* **by** *presburger*

then show *?thesis*

by (*metis a-comp-simp a-d-mult-closure am-add-op-def am-d-def local.am2*)

qed

end

class *antidomain-monoid* = *antidomain-left-monoid* +
assumes *am6* [*simp*]: $x \cdot ad\ 1 = ad\ 1$

begin

lemma *kat-3-equiv*: $d\ x \cdot y \cdot ad\ z = 0 \iff d\ x \cdot y = d\ x \cdot y \cdot d\ z$
apply *standard*
apply (*metis kat-3'*)
by (*simp add: mult-assoc a-zero-def am-d-def*)

no-notation *a-zero* (0)

no-notation *am-d* (d)

end

3.2 Antidomain Near-Semirings

We define antidomain near-semirings. We do not consider units separately. The axioms are taken from [6].

notation *zero-class.zero* (0)

class *antidomain-near-semiring* = *ab-near-semiring-one-zero1* + *antidomain-op* +
plus-ord +

assumes *ans1* [*simp*]: $ad\ x \cdot x = 0$

and *ans2* [*simp*]: $ad (x \cdot y) + ad (x \cdot ad (ad\ y)) = ad (x \cdot ad (ad\ y))$

and *ans3* [*simp*]: $ad (ad\ x) + ad\ x = 1$

and *ans4* [*simp*]: $ad (x + y) = ad\ x \cdot ad\ y$

begin

definition *ans-d* :: $'a \Rightarrow 'a\ (d)$ **where**

$$d\ x = ad\ (ad\ x)$$

lemma *a-a-one* [*simp*]: $d\ 1 = 1$

proof –

have $d\ 1 = d\ 1 + 0$

by *simp*

also have $\dots = d\ 1 + ad\ 1$

by (*metis ans1 mult-1-right*)

finally show *?thesis*

by (*simp add: ans-d-def*)

qed

lemma *a-very-costrict'*: $ad\ x = 1 \iff x = 0$

proof

assume $ad\ x = 1$

hence $x = ad\ x \cdot x$

by *simp*

thus $x = 0$

by *auto*

next

assume $x = 0$

hence $ad\ x = ad\ 0$

by *blast*

thus $ad\ x = 1$

by (*metis a-a-one ans-d-def local.ans1 local.mult-1-right*)

qed

lemma *one-idem* [*simp*]: $1 + 1 = 1$

proof –

have $1 + 1 = d\ 1 + d\ 1$

by *simp*

also have $\dots = ad\ (ad\ 1 \cdot 1) + ad\ (ad\ 1 \cdot d\ 1)$

using *a-a-one ans-d-def* **by** *auto*

also have $\dots = ad\ (ad\ 1 \cdot d\ 1)$

using *ans-d-def local.ans2* **by** *presburger*

also have $\dots = ad\ (ad\ 1 \cdot 1)$

by *simp*

also have $\dots = d\ 1$

by (*simp add: ans-d-def*)

finally show *?thesis*

by *simp*

qed

Every antidomain near-semiring is automatically a dioid, and therefore ordered.

subclass *near-dioid-one-zero*

proof

show $\bigwedge x. x + x = x$

proof –

```

fix  $x$ 
have  $x + x = 1 \cdot x + 1 \cdot x$ 
  by simp
also have  $\dots = (1 + 1) \cdot x$ 
  using distrib-right' by presburger
finally show  $x + x = x$ 
  by simp
qed
qed

```

```

lemma d1-a [simp]:  $d\ x \cdot x = x$ 
proof –
  have  $x = (d\ x + ad\ x) \cdot x$ 
    by (simp add: ans-d-def)
  also have  $\dots = d\ x \cdot x + ad\ x \cdot x$ 
    using distrib-right' by blast
  also have  $\dots = d\ x \cdot x + 0$ 
    by simp
  finally show ?thesis
    by auto
qed

```

```

lemma a-comm:  $ad\ x \cdot ad\ y = ad\ y \cdot ad\ x$ 
  using add-commute ans4 by fastforce

```

```

lemma a-subid:  $ad\ x \leq 1$ 
  using local.ans3 local.join.sup-ge2 by fastforce

```

```

lemma a-subid-aux1:  $ad\ x \cdot y \leq y$ 
  using a-subid mult-isor by fastforce

```

```

lemma a-subdist:  $ad\ (x + y) \leq ad\ x$ 
  by (metis a-subid-aux1 ans4 add-comm)

```

```

lemma a-antitone:  $x \leq y \implies ad\ y \leq ad\ x$ 
  using a-subdist local.order-prop by auto

```

```

lemma a-mul-d [simp]:  $ad\ x \cdot d\ x = 0$ 
  by (metis a-comm ans-d-def local.ans1)

```

```

lemma a-gla1:  $ad\ x \cdot y = 0 \implies ad\ x \leq ad\ y$ 

```

```

proof –
  assume  $ad\ x \cdot y = 0$ 
  hence  $a$ :  $ad\ x \cdot d\ y = 0$ 
    by (metis a-subid a-very-costrict' ans-d-def local.ans2 local.join.sup.order-iff)
  have  $ad\ x = (d\ y + ad\ y) \cdot ad\ x$ 
    by (simp add: ans-d-def)
  also have  $\dots = d\ y \cdot ad\ x + ad\ y \cdot ad\ x$ 
    using distrib-right' by blast

```

also have $\dots = ad\ x \cdot d\ y + ad\ x \cdot ad\ y$
using *a-comm ans-d-def* **by** *auto*
also have $\dots = ad\ x \cdot ad\ y$
by (*simp add: a*)
finally show $ad\ x \leq ad\ y$
by (*metis a-subid-aux1*)
qed

lemma *a-gla2*: $ad\ x \leq ad\ y \implies ad\ x \cdot y = 0$
proof –
assume $ad\ x \leq ad\ y$
hence $ad\ x \cdot y \leq ad\ y \cdot y$
using *mult-isor* **by** *blast*
thus *?thesis*
by (*simp add: join.le-bot*)
qed

lemma *a2-eq* [*simp*]: $ad\ (x \cdot d\ y) = ad\ (x \cdot y)$
proof (*rule order.antisym*)
show $ad\ (x \cdot y) \leq ad\ (x \cdot d\ y)$
by (*simp add: ans-d-def local.less-eq-def*)
next
show $ad\ (x \cdot d\ y) \leq ad\ (x \cdot y)$
by (*metis a-gla1 a-mul-d ans1 d1-a mult-assoc*)
qed

lemma *a-export'* [*simp*]: $ad\ (ad\ x \cdot y) = d\ x + ad\ y$
proof (*rule order.antisym*)
have $ad\ (ad\ x \cdot y) \cdot ad\ x \cdot d\ y = 0$
by (*simp add: a-gla2 local.mult.semigroup-axioms semigroup.assoc*)
hence $a: ad\ (ad\ x \cdot y) \cdot d\ y \leq ad\ (ad\ x)$
by (*metis a-comm a-gla1 ans4 mult-assoc ans-d-def*)
have $ad\ (ad\ x \cdot y) = ad\ (ad\ x \cdot y) \cdot d\ y + ad\ (ad\ x \cdot y) \cdot ad\ y$
by (*metis (no-types) add-commute ans3 ans4 distrib-right' mult-onel ans-d-def*)
thus $ad\ (ad\ x \cdot y) \leq d\ x + ad\ y$
by (*metis a-subid-aux1 a-join.sup-mono ans-d-def*)
next
show $d\ x + ad\ y \leq ad\ (ad\ x \cdot y)$
by (*metis a2-eq a-antitone a-comm a-subid-aux1 join.sup-least ans-d-def*)
qed

Every antidomain near-semiring is a domain near-semiring.

sublocale *dnsz*: *domain-near-semiring-one-zero* (+) (\cdot) 1 0 *ans-d* (\leq) ($<$)
apply (*unfold-locales*)
apply *simp*
using *a2-eq ans-d-def* **apply** *auto*[1]
apply (*simp add: a-subid ans-d-def local.join.sup-absorb2*)
apply (*simp add: ans-d-def*)
apply (*simp add: a-comm ans-d-def*)

using *a-a-one a-very-costrict' ans-d-def* by *force*

lemma *a-idem* [*simp*]: $ad\ x \cdot ad\ x = ad\ x$

proof –

have $ad\ x = (d\ x + ad\ x) \cdot ad\ x$

by (*simp add: ans-d-def*)

also have $\dots = d\ x \cdot ad\ x + ad\ x \cdot ad\ x$

using *distrib-right'* by *blast*

finally show *?thesis*

by (*simp add: ans-d-def*)

qed

lemma *a-3-var* [*simp*]: $ad\ x \cdot ad\ y \cdot (x + y) = 0$

by (*metis ans1 ans4*)

lemma *a-3* [*simp*]: $ad\ x \cdot ad\ y \cdot d\ (x + y) = 0$

by (*metis a-mul-d ans4*)

lemma *a-closure'* [*simp*]: $d\ (ad\ x) = ad\ x$

proof –

have $d\ (ad\ x) = ad\ (1 \cdot d\ x)$

by (*simp add: ans-d-def*)

also have $\dots = ad\ (1 \cdot x)$

using *a2-eq* by *blast*

finally show *?thesis*

by *simp*

qed

The following counterexamples show that some of the antidomain monoid axioms do not need to hold.

lemma $x \cdot ad\ 1 = ad\ 1$

oops

lemma $ad\ (x \cdot y) \cdot ad\ (x \cdot ad\ y) = ad\ x$

oops

lemma $ad\ (x \cdot y) \cdot ad\ (x \cdot ad\ y) = ad\ x$

oops

lemma *phl-seq-inv*: $d\ v \cdot x \cdot y \cdot ad\ w = 0 \implies \exists z. d\ v \cdot x \cdot d\ z = 0 \wedge ad\ z \cdot y \cdot ad\ w = 0$

proof –

assume $d\ v \cdot x \cdot y \cdot ad\ w = 0$

hence $d\ v \cdot x \cdot d\ (y \cdot ad\ w) = 0 \wedge ad\ (y \cdot ad\ w) \cdot y \cdot ad\ w = 0$

by (*metis dnsz.dom-weakly-local local.ans1 mult-assoc*)

thus $\exists z. d\ v \cdot x \cdot d\ z = 0 \wedge ad\ z \cdot y \cdot ad\ w = 0$

by *blast*
qed

lemma *a-fixpoint*: $ad\ x = x \implies (\forall y. y = 0)$

proof –

assume *a1*: $ad\ x = x$

{ **fix** *aa* :: 'a

have $aa = 0$

using *a1* by (*metis* (*no-types*) *a-mul-d* *ans-d-def* *local.annil* *local.ans3* *local.join.sup.idem* *local.mult-1-left*)

}

then show *?thesis*

by *blast*

qed

no-notation *ans-d* (*d*)

end

3.3 Antidomain Pre-Dioids

Antidomain pre-dioids are based on a different set of axioms, which are again taken from [6].

class *antidomain-pre-diod* = *pre-diod-one-zero* + *antidomain-op* +

assumes *apd1* [*simp*]: $ad\ x \cdot x = 0$

and *apd2* [*simp*]: $ad\ (x \cdot y) \leq ad\ (x \cdot ad\ (ad\ y))$

and *apd3* [*simp*]: $ad\ (ad\ x) + ad\ x = 1$

begin

definition *apd-d* :: 'a \Rightarrow 'a (*d*) **where**

$d\ x = ad\ (ad\ x)$

lemma *a-very-costrict'*: $ad\ x = 1 \longleftrightarrow x = 0$

by (*metis* *add-commute* *local.add-zero* *order.antisym* *local.apd1* *local.apd3* *local.join.bot-least* *local.mult-1-right* *local.phl-skip*)

lemma *a-subid'*: $ad\ x \leq 1$

using *local.apd3* *local.join.sup-ge2* **by** *fastforce*

lemma *d1-a'* [*simp*]: $d\ x \cdot x = x$

proof –

have $x = (d\ x + ad\ x) \cdot x$

by (*simp* *add: apd-d-def*)

also have $\dots = d\ x \cdot x + ad\ x \cdot x$

using *distrib-right'* **by** *blast*

also have $\dots = d\ x \cdot x + 0$

by *simp*

finally show *?thesis*

by *auto*
qed

lemma *a-subid-aux1'*: $ad\ x \cdot y \leq y$
using *a-subid' mult-isor* by *fastforce*

lemma *a-mul-d'* [*simp*]: $ad\ x \cdot d\ x = 0$
proof –
 have $1 = ad\ (ad\ x \cdot x)$
 using *a-very-costrict''* by *force*
 thus *?thesis*
 by (*metis a-subid' a-very-costrict'' apd-d-def order.antisym local.apd2*)
qed

lemma *a-d-closed* [*simp*]: $d\ (ad\ x) = ad\ x$
proof (*rule order.antisym*)
 have $d\ (ad\ x) = (d\ x + ad\ x) \cdot d\ (ad\ x)$
 by (*simp add: apd-d-def*)
 also have $\dots = ad\ (ad\ x) \cdot ad\ (d\ x) + ad\ x \cdot d\ (ad\ x)$
 using *apd-d-def local.distrib-right'* by *presburger*
 also have $\dots = ad\ x \cdot d\ (ad\ x)$
 using *a-mul-d' apd-d-def* by *auto*
 finally show $d\ (ad\ x) \leq ad\ x$
 by (*metis a-subid' mult-1-right mult-isol apd-d-def*)
next
 have $ad\ x = ad\ (1 \cdot x)$
 by *simp*
 also have $\dots \leq ad\ (1 \cdot d\ x)$
 using *apd-d-def local.apd2* by *presburger*
 also have $\dots = ad\ (d\ x)$
 by *simp*
 finally show $ad\ x \leq d\ (ad\ x)$
 by (*simp add: apd-d-def*)
qed

lemma *meet-ord-def*: $ad\ x \leq ad\ y \iff ad\ x \cdot ad\ y = ad\ x$
by (*metis a-d-closed a-subid-aux1' d1-a' order.eq-iff mult-1-right mult-isol*)

lemma *d-weak-loc*: $x \cdot y = 0 \iff x \cdot d\ y = 0$
proof –
 have $x \cdot y = 0 \iff ad\ (x \cdot y) = 1$
 by (*simp add: a-very-costrict''*)
 also have $\dots \iff ad\ (x \cdot d\ y) = 1$
 by (*metis apd1 apd2 a-subid' apd-d-def d1-a' order.eq-iff mult-1-left mult-assoc*)
 finally show *?thesis*
 by (*simp add: a-very-costrict''*)
qed

lemma *gla-1*: $ad\ x \cdot y = 0 \implies ad\ x \leq ad\ y$

proof –
assume $ad\ x \cdot y = 0$
hence $a: ad\ x \cdot d\ y = 0$
using *d-weak-loc* **by** *force*
hence $d\ y = ad\ x \cdot d\ y + d\ y$
by *simp*
also have $\dots = (1 + ad\ x) \cdot d\ y$
using *join.sup-commute* **by** *auto*
also have $\dots = (d\ x + ad\ x) \cdot d\ y$
using *apd-d-def calculation* **by** *auto*
also have $\dots = d\ x \cdot d\ y$
by (*simp add: a join.sup-commute*)
finally have $d\ y \leq d\ x$
by (*metis apd-d-def a-subid' mult-1-right mult-isol*)
hence $d\ y \cdot ad\ x = 0$
by (*metis apd-d-def a-d-closed a-mul-d' distrib-right' less-eq-def no-trivial-inverse*)
hence $ad\ x = ad\ y \cdot ad\ x$
by (*metis apd-d-def apd3 add-0-left distrib-right' mult-1-left*)
thus $ad\ x \leq ad\ y$
by (*metis add-commute apd3 mult-oner subdistl*)
qed

lemma *a2-eq' [simp]: ad (x · d y) = ad (x · y)*
proof (*rule order.antisym*)
show $ad\ (x \cdot y) \leq ad\ (x \cdot d\ y)$
by (*simp add: apd-d-def*)
next
show $ad\ (x \cdot d\ y) \leq ad\ (x \cdot y)$
by (*metis gla-1 apd1 a-mul-d' d1-a' mult-assoc*)
qed

lemma *a-supdist-var: ad (x + y) ≤ ad x*
by (*metis gla-1 apd1 join.le-bot subdistl*)

lemma *a-antitone': x ≤ y ⇒ ad y ≤ ad x*
using *a-supdist-var local.order-prop* **by** *auto*

lemma *a-comm-var: ad x · ad y ≤ ad y · ad x*
proof –
have $ad\ x \cdot ad\ y = d\ (ad\ x \cdot ad\ y) \cdot ad\ x \cdot ad\ y$
by (*simp add: mult-assoc*)
also have $\dots \leq d\ (ad\ x \cdot ad\ y) \cdot ad\ x$
using *a-subid' mult-isol* **by** *fastforce*
also have $\dots \leq d\ (ad\ y) \cdot ad\ x$
by (*simp add: a-antitone' a-subid-aux1' apd-d-def local.mult-isol*)
finally show *?thesis*
by *simp*
qed

lemma *a-comm'*: $ad\ x \cdot ad\ y = ad\ y \cdot ad\ x$
by (*simp add: a-comm-var order.eq-iff*)

lemma *a-closed* [*simp*]: $d\ (ad\ x \cdot ad\ y) = ad\ x \cdot ad\ y$
proof –

have *f1*: $\bigwedge x\ y.\ ad\ x \leq ad\ (ad\ y \cdot x)$
by (*simp add: a-antitone' a-subid-aux1'*)
have $\bigwedge x\ y.\ d\ (ad\ x \cdot y) \leq ad\ x$
by (*metis a2-eq' a-antitone' a-comm' a-d-closed apd-d-def f1*)
hence $\bigwedge x\ y.\ d\ (ad\ x \cdot y) \cdot y = ad\ x \cdot y$
by (*metis d1-a' meet-ord-def mult-assoc apd-d-def*)
thus *?thesis*
by (*metis f1 a-comm' apd-d-def meet-ord-def*)

qed

lemma *a-export''* [*simp*]: $ad\ (ad\ x \cdot y) = d\ x + ad\ y$

proof (*rule order.antisym*)

have $ad\ (ad\ x \cdot y) \cdot ad\ x \cdot d\ y = 0$
using *d-weak-loc mult-assoc by fastforce*
hence *a*: $ad\ (ad\ x \cdot y) \cdot d\ y \leq d\ x$
by (*metis a-closed a-comm' apd-d-def gla-1 mult-assoc*)
have $ad\ (ad\ x \cdot y) = ad\ (ad\ x \cdot y) \cdot d\ y + ad\ (ad\ x \cdot y) \cdot ad\ y$
by (*metis apd3 a-comm' d1-a' distrib-right' mult-1-right apd-d-def*)
thus $ad\ (ad\ x \cdot y) \leq d\ x + ad\ y$
by (*metis a-subid-aux1' a-join.sup-mono*)

next

have $ad\ y \leq ad\ (ad\ x \cdot y)$
by (*simp add: a-antitone' a-subid-aux1'*)
thus $d\ x + ad\ y \leq ad\ (ad\ x \cdot y)$
by (*metis apd-d-def a-mul-d' d1-a' gla-1 apd1 join.sup-least mult-assoc*)

qed

lemma *d1-sum-var*: $x + y \leq (d\ x + d\ y) \cdot (x + y)$

proof –

have $x + y = d\ x \cdot x + d\ y \cdot y$
by *simp*
also have $\dots \leq (d\ x + d\ y) \cdot x + (d\ x + d\ y) \cdot y$
using *local.distrib-right' local.join.sup-ge1 local.join.sup-ge2 local.join.sup-mono*
by *presburger*
finally show *?thesis*
using *order-trans subdistl-var by blast*

qed

lemma *a4'*: $ad\ (x + y) = ad\ x \cdot ad\ y$

proof (*rule order.antisym*)

show $ad\ (x + y) \leq ad\ x \cdot ad\ y$
by (*metis a-d-closed a-supdist-var add-commute d1-a' local.mult-isol-var*)
hence $ad\ x \cdot ad\ y = ad\ x \cdot ad\ y + ad\ (x + y)$
using *less-eq-def add-commute by simp*

```

also have ... = ad (ad (ad x · ad y) · (x + y))
  by (metis a-closed a-export'')
finally show ad x · ad y ≤ ad (x + y)
  using a-antitone' apd-d-def d1-sum-var by auto
qed

```

Antidomain pre-dioids are domain pre-dioids and antidomain near-semirings, but still not antidomain monoids.

```

sublocale dpdz: domain-pre-diooid-one-zero1 (+) (·) 1 0 (≤) (<) λx. ad (ad x)
  apply (unfold-locales)
  using apd-d-def d1-a' apply auto[1]
  using a2-eq' apd-d-def apply auto[1]
  apply (simp add: a-subid')
  apply (simp add: a4' apd-d-def)
  by (metis a-mul-d' a-very-costrict'' apd-d-def local.mult-one1)

```

```

subclass antidomain-near-semiring
  apply (unfold-locales)
  apply simp
  using local.apd2 local.less-eq-def apply blast
  apply simp
  by (simp add: a4')

```

```

lemma a-supdist: ad (x + y) ≤ ad x + ad y
  using a-supdist-var local.join.le-sup11 by auto

```

```

lemma a-gla: ad x · y = 0 ↔ ad x ≤ ad y
  using gla-1 a-gla2 by blast

```

```

lemma a-subid-aux2: x · ad y ≤ x
  using a-subid' mult-isol by fastforce

```

```

lemma a42-var: d x · d y ≤ ad (ad x + ad y)
  by (simp add: apd-d-def)

```

```

lemma d1-weak [simp]: (d x + d y) · x = x
proof –
  have (d x + d y) · x = (1 + d y) · x
    by simp
  thus ?thesis
  by (metis add-commute apd-d-def dpdz.dnso3 local.mult-1-left)
qed

```

```

lemma x · ad 1 = ad 1

```

oops

```

lemma ad x · (y + z) = ad x · y + ad x · z

```

oops

lemma $ad (x \cdot y) \cdot ad (x \cdot ad y) = ad x$

oops

lemma $ad (x \cdot y) \cdot ad (x \cdot ad y) = ad x$

oops

no-notation $apd-d (d)$

end

3.4 Antidomain Semirings

Antidomain semirings are direct expansions of antidomain pre-dioids, but do not require idempotency of addition. Hence we give a slightly different axiomatisation, following [7].

```
class antidomain-semiringl = semiring-one-zero + plus-ord + antidomain-op +  
  assumes as1 [simp]:  $ad x \cdot x = 0$   
  and as2 [simp]:  $ad (x \cdot y) + ad (x \cdot ad (ad y)) = ad (x \cdot ad (ad y))$   
  and as3 [simp]:  $ad (ad x) + ad x = 1$ 
```

begin

```
definition ads-d :: 'a  $\Rightarrow$  'a (d) where  
   $d x = ad (ad x)$ 
```

```
lemma one-idem':  $1 + 1 = 1$   
  by (metis as1 as2 as3 add-zero mult.right-neutral)
```

Every antidomain semiring is a dioid and an antidomain pre-dioid.

```
subclass dioid  
  by (standard, metis distrib-left mult.right-neutral one-idem')
```

```
subclass antidomain-pre-dioid  
  by (unfold-locales, auto simp: local.less-eq-def)
```

```
lemma am5-lem [simp]:  $ad (x \cdot y) \cdot ad (x \cdot ad y) = ad x$ 
```

proof –

```
  have  $ad (x \cdot y) \cdot ad (x \cdot ad y) = ad (x \cdot d y) \cdot ad (x \cdot ad y)$   
    using ads-d-def local.a2-eq' local.apd-d-def by auto  
  also have  $\dots = ad (x \cdot d y + x \cdot ad y)$   
    using ans4 by presburger  
  also have  $\dots = ad (x \cdot (d y + ad y))$   
    using distrib-left by presburger  
  finally show ?thesis
```

by (*simp add: ads-d-def*)
qed

lemma *am6-lem* [*simp*]: $ad (x \cdot y) \cdot x \cdot ad y = ad (x \cdot y) \cdot x$
proof –
fix $x y$
have $ad (x \cdot y) \cdot x \cdot ad y = ad (x \cdot y) \cdot x \cdot ad y + 0$
by *simp*
also have $\dots = ad (x \cdot y) \cdot x \cdot ad y + ad (x \cdot d y) \cdot x \cdot d y$
using *ans1 mult-assoc by presburger*
also have $\dots = ad (x \cdot y) \cdot x \cdot (ad y + d y)$
using *ads-d-def local.a2-eq' local.apd-d-def local.distrib-left by auto*
finally show $ad (x \cdot y) \cdot x \cdot ad y = ad (x \cdot y) \cdot x$
using *add-commute ads-d-def local.as3 by auto*
qed

lemma *a-zero* [*simp*]: $ad 0 = 1$
by (*simp add: local.a-very-costrict''*)

lemma *a-one* [*simp*]: $ad 1 = 0$
using *a-zero local.dpdz.dpd5 by blast*

subclass *antidomain-left-monoid*
by (*unfold-locales, auto simp: local.a-comm'*)

Every antidomain left semiring is a domain left semiring.

no-notation *domain-semiringl-class.fd* (($|$ -) -) [*61,81*] *82*)

definition *fdia* :: $'a \Rightarrow 'a \Rightarrow 'a$ (($|$ -) -) [*61,81*] *82*) **where**
 $|x\rangle y = ad (ad (x \cdot y))$

sublocale *ds: domain-semiringl* (+) (\cdot) *1 0* $\lambda x. ad (ad x)$ (\leq) ($<$)
rewrites *ds.fd* $x y \equiv fdia x y$
proof –
show *class.domain-semiringl* (+) (\cdot) *1 0* $(\lambda x. ad (ad x))$ (\leq) ($<$)
by (*unfold-locales, auto simp: local.dpdz.dpd4 ans-d-def*)
then interpret *ds: domain-semiringl* (+) (\cdot) *1 0* $\lambda x. ad (ad x)$ (\leq) ($<$) .
show *ds.fd* $x y \equiv fdia x y$
by (*auto simp: fdia-def ds.fd-def*)
qed

lemma *fd-eq-fdia* [*simp*]: *domain-semiringl.fd* (\cdot) $d x y \equiv fdia x y$
proof –
have *class.domain-semiringl* (+) (\cdot) *1 0* d (\leq) ($<$)
by (*unfold-locales, auto simp: ads-d-def local.ans-d-def*)
hence *domain-semiringl.fd* (\cdot) $d x y = d ((\cdot) x y)$
by (*rule domain-semiringl.fd-def*)
also have $\dots = ds.fd x y$
by (*simp add: ds.fd-def ads-d-def*)

```

finally show domain-semiringl.fd (·) d x y ≡ |x⟩ y
  by auto
qed

end

class antidomain-semiring = antidomain-semiringl + semiring-one-zero

begin

Every antidomain semiring is an antidomain monoid.

subclass antidomain-monoid
  by (standard, metis ans1 mult-1-right annir)

lemma a-zero = 0
  by (simp add: local.a-zero-def)

sublocale ds: domain-semiring (+) (·) 1 0 λx. ad (ad x) (≤) (<)
  rewrites ds.fd x y ≡ fdia x y
  by unfold-locales

end

```

3.5 The Boolean Algebra of Domain Elements

```

typedef (overloaded) 'a a2-element = {x :: 'a :: antidomain-semiring. x = d x}
  by (rule-tac x=1 in exI, auto simp: ads-d-def)

setup-lifting type-definition-a2-element

instantiation a2-element :: (antidomain-semiring) boolean-algebra

begin

lift-definition less-eq-a2-element :: 'a a2-element ⇒ 'a a2-element ⇒ bool is (≤)
.

lift-definition less-a2-element :: 'a a2-element ⇒ 'a a2-element ⇒ bool is (<) .

lift-definition bot-a2-element :: 'a a2-element is 0
  by (simp add: ads-d-def)

lift-definition top-a2-element :: 'a a2-element is 1
  by (simp add: ads-d-def)

lift-definition inf-a2-element :: 'a a2-element ⇒ 'a a2-element ⇒ 'a a2-element
is (·)
  by (metis (no-types, lifting) ads-d-def dpdz.dom-mult-closed)

```

lift-definition *sup-a2-element* :: 'a a2-element \Rightarrow 'a a2-element \Rightarrow 'a a2-element
is (+)

by (*metis ads-d-def ds.dsr5*)

lift-definition *minus-a2-element* :: 'a a2-element \Rightarrow 'a a2-element \Rightarrow 'a a2-element
is $\lambda x y. x \cdot ad y$

by (*metis (no-types, lifting) ads-d-def dpdz.domain-export'*)

lift-definition *uminus-a2-element* :: 'a a2-element \Rightarrow 'a a2-element **is** *antidomain-op*

by (*simp add: ads-d-def*)

instance

apply (*standard; transfer*)

apply (*simp add: less-le-not-le*)

apply *simp*

apply *auto[1]*

apply *simp*

apply (*metis a-subid-aux2 ads-d-def*)

apply (*metis a-subid-aux1' ads-d-def*)

apply (*metis (no-types, lifting) ads-d-def dpdz.dom-glb*)

apply *simp*

apply *simp*

apply *simp*

apply *simp*

apply (*metis a-subid' ads-d-def*)

apply (*metis (no-types, lifting) ads-d-def dpdz.dom-distrib*)

apply (*metis ads-d-def ans1*)

apply (*metis ads-d-def ans3*)

by *simp*

end

3.6 Further Properties

context *antidomain-semiringl*

begin

lemma *a-2-var*: $ad x \cdot d y = 0 \iff ad x \leq ad y$

using *local.a-gla local.ads-d-def local.dpdz.dom-weakly-local* **by** *auto*

The following two lemmas give the Galois connection of Heyting algebras.

lemma *da-shunt1*: $x \leq d y + z \implies x \cdot ad y \leq z$

proof –

assume $x \leq d y + z$

hence $x \cdot ad y \leq (d y + z) \cdot ad y$

using *mult-isor* **by** *blast*

also have $\dots = d y \cdot ad y + z \cdot ad y$

by *simp*
 also have $\dots \leq z$
 by (*simp add: a-subid-aux2 ads-d-def*)
 finally show $x \cdot ad\ y \leq z$
 by *simp*
 qed

lemma *da-shunt2*: $x \leq ad\ y + z \implies x \cdot d\ y \leq z$
 using *da-shunt1 local.a-add-idem local.ads-d-def am-add-op-def* by *auto*

lemma *d-a-galois1*: $d\ x \cdot ad\ y \leq d\ z \iff d\ x \leq d\ z + d\ y$
 by (*metis add-assoc local.a-gla local.ads-d-def local.am2 local.ans4 local.ans-d-def local.dnsz.dnso4*)

lemma *d-a-galois2*: $d\ x \cdot d\ y \leq d\ z \iff d\ x \leq d\ z + ad\ y$
proof –
 have $\bigwedge a\ aa. ad\ ((a::'a) \cdot ad\ (ad\ aa)) = ad\ (a \cdot aa)$
 using *local.a2-eq' local.apd-d-def* by *force*
 then show *?thesis*
 by (*metis d-a-galois1 local.a-export' local.ads-d-def local.ans-d-def*)
 qed

lemma *d-cancellation-1*: $d\ x \leq d\ y + d\ x \cdot ad\ y$
proof –
 have $a: d\ (d\ x \cdot ad\ y) = ad\ y \cdot d\ x$
 using *local.a-closure' local.ads-d-def local.am2 local.ans-d-def* by *auto*
 hence $d\ x \leq d\ (d\ x \cdot ad\ y) + d\ y$
 using *d-a-galois1 local.a-comm-var local.ads-d-def* by *fastforce*
 thus *?thesis*
 using *a add-commute local.ads-d-def local.am2* by *auto*
 qed

lemma *d-cancellation-2*: $(d\ z + d\ y) \cdot ad\ y \leq d\ z$
 by (*simp add: da-shunt1*)

lemma *a-de-morgan*: $ad\ (ad\ x \cdot ad\ y) = d\ (x + y)$
 by (*simp add: local.ads-d-def*)

lemma *a-de-morgan-var-3*: $ad\ (d\ x + d\ y) = ad\ x \cdot ad\ y$
 using *local.a-add-idem local.ads-d-def am-add-op-def* by *auto*

lemma *a-de-morgan-var-4*: $ad\ (d\ x \cdot d\ y) = ad\ x + ad\ y$
 using *local.a-add-idem local.ads-d-def am-add-op-def* by *auto*

lemma *a-4*: $ad\ x \leq ad\ (x \cdot y)$
 using *local.a-add-idem local.a-antitone' local.dpdz.domain-1'' am-add-op-def* by *fastforce*

lemma *a-6*: $ad\ (d\ x \cdot y) = ad\ x + ad\ y$

using *a-de-morgan-var-4* *local.ads-d-def* **by** *auto*

lemma *a-7*: $d x \cdot ad (d y + d z) = d x \cdot ad y \cdot ad z$
using *a-de-morgan-var-3* *local.mult.semigroup-axioms* *semigroup.assoc* **by** *fast-force*

lemma *a-d-add-closure* [*simp*]: $d (ad x + ad y) = ad x + ad y$
using *local.a-add-idem* *local.ads-d-def* *am-add-op-def* **by** *auto*

lemma *d-6* [*simp*]: $d x + ad x \cdot d y = d x + d y$
proof –
have $ad (ad x \cdot (x + ad y)) = d (x + y)$
by (*simp add: distrib-left ads-d-def*)
thus *?thesis*
by (*simp add: local.ads-d-def local.ans-d-def*)
qed

lemma *d-7* [*simp*]: $ad x + d x \cdot ad y = ad x + ad y$
by (*metis a-d-add-closure local.ads-d-def local.ans4 local.s4*)

lemma *a-mult-add*: $ad x \cdot (y + x) = ad x \cdot y$
by (*simp add: distrib-left*)

lemma *kat-2*: $y \cdot ad z \leq ad x \cdot y \implies d x \cdot y \cdot ad z = 0$
proof –
assume $a: y \cdot ad z \leq ad x \cdot y$
hence $d x \cdot y \cdot ad z \leq d x \cdot ad x \cdot y$
using *local.mult-isol* *mult-assoc* **by** *presburger*
thus *?thesis*
using *local.join.le-bot ads-d-def* **by** *auto*
qed

lemma *kat-3*: $d x \cdot y \cdot ad z = 0 \implies d x \cdot y = d x \cdot y \cdot d z$
using *local.a-zero-def* *local.ads-d-def* *local.am-d-def* *local.kat-3'* **by** *auto*

lemma *kat-4*: $d x \cdot y = d x \cdot y \cdot d z \implies d x \cdot y \leq y \cdot d z$
using *a-subid-aux1* *mult-assoc* *ads-d-def* **by** *auto*

lemma *kat-2-equiv*: $y \cdot ad z \leq ad x \cdot y \iff d x \cdot y \cdot ad z = 0$
proof
assume $y \cdot ad z \leq ad x \cdot y$
thus $d x \cdot y \cdot ad z = 0$
by (*simp add: kat-2*)
next
assume $1: d x \cdot y \cdot ad z = 0$
have $y \cdot ad z = (d x + ad x) \cdot y \cdot ad z$
by (*simp add: local.ads-d-def*)
also have $\dots = d x \cdot y \cdot ad z + ad x \cdot y \cdot ad z$
using *local.distrib-right* **by** *presburger*

also have $\dots = ad\ x \cdot y \cdot ad\ z$
using *1* **by** *auto*
also have $\dots \leq ad\ x \cdot y$
by (*simp add: local.a-subid-ax2*)
finally show $y \cdot ad\ z \leq ad\ x \cdot y$.
qed

lemma *kat-4-equiv*: $d\ x \cdot y = d\ x \cdot y \cdot d\ z \iff d\ x \cdot y \leq y \cdot d\ z$
using *local.ads-d-def local.dpdz.d-preserves-equation* **by** *auto*

lemma *kat-3-equiv-opp*: $ad\ z \cdot y \cdot d\ x = 0 \iff y \cdot d\ x = d\ z \cdot y \cdot d\ x$
proof –

have $ad\ z \cdot (y \cdot d\ x) = 0 \implies (ad\ z \cdot y \cdot d\ x = 0) = (y \cdot d\ x = d\ z \cdot y \cdot d\ x)$
by (*metis (no-types, opaque-lifting) add-commute local.add-zero local.ads-d-def local.as3 local.distrib-right' local.mult-1-left mult-assoc*)
thus *?thesis*
by (*metis a-4 local.a-add-idem local.a-gla2 local.ads-d-def mult-assoc am-add-op-def*)
qed

lemma *kat-4-equiv-opp*: $y \cdot d\ x = d\ z \cdot y \cdot d\ x \iff y \cdot d\ x \leq d\ z \cdot y$
using *kat-2-equiv kat-3-equiv-opp local.ads-d-def* **by** *auto*

3.7 Forward Box and Diamond Operators

lemma *fdemodalisation22*: $|x\rangle\ y \leq d\ z \iff ad\ z \cdot x \cdot d\ y = 0$

proof –
have $|x\rangle\ y \leq d\ z \iff d\ (x \cdot y) \leq d\ z$
by (*simp add: fdia-def ads-d-def*)
also have $\dots \iff ad\ z \cdot d\ (x \cdot y) = 0$
by (*metis add-commute local.a-gla local.ads-d-def local.ans4*)
also have $\dots \iff ad\ z \cdot x \cdot y = 0$
using *dpdz.dom-weakly-local mult-assoc ads-d-def* **by** *auto*
finally show *?thesis*
using *dpdz.dom-weakly-local ads-d-def* **by** *auto*
qed

lemma *dia-diff-var*: $|x\rangle\ y \leq |x\rangle\ (d\ y \cdot ad\ z) + |x\rangle\ z$

proof –
have $1: |x\rangle\ (d\ y \cdot d\ z) \leq |x\rangle\ (1 \cdot d\ z)$
using *dpdz.dom-glb-eq ds.fd-subdist fdia-def ads-d-def* **by** *force*
have $|x\rangle\ y = |x\rangle\ (d\ y \cdot (ad\ z + d\ z))$
by (*metis as3 add-comm ds.fdia-d-simp mult-1-right ads-d-def*)
also have $\dots = |x\rangle\ (d\ y \cdot ad\ z) + |x\rangle\ (d\ y \cdot d\ z)$
by (*simp add: local.distrib-left local.ds.fdia-add1*)
also have $\dots \leq |x\rangle\ (d\ y \cdot ad\ z) + |x\rangle\ (1 \cdot d\ z)$
using *1 local.join.sup.mono* **by** *blast*
finally show *?thesis*
by (*simp add: fdia-def ads-d-def*)
qed

lemma *dia-diff*: $|x\rangle y \cdot ad (|x\rangle z) \leq |x\rangle (d y \cdot ad z)$
using *fdia-def dia-diff-var d-a-galois2 ads-d-def* **by** *metis*

lemma *fdia-export-2*: $ad y \cdot |x\rangle z = |ad y \cdot x\rangle z$
using *local.am-d-def local.d-a-export local.fdia-def mult-assoc* **by** *auto*

lemma *fdia-split*: $|x\rangle y = d z \cdot |x\rangle y + ad z \cdot |x\rangle y$
by (*metis mult-onel ans3 distrib-right ads-d-def*)

definition *fbox* :: $'a \Rightarrow 'a \Rightarrow 'a ((|-) -)$ [61,81] 82) **where**
 $|x] y = ad (x \cdot ad y)$

The next lemmas establish the De Morgan duality between boxes and diamonds.

lemma *fdia-fbox-de-morgan-2*: $ad (|x\rangle y) = |x] ad y$
using *fbox-def local.a-closure local.a-loc local.am-d-def local.fdia-def* **by** *auto*

lemma *fbox-simp*: $|x] y = |x] d y$
using *fbox-def local.a-add-idem local.ads-d-def am-add-op-def* **by** *auto*

lemma *fbox-dom* [*simp*]: $|x] 0 = ad x$
by (*simp add: fbox-def*)

lemma *fbox-add1*: $|x] (d y \cdot d z) = |x] y \cdot |x] z$
using *a-de-morgan-var-4 fbox-def local.distrib-left* **by** *auto*

lemma *fbox-add2*: $|x + y] z = |x] z \cdot |y] z$
by (*simp add: fbox-def*)

lemma *fbox-mult*: $|x \cdot y] z = |x] |y] z$
using *fbox-def local.a2-eq' local.apd-d-def mult-assoc* **by** *auto*

lemma *fbox-zero* [*simp*]: $|0] x = 1$
by (*simp add: fbox-def*)

lemma *fbox-one* [*simp*]: $|1] x = d x$
by (*simp add: fbox-def ads-d-def*)

lemma *fbox-iso*: $d x \leq d y \implies |z] x \leq |z] y$

proof –

assume $d x \leq d y$

hence $ad y \leq ad x$

using *local.a-add-idem local.a-antitone' local.ads-d-def am-add-op-def* **by** *fast-force*

hence $z \cdot ad y \leq z \cdot ad x$

by (*simp add: mult-isol*)

thus $|z] x \leq |z] y$

by (*simp add: fbox-def a-antitone'*)

qed

lemma *fbox-antitone-var*: $x \leq y \implies |y| z \leq |x| z$
by (*simp add: fbox-def a-antitone mult-isor*)

lemma *fbox-subdist-1*: $|x| (d y \cdot d z) \leq |x| y$
using *a-de-morgan-var-4 fbox-def local.a-supdist-var local.distrib-left* **by** *force*

lemma *fbox-subdist-2*: $|x| y \leq |x| (d y + d z)$
by (*simp add: fbox-iso ads-d-def*)

lemma *fbox-subdist*: $|x| d y + |x| d z \leq |x| (d y + d z)$
by (*simp add: fbox-iso sup-least ads-d-def*)

lemma *fbox-diff-var*: $|x| (d y + ad z) \cdot |x| z \leq |x| y$
proof –

have $ad (ad y) \cdot ad (ad z) = ad (ad z + ad y)$
using *local.dpdz.dsg4* **by** *auto*
then have $d (d (d y + ad z) \cdot d z) \leq d y$
by (*simp add: local.a-subid-aux1' local.ads-d-def*)
then show *?thesis*
by (*metis fbox-add1 fbox-iso*)

qed

lemma *fbox-diff*: $|x| (d y + ad z) \leq |x| y + ad (|x| z)$
proof –

have $f1: \bigwedge a. ad (ad (ad (a::'a))) = ad a$
using *local.a-closure' local.ans-d-def* **by** *force*
have $f2: \bigwedge a aa. ad (ad (a::'a)) + ad aa = ad (ad a \cdot aa)$
using *local.ans-d-def* **by** *auto*
have $f3: \bigwedge a aa. ad ((a::'a) + aa) = ad (aa + a)$
by (*simp add: local.am2*)
then have $f4: \bigwedge a aa. ad (ad (ad (a::'a) \cdot aa)) = ad (ad aa + a)$
using $f2 f1$ **by** (*metis (no-types) local.ans4*)
have $f5: \bigwedge a aa ab. ad ((a::'a) \cdot (aa + ab)) = ad (a \cdot (ab + aa))$
using $f3$ *local.distrib-left* **by** *presburger*
have $f6: \bigwedge a aa. ad (ad (ad (a::'a) + aa)) = ad (ad aa \cdot a)$
using $f3 f1$ **by** *fastforce*
have $ad (x \cdot ad (y + ad z)) \leq ad (ad (x \cdot ad z) \cdot (x \cdot ad y))$
using $f5 f2 f1$ **by** (*metis (no-types) a-mult-add fbox-def fbox-subdist-1 local.a-gla2 local.ads-d-def local.ans4 local.distrib-left local.gla-1 mult-assoc*)
then show *?thesis*
using $f6 f4 f3 f1$ **by** (*simp add: fbox-def local.ads-d-def*)

qed

end

context *antidomain-semiring*

begin

lemma *kat-1*: $d x \cdot y \leq y \cdot d z \implies y \cdot ad z \leq ad x \cdot y$

proof –

assume *a*: $d x \cdot y \leq y \cdot d z$

have $y \cdot ad z = d x \cdot y \cdot ad z + ad x \cdot y \cdot ad z$

by (*metis local.ads-d-def local.as3 local.distrib-right local.mult-1-left*)

also have $\dots \leq y \cdot (d z \cdot ad z) + ad x \cdot y \cdot ad z$

by (*metis a add-iso mult-isor mult-assoc*)

also have $\dots = ad x \cdot y \cdot ad z$

by (*simp add: ads-d-def*)

finally show $y \cdot ad z \leq ad x \cdot y$

using *local.a-subid-ax2 local.dual-order.trans* **by** *blast*

qed

lemma *kat-1-equiv*: $d x \cdot y \leq y \cdot d z \iff y \cdot ad z \leq ad x \cdot y$

using *kat-1 kat-2 kat-3 kat-4* **by** *blast*

lemma *kat-3-equiv'*: $d x \cdot y \cdot ad z = 0 \iff d x \cdot y = d x \cdot y \cdot d z$

by (*simp add: kat-1-equiv local.kat-2-equiv local.kat-4-equiv*)

lemma *kat-1-equiv-opp*: $y \cdot d x \leq d z \cdot y \iff ad z \cdot y \leq y \cdot ad x$

by (*metis kat-1-equiv local.a-closure' local.ads-d-def local.ans-d-def*)

lemma *kat-2-equiv-opp*: $ad z \cdot y \leq y \cdot ad x \iff ad z \cdot y \cdot d x = 0$

by (*simp add: kat-1-equiv-opp local.kat-3-equiv-opp local.kat-4-equiv-opp*)

lemma *fbox-one-1* [*simp*]: $|x| 1 = 1$

by (*simp add: fbox-def*)

lemma *fbox-demodalisation3*: $d y \leq |x| d z \iff d y \cdot x \leq x \cdot d z$

by (*simp add: fbox-def a-gla kat-2-equiv-opp mult-assoc ads-d-def*)

end

3.8 Antidomain Kleene Algebras

class *antidomain-left-kleene-algebra* = *antidomain-semiringl* + *left-kleene-algebra-zerol*

begin

sublocale *dka*: *domain-left-kleene-algebra* (+) (\cdot) 1 0 *d* (\leq) ($<$) *star*

rewrites *domain-semiringl.fd* (\cdot) $d x y \equiv |x\rangle y$

by (*unfold-locales, auto simp add: local.ads-d-def ans-d-def*)

lemma *a-star* [*simp*]: $ad (x^*) = 0$

using *dka.dom-star local.a-very-costrict'' local.ads-d-def* **by** *force*

lemma *fbox-star-unfold* [*simp*]: $|1] z \cdot |x| x^* z = |x^*] z$

proof –
have $ad (ad z + x \cdot (x^* \cdot ad z)) = ad (x^* \cdot ad z)$
using *local.conway.dagger-unfoldl-distr mult-assoc* **by** *auto*
then show *?thesis*
using *local.a-closure' local.ans-d-def local.fbox-def local.fdia-def local.fdia-fbox-de-morgan-2*
by *fastforce*
qed

lemma *fbox-star-unfold-var* [*simp*]: $d z \cdot |x| |x^*| z = |x^*| z$
using *fbox-star-unfold* **by** *auto*

lemma *fbox-star-unfoldr* [*simp*]: $|1| z \cdot |x^*| |x| z = |x^*| z$
by (*metis fbox-star-unfold fbox-mult star-slide-var*)

lemma *fbox-star-unfoldr-var* [*simp*]: $d z \cdot |x^*| |x| z = |x^*| z$
using *fbox-star-unfoldr* **by** *auto*

lemma *fbox-star-induct-var*: $d y \leq |x| y \implies d y \leq |x^*| y$
proof –

assume *a1*: $d y \leq |x| y$
have $\bigwedge a. ad (ad (ad (a::'a))) = ad a$
using *local.a-closure' local.ans-d-def* **by** *auto*
then have $ad (ad (x^* \cdot ad y)) \leq ad y$
using *a1* **by** (*metis dka.fdia-star-induct local.a-export' local.ads-d-def local.ans4*
local.ans-d-def local.eq-refl local.fbox-def local.fdia-def local.meet-ord-def)
then have $ad (ad y + ad (x^* \cdot ad y)) = zero-class.zero$
by (*metis (no-types) add-commute local.a-2-var local.ads-d-def local.ans4*)
then show *?thesis*
using *local.a-2-var local.ads-d-def local.fbox-def* **by** *auto*
qed

lemma *fbox-star-induct*: $d y \leq d z \cdot |x| y \implies d y \leq |x^*| z$
proof –

assume *a1*: $d y \leq d z \cdot |x| y$
hence *a*: $d y \leq d z$ **and** $d y \leq |x| y$
apply (*metis local.a-subid-aux2 local.dual-order.trans local.fbox-def*)
using *a1* *dka.dom-subid-aux2 local.dual-order.trans* **by** *blast*
hence $d y \leq |x^*| y$
using *fbox-star-induct-var* **by** *blast*
thus *?thesis*
using *a* *local.fbox-iso local.order.trans* **by** *blast*
qed

lemma *fbox-star-induct-eq*: $d z \cdot |x| y = d y \implies d y \leq |x^*| z$
by (*simp add: fbox-star-induct*)

lemma *fbox-export-1*: $ad y + |x| y = |d y \cdot x| y$
by (*simp add: local.a-6 local.fbox-def mult-assoc*)

```

lemma fbox-export-2:  $d\ y + |x| y = |ad\ y \cdot x| y$ 
  by (simp add: local.ads-d-def local.ans-d-def local.fbox-def mult-assoc)

end

class antidomain-kleene-algebra = antidomain-semiring + kleene-algebra

begin

subclass antidomain-left-kleene-algebra ..

lemma  $d\ p \leq |(d\ t \cdot x)^* \cdot ad\ t| (d\ q \cdot ad\ t) \implies d\ p \leq |d\ t \cdot x| d\ q$ 

oops

end

end

```

4 Range and Antirange Semirings

```

theory Range-Semiring
imports Antidomain-Semiring
begin

```

4.1 Range Semirings

We set up the duality between domain and antidomain semirings on the one hand and range and antirange semirings on the other hand.

```

class range-op =
  fixes range-op :: 'a  $\Rightarrow$  'a (r)

class range-semiring = semiring-one-zero + plus-ord + range-op +
  assumes rsr1 [simp]:  $x + (x \cdot r\ x) = x \cdot r\ x$ 
  and rsr2 [simp]:  $r\ (r\ x \cdot y) = r(x \cdot y)$ 
  and rsr3 [simp]:  $r\ x + 1 = 1$ 
  and rsr4 [simp]:  $r\ 0 = 0$ 
  and rsr5 [simp]:  $r\ (x + y) = r\ x + r\ y$ 

begin

definition bd :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a ( $\langle \cdot | \cdot \rangle$  - [61,81] 82) where
   $\langle x | y = r\ (y \cdot x)$ 

no-notation range-op (r)

end

```

```

sublocale range-semiring  $\subseteq$  rdual: domain-semiring (+)  $\lambda x y. y \cdot x$  1 0 range-op
( $\leq$ ) ( $<$ )
  rewrites rdual.fd  $x y = \langle x | y$ 
proof -
  show class.domain-semiring (+)  $(\lambda x y. y \cdot x)$  1 0 range-op ( $\leq$ ) ( $<$ )
    by (standard, auto simp: mult-assoc distrib-left)
  then interpret rdual: domain-semiring (+)  $\lambda x y. y \cdot x$  1 0 range-op ( $\leq$ ) ( $<$ ) .
  show rdual.fd  $x y = \langle x | y$ 
    unfolding rdual.fd-def bd-def by auto
qed

```

```

sublocale domain-semiring  $\subseteq$  ddual: range-semiring (+)  $\lambda x y. y \cdot x$  1 0 domain-op
( $\leq$ ) ( $<$ )
  rewrites ddual.bd  $x y = \text{domain-semiringl-class.fd } x y$ 
proof -
  show class.range-semiring (+)  $(\lambda x y. y \cdot x)$  1 0 domain-op ( $\leq$ ) ( $<$ )
    by (standard, auto simp: mult-assoc distrib-left)
  then interpret ddual: range-semiring (+)  $\lambda x y. y \cdot x$  1 0 domain-op ( $\leq$ ) ( $<$ ) .
  show ddual.bd  $x y = \text{domain-semiringl-class.fd } x y$ 
    unfolding ddual.bd-def fd-def by auto
qed

```

4.2 Antirange Semirings

```

class antirange-op =
  fixes antirange-op :: 'a  $\Rightarrow$  'a (ar - [999] 1000)

```

```

class antirange-semiring = semiring-one-zero + plus-ord + antirange-op +
assumes ars1 [simp]:  $x \cdot \text{ar } x = 0$ 
and ars2 [simp]:  $\text{ar } (x \cdot y) + \text{ar } (\text{ar } \text{ar } x \cdot y) = \text{ar } (\text{ar } \text{ar } x \cdot y)$ 
and ars3 [simp]:  $\text{ar } \text{ar } x + \text{ar } x = 1$ 

```

begin

```

no-notation bd ( $\langle \cdot |$  - [61,81] 82)

```

```

definition ars-r :: 'a  $\Rightarrow$  'a (r) where
  r x = ar (ar x)

```

```

definition bdia :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a ( $\langle \cdot |$  - [61,81] 82) where
   $\langle x | y = \text{ar } \text{ar } (y \cdot x)$ 

```

```

definition bbox :: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a ( $[ \cdot |$  - [61,81] 82) where
   $[x | y = \text{ar } (\text{ar } y \cdot x)$ 

```

end

```

sublocale antirange-semiring  $\subseteq$  ardual: antidomain-semiring antirange-op (+)  $\lambda x$ 
y. y  $\cdot x$  1 0 ( $\leq$ ) ( $<$ )

```

```

rewrites ardual.ads-d  $x = r\ x$ 
and ardual.fdia  $x\ y = \langle x \mid y$ 
and ardual.fbox  $x\ y = [x \mid y$ 
proof -
show class.antidomain-semiring antirange-op  $(+)$   $(\lambda x\ y. y \cdot x)$   $1\ 0$   $(\leq)$   $(<)$ 
  by (standard, auto simp: mult-assoc distrib-left)
then interpret ardual: antidomain-semiring antirange-op  $(+)$   $\lambda x\ y. y \cdot x$   $1\ 0$ 
 $(\leq)$   $(<)$  .
show ardual.ads-d  $x = r\ x$ 
  by (simp add: ardual.ads-d-def local.ars-r-def)
show ardual.fdia  $x\ y = \langle x \mid y$ 
  unfolding ardual.fdia-def bdia-def by auto
show ardual.fbox  $x\ y = [x \mid y$ 
  unfolding ardual.fbox-def bbox-def by auto
qed

```

context *antirange-semiring*

begin

```

sublocale rs: range-semiring  $(+)$   $(\cdot)$   $1\ 0$   $\lambda x. ar\ ar\ x$   $(\leq)$   $(<)$ 
  by unfold-locales

```

end

```

sublocale antidomain-semiring  $\subseteq$  addual: antirange-semiring  $(+)$   $\lambda x\ y. y \cdot x$   $1\ 0$ 
antidomain-op  $(\leq)$   $(<)$ 
  rewrites addual.ars-r  $x = d\ x$ 
  and addual.bdia  $x\ y = |x\rangle\ y$ 
  and addual.bbox  $x\ y = [x]\ y$ 
proof -
show class.antirange-semiring  $(+)$   $(\lambda x\ y. y \cdot x)$   $1\ 0$  antidomain-op  $(\leq)$   $(<)$ 
  by (standard, auto simp: mult-assoc distrib-left)
then interpret addual: antirange-semiring  $(+)$   $\lambda x\ y. y \cdot x$   $1\ 0$  antidomain-op
 $(\leq)$   $(<)$  .
show addual.ars-r  $x = d\ x$ 
  by (simp add: addual.ars-r-def local.ads-d-def)
show addual.bdia  $x\ y = |x\rangle\ y$ 
  unfolding addual.bdia-def fdia-def by auto
show addual.bbox  $x\ y = [x]\ y$ 
  unfolding addual.bbox-def fbox-def by auto
qed

```

4.3 Antirange Kleene Algebras

class *antirange-kleene-algebra* = *antirange-semiring* + *kleene-algebra*

```

sublocale antirange-kleene-algebra  $\subseteq$  dual: antidomain-kleene-algebra antirange-op
 $(+)$   $\lambda x\ y. y \cdot x$   $1\ 0$   $(\leq)$   $(<)$  star

```


by (*standard, auto simp: local.star-inductr' local.star-inductl*)

sublocale *antidomain-kleene-algebra* \subseteq *dual: antirange-kleene-algebra* (+) $\lambda x y. y \cdot x$ 1 0 (\leq) ($<$) *star antidomain-op*
by (*standard, simp-all add: star-inductr star-inductl*)

Hence all range theorems have been derived by duality in a generic way.

end

5 Modal Kleene Algebras

This section studies laws that relate antidomain and antirange semirings and Kleene algebra, notably Galois connections and conjugations as those studied in [13, 7].

theory *Modal-Kleene-Algebra*
imports *Range-Semiring*
begin

class *modal-semiring* = *antidomain-semiring* + *antirange-semiring* +
assumes *domrange [simp]:* $d (r x) = r x$
and *rangedom [simp]:* $r (d x) = d x$

begin

These axioms force that the domain algebra and the range algebra coincide.

lemma *domrangefix:* $d x = x \longleftrightarrow r x = x$
by (*metis domrange rangedom*)

lemma *box-diamond-galois-1:*

assumes $d p = p$ **and** $d q = q$
shows $\langle x | p \leq q \longleftrightarrow p \leq [x] q$

proof –

have $\langle x | p \leq q \longleftrightarrow p \cdot x \leq x \cdot q$

by (*metis assms domrangefix local.ardual.ds.fdemodalisation2 local.ars-r-def*)

thus *?thesis*

by (*metis assms fbox-demodalisation3*)

qed

lemma *box-diamond-galois-2:*

assumes $d p = p$ **and** $d q = q$
shows $|x \rangle p \leq q \longleftrightarrow p \leq [x] q$

proof –

have $|x \rangle p \leq q \longleftrightarrow x \cdot p \leq q \cdot x$

by (*metis assms local.ads-d-def local.ds.fdemodalisation2*)

thus *?thesis*

by (*metis assms domrangefix local.ardual.fbox-demodalisation3*)

qed

lemma *diamond-conjugation-var-1*:

assumes $d p = p$ **and** $d q = q$

shows $|x\rangle p \leq ad q \iff \langle x| q \leq ad p$

proof –

have $|x\rangle p \leq ad q \iff x \cdot p \leq ad q \cdot x$

by (*metis assms local.ads-d-def local.ds.fdemodalisation2*)

also have $\dots \iff q \cdot x \leq x \cdot ad p$

by (*metis assms local.ads-d-def local.kat-1-equiv-opp*)

finally show *?thesis*

by (*metis assms box-diamond-galois-1 local.ads-d-def local.fbox-demodalisation3*)

qed

lemma *diamond-conjugation-aux*:

assumes $d p = p$ **and** $d q = q$

shows $\langle x| p \leq ad q \iff q \cdot \langle x| p = 0$

apply *standard*

apply (*metis assms(2) local.a-antitone' local.a-gla local.ads-d-def*)

proof –

assume *a1*: $q \cdot \langle x| p = \text{zero-class.zero}$

have $ad (ad q) = q$

using *assms(2) local.ads-d-def* **by** *fastforce*

then show $\langle x| p \leq ad q$

using *a1*

by (*metis a-closure' a-gla ardual.a-subid-aux1' bdia-def
dnsz.dsg4 dpdz.dsg1 dpdz.dsg3 ds.dsr4 order.trans*)

qed

lemma *diamond-conjugation*:

assumes $d p = p$ **and** $d q = q$

shows $p \cdot |x\rangle q = 0 \iff q \cdot \langle x| p = 0$

proof –

have $p \cdot |x\rangle q = 0 \iff |x\rangle q \leq ad p$

by (*metis assms(1) local.a-gla local.ads-d-def local.am2 local.fdia-def*)

also have $\dots \iff \langle x| p \leq ad q$

by (*simp add: assms diamond-conjugation-var-1*)

finally show *?thesis*

by (*simp add: assms diamond-conjugation-aux*)

qed

lemma *box-conjugation-var-1*:

assumes $d p = p$ **and** $d q = q$

shows $ad p \leq [x] q \iff ad q \leq [x] p$

by (*metis assms box-diamond-galois-1 box-diamond-galois-2 diamond-conjugation-var-1
local.ads-d-def*)

lemma *box-diamond-cancellation-1*: $d p = p \implies p \leq [x] \langle x| p$

using *box-diamond-galois-1 domrangefix local.ars-r-def local.bdia-def* **by** *fastforce*

```

lemma box-diamond-cancellation-2:  $d p = p \implies p \leq [x \mid |x] p$ 
  by (metis box-diamond-galois-2 local.ads-d-def local.dpdz.domain-invol local.eq-refl
local.fdia-def)

lemma box-diamond-cancellation-3:  $d p = p \implies |x [x] p \leq p$ 
  using box-diamond-galois-2 domrangefix local.ardual.fdia-fbox-de-morgan-2 local.ars-r-def local.bbox-def local.bdia-def by auto

lemma box-diamond-cancellation-4:  $d p = p \implies \langle x \mid |x \rangle p \leq p$ 
  using box-diamond-galois-1 local.ads-d-def local.fbox-def local.fdia-def local.fdia-fbox-de-morgan-2
by auto

end

class modal-kleene-algebra = modal-semiring + kleene-algebra
begin

subclass antidomain-kleene-algebra ..

subclass antirange-kleene-algebra ..

end

end

```

6 Models of Modal Kleene Algebras

```

theory Modal-Kleene-Algebra-Models
imports Kleene-Algebra.Kleene-Algebra-Models
  Modal-Kleene-Algebra

```

```

begin

```

This section develops the relation model. We also briefly develop the trace model for antidomain Kleene algebras, but not for antirange or full modal Kleene algebras. The reason is that traces are implemented as lists; we therefore expect tedious inductive proofs in the presence of range. The language model is not particularly interesting.

```

definition rel-ad :: 'a rel  $\Rightarrow$  'a rel where
  rel-ad R =  $\{(x,x) \mid x. \neg (\exists y. (x,y) \in R)\}$ 

```

```

interpretation rel-antidomain-kleene-algebra: antidomain-kleene-algebra rel-ad ( $\cup$ )
(O) Id  $\{\}$  ( $\subseteq$ ) ( $\subset$ ) rtrancl
by unfold-locales (auto simp: rel-ad-def)

```

```

definition trace-a :: ('p, 'a) trace set  $\Rightarrow$  ('p, 'a) trace set where
  trace-a X =  $\{(p,[]) \mid p. \neg (\exists x. x \in X \wedge p = \text{first } x)\}$ 

```

interpretation *trace-antidomain-kleene-algebra*: *antidomain-kleene-algebra trace-a*
 (\cup) *t-prod t-one t-zero* (\subseteq) (\subset) *t-star*

proof

show $\bigwedge x. t\text{-prod } (trace\text{-a } x) x = t\text{-zero}$

by (*auto simp: trace-a-def t-prod-def t-fusion-def t-zero-def*)

show $\bigwedge x y. trace\text{-a } (t\text{-prod } x y) \cup trace\text{-a } (t\text{-prod } x (trace\text{-a } (trace\text{-a } y))) = trace\text{-a } (t\text{-prod } x (trace\text{-a } (trace\text{-a } y)))$

by (*auto simp: trace-a-def t-prod-def t-fusion-def*)

show $\bigwedge x. trace\text{-a } (trace\text{-a } x) \cup trace\text{-a } x = t\text{-one}$

by (*auto simp: trace-a-def t-one-def*)

qed

The trace model should be extended to cover modal Kleene algebras in the future.

definition *rel-ar* :: $'a \text{ rel} \Rightarrow 'a \text{ rel}$ **where**

rel-ar $R = \{(y,y) \mid y. \neg (\exists x. (x,y) \in R)\}$

interpretation *rel-antirange-kleene-algebra*: *antirange-semiring* (\cup) (O) *Id* $\{\}$ *rel-ar*
 (\subseteq) (\subset)

apply *unfold-locales*

apply (*simp-all add: rel-ar-def*)

by *auto*

interpretation *rel-modal-kleene-algebra*: *modal-kleene-algebra* (\cup) (O) *Id* $\{\}$ (\subseteq)
 (\subset) *rtrancl rel-ad rel-ar*

apply *standard*

apply (*metis (no-types, lifting) rel-antidomain-kleene-algebra.a-d-closed rel-antidomain-kleene-algebra.a-one*

rel-antidomain-kleene-algebra.addual.ars-r-def rel-antidomain-kleene-algebra.am5-lem

rel-antidomain-kleene-algebra.am6-lem rel-antidomain-kleene-algebra.apd-d-def rel-antidomain-kleene-algebra.d

rel-antidomain-kleene-algebra.dpdz.dom-one rel-antirange-kleene-algebra.ardual.a-comm'

rel-antirange-kleene-algebra.ardual.a-d-closed rel-antirange-kleene-algebra.ardual.a-mul-d'

rel-antirange-kleene-algebra.ardual.a-mult-idem rel-antirange-kleene-algebra.ardual.a-zero

rel-antirange-kleene-algebra.ardual.ads-d-def rel-antirange-kleene-algebra.ardual.am6-lem

rel-antirange-kleene-algebra.ardual.apd-d-def rel-antirange-kleene-algebra.ardual.s4)

by (*metis rel-antidomain-kleene-algebra.a-zero rel-antidomain-kleene-algebra.addual.ars1*

rel-antidomain-kleene-algebra.addual.ars-r-def rel-antidomain-kleene-algebra.am5-lem

rel-antidomain-kleene-algebra.am6-lem rel-antidomain-kleene-algebra.ds.ddual.mult-oner

rel-antidomain-kleene-algebra.s4 rel-antirange-kleene-algebra.ardual.ads-d-def rel-antirange-kleene-algebra.ardu

rel-antirange-kleene-algebra.ardual.apd1 rel-antirange-kleene-algebra.ardual.dpdz.dns1')

end

7 Applications of Modal Kleene Algebras

theory *Modal-Kleene-Algebra-Applications*

imports *Antidomain-Semiring*

begin

This file collects some applications of the theories developed so far. These are described in [11].

context *antidomain-kleene-algebra*
begin

7.1 A Reachability Result

This example is taken from [4].

lemma *opti-iterate-var* [*simp*]: $|(ad\ y \cdot x)^* y = |x^* y$
proof (*rule order.antisym*)
show $|(ad\ y \cdot x)^* y \leq |x^* y$
by (*simp add: a-subid-aux1' ds.fd-iso2 star-iso*)
have $d\ y + |x \rangle |(ad\ y \cdot x)^* y = d\ y + ad\ y \cdot |x \rangle |(ad\ y \cdot x)^* y$
using *local.ads-d-def local.d-6 local.fdia-def* **by** *auto*
also have $\dots = d\ y + |ad\ y \cdot x \rangle |(ad\ y \cdot x)^* y$
by (*simp add: fdia-export-2*)
finally have $d\ y + |x \rangle |(ad\ y \cdot x)^* y = |(ad\ y \cdot x)^* y$
by *simp*
thus $|x^* y \leq |(ad\ y \cdot x)^* y$
using *local.dka.fd-def local.dka.fdia-star-induct-eq* **by** *auto*
qed

lemma *opti-iterate* [*simp*]: $d\ y + |(x \cdot ad\ y)^* |x \rangle y = |x^* y$
proof –
have $d\ y + |(x \cdot ad\ y)^* |x \rangle y = d\ y + |x \rangle |(ad\ y \cdot x)^* y$
by (*metis local.conway.dagger-slide local.ds.fdia-mult*)
also have $\dots = d\ y + |x \rangle |x^* y$
by *simp*
finally show *?thesis*
by *force*
qed

lemma *opti-iterate-var-2* [*simp*]: $d\ y + |ad\ y \cdot x \rangle |x^* y = |x^* y$
by (*metis local.dka.fdia-star-unfold-var opti-iterate-var*)

7.2 Derivation of Segerberg's Formula

This example is taken from [5].

definition *Alpha* :: $'a \Rightarrow 'a \Rightarrow 'a$ (*A*)
where $A\ x\ y = d\ (x \cdot y) \cdot ad\ y$

lemma *A-dom* [*simp*]: $d\ (A\ x\ y) = A\ x\ y$
using *Alpha-def local.a-d-closed local.ads-d-def local.apd-d-def* **by** *auto*

lemma *A-fdia*: $A\ x\ y = |x \rangle y \cdot ad\ y$
by (*simp add: Alpha-def local.dka.fd-def*)

lemma *A-fdia-var*: $A\ x\ y = |x \rangle d\ y \cdot ad\ y$

by (*simp add: A-fdia*)

lemma *a-A*: $ad (A x (ad y)) = |x\rangle y + ad y$
 using *Alpha-def local.a-6 local.a-d-closed local.apd-d-def local.fbox-def* by *force*

lemma *fsegerberg* [*simp*]: $d y + |x^*\rangle A x y = |x^*\rangle y$

proof (*rule order.antisym*)

have $d y + |x\rangle (d y + |x^*\rangle A x y) = d y + |x\rangle y + |x\rangle |x^*\rangle (|x\rangle y \cdot ad y)$
 by (*simp add: A-fdia add-assoc local.ds.fdia-add1*)

also have $\dots = d y + |x\rangle y \cdot ad y + |x\rangle |x^*\rangle (|x\rangle y \cdot ad y)$
 by (*metis local.am2 local.d-6 local.dka.fd-def local.fdia-def*)

finally have $d y + |x\rangle (d y + |x^*\rangle A x y) = d y + |x^*\rangle A x y$

by (*metis A-dom A-fdia add-assoc local.dka.fdia-star-unfold-var*)

thus $|x^*\rangle y \leq d y + |x^*\rangle A x y$

by (*metis local.a-d-add-closure local.ads-d-def local.dka.fdia-star-induct-eq local.fdia-def*)

have $d y + |x^*\rangle A x y = d y + |x^*\rangle (|x\rangle y \cdot ad y)$

by (*simp add: A-fdia*)

also have $\dots \leq d y + |x^*\rangle |x\rangle y$

using *local.dpdz.domain-1'' local.ds.fd-iso1 local.join.sup-mono* by *blast*

finally show $d y + |x^*\rangle A x y \leq |x^*\rangle y$

by *simp*

qed

lemma *fbox-segerberg* [*simp*]: $d y \cdot |x^*\rangle (|x\rangle y + ad y) = |x^*\rangle y$

proof –

have $|x^*\rangle (|x\rangle y + ad y) = d (|x^*\rangle (|x\rangle y + ad y))$

using *local.a-d-closed local.ads-d-def local.apd-d-def local.fbox-def* by *auto*

hence $f1: |x^*\rangle (|x\rangle y + ad y) = ad (|x^*\rangle (A x (ad y)))$

by (*simp add: a-A local.fdia-fbox-de-morgan-2*)

have $ad y + |x^*\rangle (A x (ad y)) = |x^*\rangle ad y$

by (*metis fsegerberg local.a-d-closed local.ads-d-def local.apd-d-def*)

thus *?thesis*

by (*metis f1 local.ads-d-def local.ans4 local.fbox-simp local.fdia-fbox-de-morgan-2*)

qed

7.3 Wellfoundedness and Loeb's Formula

This example is taken from [7].

definition *Omega* :: $'a \Rightarrow 'a \Rightarrow 'a (\Omega)$

where $\Omega x y = d y \cdot ad (x \cdot y)$

If y is a set, then $\Omega(x, y)$ describes those elements in y from which no further x transitions are possible.

lemma *omega-fdia*: $\Omega x y = d y \cdot ad (|x\rangle y)$

using *Omega-def local.a-d-closed local.ads-d-def local.apd-d-def local.dka.fd-def*
 by *auto*

lemma *omega-fbox*: $\Omega x y = d y \cdot |x| (ad y)$
by (*simp add: fdia-fbox-de-morgan-2 omega-fdia*)

lemma *omega-absorb1* [*simp*]: $\Omega x y \cdot ad (|x\rangle y) = \Omega x y$
by (*simp add: mult-assoc omega-fdia*)

lemma *omega-absorb2* [*simp*]: $\Omega x y \cdot ad (x \cdot y) = \Omega x y$
by (*simp add: Omega-def mult-assoc*)

lemma *omega-le-1*: $\Omega x y \leq d y$
by (*simp add: Omega-def d-a-galois1*)

lemma *omega-subid*: $\Omega x (d y) \leq d y$
by (*simp add: Omega-def local.a-subid-aux2*)

lemma *omega-le-2*: $\Omega x y \leq ad (|x\rangle y)$
by (*simp add: local.dka.dom-subid-aux2 omega-fdia*)

lemma *omega-dom* [*simp*]: $d (\Omega x y) = \Omega x y$
using *Omega-def local.a-d-closed local.ads-d-def local.apd-d-def* **by** *auto*

lemma *a-omega*: $ad (\Omega x y) = ad y + |x\rangle y$
by (*simp add: Omega-def local.a-6 local.ds.fd-def*)

lemma *omega-fdia-3* [*simp*]: $d y \cdot ad (\Omega x y) = d y \cdot |x\rangle y$
using *Omega-def local.ads-d-def local.fdia-def local.s4* **by** *presburger*

lemma *omega-zero-equiv-1*: $\Omega x y = 0 \iff d y \leq |x\rangle y$
by (*simp add: Omega-def local.a-gla local.ads-d-def local.fdia-def*)

definition *Loebian* :: 'a \Rightarrow bool
where *Loebian* x = $(\forall y. |x\rangle y \leq |x\rangle \Omega x y)$

definition *PreLoebian* :: 'a \Rightarrow bool
where *PreLoebian* x = $(\forall y. d y \leq |x^*\rangle \Omega x y)$

definition *Noetherian* :: 'a \Rightarrow bool
where *Noetherian* x = $(\forall y. \Omega x y = 0 \implies d y = 0)$

lemma *noetherian-alt*: $Noetherian x \iff (\forall y. d y \leq |x\rangle y \implies d y = 0)$
by (*simp add: Noetherian-def omega-zero-equiv-1*)

lemma *Noetherian-iff-PreLoebian*: $Noetherian x \iff PreLoebian x$
proof
assume *hyp*: *Noetherian* x
{
fix y
have $d y \cdot ad (|x^*\rangle \Omega x y) = d y \cdot ad (\Omega x y + |x\rangle |x^*\rangle \Omega x y)$
by (*metis local.dka.fdia-star-unfold-var omega-dom*)

also have $\dots = d y \cdot ad (\Omega x y) \cdot ad (|x\rangle |x^*\rangle \Omega x y)$
using *ans4 mult-assoc by presburger*
also have $\dots \leq |x\rangle d y \cdot ad (|x\rangle |x^*\rangle \Omega x y)$
by (*simp add: local.dka.dom-subid-aux2 local.mult-isor*)
also have $\dots \leq |x\rangle (d y \cdot ad (|x^*\rangle \Omega x y))$
by (*simp add: local.dia-diff*)
finally have $d y \cdot ad (|x^*\rangle \Omega x y) \leq |x\rangle (d y \cdot ad (|x^*\rangle \Omega x y))$
by *blast*
hence $d y \cdot ad (|x^*\rangle \Omega x y) = 0$
by (*metis hyp local.ads-d-def local.dpdz.dom-mult-closed local.fdia-def noetherian-alt*)
hence $d y \leq |x^*\rangle \Omega x y$
by (*simp add: local.a-gla local.ads-d-def local.dka.fd-def*)
}
thus *PreLoebian x*
using *PreLoebian-def by blast*
next
assume *a: PreLoebian x*
{
fix *y*
assume *b: $\Omega x y = 0$*
hence $d y \leq |x\rangle d y$
using *omega-zero-equiv-1 by auto*
hence $d y = 0$
by (*metis (no-types) PreLoebian-def a b a-one a-zero add-zero annir fdia-def less-eq-def*)
}
thus *Noetherian x*
by (*simp add: Noetherian-def*)
qed

lemma *Loebian-imp-Noetherian: Loebian x \implies Noetherian x*

proof –

assume *a: Loebian x*
{
fix *y*
assume *b: $\Omega x y = 0$*
hence $d y \leq |x\rangle d y$
using *omega-zero-equiv-1 by auto*
also have $\dots \leq |x\rangle (\Omega x y)$
using *Loebian-def a by auto*
finally have $d y = 0$
by (*simp add: b order.antisym fdia-def*)
}
thus *Noetherian x*
by (*simp add: Noetherian-def*)
qed

lemma *d-transitive: $(\forall y. |x\rangle |x\rangle y \leq |x\rangle y) \implies (\forall y. |x\rangle y = |x^*\rangle |x\rangle y)$*

proof –
assume $a: \forall y. |x\rangle |x\rangle y \leq |x\rangle y$
{
 fix y
 have $|x\rangle y + |x\rangle |x\rangle y \leq |x\rangle y$
 by (*simp add: a*)
 hence $|x^*\rangle |x\rangle y \leq |x\rangle y$
 using *local.dka.fd-def local.dka.fdia-star-induct-var* **by** *auto*
 have $|x\rangle y \leq |x^*\rangle |x\rangle y$
 using *local.dka.fd-def local.order-prop opti-iterate* **by** *force*
}
thus *?thesis*
using *a order.antisym dka.fd-def dka.fdia-star-induct-var* **by** *auto*
qed

lemma *d-transitive-var*: $(\forall y. |x\rangle |x\rangle y \leq |x\rangle y) \implies (\forall y. |x\rangle y = |x\rangle |x^*\rangle y)$

proof –
assume $\forall y. |x\rangle |x\rangle y \leq |x\rangle y$
then have $\forall a. |x\rangle |x^*\rangle a = |x\rangle a$
by (*metis (full-types) d-transitive local.dka.fd-def local.dka.fdia-d-simp local.star-slide-var mult-assoc*)
then show *?thesis*
by *presburger*
qed

lemma *d-transitive-PreLoebian-imp-Loebian*: $(\forall y. |x\rangle |x\rangle y \leq |x\rangle y) \implies \text{PreLoebian } x \implies \text{Loebian } x$

proof –
assume *wt*: $(\forall y. |x\rangle |x\rangle y \leq |x\rangle y)$
and *PreLoebian* x
hence $\forall y. |x\rangle y \leq |x\rangle |x^*\rangle \Omega x y$
using *PreLoebian-def local.ads-d-def local.dka.fd-def local.ds.fd-iso1* **by** *auto*
hence $\forall y. |x\rangle y \leq |x\rangle \Omega x y$
by (*metis wt d-transitive-var*)
then show *Loebian* x
using *Loebian-def fdia-def* **by** *auto*
qed

lemma *d-transitive-Noetherian-iff-Loebian*: $\forall y. |x\rangle |x\rangle y \leq |x\rangle y \implies \text{Noetherian } x \iff \text{Loebian } x$

using *Loebian-imp-Noetherian Noetherian-iff-PreLoebian d-transitive-PreLoebian-imp-Loebian*
by *blast*

lemma *Loeb-iff-box-Loeb*: $\text{Loebian } x \iff (\forall y. |x\rangle (ad (|x\rangle y) + d y) \leq |x\rangle y)$

proof –
have $\text{Loebian } x \iff (\forall y. |x\rangle y \leq |x\rangle (d y \cdot |x\rangle (ad y)))$
using *Loebian-def omega-fbox* **by** *auto*
also have $\dots \iff (\forall y. ad (|x\rangle (d y \cdot |x\rangle (ad y))) \leq ad (|x\rangle y))$
using *a-antitone' fdia-def* **by** *fastforce*

```

also have ...  $\longleftrightarrow (\forall y. |x| \text{ ad } (d y \cdot |x| (ad y)) \leq |x| (ad y))$ 
  by (simp add: fdia-fbox-de-morgan-2)
also have ...  $\longleftrightarrow (\forall y. |x| (d (ad y) + ad (|x| (ad y))) \leq |x| (ad y))$ 
  by (simp add: local.ads-d-def local.fbox-def)
finally show ?thesis
  by (metis add-commute local.a-d-closed local.ads-d-def local.apd-d-def local.fbox-def)
qed

end

```

7.4 Divergence Kleene Algebras and Separation of Termination

The notion of divergence has been added to modal Kleene algebras in [5]. More facts about divergence could be added in the future. Some could be adapted from omega algebras.

```

class nabla-op =
  fixes nabla :: 'a  $\Rightarrow$  'a ( $\nabla$ - [999] 1000)

class fdivergence-kleene-algebra = antidomain-kleene-algebra + nabla-op +
  assumes nabla-closure [simp]:  $d \nabla x = \nabla x$ 
  and nabla-unfold:  $\nabla x \leq |x\rangle \nabla x$ 
  and nabla-coinduction:  $d y \leq |x\rangle y + d z \implies d y \leq \nabla x + |x^*\rangle z$ 

begin

lemma nabla-coinduction-var:  $d y \leq |x\rangle y \implies d y \leq \nabla x$ 
proof –
  assume  $d y \leq |x\rangle y$ 
  hence  $d y \leq |x\rangle y + d 0$ 
  by simp
  hence  $d y \leq \nabla x + |x^*\rangle 0$ 
  using nabla-coinduction by blast
  thus  $d y \leq \nabla x$ 
  by (simp add: fdia-def)
qed

lemma nabla-unfold-eq [simp]:  $|x\rangle \nabla x = \nabla x$ 
proof –
  have f1:  $ad (ad (x \cdot \nabla x)) = ad (ad (x \cdot \nabla x)) + \nabla x$ 
  using local.ds.fd-def local.join.sup.order-iff local.nabla-unfold by presburger
  then have  $ad (ad (x \cdot \nabla x)) \cdot ad (ad \nabla x) = \nabla x$ 
  by (metis local.ads-d-def local.dpdz.dns5 local.dpdz.dsg4 local.join.sup-commute local.nabla-closure)
  then show ?thesis
  using f1 by (metis local.ads-d-def local.ds.fd-def local.ds.fd-subdist-2 local.join.sup.order-iff local.join.sup-commute nabla-coinduction-var)
qed

```

lemma *nabla-subdist*: $\nabla x \leq \nabla (x + y)$
proof –
 have $d \nabla x \leq \nabla (x + y)$
 by (*metis local.ds.fd-iso2 local.join.sup.cobounded1 local.nabla-closure nabla-coinduction-var nabla-unfold-eq*)
 thus *?thesis*
 by *simp*
qed

lemma *nabla-iso*: $x \leq y \implies \nabla x \leq \nabla y$
by (*metis less-eq-def nabla-subdist*)

lemma *nabla-omega*: $\Omega x (d y) = 0 \implies d y \leq \nabla x$
using *omega-zero-equiv-1 nabla-coinduction-var* **by** *fastforce*

lemma *nabla-noether*: $\nabla x = 0 \implies \text{Noetherian } x$
using *local.join.le-bot local.noetherian-alt nabla-coinduction-var* **by** *blast*

lemma *nabla-preloeb*: $\nabla x = 0 \implies \text{PreLoebian } x$
using *Noetherian-iff-PreLoebian nabla-noether* **by** *auto*

lemma *star-nabla-1* [*simp*]: $|x^*\rangle \nabla x = \nabla x$
proof (*rule order.antisym*)
 show $|x^*\rangle \nabla x \leq \nabla x$
 by (*metis dka.fdia-star-induct-var order.eq-iff nabla-closure nabla-unfold-eq*)
 show $\nabla x \leq |x^*\rangle \nabla x$
 by (*metis ds.fd-iso2 star-ext nabla-unfold-eq*)
qed

lemma *nabla-sum-expand* [*simp*]: $|x\rangle \nabla (x + y) + |y\rangle \nabla (x + y) = \nabla (x + y)$
proof –
 have $d ((x + y) \cdot \nabla(x + y)) = \nabla(x + y)$
 using *local.dka.fd-def nabla-unfold-eq* **by** *presburger*
 thus *?thesis*
 by (*simp add: local.dka.fd-def*)
qed

lemma *wagner-3*:
 assumes $d z + |x\rangle \nabla (x + y) = \nabla (x + y)$
 shows $\nabla (x + y) = \nabla x + |x^*\rangle z$
proof (*rule order.antisym*)
 have $d \nabla(x + y) \leq d z + |x\rangle \nabla(x + y)$
 by (*simp add: assms*)
 thus $\nabla (x + y) \leq \nabla x + |x^*\rangle z$
 by (*metis add-commute nabla-closure nabla-coinduction*)
 have $d z + |x\rangle \nabla (x + y) \leq \nabla (x + y)$
 using *assms* **by** *auto*
 hence $|x^*\rangle z \leq \nabla (x + y)$

by (*metis local.dka.fdia-star-induct local.nabla-closure*)
 thus $\nabla x + |x^*\rangle z \leq \nabla (x + y)$
 by (*simp add: sup-least nabla-subdist*)
qed

lemma *nabla-sum-unfold* [*simp*]: $\nabla x + |x^*\rangle |y\rangle \nabla (x + y) = \nabla (x + y)$

proof –

have $\nabla (x + y) = |x\rangle \nabla (x + y) + |y\rangle \nabla (x + y)$
 by *simp*
 thus *?thesis*
 by (*metis add-commute local.dka.fd-def local.ds.fd-def local.ds.fdia-d-simp wagner-3*)
qed

lemma *nabla-separation*: $y \cdot x \leq x \cdot (x + y)^* \implies (\nabla (x + y) = \nabla x + |x^*\rangle \nabla y)$

proof (*rule order.antisym*)

assume *quasi-comm*: $y \cdot x \leq x \cdot (x + y)^*$
 hence a : $y^* \cdot x \leq x \cdot (x + y)^*$
 using *quasicomm-var* by *blast*
 have $\nabla (x + y) = \nabla y + |y^* \cdot x\rangle \nabla (x + y)$
 by (*metis local.ds.fdia-mult local.join.sup-commute nabla-sum-unfold*)
 also have $\dots \leq \nabla y + |x \cdot (x + y)^*\rangle \nabla (x + y)$
 using *a local.ds.fd-iso2 local.join.sup.mono* by *blast*
 also have $\dots = \nabla y + |x\rangle |(x + y)^*\rangle \nabla (x + y)$
 by (*simp add: fdia-def mult-assoc*)
 finally have $\nabla (x + y) \leq \nabla y + |x\rangle \nabla (x + y)$
 by (*metis star-nabla-1*)
 thus $\nabla (x + y) \leq \nabla x + |x^*\rangle \nabla y$
 by (*metis add-commute nabla-closure nabla-coinduction*)
next
 have $\nabla x + |x^*\rangle \nabla y = \nabla x + |x^*\rangle |y\rangle \nabla y$
 by *simp*
 also have $\dots = \nabla x + |x^* \cdot y\rangle \nabla y$
 by (*simp add: fdia-def mult-assoc*)
 also have $\dots \leq \nabla x + |x^* \cdot y\rangle \nabla (x + y)$
 using *dpdz.dom-iso eq-reft fdia-def join.sup-ge2 join.sup-mono mult-isol nabla-iso*
 by *presburger*
 also have $\dots = \nabla x + |x^*\rangle |y\rangle \nabla (x + y)$
 by (*simp add: fdia-def mult-assoc*)
 finally show $\nabla x + |x^*\rangle \nabla y \leq \nabla (x + y)$
 by (*metis nabla-sum-unfold*)
qed

The next lemma is a separation of termination theorem by Bachmair and Dershowitz [2].

lemma *bachmair-dershowitz*: $y \cdot x \leq x \cdot (x + y)^* \implies \nabla x + \nabla y = 0 \iff \nabla (x + y) = 0$

proof –

assume *quasi-comm*: $y \cdot x \leq x \cdot (x + y)^*$

```

have  $\forall x. |x\rangle 0 = 0$ 
  by (simp add: fdia-def)
hence  $\nabla y = 0 \implies \nabla x + \nabla y = 0 \iff \nabla(x + y) = 0$ 
  using quasi-comm nabla-separation by presburger
thus ?thesis
  using add-commute local.join.le-bot nabla-subdist by fastforce
qed

```

The next lemma is a more complex separation of termination theorem by Doornbos, Backhouse and van der Woude [8].

lemma *separation-of-termination:*

assumes $y \cdot x \leq x \cdot (x + y)^* + y$

shows $\nabla x + \nabla y = 0 \iff \nabla(x + y) = 0$

proof

assume $xy\text{-wf}: \nabla x + \nabla y = 0$

hence $x\text{-preloeb}: \nabla(x + y) \leq |x^*\rangle \Omega x (\nabla(x + y))$

by (metis PreLoebian-def no-trivial-inverse nabla-closure nabla-preloeb)

hence $y\text{-div}: \nabla y = 0$

using add-commute no-trivial-inverse $xy\text{-wf}$ **by** blast

have $\nabla(x + y) \leq |y\rangle \nabla(x + y) + |x\rangle \nabla(x + y)$

by (simp add: local.join.sup-commute)

hence $\nabla(x + y) \cdot ad(|x\rangle \nabla(x + y)) \leq |y\rangle \nabla(x + y)$

by (simp add: local.da-shunt1 local.dka.fd-def local.join.sup-commute)

hence $\Omega x \nabla(x + y) \leq |y\rangle \nabla(x + y)$

by (simp add: fdia-def omega-fdia)

also have $\dots \leq |y\rangle |x^*\rangle (\Omega x \nabla(x + y))$

using local.dpdz.dom-iso local.ds.fd-iso1 $x\text{-preloeb}$ **by** blast

also have $\dots = |y \cdot x^*\rangle (\Omega x \nabla(x + y))$

by (simp add: fdia-def mult-assoc)

also have $\dots \leq |x \cdot (x + y)^* + y\rangle (\Omega x \nabla(x + y))$

using assms local.ds.fd-iso2 local.lazycomm-var **by** blast

also have $\dots = |x \cdot (x + y)^*\rangle (\Omega x \nabla(x + y)) + |y\rangle (\Omega x \nabla(x + y))$

by (simp add: local.dka.fd-def)

also have $\dots \leq |(x \cdot (x + y)^*)\rangle \nabla(x + y) + |y\rangle (\Omega x \nabla(x + y))$

using local.add-iso local.dpdz.domain-1'' local.ds.fd-iso1 local.omega-fdia **by**

auto

also have $\dots \leq |x\rangle |(x + y)^*\rangle \nabla(x + y) + |y\rangle (\Omega x \nabla(x + y))$

by (simp add: fdia-def mult-assoc)

finally have $\Omega x \nabla(x + y) \leq |x\rangle \nabla(x + y) + |y\rangle (\Omega x \nabla(x + y))$

by (metis star-nabla-1)

hence $\Omega x \nabla(x + y) \cdot ad(|x\rangle \nabla(x + y)) \leq |y\rangle (\Omega x \nabla(x + y))$

by (simp add: local.da-shunt1 local.dka.fd-def)

hence $\Omega x \nabla(x + y) \leq |y\rangle (\Omega x \nabla(x + y))$

by (simp add: omega-fdia mult-assoc)

hence $(\Omega x \nabla(x + y)) = 0$

by (metis noetherian-alt omega-dom nabla-noether $y\text{-div}$)

thus $\nabla(x + y) = 0$

using local.dka.fd-def local.join.le-bot $x\text{-preloeb}$ **by** *auto*

next

assume $\nabla (x + y) = 0$
thus $(\nabla x) + (\nabla y) = 0$
by (*metis local.join.le-bot local.join.sup.order-iff local.join.sup-commute nabla-subdist*)
qed

The final examples can be found in [11].

definition $T :: 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \ (- \rightsquigarrow - \rightsquigarrow - [61,61,61] 60)$
where $p \rightsquigarrow x \rightsquigarrow q \equiv ad\ p + |x| \ d\ q$

lemma $T-d$ [*simp*]: $d (p \rightsquigarrow x \rightsquigarrow q) = p \rightsquigarrow x \rightsquigarrow q$
using $T-def\ local.a-d-add-closure\ local.fbox-def$ **by** *auto*

lemma $T-p$: $d\ p \cdot (p \rightsquigarrow x \rightsquigarrow q) = d\ p \cdot |x| \ d\ q$

proof –

have $d\ p \cdot (p \rightsquigarrow x \rightsquigarrow q) = ad\ (ad\ p + ad\ (p \rightsquigarrow x \rightsquigarrow q))$

using $T-d\ local.ads-d-def$ **by** *auto*

thus *?thesis*

using $T-def\ add-commute\ local.a-mult-add\ local.ads-d-def$ **by** *auto*

qed

lemma $T-a$ [*simp*]: $ad\ p \cdot (p \rightsquigarrow x \rightsquigarrow q) = ad\ p$

by (*metis T-d T-def local.a-d-closed local.ads-d-def local.apd-d-def local.dpdz.dns5 local.join.sup.left-idem*)

lemma $T-seq$: $(p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow y \rightsquigarrow s) \leq p \rightsquigarrow x \cdot y \rightsquigarrow s$

proof –

have $f1: |x| \ q = |x| \ d\ q$

using $local.fbox-simp$ **by** *auto*

have $ad\ p \cdot ad\ (x \cdot ad\ (q \rightsquigarrow y \rightsquigarrow s)) + |x| \ d\ q \cdot |x| \ (ad\ q + |y| \ d\ s) \leq ad\ p + |x| \ d\ q \cdot |x| \ (ad\ q + |y| \ d\ s)$

using $local.a-subid-aux2\ local.add-iso$ **by** *blast*

hence $(p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow y \rightsquigarrow s) \leq ad\ p + |x|(d\ q \cdot (q \rightsquigarrow y \rightsquigarrow s))$

by (*metis T-d T-def f1 local.distrib-right' local.fbox-add1 local.fbox-def*)

also have $\dots = ad\ p + |x|(d\ q \cdot (ad\ q + |y| \ d\ s))$

by (*simp add: T-def*)

also have $\dots = ad\ p + |x|(d\ q \cdot |y| \ d\ s)$

using $T-def\ T-p$ **by** *auto*

also have $\dots \leq ad\ p + |x| \ |y| \ d\ s$

by (*metis (no-types, lifting) dka.dom-subid-aux2 dka.dsg3 order.eq-iff fbox-iso join.sup.mono*)

finally show *?thesis*

by (*simp add: T-def fbox-mult*)

qed

lemma $T-square$: $(p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \leq p \rightsquigarrow x \cdot x \rightsquigarrow p$

by (*simp add: T-seq*)

lemma $T-segerberg$ [*simp*]: $d\ p \cdot |x^*|(p \rightsquigarrow x \rightsquigarrow p) = |x^*| \ d\ p$

using $T-def\ add-commute\ local.fbox-segerberg\ local.fbox-simp$ **by** *force*

lemma lookahead [simp]: $|x^*|(d p \cdot |x| d p) = |x^*| d p$
 by (metis (full-types) local.dka.dsg3 local.fbox-add1 local.fbox-mult local.fbox-simp local.fbox-star-unfold-var local.star-slide-var local.star-trans-eq)

lemma alternation: $d p \cdot |x^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)) = |(x \cdot x)^*|(d p \cdot (q \rightsquigarrow x \rightsquigarrow p)) \cdot |x \cdot (x \cdot x)^*|(d q \cdot (p \rightsquigarrow x \rightsquigarrow q))$

proof –

have fbox-simp-2: $\bigwedge x p. |x|p = d(|x| p)$
 using local.a-d-closed local.ads-d-def local.apd-d-def local.fbox-def by fastforce
 have $|x \cdot (x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q)) \leq |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p))$
 using local.dka.domain-1'' local.fbox-iso local.order-trans by blast
 also have $\dots \leq |(x \cdot x)^*|(p \rightsquigarrow x \cdot x \rightsquigarrow p)$
 using T-seq local.dka.dom-iso local.fbox-iso by blast
 finally have 1: $|x \cdot (x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q)) \leq |(x \cdot x)^*|(p \rightsquigarrow x \cdot x \rightsquigarrow p)$.
 have $d p \cdot |x^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)) = d p \cdot |1+x| |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p))$
 by (metis (full-types) fbox-mult meyer-1)
 also have $\dots = d p \cdot |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)) \cdot |x \cdot (x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p))$
 using fbox-simp-2 fbox-mult fbox-add2 mult-assoc by auto
 also have $\dots = d p \cdot |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)) \cdot |(x \cdot x)^* \cdot x|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p))$
 by (simp add: star-slide)
 also have $\dots = d p \cdot |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)) \cdot |(x \cdot x)^*| |x|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p))$
 by (simp add: fbox-mult)
 also have $\dots = d p \cdot |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)))$
 by (metis T-d fbox-simp-2 local.dka.dom-mult-closed local.fbox-add1 mult-assoc)
 also have $\dots = d p \cdot |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$
proof –
 have f1: $d((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q)) = (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q)$
 by (metis (full-types) T-d fbox-simp-2 local.dka.dsg3)
 then have $|x \cdot (x \cdot x)^*|(d(|x|(q \rightsquigarrow x \rightsquigarrow p)) \cdot ((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))) = |x \cdot (x \cdot x)^*|d(|x|(q \rightsquigarrow x \rightsquigarrow p)) \cdot |(x \cdot x)^*|((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$
 by (metis (full-types) fbox-simp-2 local.fbox-add1)
 then have f2: $|x \cdot (x \cdot x)^*|(d(|x|(q \rightsquigarrow x \rightsquigarrow p)) \cdot ((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))) = ad((x \cdot x)^* \cdot ad((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q)) + (x \cdot x)^* \cdot ad(d(|x|(q \rightsquigarrow x \rightsquigarrow p))))$
 by (simp add: add-commute local.fbox-def)
 have $d(|x|(p \rightsquigarrow x \rightsquigarrow q)) \cdot d(|x|(q \rightsquigarrow x \rightsquigarrow p)) = |x|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p))$
 by (metis (no-types) T-d fbox-simp-2 local.fbox-add1)
 then have $d((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q)) \cdot d(d(|x|(q \rightsquigarrow x \rightsquigarrow p))) = (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p))$
 using f1 fbox-simp-2 mult-assoc by force
 then have $|x \cdot (x \cdot x)^*|(d(|x|(q \rightsquigarrow x \rightsquigarrow p)) \cdot ((q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))) =$

$q))) = |(x \cdot x)^*| ((q \rightsquigarrow x \rightsquigarrow p) \cdot |x| ((p \rightsquigarrow x \rightsquigarrow q) \cdot (q \rightsquigarrow x \rightsquigarrow p)))$
using *f2* **by** (*metis* (*no-types*) *local.ans4* *local.fbox-add1* *local.fbox-def*)
then show *?thesis*
by (*metis* (*no-types*) *T-d fbox-simp-2* *local.dka.dsg3* *local.fbox-add1* *mult-assoc*)
qed
also have $\dots = d p \cdot |(x \cdot x)^*|(p \rightsquigarrow x \rightsquigarrow p) \cdot |(x \cdot x)^*|((p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$ **using** *1*
by (*metis* *fbox-simp-2* *dka.dns5* *dka.dsg4* *join.sup.absorb2* *mult-assoc*)
also have $\dots = |(x \cdot x)^*|(d p \cdot (p \rightsquigarrow x \rightsquigarrow q) \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$
using *T-segerberg* *local.a-d-closed* *local.ads-d-def* *local.apd-d-def* *local.distrib-left* *local.fbox-def* *mult-assoc* **by** *auto*
also have $\dots = |(x \cdot x)^*|(d p \cdot |x| d q \cdot |x|(q \rightsquigarrow x \rightsquigarrow p) \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$
by (*simp add:* *T-p*)
also have $\dots = |(x \cdot x)^*|(d p \cdot |x| d q \cdot |x| |x| d p \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$
by (*metis* *T-d* *T-p* *fbox-simp-2* *fbox-add1* *fbox-simp* *mult-assoc*)
also have $\dots = |(x \cdot x)^*|(d p \cdot |x \cdot x| d p \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x| d q \cdot |x|(p \rightsquigarrow x \rightsquigarrow q))$
proof –
have *f1*: $ad (x \cdot ad (|x| d p)) = |x \cdot x| d p$
using *local.fbox-def* *local.fbox-mult* **by** *presburger*
have *f2*: $ad (d q \cdot d (x \cdot ad (d p))) = q \rightsquigarrow x \rightsquigarrow p$
by (*simp add:* *T-def* *local.a-de-morgan-var-4* *local.fbox-def*)
have $ad q + |x| d p = ad (d q \cdot d (x \cdot ad (d p)))$
by (*simp add:* *local.a-de-morgan-var-4* *local.fbox-def*)
then have $ad (x \cdot ad (|x| d p)) \cdot ((q \rightsquigarrow x \rightsquigarrow p) \cdot |x| d q) = ad (x \cdot ad (|x| d p)) \cdot ad (x \cdot ad (d q)) \cdot (ad q + |x| d p)$
using *f2* **by** (*metis* (*no-types*) *local.am2* *local.fbox-def* *mult-assoc*)
then show *?thesis*
using *f1* **by** (*simp add:* *T-def* *local.am2* *local.fbox-def* *mult-assoc*)
qed
also have $\dots = |(x \cdot x)^*|(d p \cdot |x \cdot x| d p \cdot (q \rightsquigarrow x \rightsquigarrow p) \cdot |x|(d q \cdot (p \rightsquigarrow x \rightsquigarrow q)))$
using *local.a-d-closed* *local.ads-d-def* *local.apd-d-def* *local.distrib-left* *local.fbox-def* *mult-assoc* **by** *auto*
also have $\dots = |(x \cdot x)^*|(d p \cdot |x \cdot x| d p) \cdot |(x \cdot x)^*|(q \rightsquigarrow x \rightsquigarrow p) \cdot |(x \cdot x)^*| |x|(d q \cdot (p \rightsquigarrow x \rightsquigarrow q))$
by (*metis* *T-d fbox-simp-2* *local.dka.dom-mult-closed* *local.fbox-add1*)
also have $\dots = |(x \cdot x)^*|(d p \cdot (q \rightsquigarrow x \rightsquigarrow p)) \cdot |(x \cdot x)^*| |x| (d q \cdot (p \rightsquigarrow x \rightsquigarrow q))$
by (*metis* *T-d* *local.fbox-add1* *local.fbox-simp* *lookahead*)
finally show *?thesis*
by (*metis* *fbox-mult* *star-slide*)
qed

lemma $|(x \cdot x)^*| d p \cdot |x \cdot (x \cdot x)^*| ad p = d p \cdot |x^*|((p \rightsquigarrow x \rightsquigarrow ad p) \cdot (ad p \rightsquigarrow x \rightsquigarrow p))$
using *alternation* *local.a-d-closed* *local.ads-d-def* *local.apd-d-def* **by** *auto*

lemma $|x^*| d p = d p \cdot |x^*|(p \rightsquigarrow x \rightsquigarrow p)$
by (*simp add:* *alternation*)

end

end

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