

IsaGeoCoq: Partial porting of GeoCoq 2.4.0. Case studies: Tarski's postulate of parallels implies the 5th postulate of Euclid, the postulate of Playfair and the original postulate of Euclid.

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Abstract

The GeoCoq library contains a formalization of geometry using the Coq proof assistant. It contains both proofs about the foundations of geometry [20, 15, 6, 16] and high-level proofs in the same style as in high-school. [1](Code Repository <https://github.com/GeoCoq/GeoCoq>).

Some theorems also inspired by [20] are also formalized with others ITP(Metamath, Mizar) or ATP [24, 25, 3, 23, 4, 2, 17, 5, 11, 19, 8, 9, 10].

We port a part of the GeoCoq 2.4.0 library within the Isabelle/Hol proof assistant: more precisely, the files Chap02.v to Chap13_3.v, suma.v as well as the associated definitions and some useful files for the demonstration of certain parallel postulates.

While the demonstrations in Coq are written in procedural language [26], the transcript is done in declarative language Isar[18].

The synthetic approach of the demonstrations are directly inspired by those contained in GeoCoq. Some demonstrations are credited to G.E Martin («lemma bet_le_lt:» in Ch11_angles.thy, proved by Martin as Theorem 18.17 in [14]) or Gupta H.N (Krippen Lemma, proved by Gupta in its PhD in 1965 as Theorem 3.45). (See [12]).

In this work, the proofs are not constructive. The sledeghammer tool being used to find some demonstrations.

The names of the lemmas and theorems used are kept as far as possible as well as the definitions. A different translation has been proposed when the name was already used in Isabel/Hol ("Len" is translated as "TarskiLen") or that characters were not allowed in Isabel/Hol ("anga" in Ch13_angles.v is translated as "angaP"). For some definitions the highlighting of a variable has changed the order or the position of the variables (Midpoint, Out, Inter,...).

All the lemmas are valid in absolute/neutral space defined with Tarski's axioms.

It should be noted that T.J.M. Makarios [13] has begun some demonstrations of certain proposals mainly those corresponding to SST chapters 2 and 3. It uses a definition that does not quite coincide with the definition used in Geocoq and here. As an example, Makarios introduces the axiom A11 (Axiom of continuity) in the definition of the locale "Tarski_absolute_space".

Furthermore, the definition of the locale "TarskiAbsolute" [22, 21] is not not identical to the one defined in the "Tarski_neutral_dimensionless" class of GeoCoq. Indeed this one does not contain the axiom "upper_dimension". In some cases particular, it is nevertheless to use the axiom "upper_dimension". The addition of the word "_2D" in the file indicates its presence.

In the last part, it is formalized that, in the neutral/absolute space, the axiom of the parallels of the system of Tarski implies the Playfair axiom, the 5th postulate of euclide and the postulate original from Euclid. These proofs, which are not constructive, are directly inspired by [12, 7].

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theory *Tarski-Neutral*

imports

Main

begin

1 Neutral geometry dimensionless

1.1 Tarski's axiom system for neutral geometry: dimensionless

locale *Tarski-neutral-dimensionless* =

fixes *Bet* :: '*p* ⇒ '*p* ⇒ '*p* ⇒ *bool* ((- - - -))

and *Cong* :: '*p* ⇒ '*p* ⇒ '*p* ⇒ '*p* ⇒ *bool*

and *TPA TPB TPC* :: '*p*

assumes *cong-pseudo-reflexivity*: ∀ *a b*.

Cong a b b a

and *cong-inner-transitivity*: ∀ *a b p q r s*.

Cong a b p q ∧

Cong a b r s

→

Cong p q r s

and *cong-identity*: ∀ *a b c*.

Cong a b c c

→

a = b

and *segment-construction*: ∀ *a b c q*.

∃ *x*. (*Bet q a x* ∧ *Cong a x b c*)

and *five-segment*: ∀ *a b c d a' b' c' d'*.

a ≠ *b* ∧

Bet a b c ∧

Bet a' b' c' ∧

Cong a b a' b' ∧

Cong b c b' c' ∧

Cong a d a' d' ∧

Cong b d b' d'

→

Cong c d c' d'

and *between-identity*: ∀ *a b*.

Bet a b a

→

a = b

and *inner-pasch*: ∀ *a b c p q*.

Bet a p c ∧

Bet b q c

→

(∃ *x*. *Bet p x b* ∧ *Bet q x a*)

and *lower-dim*: ¬ *Bet TPA TPB TPC* ∧ ¬ *Bet TPB TPC TPA* ∧ ¬ *Bet TPC TPA TPB*

context *Tarski-neutral-dimensionless*

begin

1.2 Definitions

definition *OFSC* ::

$[p, p, p, p, p, p, p, p] \Rightarrow \text{bool}$
(- - - - *OFSC* - - - - [99,99,99,99,99,99,99,99] 50)

where

$A\ B\ C\ D\ \text{OFSC}\ A'\ B'\ C'\ D'$

$\equiv$

$Bet\ A\ B\ C \wedge$
 $Bet\ A'\ B'\ C' \wedge$
 $Cong\ A\ B\ A'\ B' \wedge$
 $Cong\ B\ C\ B'\ C' \wedge$
 $Cong\ A\ D\ A'\ D' \wedge$
 $Cong\ B\ D\ B'\ D'$

definition *Cong3* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
(- - - *Cong3* - - - [99,99,99,99,99,99] 50)

where

$A\ B\ C\ \text{Cong3}\ A'\ B'\ C'$

$\equiv$

$Cong\ A\ B\ A'\ B' \wedge$
 $Cong\ A\ C\ A'\ C' \wedge$
 $Cong\ B\ C\ B'\ C'$

definition *Col* ::

$[p, p, p] \Rightarrow \text{bool}$
(*Col* - - - [99,99,99] 50)

where

$Col\ A\ B\ C$

$\equiv$

$Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$

definition *Bet4* ::

$[p, p, p, p] \Rightarrow \text{bool}$
(*Bet4* - - - - [99,99,99,99] 50)

where

$Bet4\ A1\ A2\ A3\ A4$

$\equiv$

$Bet\ A1\ A2\ A3 \wedge$
 $Bet\ A2\ A3\ A4 \wedge$
 $Bet\ A1\ A3\ A4 \wedge$
 $Bet\ A1\ A2\ A4$

definition *BetS* ::

$[p, p, p] \Rightarrow \text{bool}\ (BetS\ -\ -\ -\ [99,99,99]\ 50)$

where

$BetS\ A\ B\ C$

$\equiv$

$Bet\ A\ B\ C \wedge$
 $A \neq B \wedge$
 $B \neq C$

definition *SumS* ::

$[p, p, p, p, p, p, p] \Rightarrow \text{bool}$
(- - - - *SumS* - - [99,99,99,99,99,99] 50)

where

$A\ B\ C\ D\ \text{SumS}\ E\ F$

$\equiv$

$\exists\ P\ Q\ R.\ Bet\ P\ Q\ R \wedge Cong\ P\ Q\ A\ B \wedge Cong\ Q\ R\ C\ D \wedge Cong\ P\ R\ E\ F$

definition *FSC* ::
 $[p, p, p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- - - - \text{FSC} - - - - [99, 99, 99, 99, 99, 99, 99, 99] \ 50)$
where
 $A \ B \ C \ D \ \text{FSC} \ A' \ B' \ C' \ D'$
 $\equiv$
 $\text{Col } A \ B \ C \wedge$
 $A \ B \ C \ \text{Cong3} \ A' \ B' \ C' \wedge$
 $\text{Cong } A \ D \ A' \ D' \wedge$
 $\text{Cong } B \ D \ B' \ D'$

definition *IFSC* ::
 $[p, p, p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- - - - \text{IFSC} - - - - [99, 99, 99, 99, 99, 99, 99, 99] \ 50)$
where
 $A \ B \ C \ D \ \text{IFSC} \ A' \ B' \ C' \ D'$
 $\equiv$
 $\text{Bet } A \ B \ C \wedge$
 $\text{Bet } A' \ B' \ C' \wedge$
 $\text{Cong } A \ C \ A' \ C' \wedge$
 $\text{Cong } B \ C \ B' \ C' \wedge$
 $\text{Cong } A \ D \ A' \ D' \wedge$
 $\text{Cong } C \ D \ C' \ D'$

definition *Le* ::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{Le} - - [99, 99, 99, 99] \ 50)$
where
 $A \ B \ \text{Le} \ C \ D$
 $\equiv$
 $\exists \ E. (\text{Bet } C \ E \ D \wedge \text{Cong } A \ B \ C \ E)$

definition *Lt* ::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{Lt} - - [99, 99, 99, 99] \ 50)$
where
 $A \ B \ \text{Lt} \ C \ D$
 $\equiv$
 $A \ B \ \text{Le} \ C \ D \wedge \neg \text{Cong } A \ B \ C \ D$

definition *Ge* ::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{Ge} - - [99, 99, 99, 99] \ 50)$
where
 $A \ B \ \text{Ge} \ C \ D$
 $\equiv$
 $C \ D \ \text{Le} \ A \ B$

definition *Gt* ::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{Gt} - - [99, 99, 99, 99] \ 50)$
where
 $A \ B \ \text{Gt} \ C \ D$
 $\equiv$
 $C \ D \ \text{Lt} \ A \ B$

definition *Out* ::
 $[p, p, p] \Rightarrow \text{bool}$
 $(- \text{Out} - - [99, 99, 99] \ 50)$
where
 $P \ \text{Out} \ A \ B$
 $\equiv$
 $A \neq P \wedge$
 $B \neq P \wedge$
 $(\text{Bet } P \ A \ B \vee \text{Bet } P \ B \ A)$

definition *Midpoint* ::

$[p, p, p] \Rightarrow \text{bool}$
 $(- \text{Midpoint} - - [99, 99, 99] \ 50)$

where

$M \text{Midpoint} A B$

$\equiv$

$Bet A M B \wedge$

$Cong A M M B$

definition *Per* ::

$[p, p, p] \Rightarrow \text{bool}$
 $(Per - - - [99, 99, 99] \ 50)$

where

$Per A B C$

$\equiv$

$\exists C'. (B \text{Midpoint} C C' \wedge Cong A C A C')$

definition *PerpAt* ::

$[p, p, p, p, p] \Rightarrow \text{bool}$
 $(- \text{PerpAt} - - - - [99, 99, 99, 99, 99] \ 50)$

where

$X \text{PerpAt} A B C D$

$\equiv$

$A \neq B \wedge$

$C \neq D \wedge$

$Col X A B \wedge$

$Col X C D \wedge$

$(\forall U V. ((Col U A B \wedge Col V C D) \longrightarrow Per U X V))$

definition *Perp* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{Perp} - - [99, 99, 99, 99] \ 50)$

where

$A B \text{Perp} C D$

$\equiv$

$\exists X::'p. X \text{PerpAt} A B C D$

definition *Coplanar* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(\text{Coplanar} - - - - [99, 99, 99, 99] \ 50)$

where

$\text{Coplanar} A B C D$

$\equiv$

$\exists X. (Col A B X \wedge Col C D X) \vee$

$(Col A C X \wedge Col B D X) \vee$

$(Col A D X \wedge Col B C X)$

definition *TS* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{TS} - - [99, 99, 99, 99] \ 50)$

where

$A B \text{TS} P Q$

$\equiv$

$\neg Col P A B \wedge \neg Col Q A B \wedge (\exists T::'p. Col T A B \wedge Bet P T Q)$

definition *ReflectL* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{ReflectL} - - [99, 99, 99, 99] \ 50)$

where

$P' P \text{ReflectL} A B$

$\equiv$

$(\exists X. X \text{Midpoint} P P' \wedge Col A B X) \wedge (A B \text{Perp} P P' \vee P = P')$

definition *Reflect* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{Reflect} - - [99, 99, 99, 99] \ 50)$

where
 $P' P \text{ Reflect } A B$
 $\equiv$
 $(A \neq B \wedge P' P \text{ ReflectL } A B) \vee (A = B \wedge A \text{ Midpoint } P P')$

definition InAngle ::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- \text{ InAngle } - - - [99, 99, 99, 99] \ 50)$
where
 $P \text{ InAngle } A B C$
 $\equiv$
 $A \neq B \wedge C \neq B \wedge P \neq B \wedge$
 $(\exists X. \text{ Bet } A X C \wedge (X = B \vee B \text{ Out } X P))$

definition ParStrict::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{ ParStrict } - - [99, 99, 99, 99] \ 50)$
where
 $A B \text{ ParStrict } C D$
 $\equiv$
 $\text{Coplanar } A B C D \wedge$
 $\neg (\exists X. \text{ Col } X A B \wedge \text{ Col } X C D)$

definition Par::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{ Par } - - [99, 99, 99, 99] \ 50)$
where
 $A B \text{ Par } C D$
 $\equiv$
 $A B \text{ ParStrict } C D \vee (A \neq B \wedge C \neq D \wedge \text{ Col } A C D \wedge \text{ Col } B C D)$

definition Plg::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(\text{Plg } - - - [99, 99, 99, 99] \ 50)$
where
 $\text{Plg } A B C D$
 $\equiv$
 $(A \neq C \vee B \neq D) \wedge (\exists M. M \text{ Midpoint } A C \wedge M \text{ Midpoint } B D)$

definition ParallelogramStrict::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(\text{ParallelogramStrict } - - - [99, 99, 99, 99] \ 50)$
where
 $\text{ParallelogramStrict } A B A' B'$
 $\equiv$
 $A A' \text{ TS } B B' \wedge$
 $A B \text{ Par } A' B' \wedge$
 $\text{Cong } A B A' B'$

definition ParallelogramFlat::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(\text{ParallelogramFlat } - - - [99, 99, 99, 99] \ 50)$
where
 $\text{ParallelogramFlat } A B A' B'$
 $\equiv$
 $\text{Col } A B A' \wedge$
 $\text{Col } A B B' \wedge$
 $\text{Cong } A B A' B' \wedge$
 $\text{Cong } A B' A' B \wedge$
 $(A \neq A' \vee B \neq B')$

definition Parallelogram::
 $[p, p, p, p] \Rightarrow \text{bool}$
 $(\text{Parallelogram } - - - [99, 99, 99, 99] \ 50)$
where
 $\text{Parallelogram } A B A' B'$

≡
ParallelogramStrict $A B A' B' \vee \textit{ParallelogramFlat } A B A' B'$

definition *Rhombus*::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\textit{Rhombus} \text{ - - - } [99, 99, 99, 99] \ 50)$
where
 $\textit{Rhombus } A B C D$

≡
 $\textit{Plg } A B C D \wedge \textit{Cong } A B B C$

definition *Rectangle*::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\textit{Rectangle} \text{ - - - } [99, 99, 99, 99] \ 50)$
where
 $\textit{Rectangle } A B C D$

≡
 $\textit{Plg } A B C D \wedge \textit{Cong } A C B D$

definition *Square*::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\textit{Square} \text{ - - - } [99, 99, 99, 99] \ 50)$
where
 $\textit{Square } A B C D$

≡
 $\textit{Rectangle } A B C D \wedge \textit{Cong } A B B C$

definition *Kite*::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\textit{Kite} \text{ - - - } [99, 99, 99, 99] \ 50)$
where
 $\textit{Kite } A B C D$

≡
 $\textit{Cong } B C C D \wedge \textit{Cong } D A A B$

definition *Lambert*::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\textit{Lambert} \text{ - - - } [99, 99, 99, 99] \ 50)$
where
 $\textit{Lambert } A B C D$

≡
 $A \neq B \wedge B \neq C \wedge C \neq D \wedge A \neq D \wedge$
 $\textit{Per } B A D \wedge$
 $\textit{Per } A D C \wedge$
 $\textit{Per } A B C \wedge$
 $\textit{Coplanar } A B C D$

definition *OS* ::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\text{ - - OS - - } [99, 99, 99, 99] \ 50)$
where
 $A B \textit{ OS } P Q$

≡
 $\exists R::p. A B \textit{ TS } P R \wedge A B \textit{ TS } Q R$

definition *TSP* ::
 $[p, p, p, p] \Rightarrow \textit{bool}$
 $(\text{ - - TSP - - } [99, 99, 99, 99] \ 50)$
where
 $A B C \textit{ TSP } P Q$

≡
 $(\neg \textit{Coplanar } A B C P) \wedge (\neg \textit{Coplanar } A B C Q) \wedge$
 $(\exists T. \textit{Coplanar } A B C T \wedge \textit{Bet } P T Q)$

definition *OSP* ::
 $[p, p, p, p] \Rightarrow \textit{bool}$

(- - - OSP - - [99,99,99,99,99] 50)
where
 A B C OSP P Q
 $\equiv$
 $\exists R. ((A B C TSP P R) \wedge (A B C TSP Q R))$

definition Saccheri::
 ['p,'p,'p,'p] $\Rightarrow$ bool
 (Saccheri - - - - [99,99,99,99] 50)
where
 Saccheri A B C D
 $\equiv$
 Per B A D $\wedge$
 Per A D C $\wedge$
 Cong A B C D $\wedge$ A D OS B C

definition ReflectLAt ::
 ['p,'p,'p,'p,'p] $\Rightarrow$ bool
 (- ReflectLAt - - - - [99,99,99,99,99] 50)
where
 M ReflectLAt P' P A B
 $\equiv$
 (M Midpoint P P' $\wedge$ Col A B M) $\wedge$ (A B Perp P P' $\vee$ P = P')

definition ReflectAt ::
 ['p,'p,'p,'p,'p] $\Rightarrow$ bool
 (- ReflectAt - - - - [99,99,99,99,99] 50)
where
 M ReflectAt P' P A B
 $\equiv$
 (A $\neq$ B $\wedge$ M ReflectLAt P' P A B) $\vee$ (A = B $\wedge$ A = M $\wedge$ M Midpoint P P')

definition upper-dim-axiom ::
 bool
 (UpperDimAxiom [] 50)
where
 upper-dim-axiom
 $\equiv$
 $\forall A B C P Q.$
 P $\neq$ Q $\wedge$
 Cong A P A Q $\wedge$
 Cong B P B Q $\wedge$
 Cong C P C Q
 $\longrightarrow$
 (Bet A B C $\vee$ Bet B C A $\vee$ Bet C A B)

definition all-coplanar-axiom ::
 bool
 (AllCoplanarAxiom [] 50)
where
 AllCoplanarAxiom
 $\equiv$
 $\forall A B C P Q.$
 P $\neq$ Q $\wedge$
 Cong A P A Q $\wedge$
 Cong B P B Q $\wedge$
 Cong C P C Q
 $\longrightarrow$
 (Bet A B C $\vee$ Bet B C A $\vee$ Bet C A B)

definition upper-dim-3-axiom ::
 bool
where
 upper-dim-3-axiom

$\equiv$
 $\forall A B C P Q R. P \neq Q \wedge Q \neq R \wedge P \neq R \wedge$
 $\text{Cong } A P A Q \wedge \text{Cong } B P B Q \wedge \text{Cong } C P C Q \wedge$
 $\text{Cong } A P A R \wedge \text{Cong } B P B R \wedge \text{Cong } C P C R \longrightarrow$
 $(\text{Bet } A B C \vee \text{Bet } B C A \vee \text{Bet } C A B)$

definition *median-planes-axiom* ::

bool

where

median-planes-axiom

$\equiv$
 $\forall A B C D P Q. P \neq Q \wedge$
 $\text{Cong } A P A Q \wedge \text{Cong } B P B Q \wedge \text{Cong } C P C Q \wedge \text{Cong } D P D Q \longrightarrow$
 $\text{Coplanar } A B C D$

definition *plane-intersection-axiom* ::

bool

where

plane-intersection-axiom

$\equiv$
 $\forall A B C D E F P.$
 $\text{Coplanar } A B C P \wedge \text{Coplanar } D E F P \longrightarrow$
 $(\exists Q. \text{Coplanar } A B C Q \wedge \text{Coplanar } D E F Q \wedge P \neq Q)$

definition *space-separation-axiom* ::

bool

where

space-separation-axiom

$\equiv$
 $\forall A B C P Q.$
 $\neg \text{Coplanar } A B C P \wedge \neg \text{Coplanar } A B C Q \longrightarrow$
 $(A B C \text{ TSP } P Q \vee A B C \text{ OSP } P Q)$

definition *orthonormal-family-axiom* ::

bool

where

orthonormal-family-axiom

$\equiv$
 $\forall S U1' U1 U2 U3 U4.$
 $\neg (S \neq U1' \wedge \text{Bet } U1 S U1' \wedge$
 $\text{Cong } S U1 S U1' \wedge \text{Cong } S U2 S U1' \wedge \text{Cong } S U3 S U1' \wedge \text{Cong } S U4 S U1' \wedge$
 $\text{Cong } U1 U2 U1' U2 \wedge \text{Cong } U1 U3 U1' U2 \wedge \text{Cong } U1 U4 U1' U2 \wedge$
 $\text{Cong } U2 U3 U1' U2 \wedge \text{Cong } U2 U4 U1' U2 \wedge \text{Cong } U3 U4 U1' U2)$

definition *CongA* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$

$(- - - \text{CongA } - - - [99, 99, 99, 99, 99, 99] \ 50)$

where

$A B C \text{ CongA } D E F$

$\equiv$
 $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge$
 $(\exists A' C' D' F'. \text{Bet } B A A' \wedge \text{Cong } A A' E D \wedge \text{Bet } B C C' \wedge \text{Cong } C C' E F \wedge$
 $\text{Bet } E D D' \wedge \text{Cong } D D' B A \wedge \text{Bet } E F F' \wedge \text{Cong } F F' B C \wedge$
 $\text{Cong } A' C' D' F')$

definition *LeA* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$

$(- - - \text{LeA } - - - [99, 99, 99, 99, 99, 99] \ 50)$

where
 $A \ B \ C \ LeA \ D \ E \ F$
 $\equiv$
 $\exists P. (P \ InAngle \ D \ E \ F \wedge A \ B \ C \ CongA \ D \ E \ P)$

definition $LtA ::$
 $[p, p, p, p, p, p] \Rightarrow bool$
 $(- - - LtA - - - [99, 99, 99, 99, 99, 99] \ 50)$
where
 $A \ B \ C \ LtA \ D \ E \ F$
 $\equiv$
 $A \ B \ C \ LeA \ D \ E \ F \wedge \neg A \ B \ C \ CongA \ D \ E \ F$

definition $GtA ::$
 $[p, p, p, p, p, p] \Rightarrow bool$
 $(- - - GtA - - - [99, 99, 99, 99, 99, 99] \ 50)$
where
 $A \ B \ C \ GtA \ D \ E \ F$
 $\equiv$
 $D \ E \ F \ LtA \ A \ B \ C$

definition $Acute ::$
 $[p, p, p] \Rightarrow bool$
 $(Acute - - - [99, 99, 99] \ 50)$
where
 $Acute \ A \ B \ C$
 $\equiv$
 $\exists A' \ B' \ C'. (Per \ A' \ B' \ C' \wedge A \ B \ C \ LtA \ A' \ B' \ C')$

definition $Obtuse ::$
 $[p, p, p] \Rightarrow bool$
 $(Obtuse - - - [99, 99, 99] \ 50)$
where
 $Obtuse \ A \ B \ C$
 $\equiv$
 $\exists A' \ B' \ C'. (Per \ A' \ B' \ C' \wedge A' \ B' \ C' \ LtA \ A \ B \ C)$

definition $OrthAt ::$
 $[p, p, p, p, p, p] \Rightarrow bool$
 $(- \ OrthAt - - - - [99, 99, 99, 99, 99, 99] \ 50)$
where
 $X \ OrthAt \ A \ B \ C \ U \ V$
 $\equiv$
 $\neg Col \ A \ B \ C \wedge U \neq V \wedge Coplanar \ A \ B \ C \ X \wedge Col \ U \ V \ X \wedge$
 $(\forall P \ Q. (Coplanar \ A \ B \ C \ P \wedge Col \ U \ V \ Q) \longrightarrow Per \ P \ X \ Q)$

definition $Orth ::$
 $[p, p, p, p, p] \Rightarrow bool$
 $(- - - Orth - - [99, 99, 99, 99, 99] \ 50)$
where
 $A \ B \ C \ Orth \ U \ V$
 $\equiv$
 $\exists X. X \ OrthAt \ A \ B \ C \ U \ V$

definition $SuppA ::$
 $[p, p, p, p, p, p] \Rightarrow bool$
 $(- - - SuppA - - - [99, 99, 99, 99, 99, 99] \ 50)$
where
 $A \ B \ C \ SuppA \ D \ E \ F$
 $\equiv$
 $A \neq B \wedge (\exists A'. Bet \ A \ B \ A' \wedge D \ E \ F \ CongA \ C \ B \ A')$

definition $SumA ::$
 $[p, p, p, p, p, p, p, p, p, p] \Rightarrow bool$
 $(- - - - - SumA - - - [99, 99, 99, 99, 99, 99, 99, 99, 99, 99] \ 50)$
where

$A B C D E F \text{ Sum} A G H I$
 $\equiv$
 $\exists J. (C B J \text{ Cong} A D E F \wedge \neg B C \text{ OS } A J \wedge \text{ Coplanar } A B C J \wedge A B J \text{ Cong} A G H I)$

definition *TriSumA* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- - - \text{TriSumA} - - - [99, 99, 99, 99, 99, 99] \ 50)$
where
 $A B C \text{ TriSumA } D E F$
 $\equiv$
 $\exists G H I. (A B C B C A \text{ Sum} A G H I \wedge G H I C A B \text{ Sum} A D E F)$

definition *SAMS* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{SAMS} - - - - - [99, 99, 99, 99, 99, 99] \ 50)$
where
 $\text{SAMS } A B C D E F$
 $\equiv$
 $(A \neq B \wedge$
 $(E \text{ Out } D F \vee \neg \text{Bet } A B C)) \wedge$
 $(\exists J. (C B J \text{ Cong} A D E F \wedge \neg (B C \text{ OS } A J) \wedge \neg (A B \text{ TS } C J) \wedge \text{ Coplanar } A B C J))$

definition *Inter* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- \text{Inter} - - - - [99, 99, 99, 99, 99] \ 50)$
where
 $X \text{ Inter } A1 A2 B1 B2$
 $\equiv$
 $B1 \neq B2 \wedge$
 $(\exists P::p. (\text{Col } P B1 B2 \wedge \neg \text{Col } P A1 A2)) \wedge$
 $\text{Col } A1 A2 X \wedge \text{Col } B1 B2 X$

definition *Perp2* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- \text{Perp2} - - - - [99, 99, 99, 99, 99] \ 50)$
where
 $P \text{ Perp2 } A B C D$
 $\equiv$
 $\exists X Y. (\text{Col } P X Y \wedge X Y \text{ Perp } A B \wedge X Y \text{ Perp } C D)$

definition *Perp-bisect* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{PerpBisect} - - [99, 99, 99, 99] \ 50)$
where
 $P Q \text{ PerpBisect } A B$
 $\equiv$
 $A B \text{ ReflectL } P Q \wedge A \neq B$

definition *Perp-bisect-bis* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- - \text{PerpBisectBis} - - [99, 99, 99, 99] \ 50)$
where
 $P Q \text{ PerpBisectBis } A B$
 $\equiv$
 $\exists I. I \text{ PerpAt } P Q A B \wedge I \text{ Midpoint } A B$

definition *Is-on-perp-bisect* ::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- \text{IsOnPerpBisect} - - [99, 99, 99] \ 50)$
where
 $P \text{ IsOnPerpBisect } A B$
 $\equiv$
 $\text{Cong } A P P B$

definition *isosceles*::
 $[p, p, p, p, p, p] \Rightarrow \text{bool}$

(- - isosceles [99,99,99] 50)

where

$A \ B \ C \text{ isosceles}$

$\equiv$

$\text{Cong } A \ B \ B \ C$

definition *equilateral*::

$['p, 'p, 'p] \Rightarrow \text{bool}$

(- - equilateral [99,99,99] 50)

where

$A \ B \ C \text{ equilateral}$

$\equiv$

$\text{Cong } A \ B \ B \ C \wedge \text{Cong } B \ C \ C \ A$

definition *equilateralStrict*::

$['p, 'p, 'p] \Rightarrow \text{bool}$

(- - equilateralStrict [99,99,99] 50)

where

$A \ B \ C \text{ equilateralStrict}$

$\equiv$

$A \ B \ C \text{ equilateral} \wedge A \neq B$

definition *QCong*::

$(['p, 'p] \Rightarrow \text{bool}) \Rightarrow \text{bool}$

(QCong - [99] 50)

where

$QCong \ l$

$\equiv$

$\exists \ A \ B. (\forall \ X \ Y. (\text{Cong } A \ B \ X \ Y \longleftrightarrow l \ X \ Y))$

definition *TarskiLen*::

$['p, 'p, (['p, 'p] \Rightarrow \text{bool})] \Rightarrow \text{bool}$

(TarskiLen - - [99,99,99] 50)

where

$TarskiLen \ A \ B \ l$

$\equiv$

$QCong \ l \wedge l \ A \ B$

definition *QCongNull* ::

$(['p, 'p] \Rightarrow \text{bool}) \Rightarrow \text{bool}$

(QCongNull - [99] 50)

where

$QCongNull \ l$

$\equiv$

$QCong \ l \wedge (\exists \ A. l \ A \ A)$

definition *QCongA* ::

$(['p, 'p, 'p] \Rightarrow \text{bool}) \Rightarrow \text{bool}$

(QCongA - [99] 50)

where

$QCongA \ a$

$\equiv$

$\exists \ A \ B \ C. (A \neq B \wedge C \neq B \wedge (\forall \ X \ Y \ Z. A \ B \ C \text{ CongA } X \ Y \ Z \longleftrightarrow a \ X \ Y \ Z))$

definition *Ang* ::

$['p, 'p, 'p, (['p, 'p, 'p] \Rightarrow \text{bool})] \Rightarrow \text{bool}$

(- - Ang - [99,99,99,99] 50)

where

$A \ B \ C \text{ Ang } a$

$\equiv$

$QCongA \ a \wedge$

$a \ A \ B \ C$

definition *QCongAAcute* ::

$(['p, 'p, 'p] \Rightarrow \text{bool}) \Rightarrow \text{bool}$

(QCongAAcute - [99] 50)

where
 $QCongAAcute\ a$
 $\equiv$
 $\exists\ A\ B\ C. (Acute\ A\ B\ C \wedge (\forall\ X\ Y\ Z. (A\ B\ C\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z)))$

definition $AngAcute ::$
 $[p, 'p, 'p, ([p, 'p, 'p] \Rightarrow bool)] \Rightarrow bool$
 $(- - - AngAcute - [99, 99, 99, 99]\ 50)$

where
 $A\ B\ C\ AngAcute\ a$
 $\equiv$
 $((QCongAAcute\ a) \wedge (a\ A\ B\ C))$

definition $QCongANullAcute ::$
 $([p, 'p, 'p] \Rightarrow bool) \Rightarrow bool$
 $(QCongANullAcute - [99]\ 50)$

where
 $QCongANullAcute\ a$
 $\equiv$
 $QCongAAcute\ a \wedge$
 $(\forall\ A\ B\ C. (a\ A\ B\ C \longrightarrow B\ Out\ A\ C))$

definition $QCongAnNull ::$
 $([p, 'p, 'p] \Rightarrow bool) \Rightarrow bool$
 $(QCongAnNull - [99]\ 50)$

where
 $QCongAnNull\ a$
 $\equiv$
 $QCongA\ a \wedge$
 $(\forall\ A\ B\ C. (a\ A\ B\ C \longrightarrow \neg B\ Out\ A\ C))$

definition $QCongAnFlat ::$
 $([p, 'p, 'p] \Rightarrow bool) \Rightarrow bool$
 $(QCongAnFlat - [99]\ 50)$

where
 $QCongAnFlat\ a$
 $\equiv$
 $QCongA\ a \wedge$
 $(\forall\ A\ B\ C. (a\ A\ B\ C \longrightarrow \neg Bet\ A\ B\ C))$

definition $IsNullAngaP ::$
 $([p, 'p, 'p] \Rightarrow bool) \Rightarrow bool$
 $(IsNullAngaP - [99]\ 50)$

where
 $IsNullAngaP\ a$
 $\equiv$
 $QCongAAcute\ a \wedge$
 $(\exists\ A\ B\ C. (a\ A\ B\ C \wedge B\ Out\ A\ C))$

definition $QCongANull ::$
 $([p, 'p, 'p] \Rightarrow bool) \Rightarrow bool$
 $(QCongANull - [99]\ 50)$

where
 $QCongANull\ a$
 $\equiv$
 $QCongA\ a \wedge$
 $(\forall\ A\ B\ C. (a\ A\ B\ C \longrightarrow B\ Out\ A\ C))$

definition $AngFlat ::$
 $([p, 'p, 'p] \Rightarrow bool) \Rightarrow bool$
 $(AngFlat - [99]\ 50)$

where
 $AngFlat\ a$
 $\equiv$
 $QCongA\ a \wedge$
 $(\forall\ A\ B\ C. (a\ A\ B\ C \longrightarrow Bet\ A\ B\ C))$

definition *EqLTarski* ::
 ($[p, p] \Rightarrow \text{bool}$) $\Rightarrow$ ($[p, p] \Rightarrow \text{bool}$) $\Rightarrow \text{bool}$
 (- *EqLTarski* - [99,99] 50)
where
 l1 EqLTarski l2
 $\equiv$
 $\forall A B. l1 A B \longleftrightarrow l2 A B$

definition *EqA* ::
 ($[p, p, p] \Rightarrow \text{bool}$) $\Rightarrow$ ($[p, p, p] \Rightarrow \text{bool}$) $\Rightarrow \text{bool}$
 (- *EqA* - [99,99] 50)
where
 a1 EqA a2
 $\equiv$
 $\forall A B C. a1 A B C \longleftrightarrow a2 A B C$

definition *hypothesis-of-right-saccheri-quadrilaterals* ::
bool
 (*HypothesisRightSaccheriQuadrilaterals*)
where
 hypothesis-of-right-saccheri-quadrilaterals
 $\equiv$
 $\forall A B C D. \text{Saccheri } A B C D \longrightarrow \text{Per } A B C$

definition *hypothesis-of-acute-saccheri-quadrilaterals* ::
bool
 (*HypothesisAcuteSaccheriQuadrilaterals*)
where
 hypothesis-of-acute-saccheri-quadrilaterals
 $\equiv$
 $\forall A B C D. \text{Saccheri } A B C D \longrightarrow \text{Acute } A B C$

definition *hypothesis-of-obtuse-saccheri-quadrilaterals* ::
bool
 (*HypothesisObtuseSaccheriQuadrilaterals*)
where
 hypothesis-of-obtuse-saccheri-quadrilaterals
 $\equiv$
 $\forall A B C D. \text{Saccheri } A B C D \longrightarrow \text{Obtuse } A B C$

definition *Defect* ::
 ($[p, p, p, p, p] \Rightarrow \text{bool}$)
 (*Defect* - - - - - [99,99,99,99,99] 50)
where
 Defect A B C D E F
 $\equiv$
 $(\exists G H I. (A B C \text{TriSumA } G H I \wedge G H I \text{SuppA } D E F))$

definition *Ar1* ::
 ($[p, p, p, p] \Rightarrow \text{bool}$)
 (*Ar1* - - - - - [99,99,99,99,99] 50)
where
 Ar1 PO E A B C
 $\equiv$
 $PO \neq E \wedge \text{Col } PO E A \wedge \text{Col } PO E B \wedge \text{Col } PO E C$

definition *Ar2* ::
 ($[p, p, p, p, p] \Rightarrow \text{bool}$)
 (*Ar2* - - - - - [99,99,99,99,99] 50)
where
 Ar2 PO E E' A B C
 $\equiv$
 $\neg \text{Col } PO E E' \wedge \text{Col } PO E A \wedge \text{Col } PO E B \wedge \text{Col } PO E C$

definition *Pj* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(- - Pj - - [99, 99, 99, 99] \ 50)$

where

$A \ B \ Pj \ C \ D$

$\equiv$

$A \ B \ Par \ C \ D \vee C = D$

definition *Sum* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(Sum - - - - - [99, 99, 99, 99, 99, 99] \ 50)$

where

$Sum \ PO \ E \ E' \ A \ B \ C$

$\equiv$

$Ar2 \ PO \ E \ E' \ A \ B \ C \wedge$
 $(\exists \ A' \ C'. \ E \ E' \ Pj \ A \ A' \wedge Col \ PO \ E' \ A' \wedge PO \ E \ Pj \ A' \ C' \wedge$
 $PO \ E' \ Pj \ B \ C' \wedge E' \ E \ Pj \ C' \ C)$

definition *Proj* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(- - Proj - - - - [99, 99, 99, 99, 99, 99] \ 50)$

where

$P \ Q \ Proj \ A \ B \ X \ Y$

$\equiv$

$A \neq B \wedge X \neq Y \wedge \neg A \ B \ Par \ X \ Y \wedge Col \ A \ B \ Q \wedge (P \ Q \ Par \ X \ Y \vee P = Q)$

definition *Sump* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(Sump - - - - - [99, 99, 99, 99, 99, 99] \ 50)$

where

$Sump \ PO \ E \ E' \ A \ B \ C$

$\equiv$

$Col \ PO \ E \ A \wedge Col \ PO \ E \ B \wedge$
 $(\exists \ A' \ C' \ P'. \ A \ A' \ Proj \ PO \ E' \ E \ E' \wedge PO \ E \ Par \ A' \ P' \wedge$
 $B \ C' \ Proj \ A' \ P' \ PO \ E' \wedge C' \ C \ Proj \ PO \ E \ E \ E')$

definition *Prod* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(Prod - - - - - [99, 99, 99, 99, 99, 99] \ 50)$

where

$Prod \ PO \ E \ E' \ A \ B \ C$

$\equiv$

$Ar2 \ PO \ E \ E' \ A \ B \ C \wedge$
 $(\exists \ B'. \ E \ E' \ Pj \ B \ B' \wedge Col \ PO \ E' \ B' \wedge E' \ A \ Pj \ B' \ C)$

definition *Prodp* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(Prodp - - - - - [99, 99, 99, 99, 99, 99] \ 50)$

where

$Prodp \ PO \ E \ E' \ A \ B \ C$

$\equiv$

$Col \ PO \ E \ A \wedge Col \ PO \ E \ B \wedge$
 $(\exists \ B'. \ B \ B' \ Proj \ PO \ E' \ E \ E' \wedge B' \ C \ Proj \ PO \ E \ A \ E')$

definition *Opp* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(Opp - - - - - [99, 99, 99, 99, 99, 99] \ 50)$

where
 $Opp\ PO\ E\ E'\ A\ B$
 $\equiv$
 $Sum\ PO\ E\ E'\ B\ A\ PO$

definition $Diff ::$
 $[p, p, p, p, p, p] \Rightarrow bool$
 $(Diff\ -\ -\ -\ -\ -\ [99, 99, 99, 99, 99, 99]\ 50)$
where
 $Diff\ PO\ E\ E'\ A\ B\ C$
 $\equiv$
 $\exists\ B'.\ Opp\ PO\ E\ E'\ B\ B' \wedge Sum\ PO\ E\ E'\ A\ B'\ C$

definition $sum3 ::$
 $[p, p, p, p, p, p] \Rightarrow bool$
 $(sum3\ -\ -\ -\ -\ -\ [99, 99, 99, 99, 99, 99]\ 50)$
where
 $sum3\ PO\ E\ E'\ A\ B\ C\ S$
 $\equiv$
 $\exists\ AB.\ Sum\ PO\ E\ E'\ A\ B\ AB \wedge Sum\ PO\ E\ E'\ AB\ C\ S$

definition $sum4 ::$
 $[p, p, p, p, p, p, p, p] \Rightarrow bool$
 $(Sum4\ -\ -\ -\ -\ -\ -\ [99, 99, 99, 99, 99, 99, 99, 99]\ 50)$
where
 $Sum4\ PO\ E\ E'\ A\ B\ C\ D\ S$
 $\equiv$
 $\exists\ ABC.\ sum3\ PO\ E\ E'\ A\ B\ C\ ABC \wedge Sum\ PO\ E\ E'\ ABC\ D\ S$

definition $sum22 ::$
 $[p, p, p, p, p, p, p, p] \Rightarrow bool$
 $(sum22\ -\ -\ -\ -\ -\ -\ [99, 99, 99, 99, 99, 99, 99, 99]\ 50)$
where
 $sum22\ PO\ E\ E'\ A\ B\ C\ D\ S$
 $\equiv$
 $\exists\ AB\ CD.\ Sum\ PO\ E\ E'\ A\ B\ AB \wedge Sum\ PO\ E\ E'\ C\ D\ CD \wedge Sum\ PO\ E\ E'\ AB\ CD\ S$

definition $Ar2p4 ::$
 $[p, p, p, p, p, p, p, p] \Rightarrow bool$
 $(Ar2p4\ -\ -\ -\ -\ -\ -\ [99, 99, 99, 99, 99, 99, 99, 99]\ 50)$
where
 $Ar2p4\ PO\ E\ E'\ A\ B\ C\ D$
 $\equiv$
 $\neg\ Col\ PO\ E\ E' \wedge Col\ PO\ E\ A \wedge Col\ PO\ E\ B \wedge Col\ PO\ E\ C \wedge Col\ PO\ E\ D$

definition $Ps ::$
 $[p, p, p] \Rightarrow bool$
 $(Ps\ -\ -\ -\ [99, 99, 99]\ 50)$
where
 $Ps\ X\ E\ A$
 $\equiv$
 $X\ Out\ A\ E$

definition $Ng ::$
 $[p, p, p] \Rightarrow bool$
 $(Ng\ -\ -\ -\ [99, 99, 99]\ 50)$
where
 $Ng\ X\ E\ A$
 $\equiv$
 $A \neq X \wedge E \neq X \wedge Bet\ A\ X\ E$

definition *LtP* ::

$[p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{LtP} \text{ - - - - - } [99, 99, 99, 99, 99] \text{ 50})$

where

$\text{LtP } X \ E \ E' \ A \ B$

$\equiv$

$\exists \ D. \ \text{Diff } X \ E \ E' \ B \ A \ D \wedge \text{Ps } X \ E \ D$

definition *LeP* ::

$[p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{LeP} \text{ - - - - - } [99, 99, 99, 99, 99] \text{ 50})$

where

$\text{LeP } X \ E \ E' \ A \ B$

$\equiv$

$\text{LtP } X \ E \ E' \ A \ B \vee A = B$

definition *Length* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{Length} \text{ - - - - - } [99, 99, 99, 99, 99, 99] \text{ 50})$

where

$\text{Length } X \ E \ E' \ A \ B \ L$

$\equiv$

$X \neq E \wedge \text{Col } X \ E \ L \wedge \text{LeP } X \ E \ E' \ X \ L \wedge \text{Cong } X \ L \ A \ B$

definition *IsLength* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{IsLength} \text{ - - - - - } [99, 99, 99, 99, 99, 99] \text{ 50})$

where

$\text{IsLength } X \ E \ E' \ A \ B \ L$

$\equiv$

$\text{Length } X \ E \ E' \ A \ B \ L \vee (X = E \wedge X = L)$

definition *Sumg* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{Sumg} \text{ - - - - - } [99, 99, 99, 99, 99, 99] \text{ 50})$

where

$\text{Sumg } X \ E \ E' \ A \ B \ C$

$\equiv$

$\text{Sum } X \ E \ E' \ A \ B \ C \vee (\neg \text{Ar2 } X \ E \ E' \ A \ B \ B \wedge C = X)$

definition *Prodg* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{Prodg} \text{ - - - - - } [99, 99, 99, 99, 99, 99] \text{ 50})$

where

$\text{Prodg } X \ E \ E' \ A \ B \ C$

$\equiv$

$\text{Prod } X \ E \ E' \ A \ B \ C \vee (\neg \text{Ar2 } X \ E \ E' \ A \ B \ B \wedge C = X)$

definition *PythRel* ::

$[p, p, p, p, p, p] \Rightarrow \text{bool}$
 $(\text{PythRel} \text{ - - - - - } [99, 99, 99, 99, 99, 99] \text{ 50})$

where

$\text{PythRel } X \ E \ E' \ A \ B \ C$

$\equiv$

$\text{Ar2 } X \ E \ E' \ A \ B \ C \wedge ((X = B \wedge (A = C \vee \text{Opp } X \ E \ E' \ A \ C)) \vee (\exists \ B'. \ X \ B' \ \text{Perp } X \ B \wedge \text{Cong } X \ B' \ X \ B \wedge \text{Cong } X \ C \ A \ B'))$

definition *SignEq* ::

$[p, p, p, p] \Rightarrow \text{bool}$
 $(\text{SignEq} \text{ - - - - } [99, 99, 99, 99] \text{ 50})$

where

$\text{SignEq } X \ E \ A \ B$

$\equiv$

$Ps\ X\ E\ A \wedge Ps\ X\ E\ B \vee Ng\ X\ E\ A \wedge Ng\ X\ E\ B$

definition *LtPs* ::

$['p, 'p, 'p, 'p] \Rightarrow bool$
 $(LtPs\ -\ -\ -\ -\ [99,99,99,99]\ 50)$

where

$LtPs\ X\ E\ E'\ A\ B$

$\equiv$

$\exists\ D.\ Ps\ X\ E\ D \wedge Sum\ X\ E\ E'\ A\ D\ B$

definition *IsOrthocenter* ::

$['p, 'p, 'p, 'p] \Rightarrow bool$
 $(- IsOrthocenter\ -\ -\ -\ [99,99,99,99]\ 50)$

where

$H\ IsOrthocenter\ A\ B\ C \equiv \neg\ Col\ A\ B\ C \wedge$
 $A\ H\ Perp\ B\ C \wedge$
 $B\ H\ Perp\ A\ C \wedge$
 $C\ H\ Perp\ A\ B$

definition *IsCircumcenter* ::

$['p, 'p, 'p, 'p] \Rightarrow bool$
 $(- IsCircumcenter\ -\ -\ -\ [99,99,99,99]\ 50)$

where

$G\ IsCircumcenter\ A\ B\ C \equiv$
 $Cong\ A\ G\ B\ G \wedge$
 $Cong\ B\ G\ C\ G \wedge$
 $Coplanar\ G\ A\ B\ C$

definition *IsGravityCenter* ::

$['p, 'p, 'p, 'p] \Rightarrow bool$
 $(- IsGravityCenter\ -\ -\ -\ [99,99,99,99]\ 50)$

where

$G\ IsGravityCenter\ A\ B\ C \equiv \neg\ Col\ A\ B\ C \wedge$
 $(\exists\ I\ J.\ I\ Midpoint\ B\ C \wedge$
 $J\ Midpoint\ A\ C \wedge$
 $Col\ G\ A\ I \wedge$
 $Col\ G\ B\ J)$

definition *Lcos* :: $(['p, 'p] \Rightarrow bool) \Rightarrow$

$(['p, 'p] \Rightarrow bool) \Rightarrow$
 $(['p, 'p, 'p] \Rightarrow bool) \Rightarrow$
 $bool$

where

$Lcos\ lb\ lc\ a \equiv$
 $QCong\ lb \wedge QCong\ lc \wedge QCongAAcute\ a \wedge$
 $(\exists\ A\ B\ C.\ (Per\ C\ B\ A \wedge lb\ A\ B \wedge lc\ A\ C \wedge a\ B\ A\ C))$

definition *EqLcos* :: $(['p, 'p] \Rightarrow bool) \Rightarrow$

$(['p, 'p, 'p] \Rightarrow bool) \Rightarrow$
 $(['p, 'p] \Rightarrow bool) \Rightarrow$
 $(['p, 'p, 'p] \Rightarrow bool) \Rightarrow$
 $bool$

where

$EqLcos\ la\ a\ lb\ b \equiv (\exists\ lp.\ Lcos\ lp\ la\ a \wedge Lcos\ lp\ lb\ b)$

definition *Lcos2* :: $(['p, 'p] \Rightarrow bool) \Rightarrow$

$(['p, 'p] \Rightarrow bool) \Rightarrow$
 $(['p, 'p, 'p] \Rightarrow bool) \Rightarrow$
 $(['p, 'p, 'p] \Rightarrow bool) \Rightarrow$
 $bool$

where

$Lcos2\ lp\ l\ a\ b \equiv \exists\ la.\ Lcos\ la\ l\ a \wedge Lcos\ lp\ la\ b$

definition *EqLcos2* :: $(['p, 'p] \Rightarrow bool) \Rightarrow$

$([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 bool

where

$\text{EqLcos2 } l1 \ a \ b \ l2 \ c \ d \equiv (\exists \ lp. \text{Lcos2 } lp \ l1 \ a \ b \wedge \text{Lcos2 } lp \ l2 \ c \ d)$

definition $\text{Lcos3} :: ([p, p] \Rightarrow \text{bool}) \Rightarrow$

$([p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 bool

where

$\text{Lcos3 } lp \ l \ a \ b \ c \equiv \exists \ la \ lab. \text{Lcos } la \ l \ a \wedge$
 $\text{Lcos } lab \ la \ b \wedge \text{Lcos } lp \ lab \ c$

definition $\text{EqLcos3} :: ([p, p] \Rightarrow \text{bool}) \Rightarrow$

$([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 $([p, p, p] \Rightarrow \text{bool}) \Rightarrow$
 bool

where

$\text{EqLcos3 } l1 \ a \ b \ c \ l2 \ d \ e \ f \equiv (\exists \ lp. \text{Lcos3 } lp \ l1 \ a \ b \ c \wedge \text{Lcos3 } lp \ l2 \ d \ e \ f)$

definition $\text{EqV} :: p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$

$(- - \text{EqV} - - [99,99,99,99] \ 50)$

where

$A \ B \ \text{EqV} \ C \ D \equiv \text{Parallelogram } A \ B \ D \ C \vee (A = B \wedge C = D)$

definition $\text{SumV} :: p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$

$(- - - \text{SumV} - - [99,99,99,99,99,99] \ 50)$

where

$A \ B \ C \ D \ \text{SumV} \ E \ F \equiv \forall \ D'. \ C \ D \ \text{EqV} \ B \ D' \longrightarrow A \ D' \ \text{EqV} \ E \ F$

definition $\text{SumVExists} :: p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$

$(- - - \text{SumVExists} - - [99,99,99,99,99,99] \ 50)$

where

$A \ B \ C \ D \ \text{SumVExists} \ E \ F \equiv (\exists \ D'. \ B \ D' \ \text{EqV} \ C \ D \wedge A \ D' \ \text{EqV} \ E \ F)$

definition $\text{SameDir} :: p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$

$(- - \text{SameDir} - - [99,99,99,99] \ 50)$

where

$A \ B \ \text{SameDir} \ C \ D \equiv$
 $(A = B \wedge C = D) \vee (\exists \ D'. \ C \ \text{Out} \ D \ D' \wedge A \ B \ \text{EqV} \ C \ D')$

definition $\text{OppDir} :: p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$

$(- - \text{OppDir} - - [99,99,99,99] \ 50)$

where

$A \ B \ \text{OppDir} \ C \ D \equiv A \ B \ \text{SameDir} \ D \ C$

definition $\text{CongA3} :: p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$

$(- - - \text{CongA3} - - [99,99,99,99,99,99] \ 50)$

where

$A \ B \ C \ \text{CongA3} \ A' \ B' \ C' \equiv$
 $A \ B \ C \ \text{CongA} \ A' \ B' \ C' \wedge B \ C \ A \ \text{CongA} \ B' \ C' \ A' \wedge C \ A \ B \ \text{CongA} \ C' \ A' \ B'$

definition *Projp* :: 'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ bool
 (- - *Projp* - - [99,99,99,99] 50)
where
 $P \ Q \ Projp \ A \ B \equiv$
 $A \neq B \wedge ((Col \ A \ B \ Q \wedge A \ B \ Perp \ P \ Q) \vee (Col \ A \ B \ P \wedge P = Q))$

1.3 Propositions

lemma *cong-reflexivity*:
shows *Cong* *A* *B* *A* *B*
using *cong-inner-transitivity* *cong-pseudo-reflexivity* **by** *blast*

lemma *cong-symmetry*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *C* *D* *A* *B*
using *assms* *cong-inner-transitivity* *cong-reflexivity* **by** *blast*

lemma *cong-transitivity*:
assumes *Cong* *A* *B* *C* *D* **and** *Cong* *C* *D* *E* *F*
shows *Cong* *A* *B* *E* *F*
by (*meson* *assms*(1) *assms*(2) *cong-inner-transitivity* *cong-pseudo-reflexivity*)

lemma *cong-left-commutativity*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *B* *A* *C* *D*
using *assms* *cong-inner-transitivity* *cong-pseudo-reflexivity* **by** *blast*

lemma *cong-right-commutativity*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *A* *B* *D* *C*
using *assms* *cong-left-commutativity* *cong-symmetry* **by** *blast*

lemma *cong-3421*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *C* *D* *B* *A*
using *assms* *cong-left-commutativity* *cong-symmetry* **by** *blast*

lemma *cong-4312*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *D* *C* *A* *B*
using *assms* *cong-left-commutativity* *cong-symmetry* **by** *blast*

lemma *cong-4321*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *D* *C* *B* *A*
using *assms* *cong-3421* *cong-left-commutativity* **by** *blast*

lemma *cong-trivial-identity*:
shows *Cong* *A* *A* *B* *B*
using *cong-identity* *segment-construction* **by** *blast*

lemma *cong-reverse-identity*:
assumes *Cong* *A* *A* *C* *D*
shows *C* = *D*
using *assms* *cong-3421* *cong-identity* **by** *blast*

lemma *cong-commutativity*:
assumes *Cong* *A* *B* *C* *D*
shows *Cong* *B* *A* *D* *C*
using *assms* *cong-3421* **by** *blast*

lemma *not-cong-2134*:
assumes $\neg Cong \ A \ B \ C \ D$
shows $\neg Cong \ B \ A \ C \ D$

```

using assms cong-left-commutativity by blast

lemma not-cong-1243:
  assumes  $\neg \text{Cong } A B C D$ 
  shows  $\neg \text{Cong } A B D C$ 
  using assms cong-right-commutativity by blast

lemma not-cong-2143:
  assumes  $\neg \text{Cong } A B C D$ 
  shows  $\neg \text{Cong } B A D C$ 
  using assms cong-commutativity by blast

lemma not-cong-3412:
  assumes  $\neg \text{Cong } A B C D$ 
  shows  $\neg \text{Cong } C D A B$ 
  using assms cong-symmetry by blast

lemma not-cong-4312:
  assumes  $\neg \text{Cong } A B C D$ 
  shows  $\neg \text{Cong } D C A B$ 
  using assms cong-3421 by blast

lemma not-cong-3421:
  assumes  $\neg \text{Cong } A B C D$ 
  shows  $\neg \text{Cong } C D B A$ 
  using assms cong-4312 by blast

lemma not-cong-4321:
  assumes  $\neg \text{Cong } A B C D$ 
  shows  $\neg \text{Cong } D C B A$ 
  using assms cong-4321 by blast

lemma five-segment-with-def:
  assumes  $A B C D \text{ OFSC } A' B' C' D'$  and  $A \neq B$ 
  shows  $\text{Cong } C D C' D'$ 
  using assms(1) assms(2) OFSC-def five-segment by blast

lemma cong-diff:
  assumes  $A \neq B$  and  $\text{Cong } A B C D$ 
  shows  $C \neq D$ 
  using assms(1) assms(2) cong-identity by blast

lemma cong-diff-2:
  assumes  $B \neq A$  and  $\text{Cong } A B C D$ 
  shows  $C \neq D$ 
  using assms(1) assms(2) cong-identity by blast

lemma cong-diff-3:
  assumes  $C \neq D$  and  $\text{Cong } A B C D$ 
  shows  $A \neq B$ 
  using assms(1) assms(2) cong-reverse-identity by blast

lemma cong-diff-4:
  assumes  $D \neq C$  and  $\text{Cong } A B C D$ 
  shows  $A \neq B$ 
  using assms(1) assms(2) cong-reverse-identity by blast

lemma cong-3-sym:
  assumes  $A B C \text{ Cong3 } A' B' C'$ 
  shows  $A' B' C' \text{ Cong3 } A B C$ 
  using assms Cong3-def not-cong-3412 by blast

lemma cong-3-swap:
  assumes  $A B C \text{ Cong3 } A' B' C'$ 
  shows  $B A C \text{ Cong3 } B' A' C'$ 
  using assms Cong3-def cong-commutativity by blast

```



```

lemma cong-3-swap-2:
  assumes  $A\ B\ C\ Cong3\ A'\ B'\ C'$ 
  shows  $A\ C\ B\ Cong3\ A'\ C'\ B'$ 
  using assms Cong3-def cong-commutativity by blast

lemma cong3-transitivity:
  assumes  $A0\ B0\ C0\ Cong3\ A1\ B1\ C1$  and
     $A1\ B1\ C1\ Cong3\ A2\ B2\ C2$ 
  shows  $A0\ B0\ C0\ Cong3\ A2\ B2\ C2$ 
  by (meson assms(1) assms(2) Cong3-def cong-inner-transitivity not-cong-3412)

lemma eq-dec-points:
  shows  $A = B \vee \neg A = B$ 
  by simp

lemma distinct:
  assumes  $P \neq Q$ 
  shows  $R \neq P \vee R \neq Q$ 
  using assms by simp

lemma l2-11:
  assumes Bet  $A\ B\ C$  and
    Bet  $A'\ B'\ C'$  and
    Cong  $A\ B\ A'\ B'$  and
    Cong  $B\ C\ B'\ C'$ 
  shows Cong  $A\ C\ A'\ C'$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    using assms(3) assms(4) cong-reverse-identity by blast
next
  assume  $A \neq B$ 
  thus ?thesis
    using five-segment Tarski-neutral-dimensionless-axioms assms(1) assms(2) assms(3) assms(4)
      cong-commutativity cong-trivial-identity by blast
qed

lemma bet-cong3:
  assumes Bet  $A\ B\ C$  and
    Cong  $A\ B\ A'\ B'$ 
  shows  $\exists C'. A\ B\ C\ Cong3\ A'\ B'\ C'$ 
  by (meson assms(1) assms(2) Cong3-def l2-11 not-cong-3412 segment-construction)

lemma construction-uniqueness:
  assumes  $Q \neq A$  and
    Bet  $Q\ A\ X$  and
    Cong  $A\ X\ B\ C$  and
    Bet  $Q\ A\ Y$  and
    Cong  $A\ Y\ B\ C$ 
  shows  $X = Y$ 
  by (meson assms(1) assms(2) assms(3) assms(4) assms(5) cong-identity cong-inner-transitivity
    cong-reflexivity five-segment)

lemma Cong-cases:
  assumes  $Cong\ A\ B\ C\ D \vee Cong\ A\ B\ D\ C \vee Cong\ B\ A\ C\ D \vee Cong\ B\ A\ D\ C \vee Cong\ C\ D\ A\ B \vee$ 
     $Cong\ C\ D\ B\ A \vee Cong\ D\ C\ A\ B \vee Cong\ D\ C\ B\ A$ 
  shows  $Cong\ A\ B\ C\ D$ 
  using assms not-cong-3421 not-cong-4321 by blast

lemma Cong-perm :
  assumes Cong  $A\ B\ C\ D$ 
  shows  $Cong\ A\ B\ C\ D \wedge Cong\ A\ B\ D\ C \wedge Cong\ B\ A\ C\ D \wedge Cong\ B\ A\ D\ C \wedge Cong\ C\ D\ A\ B \wedge$ 
     $Cong\ C\ D\ B\ A \wedge Cong\ D\ C\ A\ B \wedge Cong\ D\ C\ B\ A$ 
  using assms not-cong-1243 not-cong-3412 by blast

```

```

lemma bet-col:
  assumes Bet A B C
  shows Col A B C
  by (simp add: assms Col-def)

lemma between-trivial:
  shows Bet A B B
  using cong-identity segment-construction by blast

lemma between-symmetry:
  assumes Bet A B C
  shows Bet C B A
  using assms between-identity between-trivial inner-pasch by blast

lemma Bet-cases:
  assumes Bet A B C  $\vee$  Bet C B A
  shows Bet A B C
  using assms between-symmetry by blast

lemma Bet-perm:
  assumes Bet A B C
  shows Bet A B C  $\wedge$  Bet C B A
  using assms Bet-cases by blast

lemma between-trivial2:
  shows Bet A A B
  using Bet-perm between-trivial by blast

lemma between-equality:
  assumes Bet A B C and Bet B A C
  shows A = B
  using assms(1) assms(2) between-identity inner-pasch by blast

lemma between-equality-2:
  assumes Bet A B C and
    Bet A C B
  shows B = C
  using assms(1) assms(2) between-equality between-symmetry by blast

lemma between-exchange3:
  assumes Bet A B C and
    Bet A C D
  shows Bet B C D
  by (metis Bet-perm assms(1) assms(2) between-identity inner-pasch)

lemma bet-neq12--neq:
  assumes Bet A B C and
    A  $\neq$  B
  shows A  $\neq$  C
  using assms(1) assms(2) between-identity by blast

lemma bet-neq21--neq:
  assumes Bet A B C and
    B  $\neq$  A
  shows A  $\neq$  C
  using assms(1) assms(2) between-identity by blast

lemma bet-neq23--neq:
  assumes Bet A B C and
    B  $\neq$  C
  shows A  $\neq$  C
  using assms(1) assms(2) between-identity by blast

lemma bet-neq32--neq:
  assumes Bet A B C and
    C  $\neq$  B

```

shows $A \neq C$
using *assms(1) assms(2) between-identity* **by** *blast*

lemma *not-bet-distincts*:
assumes $\neg \text{Bet } A \ B \ C$
shows $A \neq B \wedge B \neq C$
using *assms between-trivial between-trivial2* **by** *blast*

lemma *between-inner-transitivity*:
assumes *Bet A B D and*
Bet B C D
shows *Bet A B C*
using *assms(1) assms(2) Bet-perm between-exchange3* **by** *blast*

lemma *outer-transitivity-between2*:
assumes *Bet A B C and*
Bet B C D and
B $\neq$ C
shows *Bet A C D*
proof –
obtain *X* **where** *Bet A C X and Cong C X C D*
using *segment-construction* **by** *blast*
thus *?thesis*
using *assms(1) assms(2) assms(3) between-exchange3 cong-inner-transitivity*
construction-uniqueness **by** *blast*
qed

lemma *between-exchange2*:
assumes *Bet A B D and*
Bet B C D
shows *Bet A C D*
using *assms(1) assms(2) between-inner-transitivity outer-transitivity-between2* **by** *blast*

lemma *outer-transitivity-between*:
assumes *Bet A B C and*
Bet B C D and
B $\neq$ C
shows *Bet A B D*
using *assms(1) assms(2) assms(3) between-symmetry outer-transitivity-between2* **by** *blast*

lemma *between-exchange4*:
assumes *Bet A B C and*
Bet A C D
shows *Bet A B D*
using *assms(1) assms(2) between-exchange2 between-symmetry* **by** *blast*

lemma *l3-9-4*:
assumes *Bet4 A1 A2 A3 A4*
shows *Bet4 A4 A3 A2 A1*
using *assms Bet4-def Bet-cases* **by** *blast*

lemma *l3-17*:
assumes *Bet A B C and*
Bet A' B' C and
Bet A P A'
shows $\exists \ Q. \text{Bet } P \ Q \ C \wedge \text{Bet } B \ Q \ B'$
proof –
obtain *X* **where** *Bet B' X A and Bet P X C*
using *Bet-perm assms(2) assms(3) inner-pasch* **by** *blast*
moreover then obtain *Y* **where** *Bet X Y C and Bet B Y B'*
using *Bet-perm assms(1) inner-pasch* **by** *blast*
ultimately show *?thesis*
using *between-exchange2* **by** *blast*
qed

lemma *lower-dim-ex*:

$\exists A B C. \neg (Bet A B C \vee Bet B C A \vee Bet C A B)$
using *lower-dim* **by** *auto*

lemma *two-distinct-points*:
 $\exists X::'p. \exists Y::'p. X \neq Y$
using *lower-dim-ex not-bet-distincts* **by** *blast*

lemma *point-construction-different*:
 $\exists C. Bet A B C \wedge B \neq C$
using *two-distinct-points Tarski-neutral-dimensionless-axioms*
cong-reverse-identity segment-construction **by** *blast*

lemma *another-point*:
 $\exists B::'p. A \neq B$
using *point-construction-different* **by** *blast*

lemma *Cong-stability*:
assumes $\neg \neg Cong A B C D$
shows $Cong A B C D$
using *assms* **by** *simp*

lemma *l2-11-b*:
assumes $Bet A B C$ **and**
 $Bet A' B' C'$ **and**
 $Cong A B A' B'$ **and**
 $Cong B C B' C'$
shows $Cong A C A' C'$
using *assms(1) assms(2) assms(3) assms(4) l2-11* **by** *auto*

lemma *cong-dec-eq-dec-b*:
assumes $\neg A \neq B$
shows $A = B$
using *assms(1)* **by** *simp*

lemma *BetSEq*:
assumes $BetS A B C$
shows $Bet A B C \wedge A \neq B \wedge A \neq C \wedge B \neq C$
using *assms BetS-def between-identity* **by** *auto*

lemma *l4-2*:
assumes $A B C D IFSC A' B' C' D'$
shows $Cong B D B' D'$
proof *cases*
assume $A = C$
thus *?thesis*
by (*metis IFSC-def between-identity assms cong-diff-3*)

next
assume $A \neq C$
have $Bet A B C$ **and** $Bet A' B' C'$ **and**
 $Cong A C A' C'$ **and** $Cong B C B' C'$
 $Cong A D A' D'$ **and** $Cong C D C' D'$
using *IFSC-def assms* **by** *auto*
obtain E **where** $Bet A C E$ **and** $Cong C E A C$
using *segment-construction* **by** *blast*
obtain E' **where** $Bet A' C' E'$ **and** $Cong C' E' C E$
using *segment-construction* **by** *blast*
hence $Cong C E C' E'$
using *Cong-cases* **by** *blast*
hence $Cong E D E' D'$
using $\langle A \neq C \rangle \langle Bet A C E \rangle \langle Bet A' C' E' \rangle \langle Cong A C A' C' \rangle$
 $\langle Cong A D A' D' \rangle \langle Cong C D C' D' \rangle$
five-segment **by** *blast*
moreover
have $E \neq C$
using $\langle A \neq C \rangle \langle Cong C E A C \rangle$ *cong-reverse-identity* **by** *blast*
moreover

have $Bet\ E\ C\ B$
using $\langle Bet\ A\ B\ C \rangle \langle Bet\ A\ C\ E \rangle$ *between-exchange3 between-symmetry* **by** *blast*
moreover
have $Bet\ E'\ C'\ B'$
using $\langle Bet\ A'\ B'\ C' \rangle \langle Bet\ A'\ C'\ E' \rangle$ *between-exchange3 between-symmetry* **by** *blast*
moreover
have $Cong\ E\ C\ E'\ C'$
by (*simp add: $\langle Cong\ C\ E\ C'\ E' \rangle$ cong-commutativity*)
moreover
have $Cong\ C\ B\ C'\ B'$
using *Cong-cases $\langle Cong\ B\ C\ B'\ C' \rangle$* **by** *blast*
ultimately show *?thesis*
using $\langle Cong\ C\ D\ C'\ D' \rangle$ *five-segment* **by** *blast*
qed

lemma *l4-3*:
assumes $Bet\ A\ B\ C$ **and**
 $Bet\ A'\ B'\ C'$ **and**
 $Cong\ A\ C\ A'\ C'$
and $Cong\ B\ C\ B'\ C'$
shows $Cong\ A\ B\ A'\ B'$
proof –
have $A\ B\ C\ A\ IFSC\ A'\ B'\ C'\ A'$
using *IFSC-def* *assms(1) assms(2) assms(3) assms(4) cong-trivial-identity*
 $not-cong-2143$ **by** *blast*
thus *?thesis*
using *l4-2 not-cong-2143* **by** *blast*
qed

lemma *l4-3-1*:
assumes $Bet\ A\ B\ C$ **and**
 $Bet\ A'\ B'\ C'$ **and**
 $Cong\ A\ B\ A'\ B'$ **and**
 $Cong\ A\ C\ A'\ C'$
shows $Cong\ B\ C\ B'\ C'$
by (*meson* *assms(1) assms(2) assms(3) assms(4) between-symmetry cong-4321 l4-3*)

lemma *l4-5*:
assumes $Bet\ A\ B\ C$ **and**
 $Cong\ A\ C\ A'\ C'$
shows $\exists\ B'. (Bet\ A'\ B'\ C' \wedge A\ B\ C\ Cong3\ A'\ B'\ C')$
proof –
obtain X' **where** $Bet\ C'\ A'\ X'$ **and** $A' \neq X'$
using *point-construction-different* **by** *auto*
obtain B' **where** $Bet\ X'\ A'\ B'$ **and** $Cong\ A'\ B'\ A\ B$
using *segment-construction* **by** *blast*
obtain C'' **where** $Bet\ X'\ B'\ C''$ **and** $Cong\ B'\ C''\ B\ C$
using *segment-construction* **by** *blast*
hence $Bet\ A'\ B'\ C''$
using $\langle Bet\ X'\ A'\ B' \rangle$ *between-exchange3* **by** *blast*
moreover
have $A\ B\ C\ Cong3\ A'\ B'\ C''$
using *Cong3-def* $\langle Cong\ A'\ B'\ A\ B \rangle \langle Cong\ B'\ C''\ B\ C \rangle$ *assms(1) calculation cong-symmetry l2-11-b*
by *blast*
moreover
have $C'' = C'$
proof –
have $Bet\ X'\ A'\ C''$
using $\langle Bet\ X'\ A'\ B' \rangle \langle Bet\ X'\ B'\ C'' \rangle$ *between-exchange4* **by** *blast*
moreover have $Bet\ X'\ A'\ C'$
using *Bet-cases $\langle Bet\ C'\ A'\ X' \rangle$* **by** *auto*
moreover have $Cong\ A'\ C''\ A\ C$
using $\langle Bet\ A'\ B'\ C'' \rangle \langle Cong\ A'\ B'\ A\ B \rangle \langle Cong\ B'\ C''\ B\ C \rangle$ *assms(1) l2-11-b* **by** *blast*
ultimately show *?thesis*
by (*metis* $\langle A' \neq X' \rangle$ *assms(2) cong-symmetry construction-uniqueness*)

qed
ultimately show *?thesis*
by *auto*
qed

lemma *l4-6*:
assumes *Bet A B C* and
A B C Cong3 A' B' C'
shows *Bet A' B' C'*
proof –
obtain *x* where *P1: Bet A' x C' ∧ A B C Cong3 A' x C'*
using *Cong3-def assms(1) assms(2) l4-5* by *blast*
hence *A' x C' Cong3 A' B' C'*
using *assms(2) cong3-transitivity cong-3-sym* by *blast*
hence *A' x C' x IFSC A' x C' B'*
by (*meson Cong3-def Cong-perm IFSC-def P1 cong-reflexivity*)
hence *Cong x x x B'*
using *l4-2* by *auto*
thus *?thesis*
using *P1 cong-reverse-identity* by *blast*
qed

lemma *cong3-bet-eq*:
assumes *Bet A B C* and
A B C Cong3 A X C
shows *X = B*
proof –
have *A B C B IFSC A B C X*
by (*meson Cong3-def Cong-perm IFSC-def assms(1) assms(2) cong-reflexivity*)
thus *?thesis*
using *cong-reverse-identity l4-2* by *blast*
qed

lemma *col-permutation-1*:
assumes *Col A B C*
shows *Col B C A*
using *assms(1) Col-def* by *blast*

lemma *col-permutation-2*:
assumes *Col A B C*
shows *Col C A B*
using *assms(1) col-permutation-1* by *blast*

lemma *col-permutation-3*:
assumes *Col A B C*
shows *Col C B A*
using *assms(1) Bet-cases Col-def* by *auto*

lemma *col-permutation-4*:
assumes *Col A B C*
shows *Col B A C*
using *assms(1) Bet-perm Col-def* by *blast*

lemma *col-permutation-5*:
assumes *Col A B C*
shows *Col A C B*
using *assms(1) col-permutation-1 col-permutation-3* by *blast*

lemma *not-col-permutation-1*:
assumes $\neg \text{Col } A \ B \ C$
shows $\neg \text{Col } B \ C \ A$
using *assms col-permutation-2* by *blast*

lemma *not-col-permutation-2*:
assumes $\neg \text{Col } A \ B \ C$
shows $\neg \text{Col } C \ A \ B$

```

using assms col-permutation-1 by blast

lemma not-col-permutation-3:
  assumes  $\neg \text{Col } A \ B \ C$ 
  shows  $\neg \text{Col } C \ B \ A$ 
  using assms col-permutation-3 by blast

lemma not-col-permutation-4:
  assumes  $\neg \text{Col } A \ B \ C$ 
  shows  $\neg \text{Col } B \ A \ C$ 
  using assms col-permutation-4 by blast

lemma not-col-permutation-5:
  assumes  $\neg \text{Col } A \ B \ C$ 
  shows  $\neg \text{Col } A \ C \ B$ 
  using assms col-permutation-5 by blast

lemma Col-cases:
  assumes  $\text{Col } A \ B \ C \vee \text{Col } A \ C \ B \vee \text{Col } B \ A \ C \vee \text{Col } B \ C \ A \vee \text{Col } C \ A \ B \vee \text{Col } C \ B \ A$ 
  shows  $\text{Col } A \ B \ C$ 
  using assms not-col-permutation-4 not-col-permutation-5 by blast

lemma Col-perm:
  assumes  $\text{Col } A \ B \ C$ 
  shows  $\text{Col } A \ B \ C \wedge \text{Col } A \ C \ B \wedge \text{Col } B \ A \ C \wedge \text{Col } B \ C \ A \wedge \text{Col } C \ A \ B \wedge \text{Col } C \ B \ A$ 
  using Col-cases assms by blast

lemma col-trivial-1:
   $\text{Col } A \ A \ B$ 
  using bet-col not-bet-distincts by blast

lemma col-trivial-2:
   $\text{Col } A \ B \ B$ 
  by (simp add: Col-def between-trivial2)

lemma col-trivial-3:
   $\text{Col } A \ B \ A$ 
  by (simp add: Col-def between-trivial2)

lemma l4-13:
  assumes  $\text{Col } A \ B \ C$  and
     $A \ B \ C \text{ Cong3 } A' \ B' \ C'$ 
  shows  $\text{Col } A' \ B' \ C'$ 
  by (metis Col-def cong-3-swap cong-3-swap-2 assms(1) assms(2) l4-6)

lemma l4-14R1:
  assumes  $\text{Bet } A \ B \ C$  and
     $\text{Cong } A \ B \ A' \ B'$ 
  shows  $\exists C'. A \ B \ C \text{ Cong3 } A' \ B' \ C'$ 
  by (simp add: assms(1) assms(2) bet-cong3)

lemma l4-14R2:
  assumes  $\text{Bet } B \ C \ A$  and
     $\text{Cong } A \ B \ A' \ B'$ 
  shows  $\exists C'. A \ B \ C \text{ Cong3 } A' \ B' \ C'$ 
  by (meson assms(1) assms(2) between-symmetry cong-3-swap-2 l4-5)

lemma l4-14R3:
  assumes  $\text{Bet } C \ A \ B$  and
     $\text{Cong } A \ B \ A' \ B'$ 
  shows  $\exists C'. A \ B \ C \text{ Cong3 } A' \ B' \ C'$ 
  by (meson assms(1) assms(2) between-symmetry cong-3-swap l4-14R1 not-cong-2143)

lemma l4-14:
  assumes  $\text{Col } A \ B \ C$  and
     $\text{Cong } A \ B \ A' \ B'$ 

```

shows $\exists C'. A B C \text{ Cong3 } A' B' C'$
 using *Col-def assms(1) assms(2) l4-14R1 l4-14R2 l4-14R3* by blast

lemma *l4-16R1*:

assumes $A B C D \text{ FSC } A' B' C' D'$ and

$A \neq B$ and

$\text{Bet } A B C$

shows $\text{Cong } C D C' D'$

proof –

have $A B C \text{ Cong3 } A' B' C'$

using *FSC-def assms(1)* by blast

hence $\text{Bet } A' B' C'$

using *assms(3) l4-6* by blast

hence $A B C D \text{ OFSC } A' B' C' D'$

by (*meson Cong3-def FSC-def OFSC-def assms(1) cong-3-sym l4-6*)

thus ?thesis

using *assms(2) five-segment-with-def* by blast

qed

lemma *l4-16R2*:

assumes $A B C D \text{ FSC } A' B' C' D'$

and $\text{Bet } B C A$

shows $\text{Cong } C D C' D'$

proof –

have $A B C \text{ Cong3 } A' B' C'$

using *FSC-def assms(1)* by blast

hence $\text{Bet } B' C' A'$

using *Bet-perm assms(2) cong-3-swap-2 l4-6* by blast

hence $B C A D \text{ IFSC } B' C' A' D'$

by (*meson Cong3-def FSC-def IFSC-def assms(1) assms(2) not-cong-2143*)

thus ?thesis

using *l4-2* by auto

qed

lemma *l4-16R3*:

assumes $A B C D \text{ FSC } A' B' C' D'$ and

$A \neq B$ and

$\text{Bet } C A B$

shows $\text{Cong } C D C' D'$

proof –

have $A B C \text{ Cong3 } A' B' C'$

using *FSC-def assms(1)* by blast

hence $\text{Bet } C' A' B'$

using *assms(3) between-symmetry cong-3-swap l4-6* by blast

thus ?thesis

by (*metis Bet-cases Col-def FSC-def cong-3-swap assms(1) assms(2) assms(3) l4-16R1*)

qed

lemma *l4-16*:

assumes $A B C D \text{ FSC } A' B' C' D'$ and

$A \neq B$

shows $\text{Cong } C D C' D'$

by (*meson Col-def FSC-def assms(1) assms(2) l4-16R1 l4-16R2 l4-16R3*)

lemma *l4-17*:

assumes $A \neq B$ and

$\text{Col } A B C$ and

$\text{Cong } A P A Q$ and

$\text{Cong } B P B Q$

shows $\text{Cong } C P C Q$

proof –

{

assume $\neg \text{Bet } B C A$

hence $\exists p \text{ pa. Bet } p \text{ pa } C \wedge \text{Cong } p \text{ pa } P \text{ pa } Q \wedge \text{Cong } p P p Q \wedge p \neq \text{pa}$

using *Col-def assms(1) assms(2) assms(3) assms(4) between-symmetry* by blast

hence ?thesis


```

    using cong-reflexivity five-segment by blast
  }
  thus ?thesis
    by (meson IFSC-def assms(3) assms(4) cong-reflexivity l4-2)
qed

```

lemma l4-18:

```

  assumes A ≠ B and
    Col A B C and
    Cong A C A C' and
    Cong B C B C'
  shows C = C'
  using assms(1) assms(2) assms(3) assms(4) cong-diff-3 l4-17 by blast

```

lemma l4-19:

```

  assumes Bet A C B and
    Cong A C A C' and
    Cong B C B C'
  shows C = C'
  by (metis Col-def assms(1) assms(2) assms(3) between-equality between-trivial cong-identity
    l4-18 not-cong-3421)

```

lemma not-col-distincts:

```

  assumes ¬ Col A B C
  shows ¬ Col A B C ∧ A ≠ B ∧ B ≠ C ∧ A ≠ C
  using Col-def assms between-trivial by blast

```

lemma NCol-cases:

```

  assumes ¬ Col A B C ∨ ¬ Col A C B ∨ ¬ Col B A C ∨ ¬ Col B C A ∨ ¬ Col C A B ∨ ¬ Col C B A
  shows ¬ Col A B C
  using assms not-col-permutation-2 not-col-permutation-3 by blast

```

lemma NCol-perm:

```

  assumes ¬ Col A B C
  shows ¬ Col A B C ∧ ¬ Col A C B ∧ ¬ Col B A C ∧ ¬ Col B C A ∧ ¬ Col C A B ∧ ¬ Col C B A
  using NCol-cases assms by blast

```

lemma col-cong-3-cong-3-eq:

```

  assumes A ≠ B
    and Col A B C
    and A B C Cong3 A' B' C1
    and A B C Cong3 A' B' C2
  shows C1 = C2
  using Cong3-def cong-diff l4-18 assms(1) assms(2) assms(3) assms(4) cong-inner-transitivity
    l4-13 by meson

```

lemma l5-1:

```

  assumes A ≠ B and
    Bet A B C and
    Bet A B D
  shows Bet A C D ∨ Bet A D C
proof -
  obtain C' where Bet A D C' and Cong D C' C D
    using segment-construction by blast
  obtain D' where Bet A C D' and Cong C D' C D
    using segment-construction by blast
  obtain B' where Bet A C' B' and Cong C' B' C B
    using segment-construction by blast
  obtain B'' where Bet A D' B'' and Cong D' B'' D B
    using segment-construction by blast
  hence Cong B C' B'' C
    using assms(3) between-exchange3 between-symmetry cong-4312 cong-inner-transitivity l2-11-b
    ⟨Bet A C D'⟩ ⟨Bet A D C'⟩ ⟨Cong C D' C D⟩ ⟨Cong D C' C D⟩ by meson
  hence Cong B B' B'' B
    by (meson Bet-cases assms(2) assms(3) between-exchange4 between-inner-transitivity l2-11-b
      ⟨Bet A C D'⟩ ⟨Bet A C' B'⟩ ⟨Bet A D C'⟩ ⟨Bet A D' B''⟩ ⟨Cong C' B' C B⟩)

```

hence $B'' = B'$
by (*meson* *assms*(1) *assms*(2) *assms*(3) *between-exchange4* *cong-inner-transitivity* *construction-uniqueness* *not-cong-2134* $\langle \text{Bet } A \ C \ D' \rangle \langle \text{Bet } A \ C' \ B' \rangle \langle \text{Bet } A \ D \ C' \rangle \langle \text{Bet } A \ D' \ B'' \rangle$)
have $B \ C \ D' \text{ Cong3 } B' \ C' \ D$
proof –
have *Cong* $B \ C \ B' \ C'$
using $\langle \text{Cong } C' \ B' \ C \ B \rangle$ *not-cong-4321* **by** *blast*
moreover
have *Bet* $B' \ C' \ D$
using $\langle \text{Bet } A \ C' \ B' \rangle \langle \text{Bet } A \ D \ C' \rangle$ *between-exchange3* *between-symmetry* **by** *blast*
have *Bet* $B \ C \ D'$
using $\langle \text{Bet } A \ C \ D' \rangle$ *assms*(2) *between-exchange3* **by** *blast*
moreover **have** *Cong* $B \ D' \ B' \ D$
using *l2-11* *Cong-cases* $\langle \text{Bet } B \ C \ D' \rangle \langle \text{Bet } B' \ C' \ D \rangle \langle \text{Cong } C \ D' \ C \ D \rangle \langle \text{Cong } D \ C' \ C \ D \rangle$ *cong-transitivity* $\langle \text{Cong } B \ C \ B' \ C' \rangle$ **by** *blast*
moreover **have** *Cong* $C \ D' \ C' \ D$
by (*metis* *Cong-cases* $\langle \text{Cong } C \ D' \ C \ D \rangle \langle \text{Cong } D \ C' \ C \ D \rangle$ *cong-transitivity*)
ultimately show *?thesis*
using *Cong3-def* **by** *blast*
qed
have *Bet* $B \ C \ D'$
using *assms*(2) *between-exchange3* $\langle \text{Bet } A \ C \ D' \rangle$ **by** *blast*
have *Cong* $C \ D' \ C' \ D$
by (*metis* *Cong-perm* *cong-transitivity* $\langle \text{Cong } C \ D' \ C \ D \rangle \langle \text{Cong } D \ C' \ C \ D \rangle$)
hence $B \ C \ D' \ C' \text{ FSC } B' \ C' \ D \ C$
using *FSC-def* $\langle B \ C \ D' \text{ Cong3 } B' \ C' \ D \rangle \langle B'' = B' \rangle \langle \text{Bet } B \ C \ D' \rangle \langle \text{Cong } B \ C' \ B'' \ C \rangle$ *bet-col* *cong-pseudo-reflexivity* **by** *presburger*
hence *Cong* $D' \ C' \ D \ C$
using *cong-identity* *l4-16* $\langle B'' = B' \rangle \langle \text{Cong } C' \ B' \ C \ B \rangle \langle \text{Cong } D' \ B'' \ D \ B \rangle$ **by** *blast*
obtain E **where** *Bet* $C \ E \ C'$ **and** *Bet* $D \ E \ D'$
using *between-trivial2* *l3-17* $\langle \text{Bet } A \ C \ D' \rangle \langle \text{Bet } A \ D \ C' \rangle$ **by** *blast*
hence $D \ E \ D' \ C \text{ IFSC } D \ E \ D' \ C'$
by (*meson* *IFSC-def* *cong-reflexivity* *cong-3421* *cong-inner-transitivity* $\langle \text{Cong } C \ D' \ C \ D \rangle \langle \text{Cong } D \ C' \ C \ D \rangle \langle \text{Cong } D' \ C' \ D \ C \rangle$)
hence *Cong* $E \ C \ E \ C'$
using *l4-2* **by** *auto*
have $C \ E \ C' \ D \text{ IFSC } C \ E \ C' \ D'$
using *IFSC-def* *cong-reflexivity* *cong-3421* *cong-inner-transitivity*
by (*meson* $\langle \text{Bet } C \ E \ C' \rangle \langle \text{Cong } C \ D' \ C' \ D \rangle \langle D \ E \ D' \ C \text{ IFSC } D \ E \ D' \ C' \rangle$)
hence *Cong* $E \ D \ E \ D'$
using *l4-2* **by** *auto*
obtain P **where** *Bet* $C' \ C \ P$ **and** *Cong* $C \ P \ C \ D'$
using *segment-construction* **by** *blast*
obtain R **where** *Bet* $D' \ C \ R$ **and** *Cong* $C \ R \ C \ E$
using *segment-construction* **by** *blast*
obtain Q **where** *Bet* $P \ R \ Q$ **and** *Cong* $R \ Q \ R \ P$
using *segment-construction* **by** *blast*
have $D' \ C \ R \ P \text{ FSC } P \ C \ E \ D'$
by (*meson* *Bet-perm* *Cong3-def* *FSC-def* $\langle \text{Bet } C \ E \ C' \rangle \langle \text{Bet } C' \ C \ P \rangle \langle \text{Cong } C \ P \ C \ D' \rangle \langle \text{Bet } D' \ C \ R \rangle \langle \text{Cong } C \ R \ C \ E \rangle$ *bet-col* *between-exchange3* *cong-pseudo-reflexivity* *l2-11-b* *not-cong-4321*)
have *Cong* $R \ P \ E \ D'$
by (*metis* $\langle \text{Cong } C \ P \ C \ D' \rangle \langle \text{Cong } C \ R \ C \ E \rangle \langle D' \ C \ R \ P \text{ FSC } P \ C \ E \ D' \rangle$ *cong-commutativity* *cong-diff* *l4-16*)
have *Cong* $R \ Q \ E \ D$
by (*metis* *cong-symmetry* *cong-transitivity* $\langle \text{Cong } E \ D \ E \ D' \rangle \langle \text{Cong } R \ P \ E \ D' \rangle \langle \text{Cong } R \ Q \ R \ P \rangle$)
have $D' \ E \ D \ C \text{ FSC } P \ R \ Q \ C$
by (*meson* *Cong3-def* *FSC-def* $\langle \text{Bet } D \ E \ D' \rangle \langle \text{Bet } P \ R \ Q \rangle \langle \text{Cong } C \ P \ C \ D' \rangle \langle \text{Cong } C \ R \ C \ E \rangle \langle \text{Cong } R \ P \ E \ D' \rangle \langle \text{Cong } R \ Q \ E \ D \rangle$ *bet-col* *between-symmetry* *l2-11-b* *not-cong-3412* *not-cong-4321*)
have *Cong* $D \ C \ Q \ C$
by (*metis* $\langle \text{Cong } E \ D \ E \ D' \rangle \langle D' \ E \ D \ C \text{ FSC } P \ R \ Q \ C \rangle$ *between-trivial2* *cong-identity* *l4-16* *l4-16R2*)
have *Cong* $C \ P \ C \ Q$
by (*meson* $\langle \text{Cong } C \ D' \ C \ D \rangle \langle \text{Cong } C \ P \ C \ D' \rangle \langle \text{Cong } D \ C \ Q \ C \rangle$ *cong-transitivity* *not-cong-2143*)
have *Bet* $A \ C \ D \vee \text{Bet } A \ D \ C$
proof *cases*

```

assume  $R = C$ 
thus ?thesis
  using  $\langle \text{Bet } A \ D \ C' \rangle \langle \text{Cong } C \ R \ C \ E \rangle \langle \text{Cong } E \ C \ E \ C' \rangle$  congr-reverse-identity by blast
next
assume  $R \neq C$ 
{
  have  $\text{Cong } D' \ P \ D' \ Q$ 
  proof –
    have  $\text{Col } R \ C \ D'$ 
    by (simp add:  $\langle \text{Bet } D' \ C \ R \rangle \langle \text{Cong } C \ R \ C \ E \rangle$  bet-col between-symmetry)
    have  $\text{Cong } R \ P \ R \ Q$ 
    by (metis Cong-cases  $\langle \text{Cong } R \ Q \ R \ P \rangle$ )
    have  $\text{Cong } C \ P \ C \ Q$ 
    by (simp add:  $\langle \text{Cong } C \ P \ C \ Q \rangle$ )
    thus ?thesis
    using  $\langle \text{Col } R \ C \ D' \rangle \langle \text{Cong } R \ P \ R \ Q \rangle \langle R \neq C \rangle$  l4-17 by blast
  qed
  hence  $\text{Cong } B \ P \ B \ Q$ 
  by (metis Col-def  $\langle \text{Bet } B \ C \ D' \rangle \langle \text{Cong } C \ P \ C \ D' \rangle \langle \text{Cong } C \ P \ C \ Q \rangle$  congr-identity
    congr-reflexivity l4-17 l4-19 not-bet-distincts)
  have  $\text{Cong } B' \ P \ B' \ Q$ 
  by (metis congr-diff-2 congr-diff-4  $\langle B'' = B' \rangle \langle \text{Bet } A \ C \ D' \rangle \langle \text{Bet } A \ D' \ B'' \rangle \langle \text{Bet } C \ E \ C' \rangle$ 
     $\langle \text{Cong } C \ D' \ C \ D \rangle \langle \text{Cong } C \ D' \ C' \ D \rangle \langle \text{Cong } C \ P \ C \ Q \rangle \langle \text{Cong } C \ R \ C \ E \rangle \langle \text{Cong } D' \ P \ D' \ Q \rangle \langle R \neq C \rangle$ 
    between-exchange3 between-identity congr-reflexivity five-segment)
  have  $\text{Cong } C' \ P \ C' \ Q$ 
  proof –
    have  $\text{Bet } B \ C' \ B'$ 
    using  $\langle \text{Bet } A \ C' \ B' \rangle \langle \text{Bet } A \ D \ C' \rangle$  assms(3) between-exchange3 between-exchange4 by blast
    thus ?thesis
    by (metis Col-def  $\langle \text{Cong } B \ P \ B \ Q \rangle \langle \text{Cong } B' \ P \ B' \ Q \rangle$  between-equality l4-17
    not-bet-distincts)
  qed
  have  $\text{Cong } P \ P \ P \ Q$ 
  by (metis congr-diff  $\langle \text{Bet } C \ E \ C' \rangle \langle \text{Bet } C' \ C \ P \rangle \langle \text{Cong } C \ P \ C \ Q \rangle \langle \text{Cong } C \ R \ C \ E \rangle$ 
     $\langle \text{Cong } C' \ P \ C' \ Q \rangle \langle R \neq C \rangle$  bet-col between-equality-2 between-trivial2 l4-17))
  thus ?thesis
  using  $\langle \text{Bet } A \ C \ D' \rangle \langle \text{Bet } P \ R \ Q \rangle \langle \text{Cong } R \ P \ E \ D' \rangle \langle \text{Cong } R \ Q \ E \ D \rangle$  between-identity
    congr-reverse-identity by blast
}
hence  $R \neq C \longrightarrow \text{Bet } A \ C \ D \vee \text{Bet } A \ D \ C$  by blast
qed
thus ?thesis
  by simp
qed

```

```

lemma l5-2:
assumes  $A \neq B$  and
   $\text{Bet } A \ B \ C$  and
   $\text{Bet } A \ B \ D$ 
shows  $\text{Bet } B \ C \ D \vee \text{Bet } B \ D \ C$ 
using assms(1) assms(2) assms(3) between-exchange3 l5-1 by blast

```

```

lemma segment-construction-2:
assumes  $A \neq Q$ 
shows  $\exists X. ((\text{Bet } Q \ A \ X \vee \text{Bet } Q \ X \ A) \wedge \text{Cong } Q \ X \ B \ C)$ 
proof –
  obtain  $A'$  where  $\text{Bet } A \ Q \ A'$  and  $\text{Cong } Q \ A' \ A \ Q$ 
  using segment-construction by blast
  obtain  $X$  where  $\text{Bet } A' \ Q \ X$  and  $\text{Cong } Q \ X \ B \ C$ 
  using segment-construction by blast
  thus ?thesis
  by (metis  $\langle \text{Bet } A \ Q \ A' \rangle \langle \text{Cong } Q \ A' \ A \ Q \rangle$  congr-diff-4 between-symmetry l5-2)
qed

```

```

lemma l5-3:
assumes  $\text{Bet } A \ B \ D$  and

```

$Bet\ A\ C\ D$
shows $Bet\ A\ B\ C \vee Bet\ A\ C\ B$
by (*metis Bet-perm assms(1) assms(2) between-inner-transitivity l5-2*
point-construction-different)

lemma *bet3--bet*:
assumes $Bet\ A\ B\ E$ **and**
 $Bet\ A\ D\ E$ **and**
 $Bet\ B\ C\ D$
shows $Bet\ A\ C\ E$
by (*meson assms(1) assms(2) assms(3) between-exchange2 between-symmetry l5-3*)

lemma *le-bet*:
assumes $C\ D\ Le\ A\ B$
shows $\exists\ X. (Bet\ A\ X\ B \wedge Cong\ A\ X\ C\ D)$
by (*meson Le-def assms cong-symmetry*)

lemma *l5-5-1*:
assumes $A\ B\ Le\ C\ D$
shows $\exists\ X. (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
proof –
obtain P **where** $Bet\ C\ P\ D$ **and** $Cong\ A\ B\ C\ P$
using *Le-def assms* **by** *blast*
obtain X **where** $Bet\ A\ B\ X$ **and** $Cong\ B\ X\ P\ D$
using *segment-construction* **by** *blast*
thus *?thesis*
by (*meson* $\langle Bet\ C\ P\ D \rangle \langle Cong\ A\ B\ C\ P \rangle$ *l2-11-b*)
qed

lemma *l5-5-2*:
assumes $\exists\ X. (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
shows $A\ B\ Le\ C\ D$
proof –
obtain P **where** $Bet\ A\ B\ P$ **and** $Cong\ A\ P\ C\ D$
using *assms* **by** *blast*
then obtain B' **where** $Bet\ C\ B'\ D$ **and** $A\ B\ P\ Cong3\ C\ B'\ D$
using *l4-5* **by** *blast*
thus *?thesis*
using *Cong3-def Le-def* **by** *blast*
qed

lemma *l5-6*:
assumes $A\ B\ Le\ C\ D$ **and**
 $Cong\ A\ B\ A'\ B'$ **and**
 $Cong\ C\ D\ C'\ D'$
shows $A'\ B'\ Le\ C'\ D'$
by (*meson Cong3-def Le-def assms(1) assms(2) assms(3) cong-inner-transitivity l4-5*)

lemma *le-reflexivity*:
shows $A\ B\ Le\ A\ B$
using *between-trivial cong-reflexivity l5-5-2* **by** *blast*

lemma *le-transitivity*:
assumes $A\ B\ Le\ C\ D$ **and**
 $C\ D\ Le\ E\ F$
shows $A\ B\ Le\ E\ F$
by (*meson assms(1) assms(2) between-exchange4 cong-reflexivity l5-5-1 l5-5-2 l5-6 le-bet*)

lemma *between-cong*:
assumes $Bet\ A\ C\ B$ **and**
 $Cong\ A\ C\ A\ B$
shows $C = B$
by (*metis assms(1) assms(2) between-trivial cong-diff-2 cong-reflexivity l4-3-1*)

lemma *cong3-symmetry*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$

shows $A' B' C' \text{ Cong3 } A B C$
by (*simp add: assms cong-3-sym*)

lemma *between-cong-2*:
assumes $Bet A D B$ **and**
 $Bet A E B$ **and**
 $Cong A D A E$
shows $D = E$
using *l5-3* **by** (*metis Cong-cases assms(1) assms(2) assms(3) between-cong*)

lemma *between-cong-3*:
assumes $A \neq B$
and $Bet A B D$
and $Bet A B E$
and $Cong B D B E$
shows $D = E$
by (*meson assms(1) assms(2) assms(3) assms(4) cong-reflexivity construction-uniqueness*)

lemma *le-anti-symmetry*:
assumes $A B \text{ Le } C D$ **and**
 $C D \text{ Le } A B$
shows $Cong A B C D$

proof –
obtain Y **where** $Bet C Y D$ **and** $Cong A B C Y$
using *Le-def assms(1)* **by** *blast*
obtain T **where** $Bet C D T$ **and** $Cong C T A B$
using *assms(2) l5-5-1* **by** *blast*
have $Cong C Y C T$
by (*metis cong-transitivity ⟨Cong A B C Y⟩ ⟨Cong C T A B⟩ cong-symmetry*)
have $Bet C Y T$
using $\langle Bet C D T \rangle \langle Bet C Y D \rangle$ *between-exchange4* **by** *blast*
have $Y = T$
using $\langle Bet C Y T \rangle \langle Cong C Y C T \rangle$ *between-cong* **by** *auto*
moreover have $T = D$
using $\langle Bet C D T \rangle \langle Bet C Y D \rangle$ *between-equality-2 calculation* **by** *auto*
ultimately show *?thesis*
using $\langle Cong A B C Y \rangle$ **by** *auto*

qed

lemma *cong-dec*:
shows $Cong A B C D \vee \neg Cong A B C D$
by *simp*

lemma *bet-dec*:
shows $Bet A B C \vee \neg Bet A B C$
by *simp*

lemma *col-dec*:
shows $Col A B C \vee \neg Col A B C$
by *simp*

lemma *le-trivial*:
shows $A A \text{ Le } C D$
using *Le-def between-trivial2 cong-trivial-identity* **by** *blast*

lemma *le-cases*:
shows $A B \text{ Le } C D \vee C D \text{ Le } A B$
by (*metis Cong-cases Le-def l5-5-2 le-trivial segment-construction-2*)

lemma *le-zero*:
assumes $A B \text{ Le } C C$
shows $A = B$
by (*metis assms cong-diff-4 le-anti-symmetry le-trivial*)

lemma *le-diff*:
assumes $A \neq B$ **and** $A B \text{ Le } C D$

```

shows  $C \neq D$ 
using assms(1) assms(2) le-zero by blast

lemma lt-diff:
  assumes  $A \ B \ Lt \ C \ D$ 
  shows  $C \neq D$ 
  using Lt-def assms cong-trivial-identity le-zero by blast

lemma bet-cong-eq:
  assumes Bet  $A \ B \ C$  and
    Bet  $A \ C \ D$  and
    Cong  $B \ C \ A \ D$ 
  shows  $C = D \wedge A = B$ 
proof –
  have Bet  $C \ B \ A$ 
    using Bet-perm assms(1) by blast
  thus ?thesis
    by (metis Cong-perm Le-def assms(2) assms(3) between-cong cong-pseudo-reflexivity
      le-anti-symmetry)
qed

lemma cong--le:
  assumes Cong  $A \ B \ C \ D$ 
  shows  $A \ B \ Le \ C \ D$ 
  using Le-def assms between-trivial by blast

lemma cong--le3412:
  assumes Cong  $A \ B \ C \ D$ 
  shows  $C \ D \ Le \ A \ B$ 
  using assms cong--le cong-symmetry by blast

lemma le1221:
  shows  $A \ B \ Le \ B \ A$ 
  by (simp add: cong--le cong-pseudo-reflexivity)

lemma le-left-comm:
  assumes  $A \ B \ Le \ C \ D$ 
  shows  $B \ A \ Le \ C \ D$ 
  using assms le1221 le-transitivity by blast

lemma le-right-comm:
  assumes  $A \ B \ Le \ C \ D$ 
  shows  $A \ B \ Le \ D \ C$ 
  by (meson assms cong-right-commutativity l5-5-1 l5-5-2)

lemma le-comm:
  assumes  $A \ B \ Le \ C \ D$ 
  shows  $B \ A \ Le \ D \ C$ 
  using assms le-left-comm le-right-comm by blast

lemma ge-left-comm:
  assumes  $A \ B \ Ge \ C \ D$ 
  shows  $B \ A \ Ge \ C \ D$ 
  by (meson Ge-def assms le-right-comm)

lemma ge-right-comm:
  assumes  $A \ B \ Ge \ C \ D$ 
  shows  $A \ B \ Ge \ D \ C$ 
  using Ge-def assms le-left-comm by presburger

lemma ge-comm0:
  assumes  $A \ B \ Ge \ C \ D$ 
  shows  $B \ A \ Ge \ D \ C$ 
  by (meson assms ge-left-comm ge-right-comm)

lemma lt-right-comm:

```

assumes $A B Lt C D$
shows $A B Lt D C$
using *Lt-def assms le-right-comm not-cong-1243* **by** *blast*

lemma *lt-left-comm*:
assumes $A B Lt C D$
shows $B A Lt C D$
using *Lt-def assms le-comm lt-right-comm not-cong-2143* **by** *blast*

lemma *lt-comm*:
assumes $A B Lt C D$
shows $B A Lt D C$
using *assms lt-left-comm lt-right-comm* **by** *blast*

lemma *gt-left-comm0*:
assumes $A B Gt C D$
shows $B A Gt C D$
by (*meson Gt-def assms lt-right-comm*)

lemma *gt-right-comm*:
assumes $A B Gt C D$
shows $A B Gt D C$
using *Gt-def assms lt-left-comm* **by** *presburger*

lemma *gt-comm*:
assumes $A B Gt C D$
shows $B A Gt D C$
by (*meson assms gt-left-comm0 gt-right-comm*)

lemma *cong2-lt--lt*:
assumes $A B Lt C D$ **and**
 $Cong A B A' B'$ **and**
 $Cong C D C' D'$
shows $A' B' Lt C' D'$
by (*meson Lt-def assms(1) assms(2) assms(3) l5-6 le-anti-symmetry not-cong-3412*)

lemma *fourth-point*:
assumes $A \neq B$ **and**
 $B \neq C$ **and**
 $Col A B P$ **and**
 $Bet A B C$
shows $Bet P A B \vee Bet A P B \vee Bet B P C \vee Bet B C P$
by (*metis Col-def l5-2 assms(3) assms(4) between-symmetry*)

lemma *third-point*:
assumes $Col A B P$
shows $Bet P A B \vee Bet A P B \vee Bet A B P$
using *Col-def assms between-symmetry* **by** *blast*

lemma *l5-12-a*:
assumes $Bet A B C$
shows $A B Le A C \wedge B C Le A C$
using *assms between-symmetry cong-left-commutativity cong-reflexivity l5-5-2 le-left-comm*
by *blast*

lemma *bet--le1213*:
assumes $Bet A B C$
shows $A B Le A C$
using *assms l5-12-a* **by** *blast*

lemma *bet--le2313*:
assumes $Bet A B C$
shows $B C Le A C$
by (*simp add: assms l5-12-a*)

lemma *bet--lt1213*:

```

assumes  $B \neq C$  and
   $Bet\ A\ B\ C$ 
shows  $A\ B\ Lt\ A\ C$ 
using  $Lt-def\ assms(1)\ assms(2)\ bet--le1213\ between-cong$  by  $blast$ 

lemma  $bet--lt2313$ :
assumes  $A \neq B$  and
   $Bet\ A\ B\ C$ 
shows  $B\ C\ Lt\ A\ C$ 
using  $Lt-def\ assms(1)\ assms(2)\ bet--le2313\ bet-cong-eq\ l5-1$  by  $blast$ 

lemma  $l5-12-b$ :
assumes  $Col\ A\ B\ C$  and
   $A\ B\ Le\ A\ C$  and
   $B\ C\ Le\ A\ C$ 
shows  $Bet\ A\ B\ C$ 
proof –
{
  assume  $Bet\ B\ C\ A$ 
  hence  $?thesis$ 
  using  $assms(2)\ between-cong\ between-symmetry\ l5-12-a\ le-anti-symmetry$  by  $blast$ 
}
moreover
{
  assume  $Bet\ C\ A\ B$ 
  hence  $?thesis$ 
  by  $(metis\ assms(3)\ bet-cong-eq\ between-trivial2\ l5-12-a\ le-anti-symmetry\ le-comm)$ 
}
ultimately show  $?thesis$ 
using  $Col-def\ assms(1)$  by  $blast$ 
qed

lemma  $bet-le-eq$ :
assumes  $Bet\ A\ B\ C$ 
and  $A\ C\ Le\ B\ C$ 
shows  $A = B$ 
by  $(meson\ assms(1)\ assms(2)\ bet--le2313\ bet-cong-eq\ l5-1\ le-anti-symmetry)$ 

lemma  $or-lt-cong-gt$ :
 $A\ B\ Lt\ C\ D \vee A\ B\ Gt\ C\ D \vee Cong\ A\ B\ C\ D$ 
by  $(meson\ Gt-def\ Lt-def\ cong-symmetry\ local.le-cases)$ 

lemma  $lt--le$ :
assumes  $A\ B\ Lt\ C\ D$ 
shows  $A\ B\ Le\ C\ D$ 
using  $Lt-def\ assms$  by  $blast$ 

lemma  $le1234-lt--lt$ :
assumes  $A\ B\ Le\ C\ D$  and
   $C\ D\ Lt\ E\ F$ 
shows  $A\ B\ Lt\ E\ F$ 
by  $(meson\ Lt-def\ assms(1)\ assms(2)\ cong--le3412\ le-anti-symmetry\ le-transitivity)$ 

lemma  $le3456-lt--lt$ :
assumes  $A\ B\ Lt\ C\ D$  and
   $C\ D\ Le\ E\ F$ 
shows  $A\ B\ Lt\ E\ F$ 
by  $(meson\ Lt-def\ assms(1)\ assms(2)\ cong2-lt--lt\ cong-reflexivity\ le1234-lt--lt)$ 

lemma  $lt-transitivity$ :
assumes  $A\ B\ Lt\ C\ D$  and
   $C\ D\ Lt\ E\ F$ 
shows  $A\ B\ Lt\ E\ F$ 
using  $Lt-def\ assms(1)\ assms(2)\ le1234-lt--lt$  by  $blast$ 

lemma  $not-and-lt$ :

```


$\neg (A \ B \ Lt \ C \ D \wedge C \ D \ Lt \ A \ B)$
by (*simp add: Lt-def le-anti-symmetry*)

lemma *nlt*:
 $\neg A \ B \ Lt \ A \ B$
using *not-and-lt* **by** *blast*

lemma *le--nlt*:
assumes $A \ B \ Le \ C \ D$
shows $\neg C \ D \ Lt \ A \ B$
using *assms le3456-lt--lt nlt* **by** *blast*

lemma *cong--nlt*:
assumes $Cong \ A \ B \ C \ D$
shows $\neg A \ B \ Lt \ C \ D$
by (*simp add: Lt-def assms*)

lemma *nlt--le*:
assumes $\neg A \ B \ Lt \ C \ D$
shows $C \ D \ Le \ A \ B$
using *Lt-def assms cong--le3412 local.le-cases* **by** *blast*

lemma *lt--nle*:
assumes $A \ B \ Lt \ C \ D$
shows $\neg C \ D \ Le \ A \ B$
using *assms le--nlt* **by** *blast*

lemma *nle--lt*:
assumes $\neg A \ B \ Le \ C \ D$
shows $C \ D \ Lt \ A \ B$
using *assms nlt--le* **by** *blast*

lemma *lt1123*:
assumes $B \neq C$
shows $A \ A \ Lt \ B \ C$
using *assms le-diff nle--lt* **by** *blast*

lemma *bet2-le2--le-R1*:
assumes $Bet \ a \ P \ b$ **and**
 $Bet \ A \ Q \ B$ **and**
 $P \ a \ Le \ Q \ A$ **and**
 $P \ b \ Le \ Q \ B$ **and**
 $B = Q$
shows $a \ b \ Le \ A \ B$
by (*metis assms(3) assms(4) assms(5) le-comm le-diff*)

lemma *bet2-le2--le-R2*:
assumes $Bet \ a \ Po \ b$ **and**
 $Bet \ A \ PO \ B$ **and**
 $Po \ a \ Le \ PO \ A$ **and**
 $Po \ b \ Le \ PO \ B$ **and**
 $A \neq PO$ **and**
 $B \neq PO$
shows $a \ b \ Le \ A \ B$

proof –
obtain b' **where** $Bet \ A \ PO \ b'$ **and** $Cong \ PO \ b' \ b \ Po$
using *segment-construction* **by** *blast*
obtain a' **where** $Bet \ B \ PO \ a'$ **and** $Cong \ PO \ a' \ a \ Po$
using *segment-construction* **by** *blast*
obtain a'' **where** $Bet \ PO \ a'' \ A$ **and** $Cong \ Po \ a \ PO \ a''$
using *Le-def assms(3)* **by** *blast*
have $Cong \ PO \ a'' \ a \ Po$
using *Cong-cases* $\langle Cong \ Po \ a \ PO \ a'' \rangle$ **by** *blast*
hence $a' = a''$
by (*meson Bet-perm* $\langle Bet \ B \ PO \ a' \rangle \langle Bet \ PO \ a'' \ A \rangle \langle Cong \ PO \ a' \ a \ Po \rangle$ *assms(2) assms(6)*
between-inner-transitivity construction-uniqueness)

```

have B a' Le B A
  by (metis ‹Bet B PO a'› ‹Bet PO a'' A› ‹a' = a''› assms(2) bet--le1213 bet-le-eq le-comm
      le-cases outer-transitivity-between2)
obtain b'' where Bet PO b'' B and Cong Po b PO b''
  using Le-def assms(4) by blast
hence b' = b''
  using assms(2) assms(5) between-inner-transitivity cong-right-commutativity
      construction-uniqueness not-cong-3412 by (meson ‹Bet A PO b'› ‹Cong PO b' b Po›)
hence a' b' Le a' B
  using Bet-cases bet--le1213 between-exchange2 ‹Bet B PO a'› ‹Bet PO b'' B› by blast
hence a' b' Le A B
  using le-comm le-transitivity by (meson ‹B a' Le B A›)
have Cong a' b' a b
proof -
  have Bet a' PO b'
    using Bet-cases ‹Bet A PO b'› ‹Bet PO a'' A› ‹a' = a''› between-exchange3 by blast
  moreover have Cong a' PO a Po
    using Cong-cases ‹Cong PO a' a Po› by auto
  moreover have Cong PO b' Po b
    using ‹Cong PO b' b Po› not-cong-1243 by blast
  ultimately show ?thesis
    using assms(1) l2-11-b by blast
qed
thus ?thesis
  using ‹a' b' Le A B› cong-reflexivity l5-6 by blast
qed

lemma bet2-le2--le:
  assumes Bet a P b and
    Bet A Q B and
    P a Le Q A and
    P b Le Q B
  shows a b Le A B
proof cases
  assume A = Q
  thus ?thesis
    using assms(3) assms(4) le-diff by force
next
  assume ¬ A = Q
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) bet2-le2--le-R1 bet2-le2--le-R2 by blast
qed

lemma Le-cases:
  assumes A B Le C D ∨ B A Le C D ∨ A B Le D C ∨ B A Le D C
  shows A B Le C D
  using assms le-left-comm le-right-comm by blast

lemma Lt-cases:
  assumes A B Lt C D ∨ B A Lt C D ∨ A B Lt D C ∨ B A Lt D C
  shows A B Lt C D
  using assms lt-comm lt-left-comm by blast

lemma bet-out:
  assumes B ≠ A and
    Bet A B C
  shows A Out B C
  using Out-def assms(1) assms(2) bet-neq12--neq by fastforce

lemma bet-out-1:
  assumes B ≠ A and
    Bet C B A
  shows A Out B C
  by (simp add: assms(1) assms(2) bet-out between-symmetry)

lemma out-dec:

```

shows $P \text{ Out } A \ B \vee \neg P \text{ Out } A \ B$
 by *simp*

lemma *out-diff1*:
 assumes $A \text{ Out } B \ C$
 shows $B \neq A$
 using *Out-def assms* by *auto*

lemma *out-diff2*:
 assumes $A \text{ Out } B \ C$
 shows $C \neq A$
 using *Out-def assms* by *auto*

lemma *out-distinct*:
 assumes $A \text{ Out } B \ C$
 shows $B \neq A \wedge C \neq A$
 using *assms out-diff1 out-diff2* by *auto*

lemma *out-col*:
 assumes $A \text{ Out } B \ C$
 shows $\text{Col } A \ B \ C$
 using *Col-def Out-def assms between-symmetry* by *auto*

lemma *l6-2*:
 assumes $A \neq P$ and
 $B \neq P$ and
 $C \neq P$ and
 $\text{Bet } A \ P \ C$
 shows $\text{Bet } B \ P \ C \longleftrightarrow P \text{ Out } A \ B$

proof –
 {
 assume $\text{Bet } B \ P \ C$
 hence $P \text{ Out } A \ B$
 using *Out-def assms(1) assms(2) assms(3) assms(4) between-symmetry l5-2* by *presburger*
 }
 moreover
 {
 assume $P \text{ Out } A \ B$
 hence $\text{Bet } B \ P \ C$
 by (*metis Bet-perm Out-def between-exchange3 outer-transitivity-between2 assms(4)*)
 }
 ultimately show *?thesis*
 by *auto*
 qed

lemma *bet-out--bet*:
 assumes $\text{Bet } A \ P \ C$ and
 $P \text{ Out } A \ B$
 shows $\text{Bet } B \ P \ C$
 by (*metis l6-2 assms(1) assms(2) not-bet-distincts out-diff1*)

lemma *l6-3-1*:
 assumes $P \text{ Out } A \ B$
 shows $A \neq P \wedge B \neq P \wedge (\exists C. (C \neq P \wedge \text{Bet } A \ P \ C \wedge \text{Bet } B \ P \ C))$
 using *assms bet-out--bet out-diff1 out-diff2 point-construction-different* by *fastforce*

lemma *l6-3-2*:
 assumes $A \neq P$ and
 $B \neq P$ and
 $\exists C. (C \neq P \wedge \text{Bet } A \ P \ C \wedge \text{Bet } B \ P \ C)$
 shows $P \text{ Out } A \ B$
 using *assms(1) assms(2) assms(3) l6-2* by *blast*

lemma *l6-4-1*:
 assumes $P \text{ Out } A \ B$ and
 $\text{Col } A \ P \ B$

shows $\neg \text{Bet } A \ P \ B$
using *Out-def assms(1) between-equality between-symmetry* **by** *fastforce*

lemma *l6-4-2*:
assumes *Col A P B*
and $\neg \text{Bet } A \ P \ B$
shows *P Out A B*
by (*metis Out-def assms(1) assms(2) bet-out col-permutation-1 third-point*)

lemma *out-trivial*:
assumes $A \neq P$
shows *P Out A A*
by (*simp add: assms bet-out-1 between-trivial2*)

lemma *l6-6*:
assumes *P Out A B*
shows *P Out B A*
using *Out-def assms* **by** *auto*

lemma *l6-7*:
assumes *P Out A B and*
 P Out B C
shows *P Out A C*
by (*metis assms(1) assms(2) l6-2 l6-6 out-diff2 point-construction-different*)

lemma *bet-out-out-bet*:
assumes *Bet A B C and*
 B Out A A' and
 B Out C C'
shows *Bet A' B C'*
by (*metis Out-def assms(1) assms(2) assms(3) bet-out--bet between-inner-transitivity*
 outer-transitivity-between)

lemma *out2-bet-out*:
assumes *B Out A C and*
 B Out X P and
 Bet A X C
shows *B Out A P $\wedge$ B Out C P*

proof –
have *Bet B A C $\vee$ Bet B C A*
using *Out-def assms(1)* **by** *auto*
 {
assume *Bet B A C*
have *B Out A P $\wedge$ B Out C P*
 by (*metis Out-def between-inner-transitivity l6-7 $\langle$ Bet B A C $\rangle$ assms(1) assms(2) assms(3)*)
 }
moreover
 {
assume *Bet B C A*
hence *B Out A P*
 by (*metis Bet-perm bet3--bet bet-out-1 l5-1 l6-6 l6-7 out-distinct assms(2) assms(3)*)
 }
moreover
 {
assume *Bet B C A*
hence *B Out C P*
 using $\langle \text{Bet } B \ C \ A \rangle$ *assms(1) calculation(2) l6-6 l6-7* **by** *blast*
 }
ultimately show *?thesis*
using $\langle \text{Bet } B \ A \ C \vee \text{Bet } B \ C \ A \rangle$ **by** *blast*

qed

lemma *l6-11-uniqueness*:
assumes *A Out X R and*
 Cong A X B C and
 A Out Y R and

```

    Cong A Y B C
shows X = Y
by (metis Out-def assms(1) assms(2) assms(3) assms(4) between-cong cong-symmetry
    cong-transitivity l6-6 l6-7)

lemma l6-11-existence:
  assumes R ≠ A and
    B ≠ C
shows ∃ X. (A Out X R ∧ Cong A X B C)
by (metis Out-def assms(1) assms(2) cong-reverse-identity segment-construction-2)

lemma segment-construction-3:
  assumes A ≠ B and
    X ≠ Y
shows ∃ C. (A Out B C ∧ Cong A C X Y)
by (metis assms(1) assms(2) l6-11-existence l6-6)

lemma l6-13-1:
  assumes P Out A B and
    P A Le P B
shows Bet P A B
by (metis Out-def assms(1) assms(2) bet--lt1213 le--nlt)

lemma l6-13-2:
  assumes P Out A B and
    Bet P A B
shows P A Le P B
by (simp add: assms(2) bet--le1213)

lemma l6-16-1:
  assumes P ≠ Q and
    Col S P Q and
    Col X P Q
shows Col X P S
proof cases
  assume S = P
  thus ?thesis
    using col-trivial-2 by blast
next
  assume S ≠ P
  {
    assume Bet S P Q and Bet X P Q
    hence Col X P S
    by (metis Col-cases assms(1) col-trivial-1 l6-2 out-col)
  }
  moreover
  {
    assume Bet P Q S and Bet X P Q
    hence Col X P S
    using assms(1) bet-col outer-transitivity-between by blast
  }
  moreover
  {
    assume Bet Q S P and Bet X P Q
    hence Col X P S
    using bet-col between-inner-transitivity between-symmetry by blast
  }
  moreover
  {
    assume Bet P Q S and Bet P Q X
    hence Col X P S
    by (meson Col-def assms(1) l5-1 not-col-permutation-5)
  }
  moreover
  {

```

```

    assume Bet Q S P and Bet P Q X
    hence Col X P S
      using Col-def between-exchange4 between-symmetry by blast
  }
  moreover
  {
    assume Bet S P Q and Bet P Q X
    hence Col X P S
      using Bet-cases assms(1) bet-col outer-transitivity-between by blast
  }
  moreover
  {
    assume Bet P Q S and Bet Q X P
    hence Col X P S
      using Bet-cases Col-def between-exchange2 by blast
  }
  moreover
  {
    assume Bet Q S P and Bet Q X P
    hence Col X P S
      by (meson Col-def between-exchange3 l5-3 not-col-permutation-3)
  }
  moreover
  {
    assume Bet S P Q and Bet Q X P
    hence Col X P S
      using bet-col between-exchange3 between-symmetry by blast
  }
  ultimately show ?thesis
    by (metis Col-def assms(2) assms(3))
qed

```

```

lemma col-transitivity-1:
  assumes P ≠ Q and
    Col P Q A and
    Col P Q B
  shows Col P A B
  by (meson l6-16-1 assms(1) assms(2) assms(3) not-col-permutation-2)

```

```

lemma col-transitivity-2:
  assumes P ≠ Q and
    Col P Q A and
    Col P Q B
  shows Col Q A B
  by (metis col-transitivity-1 assms(1) assms(2) assms(3) not-col-permutation-4)

```

```

lemma l6-21:
  assumes  $\neg$  Col A B C and
    C ≠ D and
    Col A B P and
    Col A B Q and
    Col C D P and
    Col C D Q
  shows P = Q
  by (metis assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) col-transitivity-1 l6-16-1 not-col-distincts)

```

```

lemma col2--eq:
  assumes Col A X Y and
    Col B X Y and
     $\neg$  Col A X B
  shows X = Y
  using assms(1) assms(2) assms(3) l6-16-1 by blast

```

```

lemma not-col-exists:
  assumes A ≠ B

```

shows $\exists C. \neg \text{Col } A \ B \ C$
by (*metis Col-def assms col-transitivity-2 lower-dim-ex*)

lemma *col3*:
assumes $X \neq Y$ **and**
 $\text{Col } X \ Y \ A$ **and**
 $\text{Col } X \ Y \ B$ **and**
 $\text{Col } X \ Y \ C$
shows $\text{Col } A \ B \ C$
by (*metis assms(1) assms(2) assms(3) assms(4) col-transitivity-2*)

lemma *colx*:
assumes $A \neq B$ **and**
 $\text{Col } X \ Y \ A$ **and**
 $\text{Col } X \ Y \ B$ **and**
 $\text{Col } A \ B \ C$
shows $\text{Col } X \ Y \ C$
by (*metis assms(1) assms(2) assms(3) assms(4) l6-21 not-col-distincts*)

lemma *out2--bet*:
assumes $A \ \text{Out } B \ C$ **and**
 $C \ \text{Out } A \ B$
shows $\text{Bet } A \ B \ C$
by (*metis Out-def assms(1) assms(2) between-equality between-symmetry*)

lemma *bet2-le2--le1346*:
assumes $\text{Bet } A \ B \ C$ **and**
 $\text{Bet } A' \ B' \ C'$ **and**
 $A \ B \ \text{Le } A' \ B'$ **and**
 $B \ C \ \text{Le } B' \ C'$
shows $A \ C \ \text{Le } A' \ C'$
using *Le-cases assms(1) assms(2) assms(3) assms(4) bet2-le2--le* **by** *blast*

lemma *bet2-le2--le2356-R1*:
assumes $\text{Bet } A \ A \ C$ **and**
 $\text{Bet } A' \ B' \ C'$ **and**
 $A \ A \ \text{Le } A' \ B'$ **and**
 $A' \ C' \ \text{Le } A \ C$
shows $B' \ C' \ \text{Le } A \ C$
using *assms(2) assms(4) bet--le2313 le3456-lt--lt lt--nle nlt--le* **by** *blast*

lemma *bet2-le2--le2356-R2*:
assumes $A \neq B$ **and**
 $\text{Bet } A \ B \ C$ **and**
 $\text{Bet } A' \ B' \ C'$ **and**
 $A \ B \ \text{Le } A' \ B'$ **and**
 $A' \ C' \ \text{Le } A \ C$
shows $B' \ C' \ \text{Le } B \ C$
proof –
obtain $B0$ **where** $\text{Bet } A \ B \ B0$ **and** $\text{Cong } A \ B0 \ A' \ B'$
using *assms(4) l5-5-1* **by** *blast*
hence $A \neq B0$
using *assms(1) bet-neq12--neq* **by** *blast*
obtain $C0$ **where** $\text{Bet } A \ C0 \ C$ **and** $\text{Cong } A' \ C' \ A \ C0$
using *Le-def assms(5)* **by** *blast*
hence $A \neq C0$
using *assms(1) assms(3) assms(4) bet-neq12--neq cong-diff le-diff* **by** *blast*
hence $\text{Bet } A \ B0 \ C0$
by (*metis Bet-cases Cong-cases $\langle A \neq B0 \rangle \langle \text{Bet } A \ B \ B0 \rangle \langle \text{Bet } A \ C0 \ C \rangle \langle \text{Cong } A \ B0 \ A' \ B' \rangle$
 $\langle \text{Cong } A' \ C' \ A \ C0 \rangle$ *assms(1) assms(2) assms(3) bet--le1213 between-inner-transitivity*
cong--nlt l6-13-1 l6-2 le-transitivity nlt--le outer-transitivity-between
point-construction-different)
have $B0 \ C0 \ \text{Le } B \ C0$
using $\langle \text{Bet } A \ B \ B0 \rangle \langle \text{Bet } A \ B0 \ C0 \rangle$ *bet--le2313 between-exchange3* **by** *blast*
moreover have $B \ C0 \ \text{Le } B \ C$
by (*meson $\langle \text{Bet } A \ B \ B0 \rangle \langle \text{Bet } A \ B0 \ C0 \rangle \langle \text{Bet } A \ C0 \ C \rangle$ bet--le1213 between-exchange3*)*

between-exchange4)
moreover have *Cong B0 C0 B' C'*
using $\langle \text{Bet } A \ B0 \ C0 \rangle \langle \text{Cong } A \ B0 \ A' \ B' \rangle \langle \text{Cong } A' \ C' \ A \ C0 \rangle$ *assms(3) l4-3-1 not-cong-3412* **by blast**
ultimately show *?thesis*
using *cong--le3412 le-transitivity* **by blast**
qed

lemma *bet2-le2--le2356*:
assumes *Bet A B C and*
Bet A' B' C' and
A B Le A' B' and
A' C' Le A C
shows *B' C' Le B C*
proof (*cases*)
assume *A = B*
thus *?thesis*
using *assms(1) assms(2) assms(3) assms(4) bet2-le2--le2356-R1* **by blast**
next
assume $\neg A = B$
thus *?thesis*
using *assms(1) assms(2) assms(3) assms(4) bet2-le2--le2356-R2* **by blast**
qed

lemma *bet2-le2--le1245*:
assumes *Bet A B C and*
Bet A' B' C' and
B C Le B' C' and
A' C' Le A C
shows *A' B' Le A B*
using *assms(1) assms(2) assms(3) assms(4) bet2-le2--le2356 between-symmetry le-comm* **by blast**

lemma *cong-preserves-bet*:
assumes *Bet B A' A0 and*
Cong B A' E D' and
Cong B A0 E D0 and
E Out D' D0
shows *Bet E D' D0*
using *l6-13-1 Tarski-neutral-dimensionless-axioms assms(1) assms(2) assms(3) assms(4)*
bet--le1213 l5-6 **by fastforce**

lemma *out-cong-cong*:
assumes *B Out A A0 and*
E Out D D0 and
Cong B A E D and
Cong B A0 E D0
shows *Cong A A0 D D0*
by (*meson Out-def assms(1) assms(2) assms(3) assms(4) cong-4321 cong-symmetry l4-3-1 l5-6*
l6-13-1 l6-13-2)

lemma *not-out-bet*:
assumes *Col A B C and*
 $\neg B \text{ Out } A \ C$
shows *Bet A B C*
using *assms(1) assms(2) l6-4-2* **by blast**

lemma *or-bet-out*:
shows *Bet A B C $\vee$ B Out A C $\vee$ $\neg$ Col A B C*
using *not-out-bet* **by blast**

lemma *not-bet-out*:
assumes *Col A B C and*
 $\neg \text{Bet } A \ B \ C$
shows *B Out A C*
by (*simp add: assms(1) assms(2) l6-4-2*)

lemma *not-bet-and-out*:

shows $\neg (Bet\ A\ B\ C \wedge B\ Out\ A\ C)$
using *bet-col l6-4-1* **by** *blast*

lemma *out-to-bet*:
assumes *Col A' B' C' and*
B Out A C $\longleftrightarrow$ B' Out A' C' and
Bet A B C
shows *Bet A' B' C'*
using *assms(1) assms(2) assms(3) not-bet-and-out or-bet-out* **by** *blast*

lemma *col-out2-col*:
assumes *Col A B C and*
B Out A AA and
B Out C CC
shows *Col AA B CC*
by (*metis Col-cases assms(1) assms(2) assms(3) col-transitivity-1 out-col out-diff1*)

lemma *bet2-out-out*:
assumes *B $\neq$ A and*
B' $\neq$ A and
A Out C C' and
Bet A B C and
Bet A B' C'
shows *A Out B B'*
by (*meson assms(1) assms(2) assms(3) assms(4) assms(5) bet-out l6-6 l6-7*)

lemma *bet2--out*:
assumes *A $\neq$ B and*
A $\neq$ B' and
Bet A B C
and *Bet A B' C*
shows *A Out B B'*
using *Out-def assms(1) assms(2) assms(3) assms(4) l5-3* **by** *auto*

lemma *out-bet-out-1*:
assumes *P Out A C and*
Bet A B C
shows *P Out A B*
by (*metis assms(1) assms(2) not-bet-and-out out2-bet-out out-trivial*)

lemma *out-bet-out-2*:
assumes *P Out A C and*
Bet A B C
shows *P Out B C*
using *assms(1) assms(2) l6-6 l6-7 out-bet-out-1* **by** *blast*

lemma *out-bet--out*:
assumes *Bet P Q A and*
Q Out A B
shows *P Out A B*
by (*smt (verit, best) Out-def assms(1,2) bet3--bet l5-1 not-bet-and-out*
outer-transitivity-between2)

lemma *segment-reverse*:
assumes *Bet A B C*
shows $\exists B'. Bet\ A\ B'\ C \wedge Cong\ C\ B'\ A\ B$
by (*metis Bet-perm Cong-perm assms bet-cong-eq cong-reflexivity segment-construction-2*)

lemma *diff-col-ex*:
shows $\exists C. A \neq C \wedge B \neq C \wedge Col\ A\ B\ C$
by (*metis bet-col bet-neq12--neq point-construction-different*)

lemma *diff-bet-ex3*:
assumes *Bet A B C*
shows $\exists D. A \neq D \wedge B \neq D \wedge C \neq D \wedge Col\ A\ B\ D$
by (*metis Col-def bet-out-1 between-trivial2 col-transitivity-1 l6-4-1*)

point-construction-different)

lemma *diff-col-ex3*:

assumes $Col\ A\ B\ C$

shows $\exists\ D.\ A \neq D \wedge B \neq D \wedge C \neq D \wedge Col\ A\ B\ D$

by (*metis Bet-perm Col-def between-equality between-trivial2 point-construction-different*)

lemma *Out-cases*:

assumes $A\ Out\ B\ C \vee A\ Out\ C\ B$

shows $A\ Out\ B\ C$

using *assms l6-6* **by** *blast*

lemma *ex-sums*:

shows $\exists\ E\ F.\ A\ B\ C\ D\ SumS\ E\ F$

proof –

obtain R **where** $Bet\ A\ B\ R$ **and** $Cong\ B\ R\ C\ D$

using *segment-construction* **by** *blast*

thus *?thesis*

using *SumS-def cong-reflexivity* **by** *blast*

qed

lemma *sums-sym*:

assumes $A\ B\ C\ D\ SumS\ E\ F$

shows $C\ D\ A\ B\ SumS\ E\ F$

proof –

obtain $P\ Q\ R$ **where** $Bet\ P\ Q\ R$ **and** $Cong\ P\ Q\ A\ B$ **and** $Cong\ Q\ R\ C\ D$ **and** $Cong\ P\ R\ E\ F$

using *SumS-def assms* **by** *auto*

thus *?thesis*

by (*meson Cong-cases SumS-def between-symmetry*)

qed

lemma *sums2--cong56*:

assumes $A\ B\ C\ D\ SumS\ E\ F$ **and**

$A\ B\ C\ D\ SumS\ E'\ F'$

shows $Cong\ E\ F\ E'\ F'$

proof –

obtain $P\ Q\ R$ **where** $Bet\ P\ Q\ R$ **and** $Cong\ P\ Q\ A\ B$ **and** $Cong\ Q\ R\ C\ D$ **and** $Cong\ P\ R\ E\ F$

using *SumS-def assms(1)* **by** *blast*

obtain $P'\ Q'\ R'$ **where** $Bet\ P'\ Q'\ R'$ **and** $Cong\ P'\ Q'\ A\ B$ **and** $Cong\ Q'\ R'\ C\ D$ **and** $Cong\ P'\ R'\ E'\ F'$

using *SumS-def assms(2)* **by** *blast*

have $Cong\ P\ Q\ P'\ Q'$

using *Cong-cases* $\langle Cong\ P\ Q\ A\ B \rangle \langle Cong\ P'\ Q'\ A\ B \rangle$ *cong-transitivity* **by** *blast*

have $Cong\ Q\ R\ Q'\ R'$

using *Cong-cases* $\langle Cong\ Q\ R\ C\ D \rangle \langle Cong\ Q'\ R'\ C\ D \rangle$ *cong-transitivity* **by** *blast*

hence $Cong\ P\ R\ P'\ R'$

using $\langle Bet\ P\ Q\ R \rangle \langle Bet\ P'\ Q'\ R' \rangle \langle Cong\ P\ Q\ P'\ Q' \rangle$ *l2-11-b* **by** *blast*

hence $Cong\ P\ R\ E'\ F'$

using $\langle Cong\ P'\ R'\ E'\ F' \rangle$ *cong-transitivity* **by** *blast*

thus *?thesis*

using $\langle Cong\ P\ R\ E\ F \rangle$ *cong-inner-transitivity* **by** *blast*

qed

lemma *sums2--cong12*:

assumes $A\ B\ C\ D\ SumS\ E\ F$

and $A'\ B'\ C'\ D'\ SumS\ E'\ F'$

shows $Cong\ A\ B\ A'\ B'$

proof –

obtain $P\ Q\ R$ **where** $Bet\ P\ Q\ R$ **and** $Cong\ P\ Q\ A\ B$ **and** $Cong\ Q\ R\ C\ D$ **and** $Cong\ P\ R\ E\ F$

using *SumS-def assms(1)* **by** *blast*

obtain $P' Q' R'$ where $Bet P' Q' R'$ and $Cong P' Q' A' B'$ and $Cong Q' R' C D$ and $Cong P' R' E F$
 using $SumS-def\ assms(2)$ by blast
 have $Cong P R P' R'$
 using $Cong-cases \langle Cong P R E F \rangle \langle Cong P' R' E F \rangle$ *cong-transitivity* by blast
 moreover have $Cong Q R Q' R'$
 using $\langle Cong Q R C D \rangle \langle Cong Q' R' C D \rangle$ *cong-inner-transitivity cong-symmetry* by blast
 ultimately have $Cong P Q P' Q'$
 using $\langle Bet P Q R \rangle \langle Bet P' Q' R' \rangle$ $l4-3$ by blast
 hence $Cong P Q A' B'$
 using $\langle Cong P' Q' A' B' \rangle$ *cong-transitivity* by blast
 thus ?thesis
 using $\langle Cong P Q A B \rangle$ *cong-inner-transitivity* by blast
 qed

lemma *sums2--cong34*:
 assumes $A B C D SumS E F$ and
 $A B C' D' SumS E F$
 shows $Cong C D C' D'$
 using $assms(1)\ assms(2)\ sums2--cong12\ sums-sym$ by blast

lemma *cong3-sums--sums*:
 assumes $Cong A B A' B'$ and
 $Cong C D C' D'$ and
 $Cong E F E' F'$ and
 $A B C D SumS E F$
 shows $A' B' C' D' SumS E' F'$
 by (*meson SumS-def assms(1) assms(2) assms(3) assms(4) cong-inner-transitivity cong-symmetry*)

lemma *sums123312*:
 shows $A B C C SumS A B$
 using $SumS-def\ between-trivial\ cong-reflexivity\ cong-trivial-identity$ by blast

lemma *sums--cong1245*:
 assumes $A B C C SumS D E$
 shows $Cong A B D E$
 using $assms\ sums123312\ sums2--cong56$ by blast

lemma *sums--eq34*:
 assumes $A B C D SumS A B$
 shows $C = D$
 using $assms\ cong-reverse-identity\ sums123312\ sums2--cong34$ by blast

lemma *sums112323*:
 shows $A A B C SumS B C$
 by (*simp add: sums123312 sums-sym*)

lemma *sums--cong2345*:
 assumes $A A B C SumS D E$
 shows $Cong B C D E$
 using $assms\ sums112323\ sums2--cong56$ by blast

lemma *sums--eq12*:
 assumes $A B C D SumS C D$
 shows $A = B$
 using $assms\ sums--eq34\ sums-sym$ by blast

lemma *sums-left-comm*:
 assumes $A B C D SumS E F$
 shows $B A C D SumS E F$

```

using assms cong3-sums--sums cong-pseudo-reflexivity cong-reflexivity by blast

lemma sums-middle-comm:
  assumes A B C D SumS E F
  shows A B D C SumS E F
  using assms sums-left-comm sums-sym by blast

lemma sums-right-comm:
  assumes A B C D SumS E F
  shows A B C D SumS F E
  using assms cong3-sums--sums cong-pseudo-reflexivity cong-reflexivity by blast

lemma sums-comm:
  assumes A B C D SumS E F
  shows B A D C SumS F E
  using assms cong3-sums--sums cong-pseudo-reflexivity by blast

lemma bet--sums:
  assumes Bet A B C
  shows A B B C SumS A C
  using SumS-def assms cong-reflexivity by blast

lemma sums-assoc-1:
  assumes A B C D SumS G H and
    C D E F SumS I J and
    G H E F SumS K L
  shows A B I J SumS K L
proof –
  obtain P Q R where Bet P Q R and Cong P Q A B and
    Cong Q R C D and Cong P R G H
    using assms(1) SumS-def by fastforce
  obtain S where Bet P R S and Cong R S E F
    using segment-construction by blast
  hence Bet P Q S
    using  $\langle \text{Bet } P \ Q \ R \rangle$  between-exchange4 by blast
  moreover have Cong Q S I J
    using SumS-def  $\langle \text{Bet } P \ Q \ R \rangle$   $\langle \text{Bet } P \ R \ S \rangle$   $\langle \text{Cong } Q \ R \ C \ D \rangle$   $\langle \text{Cong } R \ S \ E \ F \rangle$  assms(2) between-exchange3
cong-reflexivity sums2--cong56 by blast
  moreover have Cong P S K L
    using SumS-def  $\langle \text{Bet } P \ R \ S \rangle$   $\langle \text{Cong } P \ R \ G \ H \rangle$   $\langle \text{Cong } R \ S \ E \ F \rangle$  assms(3) cong-reflexivity sums2--cong56 by blast
  ultimately show ?thesis
    using SumS-def  $\langle \text{Cong } P \ Q \ A \ B \rangle$  by blast
qed

lemma sums-assoc-2:
  assumes A B C D SumS G H and
    C D E F SumS I J and
    A B I J SumS K L
  shows G H E F SumS K L
proof –
  have E F G H SumS K L
  proof –
  have E F C D SumS I J
    by (simp add: assms(2) sums-sym)
  moreover have C D A B SumS G H
    by (simp add: assms(1) sums-sym)
  moreover have I J A B SumS K L
    using assms(3) sums-sym by blast
  ultimately show ?thesis
    using sums-assoc-1 by blast
qed
thus ?thesis
  using sums-sym by blast
qed

```

lemma *sums-assoc*:
assumes $A\ B\ C\ D\ \text{SumS}\ G\ H$ **and**
 $C\ D\ E\ F\ \text{SumS}\ I\ J$
shows $G\ H\ E\ F\ \text{SumS}\ K\ L \longleftrightarrow A\ B\ I\ J\ \text{SumS}\ K\ L$
by (*meson* *assms*(1) *assms*(2) *sums-assoc-1* *sums-assoc-2*)

lemma *sums--le1256*:
assumes $A\ B\ C\ D\ \text{SumS}\ E\ F$
shows $A\ B\ Le\ E\ F$
proof –
obtain $P\ Q\ R$ **where** $Bet\ P\ Q\ R$ **and** $Cong\ P\ Q\ A\ B$ **and**
 $Cong\ Q\ R\ C\ D$ **and** $Cong\ P\ R\ E\ F$
using *SumS-def* *assms* **by** *blast*
thus *?thesis*
using *bet--le1213* *l5-6* **by** *blast*
qed

lemma *sums--le3456*:
assumes $A\ B\ C\ D\ \text{SumS}\ E\ F$
shows $C\ D\ Le\ E\ F$
using *assms* *sums--le1256* *sums-sym* **by** *blast*

lemma *eq-sums--eq*:
assumes $A\ B\ C\ D\ \text{SumS}\ E\ E$
shows $A = B \wedge C = D$
by (*metis* *assms* *cong-identity* *le-diff* *sums--cong1245* *sums--le3456*)

lemma *sums-diff-1*:
assumes $A \neq B$ **and**
 $A\ B\ C\ D\ \text{SumS}\ E\ F$
shows $E \neq F$
using *assms*(1) *assms*(2) *eq-sums--eq* **by** *force*

lemma *sums-diff-2*:
assumes $C \neq D$ **and**
 $A\ B\ C\ D\ \text{SumS}\ E\ F$
shows $E \neq F$
using *assms*(1) *assms*(2) *eq-sums--eq* **by** *blast*

lemma *le2-sums2--le*:
assumes $A\ B\ Le\ A'\ B'$ **and**
 $C\ D\ Le\ C'\ D'$ **and**
 $A\ B\ C\ D\ \text{SumS}\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ \text{SumS}\ E'\ F'$
shows $E\ F\ Le\ E'\ F'$
proof –
obtain $P\ Q\ R$ **where** $Bet\ P\ Q\ R$ **and** $Cong\ P\ Q\ A\ B$ **and**
 $Cong\ Q\ R\ C\ D$ **and** $Cong\ P\ R\ E\ F$
using *SumS-def* *assms*(3) **by** *blast*
obtain $P'\ Q'\ R'$ **where** $Bet\ P'\ Q'\ R'$ **and** $Cong\ P'\ Q'\ A'\ B'$ **and**
 $Cong\ Q'\ R'\ C'\ D'$ **and** $Cong\ P'\ R'\ E'\ F'$
using *SumS-def* *assms*(4) **by** *blast*
have $P\ Q\ Le\ P'\ Q'$
by (*meson* $\langle Cong\ P\ Q\ A\ B \rangle \langle Cong\ P'\ Q'\ A'\ B' \rangle$ *assms*(1) *cong-symmetry* *l5-6*)
moreover **have** $Q\ R\ Le\ Q'\ R'$

by (meson Cong-perm $\langle \text{Cong } Q \ R \ C \ D \rangle \langle \text{Cong } Q' \ R' \ C' \ D' \rangle$ *assms(2) l5-6*)
 ultimately have $P \ R \ Le \ P' \ R'$
 using $\langle \text{Bet } P \ Q \ R \rangle \langle \text{Bet } P' \ Q' \ R' \rangle$ *bet2-le2--le1346* by blast
 thus ?thesis
 using $\langle \text{Cong } P \ R \ E \ F \rangle \langle \text{Cong } P' \ R' \ E' \ F' \rangle$ *l5-6* by blast
 qed

lemma *le2-sums2--cong12*:
 assumes $A \ B \ Le \ A' \ B'$ and
 $C \ D \ Le \ C' \ D'$ and
 $A \ B \ C \ D \ SumS \ E \ F$ and
 $A' \ B' \ C' \ D' \ SumS \ E \ F$
 shows $\text{Cong } A \ B \ A' \ B'$
proof –
 obtain $E' \ F'$ where $A' \ B' \ C \ D \ SumS \ E' \ F'$
 using *ex-sums* by blast
 hence $\text{Cong } E' \ F' \ E \ F$
 by (meson *assms(1) assms(2) assms(3) assms(4) le2-sums2--le le-anti-symmetry le-reflexivity*)
 hence $A' \ B' \ C \ D \ SumS \ E \ F$
 using $\langle A' \ B' \ C \ D \ SumS \ E' \ F' \rangle$ *cong3-sums--sums cong-reflexivity* by blast
 thus ?thesis
 using *assms(3) sums2--cong12* by blast
 qed

lemma *le2-sums2--cong34*:
 assumes $A \ B \ Le \ A' \ B'$ and
 $C \ D \ Le \ C' \ D'$ and
 $A \ B \ C \ D \ SumS \ E \ F$ and
 $A' \ B' \ C' \ D' \ SumS \ E \ F$
 shows $\text{Cong } C \ D \ C' \ D'$
 by (meson *assms(1) assms(2) assms(3) assms(4) cong3-sums--sums cong-reflexivity*
 le2-sums2--cong12 sums2--cong34)

lemma *le-lt12-sums2--lt*:
 assumes $A \ B \ Lt \ A' \ B'$ and
 $C \ D \ Le \ C' \ D'$ and
 $A \ B \ C \ D \ SumS \ E \ F$ and
 $A' \ B' \ C' \ D' \ SumS \ E' \ F'$
 shows $E \ F \ Lt \ E' \ F'$
proof –
 have $E \ F \ Le \ E' \ F'$
 using *Lt-def assms(1) assms(2) assms(3) assms(4) le2-sums2--le* by blast
 moreover {
 assume $\text{Cong } E \ F \ E' \ F'$
 hence $A' \ B' \ C' \ D' \ SumS \ E \ F$
 by (meson *assms(4) cong3-sums--sums cong-reflexivity not-cong-3412*)
 hence $\text{Cong } A \ B \ A' \ B'$
 using *assms(1) assms(2) assms(3) le2-sums2--cong12 lt--le* by blast
 hence *False*
 using *assms(1) cong--nlt lt--le* by blast
 }
 ultimately show ?thesis
 using *Lt-def* by blast
 qed

lemma *le-lt34-sums2--lt*:
 assumes $A \ B \ Le \ A' \ B'$ and
 $C \ D \ Lt \ C' \ D'$ and
 $A \ B \ C \ D \ SumS \ E \ F$ and
 $A' \ B' \ C' \ D' \ SumS \ E' \ F'$
 shows $E \ F \ Lt \ E' \ F'$
 using *assms(1) assms(2) assms(3) assms(4) le-lt12-sums2--lt sums-sym* by blast

lemma *lt2-sums2--lt*:
assumes $A\ B\ Lt\ A'\ B'$ **and**
 $C\ D\ Lt\ C'\ D'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $E\ F\ Lt\ E'\ F'$
using *assms(1) assms(2) assms(3) assms(4) le-lt12-sums2--lt lt--le* **by** *blast*

lemma *le2-sums2--le12*:
assumes $C'\ D'\ Le\ C\ D$ **and**
 $E\ F\ Le\ E'\ F'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $A\ B\ Le\ A'\ B'$
using *assms(1) assms(2) assms(3) assms(4) le-lt12-sums2--lt lt--nle nlt--le* **by** *blast*

lemma *le2-sums2--le34*:
assumes $A'\ B'\ Le\ A\ B$ **and**
 $E\ F\ Le\ E'\ F'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $C\ D\ Le\ C'\ D'$
by (*meson assms(1) assms(2) assms(3) assms(4) le-lt12-sums2--lt lt--nle nlt--le sums-sym*)

lemma *le-lt34-sums2--lt12*:
assumes $C'\ D'\ Lt\ C\ D$ **and**
 $E\ F\ Le\ E'\ F'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $A\ B\ Lt\ A'\ B'$
by (*meson assms(1) assms(2) assms(3) assms(4) le3456-lt--lt le-lt34-sums2--lt nle--lt nlt*)

lemma *le-lt12-sums2--lt34*:
assumes $A'\ B'\ Lt\ A\ B$ **and**
 $E\ F\ Le\ E'\ F'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $C\ D\ Lt\ C'\ D'$
using *assms(1) assms(2) assms(3) assms(4) le-lt12-sums2--lt lt--nle nlt--le* **by** *blast*

lemma *le-lt56-sums2--lt12*:
assumes $C'\ D'\ Le\ C\ D$ **and**
 $E\ F\ Lt\ E'\ F'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $A\ B\ Lt\ A'\ B'$
by (*meson assms(1) assms(2) assms(3) assms(4) le2-sums2--le le--nlt nle--lt*)

lemma *le-lt56-sums2--lt34*:
assumes $A'\ B'\ Le\ A\ B$ **and**
 $E\ F\ Lt\ E'\ F'$ **and**
 $A\ B\ C\ D\ SumS\ E\ F$ **and**
 $A'\ B'\ C'\ D'\ SumS\ E'\ F'$
shows $C\ D\ Lt\ C'\ D'$
using *assms(1) assms(2) assms(3) assms(4) le2-sums2--le lt--nle nlt--le* **by** *blast*

lemma *lt2-sums2--lt12*:
assumes $C'\ D'\ Lt\ C\ D$ **and**
 $E\ F\ Lt\ E'\ F'$ **and**

$A \ B \ C \ D \ \text{SumS} \ E \ F$ **and**
 $A' \ B' \ C' \ D' \ \text{SumS} \ E' \ F'$
shows $A \ B \ \text{Lt} \ A' \ B'$
using *assms(1) assms(2) assms(3) assms(4) le-lt34-sums2--lt nle--lt not-and-lt* **by** *blast*

lemma *lt2-sums2--lt34*:
assumes $A' \ B' \ \text{Lt} \ A \ B$ **and**
 $E \ F \ \text{Lt} \ E' \ F'$ **and**
 $A \ B \ C \ D \ \text{SumS} \ E \ F$ **and**
 $A' \ B' \ C' \ D' \ \text{SumS} \ E' \ F'$
shows $C \ D \ \text{Lt} \ C' \ D'$
by (*meson assms(1) assms(2) assms(3) assms(4) le-lt12-sums2--lt nlt--le not-and-lt*)

lemma *midpoint-dec*:
 $I \ \text{Midpoint} \ A \ B \vee \neg I \ \text{Midpoint} \ A \ B$
by *simp*

lemma *is-midpoint-id*:
assumes $A \ \text{Midpoint} \ A \ B$
shows $A = B$
using *Midpoint-def assms between-cong* **by** *blast*

lemma *is-midpoint-id-2*:
assumes $A \ \text{Midpoint} \ B \ A$
shows $A = B$
using *Midpoint-def assms cong-diff-2* **by** *blast*

lemma *l7-2*:
assumes $M \ \text{Midpoint} \ A \ B$
shows $M \ \text{Midpoint} \ B \ A$
using *Bet-perm Cong-perm Midpoint-def assms* **by** *blast*

lemma *l7-3*:
assumes $M \ \text{Midpoint} \ A \ A$
shows $M = A$
using *Midpoint-def assms bet-neq23--neq* **by** *blast*

lemma *l7-3-2*:
 $A \ \text{Midpoint} \ A \ A$
by (*simp add: Midpoint-def between-trivial2 cong-reflexivity*)

lemma *symmetric-point-construction*:
 $\exists P'. A \ \text{Midpoint} \ P \ P'$
by (*meson Midpoint-def cong--le cong--le3412 le-anti-symmetry segment-construction*)

lemma *symmetric-point-uniqueness*:
assumes $P \ \text{Midpoint} \ A \ P1$ **and**
 $P \ \text{Midpoint} \ A \ P2$
shows $P1 = P2$
by (*metis Midpoint-def assms(1) assms(2) between-cong-3 cong-diff-4 cong-inner-transitivity*)

lemma *l7-9*:
assumes $A \ \text{Midpoint} \ P \ X$ **and**
 $A \ \text{Midpoint} \ Q \ X$
shows $P = Q$
using *assms(1) assms(2) l7-2 symmetric-point-uniqueness* **by** *blast*

lemma *l7-9-bis*:
assumes $A \ \text{Midpoint} \ P \ X$ **and**
 $A \ \text{Midpoint} \ X \ Q$
shows $P = Q$
using *assms(1) assms(2) l7-2 symmetric-point-uniqueness* **by** *blast*

lemma *l7-13-R1*:
assumes $A \neq P$ **and**
 $A \ \text{Midpoint} \ P' \ P$ **and**

A Midpoint $Q' Q$
shows Cong $P Q P' Q'$
proof –
obtain X **where** $Bet P' P X$ **and** $Cong P X Q A$
using *segment-construction* **by** *blast*
obtain X' **where** $Bet X P' X'$ **and** $Cong P' X' Q A$
using *segment-construction* **by** *blast*
obtain Y **where** $Bet Q' Q Y$ **and** $Cong Q Y P A$
using *segment-construction* **by** *blast*
obtain Y' **where** $Bet Y Q' Y'$ **and** $Cong Q' Y' P A$
using *segment-construction* **by** *blast*
have $Bet Y A Q'$
using *Bet-cases* *Midpoint-def* $\langle Bet Q' Q Y \rangle$ *assms(3)* *between-exchange4* **by** *blast*
have $Bet P' A X$
using *Midpoint-def* $\langle Bet P' P X \rangle$ *assms(2)* *between-exchange4* **by** *blast*
have $Bet A P X$
using *Midpoint-def* $\langle Bet P' P X \rangle$ *assms(2)* *between-exchange3* **by** *blast*
have $Bet Y Q A$
using *Midpoint-def* $\langle Bet Q' Q Y \rangle$ *assms(3)* *between-exchange3* *between-symmetry* **by** *blast*
have $Bet A Q' Y'$
using $\langle Bet Y A Q' \rangle \langle Bet Y Q' Y' \rangle$ *between-exchange3* **by** *blast*
have $Bet X' P' A$
using $\langle Bet P' A X \rangle \langle Bet X P' X' \rangle$ *between-exchange3* *between-symmetry* **by** *blast*
hence $Bet X A X'$
using $\langle Bet P' A X \rangle \langle Bet X P' X' \rangle$ *between-symmetry* *outer-transitivity-between2* **by** *blast*
have $Bet Y A Y'$
using $\langle Bet Y A Q' \rangle \langle Bet Y Q' Y' \rangle$ *between-exchange4* **by** *blast*
have $Cong A X Y A$
using $\langle Bet A P X \rangle \langle Bet Y Q A \rangle \langle Cong P X Q A \rangle \langle Cong Q Y P A \rangle$ *l2-11-b* *not-cong-4321* **by** *blast*
have $Cong A Y' X' A$
proof –
have $Cong Q' Y' P' A$
using *Midpoint-def* $\langle Cong Q' Y' P A \rangle$ *assms(2)* *cong-4312* *cong-transitivity* **by** *blast*
have $Cong A Q' X' P'$
by (*metis* *Cong-cases* *Midpoint-def* $\langle Cong P' X' Q A \rangle$ *assms(3)* *cong-transitivity*)
thus *?thesis*
using $\langle Bet A Q' Y' \rangle \langle Bet X' P' A \rangle \langle Cong Q' Y' P' A \rangle$ *l2-11-b* **by** *force*
qed
have $Cong A Y A Y'$
proof –
have $Cong Q Y Q' Y'$
using $\langle Cong Q Y P A \rangle \langle Cong Q' Y' P A \rangle$ *cong-inner-transitivity* *cong-symmetry* **by** *blast*
thus *?thesis*
by (*meson* *Midpoint-def* $\langle Bet A Q' Y' \rangle \langle Bet Q' Q Y \rangle$ *assms(3)* *between-exchange3*
cong-left-commutativity *cong-symmetry* *l2-11-b*)
qed
have $Cong X A Y' A$
by (*metis* *cong-inner-transitivity* $\langle Cong A X Y A \rangle \langle Cong A Y A Y' \rangle$ *cong-4312*)
have $Cong A X' A Y$
using $\langle Cong A Y A Y' \rangle \langle Cong A Y' X' A \rangle$ *cong-4312* *cong-transitivity* **by** *blast*
have $Cong A X A X'$
using *Cong-cases* $\langle Cong A X Y A \rangle \langle Cong A X' A Y \rangle$ *cong-transitivity* **by** *blast*
have $X A X' Y' FSC Y' A Y X$
proof –
have $Col X A X'$
using *Col-def* $\langle Bet X A X' \rangle$ **by** *blast*
have $Cong X X' Y' Y$
using *Cong-cases* $\langle Bet X A X' \rangle \langle Bet Y A Y' \rangle \langle Cong A X Y A \rangle \langle Cong A Y' X' A \rangle$ *l2-11-b* **by** *blast*
thus *?thesis*
using *Cong3-def* *FSC-def* $\langle Col X A X' \rangle \langle Cong A X' A Y \rangle \langle Cong X A Y' A \rangle$
cong-pseudo-reflexivity *not-cong-4321* **by** *blast*
qed
hence $Y Q A X IFSC Y' Q' A X'$
by (*metis* *Bet-cases* *Cong-cases* *IFSC-def* *Midpoint-def* *bet-neq23--neq*
 $\langle Bet A P X \rangle \langle Bet A Q' Y' \rangle \langle Bet X A X' \rangle \langle Bet Y Q A \rangle \langle Cong A X A X' \rangle \langle Cong A Y A Y' \rangle$
assms(1) *assms(3)* *l4-16R1*)

hence $X P A Q$ IFSC $X' P' A Q'$
 by (metis Cong-cases IFSC-def Midpoint-def $\langle \text{Bet } A P X \rangle \langle \text{Bet } X' P' A \rangle$ assms(2)
 between-symmetry l4-2)
 thus ?thesis
 using l4-2 by force
 qed

lemma l7-13:
 assumes A Midpoint $P' P$ and
 A Midpoint $Q' Q$
 shows $\text{Cong } P Q P' Q'$
proof (cases)
 assume $A = P$
 thus ?thesis
 using Midpoint-def assms(1) assms(2) cong-3421 is-midpoint-id-2 by blast
 next
 show ?thesis
 by (metis l7-13-R1 assms(1) assms(2) cong-trivial-identity is-midpoint-id-2 not-cong-2143)
 qed

lemma l7-15:
 assumes A Midpoint $P P'$ and
 A Midpoint $Q Q'$ and
 A Midpoint $R R'$ and
 $\text{Bet } P Q R$
 shows $\text{Bet } P' Q' R'$
proof –
 have $P Q R \text{ Cong3 } P' Q' R'$
 using Cong3-def assms(1) assms(2) assms(3) l7-13 l7-2 by blast
 thus ?thesis
 using assms(4) l4-6 by blast
 qed

lemma l7-16:
 assumes A Midpoint $P P'$ and
 A Midpoint $Q Q'$ and
 A Midpoint $R R'$ and
 A Midpoint $S S'$ and
 $\text{Cong } P Q R S$
 shows $\text{Cong } P' Q' R' S'$
 by (meson assms(1) assms(2) assms(3) assms(4) assms(5) cong-transitivity l7-13 not-cong-3412)

lemma symmetry-preserves-midpoint:
 assumes Z Midpoint $A D$ and
 Z Midpoint $B E$ and
 Z Midpoint $C F$ and
 B Midpoint $A C$
 shows E Midpoint $D F$
 by (meson Midpoint-def assms(1) assms(2) assms(3) assms(4) l7-15 l7-16)

lemma Mid-cases:
 assumes A Midpoint $B C \vee A$ Midpoint $C B$
 shows A Midpoint $B C$
 using assms l7-2 by blast

lemma Mid-perm:
 assumes A Midpoint $B C$
 shows A Midpoint $B C \wedge A$ Midpoint $C B$
 by (simp add: assms l7-2)

lemma l7-17:
 assumes A Midpoint $P P'$ and
 B Midpoint $P P'$
 shows $A = B$
proof –
 have $\text{Cong } P B P' B$

using *Cong-cases Midpoint-def assms(2)* by blast
 obtain x where A Midpoint B x
 using *symmetric-point-construction* by presburger
 hence $\text{Cong } P' B P x$
 using *assms(1) l7-13 l7-2* by blast
 hence $\text{Cong } P B P x$
 using $\langle \text{Cong } P B P' B \rangle$ *cong-transitivity* by blast
 have $\text{Cong } P B P' x$
 using $\langle A \text{ Midpoint } B x \rangle$ *assms(1) cong-4321 l7-13* by blast
 have $\text{Cong } P' B P' x$
 using *cong-inner-transitivity* $\langle \text{Cong } P B P' B \rangle \langle \text{Cong } P B P' x \rangle$ by blast
 have $\text{Bet } P B P'$
 using *Midpoint-def assms(2)* by blast
 hence $B = x$
 using $\langle \text{Cong } P B P x \rangle \langle \text{Cong } P' B P' x \rangle$ *l4-19* by auto
 thus ?thesis
 using $\langle A \text{ Midpoint } B x \rangle$ *l7-3* by blast
 qed

lemma *l7-17-bis*:

assumes A Midpoint $P P'$ and
 B Midpoint $P' P$
 shows $A = B$
 by (*meson l7-17 l7-2 Tarski-neutral-dimensionless-axioms assms(1) assms(2)*)

lemma *l7-20*:

assumes $\text{Col } A M B$ and
 $\text{Cong } M A M B$
 shows $A = B \vee M \text{ Midpoint } A B$
 by (*metis Bet-cases Col-def Midpoint-def assms(1) assms(2) between-cong*
 cong-left-commutativity not-cong-3412)

lemma *l7-20-bis*:

assumes $A \neq B$ and
 $\text{Col } A M B$ and
 $\text{Cong } M A M B$
 shows $M \text{ Midpoint } A B$
 using *assms(1) assms(2) assms(3) l7-20* by blast

lemma *cong-col-mid*:

assumes $A \neq C$ and
 $\text{Col } A B C$ and
 $\text{Cong } A B B C$
 shows $B \text{ Midpoint } A C$
 using *assms(1) assms(2) assms(3) cong-left-commutativity l7-20* by blast

lemma *l7-21-R1*:

assumes $\neg \text{Col } A B C$ and
 $B \neq D$ and
 $\text{Cong } A B C D$ and
 $\text{Cong } B C D A$ and
 $\text{Col } A P C$ and
 $\text{Col } B P D$

shows $P \text{ Midpoint } A C$

proof –

obtain X where $B D P \text{ Cong3 } D B X$
 using *Col-perm assms(6) cong-pseudo-reflexivity l4-14* by blast
 hence $\text{Col } D B X$
 using *assms(6) l4-13 not-col-permutation-5* by blast
 have $B D P A \text{ FSC } D B X C$
 by (*meson Col-cases Cong-cases FSC-def* $\langle B D P \text{ Cong3 } D B X \rangle$ *assms(3) assms(4) assms(6)*)
 have $B D P C \text{ FSC } D B X A$
 using *Col-cases Cong-perm FSC-def* $\langle B D P \text{ Cong3 } D B X \rangle$ *assms(3) assms(4) assms(6)* by blast
 hence $A P C \text{ Cong3 } C X A$
 using *Cong3-def Cong-cases* $\langle B D P A \text{ FSC } D B X C \rangle$ *assms(2) cong-pseudo-reflexivity l4-16*
 by blast

hence $Col\ C\ X\ A$
using $assms(5)\ l4-13$ **by** *blast*
hence $P = X$
using $\langle Col\ D\ B\ X \rangle\ assms(1)\ assms(2)\ assms(5)\ assms(6)\ l6-21\ not-col-permutation-1$
 $not-col-permutation-5$ **by** *blast*
thus *?thesis*
by $(metis\ Col-perm\ \langle B\ D\ P\ A\ FSC\ D\ B\ X\ C \rangle\ \langle Col\ C\ X\ A \rangle\ assms(1)\ assms(2)\ l4-16\ l7-20-bis$
 $not-col-distincts)$

qed

lemma *l7-21:*

assumes $\neg\ Col\ A\ B\ C$ **and**
 $B \neq D$ **and**
 $Cong\ A\ B\ C\ D$ **and**
 $Cong\ B\ C\ D\ A$ **and**
 $Col\ A\ P\ C$ **and**
 $Col\ B\ P\ D$
shows $P\ Midpoint\ A\ C \wedge P\ Midpoint\ B\ D$
by $(metis\ Cong-cases\ Midpoint-def\ assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)\ assms(6)$
 $cong-reverse-identity\ l7-21-R1\ not-col-distincts)$

lemma *l7-22-aux-R1:*

assumes $Bet\ A1\ C\ C$ **and**
 $Bet\ B1\ C\ B2$ **and**
 $Cong\ C\ A1\ C\ B1$ **and**
 $Cong\ C\ C\ C\ B2$ **and**
 $M1\ Midpoint\ A1\ B1$ **and**
 $M2\ Midpoint\ A2\ B2$ **and**
 $C\ A1\ Le\ C\ C$
shows $Bet\ M1\ C\ M2$
by $(metis\ assms(3)\ assms(5)\ assms(7)\ cong-diff-3\ l7-3\ le-diff\ not-bet-distincts)$

lemma *l7-22-aux-R2:*

assumes $A2 \neq C$ **and**
 $Bet\ A1\ C\ A2$ **and**
 $Bet\ B1\ C\ B2$ **and**
 $Cong\ C\ A1\ C\ B1$ **and**
 $Cong\ C\ A2\ C\ B2$ **and**
 $M1\ Midpoint\ A1\ B1$ **and**
 $M2\ Midpoint\ A2\ B2$ **and**
 $C\ A1\ Le\ C\ A2$
shows $Bet\ M1\ C\ M2$

proof –

obtain X **where** $C\ Midpoint\ A2\ X$
using *symmetric-point-construction* **by** *blast*
obtain $X0$ **where** $C\ Midpoint\ B2\ X0$
using *symmetric-point-construction* **by** *blast*
obtain $X1$ **where** $C\ Midpoint\ M2\ X1$
using *symmetric-point-construction* **by** *blast*
hence $X1\ Midpoint\ X\ X0$
using $\langle C\ Midpoint\ A2\ X \rangle\ \langle C\ Midpoint\ B2\ X0 \rangle\ assms(7)\ symmetry-preserves-midpoint$ **by** *blast*
have $C\ A1\ Le\ C\ X$
by $(metis\ Midpoint-def\ cong-reflexivity\ l5-6\ \langle C\ Midpoint\ A2\ X \rangle\ assms(8)\ le-right-comm)$
hence $Bet\ C\ A1\ X$
by $(metis\ (full-types)\ Bet-cases\ Le-cases\ Midpoint-def\ \langle C\ Midpoint\ A2\ X \rangle\ assms(1,2)$
 $bet-le-eq\ l5-2)$
have $Cong\ C\ X\ C\ X0$
by $(meson\ l7-3-2\ \langle C\ Midpoint\ A2\ X \rangle\ \langle C\ Midpoint\ B2\ X0 \rangle\ assms(5)\ l7-16)$
hence $C\ B1\ Le\ C\ X0$
using $\langle C\ A1\ Le\ C\ X \rangle\ assms(4)\ l5-6$ **by** *blast*
have $Bet\ C\ B1\ X0$
proof *cases*
assume $B1 = C$
thus *?thesis*
using *between-trivial2* **by** *auto*
next

```

assume  $B1 \neq C$ 
thus ?thesis
  by (metis (full-types) Bet-cases Le-cases Midpoint-def  $\langle C B1 Le C X0 \rangle \langle C Midpoint B2 X0 \rangle$  assms(3)
    bet-cong-eq bet-le-eq l5-2)
qed
hence  $Bet X0 B1 C$ 
  using Bet-cases by blast
have  $\exists Q. Bet X1 Q C \wedge Bet A1 Q B1$ 
proof –
  have  $Bet X A1 C$ 
    using Bet-cases  $\langle Bet C A1 X \rangle$  by blast
  moreover have  $Bet X X1 X0$ 
    using Midpoint-def  $\langle X1 Midpoint X X0 \rangle$  by auto
  ultimately show ?thesis
    using  $\langle Bet X0 B1 C \rangle$  l3-17 by blast
qed
then obtain  $Q$  where  $Bet X1 Q C$  and  $Bet A1 Q B1$ 
  by blast
have  $X A1 C X1 IFSC X0 B1 C X1$ 
  by (metis Bet-cases Cong-cases IFSC-def Midpoint-def  $\langle Bet C A1 X \rangle \langle Bet X0 B1 C \rangle$ 
     $\langle Cong C X C X0 \rangle \langle X1 Midpoint X X0 \rangle$  assms(4) cong-reflexivity)
hence  $Cong A1 X1 B1 X1$ 
  using l4-2 by auto
have  $Cong Q A1 Q B1$ 
proof cases
  assume  $C = X1$ 
  thus ?thesis
    using between-identity  $\langle Bet X1 Q C \rangle$  assms(4) by blast
next
  assume  $\neg C = X1$ 
  moreover
  have  $Col C X1 Q$ 
    by (simp add: Col-def  $\langle Bet X1 Q C \rangle$ )
  moreover
  have  $Cong X1 A1 X1 B1$ 
    using Cong-cases  $\langle Cong A1 X1 B1 X1 \rangle$  by auto
  ultimately show ?thesis
    by (simp add: assms(4) l4-17)
qed
have  $Q Midpoint A1 B1$ 
  using Midpoint-def  $\langle Bet A1 Q B1 \rangle \langle Cong Q A1 Q B1 \rangle$  not-cong-2134 by blast
thus ?thesis
  by (metis Midpoint-def  $\langle Bet X1 Q C \rangle \langle C Midpoint M2 X1 \rangle$  assms(6) between-inner-transitivity
    between-symmetry l7-17)
qed

lemma l7-22-aux:
assumes  $Bet A1 C A2$  and
   $Bet B1 C B2$  and
   $Cong C A1 C B1$  and
   $Cong C A2 C B2$  and
   $M1 Midpoint A1 B1$  and
   $M2 Midpoint A2 B2$  and
   $C A1 Le C A2$ 
shows  $Bet M1 C M2$ 
by (metis assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) between-trivial
  cong-diff-4 l7-22-aux-R2 l7-3)

lemma l7-22:
assumes  $Bet A1 C A2$  and
   $Bet B1 C B2$  and
   $Cong C A1 C B1$  and
   $Cong C A2 C B2$  and
   $M1 Midpoint A1 B1$  and
   $M2 Midpoint A2 B2$ 
shows  $Bet M1 C M2$ 

```

by (*meson* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) *between-symmetry l7-22-aux le-cases*)

lemma *bet-col1*:

assumes *Bet A B D* **and**

Bet A C D

shows *Col A B C*

using *Bet-perm Col-def assms*(1) *assms*(2) *l5-3* **by** *blast*

lemma *l7-25-R1*:

assumes *Cong C A C B* **and**

Col A B C

shows $\exists X. X \text{ Midpoint } A B$

using *assms*(1) *assms*(2) *l7-20 l7-3-2 not-col-permutation-5* **by** *blast*

lemma *l7-25-R2*:

assumes *Cong C A C B* **and**

$\neg \text{Col } A B C$

shows $\exists X. X \text{ Midpoint } A B$

proof –

obtain *P* **where** *Bet C A P* **and** $A \neq P$

using *point-construction-different* **by** *auto*

obtain *Q* **where** *Bet C B Q* **and** *Cong B Q A P*

using *segment-construction* **by** *blast*

obtain *R* **where** *Bet A R Q* **and** *Bet B R P*

by (*meson* $\langle \text{Bet } C A P \rangle \langle \text{Bet } C B Q \rangle$ *l3-17 not-bet-distincts*)

obtain *X* **where** *Bet A X B* **and** *Bet R X C*

using $\langle \text{Bet } B R P \rangle \langle \text{Bet } C A P \rangle$ *inner-pasch* **by** *blast*

have *Cong X A X B*

proof –

have *Cong R A R B* $\longrightarrow$ *Cong X A X B*

proof *cases*

assume $R = C$

thus *?thesis*

using $\langle \text{Bet } R X C \rangle$ *between-identity* **by** *blast*

next

assume $\neg R = C$

have *Col R C X*

using $\langle \text{Bet } R X C \rangle$ *bet-col1 between-trivial* **by** *blast*

thus *?thesis*

using $\langle R \neq C \rangle$ *assms*(1) *l4-17* **by** *blast*

qed

have *Cong R A R B*

proof –

have *C A P B OFSC C B Q A*

by (*metis* *OFSC-def* $\langle \text{Bet } C A P \rangle \langle \text{Bet } C B Q \rangle \langle \text{Cong } B Q A P \rangle$ *assms*(1) *cong-pseudo-reflexivity not-cong-3412*)

hence *Cong P B Q A*

using *assms*(2) *col-trivial-3 five-segment-with-def* **by** *blast*

hence *Cong B P A Q*

using *Cong-cases* **by** *blast*

then obtain *R'* **where** *Bet A R' Q* **and** *B R P Cong3 A R' Q*

using $\langle \text{Bet } B R P \rangle$ *l4-5* **by** *blast*

have *B R P A IFSC A R' Q B*

using *Cong3-def IFSC-def* $\langle B R P \text{ Cong3 } A R' Q \rangle \langle \text{Bet } A R' Q \rangle \langle \text{Bet } B R P \rangle \langle \text{Cong } B Q A P \rangle$ *cong-pseudo-reflexivity not-cong-4321* **by** *blast*

hence *Cong R A R' B*

using *l4-2* **by** *auto*

have *B R P Q IFSC A R' Q P*

using *Cong3-def IFSC-def* $\langle B R P \text{ Cong3 } A R' Q \rangle \langle \text{Bet } A R' Q \rangle \langle \text{Bet } B R P \rangle \langle \text{Cong } B Q A P \rangle$ *cong-pseudo-reflexivity* **by** *blast*

hence *Cong R Q R' P*

using *l4-2* **by** *blast*

hence *A R Q Cong3 B R' P*

using *Cong3-def Cong-cases* $\langle \text{Cong } B P A Q \rangle \langle \text{Cong } R A R' B \rangle$ **by** *blast*

have *Col B R' P*

```

    using ⟨A R Q Cong3 B R' P⟩ ⟨Bet A R Q⟩ bet-col l4-13 by blast
  have R = R'
proof -
  have B ≠ P
    using Col-def ⟨Bet C A P⟩ assms(2) by blast
  moreover
  have B ≠ Q
    using ⟨A ≠ P⟩ ⟨Cong B Q A P⟩ cong-reverse-identity by force
  hence ¬ Col A Q B
    using ⟨Bet C B Q⟩ assms(2) bet-col col2--eq not-col-permutation-5 by blast
  moreover have Col A Q R
    using Bet-cases Col-def ⟨Bet A R Q⟩ by auto
  moreover have Col A Q R'
    using ⟨Bet A R' Q⟩ bet-col not-col-permutation-5 by blast
  moreover have Col B P R
    using Bet-cases Col-def ⟨Bet B R P⟩ by auto
  moreover have Col B P R'
    using ⟨Col B R' P⟩ not-col-permutation-5 by blast
  ultimately show ?thesis
    using ⟨B ≠ P⟩ ⟨¬ Col A Q B⟩ l6-21 by blast
qed
thus ?thesis
  using ⟨Cong R A R' B⟩ by auto
qed
thus ?thesis
  by (simp add: ⟨Cong R A R B ⟶ Cong X A X B⟩)
qed
thus ?thesis
  using Col-def ⟨Bet A X B⟩ between-symmetry l7-25-R1 by blast
qed

```

lemma l7-25:

```

assumes Cong C A C B
shows ∃ X. X Midpoint A B
using assms l7-25-R1 l7-25-R2 by blast

```

lemma midpoint-distinct-1:

```

assumes A ≠ B and
  I Midpoint A B
shows I ≠ A ∧ I ≠ B
using assms(1) assms(2) is-midpoint-id is-midpoint-id-2 by blast

```

lemma midpoint-distinct-2:

```

assumes I ≠ A and
  I Midpoint A B
shows A ≠ B ∧ I ≠ B
using assms(1) assms(2) is-midpoint-id l7-3 by blast

```

lemma midpoint-distinct-3:

```

assumes I ≠ B and
  I Midpoint A B
shows A ≠ B ∧ I ≠ A
using assms(1) assms(2) is-midpoint-id l7-3 by blast

```

lemma midpoint-def:

```

assumes Bet A B C and
  Cong A B B C
shows B Midpoint A C
using Midpoint-def assms(1) assms(2) by blast

```

lemma midpoint-bet:

```

assumes B Midpoint A C
shows Bet A B C
using Midpoint-def assms by blast

```

lemma midpoint-col:

assumes M Midpoint A B
shows Col M A B
using *assms bet-col col-permutation-4 midpoint-bet* **by** *blast*

lemma *midpoint-cong*:
assumes B Midpoint A C
shows $Cong$ A B B C
using *Midpoint-def assms* **by** *blast*

lemma *midpoint-out*:
assumes $A \neq C$ **and**
 B Midpoint A C
shows A Out B C
using *assms(1) assms(2) bet-out midpoint-bet midpoint-distinct-1* **by** *blast*

lemma *midpoint-out-1*:
assumes $A \neq C$ **and**
 B Midpoint A C
shows C Out A B
by (*metis midpoint-bet midpoint-distinct-1 assms(1) assms(2) bet-out-1 l6-6*)

lemma *midpoint-not-midpoint*:
assumes $A \neq B$ **and**
 I Midpoint A B
shows $\neg B$ Midpoint A I
using *assms(1) assms(2) between-equality-2 midpoint-bet midpoint-distinct-1* **by** *blast*

lemma *swap-diff*:
assumes $A \neq B$
shows $B \neq A$
using *assms* **by** *auto*

lemma *cong-cong-half-1*:
assumes M Midpoint A B **and**
 M' Midpoint A' B' **and**
 $Cong$ A B A' B'
shows $Cong$ A M A' M'

proof –
obtain M'' **where** Bet A' M'' B' **and** A M B $Cong3$ A' M'' B'
using *assms(1) assms(3) l4-5 midpoint-bet* **by** *blast*
hence M'' Midpoint A' B'
by (*meson Cong3-def assms(1) cong-inner-transitivity midpoint-cong midpoint-def*)
hence $M' = M''$
using *assms(2) l7-17* **by** *auto*
thus *?thesis*
using *Cong3-def* $\langle A$ M B $Cong3$ A' M'' $B' \rangle$ **by** *blast*

qed

lemma *cong-cong-half-2*:
assumes M Midpoint A B **and**
 M' Midpoint A' B' **and**
 $Cong$ A B A' B'
shows $Cong$ B M B' M'
using *assms(1) assms(2) assms(3) cong-cong-half-1 l7-2 not-cong-2143* **by** *blast*

lemma *cong-mid2--cong*:
assumes M Midpoint A B **and**
 M' Midpoint A' B' **and**
 $Cong$ A M A' M'
shows $Cong$ A B A' B'
by (*meson assms(1) assms(2) assms(3) cong-inner-transitivity l2-11-b midpoint-bet midpoint-cong*)

lemma *mid--lt*:
assumes $A \neq B$ **and**
 M Midpoint A B
shows A M Lt A B


```

using assms(1) assms(2) bet--lt1213 midpoint-bet midpoint-distinct-1 by blast

lemma le-mid2--le13:
  assumes M Midpoint A B and
    M' Midpoint A' B' and
    A M Le A' M'
  shows A B Le A' B'
  by (meson Midpoint-def l5-6 assms(1) assms(2) assms(3) bet2-le2--le1346)

lemma le-mid2--le12:
  assumes M Midpoint A B and
    M' Midpoint A' B'
    and A B Le A' B'
  shows A M Le A' M'
  by (meson assms(1) assms(2) assms(3) cong--le3412 cong-cong-half-1 le-anti-symmetry
    le-mid2--le13 local.le-cases)

lemma lt-mid2--lt13:
  assumes M Midpoint A B and
    M' Midpoint A' B' and
    A M Lt A' M'
  shows A B Lt A' B'
  by (meson le-mid2--le12 Tarski-neutral-dimensionless-axioms assms(1) assms(2) assms(3)
    lt--nle nlt--le)

lemma lt-mid2--lt12:
  assumes M Midpoint A B and
    M' Midpoint A' B' and
    A B Lt A' B'
  shows A M Lt A' M'
  by (meson le-mid2--le13 Tarski-neutral-dimensionless-axioms assms(1) assms(2) assms(3)
    le--nlt nle--lt)

lemma midpoint-preserves-out:
  assumes A Out B C and
    M Midpoint A A' and
    M Midpoint B B' and
    M Midpoint C C'
  shows A' Out B' C'
  using Out-def assms(1) assms(2) assms(3) assms(4) l7-15 l7-9 by fastforce

lemma col-cong-bet:
  assumes Col A B D and
    Cong A B C D and
    Bet A C B
  shows Bet C A D  $\vee$  Bet C B D
proof –
  obtain D1 where Bet B A D1 and Cong A D1 B C
    using segment-construction by blast
  obtain D2 where Bet A B D2 and Cong B D2 A C
    using segment-construction by blast
  have Cong A B C D1
    by (meson Bet-cases  $\langle$ Bet B A D1 $\rangle$   $\langle$ Cong A D1 B C $\rangle$  assms(3) between-exchange4
      cong-pseudo-reflexivity l4-3-1 not-cong-2134)
  have D = D1  $\vee$  C Midpoint D D1
proof –
  have Col D C D1
proof cases
  assume A = B
  thus ?thesis
    by (metis assms(2) col-trivial-1 cong-diff-3)
next
  assume A  $\neq$  B
  thus ?thesis
    by (meson Col-def  $\langle$ Bet B A D1 $\rangle$  assms(1) assms(3) col3 not-col-permutation-4)
qed

```

```

moreover have Cong C D C D1
  using  $\langle \text{Cong } A \ B \ C \ D1 \rangle$  assms(2) cong-inner-transitivity by blast
ultimately show ?thesis
  using l7-20 by auto
qed
{
  assume  $D = D1$ 
  hence  $\text{Bet } C \ A \ D \vee \text{Bet } C \ B \ D$ 
  using  $\langle \text{Bet } B \ A \ D1 \rangle$  assms(3) between-exchange3 between-symmetry by blast
}
moreover
{
  assume C Midpoint D D1
  have Cong B A C D2
  proof –
    have  $\text{Bet } B \ C \ A$ 
    using Bet-cases assms(3) by blast
    moreover have  $\text{Bet } C \ B \ D2$ 
    using  $\langle \text{Bet } A \ B \ D2 \rangle$  assms(3) between-exchange3 by blast
    moreover have Cong B C C B
    using cong-pseudo-reflexivity by blast
    have Cong C A B D2
    using Cong-cases  $\langle \text{Cong } B \ D2 \ A \ C \rangle$  by blast
    ultimately show ?thesis
    using  $\langle \text{Cong } B \ C \ C \ B \rangle$  l2-11-b by blast
  qed
  have C Midpoint D2 D1
  proof cases
    assume  $A = B$ 
    thus ?thesis
    by (metis  $\langle \text{C Midpoint } D \ D1 \rangle \langle \text{Cong } B \ A \ C \ D2 \rangle$  assms(2) cong-diff-3)
  next
    assume  $A \neq B$ 
    show ?thesis
    proof cases
      assume  $B = C$ 
      thus ?thesis
      using Midpoint-def  $\langle \text{Bet } A \ B \ D2 \rangle \langle \text{Cong } A \ D1 \ B \ C \rangle \langle \text{Cong } B \ D2 \ A \ C \rangle$  between-symmetry
      cong-commutativity cong-identity by blast
    next
      assume  $B \neq C$ 
      have  $\text{Bet } D1 \ C \ B$ 
      using  $\langle \text{Bet } B \ A \ D1 \rangle$  assms(3) between-exchange4 between-symmetry by blast
      have  $\text{Bet } C \ B \ D2$ 
      using  $\langle \text{Bet } A \ B \ D2 \rangle$  assms(3) between-exchange3 by blast
      thus ?thesis
      by (metis Midpoint-def cong-inner-transitivity  $\langle B \neq C \rangle \langle \text{Bet } D1 \ C \ B \rangle$ 
         $\langle \text{Cong } A \ B \ C \ D1 \rangle \langle \text{Cong } B \ A \ C \ D2 \rangle$  between-symmetry not-cong-2134
        outer-transitivity-between2)
    qed
  qed
  have  $\text{Bet } C \ A \ D \vee \text{Bet } C \ B \ D$ 
  using  $\langle \text{Bet } A \ B \ D2 \rangle \langle \text{C Midpoint } D2 \ D1 \rangle \langle D = D1 \vee \text{C Midpoint } D \ D1 \rangle$  assms(3)
    between-exchange3 calculation l7-9 by blast
}
ultimately show ?thesis
using  $\langle D = D1 \vee \text{C Midpoint } D \ D1 \rangle$  by blast
qed

lemma col-cong2-bet1:
assumes Col A B D and
  Bet A C B and
  Cong A B C D and
  Cong A C B D
shows  $\text{Bet } C \ B \ D$ 
by (metis assms(1) assms(2) assms(3) assms(4) bet--le1213 bet-cong-eq between-symmetry

```

col-cong-bet cong--le cong-left-commutativity l5-12-b l5-6 outer-transitivity-between2)

lemma *col-cong2-bet2:*

assumes *Col A B D and*

Bet A C B and

Cong A B C D and

Cong A D B C

shows *Bet C A D*

by (*metis assms(1) assms(2) assms(3) assms(4) bet-cong-eq col-cong-bet*
cong-identity not-bet-distincts not-cong-3421 outer-transitivity-between2)

lemma *col-cong2-bet3:*

assumes *Col A B D and*

Bet A B C and

Cong A B C D and

Cong A C B D

shows *Bet B C D*

by (*metis assms(1) assms(2) assms(3) assms(4) bet--le1213 bet--le2313*
bet-col col-transitivity-2 cong-diff-3 cong-reflexivity l5-12-b l5-6 not-bet-distincts)

lemma *col-cong2-bet4:*

assumes *Col A B C and*

Bet A B D and

Cong A B C D and

Cong A D B C

shows *Bet B D C*

using *assms(1) assms(2) assms(3) assms(4) col-cong2-bet3 cong-right-commutativity by blast*

lemma *col-bet2-cong1:*

assumes *Col A B D and*

Bet A C B and

Cong A B C D and

Bet C B D

shows *Cong A C D B*

by (*meson assms(2) assms(3) assms(4) between-symmetry cong-pseudo-reflexivity*
cong-right-commutativity l4-3)

lemma *col-bet2-cong2:*

assumes *Col A B D and*

Bet A C B and

Cong A B C D and

Bet C A D

shows *Cong D A B C*

by (*meson assms(2) assms(3) assms(4) between-symmetry cong-commutativity*
cong-pseudo-reflexivity cong-symmetry l4-3)

lemma *bet2-lt2--lt:*

assumes *Bet a Po b and*

Bet A PO B and

Po a Lt PO A and

Po b Lt PO B

shows *a b Lt A B*

by (*metis Lt-cases nle--lt assms(1) assms(2) assms(3) assms(4) bet2-le2--le1245 le--nlt lt--le*)

lemma *bet2-lt-le--lt:*

assumes *Bet a Po b and*

Bet A PO B and

Cong Po a PO A and

Po b Lt PO B

shows *a b Lt A B*

proof –

have *Po a Le PO A*

using *assms(3) cong--le by blast*

thus *?thesis*

by (*meson Le-cases nlt--le assms(1) assms(2) assms(4) bet2-le2--le2356 lt--nle*)

qed

lemma *per-dec*:
 $Per\ A\ B\ C \vee \neg\ Per\ A\ B\ C$
by *simp*

lemma *l8-2*:
assumes $Per\ A\ B\ C$
shows $Per\ C\ B\ A$
proof –
obtain C' **where** $B\ Midpoint\ C\ C'$ **and** $Cong\ A\ C\ A\ C'$
using *Per-def assms* **by** *blast*
obtain A' **where** $B\ Midpoint\ A\ A'$
using *symmetric-point-construction* **by** *blast*
hence $Cong\ C'\ A\ C\ A'$
using *Mid-cases* $\langle B\ Midpoint\ C\ C' \rangle$ *l7-13* **by** *blast*
thus *?thesis*
using *Per-def* $\langle B\ Midpoint\ A\ A' \rangle$ $\langle Cong\ A\ C\ A\ C' \rangle$ *cong-transitivity not-cong-2143* **by** *blast*
qed

lemma *Per-cases*:
assumes $Per\ A\ B\ C \vee Per\ C\ B\ A$
shows $Per\ A\ B\ C$
using *assms l8-2* **by** *blast*

lemma *Per-perm* :
assumes $Per\ A\ B\ C$
shows $Per\ A\ B\ C \wedge Per\ C\ B\ A$
by (*simp add: assms l8-2*)

lemma *l8-3* :
assumes $Per\ A\ B\ C$ **and**
 $A \neq B$ **and**
 $Col\ B\ A\ A'$
shows $Per\ A'\ B\ C$
by (*metis Per-def cong-left-commutativity assms(1) assms(2) assms(3) l4-17 midpoint-cong*)

lemma *l8-4*:
assumes $Per\ A\ B\ C$ **and**
 $B\ Midpoint\ C\ C'$
shows $Per\ A\ B\ C'$
by (*metis l8-2 assms(1) assms(2) l8-3 midpoint-col midpoint-distinct-1*)

lemma *l8-5*:
shows $Per\ A\ B\ B$
using *Per-def cong-reflexivity l7-3-2* **by** *blast*

lemma *l8-6*:
assumes $Per\ A\ B\ C$ **and**
 $Per\ A'\ B\ C$ **and**
 $Bet\ A\ C\ A'$
shows $B = C$
by (*metis Per-def assms(1) assms(2) assms(3) l4-19 midpoint-distinct-3 symmetric-point-uniqueness*)

lemma *l8-7*:
assumes $Per\ A\ B\ C$ **and**
 $Per\ A\ C\ B$
shows $B = C$
proof –
obtain C' **where** $P1: B\ Midpoint\ C\ C' \wedge Cong\ A\ C\ A\ C'$
using *Per-def assms(1)* **by** *blast*
obtain A' **where** $P2: C\ Midpoint\ A\ A'$
using *Per-def assms(2) l8-2* **by** *blast*
have $Per\ C'\ C\ A$
by (*metis P1 l8-3 assms(2) bet-col l8-2 midpoint-bet midpoint-distinct-3*)
hence $Cong\ A\ C'\ A'\ C'$

```

    using Cong-perm P2 Per-def symmetric-point-uniqueness by blast
  hence Cong A' C A' C'
    using P1 P2 cong-inner-transitivity midpoint-cong not-cong-2134 by blast
  hence Q4: Per A' B C
    using P1 Per-def by blast
  have Bet A' C A
    using Mid-perm P2 midpoint-bet by blast
  thus ?thesis
    using Q4 assms(1) l8-6 by blast
qed

```

```

lemma l8-8:
  assumes Per A B A
  shows A = B
  using l8-6 Tarski-neutral-dimensionless-axioms assms between-trivial2 by fastforce

```

```

lemma per-distinct:
  assumes Per A B C and
    A ≠ B
  shows A ≠ C
  using assms(1) assms(2) l8-8 by blast

```

```

lemma per-distinct-1:
  assumes Per A B C and
    B ≠ C
  shows A ≠ C
  using assms(1) assms(2) l8-8 by blast

```

```

lemma l8-9:
  assumes Per A B C and
    Col A B C
  shows A = B ∨ C = B
  using Col-cases assms(1) assms(2) l8-3 l8-8 by blast

```

```

lemma l8-10:
  assumes Per A B C and
    A B C Cong3 A' B' C'
  shows Per A' B' C'

```

```

proof -
  obtain D where B Midpoint C D and Cong A C A D
    using Per-def assms(1) by blast
  obtain D' where Bet C' B' D' and Cong B' D' B' C'
    using segment-construction by blast
  hence B' Midpoint C' D'
    by (simp add: Midpoint-def cong-4312)
  have Cong A' C' A' D'
  proof cases
    assume C = B
    thus ?thesis
      by (metis Cong3-def Cong-cases ⟨Cong B' D' B' C'⟩ assms(2) cong-reflexivity
        cong-reverse-identity)
  next
    assume ¬ C = B
    hence C B D A OFSC C' B' D' A'
      by (meson Cong3-def OFSC-def ⟨B Midpoint C D⟩ ⟨B' Midpoint C' D'⟩ assms(2) cong-commutativity
        cong-cong-half-2 cong-mid2--cong midpoint-bet)
    thus ?thesis
      by (meson Cong3-def ⟨C ≠ B⟩ ⟨Cong A C A D⟩ assms(2) cong-inner-transitivity
        five-segment-with-def not-cong-2143)
  qed
  thus ?thesis
    using Per-def ⟨B' Midpoint C' D'⟩ by blast
qed

```

```

lemma col-col-per-per:
  assumes A ≠ X and

```

```

  C ≠ X and
  Col U A X and
  Col V C X and
  Per A X C
shows Per U X V
by (meson l8-2 l8-3 Tarski-neutral-dimensionless-axioms assms(1) assms(2) assms(3) assms(4)
    assms(5) not-col-permutation-3)

lemma perp-in-dec:
  X PerpAt A B C D ∨ ¬ X PerpAt A B C D
by simp

lemma perp-distinct:
  assumes A B Perp C D
  shows A ≠ B ∧ C ≠ D
  using PerpAt-def Perp-def assms by auto

lemma l8-12:
  assumes X PerpAt A B C D
  shows X PerpAt C D A B
  using Per-perm PerpAt-def assms by auto

lemma per-col:
  assumes B ≠ C and
  Per A B C and
  Col B C D
  shows Per A B D
  by (metis l8-3 assms(1) assms(2) assms(3) l8-2)

lemma l8-13-2:
  assumes A ≠ B and
  C ≠ D and
  Col X A B and
  Col X C D and
  ∃ U. ∃ V. Col U A B ∧ Col V C D ∧ U ≠ X ∧ V ≠ X ∧ Per U X V
  shows X PerpAt A B C D
proof -
  obtain U V where Col U A B and Col V C D and U ≠ X and V ≠ X and Per U X V
  using assms(5) by blast
  {
    fix U0 V0
    assume Col U0 A B and Col V0 C D
    have Col X U U0
      using ⟨Col U A B⟩ ⟨Col U0 A B⟩ assms(1) assms(3) col3 not-col-permutation-2 by blast
    hence Per U0 X V
      using ⟨Per U X V⟩ ⟨U ≠ X⟩ l8-3 by blast
    hence Per V X U0
      using l8-2 by blast
    hence Per U0 X V0
      by (metis NCol-perm
        ⟨Col V C D⟩ ⟨Col V0 C D⟩ ⟨Per U0 X V⟩ ⟨V ≠ X⟩ assms(2) assms(4) l6-16-1 per-col)
  }
  thus ?thesis
  by (simp add: PerpAt-def assms(1) assms(2) assms(3) assms(4))
qed

lemma l8-14-1:
  ¬ A B Perp A B
  by (metis PerpAt-def Perp-def col-trivial-1 col-trivial-3 l8-8)

lemma l8-14-2-1a:
  assumes X PerpAt A B C D
  shows A B Perp C D
  using Perp-def assms by blast

lemma perp-in-distinct:

```

```

assumes  $X \text{ PerpAt } A \ B \ C \ D$ 
shows  $A \neq B \wedge C \neq D$ 
using PerpAt-def assms by blast

lemma l8-14-2-1b:
assumes  $X \text{ PerpAt } A \ B \ C \ D$  and
   $\text{Col } Y \ A \ B$  and
   $\text{Col } Y \ C \ D$ 
shows  $X = Y$ 
by (metis PerpAt-def assms(1) assms(2) assms(3) l8-13-2 l8-14-1 l8-14-2-1a)

lemma l8-14-2-1b-bis:
assumes  $A \ B \text{ Perp } C \ D$  and
   $\text{Col } X \ A \ B$  and
   $\text{Col } X \ C \ D$ 
shows  $X \text{ PerpAt } A \ B \ C \ D$ 
using Perp-def assms(1) assms(2) assms(3) l8-14-2-1b by blast

lemma l8-14-2-2:
assumes  $A \ B \text{ Perp } C \ D$  and
   $\forall Y. (\text{Col } Y \ A \ B \wedge \text{Col } Y \ C \ D) \longrightarrow X = Y$ 
shows  $X \text{ PerpAt } A \ B \ C \ D$ 
by (metis PerpAt-def Perp-def assms(1) assms(2))

lemma l8-14-3:
assumes  $X \text{ PerpAt } A \ B \ C \ D$  and
   $Y \text{ PerpAt } A \ B \ C \ D$ 
shows  $X = Y$ 
by (meson PerpAt-def assms(1) assms(2) l8-14-2-1b)

lemma l8-15-1:
assumes  $\text{Col } A \ B \ X$  and
   $A \ B \text{ Perp } C \ X$ 
shows  $X \text{ PerpAt } A \ B \ C \ X$ 
using NCol-perm assms(1) assms(2) col-trivial-3 l8-14-2-1b-bis by blast

lemma l8-15-2:
assumes  $\text{Col } A \ B \ X$  and
   $X \text{ PerpAt } A \ B \ C \ X$ 
shows  $A \ B \text{ Perp } C \ X$ 
using assms(2) l8-14-2-1a by blast

lemma perp-in-per:
assumes  $B \text{ PerpAt } A \ B \ B \ C$ 
shows  $\text{Per } A \ B \ C$ 
by (meson NCol-cases PerpAt-def assms col-trivial-3)

lemma perp-sym:
assumes  $A \ B \text{ Perp } A \ B$ 
shows  $C \ D \text{ Perp } C \ D$ 
using assms l8-14-1 by auto

lemma perp-col0:
assumes  $A \ B \text{ Perp } C \ D$  and
   $X \neq Y$  and
   $\text{Col } A \ B \ X$  and
   $\text{Col } A \ B \ Y$ 
shows  $C \ D \text{ Perp } X \ Y$ 
proof –
obtain  $X0$  where  $X0 \text{ PerpAt } A \ B \ C \ D$ 
using Perp-def assms(1) by blast
hence  $A \neq B \wedge C \neq D \wedge \text{Col } X0 \ A \ B \wedge \text{Col } X0 \ C \ D \wedge$ 
 $((\text{Col } U \ A \ B \wedge \text{Col } V \ C \ D) \longrightarrow \text{Per } U \ X0 \ V)$ 
using PerpAt-def by blast
have  $C \neq D$ 
by (simp add:  $\langle A \neq B \wedge C \neq D \wedge \text{Col } X0 \ A \ B \wedge \text{Col } X0 \ C \ D \wedge$ 

```

```

  (Col U A B  $\wedge$  Col V C D  $\longrightarrow$  Per U X0 V)›)
have X  $\neq$  Y
  by (simp add: assms(2))
have Col X0 C D
  using  $\langle A \neq B \wedge C \neq D \wedge \text{Col } X0 A B \wedge \text{Col } X0 C D \wedge (\text{Col } U A B \wedge \text{Col } V C D \longrightarrow \text{Per } U X0 V) \rangle$ 
  by blast
have Col X0 X Y
  by (meson  $\langle A \neq B \wedge C \neq D \wedge \text{Col } X0 A B \wedge \text{Col } X0 C D \wedge (\text{Col } U A B \wedge \text{Col } V C D \longrightarrow \text{Per } U X0 V) \rangle$ 
    assms(3) assms(4) col3 not-col-permutation-2)
have X0 PerpAt C D X Y
proof -
  have  $\forall U V. (\text{Col } U C D \wedge \text{Col } V X Y) \longrightarrow \text{Per } U X0 V$ 
  by (metis Per-perm PerpAt-def col-trivial-2  $\langle X0 \text{ PerpAt } A B C D \rangle$  assms(2) assms(3)
    assms(4) l6-21 not-col-permutation-1)
  thus ?thesis
  by (simp add: PerpAt-def  $\langle C \neq D \rangle \langle \text{Col } X0 C D \rangle \langle \text{Col } X0 X Y \rangle$  assms(2))
qed
thus ?thesis
  using Perp-def by auto
qed

```

```

lemma per-perp-in:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    Per A B C
  shows B PerpAt A B B C
  by (metis Col-def assms(1) assms(2) assms(3) between-trivial2 l8-13-2)

```

```

lemma per-perp:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    Per A B C
  shows A B Perp B C
  using Perp-def assms(1) assms(2) assms(3) per-perp-in by blast

```

```

lemma perp-left-comm:
  assumes A B Perp C D
  shows B A Perp C D
proof -
  obtain X where X PerpAt A B C D
  using Perp-def assms by blast
  hence X PerpAt B A C D
  using PerpAt-def col-permutation-5 by auto
  thus ?thesis
  using Perp-def by blast
qed

```

```

lemma perp-right-comm:
  assumes A B Perp C D
  shows A B Perp D C
  by (meson Perp-def assms l8-12 perp-left-comm)

```

```

lemma perp-comm:
  assumes A B Perp C D
  shows B A Perp D C
  by (simp add: assms perp-left-comm perp-right-comm)

```

```

lemma perp-in-sym:
  assumes X PerpAt A B C D
  shows X PerpAt C D A B
  by (simp add: assms l8-12)

```

```

lemma perp-in-left-comm:
  assumes X PerpAt A B C D
  shows X PerpAt B A C D
  by (metis Col-cases PerpAt-def assms)

```


lemma *perp-in-right-comm*:
assumes $X \text{ PerpAt } A \ B \ C \ D$
shows $X \text{ PerpAt } A \ B \ D \ C$
using *assms perp-in-left-comm perp-in-sym* **by** *blast*

lemma *perp-in-comm*:
assumes $X \text{ PerpAt } A \ B \ C \ D$
shows $X \text{ PerpAt } B \ A \ D \ C$
by (*simp add: assms perp-in-left-comm perp-in-right-comm*)

lemma *Perp-cases*:
assumes $A \ B \text{ Perp } C \ D \vee B \ A \text{ Perp } C \ D \vee A \ B \text{ Perp } D \ C \vee B \ A \text{ Perp } D \ C \vee C \ D \text{ Perp } A \ B \vee$
 $C \ D \text{ Perp } B \ A \vee D \ C \text{ Perp } A \ B \vee D \ C \text{ Perp } B \ A$
shows $A \ B \text{ Perp } C \ D$
by (*meson Perp-def assms perp-in-sym perp-left-comm*)

lemma *Perp-perm* :
assumes $A \ B \text{ Perp } C \ D$
shows $A \ B \text{ Perp } C \ D \wedge B \ A \text{ Perp } C \ D \wedge A \ B \text{ Perp } D \ C \wedge B \ A \text{ Perp } D \ C \wedge C \ D \text{ Perp } A \ B \wedge$
 $C \ D \text{ Perp } B \ A \wedge D \ C \text{ Perp } A \ B \wedge D \ C \text{ Perp } B \ A$
by (*meson Perp-def assms perp-in-sym perp-left-comm*)

lemma *Perp-in-cases*:
assumes $X \text{ PerpAt } A \ B \ C \ D \vee X \text{ PerpAt } B \ A \ C \ D \vee X \text{ PerpAt } A \ B \ D \ C \vee X \text{ PerpAt } B \ A \ D \ C \vee$
 $X \text{ PerpAt } C \ D \ A \ B \vee X \text{ PerpAt } C \ D \ B \ A \vee X \text{ PerpAt } D \ C \ A \ B \vee X \text{ PerpAt } D \ C \ B \ A$
shows $X \text{ PerpAt } A \ B \ C \ D$
using *assms perp-in-left-comm perp-in-sym* **by** *blast*

lemma *Perp-in-perm*:
assumes $X \text{ PerpAt } A \ B \ C \ D$
shows $X \text{ PerpAt } A \ B \ C \ D \wedge X \text{ PerpAt } B \ A \ C \ D \wedge X \text{ PerpAt } A \ B \ D \ C \wedge X \text{ PerpAt } B \ A \ D \ C \wedge$
 $X \text{ PerpAt } C \ D \ A \ B \wedge X \text{ PerpAt } C \ D \ B \ A \wedge X \text{ PerpAt } D \ C \ A \ B \wedge X \text{ PerpAt } D \ C \ B \ A$
using *Perp-in-cases assms* **by** *blast*

lemma *perp-in-col*:
assumes $X \text{ PerpAt } A \ B \ C \ D$
shows $\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X$
using *PerpAt-def assms col-permutation-2* **by** *presburger*

lemma *perp-perp-in*:
assumes $A \ B \text{ Perp } C \ A$
shows $A \text{ PerpAt } A \ B \ C \ A$
using *assms l8-15-1 not-col-distincts* **by** *blast*

lemma *perp-per-1*:
assumes $A \ B \text{ Perp } C \ A$
shows $\text{Per } B \ A \ C$
using *Perp-in-cases assms perp-in-per perp-perp-in* **by** *blast*

lemma *perp-per-2*:
assumes $A \ B \text{ Perp } A \ C$
shows $\text{Per } B \ A \ C$
by (*simp add: Perp-perm assms perp-per-1*)

lemma *perp-col*:
assumes $A \neq E$ **and**
 $A \ B \text{ Perp } C \ D$ **and**
 $\text{Col } A \ B \ E$
shows $A \ E \text{ Perp } C \ D$
using *Perp-perm assms(1) assms(2) assms(3) col-trivial-3 perp-col0* **by** *blast*

lemma *perp-col2*:
assumes $A \ B \text{ Perp } X \ Y$ **and**
 $C \neq D$ **and**
 $\text{Col } A \ B \ C$ **and**

```

    Col A B D
  shows C D Perp X Y
  using Perp-perm assms(1) assms(2) assms(3) assms(4) perp-col0 by blast

lemma perp-col4:
  assumes P ≠ Q and
    R ≠ S and
    Col A B P and
    Col A B Q and
    Col C D R and
    Col C D S and
    A B Perp C D
  shows P Q Perp R S
  using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) perp-col0 by blast

lemma perp-not-eq-1:
  assumes A B Perp C D
  shows A ≠ B
  using assms perp-distinct by auto

lemma perp-not-eq-2:
  assumes A B Perp C D
  shows C ≠ D
  using assms perp-distinct by auto

lemma diff-per-diff:
  assumes A ≠ B and
    Cong A P B R and
    Per B A P
  and Per A B R
  shows P ≠ R
  using assms(1) assms(3) assms(4) l8-2 l8-7 by blast

lemma per-not-colp:
  assumes A ≠ B and
    A ≠ P and
    B ≠ R and
    Per B A P
  and Per A B R
  shows ¬ Col P A R
  by (metis Per-cases col-permutation-4 assms(1) assms(2) assms(4) assms(5) l8-3 l8-7)

lemma per-not-col:
  assumes A ≠ B and
    B ≠ C and
    Per A B C
  shows ¬ Col A B C
  using assms(1) assms(2) assms(3) l8-9 by auto

lemma perp-not-col2:
  assumes A B Perp C D
  shows ¬ Col A B C ∨ ¬ Col A B D
  using assms l8-14-1 perp-col2 perp-distinct by blast

lemma perp-not-col:
  assumes A B Perp P A
  shows ¬ Col A B P
proof -
  have A PerpAt A B P A
  using assms perp-perp-in by auto
  hence Per P A B
  by (simp add: perp-in-per perp-in-sym)
  hence ¬ Col B A P
  by (metis NCol-perm perp-not-eq-1 perp-not-eq-2 assms per-not-col)
  thus ?thesis
  using Col-perm by blast

```

qed

lemma *perp-in-col-perp-in*:

assumes $C \neq E$ and

$Col\ C\ D\ E$ and

$P\ PerpAt\ A\ B\ C\ D$

shows $P\ PerpAt\ A\ B\ C\ E$

proof –

have $C \neq D$

using *assms(3) perp-in-distinct* by blast

have $Col\ P\ C\ D$

using *PerpAt-def assms(3)* by blast

hence $Col\ P\ C\ E$

using $\langle C \neq D \rangle$ *assms(2) col-trivial-2 colx* by blast

moreover

{

fix $U\ V$

assume $Col\ U\ A\ B$ and $Col\ V\ C\ E$

hence $Per\ U\ P\ V$

by (*metis PerpAt-def assms(1) assms(2) assms(3) col-permutation-1 col-trivial-2 colx*)

}

ultimately

show *?thesis*

using *PerpAt-def assms(1) assms(3)* by presburger

qed

lemma *perp-col2-bis*:

assumes $A\ B\ Perp\ C\ D$ and

$Col\ C\ D\ P$ and

$Col\ C\ D\ Q$ and

$P \neq Q$

shows $A\ B\ Perp\ P\ Q$

using *Perp-cases assms(1) assms(2) assms(3) assms(4) perp-col0* by blast

lemma *perp-in-perp-bis-R1*:

assumes $X \neq A$ and

$X\ PerpAt\ A\ B\ C\ D$

shows $X\ B\ Perp\ C\ D \vee A\ X\ Perp\ C\ D$

by (*metis assms(2) l8-14-2-1a perp-col perp-in-col*)

lemma *perp-in-perp-bis*:

assumes $X\ PerpAt\ A\ B\ C\ D$

shows $X\ B\ Perp\ C\ D \vee A\ X\ Perp\ C\ D$

by (*metis assms l8-14-2-1a perp-in-perp-bis-R1*)

lemma *col-per-perp*:

assumes $A \neq B$ and

$B \neq C$ and

$D \neq C$ and

$Col\ B\ C\ D$ and

$Per\ A\ B\ C$

shows $C\ D\ Perp\ A\ B$

by (*metis Perp-cases assms(1) assms(2) assms(3) assms(4) assms(5) col-trivial-2 per-perp perp-col2-bis*)

lemma *per-cong-mid-R1*:

assumes $B = H$ and

$Bet\ A\ B\ C$ and

$Cong\ A\ H\ C\ H$ and

$Per\ H\ B\ C$

shows $B\ Midpoint\ A\ C$

using *assms(1) assms(2) assms(3) midpoint-def not-cong-1243* by blast

lemma *per-cong-mid-R2*:

```

assumes
   $B \neq C$  and
   $Bet\ A\ B\ C$  and
   $Cong\ A\ H\ C\ H$  and
   $Per\ H\ B\ C$ 
shows  $B\ Midpoint\ A\ C$ 
proof –
  have  $P1: Per\ C\ B\ H$ 
    using  $Per\text{-}cases\ assms(4)$  by blast
  have  $P2: Per\ H\ B\ A$ 
    using  $assms(1)\ assms(2)\ assms(4)\ bet\text{-}col\ col\text{-}permutation\text{-}1\ per\text{-}col$  by blast
  hence  $P3: Per\ A\ B\ H$ 
    using  $Per\text{-}cases$  by blast
  obtain  $C'$  where  $P4: B\ Midpoint\ C\ C' \wedge Cong\ H\ C\ H\ C'$ 
    using  $Per\text{-}def\ assms(4)$  by blast
  obtain  $H'$  where  $P5: B\ Midpoint\ H\ H' \wedge Cong\ C\ H\ C\ H'$ 
    using  $P1\ Per\text{-}def$  by blast
  obtain  $A'$  where  $P6: B\ Midpoint\ A\ A' \wedge Cong\ H\ A\ H\ A'$ 
    using  $P2\ Per\text{-}def$  by blast
  obtain  $H''$  where  $P7: B\ Midpoint\ H\ H'' \wedge Cong\ A\ H\ A\ H'$ 
    using  $P3\ P5\ Per\text{-}def\ Tarski\text{-}neutral\text{-}dimensionless\text{-}axioms\ symmetric\text{-}point\text{-}uniqueness$ 
    by fastforce
  hence  $P8: H' = H''$ 
    using  $P5\ symmetric\text{-}point\text{-}uniqueness$  by blast
  have  $H\ B\ H'\ A\ IFSC\ H\ B\ H'\ C$ 
proof –
  have  $Q1: Bet\ H\ B\ H'$ 
    by (simp add: P7 P8 midpoint-bet)
  have  $Q2: Cong\ H\ H'\ H\ H'$ 
    by (simp add: cong-reflexivity)
  have  $Q3: Cong\ B\ H'\ B\ H'$ 
    by (simp add: cong-reflexivity)
  have  $Q4: Cong\ H\ A\ H\ C$ 
    using  $assms(3)\ not\text{-}cong\text{-}2143$  by blast
  have  $Cong\ H'\ A\ H'\ C$ 
    using  $P5\ P7\ assms(3)\ cong\text{-}commutativity\ cong\text{-}inner\text{-}transitivity$  by blast
  thus ?thesis
    by (simp add: IFSC-def Q1 Q2 Q3 Q4)
qed
thus ?thesis
  using  $assms(1)\ assms(2)\ bet\text{-}col\ bet\text{-}neq23\text{-}neq\ l4\text{-}2\ l7\text{-}20\text{-}bis$  by auto
qed

lemma per-cong-mid:
assumes  $B \neq C$  and
   $Bet\ A\ B\ C$  and
   $Cong\ A\ H\ C\ H$  and
   $Per\ H\ B\ C$ 
shows  $B\ Midpoint\ A\ C$ 
using  $assms(1)\ assms(2)\ assms(3)\ assms(4)\ per\text{-}cong\text{-}mid\text{-}R1\ per\text{-}cong\text{-}mid\text{-}R2$  by blast

lemma per-double-cong:
assumes  $Per\ A\ B\ C$  and
   $B\ Midpoint\ C\ C'$ 
shows  $Cong\ A\ C\ A\ C'$ 
using  $Mid\text{-}cases\ Per\text{-}def\ assms(1)\ assms(2)\ l7\text{-}9\text{-}bis$  by blast

lemma cong-perp-or-mid-R1:
assumes  $Col\ A\ B\ X$  and
   $A \neq B$  and
   $M\ Midpoint\ A\ B$  and
   $Cong\ A\ X\ B\ X$ 
shows  $X = M \vee \neg Col\ A\ B\ X \wedge M\ PerpAt\ X\ M\ A\ B$ 
using  $assms(1)\ assms(2)\ assms(3)\ assms(4)\ col\text{-}permutation\text{-}5\ cong\text{-}commutativity\ l7\text{-}17\text{-}bis$ 
   $l7\text{-}2\ l7\text{-}20$  by blast

```

lemma *cong-perp-or-mid-R2*:
assumes $\neg \text{Col } A \ B \ X$ **and**
 $A \neq B$ **and**
 $M \text{ Midpoint } A \ B$ **and**
 $\text{Cong } A \ X \ B \ X$
shows $X = M \vee \neg \text{Col } A \ B \ X \wedge M \text{ PerpAt } X \ M \ A \ B$

proof –
have $P1: \text{Col } M \ A \ B$
by (*simp add: assms(3) midpoint-col*)
have $\text{Per } X \ M \ A$
using *Per-def assms(3) assms(4) cong-commutativity* **by** *blast*
thus *?thesis*
by (*metis P1 assms(1) assms(2) assms(3) midpoint-distinct-1 not-col-permutation-4*
per-perp-in perp-in-col-perp-in perp-in-right-comm)

qed

lemma *cong-perp-or-mid*:
assumes $A \neq B$ **and**
 $M \text{ Midpoint } A \ B$ **and**
 $\text{Cong } A \ X \ B \ X$
shows $X = M \vee \neg \text{Col } A \ B \ X \wedge M \text{ PerpAt } X \ M \ A \ B$
using *assms(1) assms(2) assms(3) cong-perp-or-mid-R1 cong-perp-or-mid-R2* **by** *blast*

lemma *col-per2-cases*:
assumes $B \neq C$ **and**
 $B' \neq C$ **and**
 $C \neq D$ **and**
 $\text{Col } B \ C \ D$ **and**
 $\text{Per } A \ B \ C$ **and**
 $\text{Per } A \ B' \ C$
shows $B = B' \vee \neg \text{Col } B' \ C \ D$
by (*meson l8-7 Tarski-neutral-dimensionless-axioms assms(1) assms(2) assms(3) assms(4)*
assms(5) assms(6) l6-16-1 per-col)

lemma *l8-16-1*:
assumes $\text{Col } A \ B \ X$ **and**
 $\text{Col } A \ B \ U$ **and**
 $A \ B \ \text{Perp } C \ X$
shows $\neg \text{Col } A \ B \ C \wedge \text{Per } C \ X \ U$
by (*metis assms(1) assms(2) assms(3) l8-5 perp-col0 perp-left-comm perp-not-col2 perp-per-2*)

lemma *l8-16-2*:
assumes $\text{Col } A \ B \ X$ **and**
 $\text{Col } A \ B \ U$ **and**
 $U \neq X$ **and**
 $\neg \text{Col } A \ B \ C$ **and**
 $\text{Per } C \ X \ U$
shows $A \ B \ \text{Perp } C \ X$

proof –
obtain X **where** $X \text{ PerpAt } A \ B \ C \ X$
by (*metis NCol-perm assms(1) assms(2) assms(3) assms(4) assms(5) l8-13-2 l8-2*
not-col-distincts)
thus *?thesis*
by (*metis Perp-perm per-col assms(1) assms(2) assms(3) assms(4) assms(5) col3 col-per-perp*
not-col-distincts)

qed

lemma *l8-18-uniqueness*:
assumes
 $\text{Col } A \ B \ X$ **and**
 $A \ B \ \text{Perp } C \ X$ **and**
 $\text{Col } A \ B \ Y$ **and**
 $A \ B \ \text{Perp } C \ Y$
shows $X = Y$
using *assms(1) assms(2) assms(3) assms(4) l8-16-1 l8-7* **by** *blast*

```

lemma midpoint-distinct:
  assumes  $\neg \text{Col } A \ B \ C$  and
     $\text{Col } A \ B \ X$  and
     $X \text{ Midpoint } C \ C'$ 
  shows  $C \neq C'$ 
  using assms(1) assms(2) assms(3) l7-3 by auto

lemma l8-20-1-R1:
  assumes  $A = B$ 
  shows  $\text{Per } B \ A \ P$ 
  by (simp add: assms l8-2 l8-5)

lemma l8-20-1-R2:
  assumes  $A \neq B$  and
     $\text{Per } A \ B \ C$  and
     $P \text{ Midpoint } C' \ D$  and
     $A \text{ Midpoint } C' \ C$  and
     $B \text{ Midpoint } D \ C$ 
  shows  $\text{Per } B \ A \ P$ 
proof -
  obtain  $B'$  where  $P1: A \text{ Midpoint } B \ B'$ 
    using symmetric-point-construction by blast
  obtain  $D'$  where  $P2: A \text{ Midpoint } D \ D'$ 
    using symmetric-point-construction by blast
  obtain  $P'$  where  $P3: A \text{ Midpoint } P \ P'$ 
    using symmetric-point-construction by blast
  have  $P4: \text{Per } B' \ B \ C$ 
    by (metis P1 Per-cases per-col assms(1) assms(2) midpoint-col not-col-permutation-4)
  have  $P5: \text{Per } B \ B' \ C'$ 
  proof -
    have  $\text{Per } B' \ B \ C$ 
      by (simp add: P4)
    have  $B' \ B \ C \text{ Cong3 } B \ B' \ C'$ 
      by (meson Cong3-def P1 assms(4) l7-13 l7-2)
    thus ?thesis
      using P4 l8-10 by blast
  qed
  have  $P6: B' \text{ Midpoint } D' \ C'$ 
    by (meson P1 P2 assms(4) assms(5) l7-15 l7-16 l7-2 midpoint-bet midpoint-cong midpoint-def)
  have  $P7: P' \text{ Midpoint } C \ D'$ 
    using P2 P3 assms(3) assms(4) symmetry-preserves-midpoint by blast
  have  $P8: A \text{ Midpoint } P \ P'$ 
    by (simp add: P3)
  obtain  $D''$  where  $P9: B \text{ Midpoint } C \ D'' \wedge \text{Cong } B' \ C \ B' \ D$ 
    using P4 assms(5) l7-2 per-double-cong by blast
  have  $P10: D'' = D$ 
    using P9 assms(5) l7-9-bis by blast
  obtain  $D''$  where  $P11: B' \text{ Midpoint } C' \ D'' \wedge \text{Cong } B \ C' \ B \ D''$ 
    using P5 Per-def by blast
  have  $P12: D' = D''$ 
    by (meson P11 P6 l7-9-bis Tarski-neutral-dimensionless-axioms)
  have  $P13: P \text{ Midpoint } C' \ D$ 
    using assms(3) by blast
  have  $P14: \text{Cong } C \ D \ C' \ D'$ 
    using P2 assms(4) l7-13 l7-2 by blast
  have  $P15: \text{Cong } C' \ D \ C \ D'$ 
    using P2 assms(4) cong-4321 l7-13 by blast
  have  $P16: \text{Cong } P \ D \ P' \ D'$ 
    using P2 P8 cong-symmetry l7-13 by blast
  have  $P17: \text{Cong } P \ D \ P' \ C$ 
    using P16 P7 cong-3421 cong-transitivity midpoint-cong by blast
  have  $P18: C' \ P \ D \ B \text{ IFSC } D' \ P' \ C \ B$ 
    by (metis Bet-cases IFSC-def P10 P11 P12 P13 P15 P17 P7 P9 cong-commutativity
      cong-right-commutativity l7-13 l7-3-2 midpoint-bet)
  hence  $\text{Cong } B \ P \ B \ P'$ 
    using l4-2 Tarski-neutral-dimensionless-axioms not-cong-2143 by fastforce

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thus ?thesis
 using P8 Per-def by blast
 qed

lemma l8-20-1:
 assumes Per A B C and
 P Midpoint C' D and
 A Midpoint C' C and
 B Midpoint D C
 shows Per B A P
 using assms(1) assms(2) assms(3) assms(4) l8-20-1-R1 l8-20-1-R2 by fastforce

lemma l8-20-2:
 assumes P Midpoint C' D and
 A Midpoint C' C and
 B Midpoint D C and
 B $\neq$ C
 shows A $\neq$ P
 using assms(1) assms(2) assms(3) assms(4) l7-3 symmetric-point-uniqueness by blast

lemma perp-col1:
 assumes C $\neq$ X and
 A B Perp C D and
 Col C D X
 shows A B Perp C X
 using assms(1) assms(2) assms(3) col-trivial-3 perp-col2-bis by blast

lemma l8-18-existence:
 assumes $\neg$ Col A B C
 shows $\exists X. \text{Col } A B X \wedge A B \text{ Perp } C X$

proof –

obtain Y where Bet B A Y and Cong A Y A C
 using segment-construction by blast
 then obtain P where P Midpoint C Y
 using Mid-cases l7-25 by blast
 hence Per A P Y
 using Per-def $\langle \text{Cong } A Y A C \rangle$ l7-2 by blast
 obtain Z where Bet A Y Z and Cong Y Z Y P
 using segment-construction by blast
 obtain Q where Bet P Y Q and Cong Y Q Y A
 using segment-construction by blast
 obtain Q' where Bet Q Z Q' and Cong Z Q' Q Z
 using segment-construction by blast
 hence Z Midpoint Q Q'
 using midpoint-def not-cong-3412 by blast
 obtain C' where Bet Q' Y C' and Cong Y C' Y C
 using segment-construction by blast
 obtain X where X Midpoint C C'
 using $\langle \text{Cong } Y C' Y C \rangle$ l7-2 l7-25 by blast
 have A Y Z Q OFSC Q Y P A
 by (simp add: OFSC-def $\langle \text{Bet } A Y Z \rangle \langle \text{Bet } P Y Q \rangle \langle \text{Cong } Y Q Y A \rangle \langle \text{Cong } Y Z Y P \rangle$
 between-symmetry cong-4321 cong-pseudo-reflexivity)
 have A $\neq$ Y
 using $\langle \text{Cong } A Y A C \rangle$ assms is-midpoint-id l7-20 not-col-distincts by blast
 hence Cong Z Q P A
 using $\langle A Y Z Q \text{ OFSC } Q Y P A \rangle$ five-segment-with-def by auto
 hence A P Y Cong3 Q Z Y
 using Cong3-def Cong-cases $\langle \text{Cong } Y Q Y A \rangle \langle \text{Cong } Y Z Y P \rangle$ by blast
 hence Per Q Z Y
 using $\langle \text{Per } A P Y \rangle$ l8-10 by blast
 hence Per Y Z Q
 using l8-2 by blast
 have P $\neq$ Y
 by (metis Col-def $\langle \text{Bet } B A Y \rangle \langle P \text{ Midpoint } C Y \rangle$ assms between-symmetry midpoint-not-midpoint)
 obtain Q'' where Z Midpoint Q Q'' and Cong Y Q Y Q'
 using $\langle \text{Per } Y Z Q \rangle \langle Z \text{ Midpoint } Q Q' \rangle$ per-double-cong by force

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hence  $Q' = Q''$ 
  by (meson symmetric-point-uniqueness  $\langle Z \text{ Midpoint } Q \ Q' \rangle$ )
have  $Bet \ C \ Y \ C$ 
  by (metis midpoint-bet  $\langle Bet \ P \ Y \ Q \rangle \langle P \text{ Midpoint } C \ Y \rangle \langle P \neq Y \rangle$  between-symmetry
    outer-transitivity-between2)
hence  $Bet \ Z \ Y \ X$ 
  by (meson l7-22  $\langle Bet \ Q' \ Y \ C' \rangle \langle Cong \ Y \ C' \ Y \ C \rangle \langle Cong \ Y \ Q \ Y \ Q' \rangle \langle X \text{ Midpoint } C \ C' \rangle$ 
     $\langle Z \text{ Midpoint } Q \ Q' \rangle$  cong-symmetry)
have  $Q \neq Y$ 
  using  $\langle A \neq Y \rangle \langle Cong \ Y \ Q \ Y \ A \rangle$  cong-reverse-identity by blast
have  $Bet \ C \ P \ Y$ 
  using Midpoint-def  $\langle P \text{ Midpoint } C \ Y \rangle$  by auto
hence  $Per \ Y \ X \ C$ 
  using Per-def  $\langle Cong \ Y \ C' \ Y \ C \rangle \langle X \text{ Midpoint } C \ C' \rangle$  cong-inner-transitivity
    cong-reflexivity by blast
moreover
have  $Col \ P \ Y \ Q$ 
  by (simp add: Col-def  $\langle Bet \ P \ Y \ Q \rangle$ )
have  $Col \ P \ Y \ C$ 
  by (simp add: Col-def  $\langle Bet \ C \ P \ Y \rangle$ )
have  $Col \ P \ Q \ C$ 
  using  $\langle Col \ P \ Y \ C \rangle \langle Col \ P \ Y \ Q \rangle \langle P \neq Y \rangle$  col-transitivity-1 by blast
have  $Col \ Y \ Q \ C$ 
  using Bet-cases Col-def  $\langle Bet \ Q \ Y \ C \rangle$  by auto
have  $Col \ A \ Y \ B$ 
  by (simp add: Col-def  $\langle Bet \ B \ A \ Y \rangle$ )
moreover
have  $Col \ A \ Y \ Z$ 
  using Col-def  $\langle Bet \ A \ Y \ Z \rangle$  by blast
have  $Col \ A \ B \ Z$ 
  using  $\langle A \neq Y \rangle \langle Col \ A \ Y \ B \rangle \langle Col \ A \ Y \ Z \rangle$  col-transitivity-1 by blast
have  $Col \ Y \ B \ Z$ 
  using  $\langle A \neq Y \rangle \langle Col \ A \ Y \ B \rangle \langle Col \ A \ Y \ Z \rangle$  col-transitivity-2 by blast
have  $Col \ Q \ Y \ P$ 
  using Col-cases  $\langle Col \ P \ Y \ Q \rangle$  by blast
have  $Q \neq C$ 
  using  $\langle Bet \ Q \ Y \ C \rangle \langle Q \neq Y \rangle$  between-identity by blast
have  $Col \ Y \ Q' \ C'$ 
  using Col-cases Col-def  $\langle Bet \ Q' \ Y \ C' \rangle$  by blast
{
  assume  $Q = Q'$ 
  hence  $Col \ P \ B \ C$ 
    by (metis cong-reverse-identity  $\langle Col \ A \ Y \ B \rangle \langle Col \ P \ Y \ C \rangle \langle Cong \ Z \ Q \ P \ A \rangle \langle P \neq Y \rangle$ 
       $\langle Q' = Q'' \rangle \langle Z \text{ Midpoint } Q \ Q'' \rangle$  col-transitivity-1 midpoint-distinct-2)
  moreover have  $\neg Col \ P \ B \ C$ 
    by (metis  $\langle Q = Q' \rangle \langle Bet \ Q \ Z \ Q' \rangle \langle Cong \ Z \ Q \ P \ A \rangle$  assms bet-neq12--neq cong-diff-3)
  ultimately have False
    by blast
}
hence  $C \neq C'$ 
  by (metis between-cong-3  $\langle Bet \ Q \ Y \ C \rangle \langle Bet \ Q' \ Y \ C' \rangle \langle Cong \ Y \ Q \ Y \ Q' \rangle \langle P \text{ Midpoint } C \ Y \rangle$ 
     $\langle P \neq Y \rangle$  between-symmetry midpoint-distinct-3)
have  $Q \ Y \ C \ Z \ OFSC \ Q' \ Y \ C' \ Z$ 
  by (simp add: OFSC-def  $\langle Bet \ Q \ Y \ C \rangle \langle Bet \ Q' \ Y \ C' \rangle \langle Cong \ Y \ C' \ Y \ C \rangle \langle Cong \ Y \ Q \ Y \ Q' \rangle$ 
     $\langle Cong \ Z \ Q' \ Q \ Z \rangle$  cong-3421 cong-reflexivity cong-symmetry)
hence  $Cong \ C \ Z \ C' \ Z$ 
  using  $\langle Q \neq Y \rangle$  five-segment-with-def by force
have  $Col \ Z \ Y \ X$ 
  using Col-def  $\langle Bet \ Z \ Y \ X \rangle$  by blast
have  $Y \neq Z$ 
  using  $\langle Cong \ Y \ Z \ Y \ P \rangle \langle P \neq Y \rangle$  cong-diff-4 by blast
{
  assume  $X = Y$ 
  hence  $C' \neq Y$ 
    using  $\langle C \neq C' \rangle \langle Cong \ Y \ C' \ Y \ C \rangle$  cong-reverse-identity by blast
  have  $Col \ Y \ C' \ P$ 

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    by (metis Col-def Midpoint-def ⟨Bet C P Y⟩ ⟨X = Y⟩ ⟨X Midpoint C C'⟩ between-equality-2
        col-transitivity-1 not-bet-distincts)
  hence Col Y P Q'
    by (metis ⟨C' ≠ Y⟩ ⟨Col Y Q' C'⟩ col-trivial-3 colx not-col-permutation-5)
  hence Col Y Q Q'
    by (meson ⟨Col P Y Q⟩ ⟨P ≠ Y⟩ colx not-col-distincts not-col-permutation-5)
  hence False
    using l7-20 ⟨Cong Y Q Y Q'⟩ ⟨Q = Q' ⟹ False⟩ ⟨Q' = Q''⟩ ⟨Y ≠ Z⟩
        ⟨Z Midpoint Q Q''⟩ col-permutation-4 l7-17 by blast
}
hence X ≠ Y
  by auto
moreover have Col A B X
  by (meson ⟨Col A Y Z⟩ ⟨Col Y B Z⟩ ⟨Col Z Y X⟩ ⟨Y ≠ Z⟩ col3 col-permutation-3
      not-col-permutation-1)
ultimately show ?thesis
  by (metis Col-cases l8-2 assms l8-16-2)
qed

lemma l8-21-aux:
  assumes ¬ Col A B C
  shows ∃ P. ∃ T. (A B Perp P A ∧ Col A B T ∧ Bet C T P)
proof -
  obtain X where Col A B X and A B Perp C X
    using assms l8-18-existence by blast
  hence X PerpAt A B C X
    by (simp add: l8-15-1)
  have Per A X C
    using PerpAt-def ⟨X PerpAt A B C X⟩ col-trivial-1 by presburger
  obtain C' where X Midpoint C C' and Cong A C A C'
    using Per-def ⟨Per A X C⟩ by auto
  obtain C'' where A Midpoint C C''
    using symmetric-point-construction by blast
  obtain P where P Midpoint C' C''
    by (metis Cong-cases ⟨A Midpoint C C''⟩ ⟨Cong A C A C'⟩ cong-inner-transitivity
        l7-25 midpoint-cong)
  have Per X A P
  proof -
    have P Midpoint C'' C'
      using ⟨P Midpoint C' C''⟩ l7-2 by blast
    moreover have A Midpoint C'' C
      using ⟨A Midpoint C C''⟩ l7-2 by blast
    moreover have X Midpoint C' C
      by (simp add: ⟨X Midpoint C C'⟩ l7-2)
    ultimately show ?thesis
      using ⟨Per A X C⟩ l8-20-1 by blast
  qed
  have X ≠ C
    using ⟨Col A B X⟩ assms by blast
  hence A ≠ P
    using ⟨A Midpoint C C''⟩ ⟨P Midpoint C' C''⟩ ⟨X Midpoint C C'⟩ l7-9 midpoint-distinct-2
        by blast
  have ∃ T. Bet P T C ∧ Bet A T X
  proof -
    have Bet C'' A C
      using Mid-cases ⟨A Midpoint C C''⟩ midpoint-bet by blast
    moreover have Bet C' X C
      using Bet-cases Midpoint-def ⟨X Midpoint C C'⟩ by auto
    moreover have Bet C'' P C'
      using ⟨P Midpoint C' C''⟩ between-symmetry midpoint-bet by blast
    ultimately show ?thesis
      using l3-17 by fastforce
  qed
  then obtain T where Bet P T C and Bet A T X
    by blast
  show ?thesis

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proof cases
  assume  $A = X$ 
  thus ?thesis
    by (metis Bet-perm between-identity midpoint-col midpoint-not-midpoint perp-col1
       $\langle A \ B \ \text{Perp} \ C \ X \rangle \langle A \ \text{Midpoint} \ C \ C'' \rangle \langle \text{Bet} \ A \ T \ X \rangle \langle \text{Bet} \ P \ T \ C \rangle \langle \text{Col} \ A \ B \ X \rangle$ 
       $\langle P \ \text{Midpoint} \ C' \ C'' \rangle \langle X \ \text{Midpoint} \ C \ C' \rangle$  l8-20-2 perp-comm)
next
  assume  $A \neq X$ 
  have  $A \ B \ \text{Perp} \ P \ A$ 
    by (metis col3 l8-2 per-not-col  $\langle A \neq P \rangle \langle A \neq X \rangle \langle \text{Col} \ A \ B \ X \rangle \langle \text{Per} \ X \ A \ P \rangle$ 
      assms l8-16-2 not-col-distincts)
  moreover have  $\text{Col} \ A \ B \ T$ 
    by (metis Col-def  $\langle A \neq X \rangle \langle \text{Bet} \ A \ T \ X \rangle \langle \text{Col} \ A \ B \ X \rangle$  col-transitivity-2)
  moreover have  $\text{Bet} \ C \ T \ P$ 
    using Bet-cases  $\langle \text{Bet} \ P \ T \ C \rangle$  by blast
  ultimately show ?thesis
    by blast
qed
qed

lemma l8-21:
  assumes  $A \neq B$ 
  shows  $\exists \ P \ T. \ A \ B \ \text{Perp} \ P \ A \wedge \text{Col} \ A \ B \ T \wedge \text{Bet} \ C \ T \ P$ 
  by (meson assms between-trivial2 l8-21-aux not-col-exists)

lemma per-cong:
  assumes  $A \neq B$  and
     $A \neq P$  and
     $\text{Per} \ B \ A \ P$  and
     $\text{Per} \ A \ B \ R$  and
     $\text{Cong} \ A \ P \ B \ R$  and
     $\text{Col} \ A \ B \ X$  and
     $\text{Bet} \ P \ X \ R$ 
  shows  $\text{Cong} \ A \ R \ P \ B$ 
proof –
  have  $\text{Per} \ P \ A \ B$ 
    using Per-cases assms(3) by blast
  obtain  $Q$  where  $R \ \text{Midpoint} \ B \ Q$ 
    using symmetric-point-construction by auto
  have  $B \neq R$ 
    using assms(2) assms(5) cong-identity by blast
  hence  $\text{Per} \ A \ B \ Q$ 
    using Col-def Midpoint-def  $\langle R \ \text{Midpoint} \ B \ Q \rangle$  assms(4) per-col by blast
  have  $\text{Per} \ P \ A \ X$ 
    using  $\langle \text{Per} \ P \ A \ B \rangle$  assms(1) assms(6) per-col by blast
  have  $B \neq Q$ 
    using  $\langle B \neq R \rangle \langle R \ \text{Midpoint} \ B \ Q \rangle$  l7-3 by blast
  have  $\text{Per} \ R \ B \ X$ 
    by (metis col-permutation-4 per-col assms(1) assms(4) assms(6) l8-2)
  have  $X \neq A$ 
    using  $\langle B \neq R \rangle$  assms(1) assms(2) assms(3) assms(4) assms(7) bet-col per-not-colp by blast
  obtain  $P'$  where  $A \ \text{Midpoint} \ P \ P'$ 
    using Per-def assms(3) by blast
  obtain  $R'$  where  $\text{Bet} \ P' \ X \ R'$  and  $\text{Cong} \ X \ R' \ X \ R$ 
    using segment-construction by blast
  obtain  $M$  where  $M \ \text{Midpoint} \ R \ R'$ 
    using  $\langle \text{Cong} \ X \ R' \ X \ R \rangle$  l7-2 l7-25 by blast
  have  $\text{Per} \ X \ M \ R$ 
    by (metis Per-def  $\langle \text{Cong} \ X \ R' \ X \ R \rangle \langle M \ \text{Midpoint} \ R \ R' \rangle$  cong-symmetry)
  have  $\text{Cong} \ X \ P \ X \ P'$ 
    using Per-cases  $\langle A \ \text{Midpoint} \ P \ P' \rangle \langle \text{Per} \ P \ A \ X \rangle$  per-double-cong by blast
  have  $X \neq P'$ 
    using  $\langle \text{Cong} \ X \ P \ X \ P' \rangle \langle \text{Per} \ P \ A \ X \rangle$  assms(2) cong-identity l8-8 by blast
  have  $P \neq P'$ 
    using  $\langle A \ \text{Midpoint} \ P \ P' \rangle$  assms(2) l7-3 by blast
  have  $\neg \text{Col} \ X \ P \ P'$ 

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    using ⟨A Midpoint P P'⟩ ⟨Cong X P X P'⟩ ⟨P ≠ P'⟩ ⟨X ≠ A⟩ col-permutation-4 l7-17 l7-20
  by blast
have Bet A X M
  by (meson ⟨A Midpoint P P'⟩ ⟨Bet P' X R'⟩ ⟨Cong X P X P'⟩ ⟨Cong X R' X R⟩ ⟨M Midpoint R R'⟩
    assms(7) cong-symmetry l7-22)
have X ≠ R
  using ⟨B ≠ R⟩ ⟨Per R B X⟩ l8-8 by blast
have X ≠ R'
  using ⟨Cong X R' X R⟩ ⟨X ≠ R⟩ cong-diff-3 by blast
{
  assume X = M
  have Col X P P'
  proof -
    have Col X R P
      using Col-def assms(7) by blast
    moreover have Col X R P'
      by (metis Col-def ⟨Bet P' X R'⟩ ⟨M Midpoint R R'⟩ ⟨X = M⟩ ⟨X ≠ R'⟩
        col-transitivity-1 midpoint-col)
    ultimately show ?thesis
      using ⟨X ≠ R⟩ col-transitivity-1 by blast
  qed
  hence False
    by (simp add: ⟨¬ Col X P P'⟩)
}
have M = B
proof -
  have ¬ Col A X R
    by (metis ⟨B ≠ R⟩ ⟨X ≠ A⟩ assms(1) assms(4) assms(6) col-trivial-3 colx per-not-col)
  moreover have Col A X M
    using Col-def ⟨Bet A X M⟩ by auto
  moreover have A X Perp R M
    by (metis Col-cases Per-cases ⟨Per X M R⟩ ⟨X = M ⟹ False⟩ ⟨X ≠ A⟩ calculation(1)
      calculation(2) col-per-perp perp-left-comm)
  moreover have Col A X B
    using Col-cases assms(6) by blast
  moreover have A X Perp R B
    by (metis Col-cases Per-cases ⟨B ≠ R⟩ ⟨X ≠ A⟩ assms(1) assms(4) assms(6) col-per-perp)
  ultimately show ?thesis
    using l8-18-uniqueness by blast
qed
have P X R P' OFSC P' X R' P
  using Cong-cases OFSC-def ⟨Bet P' X R'⟩ ⟨Cong X P X P'⟩ ⟨Cong X R' X R⟩ assms(7)
    cong-pseudo-reflexivity by auto
hence Cong R P' R' P
  using ⟨¬ Col X P P'⟩ five-segment-with-def not-col-distincts by blast
have P' A P R IFSC R' B R P
proof -
  have M = B
    using ⟨M = B⟩ by auto
  then have B Midpoint R R'
    using ⟨M Midpoint R R'⟩ by blast
  then show ?thesis
    by (smt (z3) Bet-cases Cong-cases IFSC-def Midpoint-def ⟨A Midpoint P P'⟩
      ⟨Cong R P' R' P⟩ assms(5) cong-mid2--cong cong-pseudo-reflexivity)
qed
thus ?thesis
  using l4-2 not-cong-1243 by blast
qed

lemma perp-cong:
  assumes A ≠ B and
    A ≠ P and
    A B Perp P A and
    A B Perp R B and
    Cong A P B R and
    Col A B X and

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    Bet P X R
  shows Cong A R P B
  using Perp-cases assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) per-cong
    perp-per-1 by blast

lemma perp-exists:
  assumes A ≠ B
  shows ∃ X. PO X Perp A B
proof cases
  assume Col A B PO
  then obtain C where A ≠ C and B ≠ C and PO ≠ C and Col A B C
    using diff-col-ex3 by blast
  then obtain P T where PO C Perp P PO and Col PO C T and Bet PO T P
    using l8-21 by blast
  hence PO P Perp A B
    by (metis Perp-perm col-transitivity-2 ⟨Col A B C⟩ ⟨Col A B PO⟩ assms
      not-col-permutation-2 perp-col0)
  thus ?thesis
    by blast
next
  assume ¬ Col A B PO
  thus ?thesis
    using l8-18-existence assms col-trivial-2 col-trivial-3 l8-18-existence perp-col0 by blast
qed

lemma perp-vector:
  assumes A ≠ B
  shows ∃ X Y. A B Perp X Y
  using assms l8-21 by blast

lemma midpoint-existence-aux:
  assumes A ≠ B and
    A B Perp Q B and
    A B Perp P A and
    Col A B T and
    Bet Q T P and
    A P Le B Q
  shows ∃ X. X Midpoint A B
proof -
  obtain R where Bet B R Q and Cong A P B R
    using Le-def assms(6) by blast
  obtain X where Bet T X B and Bet R X P
    by (meson ⟨Bet B R Q⟩ assms(5) between-symmetry inner-pasch)
  have Col A B X
    by (metis Col-def ⟨Bet T X B⟩ assms(4) between-equality-2 between-trivial2 col2--eq)
  have B ≠ R
    using ⟨Cong A P B R⟩ assms(3) cong-identity perp-not-eq-2 by blast
  have ¬ Col A B Q
    using Col-cases Perp-cases assms(2) perp-not-col by blast
  have ¬ Col A B R
    using Col-def ⟨B ≠ R⟩ ⟨Bet B R Q⟩ ⟨¬ Col A B Q⟩ l6-16-1 by blast
  have P ≠ R
    using ⟨Bet R X P⟩ ⟨Col A B X⟩ ⟨¬ Col A B R⟩ between-identity by blast
  have ∃ X. X Midpoint A B
  proof cases
    assume A = P
    thus ?thesis
      using assms(3) col-trivial-3 perp-not-col2 by blast
  next
    assume ¬ A = P
    have A B Perp R B
      by (metis Col-def ⟨B ≠ R⟩ ⟨Bet B R Q⟩ assms(2) not-col-distincts perp-col2-bis)
    hence Cong A R P B
      using ⟨A ≠ P⟩ ⟨Bet R X P⟩ ⟨Col A B X⟩ ⟨Cong A P B R⟩ assms(1) assms(3)
        between-symmetry perp-cong by blast
    hence X Midpoint A B ∧ X Midpoint P R

```

```

    by (meson ⟨Bet R X P⟩ ⟨Col A B X⟩ ⟨Cong A P B R⟩ ⟨P ≠ R⟩ assms(3) bet-col
        between-symmetry cong-4312 l7-2 l7-21 not-col-permutation-2 perp-not-col)
  thus ?thesis
    by blast
qed
thus ?thesis by blast
qed

lemma midpoint-existence:
  ∃ X. X Midpoint A B
proof cases
  assume A = B
  thus ?thesis
    using l7-3-2 by blast
next
  assume P1: ¬ A = B
  obtain Q where P2: A B Perp B Q
    by (metis P1 l8-21 perp-comm)
  obtain P T where P3: A B Perp P A ∧ Col A B T ∧ Bet Q T P
    using P2 l8-21-aux not-col-distincts perp-not-col2 by blast
  have P4: A P Le B Q ∨ B Q Le A P
    by (simp add: local.le-cases)
  have P5: A P Le B Q ⟶ (∃ X. X Midpoint A B)
    by (meson P1 P2 P3 Perp-cases midpoint-existence-aux Tarski-neutral-dimensionless-axioms)
  have P6: B Q Le A P ⟶ (∃ X. X Midpoint A B)
  proof -
    {
      assume H1: B Q Le A P
      have Q6: B ≠ A
        using P1 by auto
      have Q2: B A Perp P A
        by (simp add: P3 perp-left-comm)
      have Q3: B A Perp Q B
        using P2 Perp-perm by blast
      have Q4: Col B A T
        using Col-perm P3 by blast
      have Q5: Bet P T Q
        using Bet-perm P3 by blast
      obtain X where X Midpoint B A
        using H1 Q2 Q3 Q4 Q5 Q6 midpoint-existence-aux by blast
      hence ∃ X. X Midpoint A B
        using l7-2 by blast
    }
  thus ?thesis
    by simp
qed
thus ?thesis
  using P4 P5 by blast
qed

lemma MidR-uniq-aux:
  shows ∃!x. x Midpoint A B
  using l7-17-bis midpoint-existence by blast

lemma SymR-uniq-aux:
  assumes B Midpoint A x and
    B Midpoint A y
  shows x = y
  using assms(1) assms(2) symmetric-point-uniqueness by auto

lemma perp-in-id:
  assumes X PerpAt A B C A
  shows X = A
  by (meson Col-cases assms col-trivial-3 l8-14-2-1b)

lemma l8-22:

```

```

assumes  $A \neq B$  and
   $A \neq P$  and
   $Per\ B\ A\ P$  and
   $Per\ A\ B\ R$  and
   $Cong\ A\ P\ B\ R$  and
   $Col\ A\ B\ X$  and
   $Bet\ P\ X\ R$  and
   $Cong\ A\ R\ P\ B$ 
shows  $X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$ 
by (metis assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) assms(8)
  bet-col cong-commutativity cong-diff cong-right-commutativity l7-21
  not-col-permutation-5 per-not-colp)

lemma l8-22-bis:
assumes  $A \neq B$  and
   $A \neq P$  and
   $A\ B\ Perp\ P\ A$  and
   $A\ B\ Perp\ R\ B$  and
   $Cong\ A\ P\ B\ R$  and
   $Col\ A\ B\ X$  and
   $Bet\ P\ X\ R$ 
shows  $Cong\ A\ R\ P\ B \wedge X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$ 
by (metis l8-22 Perp-cases assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
  assms(7) perp-cong perp-per-2)

lemma perp-in-perp:
assumes  $X\ PerpAt\ A\ B\ C\ D$ 
shows  $A\ B\ Perp\ C\ D$ 
using assms l8-14-2-1a by auto

lemma perp-proj:
assumes  $A\ B\ Perp\ C\ D$  and
   $\neg\ Col\ A\ C\ D$ 
shows  $\exists\ X. Col\ A\ B\ X \wedge A\ X\ Perp\ C\ D$ 
using assms(1) not-col-distincts by auto

lemma l8-24 :
assumes  $P\ A\ Perp\ A\ B$  and
   $Q\ B\ Perp\ A\ B$  and
   $Col\ A\ B\ T$  and
   $Bet\ P\ T\ Q$  and
   $Bet\ B\ R\ Q$  and
   $Cong\ A\ P\ B\ R$ 
shows  $\exists\ X. X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$ 
proof –
obtain  $X$  where  $P1: Bet\ T\ X\ B \wedge Bet\ R\ X\ P$ 
using assms(4) assms(5) inner-pasch by blast
have  $P2: Col\ A\ B\ X$ 
by (metis Out-cases  $P1$  assms(3) bet-out-1 col-out2-col not-col-distincts out-trivial)
have  $P3: A \neq B$ 
using assms(1) col-trivial-2 l8-16-1 by blast
have  $P4: A \neq P$ 
using assms(1) col-trivial-1 l8-16-1 by blast
have  $\exists\ X. X\ Midpoint\ A\ B \wedge X\ Midpoint\ P\ R$ 
proof cases
assume  $Col\ A\ B\ P$ 
thus ?thesis
using Perp-perm assms(1) perp-not-col by blast
next
assume  $Q1: \neg\ Col\ A\ B\ P$ 
have  $Q2: B \neq R$ 
using  $P4$  assms(6) cong-diff by blast
have  $Q3: Q \neq B$ 
using  $Q2$  assms(5) between-identity by blast
have  $Q4: \neg\ Col\ A\ B\ Q$ 
by (metis assms(2) col-permutation-3 l8-14-1 perp-col1 perp-not-col)

```

```

have Q5:  $\neg \text{Col } A \ B \ R$ 
  by (meson Q2 Q4 assms(5) bet-col col-transitivity-1 not-col-permutation-2)
have Q6:  $P \neq R$ 
  using P1 P2 Q5 between-identity by blast
have  $\exists X. X \text{ Midpoint } A \ B \wedge X \text{ Midpoint } P \ R$ 
proof cases
  assume  $A = P$ 
  thus ?thesis
    using P4 by blast
next
  assume R0:  $\neg A = P$ 
  have R1:  $A \ B \ \text{Perp} \ R \ B$ 
    by (metis Perp-cases Q2 bet-col1 assms(2) assms(5) bet-col col-transitivity-1 perp-col1)
  have R2:  $\text{Cong } A \ R \ P \ B$ 
    using P1 P2 P3 Perp-perm R0 R1 assms(1) assms(6) between-symmetry perp-cong by blast
  have R3:  $\neg \text{Col } A \ P \ B$ 
    using Col-perm Q1 by blast
  have R4:  $P \neq R$ 
    by (simp add: Q6)
  have R5:  $\text{Cong } A \ P \ B \ R$ 
    by (simp add: assms(6))
  have R6:  $\text{Cong } P \ B \ R \ A$ 
    using R2 not-cong-4312 by blast
  have R7:  $\text{Col } A \ X \ B$ 
    using Col-perm P2 by blast
  have R8:  $\text{Col } P \ X \ R$ 
    by (simp add: P1 bet-col between-symmetry)
  thus ?thesis using l7-21
    using R3 R4 R5 R6 R7 by blast
qed
thus ?thesis by simp
qed
thus ?thesis
  by simp
qed

```

```

lemma col-per2--per:
  assumes  $A \neq B$  and
     $\text{Col } A \ B \ C$  and
     $\text{Per } A \ X \ P$  and
     $\text{Per } B \ X \ P$ 
  shows  $\text{Per } C \ X \ P$ 
  by (meson Per-def assms(1) assms(2) assms(3) assms(4) l4-17 per-double-cong)

```

```

lemma perp-in-per-1:
  assumes  $X \ \text{PerpAt } A \ B \ C \ D$ 
  shows  $\text{Per } A \ X \ C$ 
  using PerpAt-def assms col-trivial-1 by auto

```

```

lemma perp-in-per-2:
  assumes  $X \ \text{PerpAt } A \ B \ C \ D$ 
  shows  $\text{Per } A \ X \ D$ 
  using assms perp-in-per-1 perp-in-right-comm by blast

```

```

lemma perp-in-per-3:
  assumes  $X \ \text{PerpAt } A \ B \ C \ D$ 
  shows  $\text{Per } B \ X \ C$ 
  using assms perp-in-comm perp-in-per-2 by blast

```

```

lemma perp-in-per-4:
  assumes  $X \ \text{PerpAt } A \ B \ C \ D$ 
  shows  $\text{Per } B \ X \ D$ 
  using assms perp-in-per-3 perp-in-right-comm by blast

```

```

lemma coplanar-perm-1:
  assumes  $\text{Coplanar } A \ B \ C \ D$ 

```

shows *Coplanar A B D C*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-2*:
 assumes *Coplanar A B C D*
 shows *Coplanar A C B D*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-3*:
 assumes *Coplanar A B C D*
 shows *Coplanar A C D B*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-4*:
 assumes *Coplanar A B C D*
 shows *Coplanar A D B C*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-5*:
 assumes *Coplanar A B C D*
 shows *Coplanar A D C B*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-6*:
 assumes *Coplanar A B C D*
 shows *Coplanar B A C D*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-7*:
 assumes *Coplanar A B C D*
 shows *Coplanar B A D C*
proof –
 obtain *X* where $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee (Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee (Col\ A\ D\ X \wedge Col\ B\ C\ X)$
 using *Coplanar-def assms* **by** *blast*
 thus *?thesis*
 using *Coplanar-def col-permutation-4* **by** *blast*

qed

lemma *coplanar-perm-8*:

assumes *Coplanar A B C D*

shows *Coplanar B C A D*

proof –

obtain *X* where $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$

using *Coplanar-def assms* by blast

thus ?thesis

using *Coplanar-def col-permutation-4* by blast

qed

lemma *coplanar-perm-9*:

assumes *Coplanar A B C D*

shows *Coplanar B C D A*

proof –

obtain *X* where $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$

using *Coplanar-def assms* by blast

thus ?thesis

using *Coplanar-def col-permutation-4* by blast

qed

lemma *coplanar-perm-10*:

assumes *Coplanar A B C D*

shows *Coplanar B D A C*

proof –

obtain *X* where $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$

using *Coplanar-def assms* by blast

thus ?thesis

using *Coplanar-def col-permutation-4* by blast

qed

lemma *coplanar-perm-11*:

assumes *Coplanar A B C D*

shows *Coplanar B D C A*

proof –

obtain *X* where $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$

using *Coplanar-def assms* by blast

thus ?thesis

using *Coplanar-def col-permutation-4* by blast

qed

lemma *coplanar-perm-12*:

assumes *Coplanar A B C D*

shows *Coplanar C A B D*

proof –

obtain *X* where $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$

using *Coplanar-def assms* by blast

thus ?thesis

using *Coplanar-def col-permutation-4* by blast

qed

lemma *coplanar-perm-13*:

assumes *Coplanar A B C D*

shows *Coplanar C A D B*

proof –

obtain *X* where $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$

using *Coplanar-def assms* by blast

thus ?thesis

using *Coplanar-def col-permutation-4* by blast

qed

lemma *coplanar-perm-14*:

assumes *Coplanar A B C D*

shows *Coplanar C B A D*

proof –

obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-15*:
assumes *Coplanar A B C D*
shows *Coplanar C B D A*
proof –
obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-16*:
assumes *Coplanar A B C D*
shows *Coplanar C D A B*
proof –
obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-17*:
assumes *Coplanar A B C D*
shows *Coplanar C D B A*
proof –
obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-18*:
assumes *Coplanar A B C D*
shows *Coplanar D A B C*
proof –
obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-19*:
assumes *Coplanar A B C D*
shows *Coplanar D A C B*
proof –
obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-20*:
assumes *Coplanar A B C D*
shows *Coplanar D B A C*
proof –
obtain X **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-21*:
assumes *Coplanar A B C D*
shows *Coplanar D B C A*
proof –
obtain *X* **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-22*:
assumes *Coplanar A B C D*
shows *Coplanar D C A B*
proof –
obtain *X* **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *coplanar-perm-23*:
assumes *Coplanar A B C D*
shows *Coplanar D C B A*
proof –
obtain *X* **where** $(\text{Col } A \ B \ X \wedge \text{Col } C \ D \ X) \vee (\text{Col } A \ C \ X \wedge \text{Col } B \ D \ X) \vee (\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X)$
using *Coplanar-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def col-permutation-4* **by** *blast*
qed

lemma *ncoplanar-perm-1*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } A \ B \ D \ C$
using *assms coplanar-perm-1* **by** *blast*

lemma *ncoplanar-perm-2*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } A \ C \ B \ D$
using *assms coplanar-perm-2* **by** *blast*

lemma *ncoplanar-perm-3*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } A \ C \ D \ B$
using *assms coplanar-perm-4* **by** *blast*

lemma *ncoplanar-perm-4*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } A \ D \ B \ C$
using *assms coplanar-perm-3* **by** *blast*

lemma *ncoplanar-perm-5*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } A \ D \ C \ B$
using *assms coplanar-perm-5* **by** *blast*

lemma *ncoplanar-perm-6*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } B \ A \ C \ D$
using *assms coplanar-perm-6* **by** *blast*

lemma *ncoplanar-perm-7*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } B \ A \ D \ C$
using *assms coplanar-perm-7* **by** *blast*

lemma *ncoplanar-perm-8*:

```

assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } B \ C \ A \ D$ 
using assms coplanar-perm-12 by blast

lemma ncoplanar-perm-9:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } B \ C \ D \ A$ 
using assms coplanar-perm-18 by blast

lemma ncoplanar-perm-10:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } B \ D \ A \ C$ 
using assms coplanar-perm-13 by blast

lemma ncoplanar-perm-11:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } B \ D \ C \ A$ 
using assms coplanar-perm-19 by blast

lemma ncoplanar-perm-12:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } C \ A \ B \ D$ 
using assms coplanar-perm-8 by blast

lemma ncoplanar-perm-13:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } C \ A \ D \ B$ 
using assms coplanar-perm-10 by blast

lemma ncoplanar-perm-14:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } C \ B \ A \ D$ 
using assms coplanar-perm-14 by blast

lemma ncoplanar-perm-15:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } C \ B \ D \ A$ 
using assms coplanar-perm-20 by blast

lemma ncoplanar-perm-16:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } C \ D \ A \ B$ 
using assms coplanar-perm-16 by blast

lemma ncoplanar-perm-17:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } C \ D \ B \ A$ 
using assms coplanar-perm-22 by blast

lemma ncoplanar-perm-18:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } D \ A \ B \ C$ 
using assms coplanar-perm-9 by blast

lemma ncoplanar-perm-19:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } D \ A \ C \ B$ 
using assms coplanar-perm-11 by blast

lemma ncoplanar-perm-20:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 
shows  $\neg \text{Coplanar } D \ B \ A \ C$ 
using assms coplanar-perm-15 by blast

lemma ncoplanar-perm-21:
assumes  $\neg \text{Coplanar } A \ B \ C \ D$ 

```

shows $\neg \text{Coplanar } D \ B \ C \ A$
using *assms coplanar-perm-21* **by** *blast*

lemma *ncoplanar-perm-22*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } D \ C \ A \ B$
using *assms coplanar-perm-17* **by** *blast*

lemma *ncoplanar-perm-23*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Coplanar } D \ C \ B \ A$
using *assms coplanar-perm-23* **by** *blast*

lemma *coplanar-trivial*:
shows $\text{Coplanar } A \ A \ B \ C$
using *Coplanar-def NCol-cases col-trivial-1* **by** *blast*

lemma *col--coplanar*:
assumes $\text{Col } A \ B \ C$
shows $\text{Coplanar } A \ B \ C \ D$
using *Coplanar-def assms not-col-distincts* **by** *blast*

lemma *ncop--ncol*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Col } A \ B \ C$
using *assms col--coplanar* **by** *blast*

lemma *ncop--ncols*:
assumes $\neg \text{Coplanar } A \ B \ C \ D$
shows $\neg \text{Col } A \ B \ C \wedge \neg \text{Col } A \ B \ D \wedge \neg \text{Col } A \ C \ D \wedge \neg \text{Col } B \ C \ D$
by (*meson assms col--coplanar coplanar-perm-4 ncoplanar-perm-9*)

lemma *bet--coplanar*:
assumes $\text{Bet } A \ B \ C$
shows $\text{Coplanar } A \ B \ C \ D$
using *assms bet-col ncop--ncol* **by** *blast*

lemma *out--coplanar*:
assumes $A \ \text{Out } B \ C$
shows $\text{Coplanar } A \ B \ C \ D$
using *assms col--coplanar out-col* **by** *blast*

lemma *midpoint--coplanar*:
assumes $A \ \text{Midpoint } B \ C$
shows $\text{Coplanar } A \ B \ C \ D$
using *assms midpoint-col ncop--ncol* **by** *blast*

lemma *perp--coplanar*:
assumes $A \ B \ \text{Perp } C \ D$
shows $\text{Coplanar } A \ B \ C \ D$

proof –
obtain P **where** $P \ \text{PerpAt } A \ B \ C \ D$
using *Perp-def assms* **by** *blast*
thus *?thesis*
using *Coplanar-def perp-in-col* **by** *blast*

qed

lemma *ts--coplanar*:
assumes $A \ B \ TS \ C \ D$
shows $\text{Coplanar } A \ B \ C \ D$
by (*metis Coplanar-def TS-def assms bet-col col-permutation-2 col-permutation-3*)

lemma *reflectl--coplanar*:
assumes $A \ B \ \text{ReflectL } C \ D$
shows $\text{Coplanar } A \ B \ C \ D$
by (*metis ReflectL-def perp--coplanar assms col--coplanar col-trivial-1 ncoplanar-perm-17*)

```

lemma reflect--coplanar:
  assumes A B Reflect C D
  shows Coplanar A B C D
  by (metis Reflect-def reflectl--coplanar assms col-trivial-2 ncop--ncols)

lemma inangle--coplanar:
  assumes A InAngle B C D
  shows Coplanar A B C D
proof -
  obtain X where Bet B X D  $\wedge$  (X = C  $\vee$  C Out X A)
  using InAngle-def assms by auto
  thus ?thesis
  by (meson Col-cases Coplanar-def bet-col ncop--ncols out-col)
qed

lemma pars--coplanar:
  assumes A B ParStrict C D
  shows Coplanar A B C D
  using ParStrict-def assms by auto

lemma par--coplanar:
  assumes A B Par C D
  shows Coplanar A B C D
  using Par-def assms ncop--ncols pars--coplanar by blast

lemma plg--coplanar:
  assumes Plg A B C D
  shows Coplanar A B C D
proof -
  obtain M where Bet A M C  $\wedge$  Bet B M D
  by (meson Plg-def assms midpoint-bet)
  thus ?thesis
  by (metis InAngle-def bet-out-1 inangle--coplanar ncop--ncols not-col-distincts)
qed

lemma plgs--coplanar:
  assumes ParallelogramStrict A B C D
  shows Coplanar A B C D
  using ParallelogramStrict-def assms par--coplanar by blast

lemma plgf--coplanar:
  assumes ParallelogramFlat A B C D
  shows Coplanar A B C D
  using ParallelogramFlat-def assms col--coplanar by auto

lemma parallelogram--coplanar:
  assumes Parallelogram A B C D
  shows Coplanar A B C D
  using Parallelogram-def assms plgf--coplanar plgs--coplanar by auto

lemma rhombus--coplanar:
  assumes Rhombus A B C D
  shows Coplanar A B C D
  using Rhombus-def assms plg--coplanar by blast

lemma rectangle--coplanar:
  assumes Rectangle A B C D
  shows Coplanar A B C D
  using Rectangle-def assms plg--coplanar by blast

lemma square--coplanar:
  assumes Square A B C D
  shows Coplanar A B C D
  using Square-def assms rectangle--coplanar by blast

```

```

lemma lambert--coplanar:
  assumes Lambert  $A B C D$ 
  shows Coplanar  $A B C D$ 
  using Lambert-def assms by presburger

lemma ts-distincts:
  assumes  $A B TS P Q$ 
  shows  $A \neq B \wedge A \neq P \wedge A \neq Q \wedge B \neq P \wedge B \neq Q \wedge P \neq Q$ 
  using TS-def assms bet-neq12--neq not-col-distincts by blast

lemma l9-2:
  assumes  $A B TS P Q$ 
  shows  $A B TS Q P$ 
  using TS-def assms between-symmetry by blast

lemma invert-two-sides:
  assumes  $A B TS P Q$ 
  shows  $B A TS P Q$ 
  using TS-def assms not-col-permutation-5 by blast

lemma l9-3:
  assumes  $P Q TS A C$  and
     $Col M P Q$  and
     $M$  Midpoint  $A C$  and
     $Col R P Q$  and
     $R$  Out  $A B$ 
  shows  $P Q TS B C$ 
proof –
  have  $\neg Col A P Q$ 
    using TS-def assms(1) by blast
  hence  $P \neq Q$ 
    using not-col-distincts by auto
  obtain  $T$  where  $Col T P Q$  and  $Bet A T C$ 
    using assms(2) assms(3) midpoint-bet by blast
  have  $A \neq C$ 
    using assms(1) ts-distincts by blast
  have  $T = M$ 
proof –
  have  $Bet A M C$ 
    using assms(3) midpoint-bet by blast
  hence  $Col A M C$ 
    using Col-def by force
  moreover have  $Col A T C$ 
    using  $\langle Bet A T C \rangle$  bet-col by blast
  moreover have  $Col R A B$ 
    using assms(5) out-col by auto
  ultimately show ?thesis
    by (meson l6-21  $\langle A \neq C \rangle \langle Col T P Q \rangle \langle \neg Col A P Q \rangle$  assms(2) col-permutation-3
      col-permutation-5)
qed
  have  $P Q TS B C$ 
proof cases
  assume  $C = M$ 
  thus ?thesis
    using  $\langle A \neq C \rangle$  assms(3) is-midpoint-id-2 by blast
next
  assume  $\neg C = M$ 
  have  $\neg Col B P Q$ 
    by (metis  $\langle \neg Col A P Q \rangle$  assms(4) assms(5) col-permutation-2 colx out-col out-diff2)
  have  $Bet R A B \vee Bet R B A$ 
    using Out-def assms(5) by auto
  {
    assume  $Bet R A B$ 
    obtain  $B'$  where  $M$  Midpoint  $B B'$ 
      using symmetric-point-construction by blast
    obtain  $R'$  where  $M$  Midpoint  $R R'$ 

```

```

    using symmetric-point-construction by blast
have Bet B' C R'
  using l7-15 ⟨Bet R A B⟩ ⟨M Midpoint B B'⟩ ⟨M Midpoint R R'⟩ assms(3) between-symmetry
  by fastforce
have ∃ X. Bet M X R' ∧ Bet C X B
proof -
  have Bet B M B'
    using Midpoint-def ⟨M Midpoint B B'⟩ by auto
  moreover have Bet R' C B'
    using Bet-cases ⟨Bet B' C R'⟩ by auto
  ultimately show ?thesis
    by (simp add: inner-pasch)
qed
then obtain X where Bet M X R' and Bet C X B
  by blast
have Col X P Q
proof -
  have Col P M R
    using ⟨P ≠ Q⟩ assms(2) assms(4) l6-16-1 not-col-permutation-2 by blast
  have Col Q M R
    by (metis l6-16-1 ⟨Col P M R⟩ assms(2) assms(4) col-permutation-2)
  {
    assume M = X
    hence Col X P Q
      using assms(2) by blast
  }
  hence M = X ⟶ Col X P Q by simp
  {
    assume M ≠ X
    hence M ≠ R'
      using ⟨Bet M X R'⟩ between-identity by blast
    hence M ≠ R
      using ⟨M Midpoint R R'⟩ is-midpoint-id by blast
    hence Col X P Q
      by (metis ⟨Bet M X R'⟩ ⟨M Midpoint R R'⟩ ⟨M ≠ R'⟩ assms(2) assms(4)
        bet-col col-permutation-4 col-permutation-5 colx midpoint-col)
  }
  hence M ≠ X ⟶ Col X P Q by simp
  thus ?thesis
    using assms(2) by blast
qed
have Bet B X C
  using Bet-cases ⟨Bet C X B⟩ by blast
hence P Q TS B C
  using TS-def ⟨Col X P Q⟩ ⟨¬ Col B P Q⟩ assms(1) by blast
}
hence Bet R A B ⟶ P Q TS B C by simp
{
  assume Bet R B A
  have Bet C M A
    using Bet-cases ⟨Bet A T C⟩ ⟨T = M⟩ by blast
  then obtain X where Bet B X C and Bet M X R
    using ⟨Bet R B A⟩ inner-pasch by blast
  have Col X P Q
    by (metis Col-def ⟨Bet M X R⟩ assms(2) assms(4) between-equality-2
      between-trivial2 col-transitivity-1)
  hence P Q TS B C
    using TS-def ⟨Bet B X C⟩ ⟨¬ Col B P Q⟩ assms(1) by blast
}
hence Bet R B A ⟶ P Q TS B C by simp
thus ?thesis
  using ⟨Bet R A B ⟶ P Q TS B C⟩ ⟨Bet R A B ∨ Bet R B A⟩ by blast
qed
thus ?thesis by blast
qed

```



```

lemma mid-preserves-col:
  assumes Col A B C and
    M Midpoint A A' and
    M Midpoint B B' and
    M Midpoint C C'
  shows Col A' B' C'
  using Col-def assms(1) assms(2) assms(3) assms(4) l7-15 by auto

lemma per-mid-per:
  assumes
    Per X A B and
    M Midpoint A B and
    M Midpoint X Y
  shows Cong A X B Y  $\wedge$  Per Y B A
  by (meson Cong3-def Mid-perm assms(1) assms(2) assms(3) l7-13 l8-10)

lemma sym-preserve-diff:
  assumes A  $\neq$  B and
    M Midpoint A A' and
    M Midpoint B B'
  shows A'  $\neq$  B'
  using assms(1) assms(2) assms(3) l7-9 by blast

lemma l9-4-1-aux-R1:
  assumes R = S and
    S C Le R A and
    P Q TS A C and
    Col R P Q and
    P Q Perp A R and
    Col S P Q and
    P Q Perp C S and
    M Midpoint R S
  shows  $\forall U C'. M \text{ Midpoint } U C' \longrightarrow (R \text{ Out } U A \longleftrightarrow S \text{ Out } C C')$ 
proof -
  have M = R
  using assms(1) assms(8) l7-3 by blast
  have  $\neg \text{Col } A P Q$ 
  using TS-def assms(3) by auto
  hence P  $\neq$  Q
  using not-col-distincts by blast
  obtain T where Col T P Q and Bet A T C
  using TS-def assms(3) by blast
  {
    assume  $\neg M = T$ 
    hence M PerpAt M T A M
    using perp-col2 assms(4) assms(5) not-col-permutation-3 perp-left-comm perp-perp-in
    by (metis  $\langle \text{Col } T P Q \rangle \langle M = R \rangle$ )
    hence M T Perp C M
    using  $\langle M \neq T \rangle$  assms(1) assms(4) assms(7) col-permutation-1 perp-col2  $\langle \text{Col } T P Q \rangle \langle M = R \rangle$ 
    by blast
    hence Per T M A
    using  $\langle M \text{ PerpAt } M T A M \rangle$  perp-in-per-3 by blast
    have Per T M C
    by (simp add:  $\langle M T \text{ Perp } C M \rangle$  perp-per-1)
    have M = T
    proof -
      have Per C M T
      by (simp add:  $\langle \text{Per } T M C \rangle$  l8-2)
      thus ?thesis
      using l8-6 l8-2  $\langle \text{Bet } A T C \rangle \langle \text{Per } T M A \rangle$  by blast
    qed
    hence False
    using  $\langle M \neq T \rangle$  by blast
  }
  hence M = T by blast
  have  $\forall U C'. ((M \text{ Midpoint } U C' \wedge M \text{ Out } U A) \longrightarrow M \text{ Out } C C')$ 

```

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proof -
{
  fix U C'
  assume M Midpoint U C' and M Out U A
  have C ≠ M
    using assms(1) assms(7) perp-not-eq-2 ⟨M = R⟩ by blast
  have C' ≠ M
    using midpoint-not-midpoint out-diff1 ⟨M Midpoint U C'⟩ ⟨M Out U A⟩ by blast
  have Bet U M C
    using bet-out--bet l6-6 ⟨Bet A T C⟩ ⟨M = T⟩ ⟨M Out U A⟩ by blast
  hence M Out C C'
    by (metis Out-def midpoint-bet ⟨C ≠ M⟩ ⟨C' ≠ M⟩ ⟨M Midpoint U C'⟩ ⟨M Out U A⟩ l5-2)
}
thus ?thesis by blast
qed
have ∀ U C'. ((M Midpoint U C' ∧ M Out C C') ⟶ M Out U A)
proof -
{
  fix U C'
  assume M Midpoint U C' and M Out C C'
  have C ≠ M
    using assms(1) assms(7) perp-not-eq-2 ⟨M = R⟩ by blast
  have C' ≠ M
    using l6-3-1 ⟨M Out C C'⟩ by blast
  have Bet U M C
    using Out-def between-inner-transitivity midpoint-bet outer-transitivity-between
    by (metis ⟨M Midpoint U C'⟩ ⟨M Out C C'⟩)
  hence M Out U A
    using l6-2 midpoint-distinct-1
    by (metis ⟨Bet A T C⟩ ⟨C ≠ M⟩ ⟨C' ≠ M⟩ ⟨M = R⟩ ⟨M Midpoint U C'⟩
    ⟨M ≠ T ⟹ False⟩ ⟨¬ Col A P Q⟩ assms(4))
}
thus ?thesis by blast
qed
thus ?thesis
  using ⟨M = R⟩ ⟨∀ U C'. M Midpoint U C' ∧ M Out U A ⟶ M Out C C'⟩ assms(1) by blast
qed

lemma l9-4-1-aux-R21:
  assumes R ≠ S and
    S C Le R A and
    P Q TS A C and
    Col R P Q and
    P Q Perp A R and
    Col S P Q and
    P Q Perp C S and
    M Midpoint R S
  shows ∀ U C'. M Midpoint U C' ⟶ (R Out U A ⟷ S Out C C')
proof -
  obtain D where Bet R D A and Cong S C R D
    using Le-def assms(2) by blast
  have C ≠ S
    using assms(7) perp-not-eq-2 by auto
  have R ≠ D
    using cong-identity ⟨C ≠ S⟩ ⟨Cong S C R D⟩ by blast
  have R S Perp A R
    using assms(1) assms(4) assms(5) assms(6) not-col-permutation-2 perp-col2 by blast
  have ∃ M. (M Midpoint S R ∧ M Midpoint C D)
proof -
  have ¬ Col A P Q
    using TS-def assms(3) by blast
  have P ≠ Q
    using not-col-distincts assms(3) ts-distincts by blast
  obtain T where Col T P Q and Bet A T C
    using TS-def assms(3) by blast
  have C S Perp S R

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    by (metis NCol-perm assms(1) assms(4) assms(6) assms(7) perp-col0)
  have A R Perp S R
    using Perp-perm ⟨R S Perp A R⟩ by blast
  have Col S R T
    using Col-cases assms(4) assms(6) col3 ⟨Col T P Q⟩ ⟨P ≠ Q⟩ by blast
  have Bet C T A
    using Bet-perm ⟨Bet A T C⟩ by blast
  thus ?thesis
    using l8-24 ⟨A R Perp S R⟩ ⟨Bet R D A⟩ ⟨C S Perp S R⟩ ⟨Col S R T⟩ ⟨Cong S C R D⟩ by blast
qed
then obtain M' where M' Midpoint S R and M' Midpoint C D by blast
have M = M'
  using assms(8) l7-17-bis ⟨M' Midpoint S R⟩ by blast
have ∀ U C'. (M Midpoint U C' ∧ R Out U A) ⟶ S Out C C'
proof -
  {
    fix U C'
    assume M Midpoint U C' and R Out U A
    have C' ≠ S
      using ⟨M Midpoint U C'⟩ ⟨R Out U A⟩ assms(8) l7-9 out-diff1 by blast
    have Bet S C C' ∨ Bet S C' C
      proof -
        have Bet R U A ∨ Bet R A U
          using Out-def ⟨R Out U A⟩ by auto
        {
          assume Bet R U A
          hence Bet R U D ∨ Bet R D U
            by (simp add: ⟨Bet R D A⟩ l5-3)
          hence Bet S C C' ∨ Bet S C' C
            using l7-15 l7-2 ⟨M = M'⟩ ⟨M Midpoint U C'⟩ ⟨M' Midpoint C D⟩ assms(8) by blast
        }
        hence Bet R U A ⟶ Bet S C C' ∨ Bet S C' C by simp
        have Bet R A U ⟶ Bet S C C' ∨ Bet S C' C
          using l7-15 l7-2 ⟨Bet R D A⟩ ⟨M = M'⟩ ⟨M Midpoint U C'⟩ ⟨M' Midpoint C D⟩
            assms(8) between-exchange4 by blast
        thus ?thesis
          using ⟨Bet R U A ⟶ Bet S C C' ∨ Bet S C' C⟩ ⟨Bet R A U ⟶ Bet S C C' ∨ Bet S C' C⟩ by blast
      qed
    hence S Out C C'
      using Out-def ⟨C ≠ S⟩ ⟨C' ≠ S⟩ by auto
  }
  thus ?thesis
    by blast
qed
have ∀ U C'. (M Midpoint U C' ∧ S Out C C') ⟶ R Out U A
proof -
  {
    fix U C'
    assume M Midpoint U C' and S Out C C'
    hence U ≠ R
      using l7-9-bis ⟨M = M'⟩ ⟨M' Midpoint S R⟩ out-distinct by blast
    have A ≠ R
      using assms(5) perp-distinct by auto
    have Bet S C C' ∨ Bet S C' C
      using Out-def ⟨S Out C C'⟩ by fastforce
    {
      assume Bet S C C'
      have Bet R D U
        proof -
          have M Midpoint S R
            by (simp add: ⟨M = M'⟩ ⟨M' Midpoint S R⟩)
          moreover have M Midpoint C D
            by (simp add: ⟨M = M'⟩ ⟨M' Midpoint C D⟩)
          moreover have M Midpoint C' U
            by (simp add: ⟨M Midpoint U C'⟩ l7-2)
          ultimately show ?thesis

```

```

    by (simp add: ⟨Bet S C C'⟩ l7-15)
  qed
  hence Bet R U A ∨ Bet R A U
    using ⟨Bet R D A⟩ ⟨R ≠ D⟩ l5-1 by auto
}
hence Bet S C C' ⟶ Bet R U A ∨ Bet R A U by simp
{
  assume Bet S C' C
  have Bet R U A
    using l7-15 l7-2 between-exchange4 ⟨Bet R D A⟩ ⟨M = M'⟩ ⟨M Midpoint U C'⟩
    by (meson ⟨Bet S C' C⟩ ⟨M' Midpoint C D⟩ ⟨M' Midpoint S R⟩)
}
hence Bet S C' C ⟶ Bet R U A ∨ Bet R A U by simp
hence Bet R U A ∨ Bet R A U
  using ⟨Bet S C C' ⟶ Bet R U A ∨ Bet R A U⟩ ⟨Bet S C C' ∨ Bet S C' C⟩ by blast
hence R Out U A
  by (simp add: Out-def ⟨A ≠ R⟩ ⟨U ≠ R⟩)
}
thus ?thesis by blast
qed
thus ?thesis
  using ⟨∀ U C'. M Midpoint U C' ∧ R Out U A ⟶ S Out C C'⟩ by blast
qed

```

lemma l9-4-1-aux:
assumes $S C \text{ Le } R A$ **and**
 $P Q \text{ TS } A C$ **and**
 $Col R P Q$ **and**
 $P Q \text{ Perp } A R$ **and**
 $Col S P Q$ **and**
 $P Q \text{ Perp } C S$ **and**
 $M \text{ Midpoint } R S$
shows $\forall U C'. (M \text{ Midpoint } U C' \longrightarrow (R \text{ Out } U A \longleftrightarrow S \text{ Out } C C'))$
using l9-4-1-aux-R1 l9-4-1-aux-R21 *assms* **by** fast

lemma per-col-eq:
assumes $Per A B C$ **and**
 $Col A B C$ **and**
 $B \neq C$
shows $A = B$
using *assms*(1) *assms*(2) *assms*(3) l8-9 **by** blast

lemma l9-4-1:
assumes $P Q \text{ TS } A C$ **and**
 $Col R P Q$ **and**
 $P Q \text{ Perp } A R$ **and**
 $Col S P Q$ **and**
 $P Q \text{ Perp } C S$ **and**
 $M \text{ Midpoint } R S$
shows $\forall U C'. M \text{ Midpoint } U C' \longrightarrow (R \text{ Out } U A \longleftrightarrow S \text{ Out } C C')$
proof –
 have $S C \text{ Le } R A \vee R A \text{ Le } S C$
 using *le-cases* **by** blast
 {
 assume $S C \text{ Le } R A$
 {
 fix $U C'$
 assume $M \text{ Midpoint } U C'$
 hence $(R \text{ Out } U A \longleftrightarrow S \text{ Out } C C')$
 using *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) l9-4-1-aux ⟨ $S C \text{ Le } R A$ ⟩
 by blast
 }
 hence $\forall U C'. M \text{ Midpoint } U C' \longrightarrow (R \text{ Out } U A \longleftrightarrow S \text{ Out } C C')$ **by** simp
 }
 hence $S C \text{ Le } R A \longrightarrow (\forall U C'. M \text{ Midpoint } U C' \longrightarrow (R \text{ Out } U A \longleftrightarrow S \text{ Out } C C'))$ **by** simp

```

moreover
{
  assume  $R\ A\ Le\ S\ C$ 
  {
    fix  $U\ C'$ 
    assume  $M\ Midpoint\ U\ C'$ 
    hence  $(R\ Out\ A\ U \longleftrightarrow S\ Out\ C'\ C')$ 
      using  $assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)\ assms(6)\ l7-2\ l9-2\ l9-4-1-aux$ 
      by  $(meson\ Out-cases\ le-cases)$ 
    hence  $(R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$ 
      using  $l6-6$  by  $blast$ 
  }
  hence  $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$  by  $simp$ 
}
hence  $R\ A\ Le\ S\ C \longrightarrow (\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C'))$  by  $simp$ 
thus  $?thesis$ 
  using  $\langle S\ C\ Le\ R\ A \vee R\ A\ Le\ S\ C \rangle$  calculation by  $presburger$ 
qed

```

```

lemma mid-two-sides:
assumes  $M\ Midpoint\ A\ B$  and
   $\neg\ Col\ A\ B\ X$  and
   $M\ Midpoint\ X\ Y$ 
shows  $A\ B\ TS\ X\ Y$ 
proof –
  have  $\neg\ Col\ Y\ A\ B$ 
    by  $(meson\ Mid-cases\ mid-preserves-col\ assms(1)\ assms(2)\ assms(3)\ col-permutation-3)$ 
  moreover have  $Bet\ X\ M\ Y$ 
    using  $assms(3)\ midpoint-bet$  by  $blast$ 
  ultimately show  $?thesis$ 
    by  $(metis\ TS-def\ not-col-permutation-2\ assms(1)\ assms(2)\ midpoint-col)$ 
qed

```

```

lemma col-preserves-two-sides:
assumes  $C \neq D$  and
   $Col\ A\ B\ C$  and
   $Col\ A\ B\ D$  and
   $A\ B\ TS\ X\ Y$ 
shows  $C\ D\ TS\ X\ Y$ 
proof –
  have  $\neg\ Col\ X\ A\ B$ 
    using  $TS-def\ assms(4)$  by  $blast$ 
  hence  $A \neq B$ 
    using  $not-col-distincts$  by  $blast$ 
  have  $\neg\ Col\ X\ C\ D$ 
    using  $colx\ assms(1)\ assms(2)\ assms(3)$ 
    by  $(meson\ \langle \neg\ Col\ X\ A\ B \rangle\ not-col-permutation-2)$ 
  have  $\neg\ Col\ Y\ C\ D$ 
    by  $(metis\ Col-cases\ TS-def\ colx\ assms(1)\ assms(2)\ assms(3)\ assms(4))$ 
  thus  $?thesis$ 
proof –
  obtain  $PP$  where  $\neg\ Col\ X\ A\ B$  and  $\neg\ Col\ Y\ A\ B$  and  $Col\ PP\ A\ B$  and  $Bet\ X\ PP\ Y$ 
    using  $TS-def\ assms(4)$  by  $blast$ 
  hence  $Col\ PP\ C\ D$ 
    using  $\langle A \neq B \rangle\ assms(2)\ assms(3)\ col3\ col-permutation-1$  by  $blast$ 
  thus  $?thesis$ 
    using  $TS-def\ \langle Bet\ X\ PP\ Y \rangle\ \langle \neg\ Col\ X\ C\ D \rangle\ \langle \neg\ Col\ Y\ C\ D \rangle$  by  $blast$ 
qed
qed

```

```

lemma out-out-two-sides:
assumes  $A \neq B$  and
   $A\ B\ TS\ X\ Y$  and
   $Col\ I\ A\ B$  and
   $Col\ I\ X\ Y$  and
   $I\ Out\ X\ U$  and

```

```

  I Out Y V
shows A B TS U V
proof -
  have  $\neg$  Col X A B
    using TS-def assms(2) by blast
  {
    assume Col V A B
    hence Col Y A B
      by (metis out-distinct assms(3) assms(6) col-permutation-2 colx out-col)
    have False
      using TS-def  $\langle$ Col Y A B $\rangle$  assms(2) by blast
  }
  moreover
  have  $\neg$  Col U A B
    using assms(3) assms(5) col-permutation-2 colx out-col out-distinct by (meson  $\langle$  $\neg$  Col X A B $\rangle$ )
  moreover
  obtain T where Col T A B and Bet X T Y
    using TS-def assms(2) by blast
  have I = T
  proof -
    have Col X Y I
      using assms(4) Col-cases by blast
    moreover have Col B A I
      using assms(3) Col-cases by blast
    moreover have Col B A T
      using Col-cases  $\langle$ Col T A B $\rangle$  by auto
    moreover have  $\neg$  Col X A B  $\wedge$   $\neg$  Col X B A  $\wedge$   $\neg$  Col A X B  $\wedge$   $\neg$  Col A B X  $\wedge$   $\neg$  Col B X A  $\wedge$   $\neg$  Col B A X
      using  $\langle$  $\neg$  Col X A B $\rangle$  Col-cases by blast
    moreover have  $A \neq B \wedge A \neq X \wedge A \neq Y \wedge B \neq X \wedge B \neq Y \wedge X \neq Y$ 
      using assms(2) ts-distincts by presburger
    moreover have Col X Y T
      using  $\langle$ Bet X T Y $\rangle$  bet-col1 between-trivial by blast
    ultimately show ?thesis
      using l6-21 by blast
  qed
  hence Bet U T V
    using assms(5) assms(6) bet-out-out-bet  $\langle$ Bet X T Y $\rangle$  by blast
  ultimately show ?thesis
    using TS-def  $\langle$ Col T A B $\rangle$  by blast
qed

lemma l9-4-2-aux-R1:
  assumes R = S and
    S C Le R A and
    P Q TS A C and
    Col R P Q and
    P Q Perp A R and
    Col S P Q and
    P Q Perp C S and
    R Out U A and
    S Out V C
  shows P Q TS U V
proof -
  have  $\neg$  Col A P Q
    using TS-def assms(3) by auto
  hence  $P \neq Q$ 
    using not-col-distincts by blast
  obtain T where Col T P Q and Bet A T C
    using TS-def assms(3) by blast
  have R = T
    using assms(1) assms(5) assms(6) assms(7) col-permutation-1 l8-16-1 l8-6
    by (meson  $\langle$ Bet A T C $\rangle$   $\langle$ Col T P Q $\rangle$ )
  thus ?thesis
    by (metis  $\langle$ Bet A T C $\rangle$   $\langle$ Col T P Q $\rangle$   $\langle$ P  $\neq$  Q $\rangle$  assms(1) assms(3) assms(8) assms(9)
        bet-col col-permutation-4 l6-6 out-out-two-sides)
qed

```

```

lemma l9-4-2-aux-R2:
  assumes  $R \neq S$  and
     $S \ C \ Le \ R \ A$  and
     $P \ Q \ TS \ A \ C$  and
     $Col \ R \ P \ Q$  and
     $P \ Q \ Perp \ A \ R$  and
     $Col \ S \ P \ Q$  and
     $P \ Q \ Perp \ C \ S$  and
     $R \ Out \ U \ A$  and
     $S \ Out \ V \ C$ 
  shows  $P \ Q \ TS \ U \ V$ 
proof -
  have  $P \neq Q$ 
    using assms(7) perp-distinct by auto
  have  $R \ S \ TS \ A \ C$ 
    using assms(1) assms(3) assms(4) assms(6) col-permutation-1 col-preserves-two-sides by blast
  have  $Col \ R \ S \ P$ 
    using assms(4) assms(6) col2--eq not-col-permutation-1  $\langle P \neq Q \rangle$  by blast
  have  $Col \ R \ S \ Q$ 
    using colx Tarski-neutral-dimensionless-axioms assms(4) assms(6) col-trivial-2
    by (metis  $\langle Col \ R \ S \ P \rangle$ )
  have  $R \ S \ Perp \ A \ R$ 
    using NCol-perm assms(1) assms(4) assms(5) assms(6) perp-col2 by blast
  have  $R \ S \ Perp \ C \ S$ 
    using assms(1) assms(4) assms(6) assms(7) col-permutation-1 perp-col2 by blast
  have  $\neg \ Col \ A \ R \ S$ 
    using TS-def  $\langle R \ S \ TS \ A \ C \rangle$  by force
  obtain  $T$  where  $Col \ T \ R \ S$  and  $Bet \ A \ T \ C$ 
    using TS-def  $\langle R \ S \ TS \ A \ C \rangle$  by blast
  obtain  $C'$  where  $Bet \ R \ C' \ A$  and  $Cong \ S \ C \ R \ C'$ 
    using Le-def assms(2) by blast
  have  $\exists \ X. \ X \ Midpoint \ S \ R \wedge X \ Midpoint \ C \ C'$ 
proof -
  have  $C \ S \ Perp \ S \ R$ 
    using Perp-perm  $\langle R \ S \ Perp \ C \ S \rangle$  by blast
  moreover have  $A \ R \ Perp \ S \ R$ 
    using Perp-perm  $\langle R \ S \ Perp \ A \ R \rangle$  by blast
  moreover have  $Col \ S \ R \ T$ 
    using Col-cases  $\langle Col \ T \ R \ S \rangle$  by auto
  moreover have  $Bet \ C \ T \ A$ 
    using Bet-cases  $\langle Bet \ A \ T \ C \rangle$  by blast
  ultimately show ?thesis
    using l8-24  $\langle Bet \ R \ C' \ A \rangle \langle Cong \ S \ C \ R \ C' \rangle$  by blast
qed
then obtain  $M$  where  $M \ Midpoint \ S \ R$  and  $M \ Midpoint \ C \ C'$ 
  by blast
obtain  $U'$  where  $M \ Midpoint \ U \ U'$ 
  using symmetric-point-construction by blast
have  $R \neq U$ 
  using assms(8) out-diff1 by blast
have  $R \ S \ TS \ U \ U'$ 
  by (metis Col-def Out-def invert-two-sides l6-16-1 mid-two-sides  $\langle M \ Midpoint \ S \ R \rangle$ 
     $\langle M \ Midpoint \ U \ U' \rangle \langle \neg \ Col \ A \ R \ S \rangle$  assms(8))
have  $R \ S \ TS \ V \ U$ 
proof -
  have  $Col \ M \ R \ S$ 
    using Col-def Midpoint-def  $\langle M \ Midpoint \ S \ R \rangle$  by force
  moreover have  $M \ Midpoint \ U' \ U$ 
    by (simp add:  $\langle M \ Midpoint \ U \ U' \rangle$  l7-2)
  moreover have  $S \ Out \ U' \ V$ 
    by (meson l6-7  $\langle M \ Midpoint \ S \ R \rangle \langle M \ Midpoint \ U \ U' \rangle$  assms(2) assms(3) assms(4)
      assms(5) assms(6) assms(7) assms(8) assms(9) l6-6 l7-2 l9-4-1-aux)
  ultimately show ?thesis
    using  $\langle R \ S \ TS \ U \ U' \rangle$  col-trivial-3 l9-2 l9-3 by blast
qed

```

```

thus ?thesis
using  $\langle Col\ R\ S\ P \rangle \langle Col\ R\ S\ Q \rangle \langle P \neq Q \rangle$  col-preserves-two-sides l9-2 by presburger
qed

lemma l9-4-2-aux:
assumes  $S\ C\ Le\ R\ A$  and
 $P\ Q\ TS\ A\ C$  and
 $Col\ R\ P\ Q$  and
 $P\ Q\ Perp\ A\ R$  and
 $Col\ S\ P\ Q$  and
 $P\ Q\ Perp\ C\ S$  and
 $R\ Out\ U\ A$  and
 $S\ Out\ V\ C$ 
shows  $P\ Q\ TS\ U\ V$ 
using l9-4-2-aux-R1 l9-4-2-aux-R2
by (metis assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) assms(8))

lemma l9-4-2:
assumes  $P\ Q\ TS\ A\ C$  and
 $Col\ R\ P\ Q$  and
 $P\ Q\ Perp\ A\ R$  and
 $Col\ S\ P\ Q$  and
 $P\ Q\ Perp\ C\ S$  and
 $R\ Out\ U\ A$  and
 $S\ Out\ V\ C$ 
shows  $P\ Q\ TS\ U\ V$ 
proof –
have  $S\ C\ Le\ R\ A \vee R\ A\ Le\ S\ C$ 
by (simp add: local.le-cases)
moreover have  $S\ C\ Le\ R\ A \longrightarrow P\ Q\ TS\ U\ V$ 
by (simp add: assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) l9-4-2-aux)
moreover have  $R\ A\ Le\ S\ C \longrightarrow P\ Q\ TS\ U\ V$ 
by (simp add: assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) l9-2 l9-4-2-aux)
ultimately show ?thesis
by blast
qed

lemma l9-5:
assumes  $P\ Q\ TS\ A\ C$  and
 $Col\ R\ P\ Q$  and
 $R\ Out\ A\ B$ 
shows  $P\ Q\ TS\ B\ C$ 
proof –
have  $P \neq Q$ 
using assms(1) ts-distincts by blast
obtain  $A'$  where  $Col\ P\ Q\ A'$  and  $P\ Q\ Perp\ A\ A'$ 
by (metis NCol-perm TS-def assms(1) l8-18-existence)
obtain  $C'$  where  $Col\ P\ Q\ C'$  and  $P\ Q\ Perp\ C\ C'$ 
using Col-perm TS-def assms(1) l8-18-existence by blast
obtain  $M$  where  $M\ Midpoint\ A'\ C'$ 
using midpoint-existence by blast
obtain  $D$  where  $M\ Midpoint\ A\ D$ 
using symmetric-point-construction by auto
have  $\exists B0. Col\ P\ Q\ B0 \wedge P\ Q\ Perp\ B\ B0$ 
proof –
have  $\neg Col\ P\ Q\ B$ 
using colx_perp-not-col2 Tarski-neutral-dimensionless-axioms assms(2) assms(3)
col-permutation-1 l6-3-1 out-col by (meson  $\langle Col\ P\ Q\ A' \rangle \langle P\ Q\ Perp\ A\ A' \rangle$ )
thus ?thesis
by (simp add: l8-18-existence)
qed
then obtain  $B'$  where  $Col\ P\ Q\ B'$  and  $P\ Q\ Perp\ B\ B'$  by blast
have  $P\ Q\ TS\ B\ C$ 
proof –
have  $C'\ Out\ D\ C \longleftrightarrow A'\ Out\ A\ A$ 
using Out-cases assms(1) l9-4-1 not-col-permutation-1  $\langle Col\ P\ Q\ A' \rangle \langle Col\ P\ Q\ C' \rangle$ 

```



```

    ⟨M Midpoint A D⟩ ⟨M Midpoint A' C'⟩ ⟨P Q Perp A A'⟩ ⟨P Q Perp C C'⟩ by blast
  hence C' Out D C
    using perp-not-eq-2 Tarski-neutral-dimensionless-axioms out-trivial ⟨P Q Perp A A'⟩ by blast
  have P Q TS A D
    using assms(1) col-permutation-2 l9-4-2
  by (meson ⟨C' Out D C⟩ = ⟨A' Out A A'⟩ ⟨C' Out D C⟩ ⟨Col P Q A'⟩ ⟨Col P Q C'⟩
    ⟨P Q Perp A A'⟩ ⟨P Q Perp C C'⟩)
{
  assume A' ≠ C'
  hence Col M P Q
    using col-trivial-2 l6-21 midpoint-col not-col-permutation-1
    ⟨Col P Q A'⟩ ⟨Col P Q C'⟩ ⟨M Midpoint A' C'⟩ by fast
  hence P Q TS B D
    using assms(2) assms(3) l9-3 ⟨M Midpoint A D⟩ ⟨P Q TS A D⟩ by blast
}
  hence A' ≠ C' ⟶ P Q TS B D by simp
  hence P Q TS B D
    using assms(2) assms(3) l9-3 midpoint-distinct-2 not-col-permutation-1
  by (metis ⟨Col P Q A'⟩ ⟨M Midpoint A D⟩ ⟨M Midpoint A' C'⟩ ⟨P Q TS A D⟩)
  moreover have Col B' P Q
    using ⟨Col P Q B'⟩ not-col-permutation-1 by blast
  moreover have Col C' P Q
    using ⟨Col P Q C'⟩ col-permutation-2 by blast
  moreover have P Q Perp D C'
    using Col-perm l6-3-1 out-col perp-col1 perp-right-comm
  by (metis ⟨C' Out D C⟩ ⟨P Q Perp C C'⟩)
  moreover have B' Out B B
    using ⟨P Q Perp B B'⟩ out-trivial perp-not-eq-2 by auto
  moreover have C' Out C D
    using Out-cases ⟨C' Out D C⟩ by auto
  ultimately show ?thesis
    using l9-4-2 ⟨P Q Perp B B'⟩ by blast
qed
thus ?thesis using l8-18-existence by blast
qed

lemma outer-pasch-R1:
  assumes Col P Q C and
    Bet A C P and
    Bet B Q C
  shows ∃ X. Bet A X B ∧ Bet P Q X
proof cases
  assume Bet P Q C
  thus ?thesis
    using Bet-cases assms(2) between-exchange4 between-trivial2 by blast
next
  assume ¬ Bet P Q C
  thus ?thesis
    by (metis Bet-cases assms(1) assms(3) between-exchange3 between-trivial
      outer-transitivity-between2 third-point)
qed

lemma outer-pasch-R2:
  assumes ¬ Col P Q C and
    Bet A C P and
    Bet B Q C
  shows ∃ X. Bet A X B ∧ Bet P Q X
proof cases
  assume B = Q
  thus ?thesis
    using between-trivial by blast
next
  assume B ≠ Q
  have A ≠ P
    using assms(1) assms(2) between-identity col-trivial-3 by blast
  have P ≠ Q

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  using assms(1) col-trivial-1 by blast
have P ≠ B
  using assms(1) assms(3) bet-col by blast
have P Q TS C B
proof -
  have ¬ Col C P Q
    using Col-cases assms(1) by blast
  moreover have ¬ Col B P Q
    using Col-cases colx Tarski-neutral-dimensionless-axioms assms(1) assms(3) bet-col
    col-trivial-2 ⟨B ≠ Q⟩ by fast
  moreover have ∃ T. Col T P Q ∧ Bet C T B
    using Col-cases assms(3) between-symmetry col-trivial-2 by blast
  ultimately show ?thesis
    using TS-def by blast
qed
have P Q TS A B
  using assms(1) assms(2) bet-out-1 l9-5 not-col-distincts ⟨P Q TS C B⟩ by metis
obtain X where Col X P Q and Bet A X B
  using TS-def ⟨P Q TS A B⟩ by blast
have Bet P Q X
proof -
  obtain T where Bet X T P and Bet C T B
    using assms(2) between-symmetry inner-pasch by (meson ⟨Bet A X B⟩)
  have B ≠ C
    using assms(3) bet-neq12--neq ⟨B ≠ Q⟩ by auto
  have T = Q
  proof -
    have Col Q C B
      using NCol-cases assms(3) Col-def by blast
    have Col T C B
      using ⟨Bet C T B⟩ bet-col col-permutation-4 by presburger
    have ¬ Col X P B
      by (metis Bet-cases TS-def ⟨Bet C T B⟩ ⟨Bet X T P⟩ ⟨Col X P Q⟩ ⟨P Q TS C B⟩
        assms(3) bet-col1 between-equality-2 between-trivial2 col-transitivity-2)
    thus ?thesis
      by (metis Col-def ⟨B ≠ C⟩ ⟨Bet C T B⟩ ⟨Bet X T P⟩ ⟨Col Q C B⟩ ⟨Col X P Q⟩
        between-symmetry l6-21)
  qed
  thus ?thesis
    using ⟨Bet X T P⟩ between-symmetry by blast
qed
thus ?thesis
  using ⟨Bet A X B⟩ by blast
qed

```

```

lemma outer-pasch:
  assumes Bet A C P and
    Bet B Q C
  shows ∃ X. Bet A X B ∧ Bet P Q X
  using assms(1) assms(2) outer-pasch-R1 outer-pasch-R2 by blast

```

```

lemma os-distincts:
  assumes A B OS X Y
  shows A ≠ B ∧ A ≠ X ∧ A ≠ Y ∧ B ≠ X ∧ B ≠ Y
  using OS-def assms ts-distincts by blast

```

```

lemma invert-one-side:
  assumes A B OS P Q
  shows B A OS P Q
proof -
  obtain T where A B TS P T ∧ A B TS Q T
    using OS-def assms by blast
  hence B A TS P T ∧ B A TS Q T

```

```

    using invert-two-sides by blast
  thus ?thesis
    using OS-def by blast
qed

lemma l9-8-1:
  assumes  $P \ Q \ TS \ A \ C$  and
     $P \ Q \ TS \ B \ C$ 
  shows  $P \ Q \ OS \ A \ B$ 
proof -
  have  $\exists R::'p. (P \ Q \ TS \ A \ R \wedge P \ Q \ TS \ B \ R)$ 
    using assms(1) assms(2) by blast
  thus ?thesis
    using OS-def by blast
qed

lemma not-two-sides-id:
  shows  $\neg P \ Q \ TS \ A \ A$ 
  using ts-distincts by blast

lemma l9-8-2:
  assumes  $P \ Q \ TS \ A \ C$  and
     $P \ Q \ OS \ A \ B$ 
  shows  $P \ Q \ TS \ B \ C$ 
proof -
  obtain  $D$  where  $P1: P \ Q \ TS \ A \ D \wedge P \ Q \ TS \ B \ D$ 
    using assms(2) OS-def by blast
  hence  $P \neq Q$ 
    using ts-distincts by blast
  obtain  $T$  where  $P2: Col \ T \ P \ Q \wedge Bet \ A \ T \ C$ 
    using TS-def assms(1) by blast
  obtain  $X$  where  $P3: Col \ X \ P \ Q \wedge Bet \ A \ X \ D$ 
    using TS-def  $P1$  by blast
  obtain  $Y$  where  $P4: Col \ Y \ P \ Q \wedge Bet \ B \ Y \ D$ 
    using TS-def  $P1$  by blast
  then obtain  $M$  where  $P5: Bet \ Y \ M \ A \wedge Bet \ X \ M \ B$  using  $P3$  inner-pasch by blast
  have  $P6: A \neq D$ 
    using  $P1$  ts-distincts by blast
  have  $P7: B \neq D$ 
    using  $P1$  not-two-sides-id by blast
  {
    assume  $Q0: Col \ A \ B \ D$ 
    have  $P \ Q \ TS \ B \ C$ 
    proof cases
      assume  $Q1: M = Y$ 
      have  $X = Y$ 
      proof -
        have  $S1: \neg Col \ P \ Q \ A$ 
          using TS-def assms(1) not-col-permutation-1 by blast
        have  $S3: Col \ P \ Q \ X$ 
          using Col-perm  $P3$  by blast
        have  $S4: Col \ P \ Q \ Y$ 
          using Col-perm  $P4$  by blast
        have  $S5: Col \ A \ D \ X$ 
          by (simp add: P3 bet-col col-permutation-5)
        have  $Col \ A \ D \ Y$ 
          by (metis Col-def P5 Q1 S5 Q0 between-equality between-trivial l6-16-1)
        thus ?thesis using  $S1 \ S3 \ S4 \ S5 \ P6 \ l6-21$ 
          by blast
      qed
    qed
  }
  hence  $X \ Out \ A \ B$ 
    by (metis P1 P3 P4 TS-def l6-2)
  thus ?thesis using assms(1)  $P3 \ l9-5$  by blast
next
  assume  $Z1: \neg M = Y$ 
  have  $X = Y$ 

```

```

proof -
  have S1:  $\neg \text{Col } P \ Q \ A$ 
    using TS-def assms(1) not-col-permutation-1 by blast
  have S3:  $\text{Col } P \ Q \ X$ 
    using Col-perm P3 by blast
  have S4:  $\text{Col } P \ Q \ Y$ 
    using Col-perm P4 by blast
  have S5:  $\text{Col } A \ D \ X$ 
    by (simp add: P3 bet-col col-permutation-5)
  have  $\text{Col } A \ D \ Y$ 
    by (metis Col-def P4 Q0 P7 l6-16-1)
  thus ?thesis using S1 S3 S4 S5 P6 l6-21
    by blast
qed
hence Z3:  $M \neq X$  using Z1 by blast
have Z4:  $P \ Q \ TS \ M \ C$ 
  by (meson Out-cases P4 P5 l9-5 Tarski-neutral-dimensionless-axioms Z1 assms(1) bet-out)
have  $X \text{ Out } M \ B$ 
  using P5 Z3 bet-out by auto
thus ?thesis using Z4 P3 l9-5 by blast
qed
}
hence Z99:  $\text{Col } A \ B \ D \longrightarrow P \ Q \ TS \ B \ C$  by blast
{
  assume Q0:  $\neg \text{Col } A \ B \ D$ 
  have Q1:  $P \ Q \ TS \ M \ C$ 
  proof -
    have S3:  $Y \text{ Out } A \ M$ 
    proof -
      have T1:  $A \neq Y$ 
        using Col-def P4 Q0 col-permutation-4 by blast
      have T2:  $M \neq Y$ 
      proof -
        {
          assume T3:  $M = Y$ 
          have  $\text{Col } B \ D \ X$ 
          proof -
            have U1:  $B \neq M$ 
              using P1 P4 T3 TS-def by blast
            have U2:  $\text{Col } B \ M \ D$ 
              by (simp add: P4 T3 bet-col)
            have  $\text{Col } B \ M \ X$ 
              by (simp add: P5 bet-col between-symmetry)
            thus ?thesis using U1 U2
              using col-transitivity-1 by blast
          qed
          have False
            by (metis NCol-cases P1 P3 TS-def <Col B D X> Q0 bet-col col-trivial-2 l6-21)
        }
        thus ?thesis by blast
      qed
    qed
    have  $\text{Bet } Y \ A \ M \vee \text{Bet } Y \ M \ A$  using P5 by blast
    thus ?thesis using T1 T2
      by (simp add: Out-def)
  qed
  hence  $X \text{ Out } M \ B$ 
    by (metis P1 P3 P4 P5 TS-def bet-out l9-5)
  thus ?thesis using assms(1) S3 l9-5 P3 P4 by blast
qed
have  $X \text{ Out } M \ B$ 
  by (metis P3 P5 Q1 TS-def bet-out)
hence  $P \ Q \ TS \ B \ C$  using Q1 P3 l9-5 by blast
}
hence  $\neg \text{Col } A \ B \ D \longrightarrow P \ Q \ TS \ B \ C$  by blast
thus ?thesis using Z99 by blast
qed

```

```

lemma l9-9:
  assumes  $P \ Q \ TS \ A \ B$ 
  shows  $\neg P \ Q \ OS \ A \ B$ 
  using assms l9-8-2 not-two-sides-id by blast

lemma l9-9-bis:
  assumes  $P \ Q \ OS \ A \ B$ 
  shows  $\neg P \ Q \ TS \ A \ B$ 
  using assms l9-9 by blast

lemma one-side-chara:
  assumes  $P \ Q \ OS \ A \ B$ 
  shows  $\forall X. Col \ X \ P \ Q \longrightarrow \neg Bet \ A \ X \ B$ 
proof -
  have  $\neg Col \ A \ P \ Q \wedge \neg Col \ B \ P \ Q$ 
  using OS-def TS-def assms by auto
  thus ?thesis
  using l9-9-bis TS-def assms by blast
qed

lemma l9-10:
  assumes  $\neg Col \ A \ P \ Q$ 
  shows  $\exists C. P \ Q \ TS \ A \ C$ 
  by (meson Col-perm assms mid-two-sides midpoint-existence symmetric-point-construction)

lemma one-side-reflexivity:
  assumes  $\neg Col \ A \ P \ Q$ 
  shows  $P \ Q \ OS \ A \ A$ 
  using assms l9-10 l9-8-1 by blast

lemma one-side-symmetry:
  assumes  $P \ Q \ OS \ A \ B$ 
  shows  $P \ Q \ OS \ B \ A$ 
  by (meson OS-def Tarski-neutral-dimensionless-axioms assms invert-two-sides)

lemma one-side-transitivity:
  assumes  $P \ Q \ OS \ A \ B$  and
     $P \ Q \ OS \ B \ C$ 
  shows  $P \ Q \ OS \ A \ C$ 
  by (meson OS-def l9-8-2 Tarski-neutral-dimensionless-axioms assms(1) assms(2))

lemma l9-17:
  assumes  $P \ Q \ OS \ A \ C$  and
     $Bet \ A \ B \ C$ 
  shows  $P \ Q \ OS \ A \ B$ 
proof cases
  assume  $A = C$ 
  thus ?thesis
  using assms(1) assms(2) between-identity by blast
next
  assume  $P1: \neg A = C$ 
  obtain  $D$  where  $P2: P \ Q \ TS \ A \ D \wedge P \ Q \ TS \ C \ D$ 
  using OS-def assms(1) by blast
  hence  $P3: P \neq Q$ 
  using ts-distincts by blast
  obtain  $X$  where  $P4: Col \ X \ P \ Q \wedge Bet \ A \ X \ D$ 
  using P2 TS-def by blast
  obtain  $Y$  where  $P5: Col \ Y \ P \ Q \wedge Bet \ C \ Y \ D$ 
  using P2 TS-def by blast
  obtain  $T$  where  $P6: Bet \ B \ T \ D \wedge Bet \ X \ T \ Y$ 
  using P4 P5 assms(2) l3-17 by blast
  have  $P7: P \ Q \ TS \ A \ D$ 
  by (simp add: P2)
  have  $P \ Q \ TS \ B \ D$ 
  proof -

```

```

have Q1:  $\neg \text{Col } B \ P \ Q$ 
  using assms(1) assms(2) one-side-chara by blast
have Q2:  $\neg \text{Col } D \ P \ Q$ 
  using P2 TS-def by blast
obtain T0 where  $\text{Col } T0 \ P \ Q \wedge \text{Bet } B \ T0 \ D$ 
proof -
  assume a1:  $\bigwedge T0. \text{Col } T0 \ P \ Q \wedge \text{Bet } B \ T0 \ D \implies \text{thesis}$ 
  obtain pp :: 'p where
    f2:  $\text{Bet } B \ pp \ D \wedge \text{Bet } X \ pp \ Y$ 
    using  $\langle \bigwedge \text{thesis}. (\bigwedge T. \text{Bet } B \ T \ D \wedge \text{Bet } X \ T \ Y \implies \text{thesis}) \implies \text{thesis} \rangle$  by blast
  have  $\text{Col } P \ Q \ Y$ 
    using Col-def P5 by blast
  hence  $Y = X \vee \text{Col } P \ Q \ pp$ 
    using f2 Col-def P4 colx by blast
  thus ?thesis
    using f2 a1 by (metis BetSEq BetS-def Col-def P4)
qed
thus ?thesis
  using Q1 Q2 TS-def by blast
qed
thus ?thesis using P7
  using OS-def by blast
qed

lemma l9-18-R1:
  assumes  $\text{Col } X \ Y \ P$  and
     $\text{Col } A \ B \ P$ 
  and  $X \ Y \ TS \ A \ B$ 
  shows  $\text{Bet } A \ P \ B \wedge \neg \text{Col } X \ Y \ A \wedge \neg \text{Col } X \ Y \ B$ 
  by (meson TS-def assms(1) assms(2) assms(3) col-permutation-5 l9-5 not-col-permutation-1
    not-out-bet not-two-sides-id)

lemma l9-18-R2:
  assumes  $\text{Col } X \ Y \ P$  and
     $\text{Col } A \ B \ P$  and
     $\text{Bet } A \ P \ B$  and
     $\neg \text{Col } X \ Y \ A$  and
     $\neg \text{Col } X \ Y \ B$ 
  shows  $X \ Y \ TS \ A \ B$ 
  using Col-perm TS-def assms(1) assms(3) assms(4) assms(5) by blast

lemma l9-18:
  assumes  $\text{Col } X \ Y \ P$  and
     $\text{Col } A \ B \ P$ 
  shows  $X \ Y \ TS \ A \ B \longleftrightarrow (\text{Bet } A \ P \ B \wedge \neg \text{Col } X \ Y \ A \wedge \neg \text{Col } X \ Y \ B)$ 
  using l9-18-R1 l9-18-R2 assms(1) assms(2) by blast

lemma l9-19-R1:
  assumes  $\text{Col } X \ Y \ P$  and
     $\text{Col } A \ B \ P$  and
     $X \ Y \ OS \ A \ B$ 
  shows  $P \text{ Out } A \ B \wedge \neg \text{Col } X \ Y \ A$ 
  by (meson OS-def TS-def assms(1) assms(2) assms(3) col-permutation-5 not-col-permutation-1
    not-out-bet one-side-chara)

lemma l9-19-R2:
  assumes  $\text{Col } X \ Y \ P$  and

     $P \text{ Out } A \ B$  and
     $\neg \text{Col } X \ Y \ A$ 
  shows  $X \ Y \ OS \ A \ B$ 
proof -
  obtain D where  $X \ Y \ TS \ A \ D$ 
    using Col-perm assms(3) l9-10 by blast
  thus ?thesis
    using OS-def assms(1) assms(2) l9-5 not-col-permutation-1 by blast

```

qed

lemma *l9-19*:

assumes *Col X Y P* and
Col A B P

shows $X Y OS A B \longleftrightarrow (P Out A B \wedge \neg Col X Y A)$

using *l9-19-R1 l9-19-R2 assms(1) assms(2)* by blast

lemma *one-side-not-col123*:

assumes *A B OS X Y*

shows $\neg Col A B X$

using *assms col-trivial-3 l9-19* by blast

lemma *one-side-not-col124*:

assumes *A B OS X Y*

shows $\neg Col A B Y$

using *assms one-side-not-col123 one-side-symmetry* by blast

lemma *col-two-sides*:

assumes *Col A B C* and

A $\neq$ C and

A B TS P Q

shows *A C TS P Q*

using *assms(1) assms(2) assms(3) col-preserves-two-sides col-trivial-3* by blast

lemma *col-one-side*:

assumes *Col A B C* and

A $\neq$ C and

A B OS P Q

shows *A C OS P Q*

proof –

obtain *T* where *A B TS P T $\wedge$ A B TS Q T* using *assms(1) assms(2) assms(3) OS-def* by blast

thus *?thesis*

using *col-two-sides OS-def assms(1) assms(2)* by blast

qed

lemma *out-out-one-side*:

assumes *A B OS X Y* and

A Out Y Z

shows *A B OS X Z*

by (*meson Col-cases OS-def Tarski-neutral-dimensionless-axioms assms(1) assms(2)*
col-trivial-3 l9-5)

lemma *out-one-side*:

assumes $\neg Col A B X \vee \neg Col A B Y$ and

A Out X Y

shows *A B OS X Y*

using *assms(1) assms(2) l6-6 not-col-permutation-2 one-side-reflexivity one-side-symmetry*
out-out-one-side by blast

lemma *bet--ts*:

assumes *A $\neq$ Y* and

$\neg Col A B X$ and

Bet X A Y

shows *A B TS X Y*

proof –

have $\neg Col Y A B$

using *NCol-cases assms(1) assms(2) assms(3) bet-col col2--eq* by blast

thus *?thesis*

by (*meson TS-def assms(2) assms(3) col-permutation-3 col-permutation-5 col-trivial-3*)

qed

lemma *bet-ts--ts*:

assumes *A B TS X Y* and

Bet X Y Z

shows *A B TS X Z*

```

proof –
  have  $\neg \text{Col } Z \ A \ B$ 
    using assms(1) assms(2) bet-col between-equality-2 col-permutation-1 l9-18 by blast
  thus ?thesis
    using TS-def assms(1) assms(2) between-exchange4 by blast
qed

lemma bet-ts--os:
  assumes  $A \ B \ TS \ X \ Y$  and
     $Bet \ X \ Y \ Z$ 
  shows  $A \ B \ OS \ Y \ Z$ 
  using OS-def assms(1) assms(2) bet-ts--ts l9-2 by blast

lemma l9-31 :
  assumes  $A \ X \ OS \ Y \ Z$  and
     $A \ Z \ OS \ Y \ X$ 
  shows  $A \ Y \ TS \ X \ Z$ 
proof –
  have  $A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z$ 
    using assms(1) assms(2) col-permutation-1 one-side-not-col123 one-side-not-col124
    os-distincts by blast
  obtain  $Z'$  where  $Bet \ Z \ A \ Z'$  and  $Cong \ A \ Z' \ Z \ A$ 
    using segment-construction by blast
  have  $Z' \neq A$ 
    by (metis  $\langle Cong \ A \ Z' \ Z \ A \rangle$  assms(1) cong-diff-3 os-distincts)
  have  $A \ X \ TS \ Y \ Z'$ 
    by (metis l9-8-2 one-side-symmetry  $\langle Bet \ Z \ A \ Z' \rangle \langle Z' \neq A \rangle$  assms(1) bet--ts
    one-side-not-col124)
  have  $\neg \text{Col } Y \ A \ X$ 
    using  $\langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle$  by auto
  obtain  $T$  where  $\text{Col } A \ T \ X$  and  $Bet \ Y \ T \ Z'$ 
    by (meson TS-def col-permutation-4  $\langle A \ X \ TS \ Y \ Z' \rangle$ )
  hence  $T \neq A$ 
proof –
  have  $\neg A \ Out \ Z \ Y$ 
    using Col-cases  $\langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle$  out-col by blast
  have  $A \neq Z'$ 
    using  $\langle Z' \neq A \rangle$  by auto
  thus ?thesis
    by (metis  $\langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle \langle Bet \ Y \ T \ Z' \rangle$ 
     $\langle Bet \ Z \ A \ Z' \rangle$  bet-col l6-21 not-col-distincts)
qed
have  $Y \ A \ OS \ Z' \ T$ 
  by (metis NCol-cases Out-def col3 col-trivial-3 out-one-side  $\langle A \ X \ TS \ Y \ Z' \rangle \langle Bet \ Y \ T \ Z' \rangle$ 
   $\langle \text{Col } A \ T \ X \rangle \langle T \neq A \rangle \langle \neg \text{Col } Y \ A \ X \rangle$  ts-distincts)
have  $A \ Y \ TS \ Z' \ Z$ 
  by (metis Bet-cases Col-cases  $\langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle$ 
   $\langle Bet \ Z \ A \ Z' \rangle \langle Y \ A \ OS \ Z' \ T \rangle$  bet--ts one-side-not-col123)
{
  assume  $Bet \ T \ A \ X$ 
  have  $Z' \ Z \ OS \ Y \ T$ 
    using BetSeq BetS-def TS-def l6-6 bet-col bet-out-1 col-trivial-3
    colx not-col-permutation-3 not-col-permutation-4 out-one-side
     $\langle A \ Y \ TS \ Z' \ Z \rangle \langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle$ 
     $\langle Bet \ Y \ T \ Z' \rangle \langle Bet \ Z \ A \ Z' \rangle \langle \text{Col } A \ T \ X \rangle$  by metis
  hence  $Z' \ Out \ T \ Y$ 
    by (metis  $\langle Bet \ Y \ T \ Z' \rangle$  bet-out-1 os-distincts)
  hence  $A \ Z \ OS \ Y \ T$ 
    using  $\langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle \langle Bet \ Z \ A \ Z' \rangle$ 
    bet-col invert-one-side l6-6 l9-19-R2 not-col-permutation-3 by blast
  have  $A \ Z \ TS \ X \ T$ 
proof –
  have  $\neg \text{Col } X \ A \ Z$ 
    using Col-cases  $\langle A \neq X \wedge A \neq Z \wedge \neg \text{Col } Y \ A \ X \wedge \neg \text{Col } Z \ A \ X \wedge \neg \text{Col } Y \ A \ Z \rangle$  by blast
  have  $\neg \text{Col } T \ A \ Z$ 
    using  $\langle A \ Z \ OS \ Y \ T \rangle$  col-permutation-1 one-side-not-col124 by blast

```



```

have  $\exists T0. Col\ T0\ A\ Z \wedge Bet\ X\ T0\ T$ 
proof -
  have  $Col\ A\ A\ Z$ 
    by (simp add: col-trivial-1)
  moreover have  $Bet\ X\ A\ T$ 
    using Bet-cases  $\langle Bet\ T\ A\ X \rangle$  by blast
  ultimately show ?thesis
    by auto
qed
thus ?thesis
  using TS-def  $\langle \neg Col\ T\ A\ Z \rangle \langle \neg Col\ X\ A\ Z \rangle$  by presburger
qed
have  $A\ Y\ TS\ X\ Z$ 
  by (meson l9-8-2  $\langle A\ Z\ OS\ Y\ T \rangle \langle A\ Z\ TS\ X\ T \rangle$  assms(2) l9-9 one-side-symmetry)
}
moreover
{
  assume  $Bet\ A\ X\ T$ 
  hence  $A\ Y\ OS\ Z'\ X$ 
    using Bet-cases between-equality invert-one-side not-col-permutation-4
    not-out-bet out-out-one-side
    by (meson OS-def  $\langle A\ Y\ TS\ Z'\ Z \rangle \langle Col\ A\ T\ X \rangle \langle Y\ A\ OS\ Z'\ T \rangle$  calculation)
  have  $A\ Y\ TS\ X\ Z$ 
    using  $\langle A\ Y\ OS\ Z'\ X \rangle \langle A\ Y\ TS\ Z'\ Z \rangle$  l9-8-2 by blast
}
moreover
{
  assume  $Bet\ X\ T\ A$ 
  hence  $A\ Y\ OS\ T\ X$ 
    using  $\langle A \neq X \wedge A \neq Z \wedge \neg Col\ Y\ A\ X \wedge \neg Col\ Z\ A\ X \wedge \neg Col\ Y\ A\ Z \rangle \langle T \neq A \rangle$ 
    bet-out-1 not-col-permutation-4 out-one-side by presburger
  hence  $A\ Y\ TS\ X\ Z$ 
    using  $\langle A\ Y\ TS\ Z'\ Z \rangle \langle Y\ A\ OS\ Z'\ T \rangle$  invert-two-sides l9-8-2 by blast
}
ultimately show ?thesis
  using Bet-cases  $\langle Col\ A\ T\ X \rangle$  third-point by blast
qed

lemma col123--nos:
  assumes  $Col\ P\ Q\ A$ 
  shows  $\neg P\ Q\ OS\ A\ B$ 
  using assms one-side-not-col123 by blast

lemma col124--nos:
  assumes  $Col\ P\ Q\ B$ 
  shows  $\neg P\ Q\ OS\ A\ B$ 
  using assms one-side-not-col124 by blast

lemma col2-os--os:
  assumes  $C \neq D$  and
     $Col\ A\ B\ C$  and
     $Col\ A\ B\ D$  and
     $A\ B\ OS\ X\ Y$ 
  shows  $C\ D\ OS\ X\ Y$ 
  by (metis assms(1) assms(2) assms(3) assms(4) col3 col-one-side col-trivial-3
    invert-one-side os-distincts)

lemma os-out-os:
  assumes  $Col\ A\ B\ P$  and
     $A\ B\ OS\ C\ D$  and
     $P\ Out\ C\ C'$ 
  shows  $A\ B\ OS\ C'\ D$ 
  using OS-def assms(1) assms(2) assms(3) l9-5 not-col-permutation-1 by blast

lemma ts-ts-os:
  assumes  $A\ B\ TS\ C\ D$  and

```

$C D TS A B$
shows $A C OS B D$
proof –
obtain $T1$ **where** $P1: Col T1 A B \wedge Bet C T1 D$
using $TS-def\ assms(1)$ **by** $blast$
obtain T **where** $P2: Col T C D \wedge Bet A T B$
using $TS-def\ assms(2)$ **by** $blast$
have $P3: T1 = T$
proof –
have $A \neq B$
using $assms(2)\ ts-distincts$ **by** $blast$
thus $?thesis$
proof –
have $Col T1 D C$
using $Col-def P1$ **by** $blast$
hence $f1: \forall p. (C = T1 \vee Col C p T1) \vee \neg Col C T1 p$
by $(metis\ assms(1)\ col-transitivity-1\ l6-16-1\ ts-distincts)$
have $f2: \neg Col C A B$
using $TS-def\ assms(1)$ **by** $presburger$
have $f3: (Bet B T1 A \vee Bet T1 A B) \vee Bet A B T1$
using $Col-def P1$ **by** $blast$
{
assume $T1 \neq B$
hence $C \neq T1 \wedge \neg Col C T1 B \vee (\exists p. \neg Col p T1 B \wedge Col p T1 T) \vee T \neq A \wedge T \neq B$
using $f3\ f2$ **by** $(metis\ (no-types)\ Col-def\ col-transitivity-1\ l6-16-1)$
hence $T \neq A \wedge T \neq B \vee C \neq T1 \wedge \neg Col C T1 T \vee T1 = T$
using $f3$ **by** $(meson\ Col-def\ l6-16-1)$
}
moreover
{
assume $T \neq A \wedge T \neq B$
hence $C \neq T1 \wedge \neg Col C T1 T \vee T1 = T$
using $f2$ **by** $(metis\ (no-types)\ Col-def P1 P2 \langle A \neq B \rangle\ col-transitivity-1\ l6-16-1)$
}
ultimately have $C \neq T1 \wedge \neg Col C T1 T \vee T1 = T$
using $f2\ f1\ assms(1)\ ts-distincts$ **by** $blast$
thus $?thesis$
by $(metis\ (no-types)\ Col-def P1 P2\ assms(1)\ l6-16-1\ ts-distincts)$
qed
qed
have $P4: A C OS T B$
by $(metis\ Col-cases P2 TS-def\ assms(1)\ assms(2)\ bet-out\ out-one-side)$
hence $C A OS T D$
by $(metis\ Col-cases P1 TS-def P3\ assms(2)\ bet-out\ os-distincts\ out-one-side)$
thus $?thesis$
by $(meson\ P4\ invert-one-side\ one-side-symmetry\ Tarski-neutral-dimensionless-axioms\ one-side-transitivity)$
qed

lemma $col-one-side-out$:
assumes $Col A X Y$ **and**
 $A B OS X Y$
shows $A Out X Y$
by $(meson\ assms(1)\ assms(2)\ l6-4-2\ not-col-distincts\ not-col-permutation-4\ one-side-chara)$

lemma $col-two-sides-bet$:
assumes $Col A X Y$ **and**
 $A B TS X Y$
shows $Bet X A Y$
using $Col-cases\ assms(1)\ assms(2)\ l9-8-1\ l9-9\ or-bet-out\ out-out-one-side$ **by** $blast$

lemma $os-ts1324--os$:
assumes $A X OS Y Z$ **and**
 $A Y TS X Z$
shows $A Z OS X Y$
proof –

```

obtain  $P$  where  $P1$ :  $Col\ P\ A\ Y \wedge Bet\ X\ P\ Z$ 
  using  $TS\text{-}def\ assms(2)$  by  $blast$ 
have  $P2$ :  $A\ Z\ OS\ X\ P$ 
  by ( $metis\ Col\text{-}cases\ P1\ TS\text{-}def\ assms(1)\ assms(2)\ bet\text{-}col\ bet\text{-}out\text{-}1\ col124\text{-}nos$ 
     $col\text{-}trivial\text{-}2\ l6\text{-}6\ l9\text{-}19$ )
have  $A\ Z\ OS\ P\ Y$ 
proof –
  have  $\neg\ Col\ A\ Z\ P \vee \neg\ Col\ A\ Z\ Y$ 
    using  $P2\ col124\text{-}nos$  by  $blast$ 
  moreover have  $A\ Out\ P\ Y$ 
  proof –
    have  $X\ A\ OS\ P\ Z$ 
      by ( $metis\ Col\text{-}cases\ P1\ P2\ assms(1)\ bet\text{-}out\ col123\text{-}nos\ out\text{-}one\text{-}side$ )
    hence  $A\ X\ OS\ P\ Y$ 
      by ( $meson\ invert\text{-}one\text{-}side\ one\text{-}side\text{-}symmetry\ Tarski\text{-}neutral\text{-}dimensionless\text{-}axioms$ 
         $assms(1)\ one\text{-}side\text{-}transitivity$ )
    thus  $?thesis$ 
      using  $P1\ col\text{-}one\text{-}side\text{-}out\ not\text{-}col\text{-}permutation\text{-}4$  by  $blast$ 
  qed
  ultimately show  $?thesis$ 
    by ( $simp\ add:\ out\text{-}one\text{-}side$ )
qed
thus  $?thesis$ 
  using  $P2\ one\text{-}side\text{-}transitivity$  by  $blast$ 
qed

lemma  $ts2\text{-}ex\text{-}bet2$ :
  assumes  $A\ C\ TS\ B\ D$  and
     $B\ D\ TS\ A\ C$ 
  shows  $\exists\ X.\ Bet\ A\ X\ C \wedge Bet\ B\ X\ D$ 
  by ( $metis\ TS\text{-}def\ assms(1)\ assms(2)\ bet\text{-}col\ col\text{-}permutation\text{-}5\ l9\text{-}18\text{-}R1\ not\text{-}col\text{-}permutation\text{-}2$ )

lemma  $out\text{-}one\text{-}side\text{-}1$ :
  assumes  $\neg\ Col\ A\ B\ C$  and
     $Col\ A\ B\ X$  and
     $X\ Out\ C\ D$ 
  shows  $A\ B\ OS\ C\ D$ 
  using  $assms(1)\ assms(2)\ assms(3)\ not\text{-}col\text{-}permutation\text{-}2\ one\text{-}side\text{-}reflexivity$ 
     $one\text{-}side\text{-}symmetry\ os\text{-}out\text{-}os$  by  $blast$ 

lemma  $out\text{-}two\text{-}sides\text{-}two\text{-}sides$ :
  assumes
     $Col\ A\ B\ PX$  and
     $PX\ Out\ X\ P$  and
     $A\ B\ TS\ P\ Y$ 
  shows  $A\ B\ TS\ X\ Y$ 
  using  $assms(1)\ assms(2)\ assms(3)\ l6\text{-}6\ l9\text{-}5\ not\text{-}col\text{-}permutation\text{-}1$  by  $blast$ 

lemma  $l8\text{-}21\text{-}bis$ :
  assumes  $X \neq Y$  and
     $\neg\ Col\ C\ A\ B$ 
  shows  $\exists\ P.\ Cong\ A\ P\ X\ Y \wedge A\ B\ Perp\ P\ A \wedge A\ B\ TS\ C\ P$ 
proof –
  have  $A \neq B$ 
    using  $assms(2)\ not\text{-}col\text{-}distincts$  by  $blast$ 
  hence  $\exists\ P\ T.\ A\ B\ Perp\ P\ A \wedge Col\ A\ B\ T \wedge Bet\ C\ T\ P$ 
    using  $l8\text{-}21$  by  $auto$ 
  then obtain  $P\ T$  where  $A\ B\ Perp\ P\ A$  and  $Col\ A\ B\ T$  and  $Bet\ C\ T\ P$  by  $blast$ 
  have  $A\ B\ TS\ C\ P$ 
  proof –
    have  $\neg\ Col\ P\ A\ B$ 
      using  $col\text{-}permutation\text{-}1\ perp\text{-}not\text{-}col\ \langle A\ B\ Perp\ P\ A \rangle$  by  $blast$ 
    thus  $?thesis$ 
      using  $Col\text{-}cases\ TS\text{-}def\ \langle Bet\ C\ T\ P \rangle\ \langle Col\ A\ B\ T \rangle\ assms(2)$  by  $blast$ 
  qed
  have  $P \neq A$ 

```

```

  using ⟨A B Perp P A⟩ perp-distinct by auto
obtain P' where (Bet A P P' ∨ Bet A P' P) and Cong A P' X Y
  using segment-construction-2 ⟨P ≠ A⟩ by blast
have A B Perp P' A
  by (metis Bet-cases Col-def cong-identity perp-col1 ⟨A B Perp P A⟩
    ⟨Bet A P P' ∨ Bet A P' P⟩ ⟨Cong A P' X Y⟩ assms(1) cong-symmetry perp-comm)
have ¬ Col P' A B
  using ⟨A B Perp P' A⟩ not-col-permutation-2 perp-not-col by blast
hence A B OS P P'
  using Out-def ⟨A B Perp P A⟩ ⟨A B Perp P' A⟩ ⟨Bet A P P' ∨ Bet A P' P⟩
    out-one-side perp-not-col perp-not-eq-2 by presburger
hence A B TS C P'
  using ⟨A B TS C P⟩ l9-2 l9-8-2 by blast
thus ?thesis
  using ⟨A B Perp P' A⟩ ⟨Cong A P' X Y⟩ by blast
qed

```

```

lemma ts--ncol:
  assumes A B TS X Y
  shows ¬ Col A X Y ∨ ¬ Col B X Y
  by (metis TS-def assms col-permutation-1 col-transitivity-2 ts-distincts)

```

```

lemma one-or-two-sides-aux:
  assumes ¬ Col C A B and
    ¬ Col D A B and
    Col A C X
  and Col B D X
  shows A B TS C D ∨ A B OS C D
proof -
  have P1: A ≠ X
    using assms(2) assms(4) col-permutation-2 by blast
  have P2: B ≠ X
    using assms(1) assms(3) col-permutation-4 by blast
  have P3: ¬ Col X A B
    using P1 assms(1) assms(3) col-permutation-5 col-transitivity-1 not-col-permutation-4
    by blast
  {
    assume Q0: Bet A C X ∧ Bet B D X
    hence Q1: A B OS C X
      using assms(1) bet-out not-col-distincts not-col-permutation-1 out-one-side by blast
    hence A B OS X D
      by (metis Q0 assms(2) assms(4) bet-out-1 col-permutation-2 col-permutation-3
        invert-one-side l6-4-2 not-bet-and-out not-col-distincts out-one-side)
    hence A B OS C D
      using Q1 one-side-transitivity by blast
  }
  hence P4: Bet A C X ∧ Bet B D X ⟶ A B OS C D by blast
  {
    assume Bet A C X ∧ Bet D X B
    hence A B OS C D
      by (metis P2 assms(1) bet-out bet-out-1 not-col-distincts one-side-reflexivity
        one-side-symmetry os-out-os)
  }
  hence P5: Bet A C X ∧ Bet D X B ⟶ A B OS C D by blast
  {
    assume Q0: Bet A C X ∧ Bet X B D
    have Q1: A B TS X D
      using P3 Q0 TS-def assms(2) col-trivial-3 by blast
    have A B OS X C
      using Q0 assms(1) bet-out not-col-distincts one-side-reflexivity one-side-symmetry
      out-out-one-side by blast
    hence A B TS C D
      using Q1 l9-8-2 by blast
  }
  hence P6: Bet A C X ∧ Bet X B D ⟶ A B TS C D by blast
  {

```

```

assume Q1: Bet C X A  $\wedge$  Bet B D X
hence Q2: A B OS C X
  using P1 assms(1) assms(3) between-equality-2 l6-4-2 not-col-permutation-1
    not-col-permutation-4 out-one-side by blast
have A B OS X D
  using Q1 assms(2) bet-out not-col-distincts one-side-reflexivity os-out-os by blast
hence A B OS C D using Q2
  using one-side-transitivity by blast
}
hence P7: Bet C X A  $\wedge$  Bet B D X  $\longrightarrow$  A B OS C D by blast
{
  assume Bet C X A  $\wedge$  Bet D X B
  hence A B OS C D
  by (metis Out-def P1 P2 assms(1) bet-out between-symmetry col2--eq invert-one-side
    l9-19-R2 not-col-distincts out-out-one-side)
}
hence P8: Bet C X A  $\wedge$  Bet D X B  $\longrightarrow$  A B OS C D by blast
{
  assume Q1: Bet C X A  $\wedge$  Bet X B D
  have Q2: A B TS X D
  by (metis P3 Q1 assms(2) bet--ts invert-two-sides not-col-distincts not-col-permutation-3)
  have Q3: A B OS X C
  using P1 Q1 assms(1) bet-out-1 not-col-permutation-1 out-one-side by auto
  hence A B TS C D
  using Q2 l9-8-2 by blast
}
hence P9: Bet C X A  $\wedge$  Bet X B D  $\longrightarrow$  A B TS C D by blast
{
  assume Q0: Bet X A C  $\wedge$  Bet B D X
  have Q1: A B TS X C
  by (metis P3 Q0 assms(1) bet--ts col-permutation-2 not-col-distincts)
  have A B OS X D
  by (metis NCol-cases Q0 out-one-side assms(2) assms(4) bet-out-1 invert-one-side
    l6-4-1 not-col-distincts not-out-bet)
  hence A B TS C D
  using Q1 l9-2 l9-8-2 by blast
}
hence P10: Bet X A C  $\wedge$  Bet B D X  $\longrightarrow$  A B TS C D by blast
{
  assume Q0: Bet X A C  $\wedge$  Bet D X B
  have Q1: A B TS X C
  by (metis NCol-cases P3 Q0 assms(1) bet--ts not-col-distincts)
  have A B OS X D
  by (metis P2 P3 Q0 bet-out-1 col-permutation-3 invert-one-side out-one-side)
  hence A B TS C D
  using Q1 l9-2 l9-8-2 by blast
}
hence P11: Bet X A C  $\wedge$  Bet D X B  $\longrightarrow$  A B TS C D
by blast
{
  assume Q0: Bet X A C  $\wedge$  Bet X B D
  hence Q1: A B TS C X
  by (simp add: P1 Q0 assms(1) bet--ts between-symmetry not-col-permutation-1)
  have A B TS D X
  by (simp add: P2 Q0 assms(2) bet--ts between-symmetry invert-two-sides
    not-col-permutation-3)
  hence A B OS C D
  using Q1 l9-8-1 by blast
}
hence P12: Bet X A C  $\wedge$  Bet X B D  $\longrightarrow$  A B OS C D by blast
thus ?thesis using P4 P5 P6 P7 P8 P9 P10 P11
using Col-def assms(3) assms(4) by auto
qed

```

lemma *cop--one-or-two-sides*:
assumes *Coplanar A B C D* **and**

$\neg \text{Col } C \ A \ B$ **and**
 $\neg \text{Col } D \ A \ B$
shows $A \ B \ TS \ C \ D \vee A \ B \ OS \ C \ D$
proof –
obtain X **where** $P1: \text{Col } A \ B \ X \wedge \text{Col } C \ D \ X \vee \text{Col } A \ C \ X \wedge \text{Col } B \ D \ X \vee \text{Col } A \ D \ X \wedge \text{Col } B \ C \ X$
using *Coplanar-def* *assms(1)* **by** *auto*
have $P2: \text{Col } A \ B \ X \wedge \text{Col } C \ D \ X \longrightarrow A \ B \ TS \ C \ D \vee A \ B \ OS \ C \ D$
by (*metis* *TS-def* *l9-19-R2* *assms(2)* *assms(3)* *not-col-permutation-3* *not-col-permutation-5* *not-out-bet*)
have $P3: \text{Col } A \ C \ X \wedge \text{Col } B \ D \ X \longrightarrow A \ B \ TS \ C \ D \vee A \ B \ OS \ C \ D$
using *assms(2)* *assms(3)* *one-or-two-sides-aux* **by** *blast*
have $\text{Col } A \ D \ X \wedge \text{Col } B \ C \ X \longrightarrow A \ B \ TS \ C \ D \vee A \ B \ OS \ C \ D$
using *assms(2)* *assms(3)* *l9-2* *one-or-two-sides-aux* *one-side-symmetry* **by** *blast*
thus *?thesis*
using $P1 \ P2 \ P3$ **by** *blast*
qed

lemma *os--coplanar*:

assumes $A \ B \ OS \ C \ D$
shows *Coplanar* $A \ B \ C \ D$

proof –

have $P1: \neg \text{Col } A \ B \ C$
using *assms* *one-side-not-col123* **by** *blast*
obtain C' **where** $P2: \text{Bet } C \ B \ C' \wedge \text{Cong } B \ C' \ B \ C$
using *segment-construction* **by** *presburger*
have $P3: A \ B \ TS \ D \ C'$
by (*metis* *Cong-perm* *OS-def* $P2$ *TS-def* *assms* *bet--ts* *bet-cong-eq* *invert-one-side* *l9-10* *l9-8-2* *one-side-not-col123* *ts-distincts*)
obtain T **where** $P4: \text{Col } T \ A \ B \wedge \text{Bet } D \ T \ C'$
using $P3$ *TS-def* **by** *blast*
have $P5: C' \neq T$
using $P3 \ P4$ *TS-def* **by** *blast*
have $P6: \text{Col } T \ B \ C \longrightarrow \text{Coplanar } A \ B \ C \ D$
by (*metis* *Col-def* *Coplanar-def* $P2 \ P4 \ P5$ *col-trivial-2* *l6-16-1*)
{
assume $Q0: \neg \text{Col } T \ B \ C$
{
assume $R0: \text{Bet } T \ B \ A$
have $S1: B \ C \ TS \ T \ A$
by (*metis* $P1 \ Q0 \ R0$ *bet--ts* *col-permutation-2* *not-col-distincts*)
have $C' \text{ Out } T \ D$
using $P4 \ P5$ *bet-out-1* **by** *auto*
hence $B \ C \ OS \ T \ D$
using $P2 \ Q0$ *bet-col* *invert-one-side* *not-col-permutation-3* *out-one-side-1* **by** *blast*
hence $R1: B \ C \ TS \ D \ A$
using $S1$ *l9-8-2* **by** *blast*
hence *Coplanar* $A \ B \ C \ D$
using *ncoplanar-perm-9* *ts--coplanar* **by** *blast*
}
hence $Q1: \text{Bet } T \ B \ A \longrightarrow \text{Coplanar } A \ B \ C \ D$ **by** *blast*
{
assume $R0: \neg \text{Bet } T \ B \ A$
{
have $R2: B \ C \ OS \ D \ T$
proof –
have $S1: \neg \text{Col } B \ C \ D$
by (*metis* *Col-perm* $P2 \ P3 \ P4 \ Q0$ *bet-col* *colx* *ts-distincts*)
have $S2: \text{Col } B \ C \ C'$
by (*simp* *add*: $P2$ *bet-col* *col-permutation-4*)
have $S3: C' \text{ Out } D \ T$
using $P4 \ P5$ *bet-out-1* *l6-6* **by** *auto*
thus *?thesis*
using $S1 \ S2$ *out-one-side-1* **by** *blast*
qed
have $R3: B \ C \ OS \ T \ A$
using $P4 \ Q0 \ R0$ *col-permutation-2* *col-permutation-5* *not-bet-out* *out-one-side* **by** *blast*

```

}
hence  $R1: B C OS D A$ 
  by (metis  $P2 P4 Q0$  bet-col bet-out-1 col-permutation-2 col-permutation-5 os-out-os)
hence Coplanar  $A B C D$ 
  by (simp add:  $R1$  assms coplanar-perm-19 invert-one-side l9-31 one-side-symmetry
      ts--coplanar)
}
hence  $\neg Bet T B A \longrightarrow Coplanar A B C D$  by blast
hence Coplanar  $A B C D$  using  $Q1$  by blast
}
hence  $\neg Col T B C \longrightarrow Coplanar A B C D$  by blast
thus ?thesis using  $P6$  by blast
qed

```

```

lemma coplanar-trans-1:
  assumes  $\neg Col P Q R$  and
    Coplanar  $P Q R A$  and
    Coplanar  $P Q R B$ 
  shows Coplanar  $Q R A B$ 
proof -
  have  $P1: Col Q R A \longrightarrow Coplanar Q R A B$ 
    by (simp add: col--coplanar)
  {
    assume  $T1: \neg Col Q R A$ 
    {
      assume  $T2: \neg Col Q R B$ 
      {
        have  $Col Q A B \longrightarrow Coplanar Q R A B$ 
          using ncop--ncols by blast
        {
          assume  $S1: \neg Col Q A B$ 
          have  $U1: Q R TS P A \vee Q R OS P A$ 
            by (simp add:  $T1$  assms(1) assms(2) cop--one-or-two-sides coplanar-perm-8
                not-col-permutation-2)
          have  $U2: Q R TS P B \vee Q R OS P B$ 
            using  $T2$  assms(1) assms(3) col-permutation-1 cop--one-or-two-sides
                coplanar-perm-8 by blast
          have  $W1: Q R TS P A \wedge Q R OS P A \longrightarrow Q R TS A B \vee Q R OS A B$ 
            using l9-9 by blast
          have  $W2: Q R TS P A \wedge Q R OS P B \longrightarrow Q R TS A B \vee Q R OS A B$ 
            using l9-2 l9-8-2 by blast
          have  $W3: Q R TS P B \wedge Q R OS P A \longrightarrow Q R TS A B \vee Q R OS A B$ 
            using l9-8-2 by blast
          have  $Q R TS P B \wedge Q R OS P B \longrightarrow Q R TS A B \vee Q R OS A B$ 
            using l9-9 by blast
          hence  $S2: Q R TS A B \vee Q R OS A B$  using  $U1 U2 W1 W2 W3$ 
            using OS-def l9-2 one-side-transitivity by blast
          have Coplanar  $Q R A B$ 
            using  $S2$  os--coplanar ts--coplanar by blast
        }
      }
      hence  $\neg Col Q A B \longrightarrow Coplanar Q R A B$  by blast
    }
  }
  hence Coplanar  $Q R A B$ 
    using ncop--ncols by blast
}
hence  $\neg Col Q R B \longrightarrow Coplanar Q R A B$ 
  by blast
}
hence  $\neg Col Q R A \longrightarrow Coplanar Q R A B$ 
  using ncop--ncols by blast
thus ?thesis using  $P1$  by blast
qed

```

```

lemma col-cop--cop:
  assumes Coplanar  $A B C D$  and
     $C \neq D$  and

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    Col C D E
  shows Coplanar A B C E
proof -
  have Col D A C  $\longrightarrow$  Coplanar A B C E
    by (meson assms(2) assms(3) col-permutation-1 l6-16-1 ncop--ncols)
  moreover
  {
    assume  $\neg$  Col D A C
    hence Coplanar A C B E
      by (meson assms(1) assms(3) col--coplanar coplanar-trans-1 ncoplanar-perm-11
          ncoplanar-perm-13)
    hence Coplanar A B C E
      using ncoplanar-perm-2 by blast
  }
  ultimately show ?thesis
    by blast
qed

lemma bet-cop--cop:
  assumes Coplanar A B C E and
    Bet C D E
  shows Coplanar A B C D
  by (metis NCol-perm col-cop--cop assms(1) assms(2) bet-col bet-neq12--neq)

lemma col2-cop--cop:
  assumes Coplanar A B C D and
    C  $\neq$  D and
    Col C D E and
    Col C D F
  shows Coplanar A B E F
proof cases
  assume C = E
  thus ?thesis
    using assms(1) assms(2) assms(4) col-cop--cop by blast
next
  assume C  $\neq$  E
  thus ?thesis
    by (metis assms(1) assms(2) assms(3) assms(4) col-cop--cop col-transitivity-1
        ncoplanar-perm-1 not-col-permutation-4)
qed

lemma col-cop2--cop:
  assumes U  $\neq$  V and
    Coplanar A B C U and
    Coplanar A B C V and
    Col U V P
  shows Coplanar A B C P
proof cases
  assume Col A B C
  thus ?thesis
    using ncop--ncol by blast
next
  assume  $\neg$  Col A B C
  {
    fix A0 B0 C0
    assume  $\neg$  Col A0 B0 C0 and  $\neg$  Col U A0 B0 and Coplanar A0 B0 C0 U and
      Coplanar A0 B0 C0 V and  $\neg$  Col A0 B0 C0
    have Coplanar U A0 B0 P
      by (meson col-trivial-3  $\langle$  Coplanar A0 B0 C0 U  $\rangle$   $\langle$  Coplanar A0 B0 C0 V  $\rangle$   $\langle$   $\neg$  Col A0 B0 C0  $\rangle$ 
          assms(1) assms(4) col2-cop--cop coplanar-trans-1 ncoplanar-perm-8
          not-col-permutation-1)
    hence Coplanar A0 B0 C0 P
      using  $\langle$  Coplanar A0 B0 C0 U  $\rangle$   $\langle$   $\neg$  Col U A0 B0  $\rangle$  coplanar-perm-18 coplanar-trans-1 by blast
  }
  moreover
  {

```



```

assume Col U A B and Col U A C
hence Coplanar A B C P
  by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  assms(1) assms(3) assms(4) col-cop--cop
    col-transitivity-2 coplanar-perm-14)
}
moreover
{
  assume Col U A B and  $\neg$  Col U A C
  hence Coplanar A B C P
    using calculation(1)  $\langle \neg \text{Col } A \ B \ C \rangle$  assms(2) assms(3) coplanar-perm-2
      not-col-permutation-5 by blast
}
moreover
{
  assume  $\neg$  Col U A B and Col U A C
  hence Coplanar A B C P
    using  $\langle \neg \text{Col } A \ B \ C \rangle$  assms(2) assms(3) calculation(1) by blast
}
moreover
{
  assume  $\neg$  Col U A B and  $\neg$  Col U A C
  hence Coplanar A B C P
    using  $\langle \neg \text{Col } A \ B \ C \rangle$  assms(2) assms(3) calculation(1) by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma bet-cop2--cop:
  assumes Coplanar A B C U and
    Coplanar A B C W and
    Bet U V W
  shows Coplanar A B C V
proof –
  have Col U V W
    using assms(3) bet-col by blast
  hence Col U W V
    by (meson not-col-permutation-5)
  thus ?thesis
    using assms(1) assms(2) assms(3) bet-neq23--neq col-cop2--cop by blast
qed

```

```

lemma coplanar-pseudo-trans-lem1:
  assumes  $\neg$  Col P Q R and
    Coplanar P Q R A and
    Coplanar P Q R B and
    Coplanar P Q R C
  shows Coplanar A B C R
proof cases
  assume Col R Q A
  have Coplanar B C R Q
    using assms(1) assms(3) assms(4) coplanar-perm-17 coplanar-trans-1 by blast
  thus ?thesis
    using  $\langle \text{Col } R \ Q \ A \rangle$  assms(1) col-cop--cop coplanar-perm-2 coplanar-perm-20
      not-col-distincts by blast
next
  assume  $\neg$  Col R Q A
  thus ?thesis
    by (meson assms(1) assms(2) assms(3) assms(4) coplanar-trans-1 ncoplanar-perm-18
      not-col-permutation-4)
qed

```

```

lemma coplanar-pseudo-trans:
  assumes  $\neg$  Col P Q R and
    Coplanar P Q R A and
    Coplanar P Q R B and

```

```

    Coplanar P Q R C and
    Coplanar P Q R D
shows Coplanar A B C D
proof cases
  assume Col P Q D
  moreover have P ≠ Q
    using assms(1) col-trivial-1 by blast
  moreover have Coplanar A B C Q
    using coplanar-pseudo-trans-lem1
    by (meson assms(1) assms(2) assms(3) assms(4) col-permutation-5 coplanar-perm-2)
  moreover have Coplanar A B C P
  proof -
    have ¬ Col Q R P
      using Col-cases assms(1) by blast
    moreover have Coplanar Q R P A
      using assms(2) ncoplanar-perm-12 by blast
    moreover have Coplanar Q R P B
      using assms(3) ncoplanar-perm-12 by blast
    moreover have Coplanar Q R P C
      using assms(4) ncoplanar-perm-12 by blast
    ultimately show ?thesis
      using coplanar-pseudo-trans-lem1 by blast
  qed
  ultimately show ?thesis
    using coplanar-pseudo-trans-lem1 col-cop2--cop by blast
next
  assume ¬ Col P Q D
  moreover have Coplanar P Q D A
    using NCol-cases assms(1) assms(2) assms(5) coplanar-trans-1 ncoplanar-perm-8 by blast
  moreover have Coplanar P Q D B
    using assms(1) assms(3) assms(5) col-permutation-1 coplanar-perm-12
    coplanar-trans-1 by blast
  moreover have Coplanar P Q D C
    by (meson assms(1) assms(4) assms(5) coplanar-perm-7 coplanar-trans-1
    ncoplanar-perm-14 not-col-permutation-3)
  ultimately show ?thesis
    using coplanar-pseudo-trans-lem1 by blast
qed

lemma l9-30:
  assumes ¬ Coplanar A B C P and
    ¬ Col D E F and
    Coplanar D E F P and
    Coplanar A B C X and
    Coplanar A B C Y and
    Coplanar A B C Z and
    Coplanar D E F X and
    Coplanar D E F Y and
    Coplanar D E F Z
  shows Col X Y Z
proof -
  {
    assume P1: ¬ Col X Y Z
    have P2: ¬ Col A B C
      using assms(1) col--coplanar by blast
    have Coplanar A B C P
    proof -
      have Q2: Coplanar X Y Z A
        by (meson P2 col-trivial-3 assms(4) assms(5) assms(6) coplanar-pseudo-trans ncop--ncols)
      have Q3: Coplanar X Y Z B
        using P2 assms(4) assms(5) assms(6) col-trivial-3 coplanar-pseudo-trans
        ncop--ncols by blast
      have Q4: Coplanar X Y Z C
        using P2 assms(4) assms(5) assms(6) col-trivial-2 coplanar-pseudo-trans
        ncop--ncols by blast
      have Coplanar X Y Z P

```

```

    using assms(2) assms(3) assms(7) assms(8) assms(9) coplanar-pseudo-trans by blast
  thus ?thesis using P1 Q2 Q3 Q4
    using assms(2) assms(3) assms(7) assms(8) assms(9) coplanar-pseudo-trans by blast
qed
hence False using assms(1) by blast
}
thus ?thesis by blast
qed

lemma cop-per2--col:
  assumes Coplanar A X Y Z and
    A ≠ Z and
    Per X Z A and
    Per Y Z A
  shows Col X Y Z
proof cases
  assume X = Y ∨ X = Z ∨ Y = Z
  thus ?thesis
    using not-col-distincts by blast
next
  assume ¬ (X = Y ∨ X = Z ∨ Y = Z)
  obtain B where Cong X A X B and Z Midpoint A B and Cong Y A Y B
    using Per-def assms(3) assms(4) per-double-cong by blast
  have X ≠ Y
    using ⟨¬ (X = Y ∨ X = Z ∨ Y = Z)⟩ by blast
  have X ≠ Z
    using ⟨¬ (X = Y ∨ X = Z ∨ Y = Z)⟩ by blast
  have Y ≠ Z
    using ⟨¬ (X = Y ∨ X = Z ∨ Y = Z)⟩ by blast
  obtain I where Col A X I ∧ Col Y Z I ∨ Col A Y I ∧ Col X Z I ∨ Col A Z I ∧ Col X Y I
    using Coplanar-def assms(1) by auto
  moreover
  {
    assume Col A X I and Col Y Z I
    have Col X Y Z
    proof (cases X = I)
      assume X = I
      thus ?thesis
        using Col-cases ⟨Col Y Z I⟩ by blast
    next
      assume X ≠ I
      have Col A X I ∧ Col Y Z I ⇒ Col X Y Z
        by (metis (full-types) ⟨¬ (X = Y ∨ X = Z ∨ Y = Z)⟩ assms(2,3,4) col-per2--per l8-3 l8-8
          not-col-permutation-2 not-col-permutation-4)
      moreover have Col A Y I ∧ Col X Z I ⇒ Col X Y Z
        by (simp add: ⟨Col A X I⟩ ⟨Col Y Z I⟩ calculation)
      moreover have Col A Z I ∧ Col X Y I ⇒ Col X Y Z
        by (simp add: ⟨Col A X I⟩ ⟨Col Y Z I⟩ calculation(1))
      ultimately show Col X Y Z
        using ⟨Col A X I⟩ ⟨Col Y Z I⟩ by blast
    qed
  }
  moreover have Col A Y I ∧ Col X Z I ⇒ Col X Y Z
proof cases
  assume X = I
  thus ?thesis
    by (metis Col-cases ⟨Cong X A X B⟩ ⟨Cong Y A Y B⟩ ⟨Z Midpoint A B⟩
      l4-18 midpoint-distinct-3)
next
  assume X ≠ I
  thus ?thesis
    by (metis Col-cases midpoint-bet per-double-cong ⟨Cong Y A Y B⟩ ⟨Z Midpoint A B⟩
      assms(2) assms(3) between-equality between-trivial col-trivial-3 l4-18 l8-3)
qed
moreover
{

```

```

assume  $Col\ A\ Z\ I$  and  $Col\ X\ Y\ I$ 
hence  $Z = I \longrightarrow Col\ X\ Y\ Z$ 
  by simp
moreover
{
  assume  $Z \neq I$ 
  have  $Cong\ I\ A\ I\ B$ 
    using  $l4-17$  [where  $?A = X$  and  $?B = Y$ ]
     $\langle X \neq Y \rangle \langle Col\ X\ Y\ I \rangle \langle Cong\ X\ A\ X\ B \rangle \langle Cong\ Y\ A\ Y\ B \rangle$  by auto
  have  $(Col\ A\ I\ B \wedge Cong\ I\ A\ I\ B) \longrightarrow (A = B \vee I\ Midpoint\ A\ B)$ 
    using  $l7-20-bis$  by blast
  moreover have  $A = B \longrightarrow Col\ X\ Y\ Z$ 
    using  $\langle Z\ Midpoint\ A\ B \rangle$  assms(2)  $l7-3$  by auto
  moreover have  $I\ Midpoint\ A\ B \longrightarrow ?thesis$ 
    using  $\langle Z\ Midpoint\ A\ B \rangle \langle Z \neq I \rangle$   $l7-17$  by blast
  ultimately have  $Col\ X\ Y\ Z$ 
    using Col-cases Per-def  $\langle Col\ A\ Z\ I \rangle \langle Cong\ I\ A\ I\ B \rangle \langle Z\ Midpoint\ A\ B \rangle$ 
    assms(2)  $l8-9$  by blast
}
ultimately have  $Col\ X\ Y\ Z$ 
  by blast
}
ultimately show  $?thesis$ 
  by blast
qed

```

```

lemma cop-perp2--col:
assumes  $Coplanar\ A\ B\ Y\ Z$  and
   $X\ Y\ Perp\ A\ B$  and
   $X\ Z\ Perp\ A\ B$ 
shows  $Col\ X\ Y\ Z$ 
proof cases
assume  $P1: Col\ A\ B\ X$ 
{
  assume  $Q0: X = A$ 
  hence  $Q1: X \neq B$ 
    using assms(3) perp-not-eq-2 by blast
  have  $Q2: Coplanar\ B\ Y\ Z\ X$ 
    by (simp add: Q0 assms(1) coplanar-perm-9)
  have  $Q3: Per\ Y\ X\ B$ 
    using  $Q0$  assms(2) perp-per-2 by blast
  have  $Per\ Z\ X\ B$ 
    using  $Q0$  assms(3) perp-per-2 by blast
  hence  $Col\ X\ Y\ Z$ 
    using  $Q1\ Q2\ Q3$  cop-per2--col not-col-permutation-1 by blast
}
hence  $P2: X = A \longrightarrow Col\ X\ Y\ Z$  by blast
{
  assume  $Q0: X \neq A$ 
  have  $Q1: A\ X\ Perp\ X\ Y$ 
    by (metis P1 Perp-perm Q0 assms(2) perp-col1)
  have  $Q2: A\ X\ Perp\ X\ Z$ 
    by (metis P1 Perp-perm Q0 assms(3) perp-col1)
  have  $Q3: Coplanar\ A\ Y\ Z\ X$ 
    by (metis P1 assms(1) assms(2) col-cop2--cop coplanar-perm-3 coplanar-trivial
    perp-distinct)
  have  $Q4: Per\ Y\ X\ A$ 
    using Perp-perm Q1 perp-per-2 by blast
  have  $Per\ Z\ X\ A$ 
    using  $P1\ Q0$  assms(3) perp-col1 perp-per-1 by auto
  hence  $Col\ X\ Y\ Z$ 
    using  $Q0\ Q3\ Q4$  cop-per2--col not-col-permutation-1 by blast
}
hence  $X \neq A \longrightarrow Col\ X\ Y\ Z$  by blast
thus  $?thesis$ 
  using  $P2$  by blast

```

```

next
  assume P1:  $\neg \text{Col } A \ B \ X$ 
  obtain Y0 where P2:  $Y0 \text{ PerpAt } X \ Y \ A \ B$ 
    using Perp-def assms(2) by blast
  obtain Z0 where P3:  $Z0 \text{ PerpAt } X \ Z \ A \ B$ 
    using Perp-def assms(3) by auto
  have P4:  $X \ Y0 \text{ Perp } A \ B$ 
    by (metis P1 P2 assms(2) perp-col perp-in-col)
  have P5:  $X \ Z0 \text{ Perp } A \ B$ 
    by (metis P1 P3 assms(3) perp-col perp-in-col)
  have P6:  $Y0 = Z0$ 
    by (meson P1 P2 P3 P4 P5 Perp-perm l8-18-uniqueness perp-in-col)
  have P7:  $X \neq Y0$ 
    using P4 perp-not-eq-1 by blast
  have P8:  $\text{Col } X \ Y0 \ Y$ 
    using P2 col-permutation-5 perp-in-col by blast
  have Col X Y0 Z
    using P3 P6 col-permutation-5 perp-in-col by blast
  thus ?thesis
    using P7 P8 col-transitivity-1 by blast
qed

lemma two-sides-dec:
  shows  $A \ B \ TS \ C \ D \vee \neg A \ B \ TS \ C \ D$ 
  by simp

lemma cop-nts--os:
  assumes Coplanar A B C D and
     $\neg \text{Col } C \ A \ B$  and
     $\neg \text{Col } D \ A \ B$  and
     $\neg A \ B \ TS \ C \ D$ 
  shows  $A \ B \ OS \ C \ D$ 
  using assms(1) assms(2) assms(3) assms(4) cop--one-or-two-sides by blast

lemma cop-nos--ts:
  assumes Coplanar A B C D and
     $\neg \text{Col } C \ A \ B$  and
     $\neg \text{Col } D \ A \ B$  and
     $\neg A \ B \ OS \ C \ D$ 
  shows  $A \ B \ TS \ C \ D$ 
  using assms(1) assms(2) assms(3) assms(4) cop-nts--os by blast

lemma one-side-dec:
   $A \ B \ OS \ C \ D \vee \neg A \ B \ OS \ C \ D$ 
  by simp

lemma cop-dec:
   $\text{Coplanar } A \ B \ C \ D \vee \neg \text{Coplanar } A \ B \ C \ D$ 
  by simp

lemma ex-diff-cop:
   $\exists E. \text{Coplanar } A \ B \ C \ E \wedge D \neq E$ 
  by (metis col-trivial-2 diff-col-ex ncop--ncols)

lemma ex-ncol-cop:
  assumes  $D \neq E$ 
  shows  $\exists F. \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F$ 
proof cases
  assume Col A B C
  thus ?thesis
    using assms ncop--ncols not-col-exists by blast
next
  assume P1:  $\neg \text{Col } A \ B \ C$ 
  thus ?thesis
  proof -
    have P2:  $(\text{Col } D \ E \ A \wedge \text{Col } D \ E \ B) \longrightarrow (\exists F. \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F)$ 

```

by (meson P1 assms col3 col-trivial-2 ncop--ncols)
 have P3: $(\neg \text{Col } D \ E \ A \wedge \text{Col } D \ E \ B) \longrightarrow (\exists F. \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F)$
 using col-trivial-3 ncop--ncols by blast
 have P4: $(\text{Col } D \ E \ A \wedge \neg \text{Col } D \ E \ B) \longrightarrow (\exists F. \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F)$
 using col-trivial-2 ncop--ncols by blast
 have $(\neg \text{Col } D \ E \ A \wedge \neg \text{Col } D \ E \ B) \longrightarrow (\exists F. \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F)$
 using col-trivial-3 ncop--ncols by blast
 thus ?thesis using P2 P3 P4 by blast
 qed

qed

lemma ex-ncol-cop2:

$\exists E \ F. (\text{Coplanar } A \ B \ C \ E \wedge \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F)$
 proof –
 have Coplanar A B C A
 by (simp add: coplanar-perm-3 coplanar-trivial)
 have Coplanar A B C C
 using col-trivial-2 ncop--ncols by blast
 have $\exists p. A \neq p$
 by (meson col-trivial-3 diff-col-ex3)
 moreover
 {
 assume $B \neq A$
 hence $D = B \longrightarrow (\exists p. \neg \text{Col } D \ p \ A \wedge \text{Coplanar } A \ B \ C \ p)$
 by (metis Col-cases $\langle \text{Coplanar } A \ B \ C \ C \rangle$ ncop--ncols not-col-exists)
 hence $D = B \longrightarrow (\exists p \ pa. \text{Coplanar } A \ B \ C \ p \wedge \text{Coplanar } A \ B \ C \ pa \wedge \neg \text{Col } D \ p \ pa)$
 using $\langle \text{Coplanar } A \ B \ C \ A \rangle$ by blast
 }
 moreover
 {
 assume $D \neq B$
 moreover
 {
 assume $\exists p. D \neq B \wedge \neg \text{Coplanar } A \ B \ C \ p$
 hence $D \neq B \wedge \neg \text{Col } A \ B \ C$
 using ncop--ncols by blast
 hence $\exists p. \neg \text{Col } D \ p \ B \wedge \text{Coplanar } A \ B \ C \ p$
 by (metis Col-cases $\langle \text{Coplanar } A \ B \ C \ A \rangle \langle \text{Coplanar } A \ B \ C \ C \rangle$ col2--eq)
 }
 ultimately have ?thesis
 by (meson col-trivial-3 ncop--ncols not-col-exists)
 }
 ultimately show ?thesis
 using coplanar-trivial ex-ncol-cop by blast
 qed

lemma col2-cop2--eq:

assumes $\neg \text{Coplanar } A \ B \ C \ U$ and
 $U \neq V$ and
 Coplanar A B C P and
 Coplanar A B C Q and
 Col U V P and
 Col U V Q
 shows $P = Q$
 proof –
 have Col U Q P
 by (meson assms(2) assms(5) assms(6) col-transitivity-1)
 hence Col P Q U
 using not-col-permutation-3 by blast
 thus ?thesis
 using assms(1) assms(3) assms(4) col-cop2--cop by blast
 qed

lemma cong3-cop2--col:

assumes Coplanar A B C P and
 Coplanar A B C Q and

```

  P ≠ Q and
  Cong A P A Q and
  Cong B P B Q and
  Cong C P C Q
shows Col A B C
proof cases
  assume Col A B C
  thus ?thesis by blast
next
  assume P1: ¬ Col A B C
  obtain M where P2: M Midpoint P Q
    using assms(6) l7-25 by blast
  have P3: Per A M P
    using P2 Per-def assms(4) by blast
  have P4: Per B M P
    using P2 Per-def assms(5) by blast
  have P5: Per C M P
    using P2 Per-def assms(6) by blast
  have False
proof cases
  assume Q1: A = M
  have Q2: Coplanar P B C A
    using assms(1) ncoplanar-perm-21 by blast
  have Q3: P ≠ A
    by (metis assms(3) assms(4) cong-diff-4)
  have Q4: Per B A P
    by (simp add: P4 Q1)
  have Q5: Per C A P
    by (simp add: P5 Q1)
  thus ?thesis using Q1 Q2 Q3 Q4 cop-per2--col
    using P1 not-col-permutation-1 by blast
next
  assume Q0: A ≠ M
  have Q1: Col A B M
proof —
  have R1: Coplanar A B P Q
    using P1 assms(1) assms(2) coplanar-trans-1 ncoplanar-perm-8 not-col-permutation-2
    by blast
  hence R2: Coplanar P A B M
    using P2 bet-cop--cop coplanar-perm-14 midpoint-bet ncoplanar-perm-6 by blast
  have R3: P ≠ M
    using P2 assms(3) l7-3-2 l7-9-bis by blast
  have R4: Per A M P
    by (simp add: P3)
  have R5: Per B M P
    by (simp add: P4)
  thus ?thesis
    using R2 R3 R4 cop-per2--col by blast
qed
have Col A C M
proof —
  have R1: Coplanar P A C M
    using P1 Q1 assms(1) col2-cop--cop coplanar-perm-22 ncoplanar-perm-3
    not-col-distincts by blast
  have R2: P ≠ M
    using P2 assms(3) l7-3-2 symmetric-point-uniqueness by blast
  have R3: Per A M P
    by (simp add: P3)
  have Per C M P
    by (simp add: P5)
  thus ?thesis
    using R1 R2 R3 cop-per2--col by blast
qed
thus ?thesis
  using NCol-perm P1 Q0 Q1 col-trivial-3 colx by blast
qed

```

thus ?thesis by blast

qed

lemma l9-38:

assumes $A B C TSP P Q$

shows $A B C TSP Q P$

using *Bet-perm TSP-def assms* by blast

lemma l9-39:

assumes $A B C TSP P R$ and

$Coplanar A B C D$ and

$D Out P Q$

shows $A B C TSP Q R$

proof –

have $P1: \neg Col A B C$

using *TSP-def assms(1) ncop--ncol* by blast

have $P2: \neg Coplanar A B C Q$

by (metis *TSP-def assms(1) assms(2) assms(3) col-cop2--cop l6-6 out-col out-diff2*)

have $P3: \neg Coplanar A B C R$

using *TSP-def assms(1)* by blast

obtain T where $P3A: Coplanar A B C T \wedge Bet P T R$

using *TSP-def assms(1)* by blast

have $W1: D = T \longrightarrow A B C TSP Q R$

using *P2 P3 P3A TSP-def assms(3) bet-out--bet* by blast

{

assume $V1: D \neq T$

have $V1A: \neg Col P D T$ using *P3A col-cop2--cop*

by (metis *TSP-def V1 assms(1) assms(2) col2-cop2--eq col-trivial-2*)

have $V1B: D T TS P R$

by (metis *P3 P3A V1A bet--ts invert-two-sides not-col-permutation-3*)

have $D T OS P Q$

using *V1A assms(3) not-col-permutation-1 out-one-side* by blast

hence $V2: D T TS Q R$

using *V1B l9-8-2* by blast

then obtain T' where $V3: Col T' D T \wedge Bet Q T' R$

using *TS-def* by blast

have $V4: Coplanar A B C T'$

using *Col-cases P3A V1 V3 assms(2) col-cop2--cop* by blast

hence $A B C TSP Q R$

using *P2 P3 TSP-def V3* by blast

}

hence $D \neq T \longrightarrow A B C TSP Q R$ by blast

thus ?thesis using *W1* by blast

qed

lemma l9-41-1:

assumes $A B C TSP P R$ and

$A B C TSP Q R$

shows $A B C OSP P Q$

using *OSP-def assms(1) assms(2)* by blast

lemma l9-41-2:

assumes $A B C TSP P R$ and

$A B C OSP P Q$

shows $A B C TSP Q R$

proof –

have $P1: \neg Coplanar A B C P$

using *TSP-def assms(1)* by blast

obtain S where $P2: A B C TSP P S \wedge A B C TSP Q S$

using *OSP-def assms(2)* by blast

obtain X where $P3: Coplanar A B C X \wedge Bet P X S$

using *P2 TSP-def* by blast

have $P4: \neg Coplanar A B C P \wedge \neg Coplanar A B C S$

using *P2 TSP-def* by blast

obtain Y where $P5: Coplanar A B C Y \wedge Bet Q Y S$

using *P2 TSP-def* by blast


```

have P6:  $\neg \text{Coplanar } A \ B \ C \ Q \wedge \neg \text{Coplanar } A \ B \ C \ S$ 
  using P2 TSP-def by blast
have P7:  $X \neq P \wedge S \neq X \wedge Q \neq Y \wedge S \neq Y$ 
  using P3 P4 P5 P6 by blast
{
  assume Q1:  $\text{Col } P \ Q \ S$ 
  have Q2:  $X = Y$ 
  proof -
    have R2:  $Q \neq S$ 
      using P5 P6 bet-neq12--neq by blast
    have R5:  $\text{Col } Q \ S \ X$ 
      by (metis Col-def P3 Q1 between-equality-2 l6-16-1 not-bet-distincts)
    have Col Q S Y
      by (simp add: P5 bet-col col-permutation-5)
    thus ?thesis
      using P2 P3 P5 R2 R5 TSP-def col2-cop2--eq by blast
  qed
  hence X Out P Q
    by (metis P3 P5 P7 l6-2)
  hence A B C TSP Q R
    using P3 assms(1) l9-39 by blast
}
hence P7:  $\text{Col } P \ Q \ S \longrightarrow A \ B \ C \ TSP \ Q \ R$  by blast
{
  assume Q1:  $\neg \text{Col } P \ Q \ S$ 
  obtain Z where Q2:  $\text{Bet } X \ Z \ Q \wedge \text{Bet } Y \ Z \ P$ 
    using P3 P5 inner-pasch by blast
  {
    assume X = Z
    hence False
      by (metis P2 P3 P5 Q1 Q2 TSP-def bet-col col-cop2--cop l6-16-1 not-col-permutation-5)
  }
  hence Q3:  $X \neq Z$  by blast
  have Y  $\neq$  Z
  proof -
    have X  $\neq$  Z
      by (meson  $\langle X = Z \implies \text{False} \rangle$ )
    hence Z  $\neq$  Y
      by (metis P2 P3 P5 Q2 TSP-def bet-col col-cop2--cop)
    thus ?thesis
      by meson
  qed
  hence Y Out P Z
    using Q2 bet-out l6-6 by auto
  hence Q4:  $A \ B \ C \ TSP \ Z \ R$ 
    using assms(1) P5 l9-39 by blast
  have X Out Z Q
    using Q2 Q3 bet-out by auto
  hence A B C TSP Q R
    using Q4 P3 l9-39 by blast
}
hence  $\neg \text{Col } P \ Q \ S \longrightarrow A \ B \ C \ TSP \ Q \ R$  by blast
thus ?thesis using P7 by blast
qed

lemma tsp-exists:
  assumes  $\neg \text{Coplanar } A \ B \ C \ P$ 
  shows  $\exists \ Q. A \ B \ C \ TSP \ P \ Q$ 
proof -
  obtain Q where Bet P A Q and Cong A Q A P
    using segment-construction by blast
  moreover have Coplanar A B C A
    using coplanar-trivial ncoplanar-perm-5 by blast
  moreover have  $\neg \text{Coplanar } A \ B \ C \ Q$ 
    by (metis assms bet-col bet-col1 between-cong between-symmetry calculation(1)
      calculation(2) calculation(3) col2-cop2--eq)

```

ultimately show ?thesis
 using TSP-def assms by blast
 qed

lemma osp-reflexivity:
 assumes $\neg \text{Coplanar } A \ B \ C \ P$
 shows $A \ B \ C \ \text{OSP } P \ P$
 by (meson assms l9-41-1 tsp-exists)

lemma osp-symmetry:
 assumes $A \ B \ C \ \text{OSP } P \ Q$
 shows $A \ B \ C \ \text{OSP } Q \ P$
 using OSP-def assms by auto

lemma osp-transitivity:
 assumes $A \ B \ C \ \text{OSP } P \ Q$ and
 $A \ B \ C \ \text{OSP } Q \ R$
 shows $A \ B \ C \ \text{OSP } P \ R$
 using OSP-def assms(1) assms(2) l9-41-2 by blast

lemma cop3-tsp--tsp:
 assumes $\neg \text{Col } D \ E \ F$ and
 $\text{Coplanar } A \ B \ C \ D$ and
 $\text{Coplanar } A \ B \ C \ E$ and
 $\text{Coplanar } A \ B \ C \ F$ and
 $A \ B \ C \ \text{TSP } P \ Q$
 shows $D \ E \ F \ \text{TSP } P \ Q$
 proof –
 obtain T where $\text{Coplanar } A \ B \ C \ T$ and $\text{Bet } P \ T \ Q$
 using TSP-def assms(5) by blast
 have $\neg \text{Col } A \ B \ C$
 using TSP-def assms(5) ncop--ncols by blast
 have $\text{Coplanar } D \ E \ F \ A \wedge \text{Coplanar } D \ E \ F \ B \wedge \text{Coplanar } D \ E \ F \ C \wedge \text{Coplanar } D \ E \ F \ T$
 proof –
 have $\text{Coplanar } D \ E \ F \ A$
 using assms(2) assms(3) assms(4) col-trivial-3 coplanar-pseudo-trans ncop--ncols
 by (meson $\langle \neg \text{Col } A \ B \ C \rangle$)
 moreover have $\text{Coplanar } D \ E \ F \ B$
 using assms(2) assms(3) assms(4) col-trivial-2 coplanar-pseudo-trans ncop--ncols
 by (meson $\langle \neg \text{Col } A \ B \ C \rangle$)
 moreover have $\text{Coplanar } D \ E \ F \ C$
 by (meson $\langle \neg \text{Col } A \ B \ C \rangle$ assms(2) assms(3) assms(4) coplanar-perm-16 coplanar-pseudo-trans
 coplanar-trivial)
 moreover have $\text{Coplanar } D \ E \ F \ T$
 using $\langle \neg \text{Col } A \ B \ C \rangle$ assms(2) assms(3) assms(4) coplanar-pseudo-trans $\langle \text{Coplanar } A \ B \ C \ T \rangle$
 by blast
 ultimately show ?thesis
 by simp
 qed
 hence $\neg \text{Coplanar } D \ E \ F \ P$
 using TSP-def assms(1) assms(5) coplanar-pseudo-trans by auto
 hence $\neg \text{Coplanar } D \ E \ F \ Q$
 using TSP-def assms(5) bet-col bet-col1 col2-cop2--eq $\langle \text{Bet } P \ T \ Q \rangle$ $\langle \text{Coplanar } A \ B \ C \ T \rangle$
 $\langle \text{Coplanar } D \ E \ F \ A \wedge \text{Coplanar } D \ E \ F \ B \wedge \text{Coplanar } D \ E \ F \ C \wedge \text{Coplanar } D \ E \ F \ T \rangle$ by metis
 thus ?thesis
 using TSP-def $\langle \text{Bet } P \ T \ Q \rangle$ $\langle \neg \text{Coplanar } D \ E \ F \ P \rangle$
 $\langle \text{Coplanar } D \ E \ F \ A \wedge \text{Coplanar } D \ E \ F \ B \wedge \text{Coplanar } D \ E \ F \ C \wedge \text{Coplanar } D \ E \ F \ T \rangle$ by blast
 qed

lemma cop3-osp--osp:
 assumes $\neg \text{Col } D \ E \ F$ and
 $\text{Coplanar } A \ B \ C \ D$ and
 $\text{Coplanar } A \ B \ C \ E$ and
 $\text{Coplanar } A \ B \ C \ F$ and
 $A \ B \ C \ \text{OSP } P \ Q$
 shows $D \ E \ F \ \text{OSP } P \ Q$

```

proof –
  obtain  $R$  where  $A\ B\ C\ TSP\ P\ R$  and  $A\ B\ C\ TSP\ Q\ R$ 
  using  $OSP\text{-}def\ assms(5)$  by blast
  thus ?thesis
  using  $OSP\text{-}def\ assms(1)\ assms(2)\ assms(3)\ assms(4)\ cop3\text{-}tsp\text{-}tsp$  by blast
qed

lemma ncop-distincts:
  assumes  $\neg Coplanar\ A\ B\ C\ D$ 
  shows  $A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D$ 
  using  $Coplanar\text{-}def\ assms\ col\text{-}trivial\text{-}1\ col\text{-}trivial\text{-}2$  by blast

lemma tsp-distincts:
  assumes  $A\ B\ C\ TSP\ P\ Q$ 
  shows  $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q \wedge P \neq Q$ 
proof –
  obtain  $X$  where  $\neg Coplanar\ A\ B\ C\ P \wedge \neg Coplanar\ A\ B\ C\ Q \wedge Coplanar\ A\ B\ C\ X \wedge Bet\ P\ X\ Q$ 
  by (metis TSP-def assms)
  hence  $Q \neq X$ 
  by force
  thus ?thesis
  using  $\langle \neg Coplanar\ A\ B\ C\ P \wedge \neg Coplanar\ A\ B\ C\ Q \wedge Coplanar\ A\ B\ C\ X \wedge Bet\ P\ X\ Q \rangle$ 
  bet-neq32--neq ncop-distincts by blast
qed

lemma osp-distincts:
  assumes  $A\ B\ C\ OSP\ P\ Q$ 
  shows  $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q$ 
  using  $OSP\text{-}def\ assms\ tsp\text{-}distincts$  by blast

lemma tsp--ncop1:
  assumes  $A\ B\ C\ TSP\ P\ Q$ 
  shows  $\neg Coplanar\ A\ B\ C\ P$ 
  using  $TSP\text{-}def\ assms$  by blast

lemma tsp--ncop2:
  assumes  $A\ B\ C\ TSP\ P\ Q$ 
  shows  $\neg Coplanar\ A\ B\ C\ Q$ 
  using  $TSP\text{-}def\ assms$  by auto

lemma osp--ncop1:
  assumes  $A\ B\ C\ OSP\ P\ Q$ 
  shows  $\neg Coplanar\ A\ B\ C\ P$ 
  using  $OSP\text{-}def\ TSP\text{-}def\ assms$  by blast

lemma osp--ncop2:
  assumes  $A\ B\ C\ OSP\ P\ Q$ 
  shows  $\neg Coplanar\ A\ B\ C\ Q$ 
  using  $assms\ osp\text{-}ncop1\ osp\text{-}symmetry$  by blast

lemma tsp--nosp:
  assumes  $A\ B\ C\ TSP\ P\ Q$ 
  shows  $\neg A\ B\ C\ OSP\ P\ Q$ 
  using  $assms\ l9\text{-}41\text{-}2\ tsp\text{-}distincts$  by blast

lemma osp--ntsp:
  assumes  $A\ B\ C\ OSP\ P\ Q$ 
  shows  $\neg A\ B\ C\ TSP\ P\ Q$ 
  using  $assms\ tsp\text{-}nosp$  by blast

lemma osp-bet--osp:
  assumes  $A\ B\ C\ OSP\ P\ R$  and
   $Bet\ P\ Q\ R$ 
  shows  $A\ B\ C\ OSP\ P\ Q$ 
proof –
  obtain  $S$  where  $A\ B\ C\ TSP\ P\ S$ 

```

```

    using OSP-def assms(1) by blast
then obtain Y where Coplanar A B C Y and Bet R Y S
    using TSP-def assms(1) l9-41-2 by blast
obtain X where Coplanar A B C X and Bet P X S
    by (metis TSP-def ⟨A B C TSP P S⟩)
have P ≠ X ∧ S ≠ X ∧ R ≠ Y
    using TSP-def ⟨A B C TSP P S⟩ ⟨Coplanar A B C X⟩ ⟨Coplanar A B C Y⟩ assms(1)
    osp--ncop2 by blast
{
  assume Col P R S
  have A B C TSP Q S
  proof -
    have X = Y
    proof -
      have ¬ Coplanar A B C R
        using assms(1) osp--ncop2 by blast
      moreover have R ≠ S
        using ⟨Bet R Y S⟩ ⟨P ≠ X ∧ S ≠ X ∧ R ≠ Y⟩ between-identity by blast
      moreover have Col R S X
        by (metis Bet-cases Col-cases Col-def ⟨Bet P X S⟩ ⟨Col P R S⟩ between-equality-2
            col-transitivity-1 outer-transitivity-between point-construction-different)
      moreover have Col R S Y
        using ⟨Bet R Y S⟩ bet-col1 between-trivial by blast
      ultimately show ?thesis
        using ⟨Coplanar A B C X⟩ ⟨Coplanar A B C Y⟩ col2-cop2--eq by blast
    qed
    have X Out P R
      using ⟨Bet P X S⟩ ⟨Bet R Y S⟩ ⟨P ≠ X ∧ S ≠ X ∧ R ≠ Y⟩ ⟨X = Y⟩ l6-2 by blast
    hence Y Out P Q
      using ⟨X = Y⟩ assms(2) out-bet-out-1 by blast
    thus ?thesis
      using ⟨A B C TSP P S⟩ ⟨Coplanar A B C Y⟩ l9-39 by blast
  qed
  hence A B C OSP P Q
    using OSP-def ⟨A B C TSP P S⟩ by blast
}
moreover
{
  assume ¬ Col P R S
  have X Y OS P R
  proof -
    have P ≠ X ∧ S ≠ X ∧ R ≠ Y ∧ S ≠ Y
      using ⟨A B C TSP P S⟩ ⟨Coplanar A B C Y⟩ ⟨P ≠ X ∧ S ≠ X ∧ R ≠ Y⟩ tsp--ncop2 by force
    have ¬ Col S X Y
      using bet-out-1 col-out2-col col-permutation-5 not-col-permutation-4
      by (metis ⟨Bet P X S⟩ ⟨Bet R Y S⟩ ⟨P ≠ X ∧ S ≠ X ∧ R ≠ Y ∧ S ≠ Y⟩ ⟨¬ Col P R S⟩)
    have X Y TS P S
      using Col-perm bet--ts bet-col col-transitivity-2
      by (metis ⟨Bet P X S⟩ ⟨P ≠ X ∧ S ≠ X ∧ R ≠ Y⟩ ⟨¬ Col S X Y⟩)
    have X Y TS R S
      using assms(1) bet--ts col-cop2--cop invert-two-sides not-col-distincts osp--ncop2
      ⟨Bet R Y S⟩ ⟨Coplanar A B C X⟩ ⟨Coplanar A B C Y⟩ ⟨¬ Col S X Y⟩ by metis
    thus ?thesis
      using ⟨X Y TS P S⟩ l9-8-1 by auto
  qed
  hence X Y OS P Q
    using assms(2) l9-17 by blast
  then obtain S' where X Y TS P S' and X Y TS Q S'
    using OS-def by blast
  have ¬ Col P X Y ∧ ¬ Col S' X Y ∧ (∃ T::'p. Col T X Y ∧ Bet P T S')
    using TS-def ⟨X Y TS P S'⟩ by blast
  have ¬ Col Q X Y ∧ ¬ Col S' X Y ∧ (∃ T::'p. Col T X Y ∧ Bet Q T S')
    using TS-def ⟨X Y TS Q S'⟩ by force
  obtain X' where Col X' X Y and Bet P X' S' and X Y TS Q S'
    using ⟨X Y TS Q S'⟩ ⟨¬ Col P X Y ∧ ¬ Col S' X Y ∧ (∃ T. Col T X Y ∧ Bet P T S')⟩ by blast
  obtain Y' where Col Y' X Y and Bet Q Y' S'

```

```

    using  $\langle \neg \text{Col } Q \ X \ Y \wedge \neg \text{Col } S' \ X \ Y \wedge (\exists T. \text{Col } T \ X \ Y \wedge \text{Bet } Q \ T \ S') \rangle$  by blast
  have Coplanar A B C X'
    using  $\langle \text{Coplanar } A \ B \ C \ X \rangle$  Col-cases col-cop2--cop ts-distincts
    by (metis not-col-distincts  $\langle \text{Col } X' \ X \ Y \rangle$   $\langle \text{Coplanar } A \ B \ C \ Y \rangle$ 
         $\langle \neg \text{Col } P \ X \ Y \wedge \neg \text{Col } S' \ X \ Y \wedge (\exists T. \text{Col } T \ X \ Y \wedge \text{Bet } P \ T \ S') \rangle$ )
  have Coplanar A B C Y'
    using Col-cases col-cop2--cop ts-distincts
    by (metis not-col-distincts  $\langle \text{Col } Y' \ X \ Y \rangle$   $\langle \text{Coplanar } A \ B \ C \ X \rangle$   $\langle \text{Coplanar } A \ B \ C \ Y \rangle$ 
         $\langle \neg \text{Col } P \ X \ Y \wedge \neg \text{Col } S' \ X \ Y \wedge (\exists T. \text{Col } T \ X \ Y \wedge \text{Bet } P \ T \ S') \rangle$ )
  have  $\neg \text{Coplanar } A \ B \ C \ S'$ 
    using assms(1) bet-col bet-col1 col2-cop2--eq osp--ncop1
    by (metis  $\langle \text{Bet } P \ X' \ S' \rangle$   $\langle \text{Col } X' \ X \ Y \rangle$   $\langle \text{Coplanar } A \ B \ C \ X' \rangle$ 
         $\langle \neg \text{Col } P \ X \ Y \wedge \neg \text{Col } S' \ X \ Y \wedge (\exists T. \text{Col } T \ X \ Y \wedge \text{Bet } P \ T \ S') \rangle$ )
  hence A B C OSP P Q
  proof -
    have A B C TSP P S'
      using TSP-def
      by (meson osp--ncop1  $\langle \text{Bet } P \ X' \ S' \rangle$   $\langle \text{Coplanar } A \ B \ C \ X' \rangle$   $\langle \neg \text{Coplanar } A \ B \ C \ S' \rangle$  assms(1))
    moreover have A B C TSP Q S'
      using TSP-def bet-col col-cop2--cop
      by (metis  $\langle \text{Bet } Q \ Y' \ S' \rangle$   $\langle \text{Col } Y' \ X \ Y \rangle$   $\langle \text{Coplanar } A \ B \ C \ Y' \rangle$ 
           $\langle \neg \text{Col } Q \ X \ Y \wedge \neg \text{Col } S' \ X \ Y \wedge (\exists T. \text{Col } T \ X \ Y \wedge \text{Bet } Q \ T \ S') \rangle$   $\langle \neg \text{Coplanar } A \ B \ C \ S' \rangle$ )
    ultimately show ?thesis
      using l9-41-1 by blast
  qed
}
ultimately show ?thesis
by blast
qed

```

lemma l9-18-3:

```

assumes Coplanar A B C P and
        Col X Y P
shows A B C TSP X Y  $\longleftrightarrow$  (Bet X P Y  $\wedge \neg \text{Coplanar } A \ B \ C \ X \wedge \neg \text{Coplanar } A \ B \ C \ Y$ )
by (meson TSP-def assms(1) assms(2) l9-39 not-bet-out not-col-permutation-5 tsp-distincts)

```

lemma bet-cop--tsp:

```

assumes  $\neg \text{Coplanar } A \ B \ C \ X$  and
         $P \neq Y$  and
        Coplanar A B C P and
        Bet X P Y
shows A B C TSP X Y
using TSP-def assms(1) assms(2) assms(3) assms(4) bet-col bet-col1 col2-cop2--eq by metis

```

lemma cop-out--osp:

```

assumes  $\neg \text{Coplanar } A \ B \ C \ X$  and
        Coplanar A B C P and
        P Out X Y
shows A B C OSP X Y
by (meson OSP-def assms(1) assms(2) assms(3) l9-39 tsp-exists)

```

lemma l9-19-3:

```

assumes Coplanar A B C P and
        Col X Y P
shows A B C OSP X Y  $\longleftrightarrow$  (P Out X Y  $\wedge \neg \text{Coplanar } A \ B \ C \ X$ )
by (meson assms(1) assms(2) cop-out--osp l6-4-2 l9-18-3 not-col-permutation-5 osp--ncop1
    osp--ncop2 tsp--nosp)

```

lemma cop2-ts--tsp:

```

assumes  $\neg \text{Coplanar } A \ B \ C \ X$  and Coplanar A B C D and
        Coplanar A B C E and D E TS X Y
shows A B C TSP X Y
proof -
  obtain T where Col T D E and Bet X T Y
    using TS-def assms(4) by blast
  moreover hence Coplanar A B C T

```

using *assms(2) assms(3) assms(4) col-cop2--cop not-col-permutation-2 ts-distincts* by *blast*
 ultimately show *?thesis*
 by (*metis TS-def assms(1) assms(4) bet-cop--tsp*)
 qed

lemma *cop2-os--osp*:
 assumes $\neg \text{Coplanar } A \ B \ C \ X$ and
 Coplanar A B C D and
 Coplanar A B C E and
 D E OS X Y
 shows *A B C OSP X Y*
proof –
 obtain *Z* where *D E TS X Z* and *D E TS Y Z*
 using *OS-def assms(4)* by *blast*
 hence *A B C TSP X Z*
 using *assms(1) assms(2) assms(3) cop2-ts--tsp* by *blast*
 moreover hence *A B C TSP Y Z*
 using *assms(2) assms(3) cop2-ts--tsp l9-2 tsp--ncop2 <D E TS Y Z>* by *meson*
 ultimately show *?thesis*
 using *l9-41-1* by *blast*
 qed

lemma *cop3-tsp--ts*:
 assumes $D \neq E$ and
 Coplanar A B C D and
 Coplanar A B C E and
 Coplanar D E X Y and
 A B C TSP X Y
 shows *D E TS X Y*
 by (*meson assms(1) assms(2) assms(3) assms(4) assms(5) col-cop2--cop cop2-os--osp*
 cop-nts--os not-col-permutation-2 tsp--ncop1 tsp--ncop2 tsp--nosp)

lemma *cop3-osp--os*:
 assumes $D \neq E$ and
 Coplanar A B C D and
 Coplanar A B C E and
 Coplanar D E X Y and
 A B C OSP X Y
 shows *D E OS X Y*
 by (*meson assms(1) assms(2) assms(3) assms(4) assms(5) col-cop2--cop cop2-ts--tsp*
 cop-nts--os not-col-permutation-2 osp--ncop1 osp--ncop2 tsp--nosp)

lemma *cop-tsp--ex-cop2*:
 assumes
 A B C TSP D E
 shows $\exists Q. (\text{Coplanar } A \ B \ C \ Q \wedge \text{Coplanar } D \ E \ P \ Q \wedge P \neq Q)$
proof *cases*
 assume *Col D E P*
 thus *?thesis*
 by (*meson ex-diff-cop ncop--ncols*)
next
 assume $\neg \text{Col } D \ E \ P$
 then obtain *Q* where *Coplanar A B C Q* $\wedge$ *Bet D Q E* $\wedge$ $\neg \text{Col } D \ E \ P$
 using *TSP-def assms(1)* by *blast*
 thus *?thesis*
 using *Col-perm bet-col ncop--ncols* by *blast*
 qed

lemma *cop-osp--ex-cop2*:
 assumes *Coplanar A B C P* and
 A B C OSP D E
 shows $\exists Q. \text{Coplanar } A \ B \ C \ Q \wedge \text{Coplanar } D \ E \ P \ Q \wedge P \neq Q$
proof *cases*
 assume *Col D E P*
 thus *?thesis*
 by (*metis col-trivial-3 diff-col-ex ncop--ncols*)

```

next
  assume  $P1: \neg \text{Col } D \ E \ P$ 
  obtain  $E'$  where  $P2: \text{Bet } E \ P \ E' \wedge \text{Cong } P \ E' \ P \ E$ 
    using segment-construction by blast
  have  $P3: \neg \text{Col } D \ E' \ P$ 
    by (metis  $P1 \ P2$  bet-col bet-cong-eq between-symmetry col-permutation-5 l5-2 l6-16-1)
  have  $P4: A \ B \ C \ \text{TSP } D \ E'$ 
    by (metis  $P2 \ P3$  assms(1) assms(2) bet-cop--tsp l9-41-2 not-col-distincts
      osp--ncop2 osp-symmetry)
  hence  $\neg \text{Coplanar } A \ B \ C \ D \wedge \neg \text{Coplanar } A \ B \ C \ E' \wedge (\exists \ T. \text{Coplanar } A \ B \ C \ T \wedge \text{Bet } D \ T \ E')$ 
    by (simp add: TSP-def)
  then obtain  $Q$  where  $P7: \text{Coplanar } A \ B \ C \ Q \wedge \text{Bet } D \ Q \ E'$ 
    by blast
  hence  $\text{Coplanar } D \ E' \ P \ Q$ 
    using bet-col ncop--ncols ncoplanar-perm-5 by blast
  hence  $\text{Coplanar } D \ E \ P \ Q$ 
    using Col-perm P2 P3 bet-col col-cop--cop ncoplanar-perm-5 not-col-distincts by blast
  thus ?thesis
    using  $P3 \ P7$  bet-col col-permutation-5 by blast
qed

lemma sac--coplanar:
  assumes  $\text{Saccheri } A \ B \ C \ D$ 
  shows  $\text{Coplanar } A \ B \ C \ D$ 
  using Saccheri-def assms ncoplanar-perm-4 os--coplanar by blast

lemma ex-sym:
   $\exists \ Y. (A \ B \ \text{Perp } X \ Y \vee X = Y) \wedge (\exists \ M. \text{Col } A \ B \ M \wedge M \ \text{Midpoint } X \ Y)$ 
proof cases
  assume  $\text{Col } A \ B \ X$ 
  thus ?thesis
    using l7-3-2 by blast
next
  assume  $\neg \text{Col } A \ B \ X$ 
  then obtain  $M0$  where  $\text{Col } A \ B \ M0$  and  $A \ B \ \text{Perp } X \ M0$ 
    using l8-18-existence by blast
  obtain  $Z$  where  $M0 \ \text{Midpoint } X \ Z$ 
    using symmetric-point-construction by blast
  thus ?thesis
    using Perp-cases bet-col midpoint-bet perp-col
    by (metis  $\langle A \ B \ \text{Perp } X \ M0 \rangle \langle \text{Col } A \ B \ M0 \rangle$ )
qed

lemma is-image-is-image-spec:
  assumes  $A \neq B$ 
  shows  $P' \ P \ \text{Reflect } A \ B \longleftrightarrow P' \ P \ \text{ReflectL } A \ B$ 
  by (simp add: Reflect-def assms)

lemma ex-sym1:
  assumes  $A \neq B$ 
  shows  $\exists \ Y. (A \ B \ \text{Perp } X \ Y \vee X = Y) \wedge (\exists \ M. \text{Col } A \ B \ M \wedge M \ \text{Midpoint } X \ Y \wedge X \ Y \ \text{Reflect } A \ B)$ 
proof cases
  assume  $\text{Col } A \ B \ X$ 
  thus ?thesis
    by (meson ReflectL-def Reflect-def assms l7-3-2)
next
  assume  $P0: \neg \text{Col } A \ B \ X$ 
  then obtain  $M0$  where  $P1: \text{Col } A \ B \ M0 \wedge A \ B \ \text{Perp } X \ M0$ 
    using l8-18-existence by blast
  obtain  $Z$  where  $P2: M0 \ \text{Midpoint } X \ Z$ 
    using symmetric-point-construction by blast
  have  $P3: A \ B \ \text{Perp } X \ Z$ 
  proof cases
    assume  $X = Z$ 
    thus ?thesis
      using  $P1 \ P2 \ P0$  midpoint-distinct by blast

```

```

next
  assume  $X \neq Z$ 
  hence  $P2: X Z \text{ Perp } A B$ 
  using  $P1 P2 \text{ Perp-cases bet-col midpoint-bet perp-col}$  by blast
  show ?thesis
    by (simp add: Perp-perm Tarski-neutral-dimensionless-axioms P2)
qed
have  $P10: (A B \text{ Perp } X Z \vee X = Z)$ 
  by (simp add:  $P3$ )
have  $\exists M. \text{ Col } A B M \wedge M \text{ Midpoint } X Z \wedge X Z \text{ Reflect } A B$ 
  using  $P1 P2 P3 \text{ ReflectL-def assms is-image-is-image-spec l7-2 perp-right-comm}$  by blast
thus ?thesis
  using  $P3$  by blast
qed

lemma l10-2-uniqueness:
  assumes  $P1 P \text{ Reflect } A B$  and
     $P2 P \text{ Reflect } A B$ 
  shows  $P1 = P2$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    using Reflect-def assms(1) assms(2) symmetric-point-uniqueness by auto
next
  assume  $A \neq B$ 
  hence  $P1 P \text{ ReflectL } A B$ 
    using assms(1) is-image-is-image-spec by auto
  hence  $A B \text{ Perp } P P1 \vee P = P1$ 
    using ReflectL-def by blast
  have  $P2 P \text{ ReflectL } A B$ 
    using assms(2) is-image-is-image-spec  $\langle A \neq B \rangle$  by blast
  hence  $A B \text{ Perp } P P2 \vee P = P2$ 
    using ReflectL-def by blast
  obtain  $X$  where  $X \text{ Midpoint } P P1$  and  $\text{ Col } A B X$ 
    by (metis ReflectL-def assms(1) col-trivial-1 is-image-is-image-spec midpoint-existence)
  obtain  $Y$  where  $Y \text{ Midpoint } P P2$  and  $\text{ Col } A B Y$ 
    by (metis ReflectL-def assms(2) col-trivial-1 is-image-is-image-spec midpoint-existence)
  {
    assume  $A B \text{ Perp } P P1$  and  $A B \text{ Perp } P P2$ 
    have  $P \neq X$ 
      using  $\langle A B \text{ Perp } P P1 \rangle \langle X \text{ Midpoint } P P1 \rangle$  is-midpoint-id perp-not-eq-2 by blast
    have  $P \neq Y$ 
      using  $\langle A B \text{ Perp } P P2 \rangle \langle Y \text{ Midpoint } P P2 \rangle$  is-midpoint-id perp-not-eq-2 by blast
    have  $P X \text{ Perp } A B$ 
      using Perp-perm  $\langle A B \text{ Perp } P P1 \rangle \langle P \neq X \rangle \langle X \text{ Midpoint } P P1 \rangle$  bet-col midpoint-bet not-col-permutation-5 perp-col1 by blast
    have  $P Y \text{ Perp } A B$ 
      using Perp-perm  $\langle A B \text{ Perp } P P2 \rangle \langle P \neq Y \rangle \langle Y \text{ Midpoint } P P2 \rangle$  bet-col midpoint-bet not-col-permutation-5 perp-col1 by blast
    hence  $P1 = P2$ 
      by (metis Perp-perm l7-2 l7-9-bis  $\langle \text{ Col } A B Y \rangle \langle P X \text{ Perp } A B \rangle \langle X \text{ Midpoint } P P1 \rangle$   $\langle \bigwedge \text{thesis. } (\bigwedge X. \llbracket X \text{ Midpoint } P P1; \text{ Col } A B X \rrbracket \implies \text{thesis}) \implies \text{thesis} \rangle$   $\langle Y \text{ Midpoint } P P2 \rangle$  l7-17 l8-18-uniqueness)
  }
  hence  $(A B \text{ Perp } P P1 \wedge A B \text{ Perp } P P2) \longrightarrow P1 = P2$  by blast
  moreover have  $(P = P1 \wedge A B \text{ Perp } P P2) \longrightarrow P1 = P2$ 
    by (metis  $\langle \text{ Col } A B X \rangle \langle \text{ Col } A B Y \rangle \langle X \text{ Midpoint } P P1 \rangle \langle Y \text{ Midpoint } P P2 \rangle$  colx is-midpoint-id l8-16-1 midpoint-col midpoint-distinct-2)
  moreover have  $(P = P2 \wedge A B \text{ Perp } P P1) \longrightarrow P1 = P2$ 
    by (metis  $\langle \text{ Col } A B X \rangle \langle \text{ Col } A B Y \rangle \langle X \text{ Midpoint } P P1 \rangle \langle Y \text{ Midpoint } P P2 \rangle$  l8-16-1 l8-20-2 midpoint-col not-col-distincts perp-col2)
  ultimately show ?thesis
    using  $\langle A B \text{ Perp } P P1 \vee P = P1 \rangle \langle A B \text{ Perp } P P2 \vee P = P2 \rangle$  by fastforce
qed

lemma l10-2-uniqueness-spec:

```



```

assumes  $P1 \ P \ \text{ReflectL} \ A \ B$  and
 $P2 \ P \ \text{ReflectL} \ A \ B$ 
shows  $P1 = P2$ 
proof –
  have  $A \ B \ \text{Perp} \ P \ P1 \ \vee \ P = P1$ 
    using ReflectL-def assms(1) by blast
  moreover obtain  $X1$  where  $X1 \ \text{Midpoint} \ P \ P1$  and  $\text{Col} \ A \ B \ X1$ 
    using ReflectL-def assms(1) by blast
  moreover have  $A \ B \ \text{Perp} \ P \ P2 \ \vee \ P = P2$ 
    using ReflectL-def assms(2) by blast
  moreover obtain  $X2$  where  $X2 \ \text{Midpoint} \ P \ P2$  and  $\text{Col} \ A \ B \ X2$ 
    using ReflectL-def assms(2) by blast
  {
    assume  $A \ B \ \text{Perp} \ P \ P1$  and  $A \ B \ \text{Perp} \ P \ P2$ 
    have  $P1 \ P \ \text{Reflect} \ A \ B$ 
      using Reflect-def  $\langle A \ B \ \text{Perp} \ P \ P1 \rangle \ \text{assms}(1) \ \text{perp-distinct}$  by auto
    moreover have  $P2 \ P \ \text{Reflect} \ A \ B$ 
      using  $\langle A \ B \ \text{Perp} \ P \ P2 \rangle \ \text{assms}(2) \ \text{is-image-is-image-spec} \ \text{perp-distinct}$  by auto
    ultimately have  $P1 = P2$ 
      using l10-2-uniqueness by auto
  }
  moreover
  {
    assume  $A \ B \ \text{Perp} \ P \ P1 \ \wedge \ P = P2$ 
    hence  $P1 = P2$ 
      by (metis colx perp-not-col2  $\langle \text{Col} \ A \ B \ X2 \rangle \ \langle X2 \ \text{Midpoint} \ P \ P2 \rangle \ \text{calculation}(2)$ 
        calculation(3) l8-20-2 midpoint-col)
  }
  moreover
  {
    assume  $P = P1$  and  $A \ B \ \text{Perp} \ P \ P2$ 
    hence  $P1 = P2$ 
      by (metis  $\langle \text{Col} \ A \ B \ X2 \rangle \ \langle X2 \ \text{Midpoint} \ P \ P2 \rangle \ \text{calculation}(2) \ \text{calculation}(3)$ 
        colx l8-20-2 midpoint-col perp-not-col2)
  }
  ultimately show ?thesis
    by blast
qed

```

lemma *l10-2-existence-spec:*

$\exists \ P'. \ P' \ P \ \text{ReflectL} \ A \ B$

proof *cases*

assume $\text{Col} \ A \ B \ P$

thus *?thesis*

using *ReflectL-def l7-3-2* **by** *blast*

next

assume $\neg \ \text{Col} \ A \ B \ P$

then obtain X **where** $\text{Col} \ A \ B \ X \ \wedge \ A \ B \ \text{Perp} \ P \ X$

using *l8-18-existence* **by** *blast*

moreover obtain P' **where** $X \ \text{Midpoint} \ P \ P'$

using *symmetric-point-construction* **by** *blast*

ultimately show *?thesis*

using *ReflectL-def bet-col midpoint-bet perp-col1* **by** *blast*

qed

lemma *l10-2-existence:*

$\exists \ P'. \ P' \ P \ \text{Reflect} \ A \ B$

by (*metis Reflect-def l10-2-existence-spec symmetric-point-construction*)

lemma *l10-4-spec:*

assumes $P \ P' \ \text{ReflectL} \ A \ B$

shows $P' \ P \ \text{ReflectL} \ A \ B$

proof –

obtain X **where** $X \ \text{Midpoint} \ P \ P' \ \wedge \ \text{Col} \ A \ B \ X$

using *ReflectL-def assms l7-2* **by** *blast*

thus *?thesis*

```

    using Perp-cases ReflectL-def assms by auto
qed

lemma l10-4:
  assumes  $P \ P' \ \text{Reflect } A \ B$ 
  shows  $P' \ P \ \text{Reflect } A \ B$ 
  using Reflect-def l7-2 Tarski-neutral-dimensionless-axioms assms l10-4-spec by fastforce

lemma l10-5:
  assumes  $P' \ P \ \text{Reflect } A \ B$  and
     $P'' \ P' \ \text{Reflect } A \ B$ 
  shows  $P = P''$ 
  by (meson assms(1) assms(2) l10-2-uniqueness l10-4)

lemma l10-6-uniqueness:
  assumes  $P \ P1 \ \text{Reflect } A \ B$  and
     $P \ P2 \ \text{Reflect } A \ B$ 
  shows  $P1 = P2$ 
  using assms(1) assms(2) l10-4 l10-5 by blast

lemma l10-6-uniqueness-spec:
  assumes  $P \ P1 \ \text{ReflectL } A \ B$  and
     $P \ P2 \ \text{ReflectL } A \ B$ 
  shows  $P1 = P2$ 
  using assms(1) assms(2) l10-2-uniqueness-spec l10-4-spec by blast

lemma l10-6-existence-spec:
  assumes  $A \neq B$ 
  shows  $\exists P. P' \ P \ \text{ReflectL } A \ B$ 
  using l10-2-existence-spec l10-4-spec by blast

lemma l10-6-existence:
   $\exists P. P' \ P \ \text{Reflect } A \ B$ 
  using l10-2-existence l10-4 by blast

lemma l10-7:
  assumes  $P' \ P \ \text{Reflect } A \ B$  and
     $Q' \ Q \ \text{Reflect } A \ B$  and
     $P' = Q'$ 
  shows  $P = Q$ 
  using assms(1) assms(2) assms(3) l10-6-uniqueness by blast

lemma l10-8:
  assumes  $P \ P \ \text{Reflect } A \ B$ 
  shows  $\text{Col } P \ A \ B$ 
  by (metis Col-perm assms col-trivial-2 ex-sym1 l10-6-uniqueness l7-3)

lemma col-refl:
  assumes  $\text{Col } P \ A \ B$ 
  shows  $P \ P \ \text{ReflectL } A \ B$ 
  using ReflectL-def assms col-permutation-1 l7-3-2 by blast

lemma is-image-col-cong:
  assumes  $A \neq B$  and
     $P \ P' \ \text{Reflect } A \ B$  and
     $\text{Col } A \ B \ X$ 
  shows  $\text{Cong } P \ X \ P' \ X$ 
proof -
  have  $P1: P \ P' \ \text{ReflectL } A \ B$ 
  using assms(1) assms(2) is-image-is-image-spec by blast
  obtain  $M0$  where  $P2: M0 \ \text{Midpoint } P' \ P \wedge \text{Col } A \ B \ M0$ 
  using  $P1 \ \text{ReflectL-def}$  by blast
  have  $A \ B \ \text{Perp } P' \ P \vee P' = P$ 
  using  $P1 \ \text{ReflectL-def}$  by auto
  moreover
  {

```

```

assume  $S1: A \ B \ \text{Perp} \ P' \ P$ 
hence  $A \neq B \wedge P' \neq P$ 
  using perp-distinct by blast
have  $S2: M0 = X \longrightarrow \text{Cong} \ P \ X \ P' \ X$ 
  using P2 cong-4312 midpoint-cong by blast
{
  assume  $M0 \neq X$ 
  hence  $M0 \ X \ \text{Perp} \ P' \ P$ 
    using P2 S1 assms(3) perp-col2 by blast
  hence  $\neg \text{Col} \ A \ B \ P \wedge \text{Per} \ P \ M0 \ X$ 
    by (metis Col-perm P2 S1 colx l8-2 midpoint-col midpoint-distinct-1
      per-col perp-col1 perp-not-col2 perp-per-1)
  hence  $\text{Cong} \ P \ X \ P' \ X$ 
    using P2 cong-commutativity l7-2 l8-2 per-double-cong by blast
}
hence  $\text{Cong} \ P \ X \ P' \ X$ 
  using S2 by blast
}
hence  $A \ B \ \text{Perp} \ P' \ P \longrightarrow \text{Cong} \ P \ X \ P' \ X$  by blast
moreover
{
  assume  $P = P'$ 
  hence  $\text{Cong} \ P \ X \ P' \ X$ 
    by (simp add: cong-reflexivity)
}
ultimately show ?thesis by blast
qed

```

```

lemma is-image-spec-col-cong:
assumes  $P \ P' \ \text{ReflectL} \ A \ B$  and
   $\text{Col} \ A \ B \ X$ 
shows  $\text{Cong} \ P \ X \ P' \ X$ 
by (metis Col-def Reflect-def assms(1) assms(2) between-trivial col--refl
  cong-reflexivity is-image-col-cong l10-6-uniqueness-spec)

```

```

lemma image-id:
assumes  $A \neq B$  and
   $\text{Col} \ A \ B \ T$  and
   $T \ T' \ \text{Reflect} \ A \ B$ 
shows  $T = T'$ 
using assms(1) assms(2) assms(3) cong-diff-4 is-image-col-cong by blast

```

```

lemma osym-not-col:
assumes  $P \ P' \ \text{Reflect} \ A \ B$  and
   $\neg \text{Col} \ A \ B \ P$ 
shows  $\neg \text{Col} \ A \ B \ P'$ 
using assms(1) assms(2) l10-4 local.image-id not-col-distincts by blast

```

```

lemma midpoint-preserves-image:
assumes  $A \neq B$  and
   $\text{Col} \ A \ B \ M$  and
   $P \ P' \ \text{Reflect} \ A \ B$  and
   $M \ \text{Midpoint} \ P \ Q$  and
   $M \ \text{Midpoint} \ P' \ Q'$ 
shows  $Q \ Q' \ \text{Reflect} \ A \ B$ 
proof –
obtain  $X$  where  $X \ \text{Midpoint} \ P' \ P$  and  $\text{Col} \ A \ B \ X$ 
  using ReflectL-def assms(1) assms(3) is-image-is-image-spec by blast
{
  assume  $A \ B \ \text{Perp} \ P' \ P$ 
  obtain  $Y$  where  $M \ \text{Midpoint} \ X \ Y$ 
    using symmetric-point-construction by blast
  have  $Y \ \text{Midpoint} \ Q \ Q'$ 
  proof –
    have  $X \ \text{Midpoint} \ P \ P'$ 
      using  $\langle X \ \text{Midpoint} \ P' \ P \rangle$  l7-2 by blast

```

```

thus ?thesis
  using assms(4) assms(5) symmetry-preserves-midpoint  $\langle M \text{ Midpoint } X \ Y \rangle$  by blast
qed
have  $P \neq P'$ 
  using  $\langle A \ B \ \text{Perp} \ P' \ P \rangle$  perp-distinct by blast
hence  $Q \neq Q'$ 
  using l7-9 Tarski-neutral-dimensionless-axioms assms(4) assms(5) by fastforce
have  $Y \text{ Midpoint } Q' \ Q \wedge \text{Col } A \ B \ Y$ 
  using assms(2) colx l7-2 midpoint-col midpoint-distinct-1
  by (metis  $\langle \text{Col } A \ B \ X \rangle \langle M \text{ Midpoint } X \ Y \rangle \langle Y \text{ Midpoint } Q \ Q' \rangle$ )
have  $A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q'$ 
proof –
  have  $\text{Per } M \ Y \ Q$ 
  proof –
    have  $Y \text{ Midpoint } Q \ Q'$ 
    using  $\langle Y \text{ Midpoint } Q \ Q' \rangle$  by auto
    have  $\text{Cong } M \ Q \ M \ Q'$ 
    using assms(1) assms(2) assms(3) assms(4) assms(5) cong-commutativity
    is-image-col-cong l7-16 l7-3-2 by blast
    thus ?thesis
    using Per-def  $\langle Y \text{ Midpoint } Q \ Q' \rangle$  by blast
  qed
{
  have  $X = Y \longrightarrow (A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q')$ 
  using Perp-cases assms(5) l7-3 l7-9-bis
  by (metis  $\langle A \ B \ \text{Perp} \ P' \ P \rangle \langle M \text{ Midpoint } X \ Y \rangle \langle X \text{ Midpoint } P' \ P \rangle$  assms(4))
  {
    assume  $X \neq Y$ 
    hence  $Y \text{ PerpAt } M \ Y \ Y \ Q$ 
    using midpoint-distinct-1 per-perp-in
    by (metis  $\langle M \text{ Midpoint } X \ Y \rangle \langle \text{Per } M \ Y \ Q \rangle \langle Q \neq Q' \rangle \langle Y \text{ Midpoint } Q' \ Q \wedge \text{Col } A \ B \ Y \rangle$ )
    hence  $Y \ Y \ \text{Perp} \ Y \ Q \vee M \ Y \ \text{Perp} \ Y \ Q$ 
    by (simp add: perp-in-perp-bis)
    {
      have  $Y \ Y \ \text{Perp} \ Y \ Q \longrightarrow A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q'$ 
      using perp-not-eq-1 by blast
      {
        assume  $M \ Y \ \text{Perp} \ Y \ Q$ 
        have  $Y \ Q \ \text{Perp} \ A \ B$ 
        proof cases
        assume  $A = M$ 
        thus ?thesis
        using Perp-cases  $\langle M \ Y \ \text{Perp} \ Y \ Q \rangle \langle Y \text{ Midpoint } Q' \ Q \wedge \text{Col } A \ B \ Y \rangle$ 
        assms(1) not-col-permutation-5 perp-col1 by blast
      next
        assume  $A \neq M$ 
        thus ?thesis
        by (metis  $\langle Y \text{ Midpoint } Q' \ Q \wedge \text{Col } A \ B \ Y \rangle \langle Y \ Y \ \text{Perp} \ Y \ Q \vee M \ Y \ \text{Perp} \ Y \ Q \rangle$ 
        assms(1) assms(2) col3 not-col-distincts perp-col0)
      qed
      have  $A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q'$ 
      using midpoint-col not-col-distincts perp-col0
      by (metis  $\langle Y \text{ Midpoint } Q \ Q' \rangle \langle Y \ Q \ \text{Perp} \ A \ B \rangle$ )
    }
    hence  $M \ Y \ \text{Perp} \ Y \ Q \longrightarrow A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q'$  by blast
  }
  hence  $A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q'$ 
  using perp-distinct  $\langle Y \ Y \ \text{Perp} \ Y \ Q \vee M \ Y \ \text{Perp} \ Y \ Q \rangle$  by blast
}
hence  $X \neq Y \longrightarrow (A \ B \ \text{Perp} \ Q' \ Q \vee Q = Q')$  by blast
}
thus ?thesis
using Perp-cases assms(5) l7-3 l7-9-bis
by (metis  $\langle A \ B \ \text{Perp} \ P' \ P \rangle \langle M \text{ Midpoint } X \ Y \rangle \langle Y \text{ Midpoint } Q \ Q' \rangle$  assms(4))
qed
hence  $Q \ Q' \ \text{ReflectL } A \ B$ 

```

```

    using ReflectL-def ⟨Y Midpoint Q' Q ∧ Col A B Y⟩ by blast
  }
  moreover
  {
    assume P = P'
    hence Q Q' ReflectL A B
    using assms(2) assms(4) assms(5) col--refl col-permutation-2 colx midpoint-col
      midpoint-distinct-3 symmetric-point-uniqueness by (metis ⟨Col A B X⟩ ⟨X Midpoint P' P⟩)
  }
  ultimately show ?thesis
    using ReflectL-def assms(1) assms(3) is-image-is-image-spec by auto
qed

```

```

lemma image-in-is-image-spec:
  assumes M ReflectLAT P P' A B
  shows P P' ReflectL A B

```

```

proof -
  have P1: M Midpoint P' P
    using ReflectLAT-def assms by blast
  have P2: Col A B M
    using ReflectLAT-def assms by blast
  have A B Perp P' P ∨ P' = P
    using ReflectLAT-def assms by blast
  thus ?thesis using P1 P2
    using ReflectL-def by blast
qed

```

```

lemma image-in-gen-is-image:
  assumes M ReflectAt P P' A B
  shows P P' Reflect A B
  using ReflectAt-def Reflect-def assms image-in-is-image-spec by auto

```

```

lemma image-image-in:
  assumes P ≠ P' and
    P P' ReflectL A B and
    Col A B M and
    Col P M P'
  shows M ReflectLAT P P' A B

```

```

proof -
  obtain M' where P1: M' Midpoint P' P ∧ Col A B M'
    using ReflectL-def assms(2) by blast
  have Q1: P M' Perp A B
    by (metis Col-cases P1 Perp-perm ReflectL-def assms(1) assms(2) bet-col cong-diff-3
      midpoint-bet midpoint-cong not-cong-4321 perp-col1)
  {
    assume R1: A B Perp P' P
    have R3: P ≠ M'
      using Q1 perp-not-eq-1 by auto
    have R4: A B Perp P' P
      by (simp add: R1)
    have R5: Col P P' M'
      using P1 midpoint-col not-col-permutation-3 by blast
    have R6: M' Midpoint P' P
      by (simp add: P1)
    have R7: ¬ Col A B P
      using assms(1) assms(2) col--refl col-permutation-2 l10-2-uniqueness-spec l10-4-spec
      by blast
    have R8: P ≠ P'
      by (simp add: assms(1))
    have R9: Col A B M'
      by (simp add: P1)
    have R10: Col A B M
      by (simp add: assms(3))
    have R11: Col P P' M'
      by (simp add: R5)
    have R12: Col P P' M

```

```

    using Col-perm assms(4) by blast
  have  $M = M'$ 
  proof cases
    assume  $S1: A = M'$ 
    have  $Per\ P\ M'\ A$ 
      by (simp add:  $S1$  l8-5)
    thus ?thesis using l6-21  $R8\ R9\ R10\ R11\ R12$ 
      using  $R7$  by blast
  next
    assume  $A \neq M'$ 
    thus ?thesis
      using  $R10\ R12\ R5\ R7\ R8\ R9\ l6-21$  by blast
  qed
  hence  $M\ Midpoint\ P'\ P$ 
    using  $R6$  by blast
}
hence  $Q2: A\ B\ Perp\ P'\ P \longrightarrow M\ Midpoint\ P'\ P$  by blast
have  $Q3: P' = P \longrightarrow M\ Midpoint\ P'\ P$ 
  using assms(1) by auto
have  $Q4: A\ B\ Perp\ P'\ P \vee P' = P$ 
  using ReflectL-def assms(2) by auto
hence  $M\ Midpoint\ P'\ P$ 
  using  $Q2\ Q3$  by blast
thus ?thesis
  by (simp add: ReflectLAt-def  $Q4$  assms(3))
qed

lemma image-in-col:
  assumes  $Y\ ReflectLAt\ P\ P'\ A\ B$ 
  shows  $Col\ P\ P'\ Y$ 
  using Col-perm ReflectLAt-def assms midpoint-col by blast

lemma is-image-spec-rev:
  assumes  $P\ P'\ ReflectL\ A\ B$ 
  shows  $P\ P'\ ReflectL\ B\ A$ 
  proof -
    obtain  $M0$  where  $P1: M0\ Midpoint\ P'\ P \wedge Col\ A\ B\ M0$ 
      using ReflectL-def assms by blast
    have  $P2: Col\ B\ A\ M0$ 
      using Col-cases  $P1$  by blast
    have  $A\ B\ Perp\ P'\ P \vee P' = P$ 
      using ReflectL-def assms by blast
    thus ?thesis
      using  $P1\ P2\ Perp-cases\ ReflectL-def$  by auto
  qed

lemma is-image-rev:
  assumes  $P\ P'\ Reflect\ A\ B$ 
  shows  $P\ P'\ Reflect\ B\ A$ 
  using Reflect-def assms is-image-spec-rev by auto

lemma midpoint-preserves-per:
  assumes  $Per\ A\ B\ C$  and
     $M\ Midpoint\ A\ A1$  and
     $M\ Midpoint\ B\ B1$  and
     $M\ Midpoint\ C\ C1$ 
  shows  $Per\ A1\ B1\ C1$ 
  proof -
    obtain  $C'$  where  $P1: B\ Midpoint\ C\ C' \wedge Cong\ A\ C\ A\ C'$ 
      using Per-def assms(1) by blast
    obtain  $C1'$  where  $P2: M\ Midpoint\ C'\ C1'$ 
      using symmetric-point-construction by blast
    thus ?thesis
      by (meson  $P1\ Per-def\ assms(2)\ assms(3)\ assms(4)\ l7-16\ symmetry-preserves-midpoint$ )
  qed

```

```

lemma col--image-spec:
  assumes Col A B X
  shows X X ReflectL A B
  by (simp add: assms col--refl col-permutation-2)

lemma image-triv:
  A A Reflect A B
  by (simp add: Reflect-def col--refl col-trivial-1 l7-3-2)

lemma cong-midpoint--image:
  assumes Cong A X A Y and
    B Midpoint X Y
  shows Y X Reflect A B
proof cases
  assume A = B
  thus ?thesis
    by (simp add: Reflect-def assms(2))
next
  assume S0: A  $\neq$  B
  {
    assume S1: X  $\neq$  Y
    hence X Y Perp A B
    proof -
      have T1: B  $\neq$  X
      using S1 assms(2) midpoint-distinct-1 by blast
      have T2: B  $\neq$  Y
      using S1 assms(2) midpoint-not-midpoint by blast
      have Per A B X
      using Perp-def assms(1) assms(2) by auto
      thus ?thesis
      using S0 S1 T1 T2 assms(2) col-per-perp midpoint-col by auto
    qed
    hence A B Perp X Y  $\vee$  X = Y
    using Perp-perm by blast
    hence Y X Reflect A B
    using ReflectL-def S0 assms(2) col-trivial-2 is-image-is-image-spec by blast
  }
  hence X  $\neq$  Y  $\longrightarrow$  Y X Reflect A B by blast
  thus ?thesis
    using assms(2) image-triv is-image-rev l7-3 by blast
qed

lemma col-image-spec--eq:
  assumes Col A B P and
    P P' ReflectL A B
  shows P = P'
  using assms(1) assms(2) col--image-spec l10-2-uniqueness-spec l10-4-spec by blast

lemma image-spec-triv:
  A A ReflectL B B
  using col--image-spec not-col-distincts by blast

lemma image-spec--eq:
  assumes P P' ReflectL A A
  shows P = P'
  using assms col-image-spec--eq not-col-distincts by blast

lemma image--midpoint:
  assumes P P' Reflect A A
  shows A Midpoint P' P
  using Reflect-def assms by auto

lemma is-image-spec-dec:
  A B ReflectL C D  $\vee$   $\neg$  A B ReflectL C D
  by simp

```

lemma l10-14:
 assumes $P \neq P'$ and
 $A \neq B$ and
 $P P' \text{ Reflect } A B$
 shows $A B \text{ TS } P P'$
 proof –
 have $P1: P P' \text{ ReflectL } A B$
 using $\text{assms}(2) \text{ assms}(3) \text{ is-image-is-image-spec}$ by blast
 then obtain $M0$ where $M0 \text{ Midpoint } P' P \wedge \text{Col } A B M0$
 using ReflectL-def by blast
 hence $A B \text{ Perp } P' P \longrightarrow A B \text{ TS } P P'$
 by ($\text{meson TS-def assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ between-symmetry col-permutation-2}$
 $\text{image-id midpoint-bet osym-not-col}$)
 thus ?thesis
 using $\text{assms}(1) P1 \text{ ReflectL-def}$ by blast
 qed

lemma l10-15:
 assumes $\text{Col } A B C$ and
 $\neg \text{Col } A B P$
 shows $\exists Q. A B \text{ Perp } Q C \wedge A B \text{ OS } P Q$
 proof –
 have $P1: A \neq B$
 using $\text{assms}(2) \text{ col-trivial-1}$ by auto
 obtain X where $P2: A B \text{ TS } P X$
 using $\text{assms}(2) \text{ col-permutation-1 l9-10}$ by blast
 {
 assume $Q1: A = C$
 obtain Q where $Q2: \exists T. A B \text{ Perp } Q A \wedge \text{Col } A B T \wedge \text{Bet } X T Q$
 using $P1 \text{ l8-21}$ by blast
 then obtain T where $A B \text{ Perp } Q A \wedge \text{Col } A B T \wedge \text{Bet } X T Q$ by blast
 hence $A B \text{ TS } Q X$
 by ($\text{meson } P2 \text{ TS-def between-symmetry col-permutation-2 perp-not-col}$)
 hence $Q5: A B \text{ OS } P Q$
 using $P2 \text{ l9-8-1}$ by blast
 hence $\exists Q. A B \text{ Perp } Q C \wedge A B \text{ OS } P Q$
 using $Q1 Q2$ by blast
 }
 hence $P3: A = C \longrightarrow (\exists Q. A B \text{ Perp } Q C \wedge A B \text{ OS } P Q)$ by blast
 {
 assume $Q1: A \neq C$
 then obtain Q where $Q2: \exists T. C A \text{ Perp } Q C \wedge \text{Col } C A T \wedge \text{Bet } X T Q$
 using l8-21 by presburger
 then obtain T where $Q3: C A \text{ Perp } Q C \wedge \text{Col } C A T \wedge \text{Bet } X T Q$ by blast
 have $Q4: A B \text{ Perp } Q C$
 using $\text{NCol-perm } P1 Q2 \text{ assms}(1) \text{ col-trivial-2 perp-col2}$ by blast
 have $A B \text{ TS } Q X$
 proof –
 have $R1: \neg \text{Col } Q A B$
 using $\text{Col-perm } P1 Q2 \text{ assms}(1) \text{ col-trivial-2 colx perp-not-col}$ by blast
 have $R2: \neg \text{Col } X A B$
 using $P2 \text{ TS-def}$ by auto
 have $R3: \text{Col } T A B$
 by ($\text{metis } Q1 Q3 \text{ assms}(1) \text{ col-trivial-2 colx not-col-permutation-1}$)
 have $\text{Bet } Q T X$
 using $\text{Bet-cases } Q3$ by blast
 hence $\exists T. \text{Col } T A B \wedge \text{Bet } Q T X$
 using $R3$ by blast
 thus ?thesis using $R1 R2$
 by (simp add: TS-def)
 qed
 hence $A B \text{ OS } P Q$
 using $P2 \text{ l9-8-1}$ by blast
 hence $\exists Q. A B \text{ Perp } Q C \wedge A B \text{ OS } P Q$
 using $Q4$ by blast


```

}
thus ?thesis using P3 by blast
qed

```

```

lemma ex-per-cong:
  assumes  $A \neq B$  and
     $X \neq Y$  and
     $Col\ A\ B\ C$  and
     $\neg Col\ A\ B\ D$ 
  shows  $\exists P. Per\ P\ C\ A \wedge Cong\ P\ C\ X\ Y \wedge A\ B\ OS\ P\ D$ 
proof -
  obtain  $Q$  where  $P1: A\ B\ Perp\ Q\ C \wedge A\ B\ OS\ D\ Q$ 
    using assms(3) assms(4) l10-15 by blast
  obtain  $P$  where  $P2: C\ Out\ Q\ P \wedge Cong\ C\ P\ X\ Y$ 
    by (metis P1 assms(2) perp-not-eq-2 segment-construction-3)
  have  $P3: Per\ P\ C\ A$ 
    using P1 P2 assms(3) col-trivial-3 l8-16-1 l8-3 out-col by blast
  have  $A\ B\ OS\ P\ D$ 
    using P1 P2 assms(3) one-side-symmetry os-out-os by blast
  thus ?thesis
    using P2 P3 cong-left-commutativity by blast
qed

```

```

lemma exists-cong-per:
   $\exists C. Per\ A\ B\ C \wedge Cong\ B\ C\ X\ Y$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    by (meson l8-5 Tarski-neutral-dimensionless-axioms l8-2 segment-construction)
next
  assume  $A \neq B$ 
  thus ?thesis
    by (metis Perp-perm bet-col between-trivial l8-16-1 l8-21 segment-construction)
qed

```

```

lemma upper-dim-implies-per2--col:
  assumes upper-dim-axiom
  shows  $\forall A\ B\ C\ X. (Per\ A\ X\ C \wedge X \neq C \wedge Per\ B\ X\ C) \longrightarrow Col\ A\ B\ X$ 
proof -
  {
    fix  $A\ B\ C\ X$ 
    assume  $Per\ A\ X\ C \wedge X \neq C \wedge Per\ B\ X\ C$ 
    moreover then obtain  $C'$  where  $X\ Midpoint\ C\ C'$  and  $Cong\ A\ C\ A\ C'$ 
      using Per-def by blast
    moreover obtain  $C''$  where  $X\ Midpoint\ C\ C''$  and  $Cong\ B\ C\ B\ C''$ 
      using Per-def calculation(1) by blast
    moreover hence  $C' = C''$ 
      using  $\langle X\ Midpoint\ C\ C'' \rangle$  symmetric-point-uniqueness calculation(2) by auto
    have  $C \neq C'$ 
      using calculation(1) calculation(2) l7-3 by blast
    moreover have  $Cong\ B\ C\ B\ C'$ 
      by (simp add:  $\langle C' = C'' \rangle$  calculation(5))
    moreover have  $Cong\ X\ C\ X\ C'$ 
      using calculation(2) cong-left-commutativity midpoint-cong by blast
    ultimately have  $Col\ A\ B\ X$ 
      using Col-def assms upper-dim-axiom-def by blast
  }
  thus ?thesis by blast
qed

```

```

lemma upper-dim-implies-col-perp2--col:
  assumes upper-dim-axiom
  shows  $\forall A\ B\ X\ Y\ P. (Col\ A\ B\ P \wedge A\ B\ Perp\ X\ P \wedge P\ A\ Perp\ Y\ P) \longrightarrow Col\ Y\ X\ P$ 
proof -
  {
    fix  $A\ B\ X\ Y\ P$ 

```

```

assume H1: Col A B P  $\wedge$  A B Perp X P  $\wedge$  P A Perp Y P
hence H2: P  $\neq$  A
  using perp-not-eq-1 by blast
have Col Y X P
proof –
  have T1: Per Y P A
    using H1 l8-2 perp-per-1 by blast
  moreover have Per X P A
    using H1 col-trivial-3 l8-16-1 by blast
  thus ?thesis using T1 H2
    using assms upper-dim-implies-per2--col by blast
qed
}
thus ?thesis by blast
qed

lemma upper-dim-implies-perp2--col:
assumes upper-dim-axiom
shows  $\forall$  X Y Z A B. (X Y Perp A B  $\wedge$  X Z Perp A B)  $\longrightarrow$  Col X Y Z
proof –
{
  fix X Y Z A B
  assume H1: X Y Perp A B  $\wedge$  X Z Perp A B
  hence H1A: X Y Perp A B by blast
  have H1B: X Z Perp A B using H1 by blast
  obtain C where H2: C PerpAt X Y A B
    using H1 Perp-def by blast
  obtain C' where H3: C' PerpAt X Z A B
    using H1 Perp-def by blast
  have Col X Y Z
proof cases
  assume H2: Col A B X
  {
    assume X = A
    hence Col X Y Z using upper-dim-implies-col-perp2--col
      by (metis H1 H2 Perp-cases assms col-permutation-1)
  }
  hence P1: X = A  $\longrightarrow$  Col X Y Z by blast
  {
    assume P2: X  $\neq$  A
    hence P3: A B Perp X Y using perp-sym
      using H1 Perp-perm by blast
    have Col A B X
      by (simp add: H2)
    hence P4: A X Perp X Y using perp-col
      using P2 P3 by auto
    have P5: A X Perp X Z
      by (metis H1 H2 P2 Perp-perm col-trivial-3 perp-col0)
    have P6: Col Y Z X
    proof –
      have Q1: upper-dim-axiom
        by (simp add: assms)
      have Q2: Per Y X A
        using P4 Perp-cases perp-per-2 by blast
      have Per Z X A
        by (meson P5 Perp-cases Tarski-neutral-dimensionless-axioms perp-per-2)
      thus ?thesis using Q1 Q2 P2
        using upper-dim-implies-per2--col by blast
    qed
    hence Col X Y Z
      using Col-perm by blast
  }
thus ?thesis
  using P1 by blast
next
  assume T1:  $\neg$  Col A B X

```

```

obtain Y0 where Q3: Y0 PerpAt X Y A B
  using H1 Perp-def by blast
obtain Z0 where Q4: Z0 PerpAt X Z A B
  using Perp-def H1 by blast
have Q5: X Y0 Perp A B
proof -
  have R1: X ≠ Y0
    using Q3 T1 perp-in-col by blast
  have R2: X Y Perp A B
    by (simp add: H1A)
  thus ?thesis using R1
    using Q3 perp-col perp-in-col by blast
qed
have X Z0 Perp A B
  by (metis H1B Q4 T1 perp-col perp-in-col)
hence Q7: Y0 = Z0
  by (meson Q3 Q4 Q5 T1 Perp-perm Tarski-neutral-dimensionless-axioms
    l8-18-uniqueness perp-in-col)
have Col X Y Z
proof -
  have X ≠ Y0
    using Q5 perp-distinct by auto
  moreover have Col X Y0 Y
    using Q3 not-col-permutation-5 perp-in-col by blast
  moreover have Col X Y0 Z
    using Q4 Q7 col-permutation-5 perp-in-col by blast
  ultimately show ?thesis
    using col-transitivity-1 by blast
qed
thus ?thesis
  using l8-18-uniqueness Perp-cases T1 col-trivial-3 perp-col1 perp-in-col perp-not-col
  by blast
qed
}
thus ?thesis by blast
qed

```

```

lemma upper-dim-implies-not-two-sides-one-side-aux:
  assumes upper-dim-axiom
  shows  $\forall A B X Y PX. (A \neq B \wedge PX \neq A \wedge A B \text{ Perp } X PX \wedge \text{Col } A B PX \wedge$ 
     $\neg \text{Col } X A B \wedge \neg \text{Col } Y A B \wedge \neg A B \text{ TS } X Y) \longrightarrow A B \text{ OS } X Y$ 
proof -
  {
    fix A B X Y PX
    assume A ≠ B and PX ≠ A and A B Perp X PX and
      Col A B PX and ¬ Col X A B and
      ¬ Col Y A B and ¬ A B TS X Y
    hence  $\exists P T. PX A \text{ Perp } P PX \wedge \text{Col } PX A T \wedge \text{Bet } Y T P$ 
      using l8-21 by blast
    then obtain P T where PX A Perp P PX and Col PX A T and Bet Y T P
      by blast
    have Col P X PX
      using upper-dim-implies-col-perp2--col ⟨A B Perp X PX⟩
        ⟨Col A B PX⟩ ⟨PX A Perp P PX⟩ assms by blast
    have ¬ Col P A B
      using col-trivial-2 colx not-col-permutation-3 perp-not-col
        ⟨A ≠ B⟩ ⟨Col A B PX⟩ ⟨PX A Perp P PX⟩ by blast
    have PX A TS P Y
    proof -
      have ¬ Col P PX A
        using Col-cases ⟨PX A Perp P PX⟩ perp-not-col by blast
      moreover have ¬ Col Y PX A
        by (meson ⟨Col A B PX⟩ ⟨PX ≠ A⟩ ⟨¬ Col Y A B⟩ col-transitivity-2
          not-col-permutation-1)
      moreover have Col T PX A
        using Col-cases ⟨Col PX A T⟩ by blast
    }
  }

```

```

    moreover have  $Bet\ P\ T\ Y$ 
    using  $Bet-cases\ \langle Bet\ Y\ T\ P \rangle$  by blast
    ultimately show ?thesis
    using  $TS-def$  by blast
qed
have  $X \neq PX$ 
  using  $\langle A\ B\ Perp\ X\ PX \rangle\ perp-not-eq-2$  by auto
have  $P \neq X$ 
  using  $\langle A \neq B \rangle\ \langle Col\ A\ B\ PX \rangle\ \langle PX\ A\ TS\ P\ Y \rangle\ \langle \neg\ A\ B\ TS\ X\ Y \rangle$ 
  col-permutation-2 col-preserves-two-sides not-col-distincts by blast
have  $Bet\ X\ PX\ P \vee PX\ Out\ X\ P \vee \neg\ Col\ X\ PX\ P$ 
  using l6-4-2 by blast
hence  $PX\ A\ TS\ P\ X$ 
  using Out-cases  $TS-def\ bet--ts\ between-symmetry$ 
  col-permutation-1 col-preserves-two-sides col-trivial-2 l9-5
  by (metis  $\langle Col\ A\ B\ PX \rangle\ \langle Col\ P\ X\ PX \rangle\ \langle PX\ A\ TS\ P\ Y \rangle\ \langle X \neq PX \rangle$ 
     $\langle \neg\ A\ B\ TS\ X\ Y \rangle\ \langle \neg\ Col\ P\ A\ B \rangle$ )
hence  $A\ B\ OS\ X\ Y$ 
  using col2-os--os col-trivial-2 l9-2 l9-8-1 not-col-permutation-1
  by (meson  $\langle A \neq B \rangle\ \langle Col\ A\ B\ PX \rangle\ \langle PX\ A\ TS\ P\ Y \rangle$ )
}
thus ?thesis by blast
qed

lemma upper-dim-implies-not-two-sides-one-side:
  assumes upper-dim-axiom
  shows  $\forall\ A\ B\ X\ Y. (\neg\ Col\ X\ A\ B \wedge \neg\ Col\ Y\ A\ B \wedge \neg\ A\ B\ TS\ X\ Y) \longrightarrow A\ B\ OS\ X\ Y$ 
proof -
{
  fix  $A\ B\ X\ Y$ 
  assume  $\neg\ Col\ X\ A\ B$  and  $\neg\ Col\ Y\ A\ B$  and  $\neg\ A\ B\ TS\ X\ Y$ 
  have  $A \neq B$ 
    using  $\langle \neg\ Col\ Y\ A\ B \rangle\ col-trivial-2$  by blast
  obtain  $PX$  where  $Col\ A\ B\ PX$  and  $A\ B\ Perp\ X\ PX$ 
    using Col-cases l8-18-existence  $\langle \neg\ Col\ X\ A\ B \rangle$  by blast
  have  $A\ B\ OS\ X\ Y$ 
  proof cases
    assume  $PX = A$ 
    have  $B\ A\ OS\ X\ Y$ 
    proof -
      have  $B\ A\ Perp\ X\ A$ 
        using  $\langle A\ B\ Perp\ X\ PX \rangle\ \langle PX = A \rangle\ perp-left-comm$  by blast
      moreover have  $Col\ B\ A\ A$ 
        by (simp add: col-trivial-2)
      moreover have  $\neg\ Col\ X\ B\ A$ 
        using Col-cases  $\langle \neg\ Col\ X\ A\ B \rangle$  by blast
      moreover have  $\neg\ Col\ Y\ B\ A$ 
        using Col-cases  $\langle \neg\ Col\ Y\ A\ B \rangle$  by blast
      moreover have  $\neg\ B\ A\ TS\ X\ Y$ 
        using  $\langle \neg\ A\ B\ TS\ X\ Y \rangle\ invert-two-sides$  by blast
      ultimately show ?thesis
        by (metis  $\langle A \neq B \rangle\ assms\ upper-dim-implies-not-two-sides-one-side-aux$ )
    qed
  thus ?thesis
    by (simp add: invert-one-side)
  next
    assume  $PX \neq A$ 
    thus ?thesis
      using  $\langle A\ B\ Perp\ X\ PX \rangle\ \langle A \neq B \rangle\ \langle Col\ A\ B\ PX \rangle\ \langle \neg\ A\ B\ TS\ X\ Y \rangle$ 
       $\langle \neg\ Col\ X\ A\ B \rangle\ \langle \neg\ Col\ Y\ A\ B \rangle\ assms$ 
      upper-dim-implies-not-two-sides-one-side-aux by blast
  qed
}
thus ?thesis by blast
qed

```

lemma *upper-dim-implies-not-one-side-two-sides*:
assumes *upper-dim-axiom*
shows $\forall A B X Y. (\neg \text{Col } X A B \wedge \neg \text{Col } Y A B \wedge \neg A B \text{ OS } X Y) \longrightarrow A B \text{ TS } X Y$
using *assms upper-dim-implies-not-two-sides-one-side* **by** *blast*

lemma *upper-dim-implies-one-or-two-sides*:
assumes *upper-dim-axiom*
shows $\forall A B X Y. (\neg \text{Col } X A B \wedge \neg \text{Col } Y A B) \longrightarrow (A B \text{ TS } X Y \vee A B \text{ OS } X Y)$
using *assms upper-dim-implies-not-two-sides-one-side* **by** *blast*

lemma *upper-dim-implies-all-coplanar*:
assumes *upper-dim-axiom*
shows *all-coplanar-axiom*
using *all-coplanar-axiom-def assms upper-dim-axiom-def* **by** *auto*

lemma *all-coplanar-implies-upper-dim*:
assumes *all-coplanar-axiom*
shows *upper-dim-axiom*
using *all-coplanar-axiom-def assms upper-dim-axiom-def* **by** *auto*

lemma *all-coplanar-upper-dim*:
shows *all-coplanar-axiom* $\longleftrightarrow$ *upper-dim-axiom*
using *all-coplanar-implies-upper-dim upper-dim-implies-all-coplanar* **by** *auto*

lemma *upper-dim-stab*:
shows $\neg \neg \text{upper-dim-axiom} \longrightarrow \text{upper-dim-axiom}$
by *blast*

lemma *cop--cong-on-bisect*:
assumes *Coplanar A B X P* **and**
M Midpoint A B **and**
M PerpAt A B P M **and**
Cong X A X B
shows *Col M P X*
proof –
have $X = M \vee \neg \text{Col } A B X \wedge M \text{ PerpAt } X M A B$
using *assms(2) assms(3) assms(4) cong-commutativity cong-perp-or-mid*
perp-in-distinct **by** *blast*
moreover
{
assume $\neg \text{Col } A B X$ **and** *M PerpAt X M A B*
hence *X M Perp A B*
using *perp-in-perp* **by** *blast*
have *A B Perp P M*
using *assms(3) perp-in-perp* **by** *blast*
have *Col M A B*
by (*simp add: assms(2) midpoint-col*)
hence *Col M P X*
using *cop-perp2--col Perp-perm* $\langle A B \text{ Perp } P M \rangle \langle X M \text{ Perp } A B \rangle$ *assms(1)*
coplanar-perm-1 **by** *blast*
}
ultimately show *?thesis*
using *not-col-distincts* **by** *blast*
qed

lemma *cong-cop-mid-perp--col*:
assumes *Coplanar A B X P* **and**
Cong A X B X **and**
M Midpoint A B **and**
A B Perp P M
shows *Col M P X*
proof –
have *M PerpAt A B P M*
using *Col-perm assms(3) assms(4) bet-col l8-15-1 midpoint-bet* **by** *blast*
thus *?thesis*
using *assms(1) assms(2) assms(3) cop--cong-on-bisect not-cong-2143* **by** *blast*

qed

lemma *cop-image-in2--col*:

assumes *Coplanar* $A\ B\ P\ Q$ and

$M\ \text{ReflectLAt}\ P\ P'\ A\ B$ and

$M\ \text{ReflectLAt}\ Q\ Q'\ A\ B$

shows *Col* $M\ P\ Q$

proof –

have $P1: P\ P'\ \text{ReflectL}\ A\ B$

using *assms*(2) *image-in-is-image-spec* by auto

hence $P2: A\ B\ \text{Perp}\ P'\ P \vee P' = P$

using *ReflectL-def* by auto

have $P3: Q\ Q'\ \text{ReflectL}\ A\ B$

using *assms*(3) *image-in-is-image-spec* by blast

hence $P4: A\ B\ \text{Perp}\ Q'\ Q \vee Q' = Q$

using *ReflectL-def* by auto

{
 assume $S1: A\ B\ \text{Perp}\ P'\ P \wedge A\ B\ \text{Perp}\ Q'\ Q$

 {
 assume $T1: A = M$

 have $T2: \text{Per}\ B\ A\ P$

 by (metis $P1\ \text{Perp-perm}\ S1\ T1\ \text{assms}(2)\ \text{image-in-col}\ \text{is-image-is-image-spec}\ l10-14$
 $\text{perp-col1}\ \text{perp-distinct}\ \text{perp-per-1}\ \text{ts-distincts}$)

 have $T3: \text{Per}\ B\ A\ Q$

 by (metis $S1\ T1\ \text{assms}(3)\ \text{image-in-col}\ l8-5\ \text{perp-col1}\ \text{perp-per-1}\ \text{perp-right-comm}$)

 have $T4: \text{Coplanar}\ B\ P\ Q\ A$

 using *assms*(1) *ncoplanar-perm-18* by blast

 have $T5: B \neq A$

 using $S1\ \text{perp-distinct}$ by blast

 have $T6: \text{Per}\ P\ A\ B$

 by (simp add: $T2\ l8-2$)

 have $T7: \text{Per}\ Q\ A\ B$

 using *Per-cases* $T3$ by blast

 hence *Col* $P\ Q\ A$ using $T4\ T5\ T6$

 using *cop-per2--col* by blast

 hence *Col* $A\ P\ Q$

 using *not-col-permutation-1* by blast

 hence *Col* $M\ P\ Q$

 using $T1$ by blast

}
hence $S2: A = M \longrightarrow \text{Col}\ M\ P\ Q$ by blast

{
 assume $D0: A \neq M$

 have $D1: \text{Per}\ A\ M\ P$

 proof –

 have $E1: M\ \text{Midpoint}\ P\ P'$

 using *ReflectLAt-def* *assms*(2) $l7-2$ by blast

 have *Cong* $P\ A\ P'\ A$

 using $P1\ \text{col-trivial-3}\ \text{is-image-spec-col-cong}$ by blast

 hence *Cong* $A\ P\ A\ P'$

 using *Cong-perm* by blast

 thus ?thesis

 using $E1\ \text{Per-def}$ by blast

 qed

 have $D2: \text{Per}\ A\ M\ Q$

 proof –

 have $E2: M\ \text{Midpoint}\ Q\ Q'$

 using *ReflectLAt-def* *assms*(3) $l7-2$ by blast

 have *Cong* $A\ Q\ A\ Q'$

 using $P3\ \text{col-trivial-3}\ \text{cong-commutativity}\ \text{is-image-spec-col-cong}$ by blast

 thus ?thesis

 using $E2\ \text{Per-def}$ by blast

 qed

 have *Col* $P\ Q\ M$

 proof –

 have $W1: \text{Coplanar}\ P\ Q\ A\ B$

```

    using assms(1) ncoplanar-perm-16 by blast
  have W2:  $A \neq B$ 
    using S1 perp-distinct by blast
  have Col A B M
    using ReflectLAT-def assms(2) by blast
  hence Coplanar P Q A M
    using W1 W2 col2-cop--cop col-trivial-3 by blast
  hence V1: Coplanar A P Q M
    using ncoplanar-perm-8 by blast
  have V3: Per P M A
    by (simp add: D1 l8-2)
  have Per Q M A
    using D2 Per-perm by blast
  thus ?thesis
    using V1 D0 V3 cop-per2--col by blast
qed
hence Col M P Q
  using Col-perm by blast
}
hence  $A \neq M \longrightarrow \text{Col } M \ P \ Q$  by blast
hence Col M P Q
  using S2 by blast
}
hence P5:  $(A \ B \ \text{Perp } P' \ P \wedge A \ B \ \text{Perp } Q' \ Q) \longrightarrow \text{Col } M \ P \ Q$  by blast
have P6:  $(A \ B \ \text{Perp } P' \ P \wedge (Q' = Q)) \longrightarrow \text{Col } M \ P \ Q$ 
  using ReflectLAT-def assms(3) l7-3 not-col-distincts by blast
have P7:  $(P' = P \wedge A \ B \ \text{Perp } Q' \ Q) \longrightarrow \text{Col } M \ P \ Q$ 
  using ReflectLAT-def assms(2) l7-3 not-col-distincts by blast
have  $(P' = P \wedge Q' = Q) \longrightarrow \text{Col } M \ P \ Q$ 
  using ReflectLAT-def assms(3) col-trivial-3 l7-3 by blast
thus ?thesis
  using P2 P4 P5 P6 P7 by blast
qed

lemma l10-10-spec:
  assumes  $P' \ P \ \text{ReflectL } A \ B$  and
     $Q' \ Q \ \text{ReflectL } A \ B$ 
  shows Cong P Q  $P' \ Q'$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    using assms(1) assms(2) cong-reflexivity image-spec--eq by blast
next
  assume H1:  $A \neq B$ 
  obtain X where P1:  $X \ \text{Midpoint } P \ P' \wedge \text{Col } A \ B \ X$ 
    using ReflectL-def assms(1) by blast
  obtain Y where P2:  $Y \ \text{Midpoint } Q \ Q' \wedge \text{Col } A \ B \ Y$ 
    using ReflectL-def assms(2) by blast
  obtain Z where P3:  $Z \ \text{Midpoint } X \ Y$ 
    using midpoint-existence by blast
  have P4: Col A B Z
  proof cases
    assume  $X = Y$ 
    thus ?thesis
      by (metis P2 P3 midpoint-distinct-3)
  next
    assume  $X \neq Y$ 
    thus ?thesis
      by (metis P1 P2 P3 l6-21 midpoint-col not-col-distincts)
  qed
  obtain R where P5:  $Z \ \text{Midpoint } P \ R$ 
    using symmetric-point-construction by blast
  obtain R' where P6:  $Z \ \text{Midpoint } P' \ R'$ 
    using symmetric-point-construction by blast
  have P7:  $A \ B \ \text{Perp } P \ P' \vee P = P'$ 
    using ReflectL-def assms(1) by auto

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have P8:  $A \ B \ \text{Perp} \ Q \ Q' \vee Q = Q'$ 
  using ReflectL-def assms(2) by blast
{
  assume Q1:  $A \ B \ \text{Perp} \ P \ P' \wedge A \ B \ \text{Perp} \ Q \ Q'$ 
  have Q2:  $R \ R' \ \text{ReflectL} \ A \ B$ 
  proof -
    have P P' Reflect A B
      by (simp add: H1 assms(1) is-image-is-image-spec l10-4-spec)
    hence R R' Reflect A B
      using H1 P4 P5 P6 midpoint-preserves-image by blast
    thus ?thesis
      using H1 is-image-is-image-spec by blast
  qed
  have Q3:  $R \neq R'$ 
    using P5 P6 Q1 l7-9 perp-not-eq-2 by blast
  have Q4:  $Y \ \text{Midpoint} \ R \ R'$ 
    using P1 P3 P5 P6 symmetry-preserves-midpoint by blast
  have Q5:  $\text{Cong} \ Q' \ R' \ Q \ R$ 
    using P2 Q4 l7-13 by blast
  have Q6:  $\text{Cong} \ P' \ Z \ P \ Z$ 
    using P4 assms(1) is-image-spec-col-cong by auto
  have Q7:  $\text{Cong} \ Q' \ Z \ Q \ Z$ 
    using P4 assms(2) is-image-spec-col-cong by blast
  hence Cong P Q P' Q'
  proof -
    have S1:  $\text{Cong} \ R \ Z \ R' \ Z$ 
      using P5 P6 Q6 cong-symmetry l7-16 l7-3-2 by blast
    have R  $\neq$  Z
      using Q3 S1 cong-reverse-identity by blast
    thus ?thesis
      by (meson P5 P6 Q5 Q6 Q7 S1 between-symmetry five-segment midpoint-bet
        not-cong-2143 not-cong-3412)
  qed
}
hence P9:  $(A \ B \ \text{Perp} \ P \ P' \wedge A \ B \ \text{Perp} \ Q \ Q') \longrightarrow \text{Cong} \ P \ Q \ P' \ Q'$  by blast
have P10:  $(A \ B \ \text{Perp} \ P \ P' \wedge Q = Q') \longrightarrow \text{Cong} \ P \ Q \ P' \ Q'$ 
  using P2 l7-3 Tarski-neutral-dimensionless-axioms assms(1) cong-symmetry
  is-image-spec-col-cong by fastforce
have P11:  $(P = P' \wedge A \ B \ \text{Perp} \ Q \ Q') \longrightarrow \text{Cong} \ P \ Q \ P' \ Q'$ 
  using P1 l7-3 not-cong-4321 Tarski-neutral-dimensionless-axioms assms(2)
  is-image-spec-col-cong by fastforce
have  $(P = P' \wedge Q = Q') \longrightarrow \text{Cong} \ P \ Q \ P' \ Q'$ 
  using cong-reflexivity by blast
thus ?thesis
  using P10 P11 P7 P8 P9 by blast
qed

lemma l10-10:
  assumes P' P Reflect A B and
    Q' Q Reflect A B
  shows Cong P Q P' Q'
  using Reflect-def assms(1) assms(2) cong-4321 l10-10-spec l7-13 by auto

lemma image-preserves-bet:
  assumes A A' ReflectL X Y and
    B B' ReflectL X Y and
    C C' ReflectL X Y and
    Bet A B C
  shows Bet A' B' C'
  proof -
    have A B C Cong3 A' B' C'
      using Cong3-def assms(1) assms(2) assms(3) l10-10-spec l10-4-spec by blast
    thus ?thesis
      using assms(4) l4-6 by blast
  qed

```



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lemma image-gen-preserves-bet:
  assumes A A' Reflect X Y and
    B B' Reflect X Y and
    C C' Reflect X Y and
    Bet A B C
  shows Bet A' B' C'
proof cases
  assume X = Y
  thus ?thesis
    by (metis (full-types) assms(1) assms(2) assms(3) assms(4) image--midpoint l7-15 l7-2)
next
  assume X ≠ Y
  hence A A' ReflectL X Y
    using assms(1) is-image-is-image-spec by blast
  moreover have B B' ReflectL X Y
    using assms(2) is-image-is-image-spec ⟨X ≠ Y⟩ by blast
  moreover have C C' ReflectL X Y
    using assms(3) is-image-is-image-spec ⟨X ≠ Y⟩ by blast
  ultimately show ?thesis using image-preserves-bet
    using assms(4) by blast
qed

lemma image-preserves-col:
  assumes A A' ReflectL X Y and
    B B' ReflectL X Y and
    C C' ReflectL X Y and
    Col A B C
  shows Col A' B' C' using image-preserves-bet
  using Col-def assms(1) assms(2) assms(3) assms(4) by auto

lemma image-gen-preserves-col:
  assumes A A' Reflect X Y and
    B B' Reflect X Y and
    C C' Reflect X Y and
    Col A B C
  shows Col A' B' C'
  using Col-def assms(1) assms(2) assms(3) assms(4) image-gen-preserves-bet by auto

lemma image-gen-preserves-ncol:
  assumes A A' Reflect X Y and
    B B' Reflect X Y and
    C C' Reflect X Y and
    ¬ Col A B C
  shows ¬ Col A' B' C'
  using assms(1) assms(2) assms(3) assms(4) image-gen-preserves-col l10-4 by blast

lemma image-gen-preserves-inter:
  assumes A A' Reflect X Y and
    B B' Reflect X Y and
    C C' Reflect X Y and
    D D' Reflect X Y and
    ¬ Col A B C and
    C ≠ D and
    Col A B I and
    Col C D I and
    Col A' B' I' and
    Col C' D' I'
  shows I I' Reflect X Y
proof -
  obtain I0 where I I0 Reflect X Y
    using l10-6-existence by blast
  have I0 = I'
  proof -
    have ¬ Col A' B' C'
      by (meson assms(1) assms(2) assms(3) assms(5) image-gen-preserves-col l10-4)
    moreover have C' ≠ D'

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    using assms(3) assms(4) assms(6) l10-2-uniqueness by blast
  moreover have Col A' B' I0
    using ⟨I I0 Reflect X Y⟩ assms(1) assms(2) assms(7) image-gen-preserves-col by auto
  moreover have Col C' D' I0
    using ⟨I I0 Reflect X Y⟩ assms(3) assms(4) assms(8) image-gen-preserves-col by auto
  ultimately show ?thesis
    using assms(10) assms(9) l6-21 by blast
qed
thus ?thesis
  using ⟨I I0 Reflect X Y⟩ by blast
qed

lemma intersection-with-image-gen:
  assumes A A' Reflect X Y and
    B B' Reflect X Y and
    ¬ Col A B A' and
    Col A B C and
    Col A' B' C
  shows Col C X Y
proof -
  have C C Reflect X Y
  proof -
    have A' A Reflect X Y
      using assms(1) l10-4 by blast
    moreover have B' B Reflect X Y
      by (simp add: assms(2) l10-4)
    moreover have ¬ Col A' B' A
      by (meson image-gen-preserves-col assms(1) assms(2) assms(3) l10-4)
    moreover have A' ≠ B'
      using calculation(3) col-trivial-1 by blast
    ultimately show ?thesis
      using image-gen-preserves-inter assms(1) assms(2) assms(3) assms(4) assms(5) by blast
  qed
  thus ?thesis
    using l10-8 by blast
qed

lemma image-preserves-midpoint :
  assumes A A' ReflectL X Y and
    B B' ReflectL X Y and
    C C' ReflectL X Y and
    A Midpoint B C
  shows A' Midpoint B' C'
proof -
  have Bet B' A' C' using image-preserves-bet
    using assms(1) assms(2) assms(3) assms(4) midpoint-bet by auto
  moreover
  have Cong B' A' A' C'
  proof -
    let ?C = B
    let ?D = A
    have Cong B' A' ?C ?D
      using assms(1) assms(2) l10-10-spec by auto
    moreover have Cong ?C ?D A' C'
  proof -
    let ?C' = A
    let ?D' = C
    have Cong B A ?C' ?D'
      using Midpoint-def assms(4) by auto
    moreover have Cong ?C' ?D' A' C'
      using assms(1) assms(3) cong-4321 l10-10-spec by blast
    ultimately show ?thesis
      using cong-transitivity by blast
  qed
  qed
  ultimately show ?thesis
    using cong-transitivity by blast

```

```

qed
ultimately show ?thesis
  by (simp add: Midpoint-def)
qed

lemma image-spec-preserves-per:
  assumes  $A A' \text{ ReflectL } X Y$  and
     $B B' \text{ ReflectL } X Y$  and
     $C C' \text{ ReflectL } X Y$  and
     $\text{Per } A B C$ 
  shows  $\text{Per } A' B' C'$ 
proof cases
  assume  $X = Y$ 
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) image-spec--eq by blast
next
  assume  $P1: X \neq Y$ 
  obtain  $C1$  where  $P2: B \text{ Midpoint } C C1$ 
    using symmetric-point-construction by blast
  obtain  $C1'$  where  $P3: C1 C1' \text{ ReflectL } X Y$ 
    by (meson  $P1$  l10-6-existence-spec)
  hence  $P4: B' \text{ Midpoint } C' C1'$ 
    using  $P2$  assms(2) assms(3) image-preserves-midpoint by blast
  have  $\text{Cong } A' C' A' C1'$ 
  proof -
    have  $Q1: \text{Cong } A' C' A C$ 
      using assms(1) assms(3) l10-10-spec by auto
    have  $\text{Cong } A C A' C1'$ 
      by (metis  $P2 P3$  l10-10-spec assms(1) assms(4) cong-inner-transitivity cong-symmetry
        per-double-cong)
    thus ?thesis
      using  $Q1$  cong-transitivity by blast
  qed
  thus ?thesis
    using  $P4$  Per-def by blast
qed

lemma image-preserves-per:
  assumes  $A A' \text{ Reflect } X Y$  and
     $B B' \text{ Reflect } X Y$  and
     $C C' \text{ Reflect } X Y$  and
     $\text{Per } A B C$ 
  shows  $\text{Per } A' B' C'$ 
proof cases
  assume  $X = Y$ 
  thus ?thesis using midpoint-preserves-per
    using assms(1) assms(2) assms(3) assms(4) image--midpoint l7-2 by blast
next
  assume  $P1: X \neq Y$ 
  have  $P2: X \neq Y \wedge A A' \text{ ReflectL } X Y$ 
    using  $P1$  assms(1) is-image-is-image-spec by blast
  have  $P3: X \neq Y \wedge B B' \text{ ReflectL } X Y$ 
    using  $P1$  assms(2) is-image-is-image-spec by blast
  have  $P4: X \neq Y \wedge C C' \text{ ReflectL } X Y$ 
    using  $P1$  assms(3) is-image-is-image-spec by blast
  thus ?thesis using image-spec-preserves-per
    using  $P2 P3$  assms(4) by blast
qed

lemma l10-12:
  assumes  $\text{Per } A B C$  and
     $\text{Per } A' B' C'$  and
     $\text{Cong } A B A' B'$  and
     $\text{Cong } B C B' C'$ 
  shows  $\text{Cong } A C A' C'$ 
proof cases

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assume P1: B = C
hence B' = C'
  using assms(4) cong-reverse-identity by blast
thus ?thesis
  using P1 assms(3) by blast
next
assume P2: B ≠ C
have Cong A C A' C'
proof cases
  assume A = B
  thus ?thesis
    using assms(3) assms(4) cong-diff-3 by force
next
assume P3: A ≠ B
obtain X where P4: X Midpoint B B'
  using midpoint-existence by blast
obtain A1 where P5: X Midpoint A' A1
  using Mid-perm symmetric-point-construction by blast
obtain C1 where P6: X Midpoint C' C1
  using Mid-perm symmetric-point-construction by blast
have Q1: A' B' C' Cong3 A1 B C1
  using Cong3-def P4 P5 P6 l7-13 l7-2 by blast
have Q2: Per A1 B C1
  using assms(2) Q1 l8-10 by blast
have Q3: Cong A B A1 B
  by (metis Cong3-def Q1 cong-symmetry assms(3) cong-inner-transitivity)
have Q4: Cong B C B C1
  by (metis Cong3-def Q1 cong-symmetry assms(4) cong-inner-transitivity)
obtain Y where P7: Y Midpoint C C1
  using midpoint-existence by auto
hence R1: C1 C Reflect B Y using cong-midpoint--image
  using Q4 by blast
obtain A2 where R2: A1 A2 Reflect B Y
  using l10-6-existence by blast
have R3: Cong C A2 C1 A1
  using R1 R2 l10-10 by blast
have R5: B B Reflect B Y
  using image-triv by blast
have R6: Per A2 B C using image-preserves-per
  using Q2 R1 R2 image-triv by blast
have R7: Cong A B A2 B
  using l10-10 Cong-perm Q3 R2 cong-transitivity image-triv by blast
obtain Z where R7A: Z Midpoint A A2
  using midpoint-existence by blast
have Cong B A B A2
  using Cong-perm R7 by blast
hence T1: A2 A Reflect B Z using R7A cong-midpoint--image
  by blast
obtain C0 where T2: B Midpoint C C0
  using symmetric-point-construction by blast
have T3: Cong A C A C0
  using T2 assms(1) per-double-cong by blast
have T4: Cong A2 C A2 C0
  using R6 T2 per-double-cong by blast
have T5: C0 C Reflect B Z
proof -
  have C0 C Reflect Z B
proof cases
  assume A = A2
  thus ?thesis
    by (metis R7A T2 T3 cong-midpoint--image midpoint-distinct-3)
next
  assume A ≠ A2
  thus ?thesis using l4-17 cong-midpoint--image
    by (metis R7A T2 T3 T4 midpoint-col not-col-permutation-3)
qed

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    thus ?thesis
      using is-image-rev by blast
  qed
  have T6: Cong A C A2 C0
    using T1 T5 l10-10 by auto
  have R4: Cong A C A2 C
    by (metis T4 T6 cong-symmetry cong-inner-transitivity)
  hence Q5: Cong A C A1 C1
    by (meson R3 cong-inner-transitivity not-cong-3421)
  thus ?thesis
    using Cong3-def Q1 Q5 cong-symmetry cong-transitivity by blast
  qed
  thus ?thesis by blast
qed

lemma l10-16:
  assumes  $\neg \text{Col } A \ B \ C$  and
     $\neg \text{Col } A' \ B' \ P$  and
    Cong A B A' B'
  shows  $\exists \ C'. \ A \ B \ C \text{ Cong3 } A' \ B' \ C' \wedge A' \ B' \ OS \ P \ C'$ 
proof cases
  assume A = B
  thus ?thesis
    using assms(1) not-col-distincts by auto
next
  assume P1: A  $\neq$  B
  obtain X where P2: Col A B X  $\wedge$  A B Perp C X
    using assms(1) l8-18-existence by blast
  obtain X' where P3: A B X Cong3 A' B' X'
    using P2 assms(3) l4-14 by blast
  obtain Q where P4: A' B' Perp Q X'  $\wedge$  A' B' OS P Q
    using P2 P3 assms(2) l10-15 l4-13 by blast
  obtain C' where P5: X' Out C' Q  $\wedge$  Cong X' C' X C
    by (metis P2 P4 l6-11-existence perp-distinct)
  have P6: Cong A C A' C'
  proof cases
    assume A = X
    thus ?thesis
      by (metis Cong3-def P3 P5 cong-4321 cong-commutativity cong-diff-3)
  next
    assume A  $\neq$  X
    have P7: Per A X C
      using P2 col-trivial-3 l8-16-1 l8-2 by blast
    have P8: Per A' X' C'
  proof -
    have X' PerpAt A' X' X' C'
  proof -
    have Z1: A' X' Perp X' C'
  proof -
    have W1: X'  $\neq$  C'
      using P5 out-distinct by blast
    have W2: X' Q Perp A' B'
      using P4 Perp-perm by blast
    hence X' C' Perp A' B'
      by (metis P5 Perp-perm W1 col-trivial-3 not-col-permutation-5 out-col perp-col2-bis)
    thus ?thesis
      by (metis Cong3-def P2 P3 Perp-perm  $\langle A \neq X \rangle$  col-trivial-3 cong-identity
        l4-13 perp-col2-bis)
  qed
  have Z2: Col X' A' X'
    using not-col-distincts by blast
  have Col X' X' C'
    by (simp add: col-trivial-1)
  thus ?thesis
    by (simp add: Z1 Z2 l8-14-2-1b-bis)
  qed

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    thus ?thesis
      by (simp add: perp-in-per)
qed
have P9: Cong A X A' X'
  using Cong3-def P3 by auto
have Cong X C X' C'
  using Cong-perm P5 by blast
thus ?thesis using l10-12
  using P7 P8 P9 by blast
qed
have P10: Cong B C B' C'
proof cases
  assume B = X
  thus ?thesis
    by (metis Cong3-def P3 P5 cong-4321 cong-commutativity cong-diff-3)
next
  assume B ≠ X
  have Q1: Per B X C
    using P2 col-trivial-2 l8-16-1 l8-2 by blast
  have X' PerpAt B' X' X' C'
  proof -
    have Q2: B' X' Perp X' C'
    proof -
      have R1: B' ≠ X'
        using Cong3-def P3 ⟨B ≠ X⟩ cong-identity by blast
      have X' C' Perp B' A'
      proof -
        have S1: X' ≠ C'
          using Out-def P5 by blast
        have S2: X' Q Perp B' A'
          using P4 Perp-perm by blast
        have Col X' Q C'
          using Col-perm P5 out-col by blast
        thus ?thesis
          using S1 S2 perp-col by blast
      qed
      hence R2: B' A' Perp X' C'
        using Perp-perm by blast
      have R3: Col B' A' X'
        using Col-perm P2 P3 l4-13 by blast
      thus ?thesis
        using R1 R2 perp-col by blast
    qed
  qed
  hence Q3: Col X' B' X'
    by (simp add: col-trivial-3)
  have Col X' X' C'
    by (simp add: col-trivial-1)
  thus ?thesis using l8-14-2-1b-bis
    using Q2 Q3 by blast
qed
hence Q2: Per B' X' C'
  by (simp add: perp-in-per)
have Q3: Cong B X B' X'
  using Cong3-def P3 by blast
have Q4: Cong X C X' C'
  using P5 not-cong-3412 by blast
thus ?thesis
  using Q1 Q2 Q3 l10-12 by blast
qed
have P12: A' B' OS C' Q ⟷ X' Out C' Q ∧ ¬ Col A' B' C' using l9-19 l4-13
  by (meson P2 P3 P5 one-side-not-col123 out-one-side-1)
hence P13: A' B' OS C' Q using l4-13
  by (meson P2 P3 P4 P5 l6-6 one-side-not-col124 out-one-side-1)
thus ?thesis
  using Cong3-def P10 P4 P6 assms(3) one-side-symmetry one-side-transitivity by blast
qed

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```

lemma cong-cop-image--col:
  assumes  $P \neq P'$  and
     $P P' \text{ Reflect } A B$  and
     $\text{Cong } P X P' X$  and
     $\text{Coplanar } A B P X$ 
  shows  $\text{Col } A B X$ 
proof -
  have  $P1: (A \neq B \wedge P P' \text{ ReflectL } A B) \vee (A = B \wedge A \text{ Midpoint } P' P)$ 
    by (metis assms(2) image--midpoint is-image-is-image-spec)
  {
    assume  $Q1: A \neq B \wedge P P' \text{ ReflectL } A B$ 
    then obtain  $M$  where  $Q2: M \text{ Midpoint } P' P \wedge \text{Col } A B M$ 
      using ReflectL-def by blast
    have  $\text{Col } A B X$ 
    proof cases
      assume  $R1: A = M$ 
      have  $R2: P A \text{ Perp } A B$ 
      proof -
        have  $S1: P \neq A$ 
          using  $Q2 R1$  assms(1) midpoint-distinct-2 by blast
        have  $S2: P P' \text{ Perp } A B$ 
          using Perp-perm  $Q1 \text{ ReflectL-def}$  assms(1) by blast
        have  $\text{Col } P P' A$ 
          using  $Q2 R1$  midpoint-col not-col-permutation-3 by blast
        thus ?thesis using perp-col
          using  $S1 S2$  by blast
      qed
      have  $R3: \text{Per } P A B$ 
        by (simp add:  $R2$  perp-comm perp-per-1)
      hence  $R3A: \text{Per } B A P$  using l8-2
        by blast
      have  $A \text{ Midpoint } P P' \wedge \text{Cong } X P X P'$ 
        using Cong-cases  $Q2 R1$  assms(3) l7-2 by blast
      hence  $R4: \text{Per } X A P$ 
        using Per-def by blast
      have  $R5: \text{Coplanar } P B X A$ 
        using assms(4) ncoplanar-perm-20 by blast
      have  $P \neq A$ 
        using  $R2$  perp-not-eq-1 by auto
      thus ?thesis using  $R4 R5 R3A$ 
        using cop-per2--col not-col-permutation-1 by blast
    next
      assume  $T1: A \neq M$ 
      have  $T3: P \neq M$ 
        using  $Q2$  assms(1) l7-3-2 sym-preserve-diff by blast
      have  $T2: P M \text{ Perp } M A$ 
      proof -
        have  $T4: P P' \text{ Perp } M A$ 
          using Perp-perm  $Q1 Q2 \text{ ReflectL-def}$   $T1$  assms(1) col-trivial-3 perp-col0 by blast
        have  $\text{Col } P P' M$ 
          by (simp add: Col-perm  $Q2$  midpoint-col)
        thus ?thesis using  $T3 T4$  perp-col by blast
      qed
      hence  $M P \text{ Perp } A M$ 
        using perp-comm by blast
      hence  $M \text{ PerpAt } M P A M$ 
        using perp-perp-in by blast
      hence  $M \text{ PerpAt } P M M A$ 
        by (simp add: perp-in-comm)
      hence  $U1: \text{Per } P M A$ 
        by (simp add: perp-in-per)
      have  $U2: \text{Per } X M P$  using l7-2 cong-commutativity
        using Per-def  $Q2$  assms(3) by blast
      have  $\text{Col } A X M$ 
    proof -

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```

    have W2: Coplanar P A X M
    by (meson Q1 Q2 assms(4) col-cop2--cop coplanar-perm-13 ncop-distincts)
    have Per A M P
    by (simp add: U1 l8-2)
    thus ?thesis using cop-per2--col
    using U2 T3 W2 by blast
qed
thus ?thesis
using Q2 T1 col2--eq not-col-permutation-4 by blast
qed
}
hence P2: (A ≠ B ∧ P P' ReflectL A B) ⟶ Col A B X by blast
have (A = B ∧ A Midpoint P' P) ⟶ Col A B X
using col-trivial-1 by blast
thus ?thesis using P1 P2 by blast
qed

lemma cong-cop-per2-1:
  assumes A ≠ B and
    Per A B X and
    Per A B Y and
    Cong B X B Y and
    Coplanar A B X Y
  shows X = Y ∨ B Midpoint X Y
  by (meson Per-cases assms(1) assms(2) assms(3) assms(4) assms(5) cop-per2--col
    coplanar-perm-3 l7-20-bis not-col-permutation-5)

lemma cong-cop-per2:
  assumes A ≠ B and
    Per A B X and
    Per A B Y and
    Cong B X B Y and
    Coplanar A B X Y
  shows X = Y ∨ X Y ReflectL A B
proof -
  have X = Y ∨ B Midpoint X Y
  using assms(1) assms(2) assms(3) assms(4) assms(5) cong-cop-per2-1 by blast
  thus ?thesis
  by (metis Mid-perm Per-def Reflect-def assms(1) assms(3) cong-midpoint--image
    symmetric-point-uniqueness)
qed

lemma cong-cop-per2-gen:
  assumes A ≠ B and
    Per A B X and
    Per A B Y and
    Cong B X B Y and
    Coplanar A B X Y
  shows X = Y ∨ X Y Reflect A B
proof -
  have X = Y ∨ B Midpoint X Y
  using assms(1) assms(2) assms(3) assms(4) assms(5) cong-cop-per2-1 by blast
  thus ?thesis
  using assms(2) cong-midpoint--image l10-4 per-double-cong by blast
qed

lemma ex-perp-cop:
  assumes A ≠ B
  shows ∃ Q. A B Perp Q C ∧ Coplanar A B P Q
proof -
  {
    assume Col A B C ∧ Col A B P
    hence ∃ Q. A B Perp Q C ∧ Coplanar A B P Q
    using assms ex-ncol-cop l10-15 ncop--ncols by blast
  }
  hence T1: (Col A B C ∧ Col A B P) ⟶

```



```

( $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ ) by blast
{
  assume  $\neg \text{Col } A B C \wedge \text{Col } A B P$ 
  hence  $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ 
  by (metis Perp-cases ncop--ncols not-col-distincts perp-exists)
}
hence T2:  $(\neg \text{Col } A B C \wedge \text{Col } A B P) \longrightarrow$ 
( $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ ) by blast
{
  assume  $\text{Col } A B C \wedge \neg \text{Col } A B P$ 
  hence  $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ 
  using l10-15 os--coplanar by blast
}
hence T3:  $(\text{Col } A B C \wedge \neg \text{Col } A B P) \longrightarrow$ 
( $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ ) by blast
{
  assume  $\neg \text{Col } A B C \wedge \neg \text{Col } A B P$ 
  hence  $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ 
  using l8-18-existence ncop--ncols perp-right-comm by blast
}
hence  $(\neg \text{Col } A B C \wedge \neg \text{Col } A B P) \longrightarrow$ 
( $\exists Q. A B \text{ Perp } Q C \wedge \text{Coplanar } A B P Q$ ) by blast
thus ?thesis using T1 T2 T3 by blast
qed

```

lemma hilbert-s-version-of-pasch-aux:

```

assumes Coplanar A B C P and
   $\neg \text{Col } A I P$  and
   $\neg \text{Col } B C P$  and
  Bet B I C and
   $B \neq I$  and
   $I \neq C$  and
   $B \neq C$ 
shows  $\exists X. \text{Col } I P X \wedge$ 
 $((\text{Bet } A X B \wedge A \neq X \wedge X \neq B \wedge A \neq B) \vee (\text{Bet } A X C \wedge A \neq X \wedge X \neq C \wedge A \neq C))$ 
proof –
  have T1:  $I P \text{ TS } B C$ 
  using Col-perm assms(3) assms(4) assms(5) assms(6) bet--ts bet-col
  col-transitivity-1 by blast
  have T2: Coplanar A P B I
  using assms(1) assms(4) bet-cop--cop coplanar-perm-6 ncoplanar-perm-9 by blast
  have T3:  $I P \text{ TS } A B \vee I P \text{ TS } A C$ 
  by (meson T1 T2 TS-def assms(2) cop-nos--ts coplanar-perm-21 l9-2 l9-8-2)
  have T4:  $I P \text{ TS } A B \longrightarrow (\exists X. \text{Col } I P X \wedge$ 
 $((\text{Bet } A X B \wedge A \neq X \wedge X \neq B \wedge A \neq B) \vee (\text{Bet } A X C \wedge A \neq X \wedge X \neq C \wedge A \neq C)))$ 
  by (metis TS-def not-col-permutation-2 ts-distincts)
  have  $I P \text{ TS } A C \longrightarrow (\exists X. \text{Col } I P X \wedge$ 
 $((\text{Bet } A X B \wedge A \neq X \wedge X \neq B \wedge A \neq B) \vee (\text{Bet } A X C \wedge A \neq X \wedge X \neq C \wedge A \neq C)))$ 
  by (metis TS-def not-col-permutation-2 ts-distincts)
  thus ?thesis using T3 T4 by blast
qed

```

lemma hilbert-s-version-of-pasch:

```

assumes Coplanar A B C P and
   $\neg \text{Col } C Q P$  and
   $\neg \text{Col } A B P$  and
  BetS A Q B
shows  $\exists X. \text{Col } P Q X \wedge (\text{BetS } A X C \vee \text{BetS } B X C)$ 
proof –
  obtain X where  $\text{Col } Q P X \wedge$ 
 $(\text{Bet } C X A \wedge C \neq X \wedge X \neq A \wedge C \neq A \vee \text{Bet } C X B \wedge C \neq X \wedge X \neq B \wedge C \neq B)$ 
  using BetSEq assms(1) assms(2) assms(3) assms(4) coplanar-perm-12
  hilbert-s-version-of-pasch-aux by fastforce
  thus ?thesis
  by (metis BetS-def Bet-cases Col-perm)
qed

```

lemma *two-sides-cases*:
assumes $\neg \text{Col } PO \ A \ B$ **and**
 $PO \ P \ OS \ A \ B$
shows $PO \ A \ TS \ P \ B \vee PO \ B \ TS \ P \ A$
by (*meson* *assms*(1) *assms*(2) *cop-nts--os* *l9-31* *ncoplanar-perm-3* *not-col-permutation-4*
one-side-not-col124 *one-side-symmetry* *os--coplanar*)

lemma *not-par-two-sides*:
assumes $C \neq D$ **and**
 $Col \ A \ B \ I$ **and**
 $Col \ C \ D \ I$ **and**
 $\neg \text{Col } A \ B \ C$
shows $\exists \ X \ Y. \ Col \ C \ D \ X \wedge Col \ C \ D \ Y \wedge A \ B \ TS \ X \ Y$
proof –
obtain $pp :: 'p \Rightarrow 'p \Rightarrow 'p$ **where**
 $f1: \forall p \ pa. \text{Bet } p \ pa \ (pp \ p \ pa) \wedge pa \neq (pp \ p \ pa)$
by (*meson* *point-construction-different*)
hence $f2: \forall p \ pa \ pb \ pc. (Col \ pc \ pb \ p \vee \neg \text{Col } pc \ pb \ (pp \ p \ pa)) \vee \neg \text{Col } pc \ pb \ pa$
by (*meson* *Col-def* *colx*)
have $f3: \forall p \ pa. \text{Col } pa \ p \ pa$
by (*meson* *Col-def* *between-trivial*)
have $f4: \forall p \ pa. \text{Col } pa \ p \ p$
by (*meson* *Col-def* *between-trivial*)
have $f5: \text{Col } I \ D \ C$
by (*meson* *Col-perm* *assms*(3))
have $f6: \forall p \ pa. \text{Col } (pp \ pa \ p) \ p \ pa$
using $f4 \ f3 \ f2$ **by** *blast*
hence $f7: \forall p \ pa. \text{Col } pa \ (pp \ pa \ p) \ p$
by (*meson* *Col-perm*)
hence $f8: \forall p \ pa \ pb \ pc. (pc \ pb \ TS \ p \ (pp \ p \ pa) \vee \text{Col } pc \ pb \ p) \vee \neg \text{Col } pc \ pb \ pa$
using $f2 \ f1$ **by** (*meson* *l9-18*)
have $I = D \vee \text{Col } D \ (pp \ D \ I) \ C$
using $f7 \ f5 \ f3 \ colx$ **by** *blast*
hence $I = D \vee \text{Col } C \ D \ (pp \ D \ I)$
using *Col-perm* **by** *blast*
thus *?thesis*
using $f8 \ f6 \ f3$ **by** (*metis* *Col-perm* *assms*(2) *assms*(4))

qed

lemma *cop-not-par-other-side*:
assumes $C \neq D$ **and**
 $Col \ A \ B \ I$ **and**
 $Col \ C \ D \ I$ **and**
 $\neg \text{Col } A \ B \ C$ **and**
 $\neg \text{Col } A \ B \ P$ **and**
 $Coplanar \ A \ B \ C \ P$
shows $\exists \ Q. \text{Col } C \ D \ Q \wedge A \ B \ TS \ P \ Q$
proof –
obtain $X \ Y$ **where** $P1: \text{Col } C \ D \ X \wedge \text{Col } C \ D \ Y \wedge A \ B \ TS \ X \ Y$
using *assms*(1) *assms*(2) *assms*(3) *assms*(4) *not-par-two-sides* **by** *blast*
hence $Coplanar \ C \ A \ B \ X$
using *Coplanar-def* *assms*(1) *assms*(2) *assms*(3) *col-transitivity-1* **by** *blast*
hence $Coplanar \ A \ B \ P \ X$
using *assms*(4) *assms*(6) *col-permutation-3* *coplanar-trans-1* *ncoplanar-perm-2*
ncoplanar-perm-6 **by** *blast*
thus *?thesis*
by (*meson* *P1* *l9-8-2* *TS-def* *assms*(5) *cop-nts--os* *not-col-permutation-2* *one-side-symmetry*)

qed

lemma *cop-not-par-same-side*:
assumes $C \neq D$ **and**
 $Col \ A \ B \ I$ **and**
 $Col \ C \ D \ I$ **and**
 $\neg \text{Col } A \ B \ C$ **and**
 $\neg \text{Col } A \ B \ P$ **and**

Coplanar A B C P
shows $\exists Q. \text{Col } C D Q \wedge A B \text{ OS } P Q$
proof –
obtain $X Y$ **where** $P1: \text{Col } C D X \wedge \text{Col } C D Y \wedge A B \text{ TS } X Y$
using *assms(1) assms(2) assms(3) assms(4) not-par-two-sides* **by** *blast*
hence *Coplanar C A B X*
using *Coplanar-def assms(1) assms(2) assms(3) col-transitivity-1* **by** *blast*
hence *Coplanar A B P X*
using *assms(4) assms(6) col-permutation-1 coplanar-perm-2 coplanar-trans-1 ncoplanar-perm-14* **by** *blast*
thus *?thesis*
by (*meson P1 TS-def assms(5) cop-nts--os l9-2 l9-8-1 not-col-permutation-2*)
qed

lemma *perp-bisect-equiv-defA:*
assumes $P Q \text{ PerpBisect } A B$
shows $P Q \text{ PerpBisectBis } A B$
proof –
have $P1: A B \text{ ReflectL } P Q \wedge A \neq B$
using *Perp-bisect-def assms* **by** *auto*
then obtain X **where** $P2: X \text{ Midpoint } B A \wedge \text{Col } P Q X$
using *ReflectL-def* **by** *auto*
have $P Q \text{ Perp } B A \vee B = A$
using $P1 \text{ ReflectL-def}$ **by** *auto*
have $X \text{ Midpoint } A B$
using $P2 \text{ l7-2}$ **by** *blast*
moreover
have $X \text{ PerpAt } P Q A B$
using $P1 P2 \langle P Q \text{ Perp } B A \vee B = A \rangle \text{ calculation col-permutation-2 l8-14-2-1b-bis midpoint-col perp-right-comm}$ **by** *blast*
ultimately
show *?thesis*
using *Perp-bisect-bis-def* **by** *blast*
qed

lemma *perp-bisect-equiv-defB:*
assumes $P Q \text{ PerpBisectBis } A B$
shows $P Q \text{ PerpBisect } A B$
proof –
obtain I **where** $P1: I \text{ PerpAt } P Q A B \wedge I \text{ Midpoint } A B$
using *Perp-bisect-bis-def assms* **by** *blast*
hence $A \neq B$
using *perp-in-distinct* **by** *blast*
have $P Q \text{ Perp } B A \vee B = A$
using $\langle I \text{ PerpAt } P Q A B \wedge I \text{ Midpoint } A B \rangle \text{ perp-in-perp perp-right-comm}$ **by** *blast*
moreover
have $I \text{ Midpoint } B A$
by (*meson Mid-perm P1*)
moreover
have $\text{Col } P Q I$
using $P1$ **by** (*meson perp-in-col Tarski-neutral-dimensionless-axioms*)
ultimately
have $A B \text{ ReflectL } P Q \wedge A \neq B$
using *ReflectL-def* $\langle A \neq B \rangle$ **by** *blast*
thus *?thesis*
by (*simp add: Perp-bisect-def*)
qed

lemma *perp-bisect-equiv-def:*
shows $P Q \text{ PerpBisect } A B \longleftrightarrow P Q \text{ PerpBisectBis } A B$
using *perp-bisect-equiv-defA perp-bisect-equiv-defB* **by** *blast*

lemma *perp-bisect-sym-1:*
assumes $P Q \text{ PerpBisect } A B$
shows $Q P \text{ PerpBisect } A B$
proof –

```

have P Q PerpBisectBis A B
  by (simp add: assms perp-bisect-equiv-defA)
then obtain x where x PerpAt P Q A B  $\wedge$  x Midpoint A B
  using Perp-bisect-bis-def by blast
hence Q P PerpBisectBis A B
  using Perp-bisect-bis-def Perp-in-cases by blast
thus ?thesis
  by (simp add: perp-bisect-equiv-defB)
qed

```

```

lemma perp-bisect-sym-2:
  assumes P Q PerpBisect A B
  shows P Q PerpBisect B A
proof -
  have P Q PerpBisectBis A B
    by (simp add: assms perp-bisect-equiv-defA)
  then obtain x where x PerpAt P Q A B  $\wedge$  x Midpoint A B
    using Perp-bisect-bis-def by blast
  hence P Q PerpBisectBis B A
    using Perp-bisect-bis-def l7-2 perp-in-right-comm by blast
  thus ?thesis
    by (simp add: perp-bisect-equiv-defB)
qed

```

```

lemma perp-bisect-sym-3:
  assumes A B PerpBisect C D
  shows B A PerpBisect D C
  by (meson assms perp-bisect-sym-1 perp-bisect-sym-2)

```

```

lemma perp-bisect-perp:
  assumes P Q PerpBisect A B
  shows P Q Perp A B
proof -
  have P Q PerpBisectBis A B
    by (simp add: assms perp-bisect-equiv-defA)
  then obtain x where x PerpAt P Q A B  $\wedge$  x Midpoint A B
    using Perp-bisect-bis-def by blast
  thus ?thesis
    using perp-in-perp by blast
qed

```

```

lemma perp-bisect-cong-1:
  assumes P Q PerpBisect A B
  shows Cong A P B P
proof -
  have P Q PerpBisectBis A B
    using assms perp-bisect-equiv-def by auto
  then obtain I where I PerpAt P Q A B  $\wedge$  I Midpoint A B
    using Perp-bisect-bis-def by blast
  hence Cong P A P B
    using per-double-cong perp-in-per-1 by blast
  thus ?thesis
    using Cong-perm by blast
qed

```

```

lemma perp-bisect-cong-2:
  assumes P Q PerpBisect A B
  shows Cong A Q B Q
  using assms perp-bisect-cong-1 perp-bisect-sym-1 by blast

```

```

lemma perp-bisect-cong2:
  assumes P Q PerpBisect A B
  shows Cong A P B P  $\wedge$  Cong A Q B Q
  using assms perp-bisect-cong-1 perp-bisect-cong-2 by blast

```

```

lemma perp-bisect-cong:

```

```

assumes
  Cong A pO B pO and
  Cong B pO C pO
shows Cong A pO C pO
using assms(1) assms(2) cong-transitivity by blast

lemma cong-cop-perp-bisect:
assumes  $P \neq Q$  and
  A  $\neq$  B and
  Coplanar P Q A B and
  Cong A P B P and
  Cong A Q B Q
shows P Q PerpBisect A B
proof –
obtain I where I Midpoint A B
  using midpoint-existence by blast
moreover
have Bet A I B
  by (simp add: calculation midpoint-bet)
have Cong A I B I
  using Cong-perm  $\langle I \text{ Midpoint } A \ B \rangle$  midpoint-cong by blast
have I PerpAt P Q A B
proof –
  have Per P I A
  using Per-def assms(4) calculation not-cong-2143 by blast
  have A  $\neq$  I
  using  $\langle \text{Cong } A \ I \ B \ I \rangle$  assms(2) cong-reverse-identity by blast
  hence B  $\neq$  I
  using  $\langle \text{Cong } A \ I \ B \ I \rangle$  cong-identity by blast
  thus ?thesis
proof cases
  assume P = I
  thus ?thesis
  using assms(1) assms(2) assms(5) calculation cong-perp-or-mid perp-in-left-comm by blast
next
  assume P  $\neq$  I
  have Col I P Q
  proof –
    have Coplanar A P Q I
    using  $\langle \text{Bet } A \ I \ B \rangle$  assms(3) bet-cop--cop ncoplanar-perm-8 by blast
    moreover
    have Per Q I A
    using Per-def  $\langle I \text{ Midpoint } A \ B \rangle$  assms(5) not-cong-2143 by blast
    ultimately
    show ?thesis
    using  $\langle A \neq I \rangle$   $\langle \text{Per } P \ I \ A \rangle$  col-permutation-2 cop-per2--col by blast
  qed
  moreover
  have Col I A B
  using  $\langle \text{Bet } A \ I \ B \rangle$  bet-col not-col-permutation-4 by blast
  ultimately
  show ?thesis
  by (metis assms(1) assms(2) perp-in-col-perp-in  $\langle A \neq I \rangle$   $\langle \text{Bet } A \ I \ B \rangle$   $\langle P \neq I \rangle$ 
     $\langle \text{Per } P \ I \ A \rangle$  bet-col col-per-perp l8-15-1 l8-2 not-col-permutation-2)
  qed
qed
ultimately
show ?thesis
  using Perp-bisect-bis-def perp-bisect-equiv-defB by blast
qed

lemma cong-mid-perp-bisect:
assumes  $P \neq Q$  and
  A  $\neq$  B and
  Cong A P B P and
  Q Midpoint A B

```

```

shows  $P \ Q \ \text{PerpBisect } A \ B$ 
using Perp-bisect-bis-def assms(1) assms(2) assms(3) assms(4) cong-perp-or-mid
perp-bisect-equiv-defB by blast

lemma perp-bisect-is-on-perp-bisect:
  assumes  $P \ \text{IsOnPerpBisect } A \ B$  and
     $P \ \text{IsOnPerpBisect } B \ C$ 
  shows  $P \ \text{IsOnPerpBisect } A \ C$ 
  by (metis Is-on-perp-bisect-def cong-3421 assms(1) assms(2) cong-transitivity)

lemma perp-mid-perp-bisect:
  assumes  $C \ \text{Midpoint } A \ B$  and
     $C \ D \ \text{Perp } A \ B$ 
  shows  $C \ D \ \text{PerpBisect } A \ B$ 
proof –
  have  $C \ \text{PerpAt } C \ D \ A \ B$ 
  using assms(1) assms(2) l8-14-2-1b-bis midpoint-col not-col-distincts by blast
  hence  $C \ D \ \text{PerpBisectBis } A \ B$ 
  using assms(1) Perp-bisect-bis-def by blast
  thus ?thesis
  by (simp add: perp-bisect-equiv-defB)
qed

lemma cong-cop2-perp-bisect-col:
  assumes Coplanar  $A \ C \ D \ E$  and
    Coplanar  $B \ C \ D \ E$  and
    Cong  $C \ D \ C \ E$  and
     $A \ B \ \text{PerpBisect } D \ E$ 
  shows Col  $A \ B \ C$ 
proof –
  have Cong  $D \ A \ E \ A$ 
  using assms(4) perp-bisect-cong2 by blast
  have  $A \ B \ \text{PerpBisectBis } D \ E$ 
  by (simp add: assms(4) perp-bisect-equiv-defA)
  then obtain  $F$  where  $P1: F \ \text{PerpAt } A \ B \ D \ E \wedge F \ \text{Midpoint } D \ E$ 
  using Perp-bisect-bis-def by blast
  hence  $\text{Bet } D \ F \ E \wedge \text{Cong } D \ F \ F \ E$ 
  by (simp add: midpoint-bet midpoint-cong)
  have  $D \neq E$ 
  using  $P1$  perp-in-distinct by blast
  have  $\text{Col } A \ B \ F \wedge \text{Col } D \ E \ F$  using perp-in-col
  using  $P1$  by blast
  hence  $A \ B \ \text{Perp } D \ E$ 
  using  $P1$  perp-in-perp by blast
  {
    assume  $A \neq C$ 
    hence  $A \ C \ \text{Perp } D \ E$ 
    using  $\langle \text{Cong } D \ A \ E \ A \rangle \langle D \neq E \rangle$  assms(1) assms(3) cong-cop-perp-bisect not-cong-2143
    perp-bisect-perp by blast
    hence ?thesis
    using NCol-cases  $\langle A \ B \ \text{Perp } D \ E \rangle$  assms(2) cop-perp2--col coplanar-perm-23
    perp-right-comm by blast
  }
  thus ?thesis
  using col-trivial-3 by blast
qed

lemma perp-bisect-cop2-existence:
  assumes  $A \neq B$ 
  shows  $\exists P \ Q. P \ Q \ \text{PerpBisect } A \ B \wedge \text{Coplanar } A \ B \ C \ P \wedge \text{Coplanar } A \ B \ C \ Q$ 
proof –
  obtain  $M$  where  $P1: M \ \text{Midpoint } A \ B$ 
  using midpoint-existence by blast
  have  $\text{Bet } A \ M \ B$ 
  by (simp add: P1 midpoint-bet)
  obtain  $Q$  where  $P2: A \ B \ \text{Perp } Q \ M \wedge \text{Coplanar } A \ B \ C \ Q$ 

```

```

    using ex-perp-cop assms(1) by blast
have A B ReflectL M Q
proof -
  have M Midpoint B A
    using P1 l7-2 by blast
  moreover
  have Col M Q M
    by (simp add: col-trivial-3)
  moreover
  have M Q Perp B A  $\vee$  B = A
    by (simp add: P2 Perp-perm Tarski-neutral-dimensionless-axioms)
  ultimately
  show ?thesis
    using ReflectL-def col-trivial-3 by blast
qed
moreover
have Coplanar A B C M
  using  $\langle$ Bet A M B $\rangle$  bet-col ncop--ncol ncoplanar-perm-4 by blast
ultimately
show ?thesis
  using assms(1) P2 Perp-bisect-def by blast
qed

lemma perp-bisect-existence:
  assumes A  $\neq$  B
  shows  $\exists$  P Q. P Q PerpBisect A B
proof -
  obtain P Q where P Q PerpBisect A B  $\wedge$  Coplanar A B C P  $\wedge$  Coplanar A B C Q
    using perp-bisect-cop2-existence assms(1) by blast
  thus ?thesis by blast
qed

lemma perp-bisect-existence-cop:
  assumes A  $\neq$  B
  shows  $\exists$  P Q. P Q PerpBisect A B  $\wedge$  Coplanar A B C P  $\wedge$  Coplanar A B C Q
  using perp-bisect-cop2-existence assms(1) by blast

lemma l11-3:
  assumes A B C CongA D E F
  shows  $\exists$  A' C' D' F'. B Out A' A  $\wedge$  B Out C C'  $\wedge$ 
E Out D' D  $\wedge$  E Out F F'  $\wedge$ 
A' B C' Cong3 D' E F'
proof -
  obtain A' C' D' F' where Bet B A A' and Cong A A' E D and Bet B C C' and
    Cong C C' E F and Bet E D D' and Cong D D' B A and
    Bet E F F' and Cong F F' B C and Cong A' C' D' F'
    using CongA-def assms by auto
  hence A' B C' Cong3 D' E F'
    by (meson Cong3-def between-symmetry l2-11-b not-cong-1243 not-cong-4312)
  thus ?thesis
    by (metis CongA-def l6-6  $\langle$ Bet B A A' $\rangle$   $\langle$ Bet B C C' $\rangle$   $\langle$ Bet E D D' $\rangle$ 
 $\langle$ Bet E F F' $\rangle$  assms bet-out)
qed

lemma l11-aux:
  assumes B Out A A' and
    E Out D D' and
    Cong B A' E D' and
    Bet B A A0 and
    Bet E D D0 and
    Cong A A0 E D and
    Cong D D0 B A
  shows Cong B A0 E D0  $\wedge$  Cong A' A0 D' D0
proof -
  have Cong B A0 E D0
    by (meson Bet-cases assms(4) assms(5) assms(6) assms(7) l2-11-b

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    not-cong-1243 not-cong-4312)
  moreover
  have  $Bet\ E\ D\ D' \vee Bet\ E\ D'\ D$ 
    using Out-def assms(2) by auto
  moreover
  have  $Bet\ B\ A\ A' \vee Bet\ B\ A'\ A$ 
    using Out-def assms(1) by auto
  {
    assume  $Bet\ B\ A\ A' \wedge Bet\ E\ D'\ D$ 
    have  $Cong\ A0\ A'\ D0\ D'$ 
    proof -
      have  $Bet\ B\ A'\ A0$ 
      proof -
        have  $Bet\ E\ D'\ D0$ 
        proof -
          let  $?C = D$ 
          have  $Bet\ E\ D'\ ?C$ 
            using  $\langle Bet\ B\ A\ A' \wedge Bet\ E\ D'\ D \rangle$  by auto
          moreover have  $Bet\ E\ ?C\ D0$ 
            by (simp add: assms(5))
          ultimately show  $?thesis$ 
            using between-exchange4 by blast
        qed
      moreover have  $B\ Out\ A'\ A0$ 
        by (metis Out-def  $\langle Bet\ B\ A\ A' \wedge Bet\ E\ D'\ D \rangle$  assms(1) assms(4) bet-out l5-1)
      ultimately show  $?thesis$ 
        by (metis cong-preserves-bet  $\langle Cong\ B\ A0\ E\ D0 \rangle$  assms(3) cong-symmetry)
      qed
    hence  $Bet\ A0\ A'\ B$ 
      using Bet-cases by blast
    moreover have  $Bet\ D0\ D'\ E$ 
      using Bet-cases  $\langle Bet\ B\ A\ A' \wedge Bet\ E\ D'\ D \rangle$  assms(5) between-exchange4 by blast
    ultimately show  $?thesis$ 
      by (meson l4-3  $\langle Cong\ B\ A0\ E\ D0 \rangle$  assms(3) not-cong-2143)
    qed
  hence  $Cong\ A'\ A0\ D'\ D0$ 
    using not-cong-2143 by blast
  }
  moreover
  {
    assume  $Bet\ B\ A'\ A$ 
    have  $E\ Out\ D\ D0$ 
      using assms(2) assms(5) bet-out out-diff1 by blast
    have  $Cong\ A'\ A0\ D'\ D0$ 
      using assms(2) assms(3) assms(4) between-exchange4 cong-preserves-bet l4-3-1 l6-6 l6-7
      by (meson  $\langle Bet\ B\ A'\ A \rangle \langle Cong\ B\ A0\ E\ D0 \rangle \langle E\ Out\ D\ D0 \rangle$ )
  }
  moreover
  have  $Bet\ B\ A\ A' \longrightarrow Cong\ A'\ A0\ D'\ D0$ 
    by (metis (no-types, lifting) Out-cases  $\langle Cong\ B\ A0\ E\ D0 \rangle$  assms(1) assms(2) assms(3) assms(4) assms(5) bet-out l6-7 out-cong-cong out-diff1)
  have  $Bet\ E\ D\ D' \longrightarrow Cong\ A'\ A0\ D'\ D0$ 
    using  $\langle Bet\ B\ A\ A' \longrightarrow Cong\ A'\ A0\ D'\ D0 \rangle \langle Bet\ B\ A\ A' \vee Bet\ B\ A'\ A \rangle$  calculation(4) by blast
  moreover have  $Bet\ E\ D'\ D \longrightarrow Cong\ A'\ A0\ D'\ D0$ 
    using  $\langle Bet\ B\ A\ A' \vee Bet\ B\ A'\ A \rangle \langle Bet\ B\ A\ A' \longrightarrow Cong\ A'\ A0\ D'\ D0 \rangle$  calculation(4) by blast
  ultimately show  $?thesis$ 
    using  $\langle Cong\ B\ A0\ E\ D0 \rangle$  by blast
  qed

lemma l11-3-bis:
  assumes  $\exists\ A'\ C'\ D'\ F'$ .
  ( $B\ Out\ A'\ A \wedge B\ Out\ C'\ C \wedge E\ Out\ D'\ D \wedge E\ Out\ F'\ F \wedge A'\ B\ C'\ Cong3\ D'\ E\ F'$ )
  shows  $A\ B\ C\ CongA\ D\ E\ F$ 
  proof -
    obtain  $A'\ C'\ D'\ F'$  where  $B\ Out\ A'\ A$  and  $B\ Out\ C'\ C$  and  $E\ Out\ D'\ D$  and
       $E\ Out\ F'\ F$  and  $A'\ B\ C'\ Cong3\ D'\ E\ F'$ 

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using assms by blast
obtain A0 where Bet B A A0 and Cong A A0 E D
using segment-construction by blast
obtain C0 where Bet B C C0 and Cong C C0 E F
using segment-construction by blast
obtain D0 where Bet E D D0 and Cong D D0 B A
using segment-construction by blast
obtain F0 where Bet E F F0 and Cong F F0 B C
using segment-construction by blast
have A ≠ B ∧ C ≠ B ∧ D ≠ E ∧ F ≠ E
using assms out-diff2 by blast
moreover
have Cong A0 C0 D0 F0
proof –
have Cong B A0 E D0 ∧ Cong A' A0 D' D0
proof –
have B Out A A'
by (simp add: ⟨B Out A' A⟩ l6-6)
moreover have E Out D D'
using Out-cases ⟨E Out D' D⟩ by auto
moreover have Cong B A' E D'
using Cong3-def ⟨A' B C' Cong3 D' E F'⟩ not-cong-2143 by blast
ultimately show ?thesis
using l11-aux ⟨Bet B A A0⟩ ⟨Bet E D D0⟩ ⟨Cong A A0 E D⟩ ⟨Cong D D0 B A⟩ by blast
qed
have Cong B C0 E F0 ∧ Cong C' C0 F' F0
proof –
have Cong B C0 E F0
by (meson Bet-cases ⟨Bet B C C0⟩ ⟨Bet E F F0⟩ ⟨Cong C C0 E F⟩ ⟨Cong F F0 B C⟩
l2-11-b not-cong-1243 not-cong-4321)
moreover have Cong C' C0 F' F0
by (meson Cong3-def ⟨A' B C' Cong3 D' E F'⟩ ⟨B Out C' C⟩ ⟨Bet B C C0⟩ ⟨Bet E F F0⟩
⟨Cong C C0 E F⟩ ⟨Cong F F0 B C⟩ ⟨E Out F' F⟩ l11-aux l6-6)
ultimately show ?thesis
by blast
qed
have B A' A0 Cong3 E D' D0
using Cong3-def ⟨A' B C' Cong3 D' E F'⟩ ⟨Cong B A0 E D0 ∧ Cong A' A0 D' D0⟩
cong-3-swap by blast
have B C' C0 Cong3 E F' F0
using Cong3-def ⟨A' B C' Cong3 D' E F'⟩ ⟨Cong B C0 E F0 ∧ Cong C' C0 F' F0⟩ by blast
have Cong C0 A' F0 D'
proof –
have B C' C0 A' FSC E F' F0 D'
proof –
have Col B C' C0
by (metis Bet-cases Col-def Out-def ⟨B Out C' C⟩ ⟨Bet B C C0⟩ bet-out l6-7)
moreover have Cong B A' E D'
using Cong3-def ⟨B A' A0 Cong3 E D' D0⟩ by presburger
moreover have Cong C' A' F' D'
using Cong3-def ⟨A' B C' Cong3 D' E F'⟩ not-cong-2143 by blast
ultimately show ?thesis
by (simp add: FSC-def ⟨B C' C0 Cong3 E F' F0⟩)
qed
have B ≠ C'
using ⟨B Out C' C⟩ l6-3-1 by blast
thus ?thesis
using ⟨B C' C0 A' FSC E F' F0 D'⟩ l4-16 by blast
qed
hence B A' A0 C0 FSC E D' D0 F0
using FSC-def bet-out l6-7 not-cong-2143 out-col
by (meson ⟨A ≠ B ∧ C ≠ B ∧ D ≠ E ∧ F ≠ E⟩ ⟨B A' A0 Cong3 E D' D0⟩ ⟨B Out A' A⟩
⟨Bet B A A0⟩ ⟨Cong B C0 E F0 ∧ Cong C' C0 F' F0⟩)
moreover
have B ≠ A'
using ⟨B Out A' A⟩ out-distinct by blast

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ultimately show ?thesis
  using l4-16 by auto
qed
ultimately show ?thesis
  using CongA-def ⟨Bet B A A0⟩ ⟨Bet B C C0⟩ ⟨Bet E D D0⟩ ⟨Bet E F F0⟩ ⟨Cong A A0 E D⟩
    ⟨Cong C C0 E F⟩ ⟨Cong D D0 B A⟩ ⟨Cong F F0 B C⟩ by auto
qed

lemma l11-4-1:
  assumes A B C CongA D E F and

    B Out A' A and
    B Out C' C and
    E Out D' D and
    E Out F' F and
    Cong B A' E D' and
    Cong B C' E F'
  shows Cong A' C' D' F'
proof -
  obtain A0 C0 D0 F0 where B Out A0 A and B Out C C0 and E Out D0 D and
    E Out F F0 and A0 B C0 Cong3 D0 E F0
  using assms(1) l11-3 by blast
  have B Out A' A0
  using ⟨B Out A0 A⟩ assms(2) l6-6 l6-7 by blast
  have E Out D' D0
  using Out-cases ⟨E Out D0 D⟩ assms(4) l6-7 by blast
  have Cong A' A0 D' D0
  proof -
    have Cong B A0 E D0
    using Cong3-def Cong-cases ⟨A0 B C0 Cong3 D0 E F0⟩ by blast
    thus ?thesis
    using ⟨B Out A' A0⟩ ⟨E Out D' D0⟩ assms(6) out-cong-cong by blast
  qed
  have Cong A' C0 D' F0
  proof -
    have B A0 A' C0 FSC E D0 D' F0
    using Cong3-def Cong-perm FSC-def assms(6) not-col-permutation-5 out-col
    by (meson A0 B C0 Cong3 D0 E F0) ⟨B Out A' A0⟩ ⟨Cong A' A0 D' D0⟩
    thus ?thesis
    using l4-16 Tarski-neutral-dimensionless-axioms out-diff2 ⟨B Out A' A0⟩ by blast
  qed
  have B Out C' C0
  using ⟨B Out C C0⟩ assms(3) l6-7 by blast
  have E Out F' F0
  by (meson l6-7 ⟨E Out F F0⟩ assms(5))
  hence Cong C' C0 F' F0
  using Cong3-def out-cong-cong ⟨A0 B C0 Cong3 D0 E F0⟩ ⟨B Out C' C0⟩ assms(7) by fastforce
  hence B C0 C' A' FSC E F0 F' D'
  by (meson Col-def Cong3-def FSC-def Out-def ⟨A0 B C0 Cong3 D0 E F0⟩ ⟨B Out C' C0⟩
    ⟨Cong A' C0 D' F0⟩ assms(6) assms(7) between-symmetry not-cong-2143)
  hence Cong C' A' F' D'
  using ⟨B Out C' C0⟩ l4-16 out-diff2 by blast
  thus ?thesis
  using not-cong-2143 by blast
qed

lemma l11-4-2:
  assumes A ≠ B and
    C ≠ B and
    D ≠ E and
    F ≠ E and
    ∀ A' C' D' F'. (B Out A' A ∧ B Out C' C ∧ E Out D' D ∧ E Out F' F ∧ Cong B A' E D' ∧
    Cong B C' E F' ⟶ Cong A' C' D' F')
  shows A B C CongA D E F
proof -
  obtain A' where Bet B A A' and Cong A A' E D

```

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    using segment-construction by blast
  obtain C' where Bet B C C' and Cong C C' E F
    using segment-construction by blast
  obtain D' where Bet E D D' and Cong D D' B A
    using segment-construction by blast
  obtain F' where Bet E F F' and Cong F F' B C
    using segment-construction by blast
  have Cong A' B D' E
    by (meson Bet-cases ⟨Bet B A A'⟩ ⟨Bet E D D'⟩ ⟨Cong A A' E D⟩ ⟨Cong D D' B A⟩
        l2-11-b not-cong-2134 not-cong-3421)
  have Cong B C' E F'
    using between-symmetry cong-3421 cong-right-commutativity l2-11-b
    by (meson ⟨Bet B C C'⟩ ⟨Bet E F F'⟩ ⟨Cong C C' E F⟩ ⟨Cong F F' B C⟩)
  have B Out A' A ∧ B Out C' C ∧ E Out D' D ∧ E Out F' F ∧ A' B C' Cong3 D' E F'
  proof -
    have B Out A' A
      using ⟨Bet B A A'⟩ assms(1) bet-out l6-6 by presburger
    moreover have B Out C' C
      by (simp add: ⟨Bet B C C'⟩ assms(2) bet-out l6-6)
    moreover have E Out D' D
      by (simp add: ⟨Bet E D D'⟩ assms(3) bet-out l6-6)
    moreover have E Out F' F
      by (simp add: ⟨Bet E F F'⟩ assms(4) bet-out l6-6)
    moreover have A' B C' Cong3 D' E F'
      using Cong3-def Cong-cases ⟨Cong A' B D' E⟩ ⟨Cong B C' E F'⟩ assms(5)
      calculation(1) calculation(2) calculation(3) calculation(4) by blast
    ultimately show ?thesis
      by blast
  qed
  thus ?thesis
    using l11-3-bis by blast
qed

```

```

lemma conga-refl:
  assumes A ≠ B and
    C ≠ B
  shows A B C CongA A B C
  by (meson CongA-def assms(1) assms(2) cong-reflexivity segment-construction)

```

```

lemma conga-sym:
  assumes A B C CongA A' B' C'
  shows A' B' C' CongA A B C
proof -
  obtain A0 C0 D0 F0 where Bet B A A0 and Cong A A0 B' A' and Bet B C C0 and
    Cong C C0 B' C' and Bet B' A' D0 and Cong A' D0 B A and
    Bet B' C' F0 and Cong C' F0 B C and Cong A0 C0 D0 F0
    using CongA-def assms by auto
  hence ∃ p pa pb pc. Bet B' A' p ∧ Cong A' p B A ∧ Bet B' C' pa ∧ Cong C' pa B C ∧
    Bet B A pb ∧ Cong A pb B' A' ∧ Bet B C pc ∧ Cong C pc B' C' ∧
    Cong p pa pb pc
    using cong-symmetry Tarski-neutral-dimensionless-axioms by blast
  thus ?thesis
    using CongA-def assms by auto
qed

```

```

lemma l11-10:
  assumes A B C CongA D E F and
    B Out A' A and
    B Out C' C and
    E Out D' D and
    E Out F' F
  shows A' B C' CongA D' E F'
proof -
  have A' ≠ B
    using assms(2) out-distinct by auto
  moreover have C' ≠ B

```

using *Out-def* *assms*(3) **by** *force*
 moreover **have** $D' \neq E$
 using *Out-def* *assms*(4) **by** *blast*
 moreover **have** $F' \neq E$
 using *assms*(5) *out-diff1* **by** *auto*
 moreover **have** $\forall A'0 C'0 D'0 F'0. (B \text{ Out } A'0 A' \wedge B \text{ Out } C'0 C' \wedge E \text{ Out } D'0 D' \wedge E \text{ Out } F'0 F' \wedge \text{Cong } B A'0 E D'0 \wedge \text{Cong } B C'0 E F'0)$
 $\longrightarrow \text{Cong } A'0 C'0 D'0 F'0$
 by (*meson* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *l11-4-1* *l6-7*)
 ultimately **show** *?thesis*
 using *l11-4-2* **by** *blast*
qed

lemma *out2--conga*:
 assumes $B \text{ Out } A' A$ and
 $B \text{ Out } C' C$
 shows $A B C \text{ CongA } A' B C'$
proof –
 have $A B C \text{ CongA } A B C$
 using *assms*(1) *assms*(2) *conga-refl* *out-diff2* **by** *auto*
 moreover **have** $B \text{ Out } A A$
 using *assms*(1) *out-distinct* *out-trivial* **by** *blast*
 moreover **have** $B \text{ Out } C C$
 using *assms*(2) *out-distinct* *out-trivial* **by** *blast*
 ultimately **show** *?thesis*
 using *assms*(1) *assms*(2) *l11-10* **by** *blast*
qed

lemma *cong3-diff*:
 assumes $A \neq B$ and
 $A B C \text{ Cong3 } A' B' C'$
 shows $A' \neq B'$
 using *Cong3-def* *assms*(1) *assms*(2) *cong-diff* **by** *blast*

lemma *cong3-diff2*:
 assumes $B \neq C$ and
 $A B C \text{ Cong3 } A' B' C'$
 shows $B' \neq C'$
 using *Cong3-def* *assms*(1) *assms*(2) *cong-diff* **by** *blast*

lemma *cong3-conga*:
 assumes $A \neq B$ and
 $C \neq B$ and
 $A B C \text{ Cong3 } A' B' C'$
 shows $A B C \text{ CongA } A' B' C'$
 by (*metis* *assms*(1) *assms*(2) *assms*(3) *cong3-diff* *cong3-diff2* *l11-3-bis* *out-trivial*)

lemma *cong3-conga2*:
 assumes $A B C \text{ Cong3 } A' B' C'$ and
 $A B C \text{ CongA } A'' B'' C''$
 shows $A' B' C' \text{ CongA } A'' B'' C''$
proof –
 obtain $A0 C0 A2 C2$ **where** $Bet B A A0$ and $\text{Cong } A A0 B'' A''$ and $Bet B C C0$ and
 $\text{Cong } C C0 B'' C''$ and $Bet B'' A'' A2$ and $\text{Cong } A'' A2 B A$ and
 $Bet B'' C'' C2$ and $\text{Cong } C'' C2 B C$ and $\text{Cong } A0 C0 A2 C2$
 using *CongA-def* *assms*(2) **by** *auto*
 obtain $A1$ **where** $Bet B' A' A1$ and $\text{Cong } A' A1 B'' A''$
 using *segment-construction* **by** *blast*
 obtain $C1$ **where** $Bet B' C' C1$ and $\text{Cong } C' C1 B'' C''$
 using *segment-construction* **by** *blast*
 have $\text{Cong } B'' A'' A' A1$
 using *Cong-cases* $\langle \text{Cong } A' A1 B'' A'' \rangle$ **by** *blast*
 hence $\text{Cong } A A0 A' A1$
 by (*metis* *cong-transitivity* $\langle \text{Cong } A A0 B'' A'' \rangle$)
 have $\text{Cong } B A0 B' A1$
 using *Cong3-def* *assms*(1) *cong-commutativity* *l2-11-b*

by (meson $\langle \text{Bet } B \ A \ A0 \rangle \langle \text{Bet } B' \ A' \ A1 \rangle \langle \text{Cong } A \ A0 \ A' \ A1 \rangle$)
 have Cong C C0 C' C1
 using cong-inner-transitivity cong-symmetry $\langle \text{Cong } C \ C0 \ B'' \ C'' \rangle \langle \text{Cong } C' \ C1 \ B'' \ C'' \rangle$ by blast
 have B A A0 C FSC B' A' A1 C'
 using FSC-def Cong3-def assms(1) bet-col l4-3 $\langle \text{Bet } B \ A \ A0 \rangle \langle \text{Bet } B' \ A' \ A1 \rangle$
 $\langle \text{Cong } A \ A0 \ A' \ A1 \rangle \langle \text{Cong } B \ A0 \ B' \ A1 \rangle$ by presburger
 hence Cong A0 C A1 C'
 using CongA-def assms(2) l4-16 by force
 hence B C C0 A0 FSC B' C' C1 A1
 using Cong3-def FSC-def assms(1) bet-col cong-commutativity $\langle \text{Bet } B \ C \ C0 \rangle$
 $\langle \text{Bet } B' \ C' \ C1 \rangle \langle \text{Cong } B \ A0 \ B' \ A1 \rangle \langle \text{Cong } C \ C0 \ C' \ C1 \rangle$ l2-11-b by presburger
 have Cong B A B' A'
 using Cong3-def Cong-cases assms(1) by blast
 hence Cong A'' A2 B' A'
 using $\langle \text{Cong } A'' \ A2 \ B \ A \rangle$ cong-transitivity by blast
 have Cong B C B' C'
 using Cong3-def assms(1) by blast
 hence Cong C'' C2 B' C'
 using $\langle \text{Cong } C'' \ C2 \ B \ C \rangle$ cong-transitivity by blast
 have Cong C0 A0 C2 A2
 using $\langle \text{Cong } A0 \ C0 \ A2 \ C2 \rangle$ not-cong-2143 by blast
 have Cong C1 A1 A0 C0
 by (metis CongA-def l4-16 $\langle B \ C \ C0 \ A0 \ FSC \ B' \ C' \ C1 \ A1 \rangle$ assms(2) cong-3421)
 hence Cong A1 C1 A2 C2
 using Cong-cases $\langle \text{Cong } A0 \ C0 \ A2 \ C2 \rangle$ cong-transitivity by blast
 thus ?thesis
 by (metis CongA-def assms(1) assms(2) cong3-diff cong3-diff2 $\langle \text{Bet } B' \ A' \ A1 \rangle$
 $\langle \text{Bet } B' \ C' \ C1 \rangle \langle \text{Bet } B'' \ A'' \ A2 \rangle \langle \text{Bet } B'' \ C'' \ C2 \rangle \langle \text{Cong } A' \ A1 \ B'' \ A'' \rangle$
 $\langle \text{Cong } A'' \ A2 \ B' \ A' \rangle \langle \text{Cong } C' \ C1 \ B'' \ C'' \rangle \langle \text{Cong } C'' \ C2 \ B' \ C' \rangle$)
 qed

lemma conga-diff1:
 assumes A B C CongA A' B' C'
 shows $A \neq B$
 using CongA-def assms by blast

lemma conga-diff2:
 assumes A B C CongA A' B' C'
 shows $C \neq B$
 using CongA-def assms by blast

lemma conga-diff45:
 assumes A B C CongA A' B' C'
 shows $A' \neq B'$
 using CongA-def assms by blast

lemma conga-diff56:
 assumes A B C CongA A' B' C'
 shows $C' \neq B'$
 using CongA-def assms by blast

lemma conga-trans:
 assumes A B C CongA A' B' C' and
 $A' \ B' \ C' \ \text{CongA} \ A'' \ B'' \ C''$
 shows A B C CongA A'' B'' C''

proof –
 obtain A0 C0 A1 C1 where Bet B A A0 and Cong A A0 B' A' and Bet B C C0 and
 Cong C C0 B' C' and Bet B' A' A1 and Cong A' A1 B A and
 Bet B' C' C1 and Cong C' C1 B C and Cong A0 C0 A1 C1
 using CongA-def assms(1) by auto
 have $A'' \neq B''$
 using CongA-def assms(2) by auto
 have $C'' \neq B''$
 using CongA-def assms(2) by auto
 have A1 B' C1 CongA A'' B'' C''
 proof –

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have B' Out A1 A'
  using Out-def assms(2) bet-neq12--neq conga-diff1 ⟨Bet B' A' A1⟩ by metis
moreover have B' Out C1 C'
  using Out-def assms(1) bet-neq12--neq conga-diff56 ⟨Bet B' C' C1⟩ by metis
moreover have B'' Out A'' A''
  by (simp add: ⟨A'' ≠ B''⟩ out-trivial)
moreover have B'' Out C'' C''
  by (simp add: ⟨C'' ≠ B''⟩ out-trivial)
ultimately show ?thesis
  using assms(2) l11-10 by blast
qed
have A0 B C0 CongA A' B' C'
  using Out-cases conga-diff1 conga-diff2 conga-diff45 assms(1) bet-out
  conga-diff56 l11-10 l5-3
  by (meson ⟨Bet B A A0⟩ ⟨Bet B C C0⟩ ⟨Bet B' A' A1⟩ ⟨Bet B' C' C1⟩)
have Cong B A0 B' A1
  using between-symmetry conga-diff21 l2-11-b not-cong-1243
  by (meson ⟨Bet B A A0⟩ ⟨Bet B' A' A1⟩ ⟨Cong A A0 B' A'⟩ ⟨Cong A' A1 B A⟩)
have Cong B C0 B' C1
  using between-symmetry conga-diff21 l2-11-b not-cong-1243
  by (meson ⟨Bet B C C0⟩ ⟨Bet B' C' C1⟩ ⟨Cong C C0 B' C'⟩ ⟨Cong C' C1 B C⟩)
have A0 B C0 Cong3 A1 B' C1
  using Cong3-def ⟨Cong A0 C0 A1 C1⟩ ⟨Cong B A0 B' A1⟩ ⟨Cong B C0 B' C1⟩
  not-cong-2143 by blast
hence A0 B C0 CongA A'' B'' C''
  by (meson conga3-symmetry ⟨A1 B' C1 CongA A'' B'' C''⟩ conga3-conga2)
thus ?thesis
proof -
  have B Out A A0
    using CongA-def assms(1) bet-out ⟨Bet B A A0⟩ by presburger
  moreover have B Out C C0
    using CongA-def assms(1) bet-out ⟨Bet B C C0⟩ by presburger
  moreover have B'' Out A'' A''
    by (simp add: ⟨A'' ≠ B''⟩ out-trivial)
  moreover have B'' Out C'' C''
    using ⟨C'' ≠ B''⟩ out-trivial by auto
  ultimately show ?thesis
    using ⟨A0 B C0 CongA A'' B'' C''⟩ l11-10 by blast
qed
qed

lemma conga-pseudo-refl:
  assumes A ≠ B and
    C ≠ B
  shows A B C CongA C B A
  by (meson CongA-def assms(1) assms(2) between-trivial conga-pseudo-reflexivity
    segment-construction)

lemma conga-trivial-1:
  assumes A ≠ B and
    C ≠ D
  shows A B A CongA C D C
  by (meson CongA-def assms(1) assms(2) conga-trivial-identity segment-construction)

lemma l11-13:
  assumes A B C CongA D E F and
    Bet A B A' and
    A' ≠ B and
    Bet D E D' and
    D' ≠ E
  shows A' B C CongA D' E F
proof -
  obtain A'' C'' D'' F'' where Bet B A A'' and Cong A A'' E D and Bet B C C'' and
    Cong C C'' E F and Bet E D D'' and Cong D D'' B A and Bet E F F'' and
    Cong F F'' B C and Cong A'' C'' D'' F''
  using CongA-def assms(1) by auto

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obtain  $A0$  where  $Bet\ B\ A'\ A0$  and  $Cong\ A'\ A0\ E\ D'$ 
using segment-construction by blast
obtain  $D0$  where  $Bet\ E\ D'\ D0$  and  $Cong\ D'\ D0\ B\ A'$ 
using segment-construction by blast
have  $Cong\ A0\ C''\ D0\ F''$ 
proof –
  have  $A''\ B\ A0\ C''\ OFSC\ D''\ E\ D0\ F''$ 
proof –
  have  $Bet\ A''\ B\ A0$ 
proof –
  have  $Bet\ A''\ A\ B$ 
    by (simp add: Bet B A A'' between-symmetry)
  have  $Bet\ A\ B\ A0$ 
    using  $\langle Bet\ B\ A'\ A0 \rangle\ assms(2)\ assms(3)\ outer-transitivity-between$  by blast
  have  $A \neq B$ 
    using CongA-def assms(1) by blast
  thus ?thesis
    using  $\langle Bet\ A\ B\ A0 \rangle\ \langle Bet\ A''\ A\ B \rangle\ outer-transitivity-between2$  by blast
qed
moreover have  $Bet\ D''\ E\ D0$ 
  using Bet-perm outer-transitivity-between CongA-def
  by (metis Bet E D D'' Bet E D' D0 assms(1) assms(4) assms(5))
moreover have  $Cong\ A''\ B\ D''\ E$ 
  using between-symmetry cong-3421 cong-left-commutativity l2-11-b
  by (meson Bet B A A'' Bet E D D'' Cong A A'' E D Cong D D'' B A)
moreover have  $Cong\ B\ A0\ E\ D0$ 
  using between-symmetry cong-3421 cong-right-commutativity l2-11-b
  by (meson Bet B A' A0 Bet E D' D0 Cong A' A0 E D' Cong D' D0 B A')
moreover have  $Cong\ B\ C''\ E\ F''$ 
  using between-symmetry cong-3421 cong-right-commutativity l2-11-b
  by (meson Bet B C C'' Bet E F F'' Cong C C'' E F Cong F F'' B C)
ultimately show ?thesis
  by (simp add: OFSC-def Cong A'' C'' D'' F'')
qed
have  $A'' \neq B$ 
  using  $\langle Bet\ B\ A\ A'' \rangle\ assms(1)\ bet-neq23--neq\ conga-diff1$  by blast
thus ?thesis
  using  $\langle A''\ B\ A0\ C''\ OFSC\ D''\ E\ D0\ F'' \rangle\ five-segment-with-def$  by auto
qed
thus ?thesis
  using CongA-def Bet B A' A0 Bet B C C'' Bet E D' D0 Bet E F F''
   $\langle Cong\ A'\ A0\ E\ D' \rangle\ \langle Cong\ C\ C''\ E\ F \rangle\ \langle Cong\ D'\ D0\ B\ A' \rangle\ \langle Cong\ F\ F''\ B\ C \rangle$ 
  assms(1) assms(3) assms(5) by auto
qed

lemma conga-right-comm:
assumes  $A\ B\ C\ CongA\ D\ E\ F$ 
shows  $A\ B\ C\ CongA\ F\ E\ D$ 
by (metis conga-diff45 conga-trans assms conga-diff56 conga-pseudo-refl)

lemma conga-left-comm:
assumes  $A\ B\ C\ CongA\ D\ E\ F$ 
shows  $C\ B\ A\ CongA\ D\ E\ F$ 
by (meson assms conga-right-comm conga-sym)

lemma conga-comm:
assumes  $A\ B\ C\ CongA\ D\ E\ F$ 
shows  $C\ B\ A\ CongA\ F\ E\ D$ 
by (meson conga-left-comm conga-right-comm Tarski-neutral-dimensionless-axioms assms)

lemma conga-line:
assumes  $A \neq B$  and
   $B \neq C$  and
   $A' \neq B'$  and
   $B' \neq C'$ 
and  $Bet\ A\ B\ C$  and

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$Bet\ A'\ B'\ C'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
by (*metis Bet-cases assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)*
conga-trivial-1 l11-13)

lemma l11-14:
assumes $Bet\ A\ B\ A'$ **and**
 $A \neq B$ **and**
 $A' \neq B$ **and**
 $Bet\ C\ B\ C'$ **and**
 $B \neq C$ **and**
 $B \neq C'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
by (*metis Bet-perm assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)*
conga-pseudo-refl conga-right-comm l11-13)

lemma l11-16:
assumes $Per\ A\ B\ C$ **and**
 $A \neq B$ **and**
 $C \neq B$ **and**
 $Per\ A'\ B'\ C'$ **and**
 $A' \neq B'$ **and**
 $C' \neq B'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
proof –
obtain $C0$ **where** $Bet\ B\ C\ C0$ **and** $Cong\ C\ C0\ B'\ C'$
using *segment-construction* **by** *blast*
obtain $C1$ **where** $Bet\ B'\ C'\ C1$ **and** $Cong\ C'\ C1\ B\ C$
using *segment-construction* **by** *blast*
obtain $A0$ **where** $Bet\ B\ A\ A0$ **and** $Cong\ A\ A0\ B'\ A'$
using *segment-construction* **by** *blast*
obtain $A1$ **where** $Bet\ B'\ A'\ A1$ **and** $Cong\ A'\ A1\ B\ A$
using *segment-construction* **by** *blast*
have $Cong\ A0\ C0\ A1\ C1$
proof –
have $Per\ A0\ B\ C0$
using *l8-3 assms(1) assms(2) assms(3) bet-col per-col*
by (*metis* $\langle Bet\ B\ A\ A0 \rangle \langle Bet\ B\ C\ C0 \rangle$)
moreover have $Per\ A1\ B'\ C1$
using *l8-3 assms(4) assms(5) assms(6) bet-col per-col*
by (*metis* $\langle Bet\ B'\ A'\ A1 \rangle \langle Bet\ B'\ C'\ C1 \rangle$)
moreover have $Cong\ A0\ B\ A1\ B'$
using *between-symmetry conga-3421 conga-left-commutativity l2-11-b*
by (*meson* $\langle Bet\ B\ A\ A0 \rangle \langle Bet\ B'\ A'\ A1 \rangle \langle Cong\ A\ A0\ B'\ A' \rangle \langle Cong\ A'\ A1\ B\ A \rangle$)
moreover have $Cong\ B\ C0\ B'\ C1$
using *between-symmetry conga-3421 l2-11-b not-cong-1243*
by (*meson* $\langle Bet\ B\ C\ C0 \rangle \langle Bet\ B'\ C'\ C1 \rangle \langle Cong\ C\ C0\ B'\ C' \rangle \langle Cong\ C'\ C1\ B\ C \rangle$)
ultimately show *?thesis*
using *l10-12* **by** *auto*
qed
thus *?thesis*
using *CongA-def* $\langle Bet\ B\ A\ A0 \rangle \langle Bet\ B\ C\ C0 \rangle \langle Bet\ B'\ A'\ A1 \rangle \langle Bet\ B'\ C'\ C1 \rangle$
 $\langle Cong\ A\ A0\ B'\ A' \rangle \langle Cong\ A'\ A1\ B\ A \rangle \langle Cong\ C\ C0\ B'\ C' \rangle \langle Cong\ C'\ C1\ B\ C \rangle$
 $assms(2)\ assms(3)\ assms(5)\ assms(6)$ **by** *auto*
qed

lemma l11-17:
assumes $Per\ A\ B\ C$ **and**
 $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $Per\ A'\ B'\ C'$
proof –
obtain $A0\ C0\ A1\ C1$ **where** $Bet\ B\ A\ A0$ **and** $Cong\ A\ A0\ B'\ A'$ **and** $Bet\ B\ C\ C0$ **and**
 $Cong\ C\ C0\ B'\ C'$ **and** $Bet\ B'\ A'\ A1$ **and** $Cong\ A'\ A1\ B\ A$ **and** $Bet\ B'\ C'\ C1$ **and**
 $Cong\ C'\ C1\ B\ C$ **and** $Cong\ A0\ C0\ A1\ C1$
using *CongA-def assms(2)* **by** *auto*
have $Per\ A0\ B\ C0$


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proof –
  have  $B \neq C$ 
    using assms(2) conga-diff2 by blast
  moreover have  $\text{Per } A0 \ B \ C$ 
    using l8-2 assms(1) assms(2) bet-col conga-diff1 per-col
    by (metis  $\langle \text{Bet } B \ A \ A0 \rangle$ )
  moreover have  $\text{Col } B \ C \ C0$ 
    by (simp add:  $\langle \text{Bet } B \ C \ C0 \rangle$  bet-col)
  ultimately show ?thesis
    using per-col by blast
qed
have  $\text{Per } A1 \ B' \ C1$ 
proof –
  have  $A0 \ B \ C0 \ \text{Cong3 } A1 \ B' \ C1$ 
  proof –
    have  $\text{Cong } A0 \ B \ A1 \ B'$ 
      using Bet-cases Cong3-def l2-11-b not-cong-2134 not-cong-3421
      by (meson  $\langle \text{Bet } B \ A \ A0 \rangle \langle \text{Bet } B' \ A' \ A1 \rangle \langle \text{Cong } A \ A0 \ B' \ A' \rangle \langle \text{Cong } A' \ A1 \ B \ A \rangle$ )
    moreover have  $\text{Cong } B \ C0 \ B' \ C1$ 
      by (meson  $\langle \text{Bet } B \ C \ C0 \rangle \langle \text{Bet } B' \ C' \ C1 \rangle \langle \text{Cong } C \ C0 \ B' \ C' \rangle \langle \text{Cong } C' \ C1 \ B \ C \rangle$ 
        between-symmetry cong-3421 l2-11-b not-cong-1243)
    ultimately show ?thesis
      by (simp add: Cong3-def  $\langle \text{Cong } A0 \ C0 \ A1 \ C1 \rangle$ )
  qed
  thus ?thesis
    using  $\langle \text{Per } A0 \ B \ C0 \rangle$  l8-10 by blast
qed
have  $B' \neq C1$ 
  using assms(2) between-identity conga-diff56  $\langle \text{Bet } B' \ C' \ C1 \rangle$  by blast
moreover have  $\text{Per } A' \ B' \ C1$ 
proof –
  have  $B' \neq A1$ 
    using assms(2) between-identity conga-diff45  $\langle \text{Bet } B' \ A' \ A1 \rangle$  by blast
  moreover have  $\text{Col } B' \ A1 \ A'$ 
    using Bet-cases Col-def  $\langle \text{Bet } B' \ A' \ A1 \rangle$  by blast
  ultimately show ?thesis
    using l8-3  $\langle \text{Per } A1 \ B' \ C1 \rangle$  by blast
qed
moreover have  $\text{Col } B' \ C1 \ C'$ 
  using  $\langle \text{Bet } B' \ C' \ C1 \rangle$  bet-col1 between-trivial by blast
ultimately show ?thesis
  using per-col by blast
qed

lemma l11-18-1:
  assumes Bet  $C \ B \ D$  and
     $B \neq C$  and
     $B \neq D$  and
     $A \neq B$  and
     $\text{Per } A \ B \ C$ 
  shows  $A \ B \ C \ \text{CongA } A \ B \ D$ 
proof –
  have  $\text{Per } A \ B \ D$ 
    using Bet-cases Col-def assms(1) assms(2) assms(5) per-col by blast
  thus ?thesis
    using assms(2) assms(3) assms(4) assms(5) l11-16 by auto
qed

lemma l11-18-2:
  assumes Bet  $C \ B \ D$  and
     $A \ B \ C \ \text{CongA } A \ B \ D$ 
  shows  $\text{Per } A \ B \ C$ 
proof –
  obtain  $A0 \ C0 \ A1 \ D0$  where  $P1$ :  $\text{Bet } B \ A \ A0 \wedge \text{Cong } A \ A0 \ B \ A \wedge \text{Bet } B \ C \ C0 \wedge$ 
     $\text{Cong } C \ C0 \ B \ D \wedge \text{Bet } B \ A \ A1 \wedge \text{Cong } A \ A1 \ B \ A \wedge$ 
     $\text{Bet } B \ D \ D0 \wedge \text{Cong } D \ D0 \ B \ C \wedge \text{Cong } A0 \ C0 \ A1 \ D0$ 

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  using CongA-def assms(2) by auto
have P2: A0 = A1
  by (metis P1 assms(2) conga-diff45 construction-uniqueness)
have P3: Per A0 B C0
proof -
  have Q1: Bet C0 B D0
  proof -
    have R1: Bet C0 C B
    using P1 between-symmetry by blast
    have R2: Bet C B D0
  proof -
    have S1: Bet C B D
    by (simp add: assms(1))
    have S2: Bet B D D0
    using P1 by blast
    have B ≠ D
    using assms(2) conga-diff56 by blast
    thus ?thesis
    using S1 S2 outer-transitivity-between by blast
  qed
  have C ≠ B
  using assms(2) conga-diff2 by auto
  thus ?thesis
  using R1 R2 outer-transitivity-between2 by blast
  qed
  have Q2: Cong C0 B B D0
  by (meson P1 between-symmetry cong-3421 l2-11-b not-cong-1243)
  have Cong A0 C0 A0 D0
  using P1 P2 by blast
  thus ?thesis
  using Per-def Q1 Q2 midpoint-def by blast
qed
have P4: B ≠ C0
  using P1 assms(2) bet-neq12--neq conga-diff2 by blast
have P5: Per A B C0
  by (metis P1 P3 l8-3 assms(2) bet-col bet-col1 bet-neq21--neq col-transitivity-1
    conga-diff45)
have Col B C0 C using P1
  using NCol-cases bet-col by blast
thus ?thesis
  using P4 P5 per-col by blast
qed

lemma cong3-preserves-out:
  assumes A Out B C and
    A B C Cong3 A' B' C'
  shows A' Out B' C'
  by (meson assms(1) assms(2) col-permutation-4 cong3-symmetry cong-3-swap
    l4-13 l4-6 not-bet-and-out or-bet-out out-col)

lemma l11-21-a:
  assumes B Out A C and
    A B C CongA A' B' C'
  shows B' Out A' C'
proof -
  obtain A0 C0 A1 C1 where P1: Bet B A A0 ∧
    Cong A A0 B' A' ∧ Bet B C C0 ∧
    Cong C C0 B' C' ∧ Bet B' A' A1 ∧
    Cong A' A1 B A ∧ Bet B' C' C1 ∧
    Cong C' C1 B C ∧ Cong A0 C0 A1 C1
  using CongA-def assms(2) by auto
  have P2: B Out A0 C0
  by (metis P1 assms(1) bet-out l6-6 l6-7 out-diff1)
  have P3: B' Out A1 C1
  proof -
    have B A0 C0 Cong3 B' A1 C1

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    by (meson Cong3-def P1 between-symmetry cong-right-commutativity l2-11-b not-cong-4312)
  thus ?thesis
    using P2 cong3-preserves-out by blast
qed
thus ?thesis
  by (metis P1 assms(2) bet-out conga-diff45 conga-diff56 l6-6 l6-7)
qed

lemma l11-21-b:
  assumes B Out A C and
    B' Out A' C'
  shows A B C CongA A' B' C'
  using assms(1) assms(2) between-trivial2 conga-trivial-1 l11-10 out2-bet-out out-distinct
  by meson

lemma conga-cop--or-out-ts:
  assumes Coplanar A B C C' and
    A B C CongA A B C'
  shows B Out C C'  $\vee$  A B TS C C'
proof -
  obtain A0 C0 A1 C1 where P1: Bet B A A0  $\wedge$ 
    Cong A A0 B A  $\wedge$  Bet B C C0  $\wedge$ 
    Cong C C0 B C'  $\wedge$  Bet B A A1  $\wedge$ 
    Cong A A1 B A  $\wedge$  Bet B C' C1  $\wedge$ 
    Cong C' C1 B C  $\wedge$  Cong A0 C0 A1 C1
  using CongA-def assms(2) by auto
  have P2: A0 = A1 using P1
  by (metis assms(2) conga-diff1 construction-uniqueness)
  have B Out C C'  $\vee$  A B TS C C'
proof cases
  assume C0 = C1
  thus ?thesis
    by (metis P1 assms(2) bet2--out conga-diff2 conga-diff56)
next
  assume R1: C0  $\neq$  C1
  obtain M where R2: M Midpoint C0 C1
  using midpoint-existence by blast
  have R3: Cong B C0 B C1
  by (meson Bet-cases P1 l2-11-b not-cong-2134 not-cong-3421)
  have R3A: Cong A0 C0 A0 C1
  using P1 P2 by blast
  hence R4: Per A0 M C0 using R2
  using Per-def by blast
  have R5: Per B M C0
  using Per-def R2 R3 by auto
  hence R6: Per B M C1
  using R2 l8-4 by blast
  have R7: B  $\neq$  A0
  using P1 assms(2) bet-neq12--neq conga-diff45 by blast
  hence Cong A C0 A C1
  by (meson Col-perm P1 R3 R3A bet-col l4-17)
  hence R9: Per A M C0
  using Per-def R2 by blast
  hence R10: Per A M C1
  by (meson R2 l8-4 Tarski-neutral-dimensionless-axioms)
  have R11: Col B A M
proof -
  have S1: Coplanar C0 B A M
proof -
  have Coplanar B A C0 M
proof -
  have T1: Coplanar B A C0 C1
proof -
  have Coplanar A C0 B C'
proof -
  have Coplanar A C' B C0

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proof –
  have U1: Coplanar A C' B C
    by (simp add: assms(1) coplanar-perm-4)
  have U2: B ≠ C
    using assms(2) conga-diff2 by blast
  have Col B C C0
    by (simp add: P1 bet-col)
  thus ?thesis
    by (meson col-cop--cop Tarski-neutral-dimensionless-axioms U1 U2)
qed
thus ?thesis
  using ncoplanar-perm-5 by blast
qed
thus ?thesis
  by (metis P1 col-cop--cop assms(2) bet-col conga-diff56 coplanar-perm-12)
qed
have Col C0 C1 M
  using Col-perm R2 midpoint-col by blast
thus ?thesis
  using T1 R1 col-cop--cop by blast
qed
thus ?thesis
  using ncoplanar-perm-8 by blast
qed
have C0 ≠ M
  using R1 R2 midpoint-distinct-1 by blast
thus ?thesis
  using R5 R9 S1 cop-per2--col by blast
qed
have B Out C C' ∨ A B TS C C'
proof cases
  assume Q1: B = M
  have Q2: ¬ Col A B C
    by (metis Col-def P1 Q1 R9 assms(2) conga-diff1 conga-diff2 l6-16-1 l8-9
      not-bet-and-out out-trivial)
  have Q3: ¬ Col A B C'
    by (metis Col-def P1 Q1 R10 assms(2) conga-diff1 conga-diff56 l6-16-1 l8-9
      not-bet-and-out out-trivial)
  have Q4: Col B A B
    by (simp add: col-trivial-3)
  have Bet C B C'
proof –
  have S1: Bet C1 C' B
    using Bet-cases P1 by blast
  have Bet C1 B C
proof –
  have T1: Bet C0 C B
    using Bet-cases P1 by blast
  have Bet C0 B C1
    by (simp add: Q1 R2 midpoint-bet)
  thus ?thesis
    using T1 between-exchange3 between-symmetry by blast
qed
thus ?thesis
  using S1 between-exchange3 between-symmetry by blast
qed
thus ?thesis
  by (metis Q2 Q3 Q4 bet--ts col-permutation-4 invert-two-sides)
next
assume L1: B ≠ M
have L2: B M TS C0 C1
proof –
  have M1: ¬ Col C0 B M
    by (metis (no-types) Col-perm L1 R1 R2 R5 is-midpoint-id l8-9)
  have M2: ¬ Col C1 B M
    using Col-perm L1 R1 R2 R6 l8-9 midpoint-not-midpoint by blast

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have M3: Col M B M
  using col-trivial-3 by auto
have Bet C0 M C1
  by (simp add: R2 midpoint-bet)
thus ?thesis
  using M1 M2 M3 TS-def by blast
qed
have A B TS C C'
proof -
  have W2: A B TS C C1
  proof -
    have V1: A B TS C0 C1
    using L2 P1 R11 R7 col-two-sides cong-diff invert-two-sides
      not-col-permutation-5 by blast
    have B Out C0 C
    using L2 Out-def P1 TS-def assms(2) col-trivial-1 conga-diff2 by auto
    thus ?thesis
    using V1 col-trivial-3 l9-5 by blast
  qed
  hence W1: A B TS C' C
  proof -
    have Z1: A B TS C1 C
    by (simp add: W2 l9-2)
    have Z2: Col B A B
    using not-col-distincts by blast
    have B Out C1 C'
    using L2 Out-def P1 TS-def assms(2) col-trivial-1 conga-diff56 by auto
    thus ?thesis
    using Z1 Z2 l9-5 by blast
  qed
  thus ?thesis
  by (simp add: l9-2)
qed
thus ?thesis by blast
qed
thus ?thesis by blast
qed
thus ?thesis by blast
qed

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lemma conga-os--out:
assumes $A B C$ CongA $A B C'$ **and**
 $A B OS C C'$
shows $B Out C C'$
using $assms(1)$ $assms(2)$ conga-cop--or-out-ts l9-9 os--coplanar **by** blast

lemma cong2-conga-cong:
assumes $A B C$ CongA $A' B' C'$ **and**
 $Cong A B A' B'$ **and**
 $Cong B C B' C'$
shows $Cong A C A' C'$
using $assms(1)$ $assms(2)$ $assms(3)$ cong-4321 l11-3 l11-4-1 not-cong-3412 out-distinct out-trivial
by meson

lemma angle-construction-1:
assumes $\neg Col A B C$ **and**
 $\neg Col A' B' P$
shows $\exists C'. (A B C CongA A' B' C' \wedge A' B' OS C' P)$
proof -
obtain $C0$ **where** $P1: Col B A C0 \wedge B A Perp C C0$
using $assms(1)$ col-permutation-4 l8-18-existence **by** blast
have $\exists C'. (A B C CongA A' B' C' \wedge A' B' OS C' P)$
proof cases
assume $P1A: B = C0$
obtain C' **where** $P2: Per C' B' A' \wedge Cong C' B' C B \wedge A' B' OS C' P$
by (metis $assms(1)$ $assms(2)$ col-trivial-1 col-trivial-2 ex-per-cong)

```

have P3: A B C CongA A' B' C'
  by (metis P1 P2 l8-2 os-distincts P1A assms(1) l11-16 not-col-distincts perp-per-1)
thus ?thesis using P2 by blast
next
assume P4: B ≠ C0
have ∃ C'. (A B C CongA A' B' C' ∧ A' B' OS C' P)
proof cases
  assume R1: B Out A C0
  obtain C0' where R2: B' Out A' C0' ∧ Cong B' C0' B C0
    by (metis P4 assms(2) col-trivial-1 segment-construction-3)
  have ∃ C'. Per C' C0' B' ∧ Cong C' C0' C C0 ∧ B' C0' OS C' P
  proof -
    have R4: B' ≠ C0'
      using Out-def R2 by auto
    have R5: C ≠ C0
      using P1 perp-distinct by blast
    have R6: Col B' C0' C0'
      by (simp add: col-trivial-2)
    have ¬ Col B' C0' P
      using NCol-cases R2 R4 assms(2) col-transitivity-1 out-col by blast
    hence ∃ C'. Per C' C0' B' ∧ Cong C' C0' C C0 ∧ B' C0' OS C' P
      using R4 R5 R6 ex-per-cong by blast
    thus ?thesis by auto
  qed
then obtain C' where R7: Per C' C0' B' ∧ Cong C' C0' C C0 ∧ B' C0' OS C' P
  by auto
hence R8: C0 B C Cong3 C0' B' C'
  by (meson Cong3-def P1 R2 col-trivial-2 l10-12 l8-16-1 not-col-permutation-2
    not-cong-2143 not-cong-4321)
have R9: A B C CongA A' B' C'
proof -
  have S1: C0 B C CongA C0' B' C'
    by (metis P4 R8 assms(1) cong3-conga not-col-distincts)
  have S3: B Out C C
    using assms(1) not-col-distincts out-trivial by force
  have B' ≠ C'
    using R8 assms(1) cong3-diff2 not-col-distincts by blast
  hence B' Out C' C'
    using out-trivial by auto
  thus ?thesis
    using S1 R1 S3 R2 l11-10 by blast
qed
have B' A' OS C' P
proof -
  have T1: Col B' C0' A'
    by (meson NCol-cases R2 out-col Tarski-neutral-dimensionless-axioms)
  have B' ≠ A'
    using assms(2) col-trivial-1 by auto
  thus ?thesis
    using T1 R7 col-one-side by blast
qed
hence A' B' OS C' P
  by (simp add: invert-one-side)
thus ?thesis
  using R9 by blast
next
assume U1: ¬ B Out A C0
hence U2: Bet A B C0
  using NCol-perm P1 or-bet-out by blast
obtain C0' where U3: Bet A' B' C0' ∧ Cong B' C0' B C0
  using segment-construction by blast
have U4: ∃ C'. Per C' C0' B' ∧ Cong C' C0' C C0 ∧ B' C0' OS C' P
proof -
  have V2: C ≠ C0
    using Col-cases P1 assms(1) by blast
  have B' ≠ C0'

```

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    using P4 U3 cong-diff-3 by blast
  hence  $\neg \text{Col } B' C0' P$ 
    using Col-def U3 assms(2) col-transitivity-1 by blast
  thus ?thesis using ex-per-cong
    using V2 not-col-distincts by blast
qed
then obtain C' where U5:  $\text{Per } C' C0' B' \wedge \text{Cong } C' C0' C C0 \wedge B' C0' \text{ OS } C' P$ 
  by blast
have U98:  $A B C \text{ CongA } A' B' C'$ 
proof -
  have X1:  $C0 B C \text{ Cong3 } C0' B' C'$ 
  proof -
    have X2:  $\text{Cong } C0 B C0' B'$ 
      using Cong-cases U3 by blast
    have X3:  $\text{Cong } C0 C C0' C'$ 
      using U5 not-cong-4321 by blast
    have Cong B C B' C'
      proof -
        have Y1:  $\text{Per } C C0 B$ 
          using P1 col-trivial-3 l8-16-1 by blast
        have Cong C C0 C' C0'
          using U5 not-cong-3412 by blast
        thus ?thesis
          using Cong-perm Y1 U5 X2 l10-12 by blast
      proof -
        thus ?thesis
          by (simp add: Cong3-def X2 X3)
      qed
    have X22:  $\text{Bet } C0 B A$ 
      using U2 between-symmetry by blast
    have X24:  $\text{Bet } C0' B' A'$ 
      using Bet-cases U3 by blast
    have  $A' \neq B'$ 
      using assms(2) not-col-distincts by blast
    thus ?thesis
      by (metis P4 X1 X22 X24 assms(1) cong3-conga l11-13 not-col-distincts)
  qed
  have  $A' B' \text{ OS } C' P$ 
  proof -
    have  $B' A' \text{ OS } C' P$ 
    proof -
      have W1:  $\text{Col } B' C0' A'$ 
        by (simp add: Col-def U3)
      have  $B' \neq A'$ 
        using assms(2) not-col-distincts by blast
      thus ?thesis
        using W1 U5 col-one-side by blast
    qed
    thus ?thesis
      by (simp add: invert-one-side)
  qed
  thus ?thesis
    using U98 by blast
qed
thus ?thesis by auto
qed
thus ?thesis by auto
qed

lemma angle-construction-2:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $\neg \text{Col } A' B' P$ 
  shows  $\exists C'. (A B C \text{ CongA } A' B' C' \wedge (A' B' \text{ OS } C' P \vee \text{Col } A' B' C'))$ 
  by (metis Col-def angle-construction-1 assms(1) assms(2) assms(3) col-trivial-3
    conga-line l11-21-b or-bet-out out-trivial point-construction-different)

```

lemma *ex-conga-ts*:
 assumes $\neg \text{Col } A \ B \ C$ and
 $\neg \text{Col } A' \ B' \ P$
 shows $\exists \ C'. \ A \ B \ C \text{ CongA } A' \ B' \ C' \wedge A' \ B' \ \text{TS } C' \ P$
 proof –
 obtain P' where $P1: A' \text{ Midpoint } P \ P'$
 using *symmetric-point-construction* by blast
 have $P2: \neg \text{Col } A' \ B' \ P'$
 by (metis $P1$ *assms*(2) *col-transitivity-1* *midpoint-col* *midpoint-distinct-2* *not-col-distincts*)
 obtain C' where $P3:$
 $A \ B \ C \text{ CongA } A' \ B' \ C' \wedge A' \ B' \ \text{OS } C' \ P'$
 using $P2$ *angle-construction-1* *assms*(1) by blast
 have $A' \ B' \ \text{TS } P' \ P$
 using $P1 \ P2$ *assms*(2) *bet--ts l9-2* *midpoint-bet* *not-col-distincts* by auto
 thus ?thesis
 using $P3$ *l9-8-2 one-side-symmetry* by blast
 qed

lemma *l11-15*:
 assumes $\neg \text{Col } A \ B \ C$ and
 $\neg \text{Col } D \ E \ P$
 shows
 $\exists \ F. (A \ B \ C \text{ CongA } D \ E \ F \wedge E \ D \ \text{OS } F \ P) \wedge$
 $(\forall \ F1 \ F2. ((A \ B \ C \text{ CongA } D \ E \ F1 \wedge E \ D \ \text{OS } F1 \ P) \wedge$
 $(A \ B \ C \text{ CongA } D \ E \ F2 \wedge E \ D \ \text{OS } F2 \ P))$
 $\longrightarrow E \ \text{Out } F1 \ F2)$
 proof –
 obtain F where $P1: A \ B \ C \text{ CongA } D \ E \ F \wedge D \ E \ \text{OS } F \ P$
 using *angle-construction-1* *assms*(1) *assms*(2) by blast
 hence $P2: A \ B \ C \text{ CongA } D \ E \ F \wedge E \ D \ \text{OS } F \ P$
 using *invert-one-side* by blast
 have $(\forall \ F1 \ F2. ((A \ B \ C \text{ CongA } D \ E \ F1 \wedge E \ D \ \text{OS } F1 \ P) \wedge$
 $(A \ B \ C \text{ CongA } D \ E \ F2 \wedge E \ D \ \text{OS } F2 \ P)) \longrightarrow E \ \text{Out } F1 \ F2)$
 proof –
 {
 fix $F1 \ F2$
 assume $P3: ((A \ B \ C \text{ CongA } D \ E \ F1 \wedge E \ D \ \text{OS } F1 \ P) \wedge$
 $(A \ B \ C \text{ CongA } D \ E \ F2 \wedge E \ D \ \text{OS } F2 \ P))$
 hence $P4: A \ B \ C \text{ CongA } D \ E \ F1$ by simp
 have $P5: E \ D \ \text{OS } F1 \ P$ using $P3$ by simp
 have $P6: A \ B \ C \text{ CongA } D \ E \ F2$ using $P3$ by simp
 have $P7: E \ D \ \text{OS } F2 \ P$ using $P3$ by simp
 have $P8: D \ E \ F1 \text{ CongA } D \ E \ F2$
 using $P4$ *conga-sym* $P6$ *conga-trans* by blast
 have $D \ E \ \text{OS } F1 \ F2$
 using $P5 \ P7$ *invert-one-side* *one-side-symmetry* *one-side-transitivity* by blast
 hence $E \ \text{Out } F1 \ F2$ using $P8$ *conga-os--out*
 by (meson $P3$ *conga-sym* *conga-trans*)
 }
 thus ?thesis by auto
 qed
 thus ?thesis
 using $P2$ by blast
 qed

lemma *l11-19*:
 assumes *Per* $A \ B \ P1$ and
 $\text{Per } A \ B \ P2$ and
 $A \ B \ \text{OS } P1 \ P2$
 shows $B \ \text{Out } P1 \ P2$
 proof cases
 assume $\text{Col } A \ B \ P1$
 thus ?thesis
 using *assms*(3) *col123--nos* by blast
 next


```

assume  $P1: \neg \text{Col } A \ B \ P1$ 
have  $B \text{ Out } P1 \ P2$ 
proof cases
  assume  $\text{Col } A \ B \ P2$ 
  thus ?thesis
    using  $\text{assms}(3) \text{ one-side-not-col124}$  by blast
next
  assume  $P2: \neg \text{Col } A \ B \ P2$ 
  obtain  $x$  where  $A \ B \ P1 \ \text{CongA } A \ B \ x \wedge B \ A \ \text{OS } x \ P2 \wedge$ 
 $(\forall F1 \ F2. ((A \ B \ P1 \ \text{CongA } A \ B \ F1 \wedge B \ A \ \text{OS } F1 \ P2) \wedge$ 
 $(A \ B \ P1 \ \text{CongA } A \ B \ F2 \wedge B \ A \ \text{OS } F2 \ P2)) \longrightarrow B \ \text{Out } F1 \ F2)$ 
  using  $P1 \ P2 \ \text{l11-15}$  by blast
  thus ?thesis
    by  $(\text{metis } P1 \ P2 \ \text{assms}(1) \ \text{assms}(2) \ \text{assms}(3) \ \text{conga-os--out l11-16 not-col-distincts})$ 
qed
thus ?thesis
  by simp
qed

lemma l11-22-bet:
assumes  $\text{Bet } A \ B \ C$  and
 $P' \ B' \ \text{TS } A' \ C'$  and
 $A \ B \ P \ \text{CongA } A' \ B' \ P'$  and
 $P \ B \ C \ \text{CongA } P' \ B' \ C'$ 
shows  $\text{Bet } A' \ B' \ C'$ 
proof –
obtain  $C''$  where  $P1: \text{Bet } A' \ B' \ C'' \wedge \text{Cong } B' \ C'' \ B \ C$ 
using segment-construction by blast
have  $P2: C \ B \ P \ \text{CongA } C'' \ B' \ P'$ 
  by  $(\text{metis } P1 \ \text{assms}(1) \ \text{assms}(3) \ \text{assms}(4) \ \text{cong-diff-3 conga-diff2 l11-13})$ 
have  $P3: C'' \ B' \ P' \ \text{CongA } C' \ B' \ P'$ 
  by  $(\text{meson } P2 \ \text{conga-sym Tarski-neutral-dimensionless-axioms assms}(4) \ \text{conga-comm conga-trans})$ 
have  $P4: B' \ \text{Out } C' \ C'' \vee P' \ B' \ \text{TS } C' \ C''$ 
proof –
  have  $P5: \text{Coplanar } P' \ B' \ C' \ C''$ 
    by  $(\text{meson } P1 \ \text{TS-def assms}(2) \ \text{bet--coplanar coplanar-trans-1 ncoplanar-perm-1}$ 
 $\text{ncoplanar-perm-8 ts--coplanar})$ 
  have  $P' \ B' \ C' \ \text{CongA } P' \ B' \ C''$ 
    using  $P3 \ \text{conga-comm conga-sym}$  by blast
  thus ?thesis
    by  $(\text{simp add: } P5 \ \text{conga-cop--or-out-ts})$ 
qed
have  $P6: B' \ \text{Out } C' \ C'' \longrightarrow \text{Bet } A' \ B' \ C'$ 
proof –
  {
    assume  $B' \ \text{Out } C' \ C''$ 
    hence  $\text{Bet } A' \ B' \ C'$ 
    using  $P1 \ \text{bet-out-out-bet between-exchange4 between-trivial2 col-trivial-3}$ 
 $\text{l6-6 not-bet-out}$  by blast
  }
  thus ?thesis by simp
qed
have  $P' \ B' \ \text{TS } C' \ C'' \longrightarrow \text{Bet } A' \ B' \ C'$ 
proof –
  {
    assume  $P7: P' \ B' \ \text{TS } C' \ C''$ 
    hence  $\text{Bet } A' \ B' \ C'$ 
    proof cases
      assume  $\text{Col } C' \ B' \ P'$ 
      thus ?thesis
        using  $\text{Col-perm TS-def assms}(2)$  by blast
    next
      assume  $Q1: \neg \text{Col } C' \ B' \ P'$ 
      hence  $Q2: B' \neq P'$ 
        using not-col-distincts by blast
      have  $Q3: B' \ P' \ \text{TS } A' \ C''$ 

```

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    by (metis Col-perm P1 TS-def P7 assms(2) col-trivial-3)
  have Q4:  $B' P' OS C' C''$ 
    using P7 Q3 assms(2) invert-two-sides l9-8-1 l9-9 by blast
  thus ?thesis
    using P7 invert-one-side l9-9 by blast
qed
}
thus ?thesis by simp
qed
thus ?thesis using P6 P4 by blast
qed

lemma l11-22a:
  assumes  $B P TS A C$  and
     $B' P' TS A' C'$  and
     $A B P CongA A' B' P'$  and
     $P B C CongA P' B' C'$ 
  shows  $A B C CongA A' B' C'$ 
proof -
  have P1:  $A \neq B \wedge A' \neq B' \wedge P \neq B \wedge P' \neq B' \wedge C \neq B \wedge C' \neq B'$ 
    using assms(3) assms(4) conga-diff1 conga-diff2 conga-diff45 conga-diff56 by auto
  have P2:  $A \neq C \wedge A' \neq C'$ 
    using assms(1) assms(2) not-two-sides-id by blast
  obtain A'' where P3:  $B' Out A' A'' \wedge Cong B' A'' B A$ 
    using P1 segment-construction-3 by force
  have P4:  $\neg Col A B P$ 
    using TS-def assms(1) by blast
  obtain T where P5:  $Col T B P \wedge Bet A T C$ 
    using TS-def assms(1) by blast
  have  $A B C CongA A' B' C'$ 
proof cases
  assume  $B = T$ 
  thus ?thesis
    by (metis P1 P5 assms(2) assms(3) assms(4) conga-line invert-two-sides l11-22-bet)
next
  assume P6:  $B \neq T$ 
  have  $A B C CongA A' B' C'$ 
proof cases
  assume P7A:  $Bet P B T$ 
  obtain T'' where T1:  $Bet P' B' T'' \wedge Cong B' T'' B T$ 
    using segment-construction by blast
  have  $\exists T''. Col B' P' T'' \wedge (B' Out P' T'' \longleftrightarrow B Out P T) \wedge Cong B' T'' B T$ 
proof -
  have T2:  $Col B' P' T''$  using T1
    by (simp add: bet-col col-permutation-4)
  have  $(B' Out P' T'' \longleftrightarrow B Out P T) \wedge Cong B' T'' B T$ 
    using P7A T1 not-bet-and-out by blast
  thus ?thesis using T2 by blast
qed
  then obtain T'' where T3:
     $Col B' P' T'' \wedge (B' Out P' T'' \longleftrightarrow B Out P T) \wedge Cong B' T'' B T$  by blast
  hence T4:  $B' \neq T''$ 
    using P6 conga-diff-3 by blast
  obtain C'' where T5:  $Bet A'' T'' C'' \wedge Cong T'' C'' T C$ 
    using segment-construction by blast
  have  $T B A CongA T'' B' A'$ 
proof -
  have  $P B A CongA P' B' A'$ 
    by (simp add: assms(3) conga-comm)
  moreover
  have  $Col P' B' T''$ 
    using T3 Col-cases by blast
  hence  $Bet P' B' T''$ 
    using P7A T3 not-bet-and-out not-out-bet by blast
  ultimately show ?thesis
    using T4 P6 P7A l11-13 by blast

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qed
 hence $T6: A B T \text{ Cong} A' B' T''$
 using *Out-cases P5 P6 T3 T4 P7A assms(3) between-symmetry col-permutation-4 conga-comm l11-13 l6-4-1 or-bet-out* **by** blast
 have $T7: A B T \text{ Cong} A'' B' T''$
 proof –
 have $B \text{ Out } A A$
 using *P4 not-col-distincts out-trivial* **by** presburger
 moreover have $B \text{ Out } T T$
 using *P6 not-col-distincts out-trivial* **by** presburger
 moreover have $B' \text{ Out } A'' A'$
 using *P3 l6-6* **by** blast
 ultimately show *?thesis*
 using *l11-10 T4 T6 out-trivial* **by** presburger
 qed
 hence $T8: \text{Cong } A T A'' T''$
 using *P3 T3 cong2-conga-cong cong-4321 not-cong-3412* **by** blast
 have $T9: \text{Cong } A C A'' C''$
 using *P5 T5 T8 cong-symmetry l2-11-b* **by** blast
 have $A \neq T$
 using *P4 P5* **by** auto
 hence $T10: \text{Cong } C B C'' B'$
 using *P3 T3 T5 T8 cong-commutativity cong-symmetry five-segment P5* **by** meson
 have $A B C \text{ Cong} A'' B' C''$
 using *Cong3-def P3 T10 T9 not-cong-2143 not-cong-4321* **by** blast
 hence $T11: A B C \text{ Cong} A'' B' C''$
by (*simp add: cong3-conga Tarski-neutral-dimensionless-axioms P1*)
 have $C B T \text{ Cong} C'' B' T''$
by (*simp add: Cong3-def T10 T3 T5 cong-4321 cong-symmetry*)
 hence $T12: C B T \text{ Cong} A C'' B' T''$
 using *P1 P6 cong3-conga* **by** auto
 have $T13: P B C \text{ Cong} A P' B' C''$
 proof –
 have $\text{Bet } T B P$
 using *Bet-perm P7A* **by** blast
 moreover have $\text{Bet } T'' B' P'$
 using *Col-perm P7A T3 l6-6 not-bet-and-out or-bet-out* **by** blast
 ultimately show *?thesis*
 using *P1 T12 conga-comm l11-13* **by** blast
 qed
 have $T14: P' B' C' \text{ Cong} A P' B' C''$
 proof –
 have $P' B' C' \text{ Cong} A P B C$
by (*simp add: assms(4) conga-sym*)
 thus *?thesis*
 using *T13 conga-trans* **by** blast
 qed
 have $T15: B' \text{ Out } C' C'' \vee P' B' \text{ TS } C' C''$
 proof –
 have $K7: \text{Coplanar } P' B' C' C''$
 proof –
 have $K8: \text{Coplanar } A' P' B' C'$
 using *assms(2) coplanar-perm-14 ts--coplanar* **by** blast
 have $K8A: \text{Coplanar } P' C'' B' A''$
 proof –
 have $\text{Col } P' B' T'' \wedge \text{Col } C'' A'' T''$
 using *Col-def Col-perm T3 T5* **by** blast
 hence $\text{Col } P' C'' T'' \wedge \text{Col } B' A'' T'' \vee$
 $\text{Col } P' B' T'' \wedge \text{Col } C'' A'' T'' \vee$
 $\text{Col } P' A'' T'' \wedge \text{Col } C'' B' T''$ **by** simp
 thus *?thesis*
 using *Coplanar-def* **by** auto
 qed
 hence $\text{Coplanar } A' P' B' C''$
 proof –

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    have K9:  $B' \neq A''$ 
      using P3 out-distinct by blast
    have Col  $B' A'' A'$ 
      using Col-perm P3 out-col by blast
    thus ?thesis
      using K8A K9 col-cop--cop coplanar-perm-19 by blast
  qed
  thus ?thesis
    by (meson K8 TS-def assms(2) coplanar-perm-7 coplanar-trans-1 ncoplanar-perm-2)
  qed
  thus ?thesis
    by (simp add: T14 K7 conga-cop--or-out-ts)
  qed
  have A B C CongA  $A' B' C'$ 
  proof cases
    assume  $B' Out C' C''$ 
    thus ?thesis
      using P1 P3 T11 l11-10 out-trivial by blast
  next
    assume  $W1: \neg B' Out C' C''$ 
    hence W1A:  $P' B' TS C' C''$  using T15 by simp
    have W2:  $B' P' TS A'' C'$ 
      using P3 assms(2) col-trivial-1 l9-5 by blast
    hence W3:  $B' P' OS A'' C''$ 
      using T15 W1 invert-two-sides l9-2 l9-8-1 by blast
    have W4:  $B' P' TS A'' C''$ 
  proof -
    have  $\neg Col A' B' P'$ 
      using TS-def assms(2) by auto
    thus ?thesis
      using Col-perm T3 T5 W3 one-side-chara by blast
  qed
  thus ?thesis
    using W1A W2 invert-two-sides l9-8-1 l9-9 by blast
  qed
  thus ?thesis by simp
next
  assume R1:  $\neg Bet P B T$ 
  hence R2:  $B Out P T$ 
    using Col-cases P5 l6-4-2 by blast
  have R2A:  $\exists T''. Col B' P' T'' \wedge (B' Out P' T'' \longleftrightarrow B Out P T) \wedge Cong B' T'' B T$ 
  proof -
    obtain  $T''$  where R3:  $B' Out P' T'' \wedge Cong B' T'' B T$ 
      using P1 P6 segment-construction-3 by fastforce
    thus ?thesis
      using R2 out-col by blast
  qed
  then obtain  $T''$  where R4:  $Col B' P' T'' \wedge$ 
    ( $B' Out P' T'' \longleftrightarrow B Out P T$ )  $\wedge$ 
     $Cong B' T'' B T$  by auto
  have R5:  $B' \neq T''$ 
    using P6 R4 cong-diff-3 by blast
  obtain  $C''$  where R6:  $Bet A'' T'' C'' \wedge Cong T'' C'' T C$ 
    using segment-construction by blast
  have R7:  $A B T CongA A' B' T''$ 
    using P1 R2 R4 assms(3) l11-10 l6-6 out-trivial by auto
  have R8:  $A B T CongA A'' B' T''$ 
    using P3 P4 R2 R4 assms(3) l11-10 l6-6 not-col-distincts out-trivial by blast
  have R9:  $Cong A T A'' T''$ 
    using Cong-cases P3 R4 R8 cong2-conga-cong by blast
  have R10:  $Cong A C A'' C''$ 
    using P5 R6 R9 l2-11-b not-cong-3412 by blast
  have R11:  $Cong C B C'' B'$ 
  proof -
    have  $Cong T C T'' C''$ 
      using Cong-cases R6 by blast

```

moreover have $\text{Cong } A \ B \ A'' \ B'$
 by (metis $P3 \ \text{Cong-cases}$)
 moreover have $\text{Cong } T \ B \ T'' \ B'$
 using $R4 \ \text{not-cong-4321}$ by blast
 moreover have $A \neq T$
 using $P4 \ P5$ by auto
 ultimately show ?thesis
 using $R9 \ P5 \ R6 \ \text{five-segment}$ by blast
 qed
 have $A \ B \ C \ \text{Cong3 } A'' \ B' \ C''$
 by (simp add: $\text{Cong3-def } P3 \ R10 \ R11 \ \text{cong-4321} \ \text{cong-commutativity}$)
 hence $R12: A \ B \ C \ \text{CongA } A'' \ B' \ C''$
 using $P1$ by (simp add: cong3-conga)
 have $C \ B \ T \ \text{Cong3 } C'' \ B' \ T''$
 using $\text{Cong3-def } R11 \ R4 \ R6 \ \text{not-cong-3412} \ \text{not-cong-4321}$ by blast
 hence $R13: C \ B \ T \ \text{CongA } C'' \ B' \ T''$
 using $P1 \ P6 \ \text{cong3-conga} \ \text{Tarski-neutral-dimensionless-axioms}$ by fastforce
 have $R13A: \neg \text{Col } A' \ B' \ P'$
 using $\text{TS-def} \ \text{assms}(2)$ by blast
 have $R14: B' \ \text{Out } C' \ C'' \vee P' \ B' \ \text{TS } C' \ C''$
 proof –
 have $S1: \text{Coplanar } P' \ B' \ C' \ C''$
 proof –
 have $T1: \text{Coplanar } A' \ P' \ B' \ C'$
 using $\text{assms}(2) \ \text{ncoplanar-perm-14} \ \text{ts--coplanar}$ by blast
 have $\text{Coplanar } A' \ P' \ B' \ C''$
 proof –
 have $U6: B' \neq A''$
 using $P3 \ \text{out-diff2}$ by blast
 have $\text{Coplanar } P' \ C'' \ B' \ A''$
 proof –
 have $\text{Col } P' \ B' \ T'' \wedge \text{Col } C'' \ A'' \ T''$
 using $\text{Col-def} \ \text{Col-perm } R4 \ R6$ by blast
 thus ?thesis using Coplanar-def by auto
 qed
 thus ?thesis
 by (meson $\text{Col-cases } P3 \ U6 \ \text{col-cop--cop} \ \text{ncoplanar-perm-21} \ \text{ncoplanar-perm-6} \ \text{out-col}$)
 qed
 thus ?thesis
 using $\text{NCol-cases } R13A \ T1 \ \text{coplanar-trans-1}$ by blast
 qed
 have $P' \ B' \ C' \ \text{CongA } P' \ B' \ C''$
 proof –
 have $C \ B \ P \ \text{CongA } C'' \ B' \ P'$
 using $P1 \ R12 \ R13 \ R2 \ R4 \ \text{cong-a-diff56} \ \text{l11-10} \ \text{out-trivial}$ by presburger
 hence $C' \ B' \ P' \ \text{CongA } C'' \ B' \ P'$
 by (meson $\text{cong-a-trans} \ \text{Tarski-neutral-dimensionless-axioms} \ \text{assms}(4) \ \text{cong-a-comm} \ \text{cong-a-sym}$)
 thus ?thesis
 by (simp add: cong-a-comm)
 qed
 thus ?thesis
 by (simp add: $S1 \ \text{cong-a-cop--or-out-ts}$)
 qed
 have $S1: B \ \text{Out } A \ A$
 using $P4 \ \text{not-col-distincts} \ \text{out-trivial}$ by blast
 have $S2: B \ \text{Out } C \ C$
 using $\text{TS-def} \ \text{assms}(1) \ \text{not-col-distincts} \ \text{out-trivial}$ by auto
 have $S3: B' \ \text{Out } A' \ A''$ using $P3$ by simp
 have $A \ B \ C \ \text{CongA } A' \ B' \ C'$
 proof cases
 assume $B' \ \text{Out } C' \ C''$
 thus ?thesis using $S1 \ S2 \ S3$
 using $R12 \ \text{l11-10}$ by blast
 next
 assume $\neg B' \ \text{Out } C' \ C''$

```

hence Z3:  $P' B' TS C' C''$  using R14 by simp
have Q1:  $B' P' TS A'' C'$ 
  using S3 assms(2) l9-5 not-col-distincts by blast
have Q2:  $B' P' OS A'' C''$ 
proof -
  have  $B' P' TS C'' C'$ 
  proof -
    have  $B' P' TS C' C''$  using Z3
    using invert-two-sides by blast
    thus ?thesis
    by (simp add: l9-2)
  qed
  thus ?thesis
  using Q1 l9-8-1 by blast
qed
have  $B' P' TS A'' C''$ 
  using Col-perm Q2 R4 R6 one-side-chara by blast
thus ?thesis
  using Q2 l9-9 by blast
qed
thus ?thesis using S1 S2 S3
  using R12 l11-10 by blast
qed
thus ?thesis by simp
qed
thus ?thesis by simp
qed

lemma l11-22b:
  assumes  $B P OS A C$  and
     $B' P' OS A' C'$  and
     $A B P CongA A' B' P'$  and
     $P B C CongA P' B' C'$ 
  shows  $A B C CongA A' B' C'$ 
proof -
  obtain  $D$  where  $P1: Bet A B D \wedge Cong B D A B$ 
    using segment-construction by blast
  obtain  $D'$  where  $P2: Bet A' B' D' \wedge Cong B' D' A' B'$ 
    using segment-construction by blast
  have  $P3: D B P CongA D' B' P'$ 
  proof -
    have  $Q3: D \neq B$ 
      by (metis P1 assms(1) col-trivial-3 cong-diff-3 one-side-not-col124 one-side-symmetry)
    have  $Q5: D' \neq B'$ 
      by (metis P2 assms(2) col-trivial-3 cong-diff-3 one-side-not-col124 one-side-symmetry)
    thus ?thesis
      using assms(3) P1 Q3 P2 l11-13 by blast
  qed
  have  $P5: D B C CongA D' B' C'$ 
  proof -
    have  $Q1: B P TS D C$ 
      by (metis P1 assms(1) bet--ts col-trivial-3 cong-diff-3 l9-2 l9-8-2 one-side-not-col124
        one-side-symmetry)
    have  $B' P' TS D' C'$ 
      by (metis Cong-perm P2 assms(2) bet--ts between-cong between-trivial2 l9-2 l9-8-2
        one-side-not-col123 point-construction-different ts-distincts)
    thus ?thesis
      using assms(4) Q1 P3 l11-22a by blast
  qed
  have  $P6: Bet D B A$ 
    using Bet-perm P1 by blast
  have  $P7: A \neq B$ 
    using assms(3) conga-diff1 by auto
  have  $P8: Bet D' B' A'$ 
    using Bet-cases P2 by blast
  have  $A' \neq B'$ 

```

using *assms*(3) *conga-diff45* by *blast*
 thus ?thesis
 using *P5 P6 P7 P8 l11-13* by *blast*
 qed

lemma *l11-22*:
 assumes $((B \ P \ TS \ A \ C \wedge B' \ P' \ TS \ A' \ C') \vee (B \ P \ OS \ A \ C \wedge B' \ P' \ OS \ A' \ C'))$ and
 $A \ B \ P \ CongA \ A' \ B' \ P'$ and
 $P \ B \ C \ CongA \ P' \ B' \ C'$
 shows $A \ B \ C \ CongA \ A' \ B' \ C'$
 by (meson *assms*(1) *assms*(2) *assms*(3) *l11-22a l11-22b*)

lemma *l11-24*:
 assumes $P \ InAngle \ A \ B \ C$
 shows $P \ InAngle \ C \ B \ A$
proof –
 obtain X where $Bet \ A \ X \ C$ and $X = B \vee B \ Out \ X \ P$
 using *InAngle-def assms* by *auto*
 thus ?thesis
 using *Bet-cases InAngle-def assms* by *auto*
 qed

lemma *col-in-angle*:
 assumes $A \neq B$ and
 $C \neq B$ and
 $P \neq B$ and
 $B \ Out \ A \ P \vee B \ Out \ C \ P$
 shows $P \ InAngle \ A \ B \ C$
 by (meson *InAngle-def assms*(1) *assms*(2) *assms*(3) *assms*(4) *between-trivial between-trivial2*)

lemma *out321--inangle*:
 assumes $C \neq B$ and
 $B \ Out \ A \ P$
 shows $P \ InAngle \ A \ B \ C$
 using *assms*(1) *assms*(2) *col-in-angle out-distinct* by *auto*

lemma *inangle1123*:
 assumes $A \neq B$ and
 $C \neq B$
 shows $A \ InAngle \ A \ B \ C$
 by (simp add: *assms*(1) *assms*(2) *out321--inangle out-trivial*)

lemma *out341--inangle*:
 assumes $A \neq B$ and
 $B \ Out \ C \ P$
 shows $P \ InAngle \ A \ B \ C$
 using *assms*(1) *assms*(2) *col-in-angle out-distinct* by *auto*

lemma *inangle3123*:
 assumes $A \neq B$ and
 $C \neq B$
 shows $C \ InAngle \ A \ B \ C$
 by (simp add: *assms*(1) *assms*(2) *inangle1123 l11-24*)

lemma *in-angle-two-sides*:
 assumes $\neg Col \ B \ A \ P$ and
 $\neg Col \ B \ C \ P$ and
 $P \ InAngle \ A \ B \ C$
 shows $P \ B \ TS \ A \ C$
 by (metis *InAngle-def TS-def assms*(1) *assms*(2) *assms*(3) *not-col-distincts*
 not-col-permutation-1 out-col)

lemma *in-angle-out*:
 assumes $B \ Out \ A \ C$ and
 $P \ InAngle \ A \ B \ C$
 shows $B \ Out \ A \ P$

by (metis InAngle-def assms(1) assms(2) not-bet-and-out out2-bet-out)

lemma col-in-angle-out:

assumes Col B A P and

$\neg$ Bet A B C and

P InAngle A B C

shows B Out A P

proof –

obtain X where P1: Bet A X C $\wedge$ (X = B $\vee$ B Out X P)

using InAngle-def assms(3) by auto

have B Out A P

proof cases

assume X = B

thus ?thesis

using P1 assms(2) by blast

next

assume P2: X $\neq$ B

thus ?thesis

proof –

have f1: Bet B A P $\vee$ A Out B P

by (meson assms(1) l6-4-2)

have f2: B Out X P

using P1 P2 by blast

have f3: (Bet B P A $\vee$ Bet B A P) $\vee$ Bet P B A

using f1 by (meson Bet-perm Out-def)

have f4: Bet B P X $\vee$ Bet P X B

using f2 by (meson Bet-perm Out-def)

hence f5: ((Bet B P X $\vee$ Bet X B A) $\vee$ Bet B P A) $\vee$ Bet B A P

using f3 by (meson between-exchange3)

have $\forall p. \text{Bet } p \text{ X C} \vee \neg \text{Bet } A \text{ p X}$

using P1 between-exchange3 by blast

hence f6: (P = B $\vee$ Bet B A P) $\vee$ Bet B P A

using f5 f3 by (meson Bet-perm P2 assms(2) outer-transitivity-between2)

have f7: Bet C X A

using Bet-perm P1 by blast

have P $\neq$ B

using f2 by (simp add: Out-def)

moreover

{ assume Bet B B C

hence A $\neq$ B

using assms(2) by blast }

ultimately have A $\neq$ B

using f7 f4 f1 by (meson Bet-perm Out-def P2 between-exchange3
outer-transitivity-between2)

thus ?thesis

using f6 f2 by (simp add: Out-def)

qed

qed

thus ?thesis by blast

qed

lemma l11-25-aux:

assumes P InAngle A B C and

$\neg$ Bet A B C and

B Out A' A

shows P InAngle A' B C

proof –

have P1: Bet B A' A $\vee$ Bet B A A'

using Out-def assms(3) by auto

{

assume P2: Bet B A' A

obtain X where P3: Bet A X C $\wedge$ (X = B $\vee$ B Out X P)

using InAngle-def assms(1) by auto

obtain T where P4: Bet A' T C $\wedge$ Bet X T B

using Bet-perm P2 P3 inner-pasch by blast

{


```

    assume  $X = B$ 
    hence  $P \text{ InAngle } A' B C$ 
    using  $P3 \text{ assms}(2)$  by blast
  }
  {
    assume  $B \text{ Out } X P$ 
    hence  $P \text{ InAngle } A' B C$ 
    by (metis  $\text{InAngle-def } P4 \text{ assms}(1) \text{ assms}(3) \text{ bet-out-1 l6-7 out-diff1}$ )
  }
  hence  $P \text{ InAngle } A' B C$ 
  using  $P3 \langle X = B \implies P \text{ InAngle } A' B C \rangle$  by blast
}
{
  assume  $Q0: \text{Bet } B A A'$ 
  obtain  $X$  where  $Q1: \text{Bet } A X C \wedge (X = B \vee B \text{ Out } X P)$ 
  using  $\text{InAngle-def assms}(1)$  by auto
  {
    assume  $X = B$ 
    hence  $P \text{ InAngle } A' B C$ 
    using  $Q1 \text{ assms}(2)$  by blast
  }
  {
    assume  $Q2: B \text{ Out } X P$ 
    obtain  $T$  where  $Q3: \text{Bet } A' T C \wedge \text{Bet } B X T$ 
    using  $\text{Bet-perm } Q1 Q0 \text{ outer-pasch}$  by blast
    hence  $P \text{ InAngle } A' B C$ 
    by (metis  $\text{InAngle-def } Q0 Q2 \text{ assms}(1) \text{ bet-out l6-6 l6-7 out-diff1}$ )
  }
  hence  $P \text{ InAngle } A' B C$ 
  using  $\langle X = B \implies P \text{ InAngle } A' B C \rangle Q1$  by blast
}
thus ?thesis
using  $P1 \langle \text{Bet } B A' A \implies P \text{ InAngle } A' B C \rangle$  by blast
qed

lemma l11-25:
  assumes  $P \text{ InAngle } A B C$  and
     $B \text{ Out } A' A$  and
     $B \text{ Out } C' C$  and
     $B \text{ Out } P' P$ 
  shows  $P' \text{ InAngle } A' B C'$ 
proof cases
  assume  $\text{Bet } A B C$ 
  thus ?thesis
  by (metis  $\text{Bet-perm InAngle-def assms}(2) \text{ assms}(3) \text{ assms}(4) \text{ bet-out--bet l6-6 out-distinct}$ )
next
  assume  $P1: \neg \text{Bet } A B C$ 
  have  $P2: P \text{ InAngle } A' B C$ 
  using  $P1 \text{ assms}(1) \text{ assms}(2) \text{ l11-25-aux}$  by blast
  have  $P3: P \text{ InAngle } A' B C'$ 
  proof -
    have  $P \text{ InAngle } C' B A'$  using  $\text{l11-25-aux}$ 
    using  $\text{Bet-perm } P1 P2 \text{ assms}(2) \text{ assms}(3) \text{ bet-out--bet l11-24}$  by blast
    thus ?thesis using  $\text{l11-24}$  by blast
  qed
  obtain  $X$  where  $P4: \text{Bet } A' X C' \wedge (X = B \vee B \text{ Out } X P)$ 
  using  $\text{InAngle-def } P3$  by auto
  {
    assume  $X = B$ 
    hence  $P' \text{ InAngle } A' B C'$ 
    using  $\text{InAngle-def } P3 P4 \text{ assms}(4) \text{ out-diff1}$  by auto
  }
  {
    assume  $B \text{ Out } X P$ 
    hence  $P' \text{ InAngle } A' B C'$ 
    proof -

```

```

    have  $\forall p. B \text{ Out } p \ P' \vee \neg B \text{ Out } p \ P$ 
    by (meson Out-cases assms(4) l6-7)
  thus ?thesis
    by (metis (no-types) InAngle-def P3 assms(4) out-diff1)
qed
}
thus ?thesis
  using InAngle-def P4 assms(2) assms(3) assms(4) out-distinct by auto
qed

lemma inangle-distincts:
  assumes  $P \text{ InAngle } A \ B \ C$ 
  shows  $A \neq B \wedge C \neq B \wedge P \neq B$ 
  using InAngle-def assms by auto

lemma segment-construction-0:
  shows  $\exists B'. \text{Cong } A' \ B' \ A \ B$ 
  using segment-construction by blast

lemma angle-construction-3:
  assumes  $A \neq B$  and
     $C \neq B$  and
     $A' \neq B'$ 
  shows  $\exists C'. A \ B \ C \text{ CongA } A' \ B' \ C'$ 
  by (metis angle-construction-2 assms(1) assms(2) assms(3) not-col-exists)

lemma l11-28:
  assumes  $A \ B \ C \text{ Cong3 } A' \ B' \ C'$  and
     $\text{Col } A \ C \ D$ 
  shows  $\exists D'. (\text{Cong } A \ D \ A' \ D' \wedge \text{Cong } B \ D \ B' \ D' \wedge \text{Cong } C \ D \ C' \ D')$ 
proof cases
  assume  $P1: A = C$ 
  have  $\exists D'. (\text{Cong } A \ D \ A' \ D' \wedge \text{Cong } B \ D \ B' \ D' \wedge \text{Cong } C \ D \ C' \ D')$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    by (metis P1 assms(1) cong3-diff cong3-symmetry cong-3-swap-2 not-cong-3421
      segment-construction-0)
next
  assume  $A \neq B$ 
  have  $\exists D'. (\text{Cong } A \ D \ A' \ D' \wedge \text{Cong } B \ D \ B' \ D' \wedge \text{Cong } C \ D \ C' \ D')$ 
proof cases
  assume  $A = D$ 
  thus ?thesis
    using Cong3-def P1 assms(1) cong-trivial-identity by blast
next
  assume  $A \neq D$ 
  have  $\exists D'. (\text{Cong } A \ D \ A' \ D' \wedge \text{Cong } B \ D \ B' \ D' \wedge \text{Cong } C \ D \ C' \ D')$ 
proof cases
  assume  $B = D$ 
  thus ?thesis
    using Cong3-def assms(1) cong-3-swap-2 cong-trivial-identity by blast
next
  assume  $Q1: B \neq D$ 
  obtain  $D''$  where  $Q2: B \ A \ D \text{ CongA } B' \ A' \ D''$ 
    by (metis  $\langle A \neq B \rangle \langle A \neq D \rangle$  angle-construction-3 assms(1) cong3-diff)
  obtain  $D'$  where  $Q3: A' \text{ Out } D'' \ D' \wedge \text{Cong } A' \ D' \ A \ D$ 
    by (metis Q2  $\langle A \neq D \rangle$  conga-diff56 segment-construction-3)
  have  $Q5: \text{Cong } A \ D \ A' \ D'$ 
    using Q3 not-cong-3412 by blast
  have  $B \ A \ D \text{ CongA } B' \ A' \ D'$ 
    using Q2 Q3  $\langle A \neq B \rangle \langle A \neq D \rangle$  conga-diff45 l11-10 l6-6 out-trivial by auto
  hence  $\text{Cong } B \ D \ B' \ D'$ 
    using Cong3-def Cong-perm Q5 assms(1) cong2-conga-cong by blast
  thus ?thesis
    using Cong3-def P1 Q5 assms(1) cong-reverse-identity by blast

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    qed
    thus ?thesis by simp
  qed
  thus ?thesis by simp
qed
thus ?thesis by simp
next
assume Z1:  $A \neq C$ 
have  $\exists D'. (Cong\ A\ D\ A'\ D' \wedge Cong\ B\ D\ B'\ D' \wedge Cong\ C\ D\ C'\ D')$ 
proof cases
  assume  $A = D$ 
  thus ?thesis
    using Cong3-def Cong-perm assms(1) cong-trivial-identity by blast
next
assume  $A \neq D$ 
{
  assume Bet  $A\ C\ D$ 
  obtain  $D'$  where  $W1: Bet\ A'\ C'\ D' \wedge Cong\ C'\ D'\ C\ D$ 
    using segment-construction by blast
  have  $W2: Cong\ A\ D\ A'\ D'$ 
    by (meson Cong3-def  $W1$   $\langle Bet\ A\ C\ D \rangle$  assms(1) cong-symmetry l2-11-b)
  have  $W3: Cong\ B\ D\ B'\ D'$ 
  proof -
    have  $X1: Cong\ C\ D\ C'\ D'$ 
      using  $W1$  not-cong-3412 by blast
    have  $Cong\ C\ B\ C'\ B'$ 
      using Cong3-def assms(1) cong-commutativity by presburger
    hence  $W4: A\ C\ D\ B\ OFSC\ A'\ C'\ D'\ B'$ 
      using Cong3-def OFSC-def  $W1\ X1\ \langle Bet\ A\ C\ D \rangle$  assms(1) by blast
    have  $Cong\ D\ B\ D'\ B'$ 
      using  $W4\ \langle A \neq C \rangle$  five-segment-with-def by blast
    thus ?thesis
      using Z1 not-cong-2143 by blast
  qed
  have  $Cong\ C\ D\ C'\ D'$ 
    by (simp add:  $W1$  cong-symmetry)
  hence  $\exists D'. (Cong\ A\ D\ A'\ D' \wedge Cong\ B\ D\ B'\ D' \wedge Cong\ C\ D\ C'\ D')$ 
    using  $W2\ W3$  by blast
}
{
  assume  $W3B: Bet\ C\ D\ A$ 
  then obtain  $D'$  where  $W4A: Bet\ A'\ D'\ C' \wedge A\ D\ C\ Cong3\ A'\ D'\ C'$ 
    using Bet-perm Cong3-def assms(1) l4-5 by blast
  have  $W5: Cong\ A\ D\ A'\ D'$ 
    using Cong3-def  $W4A$  by blast
  have  $A\ D\ C\ B\ IFSC\ A'\ D'\ C'\ B'$ 
    by (meson Bet-perm Cong3-def Cong-perm IFSC-def  $W4A\ W3B$  assms(1))
  hence  $Cong\ D\ B\ D'\ B'$ 
    using l4-2 by blast
  hence  $W6: Cong\ B\ D\ B'\ D'$ 
    using Cong-perm by blast
  hence  $Cong\ C\ D\ C'\ D'$ 
    using Cong3-def  $W4A$  not-cong-2143 by blast
  hence  $\exists D'. (Cong\ A\ D\ A'\ D' \wedge Cong\ B\ D\ B'\ D' \wedge Cong\ C\ D\ C'\ D')$ 
    using  $W5\ W6$  by blast
}
{
  assume  $W7: Bet\ D\ A\ C$ 
  obtain  $D'$  where  $W7A: Bet\ C'\ A'\ D' \wedge Cong\ A'\ D'\ A\ D$ 
    using segment-construction by blast
  hence  $W8: Cong\ A\ D\ A'\ D'$ 
    using Cong-cases by blast
  have  $C\ A\ D\ B\ OFSC\ C'\ A'\ D'\ B'$ 
    by (meson Bet-perm Cong3-def Cong-perm OFSC-def  $W7\ W7A$  assms(1))
  hence  $Cong\ D\ B\ D'\ B'$ 
    using Z1 five-segment-with-def by auto
}

```

```

hence w9: Cong B D B' D'
  using Cong-perm by blast
have Cong C D C' D'
proof -
  have L1: Bet C A D
    using Bet-perm W7 by blast
  have L2: Bet C' A' D'
    using Bet-perm W7
    using W7A by blast
  have Cong C A C' A' using assms(1)
    using Cong3-def assms(1) not-cong-2143 by blast
  thus ?thesis using l2-11
    using L1 L2 W8 l2-11 by blast
qed
hence  $\exists D'. (Cong A D A' D' \wedge Cong B D B' D' \wedge Cong C D C' D')$ 
  using W8 w9 by blast
}
thus ?thesis
  using Bet-cases  $\langle Bet A C D \implies \exists D'. Cong A D A' D' \wedge Cong B D B' D' \wedge Cong C D C' D' \rangle$ 
   $\langle Bet C D A \implies \exists D'. Cong A D A' D' \wedge Cong B D B' D' \wedge Cong C D C' D' \rangle$ 
  assms(2) third-point by blast
qed
thus ?thesis
  by blast
qed

```

```

lemma bet-conga--bet:
  assumes Bet A B C and
    A B C CongA A' B' C'
  shows Bet A' B' C'
proof -
  obtain A0 C0 A1 C1 where P1:
    Bet B A A0  $\wedge$  Cong A A0 B' A'  $\wedge$ 
    Bet B C C0  $\wedge$  Cong C C0 B' C'  $\wedge$ 
    Bet B' A' A1  $\wedge$  Cong A' A1 B A  $\wedge$ 
    Bet B' C' C1  $\wedge$  Cong C' C1 B C  $\wedge$ 
    Cong A0 C0 A1 C1 using CongA-def assms(2)
  by auto
  have Bet C B A0 using P1 outer-transitivity-between
    by (metis assms(1) assms(2) between-symmetry conga-diff1)
  hence Bet A0 B C
    using Bet-cases by blast
  hence P2: Bet A0 B C0
    using P1 assms(2) conga-diff2 outer-transitivity-between by blast
  have P3: A0 B C0 Cong3 A1 B' C1
  proof -
    have Q1: Cong A0 B A1 B'
      by (meson Bet-cases P1 l2-11-b not-cong-1243 not-cong-4312)
    have Q3: Cong B C0 B' C1
      using P1 between-symmetry cong-3421 l2-11-b not-cong-1243 by blast
    thus ?thesis
      by (simp add: Cong3-def Q1 P1)
  qed
  hence P4: Bet A1 B' C1 using P2 l4-6 by blast
  hence Bet A' B' C1
    using P1 Bet-cases between-exchange3 by blast
  thus ?thesis using between-inner-transitivity P1 by blast
qed

```

```

lemma in-angle-one-side:
  assumes  $\neg Col A B C$  and
     $\neg Col B A P$  and
    P InAngle A B C
  shows A B OS P C
proof -
  obtain X where P1: Bet A X C  $\wedge$  (X = B  $\vee$  B Out X P)

```

```

    using InAngle-def assms(3) by auto
  {
    assume X = B
    hence A B OS P C
      using P1 assms(1) bet-col by blast
  }
  {
    assume P2: B Out X P
    obtain C' where P2A: Bet C A C'  $\wedge$  Cong A C' C A
      using segment-construction by blast
    have A B TS X C'
    proof -
      have Q1:  $\neg$  Col X A B
        by (metis Col-def P1 assms(1) assms(2) col-transitivity-2 out-col)
      have Q2 : $\neg$  Col C' A B
        by (metis Col-def Cong-perm P2A assms(1) cong-diff l6-16-1)
      have  $\exists T. \text{Col } T A B \wedge \text{Bet } X T C'$ 
        using Bet-cases P1 P2A between-exchange3 col-trivial-1 by blast
      thus ?thesis
        by (simp add: Q1 Q2 TS-def)
    qed
    hence P3: A B TS P C'
      using P2 col-trivial-3 l9-5 by blast
    hence A B TS C C'
      using P1 P2 bet-out bet-ts--os between-trivial col123--nos col-trivial-3
        invert-two-sides l6-6 l9-2 l9-5
      by (metis P2A assms(1) bet--ts)
    hence A B OS P C
      using OS-def P3 by blast
  }
  thus ?thesis
    using P1  $\langle X = B \implies A B OS P C \rangle$  by blast
qed

lemma inangle-one-side:
  assumes  $\neg$  Col A B C and
     $\neg$  Col A B P and
     $\neg$  Col A B Q and
    P InAngle A B C and
    Q InAngle A B C
  shows A B OS P Q
  by (meson assms(1) assms(2) assms(3) assms(4) assms(5) in-angle-one-side
    not-col-permutation-4 one-side-symmetry one-side-transitivity)

lemma inangle-one-side2:
  assumes  $\neg$  Col A B C and
     $\neg$  Col A B P and
     $\neg$  Col A B Q and
     $\neg$  Col C B P and
     $\neg$  Col C B Q and
    P InAngle A B C and
    Q InAngle A B C
  shows A B OS P Q  $\wedge$  C B OS P Q
  by (meson assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7)
    inangle-one-side l11-24 not-col-permutation-3)

lemma col-conga-col:
  assumes Col A B C and
    A B C CongA D E F
  shows Col D E F
  proof -
    {
      assume Bet A B C
      hence Col D E F
        using Col-def assms(2) bet-conga--bet by blast
    }
  
```

```

{
  assume Bet B C A
  hence Col D E F
  by (meson Col-perm l11-21-a ⟨Bet A B C ⟹ Col D E F⟩ assms(1) assms(2)
    or-bet-out out-col)
}
{
  assume Bet C A B
  hence Col D E F
  by (meson Col-perm l11-21-a ⟨Bet A B C ⟹ Col D E F⟩ assms(1) assms(2)
    or-bet-out out-col)
}
thus ?thesis
  using Col-def ⟨Bet A B C ⟹ Col D E F⟩ ⟨Bet B C A ⟹ Col D E F⟩ assms(1) by blast
qed

```

```

lemma ncol-conga-ncol:
  assumes ¬ Col A B C and
    A B C CongA D E F
  shows ¬ Col D E F
  using assms(1) assms(2) col-conga-col conga-sym by blast

```

```

lemma angle-construction-4:
  assumes A ≠ B and
    C ≠ B and
    A' ≠ B'
  shows ∃ C'. (A B C CongA A' B' C' ∧ Coplanar A' B' C' P)
proof cases
  assume Col A' B' P
  thus ?thesis
    using angle-construction-3 assms(1) assms(2) assms(3) ncop--ncols by blast

```

```

next
  assume ¬ Col A' B' P
  {
    assume Col A B C
    hence ∃ C'. (A B C CongA A' B' C' ∧ Coplanar A' B' C' P)
    by (meson angle-construction-3 assms(1) assms(2) assms(3) col--coplanar col-conga-col)
  }
  {
    assume ¬ Col A B C
    then obtain C' where A B C CongA A' B' C' ∧ A' B' OS C' P
    using ⟨¬ Col A' B' P⟩ angle-construction-1 by blast
    hence ∃ C'. (A B C CongA A' B' C' ∧ Coplanar A' B' C' P)
    using os--coplanar by blast
  }
  thus ?thesis
    using ⟨Col A B C ⟹ ∃ C'. A B C CongA A' B' C' ∧ Coplanar A' B' C' P⟩ by blast
qed

```

```

lemma lea-distincts:
  assumes A B C LeA D E F
  shows A ≠ B ∧ C ≠ B ∧ D ≠ E ∧ F ≠ E
  by (metis LeA-def conga-diff1 conga-diff2 assms inangle-distincts)

```

```

lemma l11-29-a:
  assumes A B C LeA D E F
  shows ∃ Q. (C InAngle A B Q ∧ A B Q CongA D E F)
proof -
  obtain P where P1: P InAngle D E F ∧ A B C CongA D E P
  using LeA-def assms by blast
  hence P2: E ≠ D ∧ B ≠ A ∧ E ≠ F ∧ E ≠ P ∧ B ≠ C
  using conga-diff1 conga-diff2 inangle-distincts by blast
  hence P3: A ≠ B ∧ C ≠ B by blast
  {
    assume Q1: Bet A B C
    hence Q2: Bet D E P

```

```

  by (meson P1 bet-conga--bet Tarski-neutral-dimensionless-axioms)
have Q3: C InAngle A B C
  by (simp add: P3 inangle3123)
obtain X where Q4: Bet D X F  $\wedge$  (X = E  $\vee$  E Out X P)
  using InAngle-def P1 by auto
have A B C CongA D E F
proof -
{
  assume R1: X = E
  have R2: Bet E F P  $\vee$  Bet E P F
  proof -
    have R3: D  $\neq$  E using P2 by blast
    have Bet D E F
      using Col-def Col-perm P1 Q2 col-in-angle-out not-bet-and-out by blast
    have Bet D E P using Q2 by blast
    thus ?thesis using l5-2
      using R3  $\langle$ Bet D E F $\rangle$  by blast
  qed
  have A B C CongA D E F
  proof -
    have B Out A A
      using P3 out-trivial by auto
    moreover have B Out C C
      using P3 out-trivial by auto
    moreover have E Out D D
      using P2 out-trivial by auto
    moreover have E Out F P
      using Out-def P2 R2 by blast
    ultimately show ?thesis
      using P1 l11-10 by blast
  qed
}
{
  assume S1: E Out X P
  have S2: E Out P F
  proof -
    {
      assume Bet E X P
      hence E Out P F
      proof -
        have Bet E X F
          by (meson Bet-perm Q2 Q4  $\langle$ Bet E X P $\rangle$  between-exchange3)
        thus ?thesis
          by (metis Bet-perm S1 bet2--out between-equality-2 between-trivial2
            out2-bet-out out-diff1)
        qed
      }
    }
    {
      assume Bet E P X
      hence Bet D P X
        using Q2 S1 out-diff2 outer-transitivity-between2 by blast
      hence Bet D P F
        using Q4 between-exchange4 by blast
      hence E Out P F
        by (metis P2 Q2 bet-out between-exchange3)
    }
    thus ?thesis
      using Out-def S1  $\langle$ Bet E X P  $\implies$  E Out P F $\rangle$  by blast
  qed
  hence A B C CongA D E F
    by (metis Bet-perm P2 Q1 Q2 bet-out--bet conga-line)
}
thus ?thesis
  using Q4  $\langle$ X = E  $\implies$  A B C CongA D E F $\rangle$  by blast
qed
hence  $\exists$  Q. (C InAngle A B Q  $\wedge$  A B Q CongA D E F)

```

```

    using conga-diff1 conga-diff2 inangle3123 by blast
  }
  {
    assume B Out A C
    obtain Q where D E F CongA A B Q
      by (metis P2 angle-construction-3)
    hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
      by (metis conga-comm  $\langle B \text{ Out } A C \rangle$  conga-diff1 conga-sym out321--inangle)
  }
  {
    assume ZZ:  $\neg \text{Col } A B C$ 
    have Z1:  $D \neq E$ 
      using P2 by blast
    have Z2:  $F \neq E$ 
      using P2 by blast
    have Z3:  $\text{Bet } D E F \vee E \text{ Out } D F \vee \neg \text{Col } D E F$ 
      using not-bet-out by blast
    {
      assume Bet D E F
      obtain Q where Z4:  $\text{Bet } A B Q \wedge \text{Cong } B Q E F$ 
        using segment-construction by blast
      hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
        by (metis InAngle-def P3 Z1 Z2  $\langle \text{Bet } D E F \rangle$  conga-line point-construction-different)
    }
    {
      assume E Out D F
      hence Z5:  $E \text{ Out } D P$ 
        using P1 in-angle-out by blast
      have D E P CongA A B C
        by (simp add: P1 conga-sym)
      hence Z6:  $B \text{ Out } A C$  using l11-21-a Z5
        by blast
      hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
        using  $\langle B \text{ Out } A C \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$  by blast
    }
  }
  {
    assume W1:  $\neg \text{Col } D E F$ 
    obtain Q where W2:  $D E F \text{ CongA } A B Q \wedge A B Q \text{ OS } Q C$ 
      using W1 ZZ angle-construction-1 by blast
    obtain DD where W3:  $E \text{ Out } D DD \wedge \text{Cong } E DD B A$ 
      using P3 Z1 segment-construction-3 by force
    obtain FF where W4:  $E \text{ Out } F FF \wedge \text{Cong } E FF B Q$ 
      by (metis P2 W2 conga-diff56 segment-construction-3)
    hence W5:  $P \text{ InAngle } DD E FF$ 
      using Out-cases P1 P2 W3 l11-25 out-trivial by fastforce
    obtain X where W6:  $\text{Bet } DD X FF \wedge (X = E \vee E \text{ Out } X P)$ 
      using InAngle-def W5 by presburger
    {
      assume W7:  $X = E$ 
      have W8:  $\text{Bet } D E F$ 
      proof -
        have W10:  $E \text{ Out } DD D$ 
          by (simp add: W3 l6-6)
        have E Out FF F
          by (simp add: W4 l6-6)
        thus ?thesis using W6 W7 W10 bet-out-out-bet by blast
      qed
      hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
        using  $\langle \text{Bet } D E F \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$  by blast
    }
  }
  {
    assume V1:  $E \text{ Out } X P$ 
    have  $B \neq C \wedge E \neq X$ 
      using P3 V1 out-diff1 by blast
    then obtain CC where V2:  $B \text{ Out } C CC \wedge \text{Cong } B CC E X$ 
      using segment-construction-3 by blast
  }

```



```

hence V3: A B CC CongA DD E X
  using P1 P2 V1 W3 l11-10 l6-6 out-trivial by force
have V4: Cong A CC DD X
proof -
  have Cong A B DD E
    using W3 not-cong-4321 by blast
  thus ?thesis
    using V2 V3 cong2-conga-cong by blast
qed
have V5: A B Q CongA DD E FF
proof -
  have U1: D E F CongA A B Q
    by (simp add: W2)
  hence U1A: A B Q CongA D E F
    by (simp add: conga-sym)
  have U2: B Out A A
    by (simp add: P3 out-trivial)
  have U3: B Out Q Q
    using W2 conga-diff56 out-trivial by blast
  have U4: E Out DD D
    using W3 l6-6 by blast
  have E Out FF F
    by (simp add: W4 l6-6)
  thus ?thesis using l11-10
    using U1A U2 U3 U4 by blast
qed
hence V6: Cong A Q DD FF
  using Cong-perm W3 W4 cong2-conga-cong by blast
have CC B Q CongA X E FF
proof -
  have U1: B A OS CC Q
    by (metis V2 W2 col124--nos invert-one-side one-side-symmetry
      one-side-transitivity out-one-side)
  have U2: E DD OS X FF
proof -
  have ¬ Col DD E FF
    by (metis Col-perm OS-def TS-def U1 V5 ncol-conga-ncol)
  hence ¬ Col E DD X
    by (metis Col-def V2 V4 W6 ZZ cong-identity l6-16-1 os-distincts out-one-side)
  hence DD E OS X FF
    by (metis Col-perm W6 bet-out not-col-distincts one-side-reflexivity
      out-out-one-side)
  thus ?thesis
    by (simp add: invert-one-side)
qed
have CC B A CongA X E DD
  by (simp add: V3 conga-comm)
thus ?thesis
  using U1 U2 V5 l11-22b by blast
qed
hence V8: Cong CC Q X FF
  using V2 W4 cong2-conga-cong cong-commutativity not-cong-3412 by blast
have V9: CC InAngle A B Q
proof -
  have T2: Q ≠ B
    using W2 conga-diff56 by blast
  have T3: CC ≠ B
    using V2 out-distinct by blast
  have Bet A CC Q
proof -
  have T4: DD X FF Cong3 A CC Q
    using Cong3-def V4 V6 V8 not-cong-3412 by blast
  thus ?thesis
    using W6 l4-6 by blast
qed
hence ∃ X0. Bet A X0 Q ∧ (X0 = B ∨ B Out X0 CC)

```

```

    using out-trivial by blast
  thus ?thesis
    by (simp add: InAngle-def P3 T2 T3)
qed
hence C InAngle A B Q
  using V2 inangle-distincts l11-25 out-trivial by blast
hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
  using W2 conga-sym by blast
}
hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
  using W6  $\langle X = E \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$  by blast
}
hence  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
  using Z3  $\langle E \text{ Out } D F \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$ 
   $\langle \text{Bet } D E F \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$  by blast
}
thus ?thesis
  using  $\langle B \text{ Out } A C \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$ 
   $\langle \text{Bet } A B C \implies \exists Q. C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F \rangle$  not-bet-out by blast
qed

```

lemma in-angle-line:

```

assumes P  $\neq$  B and
  A  $\neq$  B and
  C  $\neq$  B and
  Bet A B C
shows P InAngle A B C
using InAngle-def assms(1) assms(2) assms(3) assms(4) by auto

```

lemma l11-29-b:

```

assumes  $\exists Q. (C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F)$ 
shows A B C LeA D E F
proof -
  obtain Q where P1: C InAngle A B Q  $\wedge$  A B Q CongA D E F
    using assms by blast
  obtain X where P2: Bet A X Q  $\wedge$  (X = B  $\vee$  B Out X C)
    using InAngle-def P1 by auto
  {
    assume P2A: X = B
    obtain P where P3: A B C CongA D E P
      using angle-construction-3 assms conga-diff45 inangle-distincts by fastforce
    have P InAngle D E F
    proof -
      have O1: Bet D E F
        by (metis P1 P2 bet-conga--bet P2A)
      have O2: P  $\neq$  E
        using P3 conga-diff56 by auto
      have O3: D  $\neq$  E
        using P3 conga-diff45 by auto
      have F  $\neq$  E
        using P1 conga-diff56 by blast
      thus ?thesis using in-angle-line
        by (simp add: O1 O2 O3)
    qed
    hence A B C LeA D E F
      using LeA-def P3 by blast
  }
  {
    assume G1: B Out X C
    obtain DD where G2: E Out D DD  $\wedge$  Cong E DD B A
      by (metis assms conga-diff1 conga-diff45 segment-construction-3)
    have G3: D  $\neq$  E  $\wedge$  DD  $\neq$  E
      using G2 out-diff1 out-diff2 by blast
    obtain FF where G3G: E Out F FF  $\wedge$  Cong E FF B Q
      by (metis P1 conga-diff56 inangle-distincts segment-construction-3)
    hence G3A: F  $\neq$  E
  }

```

```

    using out-diff1 by blast
have G3B:  $FF \neq E$ 
    using G3G out-distinct by blast
have G4:  $Bet\ A\ B\ C \vee B\ Out\ A\ C \vee \neg\ Col\ A\ B\ C$ 
    using not-bet-out by blast
{
  assume G5:  $Bet\ A\ B\ C$ 
  have G6:  $F\ InAngle\ D\ E\ F$ 
    by (simp add: G3 G3A inangle3123)
  have A B C CongA A B Q
  proof -
    have  $A \neq B$ 
      using P1 conga-diff1 by blast
    hence B Out A A
      using out-trivial by blast
    moreover
    have B Out Q C
    proof -
      {
        assume Bet B X C
        hence Bet A B X
          using G5 between-inner-transitivity by blast
        hence Bet B X Q
          using P2 between-exchange3 by blast
        hence B Out Q C
          by (metis G1 bet-out l6-6 l6-7 out-diff1)
      }
      moreover
      {
        assume Bet B C X
        hence Bet B X Q
          by (metis G5 P2 between-exchange3 l6-3-1 outer-transitivity-between)
        hence Bet B C Q
          using  $\langle Bet\ B\ C\ X \rangle$  between-exchange4 by blast
        hence B Out Q C
          by (metis G1 Out-def between-identity)
      }
      ultimately show ?thesis
        using G1 Out-def by blast
    qed
    ultimately show ?thesis
      by (simp add: out2--conga)
  qed
  hence A B C CongA D E F
    using P1 conga-trans by blast
  hence A B C LeA D E F
    using G6 LeA-def by blast
}
{
  assume G8:  $B\ Out\ A\ C$ 
  have G9:  $D\ InAngle\ D\ E\ F$ 
    by (simp add: G3 G3A inangle1123)
  have A B C CongA D E D
    by (simp add: G3 G8 l11-21-b out-trivial)
  hence A B C LeA D E F using G9 LeA-def by blast
}
{
  assume R1:  $\neg\ Col\ A\ B\ C$ 
  have R2:  $Bet\ A\ B\ Q \vee B\ Out\ A\ Q \vee \neg\ Col\ A\ B\ Q$ 
    using not-bet-out by blast
  {
    assume R3:  $Bet\ A\ B\ Q$ 
    obtain P where R4:  $A\ B\ C\ CongA\ D\ E\ P$ 
      by (metis G3 LeA-def  $\langle Bet\ A\ B\ C \implies A\ B\ C\ LeA\ D\ E\ F \rangle$  angle-construction-3
        not-bet-distincts)
    have R5:  $P\ InAngle\ D\ E\ F$ 

```

```

proof -
  have R6:  $P \neq E$ 
    using R4 conga-diff56 by auto
  have Bet D E F
    by (metis P1 R3 bet-conga--bet)
  thus ?thesis
    by (simp add: R6 G3 G3A in-angle-line)
qed
hence A B C LeA D E F using R4 R5 LeA-def by blast
}
{
  assume S1: B Out A Q
  have S2: B Out A C
    using G1 P2 S1 l6-7 out-bet-out-1 by blast
  have S3: Col A B C
    by (simp add: Col-perm S2 out-col)
  hence A B C LeA D E F
    using R1 by blast
}
{
  assume S3B:  $\neg$  Col A B Q
  obtain P where S4: A B C CongA D E P  $\wedge$  D E OS P F
    by (meson P1 R1 ncol-conga-ncol Tarski-neutral-dimensionless-axioms
        S3B angle-construction-1)
  obtain PP where S4A: E Out P PP  $\wedge$  Cong E PP B X
    by (metis G1 S4 os-distincts out-diff1 segment-construction-3)
  have S5: P InAngle D E F
  proof -
    have PP InAngle DD E FF
    proof -
      have Z3: PP  $\neq$  E
        using S4A l6-3-1 by blast
      have Z4: Bet DD PP FF
    proof -
      have L1: C B Q CongA P E F
    proof -
      have K1: B A OS C Q
        using Col-perm P1 R1 S3B in-angle-one-side invert-one-side by blast
      have K2: E D OS P F
        by (simp add: S4 invert-one-side)
      have C B A CongA P E D
        by (simp add: S4 conga-comm)
      thus ?thesis
        using K1 K2 P1 l11-22b by auto
    qed
    have L2: Cong DD FF A Q
  proof -
    have DD E FF CongA A B Q
  proof -
    have L3: A B Q CongA D E F
      by (simp add: P1)
    hence L3A: D E F CongA A B Q
      using conga-sym by blast
    have L4: E Out DD D
      using G2 Out-cases by auto
    have L5: E Out FF F
      using G3G Out-cases by blast
    have L6: B Out A A
      using S3B not-col-distincts out-trivial by auto
    have B Out Q Q
      by (metis S3B not-col-distincts out-trivial)
    thus ?thesis using L3A L4 L5 L6 l11-10
      by blast
    qed
    have L2B: Cong DD E A B
      using Cong-perm G2 by blast

```

```

have Cong E FF B Q
  by (simp add: G3G)
thus ?thesis
  using L2B ⟨DD E FF CongA A B Q⟩ cong2-conga-cong by auto
qed
have L8: Cong A X DD PP
proof -
  have L9: A B X CongA DD E PP
  proof -
    have L9B: B Out A A
      using S3B not-col-distincts out-trivial by blast
    have L9D: E Out D D
      using G3 out-trivial by auto
    have E Out PP P
      using Out-cases S4A by blast
    thus ?thesis using l11-10 S4 L9B G1 L9D
      using G2 Out-cases by blast
  qed
  have L10: Cong A B DD E
    using G2 not-cong-4321 by blast
  have Cong B X E PP
    using Cong-perm S4A by blast
  thus ?thesis
    using L10 L9 cong2-conga-cong by blast
qed
have A X Q Cong3 DD PP FF
proof -
  have L12B: Cong A Q DD FF
    using L2 not-cong-3412 by blast
  have Cong X Q PP FF
  proof -
    have L13A: X B Q CongA PP E FF
    proof -
      have L13AC: B Out Q Q
        by (metis S3B col-trivial-2 out-trivial)
      have L13AD: E Out PP P
        by (simp add: S4A l6-6)
      have E Out FF F
        by (simp add: G3G l6-6)
      thus ?thesis
        using L1 G1 L13AC L13AD l11-10 by blast
    qed
    have L13B: Cong X B PP E
      using S4A not-cong-4321 by blast
    have Cong B Q E FF
      using G3G not-cong-3412 by blast
    thus ?thesis
      using L13A L13B cong2-conga-cong by auto
  qed
  thus ?thesis
    by (simp add: Cong3-def L12B L8)
qed
thus ?thesis using P2 l4-6 by blast
qed
have PP = E ∨ E Out PP PP
  using out-trivial by auto
thus ?thesis
  using InAngle-def G3 G3B Z3 Z4 by auto
qed
thus ?thesis
  using G2 G3G S4A l11-25 by blast
qed
hence A B C LeA D E F
  using S4 LeA-def by blast
}
hence A B C LeA D E F

```

```

    using R2  $\langle B \text{ Out } A \ Q \implies A \ B \ C \text{ LeA } D \ E \ F \rangle \langle \text{Bet } A \ B \ Q \implies A \ B \ C \text{ LeA } D \ E \ F \rangle$  by blast
  }
  hence  $A \ B \ C \text{ LeA } D \ E \ F$ 
    using G4  $\langle B \text{ Out } A \ C \implies A \ B \ C \text{ LeA } D \ E \ F \rangle \langle \text{Bet } A \ B \ C \implies A \ B \ C \text{ LeA } D \ E \ F \rangle$  by blast
  }
  thus ?thesis
    using P2  $\langle X = B \implies A \ B \ C \text{ LeA } D \ E \ F \rangle$  by blast
qed

```

lemma *bet-in-angle-bet*:

```

  assumes  $\text{Bet } A \ B \ P$  and
     $P \text{ InAngle } A \ B \ C$ 
  shows  $\text{Bet } A \ B \ C$ 
  by (metis (no-types) Col-def Col-perm assms(1) assms(2) col-in-angle-out not-bet-and-out)

```

lemma *lea-line*:

```

  assumes  $\text{Bet } A \ B \ P$  and
     $A \ B \ P \text{ LeA } A \ B \ C$ 
  shows  $\text{Bet } A \ B \ C$ 
  by (metis bet-conga--bet l11-29-a assms(1) assms(2) bet-in-angle-bet)

```

lemma *eq-conga-out*:

```

  assumes  $A \ B \ A \ \text{CongA } D \ E \ F$ 
  shows  $E \text{ Out } D \ F$ 
  by (metis CongA-def assms l11-21-a out-trivial)

```

lemma *out-conga-out*:

```

  assumes  $B \text{ Out } A \ C$  and
     $A \ B \ C \text{ CongA } D \ E \ F$ 
  shows  $E \text{ Out } D \ F$ 
  using assms(1) assms(2) l11-21-a by blast

```

lemma *conga-ex-cong3*:

```

  assumes  $A \ B \ C \text{ CongA } A' \ B' \ C'$ 
  shows  $\exists \ AA \ CC. ((B \text{ Out } A \ AA \wedge B \text{ Out } C \ CC) \longrightarrow AA \ B \ CC \text{ Cong3 } A' \ B' \ C')$ 
  using out-diff2 by blast

```

lemma *conga-preserves-in-angle*:

```

  assumes  $A \ B \ C \text{ CongA } A' \ B' \ C'$  and
     $A \ B \ I \text{ CongA } A' \ B' \ I'$  and
     $I \text{ InAngle } A \ B \ C$  and  $A' \ B' \ OS \ I' \ C'$ 
  shows  $I' \text{ InAngle } A' \ B' \ C'$ 

```

proof –

```

  have P1:  $A \neq B$ 
    using assms(1) conga-diff1 by auto
  have P2:  $B \neq C$ 
    using assms(1) conga-diff2 by blast
  have P3:  $A' \neq B'$ 
    using assms(1) conga-diff45 by auto
  have P4:  $B' \neq C'$ 
    using assms(1) conga-diff56 by blast
  have P5:  $I \neq B$ 
    using assms(2) conga-diff2 by auto
  have P6:  $I' \neq B'$ 
    using assms(2) conga-diff56 by blast
  have P7:  $\text{Bet } A \ B \ C \vee B \text{ Out } A \ C \vee \neg \text{Col } A \ B \ C$ 
    using l6-4-2 by blast
  {
    assume  $\text{Bet } A \ B \ C$ 
    have Q1:  $\text{Bet } A' \ B' \ C'$ 
      using  $\langle \text{Bet } A \ B \ C \rangle$  assms(1) assms(4) bet-col col124--nos col-conga-col by blast
    hence  $I' \text{ InAngle } A' \ B' \ C'$ 
      using assms(4) bet-col col124--nos by auto
  }
  {
    assume  $B \text{ Out } A \ C$ 

```

```

hence I' InAngle A' B' C'
  by (metis P4 assms(2) assms(3) in-angle-out l11-21-a out321--inangle)
}
{
  assume Z1:  $\neg$  Col A B C
  have Q2:  $Bet\ A\ B\ I \vee B\ Out\ A\ I \vee \neg\ Col\ A\ B\ I$ 
    by (simp add: or-bet-out)
  {
    assume Bet A B I
    hence I' InAngle A' B' C'
      using  $\langle Bet\ A\ B\ C \implies I'\ InAngle\ A'\ B'\ C' \rangle$  assms(3) bet-in-angle-bet by blast
  }
  {
    assume B Out A I
    hence I' InAngle A' B' C'
      using P4 assms(2) l11-21-a out321--inangle by auto
  }
  {
    assume  $\neg$  Col A B I
    obtain AA' where Q3:  $B'\ Out\ A'\ AA' \wedge Cong\ B'\ AA'\ B\ A$ 
      using P1 P3 segment-construction-3 by presburger
    obtain CC' where Q4:  $B'\ Out\ C'\ CC' \wedge Cong\ B'\ CC'\ B\ C$ 
      using P2 P4 segment-construction-3 by presburger
    obtain J where Q5:  $Bet\ A\ J\ C \wedge (J = B \vee B\ Out\ J\ I)$ 
      using InAngle-def assms(3) by auto
    have Q6:  $B \neq J$ 
      using Q5 Z1 bet-col by auto
    have Q7:  $\neg$  Col A B J
      using Q5 Q6  $\langle \neg\ Col\ A\ B\ I \rangle$  col-permutation-2 col-transitivity-1 out-col by blast
    have  $\neg$  Col A' B' I'
      by (metis assms(4) col123--nos)
    hence  $\exists\ C'. (A\ B\ J\ CongA\ A'\ B'\ C' \wedge A'\ B'\ OS\ C'\ I')$ 
      using Q7 angle-construction-1 by blast
    then obtain J' where Q8:  $A\ B\ J\ CongA\ A'\ B'\ J' \wedge A'\ B'\ OS\ J'\ I'$  by blast
    have Q9:  $B' \neq J'$ 
      using Q8 conga-diff56 by blast
    obtain JJ' where Q10:  $B'\ Out\ J'\ JJ' \wedge Cong\ B'\ JJ'\ B\ J$ 
      using segment-construction-3 Q6 Q9 by blast
    have Q11:  $\neg$  Col A' B' J'
      using Q8 col123--nos by blast
    have Q12:  $A' \neq JJ'$ 
      by (metis Col-perm Q10 Q11 out-col)
    have Q13:  $B' \neq JJ'$ 
      using Q10 out-distinct by blast
    have Q14:  $\neg$  Col A' B' JJ'
      using Col-perm Q10 Q11 Q13 l6-16-1 out-col by blast
    have Q15:  $A\ B\ C\ CongA\ AA'\ B'\ CC'$ 
    proof -
      have T2:  $C \neq B$  using P2 by auto
      have T3:  $AA' \neq B'$ 
        using Out-def Q3 by blast
      have T4:  $CC' \neq B'$ 
        using Q4 out-distinct by blast
      have T5:  $\forall\ A'\ C'\ D'\ F'. (B\ Out\ A'\ A \wedge B\ Out\ C'\ C \wedge B'\ Out\ D'\ AA' \wedge$ 
 $B'\ Out\ F'\ CC' \wedge Cong\ B\ A'\ B'\ D' \wedge Cong\ B\ C'\ B'\ F'$ 
 $\longrightarrow Cong\ A'\ C'\ D'\ F')$ 
        using Q3 Q4 l11-4-1 Tarski-neutral-dimensionless-axioms assms(1) l6-6 l6-7
        by blast
      thus ?thesis using P1 T2 T3 T4 l11-4-2 by blast
    qed
    have Q16:  $A'\ B'\ J'\ CongA\ A'\ B'\ JJ'$ 
    proof -
      have P9:  $B'\ Out\ A'\ A'$ 
        by (simp add: P3 out-trivial)
      have B' Out JJ' J'
        using Out-cases Q10 by auto
  }
}

```

```

thus ?thesis
  using l11-10
  by (simp add: P9 out2--conga)
qed
have Q17:  $B' \text{ Out } I' JJ' \vee A' B' TS I' JJ'$ 
proof –
  have Coplanar  $A' I' B' J'$ 
    by (metis Q8 ncoplanar-perm-3 os--coplanar)
  hence Coplanar  $A' I' B' JJ'$ 
    using Q10 Q9 col-cop--cop out-col by blast
  hence R1: Coplanar  $A' B' I' JJ'$  using coplanar-perm-2
    by blast
  have  $A' B' I' \text{ CongA } A' B' JJ'$ 
proof –
  have R2:  $A' B' I' \text{ CongA } A B I$ 
    by (simp add: assms(2) conga-sym)
  have  $A B I \text{ CongA } A' B' JJ'$ 
proof –
  have f1:  $\forall p \text{ pa pb. } \neg p \text{ Out pa pb} \wedge \neg p \text{ Out pb pa} \vee p \text{ Out pa pb}$ 
    using Out-cases by blast
  hence f2:  $B' \text{ Out } JJ' J'$ 
    using Q10 by blast
  have  $B \text{ Out } J I$ 
    by (metis Q5 Q6)
  thus ?thesis
    using f2 f1 by (meson P3 Q8 l11-10 Tarski-neutral-dimensionless-axioms
       $\langle \neg \text{ Col } A B I \rangle$  col-one-side-out col-trivial-2 one-side-reflexivity out-trivial)
qed
thus ?thesis
  using R2 conga-trans by blast
qed
thus ?thesis using R1 conga-cop--or-out-ts by blast
qed
{
  assume Z2:  $B' \text{ Out } I' JJ'$ 
  have Z3:  $J B C \text{ CongA } J' B' C'$ 
proof –
  have R1:  $B A OS J C$ 
    by (metis Q5 Q7 Z1 bet-out invert-one-side not-col-distincts out-one-side)
  have R2:  $B' A' OS J' C'$ 
    by (meson Q10 Z2 assms(4) invert-one-side l6-6 one-side-symmetry out-out-one-side)
  have  $J B A \text{ CongA } J' B' A'$ 
    using Q8 conga-comm by blast
  thus ?thesis using assms(1) R1 R2 l11-22b by blast
qed
hence  $I' \text{ InAngle } A' B' C'$ 
proof –
  have  $A J C \text{ Cong3 } AA' JJ' CC'$ 
proof –
  have R8:  $\text{Cong } A J AA' JJ'$ 
proof –
  have R8A:  $A B J \text{ CongA } AA' B' JJ'$ 
proof –
  have R8AB:  $B \text{ Out } A A$ 
    by (simp add: P1 out-trivial)
  have R8AC:  $B \text{ Out } J I$ 
    using Q5 Q6 by auto
  have R8AD:  $B' \text{ Out } AA' A'$ 
    using Out-cases Q3 by auto
  have  $B' \text{ Out } JJ' I'$ 
    using Out-cases Z2 by blast
  thus ?thesis
    using assms(2) R8AB R8AC R8AD l11-10 by blast
qed
have R8B:  $\text{Cong } A B AA' B'$ 
  using Q3 not-cong-4321 by blast

```



```

have R8C: Cong B J B' JJ'
  using Q10 not-cong-3412 by blast
thus ?thesis
  using R8A R8B cong2-conga-cong by blast
qed
have LR8A: Cong A C AA' CC'
  using Q15 Q3 Q4 cong2-conga-cong cong-4321 cong-symmetry by blast
have Cong J C JJ' CC'
proof -
  have K1:B' Out JJ' J'
    using Out-cases Q10 by auto
  have B' Out CC' C'
    using Out-cases Q4 by auto
  hence J' B' C' CongA JJ' B' CC' using K1
    by (simp add: out2--conga)
  hence LR9A: J B C CongA JJ' B' CC'
    using Z3 conga-trans by blast
  have LR9B: Cong J B JJ' B'
    using Q10 not-cong-4321 by blast
  have Cong B C B' CC'
    using Q4 not-cong-3412 by blast
  thus ?thesis
    using LR9A LR9B cong2-conga-cong by blast
qed
thus ?thesis using R8 LR8A
  by (simp add: Cong3-def)
qed
hence R10: Bet AA' JJ' CC' using Q5 l4-6 by blast
have JJ' InAngle AA' B' CC'
proof -
  have R11: AA' ≠ B'
    using Out-def Q3 by auto
  have R12: CC' ≠ B'
    using Out-def Q4 by blast
  have Bet AA' JJ' CC' ∧ (JJ' = B' ∨ B' Out JJ' JJ')
    using R10 out-trivial by auto
  thus ?thesis
    using InAngle-def Q13 R11 R12 by auto
qed
thus ?thesis
  using Z2 Q3 Q4 l11-25 by blast
qed
}
{
  assume X1: A' B' TS I' JJ'
  have A' B' OS I' J'
    by (simp add: Q8 one-side-symmetry)
  hence X2: B' A' OS I' JJ'
    using Q10 invert-one-side out-out-one-side by blast
  hence I' InAngle A' B' C'
    using X1 invert-one-side l9-9 by blast
}
hence I' InAngle A' B' C'
  using Q17 ⟨B' Out I' JJ' ⟹ I' InAngle A' B' C'⟩ by blast
}
hence I' InAngle A' B' C'
  using Q2 ⟨B Out A I ⟹ I' InAngle A' B' C'⟩ ⟨Bet A B I ⟹ I' InAngle A' B' C'⟩ by blast
}
thus ?thesis
  using P7 ⟨B Out A C ⟹ I' InAngle A' B' C'⟩ ⟨Bet A B C ⟹ I' InAngle A' B' C'⟩ by blast
qed

lemma l11-30:
  assumes A B C LeA D E F and
    A B C CongA A' B' C' and
    D E F CongA D' E' F'

```

```

shows  $A' B' C' \text{ LeA } D' E' F'$ 
proof -
obtain  $Q$  where  $P1: C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F$ 
  using assms(1) l11-29-a by blast
have  $P1A: C \text{ InAngle } A B Q$ 
  using  $P1$  by simp
have  $P1B: A B Q \text{ CongA } D E F$ 
  using  $P1$  by simp
have  $P2: A \neq B$ 
  using  $P1A$  inangle-distincts by auto
have  $P3: C \neq B$ 
  using  $P1A$  inangle-distincts by blast
have  $P4: A' \neq B'$ 
  using CongA-def assms(2) by blast
have  $P5: C' \neq B'$ 
  using CongA-def assms(2) by auto
have  $P6: D \neq E$ 
  using CongA-def P1B by blast
have  $P7: F \neq E$ 
  using CongA-def P1B by blast
have  $P8: D' \neq E'$ 
  using CongA-def assms(3) by blast
have  $P9: F' \neq E'$ 
  using CongA-def assms(3) by blast
have  $P10: \text{Bet } A' B' C' \vee B' \text{ Out } A' C' \vee \neg \text{Col } A' B' C'$ 
  using or-bet-out by blast
{
  assume  $\text{Bet } A' B' C'$ 
  hence  $\exists Q'. (C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F')$ 
    by (metis P1 P4 P5 P8 P9 assms(2) assms(3) bet-conga--bet bet-in-angle-bet
        conga-line conga-sym inangle3123)
}
{
  assume  $R1: B' \text{ Out } A' C'$ 
  obtain  $Q'$  where  $R2: D' E' F' \text{ CongA } A' B' Q'$ 
    using  $P4 P8 P9$  angle-construction-3 by blast
  hence  $C' \text{ InAngle } A' B' Q'$ 
    using col-in-angle P1 R1 conga-diff56 out321--inangle by auto
  hence  $\exists Q'. (C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F')$ 
    using  $R2$  conga-sym by blast
}
{
  assume  $R3: \neg \text{Col } A' B' C'$ 
  have  $R3A: \text{Bet } D' E' F' \vee E' \text{ Out } D' F' \vee \neg \text{Col } D' E' F'$ 
    using or-bet-out by blast
  {
    assume  $\text{Bet } D' E' F'$ 
    have  $\exists Q'. (C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F')$ 
      by (metis P4 P5 P8 P9 <Bet D' E' F'> conga-line in-angle-line
          point-construction-different)
  }
  {
    assume  $R4A: E' \text{ Out } D' F'$ 
    obtain  $Q'$  where  $R4: D' E' F' \text{ CongA } A' B' Q'$ 
      using  $P4 P8 P9$  angle-construction-3 by blast
    hence  $R5: B' \text{ Out } A' Q'$  using out-conga-out R4A by blast
    have  $R6: A B Q \text{ CongA } D' E' F'$ 
      using  $P1$  assms(3) conga-trans by blast
    hence  $R7: B \text{ Out } A Q$  using out-conga-out R4A R4
      using conga-sym by blast
    have  $R8: B \text{ Out } A C$ 
      using  $P1A R7$  in-angle-out by blast
    hence  $R9: B' \text{ Out } A' C'$  using out-conga-out assms(2)
      by blast
    have  $\exists Q'. (C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F')$ 
      by (simp add: R9 <B' Out A' C' ==> \exists Q'. C' InAngle A' B' Q' \wedge A' B' Q' CongA D' E' F'>)
  }
}

```

```

}
{
  assume  $\neg \text{Col } D' E' F'$ 
  obtain  $QQ$  where  $S1: D' E' F' \text{ CongA } A' B' QQ \wedge A' B' OS QQ C'$ 
    using  $R3 \langle \neg \text{Col } D' E' F' \rangle \text{ angle-construction-1}$  by blast
  have  $S1A: A B Q \text{ CongA } A' B' QQ$  using  $S1$ 
    using  $P1 \text{ assms}(3) \text{ conga-trans}$  by blast
  have  $A' B' OS C' QQ$  using  $S1$ 
    by (simp add:  $S1 \text{ one-side-symmetry}$ )
  hence  $S2: C' \text{ InAngle } A' B' QQ$  using  $\text{conga-preserves-in-angle } S1A$ 
    using  $P1A \text{ assms}(2)$  by blast
  have  $S3: A' B' QQ \text{ CongA } D' E' F'$ 
    by (simp add:  $S1 \text{ conga-sym}$ )
  hence  $\exists Q'. (C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F')$ 
    using  $S2$  by auto
}
hence  $\exists Q'. (C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F')$ 
  using  $R3A \langle E' \text{ Out } D' F' \implies \exists Q'. C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F' \rangle$ 
     $\langle \text{Bet } D' E' F' \implies \exists Q'. C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F' \rangle$  by blast
}
thus ?thesis using  $l11-29-b$ 
  using  $P10 \langle B' \text{ Out } A' C' \implies \exists Q'. C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F' \rangle$ 
     $\langle \text{Bet } A' B' C' \implies \exists Q'. C' \text{ InAngle } A' B' Q' \wedge A' B' Q' \text{ CongA } D' E' F' \rangle$  by blast
qed

```

lemma $l11-31-1$:
assumes $B \text{ Out } A C$ **and**
 $D \neq E$ **and**
 $F \neq E$
shows $A B C \text{ LeA } D E F$
by (*metis* (*full-types*) *LeA-def* *assms*(1) *assms*(2) *assms*(3) *l11-21-b* *out321--inangle*
segment-construction-3)

lemma $l11-31-2$:
assumes $A \neq B$ **and**
 $C \neq B$ **and**
 $D \neq E$ **and**
 $F \neq E$ **and**
 $\text{Bet } D E F$
shows $A B C \text{ LeA } D E F$
by (*metis* *LeA-def* *angle-construction-3* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5)
conga-diff56 *in-angle-line*)

lemma *lea-refl*:
assumes $A \neq B$ **and**
 $C \neq B$
shows $A B C \text{ LeA } A B C$
by (*meson* *assms*(1) *assms*(2) *conga-refl* *l11-29-b* *out341--inangle* *out-trivial*)

lemma *conga--lea*:
assumes $A B C \text{ CongA } D E F$
shows $A B C \text{ LeA } D E F$
by (*metis* *conga-diff1* *conga-diff2* *l11-30* *assms* *conga-refl* *lea-refl*)

lemma *conga--lea456123*:
assumes $A B C \text{ CongA } D E F$
shows $D E F \text{ LeA } A B C$
by (*simp* add: *conga--lea* *Tarski-neutral-dimensionless-axioms* *assms* *conga-sym*)

lemma *lea-left-comm*:
assumes $A B C \text{ LeA } D E F$
shows $C B A \text{ LeA } D E F$
by (*metis* *assms* *conga-pseudo-refl* *conga-refl* *l11-30* *lea-distincts*)

lemma *lea-right-comm*:
assumes $A B C \text{ LeA } D E F$

shows $A B C LeA F E D$
 by (*meson assms conga-right-comm l11-29-a l11-29-b*)

lemma *lea-comm*:
 assumes $A B C LeA D E F$
 shows $C B A LeA F E D$
 using *assms lea-left-comm lea-right-comm* by blast

lemma *lta-left-comm*:
 assumes $A B C LtA D E F$
 shows $C B A LtA D E F$
 by (*meson LtA-def conga-left-comm lea-left-comm assms*)

lemma *lta-right-comm*:
 assumes $A B C LtA D E F$
 shows $A B C LtA F E D$
 by (*meson LtA-def conga-comm lea-comm lta-left-comm assms*)

lemma *lta-comm*:
 assumes $A B C LtA D E F$
 shows $C B A LtA F E D$
 using *assms lta-left-comm lta-right-comm* by blast

lemma *lea-out4--lea*:
 assumes $A B C LeA D E F$ and
 $B Out A A'$ and
 $B Out C C'$ and
 $E Out D D'$ and
 $E Out F F'$
 shows $A' B C' LeA D' E F'$
 using *assms(1) assms(2) assms(3) assms(4) assms(5) l11-30 l6-6 out2--conga* by auto

lemma *lea121345*:
 assumes $A \neq B$ and
 $C \neq D$ and
 $D \neq E$
 shows $A B A LeA C D E$
 using *assms(1) assms(2) assms(3) l11-31-1 out-trivial* by auto

lemma *inangle--lea*:
 assumes $P InAngle A B C$
 shows $A B P LeA A B C$
 by (*metis l11-29-b assms conga-refl inangle-distincts*)

lemma *inangle--lea-1*:
 assumes $P InAngle A B C$
 shows $P B C LeA A B C$
 by (*simp add: inangle--lea lea-comm assms l11-24*)

lemma *inangle--lta*:
 assumes $\neg Col P B C$ and
 $P InAngle A B C$
 shows $A B P LtA A B C$
 by (*metis LtA-def TS-def conga-cop--or-out-ts conga-os--out inangle--lea ncol-conga-ncol
 assms(1) assms(2) col-one-side-out col-trivial-3 in-angle-one-side inangle--coplanar
 invert-two-sides l11-21-b ncoplanar-perm-12 not-col-permutation-3 one-side-reflexivity*)

lemma *in-angle-trans*:
 assumes $C InAngle A B D$ and
 $D InAngle A B E$
 shows $C InAngle A B E$
 proof –
 obtain CC where $P1: Bet A CC D \wedge (CC = B \vee B Out CC C)$
 using *InAngle-def assms(1)* by auto
 obtain DD where $P2: Bet A DD E \wedge (DD = B \vee B Out DD D)$

```

    using InAngle-def assms(2) by auto
  hence P3: Bet A DD E by simp
  have P4: DD = B ∨ B Out DD D using P2 by simp
  {
    assume CC = B ∧ DD = B
    hence C InAngle A B E
      using InAngle-def P2 assms(1) assms(2) by auto
  }
  {
    assume CC = B ∧ B Out DD D
    hence C InAngle A B E
      by (metis InAngle-def P1 assms(1) assms(2) bet-in-angle-bet)
  }
  {
    assume B Out CC C ∧ DD = B
    hence C InAngle A B E
      by (metis Out-def P2 assms(2) in-angle-line inangle-distincts)
  }
  {
    assume P3: B Out CC C ∧ B Out DD D
    hence P3A: B Out CC C by simp
    have P3B: B Out DD D using P3 by simp
    have C InAngle A B DD
      using P3 assms(1) inangle-distincts l11-25 out-trivial by blast
    then obtain CC' where T1: Bet A CC' DD ∧ (CC' = B ∨ B Out CC' C)
      using InAngle-def by auto
    {
      assume CC' = B
      hence C InAngle A B E
        by (metis P2 P3 T1 assms(2) between-exchange4 in-angle-line inangle-distincts
          out-diff2)
    }
    {
      assume B Out CC' C
      hence C InAngle A B E
        by (metis InAngle-def P2 T1 assms(1) assms(2) between-exchange4)
    }
    hence C InAngle A B E
      using T1 ⟨CC' = B ⟹ C InAngle A B E⟩ by blast
  }
  thus ?thesis
    using P1 P2 ⟨B Out CC C ∧ DD = B ⟹ C InAngle A B E⟩
      ⟨CC = B ∧ B Out DD D ⟹ C InAngle A B E⟩
      ⟨CC = B ∧ DD = B ⟹ C InAngle A B E⟩ by blast
qed

```

lemma lea-trans:

assumes A B C LeA A1 B1 C1 and

A1 B1 C1 LeA A2 B2 C2

shows A B C LeA A2 B2 C2

proof –

obtain P1 where T1: P1 InAngle A1 B1 C1 ∧ A B C CongA A1 B1 P1

using LeA-def assms(1) by auto

obtain P2 where T2: P2 InAngle A2 B2 C2 ∧ A1 B1 C1 CongA A2 B2 P2

using LeA-def assms(2) by blast

have T3: A ≠ B

using CongA-def T1 by auto

have T4: C ≠ B

using CongA-def T1 by blast

have T5: A1 ≠ B1

using T1 inangle-distincts by blast

have T6: C1 ≠ B1

using T1 inangle-distincts by blast

have T7: A2 ≠ B2

using T2 inangle-distincts by blast

have T8: C2 ≠ B2

```

using T2 inangle-distincts by blast
have T9: Bet A B C  $\vee$  B Out A C  $\vee$   $\neg$  Col A B C
using not-out-bet by auto
{
  assume Bet A B C
  hence A B C LeA A2 B2 C2
    by (metis T1 T2 T3 T4 T7 T8 bet-conga--bet bet-in-angle-bet l11-31-2)
}
{
  assume B Out A C
  hence A B C LeA A2 B2 C2
    by (simp add: T7 T8 l11-31-1)
}
{
  assume H1:  $\neg$  Col A B C
  have T10: Bet A2 B2 C2  $\vee$  B2 Out A2 C2  $\vee$   $\neg$  Col A2 B2 C2
  using not-out-bet by auto
  {
    assume Bet A2 B2 C2
    hence A B C LeA A2 B2 C2
      by (simp add: T3 T4 T7 T8 l11-31-2)
  }
  {
    assume T10A: B2 Out A2 C2
    have B Out A C
    proof -
      have B1 Out A1 P1
      proof -
        have B1 Out A1 C1 using T2 conga-sym T2 T10A in-angle-out out-conga-out by blast
        thus ?thesis using T1 in-angle-out by blast
      qed
      thus ?thesis using T1 conga-sym l11-21-a by blast
    qed
    hence A B C LeA A2 B2 C2
      using  $\langle B \text{ Out } A \ C \implies A \ B \ C \text{ LeA } A2 \ B2 \ C2 \rangle$  by blast
  }
}
{
  assume T12:  $\neg$  Col A2 B2 C2
  obtain P where T13: A B C CongA A2 B2 P  $\wedge$  A2 B2 OS P C2
  using T12 H1 angle-construction-1 by blast
  have T14: A2 B2 OS P2 C2
  proof -
    have  $\neg$  Col B2 A2 P2
    proof -
      have B2  $\neq$  A2
      using T7 by auto
      {
        assume H2: P2 = A2
        have A2 B2 A2 CongA A1 B1 C1
        using T2 H2 conga-sym by blast
        hence B1 Out A1 C1
        using eq-conga-out by blast
        hence B1 Out A1 P1
        using T1 in-angle-out by blast
        hence B Out A C
        using T1 conga-sym out-conga-out by blast
        hence False
        using Col-cases H1 out-col by blast
      }
    qed
    hence P2  $\neq$  A2 by blast
    have Bet A2 B2 P2  $\vee$  B2 Out A2 P2  $\vee$   $\neg$  Col A2 B2 P2
    using not-out-bet by auto
    {
      assume H4: Bet A2 B2 P2
      hence Bet A2 B2 C2
      using T2 bet-in-angle-bet by blast
    }
  }
}

```

```

    hence Col B2 A2 P2  $\longrightarrow$  False
    using Col-def T12 by blast
    hence  $\neg$  Col B2 A2 P2
    using H4 bet-col not-col-permutation-4 by blast
  }
  {
    assume H5: B2 Out A2 P2
    hence B1 Out A1 C1
    using T2 conga-sym out-conga-out by blast
    hence B1 Out A1 P1
    using T1 in-angle-out by blast
    hence B Out A C
    using H1 T1 ncol-conga-ncol not-col-permutation-4 out-col by blast
    hence  $\neg$  Col B2 A2 P2
    using Col-perm H1 out-col by blast
  }
  {
    assume  $\neg$  Col A2 B2 P2
    hence  $\neg$  Col B2 A2 P2
    using Col-perm by blast
  }
  thus ?thesis
  using  $\langle B2 \text{ Out } A2 \text{ P2} \implies \neg \text{ Col } B2 \text{ A2 } P2 \rangle \langle \text{Bet } A2 \text{ B2 } P2 \implies \neg \text{ Col } B2 \text{ A2 } P2 \rangle$ 
     $\langle \text{Bet } A2 \text{ B2 } P2 \vee B2 \text{ Out } A2 \text{ P2} \vee \neg \text{ Col } A2 \text{ B2 } P2 \rangle$  by blast
qed
thus ?thesis
  by (simp add: T2 T12 in-angle-one-side)
qed
have S1: A2 B2 OS P P2
  using T13 T14 one-side-symmetry one-side-transitivity by blast
have A1 B1 P1 CongA A2 B2 P
  using conga-trans conga-sym T1 T13 by blast
hence P InAngle A2 B2 P2
  using conga-preserves-in-angle T2 T1 S1 by blast
hence P InAngle A2 B2 C2
  using T2 in-angle-trans by blast
hence A B C LeA A2 B2 C2
  using T13 LeA-def by blast
}
hence A B C LeA A2 B2 C2
  using T10  $\langle B2 \text{ Out } A2 \text{ C2} \implies A \text{ B } C \text{ LeA } A2 \text{ B2 } C2 \rangle$ 
     $\langle \text{Bet } A2 \text{ B2 } C2 \implies A \text{ B } C \text{ LeA } A2 \text{ B2 } C2 \rangle$  by blast
}
thus ?thesis
  using T9  $\langle B \text{ Out } A \text{ C} \implies A \text{ B } C \text{ LeA } A2 \text{ B2 } C2 \rangle$ 
     $\langle \text{Bet } A \text{ B } C \implies A \text{ B } C \text{ LeA } A2 \text{ B2 } C2 \rangle$  by blast
qed

lemma in-angle-asym:
  assumes D InAngle A B C and
    C InAngle A B D
  shows A B C CongA A B D
proof -
  obtain CC where P1: Bet A CC D  $\wedge$  (CC = B  $\vee$  B Out CC C)
    using InAngle-def assms(2) by auto
  obtain DD where P2: Bet A DD C  $\wedge$  (DD = B  $\vee$  B Out DD D)
    using InAngle-def assms(1) by auto
  {
    assume (CC = B)  $\wedge$  (DD = B)
    hence A B C CongA A B D
    by (metis P1 P2 assms(2) conga-line inangle-distincts)
  }
  {
    assume (CC = B)  $\wedge$  (B Out DD D)
    hence A B C CongA A B D
    by (metis P1 assms(1) bet-in-angle-bet conga-line inangle-distincts)
  }

```

```

}
{
  assume (B Out CC C) ∧ (DD = B)
  hence A B C CongA A B D
    by (metis P2 assms(2) bet-in-angle-bet conga-line inangle-distincts)
}
{
  assume V1: (B Out CC C) ∧ (B Out DD D)
  obtain X where P3: Bet CC X C ∧ Bet DD X D
    using P1 P2 between-symmetry inner-pasch by blast
  hence B Out X D
    using V1 out-bet-out-2 by blast
  hence B Out C D
    using P3 V1 out2-bet-out by blast
  hence A B C CongA A B D
    using assms(2) inangle-distincts l6-6 out2--conga out-trivial by blast
}
thus ?thesis using P1 P2
  using ⟨B Out CC C ∧ DD = B ⟹ A B C CongA A B D⟩
    ⟨CC = B ∧ B Out DD D ⟹ A B C CongA A B D⟩
    ⟨CC = B ∧ DD = B ⟹ A B C CongA A B D⟩ by blast
qed

lemma lea-asy:
  assumes A B C LeA D E F and
    D E F LeA A B C
  shows A B C CongA D E F
proof cases
  assume P1: Col A B C
  {
    assume P1A: Bet A B C
    have P2: D ≠ E
      using assms(1) lea-distincts by blast
    have P3: F ≠ E
      using assms(2) lea-distincts by auto
    have P4: A ≠ B
      using assms(1) lea-distincts by auto
    have P5: C ≠ B
      using assms(2) lea-distincts by blast
    obtain P where P6: P InAngle D E F ∧ A B C CongA D E P
      using LeA-def assms(1) by blast
    hence A B C CongA D E P by simp
    hence Bet D E P using P1 P1A bet-conga--bet
      by blast
    hence Bet D E F
      using P6 bet-in-angle-bet by blast
    hence A B C CongA D E F
      by (metis bet-conga--bet conga-line l11-29-a P2 P3 P4 P5 assms(2) bet-in-angle-bet)
  }
  {
    assume T1: ¬ Bet A B C
    hence T2: B Out A C
      using P1 not-out-bet by auto
    obtain P where T3: P InAngle A B C ∧ D E F CongA A B P
      using LeA-def assms(2) by blast
    hence T3A: P InAngle A B C by simp
    have T3B: D E F CongA A B P using T3 by simp
    have T4: E Out D F
      proof –
        have T4A: B Out A P
          using T2 T3 in-angle-out by blast
        have A B P CongA D E F
          by (simp add: T3 conga-sym)
        thus ?thesis
          using T4A l11-21-a by blast
      qed
  }
qed

```



```

  hence  $A B C \text{ CongA } D E F$ 
    by (simp add: T2 l11-21-b)
}
thus ?thesis
  using  $\langle \text{Bet } A B C \implies A B C \text{ CongA } D E F \rangle$  by blast
next
  assume T5:  $\neg \text{Col } A B C$ 
  obtain Q where T6:  $C \text{ InAngle } A B Q \wedge A B Q \text{ CongA } D E F$ 
    using assms(1) l11-29-a by blast
  hence T6A:  $C \text{ InAngle } A B Q$  by simp
  have T6B:  $A B Q \text{ CongA } D E F$  by (simp add: T6)
  obtain P where T7:  $P \text{ InAngle } A B C \wedge D E F \text{ CongA } A B P$ 
    using LeA-def assms(2) by blast
  hence T7A:  $P \text{ InAngle } A B C$  by simp
  have T7B:  $D E F \text{ CongA } A B P$  by (simp add: T7)
  have T13:  $A B Q \text{ CongA } A B P$ 
    using T6 T7 conga-trans by blast
  have T14:  $\text{Bet } A B Q \vee B \text{ Out } A Q \vee \neg \text{Col } A B Q$ 
    using not-out-bet by auto
  {
    assume R1:  $\text{Bet } A B Q$ 
    hence  $A B C \text{ CongA } D E F$ 
      using T13 T5 T7 bet-col bet-conga--bet bet-in-angle-bet by blast
  }
  {
    assume R2:  $B \text{ Out } A Q$ 
    hence  $A B C \text{ CongA } D E F$ 
      using T6 in-angle-out l11-21-a l11-21-b by blast
  }
  {
    assume R3:  $\neg \text{Col } A B Q$ 
    have R3A:  $\text{Bet } A B P \vee B \text{ Out } A P \vee \neg \text{Col } A B P$ 
      using not-out-bet by blast
    {
      assume R3AA:  $\text{Bet } A B P$ 
      hence  $A B C \text{ CongA } D E F$ 
        using T5 T7 bet-col bet-in-angle-bet by blast
    }
    {
      assume R3AB:  $B \text{ Out } A P$ 
      hence  $A B C \text{ CongA } D E F$ 
        by (meson Col-cases R3 T13 ncol-conga-ncol out-col)
    }
    {
      assume R3AC:  $\neg \text{Col } A B P$ 
      have R3AD:  $B \text{ Out } P Q \vee A B \text{ TS } P Q$ 
        proof -
          have Coplanar  $A B P Q$ 
            using T6A T7A coplanar-perm-8 in-angle-trans inangle--coplanar by blast
          thus ?thesis
            by (simp add: T13 conga-sym conga-cop--or-out-ts)
        qed
      {
        assume B Out  $P Q$ 
        hence  $C \text{ InAngle } A B P$ 
          by (meson R3 T6A bet-col between-symmetry l11-24 l11-25-aux)
        hence  $A B C \text{ CongA } A B P$ 
          by (simp add: T7A in-angle-asy)
        hence  $A B C \text{ CongA } D E F$ 
          by (meson T7B conga-sym conga-trans Tarski-neutral-dimensionless-axioms)
      }
    }
    {
      assume W1:  $A B \text{ TS } P Q$ 
      have  $A B \text{ OS } P Q$ 
        using Col-perm R3 R3AC T6A T7A in-angle-one-side in-angle-trans by blast
      hence  $A B C \text{ CongA } D E F$ 

```

```

    using W1 l9-9 by blast
  }
  hence A B C CongA D E F
    using R3AD ⟨B Out P Q ⟹ A B C CongA D E F⟩ by blast
}
hence A B C CongA D E F
  using R3A ⟨B Out A P ⟹ A B C CongA D E F⟩
    ⟨Bet A B P ⟹ A B C CongA D E F⟩ by blast
}
thus ?thesis
  using T14 ⟨B Out A Q ⟹ A B C CongA D E F⟩
    ⟨Bet A B Q ⟹ A B C CongA D E F⟩ by blast
qed

lemma col-lta--bet:
  assumes Col X Y Z and
    A B C LtA X Y Z
  shows Bet X Y Z
proof -
  have A B C LeA X Y Z ∧ ¬ A B C CongA X Y Z
    using LtA-def assms(2) by auto
  hence Y Out X Z ⟶ False
    using lea-asymp lea-distincts Tarski-neutral-dimensionless-axioms l11-31-1
    by fastforce
  thus ?thesis using not-out-bet assms(1)
    by blast
qed

lemma col-lta--out:
  assumes Col A B C and
    A B C LtA X Y Z
  shows B Out A C
proof -
  have A B C LeA X Y Z ∧ ¬ A B C CongA X Y Z
    using LtA-def assms(2) by auto
  thus ?thesis
    by (metis assms(1) l11-31-2 lea-asymp lea-distincts or-bet-out)
qed

lemma lta-distincts:
  assumes A B C LtA D E F
  shows A ≠ B ∧ C ≠ B ∧ D ≠ E ∧ F ≠ E ∧ D ≠ F
  by (metis LtA-def assms bet-neq12--neq col-lta--bet lea-distincts not-col-distincts)

lemma gta-distincts:
  assumes A B C GtA D E F
  shows A ≠ B ∧ C ≠ B ∧ D ≠ E ∧ F ≠ E ∧ A ≠ C
  using GtA-def assms lta-distincts by presburger

lemma acute-distincts:
  assumes Acute A B C
  shows A ≠ B ∧ C ≠ B
  using Acute-def assms lta-distincts by blast

lemma obtuse-distincts:
  assumes Obtuse A B C
  shows A ≠ B ∧ C ≠ B ∧ A ≠ C
  using Obtuse-def assms lta-distincts by blast

lemma two-sides-in-angle:
  assumes B ≠ P' and
    B P TS A C and
    Bet P B P'
  shows P InAngle A B C ∨ P' InAngle A B C
proof -
  obtain T where P1: Col T B P ∧ Bet A T C

```

```

    using TS-def assms(2) by auto
  have P2:  $A \neq B$ 
    using assms(2) ts-distincts by blast
  have P3:  $C \neq B$ 
    using assms(2) ts-distincts by blast
  show ?thesis
  proof cases
    assume  $B = T$ 
    thus ?thesis
      using P1 P2 P3 assms(1) in-angle-line by auto
  next
    assume  $B \neq T$ 
    thus ?thesis
      by (metis InAngle-def P1 assms(1) assms(2) assms(3) between-symmetry l6-3-2
        or-bet-out ts-distincts)
  qed
qed

lemma in-angle-reverse:
  assumes  $A' \neq B$  and
     $Bet\ A\ B\ A'$  and
     $C\ InAngle\ A\ B\ D$ 
  shows  $D\ InAngle\ A'\ B\ C$ 
proof -
  have P1:  $A \neq B$ 
    using assms(3) inangle-distincts by auto
  have P2:  $D \neq B$ 
    using assms(3) inangle-distincts by blast
  have P3:  $C \neq B$ 
    using assms(3) inangle-distincts by auto
  show ?thesis
  proof cases
    assume Col B A C
    {
      assume  $Bet\ C\ B\ A$ 
      hence  $Bet\ A\ B\ D$ 
        using Bet-cases assms(3) bet-in-angle-bet by blast
      hence  $D\ InAngle\ A'\ B\ C$ 
        by (metis P1 P2 P3  $\langle Bet\ C\ B\ A \rangle$  assms(1) between-symmetry l6-2 out341--inangle)
    }
    moreover
    {
      assume  $B\ Out\ C\ A$ 
      hence  $D\ InAngle\ A'\ B\ C$ 
        by (meson InAngle-def Out-cases P2 P3 assms(1) assms(2) bet-out--bet l11-24)
    }
    moreover
    {
      assume  $\neg Col\ C\ B\ A$ 
      hence  $D\ InAngle\ A'\ B\ C$ 
        using  $\langle Col\ B\ A\ C \rangle$  not-col-permutation-1 by blast
    }
    ultimately show ?thesis
      using l6-4-2 by blast
  next
    assume P4:  $\neg Col\ B\ A\ C$ 
    thus ?thesis
      proof cases
        assume Col B D C
        {
          assume  $Bet\ C\ B\ D$ 
          hence  $D\ InAngle\ A'\ B\ C$ 
            by (meson Col-def P2 P4 assms(3) bet-in-angle-bet between-symmetry
              col-transitivity-2 l11-24)
        }
        moreover

```

```

{
  assume  $B \text{ Out } C D$ 
  hence  $D \text{ InAngle } A' B C$ 
    using assms(1) out341--inangle by blast
}
moreover
{
  assume  $\neg \text{Col } C B D$ 
  hence  $D \text{ InAngle } A' B C$ 
    using  $\langle \text{Col } B D C \rangle$  col-permutation-2 by blast
}
ultimately show ?thesis
  using not-out-bet by blast
next
assume  $P5: \neg \text{Col } B D C$ 
have  $P6: C B \text{ TS } A D$ 
  using  $P4 P5$  assms(3) in-angle-two-sides by auto
obtain  $X$  where  $P7: \text{Bet } A X D \wedge (X = B \vee B \text{ Out } X C)$ 
  using InAngle-def assms(3) by auto
have  $P8: X = B \implies D \text{ InAngle } A' B C$ 
  using Out-def P1 P2 P3 P7 assms(1) assms(2) l5-2 out321--inangle by auto
{
  assume  $P9: B \text{ Out } X C$ 
  have  $P10: C \neq B$ 
    by (simp add: P3)
  have  $P10A: \neg \text{Col } B A C$ 
    by (simp add: P4)
  have  $P10B: \neg \text{Col } B D C$ 
    by (simp add: P5)
  have  $P10C: C \text{ InAngle } D B A$ 
    by (simp add: assms(3) l11-24)
  {
    assume  $\text{Col } D B A$ 
    have  $\text{Col } B A C$ 
    proof -
      have  $B \neq X$ 
        using  $P9$  out-distinct by blast
      have  $\text{Col } B X A$ 
        by (meson Bet-perm P10C P5 P7  $\langle \text{Col } D B A \rangle$  bet-col1 col-permutation-3 in-angle-out or-bet-out out-col)
      have  $\text{Col } B X C$ 
        by (simp add: P9 out-col)
      thus ?thesis
        using  $\langle B \neq X \rangle \langle \text{Col } B X A \rangle$  col-transitivity-1 by blast
    qed
    hence False
      by (simp add: P4)
  }
  hence  $P10E: \neg \text{Col } D B A$  by auto
  have  $P11: D B \text{ OS } C A$ 
    by (simp add: P10C P10E P5 in-angle-one-side)
  have  $P12: \neg \text{Col } A D B$ 
    using Col-cases P10E by auto
  have  $P13: \neg \text{Col } A' D B$ 
    by (metis Col-def  $\langle \text{Col } D B A \implies \text{False} \rangle$  assms(1) assms(2) col-transitivity-1)
  have  $P14: D B \text{ TS } A A'$ 
    using  $P12 P13$  TS-def assms(2) col-trivial-3 by blast
  have  $P15: D B \text{ TS } C A'$ 
    using  $P11 P14$  l9-8-2 one-side-symmetry by blast
  have  $P16: \neg \text{Col } C D B$ 
    by (simp add: P5 not-col-permutation-3)
  obtain  $Y$  where  $P17: \text{Col } Y D B \wedge \text{Bet } C Y A'$ 
    using  $P15$  TS-def by auto
  have  $P18: \text{Bet } A' Y C$ 
    using Bet-perm P17 by blast
  {

```

```

assume  $S1: Y \neq B$ 
have  $S2: Col\ D\ B\ Y$ 
  using  $P17\ not-col-permutation-2$  by blast
hence  $S3: Bet\ D\ B\ Y \vee Bet\ B\ Y\ D \vee Bet\ Y\ D\ B$ 
  using  $Col-def\ S2$  by auto
{
  assume  $S4: Bet\ D\ B\ Y$ 
  have  $S5: C\ B\ OS\ A'\ Y$ 
    by (metis  $P17\ P18\ P5\ S1\ bet-out-1\ col-transitivity-2\ l6-6$ 
      not-col-permutation-3\ not-col-permutation-5\ out-one-side)
  have  $S6: C\ B\ TS\ Y\ D$ 
    by (metis  $Bet-perm\ P16\ P17\ S1\ S4\ bet--ts\ col3\ col-trivial-3$ 
      invert-two-sides\ not-col-permutation-1)
  have  $C\ B\ TS\ A\ A'$ 
    by (metis  $P4\ assms(1)\ assms(2)\ bet--ts\ invert-two-sides$ 
      not-col-permutation-5)
  hence  $C\ B\ TS\ Y\ A$ 
    using  $S5\ l9-2\ l9-8-2$  by blast
  hence  $S9: C\ B\ OS\ A\ D$ 
    using  $P6\ S6\ l9-8-1\ l9-9$  by blast
  hence  $B\ Out\ Y\ D$ 
    using  $P6\ S9\ l9-9$  by auto
}
{
  assume  $Bet\ B\ Y\ D$ 
  hence  $B\ Out\ Y\ D$ 
    by (simp add: S1 bet-out)
}
{
  assume  $Bet\ Y\ D\ B$ 
  hence  $B\ Out\ Y\ D$ 
    by (simp add: P2 bet-out-1 l6-6)
}
have  $B\ Out\ Y\ D$ 
  using  $S3\ \langle Bet\ B\ Y\ D \implies B\ Out\ Y\ D \rangle\ \langle Bet\ D\ B\ Y \implies B\ Out\ Y\ D \rangle$ 
   $\langle Bet\ Y\ D\ B \implies B\ Out\ Y\ D \rangle$  by blast
}
hence  $P19: (Y = B \vee B\ Out\ Y\ D)$  by auto
have  $D\ InAngle\ A'\ B\ C$ 
  using  $InAngle-def\ P18\ P19\ P2\ P3\ assms(1)$  by auto
}
thus ?thesis using  $P7\ P8$  by blast
qed
qed
qed

```

lemma *in-angle-trans2*:

assumes $C\ InAngle\ A\ B\ D$ **and**
 $D\ InAngle\ A\ B\ E$

shows $D\ InAngle\ C\ B\ E$

by (*smt* (*verit, ccfv-threshold*) *Bet-cases assms(1,2) in-angle-reverse in-angle-trans*
inangle-distincts l11-24 point-construction-different)

lemma *l11-36-aux1*:

assumes $A \neq B$ **and**
 $A' \neq B$ **and**
 $D \neq E$ **and**
 $D' \neq E$ **and**
 $Bet\ A\ B\ A'$ **and**
 $Bet\ D\ E\ D'$ **and**
 $A\ B\ C\ LeA\ D\ E\ F$

shows $D'\ E\ F\ LeA\ A'\ B\ C$

proof –

obtain P **where** $P1: C\ InAngle\ A\ B\ P \wedge$
 $A\ B\ P\ CongA\ D\ E\ F$

using $assms(7)\ l11-29-a$ **by** *blast*

thus ?thesis
 by (metis LeA-def l11-13 assms(2) assms(4) assms(5) assms(6) conga-sym in-angle-reverse)
 qed

lemma l11-36-aux2:

assumes $A \neq B$ and

$A' \neq B$ and

$D \neq E$ and

$D' \neq E$ and

$Bet\ A\ B\ A'$ and

$Bet\ D\ E\ D'$ and

$D' E F LeA\ A' B C$

shows $A B C LeA\ D E F$

by (metis Bet-cases assms(1) assms(3) assms(5) assms(6) assms(7) l11-36-aux1 lea-distincts)

lemma l11-36:

assumes $A \neq B$ and

$A' \neq B$ and

$D \neq E$ and

$D' \neq E$ and

$Bet\ A\ B\ A'$ and

$Bet\ D\ E\ D'$

shows $A B C LeA\ D E F \longleftrightarrow D' E F LeA\ A' B C$

using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) l11-36-aux1 l11-36-aux2 by auto

lemma l11-41-aux:

assumes $\neg Col\ A\ B\ C$ and

$Bet\ B\ A\ D$ and

$A \neq D$

shows $A C B LtA\ C A D$

proof –

obtain M where $P1: M\ Midpoint\ A\ C$

using midpoint-existence by auto

obtain P where $P2: M\ Midpoint\ B\ P$

using symmetric-point-construction by auto

have $P3: A C B Cong3\ C A P$

using Cong3-def P1 P2 assms(1) l7-13-R1 l7-2 midpoint-distinct-1 not-col-distincts
 by meson

have $P4: A \neq C$

using assms(1) col-trivial-3 by blast

have $P5: B \neq C$

using assms(1) col-trivial-2 by blast

have $P7: A \neq M$

using P1 P4 is-midpoint-id by blast

have $P8: A C B CongA\ C A P$

by (simp add: P3 P4 P5 cong3-conga)

have $P8A: Bet\ D\ A\ B$

using Bet-perm assms(2) by blast

have $P8B: Bet\ P\ M\ B$

by (simp add: P2 between-symmetry midpoint-bet)

then obtain X where $P9: Bet\ A\ X\ P \wedge Bet\ M\ X\ D$ using P8A inner-pasch by blast

have $P9A: Bet\ A\ X\ P$ by (simp add: P9)

have $P9B: Bet\ M\ X\ D$ by (simp add: P9)

have $P10A: P\ InAngle\ C\ A\ D$

proof –

have $K1: P\ InAngle\ M\ A\ D$

by (metis InAngle-def P3 P5 P7 P9 assms(3) bet-out cong3-diff2)

have $K2: A\ Out\ C\ M$

using Out-def P1 P4 P7 midpoint-bet by auto

have $K3: A\ Out\ D\ D$

using assms(3) out-trivial by auto

have $A\ Out\ P\ P$

using K1 inangle-distincts out-trivial by auto

thus ?thesis

using K1 K2 K3 l11-25 by blast

qed

```

hence  $P10$ :  $A \ C \ B \ LeA \ C \ A \ D$ 
  using  $LeA$ -def  $P8$  by auto
{
  assume  $K5$ :  $A \ C \ B \ CongA \ C \ A \ D$ 
  hence  $K6$ :  $C \ A \ D \ CongA \ C \ A \ P$ 
    using  $P8$  conga-sym conga-trans by blast
  have  $K7$ :  $Coplanar \ C \ A \ D \ P$ 
    using  $P10A$  inangle--coplanar ncoplanar-perm-18 by blast
  hence  $K8$ :  $A \ Out \ D \ P \vee C \ A \ TS \ D \ P$ 
    by (simp add: K6 conga-cop--or-out-ts)
  {
    assume  $A \ Out \ D \ P$ 
    hence  $Col \ M \ B \ A$ 
      by (meson P8A P8B bet-col1 bet-out--bet between-symmetry not-col-permutation-4)
    hence  $K8F$ :  $Col \ A \ M \ B$ 
      using not-col-permutation-1 by blast
    have  $Col \ A \ M \ C$ 
      by (simp add: P1 bet-col midpoint-bet)
    hence False
      using  $K8F \ P7 \ assms(1) \ col-transitivity-1$  by blast
  }
hence  $K9$ :  $\neg A \ Out \ D \ P$  by auto
{
  assume  $V1$ :  $C \ A \ TS \ D \ P$ 
  hence  $V3$ :  $A \ C \ TS \ B \ P$ 
    by (metis P10A P8A assms(1) col-trivial-1 col-trivial-2 in-angle-reverse
      in-angle-two-sides invert-two-sides l11-24 l9-18 not-col-permutation-5)
  have  $A \ C \ TS \ B \ D$ 
    by (simp add: assms(1) assms(2) assms(3) bet--ts not-col-permutation-5)
  hence  $A \ C \ OS \ D \ P$ 
    using  $V1 \ V3 \ invert-two-sides \ l9-8-1 \ l9-9$  by blast
  hence False
    using  $V1 \ invert-one-side \ l9-9$  by blast
}
hence  $\neg C \ A \ TS \ D \ P$  by auto
hence False using  $K8 \ K9$  by auto
}
hence  $\neg A \ C \ B \ CongA \ C \ A \ D$  by auto
thus ?thesis
  by (simp add: LtA-def P10)
qed

```

```

lemma  $l11-41$ :
  assumes  $\neg Col \ A \ B \ C$  and
     $Bet \ B \ A \ D$  and
     $A \neq D$ 
  shows  $A \ C \ B \ LtA \ C \ A \ D \wedge A \ B \ C \ LtA \ C \ A \ D$ 
proof –
  have  $P1$ :  $A \ C \ B \ LtA \ C \ A \ D$ 
    using  $assms(1) \ assms(2) \ assms(3) \ l11-41$ -aux by auto
  have  $A \ B \ C \ LtA \ C \ A \ D$ 
proof –
  obtain  $E$  where  $T1$ :  $Bet \ C \ A \ E \wedge Cong \ A \ E \ C \ A$ 
    using segment-construction by blast
  have  $T1A$ :  $Bet \ C \ A \ E$  using  $T1$  by simp
  have  $T1B$ :  $Cong \ A \ E \ C \ A$  using  $T1$  by simp
  have  $T2$ :  $A \ B \ C \ LtA \ B \ A \ E$ 
    using  $T1 \ assms(1) \ cong-reverse-identity \ l11-41$ -aux not-col-distincts
      not-col-permutation-5 by blast
  have  $T3$ :  $B \ A \ C \ CongA \ C \ A \ B$ 
    by (metis assms(1) conga-pseudo-refl not-col-distincts)
  have  $T3A$ :  $D \ A \ C \ CongA \ E \ A \ B$ 
    by (metis CongA-def T1 T3 assms(2) assms(3) cong-reverse-identity l11-13)
  hence  $T4$ :  $B \ A \ E \ CongA \ C \ A \ D$ 
    using conga-comm conga-sym by blast
  have  $A \ B \ C \ CongA \ A \ B \ C$ 

```

```

    using T2 conga-refl lta-distincts by blast
  hence T5: A B C LeA C A D
    by (meson T2 T4 LtA-def l11-30)
  have  $\neg$  A B C CongA C A D
    by (meson T2 LtA-def conga-right-comm conga-trans T3A)
  thus ?thesis
    by (simp add: LtA-def T5)
qed
thus ?thesis by (simp add: P1)
qed

```

```

lemma not-conga:
  assumes A B C CongA A' B' C' and
     $\neg$  A B C CongA D E F
  shows  $\neg$  A' B' C' CongA D E F
  by (meson assms(1) assms(2) conga-trans)

```

```

lemma not-conga-sym:
  assumes  $\neg$  A B C CongA D E F
  shows  $\neg$  D E F CongA A B C
  using assms conga-sym by blast

```

```

lemma not-and-lta:
  shows  $\neg$  (A B C LtA D E F  $\wedge$  D E F LtA A B C)
proof -
  {
    assume A B C LtA D E F and D E F LtA A B C
    hence A B C CongA D E F
      using LtA-def lea-asymp by blast
    hence False
      using LtA-def  $\langle$ D E F LtA A B C $\rangle$  not-conga-sym by blast
  }
  thus ?thesis by auto
qed

```

```

lemma conga-preserves-lta:
  assumes A B C CongA A' B' C' and
    D E F CongA D' E' F' and
    A B C LtA D E F
  shows A' B' C' LtA D' E' F'
  by (meson LtA-def conga-trans l11-30 not-conga-sym assms(1) assms(2) assms(3))

```

```

lemma lta-trans:
  assumes A B C LtA A1 B1 C1 and
    A1 B1 C1 LtA A2 B2 C2
  shows A B C LtA A2 B2 C2
proof -
  have P1: A B C LeA A2 B2 C2
    by (meson LtA-def assms(1) assms(2) lea-trans)
  {
    assume A B C CongA A2 B2 C2
    hence False
      by (meson LtA-def lea-asymp lea-trans assms(1) assms(2) conga--lea456123)
  }
  thus ?thesis
    using LtA-def P1 by blast
qed

```

```

lemma obtuse-sym:
  assumes Obtuse A B C
  shows Obtuse C B A
  by (meson Obtuse-def lta-right-comm assms)

```

```

lemma acute-sym:
  assumes Acute A B C
  shows Acute C B A

```



```

by (meson Acute-def lta-left-comm assms)

lemma acute-col--out:
  assumes Col A B C and
    Acute A B C
  shows B Out A C
  by (meson Acute-def assms(1) assms(2) col-lta--out)

lemma col-obtuse--bet:
  assumes Col A B C and
    Obtuse A B C
  shows Bet A B C
  using Obtuse-def assms(1) assms(2) col-lta--bet by blast

lemma out--acute:
  assumes B Out A C
  shows Acute A B C
proof -
  have P1: A ≠ B
    using assms out-diff1 by auto
  then obtain D where P3: B D Perp A B
    using perp-exists by blast
  hence P4: B ≠ D
    using perp-distinct by auto
  have P5: Per A B D
    by (simp add: P3 l8-2 perp-per-1)
  have P6: A B C LeA A B D
    using P1 P4 assms l11-31-1 by auto
  {
    assume A B C CongA A B D
    hence False
      by (metis Col-cases P1 P4 P5 assms col-conga-col l8-9 out-col)
  }
  hence A B C LtA A B D
    using LtA-def P6 by auto
  thus ?thesis
    using P5 Acute-def by auto
qed

lemma bet--obtuse:
  assumes Bet A B C and
    A ≠ B and B ≠ C
  shows Obtuse A B C
proof -
  obtain D where P1: B D Perp A B
    using assms(2) perp-exists by blast
  have P5: B ≠ D
    using P1 perp-not-eq-1 by auto
  have P6: Per A B D
    using P1 Perp-cases perp-per-1 by blast
  have P7: A B D LeA A B C
    using assms(2) assms(3) P5 assms(1) l11-31-2 by auto
  {
    assume A B D CongA A B C
    hence False
      using assms(2) P5 P6 assms(1) bet-col ncol-conga-ncol per-not-col by blast
  }
  hence A B D LtA A B C
    using LtA-def P7 by blast
  thus ?thesis
    using Obtuse-def P6 by blast
qed

lemma l11-43-aux:
  assumes A ≠ B and
    A ≠ C and

```

```

    Per B A C  $\vee$  Obtuse B A C
shows Acute A B C
proof cases
  assume P1: Col A B C
  {
    assume Per B A C
    hence Acute A B C
      using Col-cases P1 assms(1) assms(2) per-col-eq by blast
  }
  {
    assume Obtuse B A C
    hence Bet B A C
      using P1 col-obtuse--bet col-permutation-4 by blast
    hence Acute A B C
      by (simp add: assms(1) bet-out out--acute)
  }
  thus ?thesis
    using  $\langle \text{Per } B \ A \ C \implies \text{Acute } A \ B \ C \rangle$  assms(3) by blast
next
  assume P2:  $\neg \text{Col } A \ B \ C$ 
  hence P3:  $B \neq C$ 
    using col-trivial-2 by auto
  obtain B' where P4:  $\text{Bet } B \ A \ B' \wedge \text{Cong } A \ B' \ B \ A$ 
    using segment-construction by blast
  have P5:  $\neg \text{Col } B' \ A \ C$ 
    by (metis Col-def P2 P4 col-transitivity-2 cong-reverse-identity)
  hence P6:  $B' \neq A \wedge B' \neq C$ 
    using not-col-distincts by blast
  hence P7:  $A \ C \ B \ \text{LtA } C \ A \ B' \wedge A \ B \ C \ \text{LtA } C \ A \ B'$ 
    using P2 P4 l11-41 by auto
  hence P7A:  $A \ C \ B \ \text{LtA } C \ A \ B'$  by simp
  have P7B:  $A \ B \ C \ \text{LtA } C \ A \ B'$  by (simp add: P7)
  {
    assume Per B A C
    have Acute A B C
      by (metis Acute-def P4 P7B  $\langle \text{Per } B \ A \ C \rangle$  assms(1) bet-col col-per2--per
        col-trivial-3 l8-3 lta-right-comm)
  }
  {
    assume T1: Obtuse B A C
    then obtain a b c where T2:  $\text{Per } a \ b \ c \wedge a \ b \ c \ \text{LtA } B \ A \ C$ 
      using Obtuse-def by blast
    hence T2A:  $\text{Per } a \ b \ c$  by simp
    have T2B:  $a \ b \ c \ \text{LtA } B \ A \ C$  by (simp add: T2)
    hence T3:  $a \ b \ c \ \text{LeA } B \ A \ C \wedge \neg a \ b \ c \ \text{CongA } B \ A \ C$ 
      by (simp add: LtA-def)
    hence T3A:  $a \ b \ c \ \text{LeA } B \ A \ C$  by simp
    have T3B:  $\neg a \ b \ c \ \text{CongA } B \ A \ C$  by (simp add: T3)
    obtain P where T4:  $P \ \text{InAngle } B \ A \ C \wedge a \ b \ c \ \text{CongA } B \ A \ P$ 
      using LeA-def T3 by blast
    hence T5:  $\text{Per } B \ A \ P$  using T4 T2 l11-17 by blast
    hence T6:  $\text{Per } P \ A \ B$ 
      using l8-2 by blast
    have Col A B B'
      by (simp add: P4 bet-col col-permutation-4)
    hence Per P A B'
      using T6 assms(1) per-col by blast
    hence S3:  $B \ A \ P \ \text{CongA } B' \ A \ P$ 
      using l8-2 P6 T5 T4 CongA-def assms(1) l11-16 by auto
    have C A B' LtA P A B
    proof -
      have S4:  $B \ A \ P \ \text{LeA } B \ A \ C \longleftrightarrow B' \ A \ C \ \text{LeA } B' \ A \ P$ 
        using P4 P6 assms(1) l11-36 by auto
      have S5:  $C \ A \ B' \ \text{LeA } P \ A \ B$ 
      proof -
        have S6:  $B \ A \ P \ \text{LeA } B \ A \ C$ 

```

```

    using T4 inangle--lea by auto
  have B' A P CongA P A B
    using S3 conga-left-comm not-conga-sym by blast
  thus ?thesis
    using P6 S4 S6 assms(2) conga-pseudo-refl l11-30 by auto
qed
{
  assume T10: C A B' CongA P A B
  have Per B' A C
  proof -
    have B A P CongA B' A C
      using T10 conga-comm conga-sym by blast
    thus ?thesis
      using T5 l11-17 by blast
  qed
  hence Per C A B
    using Col-cases P6 ⟨Col A B B'⟩ l8-2 l8-3 by blast
  have a b c CongA B A C
  proof -
    have a ≠ b
      using T3A lea-distincts by auto
    have c ≠ b
      using T2B lta-distincts by blast
    have Per B A C
      using Per-cases ⟨Per C A B⟩ by blast
    thus ?thesis
      using T2 ⟨a ≠ b⟩ ⟨c ≠ b⟩ assms(1) assms(2) l11-16 by auto
  qed
  hence False
    using T3B by blast
}
hence ¬ C A B' CongA P A B by blast
thus ?thesis
  by (simp add: LtA-def S5)
qed
hence A B C LtA B A P
  by (meson P7 lta-right-comm lta-trans)
hence Acute A B C using T5
  using Acute-def by blast
}
thus ?thesis
  using ⟨Per B A C ⟹ Acute A B C⟩ assms(3) by blast
qed

lemma l11-43:
  assumes A ≠ B and
    A ≠ C and
    Per B A C ∨ Obtuse B A C
  shows Acute A B C ∧ Acute A C B
  using Per-perm assms(1) assms(2) assms(3) l11-43-aux obtuse-sym by blast

lemma acute-lea-acute:
  assumes Acute D E F and
    A B C LeA D E F
  shows Acute A B C
proof -
  obtain A' B' C' where P1: Per A' B' C' ∧ D E F LtA A' B' C'
    using Acute-def assms(1) by auto
  have P2: A B C LeA A' B' C'
    using LtA-def P1 assms(2) lea-trans by blast
  have ¬ A B C CongA A' B' C'
    by (meson LtA-def P1 assms(2) conga--lea456123 lea-asymp lea-trans)
  hence A B C LtA A' B' C'
    by (simp add: LtA-def P2)
  thus ?thesis
    using Acute-def P1 by auto

```

qed

lemma *lea-obtuse-obtuse*:

assumes *Obtuse D E F* and

D E F LeA A B C

shows *Obtuse A B C*

proof –

obtain *A' B' C'* where *P1: Per A' B' C' $\wedge$ A' B' C' LtA D E F*

using *Obtuse-def assms(1)* by auto

hence *P2: A' B' C' LeA A B C*

using *LtA-def assms(2) lea-trans* by blast

have $\neg A' B' C' CongA A B C$

by (*meson LtA-def P1 assms(2) conga--lea456123 lea-asymp lea-trans*)

hence *A' B' C' LtA A B C*

by (*simp add: LtA-def P2*)

thus ?thesis

using *Obtuse-def P1* by auto

qed

lemma *l11-44-1-a*:

assumes *A $\neq$ B* and

A $\neq$ C and

Cong B A B C

shows *B A C CongA B C A*

using *Cong3-def assms(1) assms(2) assms(3) cong3-conga cong-inner-transitivity cong-pseudo-reflexivity* by metis

lemma *l11-44-2-a*:

assumes $\neg Col A B C$ and

B A Lt B C

shows *B C A LtA B A C*

proof –

have *T1: A $\neq$ B*

using *assms(1) col-trivial-1* by auto

have *T3: A $\neq$ C*

using *assms(1) col-trivial-3* by auto

have *B A Le B C*

by (*simp add: assms(2) lt--le*)

then obtain *C'* where *P1: Bet B C' C $\wedge$ Cong B A B C'*

using *assms(2) Le-def* by blast

have *T5: C $\neq$ C'*

using *P1 assms(2) cong--nlt* by blast

have *T5A: C' $\neq$ A*

using *Col-def Col-perm P1 assms(1)* by blast

hence *T6: C' InAngle B A C*

using *InAngle-def P1 T1 T3 out-trivial* by auto

have *T7: C' A C LtA A C' B $\wedge$ C' C A LtA A C' B*

proof –

have *W1: $\neg Col C' C A$*

by (*metis Col-def P1 T5 assms(1) col-transitivity-2*)

have *W2: Bet C C' B*

using *Bet-perm P1* by blast

have *C' $\neq$ B*

using *P1 T1 cong-identity* by blast

thus ?thesis

using *l11-41 W1 W2* by simp

qed

have *T90: B A C' LtA B A C*

proof –

have *T90A: B A C' LeA B A C*

by (*simp add: T6 inangle--lea*)

have *B A C' CongA B A C'*

using *T1 T5A conga-refl* by auto

{

assume *B A C' CongA B A C*

hence *R1: A Out C' C*

```

    by (metis P1 T7 assms(1) bet-out conga-os--out lta-distincts not-col-permutation-4
        out-one-side)
  have B A OS C' C
    by (metis Col-perm P1 T1 assms(1) bet-out cong-diff-2 out-one-side)
  hence False
    using Col-perm P1 T5 R1 bet-col col2--eq one-side-not-col123 out-col by blast
}
hence  $\neg B A C' CongA B A C$  by blast
thus ?thesis
  by (simp add: LtA-def T90A)
qed
have B A C' CongA B C' A
  using P1 T1 T5A l11-44-1-a by auto
hence K2:  $A C' B CongA B A C'$ 
  using conga-left-comm not-conga-sym by blast
have B C A LtA B A C'
proof -
  have K1:  $B C A CongA B C A$ 
    using assms(1) conga-refl not-col-distincts by blast
  have B C A LtA A C' B
proof -
  have C' C A CongA B C A
proof -
  have K2:  $C Out B C'$ 
    using P1 T5 bet-out-1 l6-6 by auto
  have C Out A A
    by (simp add: T3 out-trivial)
  thus ?thesis
    by (simp add: K2 out2--conga)
qed
have A C' B CongA A C' B
  using CongA-def K2 conga-refl by auto
thus ?thesis
  using T7  $\langle C' C A CongA B C A \rangle$  conga-preserves-lta by auto
qed
thus ?thesis
  using K1 K2 conga-preserves-lta by auto
qed
thus ?thesis
  using T90 lta-trans by blast
qed

lemma not-lta-and-conga:
 $\neg (A B C LtA D E F \wedge A B C CongA D E F)$ 
  by (simp add: LtA-def)

lemma conga-sym-equiv:
 $A B C CongA A' B' C' \longleftrightarrow A' B' C' CongA A B C$ 
  using not-conga-sym by blast

lemma conga-dec:
 $A B C CongA D E F \vee \neg A B C CongA D E F$ 
  by auto

lemma lta-not-conga:
  assumes  $A B C LtA D E F$ 
  shows  $\neg A B C CongA D E F$ 
  using assms not-lta-and-conga by auto

lemma lta--lea:
  assumes  $A B C LtA D E F$ 
  shows  $A B C LeA D E F$ 
  using LtA-def assms by auto

lemma nltA:
 $\neg A B C LtA A B C$ 

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using not-and-lta by blast

lemma lea--nlta:
  assumes A B C LeA D E F
  shows  $\neg D E F LtA A B C$ 
  by (meson lea-asymp not-lta-and-conga assms lta--lea)

lemma lta--nlea:
  assumes A B C LtA D E F
  shows  $\neg D E F LeA A B C$ 
  using assms lea--nlta by blast

lemma l11-44-1-b:
  assumes  $\neg Col A B C$  and
    B A C CongA B C A
  shows Cong B A B C
proof -
  have B A Lt B C  $\vee$  B A Gt B C  $\vee$  Cong B A B C
  by (simp add: or-lt-cong-gt)
  thus ?thesis
  by (meson Gt-def assms(1) assms(2) conga-sym l11-44-2-a not-col-permutation-3
    not-lta-and-conga)
qed

lemma l11-44-2-b:
  assumes B A C LtA B C A
  shows B C Lt B A
proof cases
  assume Col A B C
  thus ?thesis
  using Col-perm assms bet--lt1213 col-lta--bet lta-distincts by blast
next
  assume P1:  $\neg Col A B C$ 
  hence P2:  $A \neq B$ 
  using col-trivial-1 by blast
  have P3:  $A \neq C$ 
  using P1 col-trivial-3 by auto
  have B A Lt B C  $\vee$  B A Gt B C  $\vee$  Cong B A B C
  by (simp add: or-lt-cong-gt)
  {
    assume B A Lt B C
    hence B C Lt B A
    using P1 assms l11-44-2-a not-and-lta by blast
  }
  {
    assume B A Gt B C
    hence B C Lt B A
    using Gt-def P1 assms l11-44-2-a not-and-lta by blast
  }
  {
    assume Cong B A B C
    hence B A C CongA B C A
    by (simp add: P2 P3 l11-44-1-a)
    hence B C Lt B A
    using assms not-lta-and-conga by blast
  }
  thus ?thesis
  by (meson P1 not-and-lta
     $\langle B A Gt B C \implies B C Lt B A \rangle \langle B A Lt B C \vee B A Gt B C \vee Cong B A B C \rangle$  assms l11-44-2-a)
qed

lemma l11-44-1:
  assumes  $\neg Col A B C$ 
  shows B A C CongA B C A  $\longleftrightarrow$  Cong B A B C
  using assms l11-44-1-a l11-44-1-b not-col-distincts by blast

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lemma l11-44-2:
  assumes  $\neg \text{Col } A \ B \ C$ 
  shows  $B \ A \ C \text{ Lt } A \ B \ C \ A \longleftrightarrow B \ C \text{ Lt } B \ A$ 
  using assms l11-44-2-a l11-44-2-b not-col-permutation-3 by blast

lemma l11-44-2bis:
  assumes  $\neg \text{Col } A \ B \ C$ 
  shows  $B \ A \ C \text{ Le } A \ B \ C \ A \longleftrightarrow B \ C \text{ Le } B \ A$ 
proof -
  {
    assume P1:  $B \ A \ C \text{ Le } A \ B \ C \ A$ 
    {
      assume  $B \ A \text{ Lt } B \ C$ 
      hence  $B \ C \ A \text{ Lt } A \ B \ A \ C$ 
      by (simp add: assms l11-44-2-a)
      hence False
      using P1 lta--nlea by auto
    }
    hence  $\neg B \ A \text{ Lt } B \ C$  by blast
    have  $B \ C \text{ Le } B \ A$ 
    using  $\langle \neg B \ A \text{ Lt } B \ C \rangle \text{ nle--lt}$  by blast
  }
  {
    assume P2:  $B \ C \text{ Le } B \ A$ 
    have  $B \ A \ C \text{ Le } A \ B \ C \ A$ 
    proof cases
      assume Cong  $B \ C \ B \ A$ 
      hence  $B \ A \ C \text{ Cong } A \ B \ C \ A$ 
      by (metis assms conga-sym l11-44-1-a not-col-distincts)
      thus ?thesis
      by (simp add: conga--lea)
    next
      assume  $\neg \text{Cong } B \ C \ B \ A$ 
      hence  $B \ A \ C \text{ Lt } A \ B \ C \ A$ 
      by (simp add: l11-44-2 assms Lt-def P2)
      thus ?thesis
      by (simp add: lta--lea)
    qed
  }
  thus ?thesis
  using  $\langle B \ A \ C \text{ Le } A \ B \ C \ A \implies B \ C \text{ Le } B \ A \rangle$  by blast
qed

lemma l11-46:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $\text{Per } A \ B \ C \vee \text{Obtuse } A \ B \ C$ 
  shows  $B \ A \text{ Lt } A \ C \wedge B \ C \text{ Lt } A \ C$ 
proof cases
  assume  $\text{Col } A \ B \ C$ 
  thus ?thesis
  by (meson assms(1) assms(2) assms(3) bet--lt1213 bet--lt2313 col-obtuse--bet lt-left-comm per-not-col)
next
  assume P1:  $\neg \text{Col } A \ B \ C$ 
  have P2:  $A \neq C$ 
  using P1 col-trivial-3 by auto
  have P3:  $\text{Acute } B \ A \ C \wedge \text{Acute } B \ C \ A$ 
  using assms(1) assms(2) assms(3) l11-43 by auto
  then obtain A' B' C' where P4:  $\text{Per } A' \ B' \ C' \wedge B \ C \ A \text{ Lt } A' \ B' \ C'$ 
  using Acute-def P3 by auto
  {
    assume P5:  $\text{Per } A \ B \ C$ 
    have P5A:  $A \ C \ B \text{ Cong } A \ A \ C \ B$ 
    by (simp add: P2 assms(2) conga-refl)
    have S1:  $A \neq B$ 

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    by (simp add: assms(1))
  have S2:  $B \neq C$ 
    by (simp add: assms(2))
  have S3:  $A' \neq B'$ 
    using P4 lta-distincts by blast
  have S4:  $B' \neq C'$ 
    using P4 lta-distincts by blast
  hence A' B' C' CongA A B C using l11-16
    using S1 S2 S3 S4 P4 P5 by blast
  hence A C B LtA A B C
    using P5A P4 conga-preserves-lta lta-left-comm by blast
}
{
  assume Obtuse A B C
  obtain A'' B'' C'' where P6:  $\text{Per } A'' B'' C'' \wedge A'' B'' C'' \text{ LtA } A B C$ 
    using Obtuse-def ⟨Obtuse A B C⟩ by auto
  have B C A LtA A' B' C'
    by (simp add: P4)
  hence P7:  $A C B \text{ LtA } A' B' C'$ 
    by (simp add: lta-left-comm)
  have A' B' C' LtA A B C
  proof -
    have U1:  $A'' B'' C'' \text{ CongA } A' B' C'$ 
    proof -
      have V2:  $A'' \neq B''$ 
        using P6 lta-distincts by blast
      have V3:  $C'' \neq B''$ 
        using P6 lta-distincts by blast
      have V5:  $A' \neq B'$ 
        using P7 lta-distincts by blast
      have C'  $\neq B'$ 
        using P4 lta-distincts by blast
      thus ?thesis using P6 V2 V3 P4 V5
        by (simp add: l11-16)
    qed
  have U2:  $A B C \text{ CongA } A B C$ 
    using assms(1) assms(2) conga-refl by auto
  have U3:  $A'' B'' C'' \text{ LtA } A B C$ 
    by (simp add: P6)
  thus ?thesis
    using U1 U2 conga-preserves-lta by auto
  qed
  hence A C B LtA A B C
    using P7 lta-trans by blast
}
hence A C B LtA A B C
  using ⟨Per A B C  $\implies$  A C B LtA A B C⟩ assms(3) by blast
hence A B Lt A C
  by (simp add: l11-44-2-b)
hence B A Lt A C
  using Lt-cases by blast
have C A B LtA C B A
proof -
  obtain A' B' C' where U4:  $\text{Per } A' B' C' \wedge B A C \text{ LtA } A' B' C'$ 
    using Acute-def P3 by blast
  {
    assume Per A B C
    hence W3:  $A' B' C' \text{ CongA } C B A$ 
      using U4 assms(2) l11-16 l8-2 lta-distincts by blast
    have W2:  $C A B \text{ CongA } C A B$ 
      using P2 assms(1) conga-refl by auto
    have C A B LtA A' B' C'
      by (simp add: U4 lta-left-comm)
    hence C A B LtA C B A
      using W2 W3 conga-preserves-lta by blast
  }
}

```



```

{
  assume Obtuse A B C
  then obtain A'' B'' C'' where W4: Per A'' B'' C''  $\wedge$  A'' B'' C'' LtA A B C
    using Obtuse-def by auto
  have W5: C A B LtA A' B' C'
    by (simp add: U4 lta-left-comm)
  have A' B' C' LtA C B A
  proof -
    have W6: A'' B'' C'' CongA A' B' C' using l11-16 W4 U4
      using lta-distincts by blast
    have C B A CongA C B A
      using assms(1) assms(2) conga-refl by auto
    thus ?thesis
      using W4 W6 conga-left-comm conga-preserves-lta by blast
  qed
  hence C A B LtA C B A
    using W5 lta-trans by blast
}
thus ?thesis
  using  $\langle \text{Per } A \ B \ C \implies C \ A \ B \ LtA \ C \ B \ A \rangle$  assms(3) by blast
qed
hence C B Lt C A
  by (simp add: l11-44-2-b)
hence C B Lt A C
  using Lt-cases by auto
hence B C Lt A C
  using Lt-cases by blast
thus ?thesis
  by (simp add:  $\langle B \ A \ Lt \ A \ C \rangle$ )
qed

lemma l11-47:
  assumes Per A C B and
    H PerpAt C H A B
  shows Bet A H B  $\wedge$  A  $\neq$  H  $\wedge$  B  $\neq$  H
proof -
  have P1: Per C H A
    using assms(2) perp-in-per-1 by auto
  have P2: C H Perp A B
    using assms(2) perp-in-perp by auto
  thus ?thesis
  proof cases
    assume Col A C B
    thus ?thesis
      by (metis P1 assms(1) assms(2) per-distinct-1 per-not-col perp-in-distinct perp-in-id)
  next
    assume P3:  $\neg$  Col A C B
    have P4: A  $\neq$  H
      by (metis P2 Per-perm l8-7 assms(1) assms(2) col-trivial-1 perp-in-per-2 perp-not-col2)
    have P5: Per C H B
      using assms(2) perp-in-per-2 by auto
    have P6: B  $\neq$  H
      using P1 P2 assms(1) l8-2 l8-7 perp-not-eq-1 by blast
    have P7: H A Lt A C  $\wedge$  H C Lt A C
      by (metis P1 P2 P4 l11-46 l8-2 perp-distinct)
    have P8: C A Lt A B  $\wedge$  C B Lt A B
      using P3 assms(1) l11-46 not-col-distincts by blast
    have P9: H B Lt B C  $\wedge$  H C Lt B C
      by (metis P2 P5 P6 Per-cases l11-46 perp-not-eq-1)
    have P10: Bet A H B
  proof -
    have T1: Col A H B
      using assms(2) col-permutation-5 perp-in-col by blast
    have T2: A H Le A B using P7 P8
      by (meson lt-comm lt-transitivity nlt--le not-and-lt)
    have H B Le A B

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    by (meson Lt-cases P8 P9 le-transitivity le-cases lt--nle)
  thus ?thesis
    using T1 T2 l5-12-b by blast
qed
thus ?thesis
  by (simp add: P4 P6)
qed
qed

lemma l11-49:
  assumes A B C CongA A' B' C' and
    Cong B A B' A' and
    Cong B C B' C'
  shows Cong A C A' C'  $\wedge$  (A  $\neq$  C  $\longrightarrow$  (B A C CongA B' A' C'  $\wedge$  B C A CongA B' C' A'))
proof -
  have Cong A C A' C'
    using assms(1) assms(2) assms(3) cong2-conga-cong not-cong-2143 by blast
  {
    assume A  $\neq$  C
    have A  $\neq$  B
      using CongA-def assms(1) by blast
    have C  $\neq$  B
      using CongA-def assms(1) by blast
    have B A C Cong3 B' A' C'
      using Cong3-def  $\langle$  Cong A C A' C'  $\rangle$  assms(2) assms(3) by blast
    hence B A C CongA B' A' C'
      using  $\langle$  A  $\neq$  B  $\rangle$   $\langle$  A  $\neq$  C  $\rangle$  cong3-conga by auto
    moreover have B C A Cong3 B' C' A'
      using  $\langle$  B A C Cong3 B' A' C'  $\rangle$  cong-3-swap-2 by blast
    hence B C A CongA B' C' A'
      using  $\langle$  A  $\neq$  C  $\rangle$   $\langle$  C  $\neq$  B  $\rangle$  cong3-conga by auto
    ultimately have B A C CongA B' A' C'  $\wedge$  B C A CongA B' C' A'
      by blast
  }
  thus ?thesis
    using  $\langle$  Cong A C A' C'  $\rangle$  by blast
qed

lemma l11-50-1:
  assumes  $\neg$  Col A B C and
    B A C CongA B' A' C' and
    A B C CongA A' B' C' and
    Cong A B A' B'
  shows Cong A C A' C'  $\wedge$  Cong B C B' C'  $\wedge$  A C B CongA A' C' B'
proof -
  obtain C'' where P1: B' Out C'' C'  $\wedge$  Cong B' C'' B C
    by (metis Col-perm assms(1) assms(3) col-trivial-3 conga-diff56 l6-11-existence)
  have P2: B'  $\neq$  C''
    using P1 out-diff1 by auto
  have P3:  $\neg$  Col A' B' C'
    using assms(1) assms(3) ncol-conga-ncol by blast
  have P4:  $\neg$  Col A' B' C''
    by (meson P1 P2 P3 col-transitivity-1 not-col-permutation-2 out-col)
  have P5: Cong A C A' C''
  proof -
    have Q1: B Out A A
      using assms(1) not-col-distincts out-trivial by auto
    have Q2: B Out C C
      using assms(1) col-trivial-2 out-trivial by force
    have Q3: B' Out A' A'
      using P3 not-col-distincts out-trivial by auto
    have Q5: Cong B A B' A'
      using assms(4) not-cong-2143 by blast
    have Cong B C B' C''
      using P1 not-cong-3412 by blast
    thus ?thesis

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    using l11-4-1 P1 Q1 Q2 Q3 Q5 assms(3) by blast
qed
have P6: B A C Cong3 B' A' C''
    using Cong3-def Cong-perm P1 P5 assms(4) by blast
have P7: B A C CongA B' A' C''
    by (metis P6 assms(1) cong3-conga not-col-distincts)
have P8: B' A' C' CongA B' A' C''
    by (meson P7 assms(2) conga-sym conga-trans)
have B' A' OS C' C''
    using Col-perm Out-cases P1 P3 out-one-side by blast
hence A' Out C' C''
    using P8 conga-os--out by auto
hence Col A' C' C''
    using out-col by auto
hence P9: C' = C''
    using Col-perm P1 out-col P3 col-transitivity-1 by blast
have T1: Cong A C A' C'
    by (simp add: P5 P9)
have T2: Cong B C B' C'
    using Cong-perm P1 P9 by blast
hence A C B CongA A' C' B'
    using T1 assms(1) assms(2) assms(4) col-trivial-2 l11-49 by blast
thus ?thesis using T1 T2 by blast
qed

lemma l11-50-2:
  assumes  $\neg$  Col A B C and
    B C A CongA B' C' A' and
    A B C CongA A' B' C' and
    Cong A B A' B'
  shows Cong A C A' C'  $\wedge$  Cong B C B' C'  $\wedge$  C A B CongA C' A' B'
proof -
  have P1: A  $\neq$  B
    using assms(1) col-trivial-1 by auto
  have P2: B  $\neq$  C
    using assms(1) col-trivial-2 by auto
  have P3: A'  $\neq$  B'
    using P1 assms(4) cong-diff by blast
  have P4: B'  $\neq$  C'
    using assms(2) conga-diff45 by auto
  then obtain C'' where P5: B' Out C'' C'  $\wedge$  Cong B' C'' B C
    using P2 l6-11-existence by presburger
  have P5BIS: B'  $\neq$  C''
    using P5 out-diff1 by auto
  have P5A: Col B' C'' C'
    using P5 out-col by auto
  have P6:  $\neg$  Col A' B' C'
    using assms(1) assms(3) ncol-conga-ncol by blast
  {
    assume Col A' B' C''
    hence Col B' C'' A'
      using not-col-permutation-2 by blast
    hence Col B' C' A' using col-transitivity-1 P5BIS P5A by blast
    hence Col A' B' C'
      using Col-perm by blast
    hence False
      using P6 by auto
  }
  hence P7:  $\neg$  Col A' B' C'' by blast
  have P8: Cong A C A' C''
proof -
  have B Out A A
    by (simp add: P1 out-trivial)
  have K1: B Out C C
    using P2 out-trivial by auto
  have K2: B' Out A' A'

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    using P3 out-trivial by auto
have Cong B A B' A'
  by (simp add: Cong-perm assms(4))
have Cong B C B' C''
  using Cong-perm P5 by blast
thus ?thesis
  using P5 ⟨Cong B A B' A'⟩ P1 out-trivial K1 K2 assms(3) l11-4-1 by blast
qed
have P9: B C A Cong3 B' C'' A'
  using Cong3-def Cong-perm P5 P8 assms(4) by blast
hence P10: B C A CongA B' C'' A'
  using assms(1) cong3-conga not-col-distincts by auto
have P11: B' C' A' CongA B' C'' A'
  using P9 assms(2) cong3-conga2 conga-sym by blast
show ?thesis
proof cases
  assume L1: C' = C''
  hence L2: Cong A C A' C'
    by (simp add: P8)
  have L3: Cong B C B' C'
    using Cong-perm L1 P5 by blast
  have C A B Cong3 C' A' B'
    by (simp add: L1 P9 cong-3-swap cong-3-swap-2)
  hence C A B CongA C' A' B'
    by (metis CongA-def P1 assms(2) cong3-conga)
  thus ?thesis using L2 L3 by auto
next
  assume R1: C' ≠ C''
  have R1A: ¬ Col C'' C' A'
    by (metis P5A P7 R1 col-permutation-2 col-trivial-2 colx)
  have R1B: Bet B' C'' C' ∨ Bet B' C' C''
    using Out-def P5 by auto
  {
    assume S1: Bet B' C'' C'
    hence S2: C'' A' C' LtA A' C'' B' ∧ C'' C' A' LtA A' C'' B'
      using P5BIS R1A between-symmetry l11-41 by blast
    have B' C' A' CongA C'' C' A'
      by (metis P11 R1 conga-comm S1 bet-out-1 conga-diff45 not-conga-sym
        out2--conga out-trivial)
    hence B' C' A' LtA A' C'' B'
      by (meson P11 conga-right-comm not-conga not-conga-sym S2 not-lta-and-conga)
    hence Cong A C A' C' ∧ Cong B C B' C'
      by (meson P11 conga-right-comm not-lta-and-conga)
  }
  {
    assume Z1: Bet B' C' C''
    have Z2: ¬ Col C' C'' A'
      by (simp add: R1A not-col-permutation-4)
    have Z3: C'' Out C' B'
      by (simp add: R1 Z1 bet-out-1)
    have Z4: C'' Out A' A'
      using P7 not-col-distincts out-trivial by blast
    hence Z4A: B' C'' A' CongA C' C'' A'
      by (simp add: Z3 out2--conga)
    have Z4B: B' C'' A' LtA A' C' B'
      proof -
        have Z5: C' C'' A' CongA B' C'' A'
          using Z4A not-conga-sym by blast
        have Z6: A' C' B' CongA A' C' B'
          using P11 P4 conga-diff2 conga-refl by blast
        have C' C'' A' LtA A' C' B'
          using P4 Z1 Z2 between-symmetry l11-41 by blast
        thus ?thesis
          using Z5 Z6 conga-preserves-lta by auto
      qed
    have B' C'' A' CongA B' C' A'

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    using P11 not-conga-sym by blast
  hence Cong A C A' C'  $\wedge$  Cong B C B' C'
    by (meson conga-right-comm Z4B not-lta-and-conga)
}
hence R2: Cong A C A' C'  $\wedge$  Cong B C B' C'
  using R1B  $\langle$ Bet B' C'' C'  $\implies$  Cong A C A' C'  $\wedge$  Cong B C B' C' $\rangle$  by blast
hence C A B CongA C' A' B'
  using P1 assms(2) l11-49 not-cong-2143 by blast
thus ?thesis using R2 by auto
qed
qed

```

```

lemma l11-51:
  assumes A  $\neq$  B and
    A  $\neq$  C and
    B  $\neq$  C and
    Cong A B A' B' and
    Cong A C A' C' and
    Cong B C B' C'
  shows
    B A C CongA B' A' C'  $\wedge$  A B C CongA A' B' C'  $\wedge$  B C A CongA B' C' A'
proof -
  have B A C Cong3 B' A' C'  $\wedge$  A B C Cong3 A' B' C'  $\wedge$  B C A Cong3 B' C' A'
    using Cong3-def Cong-perm assms(4) assms(5) assms(6) by blast
  thus ?thesis
    using assms(1) assms(2) assms(3) cong3-conga by auto
qed

```

```

lemma conga-distinct:
  assumes A B C CongA D E F
  shows A  $\neq$  B  $\wedge$  C  $\neq$  B  $\wedge$  D  $\neq$  E  $\wedge$  F  $\neq$  E
  using CongA-def assms by auto

```

```

lemma l11-52:
  assumes A B C CongA A' B' C' and
    Cong A C A' C' and
    Cong B C B' C' and
    B C Le A C
  shows Cong B A B' A'  $\wedge$  B A C CongA B' A' C'  $\wedge$  B C A CongA B' C' A'
proof -
  have P1: A  $\neq$  B
    using CongA-def assms(1) by blast
  have P2: C  $\neq$  B
    using CongA-def assms(1) by blast
  have P3: A'  $\neq$  B'
    using CongA-def assms(1) by blast
  have P4: C'  $\neq$  B'
    using assms(1) conga-diff56 by auto
  have P5: Cong B A B' A'
  proof cases
    assume P6: Col A B C
    hence P7: Bet A B C  $\vee$  Bet B C A  $\vee$  Bet C A B
      using Col-def by blast
    {
      assume P8: Bet A B C
      hence Bet A' B' C'
        using assms(1) bet-conga--bet by blast
      hence Cong B A B' A'
        using P8 assms(2) assms(3) l4-3 not-cong-2143 by blast
    }
    {
      assume P9: Bet B C A
      hence P10: B' Out A' C'
        using Out-cases P2 assms(1) bet-out l11-21-a by blast
      hence P11: Bet B' A' C'  $\vee$  Bet B' C' A'
        by (simp add: Out-def)
    }
  qed

```

```

{
  assume Bet  $B' A' C'$ 
  hence Cong  $B A B' A'$ 
    using P3 assms(2) assms(3) assms(4) bet-le-eq l5-6 by blast
}
{
  assume Bet  $B' C' A'$ 
  hence Cong  $B A B' A'$ 
    using Cong-perm P9 assms(2) assms(3) l2-11-b by blast
}
hence Cong  $B A B' A'$ 
  using P11  $\langle \textit{Bet } B' A' C' \implies \textit{Cong } B A B' A' \rangle$  by blast
}
{
  assume Bet  $C A B$ 
  hence Cong  $B A B' A'$ 
    using P1 assms(4) bet-le-eq between-symmetry by blast
}
thus ?thesis
  using P7  $\langle \textit{Bet } A B C \implies \textit{Cong } B A B' A' \rangle \langle \textit{Bet } B C A \implies \textit{Cong } B A B' A' \rangle$  by blast
next
assume Z1:  $\neg \textit{Col } A B C$ 
obtain A'' where Z2:  $B' \textit{Out } A'' A' \wedge \textit{Cong } B' A'' B A$ 
  using P1 P3 l6-11-existence by force
hence Z3:  $A' B' C' \textit{CongA } A'' B' C'$ 
  by (simp add: P4 out2--conga out-trivial)
have Z4:  $A B C \textit{CongA } A'' B' C'$ 
  using Z3 assms(1) not-conga by blast
have Z5:  $\textit{Cong } A'' C' A C$ 
  using Z2 Z4 assms(3) cong2-conga-cong cong-4321 cong-symmetry by blast
have Z6:  $A'' B' C' \textit{Cong3 } A B C$ 
  using Cong3-def Cong-perm Z2 Z5 assms(3) by blast
have Z7:  $\textit{Cong } A'' C' A' C'$ 
  using Z5 assms(2) cong-transitivity by blast
have Z8:  $\neg \textit{Col } A' B' C'$ 
  by (metis Z1 assms(1) ncol-conga-ncol)
hence Z9:  $\neg \textit{Col } A'' B' C'$ 
  by (metis Z2 col-transitivity-1 not-col-permutation-4 out-col out-diff1)
{
  assume Z9A:  $A'' \neq A'$ 
  have Z10:  $\textit{Bet } B' A'' A' \vee \textit{Bet } B' A' A''$ 
    using Out-def Z2 by auto
  {
    assume Z11:  $\textit{Bet } B' A'' A'$ 
    have Z12:  $A'' C' B' \textit{LtA } C' A'' A' \wedge A'' B' C' \textit{LtA } C' A'' A'$ 
      by (simp add: Z11 Z9 Z9A l11-41)
    have Z13:  $\textit{Cong } A' C' A'' C'$ 
      using Cong-perm Z7 by blast
    have Z14:  $\neg \textit{Col } A'' C' A'$ 
      by (metis Col-def Z11 Z9 Z9A col-transitivity-1)
    have Z15:  $C' A'' A' \textit{CongA } C' A' A'' \longleftrightarrow \textit{Cong } C' A'' C' A'$ 
      by (simp add: Z14 l11-44-1)
    have Z16:  $\textit{Cong } C' A' C' A''$ 
      using Cong-perm Z7 by blast
    hence Z17:  $\textit{Cong } C' A'' C' A'$ 
      using Cong-perm by blast
    hence Z18:  $C' A'' A' \textit{CongA } C' A' A''$ 
      by (simp add: Z15)
    have Z19:  $\neg \textit{Col } B' C' A''$ 
      using Col-perm Z9 by blast
    have Z20:  $B' A' C' \textit{CongA } A'' A' C'$ 
      by (metis col-conga-col Z11 Z3 Z9 Z9A bet-out-1 col-trivial-3 out2--conga out-trivial)
    have Z21:  $\neg \textit{Col } B' C' A'$ 
      using Col-perm Z8 by blast
    hence Z22:  $C' B' A' \textit{LtA } C' A' B' \longleftrightarrow C' A' \textit{Lt } C' B'$ 
      by (simp add: l11-44-2)
  }
}

```

```

have A'' B' C' CongA C' B' A'
  using Z3 conga-right-comm not-conga-sym by blast
hence U1: C' B' A' LtA C' A' B'
proof -
  have C' A'' A' CongA C' A' A''
    by (metis Z15 Z17)
  hence  $\neg C' B' A' LtA C' A'' A' \vee A'' B' C' LtA C' A' A''$ 
    by (meson Z3 conga-preserves-lta lta-left-comm)
  hence  $C' B' A' LtA C' A' B' \vee A'' B' C' LtA A'' A' C' \vee A'' = B'$ 
    using Z12  $\langle A'' B' C' CongA C' B' A' \rangle$  conga-preserves-lta conga-refl
    by (metis Z15 Z17 lta-right-comm)
  thus ?thesis
    by (metis conga-diff1 conga-sym Z20  $\langle A'' B' C' CongA C' B' A' \rangle$ 
        conga-preserves-lta conga-right-comm)
qed
hence Z23: C' A' Lt C' B'
  using Z22 by auto
have Z24: C' A'' Lt C' B'
  using Z16 Z23 cong2-lt--lt cong-reflexivity by blast
have Z25: C A Le C B
proof -
  have Z26: Cong C' A'' C A
    using Z5 not-cong-2143 by blast
  have Cong C' B' C B
    using assms(3) not-cong-4321 by blast
  thus ?thesis
    using l5-6 Z24 Z26 lt--le by blast
qed
hence Z27: Cong C A C B
  by (simp add: assms(4) le-anti-symmetry le-comm)
have Cong C' A'' C' B'
  by (metis Cong-perm Z13 Z27 assms(2) assms(3) cong-transitivity)
hence False
  using Z24 cong--nlt by blast
hence Cong B A B' A' by simp
}
{
  assume W1: Bet B' A' A''
  have W2: A'  $\neq$  A''
    using Z9A by auto
  have W3: A' C' B' LtA C' A' A''  $\wedge$  A' B' C' LtA C' A' A''
    using W1 Z8 Z9A l11-41 by blast
  have W4: Cong A' C' A'' C'
    using Z7 not-cong-3412 by blast
  have  $\neg Col A'' C' A'$ 
    by (metis Col-def W1 Z8 Z9A col-transitivity-1)
  hence W6: C' A'' A' CongA C' A' A''  $\longleftrightarrow$  Cong C' A'' C' A'
    using l11-44-1 by auto
  have W7: Cong C' A' C' A''
    using Z7 not-cong-4321 by blast
  hence W8: Cong C' A'' C' A'
    using W4 not-cong-4321 by blast
  have W9:  $\neg Col B' C' A''$ 
    by (simp add: Z9 not-col-permutation-1)
  have W10: B' A'' C' CongA A' A'' C'
    by (metis Out-def W1 Z9 Z9A bet-out-1 between-trivial not-col-distincts out2--conga)
  have W12: C' B' A'' LtA C' A'' B'  $\longleftrightarrow$  C' A'' Lt C' B'
    by (simp add: W9 l11-44-2)
  have W12A: C' B' A'' LtA C' A'' B'
  proof -
    have V1: A' B' C' CongA C' B' A''
      by (simp add: Z3 conga-right-comm)
    have A' A'' C' CongA B' A'' C'
      by (metis Out-def W1  $\langle \neg Col A'' C' A' \rangle$  between-equality-2 not-col-distincts
          or-bet-out out2--conga out-col)
    hence C' A' A'' CongA C' A'' B'

```

```

    by (meson W6 W8 conga-left-comm not-conga not-conga-sym)
  thus ?thesis
    using W3 V1 conga-preserves-lta by auto
qed
hence C' A'' Lt C' B' using W12 by auto
hence W14: C' A' Lt C' B'
  using W8 cong2-lt--lt cong-reflexivity by blast
have W15: C A Le C B
proof -
  have Q1: C' A'' Le C' B'
    using W12 W12A lt--le by blast
  have Q2: Cong C' A'' C A
    using Z5 not-cong-2143 by blast
  have Cong C' B' C B
    using assms(3) not-cong-4321 by blast
  thus ?thesis using Q1 Q2 l5-6 by blast
qed
have C B Le C A
  by (simp add: assms(4) le-comm)
hence Cong C A C B
  by (simp add: W15 le-anti-symmetry)
hence Cong C' A' C' B'
  by (metis Cong-perm assms(2) assms(3) cong-inner-transitivity)
hence False
  using W14 cong--nlt by blast
hence Cong B A B' A' by simp
}
then have Cong B A B' A'
  using Z10 ⟨Bet B' A'' A' ⟹ Cong B A B' A'⟩ by blast
}
{
  assume A'' = A'
  hence Cong B A B' A'
    using Z2 not-cong-3412 by blast
}
thus ?thesis
  using ⟨A'' ≠ A' ⟹ Cong B A B' A'⟩ by blast
qed
have P6: A B C Cong3 A' B' C'
  using Cong3-def Cong-perm P5 assms(2) assms(3) by blast
thus ?thesis
  using P2 P5 assms(1) assms(3) assms(4) l11-49 le-zero by blast
qed

```

lemma l11-53:

```

assumes Per D C B and
  C ≠ D and
  A ≠ B and
  B ≠ C and
  Bet A B C
shows C A D LtA C B D ∧ B D Lt A D
proof -
  have P1: C ≠ A
    using assms(3) assms(5) between-identity by blast
  have P2: ¬ Col B A D
    using assms(1) assms(2) assms(3) assms(4) assms(5) bet-col bet-col1 col3
      col-permutation-4 l8-9 by metis
  have P3: A ≠ D
    using P2 col-trivial-2 by blast
  have P4: C A D LtA C B D
  proof -
    have P4A: B D A LtA D B C ∧ B A D LtA D B C
      by (simp add: P2 assms(4) assms(5) l11-41)
    have P4AA: A Out B C
      using assms(3) assms(5) bet-out by auto
    have A Out D D

```



```

    using P3 out-trivial by auto
  hence P4B:  $C \ A \ D \ CongA \ B \ A \ D$  using P4AA
    by (simp add: out2--conga)
  hence P4C:  $B \ A \ D \ CongA \ C \ A \ D$ 
    by (simp add: P4B conga-sym)
  have D B C CongA C B D
    using assms(1) assms(4) conga-pseudo-refl per-distinct-1 by auto
  thus ?thesis
    using P4A P4C conga-preserves-lta by blast
qed
obtain B' where P5:  $C \ Midpoint \ B \ B' \wedge \ Cong \ D \ B \ D \ B'$ 
  using Per-def assms(1) by auto
have K2:  $A \neq B'$ 
  using Bet-cases P5 assms(4) assms(5) between-equality-2 midpoint-bet by blast
{
  assume Col B D B'
  hence Col B A D
    by (metis Col-cases P5 assms(1) assms(2) assms(4) col2--eq midpoint-col
        midpoint-distinct-2 per-not-col)
  hence False
    by (simp add: P2)
}
hence P6:  $\neg \ Col \ B \ D \ B'$  by blast
hence D B B' CongA D B' B  $\longleftrightarrow$  Cong D B D B'
  by (simp add: l11-44-1)
hence D B B' CongA D B' B using P5 by simp
{
  assume K1:  $Col \ A \ D \ B'$ 
  have Col B' A B
    using Col-def P5 assms(4) assms(5) midpoint-bet outer-transitivity-between by blast
  hence Col B' B D
    using K1 K2 Col-perm col-transitivity-2 by blast
  hence Col B D B'
    using Col-perm by blast
  hence False
    by (simp add: P6)
}
hence K3B:  $\neg \ Col \ A \ D \ B'$  by blast
hence K4:  $D \ A \ B' \ LtA \ D \ B' \ A \longleftrightarrow D \ B' \ Lt \ D \ A$ 
  by (simp add: l11-44-2)
have K4A:  $C \ A \ D \ LtA \ C \ B' \ D$ 
  by (metis Midpoint-def P1 P3 P4 P5 P5 P6 assms(2) assms(4) col-trivial-1
        cong-reflexivity conga-preserves-lta conga-refl l11-51 not-cong-2134)
have D B' Lt D A
proof -
  have D A B' LtA D B' A
  proof -
    have K5A:  $A \ Out \ D \ D$ 
      using P3 out-trivial by auto
    have K5AA:  $A \ Out \ B' \ C$ 
      using K2 Out-def P1 P5 assms(4) assms(5) midpoint-bet outer-transitivity-between2
      by fast
    hence K5:  $D \ A \ C \ CongA \ D \ A \ B'$ 
      by (simp add: K5A out2--conga)
    have K6A:  $B' \ Out \ D \ D$ 
      using K3B not-col-distincts out-trivial by blast
    have B' Out A C
      using P5 K5AA assms(4) assms(5) between-equality-2 l6-4-2 midpoint-bet
      midpoint-distinct-2 out-col outer-transitivity-between2 by metis
    hence K6:  $D \ B' \ C \ CongA \ D \ B' \ A$ 
      by (simp add: K6A out2--conga)
    have D A C LtA D B' C
      by (simp add: K4A lta-comm)
    thus ?thesis
      using K5 K6 conga-preserves-lta by auto
  qed
qed

```

```

    thus ?thesis
    by (simp add: K4)
qed
thus ?thesis
using P4 P5 cong2-lt--lt cong-pseudo-reflexivity not-cong-4312 by blast
qed

lemma cong2-conga-obtuse--cong-conga2:
  assumes Obtuse A B C and
    A B C CongA A' B' C' and
    Cong A C A' C' and
    Cong B C B' C'
  shows Cong B A B' A'  $\wedge$  B A C CongA B' A' C'  $\wedge$  B C A CongA B' C' A'
proof -
  have B C Le A C
  proof cases
    assume Col A B C
    thus ?thesis
    by (simp add: assms(1) col-obtuse--bet l5-12-a)
  next
    assume  $\neg$  Col A B C
    thus ?thesis
    using l11-46 assms(1) lt--le not-col-distincts by auto
  qed
  thus ?thesis
  using l11-52 assms(2) assms(3) assms(4) by blast
qed

lemma cong2-per2--cong-conga2:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    Per A B C and
    Per A' B' C' and
    Cong A C A' C' and
    Cong B C B' C'
  shows Cong B A B' A'  $\wedge$  B A C CongA B' A' C'  $\wedge$  B C A CongA B' C' A'
proof -
  have B C Le A C  $\wedge$   $\neg$  Cong B C A C
  using assms(1) assms(2) assms(3) cong--nlt l11-46 lt--le by blast
  hence A B C CongA A' B' C'
  using assms(2) assms(3) assms(4) assms(5) assms(6) cong-diff
    cong-inner-transitivity cong-symmetry l11-16 by metis
  thus ?thesis
  using  $\langle B C Le A C \wedge \neg Cong B C A C \rangle$  assms(5) assms(6) l11-52 by blast
qed

lemma cong2-per2--cong:
  assumes Per A B C and
    Per A' B' C' and
    Cong A C A' C' and
    Cong B C B' C'
  shows Cong B A B' A'
proof cases
  assume B = C
  thus ?thesis
  using assms(3) assms(4) cong-reverse-identity not-cong-2143 by blast
next
  assume B  $\neq$  C
  show ?thesis
  proof cases
    assume A = B
    thus ?thesis
    proof -
      have Cong A C B' C'
      using  $\langle A = B \rangle$  assms(4) by blast
      hence B' = A'

```

```

    by (meson Cong3-def Per-perm assms(2) assms(3) cong-inner-transitivity
        cong-pseudo-reflexivity l8-10 l8-7)
  thus ?thesis
    using  $\langle A = B \rangle$  cong-trivial-identity by blast
qed
next
  assume  $A \neq B$ 
  show ?thesis
  proof cases
    assume  $A' = B'$ 
    thus ?thesis
      by (metis Cong3-def Per-perm  $\langle A \neq B \rangle$  assms(1) assms(3) assms(4)
          cong-inner-transitivity cong-pseudo-reflexivity l8-10 l8-7)
  next
    assume  $A' \neq B'$ 
    thus ?thesis
      using cong2-per2--cong-conga2  $\langle A \neq B \rangle$   $\langle B \neq C \rangle$  assms(1) assms(2) assms(3) assms(4)
        by blast
  qed
qed
qed
qed

lemma cong2-per2--cong-3:
  assumes Per A B C
    Per A' B' C' and
    Cong A C A' C' and
    Cong B C B' C'
  shows A B C Cong3 A' B' C'
  by (metis Cong3-def assms(1) assms(2) assms(3) assms(4) cong2-per2--cong cong-3-swap)

lemma cong-lt-per2--lt:
  assumes Per A B C and
    Per A' B' C' and
    Cong A B A' B' and
    B C Lt B' C'
  shows A C Lt A' C'
  proof cases
    assume  $A = B$ 
    thus ?thesis
      using assms(3) assms(4) cong-reverse-identity by blast
  next
    assume  $A \neq B$ 
    show ?thesis
    proof cases
      assume  $B = C$ 
      have  $B' A' Lt C' A'$ 
        by (metis cong-diff-2 l11-46  $\langle A \neq B \rangle$  assms(2) assms(3) assms(4) lt-diff lt-right-comm)
      thus ?thesis
        by (metis cong2-lt--lt cong-reflexivity cong-symmetry lt-comm  $\langle B = C \rangle$  assms(3))
    next
      assume P0:  $B \neq C$ 
      have B C Lt B' C'
        by (simp add: assms(4))
      hence R1:  $B C Le B' C' \wedge \neg Cong B C B' C'$ 
        by (simp add: Lt-def)
      then obtain C0 where P1:  $Bet B' C0 C' \wedge Cong B C B' C0$ 
        using Le-def by auto
      hence P2: Per A' B' C0
        by (metis Col-def Per-cases assms(2) bet-out-1 col-col-per-per col-trivial-1
            l8-5 out-diff2)
      have C0 A' Lt C' A' using l11-53
        by (metis P1 P2 R1 P0 bet--lt2313 between-symmetry cong-diff)
      hence P3:  $A' C0 Lt A' C'$ 
        using Lt-cases by blast
      have P4: Cong A' C0 A C
        using P1 P2 assms(1) assms(3) l10-12 not-cong-3412 by blast

```

```

have Cong A' C' A' C'
  by (simp add: cong-reflexivity)
thus ?thesis
  using cong2-lt-lt P3 P4 by blast
qed
qed

```

```

lemma cong-le-per2--le:
  assumes Per A B C and
    Per A' B' C' and
    Cong A B A' B' and
    B C Le B' C'
  shows A C Le A' C'
proof cases
  assume Cong B C B' C'
  thus ?thesis
    using assms(1) assms(2) assms(3) cong--le l10-12 by blast
next
  assume ¬ Cong B C B' C'
  hence B C Lt B' C'
    using Lt-def assms(4) by blast
  thus ?thesis
    using assms(1) assms(2) assms(3) cong-lt-per2--lt lt--le by auto
qed

```

```

lemma lt2-per2--lt:
  assumes Per A B C and
    Per A' B' C' and
    A B Lt A' B' and
    B C Lt B' C'
  shows A C Lt A' C'
proof -
  have P2: B A Lt B' A'
    by (simp add: assms(3) lt-comm)
  have P3: B C Le B' C' ∧ ¬ Cong B C B' C'
    using assms(4) cong--nlt lt--le by auto
  then obtain C0 where P4: Bet B' C0 C' ∧ Cong B C B' C0
    using Le-def by auto
  have P4A: B' ≠ C'
    using assms(4) lt-diff by auto
  have Col B' C' C0
    using P4 bet-col not-col-permutation-5 by blast
  hence P5: Per A' B' C0
    using assms(2) P4A per-col by blast
  have P6: A C Lt A' C0
    by (meson P2 P4 P5 assms(1) cong-lt-per2--lt l8-2 lt-comm not-cong-2143)
  have B' C0 Lt B' C'
    by (metis P4 assms(4) bet--lt1213 cong--nlt)
  hence A' C0 Lt A' C'
    using P5 assms(2) cong-lt-per2--lt cong-reflexivity by blast
  thus ?thesis
    using P6 lt-transitivity by blast
qed

```

```

lemma le-lt-per2--lt:
  assumes Per A B C and
    Per A' B' C' and
    A B Le A' B' and
    B C Lt B' C'
  shows A C Lt A' C'
  using Lt-def assms(1) assms(2) assms(3) assms(4) cong-lt-per2--lt lt2-per2--lt by blast

```

```

lemma le2-per2--le:
  assumes Per A B C and
    Per A' B' C' and
    A B Le A' B' and

```

$B C Le B' C'$
shows $A C Le A' C'$
proof *cases*
assume $Cong B C B' C'$
thus *?thesis*
by (*meson Per-cases cong-le-per2--le assms(1) assms(2) assms(3) le-comm not-cong-2143*)
next
assume $\neg Cong B C B' C'$
hence $B C Lt B' C'$
by (*simp add: Lt-def assms(4)*)
thus *?thesis*
using $assms(1) assms(2) assms(3) le-lt-per2--lt lt--le$ **by** *blast*
qed

lemma *cong-lt-per2--lt-1*:
assumes $Per A B C$ **and**
 $Per A' B' C'$ **and**
 $A B Lt A' B'$ **and**
 $Cong A C A' C'$
shows $B' C' Lt B C$
by (*meson Gt-def assms(1) assms(2) assms(3) assms(4) cong2-per2--cong cong-4321*
cong--nlt cong-symmetry lt2-per2--lt or-lt-cong-gt)

lemma *symmetry-preserves-conga*:
assumes $A \neq B$ **and** $C \neq B$ **and**
 M *Midpoint* $A A'$ **and**
 M *Midpoint* $B B'$ **and**
 M *Midpoint* $C C'$
shows $A B C CongA A' B' C'$
by (*metis Mid-perm assms(1) assms(2) assms(3) assms(4) assms(5) conga-trivial-1*
l11-51 l7-13 symmetric-point-uniqueness)

lemma *l11-57*:
assumes $A A' OS B B'$ **and**
 $Per B A A'$ **and**
 $Per B' A' A$ **and**
 $A A' OS C C'$ **and**
 $Per C A A'$ **and**
 $Per C' A' A$
shows $B A C CongA B' A' C'$
proof –
obtain M **where** $P1: M$ *Midpoint* $A A'$
using *midpoint-existence* **by** *auto*
obtain B'' **where** $P2: M$ *Midpoint* $B B''$
using *symmetric-point-construction* **by** *auto*
obtain C'' **where** $P3: M$ *Midpoint* $C C''$
using *symmetric-point-construction* **by** *auto*
have $P4: \neg Col A A' B$
using $assms(1) col123--nos$ **by** *auto*
have $P5: \neg Col A A' C$
using $assms(4) col123--nos$ **by** *auto*
have $P6: B A C CongA B'' A' C''$
by (*metis P1 P2 P3 assms(1) assms(4) os-distincts symmetry-preserves-conga*)
have $B'' A' C'' CongA B' A' C'$
proof –
have $B \neq M$
using $P1 P4$ *midpoint-col not-col-permutation-2* **by** *blast*
hence $P7: \neg Col B'' A A'$
using $Mid-cases P1 P2 P4$ *mid-preserves-col not-col-permutation-3* **by** *blast*
have $K3: Bet B'' A' B'$
proof –
have $Per B'' A' A$
using $P1 P2 assms(2)$ *per-mid-per* **by** *blast*
have $Col B B'' M \wedge Col A A' M$
using $P1 P2$ *midpoint-col not-col-permutation-2* **by** *blast*
hence *Coplanar* $B A A' B''$

```

    using Coplanar-def by auto
  hence Coplanar A B' B'' A'
    by (meson assms(1) between-trivial2 coplanar-trans-1 ncoplanar-perm-4
        ncoplanar-perm-8 one-side-chara os--coplanar)
  hence P8: Col B' B'' A'
    using cop-per2--col P1 P2 P7 assms(2) assms(3) not-col-distincts per-mid-per
    by blast
  have A A' TS B B''
    using P1 P2 P4 mid-two-sides by auto
  hence A' A TS B'' B'
    using assms(1) invert-two-sides l9-2 l9-8-2 by blast
  thus ?thesis
    using Col-cases P8 col-two-sides-bet by blast
qed
have  $\neg$  Col C'' A A'
  using Col-def P1 P3 P5 l7-15 l7-2 not-col-permutation-5 by meson
have Bet C'' A' C'
proof -
  have Z2: Col C' C'' A'
  proof -
    have Col C C'' M  $\wedge$  Col A A' M
      using P1 P3 col-permutation-1 midpoint-col by blast
    hence Coplanar C A A' C''
      using Coplanar-def by blast
    hence Z1: Coplanar A C' C'' A'
      by (meson assms(4) between-trivial2 coplanar-trans-1 ncoplanar-perm-4
          ncoplanar-perm-8 one-side-chara os--coplanar)
    have Per C'' A' A
      using P1 P3 assms(5) per-mid-per by blast
    thus ?thesis
      using Z1 P5 assms(6) col-trivial-1 cop-per2--col by blast
  qed
  have A A' TS C C''
    using P1 P3 P5 mid-two-sides by auto
  hence A' A TS C'' C'
    using assms(4) invert-two-sides l9-2 l9-8-2 by blast
  thus ?thesis
    using Col-cases Z2 col-two-sides-bet by blast
qed
thus ?thesis
  by (metis P6 K3 assms(1) assms(4) conga-diff45 conga-diff56 l11-14 os-distincts)
qed
thus ?thesis
  using P6 conga-trans by blast
qed

lemma cop3-orth-at--orth-at:
  assumes  $\neg$  Col D E F and
    Coplanar A B C D and
    Coplanar A B C E and
    Coplanar A B C F and
    X OrthAt A B C U V
  shows X OrthAt D E F U V
proof -
  have P1:  $\neg$  Col A B C  $\wedge$  Coplanar A B C X
    using OrthAt-def assms(5) by blast
  hence P2: Coplanar D E F X
    using assms(2) assms(3) assms(4) coplanar-pseudo-trans by blast
  {
    fix M
    assume Coplanar A B C M
    hence Coplanar D E F M
      using P1 assms(2) assms(3) assms(4) coplanar-pseudo-trans by blast
  }
  have T1:  $U \neq V$ 
    using OrthAt-def assms(5) by blast

```

```

have T2: Col U V X
  using OrthAt-def assms(5) by auto
{
  fix P Q
  assume P7: Coplanar D E F P  $\wedge$  Col U V Q
  hence Coplanar A B C P
    by (meson  $\langle \wedge M. \text{Coplanar } A B C M \implies \text{Coplanar } D E F M \rangle$  assms(1) assms(2)
        assms(3) assms(4) l9-30)
  hence Per P X Q using P7 OrthAt-def assms(5) by blast
}
thus ?thesis using assms(1)
  by (simp add: OrthAt-def P2 T1 T2)
qed

```

```

lemma col2-orth-at--orth-at:
  assumes U  $\neq$  V and
    Col P Q U and
    Col P Q V and
    X OrthAt A B C P Q
  shows X OrthAt A B C U V
proof -
  have Col P Q X
    using OrthAt-def assms(4) by auto
  hence Col U V X
    by (metis OrthAt-def assms(2) assms(3) assms(4) col3)
  thus ?thesis
    using OrthAt-def assms(1) assms(2) assms(3) assms(4) colx by presburger
qed

```

```

lemma col-orth-at--orth-at:
  assumes U  $\neq$  W and
    Col U V W and
    X OrthAt A B C U V
  shows X OrthAt A B C U W
  using assms(1) assms(2) assms(3) col2-orth-at--orth-at col-trivial-3 by blast

```

```

lemma orth-at-symmetry:
  assumes X OrthAt A B C U V
  shows X OrthAt A B C V U
  by (metis assms col2-orth-at--orth-at col-trivial-2 col-trivial-3)

```

```

lemma orth-at-distincts:
  assumes X OrthAt A B C U V
  shows A  $\neq$  B  $\wedge$  B  $\neq$  C  $\wedge$  A  $\neq$  C  $\wedge$  U  $\neq$  V
  using OrthAt-def assms not-col-distincts by fastforce

```

```

lemma orth-at-chara:
  X OrthAt A B C X P  $\longleftrightarrow$ 
( $\neg$  Col A B C  $\wedge$  X  $\neq$  P  $\wedge$  Coplanar A B C X  $\wedge$  ( $\forall D. (\text{Coplanar } A B C D \longrightarrow \text{Per } D X P)$ ))
proof -
  {
    assume X OrthAt A B C X P
    hence  $\neg$  Col A B C  $\wedge$  X  $\neq$  P  $\wedge$  Coplanar A B C X  $\wedge$  ( $\forall D. (\text{Coplanar } A B C D \longrightarrow \text{Per } D X P)$ )
      using OrthAt-def col-trivial-2 by auto
  }
  {
    assume T1:  $\neg$  Col A B C  $\wedge$  X  $\neq$  P  $\wedge$ 
      Coplanar A B C X  $\wedge$  ( $\forall D. (\text{Coplanar } A B C D \longrightarrow \text{Per } D X P)$ )
    {
      fix P0 Q
      assume Coplanar A B C P0  $\wedge$  Col X P Q
      hence Per P0 X Q using T1 OrthAt-def per-col by auto
    }
    hence X OrthAt A B C X P
      by (simp add: T1  $\langle \wedge Q P0. \text{Coplanar } A B C P0 \wedge \text{Col } X P Q \implies \text{Per } P0 X Q \rangle$ 
          OrthAt-def col-trivial-3)
  }

```

```

}
thus ?thesis
  using  $\langle X \text{ OrthAt } A \ B \ C \ X \ P \implies \neg \text{Col } A \ B \ C \wedge X \neq P \wedge$ 
   $\text{Coplanar } A \ B \ C \ X \wedge (\forall D. \text{Coplanar } A \ B \ C \ D \longrightarrow \text{Per } D \ X \ P) \rangle$ 
  by blast
qed

lemma cop3-orth--orth:
assumes  $\neg \text{Col } D \ E \ F$  and
   $\text{Coplanar } A \ B \ C \ D$  and
   $\text{Coplanar } A \ B \ C \ E$  and
   $\text{Coplanar } A \ B \ C \ F$  and
   $A \ B \ C \text{ Orth } U \ V$ 
shows  $D \ E \ F \text{ Orth } U \ V$ 
using Orth-def assms(1) assms(2) assms(3) assms(4) assms(5) cop3-orth-at--orth-at by blast

lemma col2-orth--orth:
assumes  $U \neq V$  and
   $\text{Col } P \ Q \ U$  and
   $\text{Col } P \ Q \ V$  and
   $A \ B \ C \text{ Orth } P \ Q$ 
shows  $A \ B \ C \text{ Orth } U \ V$ 
by (meson Orth-def col2-orth-at--orth-at Tarski-neutral-dimensionless-axioms
  assms(1) assms(2) assms(3) assms(4))

lemma col-orth--orth:
assumes  $U \neq W$  and
   $\text{Col } U \ V \ W$  and
   $A \ B \ C \text{ Orth } U \ V$ 
shows  $A \ B \ C \text{ Orth } U \ W$ 
by (meson assms(1) assms(2) assms(3) col2-orth--orth col-trivial-3)

lemma orth-symmetry:
assumes  $A \ B \ C \text{ Orth } U \ V$ 
shows  $A \ B \ C \text{ Orth } V \ U$ 
by (meson Orth-def assms orth-at-symmetry)

lemma orth-distincts:
assumes  $A \ B \ C \text{ Orth } U \ V$ 
shows  $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$ 
using Orth-def assms orth-at-distincts by blast

lemma col-cop-orth--orth-at:
assumes  $A \ B \ C \text{ Orth } U \ V$  and
   $\text{Coplanar } A \ B \ C \ X$  and
   $\text{Col } U \ V \ X$ 
shows  $X \text{ OrthAt } A \ B \ C \ U \ V$ 
proof –
  obtain  $Y$  where  $P1$ :
     $\neg \text{Col } A \ B \ C \wedge U \neq V \wedge \text{Coplanar } A \ B \ C \ Y \wedge \text{Col } U \ V \ Y \wedge$ 
     $(\forall P \ Q. (\text{Coplanar } A \ B \ C \ P \wedge \text{Col } U \ V \ Q) \longrightarrow \text{Per } P \ Y \ Q)$ 
    by (metis OrthAt-def Orth-def assms(1))
  hence  $P2$ :  $X = Y$ 
    using assms(2) assms(3) per-distinct-1 by blast
  {
    fix  $P \ Q$ 
    assume  $\text{Coplanar } A \ B \ C \ P \wedge \text{Col } U \ V \ Q$ 
    hence  $\text{Per } P \ X \ Q$  using  $P1 \ P2$  by auto
  }
thus ?thesis
using OrthAt-def Orth-def assms(1) assms(2) assms(3) by auto
qed

lemma l11-60-aux:
assumes  $\neg \text{Col } A \ B \ C$  and
   $\text{Cong } A \ P \ A \ Q$  and

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    Cong B P B Q and
    Cong C P C Q and
    Coplanar A B C D
shows Cong D P D Q
proof -
  obtain M where P1: Bet P M Q  $\wedge$  Cong P M M Q
  by (meson Midpoint-def midpoint-existence Tarski-neutral-dimensionless-axioms)
  obtain X where P2: (Col A B X  $\wedge$  Col C D X)  $\vee$ 
(Col A C X  $\wedge$  Col B D X)  $\vee$ 
(Col A D X  $\wedge$  Col B C X)
  using assms(5) Coplanar-def by auto
  {
    assume Col A B X  $\wedge$  Col C D X
    have C  $\neq$  X
    using  $\langle$ Col A B X  $\wedge$  Col C D X $\rangle$  assms(1) by auto
    have Cong X P X Q
    by (metis l4-17 not-col-distincts  $\langle$ Col A B X  $\wedge$  Col C D X $\rangle$  assms(1) assms(2) assms(3))
    hence Cong D P D Q
    by (metis l4-17  $\langle$ C  $\neq$  X $\rangle$   $\langle$ Col A B X  $\wedge$  Col C D X $\rangle$  assms(4) not-col-permutation-5)
  }
  {
    assume Col A C X  $\wedge$  Col B D X
    have B  $\neq$  X
    using Col-cases  $\langle$ Col A C X  $\wedge$  Col B D X $\rangle$  assms(1) by blast
    have Cong X P X Q
    by (metis l4-17  $\langle$ Col A C X  $\wedge$  Col B D X $\rangle$  assms(1) assms(2) assms(4) not-col-distincts)
    hence Cong D P D Q
    by (metis l4-17  $\langle$ B  $\neq$  X $\rangle$   $\langle$ Col A C X  $\wedge$  Col B D X $\rangle$  assms(3) not-col-permutation-5)
  }
  {
    assume Col A D X  $\wedge$  Col B C X
    have A  $\neq$  X
    by (metis not-col-permutation-1  $\langle$ Col A D X  $\wedge$  Col B C X $\rangle$  assms(1))
    have Cong X P X Q
    by (metis l4-17 not-col-distincts  $\langle$ Col A D X  $\wedge$  Col B C X $\rangle$  assms(1) assms(3) assms(4))
    hence Cong D P D Q
    by (metis col-permutation-5 l4-17  $\langle$ A  $\neq$  X $\rangle$   $\langle$ Col A D X  $\wedge$  Col B C X $\rangle$  assms(2))
  }
  thus ?thesis
  using P2  $\langle$ Col A B X  $\wedge$  Col C D X  $\implies$  Cong D P D Q $\rangle$ 
 $\langle$ Col A C X  $\wedge$  Col B D X  $\implies$  Cong D P D Q $\rangle$  by blast
qed

lemma l11-60:
  assumes  $\neg$  Col A B C and
    Per A D P and
    Per B D P and
    Per C D P and
    Coplanar A B C E
  shows Per E D P
  by (meson Per-def assms(1) assms(2) assms(3) assms(4) assms(5) l11-60-aux per-double-cong)

lemma l11-60-bis:
  assumes  $\neg$  Col A B C and
    D  $\neq$  P and
    Coplanar A B C D and
    Per A D P and
    Per B D P and
    Per C D P
  shows D OrthAt A B C D P
  using assms(1) assms(2) assms(3) assms(4) assms(5) l11-60 orth-at-chara by auto

lemma l11-61:
  assumes A  $\neq$  A' and
    A  $\neq$  B and
    A  $\neq$  C and

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  Coplanar A A' B B' and
  Per B A A' and
  Per B' A' A and
  Coplanar A A' C C' and
  Per C A A' and
  Per B A C
shows Per B' A' C'
proof -
  have P1:  $\neg \text{Col } C A A'$ 
    using assms(1) assms(3) assms(8) per-col-eq by blast
  obtain C'' where P2:  $A A' \text{ Perp } C'' A' \wedge A A' \text{ OS } C C''$  using l10-15
    using Col-perm P1 col-trivial-2 by blast
  have P6:  $B' \neq A$ 
    using assms(1) assms(6) per-distinct by blast
  have P8:  $\neg \text{Col } A' A C''$ 
    using P2 not-col-permutation-4 one-side-not-col124 by blast
  have P9:  $\text{Per } A' A' B'$ 
    by (simp add: l8-2 l8-5)
  have P10:  $\text{Per } A A' B'$ 
    by (simp add: assms(6) l8-2)
  {
    fix B'
    assume A A' OS B B'  $\wedge$  Per B' A' A
    hence B A C CongA B' A' C'' using l11-17
      by (meson P2 Perp-cases l11-57 Tarski-neutral-dimensionless-axioms assms(5) assms(8) perp-per-1)
    hence Per B' A' C''
      using assms(9) l11-17 by blast
  }
  hence Q1:  $\forall B'. (A A' \text{ OS } B B' \wedge \text{Per } B' A' A) \longrightarrow \text{Per } B' A' C''$  by simp
  {
    fix B'
    assume P12: Coplanar A A' B B'  $\wedge$  Per B' A' A  $\wedge$  B'  $\neq$  A
    have Per B' A' C''
    proof cases
      assume B' = A'
      thus ?thesis
        by (simp add: Per-perm l8-5)
    next
      assume P13: B'  $\neq$  A'
      have P14:  $\neg \text{Col } B' A' A$ 
        using P12 P13 assms(1) l8-9 by auto
      have P15:  $\neg \text{Col } B A A'$ 
        using assms(1) assms(2) assms(5) per-not-col by auto
      hence Z1:  $A A' \text{ TS } B B' \vee A A' \text{ OS } B B'$ 
        using P12 P14 cop--one-or-two-sides not-col-permutation-5 by blast
      {
        assume A A' OS B B'
        hence Per B' A' C''
          by (simp add: P12  $\langle \wedge B'a. A A' \text{ OS } B B'a \wedge \text{Per } B'a A' A \implies \text{Per } B'a A' C'' \rangle$ )
      }
      {
        assume Q2: A A' TS B B'
        obtain B'' where Z2: Bet B' A' B''  $\wedge$  Cong A' B'' A' B'
          using segment-construction by blast
        have B'  $\neq$  B''
          using P13 Z2 bet-neq12--neq by blast
        hence Z4: A'  $\neq$  B''
          using Z2 cong-diff-4 by blast
        hence A A' TS B'' B'
          by (meson TS-def Z2 Q2 bet--ts invert-two-sides l9-2 not-col-permutation-1)
        hence Z5: A A' OS B B''
          using Q2 l9-8-1 by auto
        have Per B'' A' A
          using P12 P13 Z2 bet-col col-per2--per l8-2 l8-5 by blast
        hence Per C'' A' B''
          using l8-2 Q1 Z5 by blast
      }
    }
  }

```

```

    hence  $Per\ B'\ A'\ C''$ 
    by (metis Col-def Per-perm l8-3 Z2 Z4)
  }
  thus ?thesis using Z1
  using  $\langle A\ A'\ OS\ B\ B' \implies Per\ B'\ A'\ C'' \rangle$  by blast
qed
}
hence  $\forall\ B'. (Coplanar\ A\ A'\ B\ B' \wedge Per\ B'\ A'\ A \wedge B' \neq A) \longrightarrow Per\ B'\ A'\ C''$ 
  by simp
hence  $Per\ B'\ A'\ C''$ 
  using P6 assms(4) assms(6) by blast
hence P11:  $Per\ C''\ A'\ B'$ 
  using Per-cases by auto
have  $Coplanar\ A'\ A\ C''\ C'$ 
  by (meson P1 P2 assms(7) coplanar-trans-1 ncoplanar-perm-6 ncoplanar-perm-8 os--coplanar)
thus ?thesis
  using P8 P9 P10 P11 l8-2 l11-60 by blast
qed

lemma l11-61-bis:
  assumes  $D\ OrthAt\ A\ B\ C\ D\ P$  and
     $D\ E\ Perp\ E\ Q$  and
     $Coplanar\ A\ B\ C\ E$  and
     $Coplanar\ D\ E\ P\ Q$ 
  shows  $E\ OrthAt\ A\ B\ C\ E\ Q$ 
proof -
  have P4:  $D \neq E$ 
    using assms(2) perp-not-eq-1 by auto
  have P5:  $E \neq Q$ 
    using assms(2) perp-not-eq-2 by auto
  have  $\exists\ D'. (D\ E\ Perp\ D'\ D \wedge Coplanar\ A\ B\ C\ D')$ 
proof -
  obtain F where T1:  $Coplanar\ A\ B\ C\ F \wedge \neg\ Col\ D\ E\ F$ 
    using P4 ex-ncol-cop by blast
  obtain D' where T2:  $D\ E\ Perp\ D'\ D \wedge Coplanar\ D\ E\ F\ D'$ 
    using P4 ex-perp-cop by blast
  have  $Coplanar\ A\ B\ C\ D'$ 
proof -
  have T3A:  $\neg\ Col\ A\ B\ C$ 
    using OrthAt-def assms(1) by auto
  have T3B:  $Coplanar\ A\ B\ C\ D$ 
    using OrthAt-def assms(1) by blast
  hence T4:  $Coplanar\ D\ E\ F\ A$ 
    by (meson T1 T3A assms(3) coplanar-pseudo-trans ncop-distincts)
  have T5:  $Coplanar\ D\ E\ F\ B$ 
    using T1 T3A T3B assms(3) coplanar-pseudo-trans ncop-distincts by blast
  have  $Coplanar\ D\ E\ F\ C$ 
    using T1 T3A T3B assms(3) coplanar-pseudo-trans ncop-distincts by blast
  thus ?thesis
    using T1 T2 T4 T5 coplanar-pseudo-trans by blast
qed
  thus ?thesis
    using T2 by auto
qed
then obtain D' where R1:  $D\ E\ Perp\ D'\ D \wedge Coplanar\ A\ B\ C\ D'$  by auto
hence R2:  $D \neq D'$ 
  using perp-not-eq-2 by blast
{
  fix M
  assume R3:  $Coplanar\ A\ B\ C\ M$ 
  have  $Col\ D\ P\ P$ 
    by (simp add: col-trivial-2)
  hence  $Per\ E\ D\ P$ 
    using assms(1) assms(3) orth-at-chara by auto
  hence R4:  $Per\ P\ D\ E$  using l8-2 by auto
  have R5:  $Per\ Q\ E\ D$ 

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    using Perp-cases assms(2) perp-per-2 by blast
  have R6: Coplanar D E D' M
  proof -
    have S1:  $\neg$  Col A B C
    using OrthAt-def assms(1) by auto
    have Coplanar A B C D
    using OrthAt-def assms(1) by auto
    thus ?thesis
    using S1 assms(3) R1 R3 coplanar-pseudo-trans by blast
  qed
  have R7: Per D' D E
    using Perp-cases R1 perp-per-1 by blast
  have Per D' D P
    using R1 assms(1) orth-at-chara by blast
  hence Per P D D'
    using Perp-cases by blast
  hence Per Q E M
    using l11-61 R4 R5 R6 R7 OrthAt-def P4 R2 assms(1) assms(4) by blast
  hence Per M E Q using l8-2 by auto
}
{
  fix P0 Q0
  assume Coplanar A B C P0  $\wedge$  Col E Q Q0
  hence Per P0 E Q0
    using P5  $\langle \wedge M. \text{Coplanar } A \ B \ C \ M \implies \text{Per } M \ E \ Q \rangle$  per-col by blast
}
thus ?thesis
  using OrthAt-def P5 assms(1) assms(3) col-trivial-3 by auto
qed

lemma l11-62-unicity:
  assumes Coplanar A B C D and
    Coplanar A B C D' and
     $\forall E. \text{Coplanar } A \ B \ C \ E \longrightarrow \text{Per } E \ D \ P$  and
     $\forall E. \text{Coplanar } A \ B \ C \ E \longrightarrow \text{Per } E \ D' \ P$ 
  shows D = D'
  by (metis assms(1) assms(2) assms(3) assms(4) l8-8 not-col-distincts per-not-colp)

lemma l11-62-unicity-bis:
  assumes X OrthAt A B C X U and
    Y OrthAt A B C Y U
  shows X = Y
  proof -
    have P1: Coplanar A B C X
      using assms(1) orth-at-chara by blast
    have P2: Coplanar A B C Y
      using assms(2) orth-at-chara by blast
    {
      fix E
      assume Coplanar A B C E
      hence Per E X U
        using OrthAt-def assms(1) col-trivial-2 by auto
    }
    {
      fix E
      assume Coplanar A B C E
      hence Per E Y U
        using assms(2) orth-at-chara by auto
    }
    thus ?thesis
      by (meson P1 P2  $\langle \wedge E. \text{Coplanar } A \ B \ C \ E \implies \text{Per } E \ X \ U \rangle$  l8-2 l8-7)
  qed

lemma orth-at2--eq:
  assumes X OrthAt A B C U V and
    Y OrthAt A B C U V

```

```

shows  $X = Y$ 
proof -
  have  $P1: \text{Coplanar } A \ B \ C \ X$ 
    using  $\text{assms}(1)$ 
    by ( $\text{simp add: OrthAt-def}$ )
  have  $P2: \text{Coplanar } A \ B \ C \ Y$ 
    using  $\text{OrthAt-def assms}(2)$  by auto
  {
    fix  $E$ 
    assume  $\text{Coplanar } A \ B \ C \ E$ 
    hence  $\text{Per } E \ X \ U$ 
      using  $\text{OrthAt-def assms}(1) \text{ col-trivial-3}$  by auto
  }
  {
    fix  $E$ 
    assume  $\text{Coplanar } A \ B \ C \ E$ 
    hence  $\text{Per } E \ Y \ U$ 
      using  $\text{OrthAt-def assms}(2) \text{ col-trivial-3}$  by auto
  }
  thus ?thesis
    by ( $\text{meson } P1 \ P2 \ \text{Per-perm } \langle \bigwedge E. \text{Coplanar } A \ B \ C \ E \implies \text{Per } E \ X \ U \rangle \text{ l8-7}$ )
qed

lemma  $\text{col-cop-orth-at--eq}$ :
  assumes  $X \text{ OrthAt } A \ B \ C \ U \ V$  and
     $\text{Coplanar } A \ B \ C \ Y$  and
     $\text{Col } U \ V \ Y$ 
  shows  $X = Y$ 
proof -
  have  $Y \text{ OrthAt } A \ B \ C \ U \ V$ 
    using  $\text{Orth-def assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ col-cop-orth--orth-at}$  by blast
  thus ?thesis
    using  $\text{assms}(1) \text{ orth-at2--eq}$  by auto
qed

lemma  $\text{orth-at--ncop1}$ :
  assumes  $U \neq X$  and
     $X \text{ OrthAt } A \ B \ C \ U \ V$ 
  shows  $\neg \text{Coplanar } A \ B \ C \ U$ 
  using  $\text{assms}(1) \text{ assms}(2) \text{ col-cop-orth-at--eq not-col-distincts}$  by blast

lemma  $\text{orth-at--ncop2}$ :
  assumes  $V \neq X$  and
     $X \text{ OrthAt } A \ B \ C \ U \ V$ 
  shows  $\neg \text{Coplanar } A \ B \ C \ V$ 
  using  $\text{assms}(1) \text{ assms}(2) \text{ col-cop-orth-at--eq not-col-distincts}$  by blast

lemma  $\text{orth-at--ncop}$ :
  assumes  $X \text{ OrthAt } A \ B \ C \ X \ P$ 
  shows  $\neg \text{Coplanar } A \ B \ C \ P$ 
  by ( $\text{metis assms orth-at--ncop2 orth-at-distincts}$ )

lemma  $\text{l11-62-existence}$ :
 $\exists D. (\text{Coplanar } A \ B \ C \ D \wedge (\forall E. (\text{Coplanar } A \ B \ C \ E \longrightarrow \text{Per } E \ D \ P)))$ 
proof cases
  assume  $\text{Coplanar } A \ B \ C \ P$ 
  thus ?thesis
    using  $\text{l8-5}$  by auto
next
  assume  $P1: \neg \text{Coplanar } A \ B \ C \ P$ 
  hence  $P2: \neg \text{Col } A \ B \ C$ 
    using  $\text{ncop--ncol}$  by auto
  have  $\neg \text{Col } A \ B \ P$ 
    using  $P1 \text{ ncop--ncols}$  by auto
  then obtain  $D0$  where  $P4: \text{Col } A \ B \ D0 \wedge A \ B \ \text{Perp } P \ D0$  using  $\text{l8-18-existence}$  by blast
  have  $P5: \text{Coplanar } A \ B \ C \ D0$ 

```

```

  using P4 ncop--ncols by auto
have A ≠ B
  using P2 not-col-distincts by auto
then obtain D1 where P10: A B Perp D1 D0 ∧ Coplanar A B C D1
  using ex-perp-cop by blast
have P11: ¬ Col A B D1
  using P10 P4 perp-not-col2 by blast
{
  fix D
  assume Col D0 D1 D
  hence Coplanar A B C D
    by (metis P10 P5 col-cop2--cop perp-not-eq-2)
}
obtain A0 where P11: A ≠ A0 ∧ B ≠ A0 ∧ D0 ≠ A0 ∧ Col A B A0
  using P4 diff-col-ex3 by blast
have P12: Coplanar A B C A0
  using P11 ncop--ncols by blast
have P13: Per P D0 A0
  using l8-16-1 P11 P4 by blast
show ?thesis
proof cases
  assume Z1: Per P D0 D1
  {
    fix E
    assume R1: Coplanar A B C E
    have R2: ¬ Col A0 D1 D0
      by (metis P10 P11 P4 col-permutation-5 colx perp-not-col2)
    have R3: Per A0 D0 P
      by (simp add: P13 l8-2)
    have R4: Per D1 D0 P
      by (simp add: Z1 l8-2)
    have R5: Per D0 D0 P
      by (simp add: l8-2 l8-5)
    have Coplanar A0 D1 D0 E
      using R1 P2 P12 P10 P5 coplanar-pseudo-trans by blast
    hence Per E D0 P
      using l11-60 R2 R3 R4 R5 by blast
  }
  thus ?thesis using P5 by auto
next
  assume S1: ¬ Per P D0 D1
  {
    assume S2: Col D0 D1 P
    have ∀ D. Col D0 D1 D ⟶ Coplanar A B C D
      by (simp add: ⟨∧Da. Col D0 D1 Da ⟹ Coplanar A B C Da⟩)
    hence False
      using P1 S2 by blast
  }
  hence S2A: ¬ Col D0 D1 P by blast
  then obtain D where S3: Col D0 D1 D ∧ D0 D1 Perp P D
    using l8-18-existence by blast
  have S4: Coplanar A B C D
    by (simp add: S3 ⟨∧Da. Col D0 D1 Da ⟹ Coplanar A B C Da⟩)
  {
    fix E
    assume S5: Coplanar A B C E
    have S6: D ≠ D0
      using S1 S3 l8-2 perp-per-1 by blast
    have S7: Per D0 D P
      by (metis Perp-cases S3 S6 perp-col perp-per-1)
    have S8: Per D D0 A0
    proof -
      have V4: D0 ≠ D1
        using P10 perp-not-eq-2 by blast
      have V6: Per A0 D0 D1
        using P10 P11 P4 l8-16-1 l8-2 by blast

```

```

    thus ?thesis
      using S3 V4 V6 l8-2 per-col by blast
  qed
  have S9: Per A0 D P
  proof -
    obtain A0' where W1: D Midpoint A0 A0'
      using symmetric-point-construction by auto
    obtain D0' where W2: D Midpoint D0 D0'
      using symmetric-point-construction by auto
    have Cong P A0 P A0'
    proof -
      have V3: Cong P D0 P D0'
        using S7 W2 l8-2 per-double-cong by blast
      have V4: Cong D0 A0 D0' A0'
        using W1 W2 cong-4321 l7-13 by blast
      have Per P D0' A0'
    proof -
      obtain P' where V5: D Midpoint P P'
        using symmetric-point-construction by blast
      have Per P' D0 A0
    proof -
      have  $\neg$  Col P D D0
        by (metis S2A S3 S6 col2--eq col-permutation-1)
      thus ?thesis
        by (metis (full-types) P13 S3 S8 V5 S2A col-per2--per midpoint-col)
    qed
    thus ?thesis
      using midpoint-preserves-per V5 Mid-cases W1 W2 by blast
  qed
  thus ?thesis using l10-12 P13 V3 V4 by blast
  qed
  thus ?thesis
    using Per-def Per-perm W1 by blast
  qed
  have S13: Per D D P
    using Per-perm l8-5 by blast
  have S14:  $\neg$  Col D0 A0 D
    using P11 S7 S9 per-not-col Col-perm S6 S8 by blast
  have Coplanar A B C D
    using S4 by auto
  hence Coplanar D0 A0 D E
    using P12 P2 P5 S5 coplanar-pseudo-trans by blast
  hence Per E D P
    using S13 S14 S7 S9 l11-60 by blast
}
thus ?thesis using S4 by blast
qed
qed

lemma l11-62-existence-bis:
  assumes  $\neg$  Coplanar A B C P
  shows  $\exists X. X \text{ OrthAt } A B C X P$ 
proof -
  obtain X where P1: Coplanar A B C X  $\wedge$  ( $\forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E X P$ )
    using l11-62-existence by blast
  hence P2:  $X \neq P$ 
    using assms by auto
  have P3:  $\neg$  Col A B C
    using assms ncop--ncol by auto
  thus ?thesis
    using P1 P2 P3 orth-at-chara by auto
qed

lemma l11-63-aux:
  assumes Coplanar A B C D and
     $D \neq E$  and

```

$E \text{ OrthAt } A \ B \ C \ E \ P$
shows $\exists \ Q. (D \ E \ OS \ P \ Q \wedge A \ B \ C \ Orth \ D \ Q)$
proof –
 have $P1: \neg \text{Col } A \ B \ C$
 using *OrthAt-def assms(3)* **by** *blast*
 have $P2: E \neq P$
 using *OrthAt-def assms(3)* **by** *blast*
 have $P3: \text{Coplanar } A \ B \ C \ E$
 using *OrthAt-def assms(3)* **by** *blast*
 have $P4: \forall \ P0 \ Q. (\text{Coplanar } A \ B \ C \ P0 \wedge \text{Col } E \ P \ Q) \longrightarrow \text{Per } P0 \ E \ Q$
 using *OrthAt-def assms(3)* **by** *blast*
 have $P5: \neg \text{Coplanar } A \ B \ C \ P$
 using *assms(3) orth-at-ncop* **by** *auto*
 have $P6: \text{Col } D \ E \ D$
 by (*simp add: col-trivial-3*)
 have $\neg \text{Col } D \ E \ P$
 using $P3 \ P5 \text{ assms(1) assms(2) col-cop2--cop}$ **by** *blast*
then obtain $P \ Q$ **where** $P6: D \ E \ \text{Perp } Q \ D \wedge D \ E \ OS \ P \ Q$
 using $P6 \text{ l10-15}$ **by** *blast*
 have $A \ B \ C \ Orth \ D \ Q$
proof –
obtain F **where** $P7: \text{Coplanar } A \ B \ C \ F \wedge \neg \text{Col } D \ E \ F$
 using *assms(2) ex-ncol-cop* **by** *blast*
obtain D' **where** $P8: D \ E \ \text{Perp } D' \ D \wedge \text{Coplanar } D \ E \ F \ D'$
 using *assms(2) ex-perp-cop* **by** *presburger*
 have $P9: \neg \text{Col } D' \ D \ E$
 using $P8 \text{ col-permutation-1 perp-not-col}$ **by** *blast*
 have $P10: \text{Coplanar } D \ E \ F \ A$
 by (*meson P1 P3 P7 assms(1) coplanar-pseudo-trans ncop-distincts*)
 have $P11: \text{Coplanar } D \ E \ F \ B$
 by (*meson P1 P3 P7 assms(1) coplanar-pseudo-trans ncop-distincts*)
 have $P12: \text{Coplanar } D \ E \ F \ C$
 by (*meson P1 P3 P7 assms(1) coplanar-pseudo-trans ncop-distincts*)
hence $D \ OrthAt \ A \ B \ C \ D \ Q$
proof –
 have $\text{Per } D' \ D \ Q$
proof –
obtain E' **where** $Y1: D \ E \ \text{Perp } E' \ E \wedge \text{Coplanar } D \ E \ F \ E'$
 using *assms(2) ex-perp-cop* **by** *blast*
 have $Y2: E \neq E'$
 using $Y1 \text{ perp-distinct}$ **by** *auto*
 have $Y5: \text{Coplanar } E \ D \ E' \ D'$
 by (*meson P7 P8 Y1 coplanar-perm-12 coplanar-perm-7 coplanar-trans-1 not-col-permutation-2*)
 have $Y6: \text{Per } E' \ E \ D$
 by (*simp add: Perp-perm perp-per-2 Y1*)
 have $Y7: \text{Per } D' \ D \ E$
 using $P8 \text{ col-trivial-2 col-trivial-3 l8-16-1}$ **by** *blast*
 have $Y8: \text{Coplanar } E \ D \ P \ Q$
 using $P6 \text{ ncoplanar-perm-6 os--coplanar}$ **by** *blast*
 have $Y9: \text{Per } P \ E \ D$ **using** $P4$
 using *assms(1) assms(3) l8-2 orth-at-chara* **by** *blast*
 have $Y10: \text{Coplanar } A \ B \ C \ E'$
 using $P10 \ P11 \ P12 \ P7 \ Y1 \text{ coplanar-pseudo-trans}$ **by** *blast*
hence $Y11: \text{Per } E' \ E \ P$
 using $P4 \ Y10 \text{ col-trivial-2}$ **by** *auto*
 have $E \neq D$ **using** *assms(2)* **by** *blast*
thus *?thesis*
 using $\text{l11-61 } Y2 \text{ assms(2) } P2 \ Y5 \ Y6 \ Y7 \ Y8 \ Y9 \ Y10 \ Y11$ **by** *blast*
qed
hence $X1: D \ OrthAt \ D' \ D \ E \ D \ Q$ **using** l11-60-bis
 by (*metis OS-def P6 P9 Per-perm TS-def l8-5 col-trivial-3 invert-one-side ncop-distincts perp-per-1*)
 have $X3: \text{Coplanar } D' \ D \ E \ A$
 using $P10 \ P7 \ P8 \text{ coplanar-perm-14 coplanar-trans-1 not-col-permutation-3}$ **by** *blast*
 have $X4: \text{Coplanar } D' \ D \ E \ B$


```

    using P11 P7 P8 coplanar-perm-14 coplanar-trans-1 not-col-permutation-3 by blast
  have Coplanar D' D E C
    using P12 P7 P8 coplanar-perm-14 coplanar-trans-1 not-col-permutation-3 by blast
  thus ?thesis
    using X1 P1 X3 X4 cop3-orth-at--orth-at by blast
qed
thus ?thesis
  using Orth-def by blast
qed
thus ?thesis using P6 by blast
qed

```

lemma l11-63-existence:

```

assumes Coplanar A B C D and
  ¬ Coplanar A B C P
shows ∃ Q. A B C Orth D Q
using Orth-def assms(1) assms(2) l11-62-existence-bis l11-63-aux by fastforce

```

lemma l8-21-3:

```

assumes Coplanar A B C D and
  ¬ Coplanar A B C X
shows
  ∃ P T. (A B C Orth D P ∧ Coplanar A B C T ∧ Bet X T P)

```

proof –

```

obtain E where P1: E OrthAt A B C E X
  using assms(2) l11-62-existence-bis by blast
thus ?thesis
proof cases

```

```

  assume P2: D = E
  obtain Y where P3: Bet X D Y ∧ Cong D Y D X
    using segment-construction by blast
  have P4: D ≠ X
    using assms(1) assms(2) by auto
  have P5: A B C Orth D X
    using Orth-def P1 P2 by auto
  have P6: D ≠ Y
    using P3 P4 cong-reverse-identity by blast
  have Col D X Y
    using Col-def Col-perm P3 by blast
  hence A B C Orth D Y
    using P5 P6 col-orth--orth by auto
  thus ?thesis
    using P3 assms(1) by blast

```

next

```

  assume K1: D ≠ E
  then obtain P' where P7: D E OS X P' ∧ A B C Orth D P'
    using P1 assms(1) l11-63-aux by blast
  have P8: ¬ Col A B C
    using assms(2) ncop--ncol by auto
  have P9: E ≠ X
    using P7 os-distincts by auto
  have P10: ∀ P Q. (Coplanar A B C P ∧ Col E X Q) ⟶ Per P E Q
    using OrthAt-def P1 by auto
  have P11: D OrthAt A B C D P'
    by (simp add: P7 assms(1) col-cop-orth--orth-at col-trivial-3)
  have P12: D ≠ P'
    using P7 os-distincts by auto
  have P13: ¬ Coplanar A B C P'
    using P11 orth-at--ncop by auto
  have P14: ∀ P Q. (Coplanar A B C P ∧ Col D P' Q) ⟶ Per P D Q
    using OrthAt-def P11 by auto
  obtain P where P15: Bet P' D P ∧ Cong D P D P'
    using segment-construction by blast
  have P16: D E TS X P
proof –
  have P16A: D E OS P' X using P7 one-side-symmetry by blast

```

```

    hence  $D E T S P' P$ 
    by (metis  $P12 P15$  bet--ts cong-diff-3 one-side-not-col123)
    thus ?thesis using l9-8-2  $P16A$  by blast
qed
obtain  $T$  where  $P17$ :  $Col\ T\ D\ E \wedge Bet\ X\ T\ P$ 
    using  $P16$   $TS$ -def by blast
have  $P18$ :  $D \neq P$ 
    using  $P16$  ts-distincts by blast
have  $Col\ D\ P' P$ 
    using  $Col$ -def  $Col$ -perm  $P15$  by blast
hence  $A\ B\ C\ Orth\ D\ P$ 
    using  $P18\ P7$  col-orth--orth by blast
thus ?thesis using col-cop2--cop
    by (meson  $P1\ P17$  orth-at-chara  $K1$  assms(1) col-permutation-1)
qed
qed

lemma mid2-orth-at2--cong:
  assumes  $X\ OrthAt\ A\ B\ C\ X\ P$  and
     $Y\ OrthAt\ A\ B\ C\ Y\ Q$  and
     $X\ Midpoint\ P\ P'$  and
     $Y\ Midpoint\ Q\ Q'$ 
  shows  $Cong\ P\ Q\ P'\ Q'$ 
proof -
  have  $Q1$ :  $\neg Col\ A\ B\ C$ 
    using assms(1) col--coplanar orth-at--ncop by blast
  have  $Q2$ :  $X \neq P$ 
    using assms(1) orth-at-distincts by auto
  have  $Q3$ :  $Coplanar\ A\ B\ C\ X$ 
    using  $OrthAt$ -def assms(1) by auto
  have  $Q4$ :  $\forall P0\ Q. (Coplanar\ A\ B\ C\ P0 \wedge Col\ X\ P\ Q) \longrightarrow Per\ P0\ X\ Q$ 
    using  $OrthAt$ -def assms(1) by blast
  have  $Q5$ :  $Y \neq P$ 
    by (metis assms(1) assms(2) orth-at--ncop2 orth-at-chara)
  have  $Q6$ :  $Coplanar\ A\ B\ C\ Y$ 
    using  $OrthAt$ -def assms(2) by blast
  have  $Q7$ :  $\forall P\ Q0. (Coplanar\ A\ B\ C\ P \wedge Col\ Y\ Q\ Q0) \longrightarrow Per\ P\ Y\ Q0$ 
    using  $OrthAt$ -def assms(2) by blast
  obtain  $Z$  where  $P1$ :  $Z\ Midpoint\ X\ Y$ 
    using midpoint-existence by auto
  obtain  $R$  where  $P2$ :  $Z\ Midpoint\ P\ R$ 
    using symmetric-point-construction by auto
  obtain  $R'$  where  $P3$ :  $Z\ Midpoint\ P'\ R'$ 
    using symmetric-point-construction by auto
  have  $T1$ :  $Coplanar\ A\ B\ C\ Z$ 
    using  $P1\ Q3\ Q6$  bet-cop2--cop midpoint-bet by blast
  hence  $Per\ Z\ X\ P$ 
    using  $Q4$  assms(1) orth-at-chara by auto
  hence  $T2$ :  $Cong\ Z\ P\ Z\ P'$ 
    using assms(3) per-double-cong by blast
  have  $T3$ :  $Cong\ R\ Z\ R'\ Z$ 
    by (metis  $Cong$ -perm  $Midpoint$ -def  $P2\ P3\ T2$  cong-transitivity)
  have  $T4$ :  $Cong\ R\ Q\ R'\ Q'$ 
    by (meson  $P1\ P2\ P3$  assms(3) assms(4) l7-13 not-cong-4321 symmetry-preserves-midpoint)
  have  $Per\ Z\ Y\ Q$ 
    using  $Q7\ T1$  assms(2) orth-at-chara by auto
  hence  $T5$ :  $Cong\ Z\ Q\ Z\ Q'$ 
    using assms(4) per-double-cong by auto
  have  $R \neq Z$ 
    by (metis  $P2\ P3\ Q2\ T3$  assms(3) cong-diff-3 l7-17-bis midpoint-not-midpoint)
  thus ?thesis
    using  $P2\ P3\ T2\ T3\ T4\ T5$  five-segment l7-2 midpoint-bet by blast
qed

lemma orth-at2-tsp--ts:
  assumes  $P \neq Q$  and

```

```

  P OrthAt A B C P X and
  Q OrthAt A B C Q Y and
  A B C TSP X Y
shows P Q TS X Y
proof -
  obtain T where P0: Coplanar A B C T  $\wedge$  Bet X T Y
    using TSP-def assms(4) by auto
  have P1:  $\neg$  Col A B C
    using assms(4) ncop--ncol tsp--ncop1 by blast
  have P2:  $P \neq X$ 
    using assms(2) orth-at-distincts by auto
  have P3: Coplanar A B C P
    using OrthAt-def assms(2) by blast
  have P4:  $\forall D. \text{Coplanar } A B C D \longrightarrow \text{Per } D P X$ 
    using assms(2) orth-at-chara by blast
  have P5:  $Q \neq Y$ 
    using assms(3) orth-at-distincts by auto
  have P6: Coplanar A B C Q
    using OrthAt-def assms(3) by blast
  have P7:  $\forall D. \text{Coplanar } A B C D \longrightarrow \text{Per } D Q Y$ 
    using assms(3) orth-at-chara by blast
  have P8:  $\neg$  Col X P Q
    using P3 P6 assms(1) assms(4) col-cop2--cop not-col-permutation-2 tsp--ncop1 by blast
  have P9:  $\neg$  Col Y P Q
    using P3 P6 assms(1) assms(4) col-cop2--cop not-col-permutation-2 tsp--ncop2 by blast
  have Col T P Q
  proof -
    obtain X' where Q1: P Midpoint X X'
      using symmetric-point-construction by auto
    obtain Y' where Q2: Q Midpoint Y Y'
      using symmetric-point-construction by auto
    have Per T P X
      using P0 P4 by auto
    hence Q3: Cong T X T X'
      using Q1 per-double-cong by auto
    have Per T Q Y
      using P0 P7 by auto
    hence Q4: Cong T Y T Y' using Q2 per-double-cong by auto
    have Cong X Y X' Y'
      using P1 Q1 Q2 assms(2) assms(3) mid2-orth-at2--cong by blast
    hence X T Y Cong3 X' T Y'
      using Cong3-def Q3 Q4 not-cong-2143 by blast
    hence Bet X' T Y'
      using l4-6 P0 by blast
    thus ?thesis
      using Q1 Q2 Q3 Q4 Col-def P0 between-symmetry l7-22 by blast
  qed
  thus ?thesis
    using P0 P8 P9 TS-def by blast
qed

lemma orth-dec:
  shows A B C Orth U V  $\vee$   $\neg$  A B C Orth U V
  by auto

lemma orth-at-dec:
  shows X OrthAt A B C U V  $\vee$   $\neg$  X OrthAt A B C U V
  by auto

lemma tsp-dec:
  shows A B C TSP X Y  $\vee$   $\neg$  A B C TSP X Y
  by auto

lemma osp-dec:
  shows A B C OSP X Y  $\vee$   $\neg$  A B C OSP X Y
  by auto

```

```

lemma ts2--inangle:
  assumes A C TS B P and
    B P TS A C
  shows P InAngle A B C
  by (metis InAngle-def assms(1) assms(2) bet-out ts2--ex-bet2 ts-distincts)

lemma os-ts--inangle:
  assumes B P TS A C and
    B A OS C P
  shows P InAngle A B C
proof -
  have P1:  $\neg$  Col A B P
    using TS-def assms(1) by auto
  have P2:  $\neg$  Col B A C
    using assms(2) col123--nos by blast
  obtain P' where P3: B Midpoint P P'
    using symmetric-point-construction by blast
  hence P4:  $\neg$  Col B A P'
    by (metis assms(2) col-one-side col-permutation-5 midpoint-col
      midpoint-distinct-2 one-side-not-col124)
  have P5:  $(B \neq P' \wedge B P TS A C \wedge Bet P B P') \longrightarrow (P InAngle A B C \vee P' InAngle A B C)$ 
    using two-sides-in-angle by auto
  hence P6:  $P InAngle A B C \vee P' InAngle A B C$ 
    using P3 P4 assms(1) midpoint-bet not-col-distincts by blast
  {
    assume P' InAngle A B C
    hence P7: A B OS P' C
      using Col-cases P2 P4 in-angle-one-side by blast
    hence P8:  $\neg$  A B TS P' C
      using l9-9 by auto
    have B A TS P P'
      using P1 P3 P4 bet--ts midpoint-bet not-col-distincts not-col-permutation-4 by auto
    hence B A TS C P'
      using P7 assms(2) invert-one-side l9-2 l9-8-2 l9-9 by blast
    hence B A TS P' C
      using l9-2 by blast
    hence A B TS P' C
      by (simp add: invert-two-sides)
    hence P InAngle A B C
      using P8 by auto
  }
  thus ?thesis
    using P6 by blast
qed

```

```

lemma os2--inangle:
  assumes B A OS C P and
    B C OS A P
  shows P InAngle A B C
  using assms(1) assms(2) col124--nos l9-9-bis os-ts--inangle two-sides-cases by blast

```

```

lemma acute-conga--acute:
  assumes Acute A B C and
    A B C CongA D E F
  shows Acute D E F
proof -
  have D E F LeA A B C
    by (simp add: assms(2) conga--lea456123)
  thus ?thesis
    using acute-lea-acute assms(1) by blast
qed

```

```

lemma acute-out2--acute:
  assumes B Out A' A and
    B Out C' C and

```

```

  Acute A B C
shows Acute A' B C'
by (meson out2--conga acute-conga--acute assms(1) assms(2) assms(3))

lemma conga-obtuse--obtuse:
  assumes Obtuse A B C and
    A B C CongA D E F
shows Obtuse D E F
using assms(1) assms(2) conga--lea lea-obtuse-obtuse by blast

lemma obtuse-out2--obtuse:
  assumes B Out A' A and
    B Out C' C and
    Obtuse A B C
shows Obtuse A' B C'
by (meson out2--conga assms(1) assms(2) assms(3) conga-obtuse--obtuse)

lemma bet-lea--bet:
  assumes Bet A B C and
    A B C LeA D E F
shows Bet D E F
proof -
  have A B C CongA D E F
    by (metis assms(1) assms(2) l11-31-2 lea-asymp lea-distincts)
  thus ?thesis
    using assms(1) bet-conga--bet by blast
qed

lemma out-lea--out:
  assumes E Out D F and
    A B C LeA D E F
shows B Out A C
proof -
  have D E F CongA A B C
    using l11-31-1 lea-asymp lea-distincts assms(1) assms(2) by fastforce
  thus ?thesis
    using assms(1) out-conga-out by blast
qed

lemma bet2-lta--lta:
  assumes A B C LtA D E F and
    Bet A B A' and
    A' ≠ B and
    Bet D E D' and
    D' ≠ E
shows D' E F LtA A' B C
proof -
  have P1: D' E F LeA A' B C
    by (metis Bet-cases assms(1) assms(2) assms(3) assms(4) assms(5) l11-36-aux2
      lea-distincts lta--lea)
  have ¬ D' E F CongA A' B C
    by (metis assms(1) assms(2) assms(4) between-symmetry conga-sym l11-13
      lta-distincts not-lta-and-conga)
  thus ?thesis
    by (simp add: LtA-def P1)
qed

lemma lea123456-lta--lta:
  assumes A B C LeA D E F and
    D E F LtA G H I
shows A B C LtA G H I
proof -
  have ¬ G H I LeA F E D
    by (metis (no-types) lea--nlta lta-left-comm assms(2))
  hence ¬ A B C CongA G H I
    by (metis lta-distincts assms(1) assms(2) conga-pseudo-refl l11-30)

```

thus *?thesis*
by (*meson LtA-def lea-trans assms(1) assms(2)*)
qed

lemma *lea456789-lta--lta*:
assumes *A B C LtA D E F and*
D E F LeA G H I
shows *A B C LtA G H I*
by (*meson LtA-def assms(1) assms(2) conga--lea456123 lea-trans lta--nlea*)

lemma *acute-per--lta*:
assumes *Acute A B C and*
D $\neq$ E and
E $\neq$ F and
Per D E F
shows *A B C LtA D E F*
proof –
obtain *G H I where P1: Per G H I $\wedge$ A B C LtA G H I*
using *Acute-def assms(1)* **by** *auto*
hence *G H I CongA D E F*
using *assms(2) assms(3) assms(4) l11-16 lta-distincts* **by** *blast*
thus *?thesis*
by (*metis P1 conga-preserves-lta conga-refl lta-distincts*)
qed

lemma *obtuse-per--lta*:
assumes *Obtuse A B C and*
D $\neq$ E and
E $\neq$ F and
Per D E F
shows *D E F LtA A B C*
proof –
obtain *G H I where P1: Per G H I $\wedge$ G H I LtA A B C*
using *Obtuse-def assms(1)* **by** *auto*
hence *G H I CongA D E F*
using *assms(2) assms(3) assms(4) l11-16 lta-distincts* **by** *blast*
thus *?thesis*
by (*metis P1 l11-51 assms(1) cong-reflexivity conga-preserves-lta obtuse-distincts*)
qed

lemma *acute-obtuse--lta*:
assumes *Acute A B C and*
Obtuse D E F
shows *A B C LtA D E F*
by (*metis Acute-def assms(1) assms(2) lta-distincts lta-trans obtuse-per--lta*)

lemma *lea-in-angle*:
assumes *A B P LeA A B C and*
A B OS C P
shows *P InAngle A B C*
proof –
obtain *T where P3: T InAngle A B C $\wedge$ A B P CongA A B T*
using *LeA-def assms(1)* **by** *blast*
thus *?thesis*
by (*metis assms(2) conga-preserves-in-angle conga-refl not-conga-sym*
one-side-symmetry os-distincts)
qed

lemma *acute-bet--obtuse*:
assumes *Bet A B A' and*
A' $\neq$ B and
Acute A B C
shows *Obtuse A' B C*
proof *cases*
assume *P1: Col A B C*
show *?thesis*

```

proof cases
  assume Bet A B C
  thus ?thesis
    using P1 acute-col--out assms(3) not-bet-and-out by blast
next
  assume  $\neg \textit{Bet A B C}$ 
  hence B Out A C
    using P1 or-bet-out by blast
  hence Bet C B A'
    using assms(1) bet-out--bet by blast
  hence Bet A' B C
    using between-symmetry by blast
  thus ?thesis
    using  $\langle \neg \textit{Bet A B C} \rangle$  assms(2) bet--obtuse between-trivial by blast
qed
next
  assume P2:  $\neg \textit{Col A B C}$ 
  then obtain D where P3: A B Perp D B  $\wedge$  A B OS C D
    using col-trivial-2 l10-15 by blast
  {
    assume P4: Col C B D
    hence Per A B C
    proof -
      have P5: B  $\neq$  D
        using P3 perp-not-eq-2 by auto
      have Per A B D
        using P3 Perp-perm perp-per-2 by blast
      thus ?thesis
        using P4 P5 not-col-permutation-2 per-col by blast
    qed
    hence A B C LtA A B C
      by (metis Acute-def acute-per--lta assms(3) lta-distincts)
    hence False
      by (simp add: nlta)
  }
  hence P6:  $\neg \textit{Col C B D}$  by auto
  have P7: B A' OS C D
    by (metis P3 assms(1) assms(2) bet-col col2-os--os l5-3)
  have T1: Per A B D
    by (simp add: P3 perp-left-comm perp-per-1)
  have Q1: B C TS A' A
    using P2 assms(1) assms(2) bet--ts l9-2 not-col-permutation-1 by auto
  have A B C LtA A B D
    using P2 P6 T1 acute-per--lta assms(3) not-col-distincts by auto
  hence A B C LeA A B D
    by (simp add: lta--lea)
  hence C InAngle A B D
    by (simp add: P3 lea-in-angle one-side-symmetry)
  hence C InAngle D B A
    using l11-24 by blast
  hence C B TS D A
    by (simp add: P2 P6 in-angle-two-sides not-col-permutation-1 not-col-permutation-4)
  hence B C TS D A
    using invert-two-sides by blast
  hence B C OS A' D
    using Q1 l9-8-1 by auto
  hence T1A: D InAngle A' B C
    by (simp add: P7 os2--inangle)
  hence A B D CongA A' B D
    by (metis Per-cases T1 conga-comm l11-18-1 acute-distincts assms(1) assms(3) inangle-distincts)
  hence T2: A B D LeA A' B C
    using LeA-def T1A by auto
  {
    assume A B D CongA A' B C
    hence False
      by (metis OS-def P7 T1 TS-def out2--conga  $\langle A B C \textit{ LtA A B D} \rangle \langle A B D \textit{ CongA A' B D} \rangle$ )
  }

```

```

    ⟨ $\bigwedge thesis. (\bigwedge D. A B \text{ Perp } D B \wedge A B \text{ OS } C D \implies thesis) \implies thesis$ ⟩
    col-trivial-2 invert-one-side l11-17 l11-19 not-lta-and-conga out-trivial)
  }
  hence  $\neg A B D \text{ Cong } A' B C$  by auto
  hence  $A B D \text{ Lt } A' B C$ 
    using T2 LtA-def by auto
  thus ?thesis
    using T1 Obtuse-def by blast
qed

lemma bet-obtuse--acute:
  assumes Bet A B A' and
    A'  $\neq$  B and
    Obtuse A B C
  shows Acute A' B C
proof cases
  assume P1: Col A B C
  thus ?thesis
proof cases
  assume Bet A B C
  hence B Out A' C
    using Out-def assms(1) assms(2) assms(3) l5-2 obtuse-distincts by meson
  thus ?thesis
    by (simp add: out--acute)
next
  assume  $\neg Bet A B C$ 
  thus ?thesis
    using P1 assms(3) col-obtuse--bet by blast
qed
next
  assume P2:  $\neg Col A B C$ 
  then obtain D where P3:  $A B \text{ Perp } D B \wedge A B \text{ OS } C D$ 
    using col-trivial-2 l10-15 by blast
  {
    assume P3A: Col C B D
    have P3B: B  $\neq$  D
      using P3 perp-not-eq-2 by blast
    have P3C: Per A B D
      using P3 Perp-perm perp-per-2 by blast
    hence Per A B C
      using P3A P3B not-col-permutation-2 per-col by blast
    hence A B C LtA A B C
      using P2 assms(3) not-col-distincts obtuse-per--lta by auto
    hence False
      by (simp add: nlta)
  }
  hence P4:  $\neg Col C B D$  by auto
  have Col B A A'
    using Col-def Col-perm assms(1) by blast
  hence P5: B A' OS C D
    using P3 assms(2) invert-one-side col2-os--os col-trivial-3 by blast
  have P7: Per A' B D
    by (meson Col-def P3 Per-perm Tarski-neutral-dimensionless-axioms assms(1)
      col-trivial-2 l8-16-1)
  have A' B C LtA A' B D
proof –
  have P8: A' B C LeA A' B D
proof –
  have P9: C InAngle A' B D
proof –
  have P10: B A' OS D C
    by (simp add: P5 one-side-symmetry)
  have B D OS A' C
proof –
  have P6:  $\neg Col A B D$ 
    using P3 col124--nos by auto

```



```

hence P11: B D TS A' A
  using Col-perm P5 assms(1) bet--ts l9-2 os-distincts by blast
have A B D LtA A B C
proof -
  have P11A: A ≠ B
    using P2 col-trivial-1 by auto
  have P11B: B ≠ D
    using P4 col-trivial-2 by blast
  have Per A B D
    using P3 Perp-cases perp-per-2 by blast
  thus ?thesis
    by (simp add: P11A P11B obtuse-per--lta assms(3))
qed
hence A B D LeA A B C
  by (simp add: lta--lea)
hence D InAngle A B C
  by (simp add: P3 lea-in-angle)
hence D InAngle C B A
  using l11-24 by blast
hence D B TS C A
  by (simp add: P4 P6 in-angle-two-sides not-col-permutation-4)
hence B D TS C A
  by (simp add: invert-two-sides)
thus ?thesis
  using OS-def P11 by blast
qed
thus ?thesis
  by (simp add: P10 os2--inangle)
qed
have A' B C CongA A' B C
  using assms(2) assms(3) conga-refl obtuse-distincts by blast
thus ?thesis
  by (simp add: P9 inangle--lea)
qed
{
  assume A' B C CongA A' B D
  hence B Out C D
    using P5 conga-os--out invert-one-side by blast
  hence False
    using P4 not-col-permutation-4 out-col by blast
}
hence ¬ A' B C CongA A' B D by auto
thus ?thesis
  by (simp add: LtA-def P8)
qed
thus ?thesis
  using Acute-def P7 by blast
qed

lemma inangle-dec:
  P InAngle A B C ∨ ¬ P InAngle A B C
  by blast

lemma lea-dec:
  A B C LeA D E F ∨ ¬ A B C LeA D E F
  by blast

lemma lta-dec:
  A B C LtA D E F ∨ ¬ A B C LtA D E F
  by blast

lemma lea-total:
  assumes A ≠ B and
    B ≠ C and
    D ≠ E and
    E ≠ F

```

```

shows  $A B C \text{ LeA } D E F \vee D E F \text{ LeA } A B C$ 
proof cases
  assume  $P1: Col A B C$ 
  show ?thesis
  proof cases
    assume  $B \text{ Out } A C$ 
    thus ?thesis
      using assms(3) assms(4) l11-31-1 by auto
  next
    assume  $\neg B \text{ Out } A C$ 
    thus ?thesis
      by (metis P1 assms(1) assms(2) assms(3) assms(4) l11-31-2 or-bet-out)
  qed
next
  assume  $P2: \neg Col A B C$ 
  show ?thesis
  proof cases
    assume  $P3: Col D E F$ 
    show ?thesis
    proof cases
      assume  $E \text{ Out } D F$ 
      thus ?thesis
        using assms(1) assms(2) l11-31-1 by auto
    next
      assume  $\neg E \text{ Out } D F$ 
      thus ?thesis
        by (metis P3 assms(1) assms(2) assms(3) assms(4) l11-31-2 l6-4-2)
    qed
  next
    assume  $P4: \neg Col D E F$ 
    show ?thesis
    proof cases
      assume  $A B C \text{ LeA } D E F$ 
      thus ?thesis
        by simp
    next
      assume  $P5: \neg A B C \text{ LeA } D E F$ 
      obtain  $P$  where  $P6: D E F \text{ CongA } A B P \wedge A B \text{ OS } P C$ 
      using P2 P4 angle-construction-1 by blast
      hence  $P7: B A \text{ OS } C P$ 
      using invert-one-side one-side-symmetry by blast
      have  $B C \text{ OS } A P$ 
      proof -
        {
          assume  $Col P B C$ 
          hence  $P7B: B \text{ Out } C P$ 
          using Col-cases P7 col-one-side-out by blast
          have  $A B C \text{ CongA } D E F$ 
          proof -
            have  $P7C: A B P \text{ CongA } D E F$ 
            by (simp add: P6 conga-sym)
            have  $P7D: B \text{ Out } A A$ 
            by (simp add: assms(1) out-trivial)
            have  $P7E: E \text{ Out } D D$ 
            by (simp add: assms(3) out-trivial)
            have  $E \text{ Out } F F$ 
            using assms(4) out-trivial by auto
            thus ?thesis
              using P7B P7C P7D P7E l11-10 by blast
          qed
          hence  $A B C \text{ LeA } D E F$ 
          by (simp add: conga--lea)
          hence False
          by (simp add: P5)
        }
      hence  $P8: \neg Col P B C$  by auto
    }
  qed

```

```

{
  assume  $T0: B C TS A P$ 
  have  $A B C CongA D E F$ 
  proof -
    have  $T1: A B C LeA A B P$ 
    proof -
      have  $T1A: C InAngle A B P$ 
      by (simp add:  $P7 T0 one-side-symmetry os-ts--inangle$ )
      have  $A B C CongA A B C$ 
      using  $assms(1) assms(2) conga-refl$  by auto
      thus ?thesis
      by (simp add:  $T1A inangle--lea$ )
    qed
    have  $T2: A B C CongA A B C$ 
    using  $assms(1) assms(2) conga-refl$  by auto
    have  $A B P CongA D E F$ 
    by (simp add:  $P6 conga-sym$ )
    thus ?thesis
    using  $P5 T1 T2 l11-30$  by blast
  qed
  hence  $A B C LeA D E F$ 
  by (simp add:  $conga--lea$ )
  hence False
  by (simp add:  $P5$ )
}
hence  $\neg B C TS A P$  by auto
thus ?thesis
using  $Col-perm P7 P8 one-side-symmetry os-ts1324--os two-sides-cases$  by blast
qed
hence  $P InAngle A B C$ 
using  $P7 os2--inangle$  by blast
hence  $D E F LeA A B C$ 
using  $P6 LeA-def$  by blast
thus ?thesis
by simp
qed
qed
qed

lemma or-lta2-conga:
  assumes  $A \neq B$  and
     $C \neq B$  and
     $D \neq E$  and
     $F \neq E$ 
  shows  $A B C LtA D E F \vee D E F LtA A B C \vee A B C CongA D E F$ 
proof -
  have  $P1: A B C LeA D E F \vee D E F LeA A B C$ 
  using  $assms(1) assms(2) assms(3) assms(4) lea-total$  by auto
  {
    assume  $A B C LeA D E F$ 
    hence  $A B C LtA D E F \vee D E F LtA A B C \vee A B C CongA D E F$ 
    using  $LtA-def$  by blast
  }
  {
    assume  $D E F LeA A B C$ 
    hence  $A B C LtA D E F \vee D E F LtA A B C \vee A B C CongA D E F$ 
    using  $LtA-def conga-sym$  by blast
  }
  thus ?thesis
  using  $P1 \langle A B C LeA D E F \implies A B C LtA D E F \vee D E F LtA A B C \vee A B C CongA D E F \rangle$ 
  by blast
qed

lemma angle-partition:
  assumes  $A \neq B$  and
     $B \neq C$ 

```

```

shows  $Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C$ 
proof -
obtain  $A'\ B'\ D'$  where  $P1: \neg (Bet\ A'\ B'\ D' \vee Bet\ B'\ D'\ A' \vee Bet\ D'\ A'\ B')$ 
  using lower-dim by auto
hence  $\neg Col\ A'\ B'\ D'$ 
  by (simp add: Col-def)
obtain  $C'$  where  $P3: A'\ B'\ Perp\ C'\ B'$ 
  by (metis P1 Perp-perm between-trivial2 perp-exists)
hence  $P4: A\ B\ C\ LtA\ A'\ B'\ C' \vee A'\ B'\ C'\ LtA\ A\ B\ C \vee A\ B\ C\ CongA\ A'\ B'\ C'$ 
  by (metis P1 assms(1) assms(2) between-trivial2 or-lta2-conga perp-not-eq-2)
{
  assume  $A\ B\ C\ LtA\ A'\ B'\ C'$ 
  hence  $Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C$ 
    using Acute-def P3 Perp-cases perp-per-2 by blast
}
{
  assume  $A'\ B'\ C'\ LtA\ A\ B\ C$ 
  hence  $Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C$ 
    using Obtuse-def P3 Perp-cases perp-per-2 by blast
}
{
  assume  $A\ B\ C\ CongA\ A'\ B'\ C'$ 
  hence  $Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C$ 
    by (metis P3 Perp-cases conga-sym l11-17 perp-per-2)
}
thus ?thesis
  using  $P4 \langle A\ B\ C\ LtA\ A'\ B'\ C' \implies Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C \rangle$ 
     $\langle A'\ B'\ C'\ LtA\ A\ B\ C \implies Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C \rangle$  by auto
qed

```

```

lemma acute-chara-1:
  assumes  $Bet\ A\ B\ A'$  and
     $B \neq A'$  and
     $Acute\ A\ B\ C$ 
  shows  $A\ B\ C\ LtA\ A'\ B\ C$ 
proof -
  have  $Obtuse\ A'\ B\ C$ 
    using acute-bet--obtuse assms(1) assms(2) assms(3) by blast
  thus ?thesis
    by (simp add: acute-obtuse--lta assms(3))
qed

```

```

lemma acute-chara-2:
  assumes  $Bet\ A\ B\ A'$  and
     $A\ B\ C\ LtA\ A'\ B\ C$ 
  shows  $Acute\ A\ B\ C$ 
  by (metis Acute-def acute-chara-1 angle-partition assms(1) assms(2) bet-obtuse--acute
    between-symmetry lta-distincts not-and-lta)

```

```

lemma acute-chara:
  assumes  $Bet\ A\ B\ A'$  and
     $B \neq A'$ 
  shows  $Acute\ A\ B\ C \longleftrightarrow A\ B\ C\ LtA\ A'\ B\ C$ 
  using acute-chara-1 acute-chara-2 assms(1) assms(2) by blast

```

```

lemma obtuse-chara:
  assumes  $Bet\ A\ B\ A'$  and
     $B \neq A'$ 
  shows  $Obtuse\ A\ B\ C \longleftrightarrow A'\ B\ C\ LtA\ A\ B\ C$ 
  by (metis Obtuse-def acute-bet--obtuse acute-chara assms(1) assms(2)
    bet-obtuse--acute between-symmetry lta-distincts)

```

```

lemma conga--acute:
  assumes  $A\ B\ C\ CongA\ A\ C\ B$ 
  shows  $Acute\ A\ B\ C$ 
  by (metis acute-sym angle-partition assms conga-distinct conga-obtuse--obtuse l11-17 l11-43)

```

```

lemma cong--acute:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $Cong\ A\ B\ A\ C$ 
  shows  $Acute\ A\ B\ C$ 
  using angle-partition assms(1) assms(2) assms(3) cong--nlt l11-46 lt-left-comm by blast

lemma nltA--lea:
  assumes  $\neg A\ B\ C\ LtA\ D\ E\ F$  and
     $A \neq B$  and
     $B \neq C$  and
     $D \neq E$  and
     $E \neq F$ 
  shows  $D\ E\ F\ LeA\ A\ B\ C$ 
  by (metis LtA-def assms(1) assms(2) assms(3) assms(4) assms(5) conga--lea456123 or-lta2-conga)

lemma nlea--lta:
  assumes  $\neg A\ B\ C\ LeA\ D\ E\ F$  and
     $A \neq B$  and
     $B \neq C$  and
     $D \neq E$  and
     $E \neq F$ 
  shows  $D\ E\ F\ LtA\ A\ B\ C$ 
  using assms(1) assms(2) assms(3) assms(4) assms(5) nltA--lea by blast

lemma triangle-strict-inequality:
  assumes  $Bet\ A\ B\ D$  and
     $Cong\ B\ C\ B\ D$  and
     $\neg Bet\ A\ B\ C$ 
  shows  $A\ C\ Lt\ A\ D$ 
proof cases
  assume P1:  $Col\ A\ B\ C$ 
  hence P2:  $B\ Out\ A\ C$ 
  using assms(3) not-out-bet by auto
  {
    assume  $Bet\ B\ A\ C$ 
    hence P3:  $A\ C\ Le\ A\ D$ 
    by (meson assms(1) assms(2) cong--le l5-12-a le-transitivity)
    have  $\neg Cong\ A\ C\ A\ D$ 
    by (metis Out-def P1 P2 assms(1) assms(2) assms(3) l4-18)
    hence  $A\ C\ Lt\ A\ D$ 
    by (simp add: Lt-def P3)
  }
  {
    assume  $Bet\ A\ C\ B$ 
    hence P5:  $Bet\ A\ C\ D$ 
    using assms(1) between-exchange4 by blast
    hence P6:  $A\ C\ Le\ A\ D$ 
    by (simp add: bet--le1213)
    have  $\neg Cong\ A\ C\ A\ D$ 
    using P5 assms(1) assms(3) between-cong by blast
    hence  $A\ C\ Lt\ A\ D$ 
    by (simp add: Lt-def P6)
  }
  thus ?thesis
  using P1  $\langle Bet\ B\ A\ C \implies A\ C\ Lt\ A\ D \rangle$  assms(3) between-symmetry third-point by blast
next
  assume T1:  $\neg Col\ A\ B\ C$ 
  have T2:  $A \neq D$ 
  using T1 assms(1) between-identity col-trivial-1 by auto
  have T3:  $\neg Col\ A\ C\ D$ 
  by (metis Col-def T1 T2 assms(1) col-transitivity-2)
  have T4:  $Bet\ D\ B\ A$ 
  using Bet-perm assms(1) by blast
  have T5:  $C\ D\ A\ CongA\ D\ C\ B$ 

```

```

proof –
  have  $T6: C D B \text{ CongA } D C B$ 
    by (metis assms(1) assms(2) assms(3) between-trivial conga-comm l11-44-1-a not-conga-sym)
  have  $T7: D \text{ Out } C C$ 
    using  $T3 \text{ not-col-distincts out-trivial}$  by blast
  have  $T8: D \text{ Out } A B$ 
    by (metis assms(1) assms(2) assms(3) bet-out-1 cong-diff l6-6)
  have  $T9: C \text{ Out } D D$ 
    using  $T7 \text{ out-trivial}$  by force
  have  $C \text{ Out } B B$ 
    using  $T1 \text{ not-col-distincts out-trivial}$  by auto
  thus ?thesis
    using  $T6 T7 T8 T9 \text{ l11-10}$  by blast
qed
have  $A D C \text{ LtA } A C D$ 
proof –
  have  $B \text{ InAngle } D C A$ 
    by (metis InAngle-def T1 T3 T4 not-col-distincts out-trivial)
  hence  $C D A \text{ LeA } D C A$ 
    using  $T5 \text{ LeA-def}$  by auto
  hence  $T10: A D C \text{ LeA } A C D$ 
    by (simp add: lea-comm)
  have  $\neg A D C \text{ CongA } A C D$ 
    by (metis Col-perm T3 assms(1) assms(2) assms(3) bet-col l11-44-1-b l4-18
      not-bet-distincts not-cong-3412)
  thus ?thesis
    using  $\text{LtA-def } T10$  by blast
qed
thus ?thesis
  by (simp add: l11-44-2-b)
qed

lemma triangle-inequality:
  assumes  $\text{Bet } A B D$  and
     $\text{Cong } B C B D$ 
  shows  $A C \text{ Le } A D$ 
proof cases
  assume  $\text{Bet } A B C$ 
  thus ?thesis
    by (metis assms(1) assms(2) between-cong-3 cong--le le-reflexivity)
next
  assume  $\neg \text{Bet } A B C$ 
  hence  $A C \text{ Lt } A D$ 
    using assms(1) assms(2) triangle-strict-inequality by auto
  thus ?thesis
    by (simp add: Lt-def)
qed

lemma triangle-strict-inequality-2:
  assumes  $\text{Bet } A' B' C'$  and
     $\text{Cong } A B A' B'$  and
     $\text{Cong } B C B' C'$  and
     $\neg \text{Bet } A B C$ 
  shows  $A C \text{ Lt } A' C'$ 
proof –
  obtain  $D$  where  $P1: \text{Bet } A B D \wedge \text{Cong } B D B C$ 
    using segment-construction by blast
  hence  $P2: A C \text{ Lt } A D$ 
    using  $P1 \text{ assms}(4) \text{ cong-symmetry triangle-strict-inequality}$  by blast
  have  $\text{Cong } A D A' C'$ 
    using  $P1 \text{ assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ cong-transitivity l2-11-b}$  by blast
  thus ?thesis
    using  $P2 \text{ cong2-lt--lt cong-reflexivity}$  by blast
qed

lemma triangle-inequality-2:

```

```

assumes Bet  $A' B' C'$  and
  Cong  $A B A' B'$  and
  Cong  $B C B' C'$ 
shows  $A C \text{ Le } A' C'$ 
proof –
  obtain  $D$  where  $P1: \text{Bet } A B D \wedge \text{Cong } B D B C$ 
    using segment-construction by blast
  have  $P2: A C \text{ Le } A D$ 
    by (meson  $P1$  triangle-inequality not-cong-3412)
  have Cong  $A D A' C'$ 
    using  $P1$  assms(1) assms(2) assms(3) cong-transitivity l2-11-b by blast
  thus ?thesis
    using  $P2$  cong--le le-transitivity by blast
qed

lemma triangle-strict-reverse-inequality:
assumes  $A \text{ Out } B D$  and
  Cong  $A C A D$  and
   $\neg A \text{ Out } B C$ 
shows  $B D \text{ Lt } B C$ 
proof cases
assume  $\text{Col } A B C$ 
thus ?thesis
  using assms(1) assms(2) assms(3) col-permutation-4 cong-symmetry not-bet-and-out
    or-bet-out triangle-strict-inequality by blast
next
assume  $P1: \neg \text{Col } A B C$ 
show ?thesis
proof cases
assume  $B = D$ 
thus ?thesis
  using  $P1$  lt1123 not-col-distincts by auto
next
assume  $P2: B \neq D$ 
hence  $P3: \neg \text{Col } B C D$ 
  using  $P1$  assms(1) col-trivial-2 colx not-col-permutation-5 out-col by blast
have  $P4: \neg \text{Col } A C D$ 
  using  $P1$  assms(1) col2--eq col-permutation-4 out-col out-distinct by blast
have  $P5: C \neq D$ 
  using assms(1) assms(3) by auto
hence  $P6: A C D \text{ Cong } A A D C$ 
  by (metis  $P1$  assms(2) col-trivial-3 l11-44-1-a)
{
  assume  $T1: \text{Bet } A B D$ 
  hence  $T2: \text{Bet } D B A$ 
  using Bet-perm by blast
  have  $B C D \text{ Lt } A B D C$ 
  proof –
    have  $T3: D C B \text{ Cong } A B C D$ 
    by (metis  $P3$  conga-pseudo-refl not-col-distincts)
    have  $T3A: D \text{ Out } B A$ 
    by (simp add: P2 T1 bet-out-1)
    have  $T3B: C \text{ Out } D D$ 
    using  $P5$  out-trivial by auto
    have  $T3C: C \text{ Out } A A$ 
    using  $P1$  not-col-distincts out-trivial by blast
    have  $D \text{ Out } C C$ 
    by (simp add: P5 out-trivial)
    hence  $T4: D C A \text{ Cong } A B D C$  using  $T3A T3B T3C$ 
    by (meson conga-comm conga-right-comm l11-10 P6)
    have  $D C B \text{ Lt } A D C A$ 
    proof –
      have  $T4A: D C B \text{ Le } A D C A$ 
      proof –
        have  $T4AA: B \text{ InAngle } D C A$ 
        using InAngle-def P1 P5 T2 not-col-distincts out-trivial by auto

```

```

    have  $D\ C\ B\ CongA\ D\ C\ B$ 
      using  $T3\ conga-right-comm$  by blast
    thus ?thesis
      by (simp add:  $T4AA\ inangle--lea$ )
  qed
{
  assume  $T5: D\ C\ B\ CongA\ D\ C\ A$ 
  have  $D\ C\ OS\ B\ A$ 
    using  $Col-perm\ P3\ T3A\ out-one-side$  by blast
  hence  $C\ Out\ B\ A$ 
    using  $T5\ conga-os--out$  by blast
  hence False
    using  $Col-cases\ P1\ out-col$  by blast
}
hence  $\neg D\ C\ B\ CongA\ D\ C\ A$  by auto
thus ?thesis
  using  $LtA-def\ T4A$  by blast
qed
thus ?thesis
  using  $T3\ T4\ conga-preserves-lta$  by auto
qed
}
{
  assume  $Q1: Bet\ B\ D\ A$ 
  obtain  $E$  where  $Q2: Bet\ B\ C\ E \wedge Cong\ B\ C\ C\ E$ 
    using  $Cong-perm\ segment-construction$  by blast
  have  $A\ D\ C\ LtA\ E\ C\ D$ 
  proof -
    have  $Q3: D\ C\ OS\ A\ E$ 
  proof -
    have  $Q4: D\ C\ TS\ A\ B$ 
      by (metis  $Col-perm\ P2\ Q1\ P4\ bet--ts\ between-symmetry$ )
    hence  $D\ C\ TS\ E\ B$ 
      by (metis  $Col-def\ Q1\ Q2\ TS-def\ bet--ts\ cong-identity\ invert-two-sides\ l9-2$ )
    thus ?thesis
      using  $OS-def\ Q4$  by blast
  qed
  have  $Q5: A\ C\ D\ LtA\ E\ C\ D$ 
  proof -
    have  $D\ C\ A\ LeA\ D\ C\ E$ 
  proof -
    have  $B\ Out\ D\ A$ 
      using  $P2\ Q1\ bet-out$  by auto
    hence  $B\ C\ OS\ D\ A$ 
      by (simp add:  $P3\ out-one-side$ )
    hence  $C\ B\ OS\ D\ A$ 
      using  $invert-one-side$  by blast
    hence  $C\ E\ OS\ D\ A$ 
      by (metis  $Col-def\ Q2\ Q3\ col124--nos\ col-one-side\ diff-col-ex\ not-col-permutation-5$ )
    hence  $Q5A: A\ InAngle\ D\ C\ E$ 
      by (simp add:  $\langle C\ E\ OS\ D\ A \rangle\ Q3\ invert-one-side\ one-side-symmetry\ os2--inangle$ )
    have  $D\ C\ A\ CongA\ D\ C\ A$ 
      using  $CongA-def\ P6\ conga-refl$  by auto
    thus ?thesis
      by (simp add:  $Q5A\ inangle--lea$ )
  qed
  hence  $Q6: A\ C\ D\ LeA\ E\ C\ D$ 
    using  $lea-comm$  by blast
  {
    assume  $A\ C\ D\ CongA\ E\ C\ D$ 
    hence  $D\ C\ A\ CongA\ D\ C\ E$ 
      by (simp add:  $conga-comm$ )
    hence  $C\ Out\ A\ E$ 
      using  $Q3\ conga-os--out$  by auto
    hence False

```



```

    by (meson Col-def Out-cases P1 Q2 not-col-permutation-3
        one-side-not-col123 out-one-side)
  }
  hence  $\neg A C D \text{ CongA } E C D$  by auto
  thus ?thesis
    by (simp add: LtA-def Q6)
qed
have  $E C D \text{ CongA } E C D$ 
  by (metis P1 P5 Q2 cong-diff conga-refl not-col-distincts)
thus ?thesis
  using Q5 P6 conga-preserves-lta by auto
qed
hence  $B C D \text{ LtA } B D C$ 
  using Bet-cases P1 P2 Q1 Q2 bet2-lta--lta not-col-distincts by blast
}
hence  $B C D \text{ LtA } B D C$ 
  by (meson Out-def  $\langle \text{Bet } A B D \implies B C D \text{ LtA } B D C \rangle$  assms(1) between-symmetry)
thus ?thesis
  by (simp add: l11-44-2-b)
qed
qed

lemma triangle-reverse-inequality:
  assumes  $A \text{ Out } B D$  and
     $\text{Cong } A C A D$ 
  shows  $B D \text{ Le } B C$ 
proof cases
  assume  $A \text{ Out } B C$ 
  thus ?thesis
    by (metis assms(1) assms(2) bet--le1213 cong-pseudo-reflexivity l6-11-uniqueness
        l6-6 not-bet-distincts not-cong-4312)
next
  assume  $\neg A \text{ Out } B C$ 
  thus ?thesis
    using triangle-strict-reverse-inequality assms(1) assms(2) lt--le by auto
qed

lemma os3--lta:
  assumes  $A B \text{ OS } C D$  and
     $B C \text{ OS } A D$  and
     $A C \text{ OS } B D$ 
  shows  $B A C \text{ LtA } B D C$ 
proof -
  have P1:  $D \text{ InAngle } A B C$ 
    by (simp add: assms(1) assms(2) invert-one-side os2--inangle)
  then obtain  $E$  where P2:  $\text{Bet } A E C \wedge (E = B \vee B \text{ Out } E D)$ 
    using InAngle-def by auto
  have P3:  $\neg \text{Col } A B C$ 
    using assms(1) one-side-not-col123 by auto
  have P4:  $\neg \text{Col } A C D$ 
    using assms(3) col124--nos by auto
  have P5:  $\neg \text{Col } B C D$ 
    using assms(2) one-side-not-col124 by auto
  have P6:  $\neg \text{Col } A B D$ 
    using assms(1) one-side-not-col124 by auto
  {
    assume  $E = B$ 
    hence  $B A C \text{ LtA } B D C$ 
      using P2 P3 bet-col by blast
  }
  {
    assume P7:  $B \text{ Out } E D$ 
    have P8:  $A \neq E$ 
      using P6 P7 not-col-permutation-4 out-col by blast
    have P9:  $C \neq E$ 
      using P5 P7 out-col by blast
  }

```

```

have P10: B A C LtA B E C
proof -
  have P10A:  $\neg$  Col E A B
  by (metis Col-def P2 P3 P8 col-transitivity-1)
  hence P10B: E B A LtA B E C
  using P2 P9 l11-41-aux Tarski-neutral-dimensionless-axioms by fastforce
  have P10C: E A B LtA B E C
  using P2 P9 P10A l11-41 by auto
  have P11: E A B CongA B A C
  proof -
    have P11A: A Out B B
    using assms(2) os-distincts out-trivial by auto
    have A Out C E
    using P2 P8 bet-out l6-6 by auto
    thus ?thesis
    using P11A conga-right-comm out2--conga by blast
  qed
  have P12: B E C CongA B E C
  by (metis Col-def P2 P3 P9 conga-refl)
  thus ?thesis
  using P11 P10C conga-preserves-lta by auto
qed
have B E C LtA B D C
proof -
  have U1: E Out D B
  by (metis Bet-cases P2 P8 assms(3) bet-out-1 col123--nos not-bet-out one-side-chara
    one-side-reflexivity out-col)
  have U2: D  $\neq$  E
  using P2 P4 bet-col not-col-permutation-5 by blast
  have U3:  $\neg$  Col D E C
  by (metis Col-def P2 P4 P9 col-transitivity-1)
  have U4: Bet E D B
  by (simp add: P7 U1 out2--bet)
  have D C E LtA C D B
  using P5 U3 U4 l11-41-aux not-col-distincts by blast
  have U5: D E C LtA C D B
  using P7 U4 U3 l11-41 out-diff2 by auto
  have D E C CongA B E C
  by (simp add: P9 U1 l6-6 out2--conga out-trivial)
  thus ?thesis
  by (metis U5 conga-preserves-lta conga-pseudo-refl lta-distincts)
qed
hence B A C LtA B D C
  using P10 lta-trans by blast
}
thus ?thesis
  using P2  $\langle E = B \implies B A C LtA B D C \rangle$  by blast
qed

```

lemma bet-le--lt:

assumes Bet A D B and

A $\neq$ D and

D $\neq$ B and

A C Le B C

shows D C Lt B C

proof -

have P1: A $\neq$ B

using assms(1) assms(2) between-identity by blast

have C D Lt C B

proof cases

assume P3: Col A B C

thus ?thesis

proof cases

```

assume Bet C D B
thus ?thesis
  by (simp add: assms(3) bet--lt1213)
next
assume  $\neg \text{Bet } C \ D \ B$ 
hence D Out C B
  by (metis Out-def P1 P3 assms(1) col-transitivity-2 not-col-permutation-3
    not-out-bet out-col)
have  $\neg \text{Bet } A \ B \ C$ 
  by (meson P1 bet-le-eq assms(4))
have  $B \neq C$ 
  using  $\langle \neg \text{Bet } A \ B \ C \rangle \text{ assms(1) assms(2) l5-1}$  by blast
have Bet B C A
  using Col-def P3  $\langle \neg \text{Bet } A \ B \ C \rangle \langle \neg \text{Bet } C \ D \ B \rangle \text{ assms(1)}$ 
    between-inner-transitivity outer-transitivity-between2 by blast
obtain C' where C Midpoint A C'
  using symmetric-point-construction by blast
have Bet C C' B
proof –
  have C Out C' B
  proof –
    have  $C' \neq C$ 
      by (metis  $\langle C \text{ Midpoint } A \ C' \rangle \langle \neg \text{Bet } C \ D \ B \rangle \text{ assms(1) midpoint-not-midpoint}$ )
    moreover have  $A \neq C$ 
      using  $\langle \neg \text{Bet } C \ D \ B \rangle \text{ assms(1)}$  by auto
    moreover have Bet C' C A
      by (metis midpoint-bet  $\langle C \text{ Midpoint } A \ C' \rangle \text{ between-symmetry}$ )
    ultimately show ?thesis
      using  $\langle B \neq C \rangle \langle \text{Bet } B \ C \ A \rangle \text{ l6-2}$  by blast
  qed
moreover have C C' Le C B
  by (metis cong-reflexivity l5-6 le-right-comm midpoint-cong
     $\langle C \text{ Midpoint } A \ C' \rangle \text{ assms(4)}$ )
ultimately show ?thesis
  using l6-13-1 by blast
qed
thus ?thesis
  by (metis bet--lt1213 l6-2 l6-6 le3456-lt--lt le-comm not-bet-distincts
     $\langle D \text{ Out } C \ B \rangle \langle \neg \text{Bet } C \ D \ B \rangle \text{ assms(1) assms(2) assms(4) between-symmetry}$ )
qed
next
assume Q0A:  $\neg \text{Col } A \ B \ C$ 
hence Q0B:  $\neg \text{Col } B \ C \ D$ 
  by (meson Col-def assms(1) assms(3) col-transitivity-2)
have C B D LtA C D B
proof –
  have Q1: B Out C C
    using Q0A not-col-distincts out-trivial by force
  have Q2: B Out A D
    using Out-cases assms(1) assms(3) bet-out-1 by blast
  have Q3: A Out C C
    by (metis Q0A col-trivial-3 out-trivial)
  have Q4: A Out B B
    using P1 out-trivial by auto
  have C B A LeA C A B
    using Col-perm Le-cases Q0A assms(4) l11-44-2bis by blast
hence T9: C B D LeA C A B
  using Q1 Q2 Q3 Q4 lea-out4--lea by blast
have C A B LtA C D B
proof –
  have U2:  $\neg \text{Col } D \ A \ C$ 
    using Q0B assms(1) assms(2) bet-col col-transitivity-2 not-col-permutation-3
      not-col-permutation-4 by blast
  have U3: D  $\neq$  C
    using Q0B col-trivial-2 by blast
  have U4: D C A LtA C D B

```

```

    using U2 assms(1) assms(3) l11-41-aux by auto
  have U5:  $D \ A \ C \ LtA \ C \ D \ B$ 
  by (simp add: U2 assms(1) assms(3) l11-41)
  have A Out B D
  using Out-def P1 assms(1) assms(2) by auto
  hence  $D \ A \ C \ CongA \ C \ A \ B$ 
  using Q3 conga-right-comm out2--conga by blast
  thus ?thesis
  by (metis U5 U3 assms(3) conga-preserves-lta conga-refl)
qed
thus ?thesis
  using T9 lea123456-lta--lta by blast
qed
thus ?thesis
  by (simp add: l11-44-2-b)
qed
thus ?thesis
  using Lt-cases by auto
qed

lemma cong2--ncol:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $A \neq C$  and
     $Cong \ A \ P \ B \ P$  and
     $Cong \ A \ P \ C \ P$ 
  shows  $\neg \ Col \ A \ B \ C$ 
proof -
  have  $Cong \ B \ P \ C \ P$ 
  using assms(4) assms(5) cong-inner-transitivity by blast
  thus ?thesis using bet-le--lt
  by (metis assms(1) assms(2) assms(3) assms(4) assms(5) cong-le cong--nlt
    lt--nle not-col-permutation-5 third-point)
qed

lemma cong4-cop2--eq:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $A \neq C$  and
     $Cong \ A \ P \ B \ P$  and
     $Cong \ A \ P \ C \ P$  and
     $Coplanar \ A \ B \ C \ P$  and
     $Cong \ A \ Q \ B \ Q$  and
     $Cong \ A \ Q \ C \ Q$  and
     $Coplanar \ A \ B \ C \ Q$ 
  shows  $P = Q$ 
proof -
  have P1:  $\neg \ Col \ A \ B \ C$ 
  using assms(1) assms(2) assms(3) assms(4) assms(5) cong2--ncol by auto
  {
    assume P2:  $P \neq Q$ 
    have P3:  $\forall \ R. \ Col \ P \ Q \ R \longrightarrow (Cong \ A \ R \ B \ R \wedge Cong \ A \ R \ C \ R)$ 
    using P2 assms(4) assms(5) assms(7) assms(8) l4-17 not-cong-4321 by blast
    obtain D where P4:  $D \ Midpoint \ A \ B$ 
    using midpoint-existence by auto
    have P5:  $Coplanar \ A \ B \ C \ D$ 
    using P4 coplanar-perm-9 midpoint--coplanar by blast
    have P6:  $Col \ P \ Q \ D$ 
    proof -
      have P6A:  $Coplanar \ P \ Q \ D \ A$ 
      using P1 P5 assms(6) assms(9) coplanar-pseudo-trans ncop-distincts by blast
      hence P6B:  $Coplanar \ P \ Q \ D \ B$ 
      by (metis P4 col-cop--cop midpoint-col midpoint-distinct-1)
      have P6D:  $Cong \ P \ A \ P \ B$ 
      using assms(4) not-cong-2143 by blast
      have P6E:  $Cong \ Q \ A \ Q \ B$ 

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    using assms(7) not-cong-2143 by blast
  have Cong D A D B
    using Midpoint-def P4 not-cong-2134 by blast
  thus ?thesis using cong3-cop2--col P6A P6B assms(1) P6D P6E by blast
qed
obtain R1 where P7:  $P \neq R1 \wedge Q \neq R1 \wedge D \neq R1 \wedge \text{Col } P \ Q \ R1$ 
  using P6 diff-col-ex3 by blast
obtain R2 where P8:  $\text{Bet } R1 \ D \ R2 \wedge \text{Cong } D \ R2 \ R1 \ D$ 
  using segment-construction by blast
have P9:  $\text{Col } P \ Q \ R2$ 
  by (metis P6 P7 P8 bet-col colx)
have P9A:  $\text{Cong } R1 \ A \ R1 \ B$ 
  using P3 P7 not-cong-2143 by blast
hence Per R1 D A
  using P4 Per-def by auto
hence Per A D R1 using l8-2 by blast
have P10:  $\text{Cong } A \ R1 \ A \ R2$ 
proof -
  have f1:  $\text{Bet } R2 \ D \ R1 \vee \text{Bet } R1 \ R2 \ D$ 
    by (metis Col-def P7 P8 between-equality not-col-permutation-5)
  have f2:  $\text{Cong } B \ R1 \ A \ R1$ 
    using Cong-perm  $\langle \text{Cong } R1 \ A \ R1 \ B \rangle$  by blast
  have Cong B R1 A R2  $\vee \text{Bet } R1 \ R2 \ D$ 
    using f1 Cong-perm Midpoint-def P4 P8 l7-13 by blast
  thus ?thesis
    using f2 P8 bet-cong-eq cong-inner-transitivity by blast
qed
have Col A B C
proof -
  have P11:  $\text{Cong } B \ R1 \ B \ R2$ 
    by (metis Cong-perm P10 P3 P9 P9A cong-inner-transitivity)
  have P12:  $\text{Cong } C \ R1 \ C \ R2$ 
    using P10 P3 P7 P9 cong-inner-transitivity by blast
  have P12A:  $\text{Coplanar } A \ B \ C \ R1$ 
    using P2 P7 assms(6) assms(9) col-cop2--cop by blast
  have P12B:  $\text{Coplanar } A \ B \ C \ R2$ 
    using P2 P9 assms(6) assms(9) col-cop2--cop by blast
  have  $R1 \neq R2$ 
    using P7 P8 between-identity by blast
  thus ?thesis
    using P10 P11 P12A P12B P12 cong3-cop2--col by blast
qed
hence False
  by (simp add: P1)
}
thus ?thesis by auto
qed

lemma t18-18-aux:
  assumes Cong A B D E and
    Cong A C D F and
     $F \ D \ E \ \text{LtA } C \ A \ B$  and
     $\neg \text{Col } A \ B \ C$  and
     $\neg \text{Col } D \ E \ F$  and
     $D \ F \ \text{Le } D \ E$ 
  shows  $E \ F \ \text{Lt } B \ C$ 
proof -
  obtain G0 where P1:  $C \ A \ B \ \text{CongA } F \ D \ G0 \wedge F \ D \ \text{OS } G0 \ E$ 
    using angle-construction-1 assms(4) assms(5) not-col-permutation-2 by blast
  hence P2:  $\neg \text{Col } F \ D \ G0$ 
    using col123--nos by auto
  then obtain G where P3:  $D \ \text{Out } G0 \ G \wedge \text{Cong } D \ G \ A \ B$ 
    by (metis assms(4) bet-col between-trivial2 col-trivial-2 segment-construction-3)
  have P4:  $C \ A \ B \ \text{CongA } F \ D \ G$ 
  proof -
    have P4B:  $A \ \text{Out } C \ C$ 

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    by (metis assms(4) col-trivial-3 out-trivial)
  have P4C: A Out B B
    by (metis assms(4) col-trivial-1 out-trivial)
  have P4D: D Out F F
    using P2 not-col-distincts out-trivial by blast
  have D Out G G0
    by (simp add: P3 l6-6)
  thus ?thesis using P1 P4B P4C P4D
    using l11-10 by blast
qed
have D Out G G0
  by (simp add: P3 l6-6)
hence D F OS G G0
  using P2 not-col-permutation-4 out-one-side by blast
hence F D OS G G0
  by (simp add: invert-one-side)
hence P5: F D OS G E
  using P1 one-side-transitivity by blast
have P6: ¬ Col F D G
  by (meson P5 one-side-not-col123)
have P7: Cong C B F G
  using P3 P4 assms(2) cong2-conga-cong cong-commutativity cong-symmetry by blast
have P8: F E Lt F G
proof -
  have P9: F D E LtA F D G
    by (metis P4 assms(3) assms(5) col-trivial-1 col-trivial-3
      conga-preserves-lta conga-refl)
  have P10: Cong D G D E
    using P3 assms(1) cong-transitivity by blast
  {
    assume P11: Col E F G
    have P12: F D E LeA F D G
      by (simp add: P9 lta--lea)
    have P13: ¬ F D E CongA F D G
      using P9 not-lta-and-conga by blast
    have F D E CongA F D G
    proof -
      have F Out E G
        using Col-cases P11 P5 col-one-side-out l6-6 by blast
      hence E F D CongA G F D
        by (metis assms(5) conga-os--out conga-refl l6-6 not-col-distincts
          one-side-reflexivity out2--conga)
      thus ?thesis
        by (meson P10 assms(2) assms(6) cong-4321 cong-inner-transitivity l11-52 le-comm)
    qed
    hence False
      using P13 by blast
  }
hence P15: ¬ Col E F G by auto
{
  assume P20: Col D E G
  have P21: F D E LeA F D G
    by (simp add: P9 lta--lea)
  have P22: ¬ F D E CongA F D G
    using P9 not-lta-and-conga by blast
  have F D E CongA F D G
  proof -
    have D Out E G
      by (meson Out-cases P5 P20 col-one-side-out invert-one-side not-col-permutation-5)
    thus ?thesis
      using P10 P15 ⟨D Out G G0⟩ cong-inner-transitivity l6-11-uniqueness
        l6-7 not-col-distincts by blast
    qed
    hence False
      by (simp add: P22)
  }
}

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hence P16:  $\neg \text{Col } D \ E \ G$  by auto
have P17:  $E \text{ InAngle } F \ D \ G$ 
  using lea-in-angle by (simp add: P5 P9 lta--lea)
then obtain  $H$  where P18:  $\text{Bet } F \ H \ G \wedge (H = D \vee D \text{ Out } H \ E)$ 
  using InAngle-def by auto
{
  assume  $H = D$ 
  hence  $F \ G \ E \text{ LtA } F \ E \ G$ 
  using P18 P6 bet-col by blast
}
{
  assume P19:  $D \text{ Out } H \ E$ 
  have P20:  $H \neq F$ 
    using Out-cases P19 assms(5) out-col by blast
  have P21:  $H \neq G$ 
    using P19 P16 l6-6 out-col by blast
  have  $F \ D \text{ Le } G \ D$ 
    using P10 assms(6) cong-pseudo-reflexivity l5-6 not-cong-4312 by blast
  hence  $H \ D \text{ Lt } G \ D$ 
    using P18 P20 P21 bet-le--lt by blast
  hence P22:  $D \ H \text{ Lt } D \ E$ 
    using Lt-cases P10 cong2-lt--lt cong-reflexivity by blast
  hence P23:  $D \ H \text{ Le } D \ E \wedge \neg \text{Cong } D \ H \ D \ E$ 
    using Lt-def by blast
  have P24:  $H \neq E$ 
    using P23 cong-reflexivity by blast
  have P25:  $\text{Bet } D \ H \ E$ 
    by (simp add: P19 P23 l6-13-1)
  have P26:  $E \ G \text{ OS } F \ D$ 
    by (metis InAngle-def P15 P16 P18 P25 bet-out-1 between-symmetry
      in-angle-one-side not-col-distincts not-col-permutation-1)
  have  $F \ G \ E \text{ LtA } F \ E \ G$ 
proof –
  have P27:  $F \ G \ E \text{ LtA } D \ E \ G$ 
proof –
  have P28:  $D \ G \ E \text{ CongA } D \ E \ G$ 
    by (metis P10 P16 l11-44-1-a not-col-distincts)
  have  $F \ G \ E \text{ LtA } D \ G \ E$ 
proof –
  have P29:  $F \ G \ E \text{ LeA } D \ G \ E$ 
    by (metis OS-def P17 P26 P5 TS-def in-angle-one-side inangle--lea-1
      invert-one-side l11-24 os2--inangle)
  {
    assume  $F \ G \ E \text{ CongA } D \ G \ E$ 
    hence  $E \ G \ F \text{ CongA } E \ G \ D$ 
      by (simp add: conga-comm)
    hence  $G \text{ Out } F \ D$ 
      using P26 conga-os--out by auto
    hence False
      using P6 not-col-permutation-2 out-col by blast
  }
  hence  $\neg F \ G \ E \text{ CongA } D \ G \ E$  by auto
thus ?thesis
  by (simp add: LtA-def P29)
qed
thus ?thesis
  by (metis P28 P6 col-trivial-3 conga-preserves-lta conga-refl)
qed
have  $G \ E \ D \text{ LtA } G \ E \ F$ 
proof –
  have P30:  $G \ E \ D \text{ LeA } G \ E \ F$ 
proof –
  have P31:  $D \text{ InAngle } G \ E \ F$ 
    by (simp add: P16 P17 P26 assms(5) in-angle-two-sides l11-24
      not-col-permutation-5 os-ts--inangle)
  have  $G \ E \ D \text{ CongA } G \ E \ D$ 

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    by (metis P16 col-trivial-1 col-trivial-2 conga-refl)
  thus ?thesis
    using P31 inangle--lea by auto
qed
have  $\neg G E D \text{ CongA } G E F$ 
  by (metis OS-def P26 P5 TS-def conga-os--out invert-one-side out-col)
thus ?thesis
  by (simp add: LtA-def P30)
qed
hence  $D E G \text{ LtA } F E G$ 
  using lta-comm by blast
thus ?thesis
  using P27 lta-trans by blast
qed
}
hence  $F G E \text{ LtA } F E G$ 
  using P18  $\langle H = D \implies F G E \text{ LtA } F E G \rangle$  by blast
thus ?thesis
  by (simp add: l11-44-2-b)
qed
hence  $E F \text{ Lt } F G$ 
  using lt-left-comm by blast
thus ?thesis
  using P7 cong2-lt--lt cong-pseudo-reflexivity not-cong-4312 by blast
qed

lemma t18-18:
  assumes Cong A B D E and
    Cong A C D F and
    F D E LtA C A B
  shows E F Lt B C
proof -
  have P1:  $F \neq D$ 
    using assms(3) lta-distincts by blast
  have P2:  $E \neq D$ 
    using assms(3) lta-distincts by blast
  have P3:  $C \neq A$ 
    using assms(3) lta-distincts by auto
  have P4:  $B \neq A$ 
    using assms(3) lta-distincts by blast
  {
    assume P6: Col A B C
    {
      assume P7: Bet B A C
      obtain C' where P8: Bet E D C'  $\wedge$  Cong D C' A C
        using segment-construction by blast
      have P9: Cong E F E F
        by (simp add: cong-reflexivity)
      have P10: Cong E C' B C
        using P7 P8 assms(1) l2-11-b not-cong-4321 by blast
      have E F Lt E C'
      proof -
        have P11: Cong D F D C'
          using P8 assms(2) cong-transitivity not-cong-3412 by blast
        have  $\neg \text{Bet } E D F$ 
          using Bet-perm Col-def assms(3) col-lta--out not-bet-and-out by blast
        thus ?thesis
          using P11 P8 triangle-strict-inequality by blast
      qed
      hence E F Lt B C
        using P9 P10 cong2-lt--lt by blast
    }
  }
  {
    assume  $\neg \text{Bet } B A C$ 
    hence E F Lt B C
      using P6 assms(3) between-symmetry col-lta--bet col-permutation-2 by blast
  }

```



```

}
hence  $E F Lt B C$ 
  using  $\langle Bet B A C \implies E F Lt B C \rangle$  by auto
}
{
  assume  $P12: \neg Col A B C$ 
  {
    assume  $P13: Col D E F$ 
    {
      assume  $P14: Bet F D E$ 
      hence  $C A B LeA F D E$ 
        by (simp add: P1 P2 P3 P4 l11-31-2)
      hence  $F D E LtA F D E$ 
        using assms(3) lea--nlta by auto
      hence False
        by (simp add: nlta)
      hence  $E F Lt B C$  by auto
    }
  }
  {
    assume  $\neg Bet F D E$ 
    hence  $P16: D Out F E$ 
      using P13 not-col-permutation-1 not-out-bet by blast
    obtain  $F'$  where  $P17: A Out B F' \wedge Cong A F' A C$ 
      using P3 P4 segment-construction-3 by fastforce
    hence  $P18: B F' Lt B C$ 
      by (meson P12 triangle-strict-reverse-inequality not-cong-3412 out-col)
    have  $Cong B F' E F$ 
      by (meson Out-cases P16 P17 assms(1) assms(2) cong-transitivity out-cong-cong)
    hence  $E F Lt B C$ 
      using P18 cong2-lt--lt cong-reflexivity by blast
  }
  hence  $E F Lt B C$ 
    using  $\langle Bet F D E \implies E F Lt B C \rangle$  by blast
}
{
  assume  $P20: \neg Col D E F$ 
  {
    assume  $D F Le D E$ 
    hence  $E F Lt B C$ 
      by (meson P12 t18-18-aux P20 assms(1) assms(2) assms(3))
  }
  {
    assume  $D E Le D F$ 
    hence  $E F Lt B C$ 
      by (meson P12 P20 lta-comm t18-18-aux assms(1) assms(2) assms(3) lt-comm not-col-permutation-5)
  }
  hence  $E F Lt B C$ 
    using  $\langle D F Le D E \implies E F Lt B C \rangle$  local.le-cases by blast
}
hence  $E F Lt B C$ 
  using  $\langle Col D E F \implies E F Lt B C \rangle$  by blast
}
thus ?thesis
  using  $\langle Col A B C \implies E F Lt B C \rangle$  by auto
qed

lemma t18-19:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $Cong A B D E$  and
     $Cong A C D F$  and
     $E F Lt B C$ 
  shows  $F D E LtA C A B$ 
proof -
  {

```

```

assume  $P1: C A B \text{ Le}A F D E$ 
{
  assume  $C A B \text{ Cong}A F D E$ 
  hence  $\text{False}$ 
  using  $\text{Cong-perm } \text{assms}(3) \text{ assms}(4) \text{ assms}(5) \text{ cong-nlt } l11-49$  by  $\text{blast}$ 
}
{
  assume  $P2: \neg C A B \text{ Cong}A F D E$ 
  hence  $C A B \text{ Lt}A F D E$ 
  by  $(\text{metis } P1 \text{ assms}(3) \text{ assms}(4) \text{ assms}(5) \text{ cong-symmetry } \text{lea-distincts} \text{ lta--nlea not-and-lt or-lta2-conga } t18-18)$ 
  hence  $B C \text{ Lt } E F$ 
  by  $(\text{metis } P1 P2 \text{ assms}(3) \text{ assms}(4) \text{ cong-symmetry } \text{lta--nlea lta-distincts} \text{ or-lta2-conga } t18-18)$ 
  hence  $\text{False}$ 
  using  $\text{assms}(5) \text{ not-and-lt}$  by  $\text{auto}$ 
}
hence  $\text{False}$ 
using  $\langle C A B \text{ Cong}A F D E \implies \text{False} \rangle$  by  $\text{auto}$ 
}
hence  $\neg C A B \text{ Le}A F D E$  by  $\text{auto}$ 
thus  $?thesis$ 
using  $\text{assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ assms}(4) \text{ cong-identity nlea--lta}$  by  $\text{blast}$ 
qed

```

```

lemma acute-trivial:
assumes  $A \neq B$ 
shows  $\text{Acute } A B A$ 
by  $(\text{metis } \text{acute-distincts } \text{angle-partition } \text{assms } l11-43)$ 

```

```

lemma acute-not-per:
assumes  $\text{Acute } A B C$ 
shows  $\neg \text{Per } A B C$ 
proof –
obtain  $A' B' C'$  where  $\text{Per } A' B' C' \wedge A B C \text{ Lt}A A' B' C'$ 
using  $\text{Acute-def } \text{assms}$  by  $\text{auto}$ 
thus  $?thesis$ 
using  $\text{acute-distincts } \text{acute-per--lta } \text{assms } \text{nlt}A$  by  $\text{fastforce}$ 
qed

```

```

lemma angle-bisector:
assumes  $A \neq B$  and
 $C \neq B$ 
shows  $\exists P. (P \text{ InAngle } A B C \wedge P B A \text{ Cong}A P B C)$ 
proof cases
assume  $P1: \text{Col } A B C$ 
thus  $?thesis$ 
proof cases
assume  $P2: \text{Bet } A B C$ 
then obtain  $Q$  where  $P3: \neg \text{Col } A B Q$ 
using  $\text{assms}(1) \text{ not-col-exists}$  by  $\text{auto}$ 
then obtain  $P$  where  $P4: A B \text{ Perp } P B \wedge A B \text{ OS } Q P$ 
using  $P1 \text{ l10-15 os-distincts}$  by  $\text{blast}$ 
hence  $P5: P \text{ InAngle } A B C$ 
by  $(\text{metis } P2 \text{ assms}(2) \text{ in-angle-line os-distincts})$ 
have  $P B A \text{ Cong}A P B C$ 
proof –
have  $P9: P \neq B$ 
using  $P4 \text{ os-distincts}$  by  $\text{blast}$ 
have  $\text{Per } P B A$ 
by  $(\text{simp add: } P4 \text{ Perp-perm perp-per-2})$ 
thus  $?thesis$ 
using  $P2 \text{ assms}(1) \text{ assms}(2) P9 \text{ l11-18-1}$  by  $\text{auto}$ 
qed
thus  $?thesis$ 
using  $P5$  by  $\text{auto}$ 

```

```

next
  assume T1:  $\neg \text{Bet } A \ B \ C$ 
  hence T2:  $B \text{ Out } A \ C$ 
    by (simp add: P1 l6-4-2)
  have T3:  $C \text{ InAngle } A \ B \ C$ 
    by (simp add: assms(1) assms(2) inangle3123)
  have C B A CongA C B C
    using T2 between-trivial2 l6-6 out2--conga out2-bet-out by blast
  thus ?thesis
    using T3 by auto
qed
next
  assume T4:  $\neg \text{Col } A \ B \ C$ 
  obtain C0 where T5:  $B \text{ Out } C0 \ C \wedge \text{Cong } B \ C0 \ B \ A$ 
    using assms(1) assms(2) l6-11-existence by fastforce
  obtain P where T6:  $P \text{ Midpoint } A \ C0$ 
    using midpoint-existence by auto
  have T6A:  $\neg \text{Col } A \ B \ C0$ 
    by (metis T4 T5 col3 l6-3-1 not-col-distincts out-col)
  have T6B:  $P \neq B$ 
    using Col-def Midpoint-def T6 T6A by auto
  have T6D:  $P \neq A$ 
    using T6 T6A is-midpoint-id not-col-distincts by blast
  have P InAngle A B C0
    using InAngle-def T5 T6 T6B assms(1) l6-3-1 midpoint-bet out-trivial by fastforce
  hence T7:  $P \text{ InAngle } A \ B \ C$ 
    using T5 T6B in-angle-trans2 l11-24 out341--inangle by blast
  have T8:  $(P = B) \vee B \text{ Out } P \ P$ 
    using out-trivial by auto
  have T9:  $B \text{ Out } A \ A$ 
    by (simp add: assms(1) out-trivial)
  {
    assume T9A:  $B \text{ Out } P \ P$ 
    have P B A CongA P B C0  $\wedge B \ P \ A \text{ CongA } B \ P \ C0 \wedge P \ A \ B \text{ CongA } P \ C0 \ B$ 
    proof -
      have T9B:  $\text{Cong } B \ P \ B \ P$ 
        by (simp add: cong-reflexivity)
      have T9C:  $\text{Cong } B \ A \ B \ C0$ 
        using Cong-perm T5 by blast
      have Cong P A P C0
        using Midpoint-def T6 not-cong-2134 by blast
      thus ?thesis using l11-51 T6B assms(1) T9B T9C T6D by presburger
    qed
    hence P B A CongA P B C0 by auto
    hence P B A CongA P B C using l11-10 T9A T9
      by (meson T5 l6-6)
    hence  $\exists P. (P \text{ InAngle } A \ B \ C \wedge P \ B \ A \text{ CongA } P \ B \ C)$ 
      using T7 by auto
  }
  thus ?thesis
    using T6B T8 by blast
qed

lemma reflectl--conga:
  assumes A  $\neq$  B and
    B  $\neq$  P and
    P P' ReflectL A B
  shows A B P CongA A B P'
proof -
  obtain A' where P1:  $A' \text{ Midpoint } P' \ P \wedge \text{Col } A \ B \ A' \wedge (A \ B \text{ Perp } P' \ P \vee P = P')$ 
    using ReflectL-def assms(3) by auto
  {
    assume P2:  $A \ B \text{ Perp } P' \ P$ 
    hence P3:  $P \neq P'$ 
      using perp-not-eq-2 by blast
    hence P4:  $A' \neq P'$ 

```

```

    using P1 is-midpoint-id by blast
have P5:  $A' \neq P$ 
    using P1 P3 is-midpoint-id-2 by auto
have A B P CongA A B P'
proof cases
  assume P6:  $A' = B$ 
  hence P8:  $B \neq P'$ 
    using P4 by auto
  have P9: Per A B P
    using P1 P3 P6 Perp-cases col-transitivity-2 midpoint-col midpoint-distinct-1
      not-col-permutation-2 perp-col2-bis perp-per-2 by metis
  have Per A B P'
    by (metis P1 P6 P9 midpoint-col assms(2) not-col-permutation-5 per-col)
  thus ?thesis
    using l11-16 P4 P5 P6 P9 assms(1) by auto
next
assume T1:  $A' \neq B$ 
have T2: B A' P CongA B A' P'
proof -
  have T2A: Cong B P B P'
    using assms(3) col-trivial-2 is-image-spec-col-cong l10-4-spec not-cong-4321
    by blast
  hence T2B: A' B P CongA A' B P'
    by (metis Cong-perm Midpoint-def P1 P5 T1 l11-51 assms(2) cong-reflexivity)
  have A' P B CongA A' P' B
    by (simp add: P5 T2A T2B cong-reflexivity conga-comm l11-49)
  thus ?thesis
    using P5 T2A T2B cong-reflexivity l11-49 by blast
qed
have T3: Cong A' B A' B
  by (simp add: cong-reflexivity)
have Cong A' P A' P'
  using Midpoint-def P1 not-cong-4312 by blast
hence T4: A' B P CongA A' B P'  $\wedge$  A' P B CongA A' P' B using l11-49
  using assms(2) T2 T3 by blast
show ?thesis
proof cases
  assume Bet A' B A
  thus ?thesis
    using T4 assms(1) l11-13 by blast
next
assume  $\neg$  Bet A' B A
hence T5: B Out A' A
  using P1 not-col-permutation-3 or-bet-out by blast
have T6:  $B \neq P'$ 
  using T4 conga-distinct by blast
have T8: B Out A A'
  by (simp add: T5 l6-6)
have T9: B Out P P
  using assms(2) out-trivial by auto
have B Out P' P'
  using T6 out-trivial by auto
thus ?thesis
  using l11-10 T4 T8 T9 by blast
qed
qed
}
{
  assume P = P'
  hence A B P CongA A B P'
    using assms(1) assms(2) conga-refl by auto
}
}
thus ?thesis
  using P1  $\langle A B \text{ Perp } P' P \implies A B P \text{ CongA } A B P' \rangle$  by blast
qed

```

```

lemma conga-cop-out-reflectl--out:
  assumes  $\neg B$  Out A C and
    Coplanar A B C P and
    P B A CongA P B C and
    B Out A T and
    T T' ReflectL B P
  shows B Out C T'
proof -
  have P1: P B T CongA P B T'
    by (metis assms(3) assms(4) assms(5) conga-distinct is-image-spec-rev
        out-distinct reflectl--conga)
  have P2: T T' Reflect B P
    by (metis P1 assms(5) conga-distinct is-image-is-image-spec)
  have P3: B  $\neq$  T'
    using CongA-def P1 by blast
  have P4: P B C CongA P B T'
  proof -
    have P5: P B C CongA P B A
      by (simp add: assms(3) conga-sym)
    have P B A CongA P B T'
  proof -
    have P7: B Out P P
      using assms(3) conga-diff45 out-trivial by blast
    have P8: B Out A T
      by (simp add: assms(4))
    have B Out T' T'
      using P3 out-trivial by auto
    thus ?thesis
      using P1 P7 P8 l11-10 by blast
  qed
  thus ?thesis
    using P5 not-conga by blast
qed
have P B OS C T'
proof -
  have P9: P B TS A C
    using assms(1) assms(2) assms(3) conga-cop--or-out-ts coplanar-perm-20 by blast
  hence T  $\neq$  T'
    by (metis Col-perm P2 P3 TS-def assms(4) col-transitivity-2 l10-8 out-col)
  hence P B TS T T'
    by (metis P2 P4 conga-diff45 invert-two-sides l10-14)
  hence P B TS A T'
    using assms(4) col-trivial-2 out-two-sides-two-sides by blast
  thus ?thesis
    using OS-def P9 l9-2 by blast
qed
thus ?thesis
  using P4 conga-os--out by auto
qed

lemma col-conga-cop-reflectl--col:
  assumes  $\neg B$  Out A C and
    Coplanar A B C P and
    P B A CongA P B C and
    Col B A T and
    T T' ReflectL B P
  shows Col B C T'
proof cases
  assume B = T
  thus ?thesis
    using assms(5) col-image-spec--eq not-col-distincts by blast
next
  assume P1: B  $\neq$  T
  thus ?thesis
  proof cases
    assume B Out A T

```

```

thus ?thesis
using out-col conga-cop-out-reflectl--out assms(1) assms(2) assms(3) assms(5) by blast
next
assume P2:  $\neg B \text{ Out } A \ T$ 
obtain A' where P3:  $\text{Bet } A \ B \ A' \wedge \text{Cong } B \ A' \ A \ B$ 
using segment-construction by blast
obtain C' where P4:  $\text{Bet } C \ B \ C' \wedge \text{Cong } B \ C' \ C \ B$ 
using segment-construction by blast
have P5:  $B \text{ Out } C' \ T'$ 
proof –
have P6:  $\neg B \text{ Out } A' \ C'$ 
by (metis P3 P4 assms(1) between-symmetry cong-diff-2 l6-2 out-diff1 out-diff2)
have P7: Coplanar A' B C' P
proof cases
assume Col A B C
thus ?thesis
using P3 P4 assms(1) assms(2) assms(3) bet-col bet-neq32--neq
col2-cop--cop col-transitivity-1 colx conga-diff2 conga-diff56 l6-4-2
ncoplanar-perm-15 not-col-permutation-5 by meson
next
assume P7B:  $\neg \text{Col } A \ B \ C$ 
have P7C: Coplanar A B C A'
using P3 bet-col ncop--ncols by blast
have P7D: Coplanar A B C B
using ncop-distincts by blast
have Coplanar A B C C'
using P4 bet--coplanar coplanar-perm-20 by blast
thus ?thesis
using P7B P7C P7D assms(2) coplanar-pseudo-trans by blast
qed
have P8:  $P \ B \ A' \text{ CongA } P \ B \ C'$ 
by (metis CongA-def P3 P4 assms(3) cong-reverse-identity conga-left-comm
l11-13 not-conga-sym)
have Bet A B T
using Col-cases P2 assms(4) not-out-bet by blast
hence P9:  $B \text{ Out } A' \ T$ 
by (metis P1 P3 assms(3) between-symmetry cong-reverse-identity conga-distinct l6-3-2)
thus ?thesis
using P6 P7 P8 P9 assms(5) conga-cop-out-reflectl--out by blast
qed
thus ?thesis
by (metis Col-def P4 col-transitivity-1 out-col out-diff1)
qed
qed

lemma conga2-cop2--col:
assumes  $\neg B \text{ Out } A \ C$  and
 $P \ B \ A \text{ CongA } P \ B \ C$  and
 $P' \ B \ A \text{ CongA } P' \ B \ C$  and
Coplanar A B P P' and
Coplanar B C P P'
shows Col B P P'
proof –
obtain C' where P1:  $B \text{ Out } C' \ C \wedge \text{Cong } B \ C' \ B \ A$ 
by (metis assms(2) conga-distinct l6-11-existence)
have P1A:  $\text{Cong } P \ A \ P \ C' \wedge (P \neq A \longrightarrow (B \ P \ A \text{ CongA } B \ P \ C' \wedge B \ A \ P \text{ CongA } B \ C' \ P))$ 
proof –
have P2:  $P \ B \ A \text{ CongA } P \ B \ C'$ 
proof –
have P2A:  $B \text{ Out } P \ P$ 
using assms(2) conga-diff45 out-trivial by auto
have B Out A A
using assms(2) conga-distinct out-trivial by auto
thus ?thesis
using P1 P2A assms(2) l11-10 by blast
qed

```

```

have P3: Cong B P B P
  by (simp add: cong-reflexivity)
have Cong B A B C'
  using Cong-perm P1 by blast
thus ?thesis using l11-49 P2 cong-reflexivity by blast
qed
have P4: P' B A CongA P' B C'
proof -
  have P4A: B Out P' P'
    using assms(3) conga-diff1 out-trivial by auto
  have B Out A A
    using assms(2) conga-distinct out-trivial by auto
  thus ?thesis
    using P1 P4A assms(3) l11-10 by blast
qed
have P5: Cong B P' B P'
  by (simp add: cong-reflexivity)
have P5A: Cong B A B C'
  using Cong-perm P1 by blast
hence P6: P' ≠ A → (B P' A CongA B P' C' ∧ B A P' CongA B C' P')
  using P4 P5 l11-49 by blast
have P7: Coplanar B P P' A
  using assms(4) ncoplanar-perm-18 by blast
have P8: Coplanar B P P' C'
proof -
  have Coplanar P P' B C
    using assms(5) ncoplanar-perm-16 by blast
  moreover have B ≠ C
    using P1 out-diff2 by blast
  moreover have Col B C C'
    using P1 not-col-permutation-5 out-col by blast
  ultimately show ?thesis
    using col-cop--cop coplanar-perm-12 by blast
qed
have A ≠ C'
  using P1 assms(1) by auto
thus ?thesis
  using P4 P5 P7 P8 P5A P1A cong3-cop2--col l11-49 by blast
qed

lemma conga2-cop2--col-1:
  assumes ¬ Col A B C and
    P B A CongA P B C and
    P' B A CongA P' B C and
    Coplanar A B C P and
    Coplanar A B C P'
  shows Col B P P'
proof -
  have P1: ¬ B Out A C
    using Col-cases assms(1) out-col by blast
  have P2: Coplanar A B P P'
    by (meson assms(1) assms(4) assms(5) coplanar-perm-12 coplanar-trans-1
      not-col-permutation-2)
  have Coplanar B C P P'
    using assms(1) assms(4) assms(5) coplanar-trans-1 by auto
  thus ?thesis using P1 P2 conga2-cop2--col assms(2) assms(3) conga2-cop2--col by auto
qed

lemma col-conga--conga:
  assumes P B A CongA P B C and
    Col B P P' and
    B ≠ P'
  shows P' B A CongA P' B C
proof cases
  assume Bet P B P'
  thus ?thesis

```

```

    using assms(1) assms(3) l11-13 by blast
next
  assume  $\neg \text{Bet } P \ B \ P'$ 
  hence  $P1: B \text{ Out } P \ P'$ 
    using Col-cases assms(2) or-bet-out by blast
  hence  $P2: B \text{ Out } P' \ P$ 
    by (simp add: l6-6)
  have  $P3: B \text{ Out } A \ A$ 
    using CongA-def assms(1) out-trivial by auto
  have  $B \text{ Out } C \ C$ 
    using assms(1) conga-diff56 out-trivial by blast
  thus ?thesis
    using P2 P3 assms(1) l11-10 by blast
qed

lemma cop-inangle--ex-col-inangle:
  assumes  $\neg B \text{ Out } A \ C$  and
     $P \text{ InAngle } A \ B \ C$  and
     $\text{Coplanar } A \ B \ C \ Q$ 
  shows  $\exists R. (R \text{ InAngle } A \ B \ C \wedge P \neq R \wedge \text{Col } P \ Q \ R)$ 
proof -
  have  $P1: A \neq B$ 
    using assms(2) inangle-distincts by blast
  hence  $P4: A \neq C$ 
    using assms(1) out-trivial by blast
  have  $P2: C \neq B$ 
    using assms(2) inangle-distincts by auto
  have  $P3: P \neq B$ 
    using InAngle-def assms(2) by auto
  thus ?thesis
proof cases
  assume  $P = Q$ 
  thus ?thesis
    using P1 P2 P4 col-trivial-1 inangle1123 inangle3123 by blast
next
  assume  $P5: P \neq Q$ 
  thus ?thesis
proof cases
  assume  $P6: \text{Col } B \ P \ Q$ 
  obtain  $R$  where  $P7: \text{Bet } B \ P \ R \wedge \text{Cong } P \ R \ B \ P$ 
    using segment-construction by blast
  have  $P8: R \text{ InAngle } A \ B \ C$ 
    using Out-cases P1 P2 P3 P7 assms(2) bet-out l11-25 out-trivial by blast
  have  $P \neq R$ 
    using P3 P7 cong-reverse-identity by blast
  thus ?thesis
    by (metis P3 P6 P7 P8 bet-col col-transitivity-2)
next
  assume  $T1: \neg \text{Col } B \ P \ Q$ 
  thus ?thesis
proof cases
  assume  $T2: \text{Col } A \ B \ C$ 
  have  $T3: Q \text{ InAngle } A \ B \ C$ 
    by (metis P1 P2 T1 T2 assms(1) in-angle-line l6-4-2 not-col-distincts)
  thus ?thesis
    using P5 col-trivial-2 by blast
next
  assume  $Q1: \neg \text{Col } A \ B \ C$ 
  thus ?thesis
proof cases
  assume  $Q2: \text{Col } B \ C \ P$ 
  have  $Q3: \neg \text{Col } B \ A \ P$ 
    using Col-perm P3 Q1 Q2 col-transitivity-2 by blast
  have  $Q4: \text{Coplanar } B \ P \ Q \ A$ 
    using P2 Q2 assms(3) col2-cop--cop col-trivial-3 ncoplanar-perm-22
      ncoplanar-perm-3 by blast

```



```

have Q5:  $Q \neq P$ 
  using P5 by auto
have Q6:  $Col\ B\ P\ P$ 
  using not-col-distincts by blast
have Q7:  $Col\ Q\ P\ P$ 
  using not-col-distincts by auto
have  $\neg Col\ B\ P\ A$ 
  using Col-cases Q3 by auto
then obtain Q0 where P10:  $Col\ Q\ P\ Q0 \wedge B\ P\ OS\ A\ Q0$ 
  using cop-not-par-same-side Q4 Q5 Q6 Q7 T1 by blast
have P13:  $P \neq Q0$ 
  using P10 os-distincts by auto
{
  assume B A OS P Q0
  hence ?thesis
    using P10 P13 assms(2) in-angle-trans not-col-permutation-4 os2--inangle by blast
}
{
  assume V1:  $\neg B\ A\ OS\ P\ Q0$ 
  have  $\exists R. Bet\ P\ R\ Q0 \wedge Col\ P\ Q\ R \wedge Col\ B\ A\ R$ 
  proof cases
    assume V3:  $Col\ B\ A\ Q0$ 
    have  $Col\ P\ Q\ Q0$ 
      using Col-cases P10 by auto
    thus ?thesis
      using V3 between-trivial by auto
  next
    assume V4:  $\neg Col\ B\ A\ Q0$ 
    hence V5:  $\neg Col\ Q0\ B\ A$ 
      using Col-perm by blast
    have  $\neg Col\ P\ B\ A$ 
      using Col-cases Q3 by blast
    then obtain R where V8:  $Col\ R\ B\ A \wedge Bet\ P\ R\ Q0$ 
      using cop-nos--ts V1 V5
      by (meson P10 TS-def ncoplanar-perm-2 os--coplanar)
    thus ?thesis
      by (metis Col-def P10 P13 col-transitivity-2)
  qed
  then obtain R where V9:  $Bet\ P\ R\ Q0 \wedge Col\ P\ Q\ R \wedge Col\ B\ A\ R$  by auto
  have V10:  $P \neq R$ 
    using Q3 V9 by blast
  have R InAngle A B C
  proof -
    have W1:  $\neg Col\ B\ P\ Q0$ 
      using P10 P13 T1 col2--eq by blast
    have P Out Q0 R
      using V10 V9 bet-out l6-6 by auto
    hence B P OS Q0 R
      using Q6 W1 out-one-side-1 by blast
    hence B P OS A R
      using P10 one-side-transitivity by blast
    hence B Out A R
      using V9 col-one-side-out by auto
    thus ?thesis
      by (simp add: P2 out321--inangle)
  qed
  hence ?thesis
    using V10 V9 by blast
}
thus ?thesis
  using  $\langle B\ A\ OS\ P\ Q0 \implies \exists R. R\ InAngle\ A\ B\ C \wedge P \neq R \wedge Col\ P\ Q\ R \rangle$  by blast
next
  assume Z1:  $\neg Col\ B\ C\ P$ 
  hence Z6:  $\neg Col\ B\ P\ C$ 
    by (simp add: not-col-permutation-5)
  have Z3:  $Col\ B\ P\ P$ 

```

```

    by (simp add: col-trivial-2)
have Z4: Col Q P P
    by (simp add: col-trivial-2)
have Coplanar A B C P
    using Q1 assms(2) inangle--coplanar ncoplanar-perm-18 by blast
hence Coplanar B P Q C
    using Q1 assms(3) coplanar-trans-1 ncoplanar-perm-5 by blast
then obtain Q0 where Z5: Col Q P Q0  $\wedge$  B P OS C Q0
    using cop-not-par-same-side by (metis Z3 Z4 T1 Z6)
thus ?thesis
proof cases
  assume B C OS P Q0
  thus ?thesis
  proof -
    have  $\forall p. p \text{ InAngle } C B A \vee \neg p \text{ InAngle } C B P$ 
      using assms(2) in-angle-trans l11-24 by blast
    thus ?thesis
      by (metis Col-perm Z5  $\langle B C OS P Q0 \rangle$  l11-24 os2--inangle os-distincts)
  qed
next
  assume Z6:  $\neg B C OS P Q0$ 
  have Z7:  $\exists R. \text{Bet } P R Q0 \wedge \text{Col } P Q R \wedge \text{Col } B C R$ 
  proof cases
    assume Col B C Q0
    thus ?thesis
      using Col-def Col-perm Z5 between-trivial by blast
  next
    assume Z8:  $\neg \text{Col } B C Q0$ 
    have  $\exists R. \text{Col } R B C \wedge \text{Bet } P R Q0$ 
    proof -
      have Z10: Coplanar B C P Q0
        using Z5 ncoplanar-perm-2 os--coplanar by blast
      have Z11:  $\neg \text{Col } P B C$ 
        using Col-cases Z1 by blast
      have  $\neg \text{Col } Q0 B C$ 
        using Col-perm Z8 by blast
      thus ?thesis
        using cop-nos--ts Z6 Z10 Z11 by (simp add: TS-def)
    qed
    then obtain R where Col R B C  $\wedge$  Bet P R Q0 by blast
    thus ?thesis
      using Z5 bet-col col2--eq col-permutation-1 os-distincts by metis
  qed
then obtain R where Z12: Bet P R Q0  $\wedge$  Col P Q R  $\wedge$  Col B C R by blast
have Z13:  $P \neq R$ 
  using Z1 Z12 by auto
have Z14:  $\neg \text{Col } B P Q0$ 
  using Z5 one-side-not-col124 by blast
have P Out Q0 R
  using Z12 Z13 bet-out l6-6 by auto
hence B P OS Q0 R
  using Z14 Z3 out-one-side-1 by blast
hence B P OS C R
  using Z5 one-side-transitivity by blast
hence B Out C R
  using Z12 col-one-side-out by blast
hence R InAngle A B C
  using P1 out341--inangle by auto
thus ?thesis
  using Z12 Z13 by auto
qed
qed
qed
qed
qed
qed

```

```

lemma col-inangle2--out:
  assumes  $\neg$  Bet A B C and
    P InAngle A B C and
    Q InAngle A B C and
    Col B P Q
  shows B Out P Q
proof cases
  assume Col A B C
  thus ?thesis
    by (meson assms(1) assms(2) assms(3) assms(4) bet-in-angle-bet bet-out--bet
        in-angle-out l6-6 not-col-permutation-4 or-bet-out)
next
  assume P1:  $\neg$  Col A B C
  thus ?thesis
  proof cases
    assume Col B A P
    thus ?thesis
      by (meson assms(1) assms(2) assms(3) assms(4) bet-in-angle-bet bet-out--bet
          l6-6 not-col-permutation-4 or-bet-out)
  next
    assume P2:  $\neg$  Col B A P
    have  $\neg$  Col B A Q
      using P2 assms(3) assms(4) col2--eq col-permutation-4 inangle-distincts by blast
    hence B A OS P Q
      using P1 P2 assms(2) assms(3) inangle-one-side invert-one-side
        not-col-permutation-4 by auto
    thus ?thesis
      using assms(4) col-one-side-out by auto
  qed
qed

```

lemma inangle2--lea:

```

  assumes P InAngle A B C and
    Q InAngle A B C
  shows P B Q LeA A B C
proof -
  have P1: P InAngle C B A
    by (simp add: assms(1) l11-24)
  have P2: Q InAngle C B A
    by (simp add: assms(2) l11-24)
  have P3: A  $\neq$  B
    using assms(1) inangle-distincts by auto
  have P4: C  $\neq$  B
    using assms(1) inangle-distincts by blast
  have P5: P  $\neq$  B
    using assms(1) inangle-distincts by auto
  have P6: Q  $\neq$  B
    using assms(2) inangle-distincts by auto
  thus ?thesis
  proof cases
    assume P7: Col A B C
    thus ?thesis
      proof cases
        assume Bet A B C
        thus ?thesis
          by (simp add: P3 P4 P5 P6 l11-31-2)
      next
        assume  $\neg$  Bet A B C
        hence B Out A C
          using P7 not-out-bet by blast
        hence B Out P Q
          using Out-cases assms(1) assms(2) in-angle-out l6-7 by blast
        thus ?thesis
          by (simp add: P3 P4 l11-31-1)
      qed
    qed
  qed

```

```

next
  assume T1:  $\neg \text{Col } A \ B \ C$ 
  thus ?thesis
proof cases
  assume T2:  $\text{Col } B \ P \ Q$ 
  have  $\neg \text{Bet } A \ B \ C$ 
    using T1 bet-col by auto
  hence B Out P Q
    using T2 assms(1) assms(2) col-inangle2--out by auto
  thus ?thesis
    by (simp add: P3 P4 l11-31-1)
next
  assume T3:  $\neg \text{Col } B \ P \ Q$ 
  thus ?thesis
proof cases
  assume Col B A P
  hence B Out A P
    using Col-def T1 assms(1) col-in-angle-out by blast
  hence P B Q CongA A B Q
    using P6 out2--conga out-trivial by auto
  thus ?thesis
    using LeA-def assms(2) by blast
next
  assume W0:  $\neg \text{Col } B \ A \ P$ 
  show ?thesis
proof cases
  assume Col B C P
  hence B Out C P
    by (metis P1 P3 T1 bet-out-1 col-in-angle-out out-col)
  thus ?thesis
    using P3 P4 P6 lea-left-comm lea-out4--lea assms(2)
      inangle--lea-1 out-trivial by meson
next
  assume W0A:  $\neg \text{Col } B \ C \ P$ 
  show ?thesis
proof cases
  assume Col B A Q
  hence B Out A Q
    using Col-def T1 assms(2) col-in-angle-out by blast
  thus ?thesis
    using P3 P4 P5 lea-left-comm lea-out4--lea assms(1)
      inangle--lea out-trivial by meson
next
  assume W0AA:  $\neg \text{Col } B \ A \ Q$ 
  thus ?thesis
proof cases
  assume Col B C Q
  hence B Out C Q
    using Bet-cases P2 T1 bet-col col-in-angle-out by blast
  thus ?thesis
    using P1 P3 P4 P5 lea-comm lea-out4--lea inangle--lea out-trivial by meson
next
  assume W0B:  $\neg \text{Col } B \ C \ Q$ 
  have W1: Coplanar B P A Q
    by (metis Col-perm T1 assms(1) assms(2) col--coplanar
      inangle-one-side ncoplanar-perm-13 os--coplanar)
  have W2:  $\neg \text{Col } A \ B \ P$ 
    by (simp add: W0 not-col-permutation-4)
  have W3:  $\neg \text{Col } Q \ B \ P$ 
    using Col-perm T3 by blast
  hence W4:  $B \ P \ TS \ A \ Q \vee B \ P \ OS \ A \ Q$ 
    using cop--one-or-two-sides
    by (simp add: W1 W2)
  {
    assume W4A:  $B \ P \ TS \ A \ Q$ 
    have Q InAngle P B C

```

```

proof -
  have W5:  $P B OS C Q$ 
    using  $OS-def P1 W0 W0A W4A in-angle-two-sides invert-two-sides l9-2$  by blast
  have  $C B OS P Q$ 
    by (meson  $P1 P2 T1 W0A W0B inangle-one-side not-col-permutation-3$ 
      not-col-permutation-4)
  thus ?thesis
    by (simp add:  $W5 invert-one-side os2--inangle$ )
qed
hence  $P B Q LeA A B C$ 
  by (meson  $assms(1) inangle--lea inangle--lea-1 lea-trans$ )
}
{
  assume  $W6: B P OS A Q$ 
  have  $B A OS P Q$ 
    using  $Col-perm T1 W2 W0AA assms(1) assms(2) inangle-one-side$ 
      invert-one-side by blast
  hence  $Q InAngle P B A$ 
    by (simp add:  $W6 os2--inangle$ )
  hence  $P B Q LeA A B C$ 
    by (meson  $P1 inangle--lea inangle--lea-1 lea-right-comm lea-trans$ )
}
thus ?thesis
  using  $W4 \langle B P TS A Q \implies P B Q LeA A B C \rangle$  by blast
qed
qed
qed
qed
qed
qed
qed

```

lemma *conga-inangle-per--acute*:

assumes $Per A B C$ and
 $P InAngle A B C$ and
 $P B A CongA P B C$
 shows $Acute A B P$

```

proof -
  have  $P1: \neg Col A B C$ 
    using  $assms(1) assms(3) conga-diff2 conga-diff56 l8-9$  by blast
  have  $P2: A B P LeA A B C$ 
    by (simp add:  $assms(2) inangle--lea$ )
  {
    assume  $A B P CongA A B C$ 
    hence  $P3: Per A B P$ 
      by (meson  $l11-17 not-conga-sym assms(1)$ )
    have  $P4: Coplanar P C A B$ 
      using  $assms(2) inangle--coplanar ncoplanar-perm-3$  by blast
    have  $P5: P \neq B$ 
      using  $assms(2) inangle-distincts$  by blast
    have  $Per C B P$ 
      using  $P3 Per-cases assms(3) l11-17$  by blast
    hence  $False$ 
      using  $P1 P3 P4 P5 col-permutation-1 cop-per2--col$  by blast
  }
  hence  $\neg A B P CongA A B C$  by auto
  hence  $A B P LtA A B C$ 
    by (simp add:  $LtA-def P2$ )
  thus ?thesis
    using  $Acute-def assms(1)$  by blast
qed

```

lemma *conga-inangle2-per--acute*:

assumes $Per A B C$ and
 $P InAngle A B C$ and
 $P B A CongA P B C$ and

```

  Q InAngle A B C
shows Acute P B Q
proof -
  have P1: P InAngle C B A
    using assms(2) l11-24 by auto
  have P2: Q InAngle C B A
    using assms(4) l11-24 by blast
  have P3: A ≠ B
    using assms(3) conga-diff2 by auto
  have P5: P ≠ B
    using assms(2) inangle-distincts by blast
  have P7: ¬ Col A B C
    using assms(1) assms(3) conga-distinct l8-9 by blast
  have P8: Acute A B P
    using assms(1) assms(2) assms(3) conga-inangle-per--acute by auto
  {
    assume Col P B A
    hence Col P B C
      using assms(3) col-conga-col by blast
    hence False
      using Col-perm P5 P7 ⟨Col P B A⟩ col-transitivity-2 by blast
  }
  hence P9: ¬ Col P B A by auto
  have P10: ¬ Col P B C
    using ⟨Col P B A ⟹ False⟩ assms(3) ncol-conga-ncol by blast
  have P11: ¬ Bet A B C
    using P7 bet-col by blast
  show ?thesis
  proof cases
    assume Col B A Q
    hence B Out A Q
      using P11 assms(4) col-in-angle-out by auto
    thus ?thesis
      using Out-cases P5 P8 acute-out2--acute acute-sym out-trivial by blast
  next
    assume S0: ¬ Col B A Q
    show ?thesis
    proof cases
      assume S1: Col B C Q
      hence B Out C Q
        using P11 P2 between-symmetry col-in-angle-out by blast
      hence S2: B Out Q C
        using l6-6 by blast
      have S3: B Out P P
        by (simp add: P5 out-trivial)
      have B Out A A
        by (simp add: P3 out-trivial)
      hence A B P CongA P B Q
        using S2 conga-left-comm l11-10 S3 assms(3) by blast
      thus ?thesis
        using P8 acute-conga--acute by blast
    next
      assume S4: ¬ Col B C Q
      show ?thesis
      proof cases
        assume Col B P Q
        thus ?thesis
          using out--acute col-inangle2--out P11 assms(2) assms(4) by blast
      next
        assume S5: ¬ Col B P Q
        have S6: Coplanar B P A Q
          by (metis Col-perm P7 assms(2) assms(4) coplanar-trans-1
            inangle--coplanar ncoplanar-perm-12 ncoplanar-perm-21)
        have S7: ¬ Col A B P
          using Col-cases P9 by auto
        have ¬ Col Q B P

```

```

    using Col-perm S5 by blast
  hence S8:  $B P TS A Q \vee B P OS A Q$ 
    using cop--one-or-two-sides S6 S7 by blast
{
  assume S9:  $B P TS A Q$ 
  have S10:  $Acute P B C$ 
    using P8 acute-conga--acute acute-sym assms(3) by blast
  have Q InAngle P B C
  proof -
    have S11:  $P B OS C Q$ 
      by (metis Col-perm OS-def P1 P10 P9 S9 in-angle-two-sides
        invert-two-sides l9-2)
    have C B OS P Q
      by (meson P1 P10 P2 P7 S4 inangle-one-side not-col-permutation-3
        not-col-permutation-4)
    thus ?thesis
      by (simp add: S11 invert-one-side os2--inangle)
  qed
  hence P B Q LeA P B C
    by (simp add: inangle--lea)
  hence Acute P B Q
    using S10 acute-lea-acute by blast
}
{
  assume S12:  $B P OS A Q$ 
  have B A OS P Q
    using Col-perm P7 S7 S0 assms(2) assms(4) inangle-one-side
      invert-one-side by blast
  hence Q InAngle P B A
    by (simp add: S12 os2--inangle)
  hence Q B P LeA P B A
    by (simp add: P3 P5 inangle1123 inangle2--lea)
  hence P B Q LeA A B P
    by (simp add: lea-comm)
  hence Acute P B Q
    using P8 acute-lea-acute by blast
}
thus ?thesis
  using  $\langle B P TS A Q \implies Acute P B Q \rangle$  S8 by blast
qed
qed
qed
qed

lemma lta-os--ts:
  assumes
    A O1 P LtA A O1 B and
    O1 A OS B P
  shows O1 P TS A B
proof -
  have A O1 P LeA A O1 B
    by (simp add: assms(1) lta--lea)
  hence  $\exists P0. P0 InAngle A O1 B \wedge A O1 P CongA A O1 P0$ 
    by (simp add: LeA-def)
  then obtain P' where P1:  $P' InAngle A O1 B \wedge A O1 P CongA A O1 P'$  by blast
  have P2:  $\neg Col A O1 B$ 
    using assms(2) col123--nos not-col-permutation-4 by blast
  obtain R where P3:  $O1 A TS B R \wedge O1 A TS P R$ 
    using OS-def assms(2) by blast
{
  assume Col B O1 P
  hence Bet B O1 P
    by (metis out2--conga assms(1) assms(2) between-trivial col-trivial-2
      lta-not-conga one-side-chara or-bet-out out-trivial)
  hence O1 A TS B P
    using assms(2) col-trivial-1 one-side-chara by blast
}

```

```

  hence P6:  $\neg O1 A OS B P$ 
    using l9-9-bis by auto
  hence False
    using P6 assms(2) by auto
}
hence P4:  $\neg Col B O1 P$  by auto
thus ?thesis
  by (meson P3 assms(1) inangle--lta l9-8-1 not-and-lta not-col-permutation-4
      os-ts--inangle two-sides-cases)
qed

lemma bet--suppa:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $B \neq A'$  and
     $Bet A B A'$ 
  shows  $A B C SuppA C B A'$ 
proof -
  have  $C B A' CongA C B A'$ 
    using assms(2) assms(3) conga-refl by auto
  thus ?thesis using assms(4) assms(1) SuppA-def by auto
qed

lemma ex-suppa:
  assumes  $A \neq B$  and
     $B \neq C$ 
  shows  $\exists D E F. A B C SuppA D E F$ 
proof -
  obtain  $A'$  where  $Bet A B A' \wedge Cong B A' A B$ 
    using segment-construction by blast
  thus ?thesis
    by (meson assms(1) assms(2) bet--suppa point-construction-different)
qed

lemma suppa-distincts:
  assumes  $A B C SuppA D E F$ 
  shows  $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$ 
  using CongA-def SuppA-def assms by auto

lemma suppa-right-comm:
  assumes  $A B C SuppA D E F$ 
  shows  $A B C SuppA F E D$ 
  using SuppA-def assms conga-left-comm by auto

lemma suppa-left-comm:
  assumes  $A B C SuppA D E F$ 
  shows  $C B A SuppA D E F$ 
proof -
  obtain  $A'$  where  $P1: Bet A B A' \wedge D E F CongA C B A'$ 
    using SuppA-def assms by auto
  obtain  $C'$  where  $P2: Bet C B C' \wedge Cong B C' C B$ 
    using segment-construction by blast
  hence  $C B A' CongA A B C'$ 
    by (metis Bet-cases P1 SuppA-def assms cong-diff-3 conga-diff45 conga-diff56
        conga-left-comm l11-14)
  hence  $D E F CongA A B C'$ 
    using P1 conga-trans by blast
  thus ?thesis
    by (metis CongA-def P1 P2 SuppA-def)
qed

lemma suppa-comm:
  assumes  $A B C SuppA D E F$ 
  shows  $C B A SuppA F E D$ 
  using assms suppa-left-comm suppa-right-comm by blast

```


lemma *suppa-sym*:
assumes $A\ B\ C\ SuppA\ D\ E\ F$
shows $D\ E\ F\ SuppA\ A\ B\ C$
proof –
obtain A' **where** $P1: Bet\ A\ B\ A' \wedge D\ E\ F\ CongA\ C\ B\ A'$
using *SuppA-def assms* **by** *auto*
obtain D' **where** $P2: Bet\ D\ E\ D' \wedge Cong\ E\ D'\ D\ E$
using *segment-construction* **by** *blast*
have $A'\ B\ C\ CongA\ D\ E\ F$
using $P1\ conga-right-comm\ not-conga-sym$ **by** *blast*
hence $A\ B\ C\ CongA\ F\ E\ D'$
by (*metis* $P1\ P2\ conga-right-comm\ l11-13\ suppa-distincts\ assms\ between-symmetry\ cong-diff-3$)
thus *?thesis*
by (*metis* $CongA-def\ P1\ P2\ SuppA-def$)
qed

lemma *conga2-suppa--suppa*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
 $D\ E\ F\ CongA\ D'\ E'\ F'$ **and**
 $A\ B\ C\ SuppA\ D\ E\ F$
shows $A'\ B'\ C'\ SuppA\ D'\ E'\ F'$
proof –
obtain $A0$ **where** $P1: Bet\ A\ B\ A0 \wedge D\ E\ F\ CongA\ C\ B\ A0$
using *SuppA-def assms(3)* **by** *auto*
hence $A\ B\ C\ SuppA\ D'\ E'\ F'$
by (*metis* $SuppA-def\ assms(2)\ assms(3)\ conga-sym\ conga-trans$)
hence $P2: D'\ E'\ F'\ SuppA\ A\ B\ C$
by (*simp add: suppa-sym*)
then obtain $D0$ **where** $P3: Bet\ D'\ E'\ D0 \wedge A\ B\ C\ CongA\ F'\ E'\ D0$
using $P2\ SuppA-def$ **by** *auto*
have $P5: A'\ B'\ C'\ CongA\ F'\ E'\ D0$
using $P3\ assms(1)\ not-conga\ not-conga-sym$ **by** *blast*
hence $D'\ E'\ F'\ SuppA\ A'\ B'\ C'$
using $P2\ P3\ SuppA-def$ **by** *auto*
thus *?thesis*
by (*simp add: suppa-sym*)
qed

lemma *suppa2--conga456*:
assumes $A\ B\ C\ SuppA\ D\ E\ F$ **and**
 $A\ B\ C\ SuppA\ D'\ E'\ F'$
shows $D\ E\ F\ CongA\ D'\ E'\ F'$
proof –
obtain A' **where** $P1: Bet\ A\ B\ A' \wedge D\ E\ F\ CongA\ C\ B\ A'$
using *SuppA-def assms(1)* **by** *auto*
obtain A'' **where** $P2: Bet\ A\ B\ A'' \wedge D'\ E'\ F'\ CongA\ C\ B\ A''$
using *SuppA-def assms(2)* **by** *auto*
have $C\ B\ A'\ CongA\ C\ B\ A''$
proof –
have $P3: B\ Out\ C\ C$ **using** $P1$
by (*simp add: CongA-def out-trivial*)
have $B\ Out\ A''\ A'$ **using** $P1\ P2\ l6-2$
by (*metis* $assms(1)\ between-symmetry\ conga-distinct\ suppa-distincts$)
thus *?thesis*
by (*simp add: P3 out2--conga*)
qed
hence $C\ B\ A'\ CongA\ D'\ E'\ F'$
using $P2\ not-conga\ not-conga-sym$ **by** *blast*
thus *?thesis*
using $P1\ not-conga$ **by** *blast*
qed

lemma *suppa2--conga123*:
assumes $A\ B\ C\ SuppA\ D\ E\ F$ **and**
 $A'\ B'\ C'\ SuppA\ D\ E\ F$

```

shows A B C CongA A' B' C'
using assms(1) assms(2) suppa2--conga456 suppa-sym by blast

lemma bet-out--suppa:
  assumes A ≠ B and
    B ≠ C and
    Bet A B C and
    E Out D F
  shows A B C SuppA D E F
proof -
  have D E F CongA C B C
    using assms(2) assms(4) l11-21-b out-trivial by auto
  thus ?thesis
    using SuppA-def assms(1) assms(3) by blast
qed

lemma bet-suppa--out:
  assumes Bet A B C and
    A B C SuppA D E F
  shows E Out D F
proof -
  have A B C SuppA C B C
    using assms(1) assms(2) bet--suppa suppa-distincts by auto
  hence C B C CongA D E F
    using assms(2) suppa2--conga456 by auto
  thus ?thesis
    using eq-conga-out by auto
qed

lemma out-suppa--bet:
  assumes B Out A C and
    A B C SuppA D E F
  shows Bet D E F
proof -
  obtain B' where P1: Bet A B B' ∧ Cong B B' A B
    using segment-construction by blast
  have A B C SuppA A B B'
    by (metis P1 assms(1) assms(2) bet--suppa bet-cong-eq bet-out--bet
      suppa-distincts suppa-left-comm)
  hence A B B' CongA D E F
    using assms(2) suppa2--conga456 by auto
  thus ?thesis
    using P1 bet-conga--bet by blast
qed

lemma per-suppa--per:
  assumes Per A B C and
    A B C SuppA D E F
  shows Per D E F
proof -
  obtain A' where P1: Bet A B A' ∧ D E F CongA C B A'
    using SuppA-def assms(2) by auto
  have Per C B A'
  proof -
    have P2: A ≠ B
      using assms(2) suppa-distincts by auto
    have P3: Per C B A
      by (simp add: assms(1) l8-2)
    have Col B A A'
      using P1 Col-cases Col-def by blast
    thus ?thesis
      by (metis P2 P3 per-col)
  qed
  thus ?thesis
    using P1 l11-17 not-conga-sym by blast
qed

```

```

lemma per2--suppa:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $D \neq E$  and
     $E \neq F$  and
    Per  $A B C$  and
    Per  $D E F$ 
  shows  $A B C \text{ SuppA } D E F$ 
proof -
  obtain  $D' E' F'$  where  $P1: A B C \text{ SuppA } D' E' F'$ 
    using assms(1) assms(2) ex-suppa by blast
  have  $D' E' F' \text{ CongA } D E F$ 
    using  $P1$  assms(3) assms(4) assms(5) assms(6) l11-16 per-suppa--per
      suppa-distincts by blast
  thus ?thesis
    by (meson  $P1$  conga2-suppa--suppa suppa2--conga123)
qed

```

```

lemma suppa--per:
  assumes  $A B C \text{ SuppA } A B C$ 
  shows Per  $A B C$ 
proof -
  obtain  $A'$  where  $P1: \text{Bet } A B A' \wedge A B C \text{ CongA } C B A'$ 
    using SuppA-def assms by auto
  hence  $C B A \text{ CongA } C B A'$ 
    by (simp add: conga-left-comm)
  thus ?thesis
    using  $P1$  Per-perm l11-18-2 by blast
qed

```

```

lemma acute-suppa--obtuse:
  assumes Acute  $A B C$  and
     $A B C \text{ SuppA } D E F$ 
  shows Obtuse  $D E F$ 
proof -
  obtain  $A'$  where  $P1: \text{Bet } A B A' \wedge D E F \text{ CongA } C B A'$ 
    using SuppA-def assms(2) by auto
  hence Obtuse  $C B A'$ 
    by (metis obtuse-sym acute-bet--obtuse assms(1) conga-distinct)
  thus ?thesis
    by (meson  $P1$  conga-obtuse--obtuse not-conga-sym)
qed

```

```

lemma obtuse-suppa--acute:
  assumes Obtuse  $A B C$  and
     $A B C \text{ SuppA } D E F$ 
  shows Acute  $D E F$ 
proof -
  obtain  $A'$  where  $P1: \text{Bet } A B A' \wedge D E F \text{ CongA } C B A'$ 
    using SuppA-def assms(2) by auto
  hence Acute  $C B A'$ 
    using acute-sym assms(1) bet-obtuse--acute conga-distinct by blast
  thus ?thesis
    using  $P1$  acute-conga--acute not-conga-sym by blast
qed

```

```

lemma lea-suppa2--lea:
  assumes  $A B C \text{ SuppA } A' B' C'$  and
     $D E F \text{ SuppA } D' E' F'$ 
     $A B C \text{ LeA } D E F$ 
  shows  $D' E' F' \text{ LeA } A' B' C'$ 
proof -
  obtain  $A0$  where  $P1: \text{Bet } A B A0 \wedge A' B' C' \text{ CongA } C B A0$ 
    using SuppA-def assms(1) by auto
  obtain  $D0$  where  $P2: \text{Bet } D E D0 \wedge D' E' F' \text{ CongA } F E D0$ 

```

```

    using SuppA-def assms(2) by auto
have F E D0 LeA C B A0
proof -
  have P3: D0 ≠ E
    using CongA-def P2 by auto
  have P4: A0 ≠ B
    using CongA-def P1 by blast
  have P6: Bet D0 E D
    by (simp add: P2 between-symmetry)
  have Bet A0 B A
    by (simp add: P1 between-symmetry)
  thus ?thesis
    by (metis P3 P4 P6 assms(3) l11-36-aux2 lea-comm lea-distincts)
qed
thus ?thesis
  by (meson P1 P2 l11-30 not-conga-sym)
qed

```

```

lemma lta-suppa2--lta:
  assumes A B C SuppA A' B' C'
    and D E F SuppA D' E' F'
    and A B C LtA D E F
  shows D' E' F' LtA A' B' C'
proof -
  obtain A0 where P1: Bet A B A0 ∧ A' B' C' CongA C B A0
    using SuppA-def assms(1) by auto
  obtain D0 where P2: Bet D E D0 ∧ D' E' F' CongA F E D0
    using SuppA-def assms(2) by auto
  have F E D0 LtA C B A0
  proof -
    have P5: A0 ≠ B
      using CongA-def P1 by blast
    have D0 ≠ E
      using CongA-def P2 by auto
    thus ?thesis
      using assms(3) P1 P5 P2 bet2-lta--lta lta-comm by blast
  qed
  thus ?thesis
    using P1 P2 conga-preserves-lta not-conga-sym by blast
qed

```

```

lemma suppa-dec:
  A B C SuppA D E F ∨ ¬ A B C SuppA D E F
by simp

```

```

lemma acute-one-side-aux:
  assumes C A OS P B and
    Acute A C P and
    C A Perp B C
  shows C B OS A P
proof -
  obtain R where T1: C A TS P R ∧ C A TS B R
    using OS-def assms(1) by blast
  obtain A' B' C' where P1: Per A' B' C' ∧ A C P LtA A' B' C'
    using Acute-def assms(2) by auto
  have P2: Per A C B
    by (simp add: assms(3) perp-per-1)
  hence P3: A' B' C' CongA A C B
    using P1 assms(1) l11-16 lta-distincts os-distincts by blast
  have P4: A C P LtA A C B
    by (metis P2 acute-per--lta assms(1) assms(2) os-distincts)
  {
    assume P4A: Col P C B
    have Per A C P
  proof -
    have P4B: C ≠ B

```

```

    using assms(1) os-distincts by blast
  have P4C: Per A C B
    by (simp add: P2)
  have Col C B P
    using P4A Col-cases by auto
  thus ?thesis using per-col P4B P4C by blast
qed
hence False
  using acute-not-per assms(2) by auto
}
hence P5:  $\neg \text{Col } P \ C \ B$  by auto
have P6:  $\neg \text{Col } A \ C \ P$ 
  using assms(1) col123--nos not-col-permutation-4 by blast
have P7:  $C \ B \ TS \ A \ P \vee C \ B \ OS \ A \ P$ 
  using P5 assms(1) not-col-permutation-4 os-ts1324--os two-sides-cases by blast
{
  assume P8:  $C \ B \ TS \ A \ P$ 
  then obtain T where P9:  $\text{Col } T \ C \ B \wedge \text{Bet } A \ T \ P$ 
    using TS-def by blast
  hence P10:  $C \neq T$ 
    using Col-def P6 P9 by auto
  have T InAngle A C P
    by (meson P4 P5 P8 inangle--lta assms(1) not-and-lta
      not-col-permutation-3 os-ts--inangle)
  hence  $C \ A \ OS \ T \ P$ 
    by (metis P10 P9 T1 TS-def col123--nos in-angle-one-side invert-one-side
      l6-16-1 one-side-reflexivity)
  hence P13:  $C \ A \ OS \ T \ B$ 
    using assms(1) one-side-transitivity by blast
  have  $C \ B \ OS \ A \ P$ 
    by (meson P4 lta-os--ts assms(1) one-side-symmetry os-ts1324--os)
}
thus ?thesis
  using P7 by blast
qed

lemma acute-one-side-aux0:
  assumes Col A C P and
    Acute A C P and
    C A Perp B C
  shows  $C \ B \ OS \ A \ P$ 
proof -
  have Per A C B
    by (simp add: assms(3) perp-per-1)
  hence P1:  $A \ C \ P \ LtA \ A \ C \ B$ 
    using acute-per--lta acute-distincts assms(2) assms(3) perp-not-eq-2 by fastforce
  have P2:  $C \ Out \ A \ P$ 
    using acute-col--out assms(1) assms(2) by auto
  thus ?thesis
    using Perp-cases assms(3) out-one-side perp-not-col by blast
qed

lemma acute-cop-perp--one-side:
  assumes Acute A C P and
    C A Perp B C and
    Coplanar A B C P
  shows  $C \ B \ OS \ A \ P$ 
proof cases
  assume Col A C P
  thus ?thesis
    by (simp add: acute-one-side-aux0 assms(1) assms(2))
next
  assume P1:  $\neg \text{Col } A \ C \ P$ 
  have P2:  $C \ A \ TS \ P \ B \vee C \ A \ OS \ P \ B$ 
    using Col-cases P1 assms(2) assms(3) cop-nos--ts coplanar-perm-13
      perp-not-col by blast

```

```

{
  assume P3: C A TS P B
  obtain Bs where P4: C Midpoint B Bs
    using symmetric-point-construction by auto
  have C A TS Bs B
    by (metis P3 P4 assms(2) bet--ts l9-2 midpoint-bet midpoint-distinct-2
        perp-not-col ts-distincts)
  hence P6: C A OS P Bs
    using P3 l9-8-1 by auto
  have C Bs Perp A C
  proof -
    have P6A: C ≠ Bs
      using P6 os-distincts by blast
    have Col C B Bs
      using Bet-cases Col-def P4 midpoint-bet by blast
    thus ?thesis
      using Perp-cases P6A assms(2) perp-col by blast
  qed
  hence Bs C Perp C A
    using Perp-perm by blast
  hence C A Perp Bs C
    using Perp-perm by blast
  hence C B OS A P using acute-one-side-aux
    by (metis P4 P6 assms(1) assms(2) col-one-side midpoint-col
        not-col-permutation-5 perp-distinct)
}
{
  assume C A OS P B
  hence C B OS A P using acute-one-side-aux
    using assms(1) assms(2) by blast
}
thus ?thesis
  using P2 ⟨C A TS P B ⟹ C B OS A P⟩ by auto
qed

lemma acute--not-obtuse:
  assumes Acute A B C
  shows ¬ Obtuse A B C
  using acute-obtuse--lta assms nlta by blast

lemma suma-distincts:
  assumes A B C D E F SumA G H I
  shows A ≠ B ∧ B ≠ C ∧ D ≠ E ∧ E ≠ F ∧ G ≠ H ∧ H ≠ I
proof -
  obtain J where C B J CongA D E F ∧ ¬ B C OS A J ∧ Coplanar A B C J ∧ A B J CongA G H I
    using SumA-def assms by auto
  thus ?thesis
    using CongA-def by blast
qed

lemma trisuma-distincts:
  assumes A B C TriSumA D E F
  shows A ≠ B ∧ B ≠ C ∧ A ≠ C ∧ D ≠ E ∧ E ≠ F
proof -
  obtain G H I where A B C B C A SumA G H I ∧ G H I C A B SumA D E F
    using TriSumA-def assms by auto
  thus ?thesis
    using suma-distincts by blast
qed

lemma ex-suma:
  assumes A ≠ B and
    B ≠ C and
    D ≠ E and
    E ≠ F
  shows ∃ G H I. A B C D E F SumA G H I

```

```

proof -
  have  $\exists I. A B C D E F \text{ SumA } A B I$ 
  proof cases
    assume  $P1: \text{Col } A B C$ 
    obtain  $J$  where  $P2: D E F \text{ CongA } C B J \wedge \text{Coplanar } C B J A$ 
      using angle-construction-4 assms(2) assms(3) assms(4) by presburger
    have  $P3: J \neq B$ 
      using CongA-def  $P2$  by blast
    have  $\neg B C \text{ OS } A J$ 
      by (metis  $P1$  between-trivial2 one-side-chara)
    hence  $A B C D E F \text{ SumA } A B J$ 
      by (meson  $P2$   $P3$  SumA-def assms(1) conga-refl ncoplanar-perm-15 not-conga-sym)
    thus ?thesis by blast
  next
    assume  $T1: \neg \text{Col } A B C$ 
    show ?thesis
    proof cases
      assume  $T2: \text{Col } D E F$ 
      show ?thesis
      proof cases
        assume  $T3: \text{Bet } D E F$ 
        obtain  $J$  where  $T4: B \text{ Midpoint } C J$ 
          using symmetric-point-construction by blast
        have  $A B C D E F \text{ SumA } A B J$ 
        proof -
          have  $C B J \text{ CongA } D E F$ 
            by (metis  $T3$   $T4$  assms(2) assms(3) assms(4) conga-line midpoint-bet midpoint-distinct-2)
          moreover have  $\neg B C \text{ OS } A J$ 
            by (simp add:  $T4$  col124--nos midpoint-col)
          moreover have Coplanar  $A B C J$ 
            using  $T3$  bet--coplanar bet-conga--bet calculation(1) conga-sym ncoplanar-perm-15 by blast
          moreover have  $A B J \text{ CongA } A B J$ 
            using CongA-def assms(1) calculation(1) conga-refl by auto
          ultimately show ?thesis
            using SumA-def by blast
        qed
        thus ?thesis
          by auto
      next
        assume  $T5: \neg \text{Bet } D E F$ 
        have  $A B C D E F \text{ SumA } A B C$ 
        proof -
          have  $E \text{ Out } D F$ 
            using  $T2$   $T5$  l6-4-2 by auto
          hence  $C B C \text{ CongA } D E F$ 
            using assms(2) l11-21-b out-trivial by auto
          moreover have  $\neg B C \text{ OS } A C$ 
            using os-distincts by blast
          moreover have Coplanar  $A B C C$ 
            using ncop-distincts by auto
          moreover have  $A B C \text{ CongA } A B C$ 
            using assms(1) assms(2) conga-refl by auto
          ultimately show ?thesis
            using SumA-def by blast
        qed
        thus ?thesis
          by auto
      qed
    next
      assume  $T6: \neg \text{Col } D E F$ 
      then obtain  $J$  where  $T7: D E F \text{ CongA } C B J \wedge C B \text{ TS } J A$ 
        using  $T1$  ex-conga-ts not-col-permutation-4 not-col-permutation-5 by presburger
      thus ?thesis
        proof -

```

```

have C B J CongA D E F
  using T7 not-conga-sym by blast
moreover have  $\neg B C$  OS A J
  by (simp add: T7 invert-two-sides l9-2 l9-9)
moreover have Coplanar A B C J
  using T7 ncoplanar-perm-15 ts--coplanar by blast
moreover have A B J CongA A B J
  using T7 assms(1) conga-diff56 conga-refl by blast
ultimately show ?thesis
  using SumA-def by blast
qed
qed
qed
thus ?thesis
  by auto
qed

lemma suma2--conga:
  assumes A B C D E F SumA G H I and
    A B C D E F SumA G' H' I'
  shows G H I CongA G' H' I'
proof -
  obtain J where P1: C B J CongA D E F  $\wedge$   $\neg B C$  OS A J  $\wedge$  Coplanar A B C J  $\wedge$  A B J CongA G H I
    using SumA-def assms(1) by blast
  obtain J' where P2: C B J' CongA D E F  $\wedge$   $\neg B C$  OS A J'  $\wedge$  Coplanar A B C J'  $\wedge$ 
    A B J' CongA G' H' I'
    using SumA-def assms(2) by blast
  have P3: C B J CongA C B J'
  proof -
    have C B J CongA D E F
      by (simp add: P1)
    moreover have D E F CongA C B J'
      by (simp add: P2 conga-sym)
    ultimately show ?thesis
      using not-conga by blast
  qed
  have P4: A B J CongA A B J'
  proof cases
    assume P5: Col A B C
    thus ?thesis
  proof cases
    assume P6: Bet A B C
    show ?thesis
  proof -
    have C B J CongA C B J'
      by (simp add: P3)
    moreover have Bet C B A
      by (simp add: P6 between-symmetry)
    moreover have A  $\neq$  B
      using assms(1) suma-distincts by blast
    ultimately show ?thesis
      using l11-13 by blast
  qed
  next
    assume P7:  $\neg$  Bet A B C
    moreover have B Out A C
      by (simp add: P5 calculation l6-4-2)
    moreover have B  $\neq$  J
      using CongA-def P3 by blast
    then moreover have B Out J J
      using out-trivial by auto
    moreover have B  $\neq$  J'
      using CongA-def P3 by blast
    then moreover have B Out J' J'
      using out-trivial by auto
    ultimately show ?thesis

```



```

    using P3 l11-10 by blast
qed
next
assume P8:  $\neg \text{Col } A \ B \ C$ 
show ?thesis
proof cases
  assume P9:  $\text{Col } D \ E \ F$ 
  have B Out J' J
  proof cases
    assume P10:  $\text{Bet } D \ E \ F$ 
    show ?thesis
    proof -
      have D E F CongA J' B C
      using P2 conga-right-comm not-conga-sym by blast
      hence Bet J' B C
      using P10 bet-conga--bet by blast
      moreover have D E F CongA J B C
      by (simp add: P1 conga-right-comm conga-sym)
      then moreover have Bet J B C
      using P10 bet-conga--bet by blast
      ultimately show ?thesis
      by (metis CongA-def P3 l6-2)
    qed
  next
  assume P11:  $\neg \text{Bet } D \ E \ F$ 
  have P12:  $E \text{ Out } D \ F$ 
  by (simp add: P11 P9 l6-4-2)
  show ?thesis
  proof -
    have B Out J' C
    proof -
      have D E F CongA J' B C
      using P2 conga-right-comm conga-sym by blast
      thus ?thesis
      using l11-21-a P12 by blast
    qed
    moreover have B Out C J
    by (metis P3 P8 bet-conga--bet calculation col-conga-col col-out2-col
      l6-4-2 l6-6 not-col-distincts not-conga-sym out-bet-out-1 out-trivial)
    ultimately show ?thesis
    using l6-7 by blast
  qed
qed
thus ?thesis
using P8 not-col-distincts out2--conga out-trivial by blast
next
assume P13:  $\neg \text{Col } D \ E \ F$ 
show ?thesis
proof -
  have B C TS A J
  proof -
    have Coplanar B C A J
    using P1 coplanar-perm-8 by blast
    moreover have  $\neg \text{Col } A \ B \ C$ 
    by (simp add: P8)
    moreover have  $\neg B \ C \ OS \ A \ J$ 
    using P1 by simp
    moreover have  $\neg \text{Col } J \ B \ C$ 
    proof -
      have D E F CongA J B C
      using P1 conga-right-comm not-conga-sym by blast
      thus ?thesis
      using P13 ncol-conga-ncol by blast
    qed
  qed
  ultimately show ?thesis
  using cop--one-or-two-sides by blast

```

```

qed
moreover have  $B C TS A J'$ 
proof -
  have Coplanar B C A J'
    using P2 coplanar-perm-8 by blast
  moreover have  $\neg Col A B C$ 
    by (simp add: P8)
  moreover have  $\neg B C OS A J'$ 
    using P2 by simp
  moreover have  $\neg Col J' B C$ 
  proof -
    have  $D E F CongA J' B C$ 
      using P2 conga-right-comm not-conga-sym by blast
    thus ?thesis
      using P13 ncol-conga-ncol by blast
  qed
  ultimately show ?thesis
    using cop-nos--ts by blast
qed
moreover have  $A B C CongA A B C$ 
  by (metis P8 conga-pseudo-refl conga-right-comm not-col-distincts)
moreover have  $C B J CongA C B J'$ 
  by (simp add: P3)
ultimately show ?thesis
  using l11-22a by blast
qed
qed
qed
thus ?thesis
  by (meson P1 P2 not-conga not-conga-sym)
qed

lemma suma-sym:
  assumes  $A B C D E F SumA G H I$ 
  shows  $D E F A B C SumA G H I$ 
proof -
  obtain  $J$  where  $P1: C B J CongA D E F \wedge \neg B C OS A J \wedge Coplanar A B C J \wedge A B J CongA G H I$ 
    using SumA-def assms(1) by blast
  show ?thesis
  proof cases
    assume  $P2: Col A B C$ 
    thus ?thesis
  proof cases
    assume  $P3: Bet A B C$ 
    obtain  $K$  where  $P4: Bet F E K \wedge Cong F E E K$ 
      using Cong-perm segment-construction by blast
    show ?thesis
  proof -
    have  $P5: F E K CongA A B C$ 
      by (metis CongA-def P1 P3 P4 cong-diff conga-line)
    moreover have  $\neg E F OS D K$ 
      using P4 bet-col col124--nos invert-one-side by blast
    moreover have Coplanar D E F K
      using P4 bet--coplanar ncoplanar-perm-15 by blast
    moreover have  $D E K CongA G H I$ 
  proof -
    have  $D E K CongA A B J$ 
  proof -
    have  $F E D CongA C B J$ 
      by (simp add: P1 conga-left-comm conga-sym)
    moreover have Bet F E K
      by (simp add: P4)
    moreover have  $K \neq E$ 
      using P4 calculation(1) cong-identity conga-diff1 by blast
    moreover have Bet C B A
      by (simp add: Bet-perm P3)

```

```

    moreover have  $A \neq B$ 
      using CongA-def P5 by blast
    ultimately show ?thesis
      using conga-right-comm l11-13 not-conga-sym by blast
  qed
  thus ?thesis
    using P1 not-conga by blast
  qed
  ultimately show ?thesis
    using SumA-def by blast
  qed
next
  assume T1:  $\neg \text{Bet } A \ B \ C$ 
  hence T2:  $B \text{ Out } A \ C$ 
    by (simp add: P2 l6-4-2)
  show ?thesis
  proof -
    have F E F CongA A B C
      by (metis T2 assms l11-21-b out-trivial suma-distincts)
    moreover have  $\neg E \ F \ OS \ D \ F$ 
      using os-distincts by auto
    moreover have Coplanar D E F F
      using ncop-distincts by auto
    moreover have D E F CongA G H I
  proof -
    have A B J CongA D E F
  proof -
    have C B J CongA D E F
      by (simp add: P1)
    moreover have B Out A C
      by (simp add: T2)
    moreover have  $J \neq B$ 
      using calculation(1) conga-distinct by auto
    moreover have  $D \neq E$ 
      using calculation(1) conga-distinct by blast
    moreover have  $F \neq E$ 
      using calculation(1) conga-distinct by blast
    ultimately show ?thesis
      by (meson Out-cases not-conga out2--conga out-trivial)
  qed
  hence D E F CongA A B J
    using not-conga-sym by blast
  thus ?thesis
    using P1 not-conga by blast
  qed
  ultimately show ?thesis
    using SumA-def by blast
  qed
  qed
next
  assume Q1:  $\neg \text{Col } A \ B \ C$ 
  show ?thesis
  proof cases
    assume Q2:  $\text{Col } D \ E \ F$ 
    obtain K where Q3:  $A \ B \ C \text{ CongA } F \ E \ K$ 
      using P1 angle-construction-3 conga-diff1 conga-diff56 by fastforce
    show ?thesis
  proof -
    have F E K CongA A B C
      by (simp add: Q3 conga-sym)
    moreover have  $\neg E \ F \ OS \ D \ K$ 
      using Col-cases Q2 one-side-not-col123 by blast
    moreover have Coplanar D E F K
      by (simp add: Q2 col-coplanar)
    moreover have D E K CongA G H I
  proof -

```

```

have D E K CongA A B J
proof cases
  assume Bet D E F
  hence J B A CongA D E K
  by (metis P1 bet-conga--bet calculation(1) conga-diff45
    conga-right-comm l11-13 not-conga-sym)
  thus ?thesis
  using conga-right-comm not-conga-sym by blast
next
assume  $\neg$  Bet D E F
hence W2: E Out D F
  using Q2 or-bet-out by blast
have A B J CongA D E K
proof -
  have A B C CongA F E K
  by (simp add: Q3)
  moreover have A  $\neq$  B
  using Q1 col-trivial-1 by auto
  moreover have E Out D F
  by (simp add: W2)
  moreover have B Out J C
  proof -
    have D E F CongA J B C
    by (simp add: P1 conga-left-comm conga-sym)
    thus ?thesis
    using W2 out-conga-out by blast
  qed
  moreover have K  $\neq$  E
  using CongA-def Q3 by blast
  ultimately show ?thesis
  using l11-10 out-trivial by blast
qed
thus ?thesis
  using not-conga-sym by blast
qed
thus ?thesis
  using P1 not-conga by blast
qed
ultimately show ?thesis
  using SumA-def by blast
qed
next
assume W3:  $\neg$  Col D E F
then obtain K where W4: A B C CongA F E K  $\wedge$  F E TS K D
  using Q1 ex-conga-ts not-col-permutation-3 by blast
show ?thesis
proof -
  have F E K CongA A B C
  using W4 not-conga-sym by blast
  moreover have  $\neg$  E F OS D K
  proof -
    have E F TS D K
    using W4 invert-two-sides l9-2 by blast
    thus ?thesis
    using l9-9 by auto
  qed
  moreover have Coplanar D E F K
  proof -
    have E F TS D K
    using W4 invert-two-sides l9-2 by blast
    thus ?thesis
    using ncoplanar-perm-8 ts--coplanar by blast
  qed
  moreover have D E K CongA G H I
  proof -
    have A B J CongA K E D

```

```

proof -
  have  $B \ C \ TS \ A \ J$ 
proof -
  have Coplanar  $B \ C \ A \ J$ 
    using  $P1 \ ncoplanar-perm-12$  by blast
  moreover have  $\neg \ Col \ A \ B \ C$ 
    by (simp add:  $Q1$ )
  moreover have  $\neg \ B \ C \ OS \ A \ J$ 
    using  $P1$  by simp
  moreover have  $\neg \ Col \ J \ B \ C$ 
proof -
  {
    assume  $Col \ J \ B \ C$ 
    have  $Col \ D \ E \ F$ 
    proof -
      have  $Col \ C \ B \ J$ 
        using Col-perm  $\langle Col \ J \ B \ C \rangle$  by blast
      moreover have  $C \ B \ J \ CongA \ D \ E \ F$ 
        by (simp add:  $P1$ )
      ultimately show ?thesis
        using col-conga-col by blast
    qed
    hence False
      by (simp add:  $W3$ )
  }
  thus ?thesis by blast
qed
ultimately show ?thesis
  using cop-nos--ts by blast
qed
moreover have  $E \ F \ TS \ K \ D$ 
  using  $W4 \ invert-two-sides$  by blast
moreover have  $A \ B \ C \ CongA \ K \ E \ F$ 
  by (simp add:  $W4 \ conga-right-comm$ )
moreover have  $C \ B \ J \ CongA \ F \ E \ D$ 
  by (simp add:  $P1 \ conga-right-comm$ )
ultimately show ?thesis
  using  $l11-22a$  by auto
qed
hence  $D \ E \ K \ CongA \ A \ B \ J$ 
  using conga-left-comm not-conga-sym by blast
thus ?thesis
  using  $P1 \ not-conga$  by blast
qed
ultimately show ?thesis
  using SumA-def by blast
qed
qed
qed
qed

lemma conga3-suma--suma:
  assumes  $A \ B \ C \ D \ E \ F \ SumA \ G \ H \ I$  and
     $A \ B \ C \ CongA \ A' \ B' \ C'$  and
     $D \ E \ F \ CongA \ D' \ E' \ F'$  and
     $G \ H \ I \ CongA \ G' \ H' \ I'$ 
  shows  $A' \ B' \ C' \ D' \ E' \ F' \ SumA \ G' \ H' \ I'$ 
proof -
  have  $D' \ E' \ F' \ A \ B \ C \ SumA \ G' \ H' \ I'$ 
proof -
  obtain  $J$  where  $P1: C \ B \ J \ CongA \ D \ E \ F \wedge \neg \ B \ C \ OS \ A \ J \wedge \textit{Coplanar} \ A \ B \ C \ J \wedge A \ B \ J \ CongA \ G \ H \ I$ 
    using SumA-def assms(1) by blast
  have  $A \ B \ C \ D' \ E' \ F' \ SumA \ G' \ H' \ I'$ 
proof -
  have  $C \ B \ J \ CongA \ D' \ E' \ F'$ 
    using  $P1 \ assms(3) \ not-conga$  by blast

```

```

    moreover have  $\neg B C OS A J$ 
      using  $P1$  by simp
    moreover have  $Coplanar A B C J$ 
      using  $P1$  by simp
    moreover have  $A B J CongA G' H' I'$ 
      using  $P1$  assms(4) not-conga by blast
    ultimately show ?thesis
      using SumA-def by blast
  qed
  thus ?thesis
    by (simp add: suma-sym)
qed
then obtain  $J$  where  $P2: F' E' J CongA A B C \wedge \neg E' F' OS D' J \wedge$ 
 $Coplanar D' E' F' J \wedge D' E' J CongA G' H' I'$ 
  using SumA-def by blast
have  $D' E' F' A' B' C' SumA G' H' I'$ 
proof -
  have  $F' E' J CongA A' B' C'$ 
  proof -
    have  $F' E' J CongA A B C$ 
      by (simp add: P2)
    moreover have  $A B C CongA A' B' C'$ 
      by (simp add: assms(2))
    ultimately show ?thesis
      using not-conga by blast
  qed
  moreover have  $\neg E' F' OS D' J$ 
    using P2 by simp
  moreover have  $Coplanar D' E' F' J$ 
    using P2 by simp
  moreover have  $D' E' J CongA G' H' I'$ 
    by (simp add: P2)
  ultimately show ?thesis
    using SumA-def by blast
qed
thus ?thesis
  by (simp add: suma-sym)
qed

```

```

lemma out6-suma--suma:
  assumes  $A B C D E F SumA G H I$  and
     $B Out A A'$  and
     $B Out C C'$  and
     $E Out D D'$  and
     $E Out F F'$  and
     $H Out G G'$  and
     $H Out I I'$ 
  shows  $A' B C' D' E F' SumA G' H I'$ 
proof -
  have  $A B C CongA A' B C'$ 
    using Out-cases assms(2) assms(3) out2--conga by blast
  moreover have  $D E F CongA D' E F'$ 
    using Out-cases assms(4) assms(5) out2--conga by blast
  moreover have  $G H I CongA G' H I'$ 
    by (simp add: assms(6) assms(7) l6-6 out2--conga)
  ultimately show ?thesis
    using assms(1) conga3-suma--suma by blast
qed

```

```

lemma out546-suma--conga:
  assumes  $A B C D E F SumA G H I$  and
     $E Out D F$ 
  shows  $A B C CongA G H I$ 
proof -
  have  $A B C D E F SumA A B C$ 
  proof -

```

```

have C B C CongA D E F
  by (metis assms(1) assms(2) l11-21-b out-trivial suma-distincts)
moreover have  $\neg B C OS A C$ 
  using os-distincts by auto
moreover have Coplanar A B C C
  using ncop-distincts by auto
moreover have A B C CongA A B C
  by (metis suma-distincts assms(1) conga-pseudo-refl conga-right-comm)
ultimately show ?thesis
  using SumA-def by blast
qed
thus ?thesis using suma2--conga assms(1) by blast
qed

```

```

lemma out546--suma:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    E Out D F
  shows A B C D E F SumA A B C
proof -
  have P1: D  $\neq$  E
    using assms(3) out-diff1 by auto
  have P2: F  $\neq$  E
    using Out-def assms(3) by auto
  then obtain G H I where P3: A B C D E F SumA G H I
    using P1 assms(1) assms(2) ex-suma by presburger
  hence G H I CongA A B C
    by (meson conga-sym out546-suma--conga assms(3))
  thus ?thesis
    using P1 P2 P3 assms(1) assms(2) assms(3) conga3-suma--suma
      conga-refl out-diff1 by auto
qed

```

```

lemma out213-suma--conga:
  assumes A B C D E F SumA G H I and
    B Out A C
  shows D E F CongA G H I
  using assms(1) assms(2) out546-suma--conga suma-sym by blast

```

```

lemma out213--suma:
  assumes D  $\neq$  E and
    E  $\neq$  F and
    B Out A C
  shows A B C D E F SumA D E F
  by (simp add: assms(1) assms(2) assms(3) out546--suma suma-sym)

```

```

lemma suma-left-comm:
  assumes A B C D E F SumA G H I
  shows C B A D E F SumA G H I
proof -
  have A B C CongA C B A
    using assms conga-pseudo-refl suma-distincts by fastforce
  moreover have D E F CongA D E F
    by (metis assms conga-refl suma-distincts)
  moreover have G H I CongA G H I
    by (metis assms conga-refl suma-distincts)
  ultimately show ?thesis
    using assms conga3-suma--suma by blast
qed

```

```

lemma suma-middle-comm:
  assumes A B C D E F SumA G H I
  shows A B C F E D SumA G H I
  using assms suma-left-comm suma-sym by blast

```

```

lemma suma-right-comm:

```

```

assumes  $A B C D E F \text{ SumA } G H I$ 
shows  $A B C D E F \text{ SumA } I H G$ 
proof –
  have  $A B C \text{ CongA } A B C$ 
    using assms conga-refl suma-distincts by fastforce
  moreover have  $D E F \text{ CongA } D E F$ 
    by (metis assms conga-refl suma-distincts)
  moreover have  $G H I \text{ CongA } I H G$ 
    by (meson conga-right-comm suma2--conga assms)
  ultimately show ?thesis
    using assms conga3-suma--suma by blast
qed

lemma suma-comm:
assumes  $A B C D E F \text{ SumA } G H I$ 
shows  $C B A F E D \text{ SumA } I H G$ 
by (simp add: assms suma-left-comm suma-middle-comm suma-right-comm)

lemma ts--suma:
assumes  $A B \text{ TS } C D$ 
shows  $C B A A B D \text{ SumA } C B D$ 
proof –
  have  $A B D \text{ CongA } A B D$ 
    by (metis conga-right-comm assms conga-pseudo-refl ts-distincts)
  moreover have  $\neg B A \text{ OS } C D$ 
    using assms invert-one-side l9-9 by blast
  moreover have Coplanar  $C B A D$ 
    using assms ncoplanar-perm-14 ts--coplanar by blast
  moreover have  $C B D \text{ CongA } C B D$ 
    by (metis assms conga-refl ts-distincts)
  ultimately show ?thesis
    using SumA-def by blast
qed

lemma ts--suma-1:
assumes  $A B \text{ TS } C D$ 
shows  $C A B B A D \text{ SumA } C A D$ 
by (simp add: assms invert-two-sides ts--suma)

lemma inangle--suma:
assumes  $P \text{ InAngle } A B C$ 
shows  $A B P P B C \text{ SumA } A B C$ 
proof –
  have Coplanar  $A B P C$ 
    by (simp add: assms coplanar-perm-8 inangle--coplanar)
  moreover have  $\neg B P \text{ OS } A C$ 
    by (meson assms col123--nos col124--nos in-angle-two-sides invert-two-sides l9-9-bis not-col-permutation-5)
  ultimately show ?thesis
    using SumA-def assms conga-refl inangle-distincts by blast
qed

lemma bet--suma:
assumes  $A \neq B$  and
   $B \neq C$  and
   $P \neq B$  and Bet  $A B C$ 
shows  $A B P P B C \text{ SumA } A B C$ 
proof –
  have  $P \text{ InAngle } A B C$ 
    using assms(1) assms(2) assms(3) assms(4) in-angle-line by auto
  thus ?thesis
    by (simp add: inangle--suma)
qed

lemma sams-chara:
assumes  $A \neq B$  and

```



```

  A' ≠ B and
  Bet A B A'
shows SAMS A B C D E F ↔ D E F LeA C B A'
proof -
{
  assume SAMS A B C D E F
  then obtain J where C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J
    using SAMS-def by auto
  have C B J CongA D E F
    by (simp add: ⟨C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J⟩)
  have ¬ B C OS A J
    by (simp add: ⟨C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J⟩)
  have ¬ A B TS C J
    by (simp add: ⟨C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J⟩)
  have Coplanar A B C J
    by (simp add: ⟨C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J⟩)
  have A ≠ A'
    using assms(2) assms(3) between-identity by blast
  have C ≠ B
    using ⟨C B J CongA D E F⟩ conga-diff1 by auto
  have J ≠ B
    using CongA-def ⟨C B J CongA D E F⟩ by presburger
  have D ≠ E
    using ⟨C B J CongA D E F⟩ conga-diff45 by blast
  have F ≠ E
    using CongA-def ⟨C B J CongA D E F⟩ by blast
  {
    assume E Out D F
    hence D E F LeA C B A'
      using assms(2) l11-31-1 ⟨C ≠ B⟩ by force
  }
  {
    assume ¬ Bet A B C
    have D E F LeA C B A'
    proof cases
      assume Col A B C
      hence Bet C B A'
        using assms(1) assms(3) between-exchange3 outer-transitivity-between2
        third-point ⟨¬ Bet A B C⟩ by blast
      thus ?thesis
        using assms(2) l11-31-2 by (simp add: ⟨C ≠ B⟩ ⟨D ≠ E⟩ ⟨F ≠ E⟩)
    next
      assume T9: ¬ Col A B C
      show ?thesis
      proof cases
        assume T10: Col D E F
        show ?thesis
        proof cases
          assume T11: Bet D E F
          have D E F CongA C B J
            using ⟨C B J CongA D E F⟩ conga-sym-equiv by auto
          hence T12: Bet C B J
            using T11 bet-conga--bet by blast
          have A B TS C J
          proof -
            have ¬ Col J A B
              using T9 T12 bet-col col2--eq col-permutation-1 ⟨J ≠ B⟩ by blast
            moreover have ∃ T. Col T A B ∧ Bet C T J
              using T12 col-trivial-3 by blast
            ultimately show ?thesis
              using T9 TS-def col-permutation-1 by blast
          qed
          hence False
            using ⟨¬ A B TS C J⟩ by blast
          thus ?thesis by simp
        next

```

```

    assume  $\neg Bet\ D\ E\ F$ 
    thus ?thesis
    using T10  $\langle E\ Out\ D\ F \implies D\ E\ F\ LeA\ C\ B\ A' \rangle$  or-bet-out by auto
  qed
next
  assume T13:  $\neg Col\ D\ E\ F$ 
  show ?thesis
  proof -
    have  $C\ B\ J\ LeA\ C\ B\ A'$ 
    proof -
      have  $J\ InAngle\ C\ B\ A'$ 
      proof -
        have  $A' \neq B$ 
        by (simp add: assms(2))
        moreover have  $Bet\ A\ B\ A'$ 
        by (simp add: assms(3))
        moreover have  $C\ InAngle\ A\ B\ J$ 
        proof -
          have  $\neg Col\ J\ B\ C$ 
          proof -
            have  $\neg Col\ D\ E\ F$ 
            by (simp add: T13)
            moreover have  $D\ E\ F\ CongA\ J\ B\ C$ 
            using conga-left-comm not-conga-sym
               $\langle C\ B\ J\ CongA\ D\ E\ F \rangle$  by blast
            ultimately show ?thesis
            using ncol-conga-ncol by blast
          qed
          hence  $B\ C\ TS\ A\ J$ 
          using T9 cop-nos--ts coplanar-perm-8
             $\langle Coplanar\ A\ B\ C\ J \rangle \langle \neg B\ C\ OS\ A\ J \rangle$  by blast
          then obtain X where T14:  $Col\ X\ B\ C \wedge Bet\ A\ X\ J$ 
          using TS-def by blast
          {
            assume T15:  $X \neq B$ 
            have  $B\ Out\ X\ C$ 
            proof -
              have  $Col\ B\ X\ C$ 
              by (simp add: Col-perm T14)
              moreover have  $B\ A\ OS\ X\ C$ 
              proof -
                have  $A\ B\ OS\ X\ C$ 
                proof -
                  have  $A\ B\ OS\ X\ J$ 
                  using T14 T9 T15 bet-out calculation col-transitivity-2
                    col-trivial-2 l6-21 out-one-side by metis
                  moreover have  $A\ B\ OS\ J\ C$ 
                  using T14 T9 calculation cop-nts--os l5-2
                    not-col-permutation-2 one-side-chara one-side-symmetry
                     $\langle \neg A\ B\ TS\ C\ J \rangle \langle Coplanar\ A\ B\ C\ J \rangle$  by (metis)
                  ultimately show ?thesis
                  using one-side-transitivity by blast
                qed
              qed
            qed
            thus ?thesis
            by (simp add: invert-one-side)
          qed
          ultimately show ?thesis
          using col-one-side-out by auto
        qed
      }
    }
    hence  $Bet\ A\ X\ J \wedge (X = B \vee B\ Out\ X\ C)$ 
    using T14 by blast
    thus ?thesis
    using InAngle-def  $\langle C \neq B \rangle \langle J \neq B \rangle$  assms(1) by auto
  qed
  ultimately show ?thesis

```

```

    using in-angle-reverse l11-24 by blast
  qed
  moreover have  $C B J \text{ CongA } C B J$ 
    using  $\langle C \neq B \rangle \langle J \neq B \rangle \text{ conga-refl}$  by presburger
  ultimately show ?thesis
    by (simp add: inangle--lea)
  qed
  moreover have  $D E F \text{ LeA } C B J$ 
    using  $\langle C B J \text{ CongA } D E F \rangle \text{ conga--lea456123}$  by auto
  ultimately show ?thesis
    using lea-trans by blast
  qed
  qed
  qed
}
hence  $D E F \text{ LeA } C B A'$ 
  using SAMS-def  $\langle E \text{ Out } D F \implies D E F \text{ LeA } C B A' \rangle$ 
   $\langle \text{SAMS } A B C D E F \rangle$  by fastforce
}
{
  assume P1:  $D E F \text{ LeA } C B A'$ 
  have P2:  $A \neq A'$ 
    using assms(2) assms(3) between-identity by blast
  have P3:  $C \neq B$ 
    using P1 lea-distincts by auto
  have P4:  $D \neq E$ 
    using P1 lea-distincts by auto
  have P5:  $F \neq E$ 
    using P1 lea-distincts by auto
  have SAMS  $A B C D E F$ 
  proof cases
    assume P6:  $\text{Col } A B C$ 
    show ?thesis
    proof cases
      assume P7:  $\text{Bet } A B C$ 
      have  $E \text{ Out } D F$ 
      proof -
        have  $B \text{ Out } C A'$ 
          by (meson Bet-perm P3 P7 assms(1) assms(2) assms(3) l6-2)
        moreover have  $C B A' \text{ CongA } D E F$ 
          using P1 calculation l11-21-b out-lea--out by blast
        ultimately show ?thesis
          using out-conga-out by blast
      qed
      moreover have  $C B C \text{ CongA } D E F$ 
        using P3 calculation l11-21-b out-trivial by auto
      moreover have  $\neg B C \text{ OS } A C$ 
        using os-distincts by auto
      moreover have  $\neg A B \text{ TS } C C$ 
        by (simp add: not-two-sides-id)
      moreover have  $\text{Coplanar } A B C C$ 
        using ncop-distincts by auto
      ultimately show ?thesis
        using SAMS-def assms(1) by blast
    next
      assume P8:  $\neg \text{Bet } A B C$ 
      have P9:  $B \text{ Out } A C$ 
        by (simp add: P6 P8 l6-4-2)
      obtain J where P10:  $D E F \text{ CongA } C B J$ 
        using P3 P4 P5 angle-construction-3 by blast
      show ?thesis
      proof -
        have  $C B J \text{ CongA } D E F$ 
          using P10 not-conga-sym by blast
        moreover have  $\neg B C \text{ OS } A J$ 
          using Col-cases P6 one-side-not-col123 by blast
      qed
    end
  qed
}

```

```

    moreover have  $\neg A B TS C J$ 
      using Col-cases P6 TS-def by blast
    moreover have Coplanar A B C J
      using P6 col--coplanar by auto
    ultimately show ?thesis
      using P8 SAMS-def assms(1) by blast
  qed
qed
next
assume P11:  $\neg Col A B C$ 
have P12:  $\neg Col A' B C$ 
  using P11 assms(2) assms(3) bet-col bet-col1 colx by blast
show ?thesis
proof cases
  assume P13: Col D E F
  have P14: E Out D F
  proof -
    {
      assume P14: Bet D E F
      have D E F LeA C B A'
        by (simp add: P1)
      hence Bet C B A'
        using P14 bet-lea--bet by blast
      hence Col A' B C
        using Col-def Col-perm by blast
      hence False
        by (simp add: P12)
    }
    hence  $\neg Bet D E F$  by auto
  thus ?thesis
    by (simp add: P13 l6-4-2)
  qed
show ?thesis
proof -
  have C B C CongA D E F
    by (simp add: P3 P14 l11-21-b out-trivial)
  moreover have  $\neg B C OS A C$ 
    using os-distincts by auto
  moreover have  $\neg A B TS C C$ 
    by (simp add: not-two-sides-id)
  moreover have Coplanar A B C C
    using ncop-distincts by auto
  ultimately show ?thesis
    using P14 SAMS-def assms(1) by blast
  qed
next
assume P15:  $\neg Col D E F$ 
obtain J where P16: D E F CongA C B J  $\wedge$  C B TS J A
  using P11 P15 ex-conga-ts not-col-permutation-3 by presburger
show ?thesis
proof -
  have C B J CongA D E F
    by (simp add: P16 conga-sym)
  moreover have  $\neg B C OS A J$ 
  proof -
    have C B TS A J
      using P16 by (simp add: l9-2)
    thus ?thesis
      using invert-one-side l9-9 by blast
  qed
  moreover have  $\neg A B TS C J \wedge Coplanar A B C J$ 
  proof cases
    assume Col A B J
    thus ?thesis
      using TS-def ncop--ncols not-col-permutation-1 by blast
  next

```

```

assume P17:  $\neg \text{Col } A \ B \ J$ 
have  $\neg A \ B \ TS \ C \ J$ 
proof -
  have  $A' \ B \ OS \ J \ C$ 
  proof -
    have  $\neg \text{Col } A' \ B \ C$ 
      by (simp add: P12)
    moreover have  $\neg \text{Col } B \ A' \ J$ 
    proof -
      {
        assume  $\text{Col } B \ A' \ J$ 
        hence False
          by (metis P17 assms(2) assms(3) bet-col col-trivial-2 colx)
      }
      thus ?thesis by auto
    qed
  moreover have  $J \text{ InAngle } A' \ B \ C$ 
  proof -
    obtain  $K$  where P20:  $K \text{ InAngle } C \ B \ A' \wedge D \ E \ F \text{ CongA } C \ B \ K$ 
      using LeA-def P1 by blast
    have  $J \text{ InAngle } C \ B \ A'$ 
    proof -
      have  $C \ B \ A' \text{ CongA } C \ B \ A'$ 
        by (simp add: P3 assms(2) conga-pseudo-refl conga-right-comm)
      moreover have  $C \ B \ K \text{ CongA } C \ B \ J$ 
      proof -
        have  $C \ B \ K \text{ CongA } D \ E \ F$ 
          using P20 not-conga-sym by blast
        moreover have  $D \ E \ F \text{ CongA } C \ B \ J$ 
          by (simp add: P16)
        ultimately show ?thesis
          using not-conga by blast
      qed
    moreover have  $K \text{ InAngle } C \ B \ A'$ 
      using P20 by simp
    moreover have  $C \ B \ OS \ J \ A'$ 
    proof -
      have  $C \ B \ TS \ J \ A$  using P16
        by simp
      moreover have  $C \ B \ TS \ A' \ A$ 
        using Col-perm P12 assms(3) bet--ts between-symmetry
          calculation invert-two-sides ts-distincts by blast
      ultimately show ?thesis
        using OS-def by auto
    qed
    ultimately show ?thesis
      using conga-preserves-in-angle by blast
  qed
  thus ?thesis
    by (simp add: l11-24)
  qed
  ultimately show ?thesis
    by (simp add: in-angle-one-side)
  qed
  hence  $A' \ B \ OS \ C \ J$ 
    by (simp add: one-side-symmetry)
  hence  $\neg A' \ B \ TS \ C \ J$ 
    by (simp add: l9-9-bis)
  thus ?thesis
    using assms(2) assms(3) bet-col bet-col1 col-preserves-two-sides by blast
  qed
  moreover have Coplanar  $A \ B \ C \ J$ 
  proof -
    have  $C \ B \ TS \ J \ A$ 
      using P16 by simp
    thus ?thesis

```

```

      by (simp add: coplanar-perm-20 ts--coplanar)
    qed
    ultimately show ?thesis by auto
  qed
  ultimately show ?thesis
    using P11 SAMS-def assms(1) bet-col by auto
  qed
  qed
  qed
}
thus ?thesis
  using ⟨SAMS A B C D E F ⟹ D E F LeA C B A'⟩ by blast
qed

```

lemma sams-distincts:

```

  assumes SAMS A B C D E F
  shows A ≠ B ∧ B ≠ C ∧ D ≠ E ∧ E ≠ F
proof -
  obtain J where C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J
    using SAMS-def assms by auto
  thus ?thesis
    by (metis SAMS-def assms conga-distinct)
qed

```

lemma sams-sym:

```

  assumes SAMS A B C D E F
  shows SAMS D E F A B C
proof -
  have P1: A ≠ B
    using assms sams-distincts by blast
  have P3: D ≠ E
    using assms sams-distincts by blast
  obtain D' where P5: E Midpoint D D'
    using symmetric-point-construction by blast
  obtain A' where P6: B Midpoint A A'
    using symmetric-point-construction by blast
  have P8: E ≠ D'
    using P3 P5 is-midpoint-id-2 by blast
  have P9: A ≠ A'
    using P1 P6 l7-3 by auto
  hence P10: B ≠ A'
    using P6 P9 midpoint-not-midpoint by auto
  hence D E F LeA C B A'
    using P1 P6 assms midpoint-bet sams-chara by fastforce
  hence D E F LeA A' B C
    by (simp add: lea-right-comm)
  hence A B C LeA D' E F
    by (metis Mid-cases P1 P10 P3 P5 P6 P8 l11-36 midpoint-bet)
  hence A B C LeA F E D'
    by (simp add: lea-right-comm)
  moreover have D ≠ E
    by (simp add: P3)
  moreover have D' ≠ E
    using P8 by auto
  moreover have Bet D E D'
    by (simp add: P5 midpoint-bet)
  thus ?thesis
    using P3 P8 calculation(1) sams-chara by fastforce
qed

```

lemma sams-right-comm:

```

  assumes SAMS A B C D E F
  shows SAMS A B C F E D
proof -
  have P1: E Out D F ∨ ¬ Bet A B C
    using SAMS-def assms by blast

```

```

obtain  $J$  where  $P2: C\ B\ J\ CongA\ D\ E\ F \wedge \neg B\ C\ OS\ A\ J \wedge \neg A\ B\ TS\ C\ J \wedge Coplanar\ A\ B\ C\ J$ 
using  $SAMS\text{-}def\ assms$  by  $auto$ 
{
  assume  $E\ Out\ D\ F$ 
  hence  $E\ Out\ F\ D \vee \neg Bet\ A\ B\ C$ 
    by ( $simp\ add: l6-6$ )
}
{
  assume  $\neg Bet\ A\ B\ C$ 
  hence  $E\ Out\ F\ D \vee \neg Bet\ A\ B\ C$  by  $auto$ 
}
hence  $E\ Out\ F\ D \vee \neg Bet\ A\ B\ C$ 
using  $\langle E\ Out\ D\ F \implies E\ Out\ F\ D \vee \neg Bet\ A\ B\ C \rangle\ P1$  by  $auto$ 
moreover have  $C\ B\ J\ CongA\ F\ E\ D$ 
proof –
  have  $C\ B\ J\ CongA\ D\ E\ F$ 
    by ( $simp\ add: P2$ )
  thus  $?thesis$ 
    by ( $simp\ add: conga\text{-}right\text{-}comm$ )
qed
ultimately show  $?thesis$ 
using  $P2\ SAMS\text{-}def\ assms$  by  $auto$ 
qed

```

```

lemma  $sams\text{-}left\text{-}comm$ :
assumes  $SAMS\ A\ B\ C\ D\ E\ F$ 
shows  $SAMS\ C\ B\ A\ D\ E\ F$ 
proof –
  have  $SAMS\ D\ E\ F\ A\ B\ C$ 
    by ( $simp\ add: assms\ sams\text{-}sym$ )
  hence  $SAMS\ D\ E\ F\ C\ B\ A$ 
    using  $sams\text{-}right\text{-}comm$  by  $blast$ 
  thus  $?thesis$ 
    using  $sams\text{-}sym$  by  $blast$ 
qed

```

```

lemma  $sams\text{-}comm$ :
assumes  $SAMS\ A\ B\ C\ D\ E\ F$ 
shows  $SAMS\ C\ B\ A\ F\ E\ D$ 
using  $assms\ sams\text{-}left\text{-}comm\ sams\text{-}right\text{-}comm$  by  $blast$ 

```

```

lemma  $conga2\text{-}sams\text{-}sams$ :
assumes  $A\ B\ C\ CongA\ A'\ B'\ C'$  and
   $D\ E\ F\ CongA\ D'\ E'\ F'$  and
   $SAMS\ A\ B\ C\ D\ E\ F$ 
shows  $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ 
proof –
  obtain  $A0$  where  $B\ Midpoint\ A\ A0$ 
    using  $symmetric\text{-}point\text{-}construction$  by  $auto$ 
  obtain  $A'0$  where  $B'\ Midpoint\ A'\ A'0$ 
    using  $symmetric\text{-}point\text{-}construction$  by  $blast$ 
  have  $D'\ E'\ F'\ LeA\ C'\ B'\ A'0$ 
proof –
  have  $D\ E\ F\ LeA\ C\ B\ A0$ 
    by ( $metis\ \langle B\ Midpoint\ A\ A0 \rangle\ assms(1)\ assms(3)\ conga\text{-}distinct\ midpoint\text{-}bet\ midpoint\text{-}distinct\text{-}2\ sams\text{-}chara$ )
  moreover have  $D\ E\ F\ CongA\ D'\ E'\ F'$ 
    by ( $simp\ add: assms(2)$ )
  moreover have  $C\ B\ A0\ CongA\ C'\ B'\ A'0$ 
proof –
  have  $A0\ B\ C\ CongA\ A'0\ B'\ C'$ 
    by ( $metis\ \langle B\ Midpoint\ A\ A0 \rangle\ \langle B'\ Midpoint\ A'\ A'0 \rangle\ assms(1)\ calculation(1)\ conga\text{-}diff45\ l11\text{-}13\ lea\text{-}distincts\ midpoint\text{-}bet\ midpoint\text{-}not\text{-}midpoint$ )
  thus  $?thesis$ 
    using  $conga\text{-}comm$  by  $blast$ 
qed

```

```

ultimately show ?thesis
  using l11-30 by blast
qed
thus ?thesis
  by (metis ‹B' Midpoint A' A'0› assms(1) conga-distinct lea-distincts
    midpoint-bet sams-chara)
qed

lemma out546--sams:
  assumes A ≠ B and
    B ≠ C and
    E Out D F
  shows SAMS A B C D E F
proof -
  obtain A' where Bet A B A' ∧ Cong B A' A B
    using segment-construction by blast
  hence D E F LeA C B A'
    using assms(1) assms(2) assms(3) cong-diff-3 l11-31-1 by fastforce
  thus ?thesis
    using ‹Bet A B A' ∧ Cong B A' A B› assms(1) lea-distincts sams-chara by blast
qed

lemma out213--sams:
  assumes D ≠ E and
    E ≠ F and
    B Out A C
  shows SAMS A B C D E F
  by (simp add: sams-sym assms(1) assms(2) assms(3) out546--sams)

lemma bet-suma--sams:
  assumes A B C D E F SumA G H I and
    Bet G H I
  shows SAMS A B C D E F
proof -
  obtain A' where P1: C B A' CongA D E F ∧ ¬ B C OS A A' ∧ Coplanar A B C A' ∧ A B A' CongA G H I
    using SumA-def assms(1) by auto
  hence G H I CongA A B A'
    using not-conga-sym by blast
  hence Bet A B A'
    using assms(2) bet-conga--bet by blast
  show ?thesis
proof -
  have E Out D F ∨ ¬ Bet A B C
proof -
  {
    assume Bet A B C
    hence E Out D F
    proof -
      have B Out C A'
      proof -
        have C ≠ B
          using assms(1) suma-distincts by blast
        moreover have A' ≠ B
          using CongA-def ‹G H I CongA A B A'› by blast
        moreover have A ≠ B
          using CongA-def ‹G H I CongA A B A'› by blast
        moreover have Bet C B A
          by (simp add: Bet-perm ‹Bet A B C›)
        ultimately show ?thesis
          using Out-def ‹Bet A B A'› ‹Bet A B C› l5-2 by auto
      qed
    moreover have C B A' CongA D E F
      using P1 by simp
    ultimately show ?thesis
      using l11-21-a by blast
  }
  qed

```



```

    }
    thus ?thesis
    by blast
qed
moreover have  $\exists J. (C B J \text{ CongA } D E F \wedge \neg B C \text{ OS } A J \wedge \neg A B \text{ TS } C J \wedge \text{ Coplanar } A B C J)$ 
proof -
  have  $C B A' \text{ CongA } D E F$ 
  by (simp add: P1)
  moreover have  $\neg B C \text{ OS } A A'$ 
  by (simp add: P1)
  moreover have  $\neg A B \text{ TS } C A'$ 
  using Col-def TS-def  $\langle \text{Bet } A B A' \rangle$  by blast
  moreover have  $\text{Coplanar } A B C A'$ 
  by (simp add: P1)
  ultimately show ?thesis
  by blast
qed
ultimately show ?thesis
using CongA-def SAMS-def
 $\langle C B A' \text{ CongA } D E F \wedge \neg B C \text{ OS } A A' \wedge \text{ Coplanar } A B C A' \wedge A B A' \text{ CongA } G H I \rangle$ 
by auto
qed
qed

lemma bet--sams:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $P \neq B$  and
     $\text{Bet } A B C$ 
  shows  $\text{SAMS } A B P P B C$ 
  by (meson assms(1) assms(2) assms(3) assms(4) bet--suma bet-suma--sams)

lemma suppa--sams:
  assumes  $A B C \text{ SuppA } D E F$ 
  shows  $\text{SAMS } A B C D E F$ 
proof -
  obtain  $A'$  where  $P1: \text{Bet } A B A' \wedge D E F \text{ CongA } C B A'$ 
  using SuppA-def assms by auto
  hence  $\text{SAMS } A B C C B A'$ 
  by (metis assms bet--sams conga-diff45 conga-diff56 suppa2--conga123)
  thus ?thesis
  by (meson P1 assms conga2-sams--sams not-conga-sym suppa2--conga123)
qed

lemma os-ts--sams:
  assumes  $B P \text{ TS } A C$  and
     $A B \text{ OS } P C$ 
  shows  $\text{SAMS } A B P P B C$ 
proof -
  have  $B \text{ Out } P C \vee \neg \text{Bet } A B P$ 
  using assms(2) bet-col col123--nos by blast
  moreover have  $\exists J. (P B J \text{ CongA } P B C \wedge \neg B P \text{ OS } A J \wedge \neg A B \text{ TS } P J \wedge \text{ Coplanar } A B P J)$ 
  by (metis assms(1) assms(2) conga-refl l9-9 os--coplanar os-distincts)
  ultimately show ?thesis
  using SAMS-def assms(2) os-distincts by auto
qed

lemma os2--sams:
  assumes  $A B \text{ OS } P C$  and
     $C B \text{ OS } P A$ 
  shows  $\text{SAMS } A B P P B C$ 
  by (simp add: os-ts--sams assms(1) assms(2) invert-one-side l9-31)

lemma inangle--sams:
  assumes  $P \text{ InAngle } A B C$ 
  shows  $\text{SAMS } A B P P B C$ 

```

```

proof –
  have  $Bet\ A\ B\ C \vee B\ Out\ A\ C \vee \neg\ Col\ A\ B\ C$ 
    using l6-4-2 by blast
  {
    assume  $Bet\ A\ B\ C$ 
    hence  $SAMS\ A\ B\ P\ P\ B\ C$ 
      using assms bet--sams inangle-distincts by fastforce
  }
  {
    assume  $B\ Out\ A\ C$ 
    hence  $SAMS\ A\ B\ P\ P\ B\ C$ 
      by (metis assms in-angle-out inangle-distincts out213--sams)
  }
  {
    assume  $\neg\ Col\ A\ B\ C$ 
    hence  $\neg\ Bet\ A\ B\ C$ 
      using Col-def by auto
    {
      assume  $Col\ B\ A\ P$ 
      have  $SAMS\ A\ B\ P\ P\ B\ C$ 
        by (metis  $\langle Col\ B\ A\ P \rangle \langle \neg\ Bet\ A\ B\ C \rangle$  assms col-in-angle-out inangle-distincts out213--sams)
    }
    {
      assume  $\neg\ Col\ B\ A\ P$ 
      {
        assume  $Col\ B\ C\ P$ 
        have  $SAMS\ A\ B\ P\ P\ B\ C$ 
          by (metis sams-comm  $\langle Col\ B\ C\ P \rangle \langle \neg\ Bet\ A\ B\ C \rangle$  assms between-symmetry col-in-angle-out inangle-distincts l11-24 out546--sams)
      }
      {
        assume  $\neg\ Col\ B\ C\ P$ 
        have  $SAMS\ A\ B\ P\ P\ B\ C$ 
        proof –
          have  $B\ P\ TS\ A\ C$ 
            by (simp add:  $\langle \neg\ Col\ B\ A\ P \rangle \langle \neg\ Col\ B\ C\ P \rangle$  assms in-angle-two-sides invert-two-sides)
          moreover have  $A\ B\ OS\ P\ C$ 
            by (simp add:  $\langle \neg\ Col\ A\ B\ C \rangle \langle \neg\ Col\ B\ A\ P \rangle$  assms in-angle-one-side)
          ultimately show ?thesis
            by (simp add: os-ts--sams)
          qed
        }
      }
    }
    hence  $SAMS\ A\ B\ P\ P\ B\ C$ 
      using  $\langle Col\ B\ C\ P \implies SAMS\ A\ B\ P\ P\ B\ C \rangle$  by blast
  }
  hence  $SAMS\ A\ B\ P\ P\ B\ C$ 
    using  $\langle Col\ B\ A\ P \implies SAMS\ A\ B\ P\ P\ B\ C \rangle$  by blast
}
thus ?thesis
using  $\langle B\ Out\ A\ C \implies SAMS\ A\ B\ P\ P\ B\ C \rangle$ 
   $\langle Bet\ A\ B\ C \implies SAMS\ A\ B\ P\ P\ B\ C \rangle \langle Bet\ A\ B\ C \vee B\ Out\ A\ C \vee \neg\ Col\ A\ B\ C \rangle$ 
by blast
qed

```

```

lemma sams-suma--lea123789:
  assumes  $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  and
     $SAMS\ A\ B\ C\ D\ E\ F$ 
  shows  $A\ B\ C\ LeA\ G\ H\ I$ 
proof cases
  assume  $Col\ A\ B\ C$ 
  show ?thesis
  proof cases
    assume  $Bet\ A\ B\ C$ 
    have  $P1: (A \neq B \wedge (E\ Out\ D\ F \vee \neg\ Bet\ A\ B\ C)) \wedge$ 
       $(\exists\ J. (C\ B\ J\ CongA\ D\ E\ F \wedge \neg\ (B\ C\ OS\ A\ J) \wedge \neg\ (A\ B\ TS\ C\ J) \wedge Coplanar\ A\ B\ C\ J))$ 

```

```

    using SAMS-def assms(2) by auto
  {
    assume E Out D F
    hence A B C CongA G H I
    using assms(1) out546-suma--conga by auto
    hence A B C LeA G H I
    by (simp add: conga--lea)
  }
  thus ?thesis
    using P1 ⟨Bet A B C⟩ by blast
next
  assume ¬ Bet A B C
  hence B Out A C
    using ⟨Col A B C⟩ or-bet-out by auto
  thus ?thesis
    by (metis assms(1) l11-31-1 suma-distincts)
qed
next
  assume ¬ Col A B C
  show ?thesis
  proof cases
    assume Col D E F
    show ?thesis
    proof cases
      assume Bet D E F
      have SAMS D E F A B C
        using assms(2) sams-sym by auto
      hence B Out A C
        using SAMS-def ⟨Bet D E F⟩ by blast
      thus ?thesis using l11-31-1
        by (metis assms(1) suma-distincts)
    next
      assume ¬ Bet D E F
      have A B C CongA G H I
      proof -
        have A B C D E F SumA G H I
          by (simp add: assms(1))
        moreover have E Out D F
          using ⟨Col D E F⟩ ⟨¬ Bet D E F⟩ l6-4-2 by auto
        ultimately show ?thesis
          using out546-suma--conga by auto
      qed
      show ?thesis
        by (simp add: ⟨A B C CongA G H I⟩ conga--lea)
    qed
  next
    assume ¬ Col D E F
    show ?thesis
    proof cases
      assume Col G H I
      show ?thesis
      proof cases
        assume Bet G H I
        thus ?thesis
          by (metis assms(1) l11-31-2 suma-distincts)
      next
        assume ¬ Bet G H I
        hence H Out G I
          by (simp add: ⟨Col G H I⟩ l6-4-2)
        have E Out D F ∨ ¬ Bet A B C
          using ⟨¬ Col A B C⟩ bet-col by auto
        {
          assume ¬ Bet A B C
          then obtain J where P2: C B J CongA D E F ∧ ¬ B C OS A J ∧
            Coplanar A B C J ∧ A B J CongA G H I
          using SumA-def assms(1) by blast

```

```

    have  $G H I \text{ CongA } A B J$ 
    using  $P2 \text{ not-conga-sym}$  by blast
    hence  $B \text{ Out } A J$ 
    using  $\langle H \text{ Out } G I \rangle \text{ out-conga-out}$  by blast
    hence  $B C \text{ OS } A J$ 
    using  $\text{Col-perm } \langle \neg \text{Col } A B C \rangle \text{ out-one-side}$  by blast
    hence  $\text{False}$ 
    using  $\langle C B J \text{ CongA } D E F \wedge \neg B C \text{ OS } A J \wedge$ 
     $\text{Coplanar } A B C J \wedge A B J \text{ CongA } G H I \rangle$  by linarith
  }
  hence  $\text{False}$ 
  using  $\text{Col-def } \langle \neg \text{Col } A B C \rangle$  by blast
  thus ?thesis by blast
qed
next
  assume  $\neg \text{Col } G H I$ 
  obtain  $J$  where  $P_4: C B J \text{ CongA } D E F \wedge \neg B C \text{ OS } A J \wedge$ 
 $\neg A B \text{ TS } C J \wedge \text{Coplanar } A B C J$ 
  using  $\text{SAMS-def assms}(2)$  by auto
  {
    assume  $\text{Col } J B C$ 
    have  $J B C \text{ CongA } D E F$ 
    by (simp add:  $P_4 \text{ conga-left-comm}$ )
    hence  $\text{Col } D E F$ 
    using  $\text{col-conga-col } \langle \text{Col } J B C \rangle$  by blast
    hence  $\text{False}$ 
    using  $\langle \neg \text{Col } D E F \rangle$  by blast
  }
  hence  $\neg \text{Col } J B C$  by blast
  have  $A B J \text{ CongA } G H I$ 
  proof -
    have  $A B C D E F \text{ SumA } A B J$ 
    proof -
      have  $C B J \text{ CongA } D E F$ 
      using  $P_4$  by simp
      moreover have  $\neg B C \text{ OS } A J$ 
      by (simp add:  $P_4$ )
      moreover have  $\text{Coplanar } A B C J$ 
      by (simp add:  $P_4$ )
      moreover have  $A B J \text{ CongA } A B J$ 
      by (metis  $\langle \neg \text{Col } A B C \rangle \langle \neg \text{Col } J B C \rangle \text{ col-trivial-1 conga-refl}$ )
      ultimately show ?thesis
      using  $\text{SumA-def}$  by blast
    qed
    thus ?thesis
    using  $\text{assms}(1) \text{ suma2--conga}$  by auto
  qed
  have  $\neg \text{Col } J B A$ 
  using  $\langle A B J \text{ CongA } G H I \rangle \langle \neg \text{Col } G H I \rangle \text{ col-conga-col not-col-permutation-3}$  by blast
  have  $A B C \text{ LeA } A B J$ 
  proof -
    have  $C \text{ InAngle } A B J$ 
    by (metis  $\text{Col-perm } P_4 \langle \neg \text{Col } A B C \rangle \langle \neg \text{Col } J B A \rangle \langle \neg \text{Col } J B C \rangle$ 
     $\text{cop-nos--ts coplanar-perm-7 coplanar-perm-8 invert-two-sides}$ 
     $\text{l9-2 os-ts--inangle}$ )
    moreover have  $A B C \text{ CongA } A B C$ 
    using  $\text{calculation in-angle-asym inangle3123 inangle-distincts}$  by auto
    ultimately show ?thesis
    using  $\text{inangle--lea}$  by blast
  qed
  thus ?thesis
  using  $\langle A B J \text{ CongA } G H I \rangle \text{ conga--lea lea-trans}$  by blast
qed
qed
qed

```

```

lemma sams-suma--lea456789:
  assumes A B C D E F SumA G H I and
    SAMS A B C D E F
  shows D E F LeA G H I
proof -
  have D E F A B C SumA G H I
    by (simp add: assms(1) suma-sym)
  moreover have SAMS D E F A B C
    using assms(2) sams-sym by blast
  ultimately show ?thesis
    using sams-suma--lea123789 by auto
qed

lemma sams-lea2--sams:
  assumes SAMS A' B' C' D' E' F' and
    A B C LeA A' B' C' and
    D E F LeA D' E' F'
  shows SAMS A B C D E F
proof -
  obtain A0 where B Midpoint A A0
    using symmetric-point-construction by auto
  obtain A'0 where B' Midpoint A' A'0
    using symmetric-point-construction by auto
  have D E F LeA C B A0
proof -
  have D' E' F' LeA C B A0
proof -
  have D' E' F' LeA C' B' A'0
    by (metis ‹B' Midpoint A' A'0› assms(1) assms(2) lea-distincts
      midpoint-bet midpoint-distinct-2 sams-chara)
  moreover have C' B' A'0 LeA C B A0
    by (metis Mid-cases ‹B Midpoint A A0› ‹B' Midpoint A' A'0› assms(2)
      l11-36-aux2 l7-3-2 lea-comm lea-distincts midpoint-bet sym-preserve-diff)
  ultimately show ?thesis
    using lea-trans by blast
qed
  moreover have D E F LeA D' E' F'
    using assms(3) by auto
  ultimately show ?thesis
    using ‹D' E' F' LeA C B A0› assms(3) lea-trans by blast
qed
thus ?thesis
  by (metis ‹B Midpoint A A0› assms(2) lea-distincts midpoint-bet sams-chara)
qed

lemma sams-lea456-suma2--lea:
  assumes D E F LeA D' E' F' and
    SAMS A B C D' E' F' and
    A B C D E F SumA G H I and
    A B C D' E' F' SumA G' H' I'
  shows G H I LeA G' H' I'
proof cases
  assume E' Out D' F'
  have G H I CongA G' H' I'
proof -
  have G H I CongA A B C
proof -
  have A B C D E F SumA G H I
    by (simp add: assms(3))
  moreover have E Out D F
    using ‹E' Out D' F'› assms(1) out-lea--out by blast
  ultimately show ?thesis
    using conga-sym out546-suma--conga by blast
qed
  moreover have A B C CongA G' H' I'
    using ‹E' Out D' F'› assms(4) out546-suma--conga by blast

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ultimately show ?thesis
  using conga-trans by blast
qed
thus ?thesis
  by (simp add: conga--lea)
next
assume T1:  $\neg E' \text{ Out } D' F'$ 
show ?thesis
proof cases
  assume T2:  $\text{Col } A \ B \ C$ 
  have  $E' \text{ Out } D' F' \vee \neg \text{Bet } A \ B \ C$ 
    using assms(2) SAMS-def by simp
  {
    assume  $\neg \text{Bet } A \ B \ C$ 
    hence  $B \text{ Out } A \ C$ 
      by (simp add: T2 l6-4-2)
    have  $G \ H \ I \text{ LeA } G' \ H' \ I'$ 
    proof -
      have  $D \ E \ F \text{ LeA } D' \ E' \ F'$ 
        by (simp add: assms(1))
      moreover have  $D \ E \ F \text{ CongA } G \ H \ I$ 
        using  $\langle B \text{ Out } A \ C \rangle$  assms(3) out213-suma--conga by auto
      moreover have  $D' \ E' \ F' \text{ CongA } G' \ H' \ I'$ 
        using  $\langle B \text{ Out } A \ C \rangle$  assms(4) out213-suma--conga by auto
      ultimately show ?thesis
        using l11-30 by blast
    qed
  }
  thus ?thesis
    using T1  $\langle E' \text{ Out } D' F' \vee \neg \text{Bet } A \ B \ C \rangle$  by auto
next
assume  $\neg \text{Col } A \ B \ C$ 
show ?thesis
proof cases
  assume  $\text{Col } D' \ E' \ F'$ 
  have  $\text{SAMS } D' \ E' \ F' \ A \ B \ C$ 
    by (simp add: assms(2) sams-sym)
  {
    assume  $\neg \text{Bet } D' \ E' \ F'$ 
    hence  $G \ H \ I \text{ LeA } G' \ H' \ I'$ 
      using not-bet-out T1  $\langle \text{Col } D' \ E' \ F' \rangle$  by auto
  }
  thus ?thesis
    by (metis assms(2) assms(3) assms(4) bet-lea--bet l11-31-2
      sams-suma--lea456789 suma-distincts)
next
assume  $\neg \text{Col } D' \ E' \ F'$ 
show ?thesis
proof cases
  assume  $\text{Col } D \ E \ F$ 
  have  $\neg \text{Bet } D \ E \ F$ 
    using bet-lea--bet Col-def  $\langle \neg \text{Col } D' \ E' \ F' \rangle$  assms(1) by blast
  thus ?thesis
  proof -
    have  $A \ B \ C \text{ LeA } G' \ H' \ I'$ 
      using assms(2) assms(4) sams-suma--lea123789 by auto
    moreover have  $A \ B \ C \text{ CongA } G \ H \ I$ 
      by (meson  $\langle \text{Col } D \ E \ F \rangle \langle \neg \text{Bet } D \ E \ F \rangle$  assms(3) or-bet-out
        out213-suma--conga suma-sym)
    moreover have  $G' \ H' \ I' \text{ CongA } G' \ H' \ I'$ 
    proof -
      have  $G' \neq H'$ 
        using calculation(1) lea-distincts by blast
      moreover have  $H' \neq I'$ 
        using  $\langle A \ B \ C \text{ LeA } G' \ H' \ I' \rangle$  lea-distincts by blast
      ultimately show ?thesis

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    using conga-refl by auto
qed
ultimately show ?thesis
using l11-30 by blast
qed
next
assume  $\neg Col\ D\ E\ F$ 
show ?thesis
proof cases
  assume  $Col\ G'\ H'\ I'$ 
  show ?thesis
  proof cases
    assume  $Bet\ G'\ H'\ I'$ 
    show ?thesis
    proof -
      have  $G \neq H$ 
      using assms(3) suma-distincts by auto
      moreover have  $I \neq H$ 
      using assms(3) suma-distincts by blast
      moreover have  $G' \neq H'$ 
      using assms(4) suma-distincts by auto
      moreover have  $I' \neq H'$ 
      using assms(4) suma-distincts by blast
      ultimately show ?thesis
      by (simp add:  $\langle Bet\ G'\ H'\ I' \rangle$  l11-31-2)
    qed
  next
  assume  $\neg Bet\ G'\ H'\ I'$ 
  have  $B\ Out\ A\ C$ 
  proof -
    have  $H'\ Out\ G'\ I'$ 
    using  $\langle Col\ G'\ H'\ I' \rangle$  l6-4-2 by (simp add:  $\langle \neg Bet\ G'\ H'\ I' \rangle$ )
    moreover have  $A\ B\ C\ LeA\ G'\ H'\ I'$  using sams-suma-lea123789
    using assms(2) assms(4) by auto
    ultimately show ?thesis
    using out-lea--out by blast
  qed
  hence  $Col\ A\ B\ C$ 
  using Col-perm out-col by blast
  hence False
  using  $\langle \neg Col\ A\ B\ C \rangle$  by auto
  thus ?thesis by blast
qed
next
assume  $\neg Col\ G'\ H'\ I'$ 
obtain  $F'1$  where  $P5: C\ B\ F'1\ CongA\ D'\ E'\ F' \wedge \neg B\ C\ OS\ A\ F'1 \wedge$ 
 $\neg A\ B\ TS\ C\ F'1 \wedge Coplanar\ A\ B\ C\ F'1$ 
using SAMS-def assms(2) by auto
have  $P6: D\ E\ F\ LeA\ C\ B\ F'1$ 
proof -
  have  $D\ E\ F\ CongA\ D\ E\ F$ 
  using  $\langle \neg Col\ D\ E\ F \rangle$  conga-refl not-col-distincts by fastforce
  moreover have  $D'\ E'\ F'\ CongA\ C\ B\ F'1$ 
  by (simp add:  $P5\ conga-sym$ )
  ultimately show ?thesis
  using assms(1) l11-30 by blast
qed
then obtain  $F1$  where  $P6: F1\ InAngle\ C\ B\ F'1 \wedge D\ E\ F\ CongA\ C\ B\ F1$ 
using LeA-def by auto
have  $A\ B\ F'1\ CongA\ G'\ H'\ I'$ 
proof -
  have  $A\ B\ C\ D'\ E'\ F'\ SumA\ A\ B\ F'1$ 
  proof -
    have  $C\ B\ F'1\ CongA\ D'\ E'\ F'$ 
    using  $P5$  by blast
    moreover have  $\neg B\ C\ OS\ A\ F'1$ 

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    using P5 by auto
  moreover have Coplanar A B C F'1
    by (simp add: P5)
  moreover have A B F'1 CongA A B F'1
  proof -
    have A ≠ B
      using ⟨¬ Col A B C⟩ col-trivial-1 by blast
    moreover have B ≠ F'1
      using P6 inangle-distincts by auto
    ultimately show ?thesis
      using conga-refl by auto
  qed
  ultimately show ?thesis
    using SumA-def by blast
  qed
  moreover have A B C D' E' F' SumA G' H' I'
    by (simp add: assms(4))
  ultimately show ?thesis
    using suma2--conga by auto
  qed
have ¬ Col A B F'1
  using ⟨A B F'1 CongA G' H' I'⟩ ⟨¬ Col G' H' I'⟩ col-conga-col by blast
have ¬ Col C B F'1
  proof -
    have ¬ Col D' E' F'
      by (simp add: ⟨¬ Col D' E' F'⟩)
    moreover have D' E' F' CongA C B F'1
      using P5 not-conga-sym by blast
    ultimately show ?thesis
      using ncol-conga-ncol by blast
  qed
show ?thesis
  proof -
    have A B F1 LeA A B F'1
    proof -
      have F1 InAngle A B F'1
    proof -
      have F1 InAngle F'1 B A
    proof -
      have F1 InAngle F'1 B C
        by (simp add: P6 l11-24)
      moreover have C InAngle F'1 B A
    proof -
      have B C TS A F'1
        using Col-perm P5 ⟨¬ Col A B C⟩ ⟨¬ Col C B F'1⟩
          cop-nos--ts ncoplanar-perm-12 by blast
      obtain X where Col X B C ∧ Bet A X F'1
        using TS-def ⟨B C TS A F'1⟩ by auto
      have Bet F'1 X A
        using Bet-perm ⟨Col X B C ∧ Bet A X F'1⟩ by blast
      moreover have (X = B) ∨ (B Out X C)
    proof -
      have B A OS X C
    proof -
      have A B OS X F'1
        by (metis ⟨Col X B C ∧ Bet A X F'1⟩ ⟨¬ Col A B C⟩
          ⟨¬ Col A B F'1⟩ bet-out-1 calculation out-one-side)
      moreover have A B OS F'1 C
        using Col-perm P5 ⟨¬ Col A B C⟩ ⟨¬ Col A B F'1⟩
          cop-nos--ts one-side-symmetry by blast
      ultimately show ?thesis
        using invert-one-side one-side-transitivity by blast
    qed
  thus ?thesis
    using Col-cases ⟨Col X B C ∧ Bet A X F'1⟩ col-one-side-out by blast
  qed

```



```

    ultimately show ?thesis
      by (metis InAngle-def ⟨¬ Col A B C⟩ ⟨¬ Col A B F'1⟩ not-col-distincts)
  qed
  ultimately show ?thesis
    using in-angle-trans by blast
  qed
  thus ?thesis
    using l11-24 by blast
  qed
  moreover have A B F1 CongA A B F1
  proof -
    have A ≠ B
      using ⟨¬ Col A B C⟩ col-trivial-1 by blast
    moreover have B ≠ F1
      using P6 conga-diff56 by blast
    ultimately show ?thesis
      using conga-refl by auto
  qed
  ultimately show ?thesis
    by (simp add: inangle--lea)
  qed
  moreover have A B F1 CongA G H I
  proof -
    have A B C D E F SumA A B F1
  proof -
    have B C TS F1 A
  proof -
    have B C TS F'1 A
  proof -
    have B C TS A F'1
      using Col-perm P5 ⟨¬ Col A B C⟩ ⟨¬ Col C B F'1⟩
        cop-nos--ts ncoplanar-perm-12 by blast
    thus ?thesis
      using l9-2 by blast
  qed
  moreover have B C OS F'1 F1
  proof -
    have ¬ Col C B F'1
      by (simp add: ⟨¬ Col C B F'1⟩)
    moreover have ¬ Col B C F1
  proof -
    have ¬ Col D E F
      using ⟨¬ Col D E F⟩ by auto
    moreover have D E F CongA C B F1
      by (simp add: P6)
    ultimately show ?thesis
      using ncol-conga-ncol not-col-permutation-4 by blast
  qed
  moreover have F1 InAngle C B F'1 using P6 by blast
  ultimately show ?thesis
    using in-angle-one-side invert-one-side one-side-symmetry by blast
  qed
  ultimately show ?thesis
    using l9-8-2 by blast
  qed
  thus ?thesis
  proof -
    have C B F1 CongA D E F
      using P6 not-conga-sym by blast
    moreover have ¬ B C OS A F1
      using ⟨B C TS F1 A⟩ l9-9 one-side-symmetry by blast
    moreover have Coplanar A B C F1
      using ⟨B C TS F1 A⟩ ncoplanar-perm-9 ts--coplanar by blast
    moreover have A B F1 CongA A B F1
  proof -
    have A ≠ B

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    using  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-1 by blast
  moreover have  $B \neq F1$ 
    using P6 conga-diff56 by blast
  ultimately show ?thesis
    using conga-refl by auto
qed
ultimately show ?thesis
  using SumA-def by blast
qed
qed
moreover have  $A \ B \ C \ D \ E \ F \ \text{SumA} \ G \ H \ I$ 
  by (simp add: assms(3))
ultimately show ?thesis
  using suma2--conga by auto
qed
ultimately show ?thesis
  using  $\langle A \ B \ F'1 \ \text{CongA} \ G' \ H' \ I' \rangle$  l11-30 by blast
qed
qed
qed
qed
qed
qed
lemma sams-lea123-suma2--lea:
  assumes  $A \ B \ C \ \text{LeA} \ A' \ B' \ C'$  and
    SAMS  $A' \ B' \ C' \ D \ E \ F$  and
     $A \ B \ C \ D \ E \ F \ \text{SumA} \ G \ H \ I$  and
     $A' \ B' \ C' \ D \ E \ F \ \text{SumA} \ G' \ H' \ I'$ 
  shows  $G \ H \ I \ \text{LeA} \ G' \ H' \ I'$ 
  by (meson assms(1) assms(2) assms(3) assms(4) sams-lea456-suma2--lea sams-sym suma-sym)

lemma sams-lea2-suma2--lea:
  assumes  $A \ B \ C \ \text{LeA} \ A' \ B' \ C'$  and
     $D \ E \ F \ \text{LeA} \ D' \ E' \ F'$  and
    SAMS  $A' \ B' \ C' \ D' \ E' \ F'$  and
     $A \ B \ C \ D \ E \ F \ \text{SumA} \ G \ H \ I$  and
     $A' \ B' \ C' \ D' \ E' \ F' \ \text{SumA} \ G' \ H' \ I'$ 
  shows  $G \ H \ I \ \text{LeA} \ G' \ H' \ I'$ 
proof -
  obtain  $G'' \ H'' \ I''$  where  $A \ B \ C \ D' \ E' \ F' \ \text{SumA} \ G'' \ H'' \ I''$ 
    using assms(4) assms(5) ex-suma suma-distincts by presburger
  have  $G \ H \ I \ \text{LeA} \ G'' \ H'' \ I''$ 
proof -
  have  $D \ E \ F \ \text{LeA} \ D' \ E' \ F'$ 
    by (simp add: assms(2))
  moreover have SAMS  $A \ B \ C \ D' \ E' \ F'$ 
proof -
  have SAMS  $A' \ B' \ C' \ D' \ E' \ F'$ 
    by (simp add: assms(3))
  moreover have  $A \ B \ C \ \text{LeA} \ A' \ B' \ C'$ 
    using assms(1) by auto
  moreover have  $D' \ E' \ F' \ \text{LeA} \ D' \ E' \ F'$ 
    using assms(2) lea-distincts lea-refl by blast
  ultimately show ?thesis
    using sams-lea2--sams by blast
qed
moreover have  $A \ B \ C \ D \ E \ F \ \text{SumA} \ G \ H \ I$ 
  by (simp add: assms(4))
moreover have  $A \ B \ C \ D' \ E' \ F' \ \text{SumA} \ G'' \ H'' \ I''$ 
  by (simp add:  $\langle A \ B \ C \ D' \ E' \ F' \ \text{SumA} \ G'' \ H'' \ I'' \rangle$ )
ultimately show ?thesis
  using sams-lea456-suma2--lea by blast
qed
moreover have  $G'' \ H'' \ I'' \ \text{LeA} \ G' \ H' \ I'$ 
  using sams-lea123-suma2--lea assms(3) assms(5)  $\langle A \ B \ C \ D' \ E' \ F' \ \text{SumA} \ G'' \ H'' \ I'' \rangle$ 

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    assms(1) by blast
ultimately show ?thesis
using lea-trans by blast
qed

lemma sams2-suma2--conga456:
  assumes SAMS A B C D E F and
    SAMS A B C D' E' F' and
    A B C D E F SumA G H I and
    A B C D' E' F' SumA G H I
  shows D E F CongA D' E' F'
proof cases
  assume Col A B C
  show ?thesis
proof cases
  assume P2: Bet A B C
  hence E Out D F
    using assms(1) SAMS-def by blast
  moreover have E' Out D' F'
    using P2 assms(2) SAMS-def by blast
  ultimately show ?thesis
    by (simp add: l11-21-b)
next
  assume ¬ Bet A B C
  hence B Out A C
    using ⟨Col A B C⟩ or-bet-out by blast
  show ?thesis
proof —
  have D E F CongA G H I
  proof —
    have A B C D E F SumA G H I
      by (simp add: assms(3))
    thus ?thesis
      using ⟨B Out A C⟩ out213-suma--conga by auto
  qed
  moreover have G H I CongA D' E' F'
  proof —
    have A B C D' E' F' SumA G H I
      by (simp add: assms(4))
    hence D' E' F' CongA G H I
      using ⟨B Out A C⟩ out213-suma--conga by auto
    thus ?thesis
      using not-conga-sym by blast
  qed
  ultimately show ?thesis
    using not-conga by blast
qed
qed
next
  assume ¬ Col A B C
  obtain J where T1: C B J CongA D E F ∧ ¬ B C OS A J ∧ ¬ A B TS C J ∧ Coplanar A B C J
    using assms(1) SAMS-def by blast
  have T1A: C B J CongA D E F
    using T1 by simp
  have T1B: ¬ B C OS A J
    using T1 by simp
  have T1C: ¬ A B TS C J
    using T1 by simp
  have T1D: Coplanar A B C J
    using T1 by simp
  obtain J' where T2: C B J' CongA D' E' F' ∧ ¬ B C OS A J' ∧
    ¬ A B TS C J' ∧ Coplanar A B C J'
    using assms(2) SAMS-def by blast
  have T2A: C B J' CongA D' E' F'
    using T2 by simp
  have T2B: ¬ B C OS A J'

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using T2 by simp
have T2C:  $\neg A B TS C J'$ 
using T2 by simp
have T2D:  $Coplanar A B C J'$ 
using T2 by simp
have T3:  $C B J CongA C B J'$ 
proof -
  have A B J CongA A B J'
  proof -
    have A B J CongA G H I
    proof -
      have A B C D E F SumA A B J
      using SumA-def T1A T1B T1D  $\langle \neg Col A B C \rangle$  conga-distinct
      conga-refl not-col-distincts by auto
      thus ?thesis
      using assms(3) suma2--conga by blast
    qed
    moreover have G H I CongA A B J'
    proof -
      have A B C D' E' F' SumA G H I
      by (simp add: assms(4))
      moreover have A B C D' E' F' SumA A B J'
      using SumA-def T2A T2B T2D  $\langle \neg Col A B C \rangle$  conga-distinct
      conga-refl not-col-distincts by auto
      ultimately show ?thesis
      using suma2--conga by auto
    qed
    ultimately show ?thesis
    using conga-trans by blast
  qed
  have B Out J J'  $\vee A B TS J J'$ 
  proof -
    have Coplanar A B J J'
    using T1D T2D  $\langle \neg Col A B C \rangle$  coplanar-trans-1 ncoplanar-perm-8
    not-col-permutation-2 by blast
    moreover have A B J CongA A B J'
    by (simp add:  $\langle A B J CongA A B J' \rangle$ )
    ultimately show ?thesis
    by (simp add: conga-cop--or-out-ts)
  qed
  {
    assume B Out J J'
    hence C B J CongA C B J'
    by (metis Out-cases  $\langle \neg Col A B C \rangle$  bet-out between-trivial
    not-col-distincts out2--conga)
  }
  {
    assume A B TS J J'
    hence A B OS J C
    by (meson T1C T1D TS-def  $\langle \neg Col A B C \rangle$  cop-nts--os not-col-permutation-2
    one-side-symmetry)
    hence A B TS C J'
    using  $\langle A B TS J J' \rangle$  l9-8-2 by blast
    hence False
    by (simp add: T2C)
    hence C B J CongA C B J'
    by blast
  }
  thus ?thesis
  using  $\langle B Out J J' \implies C B J CongA C B J' \rangle$   $\langle B Out J J' \vee A B TS J J' \rangle$  by blast
qed
hence C B J CongA D' E' F'
using T2A not-conga by blast
thus ?thesis
using T1A not-conga not-conga-sym by blast
qed

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lemma sams2-suma2--conga123:
  assumes SAMS A B C D E F and
    SAMS A' B' C' D E F and
    A B C D E F SumA G H I and
    A' B' C' D E F SumA G H I
  shows A B C CongA A' B' C'
proof -
  have SAMS D E F A B C
    by (simp add: assms(1) sams-sym)
  moreover have SAMS D E F A' B' C'
    by (simp add: assms(2) sams-sym)
  moreover have D E F A B C SumA G H I
    by (simp add: assms(3) suma-sym)
  moreover have D E F A' B' C' SumA G H I
    using assms(4) suma-sym by blast
  ultimately show ?thesis
    using sams2-suma2--conga456 by auto
qed

lemma suma-assoc-1:
  assumes SAMS A B C D E F and
    SAMS D E F G H I and
    A B C D E F SumA A' B' C' and
    D E F G H I SumA D' E' F' and
    A' B' C' G H I SumA K L M
  shows A B C D' E' F' SumA K L M
proof -
  obtain A0 where P1: Bet A B A0  $\wedge$  Cong A B B A0
    using Cong-perm segment-construction by blast
  obtain D0 where P2: Bet D E D0  $\wedge$  Cong D E E D0
    using Cong-perm segment-construction by blast
  show ?thesis
proof cases
  assume Col A B C
  show ?thesis
proof cases
  assume Bet A B C
  hence E Out D F
    using SAMS-def assms(1) by simp
  show ?thesis
proof -
  have A' B' C' CongA A B C
    using assms(3)  $\langle E \text{ Out } D F \rangle$  conga-sym out546-suma--conga by blast
  moreover have G H I CongA D' E' F'
    using assms(4)  $\langle E \text{ Out } D F \rangle$  out213-suma--conga by auto
  ultimately show ?thesis
    by (meson conga3-suma--suma suma2--conga assms(5))
qed
next
  assume  $\neg$  Bet A B C
  hence B Out A C
    using  $\langle Col A B C \rangle$  l6-4-2 by auto
  have A  $\neq$  B
    using  $\langle B \text{ Out } A C \rangle$  out-distinct by auto
  have B  $\neq$  C
    using  $\langle \neg Bet A B C \rangle$  between-trivial by auto
  have D'  $\neq$  E'
    using assms(4) suma-distincts by blast
  have E'  $\neq$  F'
    using assms(4) suma-distincts by auto
  show ?thesis
proof -
  obtain K0 L0 M0 where P3: A B C D' E' F' SumA K0 L0 M0
    using ex-suma  $\langle A \neq B \rangle \langle B \neq C \rangle \langle D' \neq E' \rangle \langle E' \neq F' \rangle$  by presburger
  moreover have A B C CongA A B C

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    using  $\langle A \neq B \rangle \langle B \neq C \rangle$  conga-refl by auto
  moreover have  $D' E' F' \text{ CongA } D' E' F'$ 
    using  $\langle D' \neq E' \rangle \langle E' \neq F' \rangle$  conga-refl by auto
  moreover have  $K0 L0 M0 \text{ CongA } K L M$ 
  proof -
    have  $K0 L0 M0 \text{ CongA } D' E' F'$ 
      using  $P3 \langle B \text{ Out } A C \rangle$  conga-sym out213-suma--conga by blast
    moreover have  $D' E' F' \text{ CongA } K L M$ 
    proof -
      have  $D E F G H I \text{ SumA } D' E' F'$ 
        by (simp add: assms(4))
      moreover have  $D E F G H I \text{ SumA } K L M$ 
        by (meson conga3-suma--suma out213-suma--conga sams2-suma2--conga456
            suma2--conga  $\langle B \text{ Out } A C \rangle$  assms(2) assms(3) assms(5)
            calculation not-conga-sym)
      ultimately show ?thesis
        using suma2--conga by auto
    qed
  ultimately show ?thesis
    using not-conga by blast
  qed
  ultimately show ?thesis
    using conga3-suma--suma by blast
  qed
next
assume  $T1: \neg \text{Col } A B C$ 
have  $\neg \text{Col } C B A0$ 
  by (metis Col-def P1  $\langle \neg \text{Col } A B C \rangle$  cong-diff l6-16-1)
show ?thesis
proof cases
  assume  $\text{Col } D E F$ 
  show ?thesis
  proof cases
    assume  $\text{Bet } D E F$ 
    have  $H \text{ Out } G I$  using SAMS-def assms(2)  $\langle \text{Bet } D E F \rangle$  by blast
    have  $A B C D E F \text{ SumA } A' B' C'$ 
      by (simp add: assms(3))
    moreover have  $A B C \text{ CongA } A B C$ 
      by (metis  $\langle \neg \text{Col } A B C \rangle$  conga-pseudo-refl conga-right-comm not-col-distincts)
    moreover have  $D E F \text{ CongA } D' E' F'$ 
      using  $\langle H \text{ Out } G I \rangle$  assms(4) out546-suma--conga by auto
    moreover have  $A' B' C' \text{ CongA } K L M$ 
      using  $\langle H \text{ Out } G I \rangle$  assms(5) out546-suma--conga by auto
    ultimately show ?thesis
      using conga3-suma--suma by blast
  next
    assume  $\neg \text{Bet } D E F$ 
    hence  $E \text{ Out } D F$ 
      using not-bet-out by (simp add:  $\langle \text{Col } D E F \rangle$ )
    show ?thesis
    proof -
      have  $A' B' C' \text{ CongA } A B C$ 
        using assms(3)  $\langle E \text{ Out } D F \rangle$  conga-sym out546-suma--conga by blast
      moreover have  $G H I \text{ CongA } D' E' F'$ 
        using out546-suma--conga  $\langle E \text{ Out } D F \rangle$  assms(4) out213-suma--conga by auto
      moreover have  $K L M \text{ CongA } K L M$ 
      proof -
        have  $K \neq L \wedge L \neq M$ 
          using assms(5) suma-distincts by blast
        thus ?thesis
          using conga-refl by auto
      qed
    ultimately show ?thesis
      using assms(5) conga3-suma--suma by blast
  qed

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qed
next
  assume  $\neg \text{Col } D \ E \ F$ 
  hence  $\neg \text{Col } F \ E \ D0$ 
  by (metis Col-def P2 cong-diff l6-16-1 not-col-distincts)
  show ?thesis
  proof cases
    assume Col G H I
    show ?thesis
    proof cases
      assume Bet G H I
      have SAMS G H I D E F
      by (simp add: assms(2) sams-sym)
      hence E Out D F
      using SAMS-def  $\langle \text{Bet } G \ H \ I \rangle$  by blast
      hence Col D E F
      using Col-perm out-col by blast
      hence False
      using  $\langle \neg \text{Col } D \ E \ F \rangle$  by auto
      thus ?thesis by simp
    next
      assume  $\neg \text{Bet } G \ H \ I$ 
      hence H Out G I
      using SAMS-def by (simp add:  $\langle \text{Col } G \ H \ I \rangle$  l6-4-2)
      show ?thesis
      proof -
        have A B C CongA A B C
        by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  conga-refl not-col-distincts)
        moreover have D E F CongA D' E' F'
        using assms(4) out546-suma--conga  $\langle H \ \text{Out } G \ I \rangle$  by auto
        moreover have A' B' C' CongA K L M
        using  $\langle H \ \text{Out } G \ I \rangle$  assms(5) out546-suma--conga by auto
        ultimately show ?thesis
        using assms(3) conga3-suma--suma by blast
      qed
    qed
  next
    assume  $\neg \text{Col } G \ H \ I$ 
    have  $\neg B \ C \ OS \ A \ A0$ 
    using P1 col-trivial-1 one-side-chara by blast
    have E F TS D D0
    by (metis P2  $\langle \neg \text{Col } D \ E \ F \rangle$   $\langle \neg \text{Col } F \ E \ D0 \rangle$  bet--ts bet-col
        between-trivial not-col-permutation-1)
    show ?thesis
    proof cases
      assume Col A' B' C'
      show ?thesis
      proof cases
        assume Bet A' B' C'
        show ?thesis
        proof cases
          assume Col D' E' F'
          show ?thesis
          proof cases
            assume Bet D' E' F'
            have A B C CongA G H I
            proof -
              have A B C CongA D0 E F
              proof -
                have SAMS A B C D E F
                by (simp add: assms(1))
                moreover have SAMS D0 E F D E F
                by (metis P2  $\langle \neg \text{Col } D \ E \ F \rangle$   $\langle \neg \text{Col } F \ E \ D0 \rangle$  bet--sams
                    between-symmetry not-col-distincts sams-right-comm)
                moreover have A B C D E F SumA A' B' C'
                by (simp add: assms(3))

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moreover have  $D0\ E\ F\ D\ E\ F\ SumA\ A'\ B'\ C'$ 
proof –
  have  $D\ E\ F\ D0\ E\ F\ SumA\ A'\ B'\ C'$ 
  proof –
    have  $F\ E\ D0\ CongA\ D0\ E\ F$ 
    by (metis  $\langle \neg\ Col\ F\ E\ D0 \rangle$  col-trivial-1 col-trivial-2
      conga-pseudo-refl)
    moreover have  $\neg\ E\ F\ OS\ D\ D0$ 
    using P2 col-trivial-1 one-side-chara by blast
    moreover have Coplanar  $D\ E\ F\ D0$ 
    by (meson P2 bet--coplanar ncoplanar-perm-1)
    moreover have  $D\ E\ D0\ CongA\ A'\ B'\ C'$ 
    using assms(3) P2  $\langle Bet\ A'\ B'\ C' \rangle$   $\langle \neg\ Col\ F\ E\ D0 \rangle$ 
      conga-line not-col-distincts suma-distincts by auto
    ultimately show ?thesis
    using SumA-def by blast
  qed
  thus ?thesis
  by (simp add:  $\langle D\ E\ F\ D0\ E\ F\ SumA\ A'\ B'\ C' \rangle$  suma-sym)
qed
ultimately show ?thesis
using sams2-suma2--conga123 by blast
qed
moreover have  $D0\ E\ F\ CongA\ G\ H\ I$ 
proof –
  have SAMS  $D\ E\ F\ D0\ E\ F$ 
  using P2  $\langle \neg\ Col\ D\ E\ F \rangle$   $\langle \neg\ Col\ F\ E\ D0 \rangle$  bet--sams
    not-col-distincts sams-right-comm by auto
  moreover have  $D\ E\ F\ D0\ E\ F\ SumA\ D'\ E'\ F'$ 
  proof –
    have  $F\ E\ D0\ CongA\ D0\ E\ F$ 
    by (metis (no-types)  $\langle \neg\ Col\ F\ E\ D0 \rangle$  col-trivial-1
      col-trivial-2 conga-pseudo-refl)
    moreover have  $\neg\ E\ F\ OS\ D\ D0$ 
    using P2 col-trivial-1 one-side-chara by blast
    moreover have Coplanar  $D\ E\ F\ D0$ 
    using P2 bet--coplanar ncoplanar-perm-1 by blast
    moreover have  $D\ E\ D0\ CongA\ D'\ E'\ F'$ 
    using assms(3) P2  $\langle Bet\ D'\ E'\ F' \rangle$   $\langle \neg\ Col\ F\ E\ D0 \rangle$  assms(4)
      conga-line not-col-distincts suma-distincts by auto
    ultimately show ?thesis
    using SumA-def by blast
  qed
  ultimately show ?thesis
  using assms(2) assms(4) sams2-suma2--conga456 by auto
qed
ultimately show ?thesis
using conga-trans by blast
qed
hence  $G\ H\ I\ CongA\ A\ B\ C$ 
using not-conga-sym by blast
have  $D'\ E'\ F'\ A\ B\ C\ SumA\ K\ L\ M$ 
proof –
  have  $A'\ B'\ C'\ CongA\ D'\ E'\ F'$ 
  by (metis suma-distincts  $\langle Bet\ A'\ B'\ C' \rangle$   $\langle Bet\ D'\ E'\ F' \rangle$ 
    assms(4) assms(5) conga-line)
  thus ?thesis
  by (meson conga3-suma--suma suma2--conga
     $\langle G\ H\ I\ CongA\ A\ B\ C \rangle$  assms(5))
qed
thus ?thesis
by (simp add: suma-sym)
next
assume  $\neg\ Bet\ D'\ E'\ F'$ 
hence  $E'\ Out\ D'\ F'$ 
by (simp add:  $\langle Col\ D'\ E'\ F' \rangle$  l6-4-2)

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have D E F LeA D' E' F'
  using assms(2) assms(4) sams-suma--lea123789 by blast
hence E Out D F
  using ⟨E' Out D' F'⟩ out-lea--out by blast
hence Col D E F
  using Col-perm out-col by blast
hence False
  using ⟨¬ Col D E F⟩ by auto
thus ?thesis by simp
qed
next
assume ¬ Col D' E' F'
have D E F CongA C B A0
proof -
  have SAMS A B C D E F
    by (simp add: assms(1))
  moreover have SAMS A B C C B A0
    using P1 ⟨¬ Col A B C⟩ ⟨¬ Col C B A0⟩ bet--sams not-col-distincts by auto
  moreover have A B C D E F SumA A' B' C'
    by (simp add: assms(3))
  moreover have A B C C B A0 SumA A' B' C'
  proof -
    have A B C C B A0 SumA A B A0
      by (metis P1 ⟨¬ Col A B C⟩ ⟨¬ Col C B A0⟩ bet--suma
        col-trivial-1 col-trivial-2)
    moreover have A B C CongA A B C
      using ⟨SAMS A B C C B A0⟩ calculation
        sams2-suma2--conga123 by auto
    moreover have C B A0 CongA C B A0
      using ⟨SAMS A B C C B A0⟩ calculation(1)
        sams2-suma2--conga456 by auto
    moreover have A B A0 CongA A' B' C'
      using P1 ⟨Bet A' B' C'⟩ ⟨¬ Col C B A0⟩ assms(3)
        conga-line not-col-distincts suma-distincts by auto
    ultimately show ?thesis
      using conga3-suma--suma by blast
  qed
  ultimately show ?thesis
    using sams2-suma2--conga456 by blast
qed
have SAMS C B A0 G H I
proof -
  have D E F CongA C B A0
    by (simp add: ⟨D E F CongA C B A0⟩)
  moreover have G H I CongA G H I
    using ⟨¬ Col G H I⟩ conga-refl not-col-distincts by fastforce
  moreover have SAMS D E F G H I
    by (simp add: assms(2))
  ultimately show ?thesis
    using conga2-sams--sams by blast
qed
then obtain J where P3: A0 B J CongA G H I ∧ ¬ B A0 OS C J ∧
¬ C B TS A0 J ∧ Coplanar C B A0 J
  using SAMS-def by blast
obtain F1 where P4: F E F1 CongA G H I ∧ ¬ E F OS D F1 ∧
¬ D E TS F F1 ∧ Coplanar D E F F1
  using SAMS-def assms(2) by auto
have C B J CongA D' E' F'
proof -
  have C B J CongA D E F1
  proof -
    have (B A0 TS C J ∧ E F TS D F1) ∨ (B A0 OS C J ∧ E F OS D F1)
    proof -
      have B A0 TS C J
    proof -
      have Coplanar B A0 C J

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    using P3 ncoplanar-perm-12 by blast
  moreover have  $\neg \text{Col } C \ B \ A0$ 
    by (simp add:  $\langle \neg \text{Col } C \ B \ A0 \rangle$ )
  moreover have  $\neg \text{Col } J \ B \ A0$ 
    using P3  $\langle \neg \text{Col } G \ H \ I \rangle$  col-conga-col not-col-permutation-3 by blast
  moreover have  $\neg B \ A0 \ OS \ C \ J$ 
    using P3 by simp
  ultimately show ?thesis
    by (simp add: cop-nos--ts)
qed
moreover have  $E \ F \ TS \ D \ F1$ 
proof -
  have Coplanar  $E \ F \ D \ F1$ 
    using P4 ncoplanar-perm-12 by blast
  moreover have  $\neg \text{Col } D \ E \ F$ 
    by (simp add:  $\langle \neg \text{Col } D \ E \ F \rangle$ )
  moreover have  $\neg \text{Col } F1 \ E \ F$ 
    using P4  $\langle \neg \text{Col } G \ H \ I \rangle$  col-conga-col col-permutation-3 by blast
  moreover have  $\neg E \ F \ OS \ D \ F1$ 
    using P4 by auto
  ultimately show ?thesis
    by (simp add: cop-nos--ts)
qed
ultimately show ?thesis
  by simp
qed
moreover have  $C \ B \ A0 \ CongA \ D \ E \ F$ 
  using  $\langle D \ E \ F \ CongA \ C \ B \ A0 \rangle$  not-conga-sym by blast
moreover have  $A0 \ B \ J \ CongA \ F \ E \ F1$ 
proof -
  have  $A0 \ B \ J \ CongA \ G \ H \ I$ 
    by (simp add: P3)
  moreover have  $G \ H \ I \ CongA \ F \ E \ F1$ 
    using P4 not-conga-sym by blast
  ultimately show ?thesis
    using conga-trans by blast
qed
ultimately show ?thesis
  using l11-22 by auto
qed
moreover have  $D \ E \ F1 \ CongA \ D' \ E' \ F'$ 
proof -
  have  $D \ E \ F \ G \ H \ I \ SumA \ D \ E \ F1$ 
    using P4 SumA-def  $\langle \neg \text{Col } D \ E \ F \rangle$  conga-distinct
    conga-refl not-col-distincts by auto
  moreover have  $D \ E \ F \ G \ H \ I \ SumA \ D' \ E' \ F'$ 
    by (simp add: assms(4))
  ultimately show ?thesis
    using suma2--conga by auto
qed
ultimately show ?thesis
  using conga-trans by blast
qed
show ?thesis
proof -
  have  $A \ B \ C \ D' \ E' \ F' \ SumA \ A \ B \ J$ 
proof -
  have  $C \ B \ TS \ J \ A$ 
proof -
  have  $C \ B \ TS \ A0 \ A$ 
proof -
  have  $B \neq A0$ 
    using  $\langle \neg \text{Col } C \ B \ A0 \rangle$  not-col-distincts by blast
  moreover have  $\neg \text{Col } B \ C \ A$ 
    using Col-cases  $\langle \neg \text{Col } A \ B \ C \rangle$  by auto
  moreover have  $Bet \ A \ B \ A0$ 

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    by (simp add: P1)
  ultimately show ?thesis
    by (metis Bet-cases Col-cases  $\neg$  Col C B A0)
      bet--ts invert-two-sides not-col-distincts)
qed
moreover have C B OS A0 J
proof -
  have  $\neg$  Col J C B
    using  $\langle C B J \text{ CongA } D' E' F' \rangle \langle \neg \text{ Col } D' E' F' \rangle$ 
      col-conga-col not-col-permutation-2 by blast
  moreover have  $\neg$  Col A0 C B
    using Col-cases  $\langle \neg \text{ Col } C B A0 \rangle$  by blast
  ultimately show ?thesis
    using P3 cop-nos--ts by blast
qed
ultimately show ?thesis
  using l9-8-2 by blast
qed
moreover have C B J CongA D' E' F'
  by (simp add:  $\langle C B J \text{ CongA } D' E' F' \rangle$ )
moreover have  $\neg$  B C OS A J
  using calculation(1) invert-one-side l9-9-bis
    one-side-symmetry by blast
moreover have Coplanar A B C J
  using calculation(1) ncoplanar-perm-15 ts--coplanar by blast
moreover have A B J CongA A B J
proof -
  have  $A \neq B$ 
    using  $\langle \neg \text{ Col } A B C \rangle$  col-trivial-1 by auto
  moreover have  $B \neq J$ 
    using  $\langle C B \text{ TS } J A \rangle$  ts-distincts by blast
  ultimately show ?thesis
    using conga-refl by auto
qed
ultimately show ?thesis
  using SumA-def by blast
qed
moreover have A B J CongA K L M
proof -
  have A' B' C' G H I SumA A B J
  proof -
    have A B A0 CongA A' B' C'
      using P1  $\langle \text{Bet } A' B' C' \rangle \langle \neg \text{ Col } A B C \rangle \langle \neg \text{ Col } C B A0 \rangle$ 
        assms(5) conga-line not-col-distincts suma-distincts by auto
    moreover have A0 B J CongA G H I
      by (simp add: P3)
    moreover have A B A0 A0 B J SumA A B J
  proof -
    have A0 B J CongA A0 B J
  proof -
    have A0  $\neq$  B
      using  $\langle \neg \text{ Col } C B A0 \rangle$  col-trivial-2 by auto
    moreover have  $B \neq J$ 
      using CongA-def  $\langle A0 B J \text{ CongA } G H I \rangle$  by blast
    ultimately show ?thesis
      using conga-refl by auto
  qed
  moreover have  $\neg$  B A0 OS A J
    by (simp add: Col-def P1 col123--nos)
  moreover have Coplanar A B A0 J
    using P1 bet--coplanar by auto
  moreover have A B J CongA A B J
    using P1  $\langle \neg \text{ Col } A B C \rangle$  between-symmetry calculation(1)
      l11-13 not-col-distincts by blast
  ultimately show ?thesis
    using SumA-def by blast

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    qed
    ultimately show ?thesis
      by (meson conga3-suma--suma suma2--conga)
  qed
  moreover have A' B' C' G H I SumA K L M
    by (simp add: assms(5))
  ultimately show ?thesis
    using suma2--conga by auto
  qed
  ultimately show ?thesis
  proof -
    have A B C CongA A B C  $\wedge$  D' E' F' CongA D' E' F'
      using CongA-def  $\langle$  A B J CongA K L M  $\rangle$   $\langle$  C B J CongA D' E' F'  $\rangle$ 
      conga-refl by presburger
    thus ?thesis
      using  $\langle$  A B C D' E' F' SumA A B J  $\rangle$   $\langle$  A B J CongA K L M  $\rangle$ 
      conga3-suma--suma by blast
  qed
  qed
  qed
next
  assume  $\neg$  Bet A' B' C'
  have B Out A C
  proof -
    have A B C LeA A' B' C'
      using assms(1) assms(3) sams-suma--lea123789 by auto
    moreover have B' Out A' C'
      using  $\langle$  Col A' B' C'  $\rangle$   $\langle$   $\neg$  Bet A' B' C'  $\rangle$  or-bet-out by blast
    ultimately show ?thesis
      using out-lea--out by blast
  qed
  hence Col A B C
    using Col-perm out-col by blast
  hence False
    using  $\langle$   $\neg$  Col A B C  $\rangle$  by auto
  thus ?thesis by simp
  qed
next
  assume  $\neg$  Col A' B' C'
  obtain C1 where P6: C B C1 CongA D E F  $\wedge$   $\neg$  B C OS A C1  $\wedge$ 
 $\neg$  A B TS C C1  $\wedge$  Coplanar A B C C1
    using SAMS-def assms(1) by auto
  have P6A: C B C1 CongA D E F
    using P6 by simp
  have P6B:  $\neg$  B C OS A C1
    using P6 by simp
  have P6C:  $\neg$  A B TS C C1
    using P6 by simp
  have P6D: Coplanar A B C C1
    using P6 by simp
  have A' B' C' CongA A B C1
  proof -
    have A B C D E F SumA A B C1
      using P6A P6B P6D SumA-def  $\langle$   $\neg$  Col A B C  $\rangle$  conga-distinct conga-refl
      not-col-distincts by auto
    moreover have A B C D E F SumA A' B' C'
      by (simp add: assms(3))
    ultimately show ?thesis
      using suma2--conga by auto
  qed
  qed
  have B C1 OS C A
  proof -
    have B A OS C C1
  proof -
    have A B OS C C1
  proof -

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have  $\neg \text{Col } C \ A \ B$ 
  using Col-perm  $\langle \neg \text{Col } A \ B \ C \rangle$  by blast
moreover have  $\neg \text{Col } C1 \ A \ B$ 
  using  $\langle \neg \text{Col } A' \ B' \ C' \rangle \langle A' \ B' \ C' \text{ CongA } A \ B \ C1 \rangle$  col-permutation-1
    ncol-conga-ncol by blast
ultimately show ?thesis
  using P6C P6D cop-nos--ts by blast
qed
thus ?thesis
  by (simp add: invert-one-side)
qed
moreover have  $B \ C \ TS \ A \ C1$ 
proof —
  have Coplanar  $B \ C \ A \ C1$ 
    using P6D ncoplanar-perm-12 by blast
  moreover have  $\neg \text{Col } C1 \ B \ C$ 
  proof —
    have  $D \ E \ F \text{ CongA } C1 \ B \ C$ 
      using P6A conga-left-comm not-conga-sym by blast
    thus ?thesis
      using  $\langle \neg \text{Col } D \ E \ F \rangle$  ncol-conga-ncol by blast
  qed
  ultimately show ?thesis
    using T1 P6B cop-nos--ts by blast
qed
ultimately show ?thesis
  using os-ts1324--os one-side-symmetry by blast
qed
show ?thesis
proof cases
  assume  $\text{Col } D' \ E' \ F'$ 
  show ?thesis
  proof cases
    assume  $\text{Bet } D' \ E' \ F'$ 
    obtain  $C0$  where  $P7: \text{Bet } C \ B \ C0 \wedge \text{Cong } C \ B \ B \ C0$ 
      using Cong-perm segment-construction by blast
    have  $B \ C1 \ TS \ C \ C0$ 
      by (metis  $P7 \langle B \ C1 \ OS \ C \ A \rangle$  bet--ts cong-diff-2 not-col-distincts
        one-side-not-col123)
    show ?thesis
    proof —
      have  $A \ B \ C \ C \ B \ C0 \text{ SumA } A \ B \ C0$ 
      proof —
        have  $C \ B \ C0 \text{ CongA } C \ B \ C0$ 
          by (metis  $P7 \ T1 \text{ cong-diff conga-line not-col-distincts}$ )
        moreover have  $\neg B \ C \ OS \ A \ C0$ 
          using  $P7 \text{ bet-col col124--nos invert-one-side}$  by blast
        moreover have Coplanar  $A \ B \ C \ C0$ 
          using  $P7 \text{ bet--coplanar ncoplanar-perm-15}$  by blast
        moreover have  $A \ B \ C0 \text{ CongA } A \ B \ C0$ 
          by (metis  $P7 \ T1 \text{ cong-diff conga-refl not-col-distincts}$ )
        ultimately show ?thesis
          using SumA-def by blast
      qed
    qed
  moreover have  $A \ B \ C0 \text{ CongA } K \ L \ M$ 
  proof —
    have  $A' \ B' \ C' \ G \ H \ I \text{ SumA } A \ B \ C0$ 
    proof —
      have  $A \ B \ C1 \ C1 \ B \ C0 \text{ SumA } A \ B \ C0$ 
        using  $\langle B \ C1 \ TS \ C \ C0 \rangle \langle B \ C1 \ OS \ C \ A \rangle$  l9-8-2 ts--suma-1 by blast
      moreover have  $A \ B \ C1 \text{ CongA } A' \ B' \ C'$ 
        by (simp add: P6  $\langle A' \ B' \ C' \text{ CongA } A \ B \ C1 \rangle$  conga-sym)
      moreover have  $C1 \ B \ C0 \text{ CongA } G \ H \ I$ 
    proof —
      have  $SAMS \ C \ B \ C1 \ C1 \ B \ C0$ 
        by (metis  $P7 \langle B \ C1 \ TS \ C \ C0 \rangle$  bet--sams ts-distincts)

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moreover have SAMS C B C1 G H I
proof –
  have D E F CongA C B C1
    using P6A not-conga-sym by blast
  moreover have G H I CongA G H I
    using  $\langle \neg \text{Col } G \ H \ I \rangle$  conga-refl not-col-distincts by fastforce
  moreover have SAMS D E F G H I
    by (simp add: assms(2))
  ultimately show ?thesis
    using conga2-sams--sams by blast
qed
moreover have C B C1 C1 B C0 SumA C B C0
  by (simp add:  $\langle B \ C1 \ TS \ C \ C0 \rangle$  ts--suma-1)
moreover have C B C1 G H I SumA C B C0
proof –
  have D E F G H I SumA D' E' F'
    by (simp add: assms(4))
  moreover have D E F CongA C B C1
    using P6A not-conga-sym by blast
  moreover have G H I CongA G H I
    using  $\langle \neg \text{Col } G \ H \ I \rangle$  conga-refl not-col-distincts by fastforce
  moreover have D' E' F' CongA C B C0 using P7 assms(4)
    by (metis P6A suma-distincts  $\langle \text{Bet } D' \ E' \ F' \rangle$ 
      cong-diff conga-diff1 conga-line)
  ultimately show ?thesis
    using conga3-suma--suma by blast
qed
ultimately show ?thesis
  using sams2-suma2--conga456 by auto
qed
moreover have A B C0 CongA A B C0
  by (metis P7 T1 cong-diff conga-refl not-col-distincts)
ultimately show ?thesis
  using conga3-suma--suma by blast
qed
thus ?thesis
  using assms(5) suma2--conga by auto
qed
moreover have A B C CongA A B C
proof –
  have  $A \neq B \wedge B \neq C$ 
    using T1 col-trivial-1 col-trivial-2 by auto
  thus ?thesis
    using conga-refl by auto
qed
moreover have C B C0 CongA D' E' F'
proof –
  have  $C \neq B$ 
    using T1 col-trivial-2 by blast
  moreover have  $B \neq C0$ 
    using  $\langle B \ C1 \ TS \ C \ C0 \rangle$  ts-distincts by blast
  moreover have  $D' \neq E'$ 
    using assms(4) suma-distincts by blast
  moreover have  $E' \neq F'$ 
    using assms(4) suma-distincts by auto
  ultimately show ?thesis
    by (simp add: P7  $\langle \text{Bet } D' \ E' \ F' \rangle$  conga-line)
qed
ultimately show ?thesis
  using conga3-suma--suma by blast
qed
next
assume  $\neg \text{Bet } D' \ E' \ F'$ 
hence E' Out D' F'
  by (simp add:  $\langle \text{Col } D' \ E' \ F' \rangle$  l6-4-2)
have D E F LeA D' E' F'

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    using sams-suma--lea123789 assms(2) assms(4) by auto
  hence E Out D F
    using ⟨E' Out D' F'⟩ out-lea--out by blast
  hence False
    using Col-cases ⟨¬ Col D E F⟩ out-col by blast
  thus ?thesis by simp
qed
next
assume ¬ Col D' E' F'
have SAMS C B C1 G H I
proof -
  have D E F CongA C B C1
    using P6A not-conga-sym by blast
  moreover have G H I CongA G H I
    using ⟨¬ Col G H I⟩ conga-refl not-col-distincts by fastforce
  ultimately show ?thesis
    using assms(2) conga2-sams--sams by blast
qed
then obtain J where P7: C1 B J CongA G H I ∧ ¬ B C1 OS C J ∧
¬ C B TS C1 J ∧ Coplanar C B C1 J
  using SAMS-def by blast
have P7A: C1 B J CongA G H I
  using P7 by simp
have P7B: ¬ B C1 OS C J
  using P7 by simp
have P7C: ¬ C B TS C1 J
  using P7 by simp
have P7D: Coplanar C B C1 J
  using P7 by simp
obtain F1 where P8: F E F1 CongA G H I ∧ ¬ E F OS D F1 ∧
¬ D E TS F F1 ∧ Coplanar D E F F1
  using SAMS-def assms(2) by auto
have P8A: F E F1 CongA G H I
  using P8 by simp
have P8B: ¬ E F OS D F1
  using P8 by simp
have P8C: ¬ D E TS F F1
  using P8 by simp
have P8D: Coplanar D E F F1
  using P8 by simp
have C B J CongA D' E' F'
proof -
  have C B J CongA D E F1
  proof -
    have B C1 TS C J
    proof -
      have Coplanar B C1 C J
        using P7D ncoplanar-perm-12 by blast
      moreover have ¬ Col C B C1
        using ⟨B C1 OS C A⟩ not-col-permutation-2
        one-side-not-col123 by blast
      moreover have ¬ Col J B C1
        using P7 ⟨¬ Col G H I⟩ col-conga-col not-col-permutation-3 by blast
      moreover have ¬ B C1 OS C J
        by (simp add: P7B)
      ultimately show ?thesis
        by (simp add: cop-nos--ts)
    qed
  qed
  moreover have E F TS D F1
  proof -
    have Coplanar E F D F1
      using P8D ncoplanar-perm-12 by blast
    moreover have ¬ Col F1 E F
      using P8 ⟨¬ Col G H I⟩ col-conga-col
      not-col-permutation-3 by blast
    ultimately show ?thesis

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    using P8B  $\langle \neg \text{Col } D \ E \ F \rangle$  cop-nos--ts by blast
qed
moreover have C B C1 CongA D E F
  using P6A by blast
moreover have C1 B J CongA F E F1
  using P8 by (meson P7A not-conga not-conga-sym)
ultimately show ?thesis
  using l11-22a by blast
qed
moreover have D E F1 CongA D' E' F'
proof -
  have D E F G H I SumA D E F1
    using P8A P8B P8D SumA-def  $\langle \neg \text{Col } D \ E \ F \rangle$  conga-distinct
    conga-refl not-col-distincts by auto
  moreover have D E F G H I SumA D' E' F'
    by (simp add: assms(4))
  ultimately show ?thesis
    using suma2--conga by auto
qed
ultimately show ?thesis
  using conga-trans by blast
qed
have  $\neg \text{Col } C \ B \ C1$ 
  using  $\langle B \ C1 \ OS \ C \ A \rangle$  col123--nos col-permutation-1 by blast
show ?thesis
proof -
  have A B C C B J SumA A B J
  proof -
    have B C TS J A
    proof -
      have B C TS C1 A using cop-nos--ts
        using P6B P6D T1  $\langle \neg \text{Col } C \ B \ C1 \rangle$  l9-2 ncoplanar-perm-12
        not-col-permutation-3 by blast
      moreover have B C OS C1 J
      proof -
        have  $\neg \text{Col } C1 \ C \ B$ 
          using Col-perm  $\langle \neg \text{Col } C \ B \ C1 \rangle$  by blast
        moreover have  $\neg \text{Col } J \ C \ B$ 
          using  $\langle C \ B \ J \ CongA \ D' \ E' \ F' \rangle$   $\langle \neg \text{Col } D' \ E' \ F' \rangle$  col-conga-col
          col-permutation-1 by blast
        ultimately show ?thesis
          using P7C P7D cop-nos--ts invert-one-side by blast
      qed
      ultimately show ?thesis
        using l9-8-2 by blast
    qed
  thus ?thesis
    by (simp add: l9-2 ts--suma-1)
qed
moreover have A B C CongA A B C
  using T1 conga-refl not-col-distincts by fastforce
moreover have A B J CongA K L M
proof -
  have A' B' C' G H I SumA A B J
  proof -
    have A B C1 C1 B J SumA A B J
    proof -
      have B C1 TS A J
      proof -
        have B C1 TS C J
      proof -
        have Coplanar B C1 C J
          using P7D ncoplanar-perm-12 by blast
        moreover have  $\neg \text{Col } J \ B \ C1$ 
          using P7  $\langle \neg \text{Col } G \ H \ I \rangle$  col-conga-col
          not-col-permutation-3 by blast
      qed
    qed
  qed

```



```

      ultimately show ?thesis
      by (simp add:  $\langle \neg \text{Col } C \ B \ C1 \rangle \ P7B \text{ cop-nos--ts}$ )
    qed
    moreover have  $B \ C1 \ OS \ C \ A$ 
      by (simp add:  $\langle B \ C1 \ OS \ C \ A \rangle$ )
    ultimately show ?thesis
      using l9-8-2 by blast
  qed
  thus ?thesis
    by (simp add: ts--suma-1)
  qed
  moreover have  $A \ B \ C1 \ CongA \ A' \ B' \ C'$ 
    using  $\langle A' \ B' \ C' \ CongA \ A \ B \ C1 \rangle \text{ not-conga-sym}$  by blast
  moreover have  $C1 \ B \ J \ CongA \ G \ H \ I$ 
    by (simp add: P7A)
  moreover have  $A \ B \ J \ CongA \ A \ B \ J$ 
    using  $\langle A \ B \ C \ B \ J \ SumA \ A \ B \ J \rangle \text{ suma2--conga}$  by auto
  ultimately show ?thesis
    using conga3-suma--suma by blast
  qed
  moreover have  $A' \ B' \ C' \ G \ H \ I \ SumA \ K \ L \ M$ 
    using assms(5) by simp
  ultimately show ?thesis
    using suma2--conga by auto
  qed
  ultimately show ?thesis
    using  $\langle C \ B \ J \ CongA \ D' \ E' \ F' \rangle \text{ conga3-suma--suma}$  by blast
  qed
  qed
  qed
  qed
  qed
  qed
  qed
  lemma suma-assoc-2:
    assumes SAMS  $A \ B \ C \ D \ E \ F$  and
      SAMS  $D \ E \ F \ G \ H \ I$  and
       $A \ B \ C \ D \ E \ F \ SumA \ A' \ B' \ C'$  and
       $D \ E \ F \ G \ H \ I \ SumA \ D' \ E' \ F'$  and
       $A \ B \ C \ D' \ E' \ F' \ SumA \ K \ L \ M$ 
    shows  $A' \ B' \ C' \ G \ H \ I \ SumA \ K \ L \ M$ 
    by (meson assms(1) assms(2) assms(3) assms(4) assms(5) sams-sym suma-assoc-1 suma-sym)

  lemma suma-assoc:
    assumes SAMS  $A \ B \ C \ D \ E \ F$  and
      SAMS  $D \ E \ F \ G \ H \ I$  and
       $A \ B \ C \ D \ E \ F \ SumA \ A' \ B' \ C'$  and
       $D \ E \ F \ G \ H \ I \ SumA \ D' \ E' \ F'$ 
    shows
       $A' \ B' \ C' \ G \ H \ I \ SumA \ K \ L \ M \longleftrightarrow A \ B \ C \ D' \ E' \ F' \ SumA \ K \ L \ M$ 
    by (meson assms(1) assms(2) assms(3) assms(4) suma-assoc-1 suma-assoc-2)

  lemma sams-assoc-1:
    assumes SAMS  $A \ B \ C \ D \ E \ F$  and
      SAMS  $D \ E \ F \ G \ H \ I$  and
       $A \ B \ C \ D \ E \ F \ SumA \ A' \ B' \ C'$  and
       $D \ E \ F \ G \ H \ I \ SumA \ D' \ E' \ F'$  and
      SAMS  $A' \ B' \ C' \ G \ H \ I$ 
    shows SAMS  $A \ B \ C \ D' \ E' \ F'$ 
  proof cases
    assume Col  $A \ B \ C$ 
  {
    assume E Out  $D \ F$ 
    have SAMS  $A \ B \ C \ D' \ E' \ F'$ 
  }
  proof –

```

```

have A' B' C' CongA A B C
  using assms(3) ⟨E Out D F⟩ conga-sym out546-suma--conga by blast
moreover have G H I CongA D' E' F'
  using ⟨E Out D F⟩ assms(4) out213-suma--conga by blast
ultimately show ?thesis
  using assms(5) conga2-sams--sams by blast
qed
}
{
  assume ¬ Bet A B C
  hence P1: B Out A C
    using ⟨Col A B C⟩ or-bet-out by blast
  have SAMS D' E' F' A B C
  proof -
    have D' ≠ E'
      using assms(4) suma-distincts by auto
    moreover have F' E' F' CongA A B C
    proof -
      have E' ≠ F'
        using assms(4) suma-distincts by blast
      hence E' Out F' F'
        using out-trivial by auto
      thus ?thesis
        using P1 l11-21-b by blast
    qed
    moreover have ¬ E' F' OS D' F'
      using os-distincts by blast
    moreover have ¬ D' E' TS F' F'
      by (simp add: not-two-sides-id)
    moreover have Coplanar D' E' F' F'
      using ncop-distincts by blast
    ultimately show ?thesis using SAMS-def P1 by blast
  qed
  hence SAMS A B C D' E' F'
    using sams-sym by blast
}
thus ?thesis
  using SAMS-def assms(1) ⟨E Out D F ⟹ SAMS A B C D' E' F'⟩ by blast
next
assume ¬ Col A B C
show ?thesis
proof cases
  assume Col D E F
  have H Out G I ∨ ¬ Bet D E F
    using SAMS-def assms(2) by blast
  {
    assume H Out G I
    have SAMS A B C D' E' F'
    proof -
      have A B C CongA A B C
        using ⟨¬ Col A B C⟩ conga-refl not-col-distincts by fastforce
      moreover have D E F CongA D' E' F'
        using ⟨H Out G I⟩ assms(4) out546-suma--conga by blast
      ultimately show ?thesis
        using assms(1) conga2-sams--sams by blast
    qed
  }
  {
    assume ¬ Bet D E F
    hence E Out D F
      using ⟨Col D E F⟩ l6-4-2 by blast
    have SAMS A B C D' E' F'
    proof -
      have A' B' C' CongA A B C
        using out546-suma--conga ⟨E Out D F⟩ assms(3) not-conga-sym by blast
      moreover have G H I CongA D' E' F'

```

```

    using out213-suma--conga  $\langle E \text{ Out } D \ F \rangle$  assms(4) by auto
    ultimately show ?thesis
    using assms(5) conga2-sams--sams by auto
  qed
}
thus ?thesis
  using  $\langle H \text{ Out } G \ I \implies \text{SAMS } A \ B \ C \ D' \ E' \ F' \rangle \langle H \text{ Out } G \ I \vee \neg \text{Bet } D \ E \ F \rangle$  by blast
next
assume  $\neg \text{Col } D \ E \ F$ 
show ?thesis
proof cases
  assume  $\text{Col } G \ H \ I$ 
  have  $\text{SAMS } G \ H \ I \ D \ E \ F$ 
  by (simp add: assms(2) sams-sym)
  hence  $E \text{ Out } D \ F \vee \neg \text{Bet } G \ H \ I$ 
  using SAMS-def by blast
  {
    assume  $E \text{ Out } D \ F$ 
    hence False
    using Col-cases  $\langle \neg \text{Col } D \ E \ F \rangle$  out-col by blast
    hence  $\text{SAMS } A \ B \ C \ D' \ E' \ F'$ 
    by simp
  }
  {
    assume  $\neg \text{Bet } G \ H \ I$ 
    hence  $H \text{ Out } G \ I$ 
    by (simp add:  $\langle \text{Col } G \ H \ I \rangle$  l6-4-2)
    have  $\text{SAMS } A \ B \ C \ D' \ E' \ F'$ 
    proof -
      have  $A \ B \ C \text{ CongA } A \ B \ C$ 
      by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-1 col-trivial-2 conga-refl)
      moreover have  $D \ E \ F \text{ CongA } D' \ E' \ F'$ 
      using out546-suma--conga  $\langle H \text{ Out } G \ I \rangle$  assms(4) by blast
      moreover have  $\text{SAMS } A \ B \ C \ D \ E \ F$ 
      using assms(1) by auto
      ultimately show ?thesis
      using conga2-sams--sams by auto
    qed
  }
}
thus ?thesis
  using  $\langle E \text{ Out } D \ F \vee \neg \text{Bet } G \ H \ I \rangle \langle E \text{ Out } D \ F \implies \text{SAMS } A \ B \ C \ D' \ E' \ F' \rangle$  by blast
next
assume  $\neg \text{Col } G \ H \ I$ 
show ?thesis
proof -
  have  $\neg \text{Bet } A \ B \ C$ 
  using Col-def  $\langle \neg \text{Col } A \ B \ C \rangle$  by auto
  moreover have  $\exists J. (C \ B \ J \text{ CongA } D' \ E' \ F' \wedge \neg B \ C \text{ OS } A \ J \wedge$ 
 $\neg A \ B \text{ TS } C \ J \wedge \text{Coplanar } A \ B \ C \ J)$ 
  proof -
    have  $\neg \text{Col } A' \ B' \ C'$ 
    proof -
      {
        assume  $\text{Col } A' \ B' \ C'$ 
        have  $H \text{ Out } G \ I \vee \neg \text{Bet } A' \ B' \ C'$ 
        using SAMS-def assms(5) by blast
        {
          assume  $H \text{ Out } G \ I$ 
          hence False
          using Col-cases  $\langle \neg \text{Col } G \ H \ I \rangle$  out-col by blast
        }
      }
      {
        assume  $\neg \text{Bet } A' \ B' \ C'$ 
        hence  $B' \text{ Out } A' \ C'$ 
        using not-bet-out  $\langle \text{Col } A' \ B' \ C' \rangle$  by blast
        have  $A \ B \ C \text{ LeA } A' \ B' \ C'$ 

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    using assms(1) assms(3) sams-suma--lea123789 by auto
  hence B Out A C
    using ⟨B' Out A' C'⟩ out-lea--out by blast
  hence Col A B C
    using Col-perm out-col by blast
  hence False
    using ⟨¬ Col A B C⟩ by auto
}
hence False
  using ⟨H Out G I ⟹ False⟩ ⟨H Out G I ∨ ¬ Bet A' B' C'⟩ by blast
}
thus ?thesis by blast
qed
obtain C1 where P1: C B C1 CongA D E F ∧ ¬ B C OS A C1 ∧
¬ A B TS C C1 ∧ Coplanar A B C C1
  using SAMS-def assms(1) by auto
have P1A: C B C1 CongA D E F
  using P1 by simp
have P1B: ¬ B C OS A C1
  using P1 by simp
have P1C: ¬ A B TS C C1
  using P1 by simp
have P1D: Coplanar A B C C1
  using P1 by simp
have A B C1 CongA A' B' C'
proof -
  have A B C D E F SumA A B C1
    using P1A P1B P1D SumA-def ⟨¬ Col A B C⟩ conga-distinct
    conga-refl not-col-distincts by auto
  thus ?thesis
    using assms(3) suma2--conga by auto
qed
have SAMS C B C1 G H I
proof -
  have D E F CongA C B C1
    using P1A not-conga-sym by blast
  moreover have G H I CongA G H I
    using ⟨¬ Col G H I⟩ conga-refl not-col-distincts by fastforce
  ultimately show ?thesis using conga2-sams--sams
    using assms(2) by blast
qed
then obtain J where T1: C1 B J CongA G H I ∧ ¬ B C1 OS C J ∧
¬ C B TS C1 J ∧ Coplanar C B C1 J
  using SAMS-def by auto
have T1A: C1 B J CongA G H I
  using T1 by simp
have T1B: ¬ B C1 OS C J
  using T1 by simp
have T1C: ¬ C B TS C1 J
  using T1 by simp
have T1D: Coplanar C B C1 J
  using T1 by simp
have SAMS A B C1 C1 B J
proof -
  have A' B' C' CongA A B C1
    using ⟨A B C1 CongA A' B' C'⟩ not-conga-sym by blast
  moreover have G H I CongA C1 B J
    using T1A not-conga-sym by blast
  ultimately show ?thesis
    using assms(5) conga2-sams--sams by auto
qed
then obtain J' where T2: C1 B J' CongA C1 B J ∧ ¬ B C1 OS A J' ∧
¬ A B TS C1 J' ∧ Coplanar A B C1 J'
  using SAMS-def by auto
have T2A: C1 B J' CongA C1 B J
  using T2 by simp

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have T2B:  $\neg B \ C1 \ OS \ A \ J'$ 
  using T2 by simp
have T2C:  $\neg A \ B \ TS \ C1 \ J'$ 
  using T2 by simp
have T2D: Coplanar  $A \ B \ C1 \ J'$ 
  using T2 by simp
have A' B' C' CongA  $A \ B \ C1$ 
  using  $\langle A \ B \ C1 \ CongA \ A' \ B' \ C' \rangle$  not-conga-sym by blast
hence  $\neg Col \ A \ B \ C1$ 
  using ncol-conga-ncol  $\langle \neg Col \ A' \ B' \ C' \rangle$  by blast
have D E F CongA  $C \ B \ C1$ 
  using P1A not-conga-sym by blast
hence  $\neg Col \ C \ B \ C1$ 
  using ncol-conga-ncol  $\langle \neg Col \ D \ E \ F \rangle$  by blast
hence Coplanar  $C1 \ B \ A \ J$ 
  using coplanar-trans-1 P1D T1D coplanar-perm-15
  coplanar-perm-6 by blast
have Coplanar  $C1 \ B \ J' \ J$ 
  using T2D  $\langle Coplanar \ C1 \ B \ A \ J \rangle \langle \neg Col \ A \ B \ C1 \rangle$  coplanar-perm-14
  coplanar-perm-6 coplanar-trans-1 by blast
have B Out  $J' \ J \vee C1 \ B \ TS \ J' \ J$ 
  by (meson T2  $\langle Coplanar \ C1 \ B \ A \ J \rangle \langle \neg Col \ A \ B \ C1 \rangle$  conga-cop--or-out-ts
  coplanar-trans-1 ncoplanar-perm-14 ncoplanar-perm-6)
{
  assume B Out  $J' \ J$ 
  have  $\exists J. (C \ B \ J \ CongA \ D' \ E' \ F' \wedge \neg B \ C \ OS \ A \ J \wedge \neg A \ B \ TS \ C \ J \wedge Coplanar \ A \ B \ C \ J)$ 
  proof -
    have C B C1 C1 B J SumA  $C \ B \ J$ 
    proof -
      have C1 B J CongA  $C1 \ B \ J$ 
        using CongA-def T2A conga-refl by auto
      moreover have C B J CongA  $C \ B \ J$ 
        using  $\langle \neg Col \ C \ B \ C1 \rangle$  calculation conga-diff56
        conga-pseudo-refl conga-right-comm not-col-distincts by blast
      ultimately show ?thesis
        using T1D T1B SumA-def by blast
    qed
    hence D E F G H I SumA  $C \ B \ J$ 
      using conga3-suma--suma by (meson P1A T1A suma2--conga)
    hence C B J CongA  $D' \ E' \ F'$ 
      using assms(4) suma2--conga by auto
    moreover have  $\neg B \ C \ OS \ A \ J$ 
      by (metis Col-perm P1B P1D T1C  $\langle \neg Col \ A \ B \ C \rangle$ 
       $\langle \neg Col \ C \ B \ C1 \rangle$  cop-nos--ts coplanar-perm-8 invert-two-sides
      l9-2 l9-8-2)
    moreover have  $\neg A \ B \ TS \ C \ J$ 
  proof cases
    assume Col A B J
    thus ?thesis
      using TS-def invert-two-sides not-col-permutation-3 by blast
  next
    assume  $\neg Col \ A \ B \ J$ 
    have A B OS C J
    proof -
      have A B OS C C1
        by (simp add: P1C P1D  $\langle \neg Col \ A \ B \ C1 \rangle \langle \neg Col \ A \ B \ C \rangle$  cop-nts--os
        not-col-permutation-2)
      moreover have A B OS C1 J
    proof -
      have A B OS C1 J'
        by (metis T2C T2D  $\langle B \ Out \ J' \ J \rangle \langle \neg Col \ A \ B \ C1 \rangle \langle \neg Col \ A \ B \ J \rangle$ 
        col-trivial-2 colx cop-nts--os not-col-permutation-2
        out-col out-distinct)
      thus ?thesis
        using  $\langle B \ Out \ J' \ J \rangle$  invert-one-side out-out-one-side by blast
    qed
  qed
}

```

```

    ultimately show ?thesis
      using one-side-transitivity by blast
  qed
  thus ?thesis
    using l9-9 by blast
  qed
  moreover have Coplanar A B C J
    by (meson P1D ⟨Coplanar C1 B A J⟩ ⟨¬ Col A B C1⟩ coplanar-perm-18
      coplanar-perm-2 coplanar-trans-1 not-col-permutation-2)
  ultimately show ?thesis
    by blast
  qed
}
{
  assume C1 B TS J' J
  have B C1 OS C J
  proof -
    have B C1 TS C J'
  proof -
    have B C1 TS A J'
      by (meson T2B T2D TS-def ⟨C1 B TS J' J⟩ ⟨¬ Col A B C1⟩
        cop-nts--os invert-two-sides ncoplanar-perm-12)
    moreover have B C1 OS A C
      by (meson P1B P1C P1D ⟨¬ Col A B C1⟩ ⟨¬ Col A B C⟩ ⟨¬ Col C B C1⟩
        cop-nts--os invert-one-side ncoplanar-perm-12
        not-col-permutation-2 not-col-permutation-3 os-ts1324--os)
    ultimately show ?thesis
      using l9-8-2 by blast
  qed
  moreover have B C1 TS J J'
    using ⟨C1 B TS J' J⟩ invert-two-sides l9-2 by blast
  ultimately show ?thesis
    using OS-def by blast
  qed
  hence False
    by (simp add: T1B)
  hence  $\exists J. (C B J \text{ CongA } D' E' F' \wedge \neg B C OS A J \wedge \neg A B TS C J \wedge \text{Coplanar } A B C J)$ 
    by auto
}
thus ?thesis
  using ⟨B Out J' J  $\implies \exists J. C B J \text{ CongA } D' E' F' \wedge \neg B C OS A J \wedge \neg A B TS C J \wedge \text{Coplanar } A B C J$ ⟩
    ⟨B Out J' J  $\vee C1 B TS J' J$ ⟩ by blast
  qed
  ultimately show ?thesis
    using SAMS-def not-bet-distincts by auto
  qed
  qed
  qed
  qed
lemma sams-assoc-2:
  assumes SAMS A B C D E F and
    SAMS D E F G H I and
    A B C D E F SumA A' B' C' and
    D E F G H I SumA D' E' F' and
    SAMS A B C D' E' F'
  shows SAMS A' B' C' G H I
proof -
  have SAMS G H I A' B' C'
  proof -
    have SAMS G H I D E F
      by (simp add: assms(2) sams-sym)
    moreover have SAMS D E F A B C
      by (simp add: assms(1) sams-sym)
    moreover have G H I D E F SumA D' E' F'

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```

    by (simp add: assms(4) suma-sym)
  moreover have  $D E F A B C \text{ Sum} A A' B' C'$ 
    by (simp add: assms(3) suma-sym)
  moreover have  $SAMS D' E' F' A B C$ 
    by (simp add: assms(5) sams-sym)
  ultimately show ?thesis
    using sams-assoc-1 by blast
qed
thus ?thesis
  using sams-sym by blast
qed

```

```

lemma sams-assoc:
  assumes  $SAMS A B C D E F$  and
     $SAMS D E F G H I$  and
     $A B C D E F \text{ Sum} A A' B' C'$  and
     $D E F G H I \text{ Sum} A D' E' F'$ 
  shows  $(SAMS A' B' C' G H I) \longleftrightarrow (SAMS A B C D' E' F')$ 
  using sams-assoc-1 sams-assoc-2 by (meson assms(1) assms(2) assms(3) assms(4))

```

```

lemma sams-nos--nts:
  assumes  $SAMS A B C C B J$  and
     $\neg B C OS A J$ 
  shows  $\neg A B TS C J$ 
proof -
  obtain  $J'$  where  $P1: C B J' \text{ Cong} A C B J \wedge \neg B C OS A J' \wedge \neg A B TS C J' \wedge \text{Coplanar } A B C J'$ 
    using SAMS-def assms(1) by blast
  have  $P1A: C B J' \text{ Cong} A C B J$ 
    using  $P1$  by simp
  have  $P1B: \neg B C OS A J'$ 
    using  $P1$  by simp
  have  $P1C: \neg A B TS C J'$ 
    using  $P1$  by simp
  have  $P1D: \text{Coplanar } A B C J'$ 
    using  $P1$  by simp
  have  $P2: B \text{ Out } C J \vee \neg \text{Bet } A B C$ 
    using SAMS-def assms(1) by blast
  {
    assume  $A B TS C J$ 
    have  $\text{Coplanar } C B J J'$ 
    proof -
      have  $\neg \text{Col } A C B$ 
        using TS-def  $\langle A B TS C J \rangle$  not-col-permutation-4 by blast
      moreover have  $\text{Coplanar } A C B J$ 
        using  $\langle A B TS C J \rangle$  ncoplanar-perm-2 ts--coplanar by blast
      moreover have  $\text{Coplanar } A C B J'$ 
        using  $P1D$  ncoplanar-perm-2 by blast
      ultimately show ?thesis
        using coplanar-trans-1 by blast
    qed
    have  $B \text{ Out } J J' \vee C B TS J J'$ 
      by (metis  $P1 \langle A B TS C J \rangle \langle \text{Coplanar } C B J J' \rangle$  bet-conga--bet bet-out
        col-conga-col col-two-sides-bet conga-distinct conga-os--out conga-sym cop-nts--os invert-two-sides l5-2 l6-6
        not-col-permutation-3 not-col-permutation-4)
    {
      assume  $B \text{ Out } J J'$ 
      have  $\neg \text{Col } B A J \vee \neg \text{Col } B A J'$ 
        using TS-def  $\langle A B TS C J \rangle$  not-col-permutation-3 by blast
      hence  $A B OS C J'$ 
        by (metis  $\langle B \text{ Out } J J' \rangle$  Col-cases  $P1C P1D$  TS-def
           $\langle A B TS C J \rangle$  col2--eq cop-nts--os l6-3-1 out-col)
      hence  $A B TS C J'$ 
        by (meson  $\langle A B TS C J \rangle \langle B \text{ Out } J J' \rangle$  l6-6 l9-2 not-col-distincts
          out-two-sides-two-sides)
      hence False
        by (simp add:  $P1C$ )
    }
  }

```

```

}
{
  assume  $C B TS J J'$ 
  hence  $\neg Col C A B \wedge \neg Col J A B$ 
    using  $TS-def \langle A B TS C J \rangle$  by blast
  hence False
    by (metis P1B P1D TS-def  $\langle C B TS J J' \rangle$  assms(2) cop-nts--os
        invert-two-sides l9-8-1 ncoplanar-perm-12 not-col-permutation-1)
}
hence False
  using  $\langle B Out J J' \implies False \rangle \langle B Out J J' \vee C B TS J J' \rangle$  by blast
}
thus ?thesis by auto
qed

lemma conga-sams-nos--nts:
  assumes SAMS  $A B C D E F$  and
     $C B J CongA D E F$  and
     $\neg B C OS A J$ 
  shows  $\neg A B TS C J$ 
proof -
  have SAMS  $A B C C B J$ 
  proof -
    have  $A B C CongA A B C$ 
      using assms(1) conga-refl sams-distincts by fastforce
    moreover have  $D E F CongA C B J$ 
      using assms(2) not-conga-sym by blast
    ultimately show ?thesis
      using assms(1) conga2-sams--sams by auto
  qed
  thus ?thesis
    by (simp add: assms(3) sams-nos--nts)
qed

lemma sams-lea2-suma2--conga123:
  assumes  $A B C LeA A' B' C'$  and
     $D E F LeA D' E' F'$  and
    SAMS  $A' B' C' D' E' F'$  and
     $A B C D E F SumA G H I$  and
     $A' B' C' D' E' F' SumA G H I$ 
  shows  $A B C CongA A' B' C'$ 
proof -
  have SAMS  $A B C D E F$ 
    using assms(1) assms(2) assms(3) sams-lea2--sams by blast
  moreover have SAMS  $A' B' C' D E F$ 
    by (metis assms(2) assms(3) lea-refl sams-distincts sams-lea2--sams)
  moreover have  $A' B' C' D E F SumA G H I$ 
  proof -
    obtain  $G' H' I'$  where  $P1: A' B' C' D E F SumA G' H' I'$ 
      using calculation(2) ex-suma sams-distincts by blast
    show ?thesis
  proof -
    have  $A' \neq B' \wedge B' \neq C'$ 
      using assms(1) lea-distincts by blast
    hence  $A' B' C' CongA A' B' C'$ 
      using conga-refl by auto
    moreover
    have  $D \neq E \wedge E \neq F$ 
      using  $\langle SAMS A B C D E F \rangle$  sams-distincts by blast
    hence  $D E F CongA D E F$ 
      using conga-refl by auto
    moreover have  $G' H' I' CongA G H I$ 
  proof -
    have  $G' H' I' LeA G H I$ 
      using P1 assms(2) assms(3) assms(5) sams-lea456-suma2--lea by blast
    moreover have  $G H I LeA G' H' I'$ 

```



```

proof –
  have  $SAMS\ A'\ B'\ C'\ D\ E\ F$ 
    using  $\langle SAMS\ A'\ B'\ C'\ D\ E\ F \rangle$  by auto
  thus ?thesis
    using  $P1\ assms(1)\ assms(4)\ sams-lea123-suma2--lea$  by blast
qed
ultimately show ?thesis
  by (simp add: lea-asy)
qed
ultimately show ?thesis
  using  $P1\ conga3-suma--suma$  by blast
qed
qed
ultimately show ?thesis
  using  $assms(4)\ sams2-suma2--conga123$  by blast
qed

```

```

lemma sams-lea2-suma2--conga456:
assumes  $A\ B\ C\ LeA\ A'\ B'\ C'$  and
   $D\ E\ F\ LeA\ D'\ E'\ F'$  and
   $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$  and
   $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$  and
   $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G\ H\ I$ 
shows  $D\ E\ F\ CongA\ D'\ E'\ F'$ 
proof –
have  $SAMS\ D'\ E'\ F'\ A'\ B'\ C'$ 
  by (simp add: assms(3) sams-sym)
moreover have  $D\ E\ F\ A\ B\ C\ SumA\ G\ H\ I$ 
  by (simp add: assms(4) suma-sym)
moreover have  $D'\ E'\ F'\ A'\ B'\ C'\ SumA\ G\ H\ I$ 
  by (simp add: assms(5) suma-sym)
ultimately show ?thesis
  using  $assms(1)\ assms(2)\ sams-lea2-suma2--conga123$  by auto
qed

```

```

lemma sams-suma--out213:
assumes  $A\ B\ C\ D\ E\ F\ SumA\ D\ E\ F$  and
   $SAMS\ A\ B\ C\ D\ E\ F$ 
shows  $B\ Out\ A\ C$ 
proof –
have  $E \neq D$ 
  using  $assms(2)\ sams-distincts$  by blast
hence  $E\ Out\ D\ D$ 
  using out-trivial by auto
moreover have  $D\ E\ D\ CongA\ A\ B\ C$ 
proof –
  have  $D\ E\ D\ LeA\ A\ B\ C$ 
    using  $assms(1)\ lea121345\ suma-distincts$  by auto
  moreover
  have  $E \neq D \wedge E \neq F$ 
    using  $assms(2)\ sams-distincts$  by blast
  hence  $D\ E\ F\ LeA\ D\ E\ F$ 
    using lea-refl by auto
  moreover have  $D\ E\ D\ D\ E\ F\ SumA\ D\ E\ F$ 
proof –
  have  $\neg\ E\ D\ OS\ D\ F$ 
    using os-distincts by auto
  moreover have  $Coplanar\ D\ E\ D\ F$ 
    using ncop-distincts by auto
  ultimately show ?thesis
    using  $SumA-def\ \langle D\ E\ F\ LeA\ D\ E\ F \rangle\ lea-asy$  by blast
qed
ultimately show ?thesis
  using  $assms(1)\ assms(2)\ sams-lea2-suma2--conga123$  by auto
qed
ultimately show ?thesis

```

using *eq-conga-out* by *blast*
qed

lemma *sams-suma--out546*:
 assumes *A B C D E F SumA A B C* and
SAMS A B C D E F
 shows *E Out D F*
proof –
 have *D E F A B C SumA A B C*
 using *assms(1) suma-sym* by *blast*
 moreover have *SAMS D E F A B C*
 using *assms(2) sams-sym* by *blast*
 ultimately show *?thesis*
 using *sams-suma--out213* by *blast*
qed

lemma *sams-lea-lta123-suma2--lta*:
 assumes *A B C LtA A' B' C'* and
D E F LeA D' E' F' and
SAMS A' B' C' D' E' F' and
A B C D E F SumA G H I and
A' B' C' D' E' F' SumA G' H' I'
 shows *G H I LtA G' H' I'*
proof –
 have *G H I LeA G' H' I'*
proof –
 have *A B C LeA A' B' C'*
 by (*simp add: assms(1) lta--lea*)
 thus *?thesis*
 using *assms(2) assms(3) assms(4) assms(5) sams-lea2-suma2--lea* by *blast*
qed
 moreover have $\neg G H I CongA G' H' I'$
proof –
 {
 assume *G H I CongA G' H' I'*
 have *A B C CongA A' B' C'*
proof –
 have *A B C LeA A' B' C'*
 by (*simp add: assms(1) lta--lea*)
 moreover have *A' B' C' D' E' F' SumA G H I*
 by (*meson <G H I CongA G' H' I'> assms(3) assms(5) conga3-suma--suma*
conga-sym sams2-suma2--conga123 sams2-suma2--conga456)
 ultimately show *?thesis*
 using *assms(2) assms(3) assms(4) sams-lea2-suma2--conga123* by *blast*
qed
 hence *False*
 using *assms(1) lta-not-conga* by *auto*
 }
 thus *?thesis*
 by *auto*
qed
 ultimately show *?thesis*
 using *LtA-def* by *blast*
qed

lemma *sams-lea-lta456-suma2--lta*:
 assumes *A B C LeA A' B' C'* and
D E F LtA D' E' F' and
SAMS A' B' C' D' E' F' and
A B C D E F SumA G H I and
A' B' C' D' E' F' SumA G' H' I'
 shows *G H I LtA G' H' I'*
 using *sams-lea-lta123-suma2--lta*
 by (*meson assms(1) assms(2) assms(3) assms(4) assms(5) sams-sym suma-sym*)

lemma *sams-lta2-suma2--lta*:

assumes $A B C LtA A' B' C'$ **and**
 $D E F LtA D' E' F'$ **and**
 $SAMS A' B' C' D' E' F'$ **and**
 $A B C D E F SumA G H I$ **and**
 $A' B' C' D' E' F' SumA G' H' I'$
shows $G H I LtA G' H' I'$
using *sams-lea-lta123-suma2--lta*
by (*meson LtA-def assms(1) assms(2) assms(3) assms(4) assms(5)*)

lemma *sams-lea2-suma2--lea123*:
assumes $D' E' F' LeA D E F$ **and**
 $G H I LeA G' H' I'$ **and**
 $SAMS A B C D E F$ **and**
 $A B C D E F SumA G H I$ **and**
 $A' B' C' D' E' F' SumA G' H' I'$
shows $A B C LeA A' B' C'$

proof *cases*
assume $A' B' C' LtA A B C$
hence $G' H' I' LtA G H I$ **using** *sams-lea-lta123-suma2--lta*
using *assms(1) assms(3) assms(4) assms(5)* **by** *blast*
hence $\neg G H I LeA G' H' I'$
using *lea--nlta* **by** *blast*
hence *False*
using *assms(2)* **by** *auto*
thus *?thesis* **by** *auto*
next
assume $\neg A' B' C' LtA A B C$
have $A' \neq B' \wedge B' \neq C' \wedge A \neq B \wedge B \neq C$
using *assms(4) assms(5) suma-distincts* **by** *auto*
thus *?thesis*
by (*simp add: $\langle \neg A' B' C' LtA A B C \rangle$ nlta--lea*)
qed

lemma *sams-lea2-suma2--lea456*:
assumes $A' B' C' LeA A B C$ **and**
 $G H I LeA G' H' I'$ **and**
 $SAMS A B C D E F$ **and**
 $A B C D E F SumA G H I$ **and**
 $A' B' C' D' E' F' SumA G' H' I'$
shows $D E F LeA D' E' F'$
proof –
have $SAMS D E F A B C$
by (*simp add: assms(3) sams-sym*)
moreover have $D E F A B C SumA G H I$
by (*simp add: assms(4) suma-sym*)
moreover have $D' E' F' A' B' C' SumA G' H' I'$
by (*simp add: assms(5) suma-sym*)
ultimately show *?thesis*
using *assms(1) assms(2) sams-lea2-suma2--lea123* **by** *blast*
qed

lemma *sams-lea-lta456-suma2--lta123*:
assumes $D' E' F' LtA D E F$ **and**
 $G H I LeA G' H' I'$ **and**
 $SAMS A B C D E F$ **and**
 $A B C D E F SumA G H I$ **and**
 $A' B' C' D' E' F' SumA G' H' I'$
shows $A B C LtA A' B' C'$
proof *cases*
assume $A' B' C' LeA A B C$
hence $G' H' I' LtA G H I$
using *sams-lea-lta456-suma2--lta assms(1) assms(3) assms(4) assms(5)* **by** *blast*
hence $\neg G H I LeA G' H' I'$
using *lea--nlta* **by** *blast*
hence *False*
using *assms(2)* **by** *blast*

```

    thus ?thesis by blast
next
  assume  $\neg A' B' C' \text{ LeA } A B C$ 
  have  $A' \neq B' \wedge B' \neq C' \wedge A \neq B \wedge B \neq C$ 
    using assms(4) assms(5) suma-distincts by auto
  thus ?thesis using nlea--lta
    by (simp add:  $\langle \neg A' B' C' \text{ LeA } A B C \rangle$ )
qed

lemma sams-lea-lta123-suma2--lta456:
  assumes  $A' B' C' \text{ LtA } A B C$  and
     $G H I \text{ LeA } G' H' I'$  and
     $\text{SAMS } A B C D E F$  and
     $A B C D E F \text{ SumA } G H I$  and
     $A' B' C' D' E' F' \text{ SumA } G' H' I'$ 
  shows  $D E F \text{ LtA } D' E' F'$ 
proof -
  have  $\text{SAMS } D E F A B C$ 
    by (simp add: assms(3) sams-sym)
  moreover have  $D E F A B C \text{ SumA } G H I$ 
    by (simp add: assms(4) suma-sym)
  moreover have  $D' E' F' A' B' C' \text{ SumA } G' H' I'$ 
    by (simp add: assms(5) suma-sym)
  ultimately show ?thesis
    using assms(1) assms(2) sams-lea-lta456-suma2--lta123 by blast
qed

lemma sams-lea-lta789-suma2--lta123:
  assumes  $D' E' F' \text{ LeA } D E F$  and
     $G H I \text{ LtA } G' H' I'$  and
     $\text{SAMS } A B C D E F$  and
     $A B C D E F \text{ SumA } G H I$  and
     $A' B' C' D' E' F' \text{ SumA } G' H' I'$ 
  shows  $A B C \text{ LtA } A' B' C'$ 
proof cases
  assume  $A' B' C' \text{ LeA } A B C$ 
  hence  $G' H' I' \text{ LeA } G H I$ 
    using assms(1) assms(3) assms(4) assms(5) sams-lea2-suma2--lea by blast
  hence  $\neg G H I \text{ LtA } G' H' I'$ 
    by (simp add: lea--nlta)
  hence False
    using assms(2) by blast
  thus ?thesis by auto
next
  assume  $\neg A' B' C' \text{ LeA } A B C$ 
  have  $A' \neq B' \wedge B' \neq C' \wedge A \neq B \wedge B \neq C$ 
    using assms(4) assms(5) suma-distincts by auto
  thus ?thesis
    using nlea--lta by (simp add:  $\langle \neg A' B' C' \text{ LeA } A B C \rangle$ )
qed

lemma sams-lea-lta789-suma2--lta456:
  assumes  $A' B' C' \text{ LeA } A B C$  and
     $G H I \text{ LtA } G' H' I'$  and
     $\text{SAMS } A B C D E F$  and
     $A B C D E F \text{ SumA } G H I$  and
     $A' B' C' D' E' F' \text{ SumA } G' H' I'$ 
  shows  $D E F \text{ LtA } D' E' F'$ 
proof -
  have  $\text{SAMS } D E F A B C$ 
    by (simp add: assms(3) sams-sym)
  moreover have  $D E F A B C \text{ SumA } G H I$ 
    by (simp add: assms(4) suma-sym)
  moreover have  $D' E' F' A' B' C' \text{ SumA } G' H' I'$ 
    using assms(5) suma-sym by blast
  ultimately show ?thesis

```

```

    using assms(1) assms(2) sams-lea-lta789-suma2--lta123 by blast
qed

lemma sams-lta2-suma2--lta123:
  assumes  $D' E' F' \text{ LtA } D E F$  and
     $G H I \text{ LtA } G' H' I'$  and
     $SAMS A B C D E F$  and
     $A B C D E F \text{ SumA } G H I$  and
     $A' B' C' D' E' F' \text{ SumA } G' H' I'$ 
  shows  $A B C \text{ LtA } A' B' C'$ 
proof -
  have  $D' E' F' \text{ LeA } D E F$ 
    by (simp add: assms(1) lta--lea)
  thus ?thesis
    using assms(2) assms(3) assms(4) assms(5) sams-lea-lta789-suma2--lta123 by blast
qed

lemma sams-lta2-suma2--lta456:
  assumes  $A' B' C' \text{ LtA } A B C$  and
     $G H I \text{ LtA } G' H' I'$  and
     $SAMS A B C D E F$  and
     $A B C D E F \text{ SumA } G H I$  and
     $A' B' C' D' E' F' \text{ SumA } G' H' I'$ 
  shows  $D E F \text{ LtA } D' E' F'$ 
proof -
  have  $A' B' C' \text{ LeA } A B C$ 
    by (simp add: assms(1) lta--lea)
  thus ?thesis
    using assms(2) assms(3) assms(4) assms(5) sams-lea-lta789-suma2--lta456 by blast
qed

lemma sams123231:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $B \neq C$ 
  shows  $SAMS A B C B C A$ 
proof -
  obtain  $A'$  where  $B \text{ Midpoint } A A'$ 
    using symmetric-point-construction by auto
  hence  $A' \neq B$ 
    using assms(1) midpoint-not-midpoint by blast
  moreover have  $Bet A B A'$ 
    by (simp add:  $\langle B \text{ Midpoint } A A' \rangle$  midpoint-bet)
  moreover have  $B C A \text{ LeA } C B A'$ 
  proof cases
    assume  $Col A B C$ 
    show ?thesis
    proof cases
      assume  $Bet A C B$ 
      thus ?thesis
        by (metis assms(2) assms(3) between-exchange3 calculation(1)
            calculation(2) l11-31-2)
    next
      assume  $\neg Bet A C B$ 
      hence  $C \text{ Out } B A$ 
        using Col-cases  $\langle Col A B C \rangle$  l6-6 or-bet-out by blast
      thus ?thesis
        using assms(3) calculation(1) l11-31-1 by auto
    qed
  qed
next
  assume  $\neg Col A B C$ 
  thus ?thesis
    using l11-41-aux  $\langle B \text{ Midpoint } A A' \rangle$  calculation(1) lta--lea
    midpoint-bet not-col-permutation-4 by blast
qed
ultimately show ?thesis

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    using assms(1) sams-chara by blast
qed

lemma col-suma--col:
  assumes Col D E F and
    A B C B C A SumA D E F
  shows Col A B C
proof -
{
  assume  $\neg$  Col A B C
  have False
  proof cases
    assume Bet D E F
    obtain P where P1: Bet A B P  $\wedge$  Cong A B B P
    using Cong-perm segment-construction by blast
    have D E F LtA A B P
    proof -
      have A B C LeA A B C
      using  $\langle \neg$  Col A B C  $\rangle$  lea-total not-col-distincts by blast
      moreover
      have B C TS A P
      by (metis Cong-perm P1  $\langle \neg$  Col A B C  $\rangle$  bet--ts bet-col between-trivial2
        cong-reverse-identity not-col-permutation-1)
      hence B C A LtA C B P
      using Col-perm P1  $\langle \neg$  Col A B C  $\rangle$  calculation l11-41-aux ts-distincts by blast
      moreover have A B C C B P SumA A B P
      by (simp add:  $\langle$  B C TS A P  $\rangle$  ts--suma-1)
      ultimately show ?thesis
      by (meson P1 sams-lea-lta456-suma2--lta assms(2) bet-suma--sams)
    qed
    thus ?thesis
    using P1  $\langle$  Bet D E F  $\rangle$  bet2-lta--lta lta-distincts by blast
  next
    assume  $\neg$  Bet D E F
    have C Out B A
    proof -
      have E Out D F
      by (simp add:  $\langle \neg$  Bet D E F  $\rangle$  assms(1) l6-4-2)
      moreover have B C A LeA D E F
      using sams-suma--lea456789
      by (metis assms(2) sams123231 suma-distincts)
      ultimately show ?thesis
      using out-lea--out by blast
    qed
    thus ?thesis
    using Col-cases  $\langle \neg$  Col A B C  $\rangle$  out-col by blast
  qed
}
thus ?thesis by auto
qed

lemma ncol-suma--ncol:
  assumes  $\neg$  Col A B C and
    A B C B C A SumA D E F
  shows  $\neg$  Col D E F
  using col-suma--col assms(1) assms(2) by blast

lemma per2-suma--bet:
  assumes Per A B C and
    Per D E F and
    A B C D E F SumA G H I
  shows Bet G H I
proof -
  obtain A1 where P1: C B A1 CongA D E F  $\wedge$   $\neg$  B C OS A A1  $\wedge$ 
Coplanar A B C A1  $\wedge$  A B A1 CongA G H I
  using SumA-def assms(3) by blast

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hence  $D E F \text{ CongA } A1 B C$ 
  using conga-right-comm conga-sym by blast
hence  $\text{Per } A1 B C$ 
  using assms(2) l11-17 by blast
have  $\text{Bet } A B A1$ 
proof -
  have  $\text{Col } B A A1$ 
  proof -
    have  $\text{Coplanar } C A A1 B$ 
      using P1 ncoplanar-perm-10 by blast
    moreover have  $C \neq B$ 
      using  $\langle D E F \text{ CongA } A1 B C \rangle$  conga-distinct by auto
    ultimately show ?thesis
      using assms(1)  $\langle \text{Per } A1 B C \rangle$  col-permutation-2 cop-per2--col by blast
  qed
  moreover have  $B C \text{ TS } A A1$ 
  proof -
    have  $\text{Coplanar } B C A A1$ 
      using calculation ncop--ncols by blast
    moreover
    have  $A \neq B \wedge B \neq C$ 
      using CongA-def P1 by blast
    hence  $\neg \text{Col } A B C$ 
      by (simp add: assms(1) per-not-col)
    moreover
    have  $A1 \neq B \wedge B \neq C$ 
      using  $\langle D E F \text{ CongA } A1 B C \rangle$  conga-distinct by blast
    hence  $\neg \text{Col } A1 B C$ 
      using  $\langle \text{Per } A1 B C \rangle$  by (simp add: per-not-col)
    ultimately show ?thesis
      by (simp add: P1 cop-nos--ts)
  qed
  ultimately show ?thesis
    using col-two-sides-bet by blast
qed
thus ?thesis
  using P1 bet-conga--bet by blast
qed

lemma bet-per2--suma:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $D \neq E$  and
     $E \neq F$  and
     $G \neq H$  and
     $H \neq I$  and
     $\text{Per } A B C$  and
     $\text{Per } D E F$  and
     $\text{Bet } G H I$ 
  shows  $A B C D E F \text{ SumA } G H I$ 
proof -
  obtain  $G' H' I'$  where  $A B C D E F \text{ SumA } G' H' I'$ 
    using assms(1) assms(2) assms(3) assms(4) ex-suma by blast
  moreover have  $A B C \text{ CongA } A B C$ 
    using assms(1) assms(2) conga-refl by auto
  moreover have  $D E F \text{ CongA } D E F$ 
    using assms(3) assms(4) conga-refl by auto
  moreover have  $G' H' I' \text{ CongA } G H I$ 
  proof -
    have  $G' \neq H'$ 
      using calculation(1) suma-distincts by auto
    moreover have  $H' \neq I'$ 
      using  $\langle A B C D E F \text{ SumA } G' H' I' \rangle$  suma-distincts by blast
    moreover have  $\text{Bet } G' H' I'$ 
      using  $\langle A B C D E F \text{ SumA } G' H' I' \rangle$  assms(7) assms(8) per2-suma--bet by auto
    ultimately show ?thesis

```

```

    using conga-line by (simp add: assms(5) assms(6) assms(9))
qed
ultimately show ?thesis
    using conga3-suma--suma by blast
qed

lemma per2--sams:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $D \neq E$  and
     $E \neq F$  and
    Per  $A B C$  and
    Per  $D E F$ 
  shows SAMS  $A B C D E F$ 
proof -
  obtain  $G H I$  where  $A B C D E F \text{ SumA } G H I$ 
    using assms(1) assms(2) assms(3) assms(4) ex-suma by blast
  moreover hence Bet  $G H I$ 
    using assms(5) assms(6) per2-suma--bet by auto
  ultimately show ?thesis
    using bet-suma--sams by blast
qed

lemma bet-per-suma--per456:
  assumes Per  $A B C$  and
    Bet  $G H I$  and
     $A B C D E F \text{ SumA } G H I$ 
  shows Per  $D E F$ 
proof -
  obtain  $A1$  where  $B \text{ Midpoint } A A1$ 
    using symmetric-point-construction by auto
  have  $\neg \text{Col } A B C$ 
    using assms(1) assms(3) per-col-eq suma-distincts by blast
  have  $A B C \text{ CongA } D E F$ 
  proof -
    have SAMS  $A B C A B C$ 
      using  $\langle \neg \text{Col } A B C \rangle$  assms(1) not-col-distincts per2--sams by auto
    moreover have SAMS  $A B C D E F$ 
      using assms(2) assms(3) bet-suma--sams by blast
    moreover have  $A B C A B C \text{ SumA } G H I$ 
      using assms(1) assms(2) assms(3) bet-per2--suma suma-distincts by blast
    ultimately show ?thesis
      using assms(3) sams2-suma2--conga456 by auto
  qed
  thus ?thesis
    using assms(1) l11-17 by blast
qed

lemma bet-per-suma--per123:
  assumes Per  $D E F$  and
    Bet  $G H I$  and
     $A B C D E F \text{ SumA } G H I$ 
  shows Per  $A B C$ 
  using bet-per-suma--per456 by (meson assms(1) assms(2) assms(3) suma-sym)

lemma bet-suma--per:
  assumes Bet  $D E F$  and
     $A B C A B C \text{ SumA } D E F$ 
  shows Per  $A B C$ 
proof -
  obtain  $A'$  where  $C B A' \text{ CongA } A B C \wedge A B A' \text{ CongA } D E F$ 
    using SumA-def assms(2) by blast
  have Per  $C B A$ 
  proof -
    have Bet  $A B A'$ 
  proof -

```



```

have  $D E F \text{ CongA } A B A'$ 
  using  $\langle C B A' \text{ CongA } A B C \wedge A B A' \text{ CongA } D E F \rangle \text{ not-conga-sym}$  by blast
thus ?thesis
  using assms(1) bet-conga--bet by blast
qed
moreover have  $C B A \text{ CongA } C B A'$ 
  using conga-left-comm not-conga-sym
   $\langle C B A' \text{ CongA } A B C \wedge A B A' \text{ CongA } D E F \rangle$  by blast
ultimately show ?thesis
  using l11-18-2 by auto
qed
thus ?thesis
  using Per-cases by auto
qed

```

```

lemma acute--sams:
  assumes Acute A B C
  shows SAMS A B C A B C
proof -
  obtain  $A'$  where  $B \text{ Midpoint } A A'$ 
    using symmetric-point-construction by auto
  show ?thesis
  proof -
    have  $A \neq B \wedge A' \neq B$ 
      using  $\langle B \text{ Midpoint } A A' \rangle \text{ acute-distincts assms is-midpoint-id-2}$  by blast
    moreover have  $\text{Bet } A B A'$ 
      by (simp add:  $\langle B \text{ Midpoint } A A' \rangle \text{ midpoint-bet}$ )
    moreover
    have  $\text{Obtuse } C B A'$ 
      using acute-bet--obtuse assms calculation(1) calculation(2)
      obtuse-sym by blast
    hence  $A B C \text{ LeA } C B A'$ 
      by (metis acute--not-obtuse assms calculation(1) lea-obtuse-obtuse
        lea-total obtuse-distincts)
    ultimately show ?thesis
      using sams-chara by blast
  qed
qed

```

```

lemma acute-suma--nbet:
  assumes Acute A B C and
     $A B C A B C \text{ SumA } D E F$ 
  shows  $\neg \text{Bet } D E F$ 
proof -
  {
    assume  $\text{Bet } D E F$ 
    hence  $\text{Per } A B C$ 
      using assms(2) bet-suma--per by auto
    hence  $A B C \text{ LtA } A B C$ 
      using acute-not-per assms(1) by auto
    hence False
      by (simp add: nlta)
  }
  thus ?thesis by blast
qed

```

```

lemma acute2--sams:
  assumes Acute A B C and
    Acute D E F
  shows SAMS A B C D E F
  by (metis acute--sams acute-distincts assms(1) assms(2) lea-total sams-lea2--sams)

```

```

lemma acute2-suma--nbet-a:
  assumes Acute A B C and
     $D E F \text{ LeA } A B C$  and
     $A B C D E F \text{ SumA } G H I$ 

```

```

shows  $\neg \text{Bet } G \ H \ I$ 
proof -
{
  assume Bet G H I
  obtain  $A' \ B' \ C'$  where  $A \ B \ C \ A \ B \ C \ \text{SumA} \ A' \ B' \ C'$ 
    using acute-distincts assms(1) ex-suma bymetis
  have  $G \ H \ I \ \text{LeA} \ A' \ B' \ C'$ 
  proof -
    have  $A \ B \ C \ \text{LeA} \ A \ B \ C$ 
      using acute-distincts assms(1) lea-refl byblast
    moreover have  $\text{SAMS} \ A \ B \ C \ A \ B \ C$ 
      by (simp add: acute--sams assms(1))
    ultimately show ?thesis
      using  $\langle A \ B \ C \ A \ B \ C \ \text{SumA} \ A' \ B' \ C' \rangle \ \text{assms}(1) \ \text{assms}(2) \ \text{assms}(3)$ 
        sams-lea456-suma2--lea byblast
  qed
  hence  $\text{Bet} \ A' \ B' \ C'$ 
    using  $\langle \text{Bet } G \ H \ I \rangle \ \text{bet-lea--bet byblast}$ 
  hence False
    using acute-suma--nbet assms(1) assms(3)  $\langle A \ B \ C \ A \ B \ C \ \text{SumA} \ A' \ B' \ C' \rangle$  byblast
}
thus ?thesis by blast
qed

```

```

lemma acute2-suma--nbet:
  assumes Acute A B C and
    Acute D E F and
     $A \ B \ C \ D \ E \ F \ \text{SumA} \ G \ H \ I$ 
  shows  $\neg \text{Bet } G \ H \ I$ 
proof -
  have  $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$ 
    using assms(3) suma-distincts byauto
  hence  $A \ B \ C \ \text{LeA} \ D \ E \ F \vee D \ E \ F \ \text{LeA} \ A \ B \ C$ 
    by (simp add: lea-total)
  moreover
  {
    assume  $P3: A \ B \ C \ \text{LeA} \ D \ E \ F$ 
    have  $D \ E \ F \ A \ B \ C \ \text{SumA} \ G \ H \ I$ 
      by (simp add: assms(3) suma-sym)
    hence  $\neg \text{Bet } G \ H \ I$ 
      using  $P3 \ \text{assms}(2) \ \text{acute2-suma--nbet-a byauto}$ 
  }
  {
    assume  $D \ E \ F \ \text{LeA} \ A \ B \ C$ 
    hence  $\neg \text{Bet } G \ H \ I$ 
      using acute2-suma--nbet-a assms(1) assms(3) byauto
  }
  thus ?thesis
    using  $\langle A \ B \ C \ \text{LeA} \ D \ E \ F \implies \neg \text{Bet } G \ H \ I \rangle \ \text{calculation byblast}$ 
  qed

```

```

lemma acute-per--sams:
  assumes  $A \neq B$  and
     $B \neq C$  and
    Per A B C and
    Acute D E F
  shows  $\text{SAMS} \ A \ B \ C \ D \ E \ F$ 
proof -
  have  $\text{SAMS} \ A \ B \ C \ A \ B \ C$ 
    by (simp add: assms(1) assms(2) assms(3) per2--sams)
  moreover have  $A \ B \ C \ \text{LeA} \ A \ B \ C$ 
    using assms(1) assms(2) lea-refl byauto
  moreover have  $D \ E \ F \ \text{LeA} \ A \ B \ C$ 
    by (metis acute-distincts acute-lea-acute acute-not-per assms(1)
      assms(2) assms(3) assms(4) lea-total)
  ultimately show ?thesis

```

using *sams-lea2--sams* by *blast*
qed

lemma *acute-per-suma--nbet*:
 assumes $A \neq B$ and
 $B \neq C$ and
 $Per\ A\ B\ C$ and
 $Acute\ D\ E\ F$ and
 $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
 shows $\neg\ Bet\ G\ H\ I$
 proof –
 {
 assume $Bet\ G\ H\ I$
 have $G\ H\ I\ LtA\ G\ H\ I$
 proof –
 have $A\ B\ C\ LeA\ A\ B\ C$
 using *assms(1) assms(2) lea-refl* by *auto*
 moreover have $D\ E\ F\ LtA\ A\ B\ C$
 by (*simp add: acute-per--lta assms(1) assms(2) assms(3) assms(4)*)
 moreover have $SAMS\ A\ B\ C\ A\ B\ C$
 by (*simp add: assms(1) assms(2) assms(3) per2--sams*)
 moreover have $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
 by (*simp add: assms(5)*)
 moreover have $A\ B\ C\ A\ B\ C\ SumA\ G\ H\ I$
 by (*meson bet-per-suma--per456 Tarski-neutral-dimensionless-axioms*
 $\langle Bet\ G\ H\ I \rangle\ acute-not-per\ assms(3)\ assms(4)\ calculation(4)$)
 ultimately show ?thesis
 using *sams-lea-lta456-suma2--lta* by *blast*
 qed
 hence *False*
 by (*simp add: nlta*)
 }
 thus ?thesis by *blast*
 qed

lemma *obtuse--nsams*:
 assumes $Obtuse\ A\ B\ C$
 shows $\neg\ SAMS\ A\ B\ C\ A\ B\ C$
 proof –
 {
 assume $SAMS\ A\ B\ C\ A\ B\ C$
 obtain A' where $B\ Midpoint\ A\ A'$
 using *symmetric-point-construction* by *auto*
 have $A\ B\ C\ LtA\ A\ B\ C$
 proof –
 have $A\ B\ C\ LeA\ A'\ B\ C$
 by (*metis* $\langle B\ Midpoint\ A\ A' \rangle\ \langle SAMS\ A\ B\ C\ A\ B\ C \rangle\ lea-right-comm$
 $midpoint-bet\ midpoint-distinct-2\ sams-chara\ sams-distincts$)
 moreover have $A'\ B\ C\ LtA\ A\ B\ C$
 using $\langle B\ Midpoint\ A\ A' \rangle\ assms\ calculation\ lea-distincts$
 $midpoint-bet\ obtuse-chara$ by *blast*
 ultimately show ?thesis
 using *lea--nlta* by *blast*
 qed
 hence *False*
 by (*simp add: nlta*)
 }
 thus ?thesis by *blast*
 qed

lemma *nbet-sams-suma--acute*:
 assumes $\neg\ Bet\ D\ E\ F$ and
 $SAMS\ A\ B\ C\ A\ B\ C$ and
 $A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F$
 shows $Acute\ A\ B\ C$
 proof –

```

have Acute A B C  $\vee$  Per A B C  $\vee$  Obtuse A B C
  by (metis angle-partition l8-20-1-R1 l8-5)
{
  assume Per A B C
  hence Bet D E F
    using assms(3) per2-suma--bet by auto
  hence False
    using assms(1) by auto
}
{
  assume Obtuse A B C
  hence  $\neg$  SAMS A B C A B C
    by (simp add: obtuse--nsams)
  hence False
    using assms(2) by auto
}
thus ?thesis
  using  $\langle$ Acute A B C  $\vee$  Per A B C  $\vee$  Obtuse A B C $\rangle$   $\langle$ Per A B C  $\implies$  False $\rangle$  by auto
qed

```

```

lemma nsams--obtuse:
  assumes A  $\neq$  B and
    B  $\neq$  C and
     $\neg$  SAMS A B C A B C
  shows Obtuse A B C
proof -
{
  assume Per A B C
  obtain A' where B Midpoint A A'
    using symmetric-point-construction by blast
  have  $\neg$  Col A B C
    using  $\langle$ Per A B C $\rangle$  assms(1) assms(2) per-col-eq by blast
  have SAMS A B C A B C
  proof -
    have C B A' CongA A B C
      using  $\langle$ Per A B C $\rangle$  assms(1) assms(2) assms(3) per2--sams by blast
    moreover have  $\neg$  B C OS A A'
      by (meson Col-cases  $\langle$ B Midpoint A A' $\rangle$  col-one-side-out l6-4-1
        midpoint-bet midpoint-col)
    moreover have  $\neg$  A B TS C A'
      using Col-def Midpoint-def TS-def  $\langle$ B Midpoint A A' $\rangle$  by blast
    moreover have Coplanar A B C A'
      using  $\langle$ Per A B C $\rangle$  assms(1) assms(2) assms(3) per2--sams by blast
    ultimately show ?thesis
      using SAMS-def  $\langle$  $\neg$  Col A B C $\rangle$  assms(1) bet-col by auto
  qed
  hence False
    using assms(3) by auto
}
{
  assume Acute A B C
  hence SAMS A B C A B C
    by (simp add: acute--sams)
  hence False
    using assms(3) by auto
}
thus ?thesis
  using  $\langle$ Per A B C  $\implies$  False $\rangle$  angle-partition assms(1) assms(2) by auto
qed

```

```

lemma sams2-suma2--conga:
  assumes SAMS A B C A B C and
    A B C A B C SumA D E F and
    SAMS A' B' C' A' B' C' and
    A' B' C' A' B' C' SumA D E F
  shows A B C CongA A' B' C'

```

proof –
have $A B C \text{ LeA } A' B' C' \vee A' B' C' \text{ LeA } A B C$
using $\text{assms}(1) \text{ assms}(3) \text{ lea-total sams-distincts}$ **by** *auto*
moreover
have $A B C \text{ LeA } A' B' C' \longrightarrow A B C \text{ CongA } A' B' C'$
using $\text{assms}(2) \text{ assms}(3) \text{ assms}(4) \text{ sams-lea2-suma2--conga123}$ **by** *auto*
ultimately show *?thesis*
by $(\text{meson conga-sym sams-lea2-suma2--conga123 assms}(1) \text{ assms}(2) \text{ assms}(4))$
qed

lemma *acute2-suma2--conga*:
assumes $\text{Acute } A B C$ **and**
 $A B C A B C \text{ SumA } D E F$ **and**
 $\text{Acute } A' B' C'$ **and**
 $A' B' C' A' B' C' \text{ SumA } D E F$
shows $A B C \text{ CongA } A' B' C'$

proof –
have $\text{SAMS } A B C A B C$
by $(\text{simp add: acute--sams assms}(1))$
moreover have $\text{SAMS } A' B' C' A' B' C'$
by $(\text{simp add: acute--sams assms}(3))$
ultimately show *?thesis*
using $\text{assms}(2) \text{ assms}(4) \text{ sams2-suma2--conga}$ **by** *auto*
qed

lemma *bet2-suma--out*:
assumes $\text{Bet } A B C$ **and**
 $\text{Bet } D E F$ **and**
 $A B C D E F \text{ SumA } G H I$
shows $H \text{ Out } G I$

proof –
have $A B C D E F \text{ SumA } A B A$
proof –
have $C B A \text{ CongA } D E F$
by $(\text{metis Bet-cases assms}(1) \text{ assms}(2) \text{ assms}(3) \text{ conga-line suma-distincts})$
moreover have $\neg B C \text{ OS } A A$
by $(\text{simp add: Col-def assms}(1) \text{ col124--nos})$
moreover have $\text{Coplanar } A B C A$
using *ncop-distincts* **by** *blast*
moreover have $A B A \text{ CongA } A B A$
using $\text{calculation}(1) \text{ conga-diff2 conga-trivial-1}$ **by** *auto*
ultimately show *?thesis*
using *SumA-def* **by** *blast*
qed
hence $A B A \text{ CongA } G H I$
using $\text{assms}(3) \text{ suma2--conga}$ **by** *auto*
thus *?thesis*
using *eq-conga-out* **by** *auto*
qed

lemma *col2-suma--col*:
assumes $\text{Col } A B C$ **and**
 $\text{Col } D E F$ **and**
 $A B C D E F \text{ SumA } G H I$
shows $\text{Col } G H I$

proof *cases*
assume $\text{Bet } A B C$
show *?thesis*
proof *cases*
assume $\text{Bet } D E F$
thus *?thesis* **using** *bet2-suma--out*
by $(\text{meson } \langle \text{Bet } A B C \rangle \text{ assms}(3) \text{ not-col-permutation-4 out-col})$
next
assume $\neg \text{Bet } D E F$
show *?thesis*
proof –

```

    have  $E \text{ Out } D F$ 
      using  $\langle \neg \text{Bet } D E F \rangle \text{ assms}(2) \text{ or-bet-out}$  by auto
    hence  $A B C \text{ CongA } G H I$ 
      using  $\text{assms}(3) \text{ out546-suma--conga}$  by auto
    thus ?thesis
      using  $\text{assms}(1) \text{ col-conga-col}$  by blast
  qed
qed
next
assume  $\neg \text{Bet } A B C$ 
have  $D E F \text{ CongA } G H I$ 
proof -
  have  $B \text{ Out } A C$ 
    by (simp add:  $\langle \neg \text{Bet } A B C \rangle \text{ assms}(1) \text{ l6-4-2}$ )
  thus ?thesis
    using  $\text{assms}(3) \text{ out213-suma--conga}$  by auto
  qed
  thus ?thesis
    using  $\text{assms}(2) \text{ col-conga-col}$  by blast
qed

lemma suma-suppa--bet:
  assumes  $A B C \text{ SuppA } D E F$  and
     $A B C D E F \text{ SumA } G H I$ 
  shows  $\text{Bet } G H I$ 
proof -
  obtain  $A'$  where  $P1: \text{Bet } A B A' \wedge D E F \text{ CongA } C B A'$ 
    using  $\text{SuppA-def assms}(1)$  by auto
  obtain  $A''$  where  $P2: C B A'' \text{ CongA } D E F \wedge \neg B C \text{ OS } A A'' \wedge$ 
     $\text{Coplanar } A B C A'' \wedge A B A'' \text{ CongA } G H I$ 
    using  $\text{SumA-def assms}(2)$  by auto
  have  $B \text{ Out } A' A'' \vee C B \text{ TS } A' A''$ 
  proof -
    have  $\text{Coplanar } C B A' A''$ 
    proof -
      have  $\text{Coplanar } C A'' B A$ 
        using  $P2 \text{ coplanar-perm-17}$  by blast
      moreover have  $B \neq A$ 
        using  $\text{SuppA-def assms}(1)$  by auto
      moreover have  $\text{Col } B A A' \text{ using } P1$ 
        by (simp add:  $\text{bet-col col-permutation-4}$ )
      ultimately show ?thesis
        using  $\text{col-cop--cop coplanar-perm-3}$  by blast
    qed
    moreover have  $C B A' \text{ CongA } C B A''$ 
  proof -
    have  $C B A' \text{ CongA } D E F$ 
      using  $P1 \text{ not-conga-sym}$  by blast
    moreover have  $D E F \text{ CongA } C B A''$ 
      using  $P2 \text{ not-conga-sym}$  by blast
    ultimately show ?thesis
      using  $\text{not-conga}$  by blast
  qed
  ultimately show ?thesis
    using  $\text{conga-cop--or-out-ts}$  by simp
qed
have  $\text{Bet } A B A''$ 
proof -
  have  $\neg C B \text{ TS } A' A''$ 
  proof -
    {
      assume  $C B \text{ TS } A' A''$ 
      have  $B C \text{ TS } A A'$ 
      proof -
        {
          assume  $\text{Col } A B C$ 

```

```

    hence  $Col\ A'\ C\ B$ 
    using  $P1\ assms(1)\ bet-col\ bet-col1\ col3\ suppa-distincts$  by blast
    hence  $False$ 
    using  $TS-def\ \langle C\ B\ TS\ A'\ A'' \rangle$  by auto
  }
  hence  $\neg\ Col\ A\ B\ C$  by auto
  moreover have  $\neg\ Col\ A'\ B\ C$ 
    using  $TS-def\ \langle C\ B\ TS\ A'\ A'' \rangle\ not-col-permutation-5$  by blast
  moreover
  have  $\exists\ T.\ (Col\ T\ B\ C \wedge Bet\ A\ T\ A')$ 
    using  $P1\ not-col-distincts$  by blast
  ultimately show ?thesis
    by (simp add:  $TS-def$ )
qed
hence  $B\ C\ OS\ A\ A''$ 
  using  $OS-def\ \langle C\ B\ TS\ A'\ A'' \rangle\ invert-two-sides\ l9-2$  by blast
hence  $False$ 
  using  $P2$  by simp
}
thus ?thesis by blast
qed
hence  $B\ Out\ A'\ A''$ 
  using  $\langle B\ Out\ A'\ A'' \vee C\ B\ TS\ A'\ A'' \rangle$  by auto
moreover have  $A' \neq B \wedge A'' \neq B \wedge A \neq B$ 
  using  $P2\ calculation\ conga-diff1\ out-diff1\ out-diff2$  by blast
moreover have  $Bet\ A'\ B\ A$ 
  using  $P1\ Bet-perm$  by blast
ultimately show ?thesis
  using  $bet-out--bet\ between-symmetry$  by blast
qed
moreover have  $A\ B\ A''\ CongA\ G\ H\ I$ 
  using  $P2$  by blast
ultimately show ?thesis
  using  $bet-conga--bet$  by blast
qed

lemma  $bet-suppa--suma$ :
  assumes  $G \neq H$  and
     $H \neq I$  and
     $A\ B\ C\ SuppA\ D\ E\ F$  and
     $Bet\ G\ H\ I$ 
  shows  $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ 
proof -
  obtain  $G'\ H'\ I'$  where  $A\ B\ C\ D\ E\ F\ SumA\ G'\ H'\ I'$ 
    using  $assms(3)\ ex-suma\ suppa-distincts$  by blast
  moreover have  $A\ B\ C\ CongA\ A\ B\ C$ 
    using  $calculation\ conga-refl\ suma-distincts$  by fastforce
  moreover
  have  $D \neq E \wedge E \neq F$ 
    using  $assms(3)\ suppa-distincts$  by auto
  hence  $D\ E\ F\ CongA\ D\ E\ F$ 
    using  $conga-refl$  by auto
  moreover
  have  $G'\ H'\ I'\ CongA\ G\ H\ I$ 
proof -
  have  $G' \neq H' \wedge H' \neq I'$ 
    using  $calculation(1)\ suma-distincts$  by auto
  moreover have  $Bet\ G'\ H'\ I'$ 
    using  $\langle A\ B\ C\ D\ E\ F\ SumA\ G'\ H'\ I' \rangle\ assms(3)\ suma-suppa--bet$  by blast
  ultimately show ?thesis
    using  $assms(1)\ assms(2)\ assms(4)\ conga-line$  by auto
qed
ultimately show ?thesis
  using  $conga3-suma--suma$  by blast
qed

```

lemma *bet-suma--suppa*:
assumes $A B C D E F$ *SumA* $G H I$ **and**
 $Bet\ G\ H\ I$
shows $A B C$ *SuppA* $D E F$
proof –
obtain A' **where** $C B A'$ *CongA* $D E F \wedge A B A'$ *CongA* $G H I$
using *SumA-def* *assms*(1) **by** *blast*
moreover
have $G H I$ *CongA* $A B A'$
using *calculation* *not-conga-sym* **by** *blast*
hence $Bet\ A\ B\ A'$
using *assms*(2) *bet-conga--bet* **by** *blast*
moreover **have** $D E F$ *CongA* $C B A'$
using *calculation*(1) *not-conga-sym* **by** *blast*
ultimately show *?thesis*
by (*metis* *SuppA-def* *conga-diff1*)
qed

lemma *bet2-suma--suma*:
assumes $A' \neq B$ **and**
 $D' \neq E$ **and**
 $Bet\ A\ B\ A'$ **and**
 $Bet\ D\ E\ D'$ **and**
 $A B C D E F$ *SumA* $G H I$
shows $A' B C D' E F$ *SumA* $G H I$
proof –
obtain J **where** $P1$: $C B J$ *CongA* $D E F \wedge \neg B C$ *OS* $A J \wedge$
Coplanar $A B C J \wedge A B J$ *CongA* $G H I$
using *SumA-def* *assms*(5) **by** *auto*
moreover
obtain C' **where** $P2$: $Bet\ C B C' \wedge$ *Cong* $B C' B C$
using *segment-construction* **by** *blast*
moreover
have $A B C' D' E F$ *SumA* $G H I$
proof –
have $C' B J$ *CongA* $D' E F$
by (*metis* *assms*(2) *assms*(4) *calculation*(1) *calculation*(2)
cong-diff-3 *conga-diff1* *l11-13*)
moreover **have** $\neg B C' OS\ A\ J$
by (*metis* *Col-cases* $P1\ P2$ *bet-col* *col-one-side* *cong-diff*)
moreover **have** *Coplanar* $A B C' J$
proof–
have *Coplanar* $A B C J$
using $P1$ **by** *blast*
hence *Coplanar* $A B J C$
using *coplanar-perm-1* **by** *blast*
hence *Coplanar* $A B J C'$
by (*metis* $P2$ *bet-col* *col-cop2--cop* *cong-diff* *ncop-distincts*)
thus *?thesis*
using *coplanar-perm-1* **by** *blast*
qed
ultimately show *?thesis*
using $P1$ *SumA-def* **by** *blast*
qed
moreover **have** $A B C'$ *CongA* $A' B C$
using *assms*(1) *assms*(3) *assms*(5) *between-symmetry* *calculation*(2)
calculation(3) *l11-14* *suma-distincts* **by** *auto*
moreover
have $D' \neq E \wedge E \neq F$
using *assms*(2) *calculation*(1) *conga-distinct* **by** *blast*
hence $D' E F$ *CongA* $D' E F$
using *conga-refl* **by** *auto*
moreover
have $G \neq H \wedge H \neq I$
using *assms*(5) *suma-distincts* **by** *blast*
hence $G H I$ *CongA* $G H I$


```

    using conga-refl by auto
  ultimately show ?thesis
    using conga3-suma--suma by blast
qed

lemma suma-suppa2--suma:
  assumes  $A\ B\ C\ SuppA\ A'\ B'\ C'$  and
     $D\ E\ F\ SuppA\ D'\ E'\ F'$  and
     $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ 
  shows  $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G\ H\ I$ 
proof -
  obtain  $A0$  where  $P1: Bet\ A\ B\ A0 \wedge A'\ B'\ C'\ CongA\ C\ B\ A0$ 
    using SuppA-def assms(1) by auto
  obtain  $D0$  where  $P2: Bet\ D\ E\ D0 \wedge D'\ E'\ F'\ CongA\ F\ E\ D0$ 
    using SuppA-def assms(2) by auto
  show ?thesis
proof -
  have  $A0\ B\ C\ D0\ E\ F\ SumA\ G\ H\ I$ 
proof -
  have  $A0 \neq B$ 
    using CongA-def P1 by auto
  moreover have  $D0 \neq E$ 
    using CongA-def P2 by blast
  ultimately show ?thesis
    using P1 P2 assms(3) bet2-suma--suma by auto
qed
  moreover have  $A0\ B\ C\ CongA\ A'\ B'\ C'$ 
    using P1 conga-left-comm not-conga-sym by blast
  moreover have  $D0\ E\ F\ CongA\ D'\ E'\ F'$ 
    using P2 conga-left-comm not-conga-sym by blast
  moreover
  have  $G \neq H \wedge H \neq I$ 
    using assms(3) suma-distincts by blast
  hence  $G\ H\ I\ CongA\ G\ H\ I$ 
    using conga-refl by auto
  ultimately show ?thesis
    using conga3-suma--suma by blast
qed
qed

lemma suma2-obtuse2--conga:
  assumes Obtuse  $A\ B\ C$  and
     $A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F$  and
    Obtuse  $A'\ B'\ C'$  and
     $A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F$ 
  shows  $A\ B\ C\ CongA\ A'\ B'\ C'$ 
proof -
  obtain  $A0$  where  $P1: Bet\ A\ B\ A0 \wedge Cong\ B\ A0\ A\ B$ 
    using segment-construction by blast
  moreover
  obtain  $A0'$  where  $P2: Bet\ A'\ B'\ A0' \wedge Cong\ B'\ A0'\ A'\ B'$ 
    using segment-construction by blast
  moreover
  have  $A0\ B\ C\ CongA\ A0'\ B'\ C'$ 
proof -
  have Acute  $A0\ B\ C$ 
    using assms(1) bet-obtuse--acute P1 cong-diff-3 obtuse-distincts by blast
  moreover have  $A0\ B\ C\ A0\ B\ C\ SumA\ D\ E\ F$ 
    using P1 acute-distincts assms(2) bet2-suma--suma calculation by blast
  moreover have Acute  $A0'\ B'\ C'$ 
    using P2 assms(3) bet-obtuse--acute cong-diff-3 obtuse-distincts by blast
  moreover have  $A0'\ B'\ C'\ A0'\ B'\ C'\ SumA\ D\ E\ F$ 
    by (metis P2 assms(4) bet2-suma--suma cong-diff-3)
  ultimately show ?thesis
    using acute2-suma2--conga by blast
qed

```

```

moreover have  $Bet\ A\ 0\ B\ A$ 
  using  $Bet\text{-}perm\ calculation(1)$  by  $blast$ 
moreover have  $Bet\ A\ 0'\ B'\ A'$ 
  using  $Bet\text{-}perm\ calculation(2)$  by  $blast$ 
moreover have  $A \neq B$ 
  using  $assms(1)\ obtuse\text{-}distincts$  by  $blast$ 
moreover have  $A' \neq B'$ 
  using  $assms(3)\ obtuse\text{-}distincts$  by  $blast$ 
ultimately show  $?thesis$ 
  using  $l11\text{-}13$  by  $blast$ 
qed

lemma  $bet\text{-}suma2\text{-}or\text{-}conga$ :
  assumes  $A\ 0 \neq B$  and
     $Bet\ A\ B\ A\ 0$  and
     $A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F$  and
     $A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F$ 
  shows  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C'$ 
proof –
{
  fix  $A'\ B'\ C'$ 
  assume  $Acute\ A'\ B'\ C' \wedge A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F$ 
  have  $Per\ A\ B\ C \vee Acute\ A\ B\ C \vee Obtuse\ A\ B\ C$ 
    by  $(metis\ angle\text{-}partition\ l8\text{-}20\text{-}1\text{-}R1\ l8\text{-}5)$ 
  {
    assume  $Per\ A\ B\ C$ 
    hence  $Bet\ D\ E\ F$ 
    using  $per2\text{-}suma\text{-}bet\ assms(3)$  by  $auto$ 
    hence  $False$ 
    using  $\langle Acute\ A'\ B'\ C' \wedge A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F \rangle acute\text{-}suma\text{-}nbet$  by  $blast$ 
    hence  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C'$  by  $blast$ 
  }
  {
    assume  $Acute\ A\ B\ C$ 
    have  $Acute\ A'\ B'\ C'$ 
    by  $(simp\ add: \langle Acute\ A'\ B'\ C' \wedge A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F \rangle)$ 
    moreover have  $A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F$ 
    by  $(simp\ add: \langle Acute\ A'\ B'\ C' \wedge A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F \rangle)$ 
    ultimately
    have  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C'$ 
    using  $assms(3)\ \langle Acute\ A\ B\ C \rangle acute2\text{-}suma2\text{-}conga$  by  $auto$ 
  }
  {
    assume  $Obtuse\ A\ B\ C$ 
    have  $Acute\ A\ 0\ B\ C$ 
    using  $\langle Obtuse\ A\ B\ C \rangle assms(1)\ assms(2)\ bet\text{-}obtuse\text{-}acute$  by  $auto$ 
    moreover have  $A\ 0\ B\ C\ A\ 0\ B\ C\ SumA\ D\ E\ F$ 
    using  $assms(1)\ assms(2)\ assms(3)\ bet2\text{-}suma\text{-}suma$  by  $auto$ 
    ultimately have  $A\ 0\ B\ C\ CongA\ A'\ B'\ C'$ 
    using  $\langle Acute\ A'\ B'\ C' \wedge A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F \rangle acute2\text{-}suma2\text{-}conga$  by  $auto$ 
    hence  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C'$  by  $blast$ 
  }
  hence  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C'$ 
  using  $\langle Acute\ A\ B\ C \implies A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C' \rangle$ 
   $\langle Per\ A\ B\ C \implies A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C' \rangle$ 
   $\langle Per\ A\ B\ C \vee Acute\ A\ B\ C \vee Obtuse\ A\ B\ C \rangle$  by  $blast$ 
}
hence  $P1: \forall\ A'\ B'\ C'. (Acute\ A'\ B'\ C' \wedge A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F)$ 
 $\longrightarrow (A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ 0\ B\ C\ CongA\ A'\ B'\ C')$  by  $blast$ 
have  $Per\ A'\ B'\ C' \vee Acute\ A'\ B'\ C' \vee Obtuse\ A'\ B'\ C'$ 
  by  $(metis\ angle\text{-}partition\ l8\text{-}20\text{-}1\text{-}R1\ l8\text{-}5)$ 
{
  assume  $P2: Per\ A'\ B'\ C'$ 
  have  $A\ B\ C\ CongA\ A'\ B'\ C'$ 
proof –
  have  $A \neq B \wedge B \neq C$ 

```

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    using assms(3) suma-distincts by blast
  moreover have  $A' \neq B' \wedge B' \neq C'$ 
    using assms(4) suma-distincts by auto
  moreover have  $Per\ A\ B\ C$ 
  proof -
    have  $Bet\ D\ E\ F$ 
      using P2 assms(4) per2-suma--bet by blast
    moreover have  $A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F$ 
      by (simp add: assms(3))
    ultimately show ?thesis
      using assms(3) bet-suma--per by blast
  qed
  ultimately show ?thesis
    using P2 l11-16 by blast
  qed
  hence  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A0\ B\ C\ CongA\ A'\ B'\ C'$  by blast
}
{
  assume Acute  $A'\ B'\ C'$ 
  hence  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A0\ B\ C\ CongA\ A'\ B'\ C'$ 
    using P1 assms(4) by blast
}
{
  assume Obtuse  $A'\ B'\ C'$ 
  obtain  $A0'$  where  $Bet\ A'\ B'\ A0' \wedge Cong\ B'\ A0'\ A'\ B'$ 
    using segment-construction by blast
  moreover
  have Acute  $A0'\ B'\ C'$ 
    using  $\langle Obtuse\ A'\ B'\ C' \rangle$  bet-obtuse--acute calculation cong-diff-3
    obtuse-distincts by blast
  moreover have  $A0'\ B'\ C'\ A0'\ B'\ C'\ SumA\ D\ E\ F$ 
    using acute-distincts assms(4) bet2-suma--suma calculation(1)
    calculation(2) by blast
  ultimately
  have  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A0\ B\ C\ CongA\ A'\ B'\ C'$ 
    using P1 by (metis assms(1) assms(2) assms(3) assms(4) between-symmetry
    l11-13 suma-distincts)
}
thus ?thesis
  using  $\langle Acute\ A'\ B'\ C' \implies A\ B\ C\ CongA\ A'\ B'\ C' \vee A0\ B\ C\ CongA\ A'\ B'\ C' \rangle$ 
 $\langle Per\ A'\ B'\ C' \implies A\ B\ C\ CongA\ A'\ B'\ C' \vee A0\ B\ C\ CongA\ A'\ B'\ C' \rangle$ 
 $\langle Per\ A'\ B'\ C' \vee Acute\ A'\ B'\ C' \vee Obtuse\ A'\ B'\ C' \rangle$  by blast
qed

lemma suma2--or-conga-suppa:
  assumes  $A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F$  and
     $A'\ B'\ C'\ A'\ B'\ C'\ SumA\ D\ E\ F$ 
  shows  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A\ B\ C\ SuppA\ A'\ B'\ C'$ 
  proof -
    obtain  $A0$  where  $P1: Bet\ A\ B\ A0 \wedge Cong\ B\ A0\ A\ B$ 
      using segment-construction by blast
    hence  $A0 \neq B$ 
      using assms(1) bet-cong-eq suma-distincts by blast
    hence  $A\ B\ C\ CongA\ A'\ B'\ C' \vee A0\ B\ C\ CongA\ A'\ B'\ C'$ 
      using assms(1) assms(2) P1 bet-suma2--or-conga by blast
    thus ?thesis
      by (metis P1 SuppA-def cong-diff conga-right-comm conga-sym)
  qed

lemma ex-trisuma:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $A \neq C$ 
  shows  $\exists\ D\ E\ F. A\ B\ C\ TriSumA\ D\ E\ F$ 
  proof -
    obtain  $G\ H\ I$  where  $A\ B\ C\ B\ C\ A\ SumA\ G\ H\ I$ 

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    using assms(1) assms(2) assms(3) ex-suma by presburger
  moreover
  then obtain D E F where G H I C A B SumA D E F
    using ex-suma suma-distincts by presburger
  ultimately show ?thesis
    using TriSumA-def by blast
qed

lemma trisuma-perm-231:
  assumes A B C TriSumA D E F
  shows B C A TriSumA D E F
proof -
  obtain A1 B1 C1 where P1: A B C B C A SumA A1 B1 C1  $\wedge$  A1 B1 C1 C A B SumA D E F
    using TriSumA-def assms(1) by auto
  obtain A2 B2 C2 where P2: B C A C A B SumA B2 C2 A2
    using P1 ex-suma suma-distincts by fastforce
  have A B C B2 C2 A2 SumA D E F
  proof -
    have SAMS A B C B C A
      using assms sams123231 trisuma-distincts by auto
    moreover have SAMS B C A C A B
      using P2 sams123231 suma-distincts by fastforce
    ultimately show ?thesis
      using P1 P2 suma-assoc by blast
  qed
  thus ?thesis
    using P2 TriSumA-def suma-sym by blast
qed

lemma trisuma-perm-312:
  assumes A B C TriSumA D E F
  shows C A B TriSumA D E F
  by (simp add: assms trisuma-perm-231)

lemma trisuma-perm-321:
  assumes A B C TriSumA D E F
  shows C B A TriSumA D E F
proof -
  obtain G H I where A B C B C A SumA G H I  $\wedge$  G H I C A B SumA D E F
    using TriSumA-def assms(1) by auto
  thus ?thesis
    by (meson TriSumA-def suma-comm suma-right-comm suma-sym trisuma-perm-231)
qed

lemma trisuma-perm-213:
  assumes A B C TriSumA D E F
  shows B A C TriSumA D E F
  using assms trisuma-perm-231 trisuma-perm-321 by blast

lemma trisuma-perm-132:
  assumes A B C TriSumA D E F
  shows A C B TriSumA D E F
  using assms trisuma-perm-312 trisuma-perm-321 by blast

lemma conga-trisuma--trisuma:
  assumes A B C TriSumA D E F and
    D E F CongA D' E' F'
  shows A B C TriSumA D' E' F'
proof -
  obtain G H I where P1: A B C B C A SumA G H I  $\wedge$  G H I C A B SumA D E F
    using TriSumA-def assms(1) by auto
  have G H I C A B SumA D' E' F'
  proof -
    have f1: B  $\neq$  A
      by (metis P1 suma-distincts)
    have f2: C  $\neq$  A

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    using P1 suma-distincts by blast
  have G H I CongA G H I
    by (metis (full-types) P1 conga-refl suma-distincts)
  thus ?thesis
    using f2 f1 by (meson P1 assms(2) conga3-suma--suma conga-refl)
qed
thus ?thesis using P1 TriSumA-def by blast
qed

lemma trisuma2--conga:
  assumes A B C TriSumA D E F and
    A B C TriSumA D' E' F'
  shows D E F CongA D' E' F'
proof -
  obtain G H I where P1: A B C B C A SumA G H I  $\wedge$  G H I C A B SumA D E F
    using TriSumA-def assms(1) by auto
  hence P1A: G H I C A B SumA D E F by simp
  obtain G' H' I' where P2: A B C B C A SumA G' H' I'  $\wedge$  G' H' I' C A B SumA D' E' F'
    using TriSumA-def assms(2) by auto
  have G' H' I' C A B SumA D E F
  proof -
    have G H I CongA G' H' I' using P1 P2 suma2--conga by blast
    moreover have D E F CongA D E F  $\wedge$  C A B CongA C A B
      by (metis assms(1) conga-refl trisuma-distincts)
    ultimately show ?thesis
      by (meson P1 conga3-suma--suma)
  qed
  thus ?thesis
    using P2 suma2--conga by auto
qed

lemma conga3-trisuma--trisuma:
  assumes A B C TriSumA D E F and
    A B C CongA A' B' C' and
    B C A CongA B' C' A' and
    C A B CongA C' A' B'
  shows A' B' C' TriSumA D E F
proof -
  obtain G H I where P1: A B C B C A SumA G H I  $\wedge$  G H I C A B SumA D E F
    using TriSumA-def assms(1) by auto
  thus ?thesis
  proof -
    have A' B' C' B' C' A' SumA G H I
      using conga3-suma--suma P1 by (meson assms(2) assms(3) suma2--conga)
    moreover have G H I C' A' B' SumA D E F
      using conga3-suma--suma P1 by (meson P1 assms(4) suma2--conga)
    ultimately show ?thesis
      using TriSumA-def by blast
  qed
qed

lemma col-trisuma--bet:
  assumes Col A B C and
    A B C TriSumA P Q R
  shows Bet P Q R
proof -
  obtain D E F where P1: A B C B C A SumA D E F  $\wedge$  D E F C A B SumA P Q R
    using TriSumA-def assms(2) by auto
  {
    assume Bet A B C
    have A B C CongA P Q R
  proof -
    have A B C CongA D E F
  proof -
    have C  $\neq$  A  $\wedge$  C  $\neq$  B
      using assms(2) trisuma-distincts by blast

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    hence  $C \text{ Out } B \ A$ 
    using  $\langle \text{Bet } A \ B \ C \rangle \text{ bet-out-1}$  by fastforce
    thus ?thesis
    using  $P1 \text{ out546-suma--conga}$  by auto
  qed
  moreover have  $D \ E \ F \ CongA \ P \ Q \ R$ 
  proof -
    have  $A \neq C \wedge A \neq B$ 
    using  $\text{assms}(2) \text{ trisuma-distincts}$  by blast
    hence  $A \text{ Out } C \ B$ 
    using  $\text{Out-def } \langle \text{Bet } A \ B \ C \rangle$  by auto
    thus ?thesis
    using  $P1 \text{ out546-suma--conga}$  by auto
  qed
  ultimately show ?thesis
    using  $\text{conga-trans}$  by blast
  qed
  hence  $\text{Bet } P \ Q \ R$ 
    using  $\langle \text{Bet } A \ B \ C \rangle \text{ bet-conga--bet}$  by blast
}
{
  assume  $\text{Bet } B \ C \ A$ 
  have  $B \ C \ A \ CongA \ P \ Q \ R$ 
  proof -
    have  $B \ C \ A \ CongA \ D \ E \ F$ 
    proof -
      have  $B \neq A \wedge B \neq C$ 
      using  $\text{assms}(2) \text{ trisuma-distincts}$  by blast
      hence  $B \text{ Out } A \ C$ 
      using  $\text{Out-def } \langle \text{Bet } B \ C \ A \rangle$  by auto
      thus ?thesis
      using  $P1 \text{ out213-suma--conga}$  by blast
    qed
    moreover have  $D \ E \ F \ CongA \ P \ Q \ R$ 
    proof -
      have  $A \neq C \wedge A \neq B$ 
      using  $\text{assms}(2) \text{ trisuma-distincts}$  by auto
      hence  $A \text{ Out } C \ B$ 
      using  $\langle \text{Bet } B \ C \ A \rangle \text{ bet-out-1}$  by auto
      thus ?thesis
      using  $P1 \text{ out546-suma--conga}$  by blast
    qed
    ultimately show ?thesis
      using  $\text{not-conga}$  by blast
  qed
  hence  $\text{Bet } P \ Q \ R$ 
    using  $\langle \text{Bet } B \ C \ A \rangle \text{ bet-conga--bet}$  by blast
}
{
  assume  $\text{Bet } C \ A \ B$ 
  have  $E \text{ Out } D \ F$ 
  proof -
    have  $C \text{ Out } B \ A$ 
    using  $\langle \text{Bet } C \ A \ B \rangle \text{ assms}(2) \text{ bet-out l6-6 trisuma-distincts}$  by blast
    moreover have  $B \ C \ A \ CongA \ D \ E \ F$ 
    proof -
      have  $B \neq A \wedge B \neq C$ 
      using  $P1 \text{ suma-distincts}$  by blast
      hence  $B \text{ Out } A \ C$ 
      using  $\langle \text{Bet } C \ A \ B \rangle \text{ bet-out-1}$  by auto
      thus ?thesis using  $\text{out213-suma--conga } P1$  by blast
    qed
    ultimately show ?thesis
      using  $\text{l11-21-a}$  by blast
  qed
  hence  $C \ A \ B \ CongA \ P \ Q \ R$ 

```

```

    using P1 out213-suma--conga by blast
  hence Bet P Q R
    using ⟨Bet C A B⟩ bet-conga--bet by blast
}
thus ?thesis
  using Col-def ⟨Bet A B C ⟹ Bet P Q R⟩ ⟨Bet B C A ⟹ Bet P Q R⟩ assms(1) by blast
qed

```

```

lemma suma-dec:
  A B C D E F SumA G H I  $\vee$   $\neg$  A B C D E F SumA G H I
  by simp

```

```

lemma sams-dec:
  SAMS A B C D E F  $\vee$   $\neg$  SAMS A B C D E F
  by simp

```

```

lemma trisuma-dec:
  A B C TriSumA P Q R  $\vee$   $\neg$  A B C TriSumA P Q R
  by simp

```

```

lemma acute-not-bet:
  assumes Acute A B C
  shows  $\neg$  Bet A B C
  using acute-col-out assms bet-col not-bet-and-out by blast

```

```

lemma upper-dim-3-stab:
  assumes  $\neg$   $\neg$  upper-dim-3-axiom
  shows upper-dim-3-axiom
  using assms by blast

```

```

lemma median-planes-implies-upper-dim:
  assumes median-planes-axiom
  shows upper-dim-3-axiom

```

```

proof -
{
  fix A B C P Q R
  assume P  $\neq$  Q and Q  $\neq$  R and P  $\neq$  R and
    Cong A P A Q and Cong B P B Q and Cong C P C Q and
    Cong A P A R and Cong B P B R and Cong C P C R
  have Bet A B C  $\vee$  Bet B C A  $\vee$  Bet C A B
  proof (cases Col A B C)
    case True
    thus ?thesis
      by (simp add: Col-def upper-dim-3-axiom-def)
  next
    case False
    hence  $\neg$  Col A B C
    by simp
  {
    obtain X where X Midpoint P Q
      using midpoint-existence by blast
    obtain Y where Y Midpoint P R
      using midpoint-existence by blast
    have X = Y
    proof -
      have Per X Y P
      proof -
        have  $\neg$  Col A B C
        by (simp add: False)
        moreover have Per A Y P
          using Per-def ⟨Cong A P A R⟩ ⟨Y Midpoint P R⟩ by blast
        moreover have Per B Y P
          using Per-def ⟨Cong B P B R⟩ ⟨Y Midpoint P R⟩ by blast
        moreover have Per C Y P
          using Per-def ⟨Cong C P C R⟩ ⟨Y Midpoint P R⟩ by blast
        moreover

```

```

have Cong X P X Q
  using Cong-cases ⟨X Midpoint P Q⟩ midpoint-cong by blast
hence Coplanar A B C X
  using ⟨P ≠ Q⟩ ⟨Cong A P A Q⟩ ⟨Cong B P B Q⟩ ⟨Cong C P C Q⟩ assms
    median-planes-axiom-def by blast
ultimately show ?thesis
  using l11-60 by blast
qed
moreover have Per Y X P
proof -
  have ¬ Col A B C
    by (simp add: False)
  moreover have Per A X P
    using Per-def ⟨Cong A P A Q⟩ ⟨X Midpoint P Q⟩ by blast
  moreover have Per B X P
    using Per-def ⟨Cong B P B Q⟩ ⟨X Midpoint P Q⟩ by blast
  moreover have Per C X P
    using Per-def ⟨Cong C P C Q⟩ ⟨X Midpoint P Q⟩ by blast
  moreover
  have Cong Y P Y R
    using Cong-cases ⟨Y Midpoint P R⟩ midpoint-cong by blast
  hence Coplanar A B C Y
    using ⟨P ≠ R⟩ ⟨Cong A P A R⟩ ⟨Cong B P B R⟩ ⟨Cong C P C R⟩
      assms median-planes-axiom-def by blast
  ultimately show ?thesis
    using l11-60 by blast
qed
ultimately show ?thesis
  using Per-cases l8-7 by blast
qed
hence X Midpoint P R
  by (simp add: ⟨Y Midpoint P R⟩)
hence Q = R
  using ⟨X Midpoint P Q⟩ symmetric-point-uniqueness by blast
hence False
  using ⟨Q ≠ R⟩ by blast
hence Bet A B C ∨ Bet B C A ∨ Bet C A B
  by simp
}
thus ?thesis
  by (simp add: upper-dim-3-axiom-def)
qed
}
thus ?thesis
  using upper-dim-3-axiom-def by blast
qed

```

```

lemma median-planes-aux:
  assumes ∀ A B C P Q M. P ≠ Q ∧ Cong A P A Q ∧ Cong B P B Q ∧
    Cong C P C Q ∧ M Midpoint P Q ⟶ Coplanar M A B C
  shows median-planes-axiom
proof -
  {
    fix A B C D P Q
    assume P ≠ Q and
      Cong A P A Q and
      Cong B P B Q and
      Cong C P C Q and
      Cong D P D Q
    have Coplanar A B C D
    proof (cases Col A B C)
      case True
      thus ?thesis
        using ncop--ncols by blast
    next
      case False
    end
  }

```


obtain M **where** M *Midpoint* P Q
using *midpoint-existence* **by** *blast*
obtain $A1$ $A2$ **where** *Coplanar* A B C $A1$ **and** *Coplanar* A B C $A2$ **and** $\neg$ *Col* M $A1$ $A2$
using *ex-ncol-cop2* **by** *blast*
have *Cong* $A1$ P $A1$ Q
using *False l11-60-aux* $\langle$ *Cong* A P A Q $\rangle$ $\langle$ *Cong* B P B Q $\rangle$
 $\langle$ *Cong* C P C Q $\rangle$ $\langle$ *Coplanar* A B C $A1$ $\rangle$ **by** *blast*
have *Cong* $A2$ P $A2$ Q
using *False* $\langle$ *Cong* A P A Q $\rangle$ $\langle$ *Cong* B P B Q $\rangle$ $\langle$ *Cong* C P C Q $\rangle$ $\langle$ *Coplanar* A B C $A2$ $\rangle$
l11-60-aux **by** *blast*
have *Coplanar* M A B C
using *assms* $\langle$ *M Midpoint* P Q $\rangle$ $\langle$ *Cong* A P A Q $\rangle$ $\langle$ *Cong* B P B Q $\rangle$ $\langle$ *Cong* C P C Q $\rangle$
 $\langle$ $P \neq Q$ $\rangle$ **by** *presburger*
have *Coplanar* M A B D
using *assms* $\langle$ *M Midpoint* P Q $\rangle$ $\langle$ *Cong* A P A Q $\rangle$ $\langle$ *Cong* B P B Q $\rangle$ $\langle$ *Cong* D P D Q $\rangle$
 $\langle$ $P \neq Q$ $\rangle$ **by** *blast*
have *Coplanar* A B C D
proof $-$
have *Coplanar* M $A1$ $A2$ A
by (*meson* *False coplanar-pseudo-trans* $\langle$ *Coplanar* A B C $A1$ $\rangle$ $\langle$ *Coplanar* A B C $A2$ $\rangle$
 $\langle$ *Coplanar* M A B C $\rangle$ *coplanar-perm-9 ncop-distincts*)
moreover have *Coplanar* M $A1$ $A2$ B
by (*meson* *False coplanar-pseudo-trans* $\langle$ *Coplanar* A B C $A1$ $\rangle$ $\langle$ *Coplanar* A B C $A2$ $\rangle$
 $\langle$ *Coplanar* M A B C $\rangle$ *coplanar-perm-9 ncop-distincts*)
moreover have *Coplanar* M $A1$ $A2$ C
by (*meson* *False* $\langle$ *Coplanar* A B C $A1$ $\rangle$ $\langle$ *Coplanar* A B C $A2$ $\rangle$ $\langle$ *Coplanar* M A B C $\rangle$
coplanar-perm-9 coplanar-pseudo-trans ncop-distincts)
moreover
have *Coplanar* M $A1$ $A2$ D
proof (*cases* *Col* A B M)
case *True*
hence *Col* A B M
by *blast*
have *Coplanar* P Q A B
using *Col-def Coplanar-def Midpoint-def True* $\langle$ *M Midpoint* P Q $\rangle$
ncoplanar-perm-15 **by** *blast*
show *?thesis*
proof (*cases* *Col* P Q A)
case *True*
hence *Col* P Q A
by *blast*
thus *?thesis*
using $\langle$ *Cong* $A1$ P $A1$ Q $\rangle$ $\langle$ *Cong* $A2$ P $A2$ Q $\rangle$ $\langle$ *Cong* D P D Q $\rangle$ $\langle$ *M Midpoint* P Q $\rangle$
 $\langle$ $P \neq Q$ $\rangle$ *assms* **by** *blast*
next
case *False*
hence $\neg$ *Col* P Q A
by *simp*
thus *?thesis*
using $\langle$ *Cong* $A1$ P $A1$ Q $\rangle$ $\langle$ *Cong* $A2$ P $A2$ Q $\rangle$ $\langle$ *Cong* D P D Q $\rangle$ $\langle$ *M Midpoint* P Q $\rangle$
 $\langle$ $P \neq Q$ $\rangle$ *assms* **by** *blast*
qed
next
case *False*
hence $\neg$ *Col* A B M
by *blast*
moreover have *Coplanar* M A B M
using *ncop-distincts* **by** *blast*
moreover have *Coplanar* M A B $A1$
by (*meson* $\langle$ *Coplanar* M $A1$ $A2$ A $\rangle$ $\langle$ *Coplanar* M $A1$ $A2$ B $\rangle$
 $\langle$ $\neg$ *Col* M $A1$ $A2$ $\rangle$ *coplanar-pseudo-trans ncop-distincts*)
moreover have *Coplanar* M A B $A2$
by (*meson* $\langle$ *Coplanar* M $A1$ $A2$ A $\rangle$ $\langle$ *Coplanar* M $A1$ $A2$ B $\rangle$
 $\langle$ $\neg$ *Col* M $A1$ $A2$ $\rangle$ *coplanar-pseudo-trans ncop-distincts*)
ultimately
show *?thesis*

```

    by (meson ⟨Coplanar M A B D⟩ coplanar-pseudo-trans not-col-permutation-2)
  qed
  ultimately show ?thesis
    using ⟨¬ Col M A1 A2⟩ coplanar-pseudo-trans by blast
  qed
  thus ?thesis
    by blast
  qed
}
thus ?thesis
  using median-planes-axiom-def by blast
qed

lemma orthonormal-family-aux-1:
  assumes orthonormal-family-axiom
  shows ∀ A B X P Q. ¬ Col P Q X ∧ Per A X P ∧ Per A X Q ∧ Per B X P ∧ Per B X Q ⟶ Col A B X
proof -
{
  fix A B X P Q
  assume ¬ Col P Q X and
    Per A X P and
    Per A X Q and
    Per B X P and Per B X Q
  {
    assume ¬ Col A B X
    obtain P' where Bet P X P' and Cong X P' P X
      using segment-construction by blast
    have A ≠ X
      using ⟨¬ Col A B X⟩ not-col-distincts by blast
    have X ≠ P
      using ⟨¬ Col P Q X⟩ not-col-distincts by blast
    have X ≠ P'
      using ⟨Cong X P' P X⟩ ⟨X ≠ P⟩ cong-reverse-identity by blast
    obtain Q' where Per Q' X P and Cong Q' X P' X and P X OS Q' Q
      using ex-per-cong ⟨X ≠ P'⟩ ⟨X ≠ P⟩ ⟨¬ Col P Q X⟩ col-trivial-2
      not-col-permutation-5 by metis
    have Per Q' X A
    proof (rule l11-60 [where ?A = P and ?B = Q and ?C = X], insert ⟨¬ Col P Q X⟩)
      show Per P X A
        using ⟨Per A X P⟩ l8-2 by blast
      show Per Q X A
        by (simp add: ⟨Per A X Q⟩ l8-2)
      show Per X X A
        by (simp add: l8-20-1-R1)
      show Coplanar P Q X Q'
        using ⟨P X OS Q' Q⟩ coplanar-perm-4 os--coplanar by blast
    qed
    have Per Q' X B
    proof (rule l11-60 [where ?A = P and ?B = Q and ?C = X], insert ⟨¬ Col P Q X⟩)
      show Per P X B
        by (simp add: ⟨Per B X P⟩ l8-2)
      show Per Q X B
        using ⟨Per B X Q⟩ l8-2 by blast
      show Per X X B
        by (simp add: l8-20-1-R1)
      show Coplanar P Q X Q'
        using ⟨P X OS Q' Q⟩ coplanar-perm-4 os--coplanar by blast
    qed
    have ¬ Col P X Q'
      using ⟨P X OS Q' Q⟩ col123--nos by blast
    let ?Q = Q'
    obtain A' where Bet A X A' and Cong X A' P' X
      using segment-construction by blast
    have Per P X A'
      using Col-def Per-cases ⟨A ≠ X⟩ ⟨Bet A X A'⟩ ⟨Per A X P⟩ between-symmetry l8-3 by blast
    have Per ?Q X A'

```

```

  by (metis Bet-cases Col-def  $\langle A \neq X \rangle$   $\langle \text{Bet } A \ X \ A' \rangle$   $\langle \text{Per } Q' \ X \ A \rangle$  per-col)
have  $\neg \text{Col } A' \ B \ X$ 
  by (metis  $\langle \text{Bet } A \ X \ A' \rangle$   $\langle \text{Cong } X \ A' \ P' \ X \rangle$   $\langle X \neq P' \rangle$   $\langle \neg \text{Col } A \ B \ X \rangle$ 
      bet-col cong-diff-4 l6-16-1 not-col-permutation-2)
let  $?A = A'$ 
have  $\exists B'. \text{Per } B' \ X \ ?A \wedge \text{Cong } B' \ X \ P' \ X \wedge ?A \ X \text{ OS } B' \ B$ 
proof (rule ex-per-cong)
  show  $?A \neq X$ 
    using  $\langle \neg \text{Col } A' \ B \ X \rangle$  col-trivial-3 by force
  show  $P' \neq X$ 
    using  $\langle X \neq P' \rangle$  by blast
  show  $\text{Col } ?A \ X \ X$ 
    by (simp add: col-trivial-2)
  show  $\neg \text{Col } ?A \ X \ B$ 
    using  $\langle \neg \text{Col } A' \ B \ X \rangle$  not-col-permutation-5 by blast
qed
then obtain  $B'$  where  $\text{Per } B' \ X \ ?A$  and  $\text{Cong } B' \ X \ P' \ X$  and  $?A \ X \text{ OS } B' \ B$ 
  by blast
have  $\text{Per } B' \ X \ P$ 
proof (rule l11-60 [where  $?A = ?A$  and  $?B = B$  and  $?C = X$ ], insert  $\langle \neg \text{Col } A' \ B \ X \rangle$ )
  show  $\text{Per } ?A \ X \ P$ 
    by (simp add:  $\langle \text{Per } P \ X \ A' \rangle$  l8-2)
  show  $\text{Per } B \ X \ P$ 
    by (simp add:  $\langle \text{Per } B \ X \ P \rangle$ )
  show  $\text{Per } X \ X \ P$ 
    by (simp add: l8-20-1-R1)
  show  $\text{Coplanar } ?A \ B \ X \ B'$ 
    using  $\langle A' \ X \text{ OS } B' \ B \rangle$  coplanar-perm-4 os--coplanar by blast
qed
have  $\text{Per } B' \ X \ ?Q$ 
proof (rule l11-60 [where  $?A = ?A$  and  $?B = B$  and  $?C = X$ ], insert  $\langle \neg \text{Col } A' \ B \ X \rangle$ )
  show  $\neg \text{Col } A' \ B \ X \implies \text{Per } A' \ X \ ?Q$ 
    by (simp add:  $\langle \text{Per } Q' \ X \ A' \rangle$  l8-2)
  show  $\neg \text{Col } A' \ B \ X \implies \text{Per } B \ X \ ?Q$ 
    using Per-cases  $\langle \text{Per } Q' \ X \ B \rangle$  by auto
  show  $\neg \text{Col } A' \ B \ X \implies \text{Per } X \ X \ ?Q$ 
    by (simp add: l8-20-1-R1)
  show  $\neg \text{Col } A' \ B \ X \implies \text{Coplanar } A' \ B \ X \ B'$ 
    using  $\langle A' \ X \text{ OS } B' \ B \rangle$  coplanar-perm-4 os--coplanar by blast
qed
let  $?B = B'$ 
have  $\text{Cong } ?Q \ P \ ?Q \ P'$ 
proof (rule per-double-cong [where  $?B = X$ ])
  show  $\text{Per } ?Q \ X \ P$ 
    using  $\langle \text{Per } Q' \ X \ P \rangle$  by blast
  show  $X \text{ Midpoint } P \ P'$ 
    using Cong-perm Midpoint-def  $\langle \text{Bet } P \ X \ P' \rangle$   $\langle \text{Cong } X \ P' \ P \ X \rangle$  by blast
qed
have False
proof -
  have  $\text{Cong } X \ P \ X \ P'$ 
    using Cong-cases  $\langle \text{Cong } X \ P' \ P \ X \rangle$  by blast
  moreover have  $\text{Cong } X \ ?Q \ X \ P'$ 
    using Cong-cases  $\langle \text{Cong } Q' \ X \ P' \ X \rangle$  by blast
  moreover have  $\text{Cong } X \ ?A \ X \ P'$ 
    using Cong-cases  $\langle \text{Cong } X \ A' \ P' \ X \rangle$  by blast
  moreover have  $\text{Cong } X \ ?B \ X \ P'$ 
    using Cong-cases  $\langle \text{Cong } B' \ X \ P' \ X \rangle$  by blast
  moreover have  $\text{Cong } P \ ?Q \ P' \ ?Q$ 
    using Cong-cases  $\langle \text{Cong } Q' \ P \ Q' \ P' \rangle$  by blast
  moreover
  have  $\text{Cong } P \ ?A \ P \ ?Q$ 
    by (meson  $\langle \text{Cong } Q' \ X \ P' \ X \rangle$   $\langle \text{Cong } X \ P' \ P \ X \rangle$   $\langle \text{Per } P \ X \ A' \rangle$   $\langle \text{Per } Q' \ X \ P \rangle$ 
        calculation(3) cong-4312 cong-inner-transitivity l10-12)
  hence  $\text{Cong } P \ ?A \ P' \ ?Q$ 
    using cong-transitivity calculation(5) by blast

```

```

moreover
have  $\text{Cong } P \text{ ?}B \text{ } P \text{ ?}Q$ 
proof (rule l10-12 [where  $?B = X$  and  $?B' = X$ ])
  show  $\text{Per } P \text{ } X \text{ } B'$ 
    by (simp add:  $\langle \text{Per } B' \text{ } X \text{ } P \rangle$  l8-2)
  show  $\text{Per } P \text{ } X \text{ } Q'$ 
    by (simp add:  $\langle \text{Per } Q' \text{ } X \text{ } P \rangle$  l8-2)
  show  $\text{Cong } P \text{ } X \text{ } P \text{ } X$ 
    by (simp add: cong-reflexivity)
  show  $\text{Cong } X \text{ } B' \text{ } X \text{ } Q'$ 
    by (meson calculation(2) calculation(4) cong-3421 cong-inner-transitivity)
qed
hence  $\text{Cong } P \text{ ?}B \text{ } P' \text{ ?}Q$ 
  using  $\langle \text{Cong } P \text{ } Q' \text{ } P' \text{ } Q' \rangle$  cong-transitivity by blast
moreover have  $\text{Cong } ?Q \text{ ?}A \text{ } P' \text{ ?}Q$ 
  using l10-12
  by (metis Col-def not-cong-1243  $\langle \text{Bet } P \text{ } X \text{ } P' \rangle$   $\langle \text{Per } Q' \text{ } X \text{ } A' \rangle$   $\langle \text{Per } Q' \text{ } X \text{ } P \rangle$ 
     $\langle X \neq P \rangle$  between-symmetry calculation(3) cong-reflexivity per-col)
moreover have  $\text{Cong } ?A \text{ ?}B \text{ } P' \text{ ?}Q$ 
  by (meson  $\langle \text{Cong } X \text{ } P' \text{ } P \text{ } X \rangle$   $\langle \text{Per } B' \text{ } X \text{ } A' \rangle$   $\langle \text{Per } P \text{ } X \text{ } A' \rangle$  calculation(4) calculation(6)
    cong-pseudo-reflexivity cong-transitivity l10-12)
moreover
have  $\text{Cong } P \text{ ?}Q \text{ ?}Q \text{ ?}B$ 
proof (rule l10-12 [where  $?B = X$  and  $?B' = X$ ])
  show  $\text{Per } P \text{ } X \text{ } Q'$ 
    by (simp add:  $\langle \text{Per } Q' \text{ } X \text{ } P \rangle$  l8-2)
  show  $\text{Per } Q' \text{ } X \text{ } B'$ 
    by (simp add:  $\langle \text{Per } B' \text{ } X \text{ } Q' \rangle$  l8-2)
  show  $\text{Cong } P \text{ } X \text{ } Q' \text{ } X$ 
    using  $\langle \text{Cong } X \text{ } P' \text{ } P \text{ } X \rangle$  calculation(2) cong-3421 cong-transitivity by blast
  show  $\text{Cong } X \text{ } Q' \text{ } X \text{ } B'$ 
    by (meson calculation(2) calculation(4) cong-3421 cong-inner-transitivity)
qed
have  $\text{Cong } P' \text{ ?}Q \text{ } P \text{ ?}Q$ 
  using  $\langle \text{Cong } P \text{ } Q' \text{ } P' \text{ } Q' \rangle$  Cong-cases by auto
hence  $\text{Cong } ?Q \text{ ?}B \text{ } P' \text{ ?}Q$ 
  using  $\langle \text{Cong } P \text{ } Q' \text{ } Q' \text{ } B' \rangle$  calculation(5) cong-inner-transitivity by blast
ultimately show ?thesis
  using assms  $\langle X \neq P' \rangle$   $\langle \text{Bet } P \text{ } X \text{ } P' \rangle$  orthonormal-family-axiom-def by blast
qed
}
hence  $\text{Col } A \text{ } B \text{ } X$ 
  by blast
}
thus ?thesis
  by blast
qed

```

```

lemma orthonormal-family-aux-2:
  assumes  $\forall A \text{ } B \text{ } X \text{ } P \text{ } Q. \neg \text{Col } P \text{ } Q \text{ } X \wedge \text{Per } A \text{ } X \text{ } P \wedge \text{Per } A \text{ } X \text{ } Q \wedge$ 
     $\text{Per } B \text{ } X \text{ } P \wedge \text{Per } B \text{ } X \text{ } Q \longrightarrow \text{Col } A \text{ } B \text{ } X$ 
  shows orthonormal-family-axiom
proof –
  {
    fix  $S \text{ } U1' \text{ } U1 \text{ } U2 \text{ } U3 \text{ } U4$ 
    assume  $S \neq U1'$  and
       $\text{Bet } U1 \text{ } S \text{ } U1'$  and
       $\text{Cong } S \text{ } U1 \text{ } S \text{ } U1'$  and
       $\text{Cong } S \text{ } U2 \text{ } S \text{ } U1'$  and
       $\text{Cong } S \text{ } U3 \text{ } S \text{ } U1'$  and
       $\text{Cong } S \text{ } U4 \text{ } S \text{ } U1'$  and
       $\text{Cong } U1 \text{ } U2 \text{ } U1' \text{ } U2$  and
       $\text{Cong } U1 \text{ } U3 \text{ } U1' \text{ } U2$  and
       $\text{Cong } U1 \text{ } U4 \text{ } U1' \text{ } U2$  and
       $\text{Cong } U2 \text{ } U3 \text{ } U1' \text{ } U2$  and
       $\text{Cong } U2 \text{ } U4 \text{ } U1' \text{ } U2$  and

```

```

  Cong U3 U4 U1' U2
have S Midpoint U1 U1'
  using Cong-cases ⟨Bet U1 S U1'⟩ ⟨Cong S U1 S U1'⟩ midpoint-def by blast
hence Per U2 S U1
  using Per-def ⟨Cong U1 U2 U1' U2⟩ ⟨S Midpoint U1 U1'⟩ not-cong-2143 by blast
have S ≠ U4
  using ⟨Cong S U4 S U1'⟩ ⟨S ≠ U1'⟩ cong-diff-3 by blast
have S ≠ U3
  using ⟨Cong S U3 S U1'⟩ ⟨S ≠ U1'⟩ cong-reverse-identity by auto
have S ≠ U2
  using ⟨Cong S U2 S U1'⟩ ⟨S ≠ U1'⟩ cong-diff-3 by blast
have S ≠ U1
  using ⟨Cong S U1 S U1'⟩ ⟨S ≠ U1'⟩ cong-diff-3 by blast
have U1 ≠ U2
  using ⟨Per U2 S U1⟩ ⟨S ≠ U2⟩ per-distinct-1 by blast
have Col U2 U1 S
proof –
  have U2 S U1 CongA U4 S U3
    using l11-51
    by (metis Tarski-neutral-dimensionless.cong-transitivity
      Tarski-neutral-dimensionless-axioms ⟨Cong S U1 S U1'⟩
      ⟨Cong S U2 S U1'⟩ ⟨Cong S U3 S U1'⟩ ⟨Cong S U4 S U1'⟩
      ⟨Cong U1 U2 U1' U2⟩ ⟨Cong U3 U4 U1' U2⟩ ⟨S ≠ U1⟩ ⟨S ≠ U2⟩ ⟨U1 ≠ U2⟩ cong-3421)
  hence Per U4 S U3
    using l8-10 l11-17 ⟨Per U2 S U1⟩ by blast
  hence ¬ Col U4 S U3
    using ⟨S ≠ U3⟩ ⟨S ≠ U4⟩ per-col-eq by blast
  hence ¬ Col U3 U4 S
    using Col-cases by blast
  moreover
  have U2 S U1 CongA U2 S U3
    by (metis cong-transitivity l11-51 midpoint-cong
      ⟨Cong S U3 S U1'⟩ ⟨Cong U1 U2 U1' U2⟩ ⟨Cong U2 U3 U1' U2⟩ ⟨S Midpoint U1 U1'⟩
      ⟨S ≠ U1⟩ ⟨S ≠ U2⟩ ⟨U1 ≠ U2⟩ cong-3421 cong-pseudo-reflexivity)
  hence Per U2 S U3
    using ⟨Per U2 S U1⟩ l11-17 by blast
  moreover
  have U2 S U1 CongA U2 S U4
    by (metis cong-reflexivity l11-51
      ⟨Cong S U1 S U1'⟩ ⟨Cong S U4 S U1'⟩ ⟨Cong U1 U2 U1' U2⟩ ⟨Cong U2 U4 U1' U2⟩
      ⟨S ≠ U1⟩ ⟨S ≠ U2⟩ ⟨U1 ≠ U2⟩ cong-left-commutativity cong-symmetry)
  hence Per U2 S U4
    using ⟨Per U2 S U1⟩ l11-17 by blast
  moreover have U2 S U1 CongA U1 S U3
    by (metis cong-transitivity ⟨Cong S U1 S U1'⟩ ⟨Cong S U2 S U1'⟩
      ⟨Cong S U3 S U1'⟩ ⟨Cong U1 U2 U1' U2⟩ ⟨Cong U1 U3 U1' U2⟩ ⟨S ≠ U1⟩ ⟨S ≠ U2⟩
      ⟨U1 ≠ U2⟩ cong-left-commutativity cong-symmetry l11-51)
  hence Per U1 S U3
    using ⟨Per U2 S U1⟩ l11-17 by blast
  moreover have U2 S U1 CongA U1 S U4
    by (metis cong-transitivity ⟨Cong S U1 S U1'⟩ ⟨Cong S U2 S U1'⟩
      ⟨Cong S U4 S U1'⟩ ⟨Cong U1 U2 U1' U2⟩ ⟨Cong U1 U4 U1' U2⟩ ⟨S ≠ U1⟩
      ⟨S ≠ U2⟩ ⟨U1 ≠ U2⟩ cong-left-commutativity cong-symmetry l11-51)
  hence Per U1 S U4
    using ⟨Per U2 S U1⟩ l11-17 by blast
  ultimately show ?thesis
    using assms by blast
qed
hence False
  by (metis NCol-perm ⟨Bet U1 S U1'⟩ ⟨Cong S U1 S U1'⟩ ⟨Cong S U2 S U1'⟩
    ⟨Cong U1 U2 U1' U2⟩ ⟨S ≠ U1'⟩ bet-neq23--neq cong-3421 cong-diff-2 l4-18)
}
thus ?thesis
  using orthonormal-family-axiom-def by blast
qed

```

lemma *orthonormal-family-aux*:
shows *orthonormal-family-axiom* $\longleftrightarrow$
 $(\forall A B X P Q. \neg \text{Col } P Q X \wedge \text{Per } A X P \wedge \text{Per } A X Q \wedge \text{Per } B X P \wedge \text{Per } B X Q \longrightarrow \text{Col } A B X)$
using *orthonormal-family-aux-1* *orthonormal-family-aux-2* **by** *blast*

lemma *upper-dim-implies-orthonormal-family-axiom*:

assumes *upper-dim-3-axiom*
shows *orthonormal-family-axiom*

proof –

```

{
  fix A B X P Q
  assume  $\neg \text{Col } P Q X$  and
    Per A X P and
    Per A X Q and
    Per B X P and
    Per B X Q
  obtain Q' where Bet Q X Q' and Cong X Q' X P
    using segment-construction by blast
  have  $\neg \text{Col } P Q' X$ 
    by (metis <Bet Q X Q'> <Cong X Q' X P> < $\neg \text{Col } P Q X$ > bet-col bet-cong-eq
        between-trivial2 col-permutation-2 l6-16-1)
  have Per A X Q'
    by (metis Bet-cases Col-def <Bet Q X Q'> <Per A X Q> < $\neg \text{Col } P Q X$ > col-trivial-2 per-col)
  have Per B X Q'
    by (metis Bet-cases Col-def <Bet Q X Q'> <Per B X Q> < $\neg \text{Col } P Q X$ > col-trivial-2 per-col)
  obtain R where X Midpoint P R
    using symmetric-point-construction by blast
  have  $P \neq Q'$ 
    using < $\neg \text{Col } P Q' X$ > col-trivial-1 by blast
  have  $P \neq X$ 
    using < $\neg \text{Col } P Q X$ > not-col-distincts by blast
  hence  $P \neq R$ 
    using <X Midpoint P R> l7-3 by auto
  have Col A B X
  proof –
    have  $Q' \neq R$ 
      using Bet-cases Col-def Midpoint-def <X Midpoint P R> < $\neg \text{Col } P Q' X$ > by blast
    moreover have Cong A P A Q'
      using <Cong X Q' X P> <Per A X P> <Per A X Q'> cong-reflexivity
          l10-12 not-cong-3412 by blast
    moreover have Cong B P B Q'
      using cong-reflexivity l10-12 <Cong X Q' X P> <Per B X P>
          <Per B X Q'> cong-symmetry by blast
    moreover have Cong X P X Q'
      using <Cong X Q' X P> not-cong-3412 by blast
    moreover have Cong A P A R
      using <Per A X P> <X Midpoint P R> per-double-cong by auto
    moreover have Cong B P B R
      using <Per B X P> <X Midpoint P R> per-double-cong by blast
    moreover have Cong X P X R
      using Mid-cases <X Midpoint P R> l7-13 l7-3-2 by blast
    ultimately show ?thesis
      using Col-def < $P \neq Q'$ > < $P \neq R$ > assms upper-dim-3-axiom-def by blast
  qed
}
thus ?thesis
  using orthonormal-family-aux by blast
qed

```

lemma *orthonormal-family-axiom-implies-orth-at2--col*:

assumes *orthonormal-family-axiom*

shows $\forall A B C P Q X. X \text{OrthAt } A B C X P \wedge X \text{OrthAt } A B C X Q \longrightarrow \text{Col } P Q X$

proof –

```

{
  assume  $\forall A B X P Q. \neg \text{Col } P Q X \wedge \text{Per } A X P \wedge \text{Per } A X Q \wedge$ 
    Per B X P  $\wedge$  Per B X Q  $\longrightarrow \text{Col } A B X$ 

```

```

{
  fix A B C P Q X
  assume X OrthAt A B C X P and
    X OrthAt A B C X Q
  have 1:  $\neg \text{Col } A B C \wedge X \neq P \wedge \text{Coplanar } A B C X \wedge (\forall D. (\text{Coplanar } A B C D \longrightarrow \text{Per } D X P))$ 
    using  $\langle X \text{ OrthAt } A B C X P \rangle$  orth-at-chara by blast
  have 2:  $\neg \text{Col } A B C \wedge X \neq Q \wedge \text{Coplanar } A B C X \wedge (\forall D. (\text{Coplanar } A B C D \longrightarrow \text{Per } D X Q))$ 
    using  $\langle X \text{ OrthAt } A B C X Q \rangle$  orth-at-chara by blast
  obtain D E where Coplanar A B C D and Coplanar A B C E and  $\neg \text{Col } X D E$ 
    using ex-ncol-cop2 by blast
  have  $\neg \text{Col } D E X$ 
    using  $\langle \neg \text{Col } X D E \rangle$  not-col-permutation-1 by blast
  moreover have Per P X D
    using  $\langle \text{Coplanar } A B C D \rangle$  1 l8-2 by blast
  moreover have Per P X E
    using  $\langle \text{Coplanar } A B C E \rangle$  1 l8-2 by blast
  moreover have Per Q X D
    using  $\langle \text{Coplanar } A B C D \rangle$  2 l8-2 by blast
  moreover have Per Q X E
    using  $\langle \text{Coplanar } A B C E \rangle$  2 l8-2 by blast
  ultimately have Col P Q X
    using  $\forall A B X P Q. \neg \text{Col } P Q X \wedge \text{Per } A X P \wedge \text{Per } A X Q \wedge$ 
       $\text{Per } B X P \wedge \text{Per } B X Q \longrightarrow \text{Col } A B X$  by blast
}
hence  $\forall A B C P Q X. X \text{ OrthAt } A B C X P \wedge X \text{ OrthAt } A B C X Q \longrightarrow \text{Col } P Q X$ 
  by blast
}
thus ?thesis
  using assms orthonormal-family-aux by blast
qed

```

lemma orthonormal-family-axiom-implies-not-two-sides-one-side:

assumes orthonormal-family-axiom

shows $\forall A B C X Y. \neg \text{Coplanar } A B C X \wedge \neg \text{Coplanar } A B C Y \wedge$
 $\neg A B C \text{ TSP } X Y \longrightarrow A B C \text{ OSP } X Y$

proof –

```

{
  fix A B C X Y
  assume  $\neg \text{Coplanar } A B C X$  and
     $\neg \text{Coplanar } A B C Y$  and
     $\neg A B C \text{ TSP } X Y$ 
  obtain P where P OrthAt A B C P X
    using  $\langle \neg \text{Coplanar } A B C X \rangle$  l11-62-existence-bis by blast
  have  $\forall A B X P Q. \neg \text{Col } P Q X \wedge \text{Per } A X P \wedge \text{Per } A X Q \wedge \text{Per } B X P \wedge \text{Per } B X Q \longrightarrow \text{Col } A B X$ 
    using assms orthonormal-family-aux by blast
  have  $\neg \text{Col } A B C$ 
    using  $\langle \neg \text{Coplanar } A B C Y \rangle$  col--coplanar by auto
  have  $P \neq X$ 
    using  $\langle P \text{ OrthAt } A B C P X \rangle$  orth-at-chara by auto
  have Coplanar A B C P using  $\langle P \text{ OrthAt } A B C P X \rangle$ 
    using orth-at-chara by blast
  have  $\forall D. \text{Coplanar } A B C D \longrightarrow \text{Per } D P X$ 
    using orth-at-chara  $\langle P \text{ OrthAt } A B C P X \rangle$  by presburger
  obtain X' T where A B C Orth P X' and Coplanar A B C T and Bet Y T X'
    using l8-21-3  $\langle \text{Coplanar } A B C P \rangle$   $\langle \neg \text{Coplanar } A B C Y \rangle$  by blast
  hence P OrthAt A B C P X'
    using Col-def  $\langle \text{Coplanar } A B C P \rangle$  col-cop-orth--orth-at not-bet-distincts by blast
  have  $\neg \text{Coplanar } A B C X'$ 
    using orth-at-ncop [where ?X = P]  $\langle P \text{ OrthAt } A B C P X' \rangle$  by blast
  have A B C TSP Y X'
    using TSP-def  $\langle \text{Bet } Y T X' \rangle$   $\langle \text{Coplanar } A B C T \rangle$   $\langle \neg \text{Coplanar } A B C X' \rangle$ 
       $\langle \neg \text{Coplanar } A B C Y \rangle$  by auto
  moreover
  have Bet X P X'
  proof –
    have Col X' X P

```

```

    using ⟨P OrthAt A B C P X⟩ ⟨P OrthAt A B C P X'⟩
    orthonormal-family-axiom-implies-orth-at2--col assms by blast
  hence Col X P X'
    using col-permutation-1 by blast
  moreover
  {
    assume P Out X X'
    have A B C TSP X' Y
      by (simp add: ⟨A B C TSP Y X'⟩ l9-38)
    moreover have A B C OSP X' X
      using ⟨Coplanar A B C P⟩ ⟨P Out X X'⟩ ⟨¬ Coplanar A B C X⟩
      cop-out--osp osp-symmetry by blast
    ultimately have A B C TSP X Y
      using l9-41-2 by blast
  }
  hence ¬ P Out X X'
    using ⟨¬ A B C TSP X Y⟩ by blast
  ultimately show ?thesis
    using or-bet-out by blast
qed
hence A B C TSP X X'
  using TSP-def ⟨Coplanar A B C P⟩ ⟨¬ Coplanar A B C X'⟩ ⟨¬ Coplanar A B C X⟩ by blast
ultimately
have A B C OSP X Y
  using OSP-def by blast
}
thus ?thesis
  by blast
qed

```

```

lemma orthonormal-family-axiom-implies-space-separation:
  assumes orthonormal-family-axiom
  shows space-separation-axiom
proof -
{
  fix A B C P Q
  assume ¬ Coplanar A B C P and
    ¬ Coplanar A B C Q
  {
    assume ¬ A B C TSP P Q
    hence A B C OSP P Q
      using orthonormal-family-axiom-implies-not-two-sides-one-side
      ⟨¬ Coplanar A B C P⟩ ⟨¬ Coplanar A B C Q⟩ assms by blast
  }
  hence A B C TSP P Q ∨ A B C OSP P Q
    by blast
}
thus ?thesis
  using space-separation-axiom-def by blast
qed

```

```

lemma space-separation-implies-plane-intersection:
  assumes space-separation-axiom
  shows plane-intersection-axiom
proof -
{
  fix A B C D E P
  assume Coplanar A B C P and
    ¬ Col D E P
  have ∃ Q. Coplanar A B C Q ∧ Coplanar D E P Q ∧ P ≠ Q
  proof (cases Coplanar A B C D)
    case True
    thus ?thesis
      using ⟨¬ Col D E P⟩ col-trivial-3 ncop-distincts by blast
  next
    case False
  end
}

```



```

hence  $\neg \text{Coplanar } A B C D$ 
  by blast
show ?thesis
proof (cases  $\text{Coplanar } A B C E$ )
  case True
    thus ?thesis
      using  $\langle \neg \text{Col } D E P \rangle$  col-trivial-2 ncop-distincts by blast
  next
    case False
      thus ?thesis
        using  $\langle \text{Coplanar } A B C P \rangle \langle \neg \text{Coplanar } A B C D \rangle$  assms
          cop-osp--ex-cop2 cop-tsp--ex-cop2 space-separation-axiom-def by fastforce
      qed
    qed
  }
hence 1:  $\forall A B C D E P. \text{Coplanar } A B C P \wedge \neg \text{Col } D E P \longrightarrow$ 
   $(\exists Q. \text{Coplanar } A B C Q \wedge \text{Coplanar } D E P Q \wedge P \neq Q)$ 
  by blast
{
  fix  $A B C D E F P$ 
  assume  $\text{Coplanar } A B C P$ 
    and  $\text{Coplanar } D E F P$ 
  obtain  $D' E'$  where  $\text{Coplanar } D E F D'$  and  $\text{Coplanar } D E F E'$  and  $\neg \text{Col } P D' E'$ 
    using ex-ncol-cop2 by blast
  have  $\neg \text{Col } D' E' P$ 
    by (simp add:  $\langle \neg \text{Col } P D' E' \rangle$  not-col-permutation-1)
  then obtain  $Q$  where  $\text{Coplanar } A B C Q$  and  $\text{Coplanar } D' E' P Q$  and  $P \neq Q$ 
    using 1  $\langle \text{Coplanar } A B C P \rangle$  by blast
  hence  $\exists Q. \text{Coplanar } A B C Q \wedge \text{Coplanar } D E F Q \wedge P \neq Q$ 
    by (meson  $\langle \text{Coplanar } D E F D' \rangle \langle \text{Coplanar } D E F E' \rangle \langle \text{Coplanar } D E F P \rangle$ 
       $\langle \neg \text{Col } D' E' P \rangle$  l9-30 ncop-distincts)
}
thus ?thesis
  using plane-intersection-axiom-def by blast
qed

lemma plane-intersection-implies-space-separation:
  assumes plane-intersection-axiom
  shows space-separation-axiom
proof -
{
  fix  $A B C P Q$ 
  assume  $\neg \text{Coplanar } A B C P$  and  $\neg \text{Coplanar } A B C Q$ 
  have  $\text{Coplanar } A P Q A$ 
    using ncop-distincts by blast
  then obtain  $D$  where  $\text{Coplanar } A B C D$  and  $\text{Coplanar } A P Q D$  and  $A \neq D$ 
    using assms plane-intersection-axiom-def coplanar-perm-5 coplanar-trivial by blast
  have  $A D \text{ TS } P Q \vee A D \text{ OS } P Q$ 
  proof (rule cop--one-or-two-sides)
    show  $\text{Coplanar } A D P Q$ 
      by (simp add:  $\langle \text{Coplanar } A P Q D \rangle$  coplanar-perm-4)
    show  $\neg \text{Col } P A D$ 
      by (metis  $\langle A \neq D \rangle \langle \text{Coplanar } A B C D \rangle \langle \neg \text{Coplanar } A B C P \rangle$ 
        col-cop2--cop col-permutation-3 ncop-distincts)
    show  $\neg \text{Col } Q A D$ 
      by (meson  $\langle A \neq D \rangle \langle \text{Coplanar } A B C D \rangle \langle \neg \text{Coplanar } A B C Q \rangle$ 
        col-cop--cop col-permutation-1 ncoplanar-perm-8)
  qed
  moreover
  have  $A D \text{ TS } P Q \longrightarrow A B C \text{ TSP } P Q$ 
    using  $\langle \text{Coplanar } A B C D \rangle \langle \neg \text{Coplanar } A B C P \rangle$  cop2-ts--tsp ncop-distincts by blast
  moreover have  $A D \text{ OS } P Q \longrightarrow A B C \text{ OSP } P Q$ 
    using  $\langle \text{Coplanar } A B C D \rangle \langle \neg \text{Coplanar } A B C P \rangle$  cop2-os--osp ncop-distincts by blast
  ultimately have  $A B C \text{ TSP } P Q \vee A B C \text{ OSP } P Q$ 
    by blast
}

```

```

thus ?thesis
  using space-separation-axiom-def by blast
qed

lemma space-separation-implies-median-planes:
  assumes space-separation-axiom
  shows median-planes-axiom
proof -
{
  fix A B C P Q M
  assume P ≠ Q and
    Cong A P A Q and
    Cong B P B Q and
    Cong C P C Q and
    M Midpoint P Q
{
  fix X P Q M
  assume P ≠ Q and
    Cong A P A Q and
    Cong B P B Q and
    M Midpoint P Q and
    M A B TSP Q X and
    Cong X P X Q
  have ¬ Coplanar M A B Q
    using ⟨M A B TSP Q X⟩ tsp--ncop1 by blast
  have ¬ Coplanar M A B X
    using ⟨M A B TSP Q X⟩ tsp--ncop2 by auto
  obtain T where Coplanar M A B T and Bet Q T X
    using TSP-def ⟨M A B TSP Q X⟩ by blast
  {
    fix C
    assume Coplanar M A B C
    have ¬ Col M A B
      using ⟨M A B TSP Q X⟩ ncop--ncol tsp--ncop2 by blast
    moreover have Cong M P M Q
      using midpoint-cong ⟨M Midpoint P Q⟩ cong-left-commutativity by blast
    ultimately have Cong C P C Q
      using ⟨Coplanar M A B C⟩ ⟨Cong A P A Q⟩ ⟨Cong B P B Q⟩ l11-60-aux by blast
  }
  hence ∀ C. Coplanar M A B C ⟶ Cong C P C Q
    by blast
  {
    assume Bet X T P
    have ¬ Coplanar M A B P
      by (metis ⟨P ≠ Q⟩ ⟨∀ C. Coplanar M A B C ⟶ Cong C P C Q⟩ cong-diff-4)
    have ∀ Z. ¬ Coplanar M A B Z ⟶ Z ≠ T
      using ⟨Coplanar M A B T⟩ by blast
    have X ≠ T
      using ⟨Coplanar M A B T⟩ ⟨¬ Coplanar M A B X⟩ by blast
    hence M Out P Q
      by (metis ⟨Bet Q T X⟩ ⟨Bet X T P⟩ ⟨Cong X P X Q⟩ ⟨P ≠ Q⟩
        between-cong between-symmetry l5-1 not-cong-3412)
    moreover
    have Bet P M Q
      using Midpoint-def ⟨M Midpoint P Q⟩ by auto
    ultimately have False
      using not-bet-and-out by auto
  }
  moreover
  have Cong T P T Q
    using ⟨Coplanar M A B T⟩ ⟨∀ C. Coplanar M A B C ⟶ Cong C P C Q⟩ by blast
  ultimately have False
    using triangle-strict-inequality ⟨Bet Q T X⟩ ⟨Cong X P X Q⟩
      between-symmetry cong--nlt by blast
  }
}
hence 1: ∀ X P Q M. P ≠ Q ∧ Cong A P A Q ∧ Cong B P B Q ∧ M Midpoint P Q ∧

```

```

M A B TSP Q X  $\wedge$  Cong X P X Q  $\longrightarrow$  False
  by blast
{
  assume  $\neg$  Coplanar M A B C
  have  $\neg$  Col M A B
    using  $\langle \neg$  Coplanar M A B C  $\rangle$  ncop--ncols by blast
  {
    assume Coplanar M A B Q
    have Cong M P M Q
      using Cong-cases  $\langle$  M Midpoint P Q  $\rangle$  midpoint-cong by blast
    hence False
      using l11-60-aux  $\langle$  Cong A P A Q  $\rangle$   $\langle$  Cong B P B Q  $\rangle$   $\langle$  Coplanar M A B Q  $\rangle$ 
         $\langle$  P  $\neq$  Q  $\rangle$   $\langle \neg$  Col M A B  $\rangle$  cong-diff-2 by blast
  }
  hence  $\neg$  Coplanar M A B Q
    by blast
  {
    assume M A B TSP Q C
    hence False
      using 1  $\langle$  Cong A P A Q  $\rangle$   $\langle$  Cong B P B Q  $\rangle$   $\langle$  Cong C P C Q  $\rangle$   $\langle$  M Midpoint P Q  $\rangle$   $\langle$  P  $\neq$  Q  $\rangle$  by blast
  }
  moreover
  {
    assume M A B OSP Q C
    have M Midpoint Q P
      using Mid-cases  $\langle$  M Midpoint P Q  $\rangle$  by auto
    have M A B TSP P C
    proof -
      have M A B TSP Q P
        by (metis Bet-cases Midpoint-def  $\langle$  Coplanar M A B Q  $\implies$  False  $\rangle$   $\langle$  M Midpoint P Q  $\rangle$ 
          bet-cop--tsp midpoint-distinct-3 ncop-distincts)
      thus ?thesis
        using  $\langle$  M A B OSP Q C  $\rangle$  l9-38 l9-41-2 by blast
    qed
    hence False
      using 1 by (metis  $\langle$  Cong A P A Q  $\rangle$   $\langle$  Cong B P B Q  $\rangle$   $\langle$  Cong C P C Q  $\rangle$ 
         $\langle$  M Midpoint Q P  $\rangle$   $\langle$  P  $\neq$  Q  $\rangle$  not-cong-3412)
  }
  ultimately have False
    using  $\langle \neg$  Coplanar M A B C  $\rangle$   $\langle \neg$  Coplanar M A B Q  $\rangle$  assms
      space-separation-axiom-def by blast
  }
  hence Coplanar M A B C
    by blast
}
thus ?thesis
  using median-planes-aux by blast
qed

```

theorem upper-dim-3-equivalent-axioms:

```

shows (upper-dim-3-axiom  $\longleftrightarrow$  orthonormal-family-axiom)  $\wedge$ 
      (orthonormal-family-axiom  $\longleftrightarrow$  space-separation-axiom)  $\wedge$ 
      (space-separation-axiom  $\longleftrightarrow$  plane-intersection-axiom)  $\wedge$ 
      (plane-intersection-axiom  $\longleftrightarrow$  median-planes-axiom)
using median-planes-implies-upper-dim orthonormal-family-axiom-implies-space-separation
      plane-intersection-implies-space-separation space-separation-implies-median-planes
      space-separation-implies-plane-intersection
      upper-dim-implies-orthonormal-family-axiom by fastforce

```

lemma par-reflexivity:

```

assumes A  $\neq$  B
shows A B Par A B
using Par-def assms not-col-distincts by blast

```

lemma par-strict-irreflexivity:

```

 $\neg$  A B ParStrict A B

```

```

using ParStrict-def col-trivial-3 by blast

lemma not-par-strict-id:
   $\neg A B \text{ ParStrict } A C$ 
  using ParStrict-def col-trivial-1 by blast

lemma par-id:
  assumes  $A B \text{ Par } A C$ 
  shows  $\text{Col } A B C$ 
  using Col-cases Par-def assms not-par-strict-id by auto

lemma par-strict-not-col-1:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $\neg \text{Col } A B C$ 
  using Col-perm ParStrict-def assms col-trivial-1 by blast

lemma par-strict-not-col-2:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $\neg \text{Col } B C D$ 
  using ParStrict-def assms col-trivial-3 by auto

lemma par-strict-not-col-3:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $\neg \text{Col } C D A$ 
  using Col-perm ParStrict-def assms col-trivial-1 by blast

lemma par-strict-not-col-4:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $\neg \text{Col } A B D$ 
  using Col-perm ParStrict-def assms col-trivial-3 by blast

lemma par-id-1:
  assumes  $A B \text{ Par } A C$ 
  shows  $\text{Col } B A C$ 
  using Par-def assms not-par-strict-id by auto

lemma par-id-2:
  assumes  $A B \text{ Par } A C$ 
  shows  $\text{Col } B C A$ 
  using Col-perm assms par-id-1 by blast

lemma par-id-3:
  assumes  $A B \text{ Par } A C$ 
  shows  $\text{Col } A C B$ 
  using Col-perm assms par-id-2 by blast

lemma par-id-4:
  assumes  $A B \text{ Par } A C$ 
  shows  $\text{Col } C B A$ 
  using Col-perm assms par-id-2 by blast

lemma par-id-5:
  assumes  $A B \text{ Par } A C$ 
  shows  $\text{Col } C A B$ 
  using assms col-permutation-2 par-id by blast

lemma par-strict-symmetry:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $C D \text{ ParStrict } A B$ 
  using ParStrict-def assms coplanar-perm-16 by blast

lemma par-symmetry:
  assumes  $A B \text{ Par } C D$ 
  shows  $C D \text{ Par } A B$ 
proof -
  have  $A B \text{ ParStrict } C D \vee (A \neq B \wedge C \neq D \wedge \text{Col } A C D \wedge \text{Col } B C D)$ 

```

```

    using Par-def assms by auto
  thus ?thesis
    by (metis (full-types) Col-cases Par-def col2--eq par-strict-symmetry)
qed

lemma par-left-comm:
  assumes A B Par C D
  shows B A Par C D
  by (metis ParStrict-def Par-def assms ncoplanar-perm-6 not-col-permutation-5)

lemma par-right-comm:
  assumes A B Par C D
  shows A B Par D C
  using assms par-left-comm par-symmetry by blast

lemma par-comm:
  assumes A B Par C D
  shows B A Par D C
  using assms par-left-comm par-right-comm by blast

lemma par-strict-left-comm:
  assumes A B ParStrict C D
  shows B A ParStrict C D
  using ParStrict-def assms ncoplanar-perm-6 not-col-permutation-5 by blast

lemma par-strict-right-comm:
  assumes A B ParStrict C D
  shows A B ParStrict D C
  using assms par-strict-left-comm par-strict-symmetry by blast

lemma par-strict-comm:
  assumes A B ParStrict C D
  shows B A ParStrict D C
  by (simp add: assms par-strict-left-comm par-strict-right-comm)

lemma par-strict-neq1:
  assumes A B ParStrict C D
  shows A  $\neq$  B
  using assms col-trivial-1 par-strict-not-col-4 by blast

lemma par-strict-neq2:
  assumes A B ParStrict C D
  shows C  $\neq$  D
  using assms col-trivial-2 par-strict-not-col-2 by blast

lemma par-neq1:
  assumes A B Par C D
  shows A  $\neq$  B
  using Par-def assms par-strict-neq1 by blast

lemma par-neq2:
  assumes A B Par C D
  shows C  $\neq$  D
  using assms par-neq1 par-symmetry by blast

lemma Par-cases:
  assumes A B Par C D  $\vee$  B A Par C D  $\vee$  A B Par D C  $\vee$  B A Par D C  $\vee$ 
    C D Par A B  $\vee$  C D Par B A  $\vee$  D C Par A B  $\vee$  D C Par B A
  shows A B Par C D
  using assms par-right-comm par-symmetry by blast

lemma Par-perm:
  assumes A B Par C D
  shows A B Par C D  $\wedge$  B A Par C D  $\wedge$  A B Par D C  $\wedge$  B A Par D C  $\wedge$ 
    C D Par A B  $\wedge$  C D Par B A  $\wedge$  D C Par A B  $\wedge$  D C Par B A
  using Par-cases assms by blast

```

```

lemma Par-strict-cases:
  assumes  $A B \text{ ParStrict } C D \vee B A \text{ ParStrict } C D \vee A B \text{ ParStrict } D C \vee$ 
     $B A \text{ ParStrict } D C \vee C D \text{ ParStrict } A B \vee C D \text{ ParStrict } B A \vee$ 
     $D C \text{ ParStrict } A B \vee D C \text{ ParStrict } B A$ 
  shows  $A B \text{ ParStrict } C D$ 
  using assms par-strict-right-comm par-strict-symmetry by blast

lemma Par-strict-perm:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $A B \text{ ParStrict } C D \wedge B A \text{ ParStrict } C D \wedge A B \text{ ParStrict } D C \wedge$ 
     $B A \text{ ParStrict } D C \wedge C D \text{ ParStrict } A B \wedge C D \text{ ParStrict } B A \wedge$ 
     $D C \text{ ParStrict } A B \wedge D C \text{ ParStrict } B A$ 
  using Par-strict-cases assms by blast

lemma l12-6:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $A B \text{ OS } C D$ 
  by (metis Col-def ParStrict-def Par-strict-perm TS-def assms cop-nts--os
    par-strict-not-col-2)

lemma pars--os3412:
  assumes  $A B \text{ ParStrict } C D$ 
  shows  $C D \text{ OS } A B$ 
  by (simp add: assms l12-6 par-strict-symmetry)

lemma perp-dec:
   $A B \text{ Perp } C D \vee \neg A B \text{ Perp } C D$ 
  by simp

lemma col-cop2-perp2--col:
  assumes  $X1 X2 \text{ Perp } A B$  and
     $Y1 Y2 \text{ Perp } A B$  and
     $\text{Col } X1 Y1 Y2$  and
     $\text{Coplanar } A B X2 Y1$  and
     $\text{Coplanar } A B X2 Y2$ 
  shows  $\text{Col } X2 Y1 Y2$ 
proof cases
  assume  $X1 = Y2$ 
  thus ?thesis
    using assms(1) assms(2) assms(4) cop-perp2--col not-col-permutation-2
      perp-left-comm by blast
next
  assume  $X1 \neq Y2$ 
  hence  $Y2 X1 \text{ Perp } A B$ 
    by (metis Col-cases assms(2) assms(3) perp-col perp-comm perp-right-comm)
  hence  $P1: X1 Y2 \text{ Perp } A B$ 
    using Perp-perm by blast
  thus ?thesis
proof cases
  assume  $X1 = Y1$ 
  thus ?thesis
    using assms(1) assms(2) assms(5) cop-perp2--col not-col-permutation-4 by blast
next
  assume  $X1 \neq Y1$ 
  hence  $X1 Y1 \text{ Perp } A B$ 
    using Col-cases P1 assms(3) perp-col by blast
  thus ?thesis
    using P1 assms(1) assms(4) assms(5) col-transitivity-2 cop-perp2--col
      perp-not-eq-1 by blast
qed
qed

lemma col-perp2-ncol-col:
  assumes  $X1 X2 \text{ Perp } A B$  and
     $Y1 Y2 \text{ Perp } A B$  and

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```

    Col X1 Y1 Y2 and
    ¬ Col X1 A B
shows Col X2 Y1 Y2
proof (cases Y1 = Y2)
  case True
    thus ?thesis
      using col-trivial-2 by blast
  next
    case False
    hence Y1 ≠ Y2
      by blast
    show ?thesis
    proof (cases X2 = Y1)
      case True
      thus ?thesis
        by (simp add: col-trivial-1)
    next
      case False
      hence X2 ≠ Y1
        by blast
      show ?thesis
      proof (cases X1 = Y1)
        case True
        hence Coplanar A B X2 Y1
          using assms(1) coplanar-perm-17 perp--coplanar by blast
        hence Coplanar A B X2 Y2
          using True assms(2) assms(4) coplanar-perm-18 coplanar-perm-3
            coplanar-trans-1 perp--coplanar by blast
        thus ?thesis
          using True assms(1) assms(2) col-permutation-4 cop-perp2--col by blast
      next
        case False
        hence X1 ≠ Y1
          by blast
        hence Y1 X1 Perp A B
          by (metis Col-cases assms(2) assms(3) perp-col)
        hence Coplanar A B X2 Y1
          by (meson assms(1) assms(4) coplanar-perm-3 coplanar-trans-1
            ncoplanar-perm-18 perp--coplanar)
        hence Coplanar A B X2 Y2
          by (meson False assms(1) assms(3) col-cop2--cop ncoplanar-perm-22 perp--coplanar)
        thus ?thesis
          by (metis ⟨Coplanar A B X2 Y1⟩ ⟨Y1 X1 Perp A B⟩ assms(1) assms(2)
            col-permutation-2 cop-perp2--col not-col-distincts perp-col2)
      qed
    qed
  qed
qed
lemma l12-9:
  assumes
    Coplanar C1 C2 A1 B1 and
    Coplanar C1 C2 A1 B2 and
    Coplanar C1 C2 A2 B1 and
    Coplanar C1 C2 A2 B2 and
    A1 A2 Perp C1 C2 and
    B1 B2 Perp C1 C2
  shows A1 A2 Par B1 B2
proof (cases)
  assume Col A1 B1 B2
  moreover have A1 ≠ A2
    using assms(5) perp-distinct by auto
  moreover have B1 ≠ B2
    using assms(6) perp-distinct by auto
  ultimately show ?thesis
    using Par-def assms(3) assms(4) assms(5) assms(6) col-cop2-perp2--col by blast
next

```

```

assume  $\neg \text{Col } A1 \ B1 \ B2$ 
{
  assume  $\neg \text{Col } C1 \ C2 \ A1$ 
  hence Coplanar  $A1 \ A2 \ B1 \ B2$ 
  proof -
    have Coplanar  $C1 \ C2 \ A1 \ A1$ 
      using ncop-distincts by blast
    moreover have  $C1 \ C2 \ \text{Perp } A1 \ A2$ 
      using Perp-perm assms(5) by blast
    hence Coplanar  $C1 \ C2 \ A1 \ A2$ 
      using perp-coplanar by auto
    ultimately show ?thesis
      by (meson coplanar-pseudo-trans  $\langle \neg \text{Col } C1 \ C2 \ A1 \rangle$  assms(1) assms(2))
  qed
}
moreover
{
  assume  $\neg \text{Col } C1 \ C2 \ A2$ 
  have Coplanar  $A1 \ A2 \ B1 \ B2$ 
  proof -
    have  $C1 \ C2 \ \text{Perp } A2 \ A1$ 
      using Perp-perm assms(5) by blast
    hence Coplanar  $C1 \ C2 \ A2 \ A1$ 
      using perp-coplanar by auto
    moreover have Coplanar  $C1 \ C2 \ A2 \ A2$ 
      using ncop-distincts by blast
    ultimately show ?thesis
      by (meson  $\langle \neg \text{Col } C1 \ C2 \ A2 \rangle$  assms(3) assms(4) coplanar-perm-5 coplanar-pseudo-trans-lem1)
  qed
}
moreover
{
  assume  $\exists X. \text{Col } X \ A1 \ A2 \wedge \text{Col } X \ B1 \ B2$ 
  then obtain AB where Col  $AB \ A1 \ A2$  and Col  $AB \ B1 \ B2$  by auto
  have False
  proof cases
    assume  $AB = A1$ 
    thus ?thesis
      using  $\langle \neg \text{Col } A1 \ B1 \ B2 \rangle \langle \text{Col } AB \ B1 \ B2 \rangle$  by blast
  next
    assume  $AB \neq A1$ 
    hence  $A1 \ AB \ \text{Perp } C1 \ C2$ 
      using assms(5) not-col-permutation-2 perp-col by (metis  $\langle \text{Col } AB \ A1 \ A2 \rangle$ )
    hence  $AB \ A1 \ \text{Perp } C1 \ C2$ 
      by (simp add: perp-left-comm)
    thus ?thesis
      using assms(1) assms(2) assms(6) col-cop2-perp2--col
         $\langle \text{Col } AB \ B1 \ B2 \rangle \langle \neg \text{Col } A1 \ B1 \ B2 \rangle$  by blast
  qed
}
moreover
have  $C1 \ C2 \ \text{Perp } A1 \ A2$ 
  using Perp-cases assms(5) by blast
hence  $\neg \text{Col } C1 \ C2 \ A1 \vee \neg \text{Col } C1 \ C2 \ A2$ 
  using perp-not-col2 by blast
ultimately show ?thesis
  using ParStrict-def Par-def by blast
qed

lemma parallel-existence:
  assumes  $A \neq B$ 
  shows  $\exists C \ D. \ C \neq D \wedge A \ B \ \text{Par } C \ D \wedge \text{Col } P \ C \ D$ 
proof cases
  assume Col  $A \ B \ P$ 
  thus ?thesis
    using Col-perm assms par-reflexivity by blast

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next
  assume  $\neg \text{Col } A \ B \ P$ 
  then obtain  $P'$  where  $\text{Col } A \ B \ P'$  and  $A \ B \ \text{Perp } P \ P'$ 
    using l8-18-existence by blast
  hence  $P \neq P'$ 
    using  $\langle \neg \text{Col } A \ B \ P \rangle$  by auto
  show ?thesis
  proof cases
    assume  $P' = A$ 
    have  $\exists Q. \text{Per } Q \ P \ A \wedge \text{Cong } Q \ P \ A \ B \wedge A \ P \ \text{OS } Q \ B$ 
    proof -
      have  $\text{Col } A \ P \ P$ 
        using not-col-distincts by auto
      moreover have  $\neg \text{Col } A \ P \ B$ 
        using  $\langle \neg \text{Col } A \ B \ P \rangle$  not-col-permutation-5 by blast
      ultimately show ?thesis
        using ex-per-cong not-col-distincts by presburger
    qed
  then obtain  $Q$  where  $\text{Per } Q \ P \ A$  and  $\text{Cong } Q \ P \ A \ B$  and  $A \ P \ \text{OS } Q \ B$  by auto
  hence  $P \neq Q$ 
    using os-distincts by auto
  moreover have  $A \ B \ \text{Par } P \ Q$ 
  proof -
    have  $P \ Q \ \text{Perp } P \ A$ 
      by (metis Perp-cases  $\langle P \neq P' \rangle \langle P \neq Q \rangle \langle P' = A \rangle \langle \text{Per } Q \ P \ A \rangle$  per-perp)
    moreover have  $\text{Coplanar } P \ A \ A \ P$ 
      using ncop-distincts by auto
    moreover have  $\text{Coplanar } P \ A \ B \ P$ 
      using ncop-distincts by auto
    moreover have  $\text{Coplanar } P \ A \ B \ Q$ 
      using  $\langle A \ P \ \text{OS } Q \ B \rangle$  coplanar-perm-7 os--coplanar by blast
    moreover have  $A \ B \ \text{Perp } P \ A$ 
      using  $\langle A \ B \ \text{Perp } P \ P' \rangle \langle P' = A \rangle$  by auto
    ultimately show ?thesis
      using l12-9 ncop-distincts by blast
  qed
  ultimately show ?thesis
    using col-trivial-1 by auto
next
  assume  $P' \neq A$ 
  have  $\exists Q. \text{Per } Q \ P \ P' \wedge \text{Cong } Q \ P \ A \ B \wedge P' \ P \ \text{OS } Q \ A$ 
  proof -
    have  $P' \neq P$ 
      using  $\langle P \neq P' \rangle$  by auto
    moreover have  $A \neq B$ 
      by (simp add: assms)
    moreover have  $\text{Col } P' \ P \ P$ 
      using not-col-distincts by blast
    moreover have  $\neg \text{Col } P' \ P \ A$ 
      using Col-perm  $\langle \text{Col } A \ B \ P' \rangle \langle \neg \text{Col } A \ B \ P \rangle$  col-transitivity-2  $\langle P' \neq A \rangle$  by blast
    ultimately show ?thesis
      using ex-per-cong by blast
  qed
  then obtain  $Q$  where  $\text{Per } Q \ P \ P'$  and  $\text{Cong } Q \ P \ A \ B$  and  $P' \ P \ \text{OS } Q \ A$  by blast
  hence  $P \neq Q$ 
    using os-distincts by blast
  moreover have  $A \ B \ \text{Par } P \ Q$ 
  proof -
    have  $\text{Coplanar } P \ P' \ A \ P$ 
      using ncop-distincts by auto
    moreover have  $\text{Coplanar } P \ P' \ A \ Q$ 
      using ncoplanar-perm-7 os--coplanar  $\langle P' \ P \ \text{OS } Q \ A \rangle$  by blast
    moreover
      have  $\text{Coplanar } A \ P \ B \ Q$ 
        using  $\langle \text{Coplanar } P \ P' \ A \ Q \rangle$  col-cop2--cop coplanar-perm-9
        ncop-distincts ncoplanar-perm-17 not-col-permutation-5

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    by (metis ‹Col A B P'› ‹P' ≠ A›)
  hence Coplanar P P' B Q
    using assms col2-cop--cop col-trivial-2 coplanar-perm-10 coplanar-perm-5
      calculation(1) by (meson ‹Col A B P'›)
  moreover have Coplanar P P' B P
    using ncop-distincts by auto
  moreover have P Q Perp P P'
    using Perp-perm per-perp by (metis ‹P ≠ P'› ‹P ≠ Q› ‹Per Q P P'›)
  ultimately show ?thesis
    using l12-9 ‹A B Perp P P'› by blast
qed
moreover have Col P P Q
  by (simp add: col-trivial-1)
ultimately show ?thesis
  by blast
qed
qed

lemma par-col-par:
  assumes C ≠ D' and
    A B Par C D and
    Col C D D'
  shows A B Par C D'
proof -
{
  assume P1: A B ParStrict C D
  have Coplanar A B C D'
    using assms(2) assms(3) col2--eq col2-cop--cop par--coplanar par-neq2 by blast
  {
    assume ∃ X. Col X A B ∧ Col X C D'
    then obtain X where Col X A B and Col X C D'
      by blast
    hence Col X C D
      using Col-cases assms(1) assms(3) col2--eq by blast
    hence False
      using P1 ParStrict-def ‹Col X A B› by auto
  }
  hence A B Par C D'
    using ParStrict-def Par-def ‹Coplanar A B C D'› by blast
}
{
  assume A ≠ B ∧ C ≠ D ∧ Col A C D ∧ Col B C D
  hence A B Par C D'
    using Par-def assms(1) assms(3) col2--eq col-permutation-2 by blast
}
thus ?thesis
  using Par-def ‹A B ParStrict C D ⟹ A B Par C D'› assms(2) by auto
qed

lemma parallel-existence1:
  assumes A ≠ B
  shows ∃ Q. A B Par P Q
proof -
  obtain C D where C ≠ D ∧ A B Par C D ∧ Col P C D
    using assms parallel-existence by blast
  thus ?thesis
    by (metis Col-cases Par-cases par-col-par)
qed

lemma par-not-col:
  assumes A B ParStrict C D and
    Col X A B
  shows ¬ Col X C D
  using ParStrict-def assms(1) assms(2) by blast

lemma not-strict-par1:

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```

assumes  $A \ B \ Par \ C \ D$  and
   $Col \ A \ B \ X$  and
   $Col \ C \ D \ X$ 
shows  $Col \ A \ B \ C$ 
proof –
  have  $A \ B \ ParStrict \ C \ D \vee (A \neq B \wedge C \neq D \wedge Col \ A \ C \ D \wedge Col \ B \ C \ D)$ 
    using Par-def assms(1) by blast
  thus ?thesis
    by (meson assms(2) assms(3) col2--eq col-permutation-2 par-not-col)
qed

lemma not-strict-par2:
assumes  $A \ B \ Par \ C \ D$  and
   $Col \ A \ B \ X$  and
   $Col \ C \ D \ X$ 
shows  $Col \ A \ B \ D$ 
using Par-cases assms(1) assms(2) assms(3) not-col-permutation-4
  not-strict-par1 by blast

lemma not-strict-par:
assumes  $A \ B \ Par \ C \ D$  and
   $Col \ A \ B \ X$  and
   $Col \ C \ D \ X$ 
shows  $Col \ A \ B \ C \wedge Col \ A \ B \ D$ 
using assms(1) assms(2) assms(3) not-strict-par1 not-strict-par2 by blast

lemma not-par-not-col:
assumes  $A \neq B$  and
   $A \neq C$  and
   $\neg A \ B \ Par \ A \ C$ 
shows  $\neg Col \ A \ B \ C$ 
using Par-def assms(1) assms(2) assms(3) not-col-distincts
  not-col-permutation-4 by blast

lemma not-par-inter-uniqueness:
assumes  $A \neq B$  and
   $C \neq D$  and
   $\neg A \ B \ Par \ C \ D$  and
   $Col \ A \ B \ X$  and
   $Col \ C \ D \ X$  and
   $Col \ A \ B \ Y$  and
   $Col \ C \ D \ Y$ 
shows  $X = Y$ 
proof cases
assume  $P1: C = Y$ 
thus ?thesis
proof cases
assume  $P2: C = X$ 
thus ?thesis
  using  $P1$  by auto
next
assume  $C \neq X$ 
hence  $\neg Col \ C \ D \ A$ 
  by (metis  $P1$  Par-def assms(1) assms(2) assms(3) assms(4) assms(5)
    assms(6) col-permutation-1 col-transitivity-2)
  thus ?thesis
  using assms(1) assms(4) assms(5) assms(6) assms(7) l6-21 by blast
qed
next
assume  $C \neq Y$ 
hence  $\neg Col \ A \ B \ C$ 
  by (metis Par-def assms(1) assms(2) assms(3) assms(6) assms(7)
    col-transitivity-1 col-transitivity-2 colx not-col-permutation-5)
  thus ?thesis
  using assms(2) assms(4) assms(5) assms(6) assms(7) l6-21 by blast
qed

```

```

lemma inter-uniqueness-not-par:
  assumes  $\neg \text{Col } A \ B \ C$  and
     $\text{Col } A \ B \ P$  and
     $\text{Col } C \ D \ P$ 
  shows  $\neg A \ B \ \text{Par } C \ D$ 
  using assms(1) assms(2) assms(3) not-strict-par1 by blast

lemma col-not-col-not-par:
  assumes  $\exists P. \text{Col } A \ B \ P \wedge \text{Col } C \ D \ P$  and
     $\exists Q. \text{Col } C \ D \ Q \wedge \neg \text{Col } A \ B \ Q$ 
  shows  $\neg A \ B \ \text{Par } C \ D$ 
  using assms(1) assms(2) colx not-strict-par par-neq2 by blast

lemma par-distincts:
  assumes  $A \ B \ \text{Par } C \ D$ 
  shows  $A \ B \ \text{Par } C \ D \wedge A \neq B \wedge C \neq D$ 
  using assms par-neq1 par-neq2 by blast

lemma par-not-col-strict:
  assumes  $A \ B \ \text{Par } C \ D$  and
     $\text{Col } C \ D \ P$  and
     $\neg \text{Col } A \ B \ P$ 
  shows  $A \ B \ \text{ParStrict } C \ D$ 
  using Col-cases Par-def assms(1) assms(2) assms(3) col3 by blast

lemma col-cop-perp2--pars:
  assumes  $\neg \text{Col } A \ B \ P$  and
     $\text{Col } C \ D \ P$  and
     $\text{Coplanar } A \ B \ C \ D$  and
     $A \ B \ \text{Perp } P \ Q$  and
     $C \ D \ \text{Perp } P \ Q$ 
  shows  $A \ B \ \text{ParStrict } C \ D$ 
proof -
  have P1:  $C \neq D$ 
  using assms(5) perp-not-eq-1 by auto
  hence P2:  $\text{Coplanar } A \ B \ C \ P$ 
  using col-cop--cop assms(2) assms(3) by blast
  moreover have P3:  $\text{Coplanar } A \ B \ D \ P$  using col-cop--cop
  using P1 assms(2) assms(3) col2-cop--cop col-trivial-2 by blast
  have  $A \ B \ \text{Par } C \ D$ 
proof -
  have  $\text{Coplanar } P \ A \ Q \ C$ 
proof -
  have  $\neg \text{Col } B \ P \ A$ 
  by (simp add: assms(1) not-col-permutation-1)
  moreover have  $\text{Coplanar } B \ P \ A \ Q$ 
  by (meson assms(4) ncoplanar-perm-12 perp--coplanar)
  moreover have  $\text{Coplanar } B \ P \ A \ C$ 
  using P2 ncoplanar-perm-13 by blast
  ultimately show ?thesis
  using coplanar-trans-1 by auto
qed
hence P4:  $\text{Coplanar } P \ Q \ A \ C$ 
using ncoplanar-perm-2 by blast
have  $\text{Coplanar } P \ A \ Q \ D$ 
proof -
  have  $\neg \text{Col } B \ P \ A$ 
  by (simp add: assms(1) not-col-permutation-1)
  moreover have  $\text{Coplanar } B \ P \ A \ Q$ 
  by (meson assms(4) ncoplanar-perm-12 perp--coplanar)
  moreover have  $\text{Coplanar } B \ P \ A \ D$ 
  using P3 ncoplanar-perm-13 by blast
  ultimately show ?thesis
  using coplanar-trans-1 by blast
qed

```

```

then moreover have Coplanar P Q A D
  using ncoplanar-perm-2 by blast
moreover have Coplanar P Q B C
  using P2 assms(1) assms(4) coplanar-perm-1 coplanar-perm-10 coplanar-trans-1
  perp--coplanar by blast
moreover have Coplanar P Q B D
  by (meson P3 assms(1) assms(4) coplanar-trans-1 ncoplanar-perm-1
      ncoplanar-perm-13 perp--coplanar)
ultimately show ?thesis
  using assms(4) assms(5) l12-9 P4 by auto
qed
thus ?thesis
  using assms(1) assms(2) par-not-col-strict by auto
qed

lemma all-one-side-par-strict:
  assumes C ≠ D and
    ∀ P. Col C D P ⟶ A B OS C P
  shows A B ParStrict C D
proof -
  have P1: Coplanar A B C D
    by (simp add: assms(2) col-trivial-2 os--coplanar)
  {
    assume ∃ X. Col X A B ∧ Col X C D
    then obtain X where P2: Col X A B ∧ Col X C D by blast
    have A B OS C X
      by (simp add: P2 Col-perm assms(2))
    then obtain M where A B TS C M ∧ A B TS X M
      by (meson Col-cases P2 col124--nos)
    hence False
      using P2 TS-def by blast
  }
  thus ?thesis
    using P1 ParStrict-def by auto
qed

lemma par-col-par-2:
  assumes A ≠ P and
    Col A B P and
    A B Par C D
  shows A P Par C D
  using assms(1) assms(2) assms(3) par-col-par par-symmetry by blast

lemma par-col2-par:
  assumes E ≠ F and
    A B Par C D and
    Col C D E and
    Col C D F
  shows A B Par E F
  by (metis assms(1) assms(2) assms(3) assms(4) col-transitivity-2 not-col-permutation-4
      par-col-par par-distincts par-right-comm)

lemma par-col2-par-bis:
  assumes E ≠ F and
    A B Par C D and
    Col E F C and
    Col E F D
  shows A B Par E F
  by (metis assms(1) assms(2) assms(3) assms(4) col-transitivity-1 not-col-permutation-2
      par-col2-par)

lemma par-strict-col-par-strict:
  assumes C ≠ E and
    A B ParStrict C D and
    Col C D E
  shows A B ParStrict C E

```

```

proof –
  have  $P1: C \ E \ Par \ A \ B$ 
    using  $Par\text{-}def \ Par\text{-}perm \ assms(1) \ assms(2) \ assms(3) \ par\text{-}col\text{-}par\text{-}2$  by  $blast$ 
  {
    assume  $C \ E \ ParStrict \ A \ B$ 
    hence  $A \ B \ ParStrict \ C \ E$ 
      by  $(metis \ par\text{-}strict\text{-}symmetry)$ 
  }
  thus  $?thesis$ 
    using  $Col\text{-}cases \ Par\text{-}def \ P1 \ assms(2) \ par\text{-}strict\text{-}not\text{-}col\text{-}1$  by  $blast$ 
qed

```

lemma $par\text{-}strict\text{-}col2\text{-}par\text{-}strict$:

```

assumes  $E \neq F$  and
   $A \ B \ ParStrict \ C \ D$  and
   $Col \ C \ D \ E$  and
   $Col \ C \ D \ F$ 
shows  $A \ B \ ParStrict \ E \ F$ 

```

```

proof –
  have  $Coplanar \ A \ B \ E \ F$ 
    by  $(metis \ ParStrict\text{-}def \ Par\text{-}strict\text{-}perm \ col2\text{-}cop\text{-}cop \ assms(2) \ assms(3) \ assms(4) \ par\text{-}strict\text{-}neq1)$ 
  moreover
  {
    assume  $\exists \ X. \ Col \ X \ A \ B \wedge \ Col \ X \ E \ F$ 
    then obtain  $X$  where  $Col \ X \ A \ B$  and  $Col \ X \ E \ F$ 
      by  $blast$ 
    hence  $Col \ X \ C \ D$ 
      by  $(meson \ assms(1) \ assms(3) \ assms(4) \ colx \ not\text{-}col\text{-}permutation\text{-}2)$ 
    hence  $False$ 
      using  $ParStrict\text{-}def \ \langle Col \ X \ A \ B \rangle \ assms(2)$  by  $blast$ 
  }
  ultimately show  $?thesis$ 
    using  $ParStrict\text{-}def$  by  $blast$ 
qed

```

lemma $line\text{-}dec$:

```

 $(Col \ C1 \ B1 \ B2 \wedge \ Col \ C2 \ B1 \ B2) \vee \neg (Col \ C1 \ B1 \ B2 \wedge \ Col \ C2 \ B1 \ B2)$ 
by  $simp$ 

```

lemma $par\text{-}distinct$:

```

assumes  $A \ B \ Par \ C \ D$ 
shows  $A \neq B \wedge \ C \neq D$ 
using  $assms \ par\text{-}neq1 \ par\text{-}neq2$  by  $auto$ 

```

lemma $par\text{-}col4\text{-}par$:

```

assumes  $E \neq F$  and
   $G \neq H$  and
   $A \ B \ Par \ C \ D$  and
   $Col \ A \ B \ E$  and
   $Col \ A \ B \ F$  and
   $Col \ C \ D \ G$  and
   $Col \ C \ D \ H$ 
shows  $E \ F \ Par \ G \ H$ 

```

```

proof –
  have  $C \ D \ Par \ E \ F$ 
    using  $Par\text{-}cases \ assms(1) \ assms(3) \ assms(4) \ assms(5) \ par\text{-}col2\text{-}par$  by  $blast$ 
  hence  $E \ F \ Par \ C \ D$ 
    by  $(simp \ add: \ \langle C \ D \ Par \ E \ F \rangle \ par\text{-}symmetry)$ 
  thus  $?thesis$ 
    using  $assms(2) \ assms(6) \ assms(7) \ par\text{-}col2\text{-}par$  by  $blast$ 
qed

```

lemma $par\text{-}strict\text{-}col4\text{-}par\text{-}strict$:

```

assumes  $E \neq F$  and
   $G \neq H$  and

```

```

  A B ParStrict C D and
  Col A B E and
  Col A B F and
  Col C D G and
  Col C D H
shows E F ParStrict G H
proof -
  have C D ParStrict E F
    using Par-strict-cases assms(1) assms(3) assms(4) assms(5)
    par-strict-col2-par-strict by blast
  hence E F ParStrict C D
    by (simp add: ⟨C D ParStrict E F⟩ par-strict-symmetry)
  thus ?thesis
    using assms(2) assms(6) assms(7) par-strict-col2-par-strict by blast
qed

```

```

lemma par-strict-one-side:
  assumes A B ParStrict C D and
    Col C D P
  shows A B OS C P
proof cases
  assume C = P
  thus ?thesis
    using assms(1) assms(2) not-col-permutation-5 one-side-reflexivity
    par-not-col by blast
next
  assume C ≠ P
  thus ?thesis
    using assms(1) assms(2) l12-6 par-strict-col-par-strict by blast
qed

```

```

lemma par-strict-all-one-side:
  assumes A B ParStrict C D
  shows ∀ P. Col C D P ⟶ A B OS C P
  using assms par-strict-one-side by blast

```

```

lemma inter-trivial:
  assumes ¬ Col A B X
  shows X Inter A X B X
  by (metis Col-perm Inter-def assms col-trivial-1)

```

```

lemma inter-sym:
  assumes X Inter A B C D
  shows X Inter C D A B
proof -
  obtain P where P1: Col P C D ∧ ¬ Col P A B
    using Inter-def assms by auto
  have P2: A ≠ B
    using P1 col-trivial-2 by blast
  thus ?thesis
  proof cases
    assume A = X
    have Col B A B
      by (simp add: col-trivial-3)
    {
      assume P3: Col B C D
      have Col P A B
      proof -
        have C ≠ D
          using Inter-def assms by blast
        moreover have Col C D P
          using P1 not-col-permutation-2 by blast
        moreover have Col C D A
          using Inter-def ⟨A = X⟩ assms by auto
        moreover have Col C D B
          using P3 not-col-permutation-2 by blast

```

```

      ultimately show ?thesis
      using col3 by blast
    qed
  hence False
  by (simp add: P1)
}
hence  $\neg$  Col B C D by auto
thus ?thesis
  using Inter-def P2 assms by (meson col-trivial-3)
next
assume P5:  $A \neq X$ 
have P6: Col A A B
  using not-col-distincts by blast
{
  assume P7: Col A C D
  have Col A P X
  proof -
    have  $C \neq D$ 
    using Inter-def assms by auto
    moreover have Col C D A
    using Col-cases P7 by blast
    moreover have Col C D P
    using Col-cases P1 by auto
    moreover have Col C D X
    using Inter-def assms by auto
    ultimately show ?thesis
    using col3 by blast
  qed
  hence Col P A B
  by (metis Col-perm Inter-def P5 assms col-transitivity-2)
  hence False
  by (simp add: P1)
}
hence  $\neg$  Col A C D by auto
thus ?thesis
  by (meson Inter-def P2 assms col-trivial-1)
qed
qed

```

```

lemma inter-left-comm:
  assumes X Inter A B C D
  shows X Inter B A C D
  using Col-cases Inter-def assms by auto

```

```

lemma inter-right-comm:
  assumes X Inter A B C D
  shows X Inter A B D C
  by (metis assms inter-left-comm inter-sym)

```

```

lemma inter-comm:
  assumes X Inter A B C D
  shows X Inter B A D C
  using assms inter-left-comm inter-right-comm by blast

```

```

lemma l12-17:
  assumes  $A \neq B$  and
    P Midpoint A C and
    P Midpoint B D
  shows A B Par C D
proof cases
  assume P1: Col A B P
  thus ?thesis
proof cases
  assume  $A = P$ 

```



```

thus ?thesis
  using assms(1) assms(2) assms(3) cong-diff-2 is-midpoint-id
    midpoint-col midpoint-cong not-par-not-col by blast
next
assume P2: A ≠ P
thus ?thesis
proof cases
  assume B = P
  thus ?thesis
    by (metis assms(1) assms(2) assms(3) midpoint-col midpoint-distinct-2
      midpoint-distinct-3 not-par-not-col par-comm)
next
assume P3: B ≠ P
have P4: Col B P D
  using assms(3) midpoint-col not-col-permutation-4 by blast
have P5: Col A P C
  using assms(2) midpoint-col not-col-permutation-4 by blast
hence P6: Col B C P
  using P1 P2 col-transitivity-2 not-col-permutation-3 not-col-permutation-5 by blast
have C ≠ D
  using assms(1) assms(2) assms(3) l7-9 by blast
moreover have Col A C D
  using P1 P3 P4 P6 col3 not-col-permutation-3 not-col-permutation-5 by blast
moreover have Col B C D
  using P3 P4 P6 col-trivial-3 colx by blast
ultimately show ?thesis
  by (simp add: Par-def assms(1))
qed
qed
next
assume T1: ¬ Col A B P
then obtain E where T2: Col A B E ∧ A B Perp P E
  using l8-18-existence by blast
have T3: A ≠ P
  using T1 col-trivial-3 by blast
thus ?thesis
proof cases
  assume T4: A = E
  hence T5: Per P A B
    using T2 l8-2 perp-per-1 by blast
  obtain B' where T6: Bet B A B' ∧ Cong A B' B A
    using segment-construction by blast
  obtain D' where T7: Bet B' P D' ∧ Cong P D' B' P
    using segment-construction by blast
  have T8: C Midpoint D D'
    using T6 T7 assms(2) assms(3) midpoint-def not-cong-3412
      symmetry-preserves-midpoint by blast
  have Col A B B'
    using Col-cases Col-def T6 by blast
  hence T9: Per P A B'
    using per-col T5 assms(1) by blast
  obtain B'' where T10: A Midpoint B B'' ∧ Cong P B P B''
    using Per-def T5 by auto
  hence B' = B''
    using T6 cong-symmetry midpoint-def symmetric-point-uniqueness by blast
  hence Cong P D P D'
    by (metis Cong-perm Midpoint-def T10 T7 assms(3) cong-inner-transitivity)
  hence T12: Per P C D
    using Per-def T8 by auto
  hence T13: C PerpAt P C C D
    by (metis T3 assms(1) assms(2) assms(3) l7-3-2 per-perp-in sym-preserve-diff)
  have T14: P ≠ C
    using T3 assms(2) is-midpoint-id-2 by auto
  have T15: C ≠ D
    using assms(1) assms(2) assms(3) l7-9 by auto
  have T15A: C C Perp C D ∨ P C Perp C D

```

```

    using T12 T14 T15 per-perp by auto
  {
    assume C C Perp C D
    hence A B Par C D
      using perp-distinct by auto
  }
  {
    assume P C Perp C D
    have A B Par C D
    proof -
      have Coplanar P A A C
        using ncop-distincts by blast
      moreover have Coplanar P A A D
        using ncop-distincts by blast
      moreover have Coplanar P A B C
        by (simp add: assms(2) coplanar-perm-1 midpoint--coplanar)
      moreover have Coplanar P A B D
        using assms(3) midpoint-col ncop--ncols by blast
      moreover have A B Perp P A
        using T2 T4 by auto
      moreover have C D Perp P A
    proof -
      have P A Perp C D
    proof -
      have P ≠ A
        using T3 by auto
      moreover have P C Perp C D
        using T14 T15 T12 per-perp by blast
      moreover have Col P C A
        by (simp add: assms(2) l7-2 midpoint-col)
      ultimately show ?thesis
        using perp-col by blast
    qed
    thus ?thesis
      using Perp-perm by blast
    qed
    ultimately show ?thesis using l12-9 by blast
  qed
}
thus ?thesis using T15A
  using ⟨C C Perp C D ⟹ A B Par C D⟩ by blast
next
assume S1B: A ≠ E
obtain F where S2: Bet E P F ∧ Cong P F E P
  using segment-construction by blast
hence S2A: P Midpoint E F
  using midpoint-def not-cong-3412 by blast
hence S3: Col C D F
  using T2 assms(2) assms(3) mid-preserves-col by blast
obtain A' where S4: Bet A E A' ∧ Cong E A' A E
  using segment-construction by blast
obtain C' where S5: Bet A' P C' ∧ Cong P C' A' P
  using segment-construction by blast
have S6: F Midpoint C C'
  using S4 S5 S2A assms(2) midpoint-def not-cong-3412
    symmetry-preserves-midpoint by blast
have S7: Per P E A
  using T2 col-trivial-3 l8-16-1 by blast
have S8: Cong P C P C'
proof -
  have Cong P C P A
    using Cong-perm Midpoint-def assms(2) by blast
  moreover have Cong P A P C'
proof -
  obtain A'' where S9: E Midpoint A A'' ∧ Cong P A P A''
    using Per-def S7 by blast

```

```

have S10:  $A' = A''$ 
  using Cong-perm Midpoint-def S4 S9 symmetric-point-uniqueness by blast
hence Cong P A P A' using S9 by auto
moreover have Cong P A' P C'
  using Cong-perm S5 by blast
ultimately show ?thesis
  using cong-transitivity by blast
qed
ultimately show ?thesis
  using cong-transitivity by blast
qed
hence S9:  $\text{Per } P \ F \ C$ 
  using S6 Per-def by blast
hence F PerpAt P F F C
  by (metis S2 S2A T1 T2 S1B assms(2) cong-diff-3 l7-9 per-perp-in)
hence F PerpAt F P C F
  using Perp-in-perm by blast
hence S10:  $F \ P \ \text{Perp} \ C \ F \vee F \ F \ \text{Perp} \ C \ F$ 
  using l8-15-2 perp-in-col by blast
{
  assume S11:  $F \ P \ \text{Perp} \ C \ F$ 
  have Coplanar P E A C
  proof -
    have Col P E P  $\wedge$  Col A C P
      using assms(2) col-trivial-3 midpoint-col not-col-permutation-2 by blast
    thus ?thesis
      using Coplanar-def by blast
  qed
  moreover have Coplanar P E A D
  proof -
    have Col P D B  $\wedge$  Col E A B
      using Mid-cases T2 assms(3) midpoint-col not-col-permutation-1 by blast
    thus ?thesis
      using Coplanar-def by blast
  qed
  moreover have Coplanar P E B C
    by (metis S1B T2 calculation(1) col2-cop--cop col-transitivity-1
      ncoplanar-perm-5 not-col-permutation-5)
  moreover have Coplanar P E B D
    by (metis S1B T2 calculation(2) col2-cop--cop col-transitivity-1
      ncoplanar-perm-5 not-col-permutation-5)
  moreover have P E Perp C D
  proof -
    have  $P \neq E$ 
      using T2 perp-distinct by blast
    moreover have  $C \neq D$ 
      using assms(1) assms(2) assms(3) sym-preserve-diff by blast
    moreover have P F Perp C F
      using Perp-perm S11 by blast
    hence F P Perp C F
      by (simp add: S11)
    moreover have Col P F E
      by (simp add: Col-def S2)
    moreover have Col C F D
      using Col-perm S3 by blast
    ultimately show ?thesis
      using T1 perp-col4  $\langle P \ F \ \text{Perp} \ C \ F \rangle$  col-trivial-3 by meson
  qed
  moreover hence C D Perp P E
    using Perp-cases by blast
  ultimately have A B Par C D
    using T2 l12-9 by blast
}
{
  assume F F Perp C F
  hence A B Par C D

```

```

    using perp-distinct by blast
  }
  thus ?thesis
    using S10  $\langle F \ P \ \text{Perp} \ C \ F \implies A \ B \ \text{Par} \ C \ D \rangle$  by blast
qed
qed

lemma l12-18-a:
  assumes Cong A B C D and
    Cong B C D A and
     $\neg$  Col A B C and
    B  $\neq$  D and
    Col A P C and
    Col B P D
  shows A B Par C D
proof -
  have P Midpoint A C  $\wedge$  P Midpoint B D
    using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) l7-21 by blast
  thus ?thesis
    using assms(3) l12-17 not-col-distincts by blast
qed

lemma l12-18-b:
  assumes Cong A B C D and
    Cong B C D A and
     $\neg$  Col A B C and
    B  $\neq$  D and
    Col A P C and
    Col B P D
  shows B C Par D A
proof -
  have P Midpoint A C
    using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) l7-21-R1 by blast
  have Cong B C D A
    by (simp add: assms(2))
  moreover have Cong C D A B
    using Cong-cases assms(1) by auto
  moreover
  {
    assume Col B C D
    hence Col B C P
      using NCol-perm assms(4) assms(6) col-trivial-3 colx by blast
    hence Col C A B
  }
  proof -
    {
      assume C = P
      hence False
        using  $\langle P \ \text{Midpoint} \ A \ C \rangle$  assms(3) col-trivial-3 is-midpoint-id-2 by blast
    }
    moreover have Col C P A
      using Col-cases assms(5) by blast
    moreover have Col C P B
      using Col-cases  $\langle Col \ B \ C \ P \rangle$  by blast
    ultimately show ?thesis
      using col-transitivity-1 by blast
  }
  qed
  hence False
    using NCol-cases assms(3) by blast
}
moreover have C  $\neq$  A
  using assms(3) not-col-distincts by blast
moreover have Col B P D
  by (simp add: assms(6))
moreover have Col C P A
  using Col-cases assms(5) by blast
ultimately show ?thesis

```

using *l12-18-a* by *blast*

qed

lemma *l12-18-c*:

assumes *Cong A B C D* and

Cong B C D A and

$\neg$ *Col A B C* and

B $\neq$ *D* and

Col A P C and

Col B P D

shows *B D TS A C*

proof –

have *P Midpoint A C* $\wedge$ *P Midpoint B D*

using *assms(1)* *assms(2)* *assms(3)* *assms(4)* *assms(5)* *assms(6)* *l7-21* by *blast*

thus ?thesis

proof –

have *A C TS B D*

by (*metis Col-cases mid-two-sides* $\langle$ *P Midpoint A C* $\wedge$ *P Midpoint B D* $\rangle$ *assms(3)*)

hence $\neg$ *Col B D A*

by (*meson Col-cases mid-preserves-col ts--ncol* $\langle$ *P Midpoint A C* $\wedge$ *P Midpoint B D* $\rangle$ *l7-2*)

thus ?thesis

by (*meson mid-two-sides* $\langle$ *P Midpoint A C* $\wedge$ *P Midpoint B D* $\rangle$)

qed

qed

lemma *l12-18-d*:

assumes *Cong A B C D* and

Cong B C D A and

$\neg$ *Col A B C* and

B $\neq$ *D* and

Col A P C and

Col B P D

shows *A C TS B D*

by (*metis Col-cases TS-def* *assms(1)* *assms(2)* *assms(3)*

assms(4) *assms(5)* *assms(6)* *l12-18-c not-col-distincts not-cong-2143 not-cong-4321*)

lemma *l12-18*:

assumes *Cong A B C D* and

Cong B C D A and

$\neg$ *Col A B C* and

B $\neq$ *D* and

Col A P C and

Col B P D

shows *A B Par C D* $\wedge$ *B C Par D A* $\wedge$ *B D TS A C* $\wedge$ *A C TS B D*

using *assms(1)* *assms(2)* *assms(3)* *assms(4)* *assms(5)* *assms(6)*

l12-18-a l12-18-b l12-18-c l12-18-d by *auto*

lemma *par-two-sides-two-sides*:

assumes *A B Par C D* and

B D TS A C

shows *A C TS B D*

by (*metis Par-def TS-def* *assms(1)* *assms(2)* *invert-one-side invert-two-sides*

l12-6 l9-31 not-col-permutation-4 one-side-symmetry os-ts1324--os pars--os3412)

lemma *par-one-or-two-sides*:

assumes *A B ParStrict C D*

shows $(A C TS B D \wedge B D TS A C) \vee (A C OS B D \wedge B D OS A C)$

proof (cases)

assume *A C TS B D*

hence *B D TS A C*

by (*metis Par-strict-cases* *assms invert-one-side invert-two-sides l12-6*

l9-31 os-ts1324--os)

thus ?thesis

using $\langle$ *A C TS B D* $\rangle$ by *blast*

next

assume $\neg$ *A C TS B D*

hence $A \ C \ OS \ B \ D$
 by (metis Col-perm ParStrict-def assms cop-nos--ts l9-2 ncoplanar-perm-4
 one-side-symmetry par-strict-not-col-1 par-strict-not-col-3)
 hence $B \ D \ OS \ A \ C$
 by (meson assms invert-one-side invert-two-sides l12-6 l9-31 one-side-symmetry
 os-ts1324--os pars--os3412)
 thus ?thesis
 using $\langle A \ C \ OS \ B \ D \rangle$ by auto
 qed

lemma l12-21-b:
 assumes $A \ C \ TS \ B \ D$ and
 $B \ A \ C \ CongA \ D \ C \ A$
 shows $A \ B \ Par \ C \ D$
 proof –
 have $P1: \neg \text{Col } A \ B \ C$
 using TS-def assms(1) not-col-permutation-4 by blast
 hence $P2: A \neq B$
 using col-trivial-1 by auto
 have $P3: C \neq D$
 using assms(1) ts-distincts by blast
 then obtain D' where $P4: C \text{ Out } D \ D' \wedge \text{Cong } C \ D' \ A \ B$
 using P2 segment-construction-3 by blast
 have $P5: B \ A \ C \ CongA \ D' \ C \ A$
 proof –
 have $A \text{ Out } B \ B$
 using P2 out-trivial by auto
 moreover have $A \text{ Out } C \ C$
 using P1 col-trivial-3 out-trivial by force
 moreover have $C \text{ Out } D' \ D$
 by (simp add: P4 l6-6)
 moreover have $C \text{ Out } A \ A$
 using P1 not-col-distincts out-trivial by auto
 ultimately show ?thesis
 using assms(2) l11-10 by blast
 qed
 hence $P6: \text{Cong } D' \ A \ B \ C$
 using Cong-perm P4 cong-pseudo-reflexivity l11-49 by blast
 have $P7: A \ C \ TS \ D' \ B$
 proof –
 have $A \ C \ TS \ D \ B$
 by (simp add: assms(1) l9-2)
 moreover have $\text{Col } C \ A \ C$
 using col-trivial-3 by auto
 ultimately show ?thesis
 using P4 l9-5 by blast
 qed
 then obtain M where $P8: \text{Col } M \ A \ C \wedge \text{Bet } D' \ M \ B$
 using TS-def by blast
 have $B \neq D'$
 using P7 not-two-sides-id by blast
 hence $M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D'$
 by (metis Col-cases P1 P4 P6 P8 bet-col l7-21 not-cong-3412)
 hence $A \ B \ Par \ C \ D'$
 using P2 l12-17 by blast
 thus ?thesis
 by (meson P3 P4 par-col-par l6-6 out-col)
 qed

lemma l12-22-aux:
 assumes $P \neq A$ and
 $A \neq C$ and
 $\text{Bet } P \ A \ C$ and
 $P \ A \ OS \ B \ D$ and
 $B \ A \ P \ CongA \ D \ C \ P$
 shows $A \ B \ Par \ C \ D$

```

proof -
  have P1:  $P \neq C$ 
    using CongA-def assms(5) by blast
  obtain B' where P2:  $Bet\ B\ A\ B' \wedge Cong\ A\ B'\ B\ A$ 
    using segment-construction by blast
  have P3:  $P\ A\ B\ CongA\ C\ A\ B'$ 
    by (metis CongA-def P2 assms(2) assms(3) assms(5) cong-reverse-identity l11-14)
  have P4:  $D\ C\ A\ CongA\ D\ C\ P$ 
    by (metis Col-def assms(2) assms(3) assms(4) bet-out-1 col124--nos l6-6
        out2--conga out-trivial)
  have P5:  $A\ B'\ Par\ C\ D$ 
proof -
  have  $\neg\ Col\ B\ P\ A$ 
    using assms(4) col123--nos not-col-permutation-2 by blast
  hence P A TS B B'
    by (metis P2 assms(4) bet--ts cong-reverse-identity invert-two-sides
        not-col-permutation-3 os-distincts)
  hence A C TS B' D
    by (meson assms(2) assms(3) assms(4) bet-col bet-col1 col-preserves-two-sides
        l9-2 l9-8-2)
  moreover have B' A C CongA D C A
proof -
  have B' A C CongA B A P
    by (simp add: P3 conga-comm conga-sym)
  moreover have B A P CongA D C A
    using P4 assms(5) not-conga not-conga-sym by blast
  ultimately show ?thesis
    using not-conga by blast
qed
ultimately show ?thesis
  using l12-21-b by blast
qed
have C D Par A B
proof -
  have  $A \neq B$ 
    using assms(4) os-distincts by blast
  moreover have C D Par A B'
    using P5 par-symmetry by blast
  moreover have Col A B' B
    by (simp add: Col-def P2)
  ultimately show ?thesis
    using par-col-par by blast
qed
thus ?thesis
  using Par-cases by blast
qed

lemma l12-22-b:
  assumes P Out A C and
    P A OS B D and
    B A P CongA D C P
  shows A B Par C D
proof cases
  assume A = C
  thus ?thesis
    using assms(2) assms(3) conga-comm conga-os--out not-par-not-col
        os-distincts out-col by blast
next
  assume P1:  $A \neq C$ 
  {
    assume Bet P A C
    hence A B Par C D
      using P1 assms(2) assms(3) conga-diff2 l12-22-aux by blast
  }
  {
    assume P2:  $Bet\ P\ C\ A$ 

```

```

have C D Par A B
proof -
  have P C OS D B
    using assms(1) assms(2) col-one-side one-side-symmetry
    out-col out-diff2 by blast
  moreover have D C P CongA B A P
    using assms(3) not-conga-sym by blast
  thus ?thesis
    by (metis P1 P2 assms(1) calculation l12-22-aux out-distinct)
qed
hence A B Par C D
  using Par-cases by auto
}
thus ?thesis
  using Out-def ‹Bet P A C  $\implies$  A B Par C D› assms(1) by blast
qed

lemma par-strict-par:
  assumes A B ParStrict C D
  shows A B Par C D
  using Par-def assms by auto

lemma par-strict-distinct:
  assumes A B ParStrict C D
  shows A  $\neq$  B  $\wedge$  C  $\neq$  D
  using assms par-strict-neq1 par-strict-neq2 by auto

lemma col-par:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    Col A B C
  shows A B Par B C
  by (simp add: Par-def assms(1) assms(2) assms(3) col-trivial-1)

lemma acute-col-perp--out:
  assumes Acute A B C and
    Col B C A' and
    B C Perp A A'
  shows B Out A' C
proof -
  {
    assume P1:  $\neg$  Col B C A
    then obtain B' where P2: B C Perp B' B  $\wedge$  B C OS A B'
      using assms(2) l10-15 os-distincts by blast
    have P3:  $\neg$  Col B' B C
      using P2 col124--nos col-permutation-1 by blast
    {
      assume Col B B' A
      hence A B C LtA A B C
        using P2 acute-one-side-aux acute-sym assms(1) one-side-not-col124 by blast
      hence False
        by (simp add: nlta)
    }
  }
  hence P4:  $\neg$  Col B B' A by auto
  have P5: B B' ParStrict A A'
  proof -
    have B B' Par A A'
    proof -
      have Coplanar B C B A
        using ncop-distincts by blast
      moreover have Coplanar B C B A'
        using ncop-distincts by blast
      moreover have Coplanar B C B' A
        using P2 coplanar-perm-1 os--coplanar by blast
      moreover have Coplanar B C B' A'
        using assms(2) ncop--ncols by auto
    
```



```

    moreover have  $B B' \text{ Perp } B C$ 
      using  $P2 \text{ Perp-perm}$  by blast
    moreover have  $A A' \text{ Perp } B C$ 
      using  $\text{Perp-perm assms}(3)$  by blast
    ultimately show ?thesis
      using  $l12-9$  by auto
  qed
  moreover have  $\text{Col } A A' A$ 
    by (simp add:  $\text{col-trivial-3}$ )
  moreover have  $\neg \text{Col } B B' A$ 
    by (simp add:  $P4$ )
  ultimately show ?thesis
    using  $\text{par-not-col-strict}$  by auto
  qed
  hence  $P6: \neg \text{Col } B B' A'$ 
    using  $P5 \text{ par-strict-not-col-4}$  by auto
  hence  $B B' \text{ OS } A' C$ 
  proof -
    have  $B B' \text{ OS } A' A$ 
      using  $P5 \text{ l12-6 one-side-symmetry}$  by blast
    moreover have  $B B' \text{ OS } A C$ 
      using  $P2 \text{ acute-one-side-aux acute-sym assms}(1) \text{ one-side-symmetry}$  by blast
    ultimately show ?thesis
      using  $\text{one-side-transitivity}$  by blast
  qed
  hence  $B \text{ Out } A' C$ 
    using  $\text{Col-cases assms}(2) \text{ col-one-side-out}$  by blast
}
thus ?thesis
  using  $\text{assms}(2) \text{ assms}(3) \text{ perp-not-col2}$  by blast
qed

lemma acute-col-perp--out-1:
  assumes  $\text{Acute } A B C$  and
     $\text{Col } B C A'$  and
     $B A \text{ Perp } A A'$ 
  shows  $B \text{ Out } A' C$ 
proof -
  obtain  $A0$  where  $P1: \text{Bet } A B A0 \wedge \text{Cong } B A0 A B$ 
    using  $\text{segment-construction}$  by blast
  obtain  $C0$  where  $P2: \text{Bet } C B C0 \wedge \text{Cong } B C0 C B$ 
    using  $\text{segment-construction}$  by blast
  have  $P3: \neg \text{Col } B A A'$ 
    using  $\text{assms}(3) \text{ col-trivial-2 perp-not-col2}$  by blast
  have  $\text{Bet } A' B C0$ 
  proof -
    have  $P4: \text{Col } A' B C0$ 
      using  $P2 \text{ acute-distincts assms}(1) \text{ assms}(2) \text{ bet-col col-transitivity-2}$ 
         $\text{not-col-permutation-4}$  by blast
    {
      assume  $P5: B \text{ Out } A' C0$ 
      have  $B \text{ Out } A A0$ 
      proof -
        have  $\text{Bet } C B A'$ 
          using  $\text{Bet-perm Col-def } P2 P5 \text{ assms}(2) \text{ between-exchange3}$ 
             $\text{not-bet-and-out outer-transitivity-between2}$  by metis
        hence  $A B C \text{ CongA } A0 B A'$ 
          using  $P1 P3 \text{ acute-distincts assms}(1) \text{ cong-diff-4 l11-14 not-col-distincts}$  by blast
        hence  $\text{Acute } A' B A0$ 
          using  $\text{acute-conga--acute acute-sym assms}(1)$  by blast
        moreover have  $B A0 \text{ Perp } A' A$ 
        proof -
          have  $B \neq A0$ 
            using  $P1 P3 \text{ col-trivial-1 cong-reverse-identity}$  by blast
          moreover have  $B A \text{ Perp } A' A$ 
            using  $\text{Perp-perm assms}(3)$  by blast

```

```

    moreover have  $Col\ B\ A\ A0$ 
      using  $P1\ bet-col\ not-col-permutation-4$  by blast
    ultimately show  $?thesis$ 
      using  $perp-col$  by blast
  qed
  ultimately show  $?thesis$ 
    using  $Col-cases\ P1\ acute-col-perp--out\ bet-col$  by blast
  qed
  hence  $False$ 
    using  $P1\ not-bet-and-out$  by blast
}
thus  $?thesis$ 
  using  $l6-4-2\ P4$  by blast
qed
thus  $?thesis$ 
  by (metis  $P2\ P3\ acute-distincts\ assms(1)\ cong-diff-3\ l6-2\ not-col-distincts$ )
qed

lemma  $conga-inangle-per2--inangle$ :
  assumes  $Per\ A\ B\ C$  and
     $T\ InAngle\ A\ B\ C$  and
     $P\ B\ A\ CongA\ P\ B\ C$  and
     $Per\ B\ P\ T$  and
     $Coplanar\ A\ B\ C\ P$ 
  shows  $P\ InAngle\ A\ B\ C$ 
proof cases
  assume  $P = T$ 
  thus  $?thesis$ 
    by (simp add:  $assms(2)$ )
next
  assume  $P1: P \neq T$ 
  obtain  $P'$  where  $P2: P' InAngle\ A\ B\ C \wedge P' B A CongA\ P' B C$ 
    using  $CongA-def\ angle-bisector\ assms(3)$  by presburger
  have  $P3: Acute\ P' B A$ 
    using  $P2\ acute-sym\ assms(1)\ conga-inangle-per--acute$  by blast
  have  $P4: \neg Col\ A\ B\ C$ 
    using  $assms(1)\ assms(3)\ conga-diff2\ conga-diff56\ l8-9$  by blast
  have  $P5: Col\ B\ P\ P'$ 
  proof -
    have  $\neg B Out\ A\ C$ 
      using  $Col-cases\ P4\ out-col$  by blast
    moreover have  $Coplanar\ A\ B\ P\ P'$ 
  proof -
    have  $T1: \neg Col\ C\ A\ B$ 
      using  $Col-perm\ P4$  by blast
    moreover have  $Coplanar\ C\ A\ B\ P$ 
      using  $assms(5)\ ncoplanar-perm-8$  by blast
    moreover have  $Coplanar\ C\ A\ B\ P'$ 
      using  $P2\ inangle--coplanar\ ncoplanar-perm-21$  by blast
    ultimately show  $?thesis$ 
      using  $coplanar-trans-1$  by blast
  qed
  moreover have  $Coplanar\ B\ C\ P\ P'$ 
proof -
  have  $Coplanar\ A\ B\ C\ P$ 
    by (meson  $P2\ bet--coplanar\ calculation(1)\ calculation(2)\ col-in-angle-out\ coplanar-perm-18\ coplanar-trans-1\ inangle--coplanar\ l11-21-a\ l6-6\ l6-7\ not-col-permutation-4\ not-col-permutation-5$ )
  have  $Coplanar\ A\ B\ C\ P'$ 
    using  $P2\ inangle--coplanar\ ncoplanar-perm-18$  by blast
  thus  $?thesis$ 
    using  $P4\ \langle Coplanar\ A\ B\ C\ P \rangle\ coplanar-trans-1$  by blast
qed
ultimately show  $?thesis$  using  $conga2-cop2--col\ P2\ assms(3)$  by blast
qed
have  $B Out\ P\ P'$ 

```

```

proof –
  have Acute T B P'
    using P2 acute-sym assms(1) assms(2) conga-inangle2-per--acute by blast
  moreover have B P' Perp T P
    by (metis P1 P5 acute-distincts assms(3) assms(4) calculation
      col-per-perp conga-distinct l8-2 not-col-permutation-4)
  ultimately show ?thesis
    using Col-cases P5 acute-col-perp--out by blast
qed
thus ?thesis
  using Out-cases P2 in-angle-trans inangle-distincts out341--inangle by blast
qed

```

```

lemma perp-not-par:
  assumes A B Perp X Y
  shows  $\neg A B \text{ Par } X Y$ 
proof –
  obtain P where P1: P PerpAt A B X Y
    using Perp-def assms by blast
  {
    assume P2: A B Par X Y
    {
      assume P3: A B ParStrict X Y
      hence False
      proof –
        have Col P A B
          using Col-perm P1 perp-in-col by blast
        moreover have Col P X Y
          using P1 col-permutation-2 perp-in-col by blast
        ultimately show ?thesis
          using P3 par-not-col by blast
      qed
    }
  }
  {
    assume P4: A  $\neq$  B  $\wedge$  X  $\neq$  Y  $\wedge$  Col A X Y  $\wedge$  Col B X Y
    hence False
    proof cases
      assume A = Y
      thus ?thesis
        using P4 assms not-col-permutation-1 perp-not-col by blast
    next
      assume A  $\neq$  Y
      thus ?thesis
        using Col-perm P4 Perp-perm assms perp-not-col2 by blast
    qed
  }
  hence False
  using Par-def P2  $\langle A B \text{ ParStrict } X Y \implies \text{False} \rangle$  by auto
}
thus ?thesis by auto
qed

```

```

lemma cong-conga-perp:
  assumes B P TS A C and
    Cong A B C B and
    A B P CongA C B P
  shows A C Perp B P
proof –
  have P1:  $\neg \text{Col } A B P$ 
    using TS-def assms(1) by blast
  hence P2: B  $\neq$  P
    using col-trivial-2 by blast
  have P3: A  $\neq$  B
    using assms(1) ts-distincts by blast
  have P4: C  $\neq$  B
    using assms(1) ts-distincts by auto

```

```

have P5:  $A \neq C$ 
  using assms(1) not-two-sides-id by auto
show ?thesis
proof cases
  assume P6:  $Bet\ A\ B\ C$ 
  hence Per P B A
    by (meson conga-comm Tarski-neutral-dimensionless-axioms assms(3) l11-18-2)
  thus ?thesis
    using P2 P3 P5 Per-perm P6 bet-col per-perp perp-col by blast
next
  assume P7:  $\neg\ Bet\ A\ B\ C$ 
  obtain T where P7A:  $Col\ T\ B\ P \wedge Bet\ A\ T\ C$ 
    using TS-def assms(1) by auto
  hence P8:  $B \neq T$ 
    using P7 by blast
  hence P9:  $T\ B\ A\ CongA\ T\ B\ C$ 
    by (meson Col-cases P7A col-conga--conga conga-comm assms(3))
  hence P10:  $Cong\ T\ A\ T\ C$ 
    using assms(2) cong2-conga-cong cong-reflexivity not-cong-2143 by blast
  hence P11:  $T\ Midpoint\ A\ C$ 
    using P7A midpoint-def not-cong-2134 by blast
  have P12:  $Per\ B\ T\ A$ 
    using P11 Per-def assms(2) not-cong-2143 by blast
  thus ?thesis
  proof –
    have A C Perp B T
      by (metis P11 P12 P5 P8 col-per-perp midpoint-col midpoint-distinct-1)
    moreover have  $B \neq T$ 
      by (simp add: P8)
    moreover have  $T \neq A$ 
      using P1 P7A by blast
    moreover have  $C \neq T$ 
      using P10 P5 cong-identity by blast
    moreover have  $C \neq A$ 
      using P5 by auto
    moreover have  $Col\ T\ A\ C$ 
      by (meson P7A bet-col not-col-permutation-4)
    ultimately show ?thesis
      using P2 P7A not-col-permutation-4 perp-col1 by blast
  qed
qed
qed

```

```

lemma perp-inter-exists:
  assumes A B Perp C D
  shows  $\exists\ P. Col\ A\ B\ P \wedge Col\ C\ D\ P$ 
proof –
  obtain P where P PerpAt A B C D
    using Perp-def assms by auto
  thus ?thesis
    using perp-in-col by blast
qed

```

```

lemma perp-inter-perp-in:
  assumes A B Perp C D
  shows  $\exists\ P. Col\ A\ B\ P \wedge Col\ C\ D\ P \wedge P\ PerpAt\ A\ B\ C\ D$ 
  by (meson Perp-def perp-in-col Tarski-neutral-dimensionless-axioms assms)

```

```

lemma cop-npars--inter-exists:
  assumes Coplanar A1 B1 A2 B2 and
     $\neg\ A1\ B1\ ParStrict\ A2\ B2$ 
  shows  $\exists\ X. Col\ X\ A1\ B1 \wedge Col\ X\ A2\ B2$ 
  using ParStrict-def assms(1) assms(2) by blast

```

```

lemma cop-npar--inter-exists:
  assumes Coplanar A1 B1 A2 B2 and

```

```

  ¬ A1 B1 Par A2 B2
shows ∃ X. Col X A1 B1 ∧ Col X A2 B2
proof -
  have ¬ A1 A2 ParStrict A2 B2
  using not-par-strict-id par-strict-left-comm by blast
  thus ?thesis
  using assms(1) assms(2) cop-npars--inter-exists par-strict-par by blast
qed

```

```

lemma cop-npar--inter:
  assumes A1 ≠ B1 and
    A2 ≠ B2 and
    Coplanar A1 B1 A2 B2 and
    ¬ A1 B1 Par A2 B2
  shows ∃ X. X Inter A1 B1 A2 B2
proof -
  obtain X where Col X A1 B1 and Col X A2 B2
  using cop-npar--inter-exists assms(3) assms(4) by blast
  have ∃ P. Col P A2 B2 ∧ ¬ Col P A1 B1
  proof (cases Col A2 A1 B1)
    case True
    have Col B2 A2 B2
    by (simp add: col-trivial-3)
    moreover
    have ¬ Col B2 A1 B1
    using Par-cases True assms(1) assms(2) assms(4) calculation col-par
    not-col-permutation-3 par-col2-par-bis by metis
    ultimately
    show ?thesis
    by blast
  next
    case False
    thus ?thesis
    using col-trivial-1 by blast
  qed
  moreover
  have Col A1 B1 X ∧ Col A2 B2 X
  using Col-cases ⟨Col X A1 B1⟩ ⟨Col X A2 B2⟩ by blast
  ultimately
  show ?thesis
  using assms(2) Inter-def by auto
qed

```

```

lemma inter--npar:
  assumes X Inter A1 A2 B1 B2
  shows ¬ A1 A2 Par B1 B2
proof -
  {
    assume A1 A2 ParStrict B1 B2
    hence False
    by (meson Inter-def assms not-strict-par par-strict-not-col-1 par-strict-par)
  }
  moreover
  {
    assume A1 ≠ A2 ∧ B1 ≠ B2 ∧ Col A1 B1 B2 ∧ Col A2 B1 B2
    hence A1 A2 Par B1 B2
    using Par-def by auto
    hence False
    by (meson Inter-def assms col-not-col-not-par not-col-permutation-2)
  }
  ultimately
  show ?thesis
  using Par-def by presburger
qed

```

```

lemma cong-identity-inv:

```

assumes $A \neq B$
 shows $\neg \text{Cong } A B C C$
 using *assms cong-diff* by *blast*

lemma *midpoint-midpoint-col*:

assumes $A \neq B$ and
 $M \text{ Midpoint } A A'$ and
 $M \text{ Midpoint } B B'$ and
 $\text{Col } A B B'$
 shows $A' \neq B' \wedge \text{Col } A A' B' \wedge \text{Col } B A' B'$
 proof –
 have $f1: \forall p. p \text{ Midpoint } p p$
 by (*metis l7-3-2*)
 hence $f2: B = B' \vee \text{Col } B' A M$
 by (*metis (no-types) Col-perm assms(3) assms(4) col-transitivity-2 mid-two-sides not-two-sides-id*)
 have $f3: \forall p. (M = A \vee \text{Col } A p A') \vee \neg \text{Col } M A p$
 using $f1$ by (*metis (no-types) Col-perm assms(2) col-transitivity-2 mid-two-sides not-two-sides-id*)
 hence $(M = A \vee B = B') \vee \text{Col } B A' B'$
 using $f1$ by (*metis Col-perm assms(3) assms(4) col-transitivity-2 mid-two-sides not-two-sides-id*)
 thus ?thesis
 using $f3 f2 f1$ by (*metis (full-types) Col-perm Mid-perm assms(1) assms(2) assms(3) assms(4) col-transitivity-2 l7-17-bis mid-two-sides not-two-sides-id symmetric-point-uniqueness*)
 qed

lemma *midpoint-par-strict*:

assumes $\neg \text{Col } A B B'$ and
 $M \text{ Midpoint } A A'$ and
 $M \text{ Midpoint } B B'$
 shows $A B \text{ ParStrict } A' B'$
 by (*metis assms(1) assms(2) assms(3) col-trivial-1 col-trivial-2 l12-17 par-not-col-strict*)

lemma *bet3-cong3-bet*:

assumes $A \neq B$ and
 $A \neq C$ and
 $A \neq D$ and
 $\text{Bet } D A C$ and
 $\text{Bet } A C B$ and
 $\text{Bet } D C D'$ and
 $\text{Cong } A B C D$ and
 $\text{Cong } A D B C$ and
 $\text{Cong } D C C D'$
 shows $\text{Bet } C B D'$
 proof –
 have $\text{Bet } D C B$
 using $\text{assms}(2) \text{ assms}(4) \text{ assms}(5) \text{ outer-transitivity-between2}$ by *blast*
 have $B \neq C$
 using $\text{assms}(3) \text{ assms}(8) \text{ cong-identity-inv}$ by *auto*
 have $D' \neq C$
 using $\text{assms}(2) \text{ assms}(4) \text{ assms}(9) \text{ bet-neq23--neq cong-diff}$ by *blast*
 have $D \neq C$
 using $\langle D' \neq C \rangle \text{ assms}(9) \text{ cong-reverse-identity}$ by *blast*
 have $C \text{ Out } B D'$
 using $\text{Out-def } \langle B \neq C \rangle \langle \text{Bet } D C B \rangle \langle D \neq C \rangle \langle D' \neq C \rangle \text{ assms}(6) \text{ l5-2}$ by *auto*
 hence $D \text{ Out } B D'$
 using $\langle \text{Bet } D C B \rangle \text{ out-bet--out}$ by *blast*
 have $D A \text{ Le } D C$
 using $\text{assms}(4) \text{ cong-reflexivity l5-5-2}$ by *blast*
 hence $D B \text{ Le } D D'$
 by (*meson* $\langle \text{Bet } D C B \rangle \text{ assms}(6) \text{ assms}(8) \text{ assms}(9) \text{ bet2-le2--le cong-commutativity l5-6 le-reflexivity}$)
 thus ?thesis
 by (*metis l6-13-1* $\langle \text{Bet } D C B \rangle \langle D \text{ Out } B D' \rangle \text{ between-exchange3}$)

qed

lemma *bet-double-bet*:

assumes B' Midpoint A B and

C' Midpoint A C and

Bet A B' C'

shows Bet A B C

proof –

have A B' Le A C'

by (simp add: assms(3) bet--le1213)

hence B' B Le C' C

using assms(1) assms(2) l5-6 midpoint-cong by blast

have A B Le A C

using $\langle A$ B' Le A $C' \rangle$ assms(1) assms(2) le-mid2--le13 by auto

thus ?thesis

proof cases

assume $A = B'$

thus ?thesis

using assms(1) between-trivial2 is-midpoint-id by blast

next

assume $A \neq B'$

thus ?thesis

by (metis Midpoint-def $\langle A$ B Le A $C \rangle$ assms(1) assms(2) assms(3) bet-out
l6-13-1 l6-6 l6-7 out-diff1)

qed

qed

lemma *bet-half-bet*:

assumes Bet A B C and

B' Midpoint A B and

C' Midpoint A C

shows Bet A B' C'

proof cases

assume $A = B$

thus ?thesis

using assms(2) l8-20-2 not-bet-distincts by blast

next

assume $A \neq B$

thus ?thesis

by (metis midpoint-bet assms(1) assms(2) assms(3) bet-double-bet
between-equality-2 between-exchange4 l5-3 l7-17)

qed

lemma *midpoint-preserves-bet*:

assumes B' Midpoint A B and

C' Midpoint A C

shows Bet A B $C \longleftrightarrow Bet$ A B' C'

using assms(1) assms(2) bet-double-bet bet-half-bet by blast

lemma *symmetry-preseves-bet1*:

assumes M Midpoint A A' and

M Midpoint B B' and

Bet M A B

shows Bet M A' B'

using assms(1) assms(2) assms(3) l7-15 l7-3-2 by blast

lemma *symmetry-preseves-bet2*:

assumes M Midpoint A A' and

M Midpoint B B' and

Bet M A' B'

shows Bet M A B

using Mid-cases assms(1) assms(2) assms(3) symmetry-preseves-bet1 by blast

lemma *symmetry-preserves-bet*:

assumes M Midpoint A A' and

M Midpoint B B'

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shows  $Bet\ M\ A'\ B' \longleftrightarrow Bet\ M\ A\ B$ 
using assms(1) assms(2) symmetry-preseves-bet1 symmetry-preseves-bet2 by blast

lemma bet-cong-bet:
  assumes  $A \neq B$  and
     $Bet\ A\ B\ C$  and
     $Bet\ A\ B\ D$  and
     $Cong\ A\ C\ B\ D$ 
  shows  $Bet\ B\ C\ D$ 
by (meson assms(1) assms(2) assms(3) assms(4) bet-col between-equality
    col-cong-bet col-transitivity-1)

lemma col-cong-mid :
  assumes  $A\ B\ Par\ A'\ B'$  and
     $\neg\ A\ B\ ParStrict\ A'\ B'$  and
     $Cong\ A\ B\ A'\ B'$ 
  shows  $\exists\ M. ((M\ Midpoint\ A\ A' \wedge M\ Midpoint\ B\ B') \vee$ 
     $(M\ Midpoint\ A\ B' \wedge M\ Midpoint\ B\ A'))$ 
proof cases
  assume  $A = A'$ 
  thus ?thesis
  by (metis Mid-perm Par-def midpoint-existence assms(1) assms(2) assms(3) l7-20 l7-3-2)
next
  assume  $A \neq A'$ 
  thus ?thesis
proof cases
  assume  $B = B'$ 
  thus ?thesis
  by (metis  $\langle A \neq A' \rangle$  assms(1) assms(3) bet-cong-eq between-trivial
    l7-3-2 midpoint-def not-cong-1243 not-cong-3412 par-comm par-id-1 third-point)
next
  assume  $B \neq B'$ 
  thus ?thesis
proof cases
  assume  $A = B'$ 
  thus ?thesis
  by (metis Cong-perm Mid-perm Midpoint-def Par-perm assms(1) assms(3)
    between-cong-2 between-trivial l7-3-2 midpoint-existence par-id third-point)
next
  assume  $A \neq B'$ 
  thus ?thesis
proof cases
  assume  $A' = B$ 
  thus ?thesis
  using Par-def  $\langle A \neq B' \rangle$  assms(1) assms(2) assms(3) cong-col-mid l7-3-2 by blast
next
  assume  $A' \neq B$ 
  have  $Col\ A\ B\ A'$ 
  using Par-def assms(1) assms(2) col2-eq col-permutation-5 by blast
  have  $Col\ A\ B\ B'$ 
  using assms(1) assms(2) l8-18-existence not-strict-par2 par-not-col-strict by blast
  {
    assume  $Bet\ A\ B\ B'$ 
    {
      assume  $Bet\ B\ A'\ B'$ 
      obtain  $M$  where  $M\ Midpoint\ A'\ B$ 
      using midpoint-existence by blast
      hence  $Cong\ A'\ M\ B\ M$ 
      using cong-right-commutativity midpoint-cong by blast
      have  $Bet\ B\ M\ B'$ 
      using Mid-cases  $\langle Bet\ B\ A'\ B' \rangle \langle M\ Midpoint\ A'\ B \rangle$  between-exchange4
        midpoint-bet by blast
      hence  $Bet\ A\ M\ B'$ 
      using  $\langle Bet\ A\ B\ B' \rangle$  between-exchange2 by blast
      have  $Cong\ A\ M\ B'\ M$  using l2-11
      by (meson Midpoint-def  $\langle Bet\ A\ B\ B' \rangle \langle Bet\ B\ A'\ B' \rangle \langle M\ Midpoint\ A'\ B \rangle$ )
    }
  }

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      assms(3) between-inner-transitivity between-symmetry not-cong-3412
      not-cong-3421)
hence M Midpoint A B'
  using ⟨Bet A M B'⟩ midpoint-def not-cong-1243 by blast
hence M Midpoint A B' ∧ M Midpoint B A'
  using Mid-perm ⟨M Midpoint A' B⟩ by blast
hence ∃ M. ((M Midpoint A A' ∧ M Midpoint B B') ∨
(M Midpoint A B' ∧ M Midpoint B A')) by auto
}
{
  assume Bet A' B' B
  obtain M where P2: M Midpoint B B'
    using midpoint-existence by blast
  hence Bet A' M B
    using ⟨Bet A' B' B⟩ between-exchange4 between-symmetry midpoint-bet by blast
  hence Bet M B A
    using ⟨B ≠ B'⟩ ⟨Bet A B B'⟩ ⟨Bet A' B' B⟩ between-inner-transitivity
      between-symmetry outer-transitivity-between by blast
  hence Bet A' M A
    using ⟨B ≠ B'⟩ ⟨Bet A B B'⟩ ⟨Bet A' B' B⟩ ⟨Bet A' M B⟩ between-symmetry
      outer-transitivity-between by blast
  have Cong B M B' M
    using Cong-perm P2 midpoint-cong by blast
  hence Cong A M A' M
    using l2-11 by (meson P2 ⟨Bet A' B' B⟩ ⟨Bet M B A⟩ assms(3)
      between-exchange3 between-symmetry midpoint-bet)
  hence M Midpoint A A'
    using ⟨Bet A' M A⟩ between-symmetry midpoint-def not-cong-1243 by blast
  hence M Midpoint A A' ∧ M Midpoint B B'
    using ⟨M Midpoint B B'⟩ by blast
  hence ∃ M. ((M Midpoint A A' ∧ M Midpoint B B') ∨
(M Midpoint A B' ∧ M Midpoint B A')) by auto
}
{
  assume Bet B' B A'
  hence Bet B A' A
    by (metis ⟨B ≠ B'⟩ ⟨Bet A B B'⟩ assms(3) bet-cong-eq between-symmetry l5-2)
  obtain M where M Midpoint A' B
    using midpoint-existence by blast
  hence Bet A' M B
    by (simp add: midpoint-bet)
  have Cong A' M B M
    using Cong-perm ⟨M Midpoint A' B⟩ midpoint-cong by blast
  have M Midpoint A B'
  proof -
    have Bet B M A
      using Mid-cases ⟨Bet B A' A⟩ ⟨M Midpoint A' B⟩ between-exchange4
        midpoint-bet by blast
    hence Bet B' B M
      using ⟨Bet A B B'⟩ between-inner-transitivity between-symmetry by blast
    have Bet M A' A
      using ⟨Bet A' M B⟩ ⟨Bet B A' A⟩ between-inner-transitivity
        between-symmetry by blast
    have Bet B' M A'
      using ⟨Bet A' M B⟩ ⟨Bet B' B A'⟩ bet3--bet between-trivial by blast
    have Cong A M B' M
      by (meson ⟨Bet B A' A⟩ ⟨Bet B' B A'⟩ ⟨Bet B' B M⟩ ⟨Bet M A' A⟩
        ⟨Col A B B'⟩ ⟨Cong A' M B M⟩ assms(3) between-symmetry
        col-bet2-cong1 l2-11-b)
    thus ?thesis
      by (metis Bet-cases midpoint-distinct-1 ⟨A' ≠ B⟩ ⟨Bet B' M A'⟩
        ⟨Bet M A' A⟩ ⟨M Midpoint A' B⟩ midpoint-def not-cong-1243
        outer-transitivity-between)
  qed
hence M Midpoint A B' ∧ M Midpoint B A'
  by (simp add: ⟨M Midpoint A B'⟩ Mid-perm ⟨M Midpoint A' B⟩)

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    hence  $\exists M. ((M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee$ 
    ( $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ ) by auto
  }
  have  $\exists M. ((M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee$ 
  ( $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ )
    using Par-def  $\langle \text{Bet } A' B' B \implies$ 
     $\exists M. M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B' \vee$ 
     $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A' \rangle$ 
     $\langle \text{Bet } B A' B' \implies \exists M. M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B' \vee$ 
     $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A' \rangle$ 
     $\langle \text{Bet } B' B A' \implies \exists M. M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B' \vee$ 
     $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A' \rangle$ 
    assms(1) assms(2) between-symmetry third-point by blast
  }
  {
    assume Bet B B' A
    have P3: Col A A' B'
    using Par-def assms(1) assms(2) by blast
    {
      assume Bet A A' B'
      hence  $A = A' \wedge B = B'$ 
      using  $\langle \text{Bet } B B' A \rangle$  assms(3) bet-cong-eq between-symmetry cong-symmetry by blast
      hence  $\exists M. ((M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee$ 
      ( $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ )
      using  $\langle B \neq B' \rangle$  by linarith
    }
  }
  {
    assume Bet A' B' A
    hence Bet B' B A'
    by (metis Bet-perm Cong-perm  $\langle A \neq B' \rangle \langle \text{Bet } B B' A \rangle$  assms(3)
    bet-cong-eq between-inner-transitivity l5-1)
    obtain M where M Midpoint B B'
    using midpoint-existence by blast
    hence Bet B M B'
    using midpoint-bet by blast
    have  $M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B'$ 
    proof –
    have Bet A M A'
    by (meson Bet-perm  $\langle \text{Bet } A' B' A \rangle \langle \text{Bet } B M B' \rangle \langle \text{Bet } B' B A' \rangle$  between-exchange4)
    moreover
    have Cong M B M B'
    using Midpoint-def  $\langle M \text{ Midpoint } B B' \rangle$  not-cong-2134 by blast
    hence Cong A M A' M
    using Bet-cases between-inner-transitivity  $\langle \text{Bet } B B' A \rangle \langle \text{Bet } B M B' \rangle$ 
    assms(3) between-exchange2 calculation l4-3 by meson
    ultimately
    show ?thesis
    using Cong-cases  $\langle M \text{ Midpoint } B B' \rangle$  midpoint-def by blast
  }
  qed
  hence  $\exists M. ((M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee$ 
  ( $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ ) by auto
}
{
  assume Bet B' A A'
  obtain M where M Midpoint A B'
  using midpoint-existence by blast
  hence Bet A M B'
  by (simp add: midpoint-bet)
  have Cong A M B' M
  using Cong-perm  $\langle M \text{ Midpoint } A B' \rangle$  midpoint-cong by blast
  have  $M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A'$ 
  proof –
  have Bet A' A M
  using  $\langle \text{Bet } A M B' \rangle \langle \text{Bet } B' A A' \rangle$  between-inner-transitivity
  between-symmetry by blast
  have Bet B M A'

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    using  $\langle A \neq B' \rangle \langle \text{Bet } A \ M \ B' \rangle \langle \text{Bet } B \ B' \ A \rangle \langle \text{Bet } B' \ A \ A' \rangle$ 
    bet3--bet outer-transitivity-between outer-transitivity-between2 by blast
  moreover
  have Cong B M A' M
  proof -
    let ?C = A
    let ?C' = B'
    have Bet B M ?C
    by (meson  $\langle B \neq B' \rangle \langle \text{Bet } A \ M \ B' \rangle \langle \text{Bet } B \ B' \ A \rangle$  bet3--bet l5-1)
    moreover have Bet A' M ?C'
    using Bet-cases  $\langle \text{Bet } A \ M \ B' \rangle \langle \text{Bet } A' \ A \ M \rangle \langle \text{Bet } B' \ A \ A' \rangle$  outer-transitivity-between2 by blast
    moreover have Cong B ?C A' ?C'
    using assms(3) not-cong-2134 by blast
    moreover have Cong M ?C M ?C'
    using  $\langle \text{Cong } A \ M \ B' \ M \rangle$  not-cong-2143 by blast
    ultimately show ?thesis
    using l4-3 by force
  qed
  ultimately
  show ?thesis
  by (simp add:  $\langle M \ \text{Midpoint } A \ B' \rangle$  cong-right-commutativity midpoint-def)
  qed
  have  $\exists M. ((M \ \text{Midpoint } A \ A' \wedge M \ \text{Midpoint } B \ B') \vee$ 
  ( $M \ \text{Midpoint } A \ B' \wedge M \ \text{Midpoint } B \ A')$ )
  using  $\langle M \ \text{Midpoint } A \ B' \wedge M \ \text{Midpoint } B \ A' \rangle$  by blast
}
  hence  $\exists M. ((M \ \text{Midpoint } A \ A' \wedge M \ \text{Midpoint } B \ B') \vee$ 
  ( $M \ \text{Midpoint } A \ B' \wedge M \ \text{Midpoint } B \ A')$ )
  using Col-def P3  $\langle \text{Bet } A \ A' \ B' \implies \exists M. M \ \text{Midpoint } A \ A' \wedge M \ \text{Midpoint } B \ B' \vee$ 
   $M \ \text{Midpoint } A \ B' \wedge M \ \text{Midpoint } B \ A' \rangle$ 
   $\langle \text{Bet } A' \ B' \ A \implies \exists M. M \ \text{Midpoint } A \ A' \wedge M \ \text{Midpoint } B \ B' \vee$ 
   $M \ \text{Midpoint } A \ B' \wedge M \ \text{Midpoint } B \ A' \rangle$  by blast
}
{
  assume Bet B' A B
  {
    assume Bet A B A'
    hence  $\exists M. ((M \ \text{Midpoint } A \ A' \wedge M \ \text{Midpoint } B \ B') \vee$ 
  ( $M \ \text{Midpoint } A \ B' \wedge M \ \text{Midpoint } B \ A')$ )
    using  $\langle A' \neq B \rangle \langle \text{Bet } B' \ A \ B \rangle$  assms(1) assms(3) bet-cong-eq
    not-cong-1243 outer-transitivity-between2 par-distincts by blast
  }
  {
    assume Bet B A' A
    obtain M where M Midpoint A A'
    using midpoint-existence by blast
    hence Bet A M A'
    by (simp add: midpoint-bet)
    have Cong A M A' M
    using Cong-perm  $\langle M \ \text{Midpoint } A \ A' \rangle$  midpoint-cong by blast
    have M Midpoint B B'
  proof -
    have Bet B M B'
    by (meson Bet-perm  $\langle \text{Bet } A \ M \ A' \rangle \langle \text{Bet } B \ A' \ A \rangle \langle \text{Bet } B' \ A \ B \rangle$  between-exchange4)
    moreover
    have Cong B M B' M
  proof -
    let ?C = A
    let ?C' = A'
    have Bet B M ?C
    using Bet-cases  $\langle \text{Bet } A \ M \ A' \rangle \langle \text{Bet } B \ A' \ A \rangle$  between-exchange2 by blast
    moreover have Bet B' M ?C'
    by (metis  $\langle \text{Bet } A \ M \ A' \rangle \langle \text{Bet } B' \ A \ B \rangle \langle M \ \text{Midpoint } A \ A' \rangle$ 
    between-inner-transitivity between-symmetry calculation
    midpoint-distinct-3 outer-transitivity-between2)
    moreover have Cong B ?C B' ?C'

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    using assms(3) not-cong-2143 by blast
  moreover have Cong  $M \ ?C \ M \ ?C'$ 
    using  $\langle Cong \ A \ M \ A' \ M \rangle$  cong-commutativity by blast
  ultimately show ?thesis
    using l4-3 by blast
qed
ultimately
show ?thesis
  using midpoint-def not-cong-1243 by blast
qed
hence  $\exists M. ((M \ Midpoint \ A \ A' \wedge M \ Midpoint \ B \ B') \vee$ 
( $M \ Midpoint \ A \ B' \wedge M \ Midpoint \ B \ A')$ )
  using  $\langle M \ Midpoint \ A \ A' \rangle$  by blast
}
{
  assume Bet  $A' \ A \ B$ 
  have Bet  $A \ B' \ A' \vee Bet \ A \ A' \ B'$ 
    by (metis  $\langle Bet \ A' \ A \ B \rangle \langle Bet \ B' \ A \ B \rangle$  assms(1) between-symmetry l5-2 par-distincts)
  {
    assume Bet  $A \ B' \ A'$ 
    obtain M where M Midpoint  $A \ B'$ 
      using midpoint-existence by presburger
    have M Midpoint  $B \ A'$ 
    proof -
      have Bet  $B \ M \ A'$ 
        using  $\langle A \neq B' \rangle \langle Bet \ A \ B' \ A' \rangle \langle Bet \ B' \ A \ B \rangle \langle M \ Midpoint \ A \ B' \rangle$ 
          bet3--bet between-symmetry midpoint-bet
          outer-transitivity-between by metis
      moreover
      have Cong  $A \ M \ B' \ M$ 
        using Midpoint-def  $\langle M \ Midpoint \ A \ B' \rangle$  cong-right-commutativity by blast
      hence Cong  $B \ M \ A' \ M$ 
        by (meson  $\langle A \neq B' \rangle \langle Bet \ A \ B' \ A' \rangle \langle Bet \ B' \ A \ B \rangle$  assms(3) cong-4312
          cong-reflexivity cong-symmetry five-segment)
      ultimately
      show ?thesis
        using midpoint-def not-cong-1243 by blast
    qed
    hence  $\exists M. ((M \ Midpoint \ A \ A' \wedge M \ Midpoint \ B \ B') \vee$ 
( $M \ Midpoint \ A \ B' \wedge M \ Midpoint \ B \ A')$ )
      using  $\langle M \ Midpoint \ A \ B' \rangle$  by blast
  }
}
{
  assume Bet  $A \ A' \ B'$ 
  obtain M where M Midpoint  $A \ A'$ 
    using midpoint-existence by blast
  have M Midpoint  $B \ B'$ 
  proof -
    have Bet  $B \ M \ B'$ 
    proof -
      have Bet  $B \ A \ B'$ 
        using Bet-cases  $\langle Bet \ B' \ A \ B \rangle$  by blast
      moreover have Bet  $A \ M \ B'$ 
        using Midpoint-def  $\langle Bet \ A \ A' \ B' \rangle \langle M \ Midpoint \ A \ A' \rangle$  between-exchange4
        by blast
      ultimately show ?thesis
        using between-exchange2 by blast
    qed
  moreover
  have Cong  $A \ M \ A' \ M$ 
    using Cong-perm  $\langle M \ Midpoint \ A \ A' \rangle$  midpoint-cong by blast
  hence Cong  $B \ M \ B' \ M$ 
    using Cong-cases  $\langle A \neq A' \rangle \langle Bet \ A \ A' \ B' \rangle \langle Bet \ A' \ A \ B \rangle$  assms(3)
      cong-reflexivity five-segment by blast
  ultimately
  show ?thesis

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    using midpoint-def not-cong-1243 by blast
  qed
  hence  $\exists M. ((M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B') \vee$ 
     $(M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A'))$ 
    using  $\langle M \text{ Midpoint } A \ A' \rangle$  by blast
}
  hence  $\exists M. ((M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B') \vee$ 
     $(M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A'))$ 
    using  $\langle \text{Bet } A \ B' \ A' \implies \exists M. M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B' \vee$ 
       $M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A' \rangle$ 
       $\langle \text{Bet } A \ B' \ A' \vee \text{Bet } A \ A' \ B' \rangle$  by blast
}
  hence  $\exists M. ((M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B') \vee$ 
     $(M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A'))$ 
    using Col-def  $\langle \text{Bet } A \ B \ A' \implies \exists M. M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B' \vee$ 
       $M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A' \rangle$ 
       $\langle \text{Bet } B \ A' \ A \implies \exists M. M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B' \vee$ 
       $M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A' \rangle \langle \text{Col } A \ B \ A' \rangle$  by blast
}
  thus ?thesis
    using Col-def  $\langle \text{Bet } A \ B \ B' \implies \exists M. M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B' \vee$ 
       $M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A' \rangle$ 
       $\langle \text{Bet } B \ B' \ A \implies \exists M. M \text{ Midpoint } A \ A' \wedge M \text{ Midpoint } B \ B' \vee$ 
       $M \text{ Midpoint } A \ B' \wedge M \text{ Midpoint } B \ A' \rangle \langle \text{Col } A \ B \ B' \rangle$  by blast
  qed
qed
qed
qed

```

```

lemma mid-par-cong1:
  assumes  $A \neq B$  and
     $M \text{ Midpoint } A \ A'$  and
     $M \text{ Midpoint } B \ B'$ 
  shows  $\text{Cong } A \ B \ A' \ B' \wedge A \ B \ \text{Par } A' \ B'$ 
  by (metis assms(1) assms(2) assms(3) cong-4321 cong-right-commutativity l12-17
    l7-13-R1 midpoint-distinct-2)

```

```

lemma mid-par-cong2:
  assumes  $A \neq B'$  and
     $M \text{ Midpoint } A \ A'$  and
     $M \text{ Midpoint } B \ B'$ 
  shows  $\text{Cong } A \ B' \ A' \ B \wedge A \ B' \ \text{Par } A' \ B$ 
  using assms(1) assms(2) assms(3) l7-2 mid-par-cong1 by blast

```

```

lemma mid-par-cong:
  assumes  $A \neq B$  and
     $A \neq B'$  and
     $M \text{ Midpoint } A \ A'$  and
     $M \text{ Midpoint } B \ B'$ 
  shows  $\text{Cong } A \ B \ A' \ B' \wedge \text{Cong } A \ B' \ A' \ B \wedge A \ B \ \text{Par } A' \ B' \wedge A \ B' \ \text{Par } A' \ B$ 
  using assms(1) assms(2) assms(3) assms(4) mid-par-cong1 mid-par-cong2 by auto

```

```

lemma Parallelogram-strict-Parallelogram:
  assumes ParallelogramStrict  $A \ B \ C \ D$ 
  shows Parallelogram  $A \ B \ C \ D$ 
  by (simp add: Parallelogram-def assms)

```

```

lemma plgf-permut:
  assumes ParallelogramFlat  $A \ B \ C \ D$ 
  shows ParallelogramFlat  $B \ C \ D \ A$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    by (metis ParallelogramFlat-def assms cong-diff-3 not-col-distincts)
next
  assume  $A \neq B$ 

```

```

thus ?thesis
proof -
  have f1:  $A \neq B$ 
  using  $\langle A \neq B \rangle$  by force
  have f2:  $\forall p \text{ pa pb pc. } \neg \text{Cong } p \text{ pa pb pc} \vee \text{Cong } pc \text{ pb pa } p$ 
  using not-cong-4321 by blast
  have ParallelogramFlat  $A \ B \ C \ D$ 
  using assms by force
  then show ?thesis
  using f2 f1 by (smt (z3) NCol-cases ParallelogramFlat-def col-transitivity-2 not-cong-3412)
qed
qed

```

```

lemma plgf-sym:
  assumes ParallelogramFlat  $A \ B \ C \ D$ 
  shows ParallelogramFlat  $C \ D \ A \ B$ 
  by (simp add: assms plgf-permut)

```

```

lemma plgf-irreflexive:
  shows  $\neg \text{ParallelogramFlat } A \ B \ A \ B$ 
  by (simp add: ParallelogramFlat-def)

```

```

lemma plgs-irreflexive:
  shows  $\neg \text{ParallelogramStrict } A \ B \ A \ B$ 
proof -
  {
    assume ParallelogramStrict  $A \ B \ A \ B$ 
    hence  $A \ A \text{ TS } B \ B \wedge A \ B \text{ Par } A \ B \wedge \text{Cong } A \ B \ A \ B$ 
    using ParallelogramStrict-def by auto
    hence False
    using not-two-sides-id by blast
  }
  thus ?thesis by blast
qed

```

```

lemma plg-irreflexive:
  shows  $\neg \text{Parallelogram } A \ B \ A \ B$ 
proof -
  {
    assume Parallelogram  $A \ B \ A \ B$ 
    hence  $\text{ParallelogramStrict } A \ B \ A \ B \vee \text{ParallelogramFlat } A \ B \ A \ B$ 
    by (simp add: Parallelogram-def)
    hence False
    using plgf-irreflexive plgs-irreflexive by blast
  }
  thus ?thesis by auto
qed

```

```

lemma plgf-mid:
  assumes ParallelogramFlat  $A \ B \ C \ D$ 
  shows  $\exists M. M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$ 
proof -
  have P1:  $\text{Col } A \ B \ C \wedge \text{Col } A \ B \ D \wedge \text{Cong } A \ B \ C \ D \wedge \text{Cong } A \ D \ C \ B \wedge (A \neq C \vee B \neq D)$ 
  using ParallelogramFlat-def assms by blast
  have  $\exists M. M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$ 
  proof cases
    assume  $A = B$ 
    obtain  $M$  where  $M \text{ Midpoint } A \ C$ 
    using midpoint-existence by blast
    moreover
    have  $M \text{ Midpoint } B \ D$ 
    proof -
      have  $\text{Bet } A \ M \ C$ 
      by (simp add: calculation midpoint-bet)
      hence  $\text{Bet } B \ M \ D$ 
      using P1  $\langle A = B \rangle$  cong-reverse-identity by blast
    qed
  qed

```

```

moreover
have Cong B M D M
  using P1  $\langle A = B \rangle$  M Midpoint A C cong-reverse-identity l7-13 l7-3-2
    not-cong-4321 by blast
ultimately
show ?thesis
  using Cong-perm midpoint-def by blast
qed
ultimately
show ?thesis by auto
next
assume  $A \neq B$ 
hence  $C \neq D$ 
  using P1 cong-identity by blast
hence  $A B \text{ Par } C D$ 
  using P1  $\langle A \neq B \rangle$  par-col2-par par-reflexivity by blast
have  $\neg A B \text{ ParStrict } C D$ 
  using P1 par-strict-not-col-1 by blast
then obtain M where P2:  $M \text{ Midpoint } A C \wedge M \text{ Midpoint } B D \vee$ 
 $M \text{ Midpoint } A D \wedge M \text{ Midpoint } B C$ 
  using P1  $\langle A B \text{ Par } C D \rangle$  col-cong-mid by blast
{
  assume  $M \text{ Midpoint } A D$  and  $M \text{ Midpoint } B C$  and  $B \neq C$ 
  have  $Col A B M$ 
  proof –
    have  $A \neq D$ 
      using P1  $\langle B \neq C \rangle$  cong-reverse-identity by blast
    have  $Col A D B$ 
      using P1 col-permutation-5 by blast
    moreover have  $Col A D M$ 
      by (metis Col-def Midpoint-def P2 between-equality-2 calculation
        col2--eq not-bet-distincts)
    ultimately show ?thesis
      using  $\langle A \neq D \rangle$  col-transitivity-1 by blast
  qed
  hence False
  by (metis (full-types) Cong-perm Mid-cases P1  $\langle A \neq B \rangle$   $\langle Col A B M \rangle$ 
     $\langle M \text{ Midpoint } A D \rangle$   $\langle M \text{ Midpoint } B C \rangle$ 
    cong-col-mid cong-cong-half-1 l7-9 not-col-permutation-2)
}
moreover
{
  assume  $M \text{ Midpoint } A D$  and  $M \text{ Midpoint } B C$  and  $B = C$ 
  hence  $\exists M. M \text{ Midpoint } A C \wedge M \text{ Midpoint } B D$ 
    using P1 P2 cong-identity l7-3 by blast
}
ultimately show ?thesis
  using P2 by auto
qed
thus ?thesis
  by simp
qed

lemma mid-plgs:
assumes  $\neg Col A B C$  and
   $M \text{ Midpoint } A C$  and
   $M \text{ Midpoint } B D$ 
shows ParallelogramStrict A B C D
proof –
  have  $A \neq C$ 
    using assms(1) col-trivial-3 by auto
  hence  $M \neq A$ 
    using assms(2) is-midpoint-id by auto
  have  $B \neq D$ 
    using assms(1) assms(2) assms(3) bet-col l8-20-2 midpoint-bet by blast
  hence  $M \neq D$ 

```

```

    using assms(3) is-midpoint-id-2 by auto
  have Bet A M C
    by (simp add: assms(2) midpoint-bet)
  have Bet B M D
    by (simp add: assms(3) midpoint-bet)
  thus ?thesis
proof -
  have A C TS B D
    using NCol-perm assms(1) assms(2) assms(3) mid-two-sides by blast
  moreover
  have A B Par C D
    using assms(1) assms(2) assms(3) mid-par-cong1 not-col-distincts by blast
  moreover
  have Cong A B C D
    using assms(1) assms(2) assms(3) mid-par-cong1 not-col-distincts by blast
  ultimately
  show ?thesis
    using ParallelogramStrict-def by blast
qed
qed

lemma mid-plgf-aux:
  assumes A  $\neq$  C and
    Col A B C and
    M Midpoint A C and
    M Midpoint B D
  shows ParallelogramFlat A B C D
proof cases
  assume B = D
  thus ?thesis
    by (metis ParallelogramFlat-def Cong-perm col-trivial-2 midpoint-cong
      assms(1) assms(2) assms(3) assms(4) l8-20-2)
next
  assume B  $\neq$  D
  thus ?thesis
  proof -
    have M Midpoint C A
      by (simp add: assms(3) l7-2)
    then have Cong A D C B
      by (simp add: assms(4) l7-13)
    then show ?thesis
      by (metis Col-cases ParallelogramFlat-def  $\langle B \neq D \rangle$  assms(2,3,4) col-trivial-1 l7-13 l7-2 midpoint-midpoint-col)
  qed
qed

lemma mid-plgf-1:
  assumes A  $\neq$  C

  Col A B C and
  M Midpoint A C and
  M Midpoint B D
  shows ParallelogramFlat A B C D
  using assms(1) assms(2) assms(3) assms(4) mid-plgf-aux by auto

lemma mid-plgf-2:
  assumes B  $\neq$  D
  Col A B C and
  M Midpoint A C and
  M Midpoint B D
  shows ParallelogramFlat A B C D
  by (metis Col-def assms(1) assms(2) assms(3) assms(4) l7-3 mid-plgf-aux
    midpoint-bet plgf-permut)

lemma mid-plgf:
  assumes A  $\neq$  C  $\vee$  B  $\neq$  D
  Col A B C and

```



```

    M Midpoint A C and
    M Midpoint B D
  shows ParallelogramFlat A B C D
  using assms(1) assms(2) assms(3) assms(4) mid-plgf-2 mid-plgf-aux by blast

lemma mid-plg:
  assumes A ≠ C ∨ B ≠ D and
    M Midpoint A C and
    M Midpoint B D
  shows Parallelogram A B C D
  using Parallelogram-def assms(1) assms(2) assms(3) mid-plgf-2 mid-plgf-aux mid-plgs by blast

lemma midpoint-cong-uniqueness:
  assumes Col A B C and
    M Midpoint A B and
    M Midpoint C D and
    Cong A B C D
  shows A = C ∧ B = D ∨ A = D ∧ B = C
proof cases
  assume A = B
  thus ?thesis
    using assms(2) assms(3) assms(4) cong-reverse-identity l7-3 by blast
next
  assume A ≠ B
  thus ?thesis
proof cases
  assume A = C
  thus ?thesis
    using assms(2) assms(3) symmetric-point-uniqueness by blast
next
  assume A ≠ C
  have M Midpoint A C
  proof -
    have Cong A M C M
      using assms(2) assms(3) assms(4) cong-cong-half-1 by blast
    moreover have Col A M C
      by (meson l6-16-1 midpoint-col ⟨A ≠ B⟩ assms(1) assms(2) col-permutation-2)
    hence Bet A M C
      by (metis col-trivial-2 cong-right-commutativity ⟨A ≠ C⟩ bet-cong-eq
        calculation col-cong2-bet2 not-bet-distincts third-point)
    ultimately show ?thesis
      using Cong-cases midpoint-def by blast
  qed
  thus ?thesis
    by (metis assms(2) assms(3) l7-9-bis)
qed
qed

lemma plgf-not-comm1:
  assumes A ≠ B and
    ParallelogramFlat A B C D
  shows ¬ ParallelogramFlat A B D C
proof -
  obtain M where P1: M Midpoint A C ∧ M Midpoint B D
    using assms(2) plgf-mid by blast
  thus ?thesis
    by (metis ParallelogramFlat-def assms(1) l7-2 midpoint-cong-uniqueness
      midpoint-midpoint-col)
qed

lemma plgf-not-comm2:
  assumes A ≠ B and
    ParallelogramFlat A B C D
  shows ¬ ParallelogramFlat B A C D
proof -
  have ParallelogramFlat C D A B

```

```

    by (metis assms(2) plgf-sym)
  thus ?thesis
    by (metis assms(1) plgf-not-comm1 plgf-sym)
qed

lemma plgf-not-comm:
  assumes  $A \neq B$  and
    ParallelogramFlat  $A B C D$ 
  shows  $\neg$  ParallelogramFlat  $A B D C \wedge \neg$  ParallelogramFlat  $B A C D$ 
  by (simp add: assms(1) assms(2) plgf-not-comm1 plgf-not-comm2)

lemma plgf-cong:
  assumes ParallelogramFlat  $A B C D$ 
  shows  $Cong A B C D \wedge Cong A D B C$ 
  using Cong-perm ParallelogramFlat-def assms by blast

lemma plg-to-parallelogram:
  assumes Plg  $A B C D$ 
  shows Parallelogram  $A B C D$ 
proof -
  obtain  $M$  where  $M$  Midpoint  $A C \wedge M$  Midpoint  $B D$ 
    using Plg-def assms by auto
  thus ?thesis
    using Plg-def assms mid-plg by auto
qed

lemma plgs-one-side:
  assumes ParallelogramStrict  $A B C D$ 
  shows  $A B OS C D \wedge C D OS A B$ 
proof -
  have  $A C TS B D \wedge A B Par C D \wedge Cong A B C D$ 
    using ParallelogramStrict-def assms by auto
  show ?thesis using l12-6
    by (simp add:  $\langle A C TS B D \wedge A B Par C D \wedge Cong A B C D \rangle$ 
       $\langle \wedge D C Ba A. A Ba ParStrict C D \implies A Ba OS C D \rangle$ 
      Par-perm invert-two-sides l9-2 par-two-sides-two-sides ts-ts-os)
qed

lemma parallelogram-strict-not-col:
  assumes ParallelogramStrict  $A B C D$ 
  shows  $\neg Col A B C$ 
  by (metis assms one-side-not-col123 plgs-one-side)

lemma parallelogram-strict-not-col-2:
  assumes ParallelogramStrict  $A B C D$ 
  shows  $\neg Col B C D$ 
  using Col-perm assms col124--nos plgs-one-side by blast

lemma parallelogram-strict-not-col-3:
  assumes ParallelogramStrict  $A B C D$ 
  shows  $\neg Col C D A$ 
  using assms col123--nos plgs-one-side by blast

lemma parallelogram-strict-not-col-4:
  assumes ParallelogramStrict  $A B C D$ 
  shows  $\neg Col A B D$ 
  by (metis (no-types) assms col124--nos plgs-one-side)

lemma plgs--pars:
  assumes ParallelogramStrict  $A B C D$ 
  shows  $A B ParStrict C D$ 
proof -
  have  $A C TS B D \wedge A B Par C D \wedge Cong A B C D$ 
    using ParallelogramStrict-def assms by auto
  thus ?thesis
    using Par-def assms parallelogram-strict-not-col-2 by auto

```

qed

lemma *plgs-sym*:

assumes *ParallelogramStrict* $A B C D$

shows *ParallelogramStrict* $C D A B$

proof –

have $A C TS B D \wedge A B Par C D \wedge Cong A B C D$

using *ParallelogramStrict-def* *assms* by auto

thus ?thesis

by (simp add: $\langle A C TS B D \wedge A B Par C D \wedge Cong A B C D \rangle$
Cong-perm Par-perm ParallelogramStrict-def invert-two-sides l9-2)

qed

lemma *plg-sym*:

assumes *Parallelogram* $A B C D$

shows *Parallelogram* $C D A B$

using *Parallelogram-def* *assms* *plgf-sym* *plgs-sym* by blast

lemma *Rhombus-Plg*:

assumes *Rhombus* $A B C D$

shows *Plg* $A B C D$

using *Rhombus-def* *assms* by auto

lemma *Rectangle-Plg*:

assumes *Rectangle* $A B C D$

shows *Plg* $A B C D$

using *Rectangle-def* *assms* by auto

lemma *Rectangle-Parallelogram*:

assumes *Rectangle* $A B C D$

shows *Parallelogram* $A B C D$

using *Rectangle-def* *assms* *plg-to-parallelogram* by auto

lemma *plg-cong-rectangle*:

assumes *Plg* $A B C D$ and

Cong $A C B D$

shows *Rectangle* $A B C D$

by (simp add: *Rectangle-def* *assms*(1) *assms*(2))

lemma *plg-trivial*:

assumes $A \neq B$

shows *Parallelogram* $A B B A$

by (simp add: *ParallelogramFlat-def* *Parallelogram-def* *assms* *col-trivial-2*
col-trivial-3 *cong-pseudo-reflexivity* *cong-trivial-identity*)

lemma *plg-trivial1*:

assumes $A \neq B$

shows *Parallelogram* $A A B B$

by (simp add: *ParallelogramFlat-def* *Parallelogram-def* *assms* *col-trivial-1*
cong-pseudo-reflexivity *cong-trivial-identity*)

lemma *col-not-plgs*:

assumes *Col* $A B C$

shows $\neg$ *ParallelogramStrict* $A B C D$

using *assms* *parallelogram-strict-not-col* by auto

lemma *plg-col-plgf*:

assumes *Col* $A B C$ and

Parallelogram $A B C D$

shows *ParallelogramFlat* $A B C D$

using *Parallelogram-def* *assms*(1) *assms*(2) *parallelogram-strict-not-col* by blast

lemma *plg-bet1*:

assumes *Parallelogram* $A B C D$ and

Bet $A C B$

shows *Bet* $D A C$

```

proof –
  have Col A B C
    by (meson Bet-perm assms(2) bet-col1 between-exchange2 point-construction-different)
  moreover
  have ParallelogramStrict A B C D  $\vee$  ParallelogramFlat A B C D
    using Parallelogram-def assms(1) by auto
  ultimately
  show ?thesis
    by (metis (no-types) Bet-perm ParallelogramFlat-def assms(2) col-cong2-bet2
      col-not-plgs plgf-cong)
qed

lemma plgf-trivial1:
  assumes A  $\neq$  B
  shows ParallelogramFlat A B B A
  by (meson ParallelogramFlat-def not-col-distincts assms cong-pseudo-reflexivity
    cong-trivial-identity)

lemma plgf-trivial2:
  assumes A  $\neq$  B
  shows ParallelogramFlat A A B B
  by (meson assms plgf-permut plgf-trivial1)

lemma plgf-not-point:
  assumes ParallelogramFlat A A B B
  shows A  $\neq$  B
  using assms plgf-irreflexive by blast

lemma plgf-trivial-neq:
  assumes ParallelogramFlat A A C D
  shows C = D  $\wedge$  A  $\neq$  C
  using assms cong-reverse-identity plgf-cong plgf-irreflexive by blast

lemma plgf-trivial-trans:
  assumes ParallelogramFlat A A B B and
    ParallelogramFlat B B C C
  shows ParallelogramFlat A A C C  $\vee$  A = C
  using plgf-trivial2 by auto

lemma plgf-trivial:
  assumes A  $\neq$  B
  shows ParallelogramFlat A B B A
  using assms plgf-trivial1 by auto

lemma plgf3-mid:
  assumes ParallelogramFlat A B A C
  shows A Midpoint B C
  by (metis assms midpoint-distinct-3 plgf-mid)

lemma cong3-id:
  assumes A  $\neq$  B and
    Col A B C and
    Col A B D and
    Cong A B C D and
    Cong A D B C and
    Cong A C B D
  shows A = D  $\wedge$  B = C  $\vee$  A = C  $\wedge$  B = D
proof cases
  assume A = C
  thus ?thesis
    using assms(6) cong-reverse-identity by presburger
next
  assume A  $\neq$  C
  have  $\exists M. M \text{ Midpoint } A B \wedge M \text{ Midpoint } C D \vee M \text{ Midpoint } A D \wedge M \text{ Midpoint } C B$ 
  proof –
    have A C Par B D

```

```

    by (metis col3 col-trivial-3 cong-identity par-col2-par par-reflexivity
         $\langle A \neq C \rangle$  assms(1) assms(2) assms(3) assms(6) not-col-permutation-5)
  moreover have  $\neg A \ C \text{ ParStrict } B \ D$ 
    using Col-cases assms(3) par-strict-not-col-3 by blast
  ultimately show ?thesis
    using assms(6) col-cong-mid by blast
qed
then obtain M where M Midpoint A B  $\wedge$  M Midpoint C D  $\vee$ 
M Midpoint A D  $\wedge$  M Midpoint C B
  by blast
{
  assume M Midpoint A B  $\wedge$  M Midpoint C D
  hence  $A = D \wedge B = C \vee A = C \wedge B = D$ 
    using assms(2) assms(4) midpoint-cong-uniqueness by blast
}
moreover
{
  assume M Midpoint A D  $\wedge$  M Midpoint C B
  hence  $A = D \wedge B = C \vee A = C \wedge B = D$ 
    by (metis assms(1) assms(3) assms(5) l7-2 midpoint-cong-uniqueness
        not-col-permutation-5)
}
ultimately show ?thesis
  using  $\langle M \text{ Midpoint } A \ B \wedge M \text{ Midpoint } C \ D \vee M \text{ Midpoint } A \ D \wedge M \text{ Midpoint } C \ B \rangle$ 
  by fastforce
qed

lemma col-cong-mid1:
  assumes  $A \neq D$  and
    Col A B C and
    Col A B D and
    Cong A B C D and
    Cong A C B D
  shows  $\exists M. M \text{ Midpoint } A \ D \wedge M \text{ Midpoint } B \ C$ 
proof -
  have  $\exists M. (M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D \vee$ 
    M Midpoint A D  $\wedge$  M Midpoint B C)
  proof cases
    assume  $A = B$ 
    thus ?thesis
      using assms(4) cong-reverse-identity midpoint-existence by blast
  next
    assume  $A \neq B$ 
    have  $C \neq D$ 
      using  $\langle A \neq B \rangle$  assms(4) cong-identity by blast
    hence  $A \ B \text{ Par } C \ D$ 
      by (meson  $\langle A \neq B \rangle$  assms(2) assms(3) not-col-distincts not-par-inter-uniqueness)
    moreover have  $\neg A \ B \text{ ParStrict } C \ D$ 
      using assms(3) par-strict-not-col-4 by blast
    ultimately show ?thesis
      using assms(4) col-cong-mid by auto
  qed
then obtain M where P3:  $(M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D) \vee$ 
 $(M \text{ Midpoint } A \ D \wedge M \text{ Midpoint } B \ C)$  by auto
{
  assume M Midpoint A C  $\wedge$  M Midpoint B D
  hence M Midpoint A D  $\wedge$  M Midpoint B C
    using assms(1) assms(2) assms(5) col-permutation-5 midpoint-cong-uniqueness by blast
}
thus ?thesis
  using P3 by blast
qed

lemma col-cong-mid2:
  assumes  $A \neq C$  and
    Col A B C and

```

```

    Col A B D and
    Cong A B C D and
    Cong A D B C
shows  $\exists M. M \text{ Midpoint } A C \wedge M \text{ Midpoint } B D$ 
by (simp add: assms(1) assms(2) assms(3) assms(4) assms(5) col-cong-mid1
    cong-right-commutativity)

lemma plgs-not-col:
  assumes ParallelogramStrict A B C D
  shows  $\neg Col A B C \wedge \neg Col B C D \wedge \neg Col C D A \wedge \neg Col A B D$ 
  using assms col-not-plgs parallelogram-strict-not-col-2 parallelogram-strict-not-col-3
    parallelogram-strict-not-col-4 by blast

lemma not-col-sym-not-col:
  assumes  $\neg Col A B C$  and
    A Midpoint B B'
  shows  $\neg Col A B' C$ 
  by (metis assms(1) assms(2) col-transitivity-1 midpoint-col midpoint-distinct-2
    not-col-permutation-5)

lemma plg-existence:
  assumes  $A \neq B$ 
  shows  $\exists D. Parallelogram A B C D$ 
proof -
  obtain M where M Midpoint A C
  using midpoint-existence by auto
  thus ?thesis
  by (metis assms l7-17-bis l7-3-2 mid-plg symmetric-point-construction)
qed

lemma plgs-diff:
  assumes ParallelogramStrict A B C D
  shows  $ParallelogramStrict A B C D \wedge A \neq B \wedge B \neq C \wedge C \neq D \wedge D \neq A \wedge A \neq C \wedge B \neq D$ 
  using assms not-col-distincts parallelogram-strict-not-col-2
    parallelogram-strict-not-col-3 parallelogram-strict-not-col-4 by blast

lemma sym-par:
  assumes  $A \neq B$  and
    M Midpoint A A' and
    M Midpoint B B'
  shows  $A B \text{ Par } A' B'$ 
  using l12-17 assms(1) assms(2) assms(3) by auto

lemma symmetry-preserves-two-sides:
  assumes  $Col X Y M$  and
    X Y TS A B and
    M Midpoint A A' and
    M Midpoint B B'
  shows  $X Y TS A' B'$ 
proof -
  have Bet A M A'
  by (simp add: assms(3) midpoint-bet)
  have Bet B M B'
  by (simp add: assms(4) midpoint-bet)
  have  $X \neq Y$ 
  using assms(2) ts-distincts by blast
  have  $\neg Col A X Y$ 
  using TS-def assms(2) by blast
  have  $\neg Col B X Y$ 
  using TS-def assms(2) by blast
  have  $A \neq M$ 
  using Col-perm  $\langle \neg Col A X Y \rangle$  assms(1) by blast
  hence  $A' \neq M$ 
  using assms(3) cong-identity midpoint-cong by blast
  have  $B \neq M$ 
  using Col-perm  $\langle \neg Col B X Y \rangle$  assms(1) by blast

```

hence $B' \neq M$
 using *assms(4) midpoint-not-midpoint* by blast
 have $X \ Y \ TS \ A \ A'$
 using *NCol-cases $\langle A' \neq M \rangle \langle \neg Col \ A \ X \ Y \rangle$ assms(1) assms(3) colx*
l9-18-R2 midpoint-bet midpoint-col by meson
 have $\neg Col \ B' \ X \ Y$
 using $\langle B' \neq M \rangle \langle Bet \ B \ M \ B' \rangle$ *assms(1) bet-col bet-col1 $\langle \neg Col \ B \ X \ Y \rangle$*
l6-21 not-col-permutation-3 not-col-permutation-4 by metis
 hence $X \ Y \ TS \ B \ B'$
 using *TS-def $\langle \neg Col \ B \ X \ Y \rangle \langle Bet \ B \ M \ B' \rangle$ assms(1) not-col-permutation-1* by blast
 moreover
 have $X \ Y \ OS \ A' \ B$
 by (*meson $\langle X \ Y \ TS \ A \ A' \rangle$ assms(2) l9-2 l9-8-1*)
 ultimately
 show ?thesis
 using *l9-8-2 one-side-symmetry* by blast
 qed

lemma *symmetry-preserves-one-side*:

assumes *Col X Y M* and
X Y OS A B and
M Midpoint A A' and
M Midpoint B B'
 shows $X \ Y \ OS \ A' \ B'$
 proof –
 obtain P where $X \ Y \ TS \ A \ P$ and $X \ Y \ TS \ B \ P$
 using *OS-def assms(2)* by blast
 have $X \neq Y$
 using *assms(2) os-distincts* by blast
 have $\neg Col \ A \ X \ Y$
 using *TS-def $\langle X \ Y \ TS \ A \ P \rangle$* by blast
 have $\neg Col \ B \ X \ Y$
 using *assms(2) between-trivial one-side-chara* by blast
 have $\neg Col \ P \ X \ Y$
 using *TS-def $\langle X \ Y \ TS \ A \ P \rangle$* by blast
 have $A \neq M$
 using *assms(1) assms(2) col123--nos* by blast
 hence $A' \neq M$
 using *assms(3) midpoint-not-midpoint* by blast
 have $B \neq M$
 using *assms(1) assms(2) col124--nos* by blast
 hence $B' \neq M$
 using *assms(4) midpoint-not-midpoint* by blast
 have $X \ Y \ TS \ A \ A'$
 proof –
 {
 assume *Col A' X Y*
 have *Col A M A'*
 using *Col-def Midpoint-def assms(3)* by auto
 have *Col M A' X*
 using $\langle Col \ A' \ X \ Y \rangle \langle X \neq Y \rangle$ *assms(1) col-permutation-2 col-trivial-2 colx* by blast
 hence *Col A X Y*
 by (*metis $\langle A' \neq M \rangle \langle Col \ A \ M \ A' \rangle \langle Col \ A' \ X \ Y \rangle$ assms(1) col-permutation-3*
col-permutation-4 col-trivial-2 l6-21)
 hence *False*
 using $\langle \neg Col \ A \ X \ Y \rangle$ by auto
 }
 moreover have $\exists \ T. \ Col \ T \ X \ Y \wedge \ Bet \ A \ T \ A'$
 proof –
 have *Col M X Y*
 using *assms(1) not-col-permutation-1* by blast
 moreover have *Bet A M A'*
 by (*simp add: assms(3) midpoint-bet*)
 ultimately show ?thesis
 by blast
 qed

```

ultimately show ?thesis
  using TS-def ‹ $\neg \text{Col } A \ X \ Y$ › by blast
qed
moreover
have  $X \ Y \ TS \ B \ B'$ 
  by (meson ‹ $B' \neq M$ › ‹ $\neg \text{Col } B \ X \ Y$ › assms(1) assms(4) colx l9-18-R2
    midpoint-bet midpoint-col not-col-permutation-1)
ultimately
show ?thesis
  by (meson assms(2) l9-2 l9-8-1 l9-8-2)
qed

lemma plgf-bet:
  assumes ParallelogramFlat  $A \ B \ B' \ A'$ 
  shows  $\text{Bet } A' \ B' \ A \wedge \text{Bet } B' \ A \ B$ 
     $\vee \text{Bet } A' \ A \ B' \wedge \text{Bet } A \ B' \ B$ 
     $\vee \text{Bet } A \ A' \ B \wedge \text{Bet } A' \ B \ B'$ 
     $\vee \text{Bet } A \ B \ A' \wedge \text{Bet } B \ A' \ B'$ 
proof -
  let ?t =  $\text{Bet } A' \ B' \ A \wedge \text{Bet } B' \ A \ B$ 
     $\vee \text{Bet } A' \ A \ B' \wedge \text{Bet } A \ B' \ B$ 
     $\vee \text{Bet } A \ A' \ B \wedge \text{Bet } A' \ B \ B'$ 
     $\vee \text{Bet } A \ B \ A' \wedge \text{Bet } B \ A' \ B'$ 
  have  $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge (A \neq B' \vee B \neq A')$ 
    using ParallelogramFlat-def assms by blast
  show ?thesis
  proof cases
    assume  $A = B$ 
    thus ?thesis
      using ‹ $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge$ 
         $(A \neq B' \vee B \neq A')$ › between-trivial between-trivial2
        cong-reverse-identity by blast
  next
    assume  $A \neq B$ 
    have  $A' \neq B'$ 
      using ‹ $A \neq B$ › assms plgf-not-comm1 by blast
    have  $\text{Col } A' \ B' \ A$ 
      by (meson ‹ $A \neq B$ ›
        ‹ $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge (A \neq B' \vee B \neq A')$ ›
        col-transitivity-1 not-col-permutation-3)
    have  $\text{Col } A' \ B' \ B$ 
      by (meson ‹ $A \neq B$ ›
        ‹ $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge (A \neq B' \vee B \neq A')$ ›
        col-transitivity-2 not-col-permutation-3)
    {
      assume  $\text{Bet } A \ B \ A'$ 
      hence ?t
        using ‹ $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge (A \neq B' \vee B \neq A')$ ›
          col-cong2-bet4 cong-right-commutativity by blast
    }
    moreover
    {
      assume  $\text{Bet } B \ A' \ A$ 
      hence ?t
        by (metis Bet-cases
          ‹ $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge (A \neq B' \vee B \neq A')$ ›
          ‹ $\text{Col } A' \ B' \ A$ › col-cong2-bet4 cong-right-commutativity not-col-permutation-1)
    }
    moreover
    {
      assume  $\text{Bet } A' \ A \ B$ 
      {
        assume  $\text{Bet } A \ B \ B'$ 
        hence ?t
          by (metis bet-cong-eq outer-transitivity-between2 ‹ $A \neq B$ › ‹ $\text{Bet } A' \ A \ B$ ›
            ‹ $\text{Col } A \ B \ B' \wedge \text{Col } A \ B \ A' \wedge \text{Cong } A \ B \ B' \ A' \wedge \text{Cong } A \ A' \ B' \ B \wedge (A \neq B' \vee B \neq A')$ ›)
      }
    }
  }

```



```

    cong-right-commutativity)
  }
  moreover
  {
    assume Bet B B' A
    hence ?t
    by (metis between-symmetry ⟨Bet A' A B⟩ between-inner-transitivity)
  }
  moreover
  {
    assume Bet B' A B
    {
      assume A = A'
      hence ?t
      using ⟨Bet B A' A ⟹
        Bet A' B' A ∧ Bet B' A B ∨
        Bet A' A B' ∧ Bet A B' B ∨
        Bet A A' B ∧ Bet A' B B' ∨ Bet A B A' ∧ Bet B A' B'⟩
      not-bet-distincts by blast
    }
    moreover
    {
      assume A ≠ A'
      have Bet A B' A' ∨ Bet A A' B'
      by (metis ⟨A ≠ B⟩ ⟨Bet A' A B⟩ ⟨Bet B' A B⟩ between-symmetry l5-2)
      hence Bet A' B' A
      by (metis Cong-cases ⟨A ≠ B⟩ ⟨Bet A' A B⟩ ⟨Bet B' A B⟩
        ⟨Col A B B' ∧ Col A B A' ∧ Cong A B B' A' ∧ Cong A A' B' B ∧ (A ≠ B' ∨ B ≠ A')⟩
        bet-cong-bet between-symmetry)
      hence ?t
      using ⟨Bet B' A B⟩ by blast
    }
    hence ?t
    using calculation by fastforce
  }
  moreover
  have ?t
  using Col-def
    ⟨Col A B B' ∧ Col A B A' ∧ Cong A B B' A' ∧ Cong A A' B' B ∧ (A ≠ B' ∨ B ≠ A')⟩
    calculation(1) calculation(2) calculation(3) by force
}
ultimately show ?thesis
using Col-def
  ⟨Col A B B' ∧ Col A B A' ∧ Cong A B B' A' ∧ Cong A A' B' B ∧ (A ≠ B' ∨ B ≠ A')⟩
  by blast
qed
qed

```

lemma plgs-existence:

assumes $A \neq B$
 shows $\exists C. \exists D. \text{ParallelogramStrict } A B C D$

proof –

```

  obtain C where ¬ Col A B C
  using assms not-col-exists by presburger
  obtain D where Parallelogram A B C D
  using assms plg-existence by blast
  {
    assume ParallelogramFlat A B C D
    have False
    using ParallelogramFlat-def ⟨ParallelogramFlat A B C D⟩ ⟨¬ Col A B C⟩ by force
  }
  thus ?thesis
  using Parallelogram-def ⟨Parallelogram A B C D⟩ by blast
qed

```

lemma Rectangle-not-triv:

shows $\neg \text{Rectangle } A A A A$
using *Rectangle-Parallelogram plg-irreflexive* **by** *blast*

lemma *Plg-triv*:
assumes $A \neq B$
shows $\text{Plg } A A B B$
proof –
have $\exists M. M \text{ Midpoint } A B \wedge M \text{ Midpoint } A B$
by (*simp add: midpoint-existence*)
thus *?thesis*
by (*simp add: Plg-def assms*)
qed

lemma *Rectangle-triv*:
assumes $A \neq B$
shows $\text{Rectangle } A A B B$
proof –
have $\text{Plg } A A B B$
by (*simp add: Plg-triv assms*)
moreover
have $\text{Cong } A B A B$
by (*simp add: cong-reflexivity*)
ultimately
show *?thesis*
by (*simp add: plg-cong-rectangle*)
qed

lemma *Rectangle-not-triv-2*:
shows $\neg \text{Rectangle } A B A B$
using *Rectangle-Parallelogram plg-irreflexive* **by** *blast*

lemma *Square-not-triv*:
shows $\neg \text{Square } A A A A$
by (*simp add: Rectangle-not-triv-2 Square-def Tarski-neutral-dimensionless-axioms*)

lemma *Square-not-triv-2*:
shows $\neg \text{Square } A A B B$
by (*metis Square-def Square-not-triv cong-diff-3*)

lemma *Square-not-triv-3*:
shows $\neg \text{Square } A B A B$
by (*simp add: Rectangle-not-triv-2 Square-def*)

lemma *Square-Rectangle*:
assumes $\text{Square } A B C D$
shows $\text{Rectangle } A B C D$
using *Square-def assms* **by** *blast*

lemma *Square-Parallelogram*:
assumes $\text{Square } A B C D$
shows $\text{Parallelogram } A B C D$
by (*simp add: Rectangle-Parallelogram Square-Rectangle assms*)

lemma *Rhombus-Rectangle-Square*:
assumes $\text{Rhombus } A B C D$ **and**
 $\text{Rectangle } A B C D$
shows $\text{Square } A B C D$
using *Rhombus-def Square-def assms(1) assms(2)* **by** *blast*

lemma *rhombus-cong-square*:
assumes $\text{Rhombus } A B C D$ **and**
 $\text{Cong } A C B D$
shows $\text{Square } A B C D$
by (*simp add: Rhombus-Plg Rhombus-Rectangle-Square assms(1) assms(2) plg-cong-rectangle*)

lemma *Kite-comm*:

```

assumes Kite A B C D
shows Kite C D A B
using Kite-def assms by auto

lemma per2-col-eq:
assumes A  $\neq$  P and
  A  $\neq$  P' and
  Per A P B and
  Per A P' B and
  Col P A P'
shows P = P'
by (metis assms(1) assms(2) assms(3) assms(4) assms(5) col-per2-cases
  l6-16-1 l8-2 not-col-permutation-3)

lemma per2-preserves-diff:
assumes PO  $\neq$  A' and
  PO  $\neq$  B' and
  Col PO A' B' and
  Per PO A' A and
  Per PO B' B and
  A'  $\neq$  B'
shows A  $\neq$  B
using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) not-col-permutation-4
  per2-col-eq by blast

lemma per23-preserves-bet:
assumes Bet A B C and
  A  $\neq$  B' and A  $\neq$  C' and
  Col A B' C' and
  Per A B' B and
  Per A C' C
shows Bet A B' C'
proof -
have P1: Col A B C
  by (simp add: assms(1) bet-col)
show ?thesis
proof cases
assume P2: B = B'
hence Col A C' C
  using P1 assms(2) assms(4) col-transitivity-1 by blast
hence P4: A = C'  $\vee$  C = C'
  by (simp add: assms(6) l8-9)
{
  assume A = C'
  hence Bet A B' C'
  using assms(3) by auto
}
{
  assume C = C'
  hence Bet A B' C'
  using P2 assms(1) by auto
}
thus ?thesis
using P4 assms(3) by auto
next
assume T1: B  $\neq$  B'
have T2: A  $\neq$  C
  using assms(3) assms(6) l8-8 by auto
have T3: C  $\neq$  C'
  using P1 T1 assms(2) assms(3) assms(4) assms(5) col-trivial-3 colx l8-9
  not-col-permutation-5 by blast
have T3A: A B' Perp B' B
  using T1 assms(2) assms(5) per-perp by auto
have T3B: A C' Perp C' C
  using T3 assms(3) assms(6) per-perp by auto
have T4: B B' Par C C'

```

```

proof –
  have Coplanar A B' B C
    using P1 ncop--ncols by blast
  moreover have Coplanar A B' B C'
    using assms(4) ncop--ncols by blast
  moreover have Coplanar A B' B' C
    using ncop-distincts by blast
  moreover have B B' Perp A B'
    using Perp-perm ⟨A B' Perp B' B⟩ by blast
  moreover have C C' Perp A B'
    using Col-cases Perp-cases T3B assms(2) assms(4) perp-col1 by blast
  ultimately show ?thesis
    using l12-9 bet--coplanar between-trivial by auto
qed
moreover have Bet A B' C'
proof cases
  assume B = C
  thus ?thesis
    by (metis T1 per-col-eq assms(4) assms(5) calculation l6-16-1 l6-6
        or-bet-out out-diff1 par-id)
next
  assume T6: B ≠ C
  have T7: ¬ Col A B' B
    using T1 assms(2) assms(5) l8-9 by blast
  have T8: ¬ Col A C' C
    using T3 assms(3) assms(6) l8-9 by blast
  have T9: B' ≠ C'
    using P1 T6 assms(2) assms(5) assms(6) col-per2--per col-permutation-1
        l8-2 l8-8 by blast
  have T10: B B' ParStrict C C' ∨ (B ≠ B' ∧ C ≠ C' ∧ Col B C C' ∧ Col B' C C')
    using Par-def calculation by blast
  {
    assume T11: B B' ParStrict C C'
    hence T12: B B' OS C' C
      using l12-6 one-side-symmetry by blast
    have B B' TS A C
      using Col-cases T6 T7 assms(1) bet--ts by blast
    hence Bet A B' C'
      using T12 assms(4) l9-5 l9-9 not-col-distincts or-bet-out by blast
  }
  {
    assume B ≠ B' ∧ C ≠ C' ∧ Col B C C' ∧ Col B' C C'
    hence Bet A B' C'
      using Col-def T6 T8 assms(1) col-transitivity-2 by blast
  }
  thus ?thesis
    using T10 ⟨B B' ParStrict C C' ⟹ Bet A B' C'⟩ by blast
qed
ultimately show ?thesis
  using P1 Par-def T1 T2 assms(4) col-transitivity-2 not-col-permutation-1
      par-strict-not-col-2 by blast
qed
qed

lemma per23-preserves-bet-inv:
  assumes Bet A B' C' and
    A ≠ B' and
    Col A B C and
    Per A B' B and
    Per A C' C
  shows Bet A B C
proof cases
  assume T1: B = B'
  hence Col A C' C
    using Col-def assms(1) assms(2) assms(3) col-transitivity-1 by blast
  hence T2: A = C' ∨ C = C'

```

```

    by (simp add: assms(5) l8-9)
  {
    assume A = C'
    hence Bet A B C
      using assms(1) assms(2) between-identity by blast
  }
  {
    assume C = C'
    hence Bet A B C
      by (simp add: T1 assms(1))
  }
  thus ?thesis
    using T2 ⟨A = C' ⟹ Bet A B C⟩ by auto
next
  assume P1: B ≠ B'
  hence P2: A B' Perp B' B
    using assms(2) assms(4) per-perp by auto
  have Per A C' C
    by (simp add: assms(5))
  hence P2: C' PerpAt A C' C' C
    by (metis (mono-tags, lifting) Col-cases P1 assms(1) assms(2) assms(3) assms(4)
        bet-col bet-neq12--neq col-transitivity-1 l8-9 per-perp-in)
  hence P3: A C' Perp C' C
    using perp-in-perp by auto
  hence C' ≠ C
    using ⟨A C' Perp C' C⟩ perp-not-eq-2 by auto
  have C' PerpAt C' A C C'
    by (simp add: Perp-in-perm P2)
  hence (C' A Perp C C') ∨ (C' C' Perp C C')
    using Perp-def by blast
  have A ≠ C'
    using assms(1) assms(2) between-identity by blast
  {
    assume C' A Perp C C'
    have Col A B' C' using assms(1)
      by (simp add: Col-def)
    have A B' Perp C' C
      using Col-cases ⟨A C' Perp C' C⟩ ⟨Col A B' C'⟩ assms(2) perp-col by blast
    have P7: B' B Par C' C
      proof -
        have Coplanar A B' B' C'
          using ncop-distincts by blast
        moreover have Coplanar A B' B' C
          using ncop-distincts by auto
        moreover have Coplanar A B' B C'
          using Bet-perm assms(1) bet--coplanar ncoplanar-perm-20 by blast
        moreover have Coplanar A B' B C
          using assms(3) ncop--ncols by auto
        moreover have B' B Perp A B'
          by (metis P1 Perp-perm assms(2) assms(4) per-perp)
        moreover have C' C Perp A B'
          using Perp-cases ⟨A B' Perp C' C⟩ by auto
        ultimately show ?thesis using l12-9 by blast
      qed
    have Bet A B C
      proof cases
        assume B = C
        thus ?thesis
          by (simp add: between-trivial)
      next
        assume T1: B ≠ C
        have T2: B' B ParStrict C' C ∨ (B' ≠ B ∧ C' ≠ C ∧ Col B' C' C ∧ Col B C' C)
          using P7 Par-def by auto
        {
          assume T3: B' B ParStrict C' C
          hence B' ≠ C'

```

```

    using not-par-strict-id by auto
  have  $\exists X. \text{Col } X \ B' \ B \wedge \text{Col } X \ B' \ C$ 
    using col-trivial-1 by blast
  have  $B' \ B \ OS \ C' \ C$ 
    by (simp add: T3 l12-6)
  have  $B' \ B \ TS \ A \ C'$ 
    by (metis Bet-cases T3 assms(1) assms(2) bet--ts l9-2 par-strict-not-col-1)
  hence T8:  $B' \ B \ TS \ C \ A$ 
    using  $\langle B' \ B \ OS \ C' \ C \rangle$  l9-2 l9-8-2 by blast
  then obtain T where T9:  $\text{Col } T \ B' \ B \wedge \text{Bet } C \ T \ A$ 
    using TS-def by auto
  have  $\neg \text{Col } A \ C \ B'$ 
    using T8 assms(3) not-col-permutation-2 not-col-permutation-3 ts--ncol by blast
  hence  $T = B$ 
    by (metis Col-def Col-perm T9 assms(3) colx)
  hence  $\text{Bet } A \ B \ C$ 
    using Bet-cases T9 by auto
}
{
  assume  $B' \neq B \wedge C' \neq C \wedge \text{Col } B' \ C' \ C \wedge \text{Col } B \ C' \ C$ 
  hence  $\text{Col } A \ B' \ B$ 
    by (metis Col-perm T1 assms(3) l6-16-1)
  hence  $A = B' \vee B = B'$ 
    using assms(4) l8-9 by auto
  hence  $\text{Bet } A \ B \ C$ 
    by (simp add: P1 assms(2))
}
}
thus ?thesis
  using T2  $\langle B' \ B \ \text{ParStrict } C' \ C \implies \text{Bet } A \ B \ C \rangle$  by auto
qed
}
thus ?thesis
  by (simp add: P3 perp-comm)
qed

```

lemma *per13-preserves-bet:*

assumes $\text{Bet } A \ B \ C$ **and**

$B \neq A'$ **and**

$B \neq C'$ **and**

$\text{Col } A' \ B \ C'$ **and**

$\text{Per } B \ A' \ A$ **and**

$\text{Per } B \ C' \ C$

shows $\text{Bet } A' \ B \ C'$

by (smt (verit, ccfv-SIG) NCol-perm assms(1,3,4,5,6) bet-col1 bet-out-1

between-exchange2 l8-20-1-R1

l8-7 not-bet-and-out per23-preserves-bet-inv point-construction-different third-point)

lemma *per13-preserves-bet-inv:*

assumes $\text{Bet } A' \ B \ C'$ **and**

$B \neq A'$ **and**

$B \neq C'$ **and**

$\text{Col } A \ B \ C$ **and**

$\text{Per } B \ A' \ A$ **and**

$\text{Per } B \ C' \ C$

shows $\text{Bet } A \ B \ C$

by (meson Col-def assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)

between-equality between-symmetry per23-preserves-bet)

lemma *per3-preserves-bet1:*

assumes $\text{Col } PO \ A \ B$ **and**

$\text{Bet } A \ B \ C$ **and**

$PO \neq A'$ **and**

$PO \neq B'$ **and**

$PO \neq C'$ **and**

$\text{Per } PO \ A' \ A$ **and**

$\text{Per } PO \ B' \ B$ **and**

```

    Per PO C' C and
    Col A' B' C' and
    Col PO A' B'
  shows Bet A' B' C'
proof cases
  assume A = B
  thus ?thesis
    using assms(10) assms(3) assms(4) assms(6) assms(7) between-trivial2
      per2-preserves-diff by blast
next
  assume P1: A ≠ B
  show ?thesis
  proof cases
    assume P2: A = A'
    show ?thesis
    proof cases
      assume P3: B = B'
      hence Col PO C C'
        using P1 P2 assms(10) assms(2) assms(9) bet-col col3 col-permutation-1 by blast
      hence C = C'
        using assms(5) assms(8) l8-9 not-col-permutation-5 by blast
      thus ?thesis
        using P2 P3 assms(2) by blast
    next
      assume P4: B ≠ B'
      show ?thesis
      proof cases
        assume A = B'
        thus ?thesis
          using P2 between-trivial2 by auto
      next
        assume A ≠ B'
        have A ≠ C
          using P1 assms(2) between-identity by blast
        have P7: ¬ Col PO B' B
          using P4 assms(4) assms(7) l8-9 by blast
        show ?thesis
          using P2 P7 assms(1) assms(10) assms(3) col-transitivity-1 by blast
      qed
    qed
  next
    assume R1: A ≠ A'
    show ?thesis
    proof cases
      assume R2: A' = B'
      thus ?thesis
        by (simp add: between-trivial2)
    next
      assume R3: A' ≠ B'
      show ?thesis
      proof cases
        assume B = C
        have B' = C'
          by (metis per2-col-eq ⟨A' ≠ B'⟩ ⟨B = C⟩ assms(10) assms(4) assms(5)
            assms(7) assms(8) assms(9) col-transitivity-2 not-col-permutation-2)
        thus ?thesis
          by (simp add: between-trivial)
      next
        assume R4: B ≠ C
        show ?thesis
        proof cases
          assume B = B'
          thus ?thesis
            by (metis R1 assms(1) assms(10) assms(3) assms(4) assms(6)
              l6-16-1 l8-9 not-col-permutation-2)
        next

```

```

assume  $R5: B \neq B'$ 
show ?thesis
proof cases
  assume  $A' = B$ 
  thus ?thesis
    using  $R5$   $assms(10)$   $assms(4)$   $assms(7)$  col-permutation-5 l8-9 by blast
next
  assume  $R5A: A' \neq B$ 
  have  $R6: C \neq C'$ 
    by (metis  $P1$   $R1$   $R3$   $assms(1)$   $assms(10)$   $assms(2)$   $assms(3)$ 
       $assms(5)$   $assms(6)$   $assms(9)$  bet-col col-permutation-1
      col-trivial-2 l6-21 l8-9)
  have  $R7: A \ A' \ \text{Perp} \ PO \ A'$ 
    by (metis Perp-cases  $R1$   $assms(3)$   $assms(6)$  per-perp)
  have  $R8: C \ C' \ \text{Perp} \ PO \ A'$ 
proof –
  have  $C' \ C' \ \text{Perp} \ C' \ C \vee PO \ C' \ \text{Perp} \ C' \ C$ 
    using  $R6$   $assms(5)$   $assms(8)$  per-perp by auto
moreover
  {
    assume  $C' \ C' \ \text{Perp} \ C' \ C$ 
    hence  $C \ C' \ \text{Perp} \ PO \ A'$ 
      using  $\langle C' \ C' \ \text{Perp} \ C' \ C \rangle$  perp-not-eq-1 by blast
  }
moreover
  {
    assume  $PO \ C' \ \text{Perp} \ C' \ C$ 
    hence  $C \ C' \ \text{Perp} \ PO \ A'$ 
      by (metis Col-cases  $R3$  Perp-cases  $assms(10)$   $assms(3)$   $assms(9)$  col3
        col-trivial-3 perp-col1)
  }
ultimately show ?thesis
  by blast
qed
have  $A \ A' \ \text{Par} \ C \ C'$ 
proof –
  have Coplanar  $PO \ A' \ A \ C$ 
    using  $P1$   $assms(1)$   $assms(2)$  bet-col col-trivial-2 colx ncop--ncols by blast
moreover have Coplanar  $PO \ A' \ A \ C'$ 
    using  $R3$   $assms(10)$   $assms(9)$  col-trivial-2 colx ncop--ncols by blast
moreover have Coplanar  $PO \ A' \ A' \ C$ 
    using ncop-distincts by blast
moreover have Coplanar  $PO \ A' \ A' \ C'$ 
    using ncop-distincts by blast
ultimately show ?thesis using l12-9  $R7$   $R8$  by blast
qed
have  $S1: B \ B' \ \text{Perp} \ PO \ A'$ 
  by (metis Col-cases Per-cases Perp-perm  $R5$   $assms(10)$   $assms(3)$ 
     $assms(4)$   $assms(7)$  col-per-perp)
have  $A \ A' \ \text{Par} \ B \ B'$ 
proof –
  have Coplanar  $PO \ A' \ A \ B$ 
    using  $assms(1)$  ncop--ncols by auto
moreover have Coplanar  $PO \ A' \ A \ B'$ 
    using  $assms(10)$  ncop--ncols by auto
moreover have Coplanar  $PO \ A' \ A' \ B$ 
    using ncop-distincts by auto
moreover have Coplanar  $PO \ A' \ A' \ B'$ 
    using ncop-distincts by auto
moreover have  $A \ A' \ \text{Perp} \ PO \ A'$ 
    by (simp add:  $R7$ )
moreover have  $B \ B' \ \text{Perp} \ PO \ A'$ 
    by (simp add:  $S1$ )
ultimately show ?thesis
  using l12-9 by blast
qed

```



```

{
  assume  $A A' \text{ ParStrict } B B'$ 
  hence  $A A' \text{ OS } B B'$ 
    by (simp add: l12-6)
  have  $B B' \text{ TS } A C$ 
    using  $R4 \langle A A' \text{ ParStrict } B B' \rangle \text{ assms}(2) \text{ bet--ts par-strict-not-col-3}$  by auto
  have  $B B' \text{ OS } A A'$ 
    using  $\langle A A' \text{ ParStrict } B B' \rangle \text{ pars--os3412}$  by auto
  have  $B B' \text{ TS } A' C$ 
    using  $\langle B B' \text{ OS } A A' \rangle \langle B B' \text{ TS } A C \rangle \text{ l9-8-2}$  by blast
  have  $\text{Bet } A' B' C'$ 
  proof cases
    assume  $C = C'$ 
    thus ?thesis
      using  $R6$  by auto
  next
    assume  $C \neq C'$ 
    have  $C C' \text{ Perp } PO A'$ 
      by (simp add:  $R8$ )
    have  $Q2: B B' \text{ Par } C C'$ 
  proof -
    have  $\text{Coplanar } PO A' B C$ 
      by (metis  $P1 \text{ assms}(1) \text{ assms}(2) \text{ bet-col col-transitivity-1}$ 
         $\text{colx ncop--ncols not-col-permutation-5}$ )
    moreover have  $\text{Coplanar } PO A' B C'$ 
      using  $R3 \text{ assms}(10) \text{ assms}(9) \text{ col-trivial-2 colx ncop--ncols}$  by blast
    moreover have  $\text{Coplanar } PO A' B' C$ 
      by (simp add:  $\text{assms}(10) \text{ col--coplanar}$ )
    moreover have  $\text{Coplanar } PO A' B' C'$ 
      using  $\text{assms}(10) \text{ col--coplanar}$  by auto
    moreover have  $B B' \text{ Perp } PO A'$ 
      by (simp add:  $S1$ )
    moreover have  $C C' \text{ Perp } PO A'$ 
      by (simp add:  $R8$ )
    ultimately show ?thesis
      using  $\text{l12-9}$  by auto
  qed
  hence  $Q3: (B B' \text{ ParStrict } C C') \vee (B \neq B' \wedge C \neq C' \wedge \text{Col } B C C' \wedge \text{Col } B' C C')$ 
    by (simp add:  $\text{Par-def}$ )
  {
    assume  $B B' \text{ ParStrict } C C'$ 
    hence  $B B' \text{ OS } C C'$ 
      using  $\text{l12-6}$  by auto
    hence  $B B' \text{ TS } C' A'$ 
      using  $\langle B B' \text{ TS } A' C \rangle \text{ l9-2 l9-8-2}$  by blast
    then obtain  $T$  where  $Q4: \text{Col } T B B' \wedge \text{Bet } C' T A'$ 
      using  $\text{TS-def}$  by blast
    have  $T = B'$ 
  proof -
    have  $\neg \text{Col } B B' A'$ 
      using  $\langle B B' \text{ OS } A A' \rangle \text{ col124--nos}$  by auto
    moreover have  $A' \neq C'$ 
      using  $\langle B B' \text{ TS } C' A' \rangle \text{ not-two-sides-id}$  by auto
    moreover have  $\text{Col } B B' T$ 
      using  $\text{Col-cases } Q4$  by auto
    moreover have  $\text{Col } B B' B'$ 
      using  $\text{not-col-distincts}$  by blast
    moreover have  $\text{Col } A' C' T$ 
      by (simp add:  $\text{Col-def } Q4$ )
    ultimately show ?thesis
      by (meson  $\text{assms}(9) \text{ col-permutation-5 l6-21}$ )
  qed
  hence  $\text{Bet } A' B' C'$ 
    using  $Q4 \text{ between-symmetry}$  by blast
  }
}

```

```

    assume  $B \neq B' \wedge C \neq C' \wedge \text{Col } B \ C \ C' \wedge \text{Col } B' \ C \ C'$ 
    hence  $\text{Bet } A' \ B' \ C'$ 
    using  $\text{TS-def } \langle B \ B' \ \text{TS } A \ C \rangle \text{ l6-16-1 not-col-permutation-2}$  by blast
  }
  thus ?thesis
  using  $Q3 \ \langle B \ B' \ \text{ParStrict } C \ C' \implies \text{Bet } A' \ B' \ C' \rangle$  by blast
qed
}
{
  assume  $R8: A \neq A' \wedge B \neq B' \wedge \text{Col } A \ B \ B' \wedge \text{Col } A' \ B \ B'$ 
  have  $A' \ A \ \text{Perp } PO \ A'$ 
  by (simp add:  $R7 \ \text{perp-left-comm}$ )
  have  $\neg \text{Col } A' \ A \ PO$ 
  using  $\text{Col-cases } R8 \ \text{assms}(3) \ \text{assms}(6) \ \text{l8-9}$  by blast
  hence  $\text{Bet } A' \ B' \ C'$ 
  using  $\text{Col-perm } P1 \ R8 \ \text{assms}(1) \ \text{l6-16-1}$  by blast
}
thus ?thesis
using  $\text{Par-def } \langle A \ A' \ \text{Par } B \ B' \rangle \ \langle A \ A' \ \text{ParStrict } B \ B' \implies \text{Bet } A' \ B' \ C' \rangle$  by auto
qed
qed
qed
qed
qed
qed

```

lemma *per3-preserves-bet2-aux*:

```

  assumes  $\text{Col } PO \ A \ C$  and
     $A \neq C'$  and
     $\text{Bet } A \ B' \ C'$  and
     $PO \neq A$  and
     $PO \neq B'$  and
     $PO \neq C'$  and
     $\text{Per } PO \ B' \ B$  and
     $\text{Per } PO \ C' \ C$  and
     $\text{Col } A \ B \ C$  and
     $\text{Col } PO \ A \ C'$ 
  shows  $\text{Bet } A \ B \ C$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
  by (simp add: between-trivial2)
next
  assume  $P1: A \neq B$ 
  show ?thesis
proof cases
  assume  $B = C$ 
  thus ?thesis
  by (simp add: between-trivial)
next
  assume  $P2: B \neq C$ 
  have  $P3: \text{Col } PO \ A \ B'$ 
  by (metis  $\text{Col-def } \text{assms}(10) \ \text{assms}(2) \ \text{assms}(3) \ \text{l6-16-1}$ )
  hence  $P4: \text{Col } PO \ B' \ C'$ 
  using  $\text{assms}(10) \ \text{assms}(4) \ \text{col-transitivity-1}$  by blast
  show ?thesis
proof cases
  assume  $B = B'$ 
  thus ?thesis
  by (metis  $\text{per-col-eq } \text{assms}(1) \ \text{assms}(10) \ \text{assms}(3) \ \text{assms}(4) \ \text{assms}(6) \ \text{assms}(8) \ \text{col-transitivity-1}$ )
next
  assume  $P5: B \neq B'$ 
  have  $P6: C = C'$ 
  using  $\text{assms}(1) \ \text{assms}(10) \ \text{assms}(4) \ \text{assms}(6) \ \text{assms}(8) \ \text{col-transitivity-1} \ \text{l8-9}$  by blast
  hence False

```

```

    by (metis P3 P5 P6 per-col-eq assms(1) assms(2) assms(4) assms(5)
        assms(7) assms(9) col-transitivity-1 l6-16-1 not-col-permutation-4)
  thus ?thesis by blast
qed
qed
qed

lemma per3-preserves-bet2:
  assumes Col PO A C and
    A'  $\neq$  C' and
    Bet A' B' C' and
    PO  $\neq$  A' and
    PO  $\neq$  B' and
    PO  $\neq$  C' and
    Per PO A' A and
    Per PO B' B and
    Per PO C' C and
    Col A B C and
    Col PO A' C'
  shows Bet A B C
proof cases
  assume A = A'
  thus ?thesis
    using assms(1) assms(10) assms(11) assms(2) assms(3) assms(4) assms(5)
        assms(6) assms(8) assms(9) per3-preserves-bet2-aux by blast
next
  assume P1: A  $\neq$  A'
  show ?thesis
  proof cases
    assume C = C'
    thus ?thesis
      by (metis P1 assms(1) assms(11) assms(4) assms(6) assms(7) col-trivial-3
          l6-21 l8-9 not-col-permutation-2)
  next
    assume P2: C  $\neq$  C'
    hence P3: PO A' Perp C C'
      by (metis assms(11) assms(4) assms(6) assms(9) col-per-perp l8-2 not-col-permutation-1)
    have P4: PO A' Perp A A'
      using P1 assms(4) assms(7) per-perp perp-right-comm by auto
    have A A' Par C C'
    proof -
      have Coplanar PO A' A C
        using assms(1) ncop-ncols by blast
      moreover have Coplanar PO A' A C'
        by (meson assms(11) ncop-ncols)
      moreover have Coplanar PO A' A' C
        using ncop-distincts by blast
      moreover have Coplanar PO A' A' C'
        using ncop-distincts by blast
      moreover have A A' Perp PO A'
        using P4 Perp-cases by blast
      moreover have C C' Perp PO A'
        using P3 Perp-cases by auto
      ultimately show ?thesis
        using l12-9 by blast
    qed
  {
    assume P5: A A' ParStrict C C'
    hence P6: A A' OS C C'
      by (simp add: l12-6)
    have P7: C C' OS A A'
      by (simp add: P5 pars--os3412)
    have Bet A B C
    proof cases
      assume P8: B = B'
      hence A' A OS B C'

```

```

    by (metis P6 assms(10) assms(3) bet-out col123--nos col124--nos
        invert-one-side out-one-side)
  hence A A' OS B C'
    by (simp add: invert-one-side)
  hence A A' OS B C
    using P6 one-side-symmetry one-side-transitivity by blast
  hence P12: A Out B C
    using assms(10) col-one-side-out by blast
  have C' C OS B A'
    by (metis Col-perm P5 P7 P8 assms(10) assms(3) bet-out-1 col123--nos
        out-one-side par-strict-not-col-2)
  hence C C' OS B A
    by (meson P7 invert-one-side one-side-symmetry one-side-transitivity)
  hence C C' OS A B
    using one-side-symmetry by blast
  hence C Out A B
    using assms(10) col-one-side-out col-permutation-2 by blast
  thus ?thesis
    by (simp add: P12 out2--bet)
next
assume T1: B ≠ B'
have T2: PO A' Perp B B'
proof -
  have Per PO B' B
    by (simp add: assms(8))
  hence B' PerpAt PO B' B' B
    using T1 assms(5) per-perp-in by auto
  hence B' PerpAt B' PO B B'
    by (simp add: perp-in-comm)
  hence T4: B' PO Perp B B' ∨ B' B' Perp B B'
    using Perp-def by auto
  {
    assume T5: B' PO Perp B B'
    have Col A' B' C'
      by (simp add: assms(3) bet-col)
    hence Col PO B' A'
      using assms(11) assms(2) col2--eq col-permutation-4
        col-permutation-5 by blast
    hence PO A' Perp B B'
      by (metis T5 assms(4) col-trivial-3 perp-col2 perp-comm)
  }
  {
    assume B' B' Perp B B'
    hence PO A' Perp B B'
      using perp-distinct by auto
  }
  thus ?thesis
    using T4 ⟨B' PO Perp B B' ⟹ PO A' Perp B B'⟩ by linarith
qed
have T6: B B' Par A A'
proof -
  have Coplanar PO A' B A
    by (metis Col-cases P7 assms(1) assms(10) col-transitivity-2
        ncop--ncols os-distincts)
  moreover have Coplanar PO A' B A'
    using ncop-distincts by blast
  moreover have Coplanar PO A' B' A
  proof -
    have (Bet PO A' C' ∨ Bet PO C' A') ∨ Bet C' PO A'
      by (meson assms(11) third-point)
    thus ?thesis
      by (meson Bet-perm assms(3) bet--coplanar between-exchange2
          l5-3 ncoplanar-perm-8)
  qed
  moreover have Coplanar PO A' B' A'
    using ncop-distincts by auto

```

```

moreover have  $B B' \text{ Perp } PO A'$ 
  using Perp-cases T2 by blast
moreover have  $A A' \text{ Perp } PO A'$ 
  using P4 Perp-cases by blast
ultimately show ?thesis
  using l12-9 by blast
qed
{
  assume  $B B' \text{ ParStrict } A A'$ 
  hence  $B B' \text{ OS } A A'$ 
    by (simp add: l12-6)
  have  $B B' \text{ Par } C C'$ 
  proof –
    have Coplanar PO A' B C
      by (metis Col-cases P7 assms(1) assms(10) col2--eq ncop--ncols os-distincts)
    moreover have Coplanar PO A' B C'
      using assms(11) ncop--ncols by auto
    moreover have Coplanar PO A' B' C
      by (metis Out-def assms(11) assms(2) assms(3) col-trivial-2
        l6-16-1 ncop--ncols not-col-permutation-1 out-col)
    moreover have Coplanar PO A' B' C'
      using assms(11) ncop--ncols by blast
    moreover have  $B B' \text{ Perp } PO A'$ 
      using Perp-cases T2 by blast
    moreover have  $C C' \text{ Perp } PO A'$ 
      using P3 Perp-cases by auto
    ultimately show ?thesis
      using l12-9 by blast
  qed
{
  assume  $T9: B B' \text{ ParStrict } C C'$ 
  hence  $T10: B B' \text{ OS } C C'$ 
    by (simp add: l12-6)
  have  $T11: B B' \text{ TS } A' C'$ 
    by (metis Col-cases T10  $\langle B B' \text{ ParStrict } A A' \rangle$  assms(3) bet--ts
      invert-two-sides os-distincts par-strict-not-col-4)
  have  $T12: B B' \text{ TS } A C'$ 
    using  $\langle B B' \text{ OS } A A' \rangle \langle B B' \text{ TS } A' C' \rangle$  l9-8-2
      one-side-symmetry by blast
  hence  $T12A: B B' \text{ TS } C A$ 
    using T10 l9-2 l9-8-2 one-side-symmetry by blast
  then obtain  $T$  where  $T13: \text{Col } T B B' \wedge \text{Bet } C T A$ 
    using TS-def by auto
  hence  $B = T$ 
    by (metis Col-perm TS-def T12A assms(10) bet-col1 col-transitivity-2
      col-two-sides-bet)
  hence  $\text{Bet } A B C$ 
    using Bet-perm T13 by blast
}
{
  assume  $B \neq B' \wedge C \neq C' \wedge \text{Col } B C C' \wedge \text{Col } B' C C'$ 
  hence  $\text{Bet } A B C$ 
    by (metis Col-cases P5 assms(10) col3 col-trivial-2
      not-bet-distincts par-strict-not-col-3)
}
hence  $\text{Bet } A B C$ 
  using Par-def  $\langle B B' \text{ Par } C C' \rangle \langle B B' \text{ ParStrict } C C' \rangle \implies \text{Bet } A B C$  by auto
}
{
  assume  $B \neq B' \wedge A \neq A' \wedge \text{Col } B A A' \wedge \text{Col } B' A A'$ 
  hence  $\text{Bet } A B C$ 
  proof cases
    assume  $A = B$ 
    thus ?thesis
      using between-trivial2 by auto
  next

```

```

    assume  $A \neq B$ 
    thus ?thesis
    by (metis Col-cases P5
         $\langle B \neq B' \wedge A \neq A' \wedge \text{Col } B \ A \ A' \wedge \text{Col } B' \ A \ A' \rangle$  assms(10)
        col-transitivity-2 par-strict-not-col-1)
  qed
}
thus ?thesis
  using Par-def T6  $\langle B \ B' \ \text{ParStrict } A \ A' \implies \text{Bet } A \ B \ C \rangle$  by auto
qed
}
{
  assume  $A \neq A' \wedge C \neq C' \wedge \text{Col } A \ C \ C' \wedge \text{Col } A' \ C \ C'$ 
  hence  $\text{Bet } A \ B \ C$ 
  by (metis Col-perm P3 Par-def assms(11) assms(2) assms(4)
      col-transitivity-1 perp-not-par)
}
thus ?thesis
  using Par-def  $\langle A \ A' \ \text{Par } C \ C' \rangle \langle A \ A' \ \text{ParStrict } C \ C' \implies \text{Bet } A \ B \ C \rangle$  by auto
qed
qed

```

lemma symmetry-preserves-per:

```

  assumes Per B P A and
    B Midpoint A A' and
    B Midpoint P P'
  shows Per B P' A'
proof -
  obtain C where P1: P Midpoint A C
  using symmetric-point-construction by blast
  obtain C' where P2: B Midpoint C C'
  using symmetric-point-construction by blast
  have P3: P' Midpoint A' C'
  using P1 P2 assms(2) assms(3) symmetry-preserves-midpoint by blast
  have Cong B A' B C'
  by (meson P1 P2 assms(1) assms(2) l7-16 l7-3-2 per-double-cong)
  thus ?thesis
  using P3 Per-def by blast
qed

```

lemma l13-1-aux:

```

  assumes  $\neg \text{Col } A \ B \ C$  and
    P Midpoint B C and
    Q Midpoint A C and
    R Midpoint A B
  shows
     $\exists X \ Y. (R \text{ PerpAt } X \ Y \ A \ B \wedge X \ Y \text{ Perp } P \ Q \wedge \text{Coplanar } A \ B \ C \ X \wedge \text{Coplanar } A \ B \ C \ Y)$ 
proof -
  have P1:  $Q \neq C$ 
  using assms(1) assms(3) midpoint-not-midpoint not-col-distincts by blast
  have P2:  $P \neq C$ 
  using assms(1) assms(2) is-midpoint-id-2 not-col-distincts by blast
  hence  $Q \neq R$ 
  using assms(2) assms(3) assms(4) l7-3 symmetric-point-uniqueness by blast
  have  $R \neq B$ 
  using assms(1) assms(4) midpoint-not-midpoint not-col-distincts by blast
  {
    assume V1:  $\text{Col } P \ Q \ C$ 
    have V2:  $\text{Col } B \ P \ C$ 
    by (simp add: assms(2) bet-col midpoint-bet)
    have V3:  $\text{Col } A \ Q \ C$ 
    by (simp add: assms(3) bet-col midpoint-bet)
    have  $\text{Col } A \ R \ B$ 
    using assms(4) midpoint-col not-col-permutation-4 by blast
    hence  $\text{Col } A \ B \ C$  using V1 V2 V3
    by (metis P1 P2 col2--eq col-permutation-5)
  }

```

```

    hence False
      using assms(1) by auto
  }
  hence P2A:  $\neg \text{Col } P \ Q \ C$  by auto
  then obtain C' where P3:  $\text{Col } P \ Q \ C' \wedge P \ Q \ \text{Perp } C \ C'$ 
    using l8-18-existence by blast
  obtain A' where P4:  $Q \ \text{Midpoint } C' \ A'$ 
    using symmetric-point-construction by auto
  obtain B' where P5:  $P \ \text{Midpoint } C' \ B'$ 
    using symmetric-point-construction by auto
  have P6:  $\text{Cong } C \ C' \ B \ B'$ 
    using Mid-cases P5 assms(2) l7-13 by blast
  have P7:  $\text{Cong } C \ C' \ A \ A'$ 
    using P4 assms(3) l7-13 l7-2 by blast
  have P8:  $\text{Per } P \ B' \ B$ 
  proof cases
    assume P = C'
    thus ?thesis
      using P5 Per-cases is-midpoint-id l8-5 by blast
  next
    assume P  $\neq$  C'
    hence P C' Perp C C'
      using P3 perp-col by blast
    hence Per P C' C
      using Perp-perm perp-per-2 by blast
    thus ?thesis
      using symmetry-preserves-per Mid-perm P5 assms(2) by blast
  qed
  have P8A:  $\text{Per } Q \ A' \ A$ 
  proof cases
    assume Q = C'
    thus ?thesis
      using P4 Per-cases is-midpoint-id l8-5 by blast
  next
    assume Q  $\neq$  C'
    hence C' Q Perp C C'
      using P3 col-trivial-2 perp-col2 by auto
    hence Per Q C' C
      by (simp add: perp-per-1)
    thus ?thesis
      by (meson Mid-cases P4 assms(3) l7-3-2 midpoint-preserves-per)
  qed
  have P9:  $\text{Col } A' \ C' \ Q$ 
    using P4 midpoint-col not-col-permutation-3 by blast
  have P10:  $\text{Col } B' \ C' \ P$ 
    using P5 midpoint-col not-col-permutation-3 by blast
  have P11:  $P \neq Q$ 
    using P2A col-trivial-1 by auto
  hence P12:  $A' \neq B'$ 
    using P4 P5 l7-17 by blast
  have P13:  $\text{Col } A' \ B' \ P$ 
    by (metis P10 P3 P4 P5 P9 col2--eq col-permutation-5
      midpoint-distinct-1 not-col-distincts)
  have P14:  $\text{Col } A' \ B' \ Q$ 
    using P10 P3 P4 P5 P9 col2--eq col-permutation-5 midpoint-distinct-1
      not-col-distincts by metis
  have P15:  $\text{Col } A' \ B' \ C'$ 
    using P11 P13 P14 P3 colx by blast
  have P16:  $C \neq C'$ 
    using P2A P3 by blast
  hence P17:  $A \neq A'$ 
    using P7 cong-diff by blast
  have P18:  $B \neq B'$ 
    using P16 P6 cong-diff by blast
  have P19:  $\text{Per } P \ A' \ A$ 
  proof cases

```

```

assume P20:  $A' = Q$ 
hence  $A' P \text{ Perp } C A'$ 
  by (metis P3 P4 Perp-cases midpoint-not-midpoint)
hence  $\text{Per } P A' C$ 
  by (simp add: perp-per-1)
thus ?thesis
  using P20 assms(3) l7-2 l8-4 by blast
next
assume  $A' \neq Q$ 
thus ?thesis
  by (meson P12 P13 P14 P8A col-transitivity-1 l8-2 per-col)
qed
have  $\text{Per } Q B' B$ 
proof cases
  assume P21:  $P = B'$ 
  hence P22:  $C' = B'$ 
    using P5 is-midpoint-id-2 by auto
  hence  $\text{Per } Q B' C$ 
    using P3 P21 perp-per-1 by auto
  thus ?thesis
    by (metis Col-perm P16 P21 P22 assms(2) midpoint-col per-col)
next
assume P23:  $P \neq B'$ 
have  $\text{Col } B' P Q$ 
  using P12 P13 P14 col-transitivity-2 by blast
hence  $\text{Per } B B' Q$ 
  using P8 P23 l8-2 l8-3 by blast
thus ?thesis
  using Per-perm by blast
qed
hence P24:  $\text{Per } A' B' B$ 
  using P11 P13 P14 P8 l8-3 not-col-permutation-2 by blast
have P25:  $\text{Per } A A' B'$ 
  using P11 P13 P14 P19 P8A l8-2 l8-3 not-col-permutation-5 by blast
hence  $\text{Per } B' A' A$ 
  using Per-perm by blast
hence  $\neg \text{Col } B' A' A$ 
  using P12 P17 P25 per-not-col by auto
hence P26:  $\neg \text{Col } A' B' A$ 
  using Col-cases by auto
have  $\neg \text{Col } A' B' B$ 
  using P12 P18 P24 l8-9 by auto
obtain  $X$  where P28:  $X \text{ Midpoint } A' B'$ 
  using midpoint-existence by blast
hence P28A:  $\text{Col } A' B' X$ 
  using midpoint-col not-col-permutation-2 by blast
hence  $\exists Q. A' B' \text{ Perp } Q X \wedge A' B' \text{ OS } A Q$ 
  by (simp add: P26 l10-15)
then obtain  $y$  where P29:  $A' B' \text{ Perp } y X \wedge A' B' \text{ OS } A y$  by blast
then obtain  $B''$  where P30:  $(X y \text{ Perp } A B'' \vee A = B'') \wedge$ 
 $(\exists M. (\text{Col } X y M \wedge M \text{ Midpoint } A B''))$ 
  using ex-sym by blast
hence P31:  $B'' A \text{ ReflectL } X y$ 
  using P30 ReflectL-def by blast
have P32:  $X \neq y$ 
  using P29 P28A col124--nos by blast
hence  $X \neq y \wedge B'' A \text{ ReflectL } X y \vee X = y \wedge X \text{ Midpoint } A B''$ 
  using P31 by auto
hence P33:  $B'' A \text{ Reflect } X y$ 
  by (simp add: Reflect-def)
have P33A:  $X \neq y \wedge A' B' \text{ ReflectL } X y$ 
  using P28 P29 Perp-cases ReflectL-def P32 col-trivial-3 l10-4-spec by blast
hence P34:  $A' B' \text{ Reflect } X y$ 
  using Reflect-def by blast
have P34A:  $A B'' \text{ Reflect } X y$ 
  using P33 l10-4 by blast

```



```

hence P35: Cong B'' B' A A'
  using P34 l10-10 by auto
have Per A' B' B''
proof -
  have R1:  $X \neq y \wedge A B'' \text{ ReflectL } X y \vee X = y \wedge X \text{ Midpoint } B'' A$ 
    by (simp add: P31 P32 l10-4-spec)
  have R2:  $X \neq y \wedge A' B' \text{ ReflectL } X y \vee X = y \wedge X \text{ Midpoint } B' A'$ 
    using P33A by linarith
  {
    assume  $X \neq y \wedge A B'' \text{ ReflectL } X y \wedge X \neq y \wedge A' B' \text{ ReflectL } X y$ 
    hence Per A' B' B''
      using ⟨Per B' A' A⟩ image-spec-preserves-per l10-4-spec by blast
  }
  {
    assume  $X \neq y \wedge A B'' \text{ ReflectL } X y \wedge X = y \wedge X \text{ Midpoint } B' A'$ 
    hence Per A' B' B'' by blast
  }
  {
    assume  $X = y \wedge X \text{ Midpoint } B'' A \wedge X \neq y \wedge A' B' \text{ ReflectL } X y$ 
    hence Per A' B' B'' by blast
  }
  {
    assume  $X = y \wedge X \text{ Midpoint } B'' A \wedge X = y \wedge X \text{ Midpoint } B' A'$ 
    hence Per A' B' B''
      using P32 by blast
  }
  thus ?thesis using R1 R2
    using ⟨ $X \neq y \wedge A B'' \text{ ReflectL } X y \wedge X \neq y \wedge A' B' \text{ ReflectL } X y$ ⟩
    ⇒ Per A' B' B''
    by auto
qed
have A' B' OS A B''
proof -
  {
    assume S1:  $X y \text{ Perp } A B''$ 
    have Coplanar A y A' X
      by (metis P28A P29 col-one-side coplanar-perm-16 ncop-distincts
        os--coplanar)
    have A' B' OS A y
      using P29 by blast
    hence Coplanar A' B' A y
      using os--coplanar by auto
    hence Coplanar A y B' X
      using P12 P28A col2-cop--cop col-trivial-2 ncoplanar-perm-16 by blast
    have S2:  $\neg \text{Col } A X y$ 
      using Col-perm P34A S1 local.image-id perp-distinct by blast
    have A' B' Par A B''
    proof -
      have Coplanar X y A' A
        using ⟨Coplanar A y A' X⟩ ncoplanar-perm-21 by blast
      moreover have Coplanar X y A' B''
      proof -
        have Coplanar A X y A'
          using ⟨Coplanar X y A' A⟩ ncoplanar-perm-9 by blast
        moreover have Coplanar A X y B''
          using Coplanar-def S1 perp-inter-exists by blast
        ultimately show ?thesis
          using S2 coplanar-trans-1 by auto
      qed
      moreover have Coplanar X y B' A
      proof -
        have  $\neg \text{Col } A X y$ 
          by (simp add: S2)
        moreover have Coplanar A X y B'
          using ⟨Coplanar A y B' X⟩ ncoplanar-perm-3 by blast
        moreover have Coplanar A X y B''

```

```

    using Coplanar-def S1 perp-inter-exists by blast
    ultimately show ?thesis
    using ncoplanar-perm-18 by blast
qed
moreover have Coplanar X y B' B''
proof -
  have  $\neg \text{Col } A \ X \ y$ 
  by (simp add: S2)
  moreover have Coplanar A X y B'
  using  $\langle \text{Coplanar } X \ y \ B' \ A \rangle$  ncoplanar-perm-9 by blast
  moreover have Coplanar A X y B''
  using Coplanar-def S1 perp-inter-exists by blast
  ultimately show ?thesis
  using coplanar-trans-1 by blast
qed
ultimately show ?thesis using l12-9
using P29 Perp-cases S1 by blast
qed
have A' B' OS A B''
proof -
  {
    assume A' B' ParStrict A B''
    have A' B' OS A B'' using l12-6
    using  $\langle A' \ B' \ \text{ParStrict } A \ B'' \rangle$  by blast
  }
  {
    assume  $A' \neq B' \wedge A \neq B'' \wedge \text{Col } A' \ A \ B'' \wedge \text{Col } B' \ A \ B''$ 
    have A' B' OS A B''
    using P26  $\langle A' \ B' \ \text{Par } A \ B'' \rangle$ 
     $\langle A' \ B' \ \text{ParStrict } A \ B'' \implies A' \ B' \ \text{OS } A \ B'' \rangle$ 
    col-trivial-3 par-not-col-strict by blast
  }
  thus ?thesis
  using Par-def  $\langle A' \ B' \ \text{Par } A \ B'' \rangle$ 
   $\langle A' \ B' \ \text{ParStrict } A \ B'' \implies A' \ B' \ \text{OS } A \ B'' \rangle$ 
  by auto
qed
}
{
  assume A = B''
  hence A' B' OS A B''
  using P12 P25  $\langle \text{Per } A' \ B' \ B'' \rangle$  l8-2 l8-7 by blast
}
thus ?thesis
using P30  $\langle X \ y \ \text{Perp } A \ B'' \implies A' \ B' \ \text{OS } A \ B'' \rangle$  by blast
qed
have A' B' OS A B
proof -
  have A' B' TS A C
  proof -
    have  $\neg \text{Col } A \ A' \ B'$ 
    using Col-perm  $\langle \neg \text{Col } B' \ A' \ A \rangle$  by blast
    moreover have  $\neg \text{Col } C \ A' \ B'$ 
    by (metis P13 P14 P2A  $\langle \neg \text{Col } B' \ A' \ A \rangle$  col3 not-col-distincts
    not-col-permutation-3 not-col-permutation-4)
    moreover have  $\exists T. \text{Col } T \ A' \ B' \wedge \text{Bet } A \ T \ C$ 
    using P14 assms(3) midpoint-bet not-col-permutation-1 by blast
    ultimately show ?thesis
    by (simp add: TS-def)
  qed
  moreover have A' B' TS B C
  by (metis Col-cases P13 TS-def  $\langle \neg \text{Col } A' \ B' \ B \rangle$  assms(2)
  calculation midpoint-bet)
  ultimately show ?thesis
  using OS-def by blast
qed

```

```

have Col B B'' B'
proof -
  have Coplanar A' B B'' B'
  proof -
    have Coplanar A' B' B B''
    proof -
      have  $\neg$  Col A A' B'
      using Col-perm  $\langle \neg$  Col B' A' A  $\rangle$  by blast
    moreover have Coplanar A A' B' B
    using  $\langle$  A' B' OS A B  $\rangle$  ncoplanar-perm-8 os--coplanar by blast
    moreover have Coplanar A A' B' B''
    using  $\langle$  A' B' OS A B''  $\rangle$  ncoplanar-perm-8 os--coplanar by blast
    ultimately show ?thesis
    using coplanar-trans-1 by blast
  qed
  thus ?thesis
  using ncoplanar-perm-4 by blast
qed
moreover have A'  $\neq$  B'
  by (simp add: P12)
moreover have Per B B' A'
  by (simp add: P24 l8-2)
moreover have Per B'' B' A'
  using Per-cases  $\langle$  Per A' B' B''  $\rangle$  by auto
ultimately show ?thesis
  using cop-per2--col by blast
qed
have Cong B B' A'
  using P6 P7 cong-inner-transitivity by blast
have B = B''  $\vee$  B' Midpoint B B''
proof -
  have Col B B' B''
  using  $\langle$  Col B B'' B'  $\rangle$  not-col-permutation-5 by blast
  moreover have Cong B' B B' B''
  by (metis Cong-perm P35 P6 P7 cong-inner-transitivity)
  ultimately show ?thesis
  using l7-20 by simp
qed
{
  assume B = B''
  then obtain M where S1: Col X y M  $\wedge$  M Midpoint A B
  using P30 by blast
  hence R = M
  using assms(4) l7-17 by auto
  have A  $\neq$  B
  using assms(1) col-trivial-1 by auto
  have Col R A B
  by (simp add: assms(4) midpoint-col)
  have X  $\neq$  R
  using Midpoint-def P28  $\langle$  A' B' OS A B''  $\rangle$   $\langle$  B = B''  $\rangle$  assms(4)
  midpoint-col one-side-chara by auto
  hence  $\exists$  X Y. (R PerpAt X Y A B  $\wedge$  X Y Perp P Q  $\wedge$ 
  Coplanar A B C X  $\wedge$  Coplanar A B C Y)
  proof -
    have R PerpAt R X A B
    proof -
      have R X Perp A B
      using P30 S1  $\langle$  A  $\neq$  B  $\rangle$   $\langle$  B = B''  $\rangle$   $\langle$  R = M  $\rangle$   $\langle$  X  $\neq$  R  $\rangle$ 
      perp-col perp-left-comm by blast
    thus ?thesis
    using  $\langle$  Col R A B  $\rangle$  l8-14-2-1b-bis not-col-distincts by blast
  qed
  moreover have R X Perp P Q
  proof -
    have X R Perp P Q
    proof -

```

```

have X y Perp P Q
proof -
  have P Q Perp X y
    using P11 P13 P14 P29 P33A col-trivial-2 col-trivial-3
      perp-col4 by blast
  thus ?thesis
    using Perp-perm by blast
qed
moreover have Col X y R
  by (simp add: S1  $\langle R = M \rangle$ )
ultimately show ?thesis
  using  $\langle X \neq R \rangle$  perp-col by blast
qed
thus ?thesis
  using Perp-perm by blast
qed
moreover have Coplanar A B C R
  using  $\langle Col R A B \rangle$  ncop--ncols not-col-permutation-2 by blast
moreover have Coplanar A B C X
proof -
  have Col P Q X
    using P12 P13 P14 P28A col3 by blast
  moreover have  $\neg Col P Q C$ 
    by (simp add: P2A)
  moreover have Coplanar P Q C A
    using assms(3) coplanar-perm-19 midpoint--coplanar by blast
  moreover have Coplanar P Q C B
    using assms(2) midpoint-col ncop--ncols
      not-col-permutation-5 by blast
  moreover have Coplanar P Q C C
    using ncop-distincts by auto
  moreover have Coplanar P Q C X
    using calculation(1) ncop--ncols by blast
  ultimately show ?thesis
    using coplanar-pseudo-trans by blast
qed
ultimately show ?thesis by blast
qed
}
{
  assume B' Midpoint B B''
  have A' B' TS B B''
  proof -
    have  $\neg Col B A' B'$ 
      using Col-perm  $\langle \neg Col A' B' B \rangle$  by blast
    moreover have  $\neg Col B'' A' B'$ 
      using  $\langle A' B' OS A B'' \rangle$  col124--nos not-col-permutation-2 by blast
    moreover have  $\exists T. Col T A' B' \wedge Bet B T B''$ 
      using  $\langle B' Midpoint B B'' \rangle$  col-trivial-3 midpoint-bet by blast
    ultimately show ?thesis
      by (simp add: TS-def)
  qed
  have A' B' OS B B''
    using  $\langle A' B' OS A B'' \rangle$   $\langle A' B' OS A B \rangle$  one-side-symmetry
      one-side-transitivity by blast
  have  $\neg A' B' OS B B''$ 
    using  $\langle A' B' TS B B'' \rangle$  l9-9-bis by blast
  hence False
    by (simp add:  $\langle A' B' OS B B'' \rangle$ )
  hence  $\exists X Y. (R PerpAt X Y A B \wedge X Y Perp P Q \wedge$ 
    Coplanar A B C X  $\wedge$  Coplanar A B C Y)
    by auto
}
thus ?thesis
  using  $\langle B = B'' \implies \exists X Y. R PerpAt X Y A B \wedge X Y Perp P Q \wedge$ 
    Coplanar A B C X  $\wedge$  Coplanar A B C Y

```

$\langle B = B'' \vee B' \text{ Midpoint } B B'' \rangle$ **by** blast
qed

lemma l13-1:
assumes $\neg \text{Col } A B C$ **and**
 $P \text{ Midpoint } B C$ **and**
 $Q \text{ Midpoint } A C$ **and**
 $R \text{ Midpoint } A B$
shows
 $\exists X Y. (R \text{ PerpAt } X Y A B \wedge X Y \text{ Perp } P Q)$
proof –
obtain $X Y$ **where** $R \text{ PerpAt } X Y A B$ **and** $X Y \text{ Perp } P Q$ **and**
 $\text{Coplanar } A B C X$ **and** $\text{Coplanar } A B C Y$
using l13-1-aux assms(1) assms(2) assms(3) assms(4) **by** blast
thus ?thesis **by** blast
qed

lemma per-lt:
assumes $A \neq B$ **and**
 $C \neq B$ **and**
 $\text{Per } A B C$
shows $A B \text{ Lt } A C \wedge C B \text{ Lt } A C$
proof –
have $B A \text{ Lt } A C \wedge B C \text{ Lt } A C$
using assms(1) assms(2) assms(3) l11-46 **by** auto
thus ?thesis
using lt-left-comm **by** blast
qed

lemma cong-perp-conga:
assumes $\text{Cong } A B C B$ **and**
 $A C \text{ Perp } B P$
shows $A B P \text{ Cong } A C B P \wedge B P \text{ TS } A C$
proof –
have $P1: A \neq C$
using assms(2) perp-distinct **by** auto
have $P2: B \neq P$
using assms(2) perp-distinct **by** auto
have $P3: A \neq B$
by (metis P1 assms(1) cong-diff-3)
have $P4: C \neq B$
using P3 assms(1) cong-diff **by** blast
show ?thesis
proof cases
assume $P5: \text{Col } A B C$
have $P6: \neg \text{Col } B A P$
using P3 P5 assms(2) col-transitivity-1 not-col-permutation-4
not-col-permutation-5 perp-not-col2 **by** blast
have $\text{Per } P B A$
using P3 P5 Perp-perm assms(2) not-col-permutation-5 perp-col1 perp-per-1 **by** blast
hence $P8: \text{Per } A B P$
using Per-cases **by** blast
have $\text{Per } P B C$
using P3 P5 P8 col-per2--per l8-2 l8-5 **by** blast
hence $P10: \text{Per } C B P$
using Per-perm **by** blast
show ?thesis
proof –
have $A B P \text{ Cong } A C B P$
using P2 P3 P4 P8 P10 l11-16 **by** auto
moreover **have** $B P \text{ TS } A C$
by (metis Col-cases P1 P5 P6 assms(1) bet--ts between-cong
not-cong-2143 not-cong-4321 third-point)
ultimately **show** ?thesis
by simp
qed

```

next
  assume T1:  $\neg \text{Col } A \ B \ C$ 
  obtain T where T2:  $T \text{ PerpAt } A \ C \ B \ P$ 
    using assms(2) perp-inter-perp-in by blast
  hence T3:  $\text{Col } A \ C \ T \wedge \text{Col } B \ P \ T$ 
    using perp-in-col by auto
  have T4:  $B \neq T$ 
    using Col-perm T1 T3 by blast
  have T5:  $B \ T \text{ Perp } A \ C$ 
    using Perp-cases T3 T4 assms(2) perp-col1 by blast
  {
    assume T5-1:  $A = T$ 
    have B A Lt B C  $\wedge$  C A Lt B C
    proof -
      have B  $\neq$  A
        using P3 by auto
      moreover have C  $\neq$  A
        using P1 by auto
      moreover have Per B A C
        using T5 T5-1 perp-comm perp-per-1 by blast
      ultimately show ?thesis
        by (simp add: per-lt)
    qed
    hence False
      using Cong-perm assms(1) cong--nlt by blast
  }
  hence T6:  $A \neq T$  by auto
  {
    assume T6-1:  $C = T$ 
    have B C Lt B A  $\wedge$  A C Lt B A
    proof -
      have B  $\neq$  C
        using P4 by auto
      moreover have A  $\neq$  C
        by (simp add: P1)
      moreover have Per B C A
        using T5 T6-1 perp-left-comm perp-per-1 by blast
      ultimately show ?thesis
        by (simp add: per-lt)
    qed
    hence False
      using Cong-perm assms(1) cong--nlt by blast
  }
  hence T7:  $C \neq T$  by auto
  have T8:  $T \text{ PerpAt } B \ T \ T \ A$ 
    by (metis Perp-in-cases T2 T3 T4 T6 perp-in-col-perp-in)
  have T9:  $T \text{ PerpAt } B \ T \ T \ C$ 
    by (metis Col-cases T3 T7 T8 perp-in-col-perp-in)
  have T10:  $\text{Cong } T \ A \ T \ C \wedge T \ A \ B \text{ CongA } T \ C \ B \wedge T \ B \ A \text{ CongA } T \ B \ C$ 
  proof -
    have A T B CongA C T B
    proof -
      have Per A T B
        using T2 perp-in-per-1 by auto
      moreover have Per C T B
        using T2 perp-in-per-3 by auto
      ultimately show ?thesis
        by (simp add: T4 T6 T7 l11-16)
    qed
    moreover have Cong A B C B
      by (simp add: assms(1))
    moreover have Cong T B T B
      by (simp add: cong-reflexivity)
    moreover have T B Le A B
  proof -
    have Per B T A

```

```

    using T8 perp-in-per by auto
    hence B T Lt B A  $\wedge$  A T Lt B A
    using T4 T6 per-lt by blast
    thus ?thesis
    using Le-cases Lt-def by blast
qed
ultimately show ?thesis
    using l11-52 by blast
qed
show ?thesis
proof -
    have T11: A B P CongA C B P
    proof -
        have P B A CongA P B C
        using Col-cases P2 T10 T3 col-conga--conga by blast
        thus ?thesis
        using conga-comm by blast
    qed
    moreover have B P TS A C
    proof -
        have T12: A = C  $\vee$  T Midpoint A C
        using T10 T3 l7-20-bis not-col-permutation-5 by blast
        {
            assume T Midpoint A C
            hence B P TS A C
            by (metis (mono-tags, opaque-lifting) T2 T3 l8-14-2-1b l9-18
                 $\langle A = T \implies \text{False} \rangle$   $\langle C = T \implies \text{False} \rangle$ 
                col-trivial-1 midpoint-bet not-col-permutation-1)
        }
        thus ?thesis
        using P1 T12 by auto
    qed
    ultimately show ?thesis
    by simp
qed
qed
qed

lemma perp-per-bet:
  assumes  $\neg$  Col A B C and
    Per A B C and
    P PerpAt P B A C
  shows Bet A P C
proof -
  have A  $\neq$  C
  using assms(1) col-trivial-3 by auto
  thus ?thesis
  using assms(2) assms(3) l11-47 perp-in-left-comm by blast
qed

lemma ts-per-per-ts:
  assumes A B TS C D and
    Per B C A and
    Per B D A
  shows C D TS A B
proof -
  have P1:  $\neg$  Col C A B
  using TS-def assms(1) by blast
  have P2: A  $\neq$  B
  using P1 col-trivial-2 by auto
  obtain P where P3: Col P A B  $\wedge$  Bet C P D
  using TS-def assms(1) by blast
  have P4: C  $\neq$  D
  using assms(1) not-two-sides-id by auto
  show ?thesis

```

```

proof -
{
  assume Col A C D
  hence C = D
  by (metis assms(1) assms(2) assms(3) col-per2-cases col-permutation-2 not-col-distincts ts-distincts)
  hence False
  using P4 by auto
}
hence  $\neg$  Col A C D by auto
moreover have  $\neg$  Col B C D
  using assms(1) assms(2) assms(3) per2-preserves-diff ts-distincts by blast
moreover have  $\exists T. \text{Col } T C D \wedge \text{Bet } A T B$ 
proof -
  have Col P C D
    using Col-def Col-perm P3 by blast
  moreover have Bet A P B
  proof -
    have  $\exists X. \text{Col } A B X \wedge A B \text{Perp } C X$ 
      using Col-perm P1 l8-18-existence by blast
    then obtain C' where P5:  $\text{Col } A B C' \wedge A B \text{Perp } C C'$  by blast
    have  $\exists X. \text{Col } A B X \wedge A B \text{Perp } D X$ 
      by (metis (no-types) Col-perm TS-def assms(1) l8-18-existence)
    then obtain D' where P6:  $\text{Col } A B D' \wedge A B \text{Perp } D D'$  by blast
    have P7:  $A \neq C'$ 
      using P5 assms(2) l8-7 perp-not-eq-2 perp-per-1 by blast
    have P8:  $A \neq D'$ 
      using P6 assms(3) l8-7 perp-not-eq-2 perp-per-1 by blast
    have P9: Bet A C' B
    proof -
      have  $\neg \text{Col } A C B$ 
        using Col-cases P1 by blast
      moreover have Per A C B
        by (simp add: assms(2) l8-2)
      moreover have C' PerpAt C' C A B
        using P5 Perp-in-perm l8-15-1 by blast
      ultimately show ?thesis
        using perp-per-bet by blast
    qed
    have P10: Bet A D' B
  proof -
    have  $\neg \text{Col } A D B$ 
      using P6 col-permutation-5 perp-not-col2 by blast
    moreover have Per A D B
      by (simp add: assms(3) l8-2)
    moreover have D' PerpAt D' D A B
      using P6 Perp-in-perm l8-15-1 by blast
    ultimately show ?thesis
      using perp-per-bet by blast
    qed
  show ?thesis
  proof cases
    assume P = C'
    thus ?thesis
      by (simp add: P9)
  next
    assume P  $\neq$  C'
    show ?thesis
    proof cases
      assume P = D'
      thus ?thesis
        by (simp add: P10)
    next
      assume P  $\neq$  D'
      show ?thesis
    proof cases
      assume A = P

```



```

    thus ?thesis
      by (simp add: between-trivial2)
next
assume  $A \neq P$ 
show ?thesis
proof cases
  assume  $B = P$ 
  thus ?thesis
    using between-trivial by auto
next
assume  $B \neq P$ 
have  $Bet\ C'\ P\ D'$ 
proof -
  have  $Bet\ C\ P\ D$ 
    by (simp add: P3)
  moreover have  $P \neq C'$ 
    by (simp add:  $\langle P \neq C' \rangle$ )
  moreover have  $P \neq D'$ 
    by (simp add:  $\langle P \neq D' \rangle$ )
  moreover have  $Col\ C'\ P\ D'$ 
    by (meson P2 P3 P5 P6 col3 col-permutation-2)
  moreover have  $Per\ P\ C'\ C$ 
    using P3 P5 l8-16-1 l8-2 not-col-permutation-3 not-col-permutation-4 by blast
  moreover have  $Per\ P\ D'\ D$ 
    by (metis P3 P6 calculation(3) not-col-permutation-2 perp-col2 perp-per-1)
  ultimately show ?thesis
    using per13-preserves-bet by blast
qed
thus ?thesis
  using P10 P9 bet3--bet by blast
qed
qed
qed
qed
qed
ultimately show ?thesis
  by auto
qed
ultimately show ?thesis
  by (simp add: TS-def)
qed
qed

```

lemma l13-2-1:

```

assumes  $A\ B\ TS\ C\ D$  and
   $Per\ B\ C\ A$  and
   $Per\ B\ D\ A$  and
   $Col\ C\ D\ E$  and
   $A\ E\ Perp\ C\ D$  and
   $C\ A\ B\ CongA\ D\ A\ B$ 
shows  $B\ A\ C\ CongA\ D\ A\ E \wedge B\ A\ D\ CongA\ C\ A\ E \wedge Bet\ C\ E\ D$ 

```

proof -

```

have P1:  $\neg\ Col\ C\ A\ B$ 
  using TS-def assms(1) by auto
have P2:  $A \neq C$ 
  using P1 col-trivial-1 by blast
have P3:  $A \neq B$ 
  using P1 col-trivial-2 by auto
have P4:  $A \neq D$ 
  using assms(1) ts-distincts by auto
have P5:  $Cong\ B\ C\ B\ D \wedge Cong\ A\ C\ A\ D \wedge C\ B\ A\ CongA\ D\ B\ A$ 
proof -
  have  $\neg\ Col\ B\ A\ C$ 
    by (simp add: P1 not-col-permutation-3)
  moreover have  $A\ C\ B\ CongA\ A\ D\ B$ 
    using assms(1) assms(2) assms(3) l11-16 l8-2 ts-distincts by blast

```

```

moreover have  $B\ A\ C\ CongA\ B\ A\ D$ 
  by (simp add: assms(6) conga-comm)
moreover have  $Cong\ B\ A\ B\ A$ 
  by (simp add: cong-reflexivity)
ultimately show ?thesis
  using l11-50-2 by blast
qed
hence  $P6: C\ D\ Perp\ A\ B$ 
  using assms(1) assms(6) cong-conga-perp not-cong-2143 by blast
hence  $P7: C\ D\ TS\ A\ B$ 
  by (simp add: assms(1) assms(2) assms(3) ts-per-per-ts)
obtain  $T1$  where  $P8: Col\ T1\ C\ D \wedge Bet\ A\ T1\ B$ 
  using P7 TS-def by auto
obtain  $T$  where  $P9: Col\ T\ A\ B \wedge Bet\ C\ T\ D$ 
  using TS-def assms(1) by blast
have  $P10: T1 = T$ 
  by (metis (no-types) Col-def P1 P3 P8 P9 between-equality-2 between-trivial2 l6-16-1)
have  $P11: T = E$ 
proof –
  have  $\neg Col\ A\ B\ C$ 
    using Col-perm P1 by blast
  moreover have  $C \neq D$ 
    using assms(1) ts-distincts by blast
  moreover have  $Col\ A\ B\ T$ 
    using Col-cases P9 by auto
  moreover have  $Col\ A\ B\ E$ 
    by (metis P7 Perp-cases P6 assms(1) assms(5) col-perp2-ncol-col
      col-trivial-3 not-col-permutation-3 one-side-not-col123 os-ts1324--os ts-ts-os)
  moreover have  $Col\ C\ D\ T$ 
    using NCol-cases P9 bet-col by blast
  moreover have  $Col\ C\ D\ E$ 
    by (simp add: assms(4))
  ultimately show ?thesis
    using l6-21 by blast
qed
show ?thesis
proof –
  have  $B\ A\ C\ CongA\ D\ A\ E$ 
  proof –
    have  $A\ Out\ C\ C$ 
      using P2 out-trivial by auto
    moreover have  $A\ Out\ B\ B$ 
      using P3 out-trivial by auto
    moreover have  $A\ Out\ D\ D$ 
      using P4 out-trivial by auto
    moreover have  $A\ Out\ E\ B$ 
      by (metis P10 P11 P7 P8 TS-def bet-out)
    ultimately show ?thesis
      by (meson assms(6) conga-comm conga-right-comm l11-10)
  qed
  moreover have  $B\ A\ D\ CongA\ C\ A\ E$ 
  proof –
    have  $C\ A\ E\ CongA\ D\ A\ B$ 
      by (meson Perp-cases P5 assms(5) assms(6) calculation cong-perp-conga
        conga-right-comm conga-trans not-cong-2143 not-conga-sym)
    hence  $C\ A\ E\ CongA\ B\ A\ D$ 
      by (simp add: conga-right-comm)
    thus ?thesis
      by (simp add: conga-sym)
  qed
  moreover have  $Bet\ C\ E\ D$ 
    using P11 P9 by auto
  ultimately show ?thesis by simp
qed
qed

```

lemma *triangle-mid-par-lem*:
assumes $\neg \text{Col } A \ B \ C$ **and**
 $P \text{ Midpoint } B \ C$ **and**
 $Q \text{ Midpoint } A \ C$
shows $A \ B \text{ Par } P \ Q$
proof –
obtain R **where** $R \text{ Midpoint } A \ B$
using *midpoint-existence* **by** *auto*
then obtain $X \ Y$ **where** $R \text{ PerpAt } X \ Y \ A \ B$ **and** $X \ Y \text{ Perp } P \ Q$ **and**
 $\text{Coplanar } A \ B \ C \ X$ **and** $\text{Coplanar } A \ B \ C \ Y$
using *l13-1-aux* *assms(1)* *assms(2)* *assms(3)* **by** *blast*
have $\text{Coplanar } A \ B \ C \ A$
using *ncop-distincts* **by** *auto*
have $\text{Coplanar } A \ B \ C \ B$
using *ncop-distincts* **by** *auto*
have $\text{Coplanar } A \ B \ C \ P$
using *assms(2)* *coplanar-perm-21* *midpoint--coplanar* **by** *blast*
have $\text{Coplanar } A \ B \ C \ Q$
using *assms(3)* *coplanar-perm-11* *midpoint--coplanar* **by** *blast*
have $\text{Coplanar } X \ Y \ A \ P$
using $\langle \text{Coplanar } A \ B \ C \ A \rangle \langle \text{Coplanar } A \ B \ C \ P \rangle \langle \text{Coplanar } A \ B \ C \ X \rangle$
 $\langle \text{Coplanar } A \ B \ C \ Y \rangle$ *assms(1)* *coplanar-pseudo-trans* **by** *blast*
moreover have $\text{Coplanar } X \ Y \ A \ Q$
using $\langle \text{Coplanar } A \ B \ C \ A \rangle \langle \text{Coplanar } A \ B \ C \ Q \rangle \langle \text{Coplanar } A \ B \ C \ X \rangle$
 $\langle \text{Coplanar } A \ B \ C \ Y \rangle$ *assms(1)* *coplanar-pseudo-trans* **by** *blast*
moreover have $\text{Coplanar } X \ Y \ B \ P$
using $\langle \text{Coplanar } A \ B \ C \ B \rangle \langle \text{Coplanar } A \ B \ C \ P \rangle \langle \text{Coplanar } A \ B \ C \ X \rangle$
 $\langle \text{Coplanar } A \ B \ C \ Y \rangle$ *assms(1)* *coplanar-pseudo-trans* **by** *blast*
moreover have $\text{Coplanar } X \ Y \ B \ Q$
using $\langle \text{Coplanar } A \ B \ C \ B \rangle \langle \text{Coplanar } A \ B \ C \ Q \rangle \langle \text{Coplanar } A \ B \ C \ X \rangle$
 $\langle \text{Coplanar } A \ B \ C \ Y \rangle$ *assms(1)* *coplanar-pseudo-trans* **by** *presburger*
moreover have $\text{Col } X \ Y \ R$ **and** $\text{Col } A \ B \ R$
using *perp-in-col* $\langle R \text{ PerpAt } X \ Y \ A \ B \rangle$ *perp-in-col* **by** *auto*
hence $\text{Col } X \ R \ Y$
using *Col-perm* **by** *blast*
hence $X \ Y \text{ Perp } A \ B$
using *perp-col* $\langle R \text{ PerpAt } X \ Y \ A \ B \rangle$ *perp-in-perp* **by** *auto*
ultimately show *?thesis*
by (*meson* *Perp-cases* $\langle X \ Y \text{ Perp } P \ Q \rangle$ *l12-9*)
qed

lemma *triangle-mid-par*:
assumes $\neg \text{Col } A \ B \ C$ **and**
 $P \text{ Midpoint } B \ C$ **and**
 $Q \text{ Midpoint } A \ C$
shows $A \ B \text{ ParStrict } Q \ P$
proof –
have $A \ B \text{ Par } P \ Q$
using *assms(1)* *assms(2)* *assms(3)* *triangle-mid-par-lem* **by** *blast*
hence $A \ B \text{ Par } Q \ P$
using *Par-cases* **by** *blast*
thus *?thesis*
by (*metis* *assms(1)* *assms(3)* *col-transitivity-2* *midpoint-col* *midpoint-distinct-3*
not-col-distincts *par-not-col-strict*)
qed

lemma *cop4-perp-in2--col*:
assumes $\text{Coplanar } X \ Y \ A \ A'$ **and**
 $\text{Coplanar } X \ Y \ A \ B'$ **and**
 $\text{Coplanar } X \ Y \ B \ A'$ **and**
 $\text{Coplanar } X \ Y \ B \ B'$ **and**
 $P \text{ PerpAt } A \ B \ X \ Y$ **and**
 $P \text{ PerpAt } A' \ B' \ X \ Y$
shows $\text{Col } A \ B \ A'$
proof –
have $P1: \text{Col } A \ B \ P \wedge \text{Col } X \ Y \ P$

```

    using assms(5) perp-in-col by auto
show ?thesis
proof cases
  assume P2:  $A = P$ 
  show ?thesis
  proof cases
    assume P3:  $P = X$ 
    have Col B A' P
    proof -
      have Coplanar Y B A' P
      using P3 assms(3) ncoplanar-perm-18 by blast
      moreover have  $Y \neq P$ 
      using P3 assms(6) perp-in-distinct by blast
      moreover have Per B P Y
      using assms(5) perp-in-per-4 by auto
      moreover have Per A' P Y
      using assms(6) perp-in-per-2 by auto
      ultimately show ?thesis
      using cop-per2--col by auto
    qed
  thus ?thesis
  using Col-perm P2 by blast
next
  assume P4:  $P \neq X$ 
  have Col B A' P
  proof -
    have Coplanar X B A' P
    by (metis P1 assms(3) assms(6) col2-cop--cop col-trivial-3
      ncoplanar-perm-9 perp-in-distinct)
    moreover have Per B P X
    using assms(5) perp-in-per-3 by auto
    moreover have Per A' P X
    using assms(6) perp-in-per-1 by auto
    ultimately show ?thesis
    using cop-per2--col P4 by auto
  qed
  thus ?thesis
  using Col-perm P2 by blast
qed
next
  assume P5:  $A \neq P$ 
  have P6: Per A P Y
  using assms(5) perp-in-per-2 by auto
show ?thesis
proof cases
  assume P7:  $P = A'$ 
  have P8: Per B' P Y
  using assms(6) perp-in-per-4 by auto
  have Col A B' P
  proof -
    have Coplanar Y A B' P
    using assms(2) by (metis P1 assms(6) col-transitivity-2 coplanar-trans-1
      ncop--ncols perp-in-distinct)
    thus ?thesis using P6 P8 cop-per2--col
    by (metis assms(2) assms(5) assms(6) col-permutation-4 coplanar-perm-5
      perp-in-distinct perp-in-per-1 perp-in-per-3)
  qed
  thus ?thesis
  using P1 P7 by auto
next
  assume T1:  $P \neq A'$ 
  show ?thesis
  proof cases
    assume T2:  $Y = P$ 
    {
      assume R1: Coplanar X P A A'  $\wedge$  P PerpAt A B X P  $\wedge$  P PerpAt A' B' X P  $\wedge$  A  $\neq$  P
    }
  }

```

```

    hence  $R2: \text{Per } A \ P \ X$ 
    using perp-in-per-1 by auto
    have  $\text{Per } A' \ P \ X$ 
    using R1 perp-in-per-1 by auto
    hence  $\text{Col } A \ B \ A'$ 
    by (metis R1 R2 PerpAt-def col-permutation-3 col-transitivity-2
        cop-per2--col ncoplanar-perm-5)
  }
  thus ?thesis
  using  $P5 \ T1 \ T2 \ \text{assms}(1) \ \text{assms}(2) \ \text{assms}(3) \ \text{assms}(4) \ \text{assms}(5)$ 
     $\text{assms}(6)$  by blast
next
  assume  $P10: Y \neq P$ 
  have  $\text{Col } A \ A' \ P$ 
  proof -
    have Coplanar  $Y \ A \ A' \ P$ 
    by (metis P1 assms(1) assms(6) col2-cop--cop col-trivial-2
        ncoplanar-perm-9 perp-in-distinct)
    moreover have  $\text{Per } A \ P \ Y$ 
    by (simp add: P6)
    moreover have  $\text{Per } A' \ P \ Y$ 
    using  $\text{assms}(6)$  perp-in-per-2 by auto
    ultimately show ?thesis
    using cop-per2--col P10 by auto
  qed
  thus ?thesis
  using  $P1 \ P5 \ \text{col2--eq} \ \text{col-permutation-4}$  by blast
qed
qed
qed
qed
qed

lemma l13-2:
  assumes  $A \ B \ TS \ C \ D$  and
     $\text{Per } B \ C \ A$  and
     $\text{Per } B \ D \ A$  and
     $\text{Col } C \ D \ E$  and
     $A \ E \ \text{Perp } C \ D$ 
  shows  $B \ A \ C \ \text{CongA } D \ A \ E \wedge B \ A \ D \ \text{CongA } C \ A \ E \wedge \text{Bet } C \ E \ D$ 
  proof -
    have  $P2: \neg \text{Col } C \ A \ B$ 
    using TS-def assms(1) by auto
    have  $P3: C \neq D$ 
    using  $\text{assms}(1)$  not-two-sides-id by blast
    have  $P4: \exists \ C'. B \ A \ C \ \text{CongA } D \ A \ C' \wedge D \ A \ OS \ C' \ B$ 
    proof -
      have  $\neg \text{Col } B \ A \ C$ 
      using Col-cases P2 by auto
      moreover have  $\neg \text{Col } D \ A \ B$ 
      using TS-def assms(1) by blast
      ultimately show ?thesis
      by (simp add: angle-construction-1)
    qed
  then obtain  $E'$  where  $P5: B \ A \ C \ \text{CongA } D \ A \ E' \wedge D \ A \ OS \ E' \ B$  by blast
  have  $P6: A \neq B$ 
  using  $P2$  not-col-distincts by blast
  have  $P7: A \neq C$ 
  using  $P2$  not-col-distincts by blast
  have  $P8: A \neq D$ 
  using  $P5$  os-distincts by blast
  have  $P9: ((A \ B \ TS \ C \ E' \wedge A \ E' \ TS \ D \ B) \vee$ 
     $(A \ B \ OS \ C \ E' \wedge A \ E' \ OS \ D \ B \wedge C \ A \ B \ \text{CongA } D \ A \ E' \wedge B \ A \ E' \ \text{CongA } E' \ A \ B))$ 
     $\longrightarrow C \ A \ E' \ \text{CongA } D \ A \ B$ 
  by (metis P5 P6 conga-diff56 conga-left-comm conga-pseudo-refl l11-22)
  have  $P10: C \ D \ TS \ A \ B$ 
  by (simp add: assms(1) assms(2) assms(3) ts-per-per-ts)

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have P11:  $\neg \text{Col } A \ C \ D$ 
  using P10 TS-def by auto
obtain T where P12:  $\text{Col } T \ A \ B \wedge \text{Bet } C \ T \ D$ 
  using TS-def assms(1) by blast
obtain T2 where P13:  $\text{Col } T2 \ C \ D \wedge \text{Bet } A \ T2 \ B$ 
  using P10 TS-def by auto
hence P14:  $T = T2$ 
  by (metis Col-def Col-perm P12 P2 P3 P6 l6-16-1)
have P15:  $B \text{ InAngle } D \ A \ C$ 
  using P10 assms(1) l11-24 ts2--inangle by blast
have P16:  $C \ A \ B \text{ LeA } C \ A \ D$ 
  by (simp add: P10 assms(1) inangle--lea ts2--inangle)
have P17:  $E' \text{ InAngle } D \ A \ C$ 
proof -
  have D A E' LeA D A C
    using P16 P5 P7 P8 conga-left-comm conga-pseudo-refl l11-30 by presburger
  moreover have D A OS C E'
    by (meson P11 P15 P5 col124--nos in-angle-one-side invert-one-side
      not-col-permutation-2 one-side-symmetry one-side-transitivity)
  ultimately show ?thesis
    by (simp add: lea-in-angle)
qed
obtain E'' where P18:  $\text{Bet } D \ E'' \ C \wedge (E'' = A \vee A \text{ Out } E'' \ E')$ 
  using InAngle-def P17 by auto
{
  assume E'' = A
  hence B A C CongA D A E  $\wedge$  B A D CongA C A E  $\wedge$  Bet C E D
    using Col-def P11 P18 by auto
}
{
  assume P19:  $A \text{ Out } E'' \ E'$ 
  hence P20:  $B \ A \ C \text{ CongA } D \ A \ E''$ 
    by (meson OS-def P5 out2--conga Tarski-neutral-dimensionless-axioms
      col-one-side-out col-trivial-2 l9-18-R1 not-conga one-side-reflexivity)
  have P21:  $A \neq T$ 
    using P11 P13 P14 by auto
  have B A C CongA D A E  $\wedge$  B A D CongA C A E  $\wedge$  Bet C E D
  proof cases
    assume P22:  $E'' = T$ 
    have P23:  $C \ A \ B \text{ CongA } D \ A \ B$ 
    proof -
      have C A B CongA D A T
        using P22 P20 conga-left-comm by blast
      moreover have A Out C C
        using P7 out-trivial by presburger
      moreover have A Out B B
        using P6 out-trivial by auto
      moreover have A Out D D
        using P8 out-trivial by auto
      moreover have A Out B T
        using Out-def P13 P14 P6 P21 by blast
      ultimately show ?thesis
        using l11-10 by blast
    qed
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) assms(5) l13-2-1 by blast
next
  assume P23A:  $E'' \neq T$ 
  have P24:  $D \neq E''$ 
    using P2 P20 col-trivial-3 ncol-conga-ncol not-col-permutation-3 by blast
  {
    assume P24A:  $C = E''$ 
    have P24B:  $C \ A \ OS \ B \ D$ 
      by (meson P10 assms(1) invert-one-side ts-ts-os)
    have P24C:  $A \text{ Out } B \ D$ 
  }
  proof -

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have C A B CongA C A D
  using P20 P24A conga-comm by blast
moreover have C A OS B D
  by (simp add: P24B)
ultimately show ?thesis
  using conga-os--out by blast
qed
hence False
  using Col-def P5 one-side-not-col124 out-col by blast
}
hence P25: C ≠ E'' by auto
have P26: A ≠ E''
  using P19 out-diff1 by auto
{
  assume Col E'' A B
  hence E'' = T
    by (metis P13 P14 P18 P2 P3 bet-col l6-21 not-col-permutation-3 not-col-permutation-4)
  hence False
    using P23A by auto
}
hence P27: ¬ Col E'' A B by auto
have (A B TS C E'' ∧ A E'' TS D B) ∨
(A B OS C E'' ∧ A E'' OS D B ∧ C A B CongA D A E'' ∧ B A E'' CongA E'' A B)
proof cases
  assume P27-0: A B OS C E''
  have A E'' OS D B
  proof -
    have P27-1: A E'' TS D C
      by (metis Col-def P10 P18 P24 TS-def P25 bet--ts invert-two-sides l6-16-1)
    moreover have A E'' TS B C
    proof -
      have A E'' TS T C
      proof -
        have ¬ Col T A E''
          by (metis NCol-cases P13 P14 P21 P27 bet-col col3 col-trivial-2)
        moreover have ¬ Col C A E''
          using P27-1 TS-def by auto
        moreover have ∃ T0. (Col T0 A E'' ∧ Bet T T0 C)
          by (meson P12 P18 P27-0 between-symmetry col-trivial-3 l5-3 one-side-chara)
        ultimately show ?thesis
          by (simp add: TS-def)
      qed
    qed
    moreover have A Out T B
      using Out-def P13 P14 P21 P6 by auto
    ultimately show ?thesis
      using col-trivial-1 l9-5 by blast
  qed
  ultimately show ?thesis
    using OS-def by auto
qed
thus ?thesis
  using P20 P27-0 conga-distinct conga-left-comm conga-pseudo-refl by blast
next
assume P27-2: ¬ A B OS C E''
show ?thesis
proof -
  have P27-3: A B TS C E''
    using P18 P2 P27-2 P27 assms(1) bet-cop--cop between-symmetry
    cop-nos--ts ts--coplanar by blast
  moreover have A E'' TS D B
  proof -
    have P27-3: A B OS D E''
      using P18 bet-ts--os between-symmetry calculation one-side-symmetry by blast
    have P27-4: A E'' TS T D
    proof -
      have ¬ Col T A E''

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    by (metis NCol-cases P13 P14 P21 P27 bet-col col3 col-trivial-2)
  moreover have  $\neg \text{Col } D \ A \ E''$ 
    by (metis P11 P18 P24 bet-col l6-16-1 not-col-permutation-2)
  moreover have  $\exists \ T0. (\text{Col } T0 \ A \ E'' \wedge \text{Bet } T \ T0 \ D)$ 
    by (meson Bet-perm P12 P18 P27-3 bet-col1 bet-out--bet
        between-exchange3 col-trivial-3 not-bet-out one-side-chara)
  ultimately show ?thesis
    by (simp add: TS-def)
qed
have  $A \ E'' \ TS \ B \ D$ 
proof -
  have  $A \ E'' \ TS \ T \ D$ 
    using P27-4 by simp
  moreover have  $\text{Col } A \ A \ E''$ 
    using col-trivial-1 by auto
  moreover have  $A \ \text{Out } T \ B$ 
    using P13 P14 P21 bet-out by auto
  ultimately show ?thesis
    using l9-5 by blast
qed
thus ?thesis
  by (simp add: l9-2)
qed
ultimately show ?thesis
  by simp
qed
qed
hence P28:  $C \ A \ E'' \ \text{CongA } D \ A \ B$  using l11-22
  by (metis P20 P26 P6 conga-left-comm conga-pseudo-refl)
obtain  $C'$  where P29:  $\text{Bet } B \ C \ C' \wedge \text{Cong } C \ C' \ B \ C$ 
  using segment-construction by blast
obtain  $D'$  where P30:  $\text{Bet } B \ D \ D' \wedge \text{Cong } D \ D' \ B \ D$ 
  using segment-construction by blast
have P31:  $B \ A \ D \ \text{Cong3 } D' \ A \ D$ 
proof -
  have  $\text{Per } A \ D \ B$ 
    by (simp add: assms(3) l8-2)
  then obtain  $D''$  where P31-2:  $D \ \text{Midpoint } B \ D'' \wedge \text{Cong } A \ B \ A \ D''$ 
    using Per-def by auto
  have  $D \ \text{Midpoint } B \ D'$ 
    using Cong-perm Midpoint-def P30 by blast
  hence  $D' = D''$ 
    using P31-2 symmetric-point-uniqueness by auto
  thus ?thesis
    using Cong3-def Cong-perm P30 P31-2 cong-reflexivity by blast
qed
hence P32:  $B \ A \ D \ \text{CongA } D' \ A \ D$ 
  using P6 P8 cong3-conga by auto
have  $B \ A \ C \ \text{Cong3 } C' \ A \ C$ 
proof -
  obtain  $C''$  where P33-1:  $C \ \text{Midpoint } B \ C'' \wedge \text{Cong } A \ B \ A \ C''$ 
    using Per-def assms(2) l8-2 by blast
  have  $C \ \text{Midpoint } B \ C'$ 
    using Cong-perm Midpoint-def P29 by blast
  hence  $C' = C''$ 
    using P33-1 symmetric-point-uniqueness by auto
  thus ?thesis
    using Cong3-def Cong-perm P29 P33-1 cong-reflexivity by blast
qed
hence P34:  $B \ A \ C \ \text{CongA } C' \ A \ C$ 
  using P6 P7 cong3-conga by auto
have P35:  $E'' \ A \ C' \ \text{CongA } D' \ A \ E''$ 
proof -
  have  $(A \ C \ TS \ E'' \ C' \wedge A \ D \ TS \ D' \ E'') \vee (A \ C \ OS \ E'' \ C' \wedge A \ D \ OS \ D' \ E'')$ 
  proof -
    have P35-1:  $C \ A \ OS \ D \ E''$ 

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    by (metis Col-perm P11 P18 P25 bet-out between-symmetry
        one-side-symmetry out-one-side)
  have P35-2: C A OS B D
    using P10 assms(1) one-side-symmetry ts-ts-os by blast
  have P35-3: C A TS B C'
    by (metis P2 P29 bet--ts cong-diff-4 not-col-distincts)
  have P35-4: C A OS B E''
    using P35-1 P35-2 one-side-transitivity by blast
  have P35-5: D A OS C E''
    by (metis Col-perm P18 P24 P35-1 bet2--out l5-1 one-side-not-col123 out-one-side)
  have P35-6: D A OS B C
    by (simp add: P10 assms(1) invert-two-sides l9-2 one-side-symmetry ts-ts-os)
  have P35-7: D A TS B D'
    by (metis P30 TS-def assms(1) bet--ts cong-diff-3 ts-distincts)
  have P35-8: D A OS B E''
    using P35-5 P35-6 one-side-transitivity by blast
  have P35-9: A C TS E'' C'
    using P35-3 P35-4 invert-two-sides l9-8-2 by blast
  have A D TS D' E''
    using P35-7 P35-8 invert-two-sides l9-2 l9-8-2 by blast
  thus ?thesis
    using P35-9 by simp
qed
moreover have E'' A C CongA D' A D
proof -
  have E'' A C CongA B A D
    by (simp add: P28 conga-comm)
  moreover have B A D CongA D' A D
    by (simp add: P32)
  ultimately show ?thesis
    using conga-trans by blast
qed
moreover have C A C' CongA D A E''
proof -
  have D A E'' CongA C A C'
  proof -
    have D A E'' CongA B A C
      by (simp add: P20 conga-sym)
    moreover have B A C CongA C A C'
      by (simp add: P34 conga-right-comm)
    ultimately show ?thesis
      using conga-trans by blast
  qed
  thus ?thesis
    using not-conga-sym by blast
qed
ultimately show ?thesis
  using l11-22 by auto
qed
have P36: D' ≠ B
  using P30 assms(1) bet-neq32--neq ts-distincts by blast
have P37: C' ≠ B
  using P29 assms(1) bet-neq32--neq ts-distincts by blast
hence P38: ¬ Col C' D' B
  by (metis Col-def P10 P29 P30 P36 TS-def col-transitivity-2)
have P39: C' D' ParStrict C D
proof -
  have ¬ Col C' D' B
    by (simp add: P38)
  moreover have D Midpoint D' B
    using P30 l7-2 midpoint-def not-cong-3412 by blast
  moreover have C Midpoint C' B
    using P29 l7-2 midpoint-def not-cong-3412 by blast
  ultimately show ?thesis
    using triangle-mid-par by auto
qed

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have P40: A E'' TS C D
  by (metis Bet-perm Col-def P10 P18 P24 TS-def ⟨C = E'' ⟹ False⟩
      bet--ts col-transitivity-2 invert-two-sides)
have P41: B A TS C D
  by (simp add: assms(1) invert-two-sides)
have P42: A B OS C C'
proof -
  have ¬ Col A B C
    by (simp add: P2 not-col-permutation-1)
  moreover have Col A B B
    by (simp add: col-trivial-2)
  moreover have B Out C C'
    by (metis P29 P37 bet-out cong-identity)
  ultimately show ?thesis
    using out-one-side-1 by blast
qed
have P43: A B OS D D' using out-one-side-1
  by (metis Col-perm P30 TS-def assms(1) bet-out col-trivial-1)
hence P44: A B OS D D' using invert-two-sides by blast
have P45: A B TS C' D
  using P42 assms(1) l9-8-2 by blast
hence P46: A B TS C' D'
  using P44 l9-2 l9-8-2 by blast
have P47: C' D' Perp A E''
proof -
  have A E'' TS C' D'
  proof -
    have A Out C' D' ∨ E'' A TS C' D'
    proof -
      have E'' A C' CongA E'' A D'
        by (simp add: P35 conga-right-comm)
      moreover have Coplanar E'' A C' D'
      proof -
        have f1: B A OS C C'
          by (metis P42 invert-one-side)
        have f2: Coplanar B A C' C
          by (meson P42 ncoplanar-perm-7 os--coplanar)
        have f3: Coplanar D' A C' D
          by (meson P44 P46 col124--nos coplanar-trans-1 invert-one-side
              ncoplanar-perm-7 os--coplanar ts--coplanar)
        have Coplanar D' A C' C
          using f2 f1 by (meson P46 col124--nos coplanar-trans-1
              ncoplanar-perm-6 ncoplanar-perm-8 ts--coplanar)
        thus ?thesis
          using f3 by (meson P18 bet-cop2--cop ncoplanar-perm-6
              ncoplanar-perm-7 ncoplanar-perm-8)
      qed
      ultimately show ?thesis using conga-cop--or-out-ts
        by simp
    qed
  qed
  thus ?thesis
    using P46 col-two-sides-bet invert-two-sides not-bet-and-out out-col by blast
qed
moreover have Cong C' A D' A
  using Cong3-def P31 ⟨B A C Cong3 C' A C⟩ cong-inner-transitivity by blast
moreover have C' A E'' CongA D' A E''
  by (simp add: P35 conga-left-comm)
ultimately show ?thesis
  by (simp add: cong-conga-perp)
qed
have T1: Cong A C' A D'
proof -
  have Cong A C' A B
    using Cong3-def Cong-perm ⟨B A C Cong3 C' A C⟩ by blast
  moreover have Cong A D' A B
    using Cong3-def P31 not-cong-4321 by blast

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ultimately show ?thesis
 using Cong-perm $\langle \text{Cong } A \ C' \ A \ B \rangle \langle \text{Cong } A \ D' \ A \ B \rangle$ cong-inner-transitivity by blast

qed

obtain R where T2: R Midpoint C' D'
 using midpoint-existence by auto
 have $\exists X \ Y. (R \text{ PerpAt } X \ Y \ C' \ D' \wedge X \ Y \text{ Perp } D \ C \wedge$
 Coplanar C' D' B X $\wedge$ Coplanar C' D' B Y)
 proof –
 have $\neg \text{Col } C' \ D' \ B$
 by (simp add: P38)
 moreover have D Midpoint D' B
 using P30 l7-2 midpoint-def not-cong-3412 by blast
 moreover have C Midpoint C' B
 using Cong-perm Mid-perm Midpoint-def P29 by blast
 moreover have R Midpoint C' D'
 by (simp add: T2)
 ultimately show ?thesis using l13-1-aux by blast

qed

then obtain X Y where T3: R PerpAt X Y C' D' $\wedge$ X Y Perp D C $\wedge$
 Coplanar C' D' B X $\wedge$ Coplanar C' D' B Y
 by blast
 hence $X \neq Y$
 using perp-not-eq-1 by blast
 have C D Perp A E''
 proof cases
 assume A = R
 hence W1: A PerpAt C' D' A E''
 using Col-def P47 T2 between-trivial2 l8-14-2-1b-bis midpoint-col by blast
 have Coplanar B C' D' E''
 proof –
 have $\neg \text{Col } B \ C \ D$
 using P10 TS-def by auto
 moreover have Coplanar B C D B
 using ncop-distincts by auto
 moreover have Coplanar B C D C'
 using P29 bet-col ncop--ncols by blast
 moreover have Coplanar B C D D'
 using P30 bet-col ncop--ncols by blast
 moreover have Coplanar B C D E''
 by (simp add: P18 bet--coplanar coplanar-perm-22)
 ultimately show ?thesis
 using coplanar-pseudo-trans by blast

qed

have Coplanar C' D' X E''
 proof –
 have $\neg \text{Col } B \ C' \ D'$
 by (simp add: P38 not-col-permutation-2)
 moreover have Coplanar B C' D' X
 using T3 ncoplanar-perm-8 by blast
 moreover have Coplanar B C' D' E''
 by (simp add: $\langle \text{Coplanar } B \ C' \ D' \ E'' \rangle$)
 ultimately show ?thesis
 using coplanar-trans-1 by blast

qed

have Coplanar C' D' Y E''
 proof –
 have $\neg \text{Col } B \ C' \ D'$
 by (simp add: P38 not-col-permutation-2)
 moreover have Coplanar B C' D' Y
 by (simp add: T3 coplanar-perm-12)
 moreover have Coplanar B C' D' E''
 by (simp add: $\langle \text{Coplanar } B \ C' \ D' \ E'' \rangle$)
 ultimately show ?thesis
 using coplanar-trans-1 by blast

qed

have Coplanar C' D' X A

```

proof -
  have Col C' D' A
    using T2 ⟨A = R⟩ midpoint-col not-col-permutation-2 by blast
  moreover have Col X A A
    by (simp add: col-trivial-2)
  ultimately show ?thesis
    using ncop--ncols by blast
qed
have Coplanar C' D' Y A
proof -
  have Col C' D' A
    using T2 ⟨A = R⟩ midpoint-col not-col-permutation-2 by blast
  moreover have Col Y A A
    by (simp add: col-trivial-2)
  ultimately show ?thesis
    using ncop--ncols by blast
qed
have Col X Y A
proof -
  have Coplanar C' D' X A
    by (simp add: ⟨Coplanar C' D' X A⟩)
  moreover have Coplanar C' D' X E''
    by (simp add: ⟨Coplanar C' D' X E''⟩)
  moreover have Coplanar C' D' Y A
    by (simp add: ⟨Coplanar C' D' Y A⟩)
  moreover have Coplanar C' D' Y E''
    by (simp add: ⟨Coplanar C' D' Y E''⟩)
  moreover have A PerpAt X Y C' D'
    using T3 ⟨A = R⟩ Perp-in-cases by auto
  moreover have A PerpAt A E'' C' D'
    using Perp-in-cases ⟨A PerpAt C' D' A E''⟩ by blast
  ultimately show ?thesis
    using cop4-perp-in2--col by blast
qed
have Col X Y E''
proof -
  have Coplanar C' D' X E''
    using ⟨Coplanar C' D' X E''⟩ by auto
  moreover have Coplanar C' D' X A
    by (simp add: ⟨Coplanar C' D' X A⟩)
  moreover have Coplanar C' D' Y E''
    by (simp add: ⟨Coplanar C' D' Y E''⟩)
  moreover have Coplanar C' D' Y A
    using ⟨Coplanar C' D' Y A⟩ by auto
  moreover have A PerpAt X Y C' D'
    using T3 ⟨A = R⟩ Perp-in-cases by auto
  moreover have A PerpAt E'' A C' D'
    using Perp-in-perm W1 by blast
  ultimately show ?thesis
    using cop4-perp-in2--col by blast
qed
have A E'' Perp C D
proof cases
  assume Y = A
  show ?thesis
  proof -
    have A ≠ E''
      by (simp add: P26)
    moreover have A X Perp C D
      using T3 Perp-cases ⟨Y = A⟩ by blast
    moreover have Col A X E''
      using Col-perm ⟨Col X Y E''⟩ ⟨Y = A⟩ by blast
    ultimately show ?thesis
      using perp-col by blast
  qed
next

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assume  $Y \neq A$ 
show ?thesis
proof -
  have  $A \neq E''$ 
    by (simp add: P26)
  moreover have  $A \ Y \text{ Perp } C \ D$ 
  proof -
    have  $Y \ X \text{ Perp } C \ D$ 
      using T3 by (simp add: perp-comm)
    hence  $Y \ A \text{ Perp } C \ D$ 
      using  $\langle \text{Col } X \ Y \ A \rangle \langle Y \neq A \rangle$  col-trivial-2 perp-col2 perp-left-comm by blast
    thus ?thesis
      using Perp-cases by blast
  qed
  moreover have  $\text{Col } A \ Y \ E''$ 
    using Col-perm  $\langle \text{Col } X \ Y \ A \rangle \langle \text{Col } X \ Y \ E'' \rangle \langle X \neq Y \rangle$  col-transitivity-2 by blast
  ultimately show ?thesis
    using perp-col by blast
  qed
qed
thus ?thesis
  using Perp-perm by blast
next
assume  $A \neq R$ 
have  $R \neq C'$ 
  using P46 T2 is-midpoint-id ts-distincts by blast
have  $\text{Per } A \ R \ C'$  using T1 T2 Per-def by blast
hence  $R \text{ PerpAt } A \ R \ R \ C'$ 
  by (simp add:  $\langle A \neq R \rangle \langle R \neq C' \rangle$  per-perp-in)
hence  $R \text{ PerpAt } R \ C' \ A \ R$ 
  using Perp-in-perm by blast
hence  $R \ C' \text{ Perp } A \ R \vee R \ R \text{ Perp } A \ R$ 
  using perp-in-perp by auto
{
  assume  $R \ C' \text{ Perp } A \ R$ 
  hence  $C' \ R \text{ Perp } A \ R$ 
    by (simp add:  $\langle R \ C' \text{ Perp } A \ R \rangle$  Perp-perm)
  have  $C' \ D' \text{ Perp } R \ A$ 
    by (metis P47 T2  $\langle A \neq R \rangle \langle \text{Per } A \ R \ C' \rangle \langle R \neq C' \rangle$  col-per-perp
      midpoint-col perp-distinct perp-right-comm)
  hence  $R \text{ PerpAt } C' \ D' \ R \ A$ 
    using T2 l8-14-2-1b-bis midpoint-col not-col-distincts by blast
  have  $\text{Col } B \ D \ D'$ 
    by (simp add: Col-def P30)
  have  $\text{Col } B \ C \ C'$ 
    using Col-def P29 by auto
  have  $\text{Col } D \ E'' \ C$ 
    using P18 bet-col by auto
  have  $\text{Col } R \ C' \ D'$ 
    using  $\langle R \text{ PerpAt } C' \ D' \ R \ A \rangle$  by (simp add: T2 midpoint-col)
  have  $\text{Col } A \ E'' \ E'$ 
    by (simp add: P19 out-col)
  have  $\text{Coplanar } C' \ D' \ X \ A$ 
  proof -
    have  $\neg \text{Col } B \ C' \ D'$ 
      using Col-perm P38 by blast
    moreover have  $\text{Coplanar } B \ C' \ D' \ X$ 
      using T3 ncoplanar-perm-8 by blast
    moreover have  $\text{Coplanar } B \ C' \ D' \ A$ 
      using P46 ncoplanar-perm-18 ts-coplanar by blast
    ultimately show ?thesis
      using coplanar-trans-1 by auto
  qed
  have  $\text{Coplanar } C' \ D' \ Y \ A$ 
  proof -
    have  $\neg \text{Col } B \ C' \ D'$ 

```

```

    using Col-perm P38 by blast
  moreover have Coplanar B C' D' Y
    using T3 ncoplanar-perm-8 by blast
  moreover have Coplanar B C' D' A
    using P46 ncoplanar-perm-18 ts--coplanar by blast
  ultimately show ?thesis
    using coplanar-trans-1 by auto
qed
have Coplanar C' D' X R
proof -
  have Col C' D' R
    using Col-perm ⟨Col R C' D'⟩ by blast
  moreover have Col X R R
    by (simp add: col-trivial-2)
  ultimately show ?thesis
    using ncop--ncols by blast
qed
have Coplanar C' D' Y R
  using Col-perm T2 midpoint-col ncop--ncols by blast
have Col X Y A
proof -
  have R PerpAt X Y C' D'
    using T3 by simp
  moreover have R PerpAt A R C' D'
    using Perp-in-perm ⟨R PerpAt C' D' R A⟩ by blast
  ultimately show ?thesis
    using ⟨Coplanar C' D' Y R⟩ ⟨Coplanar C' D' X R⟩ cop4-perp-in2--col
      ⟨Coplanar C' D' X A⟩ ⟨Coplanar C' D' Y A⟩ by blast
qed
have Z1: Col X Y R
  using T3 perp-in-col by blast
have Col A E'' R
proof -
  have Coplanar C' D' E'' R
    using Col-cases ⟨Col R C' D'⟩ ncop--ncols by blast
  moreover have A E'' Perp C' D'
    using P47 Perp-perm by blast
  moreover have A R Perp C' D'
    using Perp-perm ⟨C' D' Perp R A⟩ by blast
  ultimately show ?thesis
    using cop-perp2--col by blast
qed
hence Col X Y E'' using Z1
  by (metis (full-types) ⟨A ≠ R⟩ ⟨Col X Y A⟩ col-permutation-4 col-trivial-2 l6-21)
have Col A E'' R
proof -
  have Coplanar C' D' E'' R
    using Col-cases ⟨Col R C' D'⟩ ncop--ncols by blast
  moreover have A E'' Perp C' D'
    using P47 Perp-perm by blast
  moreover have A R Perp C' D'
    using Perp-perm ⟨C' D' Perp R A⟩ by blast
  ultimately show ?thesis
    using cop-perp2--col by blast
qed
have Col A R X
  using ⟨Col X Y A⟩ ⟨Col X Y R⟩ ⟨X ≠ Y⟩ col-transitivity-1
    not-col-permutation-3 by blast
have Col A R Y
  using ⟨Col X Y A⟩ ⟨Col X Y R⟩ ⟨X ≠ Y⟩ col-transitivity-2
    not-col-permutation-3 by blast
have A E'' Perp C D
proof cases
  assume X = A
  show ?thesis
proof -

```

```

have  $A \neq E''$ 
  by (simp add: P26)
moreover have  $A \perp Y \text{ Perp } C \ D$ 
  using T3  $\langle X = A \rangle \text{ perp-right-comm}$  by blast
moreover have  $\text{Col } A \ Y \ E''$ 
  using Col-perm  $\langle A \neq R \rangle \langle \text{Col } A \ E'' \ R \rangle \langle \text{Col } A \ R \ Y \rangle$ 
    col-transitivity-1 by blast
ultimately show ?thesis
  using perp-col by auto
qed
next
assume  $X \neq A$ 
show ?thesis
proof -
  have  $X \ Y \ \text{Perp } D \ C$ 
    using T3 by blast
  hence  $X \ Y \ \text{Perp } C \ D$ 
    using perp-right-comm by blast
  hence  $A \ X \ \text{Perp } C \ D$ 
    using Perp-perm  $\langle \text{Col } X \ Y \ A \rangle \langle X \neq A \rangle \text{ col-trivial-3 perp-col2-bis}$  by blast
  moreover have  $\text{Col } A \ X \ E''$ 
    using Col-perm  $\langle A \neq R \rangle \langle \text{Col } A \ E'' \ R \rangle \langle \text{Col } A \ R \ X \rangle$ 
      col-transitivity-1 by blast
  ultimately show ?thesis
    using P26 perp-col by blast
  qed
qed
}
{
  assume  $R \ R \ \text{Perp } A \ R$ 
  hence  $A \ E'' \ \text{Perp } C \ D$ 
    using perp-distinct by blast
}
hence  $A \ E'' \ \text{Perp } C \ D$ 
  using Perp-cases  $\langle R \ C' \ \text{Perp } A \ R \implies A \ E'' \ \text{Perp } C \ D \rangle$ 
     $\langle R \ C' \ \text{Perp } A \ R \vee R \ R \ \text{Perp } A \ R \rangle$  by auto
thus ?thesis
  using Perp-perm by blast
qed
show ?thesis
proof -
  have  $\text{Col } A \ E \ E''$ 
  proof -
    have Coplanar  $C \ D \ E \ E'$ 
      using assms(4) col-coplanar by auto
    moreover have  $A \ E \ \text{Perp } C \ D$ 
      using assms(5) by auto
    moreover have  $A \ E'' \ \text{Perp } C \ D$ 
      using Perp-perm  $\langle C \ D \ \text{Perp } A \ E'' \rangle$  by blast
    ultimately show ?thesis
      by (meson P11 col-perp2-ncol-col col-trivial-3 not-col-permutation-2)
  qed
  moreover have  $E'' = E$ 
  proof -
    have f1:  $C = E'' \vee \text{Col } C \ E'' \ D$ 
      by (metis P18 bet-out-1 out-col)
    hence f2:  $C = E'' \vee \text{Col } C \ E'' \ E$ 
      using Col-perm P3 assms(4) col-transitivity-1 by blast
    have  $\forall p. (C = E'' \vee \text{Col } C \ p \ D) \vee \neg \text{Col } C \ E'' \ p$ 
      using f1 by (meson col-transitivity-1)
    hence  $\exists p. \neg \text{Col } E'' \ p \ A \wedge \text{Col } E'' \ E \ p$ 
      using f2 by (metis (no-types) Col-perm P11 assms(4))
    thus ?thesis
      using Col-perm calculation col-transitivity-1 by blast
  qed
ultimately show ?thesis

```

```

    by (metis Bet-perm P18 P20 P28 conga-left-comm not-conga-sym)
  qed
  qed
  hence  $B A C \text{ CongA } D A E \wedge B A D \text{ CongA } C A E \wedge \text{Bet } C E D$ 
    by blast
}
thus ?thesis
  using P18  $\langle E'' = A \implies B A C \text{ CongA } D A E \wedge B A D \text{ CongA } C A E \wedge \text{Bet } C E D \rangle$  by blast
qed

```

```

lemma perp2-refl:
  assumes  $A \neq B$ 
  shows  $P \text{ Perp2 } A B A B$ 
proof cases
  assume  $\text{Col } A B P$ 
  obtain  $X$  where  $\neg \text{Col } A B X$ 
    using assms not-col-exists by blast
  then obtain  $Q$  where  $A B \text{ Perp } Q P \wedge A B \text{ OS } X Q$ 
    using  $\langle \text{Col } A B P \rangle$  l10-15 by blast
  thus ?thesis
    using Perp2-def Perp-cases col-trivial-3 by blast
next
  assume  $\neg \text{Col } A B P$ 
  then obtain  $Q$  where  $\text{Col } A B Q \wedge A B \text{ Perp } P Q$ 
    using l8-18-existence by blast
  thus ?thesis
    using Perp2-def Perp-cases col-trivial-3 by blast
qed

```

```

lemma perp2-sym:
  assumes  $P \text{ Perp2 } A B C D$ 
  shows  $P \text{ Perp2 } C D A B$ 
proof -
  obtain  $X Y$  where  $\text{Col } P X Y \wedge X Y \text{ Perp } A B \wedge X Y \text{ Perp } C D$ 
    using Perp2-def assms by auto
  thus ?thesis
    using Perp2-def by blast
qed

```

```

lemma perp2-left-comm:
  assumes  $P \text{ Perp2 } A B C D$ 
  shows  $P \text{ Perp2 } B A C D$ 
proof -
  obtain  $X Y$  where  $\text{Col } P X Y \wedge X Y \text{ Perp } A B \wedge X Y \text{ Perp } C D$ 
    using Perp2-def assms by auto
  thus ?thesis
    using Perp2-def perp-right-comm by blast
qed

```

```

lemma perp2-right-comm:
  assumes  $P \text{ Perp2 } A B C D$ 
  shows  $P \text{ Perp2 } A B D C$ 
proof -
  obtain  $X Y$  where  $\text{Col } P X Y \wedge X Y \text{ Perp } A B \wedge X Y \text{ Perp } C D$ 
    using Perp2-def assms by auto
  thus ?thesis
    using Perp2-def perp-right-comm by blast
qed

```

```

lemma perp2-comm:
  assumes  $P \text{ Perp2 } A B C D$ 
  shows  $P \text{ Perp2 } B A D C$ 
proof -
  obtain  $X Y$  where  $\text{Col } P X Y \wedge X Y \text{ Perp } A B \wedge X Y \text{ Perp } C D$ 
    using Perp2-def assms by auto
  thus ?thesis

```



```

    using assms perp2-left-comm perp2-right-comm by blast
qed

lemma perp2-pseudo-trans:
  assumes P Perp2 A B C D and
    P Perp2 C D E F and
     $\neg$  Col C D P
  shows P Perp2 A B E F
proof -
  obtain X Y where P1: Col P X Y  $\wedge$  X Y Perp A B  $\wedge$  X Y Perp C D
    using Perp2-def assms(1) by auto
  obtain X' Y' where P2: Col P X' Y'  $\wedge$  X' Y' Perp C D  $\wedge$  X' Y' Perp E F
    using Perp2-def assms(2) by auto
  have X Y Par X' Y'
  proof -
    have Coplanar P C D X
    proof cases
      assume X = P
      thus ?thesis
        using ncop-distincts by blast
    next
      assume X  $\neq$  P
      hence X P Perp C D
        using Col-cases P1 perp-col by blast
      hence Coplanar X P C D
        by (simp add: perp--coplanar)
      thus ?thesis
        using ncoplanar-perm-18 by blast
    qed
  have Coplanar P C D Y
  proof cases
    assume Y = P
    thus ?thesis
      using ncop-distincts by blast
  next
    assume Y  $\neq$  P
    hence Y P Perp C D
      by (metis (full-types) Col-cases P1 Perp-cases col-transitivity-2 perp-col2)
    hence Coplanar Y P C D
      by (simp add: perp--coplanar)
    thus ?thesis
      using ncoplanar-perm-18 by blast
  qed
  have Coplanar P C D X'
  proof cases
    assume X' = P
    thus ?thesis
      using ncop-distincts by blast
  next
    assume X'  $\neq$  P
    hence X' P Perp C D
      using Col-cases P2 perp-col by blast
    hence Coplanar X' P C D
      by (simp add: perp--coplanar)
    thus ?thesis
      using ncoplanar-perm-18 by blast
  qed
  have Coplanar P C D Y'
  proof cases
    assume Y' = P
    thus ?thesis
      using ncop-distincts by blast
  next
    assume Y'  $\neq$  P
    hence Y' P Perp C D
      by (metis (full-types) Col-cases P2 Perp-cases col-transitivity-2 perp-col2)

```

```

hence Coplanar  $Y' P C D$ 
  by (simp add: perp--coplanar)
thus ?thesis
  using ncoplanar-perm-18 by blast
qed
show ?thesis
proof -
  have Coplanar  $C D X X'$ 
    using Col-cases  $\langle \text{Coplanar } P C D X' \rangle \langle \text{Coplanar } P C D X \rangle$  assms(3)
    coplanar-trans-1 by blast
  moreover have Coplanar  $C D X Y'$ 
    using Col-cases  $\langle \text{Coplanar } P C D X \rangle \langle \text{Coplanar } P C D Y' \rangle$  assms(3)
    coplanar-trans-1 by blast
  moreover have Coplanar  $C D Y X'$ 
    using Col-cases  $\langle \text{Coplanar } P C D X' \rangle \langle \text{Coplanar } P C D Y \rangle$  assms(3)
    coplanar-trans-1 by blast
  moreover have Coplanar  $C D Y Y'$ 
    using Col-cases  $\langle \text{Coplanar } P C D Y' \rangle \langle \text{Coplanar } P C D Y \rangle$  assms(3)
    coplanar-trans-1 by blast
  ultimately show ?thesis
    using l12-9 P1 P2 by blast
qed
qed
thus ?thesis
proof -
  {
    assume  $X Y \text{ ParStrict } X' Y'$ 
    hence Col  $X X' Y'$ 
      using P1 P2  $\langle X Y \text{ ParStrict } X' Y' \rangle$  par-not-col by blast
  }
  hence Col  $X X' Y'$ 
    using Par-def  $\langle X Y \text{ Par } X' Y' \rangle$  by blast
  moreover have Col  $Y X' Y'$ 
  proof -
    {
      assume  $X Y \text{ ParStrict } X' Y'$ 
      hence Col  $Y X' Y'$ 
        using P1 P2  $\langle X Y \text{ ParStrict } X' Y' \rangle$  par-not-col by blast
    }
    thus ?thesis
      using Par-def  $\langle X Y \text{ Par } X' Y' \rangle$  by blast
  qed
  moreover have  $X \neq Y$ 
    using P1 perp-not-eq-1 by auto
  ultimately show ?thesis
    by (meson Perp2-def P1 P2 col-permutation-1 perp-col2)
qed
qed

lemma col-cop-perp2--pars-bis:
  assumes  $\neg \text{Col } A B P$  and
    Col  $C D P$  and
    Coplanar  $A B C D$  and
     $P \text{ Perp2 } A B C D$ 
  shows  $A B \text{ ParStrict } C D$ 
proof -
  obtain  $X Y$  where Col  $P X Y$  and  $X Y \text{ Perp } A B$  and  $X Y \text{ Perp } C D$ 
    using Perp2-def assms(4) by auto
  hence Col  $X Y P$ 
    using Col-perm by blast
  obtain  $Q$  where  $X \neq Q$  and  $Y \neq Q$  and  $P \neq Q$  and Col  $X Y Q$ 
    using  $\langle \text{Col } X Y P \rangle$  diff-col-ex3 by blast
  have  $A B \text{ Perp } P Q$ 
    using  $\langle \text{Col } X Y P \rangle \langle \text{Col } X Y Q \rangle \langle P \neq Q \rangle \langle X Y \text{ Perp } A B \rangle$  perp-col0 by blast
  moreover have  $C D \text{ Perp } P Q$ 
    using  $\langle \text{Col } X Y P \rangle \langle \text{Col } X Y Q \rangle \langle P \neq Q \rangle \langle X Y \text{ Perp } C D \rangle$  perp-col0 by blast

```

```

ultimately show ?thesis
  by (meson col-cop-perp2--pars assms(1) assms(2) assms(3))
qed

lemma perp2-preserves-bet23:
  assumes Bet PO A B and
    Col PO A' B' and
     $\neg$  Col PO A A' and
    PO Perp2 A A' B B'
  shows Bet PO A' B'
proof -
  have A  $\neq$  A'
  using assms(3) not-col-distincts by auto
  show ?thesis
  proof cases
    assume A' = B'
    thus ?thesis
      using between-trivial by auto
  next
    assume A'  $\neq$  B'
    {
      assume A = B
      then obtain X Y where P1: Col PO X Y  $\wedge$  X Y Perp A A'  $\wedge$  X Y Perp A B'
      using Perp2-def assms(4) by blast
      have Col A A' B'
      proof -
        have Coplanar X Y A' B'
        using Col-cases Coplanar-def P1 assms(2) by auto
        moreover have A A' Perp X Y
        using P1 Perp-perm by blast
        moreover have A B' Perp X Y
        using P1 Perp-perm by blast
        ultimately show ?thesis
        using cop-perp2--col by blast
      qed
      hence False
      using Col-perm  $\langle A' \neq B' \rangle$  assms(2) assms(3) l6-16-1 by blast
    }
    hence A  $\neq$  B by auto
    have A A' Par B B'
  proof -
    obtain X Y where P2: Col PO X Y  $\wedge$  X Y Perp A A'  $\wedge$  X Y Perp B B'
    using Perp2-def assms(4) by auto
    hence Coplanar X Y A B
    using Coplanar-def assms(1) bet-col not-col-permutation-2 by blast
    show ?thesis
    proof -
      have Coplanar X Y A B'
      by (metis (full-types) Col-cases P2 assms(2) assms(3) col-cop2--cop
        col-trivial-3 ncop--ncols perp--coplanar)
      moreover have Coplanar X Y A' B
    proof cases
      assume Col A X Y
      hence Col Y X A
      by (metis (no-types) Col-cases)
      thus ?thesis
      by (metis Col-cases P2 assms(1) assms(3) bet-col colx
        ncop--ncols not-col-distincts)
    next
      assume  $\neg$  Col A X Y
      moreover have Coplanar A X Y A'
      using Coplanar-def P2 perp-inter-exists by blast
      moreover have Coplanar A X Y B
      using  $\langle$  Coplanar X Y A B  $\rangle$  ncoplanar-perm-8 by blast
      ultimately show ?thesis
      using coplanar-trans-1 by auto
    end
  end
end

```

```

qed
moreover have Coplanar X Y A' B'
  using Col-cases Coplanar-def P2 assms(2) by auto
moreover have A A' Perp X Y
  using P2 Perp-perm by blast
moreover have B B' Perp X Y
  using P2 Perp-perm by blast
ultimately show ?thesis
  using ⟨Coplanar X Y A B⟩ l12-9 by auto
qed
qed
{
  assume A A' ParStrict B B'
  hence A A' OS B B'
    by (simp add: l12-6)
  have A A' TS PO B
    using Col-cases ⟨A ≠ B⟩ assms(1) assms(3) bet--ts by blast
  hence A A' TS B' PO
    using ⟨A A' OS B B'⟩ l9-2 l9-8-2 by blast
  hence Bet PO A' B'
    using Col-cases assms(2) between-symmetry col-two-sides-bet
      invert-two-sides by blast
}
thus ?thesis
  by (metis Col-cases Par-def ⟨A A' Par B B'⟩ ⟨A ≠ B⟩ assms(1) assms(3)
    bet-col col-trivial-3 l6-21)
qed
qed

lemma perp2-preserves-bet13:
  assumes Bet B PO C and
    Col PO B' C' and
    ¬ Col PO B B' and
    PO Perp2 B C' C B'
  shows Bet B' PO C'
proof cases
  assume C' = PO
  thus ?thesis
    using not-bet-distincts by blast
next
  assume C' ≠ PO
  show ?thesis
proof cases
  assume B' = PO
  thus ?thesis
    using between-trivial2 by auto
next
  assume B' ≠ PO
  have B ≠ PO
    using assms(3) col-trivial-1 by auto
  have Col B PO C
    by (simp add: Col-def assms(1))
  show ?thesis
proof cases
  assume B = C
  thus ?thesis
    using ⟨B = C⟩ ⟨B ≠ PO⟩ assms(1) between-identity by blast
next
  assume B ≠ C
  have B C' Par C B'
  proof –
    obtain X Y where P1: Col PO X Y ∧ X Y Perp B C' ∧ X Y Perp C B'
      using Perp2-def assms(4) by auto
    have Coplanar X Y B C
      by (meson P1 ⟨Col B PO C⟩ assms(1) l9-18-R2 ncop--ncols
        not-col-permutation-2 not-col-permutation-5 ts--coplanar)

```

```

have Coplanar X Y C' B'
  using Col-cases Coplanar-def P1 assms(2) by auto
have Coplanar X Y B B'
  by (metis P1 ⟨C' ≠ PO⟩ ⟨Coplanar X Y B C⟩ assms(1) assms(2)
      bet-cop--cop col-cop2--cop not-col-permutation-5 perp--coplanar)
moreover
have Coplanar X Y C' C
proof cases
  assume Col B X Y
  thus ?thesis
    by (meson P1 ⟨B ≠ PO⟩ ⟨Col B PO C⟩ col-permutation-1 colx ncop--ncols)
next
  assume ¬ Col B X Y
  thus ?thesis
    by (meson P1 ⟨Coplanar X Y B C⟩ coplanar-perm-8 coplanar-pseudo-trans
        ncop-distincts perp--coplanar)
qed
ultimately show ?thesis using l12-9
  by (meson P1 Perp-cases ⟨Coplanar X Y B C⟩ ⟨Coplanar X Y C' B'⟩)
qed
have B C' ParStrict C B'
  by (metis Out-def Par-def ⟨B C' Par C B'⟩ ⟨B ≠ C⟩ ⟨B ≠ PO⟩ assms(1)
      assms(3) col-transitivity-1 not-col-permutation-4 out-col)
have B' ≠ PO
  by (simp add: ⟨B' ≠ PO⟩)
obtain X Y where P5: Col PO X Y ∧ X Y Perp B C' ∧ X Y Perp C B'
  using Perp2-def assms(4) by auto
have X ≠ Y
  using P5 perp-not-eq-1 by auto
show ?thesis
proof cases
  assume Col X Y B
  have Col X Y C
    using P5 ⟨B ≠ PO⟩ ⟨Col B PO C⟩ ⟨Col X Y B⟩ col-permutation-1 colx by blast
  show ?thesis
  proof -
    have Col B' PO C'
      using Col-cases assms(2) by auto
    moreover have Per PO C B'
      by (metis P5 ⟨B C' ParStrict C B'⟩ ⟨Col X Y C⟩ assms(2)
          col-permutation-2 par-strict-not-col-2 perp-col2 perp-per-2)
    moreover have Per PO B C'
      using P5 ⟨B ≠ PO⟩ ⟨Col X Y B⟩ col-permutation-1 perp-col2 perp-per-2 by blast
    ultimately show ?thesis
      by (metis per13-preserves-bet-inv ⟨B C' ParStrict C B'⟩ assms(1) assms(3)
          between-symmetry not-col-distincts not-col-permutation-3 par-strict-not-col-2)
  qed
next
  assume ¬ Col X Y B
  then obtain B0 where U1: Col X Y B0 ∧ X Y Perp B B0
    using l8-18-existence by blast
  have ¬ Col X Y C
    using P5 ⟨B C' ParStrict C B'⟩ ⟨Col B PO C⟩ ⟨¬ Col X Y B⟩ assms(2)
        col-permutation-2 colx par-strict-not-col-2 by metis
  then obtain C0 where U2: Col X Y C0 ∧ X Y Perp C C0
    using l8-18-existence by blast
  have B0 ≠ PO
    by (metis P5 Perp-perm ⟨Col B PO C⟩ ⟨Col X Y B0 ∧ X Y Perp B B0⟩ ⟨¬ Col X Y C⟩
        assms(3) col-permutation-2 col-permutation-3 col-perp2-ncol-col)
  {
    assume C0 = PO
    hence C PO Par C B'
      by (metis P5 Par-def Perp-cases ⟨Col X Y C0 ∧ X Y Perp C C0⟩ ⟨¬ Col X Y C⟩
          col-perp2-ncol-col not-col-distincts not-col-permutation-3 perp-distinct)
    hence False
      by (metis ⟨B C' ParStrict C B'⟩ assms(2) assms(3) col3 not-col-distincts)
  }

```

```

    par-id-2 par-strict-not-col-2)
  }
  hence  $C0 \neq PO$  by auto
  have  $Bet\ B0\ PO\ C0$ 
  proof -
    have  $Bet\ B\ PO\ C$ 
    by (simp add: assms(1))
    moreover have  $PO \neq B0$ 
    using  $\langle B0 \neq PO \rangle$  by auto
    moreover have  $PO \neq C0$ 
    using  $\langle C0 \neq PO \rangle$  by auto
    moreover have  $Col\ B0\ PO\ C0$ 
    using  $U1\ U2\ P5\ \langle X \neq Y \rangle\ col3\ not-col-permutation-2$  by blast
    moreover have  $Per\ PO\ B0\ B$ 
    proof -
      have  $B0\ PerpAt\ PO\ B0\ B0\ B$ 
      proof cases
        assume  $X = B0$ 
        have  $B0\ PO\ Perp\ B\ B0$ 
        by (metis  $P5\ U1\ calculation(2)\ col3\ col-trivial-2\ col-trivial-3\ perp-col2$ )
        show ?thesis
        proof -
          have  $B0 \neq PO$ 
          using  $calculation(2)$  by auto
          moreover have  $B0\ Y\ Perp\ B\ B0$ 
          using  $U1\ \langle X = B0 \rangle$  by auto
          moreover have  $Col\ B0\ Y\ PO$ 
          using  $Col-perm\ P5\ \langle X = B0 \rangle$  by blast
          ultimately show ?thesis
          using  $\langle B0\ PO\ Perp\ B\ B0 \rangle\ perp-in-comm\ perp-perp-in$  by blast
        qed
      qed
    next
      assume  $X \neq B0$ 
      have  $X\ B0\ Perp\ B\ B0$ 
      using  $U1\ \langle X \neq B0 \rangle\ perp-col$  by blast
      have  $B0\ PO\ Perp\ B\ B0$ 
      by (metis  $P5\ U1\ calculation(2)\ not-col-permutation-2\ perp-col2$ )
      hence  $B0\ PerpAt\ B0\ PO\ B\ B0$ 
      by (simp add: perp-perp-in)
      thus ?thesis
      using  $Perp-in-perm$  by blast
    qed
  thus ?thesis
  by (simp add: perp-in-per)
qed
moreover have  $Per\ PO\ C0\ C$ 
proof -
  have  $C0\ PO\ Perp\ C\ C0$ 
  by (metis  $P5\ U2\ calculation(3)\ col3\ col-trivial-2\ col-trivial-3\ perp-col2$ )
  hence  $C0\ PerpAt\ PO\ C0\ C0\ C$ 
  by (simp add: perp-in-comm perp-perp-in)
  thus ?thesis
  using  $perp-in-per-2$  by auto
qed
ultimately show ?thesis
using  $per13-preserves-bet$  by blast
qed
show ?thesis
proof cases
  assume  $C' = B0$ 
  have  $B' = C0$ 
  proof -
    have  $\neg\ Col\ C'\ PO\ C$ 
    using  $P5\ U1\ \langle B0 \neq PO \rangle\ \langle C' = B0 \rangle\ \langle \neg\ Col\ X\ Y\ C \rangle\ colx\ not-col-permutation-3\ not-col-permutation-4$  by blast
    moreover have  $C \neq C0$ 

```

```

    using U2  $\langle \neg \text{Col } X \ Y \ C \rangle$  by auto
  moreover have  $\text{Col } C \ C0 \ B'$ 
proof -
  have  $\text{Coplanar } X \ Y \ C0 \ B'$ 
proof -
  have  $\text{Col } X \ Y \ C0$ 
    by (simp add: U2)
  moreover have  $\text{Col } C0 \ B' \ C0$ 
    by (simp add: col-trivial-3)
  ultimately show ?thesis
    using ncop--ncols by blast
qed
moreover have  $C \ C0 \ \text{Perp } X \ Y$ 
  using Perp-perm U2 by blast
moreover have  $C \ B' \ \text{Perp } X \ Y$ 
  using P5 Perp-perm by blast
ultimately show ?thesis
  using cop-perp2--col by auto
qed
ultimately show ?thesis
  by (metis Col-def  $\langle C' = B0 \rangle \langle \text{Bet } B0 \ PO \ C0 \rangle \text{assms}(2) \text{col}x$ )
qed
show ?thesis
  using Bet-cases  $\langle B' = C0 \rangle \langle C' = B0 \rangle \langle \text{Bet } B0 \ PO \ C0 \rangle$  by blast
next
assume  $C' \neq B0$ 
hence  $B' \neq C0$ 
  by (metis P5 U1 U2  $\langle C0 \neq PO \rangle \text{assms}(2) \text{col-permutation-1 col}x \text{l8-18-uniqueness}$ )
have  $B \ C' \ \text{Par } B \ B0$ 
proof -
  have  $\text{Coplanar } X \ Y \ B \ B$ 
    using ncop--distincts by auto
  moreover have  $\text{Coplanar } X \ Y \ B \ B0$ 
    using U1 ncop--ncols by blast
  moreover have  $\text{Coplanar } X \ Y \ C' \ B$ 
    using P5 ncoplanar-perm-1 perp--coplanar by blast
  moreover have  $\text{Coplanar } X \ Y \ C' \ B0$ 
    using  $\langle \neg \text{Col } X \ Y \ B \rangle \text{calculation}(2) \text{calculation}(3) \text{col-permutation-1}$ 
      coplanar-perm-12 coplanar-perm-18 coplanar-trans-1 by blast
  moreover have  $B \ C' \ \text{Perp } X \ Y$ 
    using P5 Perp-perm by blast
  moreover have  $B \ B0 \ \text{Perp } X \ Y$ 
    using Perp-perm U1 by blast
  ultimately show ?thesis
    using l12-9 by blast
qed
{
  assume  $B \ C' \ \text{ParStrict } B \ B0$ 
  have  $\text{Col } B \ B0 \ C'$ 
    by (simp add:  $\langle B \ C' \ \text{Par } B \ B0 \rangle \text{par-id-3}$ )
}
hence  $\text{Col } B \ B0 \ C'$ 
  using  $\langle B \ C' \ \text{Par } B \ B0 \rangle \text{par-id-3}$  by blast
have  $\text{Col } C \ C0 \ B'$ 
proof -
  have  $\text{Coplanar } X \ Y \ C0 \ B'$ 
    by (simp add: U2 col--coplanar)
  moreover have  $C \ C0 \ \text{Perp } X \ Y$ 
    by (simp add: Perp-perm U2)
  moreover have  $C \ B' \ \text{Perp } X \ Y$ 
    using P5 Perp-perm by blast
  ultimately show ?thesis
    using cop-perp2--col by auto
qed
show ?thesis
proof -

```

```

have Col B' PO C'
  using assms(2) not-col-permutation-4 by blast
moreover have Per PO C0 B'
proof -
  have C0 PerpAt PO C0 C0 B'
  proof cases
    assume X = C0
    have C0 PO Perp C B'
    proof -
      have C0 ≠ PO
        by (simp add: ⟨C0 ≠ PO⟩)
      moreover have C0 Y Perp C B'
        using P5 ⟨X = C0⟩ by auto
      moreover have Col C0 Y PO
        using Col-perm P5 ⟨X = C0⟩ by blast
      ultimately show ?thesis
        using perp-col by blast
    qed
  hence B' C0 Perp C0 PO
    using Perp-perm ⟨B' ≠ C0⟩ ⟨Col C C0 B'⟩ not-col-permutation-1
    perp-col1 by blast
  hence C0 PerpAt C0 B' PO C0
    using Perp-perm perp-perp-in by blast
  thus ?thesis
    using Perp-in-perm by blast
next
assume X ≠ C0
hence X C0 Perp C B'
  using P5 U2 perp-col by blast
have C0 PO Perp C B'
  using Col-cases P5 U2 ⟨C0 ≠ PO⟩ perp-col2 by blast
hence B' C0 Perp C0 PO
  using Perp-cases ⟨B' ≠ C0⟩ ⟨Col C C0 B'⟩ col-permutation-2 perp-col by blast
thus ?thesis
  using Perp-in-perm Perp-perm perp-perp-in by blast
qed
thus ?thesis
  using perp-in-per-2 by auto
qed
moreover have Per PO B0 C'
proof -
  have B0 PerpAt PO B0 B0 C'
  proof -
    have Col C' B B0
      using Col-cases ⟨Col B B0 C'⟩ by blast
    hence C' B0 Perp X Y using perp-col P5 Perp-cases ⟨C' ≠ B0⟩ by blast
    show ?thesis
  proof -
    have PO B0 Perp B0 C'
    proof -
      have X Y Perp B0 C'
        using P5 Perp-cases ⟨C' B0 Perp X Y⟩ by blast
      moreover have Col X Y PO
        using Col-cases P5 by blast
      ultimately show ?thesis
        using U1 ⟨B0 ≠ PO⟩ perp-col2 by presburger
    qed
    thus ?thesis
      using Perp-in-cases Perp-perm perp-perp-in by blast
    qed
  qed
  thus ?thesis
    by (simp add: perp-in-per)
qed
ultimately show ?thesis
  using ⟨B0 ≠ PO⟩ ⟨C0 ≠ PO⟩ ⟨Bet B0 PO C0⟩ between-symmetry

```



```

      per13-preserves-bet-inv by blast
    qed
  qed
  qed
  qed
  qed
  qed
lemma is-image-perp-in:
  assumes  $A \neq A'$  and
     $X \neq Y$  and
     $A A' \text{ Reflect } X Y$ 
  shows  $\exists P. P \text{ PerpAt } A A' X Y$ 
  by (metis Perp-def Perp-perm assms(1) assms(2) assms(3) ex-sym1 l10-6-uniqueness)

lemma perp-inter-perp-in-n:
  assumes  $A B \text{ Perp } C D$ 
  shows  $\exists P. \text{Col } A B P \wedge \text{Col } C D P \wedge P \text{ PerpAt } A B C D$ 
  by (simp add: assms perp-inter-perp-in)

lemma perp2-perp-in:
  assumes  $PO \text{ Perp2 } A B C D$  and
     $\neg \text{Col } PO A B$  and
     $\neg \text{Col } PO C D$ 
  shows  $\exists P Q. \text{Col } A B P \wedge \text{Col } C D Q \wedge \text{Col } PO P Q \wedge P \text{ PerpAt } PO P A B \wedge Q \text{ PerpAt } PO Q C D$ 
proof -
  obtain  $X Y$  where  $P1: \text{Col } PO X Y \wedge X Y \text{ Perp } A B \wedge X Y \text{ Perp } C D$ 
  using Perp2-def assms(1) by blast
  have  $X \neq Y$ 
  using  $P1$  perp-not-eq-1 by auto
  obtain  $P$  where  $P2: \text{Col } X Y P \wedge \text{Col } A B P \wedge P \text{ PerpAt } X Y A B$ 
  using  $P1$  perp-inter-perp-in-n by blast
  obtain  $Q$  where  $P3: \text{Col } X Y Q \wedge \text{Col } C D Q \wedge Q \text{ PerpAt } X Y C D$ 
  using  $P1$  perp-inter-perp-in-n by blast
  have  $\text{Col } A B P$ 
  using  $P2$  by simp
  moreover have  $\text{Col } C D Q$ 
  using  $P3$  by simp
  moreover have  $\text{Col } PO P Q$ 
  using  $P2 P3 P1 \langle X \neq Y \rangle \text{ col3 not-col-permutation-2}$  by blast
  moreover have  $P \text{ PerpAt } PO P A B$ 
  proof cases
    assume  $X = PO$ 
    thus ?thesis
      by (metis  $P2$  assms(2) not-col-permutation-3 not-col-permutation-4
        perp-in-col-perp-in perp-in-sym)
  next
    assume  $X \neq PO$ 
    hence  $P \text{ PerpAt } A B X PO$ 
    by (meson Col-cases  $P1 P2$  perp-in-col-perp-in perp-in-sym)
    hence  $P \text{ PerpAt } A B PO X$ 
    using Perp-in-perm by blast
    hence  $P \text{ PerpAt } A B PO P$ 
    by (metis Col-cases assms(2) perp-in-col perp-in-col-perp-in)
    thus ?thesis
      by (simp add: perp-in-sym)
  qed
  moreover have  $Q \text{ PerpAt } PO Q C D$ 
  by (metis  $P1 P3 \langle X \neq Y \rangle$  assms(3) col-trivial-2 colx not-col-permutation-3
    not-col-permutation-4 perp-in-col-perp-in perp-in-right-comm perp-in-sym)
  ultimately show ?thesis
    by blast
qed

lemma l13-8:
  assumes  $U \neq PO$  and

```

```

   $V \neq PO$  and
  Col  $PO\ P\ Q$  and
  Col  $PO\ U\ V$  and
  Per  $P\ U\ PO$  and
  Per  $Q\ V\ PO$ 
shows  $PO\ Out\ P\ Q \longleftrightarrow PO\ Out\ U\ V$ 
proof cases
  assume  $P = U$ 
  thus ?thesis
    by (metis assms(1) assms(2) assms(3) assms(4) assms(6) col-permutation-3
        col-transitivity-1 l8-9)
next
  assume  $P \neq U$ 
  hence  $Q \neq V$ 
  by (metis assms(1) assms(2) assms(3) assms(4) assms(5) col-trivial-3
      colx not-col-permutation-2 per-col-eq)
{
  assume  $PO\ Out\ P\ Q$ 
  hence  $Bet\ PO\ P\ Q \vee Bet\ PO\ Q\ P$ 
    using Out-def by force
  moreover
  {
    assume  $Bet\ PO\ P\ Q$ 
    hence  $Bet\ PO\ U\ V$ 
      by (metis l8-2 assms(1) assms(2) assms(4) assms(5) assms(6) per23-preserves-bet)
    hence  $PO\ Out\ U\ V$ 
      by (simp add: assms(1) bet-out)
  }
  moreover
  {
    assume  $Bet\ PO\ Q\ P$ 
    hence  $Bet\ U\ V\ PO$ 
      by (metis l8-2 assms(1) assms(2) assms(4) assms(5) assms(6) between-symmetry
          not-col-permutation-5 per23-preserves-bet)
    hence  $PO\ Out\ U\ V$ 
      by (simp add: assms(2) bet-out-1 l6-6)
  }
  ultimately have  $PO\ Out\ U\ V$ 
    by blast
}
moreover
{
  assume  $PO\ Out\ U\ V$ 
  have  $P \neq PO$ 
    by (metis  $\langle P \neq U \rangle$  assms(5) l8-20-1-R1 l8-7)
  moreover have  $Q \neq PO$ 
    using assms(2) assms(6) per-distinct by blast
  moreover have  $Bet\ PO\ P\ Q \vee Bet\ PO\ Q\ P$ 
  proof -
    {
      assume  $Bet\ PO\ U\ V$ 
      hence  $Bet\ PO\ P\ Q$ 
        by (metis l8-2 assms(1) assms(3) assms(5) assms(6) per23-preserves-bet-inv)
    }
    moreover
    {
      assume  $Bet\ PO\ V\ U$ 
      hence  $Bet\ PO\ Q\ P$ 
        by (metis l8-2 assms(2) assms(3) assms(5) assms(6) not-col-permutation-5
            per23-preserves-bet-inv)
    }
    ultimately show ?thesis
      using Out-def  $\langle PO\ Out\ U\ V \rangle$  by fastforce
  qed
  ultimately have  $PO\ Out\ P\ Q$ 
    using Out-def by blast
}

```

```

}
ultimately show ?thesis
  by blast
qed

```

lemma *perp-in-rewrite*:

```

assumes  $P \text{ PerpAt } A \ B \ C \ D$ 
shows  $P \text{ PerpAt } A \ P \ P \ C \ \vee \ P \text{ PerpAt } A \ P \ P \ D \ \vee \ P \text{ PerpAt } B \ P \ P \ C \ \vee \ P \text{ PerpAt } B \ P \ P \ D$ 
by (metis assms per-perp-in perp-in-distinct perp-in-per-1 perp-in-per-3 perp-in-per-4)

```

lemma *perp-out-acute*:

```

assumes  $B \text{ Out } A \ C'$  and
   $A \ B \text{ Perp } C \ C'$ 
shows  $\text{Acute } A \ B \ C$ 

```

proof –

```

have  $A \neq B$ 
  using assms(1) out-diff1 by auto
have  $C' \neq B$ 
  using Out-def assms(1) by auto
hence  $B \ C' \text{ Perp } C \ C'$ 
  by (metis assms(1) assms(2) out-col perp-col perp-comm perp-right-comm)
hence  $\text{Per } C \ C' \ B$ 
  using Perp-cases perp-per-2 by blast
hence  $\text{Acute } C' \ C \ B \wedge \text{Acute } C' \ B \ C$ 
  by (metis  $\langle C' \neq B \rangle$  assms(2) l11-43 perp-not-eq-2)
have  $C \neq B$ 
  using  $\langle B \ C' \text{ Perp } C \ C' \rangle$  l8-14-1 by auto
show ?thesis

```

proof –

```

have  $B \text{ Out } A \ C'$ 
  by (simp add: assms(1))
moreover have  $B \text{ Out } C \ C$ 
  by (simp add:  $\langle C \neq B \rangle$  out-trivial)
moreover have  $\text{Acute } C' \ B \ C$ 
  by (simp add:  $\langle \text{Acute } C' \ C \ B \wedge \text{Acute } C' \ B \ C \rangle$ )
ultimately show ?thesis
  using acute-out2--acute by auto

```

qed

qed

lemma *perp-bet-obtuse*:

```

assumes  $B \neq C'$  and
   $A \ B \text{ Perp } C \ C'$  and
   $\text{Bet } A \ B \ C'$ 

```

shows $\text{Obtuse } A \ B \ C$

proof –

```

have  $\text{Acute } C' \ B \ C$ 
proof –
  have  $B \text{ Out } C' \ C'$ 
    using assms(1) out-trivial by auto
  moreover have  $\text{Col } A \ B \ C'$ 
    by (simp add: Col-def assms(3))
  hence  $C' \ B \text{ Perp } C \ C'$ 
    using Out-def assms(2) assms(3) bet-col1 calculation perp-col2 by auto
  ultimately show ?thesis
    using perp-out-acute by blast

```

qed

thus ?thesis

```

  using acute-bet--obtuse assms(2) assms(3) between-symmetry perp-not-eq-1 by blast

```

qed

lemma *sac-perm*:

```

assumes  $\text{Saccheri } A \ B \ C \ D$ 
shows  $\text{Saccheri } D \ C \ B \ A$ 

```

by (meson Cong-cases Saccheri-def l8-2 Tarski-neutral-dimensionless-axioms
 assms invert-one-side one-side-symmetry)

lemma sac-distincts:
 assumes Saccheri A B C D
 shows $A \neq B \wedge B \neq C \wedge C \neq D \wedge A \neq D \wedge A \neq C \wedge B \neq D$
 proof –
 have Per B A D
 using Saccheri-def assms by blast
 have Per A D C
 using Saccheri-def assms by blast
 have A D OS B C
 using Saccheri-def assms by blast
 hence $B \neq C$
 using $\langle \text{Per A D C} \rangle \langle \text{Per B A D} \rangle$ l8-2 l8-7 os-distincts by blast
 thus ?thesis
 using $\langle \text{A D OS B C} \rangle$ os-distincts by blast
 qed

lemma lam-perm:
 assumes Lambert A B C D
 shows Lambert A D C B
 proof –
 have Per D A B
 using Lambert-def assms l8-2 by presburger
 moreover
 have Coplanar A D C B
 using assms lambert--coplanar ncoplanar-perm-5 by blast
 ultimately
 show ?thesis
 using Lambert-def assms by auto
 qed

lemma sac--cong:
 assumes Saccheri A B C D
 shows Cong A C B D
 proof –
 have Per B A D
 using Saccheri-def assms by blast
 have Per A D C
 using Saccheri-def assms by blast
 have $A \neq B$
 using assms sac-distincts by auto
 have $C \neq D$
 using assms sac-distincts by auto
 have $A \neq D$
 using assms sac-distincts by auto
 have B A D CongA C D A
 by (metis Per-cases $\langle A \neq B \rangle \langle A \neq D \rangle \langle C \neq D \rangle \langle \text{Per A D C} \rangle \langle \text{Per B A D} \rangle$ l11-16)
 moreover
 have Cong A B D C
 using assms Saccheri-def not-cong-1243 by blast
 moreover
 have Cong A D D A
 by (simp add: cong-pseudo-reflexivity)
 ultimately
 show ?thesis
 using l11-49 cong-4312 by blast
 qed

lemma sac--conga:
 assumes Saccheri A B C D
 shows A B C CongA B C D
 proof –

```

have Cong A B C D
  using Saccheri-def assms by blast
hence Cong A B D C
  using Cong-cases by auto
moreover
have Cong A C D B
  by (meson assms not-cong-1243 sac--cong)
moreover
have Cong B C C B
  by (simp add: cong-pseudo-reflexivity)
ultimately
show ?thesis
  using l11-51 by (metis sac-distincts assms conga-right-comm)
qed

```

```

lemma lam--pars1234:
  assumes Lambert A B C D
  shows A B ParStrict C D
proof -
  have A ≠ B
    using assms Lambert-def by blast
  moreover
  have A ≠ D
    using assms Lambert-def by blast
  moreover
  have Per B A D
    using assms Lambert-def by blast
  ultimately
  have ¬ Col B A D
    using per-col-eq by blast
  hence ¬ Col A B D
    using Col-cases by blast
  moreover
  have Col C D D
    by (simp add: col-trivial-2)
  moreover
  have Coplanar A B C D
    using assms Lambert-def by blast
  moreover
  have A B Perp D A
    using assms Lambert-def by (metis per-perp perp-comm)
  moreover
  have C D Perp D A
    using assms Lambert-def by (metis Perp-perm per-perp)
  ultimately
  show ?thesis
    using col-cop-perp2--pars by blast
qed

```

```

lemma lam--pars1423:
  assumes Lambert A B C D
  shows A D ParStrict B C
  by (meson assms lam--pars1234 lam-perm par-strict-right-comm)

```

```

lemma lam--par1234:
  assumes Lambert A B C D
  shows A B Par C D
  by (meson assms lam--pars1423 lam-perm par-strict-par par-strict-right-comm)

```

```

lemma lam--par1423:
  assumes Lambert A B C D
  shows A D Par B C
  by (meson assms lam--par1234 lam-perm par-right-comm)

```

```

lemma lam--os:
  assumes Lambert A B C D

```

shows $A B OS C D$
 using *l12-6 assms lam--pars1234* by blast

lemma *per2-os--pars*:
 assumes $Per B A D$ and
 $Per A D C$ and
 $A D OS B C$
 shows $A B ParStrict C D$
 proof –
 have $\neg Col A B D$
 using *assms(3) col123--nos not-col-permutation-5* by blast
 moreover have $Col C D D$
 by (*simp add: col-trivial-2*)
 moreover have $Coplanar A B C D$
 using *assms(3) coplanar-perm-3 os--coplanar* by blast
 moreover have $A B Perp D A$
 by (*metis Perp-cases per-perp assms(1) calculation(1) col-trivial-1 col-trivial-3*)
 moreover have $C D Perp D A$
 by (*metis Perp-cases os-distincts per-perp assms(2) assms(3)*)
 ultimately show ?thesis
 using *col-cop-perp2--pars* by blast
 qed

lemma *per2-os--ncol123*:
 assumes $Per B A D$ and
 $Per A D C$ and
 $A D OS B C$
 shows $\neg Col A B C$
 proof –
 have $A B ParStrict C D$
 using *assms(1) assms(2) assms(3) per2-os--pars* by blast
 thus ?thesis
 using *par-strict-not-col-1* by blast
 qed

lemma *per2-os--ncol234*:
 assumes $Per B A D$ and
 $Per A D C$ and
 $A D OS B C$
 shows $\neg Col B C D$
 proof –
 have $A B ParStrict C D$
 using *assms(1) assms(2) assms(3) per2-os--pars* by blast
 thus ?thesis
 using *par-strict-not-col-2* by blast
 qed

lemma *sac--pars1234*:
 assumes $Saccheri A B C D$
 shows $A B ParStrict C D$
 using *assms Saccheri-def per2-os--pars* by blast

lemma *sac--par1234*:
 assumes $Saccheri A B C D$
 shows $A B Par C D$
 using *assms par-strict-par sac--pars1234* by blast

lemma *lt-os-per2--lta*:
 assumes $Per B A D$ and
 $Per A D C$ and
 $A D OS B C$ and
 $A B Lt C D$

```

shows  $B C D LtA A B C$ 
proof -
  have  $A B Lt D C$ 
    by (simp add: assms(4) lt-right-comm)
  then obtain  $E$  where  $1: Bet D E C \wedge Cong A B D E \wedge \neg Cong A B D C$ 
    by (meson cong--nlt le-bet lt--le not-cong-3412)
  hence  $E \neq C$ 
    by auto
  have  $\neg Col A D B$ 
    using assms(3) one-side-not-col123 by auto
  have  $\neg Col A D C$ 
    using assms(3) col123--nos one-side-symmetry by blast
  have  $D \neq C$ 
    using  $\langle \neg Col A D C \rangle$  col-trivial-2 by force
  have  $D \neq E$ 
    using  $1 \langle \neg Col A D B \rangle$  col-trivial-3 cong-identity by blast
  have  $\neg Col A B C$ 
    using assms(1) assms(2) assms(3) per2-os--ncol123 by blast
  have  $\neg Col B C D$ 
    using assms(1) assms(2) assms(3) per2-os--ncol234 by blast
  have  $\neg Col E C B$ 
    by (metis 1 NCol-perm  $\langle \neg Col B C D \rangle$  bet-col col-trivial-2 l6-21)
  have  $A B ParStrict C D$ 
    using assms(1) assms(2) assms(3) per2-os--pars by blast
  have  $C \neq B$ 
    using  $\langle \neg Col E C B \rangle$  col-trivial-2 by auto
  have  $E \neq B$ 
    using  $\langle \neg Col E C B \rangle$  col-trivial-3 by auto
  have  $B C D LtA B E D$ 
proof -
  have  $E B C LtA B E D$ 
    using  $1 \langle D \neq E \rangle \langle \neg Col E C B \rangle$  between-symmetry l11-41-aux by blast
  have  $E C B LtA B E D$ 
    using  $1 \langle D \neq E \rangle \langle \neg Col E C B \rangle$  between-symmetry l11-41 by blast
  have  $E C B CongA B C D$ 
proof -
  have  $E C B CongA B C E$ 
    by (metis conga-pseudo-refl  $\langle C \neq B \rangle \langle E \neq C \rangle$ )
  moreover
  have  $Bet C E D$  using 1
    by (simp add: between-symmetry)
  hence  $C Out D E$ 
    by (metis 1 bet-out-1 l6-6)
  ultimately
  show ?thesis
    by (metis Out-def  $\langle C \neq B \rangle$  between-trivial conga-right-comm out2--conga)
qed
  thus ?thesis
    by (metis conga-preserves-lta nltA  $\langle D \neq E \rangle \langle E C B LtA B E D \rangle \langle E \neq B \rangle$  or-lta2-conga)
qed
moreover
have  $B E D LtA A B C$ 
proof -
  have Saccheri  $A B E D$ 
proof -
  have  $Per A D E$ 
    by (meson 1 Col-perm  $\langle D \neq C \rangle$  assms(2) bet-col per-col)
  moreover
  have  $Cong A B E D$ 
    using 1 Cong-cases by blast
  moreover
  have  $A D OS B E$ 
    by (meson 1  $\langle D \neq C \rangle \langle D \neq E \rangle$  assms(3) bet2--out invert-one-side l5-1
        out-out-one-side)
  ultimately
  show ?thesis

```

```

    using assms(1) Saccheri-def by blast
qed
hence A B E CongA B E D
  by (simp add: sac--conga)
moreover
have A B E LtA A B C
proof -
  have E InAngle A B C
  proof -
    have B A ParStrict C E
    proof -
      have B A ParStrict C D
      by (simp add: ⟨A B ParStrict C D⟩ par-strict-left-comm)
    moreover
    have Col C D E
    by (simp add: 1 Col-def)
    ultimately
    show ?thesis
    using ⟨E ≠ C⟩ par-strict-col-par-strict by auto
  qed
hence B A OS C E
  by (simp add: l12-6)
moreover
have B C OS A E
proof -
  have B C OS A D
  using ⟨A B ParStrict C D⟩ assms(3) l9-9-bis par-one-or-two-sides
    par-strict-right-comm by blast
  moreover
  have B C OS D E
  by (metis 1 Col-cases ⟨¬ Col B C D⟩ bet-col between-equality-2
    invert-one-side l6-4-2 out-one-side)
  ultimately
  show ?thesis
  using one-side-transitivity by blast
qed
ultimately
show ?thesis
  by (simp add: os2--inangle)
qed
hence A B E LeA A B C
  by (simp add: inangle--lea)
moreover
{
  assume 2: A B E CongA A B C
  have A B OS E C
  by (meson 2 NCol-cases ⟨E InAngle A B C⟩ ⟨¬ Col E C B⟩
    inangle--lta lta-not-conga)
  hence B Out E C
  using 2 conga-os--out by blast
  hence False
  using NCol-cases ⟨¬ Col E C B⟩ out-col by blast
}
hence ¬ A B E CongA A B C
  by auto
ultimately
show ?thesis
  by (simp add: LtA-def)
qed
ultimately
show ?thesis
  by (meson conga--lea456123 lea123456-lta--lta)
qed
ultimately
show ?thesis
  by (meson lta-trans)

```


qed

lemma *lt4321-os-per2--lta:*

assumes *Per B A D and*
 Per A D C and
 A D OS B C and
 D C Lt B A

shows *A B C LtA B C D*

by (*meson Per-cases assms(1) assms(2) assms(3) assms(4) invert-one-side*
 lt-os-per2--lta lta-comm one-side-symmetry)

lemma *lta-os-per2--lt:*

assumes *Per B A D and*
 Per A D C and
 A D OS B C and
 B C D LtA A B C

shows *A B Lt C D*

proof –

{
 assume *A B Gt C D*
 hence *C D Lt A B*
 by (*simp add: Gt-def*)
 hence *D C Lt B A*
 by (*simp add: lt-comm*)
 hence *A B C LtA B C D*
 using *assms(1) assms(2) assms(3) lt4321-os-per2--lta* **by** *blast*
 hence *?thesis*
 by (*meson assms(4) not-and-lta Tarski-neutral-dimensionless-axioms*)
}

moreover

{
 assume *Cong A B C D*
 have *B C D LeA A B C*
 by (*simp add: assms(4) lta--lea*)
 have $\neg B C D \text{ Cong} A A B C$
 by (*simp add: assms(4) lta-not-conga*)
 hence *?thesis*
 using *Saccheri-def <Cong A B C D> assms(1) assms(2) assms(3)*
 conga-sym sac--conga **by** *blast*
}

ultimately

show *?thesis*
 using *or-lt-cong-gt* **by** *blast*

qed

lemma *lta123234-os-per2--lt:*

assumes *Per B A D and*
 Per A D C and
 A D OS B C and
 A B C LtA B C D

shows *D C Lt B A*

by (*meson lta-os-per2--lt Tarski-neutral-dimensionless-axioms assms(1)*
 assms(2) assms(3) assms(4) invert-one-side l8-2 lta-comm one-side-symmetry)

lemma *conga-per2-os--cong:*

assumes *Per B A D and*
 Per A D C and
 A D OS B C and
 B C D CongA A B C

shows *Cong A B C D*

proof –

{
 assume *A B Lt C D*
 hence *?thesis*
 using *lt-os-per2--lta assms(1) assms(2) assms(3) assms(4) lta-not-conga* **by** *blast*
}

```

moreover
{
  assume  $A B Gt C D$ 
  hence  $C D Lt A B$ 
    by (simp add: Gt-def)
  hence  $D C Lt B A$ 
    by (meson Lt-cases)
  hence ?thesis
    using  $assms(1) assms(2) assms(3) lt4321-os-per2--lta$ 
    by (meson assms(4) lta-not-conga not-conga-sym)
}
ultimately
show ?thesis
  using or-lt-cong-gt by blast
qed

lemma mid2-sac--perp-lower:
assumes  $Saccheri A B C D$  and
   $M$  Midpoint  $B C$  and
   $N$  Midpoint  $A D$ 
shows  $A D Perp M N$ 
proof –
  have  $A B C CongA B C D$ 
    using  $assms(1) sac--conga$  by blast
  have  $A \neq B$ 
    using  $assms(1) sac-distincts$  by auto
  have  $C \neq B$ 
    using  $assms(1) sac-distincts$  by blast
  have  $C \neq D$ 
    using  $assms(1) sac-distincts$  by presburger
  have  $A \neq D$ 
    using  $assms(1) sac-distincts$  by blast
  have  $A \neq N$ 
    using  $\langle A \neq D \rangle assms(3) is-midpoint-id$  by blast
  have  $M \neq B$ 
    using  $\langle C \neq B \rangle assms(2) is-midpoint-id$  by blast
  have  $M \neq C$ 
    using  $\langle C \neq B \rangle assms(2) is-midpoint-id-2$  by force
  have  $Cong A B C D$ 
    using  $assms(1) Saccheri-def$  by blast
  have  $M B A CongA M C D$ 
proof –
  have  $C B A CongA B C D$ 
    by (simp add:  $\langle A B C CongA B C D \rangle conga-left-comm$ )
  moreover
  have  $B Out M C$ 
    using  $assms(2) Midpoint-def \langle M \neq B \rangle bet-out$  by blast
  moreover
  have  $B Out A A$ 
    by (simp add:  $\langle A \neq B \rangle out-trivial$ )
  moreover
  have  $C Out M B$  using  $assms(2) Midpoint-def \langle M \neq C \rangle bet-out-1$  by blast
  moreover
  have  $C Out D D$ 
    using  $\langle C \neq D \rangle out-trivial$  by auto
  ultimately
  show ?thesis
    using l11-10 by blast
qed
have  $A D OS B M$ 
proof –
  have  $A D OS B C$ 
    using  $assms(1) Saccheri-def$  by blast
  moreover
  have  $Bet B M C$ 

```

```

    by (simp add: assms(2) midpoint-bet)
ultimately
show ?thesis
  using l9-17 by blast
qed
hence  $\neg Col\ A\ D\ M$ 
  using col124--nos by blast
moreover
have  $Per\ M\ N\ A$ 
proof -
  have  $N\ Midpoint\ A\ D$ 
    by (simp add: assms(3))
  moreover
  have  $Cong\ M\ A\ M\ D$ 
    by (metis Mid-cases cong-left-commutativity  $\langle Cong\ A\ B\ C\ D \rangle$ 
       $\langle M\ B\ A\ CongA\ M\ C\ D \rangle$   $\langle M \neq B \rangle$  assms(2) l11-49 l7-13-R1 l7-3-2)
  ultimately
  show ?thesis
    using Per-def by auto
qed
ultimately
show ?thesis
  by (metis  $\langle A \neq D \rangle$   $\langle A \neq N \rangle$  assms(3) col-per-perp col-permutation-1 midpoint-col)
qed

```

```

lemma mid2-sac--perp-upper-R1:
  assumes Saccheri  $A\ B\ C\ D$  and
     $M\ Midpoint\ B\ C$  and
     $N\ Midpoint\ A\ D$ 
  shows  $Cong\ N\ B\ N\ C$ 
proof -
  have  $A \neq B$ 
    using assms(1) sac-distincts by blast
  have  $C \neq B$ 
    using assms(1) sac-distincts by blast
  have  $D \neq C$ 
    using assms(1) sac-distincts by blast
  have  $A \neq D$ 
    using assms(1) sac-distincts by blast
  have  $Per\ B\ A\ D$ 
    using assms(1) Saccheri-def by blast
  have  $Per\ A\ D\ C$ 
    using assms(1) Saccheri-def by blast
  have  $Cong\ A\ B\ C\ D$ 
    using assms(1) Saccheri-def by blast
  have  $A \neq N$ 
    using  $\langle A \neq D \rangle$  assms(3) is-midpoint-id by blast
  have  $D \neq N$ 
    using  $\langle A \neq N \rangle$  assms(3) is-midpoint-id-2 by blast
  have  $N\ A\ B\ CongA\ N\ D\ C$ 
  proof -
    have  $Per\ N\ A\ B$ 
    proof -
      have  $Per\ D\ A\ B$ 
        using Per-cases  $\langle Per\ B\ A\ D \rangle$  by blast
      moreover
      have  $Col\ A\ D\ N$ 
        using assms(3) by (meson midpoint-col not-col-permutation-1)
      moreover
      have  $Per\ N\ D\ C$ 
        using  $\langle A \neq D \rangle$   $\langle Per\ A\ D\ C \rangle$  calculation(2) col-permutation-4 l8-3 by blast
      ultimately
      show ?thesis
        using  $\langle A \neq D \rangle$  l8-3 by blast
    qed
  qed
  moreover

```

```

have Col D A N
  using assms(3) midpoint-col not-col-permutation-3 by blast
hence Per N D C
  using  $\langle A \neq D \rangle \langle \text{Per } A D C \rangle$  l8-3 by blast
ultimately
show ?thesis
  using  $\langle A \neq B \rangle \langle A \neq N \rangle \langle D \neq C \rangle \langle D \neq N \rangle$  l11-16 by auto
qed
moreover
have Cong A N D N
  using assms(3) midpoint-cong not-cong-1243 by blast
moreover
have Cong A B D C
  by (simp add:  $\langle \text{Cong } A B C D \rangle$  cong-right-commutativity)
ultimately
show ?thesis
  using Cong-cases cong2-conga-cong by blast
qed

lemma mid2-sac--perp-upper:
  assumes Saccheri A B C D and
    M Midpoint B C and
    N Midpoint A D
  shows B C Perp M N
proof -
  have C  $\neq$  B
    using assms(1) sac-distincts by blast
  have Cong N B N C
    using assms(1) assms(2) assms(3) mid2-sac--perp-upper-R1 by blast
  have B C Perp N M
  proof -
    have Col B C M
      using NCol-cases assms(2) midpoint-col by blast
    moreover
    have Col B C B
      by (simp add: col-trivial-3)
    moreover
    have B  $\neq$  M
      using  $\langle C \neq B \rangle$  assms(2) is-midpoint-id by blast
    moreover
    {
      assume Col B C N
      have A D OS B N
      proof -
        have A D OS B C
          using assms(1) Saccheri-def by blast
        moreover
        have Col B N C
          using Col-cases  $\langle \text{Col } B C N \rangle$  by blast
        hence N Midpoint B C
          using l7-20-bis  $\langle C \neq B \rangle \langle \text{Cong } N B N C \rangle$  by auto
        hence Bet B N C
          using midpoint-bet by blast
        ultimately
        show ?thesis
          using l9-17 by blast
      qed
      hence  $\neg$  Col A D N
        using col124--nos by blast
      hence False
        using assms(3) Col-cases midpoint-col by blast
    }
  hence  $\neg$  Col B C N
    by auto
  moreover
  have Per N M B

```

```

    using Per-def ⟨Cong N B N C⟩ assms(2) by blast
ultimately
show ?thesis
    using l8-16-2 by blast
qed
thus ?thesis
    by (simp add: perp-right-comm)
qed

```

```

lemma sac--pars1423:
  assumes Saccheri A B C D
  shows A D ParStrict B C
proof -
  obtain M where 1: M Midpoint B C
    using midpoint-existence by blast
  obtain N where 2: N Midpoint A D
    using midpoint-existence by blast
  have A D Perp M N
    using 1 2 assms mid2-sac--perp-lower by blast
  moreover
  have B C Perp M N
    using 1 2 assms mid2-sac--perp-upper by blast
  moreover
  have A D OS B M
  proof -
    have A D OS B C
      using assms Saccheri-def by blast
    moreover
    have Bet B M C
      by (simp add: 1 midpoint-bet)
    ultimately
    show ?thesis
      using l9-17 by blast
  qed
  hence ¬ Col A D M
    using one-side-not-col124 by blast
  moreover
  have Col B C M
    by (meson 1 col-permutation-1 midpoint-col)
  moreover
  have Coplanar A D B C
    by (meson assms coplanar-perm-4 sac--coplanar)
  ultimately
  show ?thesis using col-cop-perp2--pars by blast
qed

```

```

lemma sac--par1423:
  assumes Saccheri A B C D
  shows A D Par B C
proof -
  have A D ParStrict B C
    using assms(1) sac--pars1423 by blast
  thus ?thesis
    by (simp add: Par-def)
qed

```

```

lemma mid2-sac--lam6521:
  assumes Saccheri A B C D and
    M Midpoint B C and
    N Midpoint A D
  shows Lambert N M B A
proof -
  have Per B A D

```

```

  using assms(1) Saccheri-def by blast
have  $N \neq M$ 
  using assms(1) assms(2) assms(3) mid2-sac--perp-upper perp-distinct by blast
moreover
have  $M \neq B$ 
  using assms(1) assms(2) is-midpoint-id sac-distincts by blast
moreover
have  $A \neq B$ 
  using assms(1) sac-distincts by blast
moreover
have  $N \neq A$ 
  using assms(1) assms(3) is-midpoint-id sac-distincts by blast
moreover
have  $A \ D \ \text{Perp} \ M \ N$ 
  using assms(1) assms(2) assms(3) mid2-sac--perp-lower by auto
hence  $\text{Per} \ M \ N \ A$ 
  by (meson assms(3) col-permutation-1 l8-16-1 midpoint-col not-col-distincts)
moreover
have  $\text{Per} \ N \ A \ B$ 
proof -
  have  $\text{Per} \ D \ A \ B$ 
    by (simp add:  $\langle \text{Per} \ B \ A \ D \rangle$  l8-2)
  moreover
  have  $\text{Col} \ A \ D \ N$ 
    using assms(3) col-permutation-2 midpoint-col by blast
  ultimately
  show ?thesis
    by (metis l8-3 midpoint-distinct-2 assms(3))
qed
moreover
have  $M \ N \ \text{Perp} \ B \ M$ 
proof -
  have  $B \ C \ \text{Perp} \ M \ N$ 
    using assms(1) assms(2) assms(3) mid2-sac--perp-upper by blast
  hence  $M \ N \ \text{Perp} \ B \ C$ 
    using Perp-cases by blast
  moreover
  have  $\text{Col} \ B \ C \ M$ 
    using Col-perm assms(2) midpoint-col by blast
  ultimately
  show ?thesis
    using  $\langle M \neq B \rangle$  perp-col1 by auto
qed
hence  $\text{Per} \ N \ M \ B$ 
  by (simp add: perp-per-1)
moreover
have Coplanar  $N \ M \ B \ A$ 
proof -
  have  $B \neq C$ 
    using assms(1) sac-distincts by blast
  moreover
  have Coplanar  $N \ A \ B \ C$ 
proof -
  have Coplanar  $B \ C \ A \ D$ 
    using assms(1) ncoplanar-perm-12 sac--coplanar by blast
  moreover
  have  $A \neq D$ 
    using  $\langle A \ D \ \text{Perp} \ M \ N \rangle$  perp-not-eq-1 by blast
  moreover
  have  $\text{Col} \ A \ D \ N$ 
    using assms(3) NCol-cases midpoint-col by blast
  ultimately
  show ?thesis
    using col-cop--cop coplanar-perm-22 by blast
qed
moreover

```

```

have Col B C M
  using assms(2) midpoint-col not-col-permutation-2 by blast
ultimately
show ?thesis
  using coplanar-perm-5 col-cop--cop by blast
qed
ultimately
show ?thesis
  using Lambert-def by blast
qed

lemma mid2-sac--lam6534:
  assumes Saccheri A B C D and
    M Midpoint B C and
    N Midpoint A D
  shows Lambert N M C D
  by (meson assms(1) assms(2) assms(3) l7-2 mid2-sac--lam6521 sac-perm)

lemma lam6521-mid2--sac:
  assumes Lambert N M B A and
    M Midpoint B C and
    N Midpoint A D
  shows Saccheri A B C D
proof -
  have N ≠ M
    using assms(1) Lambert-def by blast
  have B ≠ M
    using assms(1) Lambert-def by blast
  have B ≠ A
    using assms(1) Lambert-def by blast
  have N ≠ A
    using assms(1) Lambert-def by blast
  have Per M N A
    using assms(1) Lambert-def by blast
  have Per N A B
    using assms(1) Lambert-def by blast
  have Per N M B
    using assms(1) Lambert-def by blast
  have D ≠ A
    using ⟨N ≠ A⟩ assms(3) l8-20-2 by blast
  have C ≠ B
    using ⟨B ≠ M⟩ assms(2) l8-20-2 by blast
  have Per D A B
    by (meson ⟨N ≠ A⟩ ⟨Per N A B⟩ assms(3) col-per2--per l8-20-1-R1 midpoint-col)
  have Per B A D
    by (simp add: ⟨Per D A B⟩ l8-2)
  moreover
  have D A ReflectL M N
  proof -
    have N Midpoint A D ∧ Col M N N
      using assms(3) not-col-distincts by blast
    have M N Perp A D ∨ A = D
      by (metis Perp-perm ⟨N ≠ A⟩ ⟨N ≠ M⟩ ⟨Per M N A⟩ assms(3)
        col-per-perp midpoint-col)
    moreover
    show ?thesis
      by (metis Mid-cases l7-13 ⟨D ≠ A⟩ ⟨N ≠ A⟩ ⟨N ≠ M⟩ ⟨Per M N A⟩ assms(3)
        calculation cong-cop-per2 l10-4-spec l7-3-2 midpoint-col
        per-col perp--coplanar)
  qed
  have C B ReflectL M N
  proof -
    have M Midpoint B C ∧ Col M N M
      using assms(2) not-col-distincts by blast
    moreover
    have M N Perp B C ∨ B = C

```

```

    by (metis Perp-perm  $\langle B \neq M \rangle \langle N \neq M \rangle \langle \text{Per } N M B \rangle$  assms(2) col-per-perp midpoint-col)
ultimately
show ?thesis
  using ReflectL-def by blast
qed
have Per A D C
proof -
  have A D ReflectL M N
    by (simp add:  $\langle D A \text{ ReflectL } M N \rangle$  l10-4-spec)
  moreover
  have B C ReflectL M N
    by (simp add:  $\langle C B \text{ ReflectL } M N \rangle$  l10-4-spec)
  ultimately
  show ?thesis
    using image-spec-preserves-per by (meson  $\langle D A \text{ ReflectL } M N \rangle \langle \text{Per } D A B \rangle$ )
qed
moreover
have Cong B A C D
  using l10-10-spec  $\langle C B \text{ ReflectL } M N \rangle \langle D A \text{ ReflectL } M N \rangle$  by blast
hence Cong A B C D
  using Cong-cases by blast
moreover
have A D OS B C
proof -
  have Col A N D
    using assms(3) Midpoint-def bet-col by blast
  moreover
  have A N ParStrict B C
  proof -
    have A N ParStrict B M
      using assms(1) lam--pars1423 par-strict-comm by blast
    moreover
    have Col B M C
      using assms(2) Midpoint-def bet-col by blast
    ultimately
    show ?thesis
      using  $\langle C \neq B \rangle$  par-strict-col-par-strict by force
  qed
hence A N OS B C
  by (simp add: l12-6)
ultimately
show ?thesis
  using  $\langle D \neq A \rangle$  col-one-side by blast
qed
ultimately
show ?thesis
  by (simp add: Saccheri-def)
qed

lemma lam6534-mid2--sac:
  assumes Lambert N M C D and
    M Midpoint B C and
    N Midpoint A D
  shows Saccheri A B C D
  by (meson Mid-cases assms(1) assms(2) assms(3) lam6521-mid2--sac sac-perm)

```

```

lemma cong-lam--per:
  assumes Lambert A B C D and
    Cong A D B C
  shows Per B C D
proof -
  have Saccheri A D C B
  proof -
    have A B ParStrict C D

```



```

    using assms(1) lam--pars1234 by auto
  hence  $A B OS D C$ 
    by (meson l12-6 one-side-symmetry)
  moreover
  have  $Per D A B$ 
    using Lambert-def assms(1) l8-2 by presburger
  moreover
  have  $Per A B C$ 
    using Lambert-def assms(1) by presburger
  moreover
  have  $Cong A D C B$ 
    using assms(2) not-cong-1243 by blast
  ultimately
  show ?thesis
    using Saccheri-def by blast
qed
hence  $A D C Cong A D C B$ 
  by (simp add: sac--conga)
moreover
have  $Per A D C$ 
  using Lambert-def assms(1) by presburger
ultimately
show ?thesis
  using l11-17 l8-2 by blast
qed

lemma lam-lt--acute:
  assumes Lambert  $A B C D$  and
     $A D Lt B C$ 
  shows Acute  $B C D$ 
proof -
  have  $Per B A D$ 
    using Lambert-def assms(1) by presburger
  have  $Per A D C$ 
    using assms(1) Lambert-def by blast
  moreover
  have  $B C D Lt A A D C$ 
proof -
  have  $Per D A B$ 
    using  $\langle Per B A D \rangle$  l8-2 by blast
  moreover
  have  $Per A B C$ 
    using Lambert-def assms(1) by presburger
  moreover
  have  $A B OS D C$ 
    by (simp add: assms(1) lam--os one-side-symmetry)
  moreover
  have  $A D Lt C B$ 
    using assms(2) lt-right-comm by blast
  ultimately
  show ?thesis
    by (simp add: lt-os-per2--lta lta-left-comm)
qed
ultimately
show ?thesis
  using Acute-def by blast
qed

lemma lam-lt--obtuse:
  assumes Lambert  $A B C D$  and
     $B C Lt A D$ 
  shows Obtuse  $B C D$ 
proof -
  have  $Per A D C$ 
    using assms(1) Lambert-def by blast
  moreover

```

```

have A D C LtA B C D
proof -
  have Per B A D
    using Lambert-def assms(1) by presburger
  moreover
  have Per C B A
    using Lambert-def assms(1) l8-2 by presburger
  moreover
  have B A OS D C
    by (meson assms(1) invert-one-side lam--os one-side-symmetry)
  moreover
  have B C Lt D A
    using assms(2) lt-right-comm by blast
  ultimately
  show ?thesis
    by (simp add: lt-os-per2--lta lta-left-comm one-side-symmetry)
qed
ultimately
show ?thesis
  using Obtuse-def by blast
qed

lemma ngta:
  shows  $\neg A B C GtA A B C$  using GtA-def nlta
  by blast

lemma lam-per--cong:
  assumes Lambert A B C D and
    Per B C D
  shows Cong A D B C
proof -
  {
    assume A D Lt B C
    hence Acute B C D
      using assms(1) lam-lt--acute by blast
    hence B C D LtA B C D
      using acute-not-per assms(2) by blast
    hence False
      by (simp add: nlta)
  }
  moreover
  {
    assume A D Gt B C
    hence Obtuse B C D
      using Gt-def assms(1) lam-lt--obtuse by blast
    hence B C D GtA B C D
      by (metis Lambert-def assms(1) assms(2) obtuse--nsams per2--sams)
    hence False
      by (simp add: ngta)
  }
  ultimately
  show ?thesis
    using or-lt-cong-gt by blast
qed

lemma acute-lam--lt:
  assumes Lambert A B C D and
    Acute B C D
  shows A D Lt B C
proof -
  {
    assume A D Gt B C
    hence B C Lt A D
      by (simp add: Gt-def)
    hence Obtuse B C D
      using assms(1) lam-lt--obtuse by blast
  }

```

```

  hence False
  using acute--not-obtuse assms(2) by blast
}
moreover
{
  assume Cong A D B C
  have Acute B C D
    by (simp add: assms(2))
  moreover
  have  $B \neq C$ 
    using acute-distincts calculation by auto
  moreover
  have  $C \neq D$ 
    using acute-distincts calculation(1) by blast
  moreover
  have Per B C D
    using  $\langle \textit{Cong A D B C} \rangle \textit{ assms(1) cong-lam--per}$  by auto
  ultimately
  have False
    using acute-not-per by blast
}
ultimately
show ?thesis
  using or-lt-cong-gt by blast
qed

```

```

lemma lam-obtuse--lt:
  assumes Lambert A B C D and
    Obtuse B C D
  shows B C Lt A D
proof -
  {
    assume Cong B C A D
    hence Per B C D
      using assms(1) cong-lam--per not-cong-3412 by blast
    moreover
    have Obtuse B C D
      by (simp add: assms(2))
    moreover
    have  $B \neq C$ 
      using calculation obtuse-distincts by blast
    moreover
    have  $C \neq D$ 
      using calculation(2) obtuse-distincts by blast
    ultimately
    have False
      using obtuse--nsams per2--sams by blast
  }
  moreover
  {
    assume B C Gt A D
    hence A D Lt B C
      by (simp add: Gt-def)
    hence Acute B C D
      using assms(1) lam-lt--acute by blast
    hence False
      using acute--not-obtuse assms(2) by blast
  }
  ultimately
  show ?thesis
    using or-lt-cong-gt by blast
qed

```

```

lemma sac--ex2-mid-mid-lam-conga:
  assumes Saccheri A B C D
  shows  $\exists M N. M \textit{ Midpoint } B C \wedge N \textit{ Midpoint } A D \wedge \textit{Lambert } N M B A \wedge A B C \textit{ CongA } M B A$ 

```

```

proof –
  obtain M where 1: M Midpoint B C
    using midpoint-existence by blast
  moreover
  obtain N where 2: N Midpoint A D
    using midpoint-existence by blast
  moreover
  have Lambert N M B A
    using 1 2 mid2-sac--lam6521 assms(1) by blast
  moreover
  have A B C CongA M B A
    by (metis 1 midpoint-distinct-1 out2--conga out-trivial sac-distincts
      assms(1) bet-out conga-left-comm midpoint-bet)
  ultimately
  show ?thesis by blast
qed

```

```

lemma cong-sac--per-aux1:
  assumes Saccheri A B C D and
    Cong A D B C
  shows Per A B C
proof –
  obtain M N where 1: M Midpoint B C  $\wedge$  N Midpoint A D  $\wedge$ 
    Lambert N M B A  $\wedge$  A B C CongA M B A
    using assms(1) sac--ex2-mid-mid-lam-conga by blast
  have M  $\neq$  B
    using 1 assms(1) is-midpoint-id sac-distincts by blast
  have Cong N A M B
    by (meson 1 assms(2) cong-cong-half-2 l7-2 not-cong-4321)
  hence Per A B M
    using Per-perm 1 cong-lam--per by blast
  moreover
  have Col B M C
    using 1 bet-col midpoint-bet by blast
  ultimately
  show ?thesis
    by (metis  $\langle M \neq B \rangle$  per-col)
qed

```

```

lemma cong-sac--per-aux2:
  assumes Saccheri A B C D and
    Per A B C
  shows Cong A D B C
proof –
  obtain M N where 1: M Midpoint B C  $\wedge$  N Midpoint A D  $\wedge$ 
    Lambert N M B A  $\wedge$  A B C CongA M B A
    using assms(1) sac--ex2-mid-mid-lam-conga by blast
  have C  $\neq$  B
    using 1 assms(1) sac-distincts by blast
  have Col B C M
    using 1 NCol-cases midpoint-col by blast
  hence Per M B A
    by (metis  $\langle C \neq B \rangle$  assms(2) l8-2 per-col)
  hence Cong A N B M
    using 1 lam-per--cong not-cong-2143 by blast
  show ?thesis
    using 1  $\langle Cong A N B M \rangle$  cong-mid2--cong by blast
qed

```

```

lemma cong-sac--per:
  assumes Saccheri A B C D
  shows Cong A D B C  $\longleftrightarrow$  Per A B C
  using assms cong-sac--per-aux1 cong-sac--per-aux2 by blast

```

lemma *lt-sac--acute-aux1*:
assumes *Saccheri A B C D* **and**
 $A D \text{ Lt } B C$
shows *Acute A B C*
proof –
obtain $M N$ **where** $1: M \text{ Midpoint } B C \wedge N \text{ Midpoint } A D \wedge$
 $\text{Lambert } N M B A \wedge A B C \text{ CongA } M B A$
using *assms(1) sac--ex2-mid-mid-lam-conga* **by** *blast*
have $N A \text{ Lt } M B$
using $1 \text{ assms}(2) \text{ lt-comm lt-mid2--lt12}$ **by** *blast*
hence *Acute M B A*
using 1 lam-lt--acute **by** *blast*
thus *?thesis*
by (*metis conga-distinct l11-17 1 acute--not-obtuse acute-not-per*
angle-partition conga-obtuse--obtuse)
qed

lemma *lt-sac--acute-aux2*:
assumes *Saccheri A B C D* **and**
 $\text{Acute } A B C$
shows $A D \text{ Lt } B C$
proof –
obtain $M N$ **where** $1: M \text{ Midpoint } B C \wedge N \text{ Midpoint } A D \wedge$
 $\text{Lambert } N M B A \wedge A B C \text{ CongA } M B A$
using *assms(1) sac--ex2-mid-mid-lam-conga* **by** *blast*
hence *Acute M B A*
using *acute-conga--acute assms(2)* **by** *blast*
hence $A N \text{ Lt } M B$
by (*simp add: 1 acute-lam--lt lt-left-comm*)
thus *?thesis*
by (*meson 1 lt-mid2--lt13 lt-right-comm*)
qed

lemma *lt-sac--acute*:
assumes *Saccheri A B C D*
shows $A D \text{ Lt } B C \longleftrightarrow \text{Acute } A B C$
by (*meson assms lt-sac--acute-aux1 lt-sac--acute-aux2*)

lemma *lt-sac--obtuse-aux1*:
assumes *Saccheri A B C D* **and**
 $B C \text{ Lt } A D$
shows *Obtuse A B C*
proof –
obtain $M N$ **where** $1: M \text{ Midpoint } B C \wedge N \text{ Midpoint } A D \wedge$
 $\text{Lambert } N M B A \wedge A B C \text{ CongA } M B A$
using *assms(1) sac--ex2-mid-mid-lam-conga* **by** *blast*
have $M \text{ Midpoint } B C$
using 1 **by** *blast*
have $N \text{ Midpoint } A D$
using 1 **by** *blast*
have $\text{Lambert } N M B A$
using 1 **by** *blast*
have $A B C \text{ CongA } M B A$
using 1 **by** *blast*
hence $M B \text{ Lt } N A$
using $1 \text{ assms}(2) \text{ lt-comm lt-mid2--lt12}$ **by** *blast*
hence *Obtuse M B A*
using $\langle \text{Lambert } N M B A \rangle \text{ lam-lt--obtuse}$ **by** *blast*
thus *?thesis*
by (*meson 1 $\langle \text{Obtuse } M B A \rangle$ conga-obtuse--obtuse not-conga-sym*)
qed

lemma *lt-sac--obtuse-aux2*:
assumes *Saccheri A B C D* **and**
 $\text{Obtuse } A B C$

```

shows  $B C Lt A D$ 
proof -
obtain  $M N$  where  $1: M \text{ Midpoint } B C \wedge N \text{ Midpoint } A D \wedge$ 
 $Lambert N M B A \wedge A B C \text{ CongA } M B A$ 
using assms(1) sac--ex2-mid-mid-lam-conga by blast
have  $M \text{ Midpoint } B C$ 
using  $1$  by blast
have  $N \text{ Midpoint } A D$ 
using  $1$  by blast
have  $Lambert N M B A$ 
using  $1$  by blast
have  $A B C \text{ CongA } M B A$ 
using  $1$  by blast
hence  $Obtuse M B A$ 
using assms(2) conga-obtuse--obtuse by blast
hence  $B M Lt A N$ 
by (simp add: <Lambert N M B A> lam-obtuse--lt lt-comm)
thus ?thesis
using  $1 \text{ lt-mid2--lt13}$  by blast
qed

```

```

lemma lt-sac--obtuse:
assumes  $Saccheri A B C D$ 
shows  $B C Lt A D \longleftrightarrow Obtuse A B C$ 
using assms lt-sac--obtuse-aux1 lt-sac--obtuse-aux2 by blast

```

```

lemma t22-7--per:
assumes  $Saccheri A B C D$  and
 $Bet B P C$  and
 $Bet A Q D$  and
 $A \neq Q$  and
 $Q \neq D$  and
 $Per P Q A$  and
 $Cong P Q A B$ 
shows  $Per A B C$ 
proof -
have  $A \neq B$ 
using assms(1) sac-distincts by blast
have  $C \neq B$ 
using assms(1) sac-distincts by blast
have  $A \neq D$ 
using assms(3) assms(5) between-identity by blast
have  $Per A D C$ 
using assms(1) Saccheri-def by blast
have  $A D OS B C$ 
using assms(1) Saccheri-def by blast
{
assume  $B = P$ 
have  $Per B A Q$ 
proof -
have  $Per B A D$ 
using assms(1) Saccheri-def by blast
moreover
have  $Col A D Q$ 
using Bet-cases Col-def assms(3) by auto
ultimately
show ?thesis
using  $\langle A \neq D \rangle \text{ per-col}$  by blast
qed
hence  $False$ 
using  $\langle B = P \rangle \text{ assms(4) assms(6) l8-7}$  by blast
}
hence  $B \neq P$ 
by auto
{
assume  $C = P$ 

```

```

hence  $Per\ C\ Q\ D$ 
  by (metis Per-cases per2-preserves-diff  $\langle Per\ A\ D\ C \rangle$  assms(3) assms(4)
    assms(5) assms(6) bet-col)
moreover
have  $Per\ C\ D\ Q$ 
  by (metis per2-preserves-diff  $\langle A \neq D \rangle$   $\langle C = P \rangle$   $\langle Per\ A\ D\ C \rangle$  assms(3) assms(5)
    assms(6) bet-col l8-2)
ultimately
have False
  using assms(5) l8-7 by blast
}
hence  $C \neq P$ 
  by blast
have Saccheri  $A\ B\ P\ Q$ 
proof –
  have  $Per\ B\ A\ Q$ 
  proof –
    have  $Per\ B\ A\ D$ 
      using assms(1) Saccheri-def by blast
    moreover
    have  $Col\ A\ D\ Q$ 
      using Col-perm assms(3) bet-col by blast
    ultimately
    show ?thesis
      using  $\langle A \neq D \rangle$  per-col by blast
  qed
moreover
have  $Per\ A\ Q\ P$ 
  using Per-perm assms(6) by blast
moreover
have  $Cong\ A\ B\ P\ Q$ 
  using assms(7) not-cong-3412 by blast
moreover
have  $A\ Q\ OS\ B\ P$ 
proof –
  have  $Col\ A\ D\ Q$ 
    using assms(3) bet-col1 between-trivial by blast
  moreover
  have  $A\ D\ OS\ B\ P$ 
    using  $\langle A\ D\ OS\ B\ C \rangle$  assms(2) l9-17 by blast
  ultimately
  show ?thesis
    using assms(4) col-one-side by blast
  qed
ultimately
show ?thesis
  using Saccheri-def by blast
qed
have Saccheri  $D\ C\ P\ Q$ 
proof –
  have  $Per\ C\ D\ Q$ 
    by (metis Per-perm l5-1  $\langle A \neq D \rangle$   $\langle Per\ A\ D\ C \rangle$  assms(3) bet-col1 col-per2--per l8-5)
  moreover
  have  $Per\ D\ Q\ P$ 
    by (meson Per-perm assms(3) assms(4) assms(6) bet-col col-per2--per l8-5)
  moreover
  have  $Cong\ A\ B\ C\ D$ 
    using assms(1) Saccheri-def by blast
  hence  $Cong\ D\ C\ P\ Q$ 
    by (meson assms(7) cong-transitivity not-cong-3421)
  moreover
  have  $D\ A\ OS\ C\ P$ 
    by (meson  $\langle A\ D\ OS\ B\ C \rangle$  assms(2) between-symmetry invert-one-side
      l9-17 one-side-symmetry)
  hence  $D\ Q\ OS\ C\ P$ 
    by (metis Col-cases assms(3) assms(5) bet-col col-one-side)

```

```

ultimately
show ?thesis
  using Saccheri-def by blast
qed
have  $P \neq Q$ 
  using  $\langle A \neq B \rangle$  assms(7) cong-reverse-identity by blast
have  $A B C A B C$  SumA B P C
proof -
  have  $B P Q$  CongA A B P
    by (meson  $\langle \text{Saccheri } A B P Q \rangle$  not-conga-sym sac--conga)
  hence  $B P Q$  CongA A B C
    by (metis CongA-def l5-1 out2--conga  $\langle C \neq B \rangle$  assms(2) bet2--out not-conga)
  moreover
  have  $Q P C$  CongA A B C
  proof -
    have  $Q P C$  CongA P C D
      by (simp add:  $\langle \text{Saccheri } D C P Q \rangle$  sac--conga sac-perm)
    moreover
    have  $B C D$  CongA A B C
      using assms(1) conga-sym sac--conga by blast
    hence  $P C D$  CongA A B C
      using l11-4-2
      by (metis Bet-cases CongA-def Out-def  $\langle C = P \implies \text{False} \rangle$ 
        assms(2) not-bet-distincts not-conga out2--conga)
    ultimately show ?thesis
      by (meson not-conga)
  qed
ultimately
show ?thesis
  by (metis  $\langle A \neq B \rangle \langle B \neq P \rangle \langle C \neq B \rangle \langle C \neq P \rangle$  assms(2) bet-per2--suma
    conga-left-comm conga-sym conga-trans l11-17 l11-18-2)
qed
thus ?thesis
  using assms(2) bet-suma--per by blast
qed

lemma t22-7--acute:
  assumes Saccheri A B C D and
    Bet B P C and
    Bet A Q D and
     $A \neq Q$  and
    Per P Q A and
     $P Q$  Lt A B
  shows Acute A B C
proof -
  have Per B A D
    using assms(1) Saccheri-def by blast
  have Per A D C
    using assms(1) Saccheri-def by blast
  have Cong A B C D
    using assms(1) Saccheri-def by blast
  have A D OS B C
    using assms(1) Saccheri-def by blast
  have A D ParStrict B C
    by (simp add: assms(1) sac--pars1423)
  have A B ParStrict C D
    by (simp add: assms(1) sac--pars1234)
  hence  $\neg \text{Col } A B C$ 
    using par-strict-not-col-1 by auto
  have  $A \neq B$ 
    using  $\langle \neg \text{Col } A B C \rangle$  col-trivial-1 by auto
  have  $C \neq B$ 
    using  $\langle \neg \text{Col } A B C \rangle$  col-trivial-2 by blast
  have  $A \neq D$ 
    using assms(1) sac-distincts by blast
  have  $C \neq D$ 

```



```

  using ⟨A D OS B C⟩ os-distincts by blast
have P ≠ Q
  by (metis ⟨A D OS B C⟩ assms(2) assms(3) bet-col not-col-permutation-4 one-side-chara)
have B ≠ P
  by (metis l5-1 l8-3 l8-7 ⟨A ≠ D⟩ ⟨Per B A D⟩ assms(3) assms(4) assms(5) bet-col1 l8-2)
have C ≠ P
  by (metis Per-cases per2-preserves-diff ⟨A ≠ D⟩ ⟨Cong A B C D⟩ ⟨Per A D C⟩
      assms(3) assms(4) assms(5) assms(6) bet-col cong--nlt cong-symmetry)
have Q ≠ D
  by (metis (no-types, lifting) bet-out-1 l8-2 out-col ⟨A ≠ D⟩ ⟨C ≠ P⟩
      ⟨Per A D C⟩ ⟨Per B A D⟩ assms(2) assms(5) col-per2--per l8-7)
have A B C LtA B P Q
proof -
  have A B P CongA A B C
    by (metis conga-sym out2--conga out-trivial ⟨A ≠ B⟩ ⟨B ≠ P⟩ assms(2) bet-out)
  moreover
  have B P Q CongA B P Q
    using ⟨B ≠ P⟩ ⟨P ≠ Q⟩ conga-refl by presburger
  moreover
  have A B P LtA B P Q
proof -
  have Col A D Q
    by (simp add: assms(3) bet-col col-permutation-5)
  hence Per B A Q
    using ⟨A ≠ D⟩ ⟨Per B A D⟩ per-col by blast
  moreover
  have Per A Q P
    using assms(5) l8-2 by blast
  moreover
  have Col A D Q
    by (simp add: ⟨Col A D Q⟩)
  hence A Q OS B P
    using ⟨A D OS B C⟩ assms(2) assms(4) col-one-side l9-17 by blast
  moreover
  have Q P Lt B A
    by (simp add: assms(6) lt-comm)
  ultimately
  show ?thesis
    using lt4321-os-per2--lta by blast
qed
ultimately
show ?thesis
  using conga-preserves-lta by force
qed
have A B C LtA C P Q
proof -
  have D C P CongA A B C
proof -
  let ?D = B
  let ?E = C
  let ?F = D
  have A B C CongA B C D
    using assms(1) sac--conga by blast
  hence A ≠ B ∧ C ≠ B ∧ ?D ≠ ?E ∧ ?F ≠ ?E ∧
    (∃ A' C' D' F'. Bet B A A' ∧ Cong A A' ?E ?D ∧ Bet B C C' ∧ Cong C C' ?E ?F ∧
      Bet ?E ?D D' ∧ Cong ?D D' B A ∧ Bet ?E ?F F' ∧ Cong ?F F' B C ∧
      Cong A' C' D' F')
    by (simp add: CongA-def)
  then obtain A' C' D' F' where Bet B A A' and Cong A A' ?E ?D and
    Bet B C C' and Cong C C' ?E ?F and Bet ?E ?D D' and
    Cong ?D D' B A and Bet ?E ?F F' and Cong ?F F' B C and
    Cong A' C' D' F' by blast
  moreover
  hence D C P CongA P C D
    using ⟨C ≠ D⟩ ⟨C ≠ P⟩ conga-pseudo-refl by presburger
  moreover have C Out P P

```

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    using  $\langle C \neq P \rangle$  out-trivial by auto
  moreover have  $C \text{ Out } B \ P$ 
    using  $\langle C \neq P \rangle$  assms(2) bet-out-1 l6-6 by presburger
  moreover have  $C \text{ Out } D \ D$ 
    using  $\langle C \neq D \rangle$  out-trivial by fastforce
  ultimately show ?thesis
    using l11-10 not-conga not-conga-sym by (meson  $\langle A \ B \ C \text{ CongA } B \ C \ D \rangle$ )
qed
moreover
have  $C \ P \ Q \text{ CongA } C \ P \ Q$ 
  using  $\langle C \neq P \rangle \ \langle P \neq Q \rangle$  conga-refl by auto
moreover
have  $D \ C \ P \text{ LtA } C \ P \ Q$ 
proof -
  have  $\text{Per } C \ D \ Q$ 
    by (metis Per-perm col-transitivity-1  $\langle A \neq D \rangle \ \langle \text{Per } A \ D \ C \rangle$  assms(3)
        bet-col bet-col1 col-per2--per l8-5)
  moreover
  have  $\text{Per } D \ Q \ P$ 
    by (meson Per-cases assms(3) assms(4) assms(5) bet-col col-per2--per l8-5)
  moreover
  have  $D \ Q \text{ OS } C \ P$ 
  proof -
    have  $\text{Col } D \ A \ Q$ 
      by (simp add: Col-def assms(3))
    moreover have  $D \neq Q$ 
      using  $\langle Q \neq D \rangle$  by auto
    moreover have  $D \ A \text{ OS } C \ P$ 
      using  $\langle A \ D \text{ OS } B \ C \rangle$  assms(2) between-symmetry invert-one-side l9-17
        one-side-symmetry by blast
    ultimately show ?thesis
      using col-one-side by blast
  qed
  moreover
  have  $Q \ P \text{ Lt } C \ D$ 
    by (meson  $\langle \text{Cong } A \ B \ C \ D \rangle$  assms(6) cong2-lt--lt cong-reflexivity lt-left-comm)
  ultimately
  show ?thesis
    using lt4321-os-per2--lta by blast
qed
ultimately
show ?thesis
  using conga-preserves-lta by blast
qed
have  $\text{Acute } B \ P \ Q \longrightarrow ?thesis$ 
  by (meson Acute-def lta-trans  $\langle A \ B \ C \text{ LtA } B \ P \ Q \rangle$ )
moreover
have  $\text{Per } B \ P \ Q \longrightarrow ?thesis$ 
  using Acute-def  $\langle A \ B \ C \text{ LtA } B \ P \ Q \rangle$  by blast
moreover
have  $\text{Obtuse } B \ P \ Q \longrightarrow ?thesis$ 
  by (metis not-and-lta  $\langle A \ B \ C \text{ LtA } C \ P \ Q \rangle \ \langle A \neq B \rangle \ \langle C \neq P \rangle$ 
        acute-obtuse--lta acute-per--lta angle-partition assms(2) bet-obtuse--acute)
ultimately
show ?thesis
  using  $\langle B \neq P \rangle \ \langle P \neq Q \rangle$  angle-partition by blast
qed

lemma t22-7--obtuse:
  assumes Saccheri  $A \ B \ C \ D$  and
    Bet  $B \ P \ C$  and
    Bet  $A \ Q \ D$  and
     $A \neq Q$  and
    Per  $P \ Q \ A$  and
     $A \ B \text{ Lt } P \ Q$ 
  shows Obtuse  $A \ B \ C$ 

```

```

proof -
  have  $Per\ B\ A\ D$ 
    using  $assms(1)\ Saccheri-def$  by blast
  have  $Per\ A\ D\ C$ 
    using  $assms(1)\ Saccheri-def$  by blast
  have  $Cong\ A\ B\ C\ D$ 
    using  $assms(1)\ Saccheri-def$  by blast
  have  $A\ D\ OS\ B\ C$ 
    using  $assms(1)\ Saccheri-def$  by blast
  have  $A\ D\ ParStrict\ B\ C$ 
    by (simp add:  $assms(1)\ sac--pars1423$ )
  have  $A\ B\ ParStrict\ C\ D$ 
    by (simp add:  $assms(1)\ sac--pars1234$ )
  hence  $\neg\ Col\ A\ B\ C$ 
    using  $par-strict-not-col-1$  by auto
  have  $A \neq B$ 
    using  $\langle \neg\ Col\ A\ B\ C \rangle\ col-trivial-1$  by auto
  have  $C \neq B$ 
    using  $\langle \neg\ Col\ A\ B\ C \rangle\ col-trivial-2$  by blast
  have  $A \neq D$ 
    using  $assms(1)\ sac-distincts$  by blast
  have  $C \neq D$ 
    using  $\langle A\ D\ OS\ B\ C \rangle\ os-distincts$  by blast
  have  $P \neq Q$ 
    by (metis  $\langle A\ D\ OS\ B\ C \rangle\ assms(2)\ assms(3)\ bet-col\ not-col-permutation-4\ one-side-chara$ )
  have  $B \neq P$ 
    by (metis  $Per-cases\ \langle P \neq Q \rangle\ assms(4)\ assms(5)\ assms(6)\ not-and-lt\ per-lt$ )
  have  $C \neq P$ 
    by (metis  $Per-cases\ per2-preserves-diff\ \langle A \neq D \rangle\ \langle Cong\ A\ B\ C\ D \rangle\ \langle Per\ A\ D\ C \rangle\ assms(3)\ assms(4)\ assms(5)\ assms(6)\ bet-col\ cong--nlt$ )
  have  $Q \neq D$ 
    by (metis  $col-per2--per\ l8-2\ l8-7\ \langle A \neq D \rangle\ \langle C \neq P \rangle\ \langle Per\ A\ D\ C \rangle\ \langle Per\ B\ A\ D \rangle\ assms(2)\ assms(5)\ bet-col\ between-symmetry$ )
  have  $B\ P\ Q\ LtA\ A\ B\ C$ 
proof -
  have  $A\ B\ P\ CongA\ A\ B\ C$ 
    by (metis  $conga-sym\ out2--conga\ out-trivial\ \langle A \neq B \rangle\ \langle B \neq P \rangle\ assms(2)\ bet-out$ )
  moreover
  have  $B\ P\ Q\ CongA\ B\ P\ Q$ 
    using  $\langle B \neq P \rangle\ \langle P \neq Q \rangle\ conga-refl$  by presburger
  moreover
  have  $B\ P\ Q\ LtA\ A\ B\ P$ 
proof -
  have  $Col\ A\ D\ Q$ 
    by (simp add:  $assms(3)\ bet-col\ col-permutation-5$ )
  hence  $Per\ B\ A\ Q$ 
    using  $\langle A \neq D \rangle\ \langle Per\ B\ A\ D \rangle\ per-col$  by blast
  moreover
  have  $Per\ A\ Q\ P$ 
    using  $assms(5)\ l8-2$  by blast
  moreover
  have  $Col\ A\ D\ Q$ 
    by (simp add:  $\langle Col\ A\ D\ Q \rangle$ )
  hence  $A\ Q\ OS\ B\ P$ 
    using  $\langle A\ D\ OS\ B\ C \rangle\ assms(2)\ assms(4)\ col-one-side\ l9-17$  by blast
  moreover
  have  $B\ A\ Lt\ Q\ P$ 
    by (simp add:  $assms(6)\ lt-comm$ )
  ultimately
  show ?thesis
    using  $assms(6)\ lt-os-per2--lta$  by blast
qed
ultimately
show ?thesis
  using  $conga-preserves-lta$  by force
qed

```

```

have C P Q LtA A B C
proof -
  have Q P C CongA C P Q
    using ⟨C ≠ P⟩ ⟨P ≠ Q⟩ conga-pseudo-refl by auto
  moreover
  have P C D CongA A B C
  proof -
    have B C D CongA A B C
      by (simp add: assms(1) conga-sym-equiv sac--conga)
    moreover have C Out P B
      using ⟨C ≠ P⟩ assms(2) bet-out-1 by auto
    moreover have C Out D D
      using ⟨C ≠ D⟩ out-trivial by auto
    moreover have B Out A A
      using ⟨A ≠ B⟩ out-trivial by blast
    moreover have B Out C C
      using ⟨C ≠ B⟩ out-trivial by auto
    ultimately show ?thesis
      by (meson l6-6 not-conga out2--conga)
  qed
  moreover
  have Q P C LtA P C D
  proof -
    have Per P Q D
      by (metis assms(3) assms(4) assms(5) bet-col col-permutation-4 per-col)
    moreover
    have Per Q D C
      by (meson ⟨A ≠ D⟩ ⟨Per A D C⟩ assms(3) bet-col1 between-trivial
        col-per2--per l8-20-1-R1)
    moreover
    have Q D OS P C
      by (metis ⟨A D OS B C⟩ ⟨Q ≠ D⟩ assms(2) assms(3) bet-col between-symmetry
        col2-os--os col-permutation-5 l9-17 not-col-distincts one-side-symmetry)
    moreover
    have D C Lt P Q
      by (meson ⟨Cong A B C D⟩ assms(6) cong2-lt--lt cong-reflexivity lt-left-comm)
    ultimately
    show ?thesis
      using lt4321-os-per2--lta by blast
  qed
  ultimately
  show ?thesis
    using conga-preserves-lta by blast
qed
have Acute B P Q ⟶ ?thesis
  by (metis not-and-lta ⟨A ≠ B⟩ ⟨C P Q LtA A B C⟩ ⟨C ≠ B⟩ ⟨C ≠ P⟩
    acute-bet--obtuse acute-obtuse--lta angle-partition assms(2)
    obtuse-per--lta)
moreover
have Per B P Q ⟶ ?thesis
  using Obtuse-def ⟨B P Q LtA A B C⟩ by blast
moreover
have Obtuse B P Q ⟶ ?thesis
  using Obtuse-def ⟨B P Q LtA A B C⟩ lta-trans by blast
ultimately
show ?thesis
  using ⟨B ≠ P⟩ ⟨P ≠ Q⟩ angle-partition by blast
qed

lemma t22-7--cong:
  assumes Saccheri A B C D and
    Bet B P C and
    Bet A Q D and
    A ≠ Q and
    Per P Q A and
    Per A B C

```

```

shows Cong P Q A B
proof -
  have A ≠ B
    using assms(1) sac-distincts by blast
  have C ≠ B
    using assms(1) sac-distincts by blast
{
  assume ¬ Cong P Q A B
  {
    assume P Q Le A B
    hence P Q Lt A B
      using Lt-def ⟨¬ Cong P Q A B⟩ by blast
    hence Acute A B C
      using assms(1) assms(2) assms(3) assms(4) assms(5) t22-7--acute by blast
    hence A B C LtA A B C
      using acute-not-per assms(6) by blast
  }
  moreover
  {
    assume A B Le P Q
    hence A B Lt P Q
      using ⟨¬ Cong P Q A B⟩ le-anti-symmetry nlt--le by blast
    hence Obtuse A B C
      using assms(1) assms(2) assms(3) assms(4) assms(5) t22-7--obtuse by blast
    hence A B C LtA A B C
      using ⟨A ≠ B⟩ ⟨C ≠ B⟩ assms(6) obtuse-per--lta by auto
  }
  ultimately
  have False
    using local.le-cases nlta by auto
}
thus ?thesis by blast
qed

```

```

lemma t22-7--lt5612:
  assumes Saccheri A B C D and
    Bet B P C and
    Bet A Q D and
    A ≠ Q and
    Q ≠ D and
    Per P Q A and
    Acute A B C
  shows P Q Lt A B
proof -
{
  assume Cong P Q A B
  have P Q Lt A B
    using ⟨Cong P Q A B⟩ acute-not-per assms(1) assms(2) assms(3) assms(4)
      assms(5) assms(6) assms(7) t22-7--per by blast
}
moreover
{
  assume ¬ Cong P Q A B
  {
    assume A B Le P Q
    hence Obtuse A B C
      by (meson ⟨¬ Cong P Q A B⟩ assms(1) assms(2) assms(3) assms(4) assms(6)
        le-anti-symmetry nle--lt t22-7--obtuse)
    hence A B C LtA A B C
      using acute--not-obtuse assms(7) by blast
    hence False
      by (simp add: nlta)
  }
  hence P Q Le A B
    using local.le-cases by blast
  hence P Q Lt A B

```

```

    by (simp add: Lt-def  $\neg$  Cong P Q A B)
  }
ultimately
show ?thesis
  by blast
qed

lemma t22-7--lt1256:
  assumes Saccheri A B C D and
    Bet B P C and
    Bet A Q D and
    A  $\neq$  Q and
    Q  $\neq$  D and
    Per P Q A and
    Obtuse A B C
  shows A B Lt P Q
proof -
  have A  $\neq$  B
  using assms(7) obtuse-distincts by blast
  have C  $\neq$  B
  using assms(7) obtuse-distincts by blast
  have  $\neg$  Cong P Q A B
  by (metis  $\langle A \neq B \rangle \langle C \neq B \rangle$  assms(1) assms(2) assms(3) assms(4) assms(5)
    assms(6) assms(7) obtuse--nsams per2--sams t22-7--per)
  moreover
  {
    assume  $\neg$  Cong P Q A B
    {
      assume P Q Le A B
      hence P Q Lt A B
      by (simp add: Lt-def calculation)
      hence Acute A B C
      using assms(1) assms(2) assms(3) assms(4) assms(6) t22-7--acute by blast
      hence A B C LtA A B C
      by (simp add: acute-obtuse--lta assms(7))
      hence False
      by (simp add: nltA)
    }
    hence A B Le P Q
    using local.le-cases by blast
    hence A B Lt P Q
    using calculation le-anti-symmetry nlt--le by blast
  }
  ultimately
  show ?thesis by blast
qed

lemma t22-8-aux-1:
  assumes Saccheri A B C D and

    Bet A D S and
    D  $\neq$  S and
    Per A S R
  shows C  $\neq$  R
proof -
  have A  $\neq$  S
  using assms(3) assms(2) bet-neq23--neq by presburger
  have A  $\neq$  D
  using assms(1) sac-distincts by blast
  have Per A D C
  using Saccheri-def assms(1) by blast
  thus ?thesis
  by (metis NCol-perm Per-cases  $\langle A \neq D \rangle \langle A \neq S \rangle \langle \text{Per } A D C \rangle$  assms(3)
    assms(4) assms(2) bet-col1 bet-out-1 col-per2-cases out-col)
qed

```

```

lemma t22-8-aux-2:
  assumes Saccheri A B C D and
    Bet B C R and
    Bet A D S and

    Per A S R and
    S Out R J and
    Cong S J A B
  shows Saccheri A B J S
proof -
  have A  $\neq$  S
    using assms(1) assms(3) bet-neq12--neq sac-distincts by presburger
  have R  $\neq$  S
    using assms(5) between-trivial2 not-bet-and-out by blast
  have A  $\neq$  D
    using assms(1) sac-distincts by blast
  have Per B A D
    using Saccheri-def assms(1) by blast
  have A D ParStrict B C
    by (simp add: assms(1) sac--pars1423)
  have Per B A S
    using  $\langle A \neq D \rangle \langle Per B A D \rangle$  assms(3) bet-col per-col by blast
  have A S OS B J
  proof -
    have A S OS B R
    proof -
      have Col A D S
        by (simp add: Col-def assms(3))
      moreover
      have A D OS B R
        using  $\langle A D ParStrict B C \rangle$  assms(2) bet-col par-strict-all-one-side by blast
      ultimately
      show ?thesis
        by (meson  $\langle A \neq S \rangle$  col-one-side)
    qed
    moreover
    have A S OS R J
      by (meson assms(5) calculation invert-one-side one-side-not-col124 out-one-side)
    ultimately
    show ?thesis
      using one-side-transitivity by blast
  qed
thus ?thesis
  by (metis Saccheri-def  $\langle Per B A S \rangle \langle R \neq S \rangle$  assms(5) assms(6) assms(4)
    cong-3421 out-col per-col)
qed

```

```

lemma t22-8-aux-3:
  assumes Saccheri A B C D and
    Bet B C R and
    Bet A D S and
    D  $\neq$  S and
    Per A S R and
    S Out R J and
    Cong S J A B
  shows Saccheri D C J S
proof -
  have A  $\neq$  S
    using assms(3) assms(4) between-identity by auto
  have R  $\neq$  S
    using assms(6) out-distinct by auto
  have A  $\neq$  D
    using assms(1) sac-distincts by blast
  have Per A D C
    using Saccheri-def assms(1) by blast
  have Cong A B C D

```

```

    using Saccheri-def assms(1) by blast
have A D ParStrict B C
  by (simp add: assms(1) sac--pars1423)
have Per D S J
  by (metis NCol-cases  $\langle A \neq S \rangle \langle R \neq S \rangle$  assms(3) assms(5) assms(6) bet-col
      l8-3 out-col per-col)
have Per C D S
  by (meson  $\langle A \neq D \rangle \langle Per A D C \rangle$  assms(3) bet-col col-per2--per l8-2 l8-5)
have Cong D C A B
  using Cong-cases  $\langle Cong A B C D \rangle$  by blast
hence Cong D C J S
  by (meson  $\langle Cong A B C D \rangle$  assms(7) cong-transitivity not-cong-4321)
moreover
have D S OS C J
proof -
  have D S OS C R
  proof -
    have Col D A S
    by (simp add: assms(3) bet-col col-permutation-4)
    moreover
    have D A OS C R
    using NCol-perm Par-strict-perm  $\langle A D ParStrict B C \rangle$  assms(2) bet-col
        par-strict-all-one-side by blast
    ultimately
    show ?thesis
    using assms(4) col-one-side by blast
  qed
  moreover
  have D S OS R J
  by (meson one-side-not-col124 assms(6) calculation invert-one-side out-one-side)
  ultimately
  show ?thesis
  using one-side-transitivity by blast
qed
ultimately
show ?thesis
by (simp add: Saccheri-def  $\langle Per C D S \rangle \langle Per D S J \rangle$ )
qed

```

lemma t22-8-aux-4:

```

assumes Saccheri A B C D and
  Bet B C R and
  Bet A D S and
   $D \neq S$  and
  Per A S R and
  S Out R J and
  Cong S J A B
shows S J OS C B
proof -
  have A  $\neq S$ 
  using assms(3) assms(4) between-identity by blast
  have R  $\neq S$ 
  using assms(6) l6-3-1 by blast
  have J  $\neq S$ 
  using assms(6) out-diff2 by auto
  have A  $\neq D$ 
  using assms(1) sac-distincts by blast
  have Per A D C
  using Saccheri-def assms(1) by blast
  have Per D S J
  by (metis NCol-cases  $\langle A \neq S \rangle \langle R \neq S \rangle$  assms(3) assms(5) assms(6)
      bet-col l8-3 out-col per-col)
  have Per C D S
  by (meson  $\langle A \neq D \rangle \langle Per A D C \rangle$  assms(3) bet-col col-per2--per l8-2 l8-5)
thus ?thesis
by (metis Per-cases l8-3  $\langle J \neq S \rangle \langle Per D S J \rangle$  assms(2) assms(4) assms(6))

```


bet-out-1 l6-6 l8-7 out-col out-one-side-1)

qed

lemma *t22-8-aux-5*:
assumes *Saccheri A B C D and*
Bet B C R and
Bet A D S and

Per A S R and
S Out R J and
Cong S J A B
shows *Coplanar A B C J*

proof –
have $A \neq D$
using *assms(1) sac-distincts by blast*
have *Per B A D*
using *Saccheri-def assms(1) by blast*
have *Per B A S*
using $\langle A \neq D \rangle \langle \text{Per } B \ A \ D \rangle \text{ assms}(3) \text{ bet-col per-col by blast}$
hence *Per S A B*
by (*simp add: l8-2*)
hence $\neg \text{Col } S \ A \ B$
using *assms(1) assms(3) bet-neq12--neq l8-9 sac-distincts by blast*
moreover
have *Coplanar S A B C*
proof –
have *Coplanar B C A D*
by (*meson sac--coplanar Tarski-neutral-dimensionless-axioms*
assms(1) coplanar-perm-8)
moreover
have *Col A D S*
by (*simp add: Col-def assms(3)*)
ultimately
show *?thesis*
by (*metis* $\langle A \neq D \rangle \text{ col2-cop--cop coplanar-perm-23 l6-16-1 ncoplanar-perm-6}$)

qed

moreover
have *Coplanar S A B J*
using *t22-8-aux-2 assms(1) assms(2) assms(3) assms(5) assms(6) assms(4)*
ncoplanar-perm-9 sac--coplanar by blast
ultimately
show *?thesis*
using *coplanar-trans-1 by presburger*

qed

lemma *t22-8-aux*:
assumes *Saccheri A B C D and*
Bet B C R and
Bet A D S and
D $\neq$ S and
Per A S R and
S Out R J and
Cong S J A B
shows $C \neq R \wedge \text{Saccheri } A \ B \ J \ S \wedge \text{Saccheri } D \ C \ J \ S \wedge S \ J \ O \ S \ C \ B \wedge \text{Coplanar } A \ B \ C \ J$
by (*metis assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7)*
t22-8-aux-1 t22-8-aux-2 t22-8-aux-3 t22-8-aux-4 t22-8-aux-5)

lemma *t22-8--per*:
assumes *Saccheri A B C D and*
Bet B C R and
Bet A D S and
D $\neq$ S and
Per A S R and
Cong R S A B
shows *Per A B C*

proof –

```

have A ≠ B
  using assms(1) sac-distincts by auto
have A ≠ D
  using assms(1) sac-distincts by blast
have Per A D C
  using Saccheri-def assms(1) by blast
have Cong A B C D
  using Saccheri-def assms(1) by blast
have B ≠ R
  using assms(1) assms(2) bet-neq12--neq sac-distincts by blast
have Saccheri A B R S
  by (metis ⟨A ≠ B⟩ assms(1) assms(2) assms(3) assms(5) assms(6)
    cong-reverse-identity not-cong-2134 out-trivial t22-8-aux-2)
have Per A B R
  by (metis t22-7--per ⟨A ≠ D⟩ ⟨Cong A B C D⟩ ⟨Per A D C⟩
    ⟨Saccheri A B R S⟩ assms(2) assms(3) assms(4) l8-2 not-cong-3412)
thus ?thesis
  by (meson Per-cases ⟨B ≠ R⟩ assms(2) bet-col col-per2--per
    l8-5 not-col-permutation-5)
qed

```

lemma t22-8--acute:

```

assumes Saccheri A B C D and
  Bet B C R and
  Bet A D S and
  D ≠ S and
  Per A S R and
  A B Lt R S
shows Acute A B C
proof -
  have A ≠ B
    using assms(1) sac-distincts by blast
  have B ≠ C
    using assms(1) sac-distincts by blast
  have A B Lt S R
    using assms(6) Lt-cases by blast
  then obtain J where Bet S J R ∧ Cong A B S J
    using Lt-def Le-def by blast
  hence Bet S J R
    by auto
  have Cong A B S J
    by (simp add: ⟨Bet S J R ∧ Cong A B S J⟩)
  {
    assume B = R
    hence C = B
      using assms(2) between-identity by blast
  }
  hence B ≠ R
    using ⟨B ≠ C⟩ by blast
  have J ≠ S
    using ⟨A ≠ B⟩ ⟨Cong A B S J⟩ cong-diff by blast
  have R ≠ S
    using ⟨A B Lt S R⟩ lt-diff by blast
  have J ≠ R
    using ⟨A B Lt S R⟩ ⟨Cong A B S J⟩ cong--nlt by blast
  have A B C Cong A B C D
    using assms(1) sac--conga by force
  have S Out R J
    by (simp add: Out-def ⟨Bet S J R⟩ ⟨J ≠ S⟩ ⟨R ≠ S⟩)
  have Cong S J A B
    using ⟨Bet S J R ∧ Cong A B S J⟩ not-cong-3412 by blast
  have C ≠ R
    using t22-8-aux-1 ⟨Cong S J A B⟩ ⟨S Out R J⟩ assms(1) assms(2) assms(3)
      assms(5) assms(4) by blast
  have Saccheri A B J S
    using t22-8-aux-2 ⟨Cong S J A B⟩ ⟨S Out R J⟩ assms(1) assms(2) assms(3)

```

```

  assms(5) assms(4) by blast
have Saccheri D C J S
  using t22-8-aux-3 ⟨Cong S J A B⟩ ⟨S Out R J⟩ assms(1) assms(2) assms(3)
  assms(5) assms(4) by blast
have S J OS C B
  using t22-8-aux-4 ⟨Cong S J A B⟩ ⟨S Out R J⟩ assms(1) assms(2) assms(3)
  assms(5) assms(4) by blast
have Coplanar A B C J
  using t22-8-aux-5 ⟨Cong S J A B⟩ ⟨S Out R J⟩ assms(1) assms(2) assms(3)
  assms(5) by blast
have B C D LtA R C D
proof –
  have A B J CongA B J S
    by (simp add: ⟨Saccheri A B J S⟩ sac--conga)
  have D C J CongA C J S
    by (simp add: ⟨Saccheri D C J S⟩ sac--conga)
  have J ≠ B
    using ⟨S J OS C B⟩ os-distincts by blast
  have A B ParStrict J S
    by (simp add: ⟨Saccheri A B J S⟩ sac--pars1234)
  have D S ParStrict C J
    by (simp add: ⟨Saccheri D C J S⟩ sac--pars1423)
  have ¬ Col B J S
    using ⟨A B ParStrict J S⟩ par-strict-not-col-2 by blast
  have C J TS D R
proof –
  have C J TS S R
    by (metis Par-strict-cases ⟨Bet S J R⟩ ⟨D S ParStrict C J⟩ ⟨J ≠ R⟩
      bet--ts invert-two-sides par-strict-not-col-1)
  moreover
  have C J OS S D
    using Par-strict-cases ⟨D S ParStrict C J⟩ l12-6 by blast
  ultimately
  show ?thesis
    using l9-8-2 by blast
qed
have J B TS C S
proof –
  have J B TS R S
    by (metis Col-perm ⟨Bet S J R⟩ ⟨J ≠ R⟩ ⟨¬ Col B J S⟩ bet--ts l9-2)
  moreover
  have J B OS R C
    by (metis TS-def ⟨B ≠ C⟩ ⟨B ≠ R⟩ assms(2) bet2--out calculation
      invert-one-side l5-1 not-col-permutation-3 out-one-side)
  ultimately
  show ?thesis
    using l9-8-2 by blast
qed
have ¬ Col J B C
  by (metis ⟨B ≠ C⟩ ⟨Bet S J R⟩ ⟨J ≠ R⟩ ⟨¬ Col B J S⟩ assms(2) bet-col
    bet-col1 l6-21 not-col-permutation-2)
have S J B LtA S J C
proof –
  have B InAngle S J C
    by (simp add: ⟨J B TS C S⟩ ⟨S J OS C B⟩ invert-one-side l9-2 os-ts--inangle)
  hence S J B LeA S J C
    by (simp add: inangle--lea)
  moreover
  {
    assume S J B CongA S J C
    have S J OS B C
      by (simp add: ⟨S J OS C B⟩ one-side-symmetry)
    hence J Out B C
      using ⟨S J B CongA S J C⟩ conga-os--out by blast
    hence Col J B C
      using out-col by auto
  }

```

```

    hence False
    using ⟨¬ Col J B C⟩ by auto
  }
  hence ¬ S J B CongA S J C
    by auto
  ultimately
  show ?thesis
    by (simp add: LtA-def)
qed
hence A B J LtA D C J
  by (meson conga-preserves-lta ⟨A B J CongA B J S⟩ ⟨D C J CongA C J S⟩
    conga-right-comm conga-sym-equiv)
moreover
have J B C LtA J C R
  by (simp add: ⟨C ≠ R⟩ ⟨¬ Col J B C⟩ assms(2) l11-41 lta-left-comm
    not-col-permutation-3)
moreover
have D C ParStrict J S
  by (simp add: ⟨Saccheri D C J S⟩ sac--pars1234)
hence D C OS J R
  using NCol-perm ⟨Bet S J R⟩ bet-col par-strict-all-one-side by blast
hence SAMS D C J J C R
  by (simp add: os-ts--sams ⟨C J TS D R⟩)
moreover
have A B J J B C SumA B C D
proof -
  have J B C CongA J B C
    using ⟨B ≠ C⟩ ⟨J ≠ B⟩ conga-refl by auto
  moreover
  have A B C CongA B C D
    using ⟨A B C CongA B C D⟩ by blast
  moreover
  have A S ParStrict B J
    by (simp add: ⟨Saccheri A B J S⟩ sac--pars1423)
  hence ¬ B J OS A C
    by (meson ⟨J B TS C S⟩ invert-two-sides l9-8-2 l9-9
      one-side-symmetry pars--os3412)
  moreover
  have Coplanar A B J C
    using ⟨Coplanar A B C J⟩ ncoplanar-perm-1 by blast
  ultimately
  show ?thesis
    using SumA-def by blast
qed
moreover
have D C J J C R SumA R C D
  by (meson ⟨C J TS D R⟩ suma-right-comm ts--suma-1)
ultimately
show ?thesis
  using sams-lta2-suma2--lta by blast
qed
hence Acute B C D
  using acute-chara-2 assms(2) by blast
thus ?thesis
  by (meson ⟨A B C CongA B C D⟩ ⟨A ≠ B⟩ ⟨B ≠ C⟩ acute--not-obtuse
    acute-not-per angle-partition conga-obtuse--obtuse l11-17)
qed

lemma t22-8--obtuse:
  assumes Saccheri A B C D and
    Bet B C R and
    Bet A D S and
    D ≠ S and
    Per A S R and
    R S Lt A B
  shows Obtuse A B C

```

```

proof-
  have  $A \neq B$ 
    using assms(1) sac-distincts by blast
  have  $C \neq B$ 
    using assms(1) sac-distincts by blast
  obtain  $I$  where  $Bet\ S\ R\ I \wedge Cong\ S\ I\ A\ B$ 
    using Lt-def by (meson Le-cases assms(6) l5-5-1)
  hence  $Bet\ S\ R\ I$ 
    by auto
  have  $Cong\ S\ I\ A\ B$ 
    by (simp add:  $\langle Bet\ S\ R\ I \wedge Cong\ S\ I\ A\ B \rangle$ )
  hence  $I \neq S$ 
    using  $\langle A \neq B \rangle$  cong-reverse-identity by blast
  have  $A\ D\ ParStrict\ B\ C$ 
    by (simp add: assms(1) sac--pars1423)
  have  $I \neq R$ 
    using  $\langle Bet\ S\ R\ I \wedge Cong\ S\ I\ A\ B \rangle$  assms(6) cong--nlt lt-left-comm by blast
  have  $S \neq R$ 
    by (metis col124--nos  $\langle A\ D\ ParStrict\ B\ C \rangle$  assms(2) assms(3)
      bet-col par-strict-all-one-side)
  have  $A\ B\ C\ CongA\ B\ C\ D$ 
    by (meson assms(1) sac--conga)
  have  $S\ Out\ R\ I$ 
    using  $\langle Bet\ S\ R\ I \rangle$   $\langle S \neq R \rangle$  bet-out by presburger
  have  $C \neq R$ 
    using t22-8-aux-1  $\langle Cong\ S\ I\ A\ B \rangle$   $\langle S\ Out\ R\ I \rangle$  assms(1) assms(2) assms(3)
      assms(5) assms(4) by blast
  have Saccheri A B I S
    using t22-8-aux-2  $\langle Cong\ S\ I\ A\ B \rangle$   $\langle S\ Out\ R\ I \rangle$  assms(1) assms(2) assms(3)
      assms(5) assms(4) by blast
  have Saccheri D C I S
    using t22-8-aux-3  $\langle Cong\ S\ I\ A\ B \rangle$   $\langle S\ Out\ R\ I \rangle$  assms(1) assms(2) assms(3)
      assms(5) assms(4) by blast
  have  $S\ I\ OS\ C\ B$ 
    using t22-8-aux-4  $\langle Cong\ S\ I\ A\ B \rangle$   $\langle S\ Out\ R\ I \rangle$  assms(1) assms(2) assms(3)
      assms(5) assms(4) by blast
  have Coplanar A B C I
    using t22-8-aux-5  $\langle Cong\ S\ I\ A\ B \rangle$   $\langle S\ Out\ R\ I \rangle$  assms(1) assms(2) assms(3)
      assms(5) by blast
  have  $R\ C\ D\ LtA\ B\ C\ D$ 
  proof -
    have  $A\ B\ I\ CongA\ B\ I\ S$ 
      by (simp add:  $\langle Saccheri\ A\ B\ I\ S \rangle$  sac--conga)
    have  $D\ C\ I\ CongA\ C\ I\ S$ 
      by (simp add:  $\langle Saccheri\ D\ C\ I\ S \rangle$  sac--conga)
    have  $I \neq C$ 
      using  $\langle S\ I\ OS\ C\ B \rangle$  os-distincts by blast
    have  $I \neq B$ 
      using  $\langle S\ I\ OS\ C\ B \rangle$  os-distincts by blast
    have  $A\ B\ ParStrict\ I\ S$ 
      by (simp add:  $\langle Saccheri\ A\ B\ I\ S \rangle$  sac--pars1234)
    have  $D\ S\ ParStrict\ C\ I$ 
      by (simp add:  $\langle Saccheri\ D\ C\ I\ S \rangle$  sac--pars1423)
    have  $C\ R\ OS\ D\ S$ 
      by (meson  $\langle A\ D\ ParStrict\ B\ C \rangle$   $\langle C \neq R \rangle$  assms(2) assms(3) assms(4)
        bet-col bet-col1 par-strict-col4--par-strict pars--os3412)
    hence  $C\ R\ OS\ S\ D$ 
      by (simp add: one-side-symmetry)
    have  $\neg Col\ C\ R\ S$ 
      using  $\langle C\ R\ OS\ S\ D \rangle$  one-side-not-col123 by blast
    have  $\neg Col\ B\ C\ I$ 
      by (metis Bet-cases  $\langle Bet\ S\ R\ I \rangle$   $\langle C \neq B \rangle$   $\langle I \neq R \rangle$   $\langle \neg Col\ C\ R\ S \rangle$  assms(2)
        bet-col1 between-exchange2 col-transitivity-2 point-construction-different)
    have  $C\ R\ TS\ S\ I$ 
      using  $\langle Bet\ S\ R\ I \rangle$   $\langle I \neq R \rangle$   $\langle \neg Col\ C\ R\ S \rangle$  bet--ts invert-two-sides
        not-col-permutation-4 by presburger
  
```

```

hence  $C R T S I D$ 
  using  $\langle C R O S S D \rangle$  l9-2 l9-8-2 by blast
have  $I B C LtA I C R$ 
  by (simp add: l11-41 Tarski-neutral-dimensionless-axioms  $\langle C \neq R \rangle$ 
     $\langle \neg Col B C I \rangle$  assms(2) lta-left-comm not-col-permutation-4)
moreover
have  $D C I LtA A B I$ 
proof –
  have  $R S O S C B$ 
    by (meson  $\langle C \neq R \rangle \langle \neg Col C R S \rangle$  assms(2) bet-out-1
      col-permutation-2 out-one-side)
  have  $S I C CongA D C I$ 
    by (meson  $\langle D C I CongA C I S \rangle$  conga-right-comm not-conga-sym)
  moreover
have  $S I B CongA A B I$ 
    by (meson  $\langle A B I CongA B I S \rangle$  conga-right-comm conga-sym)
  moreover
have  $S I C LtA S I B$ 
proof –
  have  $S I C LeA S I B$ 
proof –
  have  $C InAngle S I B$ 
proof –
  have  $I S O S B C$ 
    using  $\langle S I O S C B \rangle$  invert-one-side one-side-symmetry by blast
  moreover
have  $I B O S S C$ 
proof –
  have  $I B O S S R$ 
    by (metis NCol-perm  $\langle Bet S R I \rangle \langle I \neq R \rangle$  bet-out-1 calculation
      one-side-not-col123 one-side-symmetry out-one-side)
  moreover
have  $I B O S R C$ 
    by (metis NCol-perm Out-cases  $\langle C \neq B \rangle \langle \neg Col B C I \rangle$  assms(2)
      bet-out invert-one-side out-one-side)
  ultimately
show ?thesis
    using one-side-transitivity by blast
qed
ultimately
show ?thesis
  by (simp add: os2--inangle)
qed
moreover
have  $S I C CongA S I C$ 
  using  $\langle I \neq C \rangle \langle I \neq S \rangle$  conga-refl by auto
ultimately
show ?thesis
  by (simp add: inangle--lea)
qed
moreover
have  $\neg S I C CongA S I B$ 
  by (meson Bet-cases Col-def Out-def  $\langle S I O S C B \rangle \langle \neg Col B C I \rangle$  conga-os--out)
ultimately
show ?thesis
  by (simp add: LtA-def)
qed
ultimately
show ?thesis
  using conga-preserves-lta by auto
qed
moreover
have  $I C O S R D$ 
proof –
  have  $I C O S R S$ 
    using TS-def  $\langle Bet S R I \rangle \langle C R T S S I \rangle \langle I \neq R \rangle$  bet-out-1 out-one-side by presburger

```

```

moreover
have  $I C OS S D$ 
  using Par-strict-cases  $\langle D S ParStrict C I \rangle$  l12-6 by blast
ultimately
show ?thesis
  using one-side-transitivity by blast
qed
hence  $SAMS I C R R C D$ 
  by (simp add: os-ts--sams  $\langle C R TS I D \rangle$ )
moreover
have  $I C R R C D SumA D C I$ 
  by (simp add: ts--suma-1  $\langle C R TS I D \rangle$  suma-right-comm)
moreover
have  $I B C B C D SumA A B I$ 
proof –
  have  $C B A CongA B C D$ 
    by (simp add:  $\langle A B C CongA B C D \rangle$  conga-left-comm)
  moreover
have  $\neg B C OS I A$ 
proof –
  have  $B C OS A S$ 
    by (meson  $\langle A D ParStrict B C \rangle$  assms(3) bet-col par-strict-all-one-side
      par-strict-symmetry)
  have  $B C TS S I$ 
    by (metis  $\langle B C OS A S \rangle \langle C R TS S I \rangle$  assms(2) bet-col bet-col1 col3
      col-preserved-two-sides col-trivial-3 os-distincts)
  moreover
have  $B C OS S A$ 
    by (simp add:  $\langle B C OS A S \rangle$  one-side-symmetry)
  ultimately
show ?thesis
    by (meson l9-2 l9-8-2 l9-9-bis)
qed
moreover
have Coplanar  $I B C A$ 
  using  $\langle Coplanar A B C I \rangle$  ncoplanar-perm-21 by blast
moreover
have  $I B A CongA A B I$ 
  by (simp add:  $\langle A \neq B \rangle \langle I \neq B \rangle$  conga-refl conga-right-comm)
ultimately
show ?thesis
  using SumA-def by blast
qed
ultimately
show ?thesis
  by (meson sams-lta2-suma2--lta456)
qed
hence Obtuse  $B C D$ 
  using assms(2) lta-distincts obtuse-chara by blast
thus ?thesis
  using  $\langle A B C CongA B C D \rangle$  conga-obtuse--obtuse not-conga-sym by blast
qed

lemma t22-8--cong:
assumes Saccheri  $A B C D$  and
  Bet  $B C R$  and
  Bet  $A D S$  and
   $D \neq S$  and
  Per  $A S R$  and
  Per  $A B C$ 
shows Cong  $R S A B$ 
proof –
have  $A \neq B$ 
  using assms(1) sac-distincts by blast
have  $C \neq B$ 
  using assms(1) sac-distincts by blast

```

```

{
  assume  $\neg \text{Cong } R \ S \ A \ B$ 
  have  $A \ B \ C \text{ Lt} A \ A \ B \ C$ 
  proof -
    {
      assume  $R \ S \text{ Le } A \ B$ 
      hence  $\text{Obtuse } A \ B \ C$ 
        using assms(1) assms(2) assms(3) assms(4) assms(5) t22-8--obtuse
        by (meson  $\langle \neg \text{Cong } R \ S \ A \ B \rangle \text{ le-anti-symmetry nle--lt}$ )
      hence  $\text{Cong } R \ S \ A \ B$ 
        using assms(1) assms(6) obtuse--nsams per2--sams sac-distincts by blast
    }
  moreover
  {
    assume  $A \ B \text{ Le } R \ S$ 
    hence  $A \ B \text{ Lt } R \ S$ 
      using  $\langle \neg \text{Cong } R \ S \ A \ B \rangle \text{ le-anti-symmetry nlt--le}$  by blast
    hence  $\text{Acute } A \ B \ C$ 
      using assms(1) assms(2) assms(3) assms(4) assms(5) t22-8--acute by blast
    hence  $\text{Cong } R \ S \ A \ B$ 
      using acute-not-per assms(6) by blast
  }
  ultimately
  show ?thesis
    using  $\langle \neg \text{Cong } R \ S \ A \ B \rangle \text{ local.le-cases}$  by blast
qed
hence False
  by (simp add: nlta)
}
thus ?thesis
  by auto
qed

```

```

lemma t22-8--lt1256:
  assumes Saccheri A B C D and
    Bet B C R and
    Bet A D S and
     $D \neq S$  and
    Per A S R and
    Acute A B C
  shows  $A \ B \text{ Lt } R \ S$ 
proof -
  have  $\text{Cong } R \ S \ A \ B \longrightarrow A \ B \text{ Lt } R \ S$ 
    using acute-not-per assms(1) assms(2) assms(3) assms(4) assms(5)
    assms(6) t22-8--per by blast
  moreover
  {
    assume  $\neg \text{Cong } R \ S \ A \ B$ 
    {
      assume  $R \ S \text{ Le } A \ B$ 
      hence  $R \ S \text{ Lt } A \ B$ 
        by (simp add: Lt-def  $\langle \neg \text{Cong } R \ S \ A \ B \rangle$ )
      hence  $\text{Obtuse } A \ B \ C$ 
        using assms(1) assms(2) assms(3) assms(4) assms(5) t22-8--obtuse by blast
      hence False
        using acute--not-obtuse assms(6) by auto
    }
    hence  $A \ B \text{ Lt } R \ S$ 
      using nlt--le by blast
  }
  ultimately
  show ?thesis
    by auto
qed

```

```

lemma t22-8--lt5612:

```



```

assumes Saccheri A B C D and
  Bet B C R and
  Bet A D S and
  D ≠ S and
  Per A S R and
  Obtuse A B C
shows R S Lt A B
proof –
  have Cong R S A B → R S Lt A B
    by (metis Cong-perm assms(1) assms(2) assms(3) assms(4) assms(5)
      assms(6) cong--nlt cong-sac--per-aux2 lt-sac--obtuse-aux2 t22-8--per)
  moreover
  {
    assume  $\neg$  Cong R S A B
    {
      assume A B Le R S
      hence A B Lt R S
      using  $\langle \neg$  Cong R S A B  $\rangle$  le-anti-symmetry nlt--le by blast
      hence Acute A B C
      using assms(1) assms(2) assms(3) assms(4) assms(5) t22-8--acute by blast
      hence False
      using acute--not-obtuse assms(6) by blast
    }
    hence R S Lt A B
    using nlt--le by blast
  }
  ultimately
  show ?thesis
    by auto
qed

```

```

lemma t22-9-aux-1:
assumes Lambert N M P Q and
  Lambert N M R S and
  Bet M P R and
  Bet N Q S and
  Per S R M
shows Per Q P M
proof cases
  assume Q = S
  thus ?thesis
  proof –
    have Per P S N
      using  $\langle Q = S \rangle$  assms(1) l8-2 Lambert-def by blast
    hence R = P
      using Lambert-def by (meson assms(2) assms(3) bet-col between-symmetry
        col-per2--per l8-2 l8-7)
    thus ?thesis
      using  $\langle Q = S \rangle$  assms(5) by blast
  qed
next
  assume Q ≠ S
  obtain P' where M Midpoint P P'
    using symmetric-point-construction by blast
  hence M Midpoint P' P
    by (simp add: l7-2)
  obtain Q' where N Midpoint Q Q'
    using symmetric-point-construction by blast
  hence N Midpoint Q' Q
    by (simp add: l7-2)
  obtain R' where M Midpoint R R'
    using symmetric-point-construction by blast
  hence M Midpoint R' R
    by (simp add: l7-2)
  obtain S' where N Midpoint S S'
    using symmetric-point-construction by blast

```

```

hence  $N$  Midpoint  $S' S$ 
  by (simp add: l7-2)
have Saccheri  $S' R' R S$ 
  using  $\langle M$  Midpoint  $R' R \rangle \langle N$  Midpoint  $S' S \rangle$  assms(2) lam6534-mid2--sac by blast
hence Cong  $S' R' R S$ 
  using Saccheri-def by simp
have Saccheri  $Q' P' P Q$ 
  using  $\langle M$  Midpoint  $P' P \rangle \langle N$  Midpoint  $Q' Q \rangle$  assms(1) lam6534-mid2--sac by blast
hence Cong  $Q' P' P Q$ 
  using Saccheri-def by simp
have  $S' R' R$  CongA  $R' R S$ 
  by (simp add:  $\langle$  Saccheri  $S' R' R S \rangle$  sac--conga)
have  $Q' P' P$  CongA  $P' P Q$ 
  by (simp add:  $\langle$  Saccheri  $Q' P' P Q \rangle$  sac--conga)
have Bet  $P' P R$ 
  by (metis  $\langle M$  Midpoint  $P P' \rangle$  assms(3) bet-out--bet between-trivial2 midpoint-out)
have Bet  $Q' Q S$ 
  by (metis  $\langle N$  Midpoint  $Q Q' \rangle$  assms(4) bet-out--bet between-trivial2 midpoint-out)
have Bet  $R' P R$ 
  using  $\langle M$  Midpoint  $R' R \rangle$  assms(3) between-exchange2 midpoint-bet by blast
have Bet  $S' Q S$ 
  using  $\langle N$  Midpoint  $S' S \rangle$  assms(4) between-exchange2 midpoint-bet by blast
have Per  $N S R$ 
  using Lambert-def assms(2) by presburger
hence Per  $Q' S R$ 
  by (metis NCol-cases  $\langle$  Bet  $Q' Q S \rangle \langle Q \neq S \rangle$  assms(4) bet-col bet-out-1
    col-per2--per l8-20-1-R1 out-diff2)
have Per  $S' Q P$ 
proof -
  have Per  $N Q P$ 
    using Lambert-def assms(1) by auto
  moreover
  have  $N \neq Q$ 
    using  $\langle N$  Midpoint  $Q Q' \rangle \langle$  Saccheri  $Q' P' P Q \rangle$  is-midpoint-id
      sac-distincts by blast
  moreover
  have Col  $Q N S'$ 
    by (meson  $\langle$  Bet  $S' Q S \rangle \langle Q \neq S \rangle$  assms(4) bet-col col-permutation-1 l6-16-1)
  ultimately
  show ?thesis
    using l8-3 by blast
qed
have  $S' \neq Q$ 
  using  $\langle$  Bet  $Q' Q S \rangle \langle N$  Midpoint  $Q' Q \rangle \langle N$  Midpoint  $S S' \rangle \langle Q \neq S \rangle$ 
    bet-neq12--neq l7-9 by blast
have  $P \neq P'$ 
  using  $\langle$  Saccheri  $Q' P' P Q \rangle$  sac-distincts by blast
moreover
have Per  $S R R'$ 
  by (metis Mid-cases Per-perm  $\langle M$  Midpoint  $R R' \rangle$  assms(5) col-per2--per
    l7-3-2 l8-5 midpoint-col sym-preserve-diff)
hence Per  $S' R' R$ 
  by (meson Per-cases  $\langle S' R' R$  CongA  $R' R S \rangle$  conga-sym l11-17)
hence Cong  $P Q S' R'$ 
  by (meson  $\langle$  Bet  $R' P R \rangle \langle$  Bet  $S' Q S \rangle \langle$  Per  $S' Q P \rangle \langle S' \neq Q \rangle$ 
     $\langle$  Saccheri  $S' R' R S \rangle$  l8-2 t22-7--cong)
hence Cong  $Q' P' S' R'$ 
  using  $\langle$  Cong  $Q' P' P Q \rangle$  cong-transitivity by blast
hence Cong  $Q' P' R S$ 
  using  $\langle$  Cong  $S' R' R S \rangle$  cong-transitivity by blast
hence Cong  $R S Q' P'$ 
  by (simp add: cong-symmetry)
hence Per  $Q P P'$ 
  using t22-8--per  $\langle$  Saccheri  $Q' P' P Q \rangle \langle$  Bet  $P' P R \rangle \langle$  Bet  $Q' Q S \rangle \langle Q \neq S \rangle$ 
     $\langle$  Per  $Q' S R \rangle \langle$   $Q' P' P$  CongA  $P' P Q \rangle$  l11-17 l8-2 by blast
moreover

```

```

have Col P P' M
  by (simp add: Col-def ⟨M Midpoint P' P⟩ midpoint-bet)
ultimately
show ?thesis
  using per-col by blast
qed

lemma t22-9-aux-2:
  assumes Lambert N M P Q and
    Lambert N M R S and
    Bet M P R and
    Bet N Q S and
    Per Q P M
  shows Per S R M
proof cases
  assume Q = S
  thus ?thesis
  proof -
    have Per P S N
      using ⟨Q = S⟩ assms(1) l8-2 Lambert-def by blast
    hence R = P
      using Lambert-def
      by (meson assms(2) assms(3) bet-col between-symmetry col-per2--per l8-2 l8-7)
    thus ?thesis
      using ⟨Q = S⟩ assms(5) by blast
  qed
next
  assume Q ≠ S
  obtain P' where M Midpoint P P'
    using symmetric-point-construction by blast
  hence M Midpoint P' P
    by (simp add: l7-2)
  obtain Q' where N Midpoint Q Q'
    using symmetric-point-construction by blast
  hence N Midpoint Q' Q
    by (simp add: l7-2)
  obtain R' where M Midpoint R R'
    using symmetric-point-construction by blast
  hence M Midpoint R' R
    by (simp add: l7-2)
  obtain S' where N Midpoint S S'
    using symmetric-point-construction by blast
  hence N Midpoint S' S
    by (simp add: l7-2)
  have Saccheri S' R' R S
    using ⟨M Midpoint R' R⟩ ⟨N Midpoint S' S⟩ assms(2) lam6534-mid2--sac by blast
  hence Cong S' R' R S
    using Saccheri-def by simp
  have Saccheri Q' P' P Q
    using ⟨M Midpoint P' P⟩ ⟨N Midpoint Q' Q⟩ assms(1) lam6534-mid2--sac by blast
  hence Cong Q' P' P Q
    using Saccheri-def by simp
  have S' R' R CongA R' R S
    by (simp add: ⟨Saccheri S' R' R S⟩ sac--conga)
  have Q' P' P CongA P' P Q
    by (simp add: ⟨Saccheri Q' P' P Q⟩ sac--conga)
  have Bet P' P R
    by (metis ⟨M Midpoint P P'⟩ assms(3) bet-out--bet between-trivial2 midpoint-out)
  have Bet Q' Q S
    by (metis ⟨N Midpoint Q Q'⟩ assms(4) bet-out--bet between-trivial2 midpoint-out)
  have Bet R' P R
    using ⟨M Midpoint R' R⟩ assms(3) between-exchange2 midpoint-bet by blast
  have Bet S' Q S
    using ⟨N Midpoint S' S⟩ assms(4) between-exchange2 midpoint-bet by blast
  have Per N S R
    using Lambert-def assms(2) by presburger

```

hence $Per\ Q'\ S\ R$
by (*metis* $NCol$ -cases $\langle Bet\ Q'\ Q\ S \rangle \langle Q \neq S \rangle$ *assms*(4) *bet-col bet-out-1 col-per2--per l8-20-1-R1 out-diff2*)
have $Per\ S'\ Q\ P$
proof –
have $Per\ N\ Q\ P$
using *Lambert-def assms*(1) **by** *auto*
moreover
have $N \neq Q$
using $\langle N\ Midpoint\ Q\ Q' \rangle \langle Saccheri\ Q'\ P'\ P\ Q \rangle$ *is-midpoint-id sac-distincts* **by** *blast*
moreover
have $Col\ Q\ N\ S'$
by (*meson* $\langle Bet\ S'\ Q\ S \rangle \langle Q \neq S \rangle$ *assms*(4) *bet-col col-permutation-1 l6-16-1*)
ultimately
show *?thesis*
using *l8-3* **by** *blast*
qed
have $S' \neq Q$
using $\langle Bet\ Q'\ Q\ S \rangle \langle N\ Midpoint\ Q'\ Q \rangle \langle N\ Midpoint\ S\ S' \rangle \langle Q \neq S \rangle$
bet-neq12--neq l7-9 **by** *blast*
have $R \neq R'$
using $\langle Saccheri\ S'\ R'\ R\ S \rangle$ *sac-distincts* **by** *blast*
moreover
have $Col\ R\ R'\ M$
by (*simp add: Col-def* $\langle M\ Midpoint\ R'\ R \rangle$ *midpoint-bet*)
moreover
have $Per\ Q\ P\ P'$
by (*metis* *Per-cases cong-diff-2* $\langle M\ Midpoint\ P\ P' \rangle \langle M\ Midpoint\ P'\ P \rangle$
assms(5) *col-per2--per l8-5 midpoint-col midpoint-cong*)
hence $Per\ Q'\ P'\ P$
by (*meson* $\langle Q'\ P'\ P\ CongA\ P'\ P\ Q \rangle$ *l11-17 l8-2 not-conga-sym*)
hence $Cong\ R\ S\ Q'\ P'$
using $\langle Bet\ P'\ P\ R \rangle \langle Bet\ Q'\ Q\ S \rangle \langle Per\ Q'\ S\ R \rangle \langle Q \neq S \rangle \langle Saccheri\ Q'\ P'\ P\ Q \rangle$
t22-8--cong **by** *blast*
hence $Cong\ S'\ R'\ P'\ Q'$
using $\langle Cong\ S'\ R'\ R\ S \rangle$ *cong-transitivity not-cong-1243* **by** *blast*
hence $Cong\ S'\ R'\ P\ Q$
using *Cong-cases* $\langle Cong\ Q'\ P'\ P\ Q \rangle$ *cong-inner-transitivity* **by** *blast*
hence $Cong\ P\ Q\ S'\ R'$
using *not-cong-3412* **by** *blast*
hence $Per\ S'\ R'\ R$
using $\langle Bet\ R'\ P\ R \rangle \langle Bet\ S'\ Q\ S \rangle \langle Per\ S'\ Q\ P \rangle \langle Q \neq S \rangle \langle S' \neq Q \rangle$
 $\langle Saccheri\ S'\ R'\ R\ S \rangle$ *l8-2 t22-7--per* **by** *blast*
hence $Per\ S\ R\ R'$
using *Per-perm* $\langle S'\ R'\ R\ CongA\ R'\ R\ S \rangle$ *l11-17* **by** *blast*
ultimately
show *?thesis*
using *per-col* **by** *blast*
qed

lemma *t22-9-aux-3*:
assumes *Lambert* $N\ M\ P\ Q$ **and**
Lambert $N\ M\ R\ S$ **and**
Bet $M\ P\ R$ **and**
Bet $N\ Q\ S$ **and**
Acute $S\ R\ M$
shows *Acute* $Q\ P\ M$
proof *cases*
assume $Q = S$
thus *?thesis*
proof –
have $Per\ P\ S\ N$
using $\langle Q = S \rangle$ *assms*(1) *l8-2 Lambert-def* **by** *blast*
hence $R = P$
using *Lambert-def*
by (*meson* *assms*(2) *assms*(3) *bet-col between-symmetry col-per2--per l8-2 l8-7*)

```

    thus ?thesis
      using  $\langle Q = S \rangle$  assms(5) by blast
qed
next
  assume  $Q \neq S$ 
  obtain  $P'$  where  $M$  Midpoint  $P$   $P'$ 
    using symmetric-point-construction by blast
  hence  $M$  Midpoint  $P' P$ 
    by (simp add: l7-2)
  obtain  $Q'$  where  $N$  Midpoint  $Q$   $Q'$ 
    using symmetric-point-construction by blast
  hence  $N$  Midpoint  $Q' Q$ 
    by (simp add: l7-2)
  obtain  $R'$  where  $M$  Midpoint  $R$   $R'$ 
    using symmetric-point-construction by blast
  hence  $M$  Midpoint  $R' R$ 
    by (simp add: l7-2)
  obtain  $S'$  where  $N$  Midpoint  $S$   $S'$ 
    using symmetric-point-construction by blast
  hence  $N$  Midpoint  $S' S$ 
    by (simp add: l7-2)
  have Saccheri  $S' R' R S$ 
    using  $\langle M$  Midpoint  $R' R \rangle \langle N$  Midpoint  $S' S \rangle$  assms(2) lam6534-mid2--sac by blast
  hence Cong  $S' R' R S$ 
    using Saccheri-def by simp
  have Saccheri  $Q' P' P Q$ 
    using  $\langle M$  Midpoint  $P' P \rangle \langle N$  Midpoint  $Q' Q \rangle$  assms(1) lam6534-mid2--sac by blast
  hence Cong  $Q' P' P Q$ 
    using Saccheri-def by simp
  have  $S' R' R$  CongA  $R' R S$ 
    by (simp add:  $\langle$ Saccheri  $S' R' R S \rangle$  sac--conga)
  have  $Q' P' P$  CongA  $P' P Q$ 
    by (simp add:  $\langle$ Saccheri  $Q' P' P Q \rangle$  sac--conga)
  have Bet  $P' P R$ 
    by (metis  $\langle M$  Midpoint  $P P' \rangle$  assms(3) bet-out--bet between-trivial2 midpoint-out)
  have Bet  $Q' Q S$ 
    by (metis  $\langle N$  Midpoint  $Q Q' \rangle$  assms(4) bet-out--bet between-trivial2 midpoint-out)
  have Bet  $R' P R$ 
    using  $\langle M$  Midpoint  $R' R \rangle$  assms(3) between-exchange2 midpoint-bet by blast
  have Bet  $S' Q S$ 
    using  $\langle N$  Midpoint  $S' S \rangle$  assms(4) between-exchange2 midpoint-bet by blast
  have Per  $N S R$ 
    using Lambert-def assms(2) by presburger
  hence Per  $Q' S R$ 
    by (metis NCol-cases  $\langle$ Bet  $Q' Q S \rangle \langle Q \neq S \rangle$  assms(4) bet-col
      bet-out-1 col-per2--per l8-20-1-R1 out-diff2)
  have Per  $S' Q P$ 
  proof –
    have Per  $N Q P$ 
      using Lambert-def assms(1) by auto
    moreover
    have  $N \neq Q$ 
      using  $\langle N$  Midpoint  $Q Q' \rangle \langle$ Saccheri  $Q' P' P Q \rangle$  is-midpoint-id sac-distincts by blast
    moreover
    have Col  $Q N S'$ 
      by (meson  $\langle$ Bet  $S' Q S \rangle \langle Q \neq S \rangle$  assms(4) bet-col col-permutation-1 l6-16-1)
    ultimately
    show ?thesis
      using l8-3 by blast
  qed
  have  $S' \neq Q$ 
    using  $\langle$ Bet  $Q' Q S \rangle \langle N$  Midpoint  $Q' Q \rangle \langle N$  Midpoint  $S S' \rangle \langle Q \neq S \rangle$ 
      bet-neq12--neq l7-9 by blast
  have Acute  $Q' P' P$ 
  proof –
    have Saccheri  $Q' P' P Q$ 

```

```

  by (simp add: ⟨Saccheri Q' P' P Q⟩)
moreover
have Bet P' P R
  by (simp add: ⟨Bet P' P R⟩)
moreover
have Bet Q' Q S
  by (simp add: ⟨Bet Q' Q S⟩)
moreover
have Q ≠ S
  by (simp add: ⟨Q ≠ S⟩)
moreover
have Per Q' S R
  by (simp add: ⟨Per Q' S R⟩)
moreover
have S R M CongA S' R' R
proof -
  have S R R' CongA S' R' R
    by (meson ⟨S' R' R CongA R' R S⟩ conga-left-comm conga-sym-equiv)
  moreover
  have R Out S S
    using acute-distincts assms(5) out-trivial by blast
  moreover
  have R Out M R'
    using ⟨M Midpoint R R'⟩ ⟨S' R' R CongA R' R S⟩ conga-diff2 midpoint-out by blast
  moreover
  have R' ≠ S'
    using calculation(1) conga-diff45 by blast
  hence R' Out S' S'
    using out-trivial by presburger
  moreover
  have R' Out R R
    by (metis calculation(1) conga-diff2 out-trivial)
  moreover
  have Q' P' P CongA Q P M
proof -
  have Q' P' P CongA Q P P'
    by (simp add: ⟨Q' P' P CongA P' P Q⟩ conga-right-comm)
  moreover
  have P' ≠ Q'
    using CongA-def calculation by blast
  hence P' Out Q' Q'
    using bet-out-1 between-trivial2 by auto
  moreover
  have P' Out P P
    using ⟨Q' P' P CongA P' P Q⟩ conga-diff2 out-trivial by blast
  moreover
  have P Out Q Q
    using ⟨Q' P' P CongA P' P Q⟩ conga-diff56 out-trivial by force
  moreover
  have P Out M P'
    using ⟨M Midpoint P P'⟩ calculation(3) midpoint-out out-diff2 by blast
  ultimately
  show ?thesis
    using l11-10 by blast
qed
ultimately
show ?thesis
  using l11-10 by blast
qed
hence Acute S' R' R
  using acute-conga--acute assms(5) by blast
hence P Q Lt S' R'
  using ⟨Bet R' P R⟩ ⟨Bet S' Q S⟩ ⟨Per S' Q P⟩ ⟨S' ≠ Q⟩
    ⟨Saccheri S' R' R S⟩ calculation(4) l8-2 t22-7--lt5612 by blast
hence Q' P' Lt R S
  by (meson ⟨Cong Q' P' P Q⟩ ⟨Cong S' R' R S⟩ cong2-lt--lt lt-right-comm

```

```

      not-cong-3421)
ultimately
show ?thesis
  using t22-8--acute by blast
qed
moreover
have  $Q' P' P \text{ CongA } Q P M$ 
proof –
  have  $Q' P' P \text{ CongA } Q P P'$ 
    by (simp add:  $\langle Q' P' P \text{ CongA } P' P Q \rangle$  conga-right-comm)
  moreover
  have  $P' \neq Q'$ 
    using CongA-def calculation by blast
  hence  $P' \text{ Out } Q' Q'$ 
    using bet-out-1 between-trivial2 by auto
  moreover
  have  $P' \text{ Out } P P$ 
    using  $\langle Q' P' P \text{ CongA } P' P Q \rangle$  conga-diff2 out-trivial by blast
  moreover
  have  $P \text{ Out } Q Q$ 
    using  $\langle Q' P' P \text{ CongA } P' P Q \rangle$  conga-diff56 out-trivial by force
  moreover
  have  $P \text{ Out } M P'$ 
    using  $\langle M \text{ Midpoint } P P' \rangle$  calculation(3) midpoint-out out-diff2 by blast
  ultimately
  show ?thesis
    using l11-10 by blast
qed
ultimately
show ?thesis
  using acute-conga--acute by blast
qed

lemma t22-9-aux-4:
  assumes Lambert  $N M P Q$  and
    Lambert  $N M R S$  and
    Bet  $M P R$  and
    Bet  $N Q S$  and
    Acute  $Q P M$ 
  shows
    Acute  $S R M$ 
proof cases
  assume  $Q = S$ 
  thus ?thesis
  proof –
    have  $\text{Per } P S N$ 
      using  $\langle Q = S \rangle$  assms(1) l8-2 Lambert-def by blast
    hence  $R = P$ 
      using Lambert-def
      by (meson assms(2) assms(3) bet-col between-symmetry col-per2--per l8-2 l8-7)
    thus ?thesis
      using  $\langle Q = S \rangle$  assms(5) by blast
  qed
next
  assume  $Q \neq S$ 
  obtain  $P'$  where  $M \text{ Midpoint } P P'$ 
    using symmetric-point-construction by blast
  hence  $M \text{ Midpoint } P' P$ 
    by (simp add: l7-2)
  obtain  $Q'$  where  $N \text{ Midpoint } Q Q'$ 
    using symmetric-point-construction by blast
  hence  $N \text{ Midpoint } Q' Q$ 
    by (simp add: l7-2)
  obtain  $R'$  where  $M \text{ Midpoint } R R'$ 
    using symmetric-point-construction by blast
  hence  $M \text{ Midpoint } R' R$ 

```

```

  by (simp add: l7-2)
obtain  $S'$  where  $N$  Midpoint  $S S'$ 
  using symmetric-point-construction by blast
hence  $N$  Midpoint  $S' S$ 
  by (simp add: l7-2)
have Saccheri  $S' R' R S$ 
  using  $\langle M$  Midpoint  $R' R \rangle \langle N$  Midpoint  $S' S \rangle$  assms(2) lam6534-mid2--sac by blast
hence Cong  $S' R' R S$ 
  using Saccheri-def by simp
have Saccheri  $Q' P' P Q$ 
  using  $\langle M$  Midpoint  $P' P \rangle \langle N$  Midpoint  $Q' Q \rangle$  assms(1) lam6534-mid2--sac by blast
hence Cong  $Q' P' P Q$ 
  using Saccheri-def by simp
have  $S' R' R$  CongA  $R' R S$ 
  by (simp add:  $\langle$  Saccheri  $S' R' R S \rangle$  sac--conga)
have  $Q' P' P$  CongA  $P' P Q$ 
  by (simp add:  $\langle$  Saccheri  $Q' P' P Q \rangle$  sac--conga)
have Bet  $P' P R$ 
  by (metis  $\langle M$  Midpoint  $P P' \rangle$  assms(3) bet-out--bet between-trivial2 midpoint-out)
have Bet  $Q' Q S$ 
  by (metis  $\langle N$  Midpoint  $Q Q' \rangle$  assms(4) bet-out--bet between-trivial2 midpoint-out)
have Bet  $R' P R$ 
  using  $\langle M$  Midpoint  $R' R \rangle$  assms(3) between-exchange2 midpoint-bet by blast
have Bet  $S' Q S$ 
  using  $\langle N$  Midpoint  $S' S \rangle$  assms(4) between-exchange2 midpoint-bet by blast
have Per  $N S R$ 
  using Lambert-def assms(2) by presburger
hence Per  $Q' S R$ 
  by (metis NCol-cases  $\langle$  Bet  $Q' Q S \rangle \langle Q \neq S \rangle$  assms(4) bet-col bet-out-1
    col-per2--per l8-20-1-R1 out-diff2)
have Per  $S' Q P$ 
proof –
  have Per  $N Q P$ 
    using Lambert-def assms(1) by auto
  moreover
  have  $N \neq Q$ 
    using  $\langle N$  Midpoint  $Q Q' \rangle \langle$  Saccheri  $Q' P' P Q \rangle$  is-midpoint-id
      sac-distincts by blast
  moreover
  have Col  $Q N S'$ 
    by (meson  $\langle$  Bet  $S' Q S \rangle \langle Q \neq S \rangle$  assms(4) bet-col col-permutation-1 l6-16-1)
  ultimately
  show ?thesis
    using l8-3 by blast
qed
have  $S' \neq Q$ 
  using  $\langle$  Bet  $Q' Q S \rangle \langle N$  Midpoint  $Q' Q \rangle \langle N$  Midpoint  $S S' \rangle \langle Q \neq S \rangle$ 
    bet-neq12--neq l7-9 by blast
have  $Q' P' P$  CongA  $Q P M$ 
proof –
  have  $Q' P' P$  CongA  $Q P P'$ 
    by (simp add:  $\langle Q' P' P$  CongA  $P' P Q \rangle$  conga-right-comm)
  moreover
  have  $P' \neq Q'$ 
    using CongA-def calculation by blast
  hence  $P' Out Q' Q'$ 
    using bet-out-1 between-trivial2 by auto
  moreover
  have  $P' Out P P$ 
    using  $\langle Q' P' P$  CongA  $P' P Q \rangle$  conga-diff2 out-trivial by blast
  moreover
  have  $P Out Q Q$ 
    using  $\langle Q' P' P$  CongA  $P' P Q \rangle$  conga-diff56 out-trivial by force
  moreover
  have  $P Out M P'$ 
    using  $\langle M$  Midpoint  $P P' \rangle$  calculation(3) midpoint-out out-diff2 by blast

```


ultimately
 show ?thesis
 using l11-10 by blast
 qed
 hence $Q P M \text{ CongA } Q' P' P$
 using not-conga-sym by blast
 hence $\text{Acute } Q' P' P$
 using acute-conga--acute assms(5) by blast
 hence $Q' P' \text{ Lt } R S$
 using $\langle \text{Bet } P' P R \rangle \langle \text{Bet } Q' Q S \rangle \langle \text{Per } Q' S R \rangle \langle Q \neq S \rangle$
 $\langle \text{Saccheri } Q' P' P Q \rangle$ t22-8--lt1256 by blast
 hence $P Q \text{ Lt } S' R'$
 by (meson $\langle \text{Cong } Q' P' P Q \rangle \langle \text{Cong } S' R' R S \rangle$ cong2-lt--lt not-cong-3412)
 hence $\text{Acute } S' R' R$
 using t22-7--acute Per-cases $\langle \text{Bet } R' P R \rangle \langle \text{Bet } S' Q S \rangle \langle \text{Per } S' Q P \rangle$
 $\langle S' \neq Q \rangle \langle \text{Saccheri } S' R' R S \rangle$ by blast
 moreover
 have $S R M \text{ CongA } S' R' R$
 proof –
 have $S R R' \text{ CongA } S' R' R$
 by (meson $\langle S' R' R \text{ CongA } R' R S \rangle$ conga-left-comm conga-sym-equiv)
 moreover
 have $R \neq S$
 using $\langle S' R' R \text{ CongA } R' R S \rangle$ conga-diff56 by blast
 hence $R \text{ Out } S S$
 using bet-out-1 between-trivial2 by auto
 moreover
 have $R \text{ Out } M R'$
 using $\langle M \text{ Midpoint } R R' \rangle \langle S' R' R \text{ CongA } R' R S \rangle$ conga-diff2
 midpoint-out by blast
 moreover
 have $R' \neq S'$
 using calculation(1) conga-diff45 by blast
 hence $R' \text{ Out } S' S'$
 using out-trivial by presburger
 moreover
 have $R' \text{ Out } R R$
 by (metis calculation(1) conga-diff2 out-trivial)
 moreover
 have $Q' P' P \text{ CongA } Q P M$
 proof –
 have $Q' P' P \text{ CongA } Q P P'$
 by (simp add: $\langle Q' P' P \text{ CongA } P' P Q \rangle$ conga-right-comm)
 moreover
 have $P' \neq Q'$
 using CongA-def calculation by blast
 hence $P' \text{ Out } Q' Q'$
 using bet-out-1 between-trivial2 by auto
 moreover
 have $P' \text{ Out } P P$
 using $\langle Q' P' P \text{ CongA } P' P Q \rangle$ conga-diff2 out-trivial by blast
 moreover
 have $P \text{ Out } Q Q$
 using $\langle Q' P' P \text{ CongA } P' P Q \rangle$ conga-diff56 out-trivial by force
 moreover
 have $P \text{ Out } M P'$
 using $\langle M \text{ Midpoint } P P' \rangle$ calculation(3) midpoint-out out-diff2 by blast
 ultimately
 show ?thesis
 using l11-10 by blast
 qed
 hence $\text{Acute } S' R' R$
 using $\langle \text{Acute } S' R' R \rangle$ by blast
 hence $P Q \text{ Lt } S' R'$
 by (simp add: $\langle P Q \text{ Lt } S' R' \rangle$)
 hence $Q' P' \text{ Lt } R S$

by (meson $\langle \text{Cong } Q' P' P Q \rangle \langle \text{Cong } S' R' R S \rangle$ cong2-lt--lt
 lt-right-comm not-cong-3421)
 ultimately
 show ?thesis
 using l11-10 by blast
 qed
 hence $S' R' R \text{ CongA } S R M$
 using not-conga-sym by blast
 ultimately
 show ?thesis
 using acute-conga--acute by blast
 qed

lemma t22-9-aux:
 assumes Lambert $N M P Q$ and
 Lambert $N M R S$ and
 Bet $M P R$ and
 Bet $N Q S$
 shows $(\text{Per } S R M \longleftrightarrow \text{Per } Q P M) \wedge (\text{Acute } S R M \longleftrightarrow \text{Acute } Q P M)$
 by (metis assms(1) assms(2) assms(3) assms(4) t22-9-aux-1 t22-9-aux-2
 t22-9-aux-3 t22-9-aux-4)

lemma t22-9--per-1:
 assumes Lambert $N M P Q$ and
 Lambert $N M R S$ and
 Bet $M P R$ and
 Bet $N Q S$ and
 Per $S R M$
 shows Per $Q P M$
 using assms(1) assms(2) assms(3) assms(4) assms(5) t22-9-aux-1 by blast

lemma t22-9--per-2:
 assumes Lambert $N M P Q$ and
 Lambert $N M R S$ and
 Bet $M P R$ and
 Bet $N Q S$ and
 Per $Q P M$
 shows Per $S R M$
 using assms(1) assms(2) assms(3) assms(4) assms(5) t22-9-aux-2 by blast

lemma t22-9--per:
 assumes Lambert $N M P Q$ and
 Lambert $N M R S$ and
 Bet $M P R$ and
 Bet $N Q S$
 shows Per $S R M \longleftrightarrow \text{Per } Q P M$
 by (meson assms(1) assms(2) assms(3) assms(4) t22-9-aux-1 t22-9-aux-2)

lemma t22-9--acute-1:
 assumes Lambert $N M P Q$ and
 Lambert $N M R S$ and
 Bet $M P R$ and
 Bet $N Q S$ and
 Acute $S R M$
 shows Acute $Q P M$
 using assms(1) assms(2) assms(3) assms(4) assms(5) t22-9-aux by blast

lemma t22-9--acute-2:
 assumes Lambert $N M P Q$ and
 Lambert $N M R S$ and
 Bet $M P R$ and
 Bet $N Q S$ and
 Acute $Q P M$
 shows Acute $S R M$
 using assms(1) assms(2) assms(3) assms(4) assms(5) t22-9-aux by blast

lemma *t22-9--acute*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S
shows *Acute S R M* $\longleftrightarrow$ *Acute Q P M*
by (*meson assms(1) assms(2) assms(3) assms(4) t22-9-aux-3 t22-9-aux-4*)

lemma *t22-9--obtuse-1*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Obtuse S R M
shows *Obtuse Q P M*
proof –
have *Acute Q P M* $\vee$ *Per Q P M* $\vee$ *Obtuse Q P M*
by (*metis angle-partition l8-20-1-R1 l8-5*)
moreover
{
assume *Acute Q P M*
hence *S R M LtA S R M*
using *acute-obtuse--lta assms(1) assms(2) assms(3) assms(4)*
assms(5) t22-9-aux **by** *blast*
hence *False*
by (*simp add: nlta*)
}
moreover
{
assume *Per Q P M*
hence *S R M LtA S R M*
by (*metis assms(1) assms(2) assms(3) assms(4) assms(5)*
obtuse-distincts obtuse-per--lta t22-9-aux-2)
hence *False*
by (*simp add: nlta*)
}
ultimately
show *?thesis*
by *blast*
qed

lemma *t22-9--obtuse-2*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Obtuse Q P M
shows *Obtuse S R M*
proof –
have *Acute S R M* $\vee$ *Per S R M* $\vee$ *Obtuse S R M*
by (*metis angle-partition l8-20-1-R1 l8-5*)
moreover
{
assume *Acute S R M*
hence *Q P M LtA Q P M*
using *acute-obtuse--lta assms(1) assms(2) assms(3) assms(4)*
assms(5) t22-9-aux **by** *blast*
hence *False*
by (*simp add: nlta*)
}
moreover
{
assume *Per S R M*
hence *Q P M LtA Q P M*
by (*metis assms(1) assms(2) assms(3) assms(4) assms(5)*
obtuse-distincts obtuse-per--lta t22-9-aux-1)
}

```

    hence False
      by (simp add: nlta)
  }
  ultimately
  show ?thesis
    by blast
qed

lemma t22-9--obtuse:
  assumes Lambert N M P Q and
    Lambert N M R S and
    Bet M P R and
    Bet N Q S
  shows Obtuse S R M  $\longleftrightarrow$  Obtuse Q P M
  by (meson assms(1) assms(2) assms(3) assms(4) t22-9--obtuse-1 t22-9--obtuse-2)

lemma cong2-lam2--cong-conga-1:
  assumes Lambert N M P Q and
    Lambert N' M' P' Q' and
    Cong N Q N' Q' and
    Cong P Q P' Q'
  shows Cong N M N' M'
proof -
  have N Q ParStrict M P
    using assms(1) lam--pars1423 by auto
  have N M ParStrict P Q
    by (simp add: assms(1) lam--pars1234)
  have N' Q' ParStrict M' P'
    by (simp add: assms(2) lam--pars1423)
  have N' M' ParStrict P' Q'
    by (simp add: assms(2) lam--pars1234)
  have  $\neg$  Col N M P
    using  $\langle N M ParStrict P Q \rangle$  par-strict-not-col-1 by auto
  have  $\neg$  Col M N Q
    using  $\langle N M ParStrict P Q \rangle$  col-permutation-4 par-strict-not-col-4 by blast
  have  $\neg$  Col M' N' Q'
    using  $\langle N' M' ParStrict P' Q' \rangle$  not-col-permutation-4 par-strict-not-col-4 by blast
  {
    have Per N Q P  $\wedge$  Per N' Q' P'
      using Lambert-def assms(1) assms(2) by presburger
    hence N Q P CongA N' Q' P'
      by (metis cong-diff-3  $\langle N' M' ParStrict P' Q' \rangle$   $\langle N' Q' ParStrict M' P' \rangle$ 
        assms(3) assms(4) l11-16 par-strict-neq1 par-strict-neq2)
    moreover
    have Cong Q N Q' N'
      using assms(3) not-cong-2143 by blast
    moreover
    have Cong Q P Q' P'
      using Cong-perm assms(4) by blast
    moreover
    have Cong N P N' P'
      using calculation(1) calculation(2) calculation(3) l11-49 by blast
  }
  hence Cong N P N' P' by auto
  have Q N P CongA Q' N' P'  $\wedge$  Q P N CongA Q' P' N'
proof -
  have Per N Q P  $\wedge$  Per N' Q' P'
    using Lambert-def assms(1) assms(2) by presburger
  hence N Q P CongA N' Q' P'
    by (metis cong-diff-3  $\langle N' M' ParStrict P' Q' \rangle$   $\langle N' Q' ParStrict M' P' \rangle$ 
      assms(3) assms(4) l11-16 par-strict-neq1 par-strict-neq2)
  moreover
  have Cong Q N Q' N'
    using assms(3) not-cong-2143 by blast

```

```

moreover
have  $\text{Cong } Q \ P \ Q' \ P'$ 
  using  $\text{Cong-perm assms}(4)$  by blast
moreover
have  $N \neq P$ 
  using  $\langle \neg \text{Col } N \ M \ P \rangle$  col-trivial-3 by blast
ultimately
show ?thesis
  using l11-49  $\langle \text{Cong } N \ P \ N' \ P' \rangle$  by blast
qed
have  $M \ N \ P \ \text{CongA } M' \ N' \ P'$ 
proof –
  have  $N \ Q \ \text{OS } M \ P$ 
    by (simp add:  $\langle N \ Q \ \text{ParStrict } M \ P \rangle$  l12-6)
  moreover
  have  $N' \ Q' \ \text{OS } M' \ P'$ 
    by (simp add:  $\langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$  l12-6)
  moreover
  have  $\text{Per } M \ N \ Q \wedge \text{Per } M' \ N' \ Q'$ 
    using Lambert-def assms(1) assms(2) by presburger
  hence  $M \ N \ Q \ \text{CongA } M' \ N' \ Q'$ 
    using  $\langle N \ M \ \text{ParStrict } P \ Q \rangle \langle N \ Q \ \text{ParStrict } M \ P \rangle \langle N' \ M' \ \text{ParStrict } P' \ Q' \rangle$ 
     $\langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$  l11-16 par-strict-neq1 by blast
  moreover
  have  $Q \ N \ P \ \text{CongA } Q' \ N' \ P'$ 
    by (simp add:  $\langle Q \ N \ P \ \text{CongA } Q' \ N' \ P' \wedge Q \ P \ N \ \text{CongA } Q' \ P' \ N' \rangle$ )
  ultimately
  show ?thesis
    using l11-22b by blast
qed
have  $\neg \text{Col } P \ N \ M$ 
  using  $\langle \neg \text{Col } N \ M \ P \rangle$  col-permutation-1 by blast
moreover
have  $\text{Per } N \ M \ P \wedge \text{Per } N' \ M' \ P'$ 
  using Lambert-def assms(1) assms(2) by presburger
hence  $N \ M \ P \ \text{CongA } N' \ M' \ P'$ 
  by (metis  $\langle N \ M \ \text{ParStrict } P \ Q \rangle \langle N \ Q \ \text{ParStrict } M \ P \rangle \langle N' \ M' \ \text{ParStrict } P' \ Q' \rangle$ 
     $\langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$  l11-16 par-strict-neq1 par-strict-neq2)
moreover
have  $P \ N \ M \ \text{CongA } P' \ N' \ M'$ 
  by (simp add:  $\langle M \ N \ P \ \text{CongA } M' \ N' \ P' \rangle$  conga-comm)
moreover
have  $\text{Cong } P \ N \ P' \ N'$ 
  using  $\langle \text{Cong } N \ P \ N' \ P' \rangle$  not-cong-2143 by blast
ultimately
show ?thesis
  using l11-50-2 by blast
qed

lemma cong2-lam2--cong-conga-2:
assumes Lambert  $N \ M \ P \ Q$  and
  Lambert  $N' \ M' \ P' \ Q'$  and
  Cong  $N \ Q \ N' \ Q'$  and
  Cong  $P \ Q \ P' \ Q'$ 
shows  $M \ P \ Q \ \text{CongA } M' \ P' \ Q'$ 
proof –
  have  $N \ Q \ \text{ParStrict } M \ P$ 
    using assms(1) lam--pars1423 by auto
  have  $N \ M \ \text{ParStrict } P \ Q$ 
    by (simp add: assms(1) lam--pars1234)
  have  $N' \ Q' \ \text{ParStrict } M' \ P'$ 
    by (simp add: assms(2) lam--pars1423)
  have  $N' \ M' \ \text{ParStrict } P' \ Q'$ 
    by (simp add: assms(2) lam--pars1234)
  have  $\neg \text{Col } N \ M \ P$ 
    using  $\langle N \ M \ \text{ParStrict } P \ Q \rangle$  par-strict-not-col-1 by auto

```

```

have  $\neg \text{Col } M \ N \ Q$ 
  using  $\langle N \ M \ \text{ParStrict } P \ Q \rangle \text{ col-permutation-4 par-strict-not-col-4}$  by blast
have  $\neg \text{Col } M' \ N' \ Q'$ 
  using  $\langle N' \ M' \ \text{ParStrict } P' \ Q' \rangle \text{ not-col-permutation-4}$ 
    par-strict-not-col-4 by blast
{
  have  $\text{Per } N \ Q \ P \wedge \text{Per } N' \ Q' \ P'$ 
    using Lambert-def assms(1) assms(2) by presburger
  hence  $N \ Q \ P \ \text{CongA } N' \ Q' \ P'$ 
    by (metis cong-diff-3  $\langle N' \ M' \ \text{ParStrict } P' \ Q' \rangle \langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$ 
      assms(3) assms(4) l11-16 par-strict-neq1 par-strict-neq2)
  moreover
  have  $\text{Cong } Q \ N \ Q' \ N'$ 
    using assms(3) not-cong-2143 by blast
  moreover
  have  $\text{Cong } Q \ P \ Q' \ P'$ 
    using Cong-perm assms(4) by blast
  moreover
  have  $\text{Cong } N \ P \ N' \ P'$ 
    using calculation(1) calculation(2) calculation(3) l11-49 by blast
}
hence  $\text{Cong } N \ P \ N' \ P'$  by auto
have  $Q \ N \ P \ \text{CongA } Q' \ N' \ P' \wedge Q \ P \ N \ \text{CongA } Q' \ P' \ N'$ 
proof -
  have  $\text{Per } N \ Q \ P \wedge \text{Per } N' \ Q' \ P'$ 
    using Lambert-def assms(1) assms(2) by presburger
  hence  $N \ Q \ P \ \text{CongA } N' \ Q' \ P'$ 
    by (metis cong-diff-3  $\langle N' \ M' \ \text{ParStrict } P' \ Q' \rangle \langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$ 
      assms(3) assms(4) l11-16 par-strict-neq1 par-strict-neq2)
  moreover
  have  $\text{Cong } Q \ N \ Q' \ N'$ 
    using assms(3) not-cong-2143 by blast
  moreover
  have  $\text{Cong } Q \ P \ Q' \ P'$ 
    using Cong-perm assms(4) by blast
  moreover
  have  $N \neq P$ 
    using  $\langle \neg \text{Col } N \ M \ P \rangle \text{ col-trivial-3}$  by blast
  ultimately
  show ?thesis
    using l11-49  $\langle \text{Cong } N \ P \ N' \ P' \rangle$  by blast
qed
have  $M \ N \ P \ \text{CongA } M' \ N' \ P'$ 
proof -
  have  $N \ Q \ \text{OS } M \ P$ 
    by (simp add:  $\langle N \ Q \ \text{ParStrict } M \ P \rangle$  l12-6)
  moreover
  have  $N' \ Q' \ \text{OS } M' \ P'$ 
    by (simp add:  $\langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$  l12-6)
  moreover
  have  $\text{Per } M \ N \ Q \wedge \text{Per } M' \ N' \ Q'$ 
    using Lambert-def assms(1) assms(2) by presburger
  hence  $M \ N \ Q \ \text{CongA } M' \ N' \ Q'$ 
    using  $\langle N \ M \ \text{ParStrict } P \ Q \rangle \langle N \ Q \ \text{ParStrict } M \ P \rangle \langle N' \ M' \ \text{ParStrict } P' \ Q' \rangle$ 
       $\langle N' \ Q' \ \text{ParStrict } M' \ P' \rangle$  l11-16 par-strict-neq1 by blast
  moreover
  have  $Q \ N \ P \ \text{CongA } Q' \ N' \ P'$ 
    by (simp add:  $\langle Q \ N \ P \ \text{CongA } Q' \ N' \ P' \wedge Q \ P \ N \ \text{CongA } Q' \ P' \ N' \rangle$ )
  ultimately
  show ?thesis
    using l11-22b by blast
qed
{
  have  $\neg \text{Col } P \ N \ M$ 
    using  $\langle \neg \text{Col } N \ M \ P \rangle \text{ col-permutation-1}$  by blast
  moreover

```

```

have Per N M P  $\wedge$  Per N' M' P'
  using Lambert-def assms(1) assms(2) by presburger
hence N M P CongA N' M' P'
  by (metis  $\langle$ N M ParStrict P Q $\rangle$   $\langle$ N Q ParStrict M P $\rangle$   $\langle$ N' M' ParStrict P' Q' $\rangle$ 
     $\langle$ N' Q' ParStrict M' P' $\rangle$  l11-16 par-strict-neq1 par-strict-neq2)
moreover
have P N M CongA P' N' M'
  by (simp add:  $\langle$ M N P CongA M' N' P' $\rangle$  conga-comm)
moreover
have Cong P N P' N'
  using  $\langle$ Cong N P N' P' $\rangle$  not-cong-2143 by blast
ultimately
have M P N CongA M' P' N'
  using l11-50-2 by blast
}
moreover
have P N TS M Q
  by (metis Par-strict-cases  $\langle$ N M ParStrict P Q $\rangle$   $\langle$ N Q ParStrict M P $\rangle$ 
    l9-31 pars--os3412)
moreover
have P' N' TS M' Q'
  by (meson  $\langle$ N' M' ParStrict P' Q' $\rangle$   $\langle$ N' Q' ParStrict M' P' $\rangle$ 
    invert-two-sides l12-6 l9-31 one-side-symmetry)
moreover
have N P Q CongA N' P' Q'
  by (simp add:  $\langle$ Q N P CongA Q' N' P'  $\wedge$  Q P N CongA Q' P' N' $\rangle$  conga-comm)
ultimately
show ?thesis
  using l11-22a by blast
qed

```

```

lemma cong2-lam2--cong-conga:
  assumes Lambert N M P Q and
    Lambert N' M' P' Q' and
    Cong N Q N' Q' and
    Cong P Q P' Q'
  shows Cong N M N' M'  $\wedge$  M P Q CongA M' P' Q'
  by (meson assms(1) assms(2) assms(3) assms(4) cong2-lam2--cong-conga-1
    cong2-lam2--cong-conga-2)

```

```

lemma cong2-sac2--cong:
  assumes Saccheri A B C D and
    Saccheri A' B' C' D' and
    Cong A B A' B' and
    Cong A D A' D'
  shows Cong B C B' C'
proof -
  have Cong B D B' D'  $\wedge$  A B D CongA A' B' D'  $\wedge$  A D B CongA A' D' B'
  proof -
    have Per B A D  $\wedge$  Per B' A' D'
      using Saccheri-def assms(1) assms(2) by blast
    hence B A D CongA B' A' D'
      by (metis l11-16 assms(1) assms(2) sac-distincts)
    moreover
    have Cong A B A' B'
      by (simp add: assms(3))
    moreover
    have Cong A D A' D'
      by (meson assms(4))
    moreover
    have B  $\neq$  D
      using assms(1) sac-distincts by blast
    ultimately
    show ?thesis
      using l11-49 by blast
  qed

```

```

qed
hence  $\text{Cong } B \ D \ B' \ D'$ 
  by simp
have  $A \ B \ D \ \text{CongA } A' \ B' \ D'$ 
  by (simp add:  $\langle \text{Cong } B \ D \ B' \ D' \wedge A \ B \ D \ \text{CongA } A' \ B' \ D' \wedge A \ D \ B \ \text{CongA } A' \ D' \ B' \rangle$ )
have  $A \ D \ B \ \text{CongA } A' \ D' \ B'$ 
  by (simp add:  $\langle \text{Cong } B \ D \ B' \ D' \wedge A \ B \ D \ \text{CongA } A' \ B' \ D' \wedge A \ D \ B \ \text{CongA } A' \ D' \ B' \rangle$ )
have  $\text{Cong } D \ B \ D' \ B'$ 
  using  $\langle \text{Cong } B \ D \ B' \ D' \rangle$  not-cong-2143 by blast
moreover
have  $\text{Cong } D \ C \ D' \ C'$ 
proof -
  have  $\text{Cong } A \ B \ D \ C$ 
    using Saccheri-def assms(1) not-cong-1243 by blast
  moreover
  have  $\text{Cong } A' \ B' \ D' \ C'$ 
    using Saccheri-def assms(2) not-cong-1243 by blast
  ultimately
  show ?thesis
    by (meson assms(3) cong-inner-transitivity)
qed
moreover
have  $B \ D \ C \ \text{CongA } B' \ D' \ C'$ 
proof -
  have  $D \ A \ OS \ B \ C$ 
    using Saccheri-def assms(1) invert-one-side by blast
  moreover
  have  $D' \ A' \ OS \ B' \ C'$ 
    using Saccheri-def assms(2) invert-one-side by blast
  moreover
  have  $B \ D \ A \ \text{CongA } B' \ D' \ A'$ 
    by (simp add:  $\langle A \ D \ B \ \text{CongA } A' \ D' \ B' \rangle$  cong-comm)
  moreover
  have  $A \ D \ C \ \text{CongA } A' \ D' \ C'$ 
    by (metis Saccheri-def l11-16 assms(1) assms(2) sac-distincts)
  ultimately
  show ?thesis
    by (meson l11-22b Tarski-neutral-dimensionless-axioms)
qed
ultimately
show ?thesis
  using l11-49 by blast
qed

lemma sac--perp1214:
  assumes Saccheri  $A \ B \ C \ D$ 
  shows  $A \ B \ \text{Perp } A \ D$ 
proof -
  have  $B \ A \ \text{Perp } A \ D$ 
  proof -
    have  $A \neq B$ 
      using assms sac-distincts by blast
    moreover
    have  $A \neq D$ 
      using assms sac-distincts by blast
    moreover
    have  $\text{Per } B \ A \ D$ 
      using Saccheri-def assms by blast
    ultimately
    show ?thesis
      using per-perp by presburger
  qed
  thus ?thesis
    by (simp add: perp-left-comm)
qed

```



```

lemma sac--perp3414:
  assumes Saccheri A B C D
  shows C D Perp A D
  by (meson assms perp-comm sac--perp1214 sac-perm)

```

```

lemma cop-sac2--sac:
  assumes Saccheri A B C D and
    Saccheri A B E F and
    D ≠ F and
    Coplanar A B D F
  shows Saccheri D C E F

```

```

proof -
  have A B Perp A D
    using assms(1) sac--perp1214 by blast
  have A B Perp A F
    using assms(2) sac--perp1214 by blast
  have C D Perp A D
    using assms(1) sac--perp3414 by blast
  have E F Perp A F
    using assms(2) sac--perp3414 by blast
  have Col A D F
  proof -
    have A D Perp A B
      using Perp-perm ⟨A B Perp A D⟩ by blast
    moreover
    have A F Perp A B
      using Perp-perm ⟨A B Perp A F⟩ by blast
    ultimately
    show ?thesis
      using assms(4) cop-perp2--col by blast
  qed

```

```

have Per C D F
proof -
  have D A Perp D C
    using Perp-cases ⟨C D Perp A D⟩ by blast
  moreover
  have Col D A F
    using NCol-cases ⟨Col A D F⟩ by blast
  moreover
  have Col D A D
    by (simp add: col-trivial-3)
  ultimately
  show ?thesis
    using Perp-cases l8-16-1 by blast

```

```

qed
moreover
have Per D F E
proof -
  have F A Perp E F
    using Perp-perm ⟨E F Perp A F⟩ by blast
  moreover
  have Col F A F
    by (simp add: col-trivial-3)
  moreover
  have Col F A D
    by (simp add: ⟨Col A D F⟩ col-permutation-2)
  ultimately
  show ?thesis
    using Perp-perm l8-16-1 by blast

```

```

qed
moreover
have Cong D C E F
proof -
  have Cong D C A B
    using Saccheri-def assms(1) Cong-perm by blast
  moreover

```

```

have Cong A B E F
  using Saccheri-def assms(2) by auto
ultimately
show ?thesis
  using cong-transitivity by blast
qed
moreover
have D F OS C E
proof -
  have D F OS C B
  proof -
    have Col D A F
    using Col-perm ⟨Col A D F⟩ by blast
    moreover
    have D A OS C B
    by (meson Saccheri-def sac-perm assms(1))
    ultimately
    show ?thesis
    using assms(3) col-one-side by blast
  qed
  moreover
  have D F OS B E
  proof -
    have Col F A D
    using ⟨Col A D F⟩ not-col-permutation-1 by blast
    moreover
    have F A OS B E
    using Saccheri-def assms(2) invert-one-side by blast
    ultimately
    show ?thesis
    using assms(3) col-one-side invert-one-side by presburger
  qed
  ultimately
  show ?thesis
  using one-side-transitivity by blast
qed
ultimately
show ?thesis
  using Saccheri-def by blast
qed

```

```

lemma three-hypotheses-aux:
  assumes Saccheri A B C D and
    Saccheri A' B' C' D' and
    M Midpoint B C and
    M' Midpoint B' C' and
    N Midpoint A D and
    N' Midpoint A' D' and
    M N Le M' N'
  shows (Per A B C ⟷ Per A' B' C') ∧ (Acute A B C ⟷ Acute A' B' C')
proof -
  have Lambert N M C D
  using assms(1) assms(3) assms(5) mid2-sac--lam6534 by blast
  have Lambert N' M' C' D'
  using assms(2) assms(4) assms(6) mid2-sac--lam6534 by blast
  have A ≠ B
  using assms(1) sac-distincts by blast
  have C ≠ B
  using assms(1) sac-distincts by blast
  have C ≠ D
  using assms(1) sac-distincts by blast
  have A ≠ D
  using assms(1) sac-distincts by blast
  have A' ≠ B'

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    using assms(2) sac-distincts by blast
have C' ≠ B'
    using assms(2) sac-distincts by blast
have C' ≠ D'
    using assms(2) sac-distincts by blast
have A' ≠ D'
    using assms(2) sac-distincts by blast
have ¬ Col C D A
    using Saccheri-def assms(1) col124--nos col-permutation-3 by blast
obtain H where N Out D H ∧ Cong N H N' D'
    by (metis Mid-cases ⟨A ≠ D⟩ ⟨A' ≠ D'⟩ assms(5) assms(6)
        is-midpoint-id segment-construction-3)
have Col A D H
    by (metis col124--nos ⟨A ≠ D⟩ ⟨N Out D H ∧ Cong N H N' D'⟩
        assms(5) midpoint-out-1 one-side-reflexivity out-col out-out-one-side)
obtain G0 where A D Perp G0 H ∧ A D OS C G0
    using NCol-perm ⟨Col A D H⟩ ⟨¬ Col C D A⟩ l10-15 by blast
have N ≠ H
    using ⟨N Out D H ∧ Cong N H N' D'⟩ out-diff2 by blast
have G0 ≠ H
    using ⟨A D Perp G0 H ∧ A D OS C G0⟩ ⟨Col A D H⟩ l8-16-1 by blast
obtain G where H Out G0 G ∧ Cong H G C' D'
    using ⟨C' ≠ D'⟩ ⟨G0 ≠ H⟩ segment-construction-3 by presburger
have H ≠ G
    by (metis ⟨C' ≠ D'⟩ ⟨H Out G0 G ∧ Cong H G C' D'⟩ cong-diff-4)
have N D OS C G
proof -
    have D N OS C G
    proof -
        have Col D A N
        by (meson Mid-cases assms(5) l7-3-2 mid-two-sides not-two-sides-id)
        moreover
        have D ≠ N
        using ⟨A ≠ D⟩ assms(5) midpoint-not-midpoint by blast
        moreover
        have A D OS C G
        proof -
            have A D OS C G0
            by (simp add: ⟨A D Perp G0 H ∧ A D OS C G0⟩)
            moreover
            have ¬ Col A D G0
            using calculation col-one-side os-distincts by blast
            hence A D OS G0 G
            using l9-19-R2 ⟨Col A D H⟩ ⟨H Out G0 G ∧ Cong H G C' D'⟩ by blast
            ultimately
            show ?thesis
            using one-side-transitivity by blast
        qed
        hence D A OS C G
        by (simp add: invert-one-side)
        ultimately
        show ?thesis
        using col-one-side by blast
    qed
    thus ?thesis
    using invert-one-side by blast
qed
have A D Perp H G
proof -
    have A D Perp H G0
    by (simp add: ⟨A D Perp G0 H ∧ A D OS C G0⟩ perp-right-comm)
    moreover
    have Col H G0 G
    by (simp add: ⟨H Out G0 G ∧ Cong H G C' D'⟩ out-col)
    ultimately
    show ?thesis

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    using perp-col1  $\langle H \neq G \rangle$  by blast
qed
have  $\neg \text{Col } M \ N \ H$ 
proof -
  have  $M \neq N$ 
    using Lambert-def  $\langle \text{Lambert } N \ M \ C \ D \rangle$  by blast
  moreover
  have  $\text{Per } M \ N \ H$ 
  proof -
    have  $N \neq D$ 
      using  $\langle N \ D \ OS \ C \ G \rangle$  os-distincts by blast
    moreover
    have  $\text{Per } M \ N \ D$ 
      using Lambert-def  $\langle \text{Lambert } N \ M \ C \ D \rangle$  by presburger
    moreover
    have  $\text{Col } N \ D \ H$ 
      by (simp add:  $\langle N \ \text{Out } D \ H \wedge \text{Cong } N \ H \ N' \ D' \rangle$  out-col)
    ultimately
    show ?thesis
      using per-col by blast
  qed
  ultimately
  show ?thesis
    using l8-9  $\langle N \neq H \rangle$  by blast
qed
have Coplanar  $A \ D \ M \ G$ 
proof -
  have  $\neg \text{Col } C \ A \ D$ 
    by (simp add:  $\langle \neg \text{Col } C \ D \ A \rangle$  not-col-permutation-5)
  moreover
  have Coplanar  $C \ A \ D \ M$ 
  proof -
    have Coplanar  $A \ D \ C \ B$ 
      by (meson assms(1) par--coplanar par-right-comm sac--par1423)
    moreover
    have  $\text{Col } C \ B \ M$ 
      by (meson Mid-cases assms(3) l7-3-2 mid-two-sides not-two-sides-id)
    ultimately
    show ?thesis
      using col-cop--cop coplanar-perm-12  $\langle C \neq B \rangle$  by blast
  qed
  moreover
  have Coplanar  $C \ A \ D \ G$ 
  proof -
    have Coplanar  $C \ G \ D \ N$ 
      by (meson  $\langle N \ D \ OS \ C \ G \rangle$  coplanar-perm-17 os--coplanar)
    moreover
    have  $D \neq N$ 
      using  $\langle N \ D \ OS \ C \ G \rangle$  os-distincts by blast
    moreover
    have  $\text{Col } D \ N \ A$ 
      using Mid-cases  $\langle A \neq D \rangle$  assms(5) midpoint-out out-col by presburger
    ultimately
    show ?thesis
      by (meson col-cop--cop coplanar-perm-5)
  qed
  ultimately
  show ?thesis
    using coplanar-trans-1 by blast
qed
have  $\neg \text{Col } M \ N \ G$ 
proof -
  have  $M \ N \ \text{ParStrict } G \ H$ 
  proof -
    have  $M \ N \ \text{Par } G \ H$ 
    proof -

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```

have Coplanar A D M H
  using ⟨Col A D H⟩ ncop--ncols by blast
moreover
have Coplanar A D N G
  by (meson Col-cases assms(5) midpoint-col ncop--ncol)
moreover
have Coplanar A D N H
  by (meson ⟨Col A D H⟩ col--coplanar col-permutation-5 ncoplanar-perm-4)
moreover
have M N Perp A D
proof –
  have M N Perp D N
    using Lambert-def
    by (metis ⟨Lambert N M C D⟩ per-perp perp-right-comm)
  moreover
  have Col D N A
    by (metis Out-cases assms(5) ⟨A ≠ D⟩ midpoint-out-1 out-col)
  ultimately
  show ?thesis
    using col-trivial-3 perp-col2-bis ⟨A ≠ D⟩ by blast
qed
moreover
have G H Perp A D
  using Perp-perm ⟨A D Perp H G⟩ by blast
ultimately
show ?thesis
  using l12-9 ⟨Coplanar A D M G⟩ by blast
qed
moreover
have Col G H H
  by (simp add: col-trivial-2)
ultimately
show ?thesis
  using par-not-col-strict ⟨¬ Col M N H⟩ by blast
qed
hence M N OS G H
  by (simp add: l12-6)
thus ?thesis
  by (meson col123--nos)
qed
have M ≠ N
  using ⟨¬ Col M N G⟩ col-trivial-1 by force
have A ≠ H
  using Mid-cases ⟨N Out D H ∧ Cong N H N' D'⟩ assms(5)
    midpoint-bet not-bet-and-out by blast
have Col H A N
  by (metis OS-def Out-cases TS-def ⟨A ≠ D⟩ ⟨N Out D H ∧ Cong N H N' D'⟩
    assms(5) invert-one-side midpoint-out-1 one-side-reflexivity
    out-col out-out-one-side)
obtain L where Col M N L ∧ M N Perp G L
  using ⟨¬ Col M N G⟩ l8-18-existence by blast
have ¬ Col M A D
proof –
  have B C ParStrict A D
    using Par-strict-cases assms(1) sac--pars1423 by blast
  moreover
  have Col M B C
    by (simp add: assms(3) midpoint-col)
  ultimately
  show ?thesis
    using par-not-col by blast
qed
have Lambert N L G H
proof –
  have Per N H G
  proof –

```

```

have H A Perp H G
  by (metis ⟨A D Perp H G⟩ ⟨A ≠ H⟩ ⟨Col A D H⟩ col-trivial-3 perp-col2)
hence H A Perp G H
  using Perp-cases by blast
hence Per A H G
  by (simp add: perp-per-1)
thus ?thesis
  using ⟨A ≠ H⟩ ⟨Col H A N⟩ l8-3 by blast
qed
moreover
have ¬ Col N H G
  using ⟨H ≠ G⟩ ⟨N ≠ H⟩ calculation l8-9 by blast
have Per A N M
proof -
  have A D Perp M N
    using mid2-sac--perp-lower assms(1) assms(3) assms(5) by auto
  hence A N Perp M N
    by (metis NCol-perm midpoint-distinct-1 assms(5) midpoint-col perp-col)
  thus ?thesis
    by (simp add: perp-left-comm perp-per-1)
qed
have Per L N H
proof -
  have Per M N H
  proof -
    have N ≠ A
      using ⟨A ≠ D⟩ assms(5) is-midpoint-id by blast
    moreover
    have Per M N A
      by (simp add: ⟨Per A N M⟩ l8-2)
    moreover
    have Col N A H
      using ⟨Col H A N⟩ not-col-permutation-3 by blast
    ultimately
    show ?thesis
      using per-col by blast
  qed
  moreover
  have Col N M L
    by (simp add: ⟨Col M N L ∧ M N Perp G L⟩ col-permutation-4)
  ultimately
  show ?thesis
    using l8-3 ⟨M ≠ N⟩ by blast
qed
moreover
{
  assume L = N
  have Col G H N
  proof -
    have Coplanar M G H N
    proof -
      have ¬ Col A D M
        using (¬ Col M A D) col-permutation-2 by blast
      moreover
      have Coplanar A D M M
        using ncop-distincts by blast
      moreover
      have Coplanar A D M H
        using ⟨Col A D H⟩ coplanar-perm-1 ncop--ncol by blast
      moreover
      have Coplanar A D M N
        by (meson ⟨A ≠ H⟩ ⟨Col H A N⟩ calculation(3) col-cop--cop
          coplanar-perm-8 not-col-permutation-4)
      ultimately
      show ?thesis
        by (meson coplanar-pseudo-trans ⟨Coplanar A D M G⟩)
    qed
  }

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qed
moreover
have  $\text{Per } G \ N \ M$ 
  using  $\text{Perp-cases } \langle \text{Col } M \ N \ L \wedge M \ N \ \text{Perp } G \ L \rangle \langle L = N \rangle \text{ perp-per-1}$  by blast
moreover
have  $\text{Per } H \ N \ M$ 
  by ( $\text{metis } \langle A \neq D \rangle \langle \text{Col } H \ A \ N \rangle \langle \text{Per } A \ N \ M \rangle \text{ assms}(5) \langle M \neq N \rangle$ 
     $\text{col-col-per-per col-trivial-1 is-midpoint-id}$ )
ultimately
show ?thesis
  using  $\text{cop-per2--col } \langle M \neq N \rangle$  by blast
qed
hence  $\text{False}$ 
  using  $\text{Col-cases } \langle \neg \text{Col } N \ H \ G \rangle$  by blast
}
hence  $L \neq N$  by auto
moreover
have  $L \neq G$ 
  using  $\langle \text{Col } M \ N \ L \wedge M \ N \ \text{Perp } G \ L \rangle \langle \neg \text{Col } M \ N \ G \rangle$  by fastforce
moreover
have  $\text{Per } N \ L \ G$ 
proof -
  have  $N \neq L$ 
    using  $\text{calculation}(3)$  by auto
  moreover
  have  $N \ M \ \text{Perp } G \ L$ 
    by ( $\text{simp add: } \langle \text{Col } M \ N \ L \wedge M \ N \ \text{Perp } G \ L \rangle \text{ perp-left-comm}$ )
  moreover
  have  $\text{Col } N \ M \ L$ 
    by ( $\text{simp add: } \langle \text{Col } M \ N \ L \wedge M \ N \ \text{Perp } G \ L \rangle \text{ col-permutation-4}$ )
  ultimately
  show ?thesis
    by ( $\text{metis } \text{Perp-cases perp-col1 perp-per-1}$ )
qed
moreover
have  $\text{Coplanar } N \ L \ G \ H$ 
proof -
  have  $\neg \text{Col } A \ D \ M$ 
    using  $\text{Col-cases } \langle \neg \text{Col } M \ A \ D \rangle$  by blast
  moreover
  have  $\text{Coplanar } A \ D \ M \ N$ 
    by ( $\text{meson } \langle A \neq H \rangle \langle \text{Col } A \ D \ H \rangle \langle \text{Col } H \ A \ N \rangle \text{ col-transitivity-1}$ 
       $\text{coplanar-perm-1 ncop--ncol not-col-permutation-4 not-col-permutation-5}$ )
  moreover
  have  $\text{Col } A \ D \ N \wedge \text{Col } M \ L \ N$ 
    using  $\text{Col-cases Col-def } \langle \text{Col } M \ N \ L \wedge M \ N \ \text{Perp } G \ L \rangle$ 
       $\text{assms}(5) \text{ midpoint-bet}$  by blast
  hence  $\text{Coplanar } A \ D \ M \ L$ 
    using  $\text{Coplanar-def}$  by blast
  moreover
  have  $\text{Coplanar } A \ D \ M \ H$ 
    using  $\langle \text{Col } A \ D \ H \rangle \text{ coplanar-perm-1 ncop--ncol}$  by blast
  ultimately
  show ?thesis
    by ( $\text{meson coplanar-pseudo-trans } \langle \text{Coplanar } A \ D \ M \ G \rangle$ )
qed
ultimately
show ?thesis
  using  $\text{Lambert-def } \langle H \neq G \rangle \langle N \neq H \rangle$  by blast
qed
have  $N \ D \ \text{ParStrict } M \ C$ 
  by ( $\text{simp add: } \langle \text{Lambert } N \ M \ C \ D \rangle \text{ lam--pars1423}$ )
have  $\text{Bet } N \ M \ L$ 
proof -
  have  $\text{Cong } N' \ M' \ N \ L \wedge M' \ C' \ D' \ \text{CongA } L \ G \ H$ 
  proof -

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have Cong N' D' N H
  by (simp add: ⟨N Out D H ∧ Cong N H N' D'⟩ cong-symmetry)
moreover
have Cong C' D' G H
  using ⟨H Out G0 G ∧ Cong H G C' D'⟩ not-cong-4312 by blast
ultimately
show ?thesis
  using cong2-lam2--cong-conga ⟨Lambert N L G H⟩ ⟨Lambert N' M' C' D'⟩ by blast
qed
show ?thesis
proof -
  have N Out M L
  proof -
    have Col N M L
    by (simp add: ⟨Col M N L ∧ M N Perp G L⟩ col-permutation-4)
  moreover
  have N D OS M L
  proof -
    have N D OS M G
    using ⟨N D OS C G⟩ ⟨N D ParStrict M C⟩ l12-6
    one-side-transitivity by blast
  moreover
  have N D OS G L
  proof -
    have Col N H D
    using Out-cases ⟨N Out D H ∧ Cong N H N' D'⟩ out-col by blast
  moreover
  have N ≠ D
    using ⟨N D OS C G⟩ os-distincts by blast
  moreover
  have N H OS G L
    by (meson Par-strict-cases ⟨Lambert N L G H⟩ l12-6 lam--pars1423)
  ultimately
  show ?thesis
    using col-one-side by blast
  qed
  ultimately
  show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
    using col-one-side-out by blast
  qed
  moreover
  have N M Le N L
  proof -
    have Cong M N N M
    by (simp add: cong-pseudo-reflexivity)
  moreover
  have Cong M' N' N L
    using ⟨Cong N' M' N L ∧ M' C' D' CongA L G H⟩ not-cong-2134 by blast
  ultimately
  show ?thesis
    using l5-6 assms(7) by blast
  qed
  ultimately
  show ?thesis
    by (simp add: l6-13-1)
  qed
qed
have ¬ Col N M C
  using Lambert-def ⟨Lambert N M C D⟩ per-not-col by presburger
have ¬ Col N D M
  using ⟨N D ParStrict M C⟩ par-strict-not-col-1 by auto
have Coplanar M C D A

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proof –
  have  $M \ C \ \text{ParStrict} \ D \ N$ 
    using  $\text{Par-strict-perm} \ \langle N \ D \ \text{ParStrict} \ M \ C \rangle$  by blast
  moreover
  have  $Col \ D \ N \ A$ 
    by (metis  $\langle Col \ H \ A \ N \rangle \ \langle N \ \text{Out} \ D \ H \ \wedge \ Cong \ N \ H \ N' \ D' \rangle \ \langle N \neq H \rangle$ 
      col2--eq col-permutation-1 out-col)
  ultimately
  show ?thesis
    using  $os\text{-coplanar} \ \text{par-strict-all-one-side} \ \langle A \neq D \rangle$  by blast
qed
have  $\exists \ K. \ Col \ K \ M \ C \ \wedge \ Bet \ G \ K \ H$ 
proof cases
  assume  $L = M$ 
  have  $Col \ G \ C \ M$ 
  proof –
    have  $Coplanar \ N \ G \ C \ M$ 
    proof –
      have  $\neg \ Col \ A \ D \ M$ 
        using  $Col\text{-cases} \ \langle \neg \ Col \ M \ A \ D \rangle$  by blast
      moreover
      have  $Coplanar \ A \ D \ M \ N$ 
        using  $Col\text{-cases} \ \text{assms}(5) \ \text{midpoint-col} \ \text{ncop--ncols}$  by blast
      moreover
      have  $Coplanar \ A \ D \ M \ C$ 
        using  $\langle Coplanar \ M \ C \ D \ A \rangle \ \text{ncoplanar-perm-17}$  by blast
      moreover
      have  $Coplanar \ A \ D \ M \ M$ 
        using  $\text{ncop-distincts}$  by blast
      ultimately
      show ?thesis
        using  $\text{coplanar-pseudo-trans} \ \langle Coplanar \ A \ D \ M \ G \rangle$  by presburger
    qed
    moreover
    have  $N \neq M$ 
      using  $\langle \neg \ Col \ N \ M \ C \rangle \ \text{col-trivial-1}$  by blast
    moreover
    have  $Per \ G \ M \ N$ 
      using  $\langle Col \ M \ N \ L \ \wedge \ M \ N \ \text{Perp} \ G \ L \rangle \ \langle L = M \rangle \ \text{l8-2} \ \text{perp-per-1}$  by blast
    moreover
    have  $Per \ C \ M \ N$ 
      using  $\text{Lambert-def} \ \langle \text{Lambert} \ N \ M \ C \ D \rangle \ \text{l8-2}$  by presburger
    ultimately
    show ?thesis
      using  $\text{cop-per2--col}$  by blast
  qed
  hence  $Col \ G \ M \ C$ 
    using  $\text{col-permutation-5}$  by blast
  moreover
  have  $Bet \ G \ G \ H$ 
    by (simp add:  $\text{between-trivial2}$ )
  ultimately
  show ?thesis
    by blast
next
  assume  $L \neq M$ 
  hence  $\neg \ Col \ L \ M \ C$ 
    using  $\langle Bet \ N \ M \ L \rangle \ \langle \neg \ Col \ N \ M \ C \rangle \ \text{bet-col} \ \text{col2--eq} \ \text{col-permutation-3}$  by blast
  have  $M \ C \ TS \ G \ H$ 
  proof –
    have  $M \ C \ TS \ H \ L$ 
    proof –
      have  $M \ C \ TS \ N \ L$ 
        using  $\langle Bet \ N \ M \ L \rangle \ \langle L \neq M \rangle \ \langle \neg \ Col \ N \ M \ C \rangle \ \text{bet--ts}$ 
          not-col-permutation-1 by presburger
      moreover

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have  $M C OS N H$ 
proof -
  have  $M C ParStrict N D$ 
    using Par-strict-cases  $\langle N D ParStrict M C \rangle$  by blast
  moreover
  have  $Col N D H$ 
    by (metis  $\langle A \neq H \rangle \langle Col A D H \rangle \langle Col H A N \rangle col3$ 
      col-trivial-3 not-col-permutation-1)
  ultimately
  show ?thesis
    using par-strict-one-side  $\langle N \neq H \rangle$  by blast
qed
ultimately
show ?thesis
  using l9-8-2 by blast
qed
hence  $M C TS L H$ 
  by (simp add: l9-2)
moreover
have  $M C OS L G$ 
proof -
  have  $M C Par L G$ 
  proof -
    have  $Coplanar M N M L$ 
      using ncop-distincts by blast
    moreover
    have  $Coplanar M N M G$ 
      using ncop-distincts by blast
    moreover
    have  $Coplanar M N C L$ 
      by (simp add:  $\langle Bet N M L \rangle$  bet-col col--coplanar coplanar-perm-7)
    moreover
    have  $Coplanar M N C G$ 
      by (metis  $\langle N D OS C G \rangle \langle N D ParStrict M C \rangle$  coplanar-perm-10
        coplanar-perm-8 coplanar-trans-1 not-col-permutation-1
        one-side-not-col123 os--coplanar pars--coplanar)
    moreover
    have  $M C Perp M N$ 
      using Lambert-def by (metis Perp-perm  $\langle Lambert N M C D \rangle$  per-perp)
    moreover
    have  $L G Perp M N$ 
      by (simp add: Perp-perm  $\langle Col M N L \wedge M N Perp G L \rangle$ )
    ultimately
    show ?thesis
      using l12-9 by blast
  qed
  moreover
  have  $Col L G L$ 
    by (simp add: col-trivial-3)
  moreover
  have  $\neg Col M C L$ 
    using Col-cases  $\langle \neg Col L M C \rangle$  by blast
  ultimately
  show ?thesis
    using l12-6 par-not-col-strict by blast
qed
ultimately
show ?thesis
  using l9-8-2 by blast
qed
thus ?thesis
  by (simp add: TS-def)
qed
then obtain  $K$  where  $Col K M C \wedge Bet G K H$ 
  by auto
hence  $Col K M C$ 

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```

by auto
have Bet G K H
  by (simp add: ⟨Col K M C ∧ Bet G K H⟩)
have ¬ Col H M C
proof -
  have Col H N D
    using NCol-cases ⟨N Out D H ∧ Cong N H N' D'⟩ out-col by blast
  thus ?thesis
    using par-not-col ⟨N D ParStrict M C⟩ by blast
qed
have K ≠ H
  using ⟨Col K M C⟩ ⟨¬ Col H M C⟩ by blast
have N M ParStrict C D
  by (simp add: ⟨Lambert N M C D⟩ lam--pars1234)
hence ¬ Col N C D
  using Col-cases par-strict-not-col-3 by blast
have M N ParStrict H K
proof -
  have G H ParStrict N L
    by (metis Par-strict-cases lam--pars1234 ⟨Lambert N L G H⟩)
  hence G H ParStrict N M
    by (metis Col-cases ⟨Bet N M L⟩ ⟨M ≠ N⟩ bet-col par-strict-col-par-strict)
  hence M N ParStrict H G
    using Par-strict-perm by blast
  moreover
  have Col H G K
    by (simp add: Col-def ⟨Bet G K H⟩)
  ultimately
  show ?thesis
    using par-strict-col-par-strict ⟨K ≠ H⟩ by auto
qed
have ¬ Col N H K
  using ⟨M N ParStrict H K⟩ par-strict-not-col-2 by blast
have M N OS D K
proof -
  have M N OS D H
  proof -
    have ¬ Col N M D ∨ ¬ Col N M H
      using ⟨¬ Col N D M⟩ col-permutation-5 by blast
    moreover
    have N Out D H
      by (simp add: ⟨N Out D H ∧ Cong N H N' D'⟩)
    ultimately
    show ?thesis
      using invert-one-side out-one-side by blast
  qed
  moreover
  have M N OS H K
    by (simp add: ⟨M N ParStrict H K⟩ l12-6)
  ultimately
  show ?thesis
    using one-side-transitivity by blast
qed
hence M N OS C K
  using ⟨N M ParStrict C D⟩ invert-one-side l12-6 one-side-transitivity by blast
hence M Out C K
  using ⟨Col K M C⟩ col-one-side-out col-permutation-1 by blast
have N ≠ L
  using ⟨Bet N M L⟩ ⟨M ≠ N⟩ bet-neq23--neq by blast
have M ≠ K
  using ⟨M Out C K⟩ out-diff2 by blast
have Lambert N M K H
proof -
  have Per M N H
  proof -
    have Per L N H

```

```

    using Lambert-def ⟨Lambert N L G H⟩ by auto
  moreover
  have Col N L M
    by (meson ⟨Bet N M L⟩ bet-col1 between-trivial)
  ultimately
  show ?thesis
    using l8-3 ⟨N ≠ L⟩ by blast
qed
moreover
have Per N H K
proof -
  have Per N H G
    using Lambert-def ⟨Lambert N L G H⟩ by presburger
  moreover
  have Col H G K
    by (simp add: Col-def ⟨Bet G K H⟩)
  ultimately
  show ?thesis
    using per-col ⟨H ≠ G⟩ by blast
qed
moreover
have Per N M K
proof -
  have M ≠ C
    using ⟨¬ Col H M C⟩ col-trivial-2 by blast
  moreover
  have Per N M C
    using Lambert-def ⟨Lambert N M C D⟩ by presburger
  moreover
  have Col M C K
    using Col-cases ⟨Col K M C⟩ by blast
  ultimately
  show ?thesis
    using per-col by blast
qed
moreover
have Coplanar N M K H
  by (simp add: ⟨M N ParStrict H K⟩ l12-6 os--coplanar par-strict-comm)
ultimately
show ?thesis
  using Lambert-def ⟨K ≠ H⟩ ⟨M ≠ K⟩ ⟨M ≠ N⟩ ⟨N ≠ H⟩ by blast
qed
have A B C CongA B C D
  using assms(1) sac--conga by blast
have A' B' C' CongA H G L
proof -
  have A' B' C' CongA M' C' D'
  proof -
    have A' B' C' CongA B' C' D'
      by (simp add: assms(2) sac--conga)
    moreover
    have B' Out A' A'
      by (simp add: ⟨A' ≠ B'⟩ out-trivial)
    moreover
    have B' Out C' C'
      by (simp add: ⟨C' ≠ B'⟩ out-trivial)
    moreover
    have C' Out M' B'
      using ⟨C' ≠ B'⟩ assms(4) l7-2 midpoint-out by blast
    moreover
    have C' Out D' D'
      using ⟨C' ≠ D'⟩ out-trivial by auto
    ultimately
    show ?thesis
      using l11-10 by blast
  qed
qed

```

```

moreover
have  $M' C' D' \text{ CongA } H G L$ 
proof –
  have  $\text{Cong } N' D' N H$ 
  by (simp add:  $\langle N \text{ Out } D H \wedge \text{Cong } N H N' D' \rangle \text{ cong-symmetry}$ )
moreover
have  $\text{Cong } C' D' G H$ 
  by (simp add:  $\langle H \text{ Out } G0 G \wedge \text{Cong } H G C' D' \rangle \text{ cong-3421}$ )
ultimately
show ?thesis
  using cong2-lam2--cong-conga conga-right-comm  $\langle \text{Lambert } N' M' C' D' \rangle$ 
   $\langle \text{Lambert } N L G H \rangle$  by fastforce
qed
ultimately
show ?thesis
  by (meson not-conga Tarski-neutral-dimensionless-axioms)
qed
{
  assume  $D \neq H$ 
  assume  $\neg C D \text{ ParStrict } K H$ 
  have  $C D \text{ Par } K H$ 
  proof –
    have  $\text{Coplanar } N D C K$ 
    proof –
      have  $N D \text{ ParStrict } C M$ 
      using Par-strict-cases  $\langle N D \text{ ParStrict } M C \rangle$  by blast
    moreover
    have  $\text{Col } C M K$ 
    using Col-cases  $\langle \text{Col } K M C \rangle$  by blast
    ultimately
    show ?thesis
    using os--coplanar par-strict-all-one-side by blast
  qed
  moreover
  have  $\text{Coplanar } N D C H$ 
  by (metis  $\langle A \neq H \rangle \langle \text{Col } A D H \rangle \langle \text{Col } H A N \rangle \text{ col3 col-trivial-2}$ 
    col-trivial-3 coplanar-perm-1 ncop--ncol)
  moreover
  have  $\text{Coplanar } N D D K$ 
  by (meson ncop-distincts)
  moreover
  have  $\text{Coplanar } N D D H$ 
  using ncop-distincts by blast
  moreover
  have  $C D \text{ Perp } N D$ 
  using Lambert-def
  by (metis  $\langle \text{Lambert } N M C D \rangle \text{ col-per-perp col-trivial-3}$ )
  moreover
  have  $K H \text{ Perp } N D$ 
  proof –
    have  $H G \text{ Perp } D N$ 
    proof –
      have  $D \neq N$ 
      using  $\langle M N \text{ OS } D K \rangle \text{ os-distincts}$  by blast
    moreover
    have  $H G \text{ Perp } D A$ 
    using Perp-perm  $\langle A D \text{ Perp } H G \rangle$  by blast
    moreover
    have  $\text{Col } D A N$ 
    by (meson  $\langle A \neq H \rangle \langle \text{Col } A D H \rangle \langle \text{Col } H A N \rangle \text{ col-permutation-5}$ 
      col-transitivity-1 not-col-permutation-4)
    ultimately
    show ?thesis
    using perp-col1 by blast
  qed
  moreover

```

```

    have Col H G K
      by (simp add: Col-def ⟨Bet G K H⟩)
    ultimately
    show ?thesis
      by (metis Perp-cases perp-col1 ⟨K ≠ H⟩)
  qed
  ultimately
  show ?thesis
    using l12-9 by blast
  qed
  have Col C K H using ⟨C D Par K H⟩
    by (simp add: Par-def ⟨¬ C D ParStrict K H⟩)
  have Col D K H using ⟨C D Par K H⟩
    by (simp add: Par-def ⟨¬ C D ParStrict K H⟩)
  have D = H
  proof -
    have C D Perp N D
      using Lambert-def
      by (metis ⟨Lambert N M C D⟩ col-per-perp col-trivial-3)
    moreover
    have Col C D H
      using ⟨Col C K H⟩ ⟨Col D K H⟩ ⟨K ≠ H⟩ col2--eq col-permutation-5 by blast
    moreover
    have C D Perp N H
    proof -
      have C H Perp N H
        by (metis ⟨N Out D H ∧ Cong N H N' D'⟩ calculation(1)
          calculation(2) col-trivial-2 l8-16-1 out-col per-col-eq)
      moreover
      have Col C H D
        using Col-cases ⟨Col C D H⟩ by blast
      ultimately
      show ?thesis
        using perp-col ⟨C ≠ D⟩ by blast
    qed
    ultimately
    show ?thesis
      using col-trivial-2 l8-18-uniqueness by blast
  qed
  hence False
    using ⟨D ≠ H⟩ by auto
  hence C D ParStrict K H
    by simp
}
{
  assume Bet M C K
  have Bet N D H
  proof cases
    assume D = H
    thus ?thesis
      by (simp add: between-trivial)
  next
    assume D ≠ H
    hence C D ParStrict K H
      using ⟨[D ≠ H; ¬ C D ParStrict K H] ⟹ C D ParStrict K H⟩ by blast
    have Bet H D N
    proof -
      have Col H D N
        by (meson ⟨A ≠ H⟩ ⟨Col A D H⟩ ⟨Col H A N⟩ col-permutation-4
          col-transitivity-2 not-col-permutation-3)
      moreover
      {
        assume D Out H N
        have C Out K M
        proof -
          have Col C K M

```

```

    by (simp add: Col-def ⟨Bet M C K⟩)
  moreover
  have C D OS K M
  proof -
    have C D OS K N
    proof -
      have C D OS K H
        using ⟨C D ParStrict K H⟩ l12-6 by blast
      moreover
      have C D OS H N
        by (metis Col-cases ⟨D Out H N⟩ ⟨¬ Col N C D⟩
            invert-one-side out-one-side)
      ultimately
      show ?thesis
        by (meson one-side-transitivity)
    qed
  moreover
  have C D OS N M
    by (simp add: ⟨N M ParStrict C D⟩ pars--os3412)
  ultimately
  show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
    using col-one-side-out by blast
  qed
  hence False
    using ⟨Bet M C K⟩ l6-6 not-bet-and-out by blast
}
hence ¬ D Out H N
  by auto
ultimately
show ?thesis
  using l6-4-2 by blast
qed
thus ?thesis
  using Bet-cases by blast
qed
}
{
  assume Bet M K C
  have Bet N H D
  proof cases
    assume D = H
    thus ?thesis
      by (simp add: between-trivial)
  next
    assume D ≠ H
    hence C D ParStrict K H
      using ⟨[D ≠ H; ¬ C D ParStrict K H] ⟹ C D ParStrict K H⟩ by blast
    have Bet D H N
    proof -
      have Col D H N
        by (metis ⟨A ≠ H⟩ ⟨Col A D H⟩ ⟨Col H A N⟩ col2--eq col-permutation-1)
      moreover
      {
        assume H Out D N
        have K Out C M
        proof -
          have Col K C M
            by (simp add: Col-def ⟨Bet M K C⟩)
          moreover
          have K H OS C M
          proof -
            have K H OS C N

```

```

    proof –
      have  $K\ H\ OS\ C\ D$ 
        by (simp add:  $\langle C\ D\ ParStrict\ K\ H \rangle\ pars\text{--}os3412$ )
      moreover
      have  $K\ H\ OS\ D\ N$ 
        using  $\langle H\ Out\ D\ N \rangle\ \langle \neg\ Col\ N\ H\ K \rangle\ col\text{--}permutation\text{--}2$ 
          invert-one-side out-one-side by blast
      ultimately
      show ?thesis
        using one-side-transitivity by blast
    qed
  moreover
  have  $K\ H\ OS\ N\ M$ 
    using Par-strict-perm  $\langle M\ N\ ParStrict\ H\ K \rangle\ l12\text{--}6$  by blast
  ultimately
  show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
    using col-one-side-out by blast
  qed
  hence False
    using  $\langle Bet\ M\ K\ C \rangle\ bet\text{--}col\ l6\text{--}4\text{--}1\ l6\text{--}6$  by blast
}
ultimately
show ?thesis
  using  $l6\text{--}4\text{--}2$  by blast
qed
thus ?thesis
  using Bet-cases by blast
qed
}
{
  assume  $Per\ A\ B\ C$ 
  have  $Per\ A'\ B'\ C'$ 
  proof –
    have  $Per\ L\ G\ H$ 
    proof –
      have  $Lambert\ N\ H\ K\ M$ 
        by (simp add:  $\langle Lambert\ N\ M\ K\ H \rangle\ lam\text{--}perm$ )
      moreover
      have  $Lambert\ N\ H\ G\ L$ 
        by (simp add:  $\langle Lambert\ N\ L\ G\ H \rangle\ lam\text{--}perm$ )
      moreover
      have  $Bet\ H\ K\ G$ 
        using Bet-cases  $\langle Bet\ G\ K\ H \rangle$  by blast
      moreover
      have  $Bet\ N\ M\ L$ 
        by (simp add:  $\langle Bet\ N\ M\ L \rangle$ )
      moreover
      have  $Per\ H\ K\ M$ 
    proof –
      have  $A\ B\ C\ CongA\ D\ C\ M$ 
    proof –
      have  $A\ B\ C\ CongA\ D\ C\ B$ 
        by (simp add:  $\langle A\ B\ C\ CongA\ B\ C\ D \rangle\ conga\text{--}right\text{--}comm$ )
      moreover
      have  $B\ Out\ A\ A$ 
        by (simp add:  $\langle A \neq B \rangle\ bet\text{--}out\text{--}1\ between\text{--}trivial2$ )
      moreover
      have  $B\ Out\ C\ C$ 
        using  $\langle C \neq B \rangle\ out\text{--}trivial$  by blast
      moreover
      have  $C\ Out\ D\ D$ 
        using  $\langle C \neq D \rangle\ out\text{--}trivial$  by auto
    qed
  qed
  qed
}

```



```

    moreover
    have  $C \text{ Out } M B$ 
      by (simp add:  $\langle C \neq B \rangle$  assms(3) l7-2 midpoint-out)
    ultimately
    show ?thesis
      using l11-10 by blast
  qed
  hence  $\text{Per } D C M$ 
    using  $\langle \text{Per } A B C \rangle$  l11-17 by blast
  {
    assume  $\text{Bet } M C K$ 
    hence  $\text{Bet } N D H$ 
      by (simp add:  $\langle \text{Bet } M C K \implies \text{Bet } N D H \rangle$ )
    hence  $\text{Per } H K M$ 
      using t22-9-aux-2  $\langle \text{Per } D C M \rangle$   $\langle \text{Bet } M C K \rangle$   $\langle \text{Lambert } N M K H \rangle$ 
         $\langle \text{Lambert } N M C D \rangle$  by blast
  }
  moreover
  {
    assume  $\text{Bet } M K C$ 
    hence  $\text{Bet } N H D$ 
      by (simp add:  $\langle \text{Bet } M K C \implies \text{Bet } N H D \rangle$ )
    hence  $\text{Per } H K M$ 
      using  $\langle \text{Per } D C M \rangle$   $\langle \text{Bet } M K C \rangle$   $\langle \text{Lambert } N M K H \rangle$ 
         $\langle \text{Lambert } N M C D \rangle$  t22-9-aux-1 by blast
  }
  ultimately
  show ?thesis
    using Out-def  $\langle M \text{ Out } C K \rangle$  by blast
  qed
  hence  $\text{Per } M K H$ 
    by (simp add: l8-2)
  ultimately
  show ?thesis
    using t22-9--per by blast
  qed
  moreover
  have  $L G H \text{ CongA } A' B' C'$ 
    by (simp add:  $\langle A' B' C' \text{ CongA } H G L \rangle$  conga-right-comm conga-sym-equiv)
  ultimately
  show ?thesis
    using l11-17 by blast
  qed
}
moreover
{
  assume  $\text{Per } A' B' C'$ 
  have  $\text{Per } A B C$ 
  proof -
    have  $\text{Per } D C M$ 
  proof -
    have  $\text{Per } H K M$ 
  proof -
    have  $\text{Bet } H K G$ 
      using Bet-cases  $\langle \text{Bet } G K H \rangle$  by blast
    moreover
    have  $A' B' C' \text{ CongA } L G H$ 
      by (simp add:  $\langle A' B' C' \text{ CongA } H G L \rangle$  conga-right-comm)
    hence  $\text{Per } L G H$ 
      using l11-17  $\langle \text{Per } A' B' C' \rangle$  by blast
    ultimately
    show ?thesis
      using  $\langle \text{Bet } N M L \rangle$   $\langle \text{Lambert } N M K H \rangle$   $\langle \text{Lambert } N L G H \rangle$ 
        by (meson Per-perm lam-perm t22-9-aux-1)
  qed
  qed
}

```

```

    assume Bet M C K
    hence Bet N D H
    using  $\langle \textit{Bet M C K} \implies \textit{Bet N D H} \rangle$  by blast
    hence Per D C M
    using  $\langle \textit{Per H K M} \rangle \langle \textit{Bet M C K} \rangle \langle \textit{Lambert N M K H} \rangle$ 
       $\langle \textit{Lambert N M C D} \rangle$  t22-9-aux-1 by blast
  }
  moreover
  {
    assume Bet M K C
    hence Bet N H D
    by (simp add:  $\langle \textit{Bet M K C} \implies \textit{Bet N H D} \rangle$ )
    hence Per D C M
    using  $\langle \textit{Bet M K C} \rangle \langle \textit{Lambert N M C D} \rangle \langle \textit{Lambert N M K H} \rangle$ 
       $\langle \textit{Per H K M} \rangle$  t22-9-aux-2 by blast
  }
  ultimately
  show ?thesis
    using Out-def  $\langle \textit{M Out C K} \rangle$  by blast
qed
moreover
have D C M CongA A B C
proof –
  have D C B CongA A B C
    by (meson  $\langle \textit{A B C CongA B C D} \rangle$  conga-left-comm not-conga-sym)
  moreover
  have C Out D D
    using  $\langle \textit{C} \neq \textit{D} \rangle$  out-trivial by force
  moreover
  have C Out M B
    by (metis  $\langle \neg \textit{Col N M C} \rangle$  assms(3) bet-out-1 col-trivial-2 midpoint-bet)
  moreover
  have B Out A A
    by (simp add:  $\langle \textit{A} \neq \textit{B} \rangle$  out-trivial)
  moreover
  have B Out C C
    by (simp add:  $\langle \textit{C} \neq \textit{B} \rangle$  out-trivial)
  ultimately
  show ?thesis
    using l11-10 by blast
qed
ultimately
show ?thesis
  using l11-17 by blast
qed
}
moreover
{
  assume Acute A B C
  have Acute A' B' C'
  proof –
    have Acute L G H
    proof –
      have Lambert N H K M
        by (simp add:  $\langle \textit{Lambert N M K H} \rangle$  lam-perm)
      moreover
      have Lambert N H G L
        by (simp add:  $\langle \textit{Lambert N L G H} \rangle$  lam-perm)
      moreover
      have Bet H K G
        using Bet-cases  $\langle \textit{Bet G K H} \rangle$  by blast
      moreover
      have A B C CongA D C M
      proof –
        have A B C CongA D C B
          by (simp add:  $\langle \textit{A B C CongA B C D} \rangle$  conga-right-comm)

```

```

moreover
have  $B \text{ Out } A \ A$ 
  by (simp add:  $\langle A \neq B \rangle \text{ out-trivial}$ )
moreover
have  $B \text{ Out } C \ C$ 
  by (simp add:  $\langle C \neq B \rangle \text{ out-trivial}$ )
moreover
have  $C \text{ Out } D \ D$ 
  using  $\langle C \neq D \rangle \text{ out-trivial}$  by auto
moreover
have  $C \text{ Out } M \ B$ 
  by (metis  $\langle \neg \text{Col } N \ M \ C \rangle \text{ assms}(3) \text{ bet-out-1 col-trivial-2 midpoint-bet}$ )
ultimately
show ?thesis
  using l11-10 by blast
qed
hence  $\text{Acute } D \ C \ M$ 
  using acute-conga--acute  $\langle \text{Acute } A \ B \ C \rangle$  by blast
have  $\text{Acute } H \ K \ M$ 
proof –
  have  $\text{Bet } M \ C \ K \longrightarrow \text{Acute } H \ K \ M$ 
    using t22-9-aux-4  $\langle \text{Acute } D \ C \ M \rangle \langle \text{Bet } M \ C \ K \implies \text{Bet } N \ D \ H \rangle$ 
       $\langle \text{Lambert } N \ M \ C \ D \rangle \langle \text{Lambert } N \ M \ K \ H \rangle$  by blast
  moreover
  have  $\text{Bet } M \ K \ C \longrightarrow \text{Acute } H \ K \ M$ 
    using t22-9-aux-3  $\langle \text{Acute } D \ C \ M \rangle \langle \text{Bet } M \ K \ C \implies \text{Bet } N \ H \ D \rangle$ 
       $\langle \text{Lambert } N \ M \ C \ D \rangle \langle \text{Lambert } N \ M \ K \ H \rangle$  by blast
  ultimately
  show ?thesis
    using Out-def  $\langle M \text{ Out } C \ K \rangle$  by blast
qed
hence  $\text{Acute } M \ K \ H$ 
  by (meson acute-sym)
ultimately
show ?thesis
  using t22-9-aux-4  $\langle \text{Bet } N \ M \ L \rangle$  by blast
qed
moreover
have  $L \ G \ H \text{ CongA } A' \ B' \ C'$ 
  by (meson  $\langle A' \ B' \ C' \text{ CongA } H \ G \ L \rangle \text{ conga-left-comm conga-sym}$ )
ultimately
show ?thesis
  using acute-conga--acute by blast
qed
}
moreover
{
  assume  $\text{Acute } A' \ B' \ C'$ 
  have  $\text{Acute } A \ B \ C$ 
  proof –
    have  $\text{Acute } D \ C \ M$ 
    proof –
      have  $A' \ B' \ C' \text{ CongA } L \ G \ H$ 
        by (simp add:  $\langle A' \ B' \ C' \text{ CongA } H \ G \ L \rangle \text{ conga-right-comm}$ )
      hence  $\text{Acute } L \ G \ H$ 
        using  $\langle \text{Acute } A' \ B' \ C' \rangle \text{ acute-conga--acute}$  by blast
      hence  $\text{Acute } M \ K \ H$ 
        using  $\langle \text{Lambert } N \ M \ K \ H \rangle \langle \text{Lambert } N \ L \ G \ H \rangle$ 
        by (meson  $\langle \text{Bet } G \ K \ H \rangle \langle \text{Bet } N \ M \ L \rangle \text{ between-symmetry lam-perm t22-9-aux-3}$ )
      hence  $\text{Acute } H \ K \ M$ 
        by (simp add: acute-sym)
      have  $\text{Bet } M \ C \ K \longrightarrow \text{Acute } D \ C \ M$ 
        using  $\langle \text{Lambert } N \ M \ C \ D \rangle \langle \text{Lambert } N \ M \ K \ H \rangle \langle \text{Acute } H \ K \ M \rangle$ 
           $\langle \text{Bet } M \ C \ K \implies \text{Bet } N \ D \ H \rangle$  t22-9-aux-3 by blast
      moreover
      have  $\text{Bet } M \ K \ C \longrightarrow \text{Acute } D \ C \ M$ 

```

```

    using ⟨Lambert N M C D⟩ ⟨Lambert N M K H⟩ ⟨Acute H K M⟩
      ⟨Bet M K C  $\implies$  Bet N H D⟩ t22-9-aux-4 by blast
  ultimately
  show ?thesis
    using Out-def ⟨M Out C K⟩ by blast
qed
moreover
have D C M CongA A B C
proof –
  have D C B CongA A B C
    by (meson ⟨A B C CongA B C D⟩ conga-left-comm conga-sym-equiv)
  moreover
  have C Out D D
    using ⟨C  $\neq$  D⟩ out-trivial by auto
  moreover
  have C Out M B
    by (metis ⟨ $\neg$  Col N M C⟩ assms(3) bet-out-1 col-trivial-2 midpoint-bet)
  moreover
  have B Out A A
    by (simp add: ⟨A  $\neq$  B⟩ out-trivial)
  moreover
  have B Out C C
    by (simp add: ⟨C  $\neq$  B⟩ out-trivial)
  ultimately
  show ?thesis
    using l11-10 by blast
qed
ultimately
show ?thesis
  using acute-conga--acute by blast
qed
}
ultimately
show ?thesis by auto
qed

```

```

lemma three-hypotheses-aux-1:
  assumes Saccheri A B C D and
    Saccheri A' B' C' D' and
    M Midpoint B C and
    M' Midpoint B' C' and
    N Midpoint A D and
    N' Midpoint A' D' and
    M N Le M' N' and
    Per A B C
  shows Per A' B' C'
  using three-hypotheses-aux assms(1) assms(2) assms(3) assms(4)
    assms(5) assms(6) assms(7) assms(8) by blast

```

```

lemma three-hypotheses-aux-2:
  assumes Saccheri A B C D and
    Saccheri A' B' C' D' and
    M Midpoint B C and
    M' Midpoint B' C' and
    N Midpoint A D and
    N' Midpoint A' D' and
    M N Le M' N' and
    Per A' B' C'
  shows Per A B C
  using three-hypotheses-aux assms(1) assms(2) assms(3) assms(4)
    assms(5) assms(6) assms(7) assms(8) by blast

```

```

lemma three-hypotheses-aux-3:
  assumes Saccheri A B C D and
    Saccheri A' B' C' D' and
    M Midpoint B C and

```

M' Midpoint $B' C'$ and
 N Midpoint $A D$ and
 N' Midpoint $A' D'$ and
 $M N$ Le $M' N'$ and
 $Acute A B C$
shows $Acute A' B' C'$
using *three-hypotheses-aux* *assms*(1) *assms*(2) *assms*(3) *assms*(4)
assms(5) *assms*(6) *assms*(7) *assms*(8) **by** *blast*

lemma *three-hypotheses-aux-4*:
assumes *Saccheri A B C D* and
Saccheri A' B' C' D' and
 M Midpoint $B C$ and
 M' Midpoint $B' C'$ and
 N Midpoint $A D$ and
 N' Midpoint $A' D'$ and
 $M N$ Le $M' N'$ and
 $Acute A' B' C'$
shows $Acute A B C$
using *three-hypotheses-aux* *assms*(1) *assms*(2) *assms*(3) *assms*(4)
assms(5) *assms*(6) *assms*(7) *assms*(8) **by** *blast*

lemma *per-sac--rah*:
assumes *Saccheri A B C D* and
 $Per A B C$
shows *HypothesisRightSaccheriQuadrilaterals*
proof –
{
 fix $A' B' C' D'$
 assume *Saccheri A' B' C' D'*
 obtain M **where** M Midpoint $B C$
 using *midpoint-existence* **by** *blast*
 obtain M' **where** M' Midpoint $B' C'$
 using *midpoint-existence* **by** *blast*
 obtain N **where** N Midpoint $A D$
 using *midpoint-existence* **by** *blast*
 obtain N' **where** N' Midpoint $A' D'$
 using *midpoint-existence* **by** *blast*
 {
 assume $M N$ Le $M' N'$
 hence $Per A' B' C'$
 using $\langle M \text{ Midpoint } B C \rangle \langle M' \text{ Midpoint } B' C' \rangle \langle N \text{ Midpoint } A D \rangle$
 $\langle N' \text{ Midpoint } A' D' \rangle \langle \text{Saccheri } A' B' C' D' \rangle$ *assms*(1) *assms*(2)
 three-hypotheses-aux-1 **by** *blast*
 }
 moreover
 {
 assume $M' N' Le M N$
 hence $Per A' B' C'$
 using $\langle M \text{ Midpoint } B C \rangle \langle M' \text{ Midpoint } B' C' \rangle \langle M' N' Le M N \rangle$
 $\langle N \text{ Midpoint } A D \rangle \langle N' \text{ Midpoint } A' D' \rangle \langle \text{Saccheri } A' B' C' D' \rangle$
 assms(1) *assms*(2) *three-hypotheses-aux-2* **by** *blast*
 }
 ultimately
 have $Per A' B' C'$
 using *local.le-cases* **by** *blast*
}
thus *?thesis* **using** *hypothesis-of-right-saccheri-quadrilaterals-def* **by** *blast*
qed

lemma *acute-sac--aah*:
assumes *Saccheri A B C D* and
 $Acute A B C$

shows *hypothesis-of-acute-saccheri-quadrilaterals*
proof –
{
 fix $A' B' C' D'$
 assume *Saccheri* $A' B' C' D'$
 obtain M **where** M *Midpoint* $B C$
 using *midpoint-existence* **by** *blast*
 obtain M' **where** M' *Midpoint* $B' C'$
 using *midpoint-existence* **by** *blast*
 obtain N **where** N *Midpoint* $A D$
 using *midpoint-existence* **by** *blast*
 obtain N' **where** N' *Midpoint* $A' D'$
 using *midpoint-existence* **by** *blast*
 {
 assume $M N$ *Le* $M' N'$
 hence *Acute* $A' B' C'$
 using $\langle M$ *Midpoint* $B C \rangle \langle M'$ *Midpoint* $B' C' \rangle \langle N$ *Midpoint* $A D \rangle$
 $\langle N'$ *Midpoint* $A' D' \rangle \langle$ *Saccheri* $A' B' C' D' \rangle$ *assms*(1) *assms*(2)
 three-hypotheses-aux-3 **by** *blast*
 }
moreover
{
 assume $M' N' Le M N$
 hence *Acute* $A' B' C'$
 using $\langle M$ *Midpoint* $B C \rangle \langle M'$ *Midpoint* $B' C' \rangle \langle M' N' Le M N \rangle$
 $\langle N$ *Midpoint* $A D \rangle \langle N'$ *Midpoint* $A' D' \rangle \langle$ *Saccheri* $A' B' C' D' \rangle$
 assms(1) *assms*(2) *three-hypotheses-aux-4* **by** *blast*
}
ultimately
have *Acute* $A' B' C'$
 using *local.le-cases* **by** *blast*
}
thus *?thesis* **using** *hypothesis-of-acute-saccheri-quadrilaterals-def* **by** *blast*
qed

lemma *obtuse-sac--oah*:
assumes *Saccheri* $A B C D$ **and**
 Obtuse $A B C$
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
proof –
have $A \neq B$
 using *assms*(2) *obtuse-distincts* **by** *auto*
have $C \neq B$
 using *assms*(2) *obtuse-distincts* **by** *blast*
{
 fix $A' B' C' D'$
 assume *Saccheri* $A' B' C' D'$
 {
 assume *Acute* $A' B' C'$
 hence *hypothesis-of-acute-saccheri-quadrilaterals*
 using *acute-sac--aah* $\langle$ *Saccheri* $A' B' C' D' \rangle$ **by** *blast*
 hence *Acute* $A B C$
 using *assms*(1) *hypothesis-of-acute-saccheri-quadrilaterals-def* **by** *blast*
 hence $A B C$ *LtA* $A B C$
 by (*simp add: acute-obtuse--lta assms*(2))
 hence *False*
 using *nlta* **by** *blast*
 }
moreover
{
 assume *Per* $A' B' C'$
 hence *hypothesis-of-right-saccheri-quadrilaterals*
 using *per-sac--rah* $\langle$ *Saccheri* $A' B' C' D' \rangle$ **by** *blast*
 hence *Per* $A B C$
 using *assms*(1) *hypothesis-of-right-saccheri-quadrilaterals-def* **by** *blast*
 hence $A B C$ *LtA* $A B C$
}

```

    using  $\langle A \neq B \rangle \langle C \neq B \rangle$  assms(2) obtuse-per--lta by auto
  hence False
    using nlta by blast
}
ultimately
have Obtuse A' B' C'
  by (metis angle-partition l8-20-1-R1 l8-5)
}
thus ?thesis
  using hypothesis-of-obtuse-saccheri-quadrilaterals-def by blast
qed

lemma per--ex-saccheri:
  assumes Per B A D and
     $A \neq B$  and
     $A \neq D$ 
  shows  $\exists C. \text{Saccheri } A B C D$ 
proof -
  have  $\neg \text{Col } B A D$ 
    using assms(1) assms(2) assms(3) not-col-permutation-4 per-col-eq by blast
  obtain C0 where  $A D \text{ Perp } C0 D \wedge A D \text{ OS } B C0$ 
    using NCol-perm  $\langle \neg \text{Col } B A D \rangle$  col-trivial-2 l10-15 by blast
  have  $\neg \text{Col } A D C0$ 
    using  $\langle A D \text{ Perp } C0 D \wedge A D \text{ OS } B C0 \rangle$  one-side-not-col124 by blast
  obtain C where  $D \text{ Out } C0 C \wedge \text{Cong } D C A B$ 
    by (metis  $\langle \neg \text{Col } A D C0 \rangle$  assms(2) col-trivial-2 segment-construction-3)
  hence Cong A B C D
    using cong-3421 by blast
  moreover
  have Per A D C
  proof -
    have  $D \neq C0$ 
      using  $\langle \neg \text{Col } A D C0 \rangle$  col-trivial-2 by force
    moreover
    have Per A D C0
      using Perp-cases  $\langle A D \text{ Perp } C0 D \wedge A D \text{ OS } B C0 \rangle$  perp-per-2 by blast
    moreover
    have Col D C0 C
      by (simp add:  $\langle D \text{ Out } C0 C \wedge \text{Cong } D C A B \rangle$  out-col)
    ultimately
    show ?thesis
      using per-col by blast
  qed
  moreover
  have  $A D \text{ OS } B C$ 
    by (meson  $\langle A D \text{ Perp } C0 D \wedge A D \text{ OS } B C0 \rangle \langle D \text{ Out } C0 C \wedge \text{Cong } D C A B \rangle$  invert-one-side out-out-one-side)
  ultimately
  show ?thesis
    using Saccheri-def assms(1) by blast
qed

lemma ex-saccheri:
   $\exists A B C D. \text{Saccheri } A B C D$ 
proof -
  obtain  $A D E$  where  $\neg (\text{Bet } A D E \vee \text{Bet } D E A \vee \text{Bet } E A D)$ 
    using lower-dim-ex by blast
  hence  $\neg \text{Col } A D E$ 
    by (simp add: Col-def)
  obtain B where  $A D \text{ Perp } B A \wedge A D \text{ OS } E B$ 
    using  $\langle \neg \text{Col } A D E \rangle$  col-trivial-3 l10-15 by blast
  thus ?thesis
    using  $\langle A D \text{ Perp } B A \wedge A D \text{ OS } E B \rangle$  l8-2 os-distincts
      per--ex-saccheri perp-per-1 by metis
qed

```

```

lemma ex-lambert:
   $\exists A B C D. \text{Lambert } A B C D$ 
proof -
  obtain  $D C C' D'$  where Saccheri  $D C C' D'$ 
    using ex-saccheri by blast
  obtain  $A$  where  $A$  Midpoint  $D D'$ 
    using midpoint-existence by blast
  obtain  $B$  where  $B$  Midpoint  $C C'$ 
    using midpoint-existence by blast
  thus ?thesis
    using  $\langle \text{Saccheri } D C C' D' \rangle$  sac--ex2-mid-mid-lam-conga by blast
qed

lemma saccheri-s-three-hypotheses:
  hypothesis-of-acute-saccheri-quadrilaterals  $\vee$ 
  hypothesis-of-right-saccheri-quadrilaterals  $\vee$ 
  hypothesis-of-obtuse-saccheri-quadrilaterals
proof -
  obtain  $A B C D$  where Saccheri  $A B C D$ 
    using ex-saccheri by blast
  {
    assume Acute  $A B C$ 
    hence hypothesis-of-acute-saccheri-quadrilaterals
      using  $\langle \text{Saccheri } A B C D \rangle$  acute-sac--aah by blast
  }
  moreover
  {
    assume Per  $A B C$ 
    hence hypothesis-of-right-saccheri-quadrilaterals
      using  $\langle \text{Saccheri } A B C D \rangle$  per-sac--rah by blast
  }
  moreover
  {
    assume Obtuse  $A B C$ 
    hence hypothesis-of-obtuse-saccheri-quadrilaterals
      using  $\langle \text{Saccheri } A B C D \rangle$  obtuse-sac--oah by blast
  }
  ultimately
  show ?thesis
    using acute-sac--aah angle-partition
      hypothesis-of-right-saccheri-quadrilaterals-def
      obtuse-sac--oah sac-distincts by blast
qed

lemma not-aah:
  assumes hypothesis-of-right-saccheri-quadrilaterals  $\vee$ 
    hypothesis-of-obtuse-saccheri-quadrilaterals
  shows  $\neg$  hypothesis-of-acute-saccheri-quadrilaterals
proof -
  obtain  $A B C D$  where Saccheri  $A B C D$ 
    using ex-saccheri by blast
  {
    assume hypothesis-of-acute-saccheri-quadrilaterals
    have Acute  $A B C$ 
      using  $\langle \text{HypothesisAcuteSaccheriQuadrilaterals} \rangle$   $\langle \text{Saccheri } A B C D \rangle$ 
        hypothesis-of-acute-saccheri-quadrilaterals-def by blast
  }
  {
    assume hypothesis-of-right-saccheri-quadrilaterals
    hence Per  $A B C$ 
      using  $\langle \text{Saccheri } A B C D \rangle$  hypothesis-of-right-saccheri-quadrilaterals-def
        by blast
    hence  $A B C \text{ Lt } A B C$ 
      using  $\langle \text{Acute } A B C \rangle$  acute-not-per by blast
  }
  moreover
  {

```



```

    assume hypothesis-of-obtuse-saccheri-quadrilaterals
    hence Obtuse A B C
      using ⟨Saccheri A B C D⟩ hypothesis-of-obtuse-saccheri-quadrilaterals-def
      by blast
    hence A B C LtA A B C
      using ⟨Acute A B C⟩ acute--not-obtuse by blast
  }
  ultimately
  have A B C LtA A B C
    using assms by blast
  hence False
    by (meson nlta)
}
thus ?thesis by auto
qed

```

lemma not-rah:

```

  assumes hypothesis-of-acute-saccheri-quadrilaterals ∨
    hypothesis-of-obtuse-saccheri-quadrilaterals
  shows ¬ hypothesis-of-right-saccheri-quadrilaterals
proof -
  obtain A B C D where Saccheri A B C D
    using ex-saccheri by blast
  {
    assume hypothesis-of-right-saccheri-quadrilaterals
    have Per A B C
      using ⟨HypothesisRightSaccheriQuadrilaterals⟩ ⟨Saccheri A B C D⟩
        hypothesis-of-right-saccheri-quadrilaterals-def by blast
    {
      assume hypothesis-of-acute-saccheri-quadrilaterals
      hence Acute A B C
        using ⟨Saccheri A B C D⟩ ⟨HypothesisRightSaccheriQuadrilaterals⟩
          not-aah by blast
      hence A B C LtA A B C
        using ⟨Per A B C⟩ acute-not-per by blast
    }
    moreover
    {
      assume hypothesis-of-obtuse-saccheri-quadrilaterals
      hence Obtuse A B C
        using ⟨Saccheri A B C D⟩ hypothesis-of-obtuse-saccheri-quadrilaterals-def
        by blast
      hence A B C LtA A B C
        using ⟨Per A B C⟩ ⟨Saccheri A B C D⟩ obtuse-per--lta sac-distincts
        by blast
    }
    ultimately
    have A B C LtA A B C
      using assms by blast
    hence False
      by (meson nlta)
  }
  thus ?thesis by auto
qed

```

lemma not-oah:

```

  assumes hypothesis-of-acute-saccheri-quadrilaterals ∨
    hypothesis-of-right-saccheri-quadrilaterals
  shows ¬ hypothesis-of-obtuse-saccheri-quadrilaterals
proof -
  obtain A B C D where Saccheri A B C D
    using ex-saccheri by blast
  {
    assume hypothesis-of-obtuse-saccheri-quadrilaterals
    have Obtuse A B C
      using ⟨HypothesisObtuseSaccheriQuadrilaterals⟩ assms not-aah

```

```

    not-rah by blast
  {
    assume hypothesis-of-acute-saccheri-quadrilaterals
    hence Acute A B C
    using ⟨Saccheri A B C D⟩ ⟨HypothesisObtuseSaccheriQuadrilaterals⟩
      not-aah by blast
    hence A B C LtA A B C
    using ⟨HypothesisAcuteSaccheriQuadrilaterals⟩
      ⟨HypothesisObtuseSaccheriQuadrilaterals⟩ not-aah by blast
  }
  moreover
  {
    assume hypothesis-of-right-saccheri-quadrilaterals
    hence Per A B C
    using ⟨Saccheri A B C D⟩ ⟨HypothesisObtuseSaccheriQuadrilaterals⟩
      not-rah by blast
    hence A B C LtA A B C
    using ⟨Per A B C⟩ ⟨Saccheri A B C D⟩
      ⟨HypothesisObtuseSaccheriQuadrilaterals⟩
      ⟨HypothesisRightSaccheriQuadrilaterals⟩ not-rah by blast
  }
  ultimately
  have A B C LtA A B C
    using assms by blast
  hence False
    by (meson nlta)
}
thus ?thesis by auto
qed

lemma lam-per--rah-1:
  assumes Lambert A B C D and
    Per B C D
  shows hypothesis-of-right-saccheri-quadrilaterals
proof -
  obtain C' where B Midpoint C C'
    using symmetric-point-construction by blast
  obtain D' where A Midpoint D D'
    using symmetric-point-construction by blast
  have Saccheri D C C' D'
    using ⟨A Midpoint D D'⟩ ⟨B Midpoint C C'⟩ assms(1) lam6521-mid2--sac by auto
  moreover
  have Per D C C'
    by (metis Per-perm ⟨B Midpoint C C'⟩ assms(2) col-per2--per
      l8-20-1-R1 midpoint-col midpoint-distinct-1)
  ultimately
  show ?thesis
    using per-sac--rah by blast
qed

lemma lam-per--rah-2:
  assumes Lambert A B C D and
    hypothesis-of-right-saccheri-quadrilaterals
  shows Per B C D
proof -
  obtain C' where B Midpoint C C'
    using symmetric-point-construction by blast
  obtain D' where A Midpoint D D'
    using symmetric-point-construction by blast
  have Saccheri D C C' D'
    using ⟨A Midpoint D D'⟩ ⟨B Midpoint C C'⟩ assms(1) lam6521-mid2--sac by blast
  hence Per D C C'
    using assms(2) hypothesis-of-right-saccheri-quadrilaterals-def by blast
  hence Per C' C D
    using Per-perm by blast
  moreover

```

```

have  $C \neq C'$ 
  by (metis Lambert-def  $\langle B \text{ Midpoint } C \ C' \rangle$  assms(1) l8-20-2)
moreover
have  $Col \ C \ C' \ B$ 
  using  $\langle B \text{ Midpoint } C \ C' \rangle$  col-permutation-1 midpoint-col by blast
ultimately
show ?thesis
  using l8-3 by blast
qed

lemma lam-per--rah:
  assumes Lambert  $A \ B \ C \ D$ 
  shows  $Per \ B \ C \ D \longleftrightarrow hypothesis\text{-of-right-saccheri-quadrilaterals}$ 
  by (meson lam-per--rah-1 lam-per--rah-2 assms)

lemma lam-acute--aah-1:
  assumes Lambert  $A \ B \ C \ D$  and
    Acute  $B \ C \ D$ 
  shows hypothesis-of-acute-saccheri-quadrilaterals
proof -
  obtain  $C'$  where  $B \text{ Midpoint } C \ C'$ 
    using symmetric-point-construction by blast
  obtain  $D'$  where  $A \text{ Midpoint } D \ D'$ 
    using symmetric-point-construction by blast
  have Saccheri  $D \ C \ C' \ D'$ 
    using  $\langle A \text{ Midpoint } D \ D' \rangle \langle B \text{ Midpoint } C \ C' \rangle$  assms(1) lam6521-mid2--sac by blast
  moreover
  have Acute  $D \ C \ C'$ 
    by (metis l11-17 sac-distincts  $\langle B \text{ Midpoint } C \ C' \rangle$  acute--not-obtuse
      acute-not-per angle-partition assms(2) calculation
      conga-obtuse--obtuse l7-17-bis l7-2 sac--ex2-mid-mid-lam-conga)
  ultimately
  show ?thesis
    using acute-sac--aah by blast
qed

lemma lam-acute--aah-2:
  assumes Lambert  $A \ B \ C \ D$  and
    hypothesis-of-acute-saccheri-quadrilaterals
  shows Acute  $B \ C \ D$ 
proof -
  obtain  $C'$  where  $B \text{ Midpoint } C \ C'$ 
    using symmetric-point-construction by blast
  obtain  $D'$  where  $A \text{ Midpoint } D \ D'$ 
    using symmetric-point-construction by blast
  have Saccheri  $D \ C \ C' \ D'$ 
    using  $\langle A \text{ Midpoint } D \ D' \rangle \langle B \text{ Midpoint } C \ C' \rangle$  assms(1) lam6521-mid2--sac by auto
  hence Acute  $D \ C \ C'$ 
    using assms(2) hypothesis-of-acute-saccheri-quadrilaterals-def by blast
  moreover
  have  $D \ C \ C' \ CongA \ B \ C \ D$ 
    using  $\langle B \text{ Midpoint } C \ C' \rangle \langle Saccheri \ D \ C \ C' \ D' \rangle$  l7-17-bis l7-2
      sac--ex2-mid-mid-lam-conga by blast
  ultimately
  show ?thesis
    using acute-conga--acute by blast
qed

lemma lam-acute--aah:
  assumes Lambert  $A \ B \ C \ D$ 
  shows Acute  $B \ C \ D \longleftrightarrow hypothesis\text{-of-acute-saccheri-quadrilaterals}$ 
  by (meson lam-acute--aah-1 lam-acute--aah-2 assms)

lemma lam-obtuse--oah-1:
  assumes Lambert  $A \ B \ C \ D$  and

```

Obtuse B C D
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
proof –
obtain C' **where** B *Midpoint* $C C'$
using *symmetric-point-construction* **by** *blast*
obtain D' **where** A *Midpoint* $D D'$
using *symmetric-point-construction* **by** *blast*
have *Saccheri* $D C C' D'$
using $\langle A$ *Midpoint* $D D' \rangle \langle B$ *Midpoint* $C C' \rangle$ *assms(1) lam6521-mid2--sac* **by** *blast*
moreover
have $B C D$ *CongA* $D C C'$
by (*metis* $\langle B$ *Midpoint* $C C' \rangle$ *calculation l7-17-bis l7-2*
not-conga-sym sac--ex2-mid-mid-lam-conga)
hence *Obtuse* $D C C'$
using *assms(2) conga-obtuse--obtuse* **by** *blast*
ultimately
show *?thesis*
using *obtuse-sac--oah* **by** *blast*
qed

lemma *lam-obtuse--oah-2*:
assumes *Lambert* $A B C D$ **and**
hypothesis-of-obtuse-saccheri-quadrilaterals
shows *Obtuse* $B C D$
proof –
obtain C' **where** B *Midpoint* $C C'$
using *symmetric-point-construction* **by** *blast*
obtain D' **where** A *Midpoint* $D D'$
using *symmetric-point-construction* **by** *blast*
have *Saccheri* $D C C' D'$
using $\langle A$ *Midpoint* $D D' \rangle \langle B$ *Midpoint* $C C' \rangle$ *assms(1)*
lam6521-mid2--sac **by** *blast*
hence *Obtuse* $D C C'$
using *assms(2) hypothesis-of-obtuse-saccheri-quadrilaterals-def* **by** *blast*
moreover
have $D C C'$ *CongA* $B C D$
proof –
have C *Out* $B C'$
using $\langle B$ *Midpoint* $C C' \rangle \langle$ *Saccheri* $D C C' D' \rangle$ *midpoint-out*
sac-distincts **by** *presburger*
moreover
have C *Out* $D D$
by (*metis* *assms(1) lam--pars1234 out-trivial par-strict-neq2*)
ultimately
show *?thesis*
by (*meson conga-left-comm out2--conga*)
qed
ultimately
show *?thesis*
using *conga-obtuse--obtuse* **by** *blast*
qed

lemma *lam-obtuse--oah*:
assumes *Lambert* $A B C D$
shows *Obtuse* $B C D \longleftrightarrow$ *hypothesis-of-obtuse-saccheri-quadrilaterals*
by (*meson lam-obtuse--oah-1 lam-obtuse--oah-2 assms*)

lemma *t22-11--per-1*:
assumes *Saccheri* $A B C D$ **and**
 $A B D$ *CongA* $B D C$
shows *Per* $A B C$
proof –
have *Cong* $A D B C$
proof –
have $A B D$ *CongA* $C D B$
by (*simp add: assms(2) conga-right-comm*)

```

moreover
have  $Cong\ A\ B\ C\ D$ 
  using  $Saccheri-def\ assms(1)$  by  $blast$ 
hence  $Cong\ B\ A\ D\ C$ 
  using  $Cong-cases$  by  $blast$ 
moreover
have  $Cong\ B\ D\ D\ B$ 
  by ( $simp\ add:\ cong-pseudo-reflexivity$ )
ultimately
show  $?thesis$ 
  using  $Cong-cases\ cong2-conga-cong$  by  $blast$ 
qed
thus  $?thesis$ 
  using  $assms(1)\ cong-sac--per-aux1$  by  $blast$ 
qed

```

```

lemma  $t22-11--per-2$ :
assumes  $Saccheri\ A\ B\ C\ D$  and
   $Per\ A\ B\ C$ 
shows  $A\ B\ D\ CongA\ B\ D\ C$ 
proof –
have  $Cong\ A\ D\ B\ C$ 
  by ( $simp\ add:\ assms(1)\ assms(2)\ cong-sac--per$ )
have  $A \neq B$ 
  using  $assms(1)\ sac-distincts$  by  $blast$ 
moreover
have  $A \neq D$ 
  using  $assms(1)\ sac-distincts$  by  $blast$ 
moreover
have  $B \neq D$ 
  using  $assms(1)\ sac-distincts$  by  $blast$ 
moreover
have  $Cong\ A\ B\ C\ D$ 
  using  $assms(1)\ Saccheri-def$  by  $blast$ 
moreover
have  $Cong\ A\ D\ C\ B$ 
  using  $\langle Cong\ A\ D\ B\ C \rangle\ not-cong-1243$  by  $blast$ 
moreover
have  $Cong\ B\ D\ D\ B$ 
  by ( $simp\ add:\ cong-pseudo-reflexivity$ )
ultimately
show  $?thesis$ 
  by ( $simp\ add:\ conga-right-comm\ l11-51$ )
qed

```

```

lemma  $t22-11--per$ :
assumes  $Saccheri\ A\ B\ C\ D$ 
shows  $A\ B\ D\ CongA\ B\ D\ C \longleftrightarrow Per\ A\ B\ C$ 
using  $assms\ t22-11--per-1\ t22-11--per-2$  by  $blast$ 

```

```

lemma  $t22-11--acute-1$ :
assumes  $Saccheri\ A\ B\ C\ D$  and
   $A\ B\ D\ LtA\ B\ D\ C$ 
shows  $Acute\ A\ B\ C$ 
proof –
have  $A\ D\ Lt\ C\ B$ 
proof –
  have  $Cong\ D\ C\ B\ A$ 
    using  $assms(1)\ Saccheri-def\ not-cong-4321$  by  $blast$ 
  moreover
  have  $Cong\ D\ B\ B\ D$ 
    by ( $simp\ add:\ cong-pseudo-reflexivity$ )
  moreover
  have  $D\ B\ A\ LtA\ B\ D\ C$ 
    by ( $simp\ add:\ assms(2)\ lta-left-comm$ )
  ultimately

```

```

  show ?thesis
    using t18-18 by blast
qed
hence A D Lt B C
  using Lt-cases by blast
thus ?thesis
  using assms(1) lt-sac--acute-aux1 by blast
qed

lemma t22-11--acute-2:
  assumes Saccheri A B C D and
    Acute A B C
  shows A B D LtA B D C
proof -
  have D B A LtA B D C
proof -
  have D ≠ C
    using assms(1) sac-distincts by blast
  moreover
  have D ≠ B
    using assms(1) sac-distincts by blast
  moreover
  have Cong D C B A
    using Cong-cases Saccheri-def assms(1) by blast
  moreover
  have Cong D B B D
    by (simp add: cong-pseudo-reflexivity)
  moreover
  have A D Lt B C
    by (simp add: assms(1) assms(2) lt-sac--acute-aux2)
  hence A D Lt C B
    by (simp add: lt-right-comm)
  ultimately
  show ?thesis
    by (simp add: t18-19)
qed
thus ?thesis
  using lta-left-comm by blast
qed

```

```

lemma t22-11--acute:
  assumes Saccheri A B C D
  shows A B D LtA B D C  $\longleftrightarrow$  Acute A B C
  using assms t22-11--acute-1 t22-11--acute-2 by blast

```

```

lemma t22-11--obtuse-1:
  assumes Saccheri A B C D and
    B D C LtA A B D
  shows Obtuse A B C
proof -
  have B C Lt D A
proof -
  have Cong B D D B
    by (simp add: cong-pseudo-reflexivity)
  moreover
  have Cong B A D C
    using assms(1) Saccheri-def cong-commutativity by blast
  moreover
  have C D B LtA A B D
    using assms(2) lta-left-comm by blast
  ultimately
  show ?thesis
    using t18-18 by blast
qed
hence B C Lt A D
  using Lt-cases by blast

```

```

thus ?thesis
  using assms(1) lt-sac--obtuse by blast
qed

lemma t22-11--obtuse-2:
  assumes Saccheri A B C D and
    Obtuse A B C
  shows B D C LtA A B D
proof –
  have C D B LtA A B D
  proof –
    have  $B \neq D$ 
    using assms(1) sac-distincts by blast
    moreover
    have  $B \neq A$ 
    using assms(1) sac-distincts by blast
    moreover
    have Cong B D D B
    by (simp add: cong-pseudo-reflexivity)
    moreover
    have Cong B A D C
    using assms(1) Saccheri-def cong-commutativity by blast
    moreover
    have B C Lt A D
    by (simp add: lt-sac--obtuse-aux2 assms(1) assms(2))
    hence B C Lt D A
    using Lt-cases by blast
    ultimately
    show ?thesis
    by (simp add: t18-19)
qed
thus ?thesis
  by (meson lta-left-comm)
qed

lemma t22-11--obtuse:
  assumes Saccheri A B C D
  shows B D C LtA A B D  $\longleftrightarrow$  Obtuse A B C
  using assms t22-11--obtuse-1 t22-11--obtuse-2 by blast

lemma t22-12--rah-1:
  assumes  $A \neq B$  and
     $B \neq C$  and
    Per A B C and
     $B C A C A B \text{ SumA } A B C$ 
  shows hypothesis-of-right-saccheri-quadrilaterals
proof –
  obtain D where Saccheri B A D C
  using assms(1) assms(2) assms(3) per--ex-saccheri by blast
  have B C ParStrict A D
  by (simp add: <Saccheri B A D C> sac--pars1423)
  have B A ParStrict D C
  by (simp add: <Saccheri B A D C> sac--pars1234)
  hence C A TS B D
  by (meson Par-strict-cases <B C ParStrict A D> l9-31 pars--os3412)
  have B C D CongA A B C
  using Saccheri-def by (metis <Saccheri B A D C> l11-16 sac-distincts)
  have C A B CongA A C D
  proof –
    have SAMS B C A C A B
    by (metis assms(4) sams123231 suma-distincts)
    moreover
    have SAMS B C A A C D
    by (simp add: os-ts--sams <B C ParStrict A D> <C A TS B D> l12-6)
    moreover
    have  $B C A A C D \text{ SumA } A B C$ 

```

```

    by (meson ⟨B C D CongA A B C⟩ ⟨C A TS B D⟩ calculation(2)
        conga3-suma--suma sams2-suma2--conga123 sams2-suma2--conga456 ts--suma-1)
ultimately
show ?thesis
    using sams2-suma2--conga456 assms(4) by blast
qed
hence B A C CongA A C D
    by (simp add: conga-left-comm)
hence Per B A D
    using ⟨Saccheri B A D C⟩ t22-11--per-1 by blast
thus ?thesis
    using ⟨Saccheri B A D C⟩ per-sac--rah by blast
qed

lemma t22-12--rah-2:
  assumes A ≠ B and
    B ≠ C and
    Per A B C and
    hypothesis-of-right-saccheri-quadrilaterals
  shows B C A C A B SumA A B C
proof -
  obtain D where Saccheri B A D C
    using assms(1) assms(2) assms(3) per--ex-saccheri by blast
  have B C ParStrict A D
    by (simp add: ⟨Saccheri B A D C⟩ sac--pars1423)
  have B A ParStrict D C
    by (simp add: ⟨Saccheri B A D C⟩ sac--pars1234)
  hence C A TS B D
    by (meson Par-strict-cases ⟨B C ParStrict A D⟩ l9-31 pars--os3412)
  have B C D CongA A B C
    using Saccheri-def by (metis ⟨Saccheri B A D C⟩ l11-16 sac-distincts)
  have B C A A C D SumA B C D
    by (simp add: ⟨C A TS B D⟩ ts--suma-1)
  moreover
  have B C A CongA B C A
    by (simp add: assms(1) assms(2) assms(3) cong2-per2--cong-conga2
        cong-reflexivity)
  moreover
  have A C D CongA C A B
    using ⟨Saccheri B A D C⟩ assms(4) conga-left-comm conga-sym
        hypothesis-of-right-saccheri-quadrilaterals-def t22-11--per-2 by blast
  ultimately
  show ?thesis
    by (meson conga3-suma--suma ⟨B C D CongA A B C⟩)
qed

lemma t22-12--rah:
  assumes A ≠ B and
    B ≠ C and
    Per A B C
  shows B C A C A B SumA A B C ⟷ hypothesis-of-right-saccheri-quadrilaterals
  by (meson t22-12--rah-1 t22-12--rah-2 assms(1) assms(2) assms(3))

lemma t22-12--aah-1:
  assumes Per A B C and
    B C A C A B SumA P Q R and
    Acute P Q R
  shows hypothesis-of-acute-saccheri-quadrilaterals
proof -
  have A ≠ B
    using assms(2) suma-distincts by blast
  have B ≠ C
    using assms(2) suma-distincts by blast
  then obtain D where Saccheri B A D C
    using assms(1) ⟨A ≠ B⟩ per--ex-saccheri by blast
  have B C ParStrict A D

```



```

  by (simp add: ⟨Saccheri B A D C⟩ sac--pars1423)
have B A ParStrict D C
  by (simp add: ⟨Saccheri B A D C⟩ sac--pars1234)
have C A TS B D
  by (meson Par-strict-cases ⟨B A ParStrict D C⟩
      ⟨B C ParStrict A D⟩ l12-6 l9-31)
have B C D CongA A B C
  by (metis Perp-cases par-not-col ⟨A ≠ B⟩ ⟨B C ParStrict A D⟩ ⟨B ≠ C⟩
      ⟨Saccheri B A D C⟩ assms(1) col-permutation-1 cong-perp-conga
      l11-16 l7-2 l8-21-aux midpoint-cong perp-per-2 sac--perp3414
      symmetric-point-construction ts-distincts)
have C A B LtA A C D
proof —
  have B C A LeA B C A
    using ⟨B ≠ C⟩ assms(1) lea-refl per-distinct-1 by presburger
  moreover
  have P Q R LtA B C D
    by (metis ⟨B C D CongA A B C⟩ ⟨C A TS B D⟩ acute-conga--acute
        acute-not-per acute-obtuse--lta acute-per--lta angle-partition
        assms(1) assms(3) ts-distincts)
  moreover
  have SAMS B C A C A B
    by (metis ⟨A ≠ B⟩ ⟨B ≠ C⟩ assms(1) per-distinct-1 sams123231)
  moreover
  have B C A C A B SumA P Q R
    by (simp add: assms(2))
  moreover
  have B C A A C D SumA B C D
    by (simp add: ⟨C A TS B D⟩ ts--suma-1)
  ultimately
  show ?thesis
    by (meson sams-lea-lta789-suma2--lta456)
qed
hence B A C LtA A C D
  using lta-left-comm by blast
hence Acute B A D
  using ⟨Saccheri B A D C⟩ t22-11--acute-1 by blast
thus ?thesis
  using ⟨Saccheri B A D C⟩ acute-sac--aah by blast
qed

lemma t22-12--aah-2:
  assumes Per A B C and
    B C A C A B SumA P Q R and
    hypothesis-of-acute-saccheri-quadrilaterals
  shows Acute P Q R
proof —
  have A ≠ B
    using assms(2) suma-distincts by blast
  have B ≠ C
    using assms(2) suma-distincts by blast
  then obtain D where Saccheri B A D C
    using assms(1) ⟨A ≠ B⟩ per--ex-saccheri by blast
  have B C ParStrict A D
    by (simp add: ⟨Saccheri B A D C⟩ sac--pars1423)
  have B A ParStrict D C
    by (simp add: ⟨Saccheri B A D C⟩ sac--pars1234)
  have C A TS B D
    by (meson Par-strict-cases ⟨B A ParStrict D C⟩ ⟨B C ParStrict A D⟩ l12-6 l9-31)
  have B C D CongA A B C
    by (metis Perp-cases par-not-col ⟨A ≠ B⟩ ⟨B C ParStrict A D⟩
        ⟨B ≠ C⟩ ⟨Saccheri B A D C⟩ assms(1) col-permutation-1
        cong-perp-conga l11-16 l7-2 l8-21-aux midpoint-cong perp-per-2
        sac--perp3414 symmetric-point-construction ts-distincts)
  have Per B C D
    using ⟨B C D CongA A B C⟩ assms(1) l11-17 not-conga-sym by blast

```

```

moreover
have  $P \ Q \ R \ LtA \ B \ C \ D$ 
proof –
  have  $B \ C \ A \ LeA \ B \ C \ A$ 
    using assms(2) lea-total suma-distincts by blast
  moreover
  have  $C \ A \ B \ LtA \ A \ C \ D$ 
    using  $\langle \text{Saccheri } B \ A \ D \ C \rangle$  assms(3)
      hypothesis-of-acute-saccheri-quadrilaterals-def
      lta-left-comm t22-11--acute by blast
  moreover
  have  $SAMS \ B \ C \ A \ A \ C \ D$ 
    by (simp add: os-ts--sams  $\langle B \ C \ ParStrict \ A \ D \rangle \langle C \ A \ TS \ B \ D \rangle$  l12-6)
  moreover
  have  $B \ C \ A \ C \ A \ B \ SumA \ P \ Q \ R$ 
    by (simp add: assms(2))
  moreover
  have  $B \ C \ A \ A \ C \ D \ SumA \ B \ C \ D$ 
    by (simp add:  $\langle C \ A \ TS \ B \ D \rangle$  ts--suma-1)
  ultimately
  show ?thesis
    using sams-lea-lta456-suma2--lta by blast
qed
ultimately
show ?thesis
  using Acute-def by blast
qed

lemma t22-12--aah:
  assumes  $Per \ A \ B \ C$  and
     $B \ C \ A \ C \ A \ B \ SumA \ P \ Q \ R$ 
  shows  $Acute \ P \ Q \ R \longleftrightarrow hypothesis-of-acute-saccheri-quadrilaterals$ 
  by (meson assms(1) assms(2) t22-12--aah-1 t22-12--aah-2)

lemma t22-12--oah-1:
  assumes  $Per \ A \ B \ C$  and
     $B \ C \ A \ C \ A \ B \ SumA \ P \ Q \ R$  and
     $Obtuse \ P \ Q \ R$ 
  shows hypothesis-of-obtuse-saccheri-quadrilaterals
proof –
  have  $A \neq B$ 
    using assms(2) suma-distincts by blast
  have  $B \neq C$ 
    using assms(2) suma-distincts by blast
  then obtain  $D$  where  $Saccheri \ B \ A \ D \ C$ 
    using assms(1)  $\langle A \neq B \rangle$  per--ex-saccheri by blast
  have  $B \ C \ ParStrict \ A \ D$ 
    by (simp add:  $\langle Saccheri \ B \ A \ D \ C \rangle$  sac--pars1423)
  have  $B \ A \ ParStrict \ D \ C$ 
    by (simp add:  $\langle Saccheri \ B \ A \ D \ C \rangle$  sac--pars1234)
  have  $C \ A \ TS \ B \ D$ 
    by (meson Par-strict-cases  $\langle B \ A \ ParStrict \ D \ C \rangle \langle B \ C \ ParStrict \ A \ D \rangle$  l12-6 l9-31)
  have  $B \ C \ D \ CongA \ A \ B \ C$ 
    by (metis Perp-cases par-not-col  $\langle A \neq B \rangle \langle B \ C \ ParStrict \ A \ D \rangle$ 
       $\langle B \neq C \rangle \langle Saccheri \ B \ A \ D \ C \rangle$  assms(1) col-permutation-1
      cong-perp-conga l11-16 l7-2 l8-21-aux midpoint-cong
      perp-per-2 sac--perp3414 symmetric-point-construction ts-distincts)
  have  $Per \ B \ C \ D$ 
    using  $\langle B \ C \ D \ CongA \ A \ B \ C \rangle$  assms(1) l11-17 not-conga-sym by blast
  have  $A \ C \ D \ LtA \ C \ A \ B$ 
proof –
  have  $B \ C \ A \ LeA \ B \ C \ A$ 
    using  $\langle B \neq C \rangle$  assms(1) lea-refl per-distinct-1 by presburger
  moreover
  have  $B \ C \ D \ LtA \ P \ Q \ R$ 
    by (metis  $\langle B \ A \ ParStrict \ D \ C \rangle \langle B \neq C \rangle \langle Per \ B \ C \ D \rangle$ )

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    assms(3) obtuse-per--lta par-strict-neq2)
  moreover
  have SAMS B C A A C D
    by (simp add: os-ts--sams ⟨B C ParStrict A D⟩ ⟨C A TS B D⟩ l12-6)
  moreover
  have B C A A C D SumA B C D
    by (simp add: ⟨C A TS B D⟩ ts--suma-1)
  moreover
  have B C A C A B SumA P Q R
    by (simp add: assms(2))
  ultimately
  show ?thesis
    by (meson sams-lea-lta789-suma2--lta456)
qed
hence A C D LtA B A C
  by (meson lta-comm lta-left-comm)
hence Obtuse B A D
  using ⟨Saccheri B A D C⟩ t22-11--obtuse-1 by blast
thus ?thesis
  using ⟨Saccheri B A D C⟩ obtuse-sac--oah by blast
qed

lemma t22-12--oah-2:
  assumes Per A B C and
    B C A C A B SumA P Q R and
    hypothesis-of-obtuse-saccheri-quadrilaterals
  shows Obtuse P Q R
proof -
  have A ≠ B
    using assms(2) suma-distincts by blast
  have B ≠ C
    using assms(2) suma-distincts by blast
  then obtain D where Saccheri B A D C
    using assms(1) ⟨A ≠ B⟩ per--ex-saccheri by blast
  have B C ParStrict A D
    by (simp add: ⟨Saccheri B A D C⟩ sac--pars1423)
  have B A ParStrict D C
    by (simp add: ⟨Saccheri B A D C⟩ sac--pars1234)
  have C A TS B D
    by (meson Par-strict-cases ⟨B A ParStrict D C⟩
      ⟨B C ParStrict A D⟩ l12-6 l9-31)
  have B C D CongA A B C
    by (metis Perp-cases par-not-col ⟨A ≠ B⟩ ⟨B C ParStrict A D⟩
      ⟨B ≠ C⟩ ⟨Saccheri B A D C⟩ assms(1) col-permutation-1
      cong-perp-conga l11-16 l7-2 l8-21-aux midpoint-cong
      perp-per-2 sac--perp3414 symmetric-point-construction ts-distincts)
  have Per B C D
    using ⟨B C D CongA A B C⟩ assms(1) l11-17 not-conga-sym by blast
  moreover
  have B C D LtA P Q R
  proof -
    have B C A LeA B C A
      using ⟨B ≠ C⟩ assms(1) lea-refl per-distinct-1 by presburger
    moreover
    have Obtuse B A D
      using ⟨Saccheri B A D C⟩ assms(3)
      hypothesis-of-obtuse-saccheri-quadrilaterals-def by blast
    hence A C D LtA C A B
      using ⟨Saccheri B A D C⟩ lta-right-comm
      t22-11--obtuse-2 by blast
    moreover
    have SAMS B C A C A B
      by (metis ⟨A ≠ B⟩ ⟨B ≠ C⟩ assms(1) per-distinct-1 sams123231)
    moreover
    have B C A A C D SumA B C D
      by (simp add: ⟨C A TS B D⟩ ts--suma-1)

```

```

moreover
have  $B\ C\ A\ C\ A\ B\ SumA\ P\ Q\ R$ 
  using assms(2) by blast
ultimately
show ?thesis
  by (meson sams-lea-lta456-suma2--lta)
qed
ultimately
show ?thesis
  using Obtuse-def by blast
qed

lemma t22-12--oah:
assumes  $Per\ A\ B\ C$  and
   $B\ C\ A\ C\ A\ B\ SumA\ P\ Q\ R$ 
shows  $Obtuse\ P\ Q\ R \longleftrightarrow hypothesis-of-obtuse-saccheri-quadrilaterals$ 
using assms(1) assms(2) t22-12--oah-1 t22-12--oah-2 by blast

lemma t22-14-aux:
assumes  $\neg\ Col\ A\ B\ C$  and
   $Acute\ A\ B\ C$  and
   $Acute\ A\ C\ B$ 
shows  $\exists\ A'.\ Bet\ B\ A'\ C \wedge Per\ B\ A'\ A \wedge Per\ C\ A'\ A \wedge$ 
 $B\ A'\ A\ CongA\ C\ A'\ A \wedge A\ B\ C\ CongA\ A\ B\ A' \wedge$ 
 $B\ C\ A\ CongA\ A'\ C\ A \wedge A\ A'\ TS\ B\ C$ 
proof –
have  $\neg\ Col\ B\ C\ A$ 
  using Col-cases assms(1) by blast
then obtain  $A'$  where  $Col\ B\ C\ A' \wedge B\ C\ Perp\ A\ A'$ 
  using l8-18-existence by blast
have  $B\ Out\ A'\ C$ 
  by (meson acute-col-perp--out  $\langle Col\ B\ C\ A' \wedge B\ C\ Perp\ A\ A' \rangle$  assms(2))
have  $C\ Out\ A'\ B$ 
proof –
  have  $Acute\ A\ C\ B$ 
    by (simp add: assms(3))
  moreover
  have  $Col\ C\ B\ A'$ 
    using NCol-cases  $\langle B\ Out\ A'\ C \rangle$  out-col by blast
  moreover
  have  $C\ B\ Perp\ A\ A'$ 
    by (simp add:  $\langle Col\ B\ C\ A' \wedge B\ C\ Perp\ A\ A' \rangle$  perp-left-comm)
  ultimately
  show ?thesis
    by (meson acute-col-perp--out)
qed
hence  $Bet\ B\ A'\ C$ 
  by (simp add:  $\langle B\ Out\ A'\ C \rangle$  l6-6 out2--bet)
moreover
have  $Per\ B\ A'\ A$ 
  using Per-perm  $\langle Col\ B\ C\ A' \wedge B\ C\ Perp\ A\ A' \rangle$  col-trivial-3 l8-16-1 by blast
moreover
have  $Per\ C\ A'\ A$ 
  by (metis l8-3  $\langle B\ Out\ A'\ C \rangle$   $\langle Col\ B\ C\ A' \wedge B\ C\ Perp\ A\ A' \rangle$ 
calculation(2) not-col-permutation-1 out-diff1)
moreover
have  $B\ A'\ A\ CongA\ C\ A'\ A$ 
  by (metis Out-def  $\langle B\ Out\ A'\ C \rangle$   $\langle C\ Out\ A'\ B \rangle$ 
 $\langle Col\ B\ C\ A' \wedge B\ C\ Perp\ A\ A' \rangle$   $\langle \neg\ Col\ B\ C\ A \rangle$ 
calculation(2) calculation(3) l11-16)
moreover
have  $A\ B\ C\ CongA\ A\ B\ A'$ 
  by (metis CongA-def out2--conga out-trivial  $\langle B\ Out\ A'\ C \rangle$ 
 $\langle C\ Out\ A'\ B \rangle$  calculation(3) l8-9 out-col)
moreover
have  $B\ C\ A\ CongA\ A'\ C\ A$ 

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by (metis CongA-def out2--conga out-trivial
 $\langle B \text{ Out } A' C \rangle \langle C \text{ Out } A' B \rangle$ calculation(2) l8-9 out-col)

moreover

have $A A' TS B C$

by (metis Out-def $\langle B \text{ Out } A' C \rangle \langle C \text{ Out } A' B \rangle$
 $\langle Col B C A' \wedge B C \text{ Perp } A A' \rangle$ assms(1) bet--ts calculation(1)
 invert-two-sides l6-16-1 not-col-permutation-3 not-col-permutation-4)

ultimately

show ?thesis

by blast

qed

lemma t22-14--bet-aux:

assumes hypothesis-of-right-saccheri-quadrilaterals and
 $\neg Col A B C$ and
 $A B C \text{ TriSumA } P Q R$ and
 Acute $A B C$ and
 Acute $A C B$

shows Bet $P Q R$

proof –

have $C A B \text{ TriSumA } P Q R$

by (simp add: assms(3) trisuma-perm-312)

obtain $D E F$ where $C A B A B C \text{ SumA } D E F \wedge D E F B C A \text{ SumA } P Q R$

using TriSumA-def $\langle C A B \text{ TriSumA } P Q R \rangle$ by blast

obtain A' where Bet $B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C$

using assms(2) assms(4) assms(5) t22-14-aux by blast

have $\text{Per } B A' A$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

have $\text{Per } C A' A$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

have $B A' A \text{ CongA } C A' A$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

have $A B C \text{ CongA } A B A'$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

have $B C A \text{ CongA } A' C A$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

have $A A' TS B C$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

have Bet $B A' C$

using $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$
 $B C A \text{ CongA } A' C A \wedge A A' TS B C \rangle$ by auto

moreover

have $B A' C \text{ CongA } P Q R$

proof –

have $D E F B C A \text{ SumA } B A' C$

proof –

have SAMS $B A' A C A A'$

proof –

have $C A' A \text{ CongA } B A' A$

by (simp add: $\langle \text{Bet } B A' C \wedge \text{Per } B A' A \wedge \text{Per } C A' A \wedge$
 $B A' A \text{ CongA } C A' A \wedge A B C \text{ CongA } A B A' \wedge$

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B C A CongA A' C A  $\wedge$  A A' TS B C $\rangle$  conga-sym)
moreover
have A' A C CongA C A A'
  by (metis assms(2) calculation col-trivial-3 conga-distinct conga-pseudo-refl)
ultimately
show ?thesis
  using conga2-sams--sams conga-distinct sams123231 by presburger
qed
moreover
have SAMS C A A' B C A
proof –
  have A' A C CongA C A A'
    using calculation conga-pseudo-refl sams-distincts by presburger
  moreover
  have lem-acute-not-bet:  $\forall x y z. \text{Acute } x y z \longrightarrow \neg \text{Bet } x y z$ 
    using acute-col--out assms bet-col not-bet-and-out by blast
  have SAMS A' A C A C A'
    by (metis  $\langle B A' A \text{ CongA } C A' A \rangle$  calculation conga-distinct sams123231)
  hence A C A' CongA B C A
    by (metis out2--conga  $\langle \text{Bet } B A' C \rangle$  lem-acute-not-bet
      acute-trivial bet-col bet-col1 between-equality-2
      col-transitivity-1 col-trivial-3 conga-right-comm
      or-bet-out sams-distincts)
  ultimately
  show ?thesis
    using conga2-sams--sams conga-distinct sams123231 by auto
qed
moreover
have B A' A C A A' SumA D E F
proof –
  have C A OS A' B
    by (metis  $\langle A A' TS B C \rangle \langle \text{Bet } B A' C \rangle$  assms(2)
      bet-out-1 not-col-permutation-2 out-one-side ts-distincts)
  hence SAMS C A A' A' A B
    by (metis  $\langle A B C \text{ CongA } A B A' \rangle \langle \text{Bet } B A' C \rangle$  assms(2)
      bet-out col-permutation-4 conga-diff56 os2--sams out-one-side)
  moreover
  have SAMS A' A B A B C
  proof –
    have A' A B CongA A' A B
      using  $\langle \text{Per } B A' A \rangle \langle \text{SAMS } B A' A C A A' \rangle$  conga-refl
      per-distinct sams-distincts by presburger
    moreover
    have A B A' CongA A B C
      using  $\langle A B C \text{ CongA } A B A' \rangle$  conga-sym-equiv by auto
    moreover
    have SAMS A' A B A B A'
      by (metis  $\langle B A' A \text{ CongA } C A' A \rangle$  calculation(2)
        conga-distinct sams123231)
    ultimately
    show ?thesis
      by (meson conga2-sams--sams)
  qed
moreover
have C A A' A' A B SumA C A B
  by (simp add:  $\langle A A' TS B C \rangle$  l9-2 ts--suma-1)
moreover
have A' A B A B C SumA B A' A
proof –
  have A' A B A B A' SumA B A' A
    by (metis  $\langle B A' A \text{ CongA } C A' A \rangle \langle \text{Per } B A' A \rangle$ 
      assms(1) conga-distinct t22-12--rah-2)
  moreover
  have A' A B CongA A' A B
    using  $\langle C A A' A' A B \text{ SumA } C A B \rangle \langle \text{SAMS } C A A' A' A B \rangle$ 
    sams2-suma2--conga456 by auto

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moreover
have  $A B A' CongA A B C$ 
  by (simp add:  $\langle A B C CongA A B A' \rangle$  conga-sym)
moreover
have  $B A' A CongA B A' A$ 
  by (meson suma2--conga calculation(1))
ultimately
show ?thesis
  by (meson conga3-suma--suma)
qed
moreover
have  $C A B A B C SumA D E F$ 
  using  $\langle C A B A B C SumA D E F \wedge D E F B C A SumA P Q R \rangle$  by blast
ultimately
show ?thesis
  by (meson suma-assoc-1 suma-sym)
qed
moreover
have  $C A A' B C A SumA A A' C$ 
proof –
  have  $A' A C A C A' SumA C A' A$ 
    using  $\langle B A' A CongA C A' A \rangle \langle Per C A' A \rangle$  assms(1)
    calculation(1) conga-diff45 sams-distincts t22-12--rah by blast
  moreover
  have  $A' A C CongA C A A'$ 
    by (metis calculation conga-pseudo-refl suma-distincts)
  moreover
  have  $A C A' CongA B C A$ 
    by (meson  $\langle B C A CongA A' C A \rangle$  conga-right-comm conga-sym)
  moreover
  have  $C A' A CongA A A' C$ 
    using  $\langle B A' A CongA C A' A \rangle$  conga-distinct
    conga-pseudo-refl by blast
  ultimately
  show ?thesis
    by (meson conga3-suma--suma)
qed
moreover
have  $B A' A A A' C SumA B A' C$ 
  by (simp add:  $\langle A A' TS B C \rangle$  ts--suma)
ultimately
show ?thesis
  using suma-assoc by blast
qed
moreover
have  $D E F B C A SumA P Q R$ 
  by (simp add:  $\langle C A B A B C SumA D E F \wedge D E F B C A SumA P Q R \rangle$ )
ultimately
show ?thesis
  by (meson suma2--conga)
qed
ultimately
show ?thesis
  using bet-conga--bet by blast
qed

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lemma t22-14--bet:
  assumes hypothesis-of-right-saccheri-quadrilaterals and
     $A B C TriSumA P Q R$ 
  shows  $Bet P Q R$ 
proof cases
  assume  $Col A B C$ 
  thus ?thesis
    using assms(2) col-trisuma--bet by blast

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next
  assume  $\neg \text{Col } A \ B \ C$ 
  hence  $\text{Acute } A \ B \ C \vee \text{Per } A \ B \ C \vee \text{Obtuse } A \ B \ C$ 
    using angle-partition not-col-distincts by presburger
  have  $\text{Acute } A \ C \ B \vee \text{Per } A \ C \ B \vee \text{Obtuse } A \ C \ B$ 
    by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  angle-partition not-col-distincts)
  {
    assume  $\text{Acute } A \ B \ C$ 
    assume  $\text{Acute } A \ C \ B$ 
    have  $\text{Bet } P \ Q \ R$ 
      using  $\langle \neg \text{Col } A \ B \ C \rangle \langle \text{Acute } A \ B \ C \rangle \langle \text{Acute } A \ C \ B \rangle$  assms(1) assms(2)
        t22-14--bet-aux by blast
  }
  moreover
  {
    assume  $\text{Acute } A \ B \ C$ 
    assume  $\text{Per } A \ C \ B \vee \text{Obtuse } A \ C \ B$ 
    hence  $\text{Acute } C \ A \ B$ 
      using l11-43 by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  not-col-distincts)
    have  $\text{Acute } C \ B \ A$ 
      by (simp add:  $\langle \text{Acute } A \ B \ C \rangle$  acute-sym)
    have  $C \ A \ B \ \text{TriSumA } P \ Q \ R$ 
      by (simp add: assms(2) trisuma-perm-312)
    hence  $\text{Bet } P \ Q \ R$ 
      using  $\langle \text{Acute } C \ A \ B \rangle \langle \text{Acute } C \ B \ A \rangle$  assms(1) col-trisuma--bet
        t22-14--bet-aux by blast
  }
  moreover
  {
    assume  $\text{Per } A \ B \ C \vee \text{Obtuse } A \ B \ C$ 
    hence  $\text{Acute } B \ C \ A$ 
      by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  l11-43 not-col-distincts)
    have  $\text{Acute } B \ A \ C$  using l11-43
      by (metis  $\langle \text{Per } A \ B \ C \vee \text{Obtuse } A \ B \ C \rangle \langle \neg \text{Col } A \ B \ C \rangle$  not-col-distincts)
    have  $B \ A \ C \ \text{TriSumA } P \ Q \ R$ 
      by (simp add: assms(2) trisuma-perm-213)
    hence  $\text{Bet } P \ Q \ R$ 
      using  $\langle \text{Acute } B \ A \ C \rangle \langle \text{Acute } B \ C \ A \rangle$  assms(1) col-trisuma--bet
        t22-14--bet-aux by blast
  }
  ultimately
  show ?thesis
    using  $\langle \text{Acute } A \ B \ C \vee \text{Per } A \ B \ C \vee \text{Obtuse } A \ B \ C \rangle$ 
       $\langle \text{Acute } A \ C \ B \vee \text{Per } A \ C \ B \vee \text{Obtuse } A \ C \ B \rangle$  by fastforce
qed

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lemma *t22-14--sams-nbet-aux:*

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assumes hypothesis-of-acute-saccheri-quadrilaterals and
   $\neg \text{Col } A \ B \ C$  and
   $C \ A \ B \ A \ B \ C \ \text{SumA } D \ E \ F$  and
   $D \ E \ F \ B \ C \ A \ \text{SumA } P \ Q \ R$  and
   $\text{Acute } A \ B \ C$  and
   $\text{Acute } A \ C \ B$ 

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shows $\text{SAMS } D \ E \ F \ B \ C \ A \wedge \neg \text{Bet } P \ Q \ R$

proof –

obtain A' **where**

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   $\text{Bet } B \ A' \ C \wedge \text{Per } B \ A' \ A \wedge \text{Per } C \ A' \ A \wedge$ 
   $B \ A' \ A \ \text{CongA } C \ A' \ A \wedge A \ B \ C \ \text{CongA } A \ B \ A' \wedge$ 
   $B \ C \ A \ \text{CongA } A' \ C \ A \wedge A \ A' \ \text{TS } B \ C$ 
  using assms(2) assms(5) assms(6) t22-14-aux by blast
  have  $\text{Bet } B \ A' \ C$ 
    using  $\langle \text{Bet } B \ A' \ C \wedge \text{Per } B \ A' \ A \wedge \text{Per } C \ A' \ A \wedge$ 
       $B \ A' \ A \ \text{CongA } C \ A' \ A \wedge A \ B \ C \ \text{CongA } A \ B \ A' \wedge$ 
       $B \ C \ A \ \text{CongA } A' \ C \ A \wedge A \ A' \ \text{TS } B \ C \rangle$  by auto
  have  $\text{Per } B \ A' \ A$ 
    using  $\langle \text{Bet } B \ A' \ C \wedge \text{Per } B \ A' \ A \wedge \text{Per } C \ A' \ A \wedge$ 

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  B A' A CongA C A' A ∧ A B C CongA A B A' ∧
  B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have Per C A' A
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
  B A' A CongA C A' A ∧ A B C CongA A B A' ∧
  B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have B A' A CongA C A' A
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
  B A' A CongA C A' A ∧ A B C CongA A B A' ∧
  B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have A B C CongA A B A'
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
  B A' A CongA C A' A ∧ A B C CongA A B A' ∧
  B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have B C A CongA A' C A
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
  B A' A CongA C A' A ∧ A B C CongA A B A' ∧
  B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have A A' TS B C
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
  B A' A CongA C A' A ∧ A B C CongA A B A' ∧
  B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have A ≠ B
  using assms(2) col-trivial-1 by auto
have C ≠ B
  using assms(2) col-trivial-2 by blast
have A ≠ C
  using assms(2) col-trivial-3 by auto
have A ≠ A'
  using ⟨B A' A CongA C A' A⟩ conga-diff56 by auto
have A' ≠ B
  using ⟨A B C CongA A B A'⟩ conga-diff56 by auto
have A' ≠ C
  using ⟨B C A CongA A' C A⟩ conga-diff45 by auto
have P ≠ Q
  using assms(4) suma-distincts by blast
have R ≠ Q
  using assms(4) suma-distincts by blast
obtain G H I where B A' A C A A' SumA G H I
  using ⟨A ≠ A'⟩ ⟨A ≠ C⟩ ⟨A' ≠ B⟩ ex-suma by presburger
have D E F LtA G H I
proof –
  obtain V W X where A' A B A B C SumA V W X
    using ⟨A ≠ A'⟩ ⟨A ≠ B⟩ ⟨C ≠ B⟩ ex-suma by presburger
  have V ≠ W
    using ⟨A' A B A B C SumA V W X⟩ suma-distincts by presburger
  have X ≠ W
    using ⟨A' A B A B C SumA V W X⟩ suma-distincts by blast
  have A' A B A B A' SumA V W X
    using ⟨A ≠ A'⟩ ⟨A ≠ B⟩ ⟨A' A B A B C SumA V W X⟩
    ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧ B A' A CongA C A' A ∧
    A B C CongA A B A' ∧ B C A CongA A' C A ∧ A A' TS B C⟩
    ⟨V ≠ W⟩ ⟨X ≠ W⟩ conga3-suma--suma conga-refl by presburger
  hence V W X LtA B A' A
    using ⟨Per B A' A⟩ assms(1) ⟨A ≠ A'⟩ ⟨A' ≠ B⟩ acute-per--lta
    t22-12--aah-2 by presburger
moreover
  have SAMS C A A' B A' A
    by (metis ⟨A ≠ A'⟩ ⟨A' ≠ B⟩ ⟨A' ≠ C⟩ ⟨Per B A' A⟩ ⟨Per C A' A⟩
    acute-per--sams acute-sym l11-43 sams-sym)
moreover
  have C A A' V W X SumA D E F
proof –
  have SAMS C A A' A' A B
    by (meson os-ts--sams ⟨A A' TS B C⟩ ⟨A' ≠ C⟩ ⟨Bet B A' C⟩
    assms(2) bet-out-1 l9-2 not-col-permutation-2 out-one-side)

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moreover
have  $SAMS\ A'\ A\ B\ A\ B\ C$ 
  using  $\langle A \neq A' \rangle \langle A' \neq B \rangle \langle Per\ B\ A'\ A \rangle\ acute2\text{--}sams\ assms(5)$ 
     $l11\text{--}43$  by force
moreover
have  $C\ A\ A'\ A'\ A\ B\ SumA\ C\ A\ B$ 
  by (metis  $\langle A' \neq B \rangle \langle B\ C\ A\ CongA\ A'\ C\ A \rangle \langle Bet\ B\ A'\ C \rangle\ assms(2)$ 
    bet--ts between-symmetry col-permutation-3 col-permutation-5
    ncol-conga-ncol ts--suma)
ultimately
show ?thesis
  by (meson suma-assoc  $\langle A'\ A\ B\ A\ B\ C\ SumA\ V\ W\ X \rangle\ assms(3)$ )
qed
ultimately
show ?thesis
  by (metis not-and-lta  $\langle B\ A'\ A\ C\ A\ A'\ SumA\ G\ H\ I \rangle\ nlta\text{--}lea$ 
    sams-lea-lta456-suma2--lta123 suma-distincts suma-sym)
qed
obtain  $J\ K\ L$  where  $C\ A\ A'\ B\ C\ A\ SumA\ J\ K\ L$ 
  using  $\langle A \neq A' \rangle \langle A \neq C \rangle \langle C \neq B \rangle\ ex\text{--}suma$  by presburger
have  $J \neq K$ 
  using  $\langle C\ A\ A'\ B\ C\ A\ SumA\ J\ K\ L \rangle\ suma\text{--}distincts$  by presburger
have  $L \neq K$ 
  using  $\langle C\ A\ A'\ B\ C\ A\ SumA\ J\ K\ L \rangle\ suma\text{--}distincts$  by blast
have  $J\ K\ L\ LtA\ A\ A'\ C$ 
proof  $-$ 
  have Acute  $J\ K\ L$ 
  proof  $-$ 
    have Per  $C\ A'\ A$ 
    by (simp add:  $\langle Per\ C\ A'\ A \rangle$ )
    moreover
    have  $A'\ A\ C\ A\ C\ A'\ SumA\ J\ K\ L$ 
    proof  $-$ 
      have  $C\ A\ A'\ CongA\ A'\ A\ C$ 
      using  $\langle A \neq A' \rangle \langle A \neq C \rangle\ conga\text{--}pseudo\text{--}refl$  by force
      moreover
      have  $B\ C\ A\ CongA\ A\ C\ A'$ 
      by (simp add:  $\langle B\ C\ A\ CongA\ A'\ C\ A \rangle\ conga\text{--}right\text{--}comm$ )
      moreover
      have  $J\ K\ L\ CongA\ J\ K\ L$ 
      by (simp add:  $\langle J \neq K \rangle \langle L \neq K \rangle\ conga\text{--}refl$ )
      ultimately
      show ?thesis
      by (meson conga3-suma--suma  $\langle C\ A\ A'\ B\ C\ A\ SumA\ J\ K\ L \rangle$ )
    qed
    ultimately
    show ?thesis
    using assms(1) t22-12--aah by blast
  qed
moreover
have Per  $A\ A'\ C$ 
  by (simp add:  $\langle Per\ C\ A'\ A \rangle\ l8\text{--}2$ )
ultimately
show ?thesis
  by (simp add:  $\langle A \neq A' \rangle \langle A' \neq C \rangle\ acute\text{--}per\text{--}lta$ )
qed
have  $SAMS\ G\ H\ I\ B\ C\ A$ 
proof  $-$ 
  have  $SAMS\ B\ A'\ A\ C\ A\ A'$ 
  by (metis  $\langle A \neq A' \rangle \langle A' \neq B \rangle \langle A' \neq C \rangle \langle Per\ B\ A'\ A \rangle$ 
     $\langle Per\ C\ A'\ A \rangle\ acute\text{--}per\text{--}sams\ acute\text{--}sym\ l11\text{--}43$ )
  moreover
  have  $SAMS\ C\ A\ A'\ B\ C\ A$ 
  by (metis  $\langle A \neq A' \rangle \langle A' \neq C \rangle \langle Per\ C\ A'\ A \rangle\ acute2\text{--}sams$ 
    acute-sym assms(6) l11-43)
  moreover

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have $SAMS\ B\ A'\ A\ J\ K\ L$
by ($metis\ \langle A' \neq B \rangle\ \langle A' \neq C \rangle\ \langle Bet\ B\ A'\ C \rangle\ \langle J\ K\ L\ LtA\ A\ A'\ C \rangle$
 $lta\text{--}lea\ sams\ chara$)
ultimately
show $?thesis$
using $\langle B\ A'\ A\ C\ A\ A'\ SumA\ G\ H\ I \rangle\ \langle C\ A\ A'\ B\ C\ A\ SumA\ J\ K\ L \rangle$
 $sams\text{--}assoc$ **by** $blast$
qed
obtain $S\ T\ U$ **where** $G\ H\ I\ B\ C\ A\ SumA\ S\ T\ U$
using $\langle SAMS\ G\ H\ I\ B\ C\ A \rangle\ ex\text{--}suma\ sams\text{--}distincts$ **by** $presburger$
have $S \neq T$
using $\langle G\ H\ I\ B\ C\ A\ SumA\ S\ T\ U \rangle\ suma\text{--}distincts$ **by** $blast$
have $U \neq T$
using $\langle G\ H\ I\ B\ C\ A\ SumA\ S\ T\ U \rangle\ suma\text{--}distincts$ **by** $blast$
have $S\ T\ U\ LtA\ B\ A'\ C$
proof $-$
have $B\ A'\ A\ LeA\ B\ A'\ A$
using $\langle A \neq A' \rangle\ \langle A' \neq B \rangle\ lea\text{--}refl$ **by** $auto$
moreover
have $J\ K\ L\ LtA\ A\ A'\ C$
by ($simp\ add:$ $\langle J\ K\ L\ LtA\ A\ A'\ C \rangle$)
moreover
have $SAMS\ B\ A'\ A\ A\ A'\ C$
using $\langle A \neq A' \rangle\ \langle A' \neq B \rangle\ \langle A' \neq C \rangle\ \langle Bet\ B\ A'\ C \rangle$
 $bet\text{--}sams$ **by** $force$
moreover
have $B\ A'\ A\ J\ K\ L\ SumA\ S\ T\ U$
proof $-$
have $SAMS\ B\ A'\ A\ C\ A\ A'$
by ($metis\ \langle A \neq A' \rangle\ \langle A' \neq B \rangle\ \langle A' \neq C \rangle\ \langle Per\ B\ A'\ A \rangle$
 $\langle Per\ C\ A'\ A \rangle\ acute\text{--}per\text{--}sams\ acute\text{--}sym\ l11\text{--}43$)
moreover
have $SAMS\ C\ A\ A'\ B\ C\ A$
by ($metis\ \langle A \neq A' \rangle\ \langle A' \neq C \rangle\ \langle Per\ C\ A'\ A \rangle\ acute2\text{--}sams$
 $acute\text{--}sym\ assms(6)\ l11\text{--}43$)
ultimately
show $?thesis$
by ($meson\ \langle G\ H\ I\ B\ C\ A\ SumA\ S\ T\ U \rangle\ \langle C\ A\ A'\ B\ C\ A\ SumA\ J\ K\ L \rangle$
 $\langle B\ A'\ A\ C\ A\ A'\ SumA\ G\ H\ I \rangle\ suma\text{--}assoc\text{--}1$)
qed
moreover
have $B\ A'\ A\ A\ A'\ C\ SumA\ B\ A'\ C$
using $\langle A \neq A' \rangle\ \langle A' \neq B \rangle\ \langle A' \neq C \rangle\ \langle Bet\ B\ A'\ C \rangle$
 $bet\text{--}suma$ **by** $auto$
ultimately
show $?thesis$
by ($meson\ sams\ lea\text{--}lta456\text{--}suma2\text{--}lta$)
qed
have $SAMS\ D\ E\ F\ B\ C\ A$
proof $-$
have $D\ E\ F\ LeA\ G\ H\ I$
by ($simp\ add:$ $\langle D\ E\ F\ LtA\ G\ H\ I \rangle\ lta\text{--}lea$)
moreover
have $B\ C\ A\ LeA\ B\ C\ A$
using $\langle A \neq C \rangle\ \langle C \neq B \rangle\ lea\text{--}refl$ **by** $auto$
ultimately
show $?thesis$
using $\langle SAMS\ G\ H\ I\ B\ C\ A \rangle\ sams\ lea2\text{--}sams$ **by** $blast$
qed
moreover
{
assume $Bet\ P\ Q\ R$
have $P\ Q\ R\ LtA\ P\ Q\ R$
proof $-$
have $P\ Q\ R\ CongA\ P\ Q\ R$
by ($simp\ add:$ $\langle P \neq Q \rangle\ \langle R \neq Q \rangle\ conga\text{--}refl$)
}

```

moreover
have  $B A' C \text{ CongA } P Q R$ 
  by (metis conga-line  $\langle A' \neq B \rangle \langle A' \neq C \rangle \langle \text{Bet } B A' C \rangle$ 
     $\langle \text{Bet } P Q R \rangle \langle P \neq Q \rangle \langle R \neq Q \rangle$ )
moreover
have  $B C A \text{ LeA } B C A$ 
  using  $\langle A \neq C \rangle \langle C \neq B \rangle$  lea-refl by auto
hence  $P Q R \text{ LtA } S T U$ 
  using  $\langle D E F \text{ LtA } G H I \rangle \langle \text{SAMS } G H I B C A \rangle$  assms(4)
     $\langle G H I B C A \text{ SumA } S T U \rangle$  sams-lea-lta123-suma2--lta by blast
hence  $P Q R \text{ LtA } B A' C$ 
  using lta-trans  $\langle S T U \text{ LtA } B A' C \rangle$  by blast
ultimately
show ?thesis
  using conga-preserves-lta by blast
qed
hence False
  using nlta by blast
}
hence  $\neg \text{Bet } P Q R$ 
  by blast
ultimately
show ?thesis
  by auto
qed

```

```

lemma t22-14--sams-nbet:
assumes hypothesis-of-acute-saccheri-quadrilaterals and
   $\neg \text{Col } A B C$  and
   $C A B A B C \text{ SumA } D E F$  and
   $D E F B C A \text{ SumA } P Q R$ 
shows  $\text{SAMS } D E F B C A \wedge \neg \text{Bet } P Q R$ 
proof –
have  $A \neq B$ 
  using assms(2) col-trivial-1 by blast
have  $A \neq C$ 
  using assms(2) col-trivial-3 by blast
have  $B \neq C$ 
  using assms(2) col-trivial-2 by blast
{
assume Acute  $A B C$ 
assume Acute  $A C B$ 
hence  $\text{SAMS } D E F B C A \wedge \neg \text{Bet } P Q R$ 
  using  $\langle \text{Acute } A B C \rangle$  assms(1) assms(2) assms(3) assms(4)
    t22-14--sams-nbet-aux by blast
}
moreover
{
assume Acute  $A B C$ 
assume  $\text{Per } A C B \vee \text{Obtuse } A C B$ 
have Acute  $C A B$ 
  using l11-43 by (metis  $\langle A \neq C \rangle \langle \text{Acute } A B C \rangle$ 
     $\langle \text{Per } A C B \vee \text{Obtuse } A C B \rangle$  acute--not-obtuse acute-not-per)
have Acute  $C B A$ 
  by (simp add:  $\langle \text{Acute } A B C \rangle$  acute-sym)
obtain  $G H I$  where  $B C A C A B \text{ SumA } G H I$ 
  using  $\langle A \neq B \rangle \langle A \neq C \rangle \langle B \neq C \rangle$  ex-suma by presburger
have  $G \neq H$ 
  using  $\langle B C A C A B \text{ SumA } G H I \rangle$  suma-distincts by blast
have  $H \neq I$ 
  using  $\langle B C A C A B \text{ SumA } G H I \rangle$  suma-distincts by presburger
have  $G H I A B C \text{ SumA } P Q R$ 
proof –

```

```

have SAMS B C A C A B
  using ⟨A ≠ B⟩ ⟨A ≠ C⟩ ⟨B ≠ C⟩ sams123231 by auto
moreover
have SAMS C A B A B C
  using ⟨A ≠ B⟩ ⟨A ≠ C⟩ ⟨B ≠ C⟩ sams123231 by presburger
moreover
have B C A D E F SumA P Q R
  by (meson assms(4) suma-sym)
ultimately
show ?thesis
  using ⟨B C A C A B SumA G H I⟩ assms(3) suma-assoc by blast
qed
hence SAMS D E F B C A ∧ ¬ Bet P Q R
  by (metis ⟨A ≠ B⟩ ⟨A ≠ C⟩ ⟨Acute C A B⟩ ⟨Acute C B A⟩
    ⟨B C A C A B SumA G H I⟩ ⟨B ≠ C⟩ assms(1) assms(2) assms(3)
    not-col-permutation-2 sams123231 sams-assoc sams-sym
    t22-14--sams-nbet-aux)
}
moreover
{
  assume Per A B C ∨ Obtuse A B C
  have Acute B A C
    using l11-43 ⟨A ≠ B⟩ ⟨B ≠ C⟩ ⟨Per A B C ∨ Obtuse A B C⟩ by auto
  have Acute B C A
    using l11-43 ⟨A ≠ B⟩ ⟨B ≠ C⟩ ⟨Per A B C ∨ Obtuse A B C⟩ by auto
  hence SAMS D E F B C A ∧ ¬ Bet P Q R
    by (meson Bet-cases ⟨Acute B A C⟩ assms(1) assms(2) assms(3)
      assms(4) col-permutation-4 sams-comm suma-comm suma-sym
      t22-14--sams-nbet-aux)
}
ultimately
show ?thesis
  by (metis angle-partition l8-2 l8-5)
qed

lemma t22-14--nsams-aux:
  assumes hypothesis-of-obtuse-saccheri-quadrilaterals and
    ¬ Col A B C and
    C A B A B C SumA D E F and
    Acute A B C and
    Acute A C B
  shows ¬ SAMS D E F B C A
proof -
{
  assume SAMS D E F B C A
  obtain A' where
    Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
    B A' A CongA C A' A ∧ A B C CongA A B A' ∧
    B C A CongA A' C A ∧ A A' TS B C
    using assms(2) assms(5) assms(4) t22-14-aux by blast
  have Bet B A' C
    using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
      B A' A CongA C A' A ∧ A B C CongA A B A' ∧
      B C A CongA A' C A ∧ A A' TS B C⟩ by auto
  have Per B A' A
    using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
      B A' A CongA C A' A ∧ A B C CongA A B A' ∧
      B C A CongA A' C A ∧ A A' TS B C⟩ by auto
  have Per C A' A
    using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
      B A' A CongA C A' A ∧ A B C CongA A B A' ∧
      B C A CongA A' C A ∧ A A' TS B C⟩ by auto
  have B A' A CongA C A' A
    using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
      B A' A CongA C A' A ∧ A B C CongA A B A' ∧
      B C A CongA A' C A ∧ A A' TS B C⟩ by auto
}

```

```

have A B C CongA A B A'
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
    B A' A CongA C A' A ∧ A B C CongA A B A' ∧
    B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have B C A CongA A' C A
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
    B A' A CongA C A' A ∧ A B C CongA A B A' ∧
    B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have A A' TS B C
  using ⟨Bet B A' C ∧ Per B A' A ∧ Per C A' A ∧
    B A' A CongA C A' A ∧ A B C CongA A B A' ∧
    B C A CongA A' C A ∧ A A' TS B C⟩ by auto
have A ≠ B
  using assms(2) col-trivial-1 by auto
have C ≠ B
  using assms(2) col-trivial-2 by blast
have A ≠ C
  using assms(2) col-trivial-3 by auto
have A ≠ A'
  using ⟨B A' A CongA C A' A⟩ conga-diff56 by auto
have A' ≠ B
  using ⟨A B C CongA A B A'⟩ conga-diff56 by auto
have A' ≠ C
  using ⟨B C A CongA A' C A⟩ conga-diff45 by auto
have D ≠ E
  using ⟨SAMS D E F B C A⟩ sams-distincts by blast
have E ≠ F
  using assms(3) suma-distincts by blast
obtain P Q R where D E F B C A SumA P Q R
  using ⟨A ≠ C⟩ ⟨C ≠ B⟩ ⟨D ≠ E⟩ ⟨E ≠ F⟩ ex-suma by presburger
have P ≠ Q
  using ⟨D E F B C A SumA P Q R⟩ suma-distincts by blast
have R ≠ Q
  using ⟨D E F B C A SumA P Q R⟩ suma-distincts by blast
have B A' C LtA P Q R
proof –
  obtain G H I where B A' A C A A' SumA G H I
    using ⟨A ≠ A'⟩ ⟨A ≠ C⟩ ⟨A' ≠ B⟩ ex-suma by presburger
  have G ≠ H
    by (metis suma-distincts ⟨B A' A C A A' SumA G H I⟩)
  have I ≠ H
    by (metis suma-distincts ⟨B A' A C A A' SumA G H I⟩)
  have G H I LtA D E F
proof –
  obtain V W X where A' A B A B C SumA V W X
    using ⟨A ≠ A'⟩ ⟨A ≠ B⟩ ⟨C ≠ B⟩ ex-suma by presburger
  have V ≠ W
    by (metis suma-distincts ⟨A' A B A B C SumA V W X⟩)
  have X ≠ W
    by (metis suma-distincts ⟨A' A B A B C SumA V W X⟩)
  have SAMS C A A' A' A B
    by (meson os-ts--sams ⟨A A' TS B C⟩ ⟨A' ≠ C⟩ ⟨Bet B A' C⟩
      assms(2) bet-out-1 l9-2 not-col-permutation-2 out-one-side)
  have SAMS A' A B A B C
    using ⟨A ≠ A'⟩ ⟨A' ≠ B⟩ ⟨Per B A' A⟩ acute2--sams
      assms(4) l11-43 by force
  have C A A' A' A B SumA C A B
    by (meson ⟨A' ≠ B⟩ ⟨B C A CongA A' C A⟩ ⟨Bet B A' C⟩
      assms(2) bet--ts between-symmetry col-permutation-2
      ncol-conga-ncol not-col-permutation-5 ts--suma)
  have C A A' LeA C A A'
    using ⟨A ≠ A'⟩ ⟨A ≠ C⟩ lea-refl by auto
moreover
have A' A B A B A' SumA V W X
  by (meson ⟨A B C CongA A B A'⟩ ⟨A' A B A B C SumA V W X⟩
    ⟨SAMS A' A B A B C⟩ conga3-suma--suma)

```

sams2-suma2--conga123 suma2--conga)
 hence *Obtuse* $V W X$
 using $\langle \text{Per } B A' A \rangle$ *assms(1) t22-12--oah* **by** *blast*
 hence $B A' A \text{ LtA } V W X$
 using $\langle \text{Per } B A' A \rangle \langle A \neq A' \rangle \langle A' \neq B \rangle$
obtuse-per--lta **by** *auto*
 moreover
 have $\text{SAMS } C A A' V W X$
 by (*metis* $\langle A \neq B \rangle \langle A \neq C \rangle \langle A' A B A B C \text{ SumA } V W X \rangle$
 $\langle C A A' A' A B \text{ SumA } C A B \rangle \langle C \neq B \rangle \langle \text{SAMS } A' A B A B C \rangle$
 $\langle \text{SAMS } C A A' A' A B \rangle$ *sams123231 sams-assoc-1*)
 moreover
 have $C A A' B A' A \text{ SumA } G H I$
 by (*simp add:* $\langle B A' A C A A' \text{ SumA } G H I \rangle$ *suma-sym*)
 moreover
 have $C A A' V W X \text{ SumA } D E F$
 by (*meson* $\langle A' A B A B C \text{ SumA } V W X \rangle$
 $\langle C A A' A' A B \text{ SumA } C A B \rangle \langle \text{SAMS } A' A B A B C \rangle$
 $\langle \text{SAMS } C A A' A' A B \rangle$ *assms(3) suma-assoc-1*)
 ultimately
 show *?thesis*
 using *sams-lea-lta456-suma2--lta* **by** *blast*
 qed
 obtain $J K L$ **where** $C A A' B C A \text{ SumA } J K L$
 using $\langle A \neq A' \rangle \langle A \neq C \rangle \langle C \neq B \rangle$ *ex-suma* **by** *presburger*
 have $J \neq K$
 using $\langle C A A' B C A \text{ SumA } J K L \rangle$ *conga-distinct*
suma2--conga **by** *blast*
 have $L \neq K$
 using $\langle C A A' B C A \text{ SumA } J K L \rangle$ *conga-distinct*
suma2--conga **by** *blast*
 have $A A' C \text{ LtA } J K L$
 proof –
 have $A' A C A C A' \text{ SumA } J K L$
 proof –
 have $C A A' \text{ CongA } A' A C$
 using $\langle A \neq A' \rangle \langle A \neq C \rangle$ *conga-pseudo-refl* **by** *auto*
 moreover
 have $B C A \text{ CongA } A C A'$
 using $\langle B C A \text{ CongA } A' C A \rangle$ *conga-right-comm* **by** *blast*
 moreover
 have $J K L \text{ CongA } J K L$
 using $\langle J \neq K \rangle \langle L \neq K \rangle$ *conga-refl* **by** *presburger*
 ultimately
 show *?thesis*
 using $\langle C A A' B C A \text{ SumA } J K L \rangle$ *conga3-suma--suma* **by** *blast*
 qed
 hence *Obtuse* $J K L$
 using $\langle \text{Per } C A' A \rangle$ *assms(1) t22-12--oah* **by** *blast*
 moreover
 have $\text{Per } A A' C$
 using *Per-perm* $\langle \text{Per } C A' A \rangle$ **by** *blast*
 ultimately
 show *?thesis*
 by (*simp add:* $\langle A \neq A' \rangle \langle A' \neq C \rangle$ *obtuse-per--lta*)
 qed
 obtain $S T U$ **where** $B A' A J K L \text{ SumA } S T U$
 using $\langle A \neq A' \rangle \langle A' \neq B \rangle \langle J \neq K \rangle \langle L \neq K \rangle$ *ex-suma* **by** *fastforce*
 have $S \neq T$
 by (*metis* *suma-distincts* $\langle B A' A J K L \text{ SumA } S T U \rangle$)
 have $U \neq T$
 by (*metis* *suma-distincts* $\langle B A' A J K L \text{ SumA } S T U \rangle$)
 have $B A' C \text{ LtA } S T U$
 proof –
 have $B A' A \text{ LeA } B A' A$
 using $\langle A \neq A' \rangle \langle A' \neq B \rangle$ *lea-refl* **by** *force*

```

moreover
have  $A A' C LtA J K L$ 
  by (simp add:  $\langle A A' C LtA J K L \rangle$ )
moreover
have  $SAMS B A' A J K L$ 
proof –
  have  $SAMS B A' A C A A'$ 
    by (metis acute-sym  $\langle A \neq A' \rangle \langle A' \neq B \rangle \langle A' \neq C \rangle$ 
       $\langle Per B A' A \rangle \langle Per C A' A \rangle$  acute-per--sams l11-43)
  moreover
have  $SAMS C A A' B C A$ 
    by (metis  $\langle A \neq A' \rangle \langle A' \neq C \rangle \langle Per C A' A \rangle$ 
      acute2--sams acute-sym assms(5) l11-43)
  moreover
have  $SAMS G H I B C A$ 
proof –
  have  $SAMS D E F B C A$ 
    by (simp add:  $\langle SAMS D E F B C A \rangle$ )
  moreover
have  $G H I LeA D E F$ 
    by (simp add:  $\langle G H I LtA D E F \rangle$  lta--lea)
  moreover
have  $B C A LeA B C A$ 
    using  $\langle A \neq C \rangle \langle C \neq B \rangle$  lea-refl by auto
  ultimately
show ?thesis
    by (meson sams-lea2--sams)
qed
ultimately
show ?thesis
  by (meson  $\langle B A' A C A A' SumA G H I \rangle$ 
     $\langle C A A' B C A SumA J K L \rangle$  sams-assoc-1)
qed
moreover
have  $B A' A A A' C SumA B A' C$ 
  using  $\langle A \neq A' \rangle \langle A' \neq B \rangle \langle A' \neq C \rangle \langle Bet B A' C \rangle$ 
    bet--suma by auto
moreover
have  $B A' A J K L SumA S T U$ 
  by (simp add:  $\langle B A' A J K L SumA S T U \rangle$ )
ultimately
show ?thesis
  using sams-lea-lta456-suma2--lta by blast
qed
moreover
have  $S T U LtA P Q R$ 
proof –
  have  $G H I LtA D E F$ 
    using  $\langle G H I LtA D E F \rangle$  by blast
  moreover
have  $B C A LeA B C A$ 
    using  $\langle A \neq C \rangle \langle C \neq B \rangle$  lea-refl by auto
  moreover
have  $SAMS D E F B C A$ 
    by (simp add:  $\langle SAMS D E F B C A \rangle$ )
  moreover
have  $G H I B C A SumA S T U$ 
    by (meson  $\langle B A' C LtA S T U \rangle \langle Bet B A' C \rangle$ 
      bet-col col-lta--out not-bet-and-out)
  moreover
have  $D E F B C A SumA P Q R$ 
    by (simp add:  $\langle D E F B C A SumA P Q R \rangle$ )
  ultimately
show ?thesis
    by (meson sams-lea-lta123-suma2--lta)
qed

```



```

ultimately
show ?thesis
  using lta-trans by blast
qed
hence False
  by (meson ⟨Bet B A' C⟩ bet-col col-lta--out not-bet-and-out)
}
thus ?thesis by auto
qed

```

```

lemma t22-14--nsams:
  assumes hypothesis-of-obtuse-saccheri-quadrilaterals and
     $\neg$  Col A B C and
    C A B A B C SumA D E F
  shows  $\neg$  SAMS D E F B C A
proof -
  have A  $\neq$  B
    using assms(2) col-trivial-1 by auto
  have C  $\neq$  B
    using assms(2) not-col-distincts by blast
  have A  $\neq$  C
    using assms(2) col-trivial-3 by blast
  {
    assume Acute A B C
    {
      assume Acute A C B
      have  $\neg$  SAMS D E F B C A
        by (simp add: t22-14--nsams-aux ⟨Acute A B C⟩ ⟨Acute A C B⟩
            assms(1) assms(2) assms(3))
    }
    moreover
    {
      assume Per A C B  $\vee$  Obtuse A C B
      have Acute C A B
        using l11-43 ⟨A  $\neq$  C⟩ ⟨C  $\neq$  B⟩ ⟨Per A C B  $\vee$  Obtuse A C B⟩ by auto
      have Acute C B A
        using l11-43 ⟨A  $\neq$  C⟩ ⟨C  $\neq$  B⟩ ⟨Per A C B  $\vee$  Obtuse A C B⟩ by auto
      obtain G H I where B C A C A B SumA G H I
        using ⟨A  $\neq$  B⟩ ⟨A  $\neq$  C⟩ ⟨C  $\neq$  B⟩ ex-suma by presburger
      have G  $\neq$  H
        using ⟨B C A C A B SumA G H I⟩ suma-distincts by blast
      have I  $\neq$  H
        using ⟨B C A C A B SumA G H I⟩ suma-distincts by blast
      {
        assume SAMS D E F B C A
        have  $\neg$  SAMS G H I A B C
          by (simp add: ⟨Acute C A B⟩ ⟨Acute C B A⟩ ⟨B C A C A B SumA G H I⟩
              assms(1) assms(2) not-col-permutation-2 t22-14--nsams-aux)
        hence False
          by (metis ⟨A  $\neq$  B⟩ ⟨A  $\neq$  C⟩ ⟨B C A C A B SumA G H I⟩ ⟨C  $\neq$  B⟩
              ⟨SAMS D E F B C A⟩ assms(3) sams123231 sams-assoc sams-sym)
      }
      hence  $\neg$  SAMS D E F B C A
        by blast
    }
  }
  ultimately
  have  $\neg$  SAMS D E F B C A
    using ⟨A  $\neq$  C⟩ ⟨C  $\neq$  B⟩ angle-partition by blast
}
moreover
{
  assume Per A B C  $\vee$  Obtuse A B C
  have Acute B A C
    using l11-43 ⟨A  $\neq$  B⟩ ⟨C  $\neq$  B⟩ ⟨Per A B C  $\vee$  Obtuse A B C⟩ by force
}

```

```

have Acute B C A
  using l11-43  $\langle A \neq B \rangle \langle C \neq B \rangle \langle \text{Per } A B C \vee \text{Obtuse } A B C \rangle$  by auto
{
  assume SAMS D E F B C A
  hence SAMS D E F A C B
    by (simp add: sams-right-comm)
  have False
    by (meson t22-14--nsams-aux  $\langle \text{Acute } B A C \rangle \langle \text{Acute } B C A \rangle$ 
       $\langle \text{SAMS } D E F A C B \rangle$  assms(1) assms(2) assms(3)
      not-col-permutation-4 sams-left-comm suma-comm suma-sym)
}
hence  $\neg$  SAMS D E F B C A
  by auto
}
ultimately
show ?thesis
  by (metis  $\langle A \neq B \rangle \langle C \neq B \rangle$  angle-partition)
qed

```

```

lemma t22-14--rah:
  assumes  $\neg$  Col A B C and
    A B C TriSumA P Q R and
    Bet P Q R
  shows hypothesis-of-right-saccheri-quadrilaterals
proof -
  have C A B TriSumA P Q R
    using assms(2) trisuma-perm-312 by blast
  {
    assume hypothesis-of-acute-saccheri-quadrilaterals
    obtain D E F where C A B A B C SumA D E F
      using  $\langle C A B \text{ TriSumA } P Q R \rangle$  TriSumA-def by blast
    have  $\neg$  Bet P Q R
    proof -
      have  $\neg$  Col A B C
        by (simp add: assms(1))
      moreover
      have C A B A B C SumA D E F
        using  $\langle C A B A B C \text{ SumA } D E F \rangle$  by blast
      moreover
      have D E F B C A SumA P Q R
        by (meson TriSumA-def  $\langle C A B \text{ TriSumA } P Q R \rangle$  assms(3)
          bet-suma--sams calculation(2) conga3-suma--suma
          sams2-suma2--conga456 suma2--conga)
      ultimately
      show ?thesis
        using t22-14--sams-nbet  $\langle \text{HypothesisAcuteSaccheriQuadrilaterals} \rangle$  by blast
    qed
    hence False
      using assms(3) by auto
  }
  moreover
  {
    assume hypothesis-of-obtuse-saccheri-quadrilaterals
    obtain D E F where C A B A B C SumA D E F  $\wedge$  D E F B C A SumA P Q R
      using  $\langle C A B \text{ TriSumA } P Q R \rangle$  TriSumA-def by blast
    have C A B A B C SumA D E F
      by (simp add:  $\langle C A B A B C \text{ SumA } D E F \wedge D E F B C A \text{ SumA } P Q R \rangle$ )
    have SAMS D E F B C A
      using  $\langle C A B A B C \text{ SumA } D E F \wedge D E F B C A \text{ SumA } P Q R \rangle$ 
        assms(3) bet-suma--sams by blast
    hence False
      using  $\langle C A B A B C \text{ SumA } D E F \rangle$ 
         $\langle \text{HypothesisObtuseSaccheriQuadrilaterals} \rangle$  assms(1)
        t22-14--nsams by blast
  }

```

```

}
ultimately
show ?thesis
  using saccheri-s-three-hypotheses by blast
qed

```

lemma *t22-14--aah*:

```

assumes C A B A B C SumA D E F and
  D E F B C A SumA P Q R and
  SAMS D E F B C A and
   $\neg \text{Bet } P \ Q \ R$ 
shows hypothesis-of-acute-saccheri-quadrilaterals
proof -
{
  assume hypothesis-of-right-saccheri-quadrilaterals
  have A B C TriSumA P Q R
    using TriSumA-def assms(1) assms(2) trisuma-perm-231 by blast
  hence Bet P Q R
    using  $\langle \text{HypothesisRightSaccheriQuadrilaterals} \rangle$  t22-14--bet by blast
  hence False
    using assms(4) by auto
}
moreover
{
  assume hypothesis-of-obtuse-saccheri-quadrilaterals
  {
    assume Col A B C
    have C A B TriSumA P Q R
      using TriSumA-def assms(1) assms(2) by blast
    hence Bet P Q R
      using  $\langle \text{Col } A \ B \ C \rangle$  col-trisuma--bet not-col-permutation-1 by blast
    hence False
      using assms(4) by blast
  }
  moreover
  {
    assume  $\neg \text{Col } A \ B \ C$ 
    hence  $\neg \text{SAMS } D \ E \ F \ B \ C \ A$ 
      by (simp add: t22-14--nsams  $\langle \text{HypothesisObtuseSaccheriQuadrilaterals} \rangle$  assms(1))
    hence False
      using assms(3) by auto
  }
}
ultimately
have False
  by auto
}
ultimately
show ?thesis
  using saccheri-s-three-hypotheses by blast
qed

```

lemma *t22-14--oah*:

```

assumes C A B A B C SumA D E F and
   $\neg \text{SAMS } D \ E \ F \ B \ C \ A$ 
shows hypothesis-of-obtuse-saccheri-quadrilaterals
proof -
  have  $A \neq B$ 
    using assms(1) suma-distincts by auto
  have  $A \neq C$ 
    using assms(1) suma-distincts by blast
  have  $C \neq B$ 
    using assms(1) suma-distincts by blast

```

```

have  $D \neq E$ 
  using assms(1) suma-distincts by blast
have  $F \neq E$ 
  using assms(1) suma-distincts by blast
{
  assume Col A B C
  have SAMS D E F B C A
  proof -
    {
      assume Bet A B C
      hence SAMS D E F B C A
        by (metis  $\langle C \neq B \rangle \langle D \neq E \rangle \langle F \neq E \rangle$  bet-out between-symmetry
            out213--sams sams-sym)
    }
  moreover
  {
    assume  $\neg \textit{Bet A B C}$ 
    hence B Out A C
      using  $\langle \textit{Col A B C} \rangle$  l6-4-2 by blast
    hence C A B CongA D E F
      using assms(1) out546-suma--conga by auto
    moreover
    have B C A CongA B C A
      using  $\langle A \neq C \rangle \langle C \neq B \rangle$  conga-pseudo-refl conga-right-comm by presburger
    moreover
    have SAMS C A B B C A
      using  $\langle A \neq B \rangle \langle A \neq C \rangle \langle C \neq B \rangle$  sams123231 sams-comm by presburger
    ultimately
    have SAMS D E F B C A
      using conga2-sams--sams by blast
  }
  ultimately
  show ?thesis
    by auto
qed
hence False
  using assms(2) by auto
hence hypothesis-of-obtuse-saccheri-quadrilaterals
  by auto
}
moreover
{
  assume  $\neg \textit{Col A B C}$ 
  obtain P Q R where D E F B C A SumA P Q R
    using  $\langle A \neq C \rangle \langle C \neq B \rangle \langle D \neq E \rangle \langle F \neq E \rangle$  ex-suma by presburger
  {
    assume hypothesis-of-acute-saccheri-quadrilaterals
    hence SAMS D E F B C A
      using t22-14--sams-nbet  $\langle D E F B C A \textit{ SumA P Q R} \rangle$ 
         $\langle \neg \textit{Col A B C} \rangle$  assms(1) by blast
    hence False
      using assms(2) by auto
  }
}
moreover
{
  assume hypothesis-of-right-saccheri-quadrilaterals
  have C A B TriSumA P Q R
    using  $\langle C A B A B C \textit{ SumA D E F} \rangle$  TriSumA-def
       $\langle D E F B C A \textit{ SumA P Q R} \rangle$  by blast
  hence Bet P Q R
    using  $\langle \textit{HypothesisRightSaccheriQuadrilaterals} \rangle$ 
      t22-14--bet by blast
  hence SAMS D E F B C A
    using bet-suma--sams  $\langle D E F B C A \textit{ SumA P Q R} \rangle$  by auto
  hence False
    using assms(2) by auto
}

```

```

}
ultimately
have hypothesis-of-obtuse-saccheri-quadrilaterals
  using saccheri-s-three-hypotheses by blast
}
ultimately
show ?thesis
  by blast
qed

```

```

lemma cong-mid--suma:
  assumes  $\neg \text{Col } A \ B \ C$  and
    M Midpoint A B and
    Cong M A M C
  shows C A B A B C SumA A C B
proof -
  have  $A \neq B$ 
    using assms(1) col-trivial-1 by blast
  have  $C \neq B$ 
    using assms(1) not-col-distincts by blast
  have  $A \neq C$ 
    using assms(1) col-trivial-3 by auto
  have  $A \neq M$ 
    using  $\langle A \neq B \rangle$  assms(2) is-midpoint-id by blast
  have  $M \neq B$ 
    using  $\langle A \neq B \rangle$  assms(2) midpoint-not-midpoint by blast
  have  $M \neq C$ 
    using NCol-cases assms(1) assms(2) midpoint-col by blast
  have A B C CongA M C B
proof -
  have M B C CongA M C B
    by (metis  $\langle C \neq B \rangle$  assms(2) assms(3) cong-transitivity
      l11-44-1-a midpoint-cong not-cong-4312 not-conga-sym)
  moreover
  have B Out A M
    by (simp add:  $\langle A \neq B \rangle$  assms(2) midpoint-out-1)
  moreover
  have B Out C C
    using  $\langle C \neq B \rangle$  out-trivial by blast
  moreover
  have C Out M M
    using  $\langle M \neq C \rangle$  out-trivial by force
  moreover
  have C Out B B
    using  $\langle C \neq B \rangle$  out-trivial by force
  ultimately
  show ?thesis
    using l11-10 by blast
qed
have B A C CongA M C A
proof -
  have M A C CongA M C A
    by (simp add:  $\langle A \neq C \rangle$   $\langle A \neq M \rangle$  assms(3) l11-44-1-a)
  moreover
  have A Out B M
    using Out-def  $\langle A \neq B \rangle$   $\langle A \neq M \rangle$  assms(2) midpoint-bet by auto
  moreover
  have A Out C C
    using  $\langle A \neq C \rangle$  out-trivial by auto
  moreover
  have C Out M M
    by (simp add:  $\langle M \neq C \rangle$  out-trivial)
  moreover
  have C Out A A

```

```

    by (simp add:  $\langle A \neq C \rangle$  out-trivial)
ultimately
show ?thesis
  using l11-10 by blast
qed
have A C M M C B SumA A C B
  by (meson  $\langle B A C \text{ CongA } M C A \rangle \langle M \neq B \rangle$  assms(1) assms(2)
    bet--ts midpoint-bet ncol-conga-ncol not-col-permutation-4 ts--suma)
moreover
have A C M CongA C A B
  by (meson  $\langle B A C \text{ CongA } M C A \rangle$  conga-comm conga-sym)
moreover
have M C B CongA A B C
  using  $\langle A B C \text{ CongA } M C B \rangle$  not-conga-sym by blast
moreover
have A C B CongA A C B
  using  $\langle A \neq C \rangle \langle C \neq B \rangle$  conga-refl by force
ultimately
show ?thesis
  using conga3-suma--suma by blast
qed

lemma t22-17--rah-1:
  assumes  $\neg \text{Col } A B C$  and
    M Midpoint A B and
    Cong M A M C and
    Per A C B
  shows hypothesis-of-right-saccheri-quadrilaterals
proof -
  have C  $\neq$  B
    using assms(1) not-col-distincts by blast
  have A  $\neq$  C
    using assms(1) col-trivial-3 by auto
  have C A B A B C SumA A C B
    using assms(1) assms(2) assms(3) cong-mid--suma by blast
  obtain P Q R where A C B B C A SumA P Q R
    using  $\langle A \neq C \rangle \langle C \neq B \rangle$  ex-suma by presburger
  have C A B TriSumA P Q R
    using TriSumA-def  $\langle A C B B C A \text{ SumA } P Q R \rangle \langle C A B A B C \text{ SumA } A C B \rangle$  by blast
  moreover
  have Bet P Q R
  proof -
    have Per A C B
      by (simp add: assms(4))
    moreover
    have Per B C A
      using calculation l8-2 by presburger
    moreover
    have A C B B C A SumA P Q R
      using  $\langle A C B B C A \text{ SumA } P Q R \rangle$  by blast
    ultimately
    show ?thesis
      using per2-suma--bet by blast
  qed
ultimately
show ?thesis
  using assms(1) col-permutation-1 t22-14--rah by blast
qed

lemma t22-17--rah-2:
  assumes  $\neg \text{Col } A B C$  and
    M Midpoint A B and
    Cong M A M C and
    hypothesis-of-right-saccheri-quadrilaterals

```

shows $Per\ A\ C\ B$
proof –
 have $A \neq B$
 using *assms(1) col-trivial-1* **by** *blast*
 have $C \neq B$
 using *assms(1) not-col-distincts* **by** *blast*
 have $A \neq C$
 using *assms(1) col-trivial-3* **by** *auto*
 have $A \neq M$
 using $\langle A \neq B \rangle$ *assms(2) is-midpoint-id* **by** *blast*
 have $M \neq B$
 using $\langle A \neq B \rangle$ *assms(2) midpoint-not-midpoint* **by** *blast*
 have $M \neq C$
 using *NCol-cases assms(1) assms(2) midpoint-col* **by** *blast*
 have $C\ A\ B\ A\ B\ C\ SumA\ A\ C\ B$
 using *assms(1) assms(2) assms(3) cong-mid--suma* **by** *blast*
obtain $P\ Q\ R$ **where** $A\ C\ B\ B\ C\ A\ SumA\ P\ Q\ R$
 using $\langle A \neq C \rangle\ \langle C \neq B \rangle$ *ex-suma* **by** *presburger*
 have $C\ A\ B\ TriSumA\ P\ Q\ R$
 using *TriSumA-def* $\langle A\ C\ B\ B\ C\ A\ SumA\ P\ Q\ R \rangle\ \langle C\ A\ B\ A\ B\ C\ SumA\ A\ C\ B \rangle$ **by** *blast*
 hence $Bet\ P\ Q\ R$
 using *assms(4) t22-14--bet* **by** *blast*
moreover
 have $A\ C\ B\ A\ C\ B\ SumA\ P\ Q\ R$
 using $\langle A\ C\ B\ B\ C\ A\ SumA\ P\ Q\ R \rangle$ *suma-middle-comm* **by** *blast*
ultimately
show *?thesis*
 by (*meson bet-suma--per*)
qed

lemma *t22-17--rah*:
assumes $\neg\ Col\ A\ B\ C$ **and**
 $M\ Midpoint\ A\ B$ **and**
 $Cong\ M\ A\ M\ C$
shows $Per\ A\ C\ B \longleftrightarrow hypothesis-of-right-saccheri-quadrilaterals$
by (*meson t22-17--rah-1 t22-17--rah-2 assms(1) assms(2) assms(3)*)

lemma *t22-17--oah-1*:
assumes $\neg\ Col\ A\ B\ C$ **and**
 $M\ Midpoint\ A\ B$ **and**
 $Cong\ M\ A\ M\ C$ **and**
 $Obtuse\ A\ C\ B$
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
proof –
 have $C \neq B$
 using *assms(1) not-col-distincts* **by** *blast*
 have $A \neq C$
 using *assms(1) col-trivial-3* **by** *auto*
 have $C\ A\ B\ A\ B\ C\ SumA\ A\ C\ B$
 using *assms(1) assms(2) assms(3) cong-mid--suma* **by** *blast*
obtain $P\ Q\ R$ **where** $A\ C\ B\ B\ C\ A\ SumA\ P\ Q\ R$
 using $\langle A \neq C \rangle\ \langle C \neq B \rangle$ *ex-suma* **by** *presburger*
 have $C\ A\ B\ A\ B\ C\ SumA\ B\ C\ A$
 by (*simp add:* $\langle C\ A\ B\ A\ B\ C\ SumA\ A\ C\ B \rangle$ *suma-right-comm*)
moreover
 have $\neg\ SAMS\ B\ C\ A\ B\ C\ A$
 using *assms(4) obtuse--nsams sams-comm* **by** *blast*
ultimately
show *?thesis*
 using *t22-14--oah* **by** *blast*
qed

lemma *t22-17--oah-2*:
assumes $\neg\ Col\ A\ B\ C$ **and**
 $M\ Midpoint\ A\ B$ **and**
 $Cong\ M\ A\ M\ C$ **and**

hypothesis-of-obtuse-saccheri-quadrilaterals
shows *Obtuse A C B*
proof –
 have $C \neq B$
 using *assms(1) not-col-distincts* **by** *blast*
 have $A \neq C$
 using *assms(1) col-trivial-3* **by** *auto*
 have $C A B A B C$ *SumA A C B*
 using *assms(1) assms(2) assms(3) cong-mid--suma* **by** *blast*
 hence $\neg \text{SAMS } B C A B C A$
 by (*meson assms(1) assms(4) sams-left-comm t22-14--nsams*)
 hence *Obtuse B C A*
 using $\langle A \neq C \rangle \langle C \neq B \rangle$ *nsams--obtuse* **by** *auto*
 thus *?thesis*
 using *obtuse-sym* **by** *blast*
qed

lemma *t22-17--oah*:
 assumes $\neg \text{Col } A B C$ **and**
 M Midpoint A B **and**
 Cong M A M C
shows *Obtuse A C B* $\longleftrightarrow$ *hypothesis-of-obtuse-saccheri-quadrilaterals*
 by (*meson t22-17--oah-1 t22-17--oah-2 assms(1) assms(2) assms(3)*)

lemma *t22-17--aah-1*:
 assumes $\neg \text{Col } A B C$ **and**
 M Midpoint A B **and**
 Cong M A M C **and**
 Acute A C B
shows *hypothesis-of-acute-saccheri-quadrilaterals*
proof –
 {
 assume *Per A C B*
 hence $A C B$ *LtA A C B*
 using *acute-not-per assms(4)* **by** *auto*
 hence *False*
 by (*simp add: nlta*)
 }
moreover
 {
 assume *Obtuse A C B*
 hence $A C B$ *LtA A C B*
 using *acute--not-obtuse assms(4)* **by** *blast*
 hence *False*
 by (*simp add: nlta*)
 }
ultimately
show *?thesis*
 using *assms(1) assms(2) assms(3) saccheri-s-three-hypotheses*
 t22-17--oah-2 t22-17--rah **by** *blast*
qed

lemma *t22-17--aah-2*:
 assumes $\neg \text{Col } A B C$ **and**
 M Midpoint A B **and**
 Cong M A M C **and**
 hypothesis-of-acute-saccheri-quadrilaterals
shows *Acute A C B*
proof –
 {
 assume *Per A C B*
 hence *hypothesis-of-right-saccheri-quadrilaterals*
 using *assms(1) assms(2) assms(3) t22-17--rah-1* **by** *blast*
 hence *False*
 using *assms(4) not-aah* **by** *auto*
 }


```

}
moreover
{
  assume Obtuse A C B
  hence hypothesis-of-obtuse-saccheri-quadrilaterals
    using assms(1) assms(2) assms(3) t22-17--oah-1 by blast
  hence False
    using assms(4) not-oah by blast
}
ultimately
show ?thesis
  by (metis angle-partition assms(1) not-col-distincts)
qed

```

```

lemma t22-17--aah:
  assumes  $\neg$  Col A B C and
    M Midpoint A B and
    Cong M A M C
  shows Acute A C B  $\longleftrightarrow$  hypothesis-of-acute-saccheri-quadrilaterals
  by (meson t22-17--aah-1 t22-17--aah-2 assms(1) assms(2) assms(3))

```

```

lemma t22-20:
  assumes  $\neg$  hypothesis-of-obtuse-saccheri-quadrilaterals and
    A B C B C A SumA D E F
  shows SAMS D E F C A B
  using assms(1) assms(2) t22-14--oah by blast

```

```

lemma absolute-exterior-angle-theorem:
  assumes  $\neg$  hypothesis-of-obtuse-saccheri-quadrilaterals and
    Bet B A B' and
    A  $\neq$  B' and
    A B C B C A SumA D E F
  shows D E F LeA C A B'
proof –
  have SAMS D E F C A B
    using t22-20 by (simp add: assms(1) assms(4))
  thus ?thesis
    by (metis assms(2) assms(3) lea-left-comm sams-chara sams-comm
      sams-distincts sams-sym)
qed

```

```

lemma defect-distincts:
  assumes Defect A B C D E F
  shows A  $\neq$  B  $\wedge$  B  $\neq$  C  $\wedge$  A  $\neq$  C  $\wedge$  D  $\neq$  E  $\wedge$  E  $\neq$  F
proof –
  obtain G H I where A B C TriSumA G H I  $\wedge$  G H I SuppA D E F
    using Defect-def assms by presburger
  thus ?thesis
    by (meson suppa-distincts trisuma-distincts)
qed

```

```

lemma ex-defect:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    A  $\neq$  C
  shows  $\exists$  D E F. Defect A B C D E F
proof –
  obtain G H I where A B C TriSumA G H I
    using assms(1) assms(2) assms(3) ex-trisuma by presburger
  have G  $\neq$  H
    using  $\langle A B C TriSumA G H I \rangle$  trisuma-distincts by blast
  have H  $\neq$  I
    using  $\langle A B C TriSumA G H I \rangle$  trisuma-distincts by presburger
  obtain D E F where G H I SuppA D E F
    using ex-suppa  $\langle A B C TriSumA G H I \rangle$  trisuma-distincts by presburger
  thus ?thesis

```

```

    using Defect-def ⟨A B C TriSumA G H I⟩ by blast
qed

lemma conga-defect--defect:
  assumes Defect A B C D E F and
    D E F CongA D' E' F'
  shows Defect A B C D' E' F'
proof -
  obtain G H I where A B C TriSumA G H I ∧ G H I SuppA D E F
    using Defect-def assms(1) by blast
  have G H I CongA G H I
  proof -
    have G ≠ H
      using ⟨A B C TriSumA G H I ∧ G H I SuppA D E F⟩ trisuma-distincts by blast
    moreover
    have H ≠ I
      using ⟨A B C TriSumA G H I ∧ G H I SuppA D E F⟩ trisuma-distincts by blast
    ultimately
    show ?thesis
      using conga-pseudo-refl conga-right-comm by presburger
  qed
  thus ?thesis
    by (meson Defect-def conga2-suppa--suppa assms(2)
      ⟨A B C TriSumA G H I ∧ G H I SuppA D E F⟩)
qed

lemma defect2--conga:
  assumes Defect A B C D E F and
    Defect A B C D' E' F'
  shows D E F CongA D' E' F'
proof -
  obtain G H I where A B C TriSumA G H I ∧ G H I SuppA D E F
    using Defect-def assms(1) by blast
  obtain G' H' I' where A B C TriSumA G' H' I' ∧ G' H' I' SuppA D' E' F'
    using Defect-def assms(2) by blast
  have G H I SuppA D' E' F'
    by (meson conga2-suppa--suppa ⟨A B C TriSumA G H I ∧ G H I SuppA D E F⟩
      ⟨∧thesis. (∧G' H' I'.
        A B C TriSumA G' H' I' ∧ G' H' I' SuppA D' E' F' ⇒ thesis) ⇒ thesis⟩
        suppa2--conga456 trisuma2--conga)
  thus ?thesis
    using ⟨A B C TriSumA G H I ∧ G H I SuppA D E F⟩ suppa2--conga456 by blast
qed

lemma defect-perm-231:
  assumes Defect A B C D E F
  shows Defect B C A D E F
proof -
  obtain G H I where A B C TriSumA G H I ∧ G H I SuppA D E F
    using Defect-def assms(1) by blast
  thus ?thesis
    by (meson Defect-def trisuma-perm-231)
qed

lemma defect-perm-312:
  assumes Defect A B C D E F
  shows Defect C A B D E F
  by (simp add: assms defect-perm-231)

lemma defect-perm-321:
  assumes Defect A B C D E F
  shows Defect C B A D E F
proof -
  obtain G H I where A B C TriSumA G H I ∧ G H I SuppA D E F
    using Defect-def assms(1) by blast
  thus ?thesis

```

using *Defect-def trisuma-perm-321* by blast
 qed

lemma *defect-perm-213*:
 assumes *Defect A B C D E F*
 shows *Defect B A C D E F*
 by (meson assms *defect-perm-231 defect-perm-321*)

lemma *defect-perm-132*:
 assumes *Defect A B C D E F*
 shows *Defect A C B D E F*
 by (meson assms *defect-perm-312 defect-perm-321*)

lemma *conga3-defect--defect*:
 assumes *Defect A B C D E F* and
 A B C CongA A' B' C' and
 B C A CongA B' C' A' and
 C A B CongA C' A' B'
 shows *Defect A' B' C' D E F*
 proof –
 obtain *G H I* where *A B C TriSumA G H I* $\wedge$ *G H I SuppA D E F*
 using *Defect-def assms(1)* by blast
 thus ?thesis
 using *Defect-def assms(2) assms(3) assms(4) conga3-trisuma--trisuma* by blast
 qed

lemma *col-defect--out*:
 assumes *Col A B C* and
 Defect A B C D E F
 shows *E Out D F*
 proof –
 obtain *G H I* where *A B C TriSumA G H I* $\wedge$ *G H I SuppA D E F*
 using *Defect-def assms(2)* by blast
 thus ?thesis
 using *assms(1) bet-suppa--out col-trisuma--bet* by blast
 qed

lemma *rah-defect--out*:
 assumes *hypothesis-of-right-saccheri-quadrilaterals* and
 Defect A B C D E F
 shows *E Out D F*
 proof –
 obtain *G H I* where *A B C TriSumA G H I* $\wedge$ *G H I SuppA D E F*
 using *Defect-def assms(2)* by blast
 thus ?thesis
 using *assms(1) bet-suppa--out t22-14--bet* by blast
 qed

lemma *defect-ncol-out--rah*:
 assumes $\neg$ *Col A B C* and
 Defect A B C D E F and
 E Out D F
 shows *hypothesis-of-right-saccheri-quadrilaterals*
 proof –
 obtain *G H I* where *A B C TriSumA G H I* $\wedge$ *G H I SuppA D E F*
 using *Defect-def assms(2)* by blast
 thus ?thesis
 by (meson assms(1) assms(3) *out-suppa--bet suppa-sym t22-14--rah*)
 qed

lemma *t22-16-1*:

assumes $\neg$ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
Bet A C1 C **and**
Defect A B C1 D E F **and**
Defect B C1 C G H I **and**
D E F G H I SumA K L M
shows *SAMS D E F G H I* $\wedge$ *Defect A B C K L M*
proof –
have $A \neq B$
using *assms(3) defect-distincts* **by** *blast*
have $C1 \neq B$
using *assms(3) defect-distincts* **by** *blast*
have $A \neq C1$
using *assms(3) defect-distincts* **by** *blast*
have $D \neq E$
using *assms(3) defect-distincts* **by** *presburger*
have $F \neq E$
using *assms(3) defect-distincts* **by** *blast*
have $C1 \neq C$
using *assms(4) defect-distincts* **by** *blast*
have $C \neq B$
using *assms(4) defect-distincts* **by** *blast*
have $G \neq H$
using *assms(5) suma-distincts* **by** *blast*
have $I \neq H$
using *assms(5) suma-distincts* **by** *blast*
have *Defect C1 B A D E F*
by (*simp add: assms(3) defect-perm-321*)
obtain $P Q R$ **where** $C1 B A \text{ TriSumA } P Q R \wedge P Q R \text{ SuppA } D E F$
using *Defect-def* $\langle \text{Defect } C1 B A D E F \rangle$ **by** *presburger*
have $P Q R D E F \text{ SumA } A C1 C$
by (*simp add:* $\langle A \neq C1 \rangle \langle C1 B A \text{ TriSumA } P Q R \wedge P Q R \text{ SuppA } D E F \rangle$
 $\langle C1 \neq C \rangle$ *assms(2) bet-suppa--suma*)
obtain $S T U$ **where** 1: $C1 B A B A C1 \text{ SumA } S T U \wedge S T U A C1 B \text{ SumA } P Q R$
using *TriSumA-def* $\langle C1 B A \text{ TriSumA } P Q R \wedge P Q R \text{ SuppA } D E F \rangle$
by *blast*
have $S T U D E F \text{ SumA } B C1 C \wedge \text{SAMS } S T U D E F$
proof –
have $B C1 C A C1 B \text{ SumA } A C1 C$
by (*metis* $\langle A \neq C1 \rangle \langle C1 \neq B \rangle \langle C1 \neq C \rangle$ *assms(2) bet--suma*
between-symmetry suma-comm)
obtain $V W X$ **where** $A C1 B D E F \text{ SumA } V W X$
using *ex-suma* $\langle D \neq E \rangle \langle F \neq E \rangle \langle A \neq C1 \rangle \langle C1 \neq B \rangle$ **by** *presburger*
have $P Q R \text{ SuppA } D E F$
using $\langle C1 B A \text{ TriSumA } P Q R \wedge P Q R \text{ SuppA } D E F \rangle$ **by** *blast*
hence *SAMS P Q R D E F*
by (*simp add: suppa--sams*)
have $C1 B A B A C1 \text{ SumA } S T U$
using $\langle C1 B A B A C1 \text{ SumA } S T U \wedge S T U A C1 B \text{ SumA } P Q R \rangle$ **by** *blast*
hence *SAMS S T U A C1 B*
by (*simp add: assms(1) t22-20*)
have *SAMS A C1 B D E F*
proof –
have $A C1 B \text{ LeA } P Q R$
using $\langle C1 B A B A C1 \text{ SumA } S T U \wedge S T U A C1 B \text{ SumA } P Q R \rangle$
 $\langle \text{SAMS } S T U A C1 B \rangle$ *sams-suma--lea456789* **by** *blast*
moreover
have $D E F \text{ LeA } D E F$
by (*simp add:* $\langle D \neq E \rangle \langle F \neq E \rangle$ *lea-refl*)
ultimately
show *?thesis*
using *sams-lea2--sams* $\langle \text{SAMS } P Q R D E F \rangle$ **by** *blast*
qed
have $S T U V W X \text{ SumA } A C1 C$
by (*meson* $\langle A C1 B D E F \text{ SumA } V W X \rangle$
 $\langle C1 B A B A C1 \text{ SumA } S T U \wedge S T U A C1 B \text{ SumA } P Q R \rangle$
 $\langle P Q R D E F \text{ SumA } A C1 C \rangle \langle \text{SAMS } A C1 B D E F \rangle$)

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      ⟨SAMS S T U A C1 B⟩ suma-assoc-1)
have SAMS S T U D E F
proof –
  have S T U LeA P Q R
    using sams-suma--lea123789 ⟨SAMS S T U A C1 B⟩
      ⟨C1 B A B A C1 SumA S T U ∧ S T U A C1 B SumA P Q R⟩
    by blast
  moreover
  have D E F LeA D E F
    by (simp add: ⟨D ≠ E⟩ ⟨F ≠ E⟩ lea-refl)
  ultimately
  show ?thesis
    using sams-lea2--sams ⟨SAMS P Q R D E F⟩ by blast
qed
moreover
have S T U D E F SumA B C1 C
proof –
  obtain B' C1' C' where S T U D E F SumA B' C1' C'
    using ex-suma calculation sams-distincts by presburger
  have B' C1' C' A C1 B SumA A C1 C
  proof –
    have SAMS S T U D E F
      using calculation by auto
    moreover
    have SAMS D E F A C1 B
      by (simp add: ⟨SAMS A C1 B D E F⟩ sams-sym)
    moreover
    have D E F A C1 B SumA V W X
      by (simp add: ⟨A C1 B D E F SumA V W X⟩ suma-sym)
    ultimately
    show ?thesis
      by (meson suma-assoc-2 ⟨S T U V W X SumA A C1 C⟩
        ⟨S T U D E F SumA B' C1' C'⟩)
  qed
  have B' C1' C' CongA B C1 C
    by (meson ⟨B C1 C A C1 B SumA A C1 C⟩
      ⟨B' C1' C' A C1 B SumA A C1 C⟩ assms(2)
      bet-suma--sams sams2-suma2--conga123)
  thus ?thesis
    by (meson ⟨S T U D E F SumA B' C1' C'⟩ calculation
      conga3-suma--suma sams2-suma2--conga123 sams2-suma2--conga456)
qed
ultimately
show ?thesis
  by auto
qed
hence S T U D E F SumA B C1 C
  by blast
have SAMS S T U D E F
  using ⟨S T U D E F SumA B C1 C ∧ SAMS S T U D E F⟩ by blast
have Defect C1 C B G H I
  by (simp add: assms(4) defect-perm-231)
obtain P Q R where C1 C B TriSumA P Q R ∧ P Q R SuppA G H I
  using Defect-def ⟨Defect C1 C B G H I⟩ by blast
obtain V W X where C1 C B C B C1 SumA V W X ∧ V W X B C1 C SumA P Q R
  using ⟨C1 C B TriSumA P Q R ∧ P Q R SuppA G H I⟩ TriSumA-def by blast
have P Q R G H I SumA A C1 C
  by (simp add: ⟨A ≠ C1⟩ ⟨C1 C B TriSumA P Q R ∧ P Q R SuppA G H I⟩
    ⟨C1 ≠ C⟩ assms(2) bet-suppa--suma)
hence SAMS P Q R G H I
  by (meson bet-suma--sams assms(2))
have C1 C B C B C1 SumA V W X
  by (simp add: ⟨C1 C B C B C1 SumA V W X ∧ V W X B C1 C SumA P Q R⟩)
hence SAMS V W X B C1 C
  using assms(1) t22-14--oah by blast
have SAMS D E F G H I

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proof –
  have  $D \ E \ F \ LeA \ P \ Q \ R$ 
proof –
  have  $D \ E \ F \ LeA \ B \ C1 \ C$ 
  using sams-suma--lea456789  $\langle S \ T \ U \ D \ E \ F \ SumA \ B \ C1 \ C \rangle$ 
     $\langle SAMS \ S \ T \ U \ D \ E \ F \rangle$  by blast
  moreover
  have  $B \ C1 \ C \ LeA \ P \ Q \ R$ 
  using sams-suma--lea456789
     $\langle C1 \ C \ B \ C \ B \ C1 \ SumA \ V \ W \ X \wedge V \ W \ X \ B \ C1 \ C \ SumA \ P \ Q \ R \rangle$ 
     $\langle SAMS \ V \ W \ X \ B \ C1 \ C \rangle$  by blast
  ultimately
  show ?thesis
    by (meson lea-trans)
qed
moreover
have  $G \ H \ I \ LeA \ G \ H \ I$ 
  by (simp add:  $\langle G \neq H \rangle \langle I \neq H \rangle$  lea-refl)
ultimately
show ?thesis
  using sams-lea2--sams  $\langle SAMS \ P \ Q \ R \ G \ H \ I \rangle$  by blast
qed
have  $V \neq W$ 
  using  $\langle SAMS \ V \ W \ X \ B \ C1 \ C \rangle$  sams-distincts by blast
have  $W \neq X$ 
  using  $\langle SAMS \ V \ W \ X \ B \ C1 \ C \rangle$  sams-distincts by blast
have  $S \neq T$ 
  using  $\langle SAMS \ S \ T \ U \ D \ E \ F \rangle$  sams-distincts by blast
have  $T \neq U$ 
  using  $\langle SAMS \ S \ T \ U \ D \ E \ F \rangle$  sams-distincts by blast
obtain  $A' \ B' \ C'$  where  $V \ W \ X \ S \ T \ U \ SumA \ A' \ B' \ C'$ 
  using ex-suma  $\langle V \neq W \rangle \langle W \neq X \rangle \langle S \neq T \rangle \langle T \neq U \rangle$  by blast
have  $SAMS \ V \ W \ X \ S \ T \ U$ 
proof –
  have  $V \ W \ X \ LeA \ V \ W \ X$ 
  using  $\langle V \neq W \rangle \langle W \neq X \rangle$  lea-refl by auto
  moreover
  have  $S \ T \ U \ LeA \ B \ C1 \ C$ 
  using sams-suma--lea123789  $\langle S \ T \ U \ D \ E \ F \ SumA \ B \ C1 \ C \rangle$ 
     $\langle SAMS \ S \ T \ U \ D \ E \ F \rangle$  by blast
  ultimately
  show ?thesis
    by (meson sams-lea2--sams  $\langle SAMS \ V \ W \ X \ B \ C1 \ C \rangle$ )
qed
have  $A' \ B' \ C' \ D \ E \ F \ SumA \ P \ Q \ R$ 
proof –
  have  $SAMS \ V \ W \ X \ S \ T \ U$ 
  by (simp add:  $\langle SAMS \ V \ W \ X \ S \ T \ U \rangle$ )
  moreover
  have  $SAMS \ S \ T \ U \ D \ E \ F$ 
  using  $\langle SAMS \ S \ T \ U \ D \ E \ F \rangle$  by auto
  moreover
  have  $V \ W \ X \ S \ T \ U \ SumA \ A' \ B' \ C'$ 
  by (simp add:  $\langle V \ W \ X \ S \ T \ U \ SumA \ A' \ B' \ C' \rangle$ )
  moreover
  have  $S \ T \ U \ D \ E \ F \ SumA \ B \ C1 \ C$ 
  by (simp add:  $\langle S \ T \ U \ D \ E \ F \ SumA \ B \ C1 \ C \rangle$ )
  moreover
  have  $V \ W \ X \ B \ C1 \ C \ SumA \ P \ Q \ R$ 
  by (simp add:  $\langle C1 \ C \ B \ C \ B \ C1 \ SumA \ V \ W \ X \wedge V \ W \ X \ B \ C1 \ C \ SumA \ P \ Q \ R \rangle$ )
  ultimately
  show ?thesis
    using suma-assoc by blast
qed
have  $SAMS \ A' \ B' \ C' \ D \ E \ F$ 
  using sams-assoc  $\langle SAMS \ V \ W \ X \ S \ T \ U \rangle \langle SAMS \ S \ T \ U \ D \ E \ F \rangle$ 

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    ⟨V W X S T U SumA A' B' C'⟩ ⟨S T U D E F SumA B C1 C⟩
    ⟨SAMS V W X B C1 C⟩ by blast
hence A' B' C' K L M SumA A C1 C
  using suma-assoc assms(5) ⟨SAMS D E F G H I⟩
  ⟨A' B' C' D E F SumA P Q R⟩ ⟨P Q R G H I SumA A C1 C⟩ by blast
hence A' B' C' SuppA K L M
  using bet-suma--suppa assms(2) by blast
moreover
have A B C TriSumA A' B' C'
proof -
  have X ≠ W
    using ⟨W ≠ X⟩ by blast
  then obtain D E F where V W X C1 B A SumA D E F
    using ex-suma ⟨V ≠ W⟩ ⟨C1 ≠ B⟩ ⟨A ≠ B⟩ by presburger
  have SAMS V W X C1 B A
  proof -
    have V W X LeA V W X
      by (simp add: ⟨V ≠ W⟩ ⟨X ≠ W⟩ lea-refl)
    moreover
    have C1 B A LeA S T U
  proof -
    have C1 B A B A C1 SumA S T U
      using 1 by blast
    moreover
    have SAMS C1 B A B A C1
      using ⟨A ≠ B⟩ ⟨A ≠ C1⟩ ⟨C1 ≠ B⟩ sams123231 by auto
    ultimately
    show ?thesis
      using sams-suma--lea123789 by blast
  qed
  ultimately
  show ?thesis
    using sams-lea2--sams ⟨SAMS V W X S T U⟩ by blast
  qed
have A C B TriSumA A' B' C'
proof -
  have A C B C B A SumA D E F
  proof -
    have C1 C B CongA A C B
  proof -
    have C Out A C1
      by (simp add: ⟨C1 ≠ C⟩ assms(2) bet-out-1 l6-6)
    moreover
    have C Out B B
      using ⟨C ≠ B⟩ out-trivial by auto
    ultimately
    show ?thesis
      using out2--conga by blast
  qed
  have C1 C B C B A SumA D E F
  proof -
    have SAMS C1 C B C B C1
      by (simp add: ⟨C ≠ B⟩ ⟨C1 ≠ B⟩ ⟨C1 ≠ C⟩ sams123231)
    moreover
    have C1 InAngle C B A
      by (meson Bet-cases InAngle-def ⟨A ≠ B⟩ ⟨C ≠ B⟩
        ⟨C1 ≠ B⟩ assms(2) out-trivial)
    hence SAMS C B C1 C1 B A
      by (simp add: inangle--sams)
    moreover
    have C B C1 C1 B A SumA C B A
      by (simp add: ⟨C1 InAngle C B A⟩ inangle--suma)
    ultimately
    show ?thesis
      by (meson suma-assoc-1 ⟨V W X C1 B A SumA D E F⟩
        ⟨C1 C B C B C1 SumA V W X⟩)

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qed
moreover
have  $C B A \text{ CongA } C B A$ 
  by (simp add:  $\langle A \neq B \rangle \langle C \neq B \rangle \text{ conga-refl}$ )
moreover
have  $D \neq E$ 
  using  $\langle V W X C1 B A \text{ SumA } D E F \rangle \text{ suma-distincts}$  by presburger
have  $E \neq F$ 
  using  $\langle V W X C1 B A \text{ SumA } D E F \rangle \text{ suma-distincts}$  by presburger
have  $D E F \text{ CongA } D E F$ 
  using  $\langle D \neq E \rangle \langle E \neq F \rangle \text{ lea-asymp lea-refl}$  by auto
ultimately
show ?thesis
  by (meson conga3-suma--suma  $\langle C1 C B \text{ CongA } A C B \rangle$ )
qed
moreover
have  $D E F B A C \text{ SumA } A' B' C'$ 
proof -
  have  $B A C \text{ CongA } B A C1$ 
  proof -
    have  $A \text{ Out } B B$ 
    using  $\langle A \neq B \rangle \text{ bet-out-1 between-trivial2}$  by auto
    moreover
    have  $A \text{ Out } C1 C$ 
    using  $\langle A \neq C1 \rangle \text{ assms}(2) \text{ bet-out}$  by auto
    ultimately
    show ?thesis
      by (simp add:  $\text{out2--conga Tarski-neutral-dimensionless-axioms}$ )
  qed
  have  $D E F B A C1 \text{ SumA } A' B' C'$ 
  proof -
    have  $SAMS C1 B A B A C1$ 
    using  $\langle A \neq B \rangle \langle A \neq C1 \rangle \langle C1 \neq B \rangle \text{ sams123231}$  by auto
    moreover
    have  $C1 B A B A C1 \text{ SumA } S T U$ 
    using 1 by auto
    ultimately
    show ?thesis
      by (meson suma-assoc-2  $\langle V W X C1 B A \text{ SumA } D E F \rangle$ 
         $\langle SAMS V W X C1 B A \rangle \langle V W X S T U \text{ SumA } A' B' C' \rangle$ )
  qed
  moreover
  have  $D \neq E$ 
    using  $\langle A C B C B A \text{ SumA } D E F \rangle \text{ suma-distincts}$  by presburger
  have  $E \neq F$ 
    using  $\langle A C B C B A \text{ SumA } D E F \rangle \text{ suma-distincts}$  by presburger
  have  $D E F \text{ CongA } D E F$ 
    using  $\langle D \neq E \rangle \langle E \neq F \rangle \text{ lea-asymp lea-refl}$  by auto
  moreover
  have  $B A C1 \text{ CongA } B A C$ 
    by (simp add:  $\langle B A C \text{ CongA } B A C1 \rangle \text{ conga-sym-equiv}$ )
  moreover
  have  $A' \neq B'$ 
    using  $\langle A' B' C' \text{ SuppA } K L M \rangle \text{ suppa-distincts}$  by blast
  have  $B' \neq C'$ 
    using  $\langle A' B' C' \text{ SuppA } K L M \rangle \text{ suppa-distincts}$  by blast
  have  $A' B' C' \text{ CongA } A' B' C'$ 
    using  $\langle A' \neq B' \rangle \langle B' \neq C' \rangle \text{ conga-refl}$  by auto
  ultimately
  show ?thesis
    by (meson conga3-suma--suma)
qed
ultimately
show ?thesis
  using  $\text{TriSumA-def}$  by blast
qed

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thus ?thesis
 by (simp add: trisuma-perm-132)
 qed
 ultimately
 show ?thesis
 using Defect-def $\langle \text{SAMS } D \ E \ F \ G \ H \ I \rangle$ by blast
 qed

lemma t22-16-1bis:
 assumes $\neg$ hypothesis-of-obtuse-saccheri-quadrilaterals and
 Bet A C1 C and
 Defect A B C1 D E F and
 Defect B C1 C G H I and
 Defect A B C K L M
 shows $\text{SAMS } D \ E \ F \ G \ H \ I \wedge D \ E \ F \ G \ H \ I \text{ SumA } K \ L \ M$
 proof –
 have $D \neq E$
 using assms(3) defect-distincts by presburger
 have $E \neq F$
 using assms(3) defect-distincts by blast
 have $C1 \neq C$
 using assms(4) defect-distincts by blast
 have $C \neq B$
 using assms(4) defect-distincts by blast
 have $G \neq H$
 using assms(4) defect-distincts by blast
 have $H \neq I$
 using assms(4) defect-distincts by blast
 obtain $K' \ L' \ M'$ where $D \ E \ F \ G \ H \ I \text{ SumA } K' \ L' \ M'$
 using ex-suma $\langle D \neq E \rangle \langle E \neq F \rangle \langle G \neq H \rangle \langle H \neq I \rangle$ by presburger
 hence $\text{SAMS } D \ E \ F \ G \ H \ I \wedge \text{Defect } A \ B \ C \ K' \ L' \ M'$
 using assms(1) assms(2) assms(3) assms(4) t22-16-1 by blast
 hence $\text{SAMS } D \ E \ F \ G \ H \ I$
 by auto
 moreover
 have $D \ E \ F \ G \ H \ I \text{ SumA } K \ L \ M$
 proof –
 have $D \ E \ F \text{ CongA } D \ E \ F$
 using $\langle D \neq E \rangle \langle E \neq F \rangle$ conga-refl by auto
 moreover
 have $G \ H \ I \text{ CongA } G \ H \ I$
 using $\langle G \neq H \rangle \langle H \neq I \rangle$ conga-refl by auto
 moreover
 have $K' \ L' \ M' \text{ CongA } K \ L \ M$
 by (meson defect2--conga assms(5) $\langle \text{SAMS } D \ E \ F \ G \ H \ I \wedge \text{Defect } A \ B \ C \ K' \ L' \ M' \rangle$)
 ultimately
 show ?thesis
 by (meson conga3-suma--suma $\langle D \ E \ F \ G \ H \ I \text{ SumA } K' \ L' \ M' \rangle$)
 qed
 ultimately
 show ?thesis
 by auto
 qed

lemma t22-16-2aux:
 assumes $\neg$ hypothesis-of-obtuse-saccheri-quadrilaterals and
 Defect A B C D1 D2 D3 and
 Defect A B D C1 C2 C3 and
 Defect A D C B1 B2 B3 and
 Defect C B D A1 A2 A3 and
 Bet A PO C and
 Bet B PO D and
 Col A B C and
 D1 D2 D3 B1 B2 B3 SumA P Q R
 shows $\text{SAMS } C1 \ C2 \ C3 \ A1 \ A2 \ A3 \wedge C1 \ C2 \ C3 \ A1 \ A2 \ A3 \text{ SumA } P \ Q \ R$
 proof –

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have B ≠ D
  using assms(3) defect-distincts by auto
have C1 ≠ C2
  using assms(3) defect-distincts by blast
have C2 ≠ C3
  using assms(3) defect-distincts by auto
have A1 ≠ A2
  using assms(5) defect-distincts by blast
have A2 ≠ A3
  using assms(5) defect-distincts by blast
have A ≠ C
  using assms(2) defect-distincts by blast
have P ≠ Q
  using assms(9) suma-distincts by blast
have Q ≠ R
  using assms(9) suma-distincts by blast
have D2 Out D1 D3
  using assms(2) assms(8) col-defect--out by blast
thus ?thesis
proof (cases Col A D C)
  case True
  have Col C B D
    by (metis True ⟨A ≠ C⟩ assms(8) col-permutation-4
      col-permutation-5 col-transitivity-1)
  hence A2 Out A1 A3
    using assms(5) col-defect--out by blast
  have Col A B D
    by (meson True ⟨A ≠ C⟩ assms(8) col-permutation-5 col-transitivity-1)
  hence B2 Out B1 B3
    using True assms(4) col-defect--out by blast
  have C2 Out C1 C3
    using assms(3) by (meson True ⟨A ≠ C⟩ assms(8)
      col-defect--out col-permutation-5 col-transitivity-1)
  have SAMS C1 C2 C3 A1 A2 A3
    by (simp add: ⟨A2 Out A1 A3⟩ ⟨C1 ≠ C2⟩ ⟨C2 ≠ C3⟩ out546--sams)
  moreover
  have C1 C2 C3 A1 A2 A3 SumA P Q R
  proof –
    have D1 D2 D3 CongA C1 C2 C3
      by (simp add: ⟨C2 Out C1 C3⟩ ⟨D2 Out D1 D3⟩ l11-21-b)
    moreover
    have B1 B2 B3 CongA A1 A2 A3
      by (simp add: ⟨A2 Out A1 A3⟩ ⟨B2 Out B1 B3⟩ l11-21-b)
    moreover
    have P Q R CongA P Q R
      using ⟨P ≠ Q⟩ ⟨Q ≠ R⟩ conga-refl by auto
    ultimately
    show ?thesis
      by (meson conga3-suma--suma assms(9))
  qed
  ultimately
  show ?thesis
    by auto
next
case False
have B = PO
proof –
  have ¬ Col A C D
    using False col-permutation-5 by blast
  moreover
  have D ≠ B
    using ⟨B ≠ D⟩ by auto
  moreover
  have Col A C B
    by (simp add: assms(8) col-permutation-5)
  moreover

```

```

have Col A C PO
  using Col-perm assms(6) bet-col by blast
moreover
have Col D B B
  by (simp add: col-trivial-2)
moreover
have Col D B PO
  using Col-def assms(7) by auto
ultimately
show ?thesis
  using l6-21 by blast
qed
have B1 B2 B3 CongA P Q R
  using assms(9) out213-suma--conga ⟨D2 Out D1 D3⟩ by blast
have SAMS C1 C2 C3 A1 A2 A3 ∧ C1 C2 C3 A1 A2 A3 SumA B1 B2 B3
proof –
  have Defect A D B C1 C2 C3
    using assms(3) defect-perm-132 by blast
  moreover
  have Defect D B C A1 A2 A3
    using assms(5) defect-perm-321 by blast
  ultimately
  show ?thesis
    using ⟨B = PO⟩ assms(1) assms(4) assms(6) t22-16-1bis by blast
qed
hence SAMS C1 C2 C3 A1 A2 A3
  by blast
moreover
have C1 C2 C3 A1 A2 A3 SumA P Q R
proof –
  have C1 C2 C3 A1 A2 A3 SumA B1 B2 B3
    using ⟨SAMS C1 C2 C3 A1 A2 A3 ∧ C1 C2 C3 A1 A2 A3 SumA B1 B2 B3⟩
    by blast
  moreover
  have C1 C2 C3 CongA C1 C2 C3
    using ⟨C1 ≠ C2⟩ ⟨C2 ≠ C3⟩ conga-refl by auto
  moreover
  have A1 A2 A3 CongA A1 A2 A3
    using ⟨A1 ≠ A2⟩ ⟨A2 ≠ A3⟩ conga-refl by auto
  moreover
  have B1 B2 B3 CongA P Q R
    using ⟨D2 Out D1 D3⟩ assms(9) out213-suma--conga by auto
  ultimately
  show ?thesis
    by (meson conga3-suma--suma)
qed
ultimately
show ?thesis
  by auto
qed
qed

lemma t22-16-2aux1:
  assumes ¬ hypothesis-of-obtuse-saccheri-quadrilaterals and
    Defect A B C D1 D2 D3 and
    Defect A B D C1 C2 C3 and
    Defect A D C B1 B2 B3 and
    Defect C B D A1 A2 A3 and
    Bet A PO C and
    Bet B PO D and
    Col A B D and
    D1 D2 D3 B1 B2 B3 SumA P Q R
  shows SAMS C1 C2 C3 A1 A2 A3 ∧ C1 C2 C3 A1 A2 A3 SumA P Q R
proof –
  have B ≠ D
    using assms(3) defect-distincts by auto

```

```

have C1 ≠ C2
  using assms(3) defect-distincts by blast
have C2 ≠ C3
  using assms(3) defect-distincts by auto
have A1 ≠ A2
  using assms(5) defect-distincts by blast
have A2 ≠ A3
  using assms(5) defect-distincts by blast
have A ≠ C
  using assms(2) defect-distincts by blast
have P ≠ Q
  using assms(9) suma-distincts by blast
have Q ≠ R
  using assms(9) suma-distincts by blast
have C2 Out C1 C3
  using col-defect--out assms(3) assms(8) by blast
have SAMS C1 C2 C3 A1 A2 A3
  by (simp add: ⟨A1 ≠ A2⟩ ⟨A2 ≠ A3⟩ ⟨C2 Out C1 C3⟩ out213--sams)
moreover
have C1 C2 C3 A1 A2 A3 SumA P Q R
proof -
  have C1 C2 C3 A1 A2 A3 SumA A1 A2 A3
    using ⟨A1 ≠ A2⟩ ⟨A2 ≠ A3⟩ ⟨C2 Out C1 C3⟩ by (simp add: out213--suma)
  have D1 D2 D3 B1 B2 B3 SumA A1 A2 A3
  proof -
    have Defect B A D C1 C2 C3
      by (simp add: assms(3) defect-perm-213)
    moreover
    have Defect B A C D1 D2 D3
      using assms(2) defect-perm-213 by blast
    moreover
    have Defect B C D A1 A2 A3
      by (simp add: assms(5) defect-perm-213)
    moreover
    have Defect D A C B1 B2 B3
      using assms(4) defect-perm-213 by blast
    moreover
    have Col B A D
      using Col-perm assms(8) by blast
    ultimately
    show ?thesis
      using assms(1) t22-16-2aux assms(6) assms(7)
        ⟨C1 C2 C3 A1 A2 A3 SumA A1 A2 A3⟩ by blast
  qed
hence A1 A2 A3 CongA P Q R
  by (meson assms(9) suma2--conga)
thus ?thesis
  by (meson ⟨C1 C2 C3 A1 A2 A3 SumA A1 A2 A3⟩ calculation
    conga3-suma--suma sams2-suma2--conga123 suma2--conga)
qed
ultimately
show ?thesis
  by blast
qed

```

lemma t22-16-2:

```

assumes ¬ hypothesis-of-obtuse-saccheri-quadrilaterals and
  Defect A B C D1 D2 D3 and
  Defect A B D C1 C2 C3 and
  Defect A D C B1 B2 B3 and
  Defect C B D A1 A2 A3 and
  Bet A PO C and
  Bet B PO D and
  SAMS D1 D2 D3 B1 B2 B3 and
  D1 D2 D3 B1 B2 B3 SumA P Q R

```

```

shows SAMS C1 C2 C3 A1 A2 A3  $\wedge$  C1 C2 C3 A1 A2 A3 SumA P Q R
proof -
  have A  $\neq$  B
    using assms(3) defect-distincts by blast
  have B  $\neq$  D
    using assms(3) defect-distincts by auto
  have A  $\neq$  D
    using assms(3) defect-distincts by blast
  have C1  $\neq$  C2
    using assms(3) defect-distincts by blast
  have C2  $\neq$  C3
    using assms(3) defect-distincts by auto
  have C  $\neq$  B
    using assms(2) defect-distincts by blast
  have C  $\neq$  D
    using assms(4) defect-distincts by blast
  have A1  $\neq$  A2
    using assms(5) defect-distincts by blast
  have A2  $\neq$  A3
    using assms(5) defect-distincts by blast
  have A  $\neq$  C
    using assms(2) defect-distincts by blast
  have D1  $\neq$  D2
    using assms(8) sams-distincts by presburger
  have D2  $\neq$  D3
    using assms(8) sams-distincts by auto
  have P  $\neq$  Q
    using assms(9) suma-distincts by blast
  have Q  $\neq$  R
    using assms(9) suma-distincts by blast
  thus ?thesis
proof (cases Col A B C)
  case True
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
      assms(7) assms(9) t22-16-2aux by blast
next
  case False
  show ?thesis
  proof (cases Col A D C)
    case True
    have Bet D PO B
      by (simp add: assms(7) between-symmetry)
    moreover
    have Col A D C
      by (simp add: True)
    moreover
    have B1 B2 B3 D1 D2 D3 SumA P Q R
      by (simp add: assms(9) suma-sym)
    ultimately
    show ?thesis
      using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
        assms(7) assms(9) t22-16-2aux
        by (meson defect-perm-132)
  next
    case False
    thus ?thesis
  proof (cases Col A B D)
    case True
    thus ?thesis
      using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
        assms(7) assms(9) t22-16-2aux1 by blast
  next
    case False
    thus ?thesis
  proof (cases Col C B D)

```

```

case True
thus ?thesis
  using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
    assms(7) assms(9) t22-16-2aux1
  by (meson between-symmetry defect-perm-321 sams-sym suma-sym)
next
case False
have Col B PO D
  by (simp add: assms(7) bet-col)
have Col A PO C
  by (simp add: Col-def assms(6))
{
  assume PO = A
  hence Bet B A D
    using assms(7) by blast
  hence False
    using ⟨¬ Col A B D⟩ Bet-cases Col-def by blast
}
hence PO ≠ A
  by auto
{
  assume PO = B
  hence Bet A B C
    using assms(6) by blast
  hence False
    using ⟨¬ Col A B C⟩ bet-col by blast
}
hence PO ≠ B
  by auto
{
  assume PO = C
  hence False
    using False ⟨Col B PO D⟩ col-permutation-4 by blast
}
hence PO ≠ C
  by auto
{
  assume PO = D
  hence Bet A D C
    using assms(6) by auto
  hence False
    using ⟨¬ Col A D C⟩ bet-col by blast
}
hence PO ≠ D
  by auto
obtain S1 T1 U1 where Defect A B PO S1 T1 U1
  using ⟨A ≠ B⟩ ⟨PO ≠ A⟩ ⟨PO ≠ B⟩ ex-defect by presburger
obtain S2 T2 U2 where Defect B C PO S2 T2 U2
  using ⟨C ≠ B⟩ ⟨PO ≠ B⟩ ⟨PO ≠ C⟩ ex-defect by presburger
obtain S3 T3 U3 where Defect C D PO S3 T3 U3
  using ⟨C ≠ D⟩ ⟨PO ≠ C⟩ ⟨PO ≠ D⟩ ex-defect by presburger
obtain S4 T4 U4 where Defect A D PO S4 T4 U4
  using ⟨A ≠ D⟩ ⟨PO = A ⟹ False⟩
    ⟨PO = D ⟹ False⟩ ex-defect by presburger
have Defect B PO C S2 T2 U2
  by (simp add: ⟨Defect B C PO S2 T2 U2⟩ defect-perm-132)
hence SAMS S1 T1 U1 S2 T2 U2 ∧ S1 T1 U1 S2 T2 U2 SumA D1 D2 D3
  using assms(1) assms(2) assms(6)
    ⟨Defect A B PO S1 T1 U1⟩ t22-16-1bis by blast
have SAMS S4 T4 U4 S3 T3 U3 ∧ S4 T4 U4 S3 T3 U3 SumA B1 B2 B3
  using assms(1) assms(6) t22-16-1bis
  by (meson ⟨Defect A D PO S4 T4 U4⟩ ⟨Defect C D PO S3 T3 U3⟩
    assms(4) defect-perm-231)
have SAMS S1 T1 U1 S4 T4 U4 ∧ S1 T1 U1 S4 T4 U4 SumA C1 C2 C3
  using assms(1) t22-16-1bis
  by (meson ⟨Defect A B PO S1 T1 U1⟩ ⟨Defect A D PO S4 T4 U4⟩

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    assms(3) assms(7) defect-perm-132 defect-perm-213)
have SAMS S2 T2 U2 S3 T3 U3  $\wedge$  S2 T2 U2 S3 T3 U3 SumA A1 A2 A3
  using assms(1) t22-16-1bis
  by (meson  $\langle$ Defect B C PO S2 T2 U2 $\rangle$   $\langle$ Defect C D PO S3 T3 U3 $\rangle$ 
    assms(5) assms(7) defect-perm-132 defect-perm-213)
have S3  $\neq$  T3
  using  $\langle$ SAMS S2 T2 U2 S3 T3 U3  $\wedge$  S2 T2 U2 S3 T3 U3 SumA A1 A2 A3 $\rangle$ 
    sams-distincts by blast
have T3  $\neq$  U3
  using  $\langle$ SAMS S2 T2 U2 S3 T3 U3  $\wedge$  S2 T2 U2 S3 T3 U3 SumA A1 A2 A3 $\rangle$ 
    sams-distincts by blast
obtain V W X where D1 D2 D3 S3 T3 U3 SumA V W X
  using ex-suma  $\langle$ D1  $\neq$  D2 $\rangle$   $\langle$ D2  $\neq$  D3 $\rangle$   $\langle$ S3  $\neq$  T3 $\rangle$   $\langle$ T3  $\neq$  U3 $\rangle$  by blast
have SAMS D1 D2 D3 S3 T3 U3
proof -
  have SAMS D1 D2 D3 B1 B2 B3
    by (simp add: assms(8))
  moreover
  have D1 D2 D3 LeA D1 D2 D3
    using  $\langle$ D1  $\neq$  D2 $\rangle$   $\langle$ D2  $\neq$  D3 $\rangle$  lta--lea nlea--lta by blast
  moreover
  have S3 T3 U3 LeA B1 B2 B3
  proof -
    have S4 T4 U4 S3 T3 U3 SumA B1 B2 B3
      using  $\langle$ SAMS S4 T4 U4 S3 T3 U3  $\wedge$  S4 T4 U4 S3 T3 U3 SumA B1 B2 B3 $\rangle$ 
        by linarith
    moreover
    have SAMS S4 T4 U4 S3 T3 U3
      using  $\langle$ SAMS S4 T4 U4 S3 T3 U3  $\wedge$  S4 T4 U4 S3 T3 U3 SumA B1 B2 B3 $\rangle$ 
        by auto
    ultimately
    show ?thesis
      using sams-suma--lea456789 by blast
  qed
  ultimately
  show ?thesis
    using sams-lea2--sams by blast
qed
have V W X S4 T4 U4 SumA P Q R
proof -
  have SAMS S3 T3 U3 S4 T4 U4
    by (simp add:  $\langle$ SAMS S4 T4 U4 S3 T3 U3  $\wedge$  S4 T4 U4 S3 T3 U3 SumA B1 B2 B3 $\rangle$ 
      sams-sym)
  moreover
  have S3 T3 U3 S4 T4 U4 SumA B1 B2 B3
    by (simp add:  $\langle$ SAMS S4 T4 U4 S3 T3 U3  $\wedge$  S4 T4 U4 S3 T3 U3 SumA B1 B2 B3 $\rangle$ 
      suma-sym)
  ultimately
  show ?thesis
    using assms(9)  $\langle$ SAMS D1 D2 D3 S3 T3 U3 $\rangle$ 
       $\langle$ D1 D2 D3 S3 T3 U3 SumA V W X $\rangle$  suma-assoc-2 by blast
qed
have SAMS V W X S4 T4 U4
proof -
  have SAMS S3 T3 U3 S4 T4 U4
    by (simp add:  $\langle$ SAMS S4 T4 U4 S3 T3 U3  $\wedge$  S4 T4 U4 S3 T3 U3 SumA B1 B2 B3 $\rangle$ 
      sams-sym)
  moreover
  have S3 T3 U3 S4 T4 U4 SumA B1 B2 B3
    by (simp add:  $\langle$ SAMS S4 T4 U4 S3 T3 U3  $\wedge$  S4 T4 U4 S3 T3 U3 SumA B1 B2 B3 $\rangle$ 
      suma-sym)
  ultimately
  show ?thesis
    using assms(8)  $\langle$ SAMS D1 D2 D3 S3 T3 U3 $\rangle$   $\langle$ D1 D2 D3 S3 T3 U3 SumA V W X $\rangle$ 
      sams-assoc by blast
qed

```

```

have A1 A2 A3 S1 T1 U1 SumA V W X
proof -
  have SAMS S3 T3 U3 S2 T2 U2
    by (simp add: ⟨SAMS S2 T2 U2 S3 T3 U3 ∧ S2 T2 U2 S3 T3 U3 SumA A1 A2 A3⟩
        sams-sym)
  moreover
  have SAMS S2 T2 U2 S1 T1 U1
    using ⟨SAMS S1 T1 U1 S2 T2 U2 ∧ S1 T1 U1 S2 T2 U2 SumA D1 D2 D3⟩
    sams-sym by blast
  moreover
  have S3 T3 U3 S2 T2 U2 SumA A1 A2 A3
    by (simp add: ⟨SAMS S2 T2 U2 S3 T3 U3 ∧ S2 T2 U2 S3 T3 U3 SumA A1 A2 A3⟩
        suma-sym)
  moreover
  have S2 T2 U2 S1 T1 U1 SumA D1 D2 D3
    by (simp add: ⟨SAMS S1 T1 U1 S2 T2 U2 ∧ S1 T1 U1 S2 T2 U2 SumA D1 D2 D3⟩
        suma-sym)
  moreover
  have S3 T3 U3 D1 D2 D3 SumA V W X
    by (simp add: ⟨D1 D2 D3 S3 T3 U3 SumA V W X⟩ suma-sym)
  ultimately
  show ?thesis
    by (meson suma-assoc-2)
qed
have SAMS A1 A2 A3 S1 T1 U1
proof -
  have SAMS S3 T3 U3 S2 T2 U2
    by (simp add: ⟨SAMS S2 T2 U2 S3 T3 U3 ∧ S2 T2 U2 S3 T3 U3 SumA A1 A2 A3⟩
        sams-sym)
  moreover
  have SAMS S2 T2 U2 S1 T1 U1
    using ⟨SAMS S1 T1 U1 S2 T2 U2 ∧ S1 T1 U1 S2 T2 U2 SumA D1 D2 D3⟩
    sams-sym by blast
  moreover
  have S3 T3 U3 S2 T2 U2 SumA A1 A2 A3
    by (simp add: ⟨SAMS S2 T2 U2 S3 T3 U3 ∧ S2 T2 U2 S3 T3 U3 SumA A1 A2 A3⟩
        suma-sym)
  moreover
  have S2 T2 U2 S1 T1 U1 SumA D1 D2 D3
    by (simp add: ⟨SAMS S1 T1 U1 S2 T2 U2 ∧ S1 T1 U1 S2 T2 U2 SumA D1 D2 D3⟩
        suma-sym)
  moreover
  have SAMS S3 T3 U3 D1 D2 D3
    by (simp add: ⟨SAMS D1 D2 D3 S3 T3 U3⟩ sams-sym)
  ultimately
  show ?thesis
    using sams-assoc-2 by blast
qed
have SAMS C1 C2 C3 A1 A2 A3
proof -
  have SAMS S4 T4 U4 S1 T1 U1
    using ⟨SAMS S1 T1 U1 S4 T4 U4 ∧ S1 T1 U1 S4 T4 U4 SumA C1 C2 C3⟩
    by (simp add: sams-sym)
  moreover
  have SAMS S1 T1 U1 A1 A2 A3
    by (simp add: ⟨SAMS A1 A2 A3 S1 T1 U1⟩ sams-sym)
  moreover
  have S4 T4 U4 S1 T1 U1 SumA C1 C2 C3
    by (simp add: ⟨SAMS S1 T1 U1 S4 T4 U4 ∧ S1 T1 U1 S4 T4 U4 SumA C1 C2 C3⟩
        suma-sym)
  moreover
  have S1 T1 U1 A1 A2 A3 SumA V W X
    by (simp add: ⟨A1 A2 A3 S1 T1 U1 SumA V W X⟩ suma-sym)
  moreover
  have SAMS S4 T4 U4 V W X
    by (simp add: ⟨SAMS V W X S4 T4 U4⟩ sams-sym)

```



```

ultimately
show ?thesis
  by (meson sams-assoc-2)
qed
moreover
have C1 C2 C3 A1 A2 A3 SumA P Q R
proof -
  have SAMS S4 T4 U4 S1 T1 U1
    using ⟨SAMS S1 T1 U1 S4 T4 U4 ∧ S1 T1 U1 S4 T4 U4 SumA C1 C2 C3⟩
    by (simp add: sams-sym)
  moreover
  have SAMS S1 T1 U1 A1 A2 A3
    by (simp add: ⟨SAMS A1 A2 A3 S1 T1 U1⟩ sams-sym)
  moreover
  have S4 T4 U4 S1 T1 U1 SumA C1 C2 C3
    by (simp add: ⟨SAMS S1 T1 U1 S4 T4 U4 ∧ S1 T1 U1 S4 T4 U4 SumA C1 C2 C3⟩
      suma-sym)
  moreover
  have S1 T1 U1 A1 A2 A3 SumA V W X
    by (simp add: ⟨A1 A2 A3 S1 T1 U1 SumA V W X⟩ suma-sym)
  moreover
  have S4 T4 U4 V W X SumA P Q R
    by (simp add: ⟨V W X S4 T4 U4 SumA P Q R⟩ suma-sym)
  ultimately
  show ?thesis
    using suma-assoc by blast
qed
ultimately
show ?thesis
  by auto
qed
qed
qed
qed
qed
qed

```

```

lemma isosceles-sym :
  assumes A B C isosceles
  shows C B A isosceles
  using assms isosceles-def not-cong-4321 by blast

```

```

lemma isosceles-conga:
  assumes A ≠ C and
    A ≠ B and
    A B C isosceles
  shows C A B CongA A C B
  by (meson assms(1) assms(2) assms(3) conga-comm isosceles-def
    l11-44-1-a not-cong-2134)

```

```

lemma conga-isosceles:
  assumes ¬ Col A B C and
    C A B CongA A C B
  shows A B C isosceles
  by (meson assms(1) assms(2) cong-left-commutativity conga-comm
    isosceles-def l11-44-1-b)

```

```

lemma isosceles-foot--midpoint-conga:
  assumes A B C isosceles and
    Col H A C and
    H B Perp A C
  shows ¬ Col A B C ∧ A ≠ H ∧ C ≠ H ∧ H Midpoint A C ∧ H B A CongA H B C
proof -
  {
    assume Col A B C

```

```

  hence  $\neg H B \text{ Perp } A C$ 
    by (metis NCol-perm Perp-cases assms(2) l8-16-1)
  hence False
    by (simp add: assms(3))
}
moreover
{
  assume  $A = H$ 
  have  $A B \text{ Lt } B C \wedge A C \text{ Lt } B C$ 
    by (metis Col-def Out-cases Perp-perm l11-46  $\langle A = H \rangle$ 
        acute-col-perp--out angle-partition assms(3) bet-neq12--neq
        calculation col-trivial-3 not-bet-and-out)
  hence False
    using assms(1) cong--nlt isosceles-def by blast
}
moreover
have  $C \neq H$ 
  by (metis l11-46 assms(1) assms(3) calculation(1) cong--nlt
      isosceles-def not-col-distincts not-cong-4321 perp-per-1)
moreover
have  $H \text{ PerpAt } A C B H$ 
  using Perp-perm assms(2) assms(3) col-permutation-1 l8-15-1 by blast
hence  $\text{Per } A H B$ 
  using perp-in-per-1 by force
hence  $\text{Per } C H B$ 
  using assms(2) calculation(2) l8-3 by blast
have  $H \text{ Midpoint } A C$ 
  by (metis  $\langle \text{Per } A H B \rangle \langle \text{Per } C H B \rangle$  assms(1) assms(2) assms(3)
      cong2-per2--cong cong-left-commutativity cong-reflexivity
      isosceles-def l7-20-bis not-col-permutation-4 not-cong-2143 perp-distinct)
moreover
have  $H B A \text{ CongA } H B C$ 
  by (meson Perp-cases assms(1) assms(3) cong-perp-conga
      conga-comm isosceles-def not-cong-3421 not-conga-sym)
ultimately
show ?thesis
  by blast
qed

```

```

lemma equilateral-strict-equilateral:
  assumes  $A B C \text{ equilateralStrict}$ 
  shows  $A B C \text{ equilateral}$ 
  using assms equilateralStrict-def by blast

```

```

lemma equilateral-cong:
  assumes  $A B C \text{ equilateral}$ 
  shows  $\text{Cong } A B B C \wedge \text{Cong } B C C A \wedge \text{Cong } C A A B$ 
  by (meson assms cong-transitivity equilateral-def not-cong-3412)

```

```

lemma equilateral-rot:
  assumes  $A B C \text{ equilateral}$ 
  shows  $B C A \text{ equilateral}$ 
  using assms equilateral-cong equilateral-def by blast

```

```

lemma equilateral-swap:
  assumes  $A B C \text{ equilateral}$ 
  shows  $B A C \text{ equilateral}$ 
  by (meson assms cong-4321 equilateral-def equilateral-rot)

```

```

lemma equilateral-rot-2:
  assumes  $A B C \text{ equilateral}$ 
  shows  $C B A \text{ equilateral}$ 
  using assms equilateral-rot equilateral-swap by blast

```

```

lemma equilateral-swap-2:
  assumes  $A B C \text{ equilateral}$ 

```

```

shows  $A \ C \ B$  equilateral
using assms equilateral-rot equilateral-rot-2 by blast

lemma equilateral-swap-rot:
  assumes  $A \ B \ C$  equilateral
  shows  $C \ A \ B$  equilateral
  by (simp add: assms equilateral-rot)

lemma equilateral-isosceles-1:
  assumes  $A \ B \ C$  equilateral
  shows  $A \ B \ C$  isosceles
  using assms equilateral-cong isosceles-def by blast

lemma equilateral-isosceles-2:
  assumes  $A \ B \ C$  equilateral
  shows  $B \ C \ A$  isosceles
  using assms equilateral-cong isosceles-def by blast

lemma equilateral-isosceles-3:
  assumes  $A \ B \ C$  equilateral
  shows  $C \ A \ B$  isosceles
  using assms equilateral-cong isosceles-def by blast

lemma equilateral-strict-neq:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $A \neq B \wedge B \neq C \wedge A \neq C$ 
  by (metis assms bet-cong-eq between-trivial2 equilateralStrict-def equilateral-cong)

lemma equilateral-strict-swap-1:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $A \ C \ B$  equilateralStrict
  by (metis assms equilateralStrict-def equilateral-strict-neq equilateral-swap-2)

lemma equilateral-strict-swap-2:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $B \ A \ C$  equilateralStrict
  by (metis assms equilateralStrict-def equilateral-swap)

lemma equilateral-strict-swap-3:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $B \ C \ A$  equilateralStrict
  using assms equilateral-strict-swap-1 equilateral-strict-swap-2 by blast

lemma equilateral-strict-swap-4:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $C \ A \ B$  equilateralStrict
  by (simp add: assms equilateral-strict-swap-3)

lemma equilateral-strict-swap-5:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $C \ B \ A$  equilateralStrict
  using assms equilateral-strict-swap-1 equilateral-strict-swap-4 by blast

lemma equilateral-strict--not-col:
  assumes  $A \ B \ C$  equilateralStrict
  shows  $\neg \text{Col } A \ B \ C$ 
proof –
  have  $A \neq B$ 
  using assms equilateral-strict-neq by blast
  have  $A \neq C$ 
  using assms equilateral-strict-neq by blast
  have  $C \neq B$ 
  using assms equilateral-strict-neq by blast
  have Cong  $B \ A \ B \ C$ 
  using assms cong-left-commutativity equilateral-cong
  equilateral-strict-equilateral by blast

```

```

have Cong C A C B
  using assms cong-3421 equilateral-cong equilateral-strict-equilateral
  by blast
{
  assume Col A B C
  have B Midpoint A C
    using l7-20-bis by (simp add:  $\langle A \neq C \rangle \langle Col A B C \rangle \langle Cong B A B C \rangle$ )
  moreover
  have C Midpoint A B
    by (simp add:  $\langle A \neq B \rangle \langle Col A B C \rangle \langle Cong C A C B \rangle$  col-permutation-5 l7-20-bis)
  ultimately
  have False
    using  $\langle A \neq B \rangle$  midpoint-not-midpoint by blast
}
thus ?thesis
  by auto
qed

```

```

lemma equilateral-strict-conga-1:
  assumes A B C equilateralStrict
  shows C A B CongA A C B
proof -
  have A  $\neq$  B
    using assms equilateral-strict-neq by blast
  moreover
  have A  $\neq$  C
    using assms equilateral-strict-neq by blast
  moreover
  have A B C isosceles
    by (meson assms equilateral-isosceles-1 equilateral-strict-equilateral)
  ultimately
  show ?thesis
    by (simp add: isosceles-conga)
qed

```

```

lemma equilateral-strict-conga-2:
  assumes A B C equilateralStrict
  shows B A C CongA A B C
  by (simp add: equilateral-strict-conga-1 assms equilateral-strict-swap-1)

```

```

lemma equilateral-strict-conga-3:
  assumes A B C equilateralStrict
  shows C B A CongA B C A
  by (simp add: equilateral-strict-conga-2 assms equilateral-strict-swap-3)

```

```

lemma conga3-equilateral:
  assumes  $\neg Col A B C$  and
    B A C CongA A B C and
    A B C CongA B C A
  shows A B C equilateral
  by (meson assms(1) assms(2) assms(3) col-conga-col conga-right-comm
    conga-trans equilateral-def equilateral-rot-2 l11-44-1-b
    not-cong-2134 not-cong-3421)

```

```

lemma lg-exists:
   $\exists l. (QCong l \wedge l A B)$ 
  using QCong-def cong-pseudo-reflexivity by blast

```

```

lemma lg-cong:
  assumes QCong l and
    l A B and
    l C D
  shows Cong A B C D
  by (metis QCong-def assms(1) assms(2) assms(3) cong-inner-transitivity)

```

```

lemma lg-cong-lg:

```

```

assumes  $QCong\ l$  and
   $l\ A\ B$  and
   $Cong\ A\ B\ C\ D$ 
shows  $l\ C\ D$ 
by (metis  $QCong-def\ assms(1)\ assms(2)\ assms(3)\ cong-transitivity$ )

lemma lg-sym:
assumes  $QCong\ l$ 
  and  $l\ A\ B$ 
shows  $l\ B\ A$ 
using  $assms(1)\ assms(2)\ cong-pseudo-reflexivity\ lg-cong-lg$  by blast

lemma ex-points-lg:
assumes  $QCong\ l$ 
shows  $\exists\ A\ B. l\ A\ B$ 
using  $QCong-def\ assms\ cong-pseudo-reflexivity$  by fastforce

lemma is-len-cong:
assumes  $TarskiLen\ A\ B\ l$  and
   $TarskiLen\ C\ D\ l$ 
shows  $Cong\ A\ B\ C\ D$ 
using  $TarskiLen-def\ assms(1)\ assms(2)\ lg-cong$  by auto

lemma is-len-cong-is-len:
assumes  $TarskiLen\ A\ B\ l$  and
   $Cong\ A\ B\ C\ D$ 
shows  $TarskiLen\ C\ D\ l$ 
using  $TarskiLen-def\ assms(1)\ assms(2)\ lg-cong-lg$  by fastforce

lemma not-cong-is-len:
assumes  $\neg\ Cong\ A\ B\ C\ D$  and
   $TarskiLen\ A\ B\ l$ 
shows  $\neg\ l\ C\ D$ 
using  $TarskiLen-def\ assms(1)\ assms(2)\ lg-cong$  by auto

lemma not-cong-is-len1:
assumes  $\neg\ Cong\ A\ B\ C\ D$ 
  and  $TarskiLen\ A\ B\ l$ 
shows  $\neg\ TarskiLen\ C\ D\ l$ 
using  $assms(1)\ assms(2)\ is-len-cong$  by blast

lemma lg-null-instance:
assumes  $QCongNull\ l$ 
shows  $l\ A\ A$ 
by (metis  $QCongNull-def\ QCong-def\ assms\ cong-diff\ cong-trivial-identity$ )

lemma lg-null-trivial:
assumes  $QCong\ l$ 
  and  $l\ A\ A$ 
shows  $QCongNull\ l$ 
using  $QCongNull-def\ assms(1)\ assms(2)$  by auto

lemma lg-null-dec:

shows  $QCongNull\ l \vee \neg\ QCongNull\ l$ 
by simp

lemma ex-point-lg:
assumes  $QCong\ l$ 
shows  $\exists\ B. l\ A\ B$ 
by (metis  $QCong-def\ assms\ not-cong-3412\ segment-construction$ )

lemma ex-point-lg-out:
assumes  $A \neq P$  and
   $QCong\ l$  and
   $\neg\ QCongNull\ l$ 

```

shows $\exists B. (l A B \wedge A \text{ Out } B P)$
proof –
 obtain $X Y$ where $P1: \forall X0 Y0. (Cong X Y X0 Y0 \longleftrightarrow l X0 Y0)$
 using $QCong\text{-}def\ assms(2)$ by *auto*
 hence $l X Y$
 using *cong-reflexivity* by *auto*
 hence $X \neq Y$
 using $assms(2)\ assms(3)\ lg\text{-}null\text{-}trivial$ by *auto*
 then obtain B where $A \text{ Out } P B \wedge Cong A B X Y$
 using $assms(1)\ segment\text{-}construction\text{-}3$ by *blast*
 thus *?thesis*
 using *Cong-perm Out-cases P1* by *blast*
qed

lemma *ex-point-lg-bet*:
 assumes $QCong\ l$
 shows $\exists B. (l M B \wedge Bet A M B)$
proof –
 obtain $X Y$ where $P1: \forall X0 Y0. (Cong X Y X0 Y0 \longleftrightarrow l X0 Y0)$
 using $QCong\text{-}def\ assms$ by *auto*
 hence $l X Y$
 using *cong-reflexivity* by *blast*
 obtain B where $Bet A M B \wedge Cong M B X Y$
 using *segment-construction* by *blast*
 thus *?thesis*
 using *Cong-perm P1* by *blast*
qed

lemma *ex-points-lg-not-col*:
 assumes $QCong\ l$
 and $\neg QCongNull\ l$
 shows $\exists A B. (l A B \wedge \neg Col A B P)$
proof –
 have $\exists B::'p. A \neq B$
 using *another-point* by *blast*
 then obtain $A::'p$ where $P \neq A$
 by *metis*
 then obtain Q where $\neg Col P A Q$
 using *not-col-exists* by *auto*
 hence $A \neq Q$
 using *col-trivial-2* by *auto*
 then obtain B where $l A B \wedge A \text{ Out } B Q$
 using $assms(1)\ assms(2)\ ex\text{-}point\text{-}lg\text{-}out$ by *blast*
 thus *?thesis*
 by (*metis* $\langle \neg Col P A Q \rangle\ col\text{-}transitivity\text{-}1\ not\text{-}col\text{-}permutation\text{-}1$
out-col out-diff1)
qed

lemma *ex-eql*:
 assumes $\exists A B. (TarskiLen A B l1 \wedge TarskiLen A B l2)$
 shows $l1 = l2$
proof –
 obtain $A B$ where $P1: TarskiLen A B l1 \wedge TarskiLen A B l2$
 using *assms* by *auto*
 have $\forall A0 B0. (l1 A0 B0 \longrightarrow l2 A0 B0)$
 by (*metis* $TarskiLen\text{-}def\ \langle TarskiLen A B l1 \wedge TarskiLen A B l2 \rangle\ lg\text{-}cong\ lg\text{-}cong\text{-}lg$)
 have $\forall A0 B0. (l1 A0 B0 \longleftrightarrow l2 A0 B0)$
proof –
 have $\forall A0 B0. (l1 A0 B0 \longrightarrow l2 A0 B0)$
 by (*metis* $TarskiLen\text{-}def\ \langle TarskiLen A B l1 \wedge TarskiLen A B l2 \rangle\ lg\text{-}cong\ lg\text{-}cong\text{-}lg$)
 moreover have $\forall A0 B0. (l2 A0 B0 \longrightarrow l1 A0 B0)$
 by (*metis* $TarskiLen\text{-}def\ \langle TarskiLen A B l1 \wedge TarskiLen A B l2 \rangle\ lg\text{-}cong\ lg\text{-}cong\text{-}lg$)
 ultimately show *?thesis* by *blast*
qed

thus ?thesis by blast

qed

lemma all-eql:

assumes TarskiLen A B l1 and

TarskiLen A B l2

shows l1 EqLTarski l2

by (metis EqLTarski-def assms(1) assms(2) ex-eql)

lemma null-len:

assumes TarskiLen A A la and

TarskiLen B B lb

shows la EqLTarski lb

using all-eql assms(1) assms(2) cong-trivial-identity is-len-cong-is-len by blast

lemma eqL-equivalence:

assumes QCong la and

QCong lb and

QCong lc

shows la = la $\wedge$ (la = lb $\longrightarrow$ lb = la) $\wedge$ (la = lb $\wedge$ lb = lc $\longrightarrow$ la = lc)

by simp

lemma ex-lg:

$\exists l. (QCong l \wedge l A B)$

by (simp add: lg-exists)

lemma lg-eql-lg:

assumes QCong l1 and

l1 EqLTarski l2

shows QCong l2

using EqLTarski-def QCong-def assms(1) assms(2) by force

lemma ex-eqL:

assumes QCong l1 and

QCong l2 and

$\exists A B. (l1 A B \wedge l2 A B)$

shows l1 EqLTarski l2

using TarskiLen-def all-eql assms(1) assms(2) assms(3) by force

lemma ang-exists:

assumes $A \neq B$ and

$C \neq B$

shows $\exists a. (QCongA a \wedge a A B C)$

proof -

have $A B C CongA A B C$

by (simp add: assms(1) assms(2) conga-refl)

thus ?thesis

using QCongA-def assms(1) assms(2) by auto

qed

lemma ex-points-eng:

assumes QCongA a

shows $\exists A B C. (a A B C)$

proof -

obtain $A B C$ where $A \neq B \wedge C \neq B \wedge (\forall X Y Z. (A B C CongA X Y Z \longleftrightarrow a X Y Z))$

using QCongA-def assms by auto

thus ?thesis

using conga-pseudo-refl by blast

qed

lemma ang-conga:

assumes QCongA a and

$a A B C$ and

$a A' B' C'$

shows $A B C CongA A' B' C'$

proof -

obtain $A0\ B0\ C0$ **where** $A0 \neq B0 \wedge C0 \neq B0 \wedge$
 $(\forall\ X\ Y\ Z. (A0\ B0\ C0\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z))$
using $QCongA\text{-def}\ assms(1)$ **by** *auto*
thus *?thesis*
by $(meson\ assms(2)\ assms(3)\ not\ conga\ not\ conga\ sym)$
qed

lemma *is-ang-conga*:
assumes $A\ B\ C\ Ang\ a$ **and**
 $A'\ B'\ C'\ Ang\ a$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
using $Ang\text{-def}\ ang\ conga\ assms(1)\ assms(2)$ **by** *auto*

lemma *is-ang-conga-is-ang*:
assumes $A\ B\ C\ Ang\ a$ **and**
 $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $A'\ B'\ C'\ Ang\ a$
proof –
have $QCongA\ a$
using $Ang\text{-def}\ assms(1)$ **by** *auto*
then obtain $A0\ B0\ C0$ **where** $A0 \neq B0 \wedge C0 \neq B0 \wedge$
 $(\forall\ X\ Y\ Z. (A0\ B0\ C0\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z))$
using $QCongA\text{-def}$ **by** *auto*
thus *?thesis*
by $(metis\ Ang\text{-def}\ assms(1)\ assms(2)\ not\ conga)$
qed

lemma *not-conga-not-ang*:
assumes $QCongA\ a$ **and**
 $\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
 $a\ A\ B\ C$
shows $\neg\ a\ A'\ B'\ C'$
using $ang\ conga\ assms(1)\ assms(2)\ assms(3)$ **by** *auto*

lemma *not-conga-is-ang*:
assumes $\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
 $A\ B\ C\ Ang\ a$
shows $\neg\ a\ A'\ B'\ C'$
using $Ang\text{-def}\ ang\ conga\ assms(1)\ assms(2)$ **by** *auto*

lemma *not-cong-is-ang1*:
assumes $\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
 $A\ B\ C\ Ang\ a$
shows $\neg\ A'\ B'\ C'\ Ang\ a$
using $assms(1)\ assms(2)\ is\ ang\ conga$ **by** *blast*

lemma *ex-ega*:
assumes $\exists\ A\ B\ C. (A\ B\ C\ Ang\ a1 \wedge A\ B\ C\ Ang\ a2)$
shows $a1 = a2$
proof –
obtain $A\ B\ C$ **where** $P1: A\ B\ C\ Ang\ a1 \wedge A\ B\ C\ Ang\ a2$
using $assms$ **by** *auto*
{
 fix $x\ y\ z$
 assume $a1\ x\ y\ z$
 hence $x\ y\ z\ Ang\ a1$
 using $Ang\text{-def}\ assms$ **by** *auto*
 hence $x\ y\ z\ CongA\ A\ B\ C$
 using $P1\ not\ cong\ is\ ang1$ **by** *blast*
 hence $x\ y\ z\ Ang\ a2$
 using $P1\ is\ ang\ conga\ is\ ang\ not\ conga\ sym$ **by** *blast*
 hence $a2\ x\ y\ z$
 using $Ang\text{-def}\ assms$ **by** *auto*
}
{
 fix $x\ y\ z$


```

assume  $a2\ x\ y\ z$ 
hence  $x\ y\ z\ Ang\ a2$ 
  using Ang-def assms by auto
hence  $x\ y\ z\ CongA\ A\ B\ C$ 
  using P1 not-cong-is-ang1 by blast
hence  $x\ y\ z\ Ang\ a1$ 
  using P1 is-ang-conga-is-ang not-conga-sym by blast
hence  $a1\ x\ y\ z$ 
  using Ang-def assms by auto
}
hence  $\forall\ x\ y\ z. (a1\ x\ y\ z) \longleftrightarrow (a2\ x\ y\ z)$ 
  using  $\langle \bigwedge z\ y\ x. a1\ x\ y\ z \implies a2\ x\ y\ z \rangle$  by blast
hence  $\bigwedge x\ y. (\forall z. (a1\ x\ y)\ z = (a2\ x\ y)\ z)$ 
  by simp
hence  $\bigwedge x\ y. (a1\ x\ y) = (a2\ x\ y)$  using fun-eq-iff by auto
thus ?thesis using fun-eq-iff by auto
qed

```

```

lemma all-eqa:
  assumes  $A\ B\ C\ Ang\ a1$  and
     $A\ B\ C\ Ang\ a2$ 
  shows  $a1 = a2$ 
  using assms(1) assms(2) ex-eqa by blast

```

```

lemma is-ang-distinct:
  assumes  $A\ B\ C\ Ang\ a$ 
  shows  $A \neq B \wedge C \neq B$ 
  using assms conga-diff1 conga-diff2 is-ang-conga by blast

```

```

lemma null-ang:
  assumes  $A\ B\ A\ Ang\ a1$  and
     $C\ D\ C\ Ang\ a2$ 
  shows  $a1 = a2$ 
  using all-eqa assms(1) assms(2) conga-trivial-1
    is-ang-conga-is-ang is-ang-distinct by auto

```

```

lemma flat-ang:
  assumes  $Bet\ A\ B\ C$  and
     $Bet\ A'\ B'\ C'$  and
     $A\ B\ C\ Ang\ a1$  and
     $A'\ B'\ C'\ Ang\ a2$ 
  shows  $a1 = a2$ 
proof –
  have  $A\ B\ C\ Ang\ a2$ 
  proof –
    have  $A'\ B'\ C'\ Ang\ a2$ 
    by (simp add: assms(4))
    moreover have  $A'\ B'\ C'\ CongA\ A\ B\ C$ 
    by (metis assms(1) assms(2) assms(3) calculation
      conga-line is-ang-distinct)
    ultimately show ?thesis
    using is-ang-conga-is-ang by blast
  qed
thus ?thesis
  using assms(3) all-eqa by auto
qed

```

```

lemma ang-distinct:
  assumes  $QCongA\ a$  and
     $a\ A\ B\ C$ 
  shows  $A \neq B \wedge C \neq B$ 
proof –
  have  $A\ B\ C\ Ang\ a$ 
  by (simp add: Ang-def assms(1) assms(2))
thus ?thesis
  using is-ang-distinct by auto

```

qed

lemma *ex-ang*:

assumes $B \neq A$ and

$B \neq C$

shows $\exists a. (QCongA\ a \wedge a\ A\ B\ C)$

using *ang-exists* *assms*(1) *assms*(2) by auto

lemma *anga-exists*:

assumes $A \neq B$ and

$C \neq B$ and

Acute $A\ B\ C$

shows $\exists a. (QCongA\ Acute\ a \wedge a\ A\ B\ C)$

proof –

have $A\ B\ C\ CongA\ A\ B\ C$

by (*simp* add: *assms*(1) *assms*(2) *conga-refl*)

thus ?thesis

using *assms*(1) *QCongAAcute-def* *assms*(3) by blast

qed

lemma *anga-is-ang*:

assumes *QCongAAcute* a

shows *QCongA* a

proof –

obtain $A0\ B0\ C0$ where *Acute* $A0\ B0\ C0 \wedge$

$(\forall\ X\ Y\ Z. (A0\ B0\ C0\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z))$

using *QCongAAcute-def* *assms* by auto

thus ?thesis

using *QCongA-def* by (*metis* *acute-distincts*)

qed

lemma *ex-points-anga*:

assumes *QCongAAcute* a

shows $\exists\ A\ B\ C. a\ A\ B\ C$

by (*simp* add: *anga-is-ang* *assms* *ex-points-eng*)

lemma *anga-conga*:

assumes *QCongAAcute* a and

$a\ A\ B\ C$ and

$a\ A'\ B'\ C'$

shows $A\ B\ C\ CongA\ A'\ B'\ C'$

by (*meson* *ang-conga* *anga-is-ang* *assms*(1) *assms*(2) *assms*(3))

lemma *is-anga-to-is-ang*:

assumes $A\ B\ C\ AngAcute\ a$

shows $A\ B\ C\ Ang\ a$

using *AngAcute-def* *Ang-def* *anga-is-ang* *assms* by auto

lemma *is-anga-conga*:

assumes $A\ B\ C\ AngAcute\ a$ and

$A'\ B'\ C'\ AngAcute\ a$

shows $A\ B\ C\ CongA\ A'\ B'\ C'$

using *AngAcute-def* *anga-conga* *assms*(1) *assms*(2) by auto

lemma *is-anga-conga-is-anga*:

assumes $A\ B\ C\ AngAcute\ a$ and

$A\ B\ C\ CongA\ A'\ B'\ C'$

shows $A'\ B'\ C'\ AngAcute\ a$

using *AngAcute-def* *Ang-def* *is-ang-conga-is-ang* *assms*(1) *assms*(2)

is-anga-to-is-ang by *fastforce*

lemma *not-conga-is-anga*:

assumes $\neg\ A\ B\ C\ CongA\ A'\ B'\ C'$ and

$A\ B\ C\ AngAcute\ a$

shows $\neg\ a\ A'\ B'\ C'$

using *AngAcute-def* *anga-conga* *assms*(1) *assms*(2) by auto

```

lemma not-cong-is-anga1:
  assumes  $\neg A \ B \ C \ CongA \ A' \ B' \ C'$  and
     $A \ B \ C \ AngAcute \ a$ 
  shows  $\neg A' \ B' \ C' \ AngAcute \ a$ 
  using assms(1) assms(2) is-anga-conga by auto

lemma ex-egaa:
  assumes  $\exists A \ B \ C. (A \ B \ C \ AngAcute \ a1 \ \wedge \ A \ B \ C \ AngAcute \ a2)$ 
  shows  $a1 \ EqA \ a2$ 
proof -
  obtain  $A \ B \ C$  where  $A \ B \ C \ AngAcute \ a1$  and  $A \ B \ C \ AngAcute \ a2$ 
  using assms by blast
  {
    fix  $A0 \ B0 \ C0$ 
    assume  $a1 \ A0 \ B0 \ C0$ 
    hence  $A \ B \ C \ CongA \ A0 \ B0 \ C0$ 
      using is-ang-conga  $\langle A \ B \ C \ AngAcute \ a1 \rangle$  not-conga-is-anga by blast
    hence  $A0 \ B0 \ C0 \ Ang \ a2$ 
      using  $\langle A \ B \ C \ AngAcute \ a2 \rangle$  is-anga-conga-is-anga is-anga-to-is-ang by blast
    hence  $a2 \ A0 \ B0 \ C0$ 
      by (simp add: Ang-def)
  }
  moreover
  {
    fix  $A0 \ B0 \ C0$ 
    assume  $a2 \ A0 \ B0 \ C0$ 
    hence  $A \ B \ C \ CongA \ A0 \ B0 \ C0$ 
      using is-ang-conga  $\langle A \ B \ C \ AngAcute \ a2 \rangle$  not-conga-is-anga by blast
    hence  $A0 \ B0 \ C0 \ Ang \ a1$ 
      using  $\langle A \ B \ C \ AngAcute \ a1 \rangle$  is-anga-conga-is-anga is-anga-to-is-ang by blast
    hence  $a1 \ A0 \ B0 \ C0$ 
      by (simp add: Ang-def)
  }
  ultimately show ?thesis
    using EqA-def by blast
qed

lemma all-egaa:
  assumes  $A \ B \ C \ AngAcute \ a1$  and
     $A \ B \ C \ AngAcute \ a2$ 
  shows  $a1 \ EqA \ a2$ 
  using assms(1) assms(2) ex-egaa by blast

lemma is-anga-distinct:
  assumes  $A \ B \ C \ AngAcute \ a$ 
  shows  $A \neq B \ \wedge \ C \neq B$ 
  using assms is-ang-distinct is-anga-to-is-ang by blast

lemma null-anga:
  assumes  $A \ B \ A \ AngAcute \ a1$  and
     $C \ D \ C \ AngAcute \ a2$ 
  shows  $a1 \ EqA \ a2$ 
proof (rule all-egaa [where  $?A=A$  and  $?B=B$  and  $?C=A$ ])
  show  $A \ B \ A \ AngAcute \ a1$ 
    using assms(1) by blast
  show  $A \ B \ A \ AngAcute \ a2$ 
  proof (rule is-anga-conga-is-anga [where  $?A=C$  and  $?B=D$  and  $?C=C$ ])
    show  $C \ D \ C \ AngAcute \ a2$ 
      using assms(2) by auto
    show  $C \ D \ C \ CongA \ A \ B \ A$ 
      using assms(1) assms(2) conga-trivial-1 is-anga-distinct by force
  qed
qed

```

```

lemma anga-distinct:

```

```

assumes  $QCongAacute\ a$  and
   $a\ A\ B\ C$ 
shows  $A \neq B \wedge C \neq B$ 
using ang-distinct anga-is-ang assms(1) assms(2) by blast

lemma out-is-len-eq:
assumes  $A\ Out\ B\ C$  and
   $TarskiLen\ A\ B\ l$  and
   $TarskiLen\ A\ C\ l$ 
shows  $B = C$ 
using Out-def assms(1) assms(2) assms(3) between-cong
  not-cong-is-len1 by fastforce

lemma out-len-eq:
assumes  $QCong\ l$  and
   $A\ Out\ B\ C$  and
   $l\ A\ B$  and
   $l\ A\ C$ 
shows  $B = C$  using out-is-len-eq
using TarskiLen-def assms(1) assms(2) assms(3) assms(4) by auto

lemma ex-anga:
assumes  $Acute\ A\ B\ C$ 
shows  $\exists\ a. (QCongAacute\ a \wedge a\ A\ B\ C)$ 
using acute-distincts anga-exists assms by blast

lemma not-null-ang-ang:
assumes  $QCongAnNull\ a$ 
shows  $QCongA\ a$ 
using QCongAnNull-def assms by blast

lemma not-null-ang-def-equiv:
 $QCongAnNull\ a \longleftrightarrow (QCongA\ a \wedge (\exists\ A\ B\ C. (a\ A\ B\ C \wedge \neg B\ Out\ A\ C)))$ 
proof –
{
  assume  $QCongAnNull\ a$ 
  have  $QCongA\ a \wedge (\exists\ A\ B\ C. (a\ A\ B\ C \wedge \neg B\ Out\ A\ C))$ 
    using QCongAnNull-def ⟨QCongAnNull a⟩ ex-points-eng
    by fastforce
}
{
  assume  $QCongA\ a \wedge (\exists\ A\ B\ C. (a\ A\ B\ C \wedge \neg B\ Out\ A\ C))$ 
  have  $QCongAnNull\ a$ 
    by (metis Ang-def QCongAnNull-def l11-21-a
       $\langle QCongA\ a \wedge (\exists\ A\ B\ C. a\ A\ B\ C \wedge \neg B\ Out\ A\ C) \rangle$  not-conga-is-ang)
}
thus ?thesis
using  $\langle QCongAnNull\ a \implies QCongA\ a \wedge (\exists\ A\ B\ C. a\ A\ B\ C \wedge \neg B\ Out\ A\ C) \rangle$  by blast
qed

lemma not-flat-ang-def-equiv:
 $QCongAnFlat\ a \longleftrightarrow (QCongA\ a \wedge (\exists\ A\ B\ C. (a\ A\ B\ C \wedge \neg Bet\ A\ B\ C)))$ 
proof –
{
  assume  $QCongAnFlat\ a$ 
  hence  $QCongA\ a \wedge (\exists\ A\ B\ C. (a\ A\ B\ C \wedge \neg Bet\ A\ B\ C))$ 
    using QCongAnFlat-def ex-points-eng by fastforce
}
{
  assume  $QCongA\ a \wedge (\exists\ A\ B\ C. (a\ A\ B\ C \wedge \neg Bet\ A\ B\ C))$ 
  have  $QCongAnFlat\ a$ 
    by (metis QCongAnFlat-def ⟨QCongA a ∧ (∃ A B C. a A B C ∧ ¬ (A – B – C))⟩ ang-conga
      bet-conga--bet)
}
thus ?thesis
using  $\langle QCongAnFlat\ a \implies QCongA\ a \wedge (\exists\ A\ B\ C. a\ A\ B\ C \wedge \neg Bet\ A\ B\ C) \rangle$ 

```

by blast
qed

lemma ang-const:

assumes $QCongA\ a$ and

$A \neq B$

shows $\exists\ C.\ a\ A\ B\ C$

proof -

obtain $A0\ B0\ C0$ where $A0 \neq B0 \wedge C0 \neq B0 \wedge$

$(\forall\ X\ Y\ Z.\ (A0\ B0\ C0\ CongA\ X\ Y\ Z \longrightarrow a\ X\ Y\ Z))$

by (metis $QCongA$ -def $assms(1)$)

hence $(A0\ B0\ C0\ CongA\ A0\ B0\ C0) \longleftrightarrow a\ A0\ B0\ C0$

by (simp add: conga-refl)

hence $a\ A0\ B0\ C0$

using $\langle A0 \neq B0 \wedge C0 \neq B0 \wedge (\forall\ X\ Y\ Z.\ A0\ B0\ C0\ CongA\ X\ Y\ Z \longrightarrow a\ X\ Y\ Z) \rangle$

conga-refl by blast

thus ?thesis

using $\langle A0 \neq B0 \wedge C0 \neq B0 \wedge (\forall\ X\ Y\ Z.\ A0\ B0\ C0\ CongA\ X\ Y\ Z \longrightarrow a\ X\ Y\ Z) \rangle$

angle-construction-3 $assms(2)$ by blast

qed

lemma ang-sym:

assumes $QCongA\ a$ and

$a\ A\ B\ C$

shows $a\ C\ B\ A$

proof -

obtain $A0\ B0\ C0$ where $A0 \neq B0 \wedge C0 \neq B0 \wedge$

$(\forall\ X\ Y\ Z.\ (A0\ B0\ C0\ CongA\ X\ Y\ Z \longrightarrow a\ X\ Y\ Z))$

by (metis $QCongA$ -def $assms(1)$)

thus ?thesis

by (metis ang-conga $assms(1)$ $assms(2)$ conga-left-comm conga-refl not-conga-sym)

qed

lemma ang-not-null-lg:

assumes $QCongA\ a$ and

$QCong\ l$ and

$a\ A\ B\ C$ and

$l\ A\ B$

shows $\neg\ QCongNull\ l$

by (metis $QCongNull$ -def $TarskiLen$ -def ang-distinct $assms(1)$

$assms(3)$ $assms(4)$ conga-reverse-identity not-cong-is-len)

lemma ang-distincts:

assumes $QCongA\ a$ and

$a\ A\ B\ C$

shows $A \neq B \wedge C \neq B$

using ang-distinct $assms(1)$ $assms(2)$ by auto

lemma anga-sym:

assumes $QCongAAcute\ a$ and

$a\ A\ B\ C$

shows $a\ C\ B\ A$

by (simp add: ang-sym anga-is-ang $assms(1)$ $assms(2)$)

lemma anga-not-null-lg:

assumes $QCongAAcute\ a$ and

$QCong\ l$ and

$a\ A\ B\ C$ and

$l\ A\ B$

shows $\neg\ QCongNull\ l$

using ang-not-null-lg anga-is-ang $assms(1)$ $assms(2)$ $assms(3)$ $assms(4)$ by blast

lemma anga-distincts:

assumes $QCongAAcute\ a$ and

$a\ A\ B\ C$

shows $A \neq B \wedge C \neq B$

```

using anga-distinct assms(1) assms(2) by blast

lemma ang-const-o:
  assumes  $\neg \text{Col } A \ B \ P$  and
     $Q\text{Cong}A \ a$  and
     $Q\text{CongAnNull} \ a$  and
     $Q\text{CongAnFlat} \ a$ 
  shows  $\exists \ C. \ a \ A \ B \ C \wedge A \ B \ OS \ C \ P$ 
proof -
  obtain  $A0 \ B0 \ C0$  where  $P1: A0 \neq B0 \wedge C0 \neq B0 \wedge$ 
     $(\forall \ X \ Y \ Z. (A0 \ B0 \ C0 \text{ Cong}A \ X \ Y \ Z \longrightarrow a \ X \ Y \ Z))$ 
    by (metis  $Q\text{Cong}A\text{-def}$  assms(2))
  hence  $a \ A0 \ B0 \ C0$ 
    by (simp add: conga-refl)
  hence  $T1: A0 \neq C0$ 
    using  $P1 \ Q\text{CongAnNull}\text{-def}$  assms(3) out-trivial by fastforce
  have  $A \neq B$ 
    using assms(1) col-trivial-1 by blast
  have  $A0 \neq B0 \wedge B0 \neq C0$ 
    using  $P1$  by auto
  then obtain  $C$  where  $P2: A0 \ B0 \ C0 \text{ Cong}A \ A \ B \ C \wedge (A \ B \ OS \ C \ P \vee \text{Col } A \ B \ C)$ 
    using angle-construction-2 assms(1) by blast
  hence  $a \ A \ B \ C$ 
    by (simp add:  $P1$ )
  have  $P3: A \ B \ OS \ C \ P \vee \text{Col } A \ B \ C$ 
    using  $P2$  by simp
  have  $P4: \forall \ A \ B \ C. (a \ A \ B \ C \longrightarrow \neg B \text{ Out } A \ C)$ 
    using assms(3) by (simp add:  $Q\text{CongAnNull}\text{-def}$ )
  have  $P5: \forall \ A \ B \ C. (a \ A \ B \ C \longrightarrow \neg \text{Bet } A \ B \ C)$ 
    using assms(4)  $Q\text{CongAnFlat}\text{-def}$  by auto
  {
    assume  $\text{Col } A \ B \ C$ 
    have  $\neg B \text{ Out } A \ C$ 
      using  $P4$  by (simp add:  $\langle a \ A \ B \ C \rangle$ )
    have  $\neg \text{Bet } A \ B \ C$ 
      using  $P5$  by (simp add:  $\langle a \ A \ B \ C \rangle$ )
    hence  $A \ B \ OS \ C \ P$ 
      using  $\langle \text{Col } A \ B \ C \rangle \langle \neg B \text{ Out } A \ C \rangle$  l6-4-2 by blast
    hence  $\exists \ C1. (a \ A \ B \ C1 \wedge A \ B \ OS \ C1 \ P)$ 
      using  $\langle a \ A \ B \ C \rangle$  by blast
  }
  hence  $\exists \ C1. (a \ A \ B \ C1 \wedge A \ B \ OS \ C1 \ P)$ 
    using  $P3 \ \langle a \ A \ B \ C \rangle$  by blast
  thus ?thesis
    by simp
qed

```

```

lemma anga-const:
  assumes  $Q\text{Cong}A\text{Acute} \ a$  and
     $A \neq B$ 
  shows  $\exists \ C. \ a \ A \ B \ C$ 
  using ang-const anga-is-ang assms(1) assms(2) by fastforce

```

```

lemma null-anga-null-angaP:
   $Q\text{CongAnNullAcute} \ a \longleftrightarrow \text{IsNullAngaP} \ a$ 
proof -
  have  $Q\text{CongAnNullAcute} \ a \longrightarrow \text{IsNullAngaP} \ a$ 
    using  $\text{IsNullAngaP}\text{-def} \ Q\text{CongAnNullAcute}\text{-def}$  ex-points-anga by fastforce
  moreover have  $\text{IsNullAngaP} \ a \longrightarrow Q\text{CongAnNullAcute} \ a$ 
    by (metis  $\text{IsNullAngaP}\text{-def} \ Q\text{CongAnNull}\text{-def} \ Q\text{CongAnNullAcute}\text{-def}$ 
      anga-is-ang not-null-ang-def-equiv)
  ultimately show ?thesis
    by blast
qed

```

```

lemma is-null-anga-out:

```

```

assumes
   $a \ A \ B \ C$  and
   $QCongANullAcute \ a$ 
shows  $B \ Out \ A \ C$ 
using  $QCongANullAcute-def \ assms(1) \ assms(2)$  by auto

lemma anga-acute:
assumes  $QCongAAcute \ a$  and
   $a \ A \ B \ C$ 
shows  $Acute \ A \ B \ C$ 
using  $QCongAAcute-def \ acute-conga--acute \ assms(1) \ assms(2)$  by metis

lemma not-null-not-col:
assumes  $QCongAAcute \ a$  and
   $\neg \ QCongANullAcute \ a$  and
   $a \ A \ B \ C$ 
shows  $\neg \ Col \ A \ B \ C$ 
proof –
  have  $Acute \ A \ B \ C$ 
    using anga-acute  $assms(1) \ assms(3)$  by blast
  thus ?thesis
    using  $IsNullAngaP-def \ acute-col--out \ assms(1) \ assms(2) \ assms(3)$ 
       $null-anga-null-angaP$  by blast
qed

lemma ang-cong-ang:
assumes  $QCongA \ a$  and
   $a \ A \ B \ C$  and
   $A \ B \ C \ CongA \ A' \ B' \ C'$ 
shows  $a \ A' \ B' \ C'$ 
by (metis  $QCongA-def \ assms(1) \ assms(2) \ assms(3) \ not-conga$ )

lemma is-null-ang-out:
assumes
   $a \ A \ B \ C$  and
   $QCongANull \ a$ 
shows  $B \ Out \ A \ C$ 
proof –
  have  $a \ A \ B \ C \longrightarrow B \ Out \ A \ C$ 
    using  $QCongANull-def \ assms(2)$  by auto
  thus ?thesis
    by (simp add: assms(1))
qed

lemma out-null-ang:
assumes  $QCongA \ a$  and
   $a \ A \ B \ C$  and
   $B \ Out \ A \ C$ 
shows  $QCongANull \ a$ 
by (metis  $QCongANull-def \ QCongAnNull-def \ assms(1) \ assms(2) \ assms(3)$ 
   $not-null-ang-def-equiv$ )

lemma bet-flat-ang:
assumes  $QCongA \ a$  and
   $a \ A \ B \ C$  and
   $Bet \ A \ B \ C$ 
shows  $AngFlat \ a$ 
by (metis  $AngFlat-def \ QCongAnFlat-def \ assms(1) \ assms(2) \ assms(3)$ 
   $not-flat-ang-def-equiv$ )

lemma out-null-anga:
assumes  $QCongAAcute \ a$  and
   $a \ A \ B \ C$  and
   $B \ Out \ A \ C$ 
shows  $QCongANullAcute \ a$ 
using  $IsNullAngaP-def \ assms(1) \ assms(2) \ assms(3) \ null-anga-null-angaP$  by auto

```

lemma *anga-not-flat*:
assumes $QCongAAcute\ a$
shows $QCongAnFlat\ a$
by (*metis* (*no-types*, *lifting*) $QCongAnFlat-def\ anga-is-ang\ assms$
bet-col is-null-anga-out not-bet-and-out not-null-not-col)

lemma *anga-const-o*:
assumes $\neg Col\ A\ B\ P$ **and**
 $\neg QCongANullAcute\ a$ **and**
 $QCongAAcute\ a$
shows $\exists\ C. (a\ A\ B\ C \wedge A\ B\ OS\ C\ P)$

proof –
have $QCongA\ a$
by (*simp add: anga-is-ang assms(3)*)
moreover have $QCongAnNull\ a$
using $QCongANullAcute-def\ assms(2)\ assms(3)\ calculation$
not-null-ang-def-equiv **by** *auto*
moreover have $QCongAnFlat\ a$
by (*simp add: anga-not-flat assms(3)*)
ultimately show *?thesis*
by (*simp add: ang-const-o assms(1)*)

qed

lemma *anga-conga-anga*:
assumes $QCongAAcute\ a$ **and**
 $a\ A\ B\ C$ **and**
 $A\ B\ C\ CongA\ A'\ B'\ C'$
shows $a\ A'\ B'\ C'$
using *ang-cong-ang anga-is-ang assms(1) assms(2) assms(3)* **by** *blast*

lemma *anga-out-anga*:
assumes $QCongAAcute\ a$ **and**
 $a\ A\ B\ C$ **and**
 $B\ Out\ A\ A'$ **and**
 $B\ Out\ C\ C'$
shows $a\ A'\ B\ C'$

proof –
have $A\ B\ C\ CongA\ A'\ B\ C'$
by (*simp add: assms(3) assms(4) l6-6 out2--conga*)
thus *?thesis*
using *anga-conga-anga assms(1) assms(2)* **by** *blast*

qed

lemma *out-out-anga*:
assumes $QCongAAcute\ a$ **and**
 $B\ Out\ A\ C$ **and**
 $B'\ Out\ A'\ C'$ **and**
 $a\ A\ B\ C$
shows $a\ A'\ B'\ C'$

proof –
have $A\ B\ C\ CongA\ A'\ B'\ C'$
by (*simp add: assms(2) assms(3) l11-21-b*)
thus *?thesis*
using *anga-conga-anga assms(1) assms(4)* **by** *blast*

qed

lemma *is-null-all*:
assumes $A \neq B$ **and**
 $QCongANullAcute\ a$
shows $a\ A\ B\ A$

proof –
obtain $A0\ B0\ C0$ **where** $Acute\ A0\ B0\ C0 \wedge$
 $(\forall\ X\ Y\ Z. (A0\ B0\ C0\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z))$
using $QCongAAcute-def\ QCongANullAcute-def\ assms(2)$ **by** *auto*
hence $a\ A0\ B0\ C0$


```

    using acute-distincts conga-refl by blast
  thus ?thesis
    using QCongANullAcute-def assms(1) assms(2) out-out-anga
      out-trivial by fast
qed

```

```

lemma anga-col-out:
  assumes QCongAAcute a and
    a A B C and
    Col A B C
  shows B Out A C
proof -
  have Acute A B C
  using anga-acute assms(1) assms(2) by auto
  hence P1: Bet A B C  $\longrightarrow$  B Out A C
  using acute-not-bet by auto
  hence Bet C A B  $\longrightarrow$  B Out A C
  using assms(3) l6-4-2 by auto
  thus ?thesis
    using P1 assms(3) l6-4-2 by blast
qed

```

```

lemma ang-not-lg-null:
  assumes QCong la and
    QCong lc and
    QCongA a and
    la A B and
    lc C B and
    a A B C
  shows  $\neg$  QCongNull la  $\wedge$   $\neg$  QCongNull lc
  by (metis ang-not-null-lg ang-sym assms(1) assms(2) assms(3)
      assms(4) assms(5) assms(6))

```

```

lemma anga-not-lg-null:
  assumes
    QCongAAcute a and
    la A B and
    lc C B and
    a A B C
  shows  $\neg$  QCongNull la  $\wedge$   $\neg$  QCongNull lc
  by (metis QCongNull-def anga-not-null-lg anga-sym assms(1)
      assms(2) assms(3) assms(4))

```

```

lemma anga-col-null:
  assumes QCongAAcute a and
    a A B C and
    Col A B C
  shows B Out A C  $\wedge$  QCongANullAcute a
  using anga-col-out assms(1) assms(2) assms(3) out-null-anga by blast

```

```

lemma eqA-preserves-ang:
  assumes QCongA a and
    a EqA b
  shows QCongA b
proof -
  obtain x x0 x1 where x  $\neq$  x0 and x1  $\neq$  x0 and  $\forall$  X Y Z. x x0 x1 CongA X Y Z  $\longleftrightarrow$  a X Y Z
  using QCongA-def assms(1) by auto
  thus ?thesis
    using EqA-def QCongA-def assms(2) by auto
qed

```

```

lemma eqA-preserves-anga:
  assumes QCongAAcute a and
    a EqA b
  shows QCongAAcute b
proof -

```

```

have QCongA a
  by (simp add: anga-is-ang assms(1))
obtain A B C where a A B C
  using assms(1) ex-points-anga by blast
hence b A B C
  using EqA-def assms(2) by fastforce
have Acute A B C
  using ⟨a A B C⟩ anga-acute assms(1) by blast
moreover
{
  fix X Y Z
  assume A B C CongA X Y Z
  hence b X Y Z
    using EqA-def ⟨a A B C⟩ anga-conga-anga assms(1) assms(2) by fastforce
}
moreover
{
  fix X Y Z
  assume b X Y Z
  hence A B C CongA X Y Z
    using EqA-def ⟨a A B C⟩ anga-conga assms(1) assms(2) by auto
}
ultimately show ?thesis
  by (metis QCongAAcute-def)
qed

```

lemma l13-6:

```

assumes Lcos lc l a and
  Lcos ld l a
shows lc EqLTarski ld
proof -
  have H1: QCong lc ∧ QCong l ∧ QCongAAcute a ∧
    (∃ A B C. (Per C B A ∧ lc A B ∧ l A C ∧ a B A C))
    using Lcos-def assms(1) by force
  then obtain X1 Y1 Z1 where Per Z1 Y1 X1 and lc X1 Y1 and l X1 Z1 and a Y1 X1 Z1
    by blast
  have H2: QCong ld ∧ QCong l ∧ QCongAAcute a ∧
    (∃ A B C. (Per C B A ∧ ld A B ∧ l A C ∧ a B A C))
    using Lcos-def assms(2) by force
  then obtain X2 Y2 Z2 where Per Z2 Y2 X2 and ld X2 Y2 and l X2 Z2 and a Y2 X2 Z2
    by blast
  have TarskiLen X1 Z1 l
    by (simp add: TarskiLen-def H1 ⟨l X1 Z1⟩)
  have TarskiLen X2 Z2 l
    by (simp add: TarskiLen-def H2 ⟨l X2 Z2⟩)
  have Cong X1 Z1 X2 Z2
    using ⟨TarskiLen X1 Z1 l⟩ ⟨TarskiLen X2 Z2 l⟩ is-len-cong by auto
  have Y1 X1 Z1 AngAcute a
    using ⟨a Y1 X1 Z1⟩ H1 by (simp add: AngAcute-def)
  have Y2 X2 Z2 AngAcute a
    using H2 by (simp add: AngAcute-def ⟨a Y2 X2 Z2⟩)
  have Y1 X1 Z1 CongA Y2 X2 Z2
    using ⟨Y1 X1 Z1 AngAcute a⟩ ⟨a Y2 X2 Z2⟩ not-conga-is-anga by blast
  have TarskiLen X1 Y1 lc
    by (simp add: H1 TarskiLen-def ⟨lc X1 Y1⟩)
  have TarskiLen X2 Y2 ld
    by (simp add: H2 TarskiLen-def ⟨ld X2 Y2⟩)
  {
    assume Z1 = Y1
    hence X2 Out Y2 Z2
      using ⟨Y1 X1 Z1 CongA Y2 X2 Z2⟩ eq-conga-out by blast
    hence Col Z2 Y2 X2
      by (simp add: col-permutation-3 out-col)
    hence Y2 = Z2 ∨ X2 = Y2

```

```

    using l8-9 ⟨Per Z2 Y2 X2⟩ by blast
  hence Y2 = Z2
    using ⟨X2 Out Y2 Z2⟩ out-diff1 by blast
  have TarskiLen X1 Y1 l
    using ⟨TarskiLen X1 Z1 l⟩ ⟨Z1 = Y1⟩ by blast
  hence l EqLTarski lc
    by (metis ⟨TarskiLen X1 Y1 lc⟩ all-eql)
  have TarskiLen X2 Y2 l
    by (simp add: ⟨TarskiLen X2 Z2 l⟩ ⟨Y2 = Z2⟩)
  hence l EqLTarski ld
    by (metis ⟨TarskiLen X2 Y2 ld⟩ all-eql)
  hence ?thesis
    using EqLTarski-def⟨l EqLTarski lc⟩ by auto
}
moreover
{
  assume Z1 ≠ Y1
  {
    assume Y2 = Z2
    have Y2 X2 Y2 CongA Y1 X1 Z1
      using ⟨Y1 X1 Z1 CongA Y2 X2 Z2⟩ ⟨Y2 = Z2⟩ conga-sym-equiv by auto
    hence X1 Out Y1 Z1
      by (simp add: eq-conga-out)
    have Z1 = Y1 ∨ X1 = Y1
      using ⟨Per Z1 Y1 X1⟩ ⟨X1 Out Y1 Z1⟩ l8-2 l8-9 out-col by blast
    hence Y1 = Z1
      using ⟨X1 Out Y1 Z1⟩ out-diff1 by blast
    hence False
      using ⟨Z1 ≠ Y1⟩ by blast
  }
  hence Z2 ≠ Y2
    by blast
  hence X1 Y1 Z1 CongA X2 Y2 Z2
    by (metis ⟨Per Z1 Y1 X1⟩ ⟨Per Z2 Y2 X2⟩ ⟨Y1 X1 Z1 CongA Y2 X2 Z2⟩ ⟨Z1 ≠ Y1⟩
      conga-comm conga-diff1 conga-diff45 l11-16)
  {
    assume Col Z1 X1 Y1
    hence X1 = Y1 ∨ Z1 = Y1
      using NCol-perm ⟨Per Z1 Y1 X1⟩ l8-3 per-distinct-1 by blast
    hence False
      using CongA-def ⟨X1 Y1 Z1 CongA X2 Y2 Z2⟩ by blast
  }
  hence ¬ Col Z1 X1 Y1
    by blast
  have Z1 X1 Y1 CongA Z2 X2 Y2
    by (simp add: ⟨Y1 X1 Z1 CongA Y2 X2 Z2⟩ conga-comm)
  have Cong Z1 Y1 Z2 Y2 ∧ Cong X1 Y1 X2 Y2 ∧ Y1 Z1 X1 CongA Y2 Z2 X2
    using l11-50-2 ⟨Cong X1 Z1 X2 Z2⟩ ⟨X1 Y1 Z1 CongA X2 Y2 Z2⟩
      ⟨Z1 X1 Y1 CongA Z2 X2 Y2⟩ ⟨¬ Col Z1 X1 Y1⟩ not-cong-2143 by blast
  hence ?thesis
    by (metis EqLTarski-def⟨TarskiLen X1 Y1 lc⟩ ⟨TarskiLen X2 Y2 ld⟩ ex-eql is-len-cong-is-len)
}
ultimately show ?thesis
  by blast
qed

lemma null-lcos-eql:
  assumes Lcos lp l a and
    QCongANullAcute a
  shows l EqLTarski lp
proof -
  have H1: QCong lp ∧ QCong l ∧ QCongAAcute a ∧
    (∃ A B C. (Per C B A ∧ lp A B ∧ l A C ∧ a B A C))
    using assms(1) Lcos-def by blast
  then obtain A B C where Per C B A and lp A B and l A C and a B A C
    by blast

```

```

have A Out B C
  using ⟨a B A C⟩ assms(2) is-null-anga-out by force
hence Col A B C
  by (simp add: out-col)
hence C = B ∨ A = B
  using Per-cases ⟨Per C B A⟩ l8-9 by blast
moreover have C = B ⟶ ?thesis
proof –
{
  assume C = B
  have QCong l
    using H1 by blast
  have l A B
    using ⟨C = B⟩ ⟨l A C⟩ by auto
  hence TarskiLen A B l
    by (simp add: TarskiLen-def ⟨QCong l⟩)
  moreover have TarskiLen A B lp
    by (simp add: H1 TarskiLen-def ⟨lp A B⟩)
  ultimately have l EqLTarski lp
    using all-eql assms(1) l13-6 by blast
}
thus ?thesis by blast
qed
moreover have A = B ⟶ ?thesis
  using ⟨A Out B C⟩ out-diff1 by blast
ultimately show ?thesis
  by fastforce
qed

lemma eql-lcos-null:
  assumes Lcos l lp a and
    l EqLTarski lp
  shows QCongANullAcute a
proof –
  have H1: QCong l ∧ QCong lp ∧ QCongAAcute a ∧
    (∃ A B C. (Per C B A ∧ l A B ∧ lp A C ∧ a B A C))
    using assms(1) Lcos-def by blast
  then obtain B A C where Per C A B and l B A and lp B C and a A B C
    by blast
  have ∀ A0 B0. l A0 B0 ⟷ lp A0 B0
    using EqLTarski-def assms(2) by force
  have l B A ⟷ lp B A
    using ⟨∀ A0 B0. l A0 B0 = lp A0 B0⟩ by auto
  hence lp B A
    by (simp add: ⟨l B A⟩)
  have l B C ⟷ lp B C
    using ⟨∀ A0 B0. l A0 B0 = lp A0 B0⟩ by auto
  hence l B C
    by (simp add: ⟨lp B C⟩)
  have Cong B A B C
    using Lcos-def TarskiLen-def ⟨l B A⟩ ⟨l B C⟩ assms(1) is-len-cong by force
  obtain B' where A Midpoint B B' and Cong C B C B'
    using Per-def ⟨Per C A B⟩ by blast
  have A ≠ B ∧ C ≠ B
    using anga-distinct H1 ⟨a A B C⟩ by blast
  hence A ≠ B
    by simp
  have C ≠ B
    using ⟨A ≠ B ∧ C ≠ B⟩ by fastforce
  have A = C
    by (meson ⟨Cong B A B C⟩ ⟨Per C A B⟩ cong2-per2--cong cong-diff-4
      cong-pseudo-reflexivity cong-transitivity l8-20-1-R1)
  have QCongAAcute a
    using H1 by blast
  moreover {
    fix A0 B0 C0

```

```

    assume a A0 B0 C0
    hence A B A CongA A0 B0 C0
      using anga-conga  $\langle A = C \rangle \langle a A B C \rangle$  calculation by blast
    hence B0 Out A0 C0
      using eq-conga-out by blast
  }
  ultimately show ?thesis
    using QCongANullAcute-def by auto
qed

lemma lcos-lg-not-null:
  assumes Lcos l lp a
  shows  $\neg$  QCongNull l  $\wedge$   $\neg$  QCongNull lp
proof -
  have H1: QCong l  $\wedge$  QCong lp  $\wedge$  QCongAAcute a  $\wedge$ 
    ( $\exists$  A B C. (Per C B A  $\wedge$  l A B  $\wedge$  lp A C  $\wedge$  a B A C))
    using assms(1) Lcos-def by blast
  then obtain A B C where Per C B A and l A B and lp A C and a B A C
    by blast
  hence B  $\neq$  A  $\wedge$  C  $\neq$  A
    using H1 anga-distinct  $\langle a B A C \rangle$  by blast
  {
    assume QCong l  $\wedge$  ( $\exists$  X. l X X)
    then obtain X where l X X
      by blast
    hence Cong A B X X
      using  $\langle$  QCong l  $\wedge$  ( $\exists$  X. l X X)  $\rangle$   $\langle$  l A B  $\rangle$  lg-cong by blast
    hence False
      using  $\langle B \neq A \wedge C \neq A \rangle$  cong-identity by blast
  }
  moreover
  {
    assume QCong lp  $\wedge$  ( $\exists$  X. lp X X)
    then obtain X where lp X X
      by blast
    hence Cong A C X X
      using  $\langle$  QCong lp  $\wedge$  ( $\exists$  X. lp X X)  $\rangle$   $\langle$  lp A C  $\rangle$  lg-cong by blast
    hence False
      using  $\langle B \neq A \wedge C \neq A \rangle$  cong-diff-2 by blast
  }
  ultimately show ?thesis
    using QCongNull-def by auto
qed

lemma perp-acute-out:
  assumes Acute A B C and
    A B Perp C C' and
    Col A B C'
  shows B Out A C'
proof -
  have B Out C' A
    using acute-col-perp--out
    by (meson acute-sym assms(1) assms(2) assms(3) col-permutation-4 perp-comm perp-right-comm)
  thus ?thesis
    using l6-6 by blast
qed

lemma perp-col-out--acute-aux:
  assumes A B Perp C C' and
    B Out A C'
  shows Acute A B C
  using assms(1) assms(2) perp-out-acute by blast

lemma perp-out--acute:
  assumes A B Perp C C' and
    Col A B C'

```

```

shows Acute  $A\ B\ C \longleftrightarrow B\ \text{Out}\ A\ C'$ 
using assms(1) assms(2) perp-acute-out perp-col-out--acute-aux by blast

lemma obtuse-not-acute:
  assumes Obtuse  $A\ B\ C$ 
  shows  $\neg$  Acute  $A\ B\ C$ 
  using acute--not-obtuse assms by blast

lemma acute-not-obtuse:
  assumes Acute  $A\ B\ C$ 
  shows  $\neg$  Obtuse  $A\ B\ C$ 
  using assms obtuse-not-acute by blast

lemma perp-obtuse-bet:
  assumes  $A\ B\ \text{Perp}\ C\ C'$  and
     $\text{Col}\ A\ B\ C'$  and
    Obtuse  $A\ B\ C$ 
  shows Bet  $A\ B\ C'$ 
proof –
  obtain  $A0\ B0\ C0$  where Per  $A0\ B0\ C0$  and  $A0\ B0\ C0\ \text{LtA}\ A\ B\ C$ 
    using Obtuse-def assms(3) by blast
  have  $A0 \neq B0$ 
    using  $\langle A0\ B0\ C0\ \text{LtA}\ A\ B\ C \rangle$  lta-distincts by auto
  have  $C0 \neq B0$ 
    using  $\langle A0\ B0\ C0\ \text{LtA}\ A\ B\ C \rangle$  lta-distincts by blast
  have  $A \neq B$ 
    using assms(1) perp-not-eq-1 by blast
  have  $C \neq B$ 
    using assms(1) assms(2) col-trivial-2 l8-16-1 by blast
  {
    assume  $B = C'$ 
    hence Per  $A\ B\ C$ 
      using assms(1) perp-left-comm perp-per-1 by blast
    have  $A0\ B0\ C0\ \text{CongA}\ A\ B\ C$ 
      by (simp add:  $\langle A \neq B \rangle \langle A0 \neq B0 \rangle \langle C \neq B \rangle \langle C0 \neq B0 \rangle \langle \text{Per}\ A\ B\ C \rangle \langle \text{Per}\ A0\ B0\ C0 \rangle$  l11-16)
    hence False
      by (simp add:  $\langle A0\ B0\ C0\ \text{LtA}\ A\ B\ C \rangle$  lta-not-conga)
  }
  hence  $B \neq C'$ 
    by blast
  {
    assume Bet  $B\ C'\ A \vee \text{Bet}\ C'\ A\ B$ 
    hence Bet  $B\ A\ C' \vee \text{Bet}\ B\ C'\ A$ 
      using Bet-cases by blast
    hence  $B\ \text{Out}\ A\ C'$ 
      using Out-def  $\langle A \neq B \rangle \langle B \neq C' \rangle$  by blast
  }
  thus ?thesis using perp-out-acute
    using Col-def assms(1) assms(2) assms(3) obtuse-not-acute by blast
qed

lemma lcos-const0:
  assumes Lcos  $lp\ l\ a$  and
    QCongANullAcute  $a$ 
  shows  $\exists\ A\ B\ C. l\ A\ B \wedge lp\ B\ C \wedge a\ A\ B\ C$ 
proof –
  have  $QCong\ lp \wedge QCong\ l \wedge QCongA\ \text{Acute}\ a \wedge$ 
     $(\exists\ A\ B\ C. (\text{Per}\ C\ B\ A \wedge lp\ A\ B \wedge l\ A\ C \wedge a\ B\ A\ C))$ 
    using assms(1) Lcos-def by blast
  thus ?thesis
    using anga-sym lg-sym by blast
qed

lemma lcos-const1:
  fixes  $P::'p$ 
  assumes Lcos  $lp\ l\ a$  and

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$\neg QCongANullAcute\ a$
shows $\exists\ A\ B\ C. \neg Col\ A\ B\ P \wedge A\ B\ OS\ C\ P \wedge l\ A\ B \wedge lp\ B\ C \wedge a\ A\ B\ C$
proof –
have $H1: QCong\ lp \wedge QCong\ l \wedge QCongAAcute\ a \wedge$
 $(\exists\ A\ B\ C. (Per\ C\ B\ A \wedge lp\ A\ B \wedge l\ A\ C \wedge a\ B\ A\ C))$
using *assms(1) Lcos-def* **by** *blast*
have $\neg QCongNull\ lp \wedge \neg QCongNull\ l$
using *assms(1) lcos-lg-not-null* **by** *auto*
then obtain $A\ B$ **where** $l\ A\ B$ **and** $\neg Col\ A\ B\ P$
using $H1\ ex-points-lg-not-col$ **by** *blast*
then obtain C' **where** $a\ A\ B\ C'$ **and** $A\ B\ OS\ C'\ P$
using *anga-const-o H1* $\langle \neg Col\ A\ B\ P \rangle$ *assms(2)* **by** *blast*
have $A \neq B \wedge B \neq C'$
using $\langle A\ B\ OS\ C'\ P \rangle\ os-distincts$ **by** *blast*
obtain C **where** $lp\ B\ C$ **and** $B\ Out\ C\ C'$
using $H1\ ex-points-lg-not-col\ \langle A \neq B \wedge B \neq C' \rangle \langle \neg QCongNull\ lp \wedge \neg QCongNull\ l \rangle$
ex-point-lg-out **by** *blast*
have $A\ B\ OS\ C\ P$
using $\langle A\ B\ OS\ C'\ P \rangle\ \langle B\ Out\ C\ C' \rangle\ l6-6\ not-col-distincts\ os-out-os$ **by** *blast*
moreover have $a\ A\ B\ C$
proof –
have $B\ Out\ A\ A$
using $\langle A \neq B \wedge B \neq C' \rangle\ out-trivial$ **by** *auto*
moreover have $B\ Out\ C'\ C$
using *Out-cases* $\langle B\ Out\ C\ C' \rangle$ **by** *blast*
ultimately show *?thesis*
using $H1\ \langle a\ A\ B\ C' \rangle\ anga-out-anga$ **by** *blast*
qed
ultimately have $\exists\ C. \neg Col\ A\ B\ P \wedge A\ B\ OS\ C\ P \wedge l\ A\ B \wedge lp\ B\ C \wedge a\ A\ B\ C$
using $\langle \neg Col\ A\ B\ P \rangle\ \langle l\ A\ B \rangle\ \langle lp\ B\ C \rangle$ **by** *blast*
thus *?thesis*
by *blast*
qed

lemma *lcos-const*:
assumes $Lcos\ lp\ l\ a$
shows $\exists\ A\ B\ C. lp\ A\ B \wedge l\ B\ C \wedge a\ A\ B\ C$
proof –
have $H1: QCong\ lp \wedge QCong\ l \wedge QCongAAcute\ a \wedge$
 $(\exists\ A\ B\ C. (Per\ C\ B\ A \wedge lp\ A\ B \wedge l\ A\ C \wedge a\ B\ A\ C))$
using *assms(1) Lcos-def* **by** *blast*
then obtain $A\ B\ C$ **where** $Per\ C\ B\ A$ **and** $lp\ A\ B$ **and** $l\ A\ C$ **and** $a\ B\ A\ C$
by *blast*
hence $lp\ B\ A$
using *lg-sym H1* **by** *blast*
thus *?thesis*
using $\langle a\ B\ A\ C \rangle\ \langle l\ A\ C \rangle$ **by** *auto*
qed

lemma *lcos-lg-distincts*:
assumes $Lcos\ lp\ l\ a$ **and**
 $l\ A\ B$ **and**
 $a\ A\ B\ C$
shows $A \neq B \wedge C \neq B$
proof –
have $H1: QCong\ lp \wedge QCong\ l \wedge QCongAAcute\ a \wedge$
 $(\exists\ A\ B\ C. (Per\ C\ B\ A \wedge lp\ A\ B \wedge l\ A\ C \wedge a\ B\ A\ C))$
using *assms(1) Lcos-def* **by** *blast*
have $\neg QCongNull\ lp \wedge \neg QCongNull\ l$
using *assms(1) lcos-lg-not-null* **by** *auto*
{
assume $A = B$
hence *False*
using $H1\ \langle \neg QCongNull\ lp \wedge \neg QCongNull\ l \rangle\ assms(2)\ lg-null-trivial$ **by** *blast*
}

```

moreover
{
  assume  $C = B$ 
  hence  $False$ 
  using  $H1$  anga-distincts assms(3) by blast
}
ultimately show ?thesis
by blast
qed

lemma lcos-const-a:
assumes  $Lcos$   $lp$   $l$   $a$ 
shows  $\forall B. \exists A C. l A B \wedge lp B C \wedge a A B C$ 
proof –
{
  fix  $B$ 
  have  $H1: QCong\ lp \wedge QCong\ l \wedge QCongAAcute\ a$ 
  using assms(1) Lcos-def by blast
  then obtain  $A$  where  $l B A$ 
  using ex-point-lg by blast
  hence  $l A B$ 
  using  $H1$  lg-sym by auto
  have  $\neg QCongNull\ lp \wedge \neg QCongNull\ l$ 
  using assms(1) lcos-lg-not-null by auto
  {
    assume  $A = B$ 
    hence  $QCongNull\ l$ 
    using  $H1$   $\langle l B A \rangle$  lg-null-trivial by auto
    hence  $False$ 
    using  $\neg QCongNull\ lp \wedge \neg QCongNull\ l$  by linarith
  }
  then obtain  $C'$  where  $a A B C'$ 
  using  $H1$  anga-const by blast
  hence  $C' \neq B$ 
  using  $\langle l A B \rangle$  assms lcos-lg-distincts by auto
  then obtain  $C$  where  $lp B C$  and  $B Out\ C\ C'$ 
  using ex-point-lg-out  $H1$   $\neg QCongNull\ lp \wedge \neg QCongNull\ l$  by metis
  have  $a A B C$ 
  proof (rule anga-out-anga [where  $?A = A$  and  $?C = C'$ ], simp-all add: H1  $\langle a A B C' \rangle$ )
  show  $B Out\ A\ A$ 
  using  $\langle A = B \implies False \rangle$  out-trivial by blast
  show  $B Out\ C'\ C$ 
  using Out-cases  $\langle B Out\ C\ C' \rangle$  by auto
qed
  hence  $\exists C. l A B \wedge lp B C \wedge a A B C$ 
  using  $\langle l A B \rangle$   $\langle lp B C \rangle$  by blast
  hence  $\exists A C. l A B \wedge lp B C \wedge a A B C$ 
  by blast
}
thus ?thesis
by blast
qed

```

```

lemma lcos-const-ab:
assumes  $Lcos$   $lp$   $l$   $a$  and
 $l A B$ 
shows  $\exists C. lp B C \wedge a A B C$ 
proof –
have  $H1: QCong\ lp \wedge QCong\ l \wedge QCongAAcute\ a$ 
using assms(1) Lcos-def by blast
have  $\neg QCongNull\ lp \wedge \neg QCongNull\ l$ 
using assms(1) lcos-lg-not-null by auto
{
  assume  $A = B$ 
  hence  $QCongNull\ l$ 
  using  $H1$  assms(2) lg-null-trivial by auto

```



```

  hence False
  using  $\langle \neg QCongNull\ lp \wedge \neg QCongNull\ l \rangle$  by linarith
}
then obtain C' where a A B C'
  using H1 anga-const by blast
hence C'  $\neq$  B
  using  $\langle l\ A\ B \rangle$  assms lcos-lg-distincts by auto
then obtain C where lp B C and B Out C C'
  using ex-point-lg-out H1  $\langle \neg QCongNull\ lp \wedge \neg QCongNull\ l \rangle$  by metis
have a A B C
proof (rule anga-out-anga [where ?A = A and ?C = C'], simp-all add: H1  $\langle a\ A\ B\ C' \rangle$ )
  show B Out A A
    using  $\langle A = B \implies False \rangle$  out-trivial by blast
  show B Out C' C
    using Out-cases  $\langle B\ Out\ C\ C' \rangle$  by auto
qed
thus ?thesis
  using  $\langle lp\ B\ C \rangle$  by auto
qed

```

```

lemma lcos-const-cb:
  assumes Lcos lp l a and
    lp B C
  shows  $\exists A. l\ A\ B \wedge a\ A\ B\ C$ 
proof -
  have H1:  $QCong\ lp \wedge QCong\ l \wedge QCongAacute\ a$ 
    using assms(1) Lcos-def by blast
  have  $\neg QCongNull\ lp \wedge \neg QCongNull\ l$ 
    using assms(1) lcos-lg-not-null by auto
  {
    assume C = B
    hence QCongNull lp
      using H1 assms(2) lg-null-trivial by auto
    hence False
      using  $\langle \neg QCongNull\ lp \wedge \neg QCongNull\ l \rangle$  by linarith
  }
  then obtain A' where a C B A'
    using H1 anga-const by blast
  hence A'  $\neq$  B
    using anga-distinct H1 by blast
  then obtain A where l B A and B Out A A'
    using ex-point-lg-out H1  $\langle \neg QCongNull\ lp \wedge \neg QCongNull\ l \rangle$  by metis
  have a A B C
  proof (rule anga-out-anga [where ?A = A' and ?C = C], simp add: H1)
    show B Out A' A
      using  $\langle B\ Out\ A\ A' \rangle$  l6-6 by blast
    show B Out C C
      using  $\langle C = B \implies False \rangle$  out-trivial by blast
    show a A' B C
      using anga-sym H1  $\langle a\ C\ B\ A' \rangle$  by blast
  qed
  thus ?thesis
    using H1  $\langle l\ B\ A \rangle$  lg-sym by blast
qed

```

```

lemma lcos-lg-anga:
  assumes Lcos lp l a
  shows  $Lcos\ lp\ l\ a \wedge QCong\ l \wedge QCong\ lp \wedge QCongAacute\ a$ 
  using Lcos-def assms by blast

```

```

lemma lcos-eql-lcos:
  assumes lp1 EqLTarski lp2 and
    l1 EqLTarski l2 and
    Lcos lp1 l1 a
  shows  $Lcos\ lp2\ l2\ a$ 
proof -

```

```

have H1: QCong lp1  $\wedge$  QCong l1  $\wedge$  QCongAAcute a  $\wedge$ 
  ( $\exists$  A B C. (Per C B A  $\wedge$  lp1 A B  $\wedge$  l1 A C  $\wedge$  a B A C))
  using assms(3) Lcos-def by blast
have QCong l2
  using EqLTarski-def H1 QCong-def assms(2) by auto
moreover have QCong lp2
  using EqLTarski-def H1 QCong-def assms(1) by auto
ultimately show ?thesis
  using EqLTarski-def H1 Lcos-def assms(1) assms(2) by auto
qed

lemma lcos-not-lg-null:
  assumes Lcos lp l a
  shows  $\neg$  QCongNull lp
  using assms lcos-lg-not-null by blast

lemma lcos-const-o:
  assumes  $\neg$  Col A B P and
     $\neg$  QCongANullAcute a and
    QCong l and
    QCong lp and
    QCongAAcute a and
    l A B and
    Lcos lp l a
  shows  $\exists$  C. A B OS C P  $\wedge$  a A B C  $\wedge$  lp B C
proof -
  obtain C' where a A B C' and A B OS C' P
    using anga-const-o assms(1) assms(2) assms(5) by blast
  hence C'  $\neq$  B
    using os-distincts by blast
  have A  $\neq$  B
    using assms(1) not-col-distincts by auto
  have  $\neg$  QCongNull lp
    using assms(7) lcos-lg-not-null by blast
  have B  $\neq$  C'
    using  $\langle C' \neq B \rangle$  by auto
  obtain lc' where QCong lc' and lc' C' B
    using lg-exists by blast
  hence  $\neg$  QCongNull l  $\wedge$   $\neg$  QCongNull lc'
    using  $\langle A \neq B \rangle \langle C' \neq B \rangle$  assms(3) assms(6) cong-identity lg-cong lg-null-instance by blast
  obtain C where lp B C and B Out C C'
    using ex-point-lg-out  $\langle C' \neq B \rangle \langle \neg$  QCongNull lp  $\rangle$  assms(4) by metis
  have A B OS C P
    using Out-cases  $\langle A B OS C' P \rangle \langle B Out C C' \rangle$  col-trivial-2 os-out-os by blast
  moreover have a A B C
  proof (rule anga-out-anga [where ?A = A and ?C = C'], simp-all add: assms(5)  $\langle a A B C' \rangle$ )
    show B Out A A
      by (simp add:  $\langle A \neq B \rangle$  out-trivial)
    show B Out C' C
      by (simp add:  $\langle B Out C C' \rangle$  l6-6)
  qed
  ultimately show ?thesis
    using  $\langle lp B C \rangle$  by blast
qed

```

```

lemma flat-not-acute:
  assumes Bet A B C
  shows  $\neg$  Acute A B C
  using acute-not-bet assms by force

```

```

lemma acute-comp-not-acute:
  assumes Bet A B C and
    Acute A B D
  shows  $\neg$  Acute C B D
proof -
  {

```

```

assume  $Col\ A\ C\ D$ 
moreover have  $Bet\ A\ C\ D \longrightarrow ?thesis$ 
  using  $assms(1)\ assms(2)\ between-exchange_4\ flat-not-acute$  by blast
moreover
{
  assume  $Bet\ C\ D\ A$ 
  have  $Bet\ A\ B\ D \vee Bet\ A\ D\ B$ 
    using  $\langle Bet\ C\ D\ A \rangle\ assms(1)\ between-symmetry\ l5-3$  by blast
  moreover have  $Bet\ A\ B\ D \longrightarrow ?thesis$ 
    using  $acute-not-bet\ assms(2)$  by force
  moreover have  $Bet\ A\ D\ B \longrightarrow ?thesis$ 
    using  $acute-not-bet\ acute-sym\ assms(1)\ between-exchange_3$  by blast
  ultimately have  $?thesis$ 
    by blast
}
moreover have  $Bet\ D\ A\ C \longrightarrow ?thesis$ 
  using  $acute-not-bet\ acute-sym\ assms(1)\ between-exchange_2$  by blast
ultimately have  $?thesis$ 
  using  $Col-def$  by blast
}
moreover
{
  assume  $\neg Col\ A\ C\ D$ 
  assume  $Acute\ C\ B\ D$ 
  obtain  $A0\ B0\ C0$  where  $Per\ A0\ B0\ C0$  and  $A\ B\ D\ LtA\ A0\ B0\ C0$ 
    using  $Acute-def\ assms(2)$  by blast
  obtain  $A1\ B1\ C1$  where  $Per\ A1\ B1\ C1$  and  $C\ B\ D\ LtA\ A1\ B1\ C1$ 
    using  $Acute-def\ \langle Acute\ C\ B\ D \rangle$  by blast
  have  $A \neq B$ 
    using  $\langle A\ B\ D\ LtA\ A0\ B0\ C0 \rangle\ lta-distincts$  by auto
  have  $D \neq B$ 
    using  $\langle A\ B\ D\ LtA\ A0\ B0\ C0 \rangle\ lta-distincts$  by auto
  have  $A0 \neq B0$ 
    using  $\langle A\ B\ D\ LtA\ A0\ B0\ C0 \rangle\ lta-distincts$  by blast
  have  $C0 \neq B0$ 
    using  $\langle A\ B\ D\ LtA\ A0\ B0\ C0 \rangle\ lta-distincts$  by blast
  have  $A1 \neq B1$ 
    using  $\langle C\ B\ D\ LtA\ A1\ B1\ C1 \rangle\ lta-distincts$  by auto
  have  $C1 \neq B1$ 
    using  $\langle C\ B\ D\ LtA\ A1\ B1\ C1 \rangle\ lta-distincts$  by blast
  have  $A0\ B0\ C0\ CongA\ A1\ B1\ C1$ 
    by  $(simp\ add: \langle A0 \neq B0 \rangle\ \langle A1 \neq B1 \rangle\ \langle C0 \neq B0 \rangle\ \langle C1 \neq B1 \rangle\ \langle Per\ A0\ B0\ C0 \rangle\ \langle Per\ A1\ B1\ C1 \rangle\ l11-16)$ 
  have  $C\ B\ D\ LtA\ A0\ B0\ C0$ 
    using  $\langle A0 \neq B0 \rangle\ \langle Acute\ C\ B\ D \rangle\ \langle C0 \neq B0 \rangle\ \langle Per\ A0\ B0\ C0 \rangle\ acute-per--lta$  by auto
  have  $A \neq C$ 
    using  $\langle A \neq B \rangle\ assms(1)\ between-identity$  by blast
  have  $Col\ A\ C\ B$ 
    by  $(simp\ add: assms(1)\ bet-col\ col-permutation-5)$ 
  obtain  $P$  where  $A\ C\ Perp\ P\ B$  and  $A\ C\ OS\ D\ P$ 
    using  $\langle \neg Col\ A\ C\ D \rangle\ assms(1)\ bet-col\ l10-15\ not-col-permutation-5$  by blast
  have  $A\ B\ Perp\ P\ B$ 
    using  $\langle A\ C\ Perp\ P\ B \rangle\ \langle A \neq B \rangle\ \langle Col\ A\ C\ B \rangle\ perp-col$  by blast
  hence  $B\ PerpAt\ B\ A\ P\ B$ 
    using  $perp-left-comm\ perp-perp-in$  by blast
  hence  $Per\ A\ B\ P$ 
    by  $(simp\ add: perp-in-per-3)$ 
  have  $P \neq B$ 
    using  $\langle B\ PerpAt\ B\ A\ P\ B \rangle\ perp-in-distinct$  by blast
  have  $A\ B\ P\ CongA\ A0\ B0\ C0$ 
    by  $(simp\ add: \langle A \neq B \rangle\ \langle A0 \neq B0 \rangle\ \langle C0 \neq B0 \rangle\ \langle P \neq B \rangle\ \langle Per\ A\ B\ P \rangle\ \langle Per\ A0\ B0\ C0 \rangle\ l11-16)$ 
  have  $A\ B\ D\ CongA\ A\ B\ D$ 
    by  $(simp\ add: \langle A \neq B \rangle\ \langle D \neq B \rangle\ conga-refl)$ 
  moreover have  $A0\ B0\ C0\ CongA\ A\ B\ P$ 
    using  $\langle A\ B\ P\ CongA\ A0\ B0\ C0 \rangle\ conga-sym$  by blast
  ultimately have  $A\ B\ D\ LtA\ A\ B\ P$ 

```

```

    by (simp add: ⟨A B D LtA A0 B0 C0⟩ conga-preserves-lta)
  have C B D CongA C B D
    using ⟨Acute C B D⟩ acute-distincts conga-refl by blast
  hence C B D LtA A B P
    using ⟨A0 B0 C0 CongA A B P⟩ ⟨C B D LtA A0 B0 C0⟩ conga-preserves-lta by auto
  have A B D LeA A B P ⟷ C B P LeA C B D
    by (metis ⟨A ≠ B⟩ ⟨Acute C B D⟩ acute-not-bet assms(1) l11-36 not-bet-distincts)
  hence C B P LeA C B D
    by (metis ⟨A B D LtA A B P⟩ ⟨A ≠ B⟩ ⟨D ≠ B⟩ ⟨P ≠ B⟩ lea123456-lta--lta lea-total nlta)
  have Per C B P
    by (meson Col-cases PerpAt-def ⟨B PerpAt B A P B⟩ ⟨Col A C B⟩ col-trivial-1)
  hence A B P CongA C B P
    using ⟨A ≠ B⟩ ⟨C B D CongA C B D⟩ ⟨P ≠ B⟩ ⟨Per A B P⟩ conga-diff45 l11-16 by blast
  have C B P CongA A B P
    using ⟨A B P CongA C B P⟩ conga-sym-equiv by auto
  hence A B P LeA C B D
    using ⟨C B D CongA C B D⟩ ⟨C B P LeA C B D⟩ l11-30 by blast
  hence False
    using ⟨C B D LtA A B P⟩ lea--nlta by auto
}
ultimately show ?thesis
  by blast
qed

lemma lcos-per:
  assumes QCongAAcute a and
    QCong l and
    QCong lp and
    Lcos lp l a and
    l A C and
    lp A B and
    a B A C
  shows Per A B C
proof (cases B = C)
  case True
  thus ?thesis
    by (simp add: l8-5)
next
  case False
  obtain A0 B0 C0 where Per C0 B0 A0 and lp A0 B0 and l A0 C0 and a B0 A0 C0
    using Lcos-def assms(4) by auto
  hence B0 A0 C0 CongA B A C
    using anga-conga assms(1) assms(7) by blast
  have Cong A0 C0 A C
    using ⟨l A0 C0⟩ assms(2) assms(5) lg-cong by auto
  have Cong A0 B0 A B
    using ⟨lp A0 B0⟩ assms(3) assms(6) lg-cong by auto
  hence Cong B0 C0 B C ∧ (B0 ≠ C0 ⟶ A0 B0 C0 CongA A B C ∧ A0 C0 B0 CongA A C B)
    using ⟨B0 A0 C0 CongA B A C⟩ ⟨Cong A0 C0 A C⟩ l11-49 by blast
  hence B0 ≠ C0
    using False cong-reverse-identity by blast
  hence A0 B0 C0 CongA A B C ∧ A0 C0 B0 CongA A C B
    by (simp add: ⟨Cong B0 C0 B C ∧ (B0 ≠ C0 ⟶ A0 B0 C0 CongA A B C ∧ A0 C0 B0 CongA A C B)⟩)
  thus ?thesis
    using Per-cases ⟨Per C0 B0 A0⟩ l11-17 by blast
qed

lemma is-null-anga-dec:
  shows QCongANullAcute a ∨ ¬ QCongANullAcute a
  by blast

lemma lcos-lg:
  assumes Lcos lp l a and
    A B Perp B C and
    a B A C and

```

```

  l A C
shows lp A B
proof -
obtain A' B' C' where Per C' B' A' and lp A' B' and l A' C' and a B' A' C'
  using assms(1) Lcos-def by force
hence Cong A C A' C'
  by (metis TarskiLen-def assms(1) assms(4) is-len-cong lcos-lg-anga)
have B A C CongA B' A' C'
  using anga-conga Lcos-def ⟨a B' A' C'⟩ assms(1) assms(3) by blast
{
  assume QCongANullAcute a
  hence QCongAAcute a ∧ (∀ A B C. (a A B C ⟶ B Out A C))
    using QCongANullAcute-def by simp
  hence ∀ A B C. a A B C ⟶ B Out A C
    by blast
  have l EqLTarski lp
    using ⟨QCongANullAcute a⟩ assms(1) null-lcos-eql by blast
  have A Out B C
    by (simp add: ⟨∀ A B C. a A B C ⟶ B Out A C⟩ assms(3))
  have B A Perp C B
    by (simp add: assms(2) perp-comm)
  hence ¬ Col B A C
    by (simp add: perp-not-col)
  hence False
    using ⟨A Out B C⟩ not-col-permutation-4 out-col by blast
  hence ?thesis
    by blast
}
moreover
{
  assume ¬ QCongANullAcute a
  have A ≠ B
    using assms(2) perp-distinct by presburger
  have A' ≠ B'
    using ⟨B A C CongA B' A' C'⟩ conga-diff1 conga-sym-equiv by blast
  have C ≠ B
    using assms(2) perp-distinct by blast
  moreover
  {
    assume C' = B'
    have QCongAAcute a
      using assms(1) Lcos-def by blast
    hence False
      by (metis ⟨A' ≠ B'⟩ ⟨C' = B'⟩ ⟨¬ QCongANullAcute a⟩ ⟨a B' A' C'⟩
        bet-out not-bet-distincts out-null-anga)
  }
  hence C' ≠ B'
    by blast
  moreover have B PerpAt A B B C
    by (simp add: assms(2) col-trivial-1 col-trivial-3 l8-14-2-1b-bis)
  hence Per A B C
    using perp-in-per by blast
  moreover have Per A' B' C'
    using Per-perm ⟨Per C' B' A'⟩ by blast
  ultimately have A B C CongA A' B' C'
    by (simp add: ⟨A ≠ B⟩ ⟨A' ≠ B'⟩ l11-16)
  have Cong C B C' B' ∧ Cong A B A' B' ∧ B C A CongA B' C' A'
  proof (rule l11-50-2, simp-all add: ⟨A B C CongA A' B' C'⟩)
  {
    assume Col C A B
    have B A Perp C B
      by (simp add: assms(2) perp-comm)
    hence False
      using NCol-cases ⟨Col C A B⟩ perp-not-col by blast
  }
  thus ¬ Col C A B

```

```

    by blast
  show  $C A B \text{ CongA } C' A' B'$ 
    by (simp add:  $\langle B A C \text{ CongA } B' A' C' \rangle \text{ conga-comm}$ )
  show  $\text{Cong } C A C' A'$ 
    using  $\langle \text{Cong } A C A' C' \rangle \text{ not-cong-2143}$  by blast
qed
have  $Q\text{Cong } lp$ 
  using  $Lcos\text{-def } assms(1)$  by blast
moreover have  $lp A' B'$ 
  using  $\langle lp A' B' \rangle$  by auto
moreover have  $\text{Cong } A' B' A B$ 
  using  $\langle \text{Cong } C B C' B' \wedge \text{Cong } A B A' B' \wedge B C A \text{ CongA } B' C' A' \rangle \text{ not-cong-3412}$  by blast
ultimately have  $?thesis$ 
  using  $lg\text{-cong-}lg$  by blast
}
ultimately show  $?thesis$ 
  by blast
qed

```

lemma l13-7:

```

assumes  $Lcos \text{ la } l \text{ a}$  and
   $Lcos \text{ lb } l \text{ b}$  and
   $Lcos \text{ lab } la \text{ b}$  and
   $Lcos \text{ lba } lb \text{ a}$ 
shows  $\text{lab } EqLTarski \text{ lba}$ 
proof -
  have  $Q\text{Cong } l$ 
    using  $assms(1) \text{ lcos-}lg\text{-anga}$  by blast
  have  $Q\text{Cong } la$ 
    using  $assms(1) \text{ lcos-}lg\text{-anga}$  by blast
  have  $Q\text{CongA}Acute \text{ a}$ 
    using  $assms(4) \text{ lcos-}lg\text{-anga}$  by blast
  have  $Q\text{Cong } lb$ 
    using  $assms(2) \text{ lcos-}lg\text{-anga}$  by blast
  have  $Q\text{CongA}Acute \text{ b}$ 
    using  $assms(2) \text{ lcos-}lg\text{-anga}$  by blast
  have  $Q\text{Cong } lab$ 
    using  $assms(3) \text{ lcos-}lg\text{-anga}$  by blast
  have  $Q\text{Cong } lba$ 
    using  $assms(4) \text{ lcos-}lg\text{-anga}$  by blast
  {
    assume  $Q\text{CongA}NullAcute \text{ a}$ 
    hence  $lb \text{ EqLTarski } lba$ 
      using  $assms(4) \text{ null-lcos-eql}$  by blast
    have  $l \text{ EqLTarski } la$ 
      using  $\langle Q\text{CongA}NullAcute \text{ a} \rangle \text{ assms}(1) \text{ null-lcos-eql}$  by auto
    have  $Lcos \text{ lab } l \text{ b}$ 
      using  $EqLTarski\text{-def} \langle l \text{ EqLTarski } la \rangle \text{ assms}(3)$  by presburger
    have  $lb \text{ EqLTarski } lab$ 
      using  $\langle Lcos \text{ lab } l \text{ b} \rangle \text{ assms}(2) \text{ l13-6}$  by auto
    hence  $?thesis$ 
      by (meson  $\langle l \text{ EqLTarski } la \rangle \langle lb \text{ EqLTarski } lba \rangle \text{ assms}(2) \text{ l13-6 } \text{lcos-eql-lcos}$ )
  }
moreover
{
  assume  $\neg Q\text{CongA}NullAcute \text{ a}$ 
  have  $Q\text{CongA}NullAcute \text{ b} \longrightarrow ?thesis$ 
    by (meson  $assms(1) \text{ assms}(2) \text{ assms}(3) \text{ assms}(4) \text{ l13-6 } \text{lcos-eql-lcos } \text{null-lcos-eql}$ )
  moreover {
    assume  $\neg Q\text{CongA}NullAcute \text{ b}$ 
    obtain  $C A B$  where  $la \text{ C } A$  and  $l A B$  and  $a \text{ C } A B$ 
      using  $assms(1) \text{ lcos-const}$  by blast
    have  $C \neq A \wedge B \neq A$ 
      by (metis  $\langle l A B \rangle \langle la \text{ C } A \rangle \text{ assms}(1) \text{ assms}(3) \text{ lcos-const-ab } \text{lcos-}lg\text{-distincts}$ )
  }
  assume  $Col A B C$ 

```

```

hence Col C A B
  using Col-cases by blast
hence A Out C B  $\wedge$  QCongANullAcute a
  using anga-col-null  $\langle$ QCongAACute a $\rangle$   $\langle$ a C A B $\rangle$  by blast
hence False
  using  $\langle$  $\neg$  QCongANullAcute a $\rangle$  by blast
}
obtain P where P C Reflect B A
  using l10-2-existence by blast
hence C P Reflect B A
  using l10-4 by blast
moreover have  $\neg$  Col B A C
  using  $\langle$ Col A B C  $\implies$  False $\rangle$  col-permutation-4 by blast
ultimately have  $\neg$  Col B A P
  using osym-not-col by blast
have l B A
  by (simp add:  $\langle$ QCong l $\rangle$   $\langle$ l A B $\rangle$  lg-sym)
have  $\exists$  D. B A OS D P  $\wedge$  b B A D  $\wedge$  lb A D
  using lcos-const-o [where ?l = l]  $\langle$  $\neg$  Col B A P $\rangle$   $\langle$  $\neg$  QCongANullAcute b $\rangle$ 
   $\langle$ QCong l $\rangle$   $\langle$ QCong lb $\rangle$   $\langle$ QCongAACute b $\rangle$   $\langle$ l B A $\rangle$  assms(2) by simp
then obtain D where B A OS D P and b B A D and lb A D
  by blast
have P  $\neq$  C
  using Col-cases  $\langle$ C P Reflect B A $\rangle$   $\langle$ Col A B C  $\implies$  False $\rangle$  l10-8 by blast
hence B A TS P C
  using l10-14  $\langle$ C  $\neq$  A  $\wedge$  B  $\neq$  A $\rangle$   $\langle$ P C Reflect B A $\rangle$  by simp
have B A OS P D
  by (simp add:  $\langle$ B A OS D P $\rangle$  one-side-symmetry)
hence B A TS D C
  using l9-8-2 [where ?A = P]  $\langle$ B A TS P C $\rangle$  by blast
have la A C
  by (simp add:  $\langle$ QCong la $\rangle$   $\langle$ la C A $\rangle$  lg-sym)
hence Per A C B
  using lcos-per  $\langle$ QCong l $\rangle$   $\langle$ QCong la $\rangle$   $\langle$ QCongAACute a $\rangle$   $\langle$ a C A B $\rangle$   $\langle$ l A B $\rangle$ 
  assms(1) by blast
have b D A B
  using anga-sym  $\langle$ QCongAACute b $\rangle$   $\langle$ b B A D $\rangle$  by auto
hence Per A D B using lcos-per
  using  $\langle$ QCong l $\rangle$   $\langle$ QCong lb $\rangle$   $\langle$ QCongAACute b $\rangle$   $\langle$ l A B $\rangle$   $\langle$ lb A D $\rangle$  assms(2) by blast
{
  assume Col C D A
  hence Per B C D
    using  $\langle$ B A TS D C $\rangle$   $\langle$ Per A C B $\rangle$   $\langle$ Per A D B $\rangle$  col-permutation-3
    per2-preserves-diff ts-distincts by blast
  have Per B D C
    using  $\langle$ B A TS D C $\rangle$   $\langle$ Col C D A $\rangle$   $\langle$ Per A C B $\rangle$   $\langle$ Per A D B $\rangle$  not-col-permutation-1
    per2-preserves-diff ts-distincts by blast
  have C = D
    using  $\langle$ Per B C D $\rangle$   $\langle$ Per B D C $\rangle$  l8-7 by blast
  then obtain T where Col T B A and Bet C T C
    using  $\langle$ B A TS D C $\rangle$  ts-distincts by blast
  hence C = T
    using between-identity by blast
  hence False
    using  $\langle$ B A TS D C $\rangle$   $\langle$ C = D $\rangle$  ts-distincts by blast
}
obtain E where Col C D E and C D Perp A E
  using  $\langle$ Col C D A  $\implies$  False $\rangle$  l8-18-existence by blast
have B A C CongA D A E  $\wedge$  B A D CongA C A E  $\wedge$  Bet C E D
proof (rule l13-2, simp-all add:  $\langle$ Col C D E $\rangle$ )
  show A B TS C D
    using  $\langle$ B A TS D C $\rangle$  invert-two-sides l9-2 by blast
  show Per B C A
    using Per-perm  $\langle$ Per A C B $\rangle$  by blast
  show Per B D A
    using Per-cases  $\langle$ Per A D B $\rangle$  by blast

```

```

  show  $A \ E \ \text{Perp} \ C \ D$ 
  using  $\text{Perp-perm} \langle C \ D \ \text{Perp} \ A \ E \rangle$  by blast
qed
hence  $a \ D \ A \ E$ 
  using  $\text{anga-conga-anga} \langle Q\text{CongAACute } a \rangle \langle a \ C \ A \ B \rangle$   $\text{conga-left-comm}$  by blast
have  $b \ C \ A \ E$ 
  using  $\text{anga-conga-anga} \langle B \ A \ C \ \text{CongA} \ D \ A \ E \wedge B \ A \ D \ \text{CongA} \ C \ A \ E \wedge \text{Bet} \ C \ E \ D \rangle$ 
   $\langle Q\text{CongAACute } b \rangle \langle b \ B \ A \ D \rangle$  by blast
have  $C \ E \ \text{Perp} \ A \ E$ 
proof (rule  $\text{perp-col}$  [where  $?B = D$ ])
{
  assume  $C = E$ 
  {
    fix  $A0 \ B0 \ C0$ 
    assume  $b \ A0 \ B0 \ C0$ 
    hence  $A0 \ B0 \ C0 \ \text{CongA} \ C \ A \ C$ 
    using  $\text{anga-conga} \langle C = E \rangle \langle Q\text{CongAACute } b \rangle \langle b \ C \ A \ E \rangle$  by blast
    have  $A \ \text{Out} \ C \ C$ 
    by (simp add:  $\langle C \neq A \wedge B \neq A \rangle$   $\text{out-trivial}$ )
    hence  $B0 \ \text{Out} \ A0 \ C0$ 
    using  $\text{l11-21-a} \langle A0 \ B0 \ C0 \ \text{CongA} \ C \ A \ C \rangle$   $\text{conga-sym-equiv}$  by blast
  }
  hence  $Q\text{CongANullAcute } b$ 
  using  $\langle Q\text{CongAACute } b \rangle \ Q\text{CongANullAcute-def}$  by blast
}
thus  $C \neq E$ 
  using  $\langle \neg \ Q\text{CongANullAcute } b \rangle$  by blast
show  $C \ D \ \text{Perp} \ A \ E$ 
  by (simp add:  $\langle C \ D \ \text{Perp} \ A \ E \rangle$ )
show  $\text{Col} \ C \ D \ E$ 
  by (simp add:  $\langle \text{Col} \ C \ D \ E \rangle$ )
qed
have  $\text{lab} \ A \ E$ 
proof (rule  $\text{lcos-lg}$  [where  $?l = \text{la}$  and  $?a = b$  and  $?C = C$ ])
  show  $\text{Lcos} \ \text{lab} \ \text{la} \ b$ 
  by (simp add:  $\text{assms}(3)$ )
  show  $A \ E \ \text{Perp} \ E \ C$ 
  using  $\text{Perp-perm} \langle C \ E \ \text{Perp} \ A \ E \rangle$  by blast
  show  $b \ E \ A \ C$  using  $\text{anga-sym}$ 
  using  $\langle Q\text{CongAACute } b \rangle \langle b \ C \ A \ E \rangle$  by auto
  show  $\text{la} \ A \ C$ 
  using  $\text{lg-sym}$  by (simp add:  $\langle \text{la} \ A \ C \rangle$ )
qed
have  $D \ E \ \text{Perp} \ A \ E$ 
proof (rule  $\text{perp-col}$  [where  $?B = C$ ])
{
  assume  $D = E$ 
  {
    fix  $A0 \ B0 \ C0$ 
    assume  $a \ A0 \ B0 \ C0$ 
    have  $A0 \ B0 \ C0 \ \text{CongA} \ D \ A \ D$ 
    using  $\text{anga-conga} \langle D = E \rangle \langle a \ A0 \ B0 \ C0 \rangle \langle a \ D \ A \ E \rangle \langle Q\text{CongAACute } a \rangle$  by blast
    have  $B0 \ \text{Out} \ A0 \ C0$ 
    proof (rule  $\text{l11-21-a}$  [where  $?A=D$  and  $?B=A$  and  $?C=D$ ])
      show  $A \ \text{Out} \ D \ D$ 
      using  $\langle \text{Col} \ C \ D \ A \implies \text{False} \rangle \langle \text{Col} \ C \ D \ E \rangle \langle D = E \rangle$   $\text{out-trivial}$  by force
      show  $D \ A \ D \ \text{CongA} \ A0 \ B0 \ C0$ 
      by (simp add:  $\langle A0 \ B0 \ C0 \ \text{CongA} \ D \ A \ D \rangle$   $\text{conga-sym}$ )
    qed
  }
  hence  $Q\text{CongANullAcute } a$ 
  using  $\langle Q\text{CongAACute } a \rangle \ Q\text{CongANullAcute-def}$  by blast
  hence  $\text{False}$ 
  using  $\langle \neg \ Q\text{CongANullAcute } a \rangle$  by blast
}
thus  $D \neq E$ 

```



```

    by blast
  show  $D \ C \ \text{Perp} \ A \ E$ 
    using  $\text{Perp-cases} \langle C \ D \ \text{Perp} \ A \ E \rangle$  by blast
  show  $\text{Col} \ D \ C \ E$ 
    using  $\langle \text{Col} \ C \ D \ E \rangle \text{ not-col-permutation-4}$  by blast
qed
have  $\text{lba} \ A \ E$ 
proof (rule  $\text{lcos-lg}$  [where  $?a=a$  and  $?l = \text{l}b$  and  $?C=D$ ],
  simp-all add:  $\text{assms}(4) \langle \text{l}b \ A \ D \rangle$ )
  show  $A \ E \ \text{Perp} \ E \ D$ 
    using  $\text{Perp-cases} \langle D \ E \ \text{Perp} \ A \ E \rangle$  by blast
  show  $a \ E \ A \ D$ 
    using  $\langle a \ D \ A \ E \rangle \text{ anga-sym} \langle Q\text{Cong}A\text{ACute} \ a \rangle$  by blast
qed
hence  $\text{lab} \ E\text{qLTarski} \ \text{lba}$ 
  using  $\langle Q\text{Cong} \ \text{lab} \rangle \langle Q\text{Cong} \ \text{lba} \rangle \langle \text{lab} \ A \ E \rangle \text{ assms}(3) \text{ ex-eqL l13-6}$  by blast
}
ultimately have  $?thesis$ 
  by blast
}
ultimately show  $?thesis$ 
  by blast
qed

lemma out-acute:
  assumes  $B \ \text{Out} \ A \ C$ 
  shows  $\text{Acute} \ A \ B \ C$ 
proof -
  have  $A \neq B \wedge C \neq B$ 
    using  $\text{assms out-distinct}$  by auto
  then obtain  $Q$  where  $\neg \text{Col} \ A \ B \ Q$ 
    using  $\text{not-col-exists}$  by blast
  then obtain  $P$  where  $A \ B \ \text{Perp} \ P \ B$  and  $A \ B \ \text{OS} \ Q \ P$ 
    using  $\text{l10-15 not-col-distincts}$  by blast
  hence  $\text{Per} \ P \ B \ A$ 
    using  $\text{Perp-perm perp-per-1}$  by blast
  hence  $\text{Per} \ A \ B \ P$ 
    using  $\text{l8-2}$  by blast
  moreover have  $A \ B \ C \ \text{LtA} \ A \ B \ P$ 
  proof -
    have  $A \ B \ C \ \text{LeA} \ A \ B \ P$ 
      using  $\langle A \ B \ \text{Perp} \ P \ B \rangle \langle A \neq B \wedge C \neq B \rangle \text{ assms l11-31-1 perp-not-eq-2}$  by auto
    moreover have  $\neg A \ B \ C \ \text{CongA} \ A \ B \ P$ 
      by (metis  $\text{Col-def Out-def} \langle A \ B \ \text{OS} \ Q \ P \rangle \langle \text{Per} \ P \ B \ A \rangle \text{ assms col124--nos l11-21-a l8-20-1-R1 l8-6}$ )
    ultimately show  $?thesis$ 
      using  $\text{LtA-def}$  by blast
  qed
  ultimately show  $?thesis$ 
    using  $\text{Acute-def}$  by blast
qed

lemma perp-acute:
  assumes  $\text{Col} \ A \ C \ P$  and
     $P \ \text{PerpAt} \ B \ P \ A \ C$ 
  shows  $\text{Acute} \ A \ B \ P$ 
proof (cases  $\text{Col} \ P \ A \ B$ )
  case True
  thus  $?thesis$ 
    by (metis  $\text{acute-trivial assms}(2) \text{ l8-14-2-1b not-col-distincts not-col-permutation-2 perp-in-distinct}$ )
next
  case False
  have  $\text{Per} \ A \ P \ B$ 
    using  $\text{PerpAt-def assms}(2) \text{ col-trivial-1 l8-2}$  by presburger
  thus  $?thesis$  using  $\text{l11-43}$ 

```

by (metis False acute-sym col-trivial-1 col-trivial-3)
qed

lemma null-lcos:
 assumes $QCong\ l$ and
 $\neg QCongNull\ l$ and
 $QCongANullAcute\ a$
 shows $Lcos\ l\ l\ a$
proof –
 obtain $A\ B\ C$ where $a\ A\ B\ C$
 using ex-points-anga $QCongANullAcute-def\ assms(3)$ by presburger
 hence $B\ Out\ A\ C$
 using $assms(3)\ is-null-anga-out$ by auto
 hence $A \neq B$
 by (simp add: out-distinct)
 obtain $A'\ B'$ where $l\ A'\ B'$
 using $assms(1)\ ex-points-lg$ by blast
 obtain P where $l\ B\ P$ and $B\ Out\ P\ A$
 using ex-point-lg-out $\langle A \neq B \rangle\ assms(1)\ assms(2)$ by metis
 moreover have $Per\ P\ P\ B$
 by (simp add: l8-20-1-R1)
 moreover have $a\ P\ B\ P$
 by (metis $assms(1)\ assms(2)\ assms(3)\ calculation(1)\ is-null-all\ lg-null-trivial$)
 ultimately show ?thesis
 using $Lcos-def\ QCongANullAcute-def\ assms(1)\ assms(3)$ by auto
qed

lemma lcos-exists:
 assumes $QCongAAcute\ a$ and
 $QCong\ l$ and
 $\neg QCongNull\ l$
 shows $\exists\ lp.\ Lcos\ lp\ l\ a$
proof (cases $QCongANullAcute\ a$)
 case True
 thus ?thesis
 using $assms(2)\ assms(3)\ null-lcos$ by blast
next
 case False
 obtain $A\ B$ where $l\ A\ B$
 using $assms(2)\ ex-points-lg$ by presburger
 obtain C where $a\ A\ B\ C$
 by (metis $QCongNull-def\ \langle l\ A\ B \rangle\ anga-const\ assms(1)\ assms(2)\ assms(3)$)
 {
 assume $Col\ B\ C\ A$
 hence $B\ Out\ A\ C \wedge QCongANullAcute\ a$
 using $\langle a\ A\ B\ C \rangle\ anga-col-out\ assms(1)\ col-permutation-2\ out-null-anga$ by blast
 hence False
 using False by blast
 }
 hence $\neg Col\ B\ C\ A$
 by blast
 then obtain P where $Col\ B\ C\ P$ and $B\ C\ Perp\ A\ P$
 using l8-18-existence by blast
 obtain lp where $QCong\ lp$ and $lp\ B\ P$
 using lg-exists by fastforce
 obtain $A'\ B'\ C'$ where $Acute\ A'\ B'\ C'$ and $\forall\ X\ Y\ Z.\ A'\ B'\ C'\ CongA\ X\ Y\ Z \longleftrightarrow a\ X\ Y\ Z$
 using $QCongAAcute-def\ assms(1)$ by metis
 hence $A'\ B'\ C'\ CongA\ A\ B\ C$
 using $\langle a\ A\ B\ C \rangle$ by auto
 hence $A\ B\ C\ CongA\ A'\ B'\ C'$
 using not-conga-sym by blast
 moreover have $C'\ InAngle\ A'\ B'\ C'$
 using $\langle Acute\ A'\ B'\ C' \rangle\ acute-distincts\ inangle3123$ by blast
 ultimately have $A\ B\ C\ LeA\ A'\ B'\ C'$
 by (simp add: conga--lea)
 hence $Acute\ A\ B\ C$

```

    using ⟨Acute A' B' C'⟩ acute-lea-acute by blast
have B P Perp P A
proof (rule perp-col [where ?B = C], simp-all add:⟨Col B C P⟩)
{
  assume B = P
  hence Per A B C
    using ⟨B C Perp A P⟩ l8-2 perp-per-1 by blast
  moreover have ¬ Per A B C
    using ⟨Acute A B C⟩ by (simp add: acute-not-per)
  ultimately have False
    by simp
}
thus B ≠ P
  by blast
show B C Perp P A
  using Perp-perm ⟨B C Perp A P⟩ by blast
qed
hence Per A P B
  using Perp-cases perp-per-1 by blast
moreover have lp B P
  using ⟨lp B P⟩ by blast
moreover have l B A
  by (simp add: ⟨l A B⟩ assms(2) lg-sym)
moreover have a A B P
proof (rule anga-out-anga [where ?A=A and ?C=C], simp-all add:⟨a A B C⟩)
  show QCongAACute a
    by (simp add: assms(1))
  show B Out A A
    by (metis ⟨¬ Col B C A⟩ col-trivial-3 out-trivial)
  show B Out C P
proof (rule perp-acute-out [where ?C=A])
  show Acute C B A
    by (simp add: ⟨Acute A B C⟩ acute-sym)
  show C B Perp A P
    using Perp-perm ⟨B C Perp A P⟩ by blast
  show Col C B P
    using ⟨Col B C P⟩ not-col-permutation-4 by blast
qed
qed
hence a P B A
  using anga-sym assms(1) by blast
ultimately show ?thesis
  using Lcos-def ⟨^thesis. (∧lp. [QCong lp; lp B P] ⇒ thesis) ⇒ thesis⟩
    assms(1) assms(2) by fastforce
qed

```

```

lemma lcos-uniqueness:
  assumes Lcos l1 l a and
    Lcos l2 l a
  shows l1 EqLTarski l2
  using assms(1) assms(2) l13-6 by blast

```

```

lemma lcos-ega-lcos:
  assumes Lcos lp l a and
    a EqA b
  shows Lcos lp l b
proof -
  have Lcos lp l a
    using lcos-lg-anga assms(1) by blast
  have QCong l
    using lcos-lg-anga assms(1) by blast
  have QCong lp
    using lcos-lg-anga assms(1) by blast
  have QCongAACute a
    using lcos-lg-anga assms(1) by blast
  have ∀ A B C. a A B C ⟷ b A B C

```

```

  using EqA-def assms(2) by presburger
have QCongA a using anga-is-ang
  by (simp add: ⟨QCongAACute a⟩)
have QCongA b
  using eqA-preserves-ang ⟨QCongA a⟩ assms(2) by blast
have QCongAAcute b
  using eqA-preserves-anga ⟨QCongAACute a⟩ assms(2) by blast
obtain A B C where Per C B A and lp A B and l A C and a B A C
  using Lcos-def assms(1) by auto
thus ?thesis
  using ⟨∀ A B C. a A B C = b A B C⟩ assms(1) by presburger
qed

```

```

lemma lcos-eq-refl:
  assumes QCong la and
    ¬ QCongNull la and
    QCongAAcute a
  shows EqLcos la a la a
proof -
  obtain lp where Lcos lp la a
    using lcos-exists assms(1) assms(2) assms(3) by blast
  thus ?thesis
    using EqLcos-def by blast
qed

```

```

lemma lcos-eq-sym:
  assumes EqLcos la a lb b
  shows EqLcos lb b la a
proof -
  obtain lp where Lcos lp la a and Lcos lp lb b
    using EqLcos-def assms(1) by blast
  thus ?thesis
    using EqLcos-def by blast
qed

```

```

lemma lcos-eq-trans:
  assumes EqLcos la a lb b and
    EqLcos lb b lc c
  shows EqLcos la a lc c
proof -
  obtain lab where Lcos lab la a and Lcos lab lb b
    using EqLcos-def assms(1) by blast
  obtain lbc where Lcos lbc lb b and Lcos lbc lc c
    using EqLcos-def assms(2) by blast
  hence lab EqLTarski lbc
    using ⟨Lcos lab lb b⟩ lcos-uniqueness by auto
  hence Lcos lbc la a
    by (metis EqLTarski-def ⟨Lcos lab la a⟩ lcos-eql-lcos)
  thus ?thesis
    using EqLcos-def ⟨Lcos lbc lc c⟩ by blast
qed

```

```

lemma lcos2-comm:
  assumes Lcos2 lp l a b
  shows Lcos2 lp l b a
proof -
  obtain la where Lcos la l a and Lcos lp la b
    using Lcos2-def assms(1) by blast
  hence ¬ QCongNull la ∧ ¬ QCongNull l
    using lcos-lg-not-null by blast
  then obtain lb where Lcos lb l b
    using lcos-exists ⟨Lcos la l a⟩ ⟨Lcos lp la b⟩ lcos-lg-anga by blast
  moreover
  obtain lp' where Lcos lp' lb a
    by (meson ⟨Lcos la l a⟩ calculation lcos-exists lcos-lg-anga lcos-not-lg-null)
  hence Lcos lp lb a

```

by (meson $\langle Lcos\ la\ l\ a \rangle \langle Lcos\ lp\ la\ b \rangle$ calculation l13-6 l13-7 lcos-eql-lcos)
ultimately show ?thesis
using Lcos2-def by blast
qed

lemma lcos2-exists:
assumes $QCong\ l$ and
 $\neg QCongNull\ l$ and
 $QCongAAcute\ a$ and
 $QCongAAcute\ b$
shows $\exists\ lp. Lcos2\ lp\ l\ a\ b$
proof -
obtain la where $Lcos\ la\ l\ a$
using assms(1) assms(2) assms(3) lcos-exists by blast
hence $\neg QCongNull\ la \wedge \neg QCongNull\ l$
using lcos-lg-not-null by blast
then obtain lab where $Lcos\ lab\ la\ b$
using $\langle Lcos\ la\ l\ a \rangle$ assms(4) lcos-exists lcos-lg-anga by blast
thus ?thesis
using Lcos2-def $\langle Lcos\ la\ l\ a \rangle$ by blast
qed

lemma lcos2-exists':
assumes $QCong\ l$ and
 $\neg QCongNull\ l$ and
 $QCongAAcute\ a$ and
 $QCongAAcute\ b$
shows $\exists\ la\ lab. Lcos\ la\ l\ a \wedge Lcos\ lab\ la\ b$
proof -
obtain la where $Lcos\ la\ l\ a$
using assms(1) assms(2) assms(3) lcos-exists by blast
hence $\neg QCongNull\ la \wedge \neg QCongNull\ l$
using lcos-lg-not-null by blast
then obtain lab where $Lcos\ lab\ la\ b$
using $\langle Lcos\ la\ l\ a \rangle$ assms(4) lcos-exists lcos-lg-anga by blast
thus ?thesis
using $\langle Lcos\ la\ l\ a \rangle$ by blast
qed

lemma lcos2-eq-refl:
assumes $QCong\ l$ and
 $\neg QCongNull\ l$ and
 $QCongAAcute\ a$ and
 $QCongAAcute\ b$
shows $EqLcos2\ l\ a\ b\ l\ a\ b$
by (simp add: EqLcos2-def assms(1) assms(2) assms(3) assms(4) lcos2-exists)

lemma lcos2-eq-sym:
assumes $EqLcos2\ l1\ a\ b\ l2\ c\ d$
shows $EqLcos2\ l2\ c\ d\ l1\ a\ b$
using EqLcos2-def assms by auto

lemma lcos2-uniqueness:
assumes $Lcos2\ l1\ l\ a\ b$ and
 $Lcos2\ l2\ l\ a\ b$
shows $l1\ EqLTarski\ l2$
by (meson Lcos2-def assms(1) assms(2) l13-7 lcos2-comm)

lemma lcos2-eql-lcos2:
assumes $Lcos2\ la\ lla\ a\ b$ and
 $lla\ EqLTarski\ llb$ and
 $la\ EqLTarski\ lb$
shows $Lcos2\ lb\ llb\ a\ b$
by (meson Lcos2-def assms(1) assms(2) assms(3) l13-6 lcos-eql-lcos)

lemma lcos2-lg-anga:

```

assumes  $Lcos2\ lp\ l\ a\ b$ 
shows  $Lcos2\ lp\ l\ a\ b \wedge QCong\ lp \wedge QCong\ l \wedge QCongAacute\ a \wedge QCongAacute\ b$ 
using  $Lcos2-def\ assms\ lcos-lg-anga$  by blast

lemma  $lcos2-eq-trans$ :
assumes  $EqLcos2\ l1\ a\ b\ l2\ c\ d$  and
 $EqLcos2\ l2\ c\ d\ l3\ e\ f$ 
shows  $EqLcos2\ l1\ a\ b\ l3\ e\ f$ 
proof –
obtain  $lp$  where  $Lcos2\ lp\ l1\ a\ b$  and  $Lcos2\ lp\ l2\ c\ d$ 
using  $EqLcos2-def\ assms(1)$  by blast
obtain  $lq$  where  $Lcos2\ lq\ l2\ c\ d$  and  $Lcos2\ lq\ l3\ e\ f$ 
using  $EqLcos2-def\ assms(2)$  by blast
have  $lp\ EqLTarski\ lq$ 
using  $\langle Lcos2\ lp\ l2\ c\ d \rangle \langle Lcos2\ lq\ l2\ c\ d \rangle\ lcos2-uniqueness$  by auto
thus ?thesis
by (metis  $EqLTarski-def\ EqLcos2-def\ \langle Lcos2\ lp\ l1\ a\ b \rangle \langle Lcos2\ lq\ l3\ e\ f \rangle\ lcos2-eql-lcos2$ )
qed

lemma  $lcos-eq-lcos2-eq$ :
assumes  $QCongAacute\ c$  and
 $EqLcos\ la\ a\ lb\ b$ 
shows  $EqLcos2\ la\ a\ c\ lb\ b\ c$ 
proof –
obtain  $lp$  where  $Lcos\ lp\ la\ a$  and  $Lcos\ lp\ lb\ b$ 
using  $EqLcos-def\ assms(2)$  by blast
hence  $\neg QCongNull\ lp \wedge \neg QCongNull\ la$ 
using  $lcos-lg-not-null$  by blast
then obtain  $lq$  where  $Lcos\ lq\ lp\ c$ 
using  $lcos-exists\ lcos-lg-anga\ \langle Lcos\ lp\ la\ a \rangle\ assms(1)$  by blast
have  $Lcos2\ lq\ la\ a\ c$ 
using  $Lcos2-def\ \langle Lcos\ lp\ la\ a \rangle \langle Lcos\ lq\ lp\ c \rangle$  by blast
moreover have  $Lcos2\ lq\ lb\ b\ c$ 
using  $Lcos2-def\ \langle Lcos\ lp\ lb\ b \rangle \langle Lcos\ lq\ lp\ c \rangle$  by blast
ultimately show ?thesis
using  $EqLcos2-def$  by blast
qed

lemma  $lcos2-lg-not-null$ :
assumes  $Lcos2\ lp\ l\ a\ b$ 
shows  $\neg QCongNull\ l \wedge \neg QCongNull\ lp$ 
proof –
obtain  $la$  where  $Lcos\ la\ l\ a$  and  $Lcos\ lp\ la\ b$ 
using  $assms(1)\ Lcos2-def$  by auto
thus ?thesis
using  $lcos-lg-not-null$  by blast
qed

lemma  $lcos3-lcos-1-2-a$ :
assumes  $Lcos3\ lp\ l\ a\ b\ c$ 
shows  $\exists\ la.\ Lcos\ la\ l\ a \wedge Lcos2\ lp\ la\ b\ c$ 
using  $Lcos2-def\ Lcos3-def\ assms$  by presburger

lemma  $lcos3-lcos-1-2-b$ :
assumes  $\exists\ la.\ Lcos\ la\ l\ a \wedge Lcos2\ lp\ la\ b\ c$ 
shows  $Lcos3\ lp\ l\ a\ b\ c$ 
using  $Lcos2-def\ Lcos3-def\ assms$  by blast

lemma  $lcos3-lcos-1-2$ :
shows  $Lcos3\ lp\ l\ a\ b\ c \longleftrightarrow (\exists\ la.\ Lcos\ la\ l\ a \wedge Lcos2\ lp\ la\ b\ c)$ 
using  $lcos3-lcos-1-2-a\ lcos3-lcos-1-2-b$  by blast

lemma  $lcos3-lcos-2-1-a$ :
assumes  $Lcos3\ lp\ l\ a\ b\ c$ 
shows  $\exists\ lab.\ Lcos2\ lab\ l\ a\ b \wedge Lcos\ lp\ lab\ c$ 
using  $Lcos2-def\ assms\ lcos3-lcos-1-2$  by auto

```

```

lemma lcos3-lcos-2-1-b:
  assumes  $\exists lab. Lcos2\ lab\ l\ a\ b \wedge Lcos\ lp\ lab\ c$ 
  shows  $Lcos3\ lp\ l\ a\ b\ c$ 
  using Lcos2-def Lcos3-def assms by blast

lemma lcos3-lcos-2-1:
  shows  $Lcos3\ lp\ l\ a\ b\ c \longleftrightarrow (\exists lab. Lcos2\ lab\ l\ a\ b \wedge Lcos\ lp\ lab\ c)$ 
  using lcos3-lcos-2-1-a lcos3-lcos-2-1-b by blast

lemma lcos3-permut3:
  assumes  $Lcos3\ lp\ l\ a\ b\ c$ 
  shows  $Lcos3\ lp\ l\ b\ a\ c$ 
proof -
  have  $Lcos3\ lp\ l\ a\ b\ c \longleftrightarrow (\exists lab. Lcos2\ lab\ l\ a\ b \wedge Lcos\ lp\ lab\ c)$ 
    by (simp add: lcos3-lcos-2-1)
  then obtain lab where  $Lcos2\ lab\ l\ a\ b$  and  $Lcos\ lp\ lab\ c$ 
    using assms by blast
  thus ?thesis
    using lcos2-comm lcos3-lcos-2-1-b by blast
qed

lemma lcos3-permut1:
  assumes  $Lcos3\ lp\ l\ a\ b\ c$ 
  shows  $Lcos3\ lp\ l\ a\ c\ b$ 
proof -
  have  $Lcos3\ lp\ l\ a\ b\ c \longleftrightarrow (\exists lab. Lcos2\ lab\ l\ a\ b \wedge Lcos\ lp\ lab\ c)$ 
    by (simp add: lcos3-lcos-2-1)
  then obtain la where  $Lcos\ la\ l\ a$  and  $Lcos2\ lp\ la\ b\ c$ 
    using assms lcos3-lcos-1-2-a by blast
  thus ?thesis
    using lcos2-comm lcos3-lcos-1-2-b by blast
qed

lemma lcos3-permut2:
  assumes  $Lcos3\ lp\ l\ a\ b\ c$ 
  shows  $Lcos3\ lp\ l\ c\ b\ a$ 
  using assms lcos3-permut1 lcos3-permut3 by blast

lemma lcos3-exists:
  assumes  $QCong\ l$  and
     $\neg QCongNull\ l$  and
     $QCongAAcute\ a$  and
     $QCongAAcute\ b$  and
     $QCongAAcute\ c$ 
  shows  $\exists lp. Lcos3\ lp\ l\ a\ b\ c$ 
proof -
  obtain la where  $Lcos\ la\ l\ a$ 
    using assms(1) assms(2) assms(3) lcos2-exists' by blast
  hence  $\neg QCongNull\ la \wedge \neg QCongNull\ l$ 
    using lcos-lg-not-null by blast
  then obtain lab where  $Lcos\ lab\ la\ b$ 
    using  $\langle Lcos\ la\ l\ a \rangle$  assms(4) lcos-exists lcos-lg-anga by blast
  hence  $\neg QCongNull\ lab \wedge \neg QCongNull\ la$ 
    using lcos-lg-not-null by blast
  then obtain lp where  $Lcos\ lp\ lab\ c$ 
    using  $\langle Lcos\ lab\ la\ b \rangle$  assms(5) lcos2-exists' lcos-lg-anga by blast
  thus ?thesis
    by (meson  $\langle Lcos\ la\ l\ a \rangle$  assms(4) lcos2-exists lcos3-lcos-1-2
      lcos-lg-anga lcos-not-lg-null)
qed

lemma lcos3-eq-refl:
  assumes  $QCong\ l$  and
     $\neg QCongNull\ l$  and
     $QCongAAcute\ a$  and

```

$QCongAAcute\ b$ and
 $QCongAAcute\ c$
shows $EqLcos3\ l\ a\ b\ c\ l\ a\ b\ c$
by (*simp add: EqLcos3-def* *assms(1)* *assms(2)* *assms(3)* *assms(4)* *assms(5)* *lcos3-exists*)

lemma *lcos3-eq-sym*:
assumes $EqLcos3\ l1\ a\ b\ c\ l2\ d\ e\ f$
shows $EqLcos3\ l2\ d\ e\ f\ l1\ a\ b\ c$
using *EqLcos3-def* *assms* **by** *auto*

lemma *lcos3-uniqueness*:
assumes $Lcos3\ l1\ l\ a\ b\ c$ and
 $Lcos3\ l2\ l\ a\ b\ c$
shows $l1\ EqLTarski\ l2$
proof –
obtain $la\ lab$ **where** $Lcos\ la\ l\ a$ and $Lcos\ lab\ la\ b$ and $Lcos\ l1\ lab\ c$
using *Lcos3-def* *assms(1)* **by** *blast*
obtain $la'\ lab'$ **where** $Lcos\ la'\ l\ a$ and $Lcos\ lab'\ la'\ b$ and $Lcos\ l2\ lab'\ c$
using *Lcos3-def* *assms(2)* **by** *blast*
have $la\ EqLTarski\ la'$
using $\langle Lcos\ la\ l\ a \rangle\ \langle Lcos\ la'\ l\ a \rangle\ lcos-uniqueness$ **by** *auto*
have $Lcos\ lab'\ la\ b$
by (*meson* $\langle Lcos\ la\ l\ a \rangle\ \langle Lcos\ la'\ l\ a \rangle\ \langle Lcos\ lab'\ la'\ b \rangle\ l13-6\ lcos-eql-lcos$)
have $lab\ EqLTarski\ lab'$
using $\langle Lcos\ lab\ la\ b \rangle\ \langle Lcos\ lab'\ la\ b \rangle\ l13-6$ **by** *auto*
thus *?thesis*
using $\langle Lcos\ l1\ lab\ c \rangle\ \langle Lcos\ l2\ lab'\ c \rangle\ lcos-eql-lcos\ lcos-uniqueness$ **by** *blast*
qed

lemma *lcos3-eql-lcos3*:
assumes $Lcos3\ la\ lla\ a\ b\ c$ and
 $lla\ EqLTarski\ llb$ and
 $la\ EqLTarski\ lb$
shows $Lcos3\ lb\ llb\ a\ b\ c$
proof –
obtain $lpa\ lpab$ **where** $Lcos\ lpa\ lla\ a$ and $Lcos\ lpab\ lpa\ b$ and $Lcos\ la\ lpab\ c$
using *Lcos3-def* *assms(1)* **by** *blast*
thus *?thesis*
by (*meson* *Lcos3-def* *assms(2)* *assms(3)* *lcos-eql-lcos* *lcos-uniqueness*)
qed

lemma *lcos3-lg-anga*:
assumes $Lcos3\ lp\ l\ a\ b\ c$
shows $Lcos3\ lp\ l\ a\ b\ c \wedge QCong\ lp \wedge QCong\ l \wedge QCongAAcute\ a \wedge QCongAAcute\ b \wedge QCongAAcute\ c$
proof –
obtain $la\ lab$ **where** $Lcos\ la\ l\ a$ and $Lcos\ lab\ la\ b$ and $Lcos\ lp\ lab\ c$
using *Lcos3-def* *assms(1)* **by** *blast*
thus *?thesis*
using *assms* *lcos-lg-anga* **by** *blast*
qed

lemma *lcos3-lg-not-null*:
assumes $Lcos3\ lp\ l\ a\ b\ c$
shows $\neg QCongNull\ l \wedge \neg QCongNull\ lp$
proof –
obtain $la\ lab$ **where** $Lcos\ la\ l\ a$ and $Lcos\ lab\ la\ b$ and $Lcos\ lp\ lab\ c$
using *Lcos3-def* *assms(1)* **by** *blast*
thus *?thesis*
using *lcos-lg-not-null* **by** *blast*
qed

lemma *lcos3-eq-trans*:
assumes $EqLcos3\ l1\ a\ b\ c\ l2\ d\ e\ f$ and
 $EqLcos3\ l2\ d\ e\ f\ l3\ g\ h\ i$
shows $EqLcos3\ l1\ a\ b\ c\ l3\ g\ h\ i$
proof –

obtain lp **where** $Lcos3\ lp\ l1\ a\ b\ c$ **and** $Lcos3\ lp\ l2\ d\ e\ f$
using $EqLcos3-def\ assms(1)$ **by** *blast*
obtain lq **where** $Lcos3\ lq\ l2\ d\ e\ f$ **and** $Lcos3\ lq\ l3\ g\ h\ i$
using $EqLcos3-def\ assms(2)$ **by** *blast*
thus *?thesis*
by (*metis* $EqLTarski-def\ EqLcos3-def\ \langle Lcos3\ lp\ l1\ a\ b\ c\rangle\ \langle Lcos3\ lp\ l2\ d\ e\ f\rangle$
 $lcos3-eql-lcos3\ lcos3-uniqueness$)

qed

lemma $lcos-eq-lcos3-eq$:
assumes $QCongAAcute\ c$ **and**
 $QCongAAcute\ d$ **and**
 $EqLcos\ la\ a\ lb\ b$
shows $EqLcos3\ la\ a\ c\ d\ lb\ b\ c\ d$
proof –
obtain lp **where** $Lcos\ lp\ la\ a$ **and** $Lcos\ lp\ lb\ b$
using $EqLcos-def\ assms(3)$ **by** *blast*
hence $\neg\ QCongNull\ lp \wedge \neg\ QCongNull\ la$
using $lcos-lg-not-null$ **by** *blast*
then obtain lq **where** $Lcos\ lq\ lp\ c$
using $lcos-exists\ \langle Lcos\ lp\ lb\ b\rangle\ assms(1)\ lcos-lg-anga$ **by** *blast*
hence $\neg\ QCongNull\ lq \wedge \neg\ QCongNull\ lp$
using $lcos-lg-not-null$ **by** *blast*
then obtain lm **where** $Lcos\ lm\ lq\ d$
using $lcos-exists\ \langle Lcos\ lq\ lp\ c\rangle\ assms(2)\ lcos-lg-anga$ **by** *blast*
thus *?thesis*
using $EqLcos3-def\ Lcos3-def\ \langle Lcos\ lp\ la\ a\rangle\ \langle Lcos\ lp\ lb\ b\rangle\ \langle Lcos\ lq\ lp\ c\rangle$ **by** *blast*

qed

lemma $lcos2-eq-lcos3-eq$:
assumes $QCongAAcute\ e$ **and**
 $EqLcos2\ la\ a\ b\ lb\ c\ d$
shows $EqLcos3\ la\ a\ b\ e\ lb\ c\ d\ e$
proof –
obtain lp **where** $Lcos2\ lp\ la\ a\ b$ **and** $Lcos2\ lp\ lb\ c\ d$
using $EqLcos2-def\ assms(2)$ **by** *blast*
hence $\neg\ QCongNull\ la \wedge \neg\ QCongNull\ lp$
using $lcos2-lg-not-null$ **by** *blast*
then obtain lq **where** $Lcos\ lq\ lp\ e$
using $lcos-exists\ \langle Lcos2\ lp\ la\ a\ b\rangle\ assms(1)\ lcos2-lg-anga$ **by** *blast*
hence $\neg\ QCongNull\ lq \wedge \neg\ QCongNull\ lp$
using $lcos-lg-not-null$ **by** *blast*
have $Lcos3\ lq\ la\ a\ b\ e$
using $\langle Lcos\ lq\ lp\ e\rangle\ \langle Lcos2\ lp\ la\ a\ b\rangle\ lcos3-lcos-2-1$ **by** *blast*
moreover have $Lcos3\ lq\ lb\ c\ d\ e$
using $\langle Lcos\ lq\ lp\ e\rangle\ \langle Lcos2\ lp\ lb\ c\ d\rangle\ lcos3-lcos-2-1$ **by** *auto*
ultimately show *?thesis*
using $EqLcos3-def$ **by** *blast*

qed

lemma $projp-id$:
assumes $P\ P'\ Projp\ A\ B$ **and**
 $P\ Q'\ Projp\ A\ B$
shows $P' = Q'$
proof –
have $Col\ A\ B\ P' \wedge A\ B\ Perp\ P\ P' \longrightarrow ?thesis$
by (*metis* $Projp-def\ assms(2)\ l8-18-uniqueness\ perp-not-col2$)
moreover have $Col\ A\ B\ P \wedge P = P' \longrightarrow ?thesis$
by (*metis* $Projp-def\ assms(2)\ perp-not-col2$)
ultimately show *?thesis*
using $Projp-def\ assms(1)$ **by** *force*

qed

lemma $projp-idem$:
assumes $P\ P'\ Projp\ A\ B$
shows $P'\ P'\ Projp\ A\ B$

using *Projp-def* *assms* **by** *force*

lemma *col-projp-eq*:

assumes *Col A B P* **and**

P P' Projp A B

shows $P = P'$

by (*meson Projp-def assms(1) assms(2) perp-not-col2*)

lemma *projp-col*:

assumes *P P' Projp A B*

shows *Col A B P'*

using *Projp-def* *assms* **by** *force*

lemma *project-id*:

assumes *P P' Proj A B X Y* **and**

Col A B P

shows $P = P'$

proof –

{

assume *P P' Par X Y*

have $P' = A \longrightarrow A B Par X Y$

by (*metis Proj-def* $\langle P P' Par X Y \rangle$ *assms(1) assms(2) not-col-permutation-5*
par-col-par-2 par-left-comm)

moreover

{

assume $P' \neq A$

have *Col P' P A*

by (*metis Proj-def assms(1) assms(2) col3 not-col-distincts*)

moreover **have** *P' P Par X Y*

by (*simp add:* $\langle P P' Par X Y \rangle$ *par-left-comm*)

ultimately **have** *A P' Par X Y*

using $\langle P' \neq A \rangle$ *par-col-par-2 par-left-comm* **by** *presburger*

hence *A B Par X Y*

by (*metis Proj-def assms(1) not-col-permutation-5 par-col-par-2*)

}

ultimately **have** *A B Par X Y*

by *blast*

hence *False*

using *Proj-def assms(1)* **by** *force*

}

thus *?thesis*

using *Proj-def assms(1)* **by** *force*

qed

lemma *project-not-id*:

assumes *P P' Proj A B X Y* **and**

$\neg Col A B P$

shows $P \neq P'$

using *Proj-def assms(1) assms(2)* **by** *force*

lemma *project-col*:

assumes *P P Proj A B X Y*

shows *Col A B P*

using *assms project-not-id* **by** *blast*

lemma *project-not-col*:

assumes *P P' Proj A B X Y* **and**

$P \neq P'$

shows $\neg Col A B P$

using *assms(1) assms(2) project-id* **by** *blast*

lemma *par-col-project*:

assumes $A \neq B$ **and**

$\neg A B Par X Y$ **and**

```

  P P' Par X Y and
  Col A B P'
shows P P' Proj A B X Y
using Proj-def assms(1) assms(2) assms(3) assms(4) par-neq2 by presburger

lemma cong-conga3-cong3:
  assumes  $\neg$  Col A B C and
    Cong A B A' B' and
    A B C CongA3 A' B' C'
  shows A B C Cong3 A' B' C'
  using CongA3-def assms(1) assms(2) assms(3) l11-50-2 Cong3-def by blast

lemma project-par-dir:
  assumes  $P \neq P'$  and
    P P' Proj A B X Y
  shows P P' Par X Y
  using Proj-def assms(1) assms(2) by presburger

lemma project-idem:
  assumes P P' Proj A B X Y
  shows P' P' Proj A B X Y
  using Proj-def assms by force

lemma perp-projp:
  assumes P' PerpAt A B P P'
  shows P P' Projp A B
proof -
  have  $A \neq B$ 
  using assms perp-in-distinct by auto
  moreover have Col A B P'
  using assms perp-in-col by blast
  moreover have A B Perp P P'
  using assms perp-in-perp by auto
  ultimately show ?thesis
  using Projp-def by blast
qed

lemma proj-distinct:
  assumes P P' Projp A B
  shows  $P' \neq A \vee P' \neq B$ 
proof -
  have Col A B P'  $\wedge$  A B Perp P P'  $\longrightarrow$  ?thesis
  using perp-distinct by blast
  moreover have Col A B P  $\wedge$  P = P'  $\longrightarrow$  ?thesis
  using Projp-def assms by presburger
  ultimately show ?thesis
  using Projp-def assms by auto
qed

lemma l13-10-aux1:
  assumes Col PO A B and
    Col PO P Q and
    PO P Perp P A and
    PO Q Perp Q B and
    QCong la and
    QCong lb and
    QCong lp and
    QCong lq and
    la PO A and
    lb PO B and
    lp PO P and
    lq PO Q
  shows  $\exists a. QCongA\acute{A} a \wedge Lcos lp la a \wedge Lcos lq lb a$ 
proof -
  have Acute A PO P
  proof (rule perp-acute [where ?C=P])

```

```

show Col A P P
  using col-trivial-2 by auto
show P PerpAt PO P A P
  by (simp add: assms(3) col-trivial-2 l8-15-1 perp-right-comm)
qed
have P ≠ PO
  using assms(3) perp-distinct by blast
have A ≠ PO
  using ⟨Acute A PO P⟩ acute-distincts by presburger
have Q ≠ PO
  using assms(4) perp-distinct by blast
have B ≠ PO
  using assms(4) col-trivial-2 perp-not-col by blast
obtain a where QCongAAcute a and a A PO P
  using ⟨Acute A PO P⟩ ex-anga by blast
have ¬ PO P Par A P
  using Par-cases assms(3) perp-not-par by blast
have A ≠ P
  using assms(3) perp-distinct by blast
have PO PO Proj PO P A P
  using Proj-def ⟨A ≠ P⟩ ⟨P ≠ PO⟩ ⟨¬ PO P Par A P⟩ col-trivial-3 by presburger
have A P Proj PO P A P
  using Proj-def ⟨A ≠ P⟩ ⟨P ≠ PO⟩ ⟨¬ PO P Par A P⟩ col-trivial-2 par-reflexivity by auto
have B Q Par A P
proof (rule l12-9 [where ?C1.0=PO and ?C2.0=P])
  show Coplanar PO P B A
    using assms(1) ncop--ncols not-col-permutation-5 by blast
  show Coplanar PO P B P
    using ncop-distincts by blast
  show Coplanar PO P Q A
    using assms(2) ncop--ncols by blast
  show Coplanar PO P Q P
    using ncop-distincts by blast
  show B Q Perp PO P
    by (metis Col-cases Perp-cases ⟨P ≠ PO⟩ assms(2) assms(4) perp-col)
  show A P Perp PO P
    using Perp-perm assms(3) by blast
qed
hence B Q Proj PO P A P
  unfolding Proj-def using ⟨A ≠ P⟩ ⟨P ≠ PO⟩ ⟨¬ PO P Par A P⟩ assms(2) by blast
{
  assume Bet PO A B
  have Bet PO P Q
  proof (rule per23-preserves-bet [where ?B=A and ?C=B], simp-all add: ⟨Bet PO A B⟩ assms(2))
    show PO ≠ P
      using ⟨P ≠ PO⟩ by blast
    show PO ≠ Q
      using ⟨Q ≠ PO⟩ by blast
    show Per PO P A
      by (simp add: assms(3) perp-comm perp-per-1)
    show Per PO Q B
      using assms(4) perp-comm perp-per-1 by blast
  qed
  have A PO P CongA B PO Q
  proof (rule out2--conga)
    show PO Out B A
      by (simp add: Out-def ⟨A ≠ PO⟩ ⟨B ≠ PO⟩ ⟨Bet PO A B⟩)
    show PO Out Q P
      using Out-def ⟨Bet PO P Q⟩ ⟨P ≠ PO⟩ ⟨Q ≠ PO⟩ by blast
  qed
  hence a B PO Q
    using ⟨QCongAAcute a⟩ ⟨a A PO P⟩ anga-conga-anga by blast
}
moreover
{
  assume Bet A B PO

```

```

have Bet PO Q P
proof (rule per23-preserves-bet [where ?B=B and ?C=A])
  show Bet PO B A
    using Bet-cases ⟨Bet A B PO⟩ by blast
  show PO ≠ Q
    using ⟨Q ≠ PO⟩ by auto
  show PO ≠ P
    using ⟨P ≠ PO⟩ by blast
  show Col PO Q P
    using Col-cases assms(2) by auto
  show Per PO Q B
    by (simp add: assms(4) perp-comm perp-per-1)
  show Per PO P A
    by (simp add: assms(3) perp-left-comm perp-per-2)
qed
hence Bet P Q PO
  using Bet-cases by blast
have A PO P CongA B PO Q
proof (rule out2--conga)
  show PO Out B A
    by (simp add: ⟨B ≠ PO⟩ ⟨Bet A B PO⟩ bet-out-1)
  show PO Out Q P
    by (simp add: ⟨Bet P Q PO⟩ ⟨Q ≠ PO⟩ bet-out-1)
qed
hence a B PO Q
  using ⟨QCongAACute a⟩ ⟨a A PO P⟩ anga-conga-anga by blast
}
moreover
{
  assume Bet B PO A
  have Bet Q PO P
  proof (rule per13-preserves-bet [where ?A=B and ?C=A])
    show Bet B PO A
      by (simp add: ⟨Bet B PO A⟩)
    show PO ≠ Q
      using ⟨Q ≠ PO⟩ by auto
    show PO ≠ P
      using ⟨P ≠ PO⟩ by blast
    show Col Q PO P
      using Col-cases assms(2) by blast
    show Per PO Q B
      by (simp add: assms(4) perp-comm perp-per-1)
    show Per PO P A
      using Perp-cases assms(3) perp-per-1 by blast
  qed
  have A PO P CongA B PO Q
    using ⟨A ≠ PO⟩ ⟨B ≠ PO⟩ ⟨Bet B PO A⟩ ⟨Bet Q PO P⟩ ⟨P ≠ PO⟩ ⟨Q ≠ PO⟩
      conga-sym-equiv l11-14 by blast
  hence a B PO Q
    using ⟨QCongAACute a⟩ ⟨a A PO P⟩ anga-conga-anga by blast
}
ultimately have a B PO Q
  using Col-def assms(1) by blast
have QCongAACute a
  by (simp add: ⟨QCongAACute a⟩)
moreover have Lcos lp la a
proof —
  have Per A P PO
    using Perp-perm assms(3) perp-per-2 by blast
  moreover have a P PO A
    using ⟨a A PO P⟩ anga-sym ⟨QCongAACute a⟩ by blast
  ultimately show ?thesis
    using assms(11) assms(9) Lcos-def ⟨QCongAACute a⟩ assms(5) assms(7) by auto
qed
moreover have Lcos lq lb a
proof —

```

```

have Per B Q PO
  using Perp-perm assms(4) perp-per-1 by blast
moreover have a Q PO B
  by (simp add: ⟨QCongAACute a⟩ ⟨a B PO Q⟩ anga-sym)
ultimately show ?thesis
  using assms(10) assms(12) Lcos-def ⟨QCongAACute a⟩ assms(6) assms(8) by auto
qed
ultimately show ?thesis
  by blast
qed

lemma l13-10-aux2:
  assumes Col PO A B and
    QCong la and
    QCong lla and
    QCong lb and
    QCong llb and
    la PO A and
    lla PO A and
    lb PO B and
    llb PO B and
    A ≠ PO and
    B ≠ PO
  shows ∃ a. QCongAACute a ∧ Lcos lla la a ∧ Lcos llb lb a
proof -
  have Acute A PO A
    using acute-trivial assms(10) by auto
  then obtain a where QCongAACute a and a A PO A
    using anga-exists ex-anga by fastforce
  have Col A PO A
    using not-col-distincts by blast
  hence QCongANullAcute a
    using anga-col-null ⟨QCongAACute a⟩ ⟨a A PO A⟩ by blast
  have la EqLTarski lla
    using assms(2) assms(3) assms(6) assms(7) ex-eqL by auto
  moreover
  have Lcos la la a
  proof (rule null-lcos)
    show QCong la
      by (simp add: assms(2))
    show ¬ QCongNull la
      using assms(10) assms(2) assms(6) between-cong lg-cong
        lg-null-instance not-bet-distincts by blast
    show QCongANullAcute a
      by (simp add: ⟨QCongANullAcute a⟩)
  qed
  ultimately have Lcos lla la a
    using l13-6 lcos-eql-lcos by blast
  moreover have Lcos llb lb a
  proof -
    obtain b where QCongAACute b and b B PO B
      using anga-exists acute-trivial assms(11) by blast
    have a EqA b using null-anga
      using AngAcute-def ⟨QCongAACute a⟩ ⟨QCongAACute b⟩ ⟨a A PO A⟩ ⟨b B PO B⟩ by presburger
    have lb EqLTarski llb
      using assms(4) assms(5) assms(8) assms(9) ex-eqL by blast
    have ¬ QCongNull lb
      using assms(11) assms(4) assms(8) cong-identity lg-cong lg-null-instance by blast
    hence Lcos lb lb a
      using null-lcos assms(4) ⟨QCongANullAcute a⟩ by blast
    thus ?thesis
      using ⟨lb EqLTarski llb⟩ l13-6 lcos-eql-lcos by blast
  qed
  ultimately show ?thesis
    using ⟨QCongAACute a⟩ by blast
qed

```

```

lemma l13-6-bis:
  assumes Lcos lp l1 a and
    Lcos lp l2 a
  shows l1 EqLTarski l2
proof -
  have QCongANullAcute a  $\longrightarrow$  ?thesis
    by (metis (full-types) EqLTarski-def assms(1) assms(2) null-lcos-eql)
  moreover
  {
    assume  $\neg$  QCongANullAcute a
    obtain A B C where Per C B A and lp A B and l2 A C and a B A C
      using Lcos-def assms(2) by auto
    obtain A' B' C' where Per C' B' A' and lp A' B' and l1 A' C' and a B' A' C'
      using Lcos-def assms(1) by auto
    {
      assume B = C
      hence Col B A B
        by (simp add: col-trivial-3)
      hence Col B A C
        by (simp add:  $\langle B = C \rangle$  col-trivial-2)
      moreover
      have QCongAACute a
        using lcos-lg-anga assms(2) by blast
      hence  $\neg$  Col B A C
        using not-null-not-col [where ?a=a]  $\langle \neg$  QCongANullAcute a  $\rangle$   $\langle a B A C \rangle$  by blast
      ultimately have False
        by blast
    }
    hence B  $\neq$  C
      by blast
    {
      assume B' = C'
      hence Col B' A' B'
        by (simp add: col-trivial-3)
      hence Col B' A' C'
        by (simp add:  $\langle B' = C' \rangle$  col-trivial-2)
      moreover
      have QCongAACute a
        using lcos-lg-anga assms(2) by blast
      hence  $\neg$  Col B' A' C'
        using not-null-not-col [where ?a=a]  $\langle \neg$  QCongANullAcute a  $\rangle$   $\langle a B' A' C' \rangle$  by blast
      ultimately have False
        by blast
    }
    hence B'  $\neq$  C'
      by blast
    have A  $\neq$  B
      using  $\langle lp A B \rangle$  assms(2) lcos-const-cb lcos-lg-distincts by blast
    have A  $\neq$  C
      using  $\langle A \neq B \rangle \langle Per C B A \rangle$  per-distinct by blast
    have QCong lp
      using assms(1) Lcos-def by blast
    hence Cong A B A' B'
      using lg-cong [where ?l = lp]  $\langle lp A B \rangle \langle lp A' B' \rangle$  by blast
    have QCongAACute a
      using lcos-lg-anga assms(2) by blast
    hence B A C CongA B' A' C'
      using anga-conga [where ?a = a]  $\langle a B A C \rangle \langle a B' A' C' \rangle$  by blast
    have C B A CongA C' B' A'
      using l11-16  $\langle A \neq B \rangle \langle B \neq C \rangle \langle B' = C' \implies False \rangle \langle Cong A B A' B' \rangle \langle Per C B A \rangle$ 
         $\langle Per C' B' A' \rangle$  cong-diff by blast
    hence A B C CongA A' B' C'
      by (simp add: conga-comm)
    hence Cong A C A' C'  $\wedge$  Cong B C B' C'  $\wedge$  A C B CongA A' C' B'
      using l11-50-1  $\langle B A C CongA B' A' C' \rangle \langle Cong A B A' B' \rangle$  Col-cases
  }

```

```

  ⟨A ≠ B⟩ ⟨B ≠ C⟩ ⟨Per C B A⟩ per-col-eq by blast
have Cong A C A' C'
  using ⟨Cong A C A' C' ∧ Cong B C B' C' ∧ A C B Cong A A' C' B'⟩ by auto
have QCong l1
  using assms(1) Lcos-def by blast
have QCong l2
  using assms(2) Lcos-def by blast
have TarskiLen A' C' l1
  using ⟨l1 A' C'⟩ TarskiLen-def ⟨QCong l1⟩ by blast
moreover have l2 A' C'
  using lg-cong-lg [where ?A = A and ?B = C] ⟨QCong l2⟩ ⟨l2 A C⟩
  ⟨Cong A C A' C'⟩ by blast
hence TarskiLen A' C' l2
  using TarskiLen-def ⟨QCong l2⟩ by blast
ultimately have ?thesis
  using all-eql by force
}
ultimately show ?thesis by blast
qed

```

lemma lcos3-lcos2:

```

assumes EqLcos3 l1 a b n l2 c d n
shows EqLcos2 l1 a b l2 c d
proof -
  obtain lp where Lcos3 lp l1 a b n and Lcos3 lp l2 c d n
  using EqLcos3-def assms by blast
  obtain ll1 where Lcos2 ll1 l1 a b and Lcos lp ll1 n
  using ⟨Lcos3 lp l1 a b n⟩ lcos3-lcos-2-1 by fastforce
  obtain ll2 where Lcos2 ll2 l2 c d and Lcos lp ll2 n
  using ⟨Lcos3 lp l2 c d n⟩ lcos3-lcos-2-1 by blast
  have ll1 EqLTarski ll2
    using l13-6-bis [where ?lp = lp and ?a = n] ⟨Lcos lp ll1 n⟩ ⟨Lcos lp ll2 n⟩ by blast
  have Lcos2 ll1 l2 c d
  proof (rule lcos2-eql-lcos2 [where ?lla=l2 and ?la=ll2])
    show Lcos2 ll2 l2 c d
      by (simp add: ⟨Lcos2 ll2 l2 c d⟩)
    show l2 EqLTarski l2
      by (simp add: EqLTarski-def)
    show ll2 EqLTarski ll1
      using EqLTarski-def ⟨ll1 EqLTarski ll2⟩ by auto
  qed
  thus ?thesis
    using ⟨Lcos2 ll1 l1 a b⟩ EqLcos2-def by blast
qed

```

lemma lcos2-lcos:

```

assumes EqLcos2 l1 a c l2 b c
shows EqLcos l1 a l2 b
proof -
  obtain lp where Lcos2 lp l1 a c and Lcos2 lp l2 b c
  using EqLcos2-def assms by blast
  then obtain lx where Lcos lx l1 a and Lcos lp lx c
  using Lcos2-def by blast
  obtain ly where Lcos ly l2 b and Lcos lp ly c
  using Lcos2-def ⟨Lcos2 lp l2 b c⟩ by auto
  hence Lcos lx l2 b
    by (meson ⟨Lcos lp lx c⟩ l13-6-bis lcos-eql-lcos)
  thus ?thesis
    using ⟨Lcos lx l1 a⟩ EqLcos-def by blast
qed

```

lemma lcos-per-anga:

```

assumes Lcos lp la a and
  la PO A and
  lp PO P and
  Per A P PO

```


shows $a A PO P$

proof –

obtain $A' P' O'$ **where** $Per A' P' O'$ **and** $lp O' P'$ **and** $la O' A'$ **and** $a P' O' A'$
using $assms(1)$ $Lcos-def$ **by** $blast$
have $QCong la$
using $assms(1)$ $Lcos-def$ **by** $blast$
hence $Cong PO A O' A'$
using $lg-cong$ [**where** $?l = la$] $assms(2)$ $\langle la O' A' \rangle$ **by** $blast$
have $QCong lp$
using $assms(1)$ $Lcos-def$ **by** $blast$
hence $Cong PO P O' P'$
using $lg-cong$ [**where** $?l = lp$] $assms(3)$ $\langle lp O' P' \rangle$ **by** $blast$
have $\neg QCongNull lp$
using $lcos-lg-not-null$ $assms(1)$ **by** $blast$
have $\neg QCongNull la$
using $lcos-lg-not-null$ $assms(1)$ **by** $blast$
{
assume $A = P$
have $Cong O' A' O' P'$
using $\langle A = P \rangle \langle Cong PO A O' A' \rangle \langle Cong PO P O' P' \rangle$ $cong-inner-transitivity$ **by** $blast$
have $A' = P'$
proof –
{
assume $Col A' P' O'$
have $A' = P' \vee O' = P'$
using $\langle Col A' P' O' \rangle \langle Per A' P' O' \rangle$ $l8-9$ **by** $auto$
moreover have $O' = P' \longrightarrow A' = P'$
using $\langle Cong O' A' O' P' \rangle$ $cong-identity$ **by** $blast$
ultimately have $?thesis$
by $blast$
}
moreover
{
assume $\neg Col A' P' O'$
have $False$
by ($metis \langle Cong O' A' O' P' \rangle \langle Per A' P' O' \rangle \langle \neg Col A' P' O' \rangle$
 $cong-nlt not-col-distincts not-cong-4312 per-lt$)
}
ultimately show $?thesis$
by $blast$
qed
have $QCongANullAcute a$
proof ($rule out-null-anga$ [**where** $?A=P'$ **and** $?B=O'$ **and** $?C=P'$])
show $QCongAACute a$
using $assms(1)$ $Lcos-def$ **by** $blast$
show $a P' O' P'$
using $\langle A' = P' \rangle \langle a P' O' A' \rangle$ **by** $auto$
have $P' \neq O'$
using $\langle A' = P' \rangle \langle a P' O' P' \rangle \langle la O' A' \rangle$ $assms(1)$ $lcos-lg-distincts$ **by** $blast$
thus $O' Out P' P'$
by ($simp add: out-trivial$)
qed
moreover have $P \neq PO$
using $\langle Cong PO P O' P' \rangle \langle lp O' P' \rangle$ $assms(1)$ $cong-diff-4$ $lcos-const-cb$
 $lcos-lg-distincts$ **by** $blast$
ultimately have $?thesis$
by ($simp add: \langle A = P \rangle is-null-all$)
}
moreover
{
assume $A \neq P$
have $PO \neq P$
using $assms(1)$ $assms(3)$ $lcos-const-cb$ $lcos-lg-distincts$ **by** $blast$
hence $Col A P PO \longrightarrow ?thesis$
using $\langle A \neq P \rangle$ $assms(4)$ $l8-9$ **by** $blast$
moreover

```

{
  assume  $\neg \text{Col } A \ P \ PO$ 
  have  $P' \neq O' \vee A' \neq O'$ 
  using  $\langle A \neq P \rangle \langle \text{Cong } PO \ A \ O' \ A' \rangle \langle \text{Cong } PO \ P \ O' \ P' \rangle$  congr-identity by blast
  {
    assume  $A' = P'$ 
    have  $\text{Cong } PO \ P \ PO \ A$ 
    using  $\text{Cong-perm } \langle A' = P' \rangle \langle \text{Cong } PO \ A \ O' \ A' \rangle \langle \text{Cong } PO \ P \ O' \ P' \rangle$ 
      congr-transitivity by blast
    moreover have  $P \ A \ Lt \ A \ PO \wedge P \ PO \ Lt \ A \ PO$ 
    using  $\langle A \neq P \rangle \langle PO \neq P \rangle$  assms(4) lt-left-comm per-lt by blast
    ultimately have False
    using congr-nlt not-congr-2143 by blast
  }
  hence  $A' \neq P'$ 
  by blast
  have  $A \ P \ PO \ \text{CongA } A' \ P' \ O'$ 
  using  $\langle A \neq P \rangle \langle A' \neq P' \rangle \langle \text{Cong } PO \ P \ O' \ P' \rangle \langle PO \neq P \rangle \langle \text{Per } A' \ P' \ O' \rangle$  assms(4)
    congr-diff l11-16 by blast
  have  $\text{Cong } P \ A \ P' \ A' \wedge P \ A \ PO \ \text{CongA } P' \ A' \ O' \wedge P \ PO \ A \ \text{CongA } P' \ O' \ A'$ 
  by (metis  $\langle A \neq P \rangle \langle \text{Cong } PO \ A \ O' \ A' \rangle \langle \text{Cong } PO \ P \ O' \ P' \rangle \langle PO \neq P \rangle$ 
     $\langle \text{Per } A' \ P' \ O' \rangle$  assms(4) congr2-per2--congr l11-51 not-congr-2143 per-distinct)
  have  $Q\text{CongAACute } a$ 
  using assms(1) Lcos-def by blast
  moreover have  $a \ A' \ O' \ P'$ 
  by (simp add:  $\langle a \ P' \ O' \ A' \rangle$  anga-sym calculation)
  moreover have  $A' \ O' \ P' \ \text{CongA } A \ PO \ P$ 
  by (simp add:  $\langle \text{Cong } P \ A \ P' \ A' \wedge P \ A \ PO \ \text{CongA } P' \ A' \ O' \wedge P \ PO \ A \ \text{CongA } P' \ O' \ A' \rangle$ 
    conga-comm conga-sym-equiv)
  ultimately have ?thesis
  using anga-conga-anga by blast
}
ultimately have ?thesis
by blast
}
ultimately show ?thesis
by blast
qed

```

lemma *lcos-lcos-cop--col*:

assumes *Lcos* *lp* *la* *a* and

Lcos *lp* *lb* *b* and

la *PO* *A* and

lb *PO* *B* and

lp *PO* *P* and

a *A* *PO* *P* and

b *B* *PO* *P* and

Coplanar *PO* *A* *B* *P*

shows *Col* *A* *B* *P*

proof –

have $Q\text{Cong } la$

using *assms(1)* *Lcos-def* **by** *blast*

have $Q\text{Cong } lp$

using *assms(1)* *Lcos-def* **by** *blast*

have $Q\text{Cong } lb$

using *assms(2)* *Lcos-def* **by** *blast*

have $Q\text{CongAACute } a$

using *assms(1)* *Lcos-def* **by** *blast*

have $Q\text{CongAACute } b$

using *assms(2)* *Lcos-def* **by** *blast*

have $\text{Per } PO \ P \ A$

using *lcos-per* $\langle Q\text{Cong } la \rangle \langle Q\text{Cong } lp \rangle \langle Q\text{CongAACute } a \rangle$ *anga-sym* *assms(1)*

assms(3) *assms(5)* *assms(6)* **by** *blast*

have $\text{Per } PO \ P \ B$

using *lcos-per* $\langle Q\text{Cong } lb \rangle \langle Q\text{Cong } lp \rangle \langle Q\text{CongAACute } b \rangle$ *anga-sym* *assms(2)*

assms(4) *assms(5)* *assms(7)* **by** *blast*

```

show ?thesis
proof (rule cop-per2--col [where ?A=PO], simp add: assms(8))
  show  $PO \neq P$ 
    using assms(1) assms(5) lcos-const-cb lcos-lg-distincts by blast
  show  $Per\ A\ P\ PO$ 
    using  $\langle Per\ PO\ P\ A \rangle$  l8-2 by blast
  show  $Per\ B\ P\ PO$ 
    using Per-cases  $\langle Per\ PO\ P\ B \rangle$  by auto
qed
qed

lemma l13-10-aux3:
  assumes  $\neg Col\ PO\ A\ A'$  and
     $B \neq PO$  and
     $C \neq PO$  and
     $Col\ PO\ A\ B$  and
     $Col\ PO\ B\ C$  and
     $B' \neq PO$  and
     $C' \neq PO$  and
     $Col\ PO\ A'\ B'$  and
     $Col\ PO\ B'\ C'$  and
     $PO\ Perp2\ B\ C'\ C\ B'$  and
     $PO\ Perp2\ C\ A'\ A\ C'$  and
     $Bet\ A\ PO\ B$ 
  shows  $Bet\ A'\ PO\ B'$ 
proof -
  have  $A \neq PO$ 
    using assms(1) col-trivial-1 by fastforce
  have  $A' \neq PO$ 
    using assms(1) not-col-distincts by blast
  have  $Col\ PO\ A\ C$ 
    by (metis assms(2) assms(4) assms(5) col-trivial-3 colx)
  have  $Col\ PO\ A'\ C'$ 
    by (metis assms(6) assms(8) assms(9) col-trivial-3 colx)
  have  $Bet\ C\ A\ PO \vee Bet\ A\ C\ PO \vee Bet\ PO\ C\ B \vee Bet\ PO\ B\ C$ 
    by (metis NCol-perm  $\langle Col\ PO\ A\ C \rangle$  assms(12) fourth-point not-bet-distincts)
  moreover
  {
    assume  $Bet\ C\ A\ PO$ 
    have  $Bet\ PO\ C'\ A'$ 
    proof (rule perp2-preserves-bet23 [where ?A = A and ?B = C])
      show  $Bet\ PO\ A\ C$ 
        using Bet-cases  $\langle Bet\ C\ A\ PO \rangle$  by blast
      show  $Col\ PO\ C'\ A'$ 
        using  $\langle Col\ PO\ A'\ C' \rangle$  not-col-permutation-5 by blast
      show  $\neg Col\ PO\ A\ C'$ 
        by (metis  $\langle Col\ PO\ C'\ A' \rangle$  assms(1) assms(7) col-trivial-3 colx)
      show  $PO\ Perp2\ A\ C'\ C\ A'$ 
        by (simp add: assms(11) perp2-sym)
    qed
    moreover have  $Bet\ B'\ PO\ C'$ 
  proof (rule perp2-preserves-bet13 [where ?B=B and ?C=C])
    show  $Bet\ B\ PO\ C$ 
      using  $\langle A \neq PO \rangle$   $\langle Bet\ C\ A\ PO \rangle$  assms(12) between-symmetry
        outer-transitivity-between2 by blast
    show  $Col\ PO\ B'\ C'$ 
      by (simp add: assms(9))
    show  $\neg Col\ PO\ B\ B'$ 
      by (metis assms(1) assms(2) assms(4) assms(6) assms(8) col-trivial-3
        colx not-col-permutation-5)
    show  $PO\ Perp2\ B\ C'\ C\ B'$ 
      using assms(10) by auto
  qed
  ultimately have  $Bet\ A'\ PO\ B'$ 
    using assms(7) between-symmetry outer-transitivity-between by blast
}

```

```

moreover
{
  assume Bet A C PO
  have Bet PO A' C'
    by (metis  $\langle \text{Bet } A \ C \ PO \rangle \langle \text{Col } PO \ A' \ C' \rangle$  assms(1) assms(11) assms(3)
      bet-col between-symmetry col-transitivity-1 perp2-preserves-bet23)
  moreover have Bet B' PO C'
  proof (rule perp2-preserves-bet13 [where  $?B = B$  and  $?C = C$ ],
    simp-all add: assms(9) assms(10))
  have Bet C PO B
    using  $\langle \text{Bet } A \ C \ PO \rangle$  assms(12) between-exchange3 by blast
  thus Bet B PO C
    using Bet-cases by blast
  show  $\neg \text{Col } PO \ B \ B'$ 
    by (metis assms(1) assms(2) assms(4) assms(6) assms(8)
      colx not-col-distincts not-col-permutation-2)
  qed
  ultimately have Bet A' PO B'
    using between-inner-transitivity between-symmetry by blast
}
moreover
{
  assume Bet PO C B
  have Bet A PO C
    using  $\langle \text{Bet } PO \ C \ B \rangle$  assms(12) between-inner-transitivity by blast
  have Bet PO B' C'
    by (metis  $\langle \text{Bet } PO \ C \ B \rangle \langle \text{Col } PO \ A \ C \rangle$  assms(1) assms(10) assms(3) assms(6)
      assms(8) assms(9) colx not-col-distincts not-col-permutation-5
      perp2-preserves-bet23 perp2-sym)
  have Bet C' PO A'
  proof (rule perp2-preserves-bet13 [where  $?B = C$  and  $?C = A$ ])
  show Bet C PO A
    by (simp add:  $\langle \text{Bet } A \ PO \ C \rangle$  between-symmetry)
  show Col PO C' A'
    using  $\langle \text{Col } PO \ A' \ C' \rangle$  not-col-permutation-5 by blast
  show  $\neg \text{Col } PO \ C \ C'$ 
    by (metis  $\langle \text{Col } PO \ A \ C \rangle \langle \text{Col } PO \ C' \ A' \rangle$  assms(1) assms(3) assms(7) col-trivial-3 colx)
  show PO Perp2 C A' A C'
    using assms(11) by auto
  qed
  have Bet A' PO B'
    using  $\langle \text{Bet } C' \ PO \ A' \rangle \langle \text{Bet } PO \ B' \ C' \rangle$  between-exchange3 between-symmetry by blast
}
moreover
{
  assume Bet PO C B
  hence Bet A PO C
    using  $\langle \text{Bet } PO \ C \ B \rangle$  assms(12) between-inner-transitivity by blast
  moreover have Bet PO B' C'
  proof (rule perp2-preserves-bet23 [where  $?A = C$  and  $?B = B$ ])
  show Bet PO C B
    by (simp add:  $\langle \text{Bet } PO \ C \ B \rangle$ )
  show Col PO B' C'
    by (simp add: assms(9))
  show  $\neg \text{Col } PO \ C \ B'$ 
    by (metis  $\langle \text{Col } PO \ A \ C \rangle$  assms(1) assms(3) assms(6) assms(8) col-trivial-3 colx
      not-col-permutation-5)
  show PO Perp2 C B' B C'
    by (simp add: assms(10) perp2-sym)
  qed
  moreover have Bet C' PO A'
    using  $\langle \text{Bet } A \ PO \ C \rangle \langle \text{Col } PO \ A' \ C' \rangle$  assms(1) assms(11) between-symmetry
      perp2-preserves-bet13 perp2-sym by blast
  ultimately have Bet A' PO B'
    using  $\langle \text{Bet } PO \ C \ B \implies \text{Bet } A' \ PO \ B' \rangle \langle \text{Bet } PO \ C \ B \rangle$  by blast
}

```

```

ultimately show ?thesis
  by (metis ‹Col PO A C› ‹Col PO A' C'› assms(1) assms(10) assms(11) assms(2)
      assms(4) assms(7) assms(9) between-symmetry colx not-col-distincts
      not-col-permutation-5 outer-transitivity-between perp2-preserves-bet13
      perp2-preserves-bet23 third-point)
qed

lemma l13-10-aux4:
  assumes  $\neg$  Col PO A A' and
    B  $\neq$  PO and
    C  $\neq$  PO and
    Col PO A B and
    Col PO B C and
    B'  $\neq$  PO and
    C'  $\neq$  PO and
    Col PO A' B' and
    Col PO B' C' and
    PO Perp2 B C' C B' and
    PO Perp2 C A' A C' and
    Bet PO A B
  shows PO Out A' B'
proof -
  have A  $\neq$  PO
  using assms(1) not-col-distincts by blast
  have A'  $\neq$  PO
  using assms(1) not-col-distincts by blast
  have Col PO A C
  by (metis assms(2) assms(4) assms(5) col-trivial-3 colx)
  {
    assume A = B
    have  $\neg$  Col A C' PO
    by (metis assms(1) assms(6) assms(7) assms(8) assms(9) col2--eq
        col-permutation-4 col-permutation-5)
    have PO Perp2 C A' C B'
    proof (rule perp2-pseudo-trans [where ?C = C and ?D = B'])
      show PO Perp2 C A' C B'
      using ‹A = B› ‹ $\neg$  Col A C' PO› assms(10) assms(11) perp2-pseudo-trans by blast
      show PO Perp2 C B' C B'
      by (metis ‹A = B› assms(1) assms(5) assms(6) assms(8)
          col-transitivity-2 not-col-permutation-1 perp2-refl)
      show  $\neg$  Col C B' PO
      by (metis ‹A = B› assms(1) assms(3) assms(5) assms(6) assms(8) col2--eq
          col-permutation-4 col-permutation-5)
    qed
    have C A' Par C B'
    by (metis ‹Col PO A C› ‹PO Perp2 C A' C B'› assms(1) assms(6) assms(8)
        between-equality-2 between-trivial col-transitivity-1 not-col-permutation-5
        not-par-not-col par-reflexivity perp2-preserves-bet23 perp2-sym)
    have A' = B'
    using ‹A = B› ‹C A' Par C B'› assms(1) assms(3) assms(5) assms(8)
        col-trivial-3 l6-21 not-col-permutation-2 par-id-2 by blast
    hence ?thesis
    using assms(6) out-trivial by blast
  }
  moreover
  {
    assume A  $\neq$  B
    have Bet C PO A  $\vee$  Bet PO C A  $\vee$  Bet A C B  $\vee$  Bet A B C
    using ‹A  $\neq$  B› ‹A  $\neq$  PO› ‹Col PO A C› assms(12) fourth-point by auto
    moreover
    {
      assume Bet C PO A
      have Bet C PO B
      using ‹A  $\neq$  PO› ‹Bet C PO A› assms(12) outer-transitivity-between by blast
      hence Bet B' PO C'
      by (metis Col-cases assms(1) assms(10) assms(2) assms(4))
    }
  }

```

```

      assms(6) assms(8) assms(9) between-symmetry col2--eq perp2-preserves-bet13)
moreover have Bet A' PO C'
  by (metis ‹Bet C PO A› assms(1) assms(11) assms(6) assms(8)
      assms(9) between-symmetry col-not-col-not-par not-col-distincts
      not-par-not-col perp2-preserves-bet13 perp2-sym)
ultimately have ?thesis
  using ‹A' ≠ PO› assms(6) assms(7) l6-2 by blast
}
moreover
{
  assume Bet PO C A
  hence Bet PO C B
    using assms(12) between-exchange4 by blast
  have Bet PO B' C'
  proof (rule perp2-preserves-bet23 [where ?A = C and ?B = B])
    show Bet PO C B
      using ‹Bet PO C B› by auto
    show Col PO B' C'
      by (simp add: assms(9))
    show ¬ Col PO C B'
      by (metis (full-types) ‹Col PO A C› assms(1) assms(3) assms(6)
          assms(8) colx l6-16-1 not-col-permutation-5)
    show PO Perp2 C B' B C'
      by (simp add: assms(10) perp2-sym)
  qed
  moreover have Bet PO A' C'
    by (metis ‹Bet PO C A› ‹Col PO A C› assms(1) assms(11) assms(3) assms(6)
        assms(8) assms(9) colx not-col-distincts perp2-preserves-bet23)
  ultimately have ?thesis
    using ‹A' ≠ PO› assms(6) bet2--out by force
}
moreover
{
  assume Bet A C B
  have Bet PO A C
    using ‹Bet A C B› assms(12) between-inner-transitivity by blast
  have Bet PO C' A'
  proof (rule perp2-preserves-bet23 [where ?A = A and ?B = C])
    show Bet PO A C
      by (simp add: ‹Bet PO A C›)
    show Col PO C' A'
      by (metis Col-cases assms(6) assms(8) assms(9) col-transitivity-1)
    show ¬ Col PO A C'
      by (metis ‹Col PO C' A'› assms(1) assms(7) col-trivial-3 colx)
    show PO Perp2 A C' C A'
      by (simp add: assms(11) perp2-sym)
  qed
  have Bet PO C B
    using ‹Bet A C B› assms(12) between-exchange2 by blast
  have Bet PO B' C'
  proof (rule perp2-preserves-bet23 [where ?A = C and ?B = B])
    show Bet PO C B
      by (simp add: ‹Bet PO C B›)
    show Col PO B' C'
      by (simp add: assms(9))
    {
      assume Col PO C B'
      have False
        by (metis ‹Col PO A C› ‹Col PO C B'› assms(1) assms(3) assms(6)
            assms(8) col-transitivity-1 not-col-permutation-5)
    }
    thus ¬ Col PO C B'
      by blast
    show PO Perp2 C B' B C'
      by (simp add: assms(10) perp2-sym)
  qed
}

```

```

    hence ?thesis
    by (meson Out-def  $\langle A' \neq PO \rangle \langle Bet PO C' A' \rangle$  assms(6) between-exchange4)
  }
  moreover
  {
    assume Bet A B C
    have Bet PO A C
      using  $\langle A \neq B \rangle \langle Bet A B C \rangle$  assms(12) outer-transitivity-between by blast
    have Bet PO B C
      using  $\langle Bet A B C \rangle \langle Bet PO A C \rangle$  between-exchange2 by blast
    have Bet PO C' B'
    proof (rule perp2-preserves-bet23 [where ?A=B and ?B=C])
      show Bet PO B C
        by (simp add:  $\langle Bet PO B C \rangle$ )
      show Col PO C' B'
        using Col-cases assms(9) by blast
    {
      assume Col PO B C'
      hence Col PO A A'
        by (metis  $\langle Col PO C' B' \rangle$  assms(2) assms(4) assms(6) assms(7) assms(8)
            col-permutation-5 col-transitivity-1)
      hence False
        using assms(1) by blast
    }
    thus  $\neg Col PO B C'$ 
      by blast
    show PO Perp2 B C' C B'
      by (simp add: assms(10))
  }
  qed
  have Bet PO C' A'
  proof (rule perp2-preserves-bet23 [where ?A=A and ?B=C])
    show Bet PO A C
      by (simp add:  $\langle Bet PO A C \rangle$ )
    show Col PO C' A'
      by (metis assms(6) assms(8) assms(9) col-permutation-5 col-trivial-3 colx)
    show  $\neg Col PO A C'$ 
      by (metis  $\langle Col PO C' A' \rangle$  assms(1) assms(7) col-trivial-3 colx)
    show PO Perp2 A C' C A'
      by (simp add: assms(11) perp2-sym)
  }
  qed
  hence ?thesis
    by (metis Out-def  $\langle Bet PO C' B' \rangle$  assms(7) bet-out l5-1)
  }
  ultimately have ?thesis
    by blast
}
ultimately show ?thesis
  by blast
qed

lemma l13-10-aux5:
  assumes  $\neg Col PO A A'$  and
    B  $\neq PO$  and
    C  $\neq PO$  and
    Col PO A B and
    Col PO B C and
    B'  $\neq PO$  and
    C'  $\neq PO$  and
    Col PO A' B' and
    Col PO B' C' and
    PO Perp2 B C' C B' and
    PO Perp2 C A' A C' and
    PO Out A B
  shows PO Out A' B'
proof -
  have A'  $\neq PO$ 

```

```

    using assms(1) not-col-distincts by blast
have A ≠ PO
    using Out-def assms(12) by blast
have B ≠ PO
    using assms(2) by auto
have Bet PO A B ∨ Bet PO B A
    using Out-def assms(12) by force
moreover have Bet PO A B ⟶ ?thesis
    using assms(1) assms(10) assms(11) assms(2) assms(3) assms(4) assms(5)
        assms(6) assms(7) assms(8) assms(9) l13-10-aux4 by blast
moreover
{
    assume Bet PO B A
    have PO Out B' A'
    proof (rule l13-10-aux4 [where ?A = B and ?B = A and ?C = C and ?C' = C])
        show ¬ Col PO B B'
            by (metis assms(1) assms(2) assms(4) assms(6) assms(8) col-trivial-3
                colx not-col-permutation-5)
        show A ≠ PO
            by (simp add: ⟨A ≠ PO⟩)
        show C ≠ PO
            by (simp add: assms(3))
        show Col PO B A
            using Col-cases assms(4) by blast
        show Col PO A C
            by (metis ⟨Col PO B A⟩ assms(2) assms(5) col-transitivity-1)
        show A' ≠ PO
            by (simp add: ⟨A' ≠ PO⟩)
        show C' ≠ PO
            by (simp add: assms(7))
        show Col PO B' A'
            using assms(8) not-col-permutation-5 by blast
        show Col PO A' C'
            by (metis ⟨Col PO B' A'⟩ assms(6) assms(9) col-transitivity-1)
        show PO Perp2 A C' C A'
            by (simp add: assms(11) perp2-sym)
        show PO Perp2 C B' B C'
            by (simp add: assms(10) perp2-sym)
        show Bet PO B A
            by (simp add: ⟨Bet PO B A⟩)
    qed
    hence ?thesis
        using l6-6 by blast
}
ultimately show ?thesis
    by blast
qed

```

```

lemma cop-per2--perp-aux:
  assumes A ≠ B and
    X ≠ Y and
    B ≠ X and
    Coplanar A B X Y and
    Per A B X and
    Per A B Y
  shows A B Perp X Y
proof -
  have B B Perp B X ⟶ A B Perp X Y
    using perp-distinct by blast
  moreover
  {
    assume A B Perp B X
    hence X B Perp A B
      using Perp-cases by blast
    have Col X Y B
    proof (rule cop-per2--col [where ?A = A])

```



```

  show Coplanar A X Y B
    using assms(4) coplanar-perm-3 by blast
  show A ≠ B
    by (simp add: assms(1))
  show Per X B A
    by (simp add: assms(5) l8-2)
  show Per Y B A
    by (simp add: assms(6) l8-2)
qed
have X Y Perp A B
  using perp-col [where ?B = B]
  by (meson Col-cases ⟨Col X Y B⟩ ⟨X B Perp A B⟩ assms(2))
hence A B Perp X Y
  using Perp-cases by blast
}
ultimately show ?thesis
  using ⟨B ≠ X⟩ assms(1) assms(5) per-perp by blast
qed

lemma cop-per2--perp:
  assumes A ≠ B and
    X ≠ Y and
    B ≠ X ∨ B ≠ Y and
    Coplanar A B X Y and
    Per A B X and
    Per A B Y
  shows A B Perp X Y
proof -
  have B ≠ X ⟶ ?thesis
    by (simp add: assms(1) assms(2) assms(4) assms(5) assms(6) cop-per2--perp-aux)
  moreover hence B ≠ Y ⟶ ?thesis
    using cop-per2--perp-aux assms(1) assms(6) per-perp by force
  ultimately show ?thesis
    using assms(3) by blast
qed

lemma l13-10:
  assumes ¬ Col PO A A' and
    B ≠ PO and
    C ≠ PO and
    Col PO A B and
    Col PO B C and
    B' ≠ PO and
    C' ≠ PO and
    Col PO A' B' and
    Col PO B' C' and
    PO Perp2 B C' C B' and
    PO Perp2 C A' A C'
  shows PO Perp2 A B' B A'
proof -
  have Col PO A C
    by (metis assms(2) assms(4) assms(5) col-trivial-3 colx)
  have Col PO A' C'
    by (metis assms(6) assms(8) assms(9) col-trivial-3 colx)
  have A ≠ PO
    using assms(1) not-col-distincts by auto
  have ¬ Col A B' PO
    by (meson assms(1) assms(6) assms(8) col-trivial-3 colx
      not-col-permutation-3 not-col-permutation-5)
  have ∃ P Q. Col B C' P ∧ Col C B' Q ∧ Col PO P Q ∧
    P PerpAt PO P B C' ∧ Q PerpAt PO Q C B'
proof (rule perp2-perp-in)
  show PO Perp2 B C' C B'
    by (simp add: assms(10))
  show ¬ Col PO B C'
    by (metis NCol-perm ⟨Col PO A' C'⟩ assms(1) assms(2) assms(4))

```

```

    assms(7) colx not-col-distincts)
  show  $\neg \text{Col } PO \ C \ B'$ 
    by (meson  $\langle \text{Col } PO \ A \ C \rangle \langle \neg \text{Col } A \ B' \ PO \rangle$  assms(3) col-trivial-3 colx
        not-col-permutation-3 not-col-permutation-5)
qed
then obtain  $L \ L'$  where  $\text{Col } B \ C' \ L$  and  $\text{Col } C \ B' \ L'$  and
   $\text{Col } PO \ L \ L'$  and  $L \text{ PerpAt } PO \ L \ B \ C'$  and  $L' \text{ PerpAt } PO \ L' \ C \ B'$ 
  by blast
have  $\exists P \ Q. \text{Col } C \ A' \ P \wedge \text{Col } A \ C' \ Q \wedge \text{Col } PO \ P \ Q \wedge P \text{ PerpAt } PO \ P \ C \ A' \wedge Q \text{ PerpAt } PO \ Q \ A \ C'$ 
proof (rule perp2-perp-in)
  show  $PO \text{ Perp2 } C \ A' \ A \ C'$ 
    by (simp add: assms(11))
  show  $\neg \text{Col } PO \ C \ A'$ 
    by (metis  $\langle \text{Col } PO \ A \ C \rangle$  assms(1) assms(3) col-trivial-3 colx)
  show  $\neg \text{Col } PO \ A \ C'$ 
    using NCol-perm  $\langle \text{Col } PO \ A' \ C' \rangle$  assms(1) assms(7) col-trivial-3 colx by blast
qed
then obtain  $M \ M'$  where  $\text{Col } C \ A' \ M$  and  $\text{Col } A \ C' \ M'$  and
   $\text{Col } PO \ M \ M'$  and  $M \text{ PerpAt } PO \ M \ C \ A'$  and  $M' \text{ PerpAt } PO \ M' \ A \ C'$ 
  by blast
obtain  $N$  where  $\text{Col } A \ B' \ N$  and  $A \ B' \text{ Perp } PO \ N$ 
  using l8-18-existence  $\langle \neg \text{Col } A \ B' \ PO \rangle$  by blast
have  $\text{Col } PO \ PO \ N$ 
  using col-trivial-1 by auto
moreover have  $PO \ N \text{ Perp } A \ B'$ 
  using Perp-perm  $\langle A \ B' \text{ Perp } PO \ N \rangle$  by blast
moreover have  $PO \ N \text{ Perp } B \ A'$ 
proof -
  obtain  $la$  where  $Q\text{Cong } la$  and  $la \ PO \ A$ 
    using lg-exists by blast
  obtain  $lb$  where  $Q\text{Cong } lb$  and  $lb \ PO \ B$ 
    using lg-exists by blast
  obtain  $lc$  where  $Q\text{Cong } lc$  and  $lc \ PO \ C$ 
    using lg-exists by blast
  obtain  $la'$  where  $Q\text{Cong } la'$  and  $la' \ PO \ A'$ 
    using lg-exists by blast
  obtain  $lb'$  where  $Q\text{Cong } lb'$  and  $lb' \ PO \ B'$ 
    using lg-exists by blast
  obtain  $lc'$  where  $Q\text{Cong } lc'$  and  $lc' \ PO \ C'$ 
    using lg-exists by blast
  obtain  $ll$  where  $Q\text{Cong } ll$  and  $ll \ PO \ L$ 
    using lg-exists by blast
  obtain  $ll'$  where  $Q\text{Cong } ll'$  and  $ll' \ PO \ L'$ 
    using lg-exists by blast
  obtain  $lm$  where  $Q\text{Cong } lm$  and  $lm \ PO \ M$ 
    using lg-exists by blast
  obtain  $lm'$  where  $Q\text{Cong } lm'$  and  $lm' \ PO \ M'$ 
    using lg-exists by blast
  obtain  $ln$  where  $Q\text{Cong } ln$  and  $ln \ PO \ N$ 
    using lg-exists by blast
  have  $PO \neq L$ 
    using  $\langle L \text{ PerpAt } PO \ L \ B \ C' \rangle$  perp-in-distinct by auto
  have  $PO \neq L'$ 
    using  $\langle L' \text{ PerpAt } PO \ L' \ C \ B' \rangle$  perp-in-distinct by auto
  have  $\neg \text{Col } PO \ B \ B'$ 
    by (meson  $\langle \neg \text{Col } A \ B' \ PO \rangle$  assms(2) assms(4) col-permutation-1 col-trivial-3 colx)
  have  $\neg \text{Col } PO \ C \ C'$ 
    by (meson  $\langle \text{Col } PO \ A \ C \rangle \langle \neg \text{Col } A \ B' \ PO \rangle$  assms(3) assms(7) assms(9)
        col-transitivity-2 not-col-permutation-1)
  have  $\exists a. Q\text{CongAAcute } a \wedge L\text{cos } ll \ lb \ a \wedge L\text{cos } ll' \ lc \ a$ 
  proof (cases  $B = L$ )
    case True
      hence  $B' \neq L'$ 
        using  $\langle \text{Col } PO \ L \ L' \rangle \langle \neg \text{Col } PO \ B \ B' \rangle$  by auto
      hence  $C = L'$ 
        by (metis Col-cases True  $\langle \text{Col } C \ B' \ L' \rangle \langle \text{Col } PO \ L \ L' \rangle \langle \neg \text{Col } PO \ B \ B' \rangle$  assms(5) colx)

```

```

thus ?thesis
  using True <Col PO L L'> <PO ≠ L'> <QCong lb> <QCong lc> <QCong ll'>
    <QCong ll> <lb PO B> <lc PO C> <ll PO L> <ll' PO L'> assms(2) l13-10-aux2 by auto
next
  case False
  have PO L Perp L B
    by (meson False <L PerpAt PO L B C'> l8-15-2 perp-in-col
      perp-in-col-perp-in perp-right-comm)
  moreover have PO L' Perp L' C
    by (metis Col-cases <Col PO L L'> <L' PerpAt PO L' C B'> <PO ≠ L'> assms(5) calculation
      col-transitivity-2 per-perp perp-in-per-1 perp-not-col2)
  ultimately show ?thesis
    using l13-10-aux1
    using <Col PO L L'> <QCong lb> <QCong lc> <QCong ll'> <QCong ll>
      <lb PO B> <lc PO C> <ll PO L> <ll' PO L'> assms(5) by auto
qed
then obtain l' where QCongAAcute l' and Lcos ll lb l' and Lcos ll' lc l'
  by blast
have ∃ a. QCongAAcute a ∧ Lcos ll lc' a ∧ Lcos ll' lb' a
proof (cases C' = L)
  case True
  have C ≠ L'
    using <¬ Col PO C C'> True <Col PO L L'> not-col-permutation-5 by blast
  hence B' = L'
    using l6-21 by (metis True <Col C B' L'> <Col PO L L'> <¬ Col PO C C'>
      assms(9) col-transitivity-1 not-col-distincts)
  thus ?thesis
    using l13-10-aux2 [where ?PO = PO and ?A = A and ?B = B]
    by (metis True <QCong lb'> <QCong lc'> <QCong ll'> <QCong ll>
      <lb' PO B'> <lc' PO C'> <ll PO L> <ll' PO L'> assms(6)
      assms(7) assms(9) l13-10-aux2)
next
  case False
  show ?thesis
  proof (rule l13-10-aux1 [where ?PO=PO and ?A=C' and ?B=B' and ?P=L and ?Q=L'],
    simp-all add: <Col PO L L'> <QCong lc'> <QCong lb'> <QCong ll> <QCong ll'>
    <lc' PO C'><lb' PO B'><ll PO L><ll' PO L'>
    show Col PO C' B'
      using Col-cases assms(9) by auto
    show PO L Perp L C'
      by (metis False <Col B C' L> <L PerpAt PO L B C'> col-permutation-5
        col-transitivity-1 perp-col2-bis perp-in-perp)
    show PO L' Perp L' B'
      by (metis <Col C B' L'> <Col PO L L'> <L' PerpAt PO L' C B'>
        <PO L Perp L C'> assms(9) col-transitivity-2 not-col-permutation-1
        perp-col2-bis perp-in-perp perp-not-col2)
  qed
qed
then obtain l where QCongAAcute l and Lcos ll lc' l and Lcos ll' lb' l
  by blast
have ∃ a. QCongAAcute a ∧ Lcos lm lc a ∧ Lcos lm' la a
proof (cases C = M)
  case True
  have A = M'
  proof (rule l6-21 [where ?A=PO and ?B=C and ?C=C' and ?D=M'])
    show ¬ Col PO C C'
      using <¬ Col PO C C'> by blast
    {
      assume C' = M'
      hence False
      using True <Col PO M M'> <¬ Col PO C C'> by blast
    }
  thus C' ≠ M'
  by blast
  show Col PO C A
    using Col-cases <Col PO A C> by blast

```

```

show Col PO C M'
  by (simp add: True ⟨Col PO M M'⟩)
show Col C' M' A
  using Col-cases ⟨Col A C' M'⟩ by blast
show Col C' M' M'
  by (simp add: col-trivial-2)
qed
show ?thesis
proof (rule l13-10-aux2 [where ?PO=PO and ?A=C and ?B=A],
  simp-all add: ⟨QCong lc⟩ ⟨QCong lm⟩ ⟨QCong lm'⟩ ⟨QCong la⟩
  ⟨lc PO C⟩ ⟨la PO A⟩ assms(3) ⟨A ≠ PO⟩)
  show Col PO C A
    by (simp add: True ⟨A = M'⟩ ⟨Col PO M M'⟩)
  show lm PO C
    by (simp add: True ⟨lm PO M⟩)
  show lm' PO A
    by (simp add: ⟨A = M'⟩ ⟨lm' PO M'⟩)
qed
next
case False
hence C ≠ M
  by blast
have PO M Perp M C
  using False ⟨Col C A' M⟩ ⟨M PerpAt PO M C A'⟩ col-trivial-3
  perp-col2-bis perp-in-perp by blast
moreover have PO M' Perp M' A
  by (metis ⟨Col A C' M'⟩ ⟨Col PO A C⟩ ⟨Col PO M M'⟩ ⟨M' PerpAt PO M' A C'⟩
  calculation col3 not-col-distincts perp-col1 perp-in-perp
  perp-not-col2 perp-right-comm)
ultimately show ?thesis
  using l13-10-aux1 by (metis ⟨Col PO A C⟩ ⟨Col PO M M'⟩ ⟨QCong la⟩
  ⟨QCong lc⟩ ⟨QCong lm'⟩ ⟨QCong lm⟩ ⟨la PO A⟩ ⟨lc PO C⟩ ⟨lm PO M⟩
  ⟨lm' PO M'⟩ not-col-permutation-5)
qed
then obtain m' where QCongAAcute m' and Lcos lm lc m' and Lcos lm' la m'
  by blast
have ∃ a. QCongAAcute a ∧ Lcos lm la' a ∧ Lcos lm' lc' a
proof (cases A' = M)
case True
  have C' = M'
proof (rule l6-21 [where ?A=PO and ?B=A' and ?C=A and ?D=M'])
  show ¬ Col PO A' A
    using assms(1) not-col-permutation-5 by blast
  show A ≠ M'
    using Col-cases True ⟨Col PO M M'⟩ assms(1) by blast
  show Col PO A' C'
    by (simp add: ⟨Col PO A' C'⟩)
  show Col PO A' M'
    by (simp add: True ⟨Col PO M M'⟩)
  show Col A M' C'
    using Col-cases ⟨Col A C' M'⟩ by blast
  show Col A M' M'
    by (simp add: col-trivial-2)
qed
show ?thesis
proof (rule l13-10-aux2 [where PO=PO and ?A=A' and ?B=C'],
  simp-all add: ⟨QCong la'⟩ ⟨QCong lc'⟩ ⟨QCong lm⟩ ⟨QCong lm'⟩
  assms(7) ⟨Col PO A' C'⟩ ⟨la' PO A'⟩ ⟨lc' PO C'⟩)
  show lm PO A'
    by (simp add: True ⟨lm PO M⟩)
  show lm' PO C'
    using ⟨C' = M'⟩ ⟨lm' PO M'⟩ by auto
  show A' ≠ PO
    using assms(1) col-trivial-3 by blast
qed
next

```

```

case False
hence  $A' \neq M$ 
by simp
have  $PO \ M \ \text{Perp} \ M \ A'$ 
using False  $\langle \text{Col } C \ A' \ M \rangle \langle M \ \text{PerpAt} \ PO \ M \ C \ A' \rangle$  col-trivial-2
perp-col2-bis perp-in-perp by blast
moreover have  $PO \ M' \ \text{Perp} \ M' \ C'$ 
by (metis  $\langle \text{Col } A \ C' \ M' \rangle \langle \text{Col } PO \ M \ M' \rangle \langle M' \ \text{PerpAt} \ PO \ M' \ A \ C' \rangle$  assms(6)
assms(8) assms(9) calculation col-transitivity-1 not-col-distincts
perp-col2-bis perp-in-perp perp-not-col2)
ultimately show ?thesis
using l13-10-aux1 [where ? $PO=PO$  and ? $A=A'$  and ? $B=C'$  and ? $P=M$  and ? $Q=M'$ ]
by (simp add:  $\langle \text{Col } PO \ A' \ C' \rangle \langle \text{Col } PO \ M \ M' \rangle \langle \text{QCong } la' \rangle \langle \text{QCong } lc' \rangle$ 
 $\langle \text{QCong } lm' \rangle \langle \text{QCong } lm \rangle \langle la' \ PO \ A' \rangle \langle lc' \ PO \ C' \rangle \langle lm \ PO \ M \rangle \langle lm' \ PO \ M' \rangle$ )
qed
then obtain  $m$  where  $\text{QCongAAcute } m$  and  $\text{Lcos } lm \ la' \ m$  and  $\text{Lcos } lm' \ lc' \ m$ 
by blast
have  $\exists a. \text{QCongAAcute } a \wedge \text{Lcos } ln \ la \ a$ 
proof -
have  $\exists a. \text{QCongAAcute } a \wedge a \ A \ PO \ N$ 
proof (rule anga-exists,simp-all add:  $\langle A \neq PO \rangle$ )
show  $N \neq PO$ 
using  $\langle \text{Col } A \ B' \ N \rangle \langle \neg \text{Col } A \ B' \ PO \rangle$  by auto
show  $\text{Acute } A \ PO \ N$ 
by (meson  $\langle \text{Col } A \ B' \ N \rangle$  calculation(2) l8-14-2-1b-bis not-col-distincts
not-col-permutation-1 perp-acute)
qed
then obtain  $n'$  where  $\text{QCongAAcute } n'$  and  $n' \ A \ PO \ N$ 
by blast
have  $\text{Per } A \ N \ PO$ 
by (meson  $\langle A \ B' \ \text{Perp} \ PO \ N \rangle \langle \text{Col } A \ B' \ N \rangle$  l8-16-1 l8-2 not-col-distincts)
hence  $\text{Lcos } ln \ la \ n'$ 
by (metis  $\text{Lcos-def } \langle \text{QCong } la \rangle \langle \text{QCong } ln \rangle \langle \text{QCongAAcute } n' \rangle$ 
 $\langle la \ PO \ A \rangle \langle ln \ PO \ N \rangle \langle n' \ A \ PO \ N \rangle$  anga-sym)
thus ?thesis
using  $\langle \text{QCongAAcute } n' \rangle$  by blast
qed
then obtain  $n'$  where  $\text{QCongAAcute } n'$  and  $\text{Lcos } ln \ la \ n'$ 
by blast
have  $\exists a. \text{QCongAAcute } a \wedge \text{Lcos } ln \ lb' \ a$ 
proof -
have  $\exists a. \text{QCongAAcute } a \wedge a \ B' \ PO \ N$ 
proof (rule anga-exists, simp add: assms(6))
show  $N \neq PO$ 
using  $\langle \text{Col } A \ B' \ N \rangle \langle \neg \text{Col } A \ B' \ PO \rangle$  by blast
show  $\text{Acute } B' \ PO \ N$ 
proof (cases  $B' = N$ )
case True
thus ?thesis
using  $\langle N \neq PO \rangle$  acute-trivial by auto
next
case False
show ?thesis
proof (rule perp-acute [where ? $C = N$ ])
show  $\text{Col } B' \ N \ N$ 
using col-trivial-2 by auto
show  $N \ \text{PerpAt} \ PO \ N \ B' \ N$ 
by (meson False  $\langle A \ B' \ \text{Perp} \ PO \ N \rangle \langle \text{Col } A \ B' \ N \rangle$  l8-15-1 not-col-permutation-4
perp-in-col-perp-in perp-in-left-comm perp-in-sym)
qed
qed
qed
then obtain  $n$  where  $\text{QCongAAcute } n$  and  $n \ B' \ PO \ N$ 
by blast
have  $\text{Per } B' \ N \ PO$ 
using  $\langle A \ B' \ \text{Perp} \ PO \ N \rangle \langle \text{Col } A \ B' \ N \rangle$  col-trivial-2 l8-16-1 l8-2 by blast

```

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moreover have  $n \ N \ PO \ B'$ 
  using  $\langle QCongAACute \ n \rangle \langle n \ B' \ PO \ N \rangle$  anga-sym by blast
ultimately show ?thesis
  by (metis  $\langle ln \ PO \ N \rangle \langle lb' \ PO \ B' \rangle$  Lcos-def  $\langle QCong \ lb' \rangle \langle QCong \ ln \rangle$ 
     $\langle \exists a. \ QCongAACute \ a \wedge a \ B' \ PO \ N \rangle$  anga-sym)
qed
then obtain  $n$  where  $QCongAACute \ n$  and  $Lcos \ ln \ lb' \ n$ 
  by blast
have  $EqLcos \ lc \ l' \ lb' \ l$ 
  using EqLcos-def  $\langle Lcos \ ll' \ lb' \ l \rangle \langle Lcos \ ll' \ lc \ l' \rangle$  by auto
have  $EqLcos \ lb \ l' \ lc' \ l$ 
  using EqLcos-def  $\langle Lcos \ ll \ lb \ l' \rangle \langle Lcos \ ll \ lc' \ l \rangle$  by blast
have  $EqLcos \ lc \ m' \ la' \ m$ 
  using EqLcos-def  $\langle Lcos \ lm \ la' \ m \rangle \langle Lcos \ lm \ lc \ m' \rangle$  by auto
have  $EqLcos \ la \ m' \ lc' \ m$ 
  using EqLcos-def  $\langle Lcos \ lm' \ la \ m' \rangle \langle Lcos \ lm' \ lc' \ m \rangle$  by blast
have  $EqLcos \ la \ n' \ lb' \ n$ 
  using EqLcos-def  $\langle Lcos \ ln \ la \ n' \rangle \langle Lcos \ ln \ lb' \ n \rangle$  by blast
have  $\exists \ lp. \ Lcos \ lp \ lb \ n'$ 
proof (rule lcos-exists, simp-add:  $\langle QCongAACute \ n' \rangle \langle QCong \ lb \rangle$ )
  {
    assume  $QCongNull \ lb$ 
    then obtain  $X$  where  $lb \ X \ X$ 
    by (simp-add: lg-null-instance)
    have  $Cong \ B \ PO \ X \ X$ 
    using  $\langle QCong \ lb \rangle \langle lb \ PO \ B \rangle \langle lb \ X \ X \rangle$  lg-cong lg-sym by blast
    hence  $B = PO$ 
    by (simp-add: cong-identity)
    hence False
    by (simp-add: assms(2))
  }
thus  $\neg \ QCongNull \ lb$ 
by blast
qed
then obtain  $bn'$  where  $Lcos \ bn' \ lb \ n'$ 
  by blast
have  $\neg \ QCongNull \ ll$ 
  using  $\langle Lcos \ ll \ lb \ l' \rangle$  lcos-not-lg-null by auto
hence  $\exists \ lp. \ Lcos \ lp \ ll \ n'$ 
  using  $\langle QCong \ ll \rangle \langle QCongAACute \ n' \rangle$  lcos-exists by auto
then obtain  $bl'n'$  where  $Lcos \ bl'n' \ ll \ n'$ 
  by blast
have  $\exists \ lp. \ Lcos \ lp \ bn' \ l'$ 
proof (rule lcos-exists, simp-add:  $\langle QCongAACute \ l' \rangle$ )
  show  $QCong \ bn'$ 
    using  $\langle Lcos \ bn' \ lb \ n' \rangle$  lcos-lg-anga by blast
  show  $\neg \ QCongNull \ bn'$ 
    using  $\langle Lcos \ bn' \ lb \ n' \rangle$  lcos-lg-not-null by auto
qed
then obtain  $bn'l'$  where  $Lcos \ bn'l' \ bn' \ l'$ 
  by blast
have  $Lcos2 \ bl'n' \ lb \ l' \ n'$ 
  using Lcos2-def  $\langle Lcos \ bl'n' \ ll \ n' \rangle \langle Lcos \ ll \ lb \ l' \rangle$  by blast
have  $Lcos2 \ bn'l' \ lb \ n' \ l'$ 
  using Lcos2-def  $\langle Lcos \ bn' \ lb \ n' \rangle \langle Lcos \ bn'l' \ bn' \ l' \rangle$  by blast
have  $bl'n' \ EqLTarski \ bn'l'$ 
  using  $\langle Lcos \ bl'n' \ ll \ n' \rangle \langle Lcos \ bn' \ lb \ n' \rangle \langle Lcos \ bn'l' \ bn' \ l' \rangle$ 
     $\langle Lcos \ ll \ lb \ l' \rangle$  l13-7 by blast
have  $EqLcos2 \ lb \ l' \ n' \ lb \ n' \ l'$ 
  using EqLcos2-def  $\langle Lcos2 \ bn'l' \ lb \ n' \ l' \rangle$  lcos2-comm by blast
have  $EqLcos2 \ lb \ l' \ n' \ lc' \ l \ n'$ 
  by (simp-add:  $\langle EqLcos \ lb \ l' \ lc' \ l \rangle \langle QCongAACute \ n' \rangle$  lcos-eq-lcos2-eq)
have  $EqLcos3 \ lb \ l' \ n' \ m \ lc' \ l \ n' \ m$ 
  by (simp-add:  $\langle EqLcos \ lb \ l' \ lc' \ l \rangle \langle QCongAACute \ m \rangle \langle QCongAACute \ n' \rangle$  lcos-eq-lcos3-eq)
have  $EqLcos3 \ lb \ l' \ n' \ m \ lc' \ m \ l \ n'$ 
proof –

```

obtain lp where $Lcos3\ lp\ lb\ l'\ n'\ m$ and $Lcos3\ lp\ lc'\ l\ n'\ m$
 using $EqLcos3-def\ \langle EqLcos3\ lb\ l'\ n'\ m\ lc'\ l\ n'\ m \rangle$ by blast
 thus ?thesis
 by (meson $EqLcos3-def\ lcos3-permut1\ lcos3-permut3$)
 qed
 have $EqLcos3\ la\ m'\ l\ n'\ lc'\ m\ l\ n'$
 using $\langle EqLcos\ la\ m'\ lc'\ m \rangle\ \langle QCongAACute\ l \rangle\ \langle QCongAACute\ n' \rangle\ lcos-eq-lcos3-eq$ by force
 hence $EqLcos3\ lb\ l'\ n'\ m\ la\ m'\ l\ n'$
 by (meson $\langle EqLcos3\ lb\ l'\ n'\ m\ lc'\ m\ l\ n' \rangle$
 $lcos3-eq-sym\ lcos3-eq-trans$)
 hence $EqLcos3\ lb\ l'\ n'\ m\ la\ n'\ m'\ l$
 by (meson $EqLcos3-def\ lcos3-permut1\ lcos3-permut3$)
 have $EqLcos3\ la\ n'\ m'\ l\ lb'\ n\ m'\ l$
 using $\langle EqLcos\ la\ n'\ lb'\ n \rangle\ \langle QCongAACute\ l \rangle\ \langle QCongAACute\ m' \rangle\ lcos-eq-lcos3-eq$ by blast
 hence $EqLcos3\ lb\ l'\ n'\ m\ lb'\ n\ m'\ l$
 using $\langle EqLcos3\ lb\ l'\ n'\ m\ la\ n'\ m'\ l \rangle\ lcos3-eq-trans$ by blast
 hence $EqLcos3\ lb\ l'\ n'\ m\ lb'\ l\ n\ m'$
 by (meson $EqLcos3-def\ lcos3-permut1\ lcos3-permut3$)
 have $EqLcos3\ lb\ l'\ n'\ m\ lb'\ l\ n\ m'$
 proof –
 obtain lp where $Lcos3\ lp\ lb\ l'\ n'\ m$ and $Lcos3\ lp\ lb'\ n\ m'\ l$
 using $EqLcos3-def\ \langle EqLcos3\ lb\ l'\ n'\ m\ lb'\ n\ m'\ l \rangle$ by auto
 thus ?thesis
 using $\langle EqLcos3\ lb\ l'\ n'\ m\ lb'\ l\ n\ m' \rangle$ by blast
 qed
 have $EqLcos3\ lb'\ l\ n\ m'\ lc'\ l'\ n\ m'$
 by (simp add: $\langle EqLcos\ lc'\ l'\ lb'\ l \rangle\ \langle QCongAACute\ m' \rangle\ \langle QCongAACute\ n \rangle$
 $lcos-eq-lcos3-eq\ lcos-eq-sym$)
 have $EqLcos3\ lb\ l'\ n'\ m\ lc'\ l'\ n\ m'$
 using $\langle EqLcos3\ lb\ l'\ n'\ m\ lb'\ l\ n\ m' \rangle\ \langle EqLcos3\ lb'\ l\ n\ m'\ lc'\ l'\ n\ m' \rangle$
 $lcos3-eq-trans$ by blast
 have $EqLcos3\ lb\ l'\ n'\ m\ lc'\ m'\ l'\ n$
 by (meson $EqLcos3-def\ \langle EqLcos3\ lb\ l'\ n'\ m\ lc'\ l'\ n\ m' \rangle\ lcos3-permut1\ lcos3-permut2$)
 have $EqLcos3\ la'\ m'\ l'\ n\ lc'\ m'\ l'\ n$
 by (simp add: $\langle EqLcos\ lc'\ m'\ la'\ m \rangle\ \langle QCongAACute\ l' \rangle\ \langle QCongAACute\ n \rangle$
 $lcos-eq-lcos3-eq\ lcos-eq-sym$)
 hence $EqLcos3\ lb\ l'\ n'\ m\ la'\ m'\ l'\ n$
 by (meson $\langle EqLcos3\ lb\ l'\ n'\ m\ lc'\ m'\ l'\ n \rangle\ lcos3-eq-sym\ lcos3-eq-trans$)
 hence $EqLcos3\ lb\ l'\ n'\ m\ la'\ n\ l'\ m$
 by (meson $EqLcos3-def\ lcos3-permut2$)
 have $EqLcos2\ lb\ l'\ n'\ la'\ n\ l'$
 using $\langle EqLcos3\ lb\ l'\ n'\ m\ la'\ n\ l'\ m \rangle\ lcos3-lcos2$ by auto
 have $EqLcos2\ lb\ n'\ l'\ la'\ n\ l'$
 by (meson $\langle EqLcos2\ lb\ l'\ n'\ la'\ n\ l' \rangle\ \langle EqLcos2\ lb\ l'\ n'\ lb\ n'\ l' \rangle\ lcos2-eq-sym\ lcos2-eq-trans$)
 have $EqLcos\ lb\ n'\ la'\ n$
 using $\langle EqLcos2\ lb\ n'\ l'\ la'\ n\ l' \rangle\ lcos2-lcos$ by auto
 obtain ln' where $Lcos\ ln'\ lb\ n'$ and $Lcos\ ln'\ la'\ n$
 using $EqLcos-def\ \langle EqLcos\ lb\ n'\ la'\ n \rangle$ by blast
 have $PO \neq N$
 using $\langle Col\ A\ B'\ N \rangle\ \langle \neg\ Col\ A\ B'\ PO \rangle$ by auto
 have $Per\ A\ N\ PO$
 using $\langle A\ B'\ Perp\ PO\ N \rangle\ \langle Col\ A\ B'\ N \rangle\ col-trivial-3\ l8-16-1\ l8-2$ by blast
 hence $n'\ A\ PO\ N$
 using $lcos-per-anga$ [where ?lp=la and ?la=ln] $\langle Lcos\ ln\ la\ n' \rangle$
 $\langle la\ PO\ A \rangle\ \langle ln\ PO\ N \rangle\ lcos-per-anga$ by blast
 have $Per\ B'\ N\ PO$
 using $\langle A\ B'\ Perp\ PO\ N \rangle\ \langle Col\ A\ B'\ N \rangle\ col-trivial-2\ l8-16-1\ l8-2$ by blast
 hence $n\ B'\ PO\ N$
 using $lcos-per-anga$ [where ?lp=lb' and ?la=ln] $\langle Lcos\ ln\ lb'\ n \rangle$
 $\langle lb'\ PO\ B' \rangle\ \langle ln\ PO\ N \rangle\ lcos-per-anga$ by blast
 have $Bet\ A\ PO\ B \vee PO\ Out\ A\ B \vee \neg\ Col\ A\ PO\ B$
 using or-bet-out by presburger
 moreover
 {
 assume $Bet\ A\ PO\ B$
 have $QCong\ ln'$

```

    using ⟨Lcos ln' la' n⟩ lcos-lg-anga by blast
  then obtain N' where ln' PO N' and Bet N PO N'
    using ex-point-lg-bet by blast
{
  assume PO = N'
  hence ln' PO PO
    using ⟨ln' PO N'⟩ by auto
  hence QCongNull ln'
    using ⟨QCong ln'⟩ by (simp add: lg-null-trivial)
  moreover have ¬ QCongNull ln' ∧ ¬ QCongNull lb
    using lcos-lg-not-null ⟨Lcos ln' lb n'⟩ by force
  hence ¬ QCongNull ln'
    by blast
  ultimately have False
    by blast
}
hence PO ≠ N'
  by blast
{
  assume A' = B
  hence False
    using assms(1) assms(4) by auto
}
hence A' ≠ B
  by blast
have Per PO N' B
proof (rule lcos-per [where ?a = n' and ?l=lb and ?lp=ln'],
  simp-all add: ⟨QCongAACute n'⟩ ⟨QCong lb⟩ ⟨QCong ln'⟩ ⟨Lcos ln' lb n'⟩
  ⟨lb PO B⟩ ⟨ln' PO N'⟩)
  have N PO A CongA B PO N'
  proof (rule l11-13 [where ?A = N' and ?D = A],
    simp-all add: ⟨Bet A PO B⟩ assms(2))
    show N' PO A CongA A PO N'
      using ⟨A ≠ PO⟩ ⟨PO ≠ N'⟩ conga-pseudo-refl by force
    show Bet N' PO N
      by (simp add: ⟨Bet N PO N'⟩ between-symmetry)
    show N ≠ PO
      using ⟨PO ≠ N⟩ by blast
  qed
  thus n' N' PO B
    using ⟨QCongAACute n'⟩ ⟨n' A PO N⟩ anga-conga-anga anga-sym by blast
qed
have N PO B' CongA A' PO N'
proof (rule l11-13 [where ?A = N' and ?D = B'])
  show N' PO B' CongA B' PO N'
    using ⟨PO ≠ N'⟩ assms(6) conga-pseudo-refl by force
  show Bet N' PO N
    using Bet-cases ⟨Bet N PO N'⟩ by auto
  show N ≠ PO
    using ⟨PO ≠ N⟩ by blast
  have Bet A' PO B'
    using l13-10-aux3 [where ?A=A and ?B=B and ?C=C] ⟨Bet A PO B⟩ assms(1)
    assms(10) assms(11) assms(2) assms(3) assms(4) assms(5) assms(6)
    assms(7) assms(8) assms(9) by blast
  thus Bet B' PO A'
    using between-symmetry by blast
  show A' ≠ PO
    using assms(1) not-col-distincts by blast
qed
hence n N' PO A'
  using ⟨QCongAACute n⟩ ⟨n B' PO N⟩ anga-conga-anga anga-sym by blast
hence Per PO N' A'
  using lcos-per [where ?a=n and ?l=ln' and ?lp=la']
  ⟨Lcos ln' la' n⟩ ⟨la' PO A'⟩ ⟨ln' PO N'⟩ lcos-lg-anga lcos-per by auto
have ?thesis
proof (rule perp-col [where ?B=N'])

```



```

show  $PO \neq N$ 
  by (simp add:  $\langle PO \neq N \rangle$ )
show  $PO N' \text{ Perp } B A'$ 
proof (rule cop-per2--perp)
  show  $PO \neq N'$ 
    by (simp add:  $\langle PO \neq N' \rangle$ )
  show  $B \neq A'$ 
    using  $\langle A' \neq B \rangle$  by auto
  show  $N' \neq B \vee N' \neq A'$ 
    using  $\langle B \neq A' \rangle$  by blast
  have Coplanar  $B N PO A'$ 
proof (rule col-cop--cop [where  $?D=B$ ])
  show Coplanar  $B N PO B'$ 
    using Coplanar-def  $\langle Col A B' N \rangle$  assms(4) not-col-permutation-1
    not-col-permutation-3 by blast
  show  $PO \neq B'$ 
    using assms(6) by auto
  show Col  $PO B' A'$ 
    using Col-cases assms(8) by blast
qed
thus Coplanar  $PO N' B A'$ 
  by (metis  $\langle Bet N PO N' \rangle$   $\langle PO \neq N \rangle$  bet-col bet-col1 col2-cop--cop
    ncoplanar-perm-23 ncoplanar-perm-9)
show Per  $PO N' B$ 
  by (simp add:  $\langle Per PO N' B \rangle$ )
show Per  $PO N' A'$ 
  by (simp add:  $\langle Per PO N' A' \rangle$ )
qed
show Col  $PO N' N$ 
  by (simp add: Col-def  $\langle Bet N PO N' \rangle$ )
qed
}
moreover
{
  assume  $PO \text{ Out } A B$ 
  have  $\exists N'. ln' PO N' \wedge PO \text{ Out } N' N$ 
proof (rule ex-point-lg-out)
  show  $PO \neq N$ 
    using  $\langle PO \neq N \rangle$  by blast
  show QCong  $ln'$ 
    using  $\langle Lcos ln' la' n \rangle$  lcos-lg-anga by blast
  show  $\neg QCongNull ln'$ 
    using  $\langle Lcos ln' la' n \rangle$  lcos-lg-not-null by blast
qed
then obtain  $N'$  where  $ln' PO N'$  and  $PO \text{ Out } N' N$ 
  by blast
have Per  $PO N' B$ 
proof (rule lcos-per [where  $?a=n'$  and  $?lp=ln'$  and  $?l=lb$ ],
    simp-all add:  $\langle QCongAACute n' \rangle$   $\langle QCong lb \rangle$   $\langle Lcos ln' lb n' \rangle$ 
     $\langle lb PO B \rangle$   $\langle ln' PO N' \rangle$ )
  show QCong  $ln'$ 
    using  $\langle Lcos ln' la' n \rangle$  lcos-lg-anga by blast
  show  $n' N' PO B$ 
    using Out-cases  $\langle PO \text{ Out } A B \rangle$   $\langle PO \text{ Out } N' N \rangle$   $\langle QCongAACute n' \rangle$ 
     $\langle n' A PO N \rangle$  anga-out-anga anga-sym by blast
qed
have Per  $PO N' A'$ 
proof (rule lcos-per [where  $?a=n$  and  $?lp=ln'$  and  $?l=la'$ ])
  show QCongAACute  $n$ 
    by (simp add:  $\langle QCongAACute n \rangle$ )
  show QCong  $la'$ 
    using  $\langle QCong la' \rangle$  by auto
  show QCong  $ln'$ 
    using  $\langle Lcos ln' la' n \rangle$  lcos-lg-anga by blast
  show Lcos  $ln' la' n$ 
    by (simp add:  $\langle Lcos ln' la' n \rangle$ )

```

```

show  $la' PO A'$ 
  by (simp add:  $\langle la' PO A' \rangle$ )
show  $ln' PO N'$ 
  by (simp add:  $\langle ln' PO N' \rangle$ )
have  $B' PO N \text{ CongA } A' PO N'$ 
proof (rule out2--conga)
  show  $PO \text{ Out } A' B'$ 
    using  $\langle PO \text{ Out } A B \rangle$  assms(1) assms(10) assms(11) assms(2) assms(3)
    assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) l13-10-aux5 by blast
  show  $PO \text{ Out } N' N$ 
    by (simp add:  $\langle PO \text{ Out } N' N \rangle$ )
qed
thus  $n N' PO A'$ 
  using  $\langle Q\text{CongAACute } n \rangle \langle n B' PO N \rangle$  anga-conga-anga anga-sym by blast
qed
have ?thesis
proof (rule perp-col [where ?B = N'])
  show  $PO \neq N$ 
    by (simp add:  $\langle PO \neq N \rangle$ )
  show  $PO N' \text{ Perp } B A'$ 
proof (rule cop-per2--perp)
  show  $PO \neq N'$ 
    using  $\langle PO \text{ Out } N' N \rangle$  out-distinct by blast
  show  $B \neq A'$ 
    using assms(1) assms(4) by fastforce
  show  $N' \neq B \vee N' \neq A'$ 
    using  $\langle B \neq A' \rangle$  by force
  have Coplanar B N PO A'
proof (rule col-cop--cop [where ?D = N])
  have Coplanar B N PO A'
proof (rule col-cop--cop [where ?D = B'])
  show Coplanar B N PO B'
    using Col-cases Coplanar-def  $\langle Col A B' N \rangle$  assms(4) by auto
  show  $PO \neq B'$ 
    using assms(6) by auto
  show Col PO B' A'
    using assms(8) not-col-permutation-5 by blast
qed
thus Coplanar B A' PO N
  using coplanar-perm-5 by blast
show  $PO \neq N$ 
  by (simp add:  $\langle PO \neq N \rangle$ )
show Col PO N N'
  using Out-cases  $\langle PO \text{ Out } N' N \rangle$  out-col by blast
qed
thus Coplanar PO N' B A'
  using coplanar-perm-16 by blast
show Per PO N' B
  using  $\langle Per PO N' B \rangle$  by blast
show Per PO N' A'
  using  $\langle Per PO N' A' \rangle$  by blast
qed
show Col PO N' N
  by (simp add:  $\langle PO \text{ Out } N' N \rangle$  out-col)
qed
}
moreover have  $\neg Col A PO B \longrightarrow ?thesis$ 
  using assms(4) not-col-permutation-4 by blast
ultimately show ?thesis by blast
qed
ultimately show ?thesis
  using Perp2-def by blast
qed

```

lemma *Pj-exists*:

fixes $A B C$

shows $\exists D. A B Pj C D$
 using *Pj-def* by *blast*

lemma *project-trivial*:
 assumes $A \neq B$
 and $X \neq Y$
 and $Col A B P$
 and $\neg A B Par X Y$
 shows $P P Proj A B X Y$
 by (*simp add: Proj-def* *assms(1) assms(2) assms(3) assms(4)*)

lemma *pj-col-project*:
 assumes $A \neq B$
 and $X \neq Y$
 and $Col P' A B$
 and $\neg A B Par X Y$
 and $X Y Pj P P'$
 shows $P P' Proj A B X Y$
 by (*metis Pj-def assms(1) assms(2) assms(3) assms(4) assms(5) not-col-permutation-2*
par-col-project par-symmetry project-trivial)

lemma *pj-trivial*:
 shows $A B Pj C C$
 by (*simp add: Pj-def*)

lemma *O-not-positive*:
 shows $\neg Ps PO E PO$
 using *Ps-def out-distinct* by *blast*

lemma *col-pos-or-neg*:
 assumes $PO \neq E$
 and $PO \neq X$
 and $Col PO E X$
 shows $Ps PO E X \vee Ng PO E X$
 by (*metis NCol-cases Ng-def Ps-def assms(1,2,3) or-bet-out*)

lemma *length-cong*:
 assumes $Length PO E E' A B AB$
 shows $Cong A B PO AB$
 using *Cong-cases Length-def assms* by *auto*

lemma *triangular-equality-equiv-a* :
 assumes $\forall PO E A. PO \neq E \longrightarrow (\forall E' B C AB BC AC. Bet A B C \wedge Length PO E E' A B AB \wedge$
 $Length PO E E' B C BC \wedge$
 $Length PO E E' A C AC$
 $\longrightarrow Sum PO E E' AB BC AC)$
 shows $\forall PO E E' A B C AB BC AC. PO \neq E \wedge Bet A B C \wedge Length PO E E' A B AB \wedge$
 $Length PO E E' B C BC \wedge Length PO E E' A C AC$
 $\longrightarrow Sum PO E E' AB BC AC$
 using *assms* by *force*

lemma *triangular-equality-equiv-b* :
 assumes $\forall PO E E' A B C AB BC AC. PO \neq E \wedge Bet A B C \wedge Length PO E E' A B AB \wedge$
 $Length PO E E' B C BC \wedge Length PO E E' A C AC$
 $\longrightarrow Sum PO E E' AB BC AC$
 shows $\forall PO E A. PO \neq E \longrightarrow (\forall E' B C AB BC AC. Bet A B C \wedge Length PO E E' A B AB \wedge$
 $Length PO E E' B C BC \wedge Length PO E E' A C AC$
 $\longrightarrow Sum PO E E' AB BC AC)$
 by (*simp add: assms*)

lemma *triangular-equality-equiv* :
 shows $(\forall PO E A. PO \neq E \longrightarrow (\forall E' B C AB BC AC. Bet A B C \wedge Length PO E E' A B AB \wedge$
 $Length PO E E' B C BC \wedge Length PO E E' A C AC$

$\longrightarrow \text{Sum } PO \ E \ E' \ AB \ BC \ AC))$

$\longleftrightarrow$
 $(\forall \ PO \ E \ E' \ A \ B \ C \ AB \ BC \ AC. \ PO \neq E \wedge \text{Bet } A \ B \ C \wedge \text{Length } PO \ E \ E' \ A \ B \ AB \wedge$
 $\text{Length } PO \ E \ E' \ B \ C \ BC \wedge \text{Length } PO \ E \ E' \ A \ C \ AC$
 $\longrightarrow \text{Sum } PO \ E \ E' \ AB \ BC \ AC)$

by *blast*

lemma *sign-dec*:
assumes *Col* *PO* *E* *A*
and *PO* $\neq$ *E*
shows $A = PO \vee Ps \ PO \ E \ A \vee Ng \ PO \ E \ A$
by (*metis* *assms*(1,2) *col-pos-or-neg*)

lemma *not-neg-pos*:
assumes $E \neq PO$
and *Col* *PO* *E* *A*
and $\neg Ng \ PO \ E \ A$
shows $Ps \ PO \ E \ A \vee A = PO$
using *assms*(1,2,3) *sign-dec* **by** *blast*

lemma *Tarski-Pre-Non-Euclidean-aux-pre*:
assumes $\exists \ A \ B \ C \ D \ T. \neg ((\text{Bet } A \ D \ T \wedge \text{Bet } B \ D \ C \wedge A \neq D)$
 $\longrightarrow$
 $(\exists \ X \ Y. \text{Bet } A \ B \ X \wedge \text{Bet } A \ C \ Y \wedge \text{Bet } X \ T \ Y))$
shows $\exists \ A0 \ B0 \ C0 \ D0 \ T0. (\text{Bet } A0 \ D0 \ T0 \wedge \text{Bet } B0 \ D0 \ C0 \wedge A0 \neq D0 \wedge$
 $(\forall \ X \ Y. ((\text{Bet } A0 \ B0 \ X \wedge \text{Bet } A0 \ C0 \ Y) \longrightarrow \neg \text{Bet } X \ T0 \ Y)))$
using *assms* **by** *blast*

end
end

theory *Tarski-Neutral-Archimedes*

imports
Tarski-Neutral

begin

2 Continuity Axioms

context *Tarski-neutral-dimensionless*

begin

2.1 Definitions

definition *greenberg-s-axiom* ::
bool
 (GreenBergsAxiom)
where
 $\text{greenberg-s-axiom} \equiv \forall \ P \ Q \ R \ A \ B \ C.$
 $\neg \text{Col } A \ B \ C \wedge \text{Acute } A \ B \ C \wedge Q \neq R \wedge \text{Per } P \ Q \ R \longrightarrow (\exists \ S. \ P \ S \ Q \text{ LtA } A \ B \ C \wedge Q \text{ Out } S \ R)$

definition *aristotle-s-axiom* ::
bool
 (AristotleAxiom) **where**
 $\text{aristotle-s-axiom} \equiv \forall \ P \ Q \ A \ B \ C.$
 $\neg \text{Col } A \ B \ C \wedge \text{Acute } A \ B \ C \longrightarrow$
 $(\exists \ X \ Y. \ B \text{ Out } A \ X \wedge B \text{ Out } C \ Y \wedge \text{Per } B \ X \ Y \wedge P \ Q \text{ Lt } X \ Y)$

definition *Axiom1*:: *bool* **where** $\text{Axiom1} \equiv \forall \ A \ B \ C \ D.$
 $(\exists \ I. \text{Col } I \ A \ B \wedge \text{Col } I \ C \ D) \vee \neg (\exists \ I. \text{Col } I \ A \ B \wedge \text{Col } I \ C \ D)$

definition *PreGrad* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow \text{bool}$ **where**
 $\text{PreGrad } A \ B \ C \ D \equiv (A \neq B \wedge \text{Bet } A \ B \ C \wedge \text{Bet } A \ C \ D \wedge \text{Cong } A \ B \ C \ D)$

fun *Sym* :: 'p ⇒ 'p ⇒ 'p ⇒ 'p **where**
Sym A B C = (if (A ≠ B ∧ Bet A B C) then
 (SOME x::'p. PreGrad A B C x)
 else
 A)

fun *Gradn* :: ['p,'p] ⇒ nat ⇒ 'p **where**
Gradn A B n = (if (A = B) then
 A
 else
 (if (n = 0) then
 A
 else
 (if (n = 1) then
 B
 else
 (Sym A B (Gradn A B (n-1))))))

definition *Grad* :: ['p,'p,'p] ⇒ bool **where**
Grad A B C ≡ ∃ n. (n ≠ 0) ∧ (C = Gradn A B n)

inductive *GradI* :: ['p,'p,'p] ⇒ bool **for** A B
where
gradi-init : *GradI* A B B
| *gradi-stab* : *GradI* A B C' **if**
GradI A B C
and Bet A C C'
and Cong A B C C'

definition *Reach* :: ['p,'p,'p,'p] ⇒ bool **where**
Reach A B C D ≡ ∃ B'. *Grad* A B B' ∧ C D Le A B'

definition *archimedes-axiom* ::
bool
(*ArchimedesAxiom*) **where**
archimedes-axiom ≡ ∀ A B C D::'p.
A ≠ B → *Reach* A B C D

inductive *GradA* :: ['p,'p,'p,'p,'p,'p] ⇒ bool **for** A B C
where
grada-init : *GradA* A B C D E F **if**
A B C CongA D E F
| *grada-stab* : *GradA* A B C G H I **if**
GradA A B C D E F
and SAMS D E F A B C
and D E F A B C SumA G H I

inductive *GradAExp* :: ['p,'p,'p,'p,'p,'p] ⇒ bool **for** A B C
where
gradaexp-init : *GradAExp* A B C D E F **if**
A B C CongA D E F
| *gradaexp-stab* : *GradAExp* A B C G H I **if**
GradAExp A B C D E F
and SAMS D E F D E F
and D E F D E F SumA G H I

definition *Grad2* :: ['p,'p,'p,'p,'p,'p] ⇒ bool **where**
Grad2 A B C D E F ≡ ∃ n. (n ≠ 0) ∧ (C = Gradn A B n) ∧ (F = Gradn D E n)

fun *SymR* :: 'p ⇒ 'p ⇒ 'p **where**
SymR A B = (SOME x::'p. B Midpoint A x)

fun *GradExpn* :: 'p ⇒ 'p ⇒ nat ⇒ 'p **where**
(*GradExpn* A B n) = (if (A = B) then
A

```

      else
      (if (n = 0) then
        A
      else
      (if (n = 1) then
        B
      else
      (SymR A (GradExpn A B (n-1))))))

```

definition *GradExp* :: 'p ⇒ 'p ⇒ 'p ⇒ bool **where**
GradExp A B C ≡ ∃ n. (n ≠ 0) ∧ C = *GradExpn* A B n

definition *GradExp2* :: ['p, 'p, 'p, 'p, 'p, 'p] ⇒ bool **where**
GradExp2 A B C D E F ≡ ∃ n. (n ≠ 0) ∧ (C = *GradExpn* A B n) ∧ (F = *GradExpn* D E n)

fun *MidR* :: 'p ⇒ 'p ⇒ 'p **where**
MidR A B = (SOME x. x *Midpoint* A B)

fun *GradExpInvn* :: 'p ⇒ 'p ⇒ nat ⇒ 'p **where**
(*GradExpInvn* A B n) = (if (A = B) then
 A
 else
 (if (n = 0) then
 B
 else
 (if (n = 1) then
 (*MidR* A B)
 else
 (*MidR* A (*GradExpInvn* A B (n-1))))))

definition *GradExpInv* :: 'p ⇒ 'p ⇒ 'p ⇒ bool **where**
GradExpInv A B C ≡ ∃ n. B = *GradExpInvn* A C n

2.2 Propositions

lemma *PreGrad-lem1*:
assumes A ≠ B **and**
 B *Bet* A B C
shows ∃ x. *PreGrad* A B C x
by (meson *PreGrad-def* *assms*(1) *assms*(2) *not-cong-3412* *segment-construction*)

lemma *PreGrad-uniq*:
assumes *PreGrad* A B C x **and**
PreGrad A B C y
shows x = y
by (*metis* (*no-types*, *lifting*) *PreGrad-def* *assms*(1) *assms*(2)
bet-neq12--neq *between-cong-3* *cong-inner-transitivity*)

lemma *Diff-Mid--PreGrad*:
assumes A ≠ B **and**
 B *Midpoint* A C
shows *PreGrad* A B B C
by (*simp* *add*: *PreGrad-def* *assms*(1) *assms*(2) *between-trivial* *midpoint-bet* *midpoint-cong*)

lemma *Diff-Mid-Mid-PreGrad*:
assumes A ≠ B **and**
 B *Midpoint* A C **and**
 C *Midpoint* B D
shows *PreGrad* A B C D

proof –
have B *Bet* A B C
using *Midpoint-def* *assms*(2) **by** *presburger*
moreover have B *Bet* A C D
using *Midpoint-def* *assms*(3) *calculation* *is-midpoint-id* *outer-transitivity-between2* **by** *blast*
moreover have C *Bet* B C D
using *assms*(2) *assms*(3) *cong-transitivity* *midpoint-cong* **by** *blast*

ultimately show ?thesis
 by (simp add: assms(1) PreGrad-def)
 qed

lemma Sym-Diff--Diff:
 assumes Sym A B C = D and
 A $\neq$ D
 shows A $\neq$ B
 using assms(1) assms(2) by force

lemma Sym-Refl:
 Sym A A A = A
 by simp

lemma Diff-Mid--Sym:
 assumes A $\neq$ B and
 B Midpoint A C
 shows Sym A B B = C
 using someI-ex by (metis Diff-Mid--PreGrad Sym.elims PreGrad-uniq
 assms(1) assms(2) between-trivial)

lemma Mid-Mid--Sym:
 assumes A $\neq$ B and
 B Midpoint A C and
 C Midpoint B D
 shows Sym A B C = D
 proof –
 have PreGrad A B C D
 by (simp add: Diff-Mid-Mid-PreGrad assms(1) assms(2) assms(3))
 thus ?thesis
 using someI-ex assms(1)
 by (metis PreGrad-uniq Sym.elims assms(2) midpoint-bet)
 qed

lemma Sym-Bet--Bet-Bet:
 assumes Sym A B C = D and
 A $\neq$ B and
 Bet A B C
 shows Bet A B D $\wedge$ Bet A C D
 proof –
 have (SOME x::'p. PreGrad A B C x) = D
 using assms(1) assms(2) assms(3) by auto
 hence PreGrad A B C D
 by (metis PreGrad-lem1 assms(2) assms(3) someI2)
 thus ?thesis
 by (meson PreGrad-def between-exchange4)
 qed

lemma Sym-Bet--Cong:
 assumes Sym A B C = D and
 A $\neq$ B and
 Bet A B C
 shows Cong A B C D
 proof –
 have (SOME x::'p. PreGrad A B C x) = D
 using assms(1) assms(2) assms(3) by auto
 hence PreGrad A B C D
 by (metis PreGrad-lem1 assms(2) assms(3) someI2)
 thus ?thesis
 by (meson PreGrad-def between-exchange4)
 qed

lemma LemSym-aux:
 assumes A $\neq$ B and
 Bet A B C and
 Bet A C D and

```

  Cong A B C D
shows Sym A B C = D
proof -
  have PreGrad A B C D
  using PreGrad-def assms(1) assms(2) assms(3) assms(4) by blast
  thus ?thesis
  by (metis PreGrad-def PreGrad-uniq Sym-Bet--Bet-Bet Sym-Bet--Cong)
qed

```

```

lemma Lem-Gradn-id-n:
  Gradn A A n = A
by simp

```

```

lemma Lem-Gradn-0:
  Gradn A B 0 = A
by simp

```

```

lemma Lem-Gradn-1:
  Gradn A B 1 = B
by simp

```

```

lemma Diff--Gradn-Sym:
  assumes A ≠ B and
  n > 1
shows Gradn A B n = Sym A B (Gradn A B (n-1))
proof -
  have ¬ (n = 0 ∧ n = 1)
  by auto
  thus ?thesis
  using assms(1) assms(2) by simp
qed

```

```

lemma Diff--Bet-Gradn-Suc:
  assumes A ≠ B
shows Bet A B (Gradn A B (Suc n))
proof (induction n)
  case 0
  hence Gradn A B (Suc 0) = B
  using assms(1) by simp
  thus ?case
  by (simp add: between-trivial)
next
  case (Suc n)
  {
    assume 1: Bet A B (Gradn A B (Suc n))
    have Gradn A B (Suc (Suc n)) = Sym A B (Gradn A B (Suc n))
    by simp
    hence Bet A B (Gradn A B (Suc (Suc n))) ∧
    Bet A (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))
    using 1 Sym-Bet--Bet-Bet assms by presburger
    hence Bet A B (Gradn A B (Suc (Suc n)))
    by simp
  }
  thus ?case
  using Suc.IH by blast
qed

```

```

lemma Diff-Le-Gradn-Suc:
  assumes A ≠ B
shows A B Le A (Gradn A B (Suc n))
by (meson Diff--Bet-Gradn-Suc assms bet--le1213)

```

```

lemma Diff--Bet-Gradn:
  assumes A ≠ B and
  n ≠ 0
shows Bet A B (Gradn A B n)

```



```

using assms(1) assms(2) Diff-Bet-Gradn-Suc not0-implies-Suc by blast

lemma Diff-Le-Gradn-n:
  assumes  $A \neq B$  and
     $n \neq 0$ 
  shows  $A \leq B$  Le  $A (Gradn\ A\ B\ n)$ 
  by (meson Diff-Bet-Gradn assms(1) assms(2) l5-12-a)

lemma Diff-Bet-Gradn-Suc-Gradn-Suc2:
  assumes  $A \neq B$ 
  shows  $Bet\ A\ (Gradn\ A\ B\ (Suc\ n))\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
proof (induction n)
  case 0
  hence 1:  $Gradn\ A\ B\ (Suc\ 0) = B$ 
    using assms(1) by simp
  from assms(1)
  have  $(Gradn\ A\ B\ (Suc\ (Suc\ 0))) = (Sym\ A\ B\ (Gradn\ A\ B\ (Suc\ 0)))$ 
    by simp
  thus ?case
    by (metis 1 Diff-Bet-Gradn-Suc assms)
next
  case (Suc n)
  {
    assume 1:  $Bet\ A\ (Gradn\ A\ B\ (Suc\ n))\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
    have  $Gradn\ A\ B\ (Suc\ (Suc\ n)) = Sym\ A\ B\ (Gradn\ A\ B\ (Suc\ n))$ 
      by simp
    hence  $Bet\ A\ B\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
      using Diff-Bet-Gradn-Suc assms by blast
    have  $Gradn\ A\ B\ (Suc(Suc\ (Suc\ n))) = Sym\ A\ B\ (Gradn\ A\ B\ (Suc(Suc\ n)))$ 
      by simp
    hence  $PreGrad\ A\ B\ (Gradn\ A\ B\ (Suc(Suc\ n)))\ (Gradn\ A\ B\ (Suc(Suc\ (Suc\ n))))$ 
      using PreGrad-def Sym-Bet--Bet-Bet Sym-Bet--Cong
       $\langle Bet\ A\ B\ (Gradn\ A\ B\ (Suc\ (Suc\ n))) \rangle$  assms by presburger
    hence  $Bet\ A\ B\ (Gradn\ A\ B\ (Suc(Suc\ (Suc\ n)))) \wedge$ 
       $Bet\ A\ (Gradn\ A\ B\ (Suc(Suc\ n)))\ (Gradn\ A\ B\ (Suc(Suc\ (Suc\ n))))$ 
      by (metis Diff-Bet-Gradn-Suc Sym-Bet--Bet-Bet
         $\langle Gradn\ A\ B\ (Suc\ (Suc\ (Suc\ n))) = Sym\ A\ B\ (Gradn\ A\ B\ (Suc\ (Suc\ n))) \rangle$  assms)
    hence  $Bet\ A\ (Gradn\ A\ B\ (Suc(Suc\ n)))\ (Gradn\ A\ B\ (Suc(Suc\ (Suc\ n))))$ 
      by blast
  }
  thus ?case
    using Suc.IH by blast
qed

lemma Diff-Bet-Gradn-Gradn-SucA:
  assumes  $A \neq B$ 
  shows  $A\ (Gradn\ A\ B\ (Suc\ n))\ Le\ A\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
  by (meson Diff-Bet-Gradn-Suc-Gradn-Suc2 assms bet--le1213)

lemma Diff-Bet-Gradn-Gradn-Suc:
  assumes  $A \neq B$ 
  shows  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ (Suc\ n))$ 
proof (induction n)
  case 0
  hence  $Gradn\ A\ B\ 0 = A$  by simp
  thus ?case
    using between-trivial2 by presburger
next
  case (Suc n)
  {
    assume 1:  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ (Suc\ n))$ 
    have  $Gradn\ A\ B\ (Suc\ (Suc\ n)) = Sym\ A\ B\ (Gradn\ A\ B\ (Suc\ n))$ 
      by simp
    hence  $Bet\ A\ B\ (Gradn\ A\ B\ (Suc\ (Suc\ n))) \wedge$ 
       $Bet\ A\ (Gradn\ A\ B\ (Suc\ n))\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
      using 1 Sym-Bet--Bet-Bet assms Diff-Bet-Gradn-Suc by presburger
  }

```

```

    hence  $Bet\ A\ (Gradn\ A\ B\ (Suc\ n))\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
      by blast
  }
  thus ?case
    using Suc.IH by simp
qed

lemma Bet-Gradn-Gradn-Suc:
  shows  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ (Suc\ n))$ 
  by (metis Lem-Gradn-id-n Diff--Bet-Gradn-Gradn-Suc not-bet-distincts)

lemma Gradn-Le-Gradn-Suc:
  shows  $A\ (Gradn\ A\ B\ n)\ Le\ A\ (Gradn\ A\ B\ (Suc\ n))$ 
  using Bet-Gradn-Gradn-Suc bet--le1213 by blast

lemma Bet-Gradn-Suc-Gradn-Suc2:
  shows  $Bet\ B\ (Gradn\ A\ B\ (Suc\ n))\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
  by (metis Bet-Gradn-Gradn-Suc Diff--Bet-Gradn-Suc between-exchange3)

lemma Gradn-Suc-Le-Gradn-Suc2:
  shows  $B\ (Gradn\ A\ B\ (Suc\ n))\ Le\ B\ (Gradn\ A\ B\ (Suc\ (Suc\ n)))$ 
  using Bet-Gradn-Suc-Gradn-Suc2 bet--le1213 by blast

lemma Diff-Le--Bet-Gradn-Plus:
  assumes  $A \neq B$  and
     $n \leq m$ 
  shows  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ (k + n))$ 
proof (induction k)
  case 0
  thus ?case
    using between-trivial by auto
next
  case (Suc k)
  {
    assume  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ (k + n))$ 
    have  $Bet\ A\ (Gradn\ A\ B\ (k + n))\ (Gradn\ A\ B\ (Suc\ (k + n)))$ 
      using Diff--Bet-Gradn-Gradn-Suc assms(1) by presburger
    hence  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ ((Suc\ k) + n))$ 
      by (metis  $\langle Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ (k + n)) \rangle$ 
        add-Suc between-exchange4)
  }
  thus ?case
    using Suc.IH by blast
qed

lemma Diff-Le-Gradn-Plus:
  assumes  $A \neq B$  and
     $n \leq m$ 
  shows  $A\ (Gradn\ A\ B\ n)\ Le\ A\ (Gradn\ A\ B\ (k + n))$ 
  by (meson Diff-Le--Bet-Gradn-Plus assms(1) assms(2) l5-12-a)

lemma Diff-Le-Bet--Gradn-Gradn:
  assumes  $A \neq B$  and
     $n \leq m$ 
  shows  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ m)$ 
proof (cases  $n = 0$ )
  case True
  thus ?thesis
    using Lem-Gradn-0 between-trivial2 by presburger
next
  case False
  hence 1:  $n \neq 0$ 
    by auto
  show  $Bet\ A\ (Gradn\ A\ B\ n)\ (Gradn\ A\ B\ m)$ 
proof (cases  $n = m$ )
  case True

```

```

    thus ?thesis
    using between-trivial by presburger
next
case False
hence  $n < m$ 
    using assms(2) le-neq-implies-less by blast
then obtain  $k$  where  $m = k + n$ 
    using add.commute assms(2) le-Suc-ex by blast
have Bet  $A$  (Gradn  $A$   $B$   $n$ ) (Gradn  $A$   $B$  ( $k + n$ ))
    using Diff-Le--Bet-Gradn-Plus assms(1) by blast
thus ?thesis
    using  $\langle m = k + n \rangle$  by blast
qed
qed

lemma Diff-Le-Gradn:
  assumes  $A \neq B$  and
     $n \leq m$ 
  shows  $A$  (Gradn  $A$   $B$   $n$ ) Le  $A$  (Gradn  $A$   $B$   $m$ )
  by (metis Diff-Le-Bet--Gradn-Gradn bet--le1213 assms(1) assms(2))

lemma Diff--Cong-Gradn-Suc-Gradn-Suc2:
  assumes  $A \neq B$ 
  shows Cong  $A$   $B$  (Gradn  $A$   $B$  (Suc  $n$ )) (Gradn  $A$   $B$  (Suc (Suc  $n$ )))
proof (induction  $n$ )
case 0
hence 1: Gradn  $A$   $B$  (Suc 0) =  $B$ 
    using assms(1) by simp
from assms(1)
have (Gradn  $A$   $B$  (Suc (Suc 0))) = (Sym  $A$   $B$  (Gradn  $A$   $B$  (Suc 0))) by simp
hence (Gradn  $A$   $B$  (Suc (Suc 0))) = (Sym  $A$   $B$   $B$ )
    using 1 by presburger
obtain  $C$  where  $B$  Midpoint  $A$   $C$ 
    using symmetric-point-construction by blast
hence  $C = \text{Sym } A \ B \ B$ 
    using Diff-Mid--Sym assms by blast
hence (Gradn  $A$   $B$  (Suc (Suc 0))) =  $C$ 
    using 1  $\langle \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } 0)) = \text{Sym } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } 0)) \rangle$  by presburger
have Cong  $A$   $B$  (Gradn  $A$   $B$  (Suc 0)) (Gradn  $A$   $B$  (Suc (Suc 0)))
    using 1  $\langle B \text{ Midpoint } A \ C \rangle$   $\langle \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } 0)) = C \rangle$  midpoint-cong
    by presburger
thus ?case
    by blast
next
case (Suc  $n$ )
{
  assume Cong  $A$   $B$  (Gradn  $A$   $B$  (Suc  $n$ )) (Gradn  $A$   $B$  (Suc (Suc  $n$ )))
  have 1: Gradn  $A$   $B$  (Suc(Suc (Suc  $n$ ))) = Sym  $A$   $B$  (Gradn  $A$   $B$  (Suc(Suc  $n$ )))
    by simp
  have Bet  $A$   $B$  (Gradn  $A$   $B$  (Suc (Suc  $n$ )))
    using Diff--Bet-Gradn-Suc assms by blast
  hence PreGrad  $A$   $B$  (Gradn  $A$   $B$  (Suc(Suc  $n$ ))) (Gradn  $A$   $B$  (Suc(Suc (Suc  $n$ ))))
    using 1 assms
    by (metis PreGrad-def Sym-Bet--Cong Diff-Bet-Gradn-Suc-Gradn-Suc2)
  hence Cong  $A$   $B$  (Gradn  $A$   $B$  (Suc (Suc  $n$ ))) (Gradn  $A$   $B$  (Suc(Suc (Suc  $n$ ))))
    using 1 Sym-Bet--Cong  $\langle \text{Bet } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n))) \rangle$  assms
    by presburger
}
thus ?case
    using Suc.IH by blast
qed

lemma Cong-Gradn-Suc-Gradn-Suc2:
  shows Cong  $A$   $B$  (Gradn  $A$   $B$  (Suc  $n$ )) (Gradn  $A$   $B$  (Suc (Suc  $n$ )))
  using Diff--Cong-Gradn-Suc-Gradn-Suc2 cong-reflexivity by auto

```

```

lemma Cong-Gradn-Gradn-Suc:
  shows Cong a b (Gradn a b n) (Gradn a b (Suc n))
proof (cases a = b)
  case True
  thus ?thesis
    by (simp add: cong-trivial-identity)
next
  case False
  have 1: Gradn a b 0 = a
    by auto
  have 2: (Gradn a b (Suc 0)) = b
    by auto
  {
    assume 3: n = 0
    hence Cong a b a b  $\wedge$  (Suc n = 1)
      by (simp add: cong-reflexivity)
    hence Cong a b (Gradn a b n) (Gradn a b (Suc n))
      using 1 2 3 by presburger
  }
  moreover
  {
    assume n  $\neq$  0
    hence Cong a b (Gradn a b n) (Gradn a b (Suc n))
      by (metis False Diff--Cong-Gradn-Suc-Gradn-Suc2 old.nat.exhaust)
  }
  ultimately
  show ?thesis
    by blast
qed

```

```

lemma Diff-Bet-Bet-Cong-Gradn-Suc:
  assumes A  $\neq$  B and
    Bet A B C and
    Bet A (Gradn A B n) C and
    Cong A B (Gradn A B n) C
  shows C = (Gradn A B (Suc n))
proof (cases n = 0)
  case True
  thus ?thesis
    by (metis Lem-Gradn-0 Diff--Bet-Gradn-Suc Cong-Gradn-Gradn-Suc
      assms(1) assms(2) assms(4) between-cong)
next
  case False
  hence Bet A B (Gradn A B n)
    using Diff--Bet-Gradn assms(1) by blast
  thus ?thesis
    by (metis LemSym-aux Diff--Bet-Gradn-Gradn-Suc Cong-Gradn-Gradn-Suc
      assms(1) assms(3) assms(4))
qed

```

```

lemma grad-rec-0-1:
  shows Cong a b (Gradn a b 0) (Gradn a b 1)
  by (simp add: cong-reflexivity)

```

```

lemma grad-rec-1-2:
  shows Cong a b (Gradn a b 1) (Gradn a b 2)
  by (metis Cong-Gradn-Gradn-Suc Suc-1)

```

```

lemma grad-rec-2-3:
  shows Cong a b (Gradn a b 2) (Gradn a b 3)
proof (cases a = b)
  case True
  thus ?thesis
    using Lem-Gradn-id-n cong-reflexivity by presburger
next
  case False

```

```

thus ?thesis
  using Diff--Cong-Gradn-Suc-Gradn-Suc2 numeral-2-eq-2 numeral-3-eq-3
  by presburger
qed

lemma grad-rec-a-a:
  shows (Gradn a a n) = a
  by simp

lemma Gradn-uniq-aux-1:
  assumes A ≠ B
  shows Gradn A B n ≠ Gradn A B (Suc n)
proof –
  have Gradn A B 0 ≠ Gradn A B (Suc 0)
    by (simp add: assms)
  moreover
  have n > 0 ⟶ (Gradn A B n ≠ Gradn A B (Suc n))
    by (metis Diff--Cong-Gradn-Suc-Gradn-Suc2 assms cong-diff-2 gr0-implies-Suc)
  ultimately
  show ?thesis
    by blast
qed

lemma Gradn-uniq-aux-1-aa:
  assumes A ≠ B
  shows Gradn A B (k + n) ≠ Gradn A B (k + (Suc n))
proof (induction k)
  case 0
  show Gradn A B (0 + n) ≠ Gradn A B (0 + (Suc n))
    using Gradn-uniq-aux-1 assms plus-nat.add-0 by presburger
next
  case (Suc k)
  show Gradn A B (Suc k + n) ≠ Gradn A B (Suc k + (Suc n))
    using Gradn-uniq-aux-1 add-Suc-right assms by presburger
qed

lemma Gradn-uniq-aux-1-bb:
  assumes A ≠ B
  shows Gradn A B (k + n) ≠ Gradn A B (k + (Suc (Suc n)))
proof (induction k)
  case 0
  show Gradn A B (0 + n) ≠ Gradn A B (0 + (Suc (Suc n)))
    by (metis Gradn-uniq-aux-1 Diff--Bet-Gradn-Gradn-Suc add.left-neutral
      assms between-equality-2)
next
  case (Suc k)
  show Gradn A B ((Suc k) + n) ≠ Gradn A B ((Suc k) + (Suc (Suc n)))
    by (metis Gradn-uniq-aux-1-aa Diff--Bet-Gradn-Suc-Gradn-Suc2 add-Suc
      add-Suc-shift assms between-equality-2)
qed

lemma Gradn-aux-1-0:
  assumes A ≠ B
  shows Gradn A B (Suc n) ≠ A
  by (metis Diff--Bet-Gradn-Suc assms bet-neq32--neq)

lemma Gradn-aux-1-1:
  assumes A ≠ B and
  n ≠ 0
  shows Gradn A B (Suc n) ≠ B
proof –
  obtain m where n = Suc m
    using assms(2) not0-implies-Suc by blast
  have Gradn A B (Suc (Suc m)) ≠ B
  proof (induction m)
  show Gradn A B (Suc (Suc 0)) ≠ B

```

```

    by (metis Gradn-uniq-aux-1 Diff--Bet-Gradn-Suc
        Diff-Bet-Gradn-Suc-Gradn-Suc2 assms(1) between-equality-2)
next
fix m
assume Gradn A B (Suc(Suc m))  $\neq$  B
thus Gradn A B (Suc(Suc(Suc m)))  $\neq$  B
  by (metis Diff--Bet-Gradn-Suc Diff-Bet-Gradn-Suc-Gradn-Suc2
      assms(1) between-equality-2)
qed
thus ?thesis
  by (simp add:  $\langle n = \text{Suc } m \rangle$ )
qed

```

```

lemma Gradn-aux-1-1-bis:
  assumes A  $\neq$  B and
    n  $\neq$  1
  shows Gradn A B n  $\neq$  B
proof (cases n = 0)
case True
thus ?thesis
  using Lem-Gradn-0 assms(1) by presburger
next
case False
then obtain m where n = Suc m
  using not0-implies-Suc by presburger
hence m  $\neq$  0
  using assms(2) by force
thus ?thesis
  using Gradn-aux-1-1 assms(1)  $\langle n = \text{Suc } m \rangle$  by blast
qed

```

```

lemma Gradn-aux-1-2:
  assumes A  $\neq$  B and
    Gradn A B n = A
  shows n = 0
proof -
{
  assume n  $\neq$  0
  then obtain m where n = Suc m
    using not0-implies-Suc by presburger
  hence Gradn A B n  $\neq$  A
    using Gradn-aux-1-0 assms(1) by blast
  hence False
    using Gradn-aux-1-0 assms(1) assms(2) by blast
}
thus ?thesis by blast
qed

```

```

lemma Gradn-aux-1-3:
  assumes A  $\neq$  B and
    Gradn A B n = B
  shows n = 1
  using Gradn-aux-1-1-bis assms(1) assms(2) by blast

```

```

lemma Gradn-uniq-aux-2-a:
  assumes A  $\neq$  B and
    n  $\neq$  0
  shows Gradn A B 0  $\neq$  Gradn A B n
  by (metis Gradn-aux-1-2 Lem-Gradn-0 assms(1) assms(2))

```

```

lemma Gradn-uniq-aux-2:
  assumes A  $\neq$  B and
    n < m
  shows Gradn A B n  $\neq$  Gradn A B m
proof -
  obtain k where m = (Suc k) + n

```

```

by (metis Suc-diff-Suc add commute add-diff-cancel-left'
    assms(2) less-imp-add-positive)
have Gradn A B n  $\neq$  Gradn A B ((Suc k) + n)
proof (induction k)
  case 0
  have 0 + n = n
  by simp
  thus Gradn A B n  $\neq$  Gradn A B ((Suc 0) + n)
  using Gradn-uniq-aux-1-aa assms(1) by (metis add commute)
next
case (Suc k)
hence Gradn A B n  $\neq$  Gradn A B ((Suc k) + n)
by blast
have Suc ((Suc k) + n) = Suc(Suc(k)) + n
by simp
{
  assume Gradn A B n = Gradn A B ((Suc (Suc k)) + n)
  have Gradn A B n = Gradn A B ((Suc k) + n)
  proof (cases n = 0)
    case True
    thus ?thesis
    by (metis Gradn-aux-1-2 Lem-Gradn-0
         $\langle$ Gradn A B n = Gradn A B (Suc (Suc k) + n) $\rangle$  add-cancel-left-right
        assms(1) nat-neq-iff zero-less-Suc)
  next
  case False
  have (Suc k) + n = Suc(k + n)
  by simp
  hence Bet A B (Gradn A B ((Suc k)+n))
  using assms(1) Diff--Bet-Gradn-Suc by presburger
  hence Bet A B (Gradn A B n)
  using assms(1) Diff--Bet-Gradn-Suc
   $\langle$ Gradn A B n = Gradn A B (Suc (Suc k) + n) $\rangle$  add-Suc by presburger
  have Bet A (Gradn A B ((Suc k)+n)) (Gradn A B (Suc((Suc k) + n)))
  using Diff--Bet-Gradn-Gradn-Suc assms(1) by blast
  hence Bet A (Gradn A B ((Suc k)+n)) (Gradn A B n)
  using  $\langle$ Gradn A B n = Gradn A B ((Suc (Suc k)) + n) $\rangle$ 
   $\langle$ Suc ((Suc k) + n) = Suc(Suc(k)) + n $\rangle$  by simp
  moreover
  have Bet A (Gradn A B n) (Gradn A B ((Suc k) + n))
  using Diff-Le--Bet-Gradn-Plus assms(1) by blast
  ultimately
  show ?thesis
  using between-equality-2 by blast
qed
hence False
using Suc.IH by blast
}
thus Gradn A B n  $\neq$  Gradn A B ((Suc (Suc k)) + n)
by blast
qed
thus ?thesis
using  $\langle$ m = Suc k + n $\rangle$  by blast
qed

lemma Gradn-uniq:
  assumes A  $\neq$  B and
    Gradn A B n = Gradn A B m
  shows n = m
proof -
{
  assume n  $\neq$  m
  {
    assume n < m
    hence False
    using Gradn-uniq-aux-2 assms(1) assms(2) by blast
  }
}

```

```

}
moreover
{
  assume  $m < n$ 
  hence False
  by (metis Gradn-uniq-aux-2 assms(1) assms(2))
}
ultimately
have False
  using  $\langle n \neq m \rangle$  nat-neq-iff by blast
}
thus ?thesis by blast
qed

lemma Gradn-le-suc-1:
shows  $A \text{ (Gradn } A \ B \ n) \text{ Le } A \text{ (Gradn } A \ B \text{ (Suc } n))}$ 
using Bet-Gradn-Gradn-Suc l5-12-a by presburger

lemma Gradn-le-1:
assumes  $m \leq n$ 
shows  $A \text{ (Gradn } A \ B \ m) \text{ Le } A \text{ (Gradn } A \ B \text{ (Suc } n))}$ 
by (metis Bet-Gradn-Gradn-Suc Lem-Gradn-id-n Diff-Le-Bet--Gradn-Gradn
  assms bet--le1213 le-Suc-eq)

lemma Gradn-le-suc-2:
shows  $B \text{ (Gradn } A \ B \text{ (Suc } n)) \text{ Le } B \text{ (Gradn } A \ B \text{ (Suc(Suc } n))})$ 
by (metis Bet-Gradn-Gradn-Suc Diff--Bet-Gradn-Suc bet--le1213
  between-exchange3)

lemma grad-equiv-coq-1:
shows  $\text{Grad } A \ B \ B$ 
proof –
  have  $\text{Gradn } A \ B \text{ (Suc } 0) = B$ 
    by auto
  thus ?thesis
    by (metis Grad-def n-not-Suc-n)
qed

lemma grad-aab--ab:
assumes  $\text{Grad } A \ A \ B$ 
shows  $A = B$ 
proof –
  obtain  $n$  where  $B = \text{Gradn } A \ A \ n$ 
    using Grad-def assms by blast
  thus ?thesis
    by simp
qed

lemma grad-stab:
assumes  $\text{Grad } A \ B \ C$  and
   $\text{Bet } A \ C \ C'$  and
   $\text{Cong } A \ B \ C \ C'$ 
shows  $\text{Grad } A \ B \ C'$ 
proof (cases  $A = B$ )
  case True
  thus ?thesis
    using assms(1) assms(3) cong-reverse-identity by blast
next
  case False
  obtain  $n$  where  $n \neq 0 \wedge C = \text{Gradn } A \ B \ n$ 
    using Grad-def assms(1) by presburger
  hence  $\text{Bet } A \ B \text{ (Gradn } A \ B \ n)$ 
    using False Diff--Bet-Gradn by blast
  hence  $\text{Bet } A \ B \ C$ 
    using  $\langle n \neq 0 \wedge C = \text{Gradn } A \ B \ n \rangle$  by blast
  hence  $C' = \text{Gradn } A \ B \text{ (Suc } n)$ 

```



```

    using False Diff-Bet-Bet-Cong-Gradn-Suc  $\langle n \neq 0 \wedge C = \text{Gradn } A \ B \ n \rangle$ 
      assms(2) assms(3) between-exchange4 by blast
  thus ?thesis
    using Grad-def by blast
qed

```

```

lemma grad--bet:
  assumes Grad A B C
  shows Bet A B C
proof (cases A = B)
  case True
  thus ?thesis
    by (simp add: between-trivial2)
next
  case False
  obtain n where  $n \neq 0 \wedge C = \text{Gradn } A \ B \ n$ 
    using Grad-def assms(1) by presburger
  hence Bet A B (Gradn A B n)
    using False Diff--Bet-Gradn by blast
  thus ?thesis
    using  $\langle n \neq 0 \wedge C = \text{Gradn } A \ B \ n \rangle$  by blast
qed

```

```

lemma grad--col:
  assumes Grad A B C
  shows Col A B C
  by (simp add: assms bet-col grad--bet)

```

```

lemma grad-neq--neq13:
  assumes Grad A B C and
    A  $\neq$  B
  shows A  $\neq$  C
  using assms(1) assms(2) between-identity grad--bet by blast

```

```

lemma grad-neq--neq12:
  assumes Grad A B C and
    A  $\neq$  C
  shows A  $\neq$  B
  using Grad-def assms(1) assms(2) grad-rec-a-a by force

```

```

lemma grad112--eq:
  assumes Grad A A B
  shows A = B
  by (meson assms grad-neq--neq12)

```

```

lemma grad121--eq:
  assumes Grad A B A
  shows A = B
  using assms grad-neq--neq13 by blast

```

```

lemma grad--le:
  assumes Grad A B C
  shows A B Le A C
  using assms bet--le1213 grad--bet by blast

```

```

lemma grad2-init:
  shows Grad2 A B B C D D
proof -
  have  $(B = \text{Gradn } A \ B \ (\text{Suc } 0)) \wedge (D = \text{Gradn } C \ D \ (\text{Suc } 0))$ 
    using One-nat-def Lem-Gradn-1 by presburger
  thus ?thesis
    using Grad2-def by blast
qed

```

```

lemma Grad2-stab:
  assumes Grad2 A B C D E F and

```

Bet A C C' and
Cong A B C C' and
Bet D F F' and
Cong D E F F'
shows *Grad2 A B C' D E F'*
proof –
obtain *n* **where** $(n \neq 0) \wedge (C = \text{Gradn } A \ B \ n) \wedge (F = \text{Gradn } D \ E \ n)$
using *Grad2-def assms(1)* **by** *presburger*
have $C' = \text{Gradn } A \ B \ (\text{Suc } n)$
by *(metis Diff-Bet-Bet-Cong-Gradn-Suc Lem-Gradn-id-n Diff--Bet-Gradn*
 $\langle n \neq 0 \wedge C = \text{Gradn } A \ B \ n \wedge F = \text{Gradn } D \ E \ n \rangle$ *assms(2) assms(3)*
between-exchange4 cong-reverse-identity)
moreover
have $F' = \text{Gradn } D \ E \ (\text{Suc } n)$
by *(metis Diff-Bet-Bet-Cong-Gradn-Suc Lem-Gradn-id-n Diff--Bet-Gradn*
 $\langle n \neq 0 \wedge C = \text{Gradn } A \ B \ n \wedge F = \text{Gradn } D \ E \ n \rangle$ *assms(4) assms(5)*
between-exchange4 cong-reverse-identity)
ultimately
show *?thesis*
using *Grad2-def* **by** *blast*
qed

lemma *bet-cong2-grad--grad2-aux-1:*
assumes $C = (\text{Gradn } A \ B \ 0)$ **and**
Bet D E F and
Cong A B D E and
Cong B C E F
shows $F = \text{Gradn } D \ E \ 2$
proof –
have $(\text{Gradn } A \ B \ 0) = A$
using *Lem-Gradn-0* **by** *blast*
hence $A = C$
using *assms(1)* **by** *auto*
hence $\text{Cong } D \ E \ E \ F$
using *assms(3) assms(4) cong-transitivity not-cong-4312* **by** *blast*
thus *?thesis*
by *(metis Diff-Bet-Bet-Cong-Gradn-Suc Lem-Gradn-1 Suc-1*
 $\text{assms(2) cong-diff-3 grad-rec-1-2})$
qed

lemma *bet-cong2-grad--grad2-aux-2:*
assumes *Bet D E F and*
Cong A B D E and
Cong B (Gradn A B (Suc n)) E F
shows $F = \text{Gradn } D \ E \ (\text{Suc } n)$
proof –
have $\forall A \ B \ D \ E \ F. (\text{Bet } D \ E \ F \wedge \text{Cong } A \ B \ D \ E \wedge$
 $\text{Cong } B \ (\text{Gradn } A \ B \ (\text{Suc } n)) \ E \ F \longrightarrow F = \text{Gradn } D \ E \ (\text{Suc } n))$
proof *(induction n)*
case *0*
{
fix $A \ B \ C \ D \ E \ F$
assume $1: \text{Bet } D \ E \ F \wedge \text{Cong } A \ B \ D \ E \wedge \text{Cong } B \ (\text{Gradn } A \ B \ (\text{Suc } 0)) \ E \ F$
hence $\text{Gradn } A \ B \ (\text{Suc } 0) = B$
using *One-nat-def Lem-Gradn-1* **by** *presburger*
hence $E = F$
by *(metis 1 cong-diff-4)*
hence $F = \text{Gradn } D \ E \ (\text{Suc } 0)$
by *simp*
}
thus *?case*
by *blast*
next
case $(\text{Suc } n)$
{
assume $2: \forall A \ B \ D \ E \ F. (\text{Bet } D \ E \ F \wedge \text{Cong } A \ B \ D \ E \wedge$

```

Cong B (Gradn A B (Suc n)) E F)  $\longrightarrow$  F = Gradn D E (Suc n)
{
  fix A B D E F
  assume Bet D E F and
    Cong A B D E and
      Cong B (Gradn A B (Suc (Suc n))) E F
  have Cong A B (Gradn A B (Suc n)) (Gradn A B (Suc(Suc n)))
    using Cong-Gradn-Suc-Gradn-Suc2 by auto
  have Bet A (Gradn A B (Suc n)) (Gradn A B (Suc(Suc n)))
    using Bet-Gradn-Gradn-Suc by auto
  have F = Gradn D E (Suc (Suc n))
  proof (cases A = B)
    case True
    thus ?thesis
      by (metis  $\langle$ Cong A B D E $\rangle$   $\langle$ Cong B (Gradn A B (Suc (Suc n))) E F $\rangle$ 
        cong-reverse-identity grad-rec-a-a)
  next
    case False
    have D  $\neq$  E
      using False  $\langle$ Cong A B D E $\rangle$  cong-diff by blast
    obtain F' where Bet D E F' and Cong E F' B (Gradn A B (Suc n))
      using segment-construction by fastforce
    have Cong B (Gradn A B (Suc n)) E F'
      using  $\langle$ Cong E F' B (Gradn A B (Suc n)) $\rangle$  not-cong-3412 by blast
    hence F' = Gradn D E (Suc n)
      using  $\langle$ Bet D E F' $\rangle$   $\langle$ Cong A B D E $\rangle$  2 by blast
    thus ?thesis
  proof (cases E = F')
    case True
    thus ?thesis
      by (metis Diff-Bet-Bet-Cong-Gradn-Suc
         $\langle$ Cong A B (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n))) $\rangle$   $\langle$ D  $\neq$  E $\rangle$ 
         $\langle$ F' = Gradn D E (Suc n) $\rangle$  cong-inner-transitivity
        cong-reverse-identity  $\langle$ Bet D E F' $\rangle$   $\langle$ Cong A B D E $\rangle$ 
         $\langle$ Cong B (Gradn A B (Suc (Suc n))) E F $\rangle$ 
         $\langle$ Cong E F' B (Gradn A B (Suc n)) $\rangle$ )
    next
      case False
      have F = Sym D E (Gradn D E ((Suc(Suc n))-1))
      proof -
        have Bet D E (Gradn D E ((Suc(Suc n))-1))
          using  $\langle$ Bet D E F' $\rangle$   $\langle$ F' = Gradn D E (Suc n) $\rangle$  by auto
        moreover
        have PreGrad D E (Gradn D E ((Suc(Suc n))-1)) F
        proof -
          have Bet D E (Gradn D E (Suc n))
            using Diff--Bet-Gradn-Suc  $\langle$ D  $\neq$  E $\rangle$  by blast
          hence Bet D E (Gradn D E ((Suc(Suc n))-1))
            by fastforce
          moreover
          have Bet D (Gradn D E (Suc n)) F
          proof -
            have Bet B (Gradn A B (Suc n)) (Gradn A B (Suc(Suc n)))
              by (metis Diff--Bet-Gradn-Suc
                 $\langle$ Bet A (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n))) $\rangle$ 
                between-exchange3)
            moreover
            have Cong B (Gradn A B (Suc n)) E (Gradn D E (Suc n))
              using  $\langle$ Cong B (Gradn A B (Suc n)) E F' $\rangle$ 
                 $\langle$ F' = Gradn D E (Suc n) $\rangle$  by blast
            moreover
            have Cong B (Gradn A B (Suc(Suc n))) E (Gradn D E (Suc(Suc n)))
            proof -
              have Bet A B (Gradn A B (Suc n))
                using Diff--Bet-Gradn-Suc not-bet-distincts by blast
              moreover

```

```

have Cong D E (Gradn D E (Suc n)) (Gradn D E (Suc (Suc n)))
  using Cong-Gradn-Suc-Gradn-Suc2 by auto
moreover
have Bet A (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))
  using Bet-Gradn-Gradn-Suc by blast
moreover
have Cong D E (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))
  using ⟨Cong A B (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))⟩
    ⟨Cong A B D E⟩ cong-inner-transitivity by blast
moreover
have Bet D (Gradn D E (Suc n)) (Gradn D E (Suc (Suc n)))
  using Bet-Gradn-Gradn-Suc by blast
thus ?thesis
  using l2-11-b between-exchange3 calculation(2) cong-inner-transitivity
  by (meson ⟨Bet B (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))⟩
    ⟨Cong B (Gradn A B (Suc n)) E (Gradn D E (Suc n))⟩
    ⟨Bet D E (Gradn D E (Suc n))⟩
    calculation(3) calculation(4))
qed
moreover
have E Out (Gradn D E (Suc(Suc n))) (Gradn D E (Suc n))
  by (metis False Diff--Bet-Gradn-Gradn-Suc
    ⟨Bet D E (Gradn D E (Suc n))⟩
    ⟨D ≠ E⟩ ⟨F' = Gradn D E (Suc n)⟩
    bet-out between-exchange3 l6-6)
ultimately
show ?thesis
  using Bet-Gradn-Gradn-Suc Diff--Bet-Gradn-Suc ⟨D ≠ E⟩
    construction-uniqueness not-cong-3412
  by (metis ⟨Bet D E F⟩ ⟨Cong B (Gradn A B (Suc (Suc n))) E F⟩)
qed
hence Bet D (Gradn D E ((Suc(Suc n))−1)) F
  by simp
moreover
have Cong (Gradn A B (Suc n)) (Gradn A B (Suc(Suc n))) (Gradn D E (Suc n)) F
proof −
  let ?A = A
  let ?B = Gradn A B (Suc n)
  let ?C = Gradn A B (Suc (Suc n))
  let ?A' = D
  let ?B' = Gradn D E (Suc n)
  let ?C' = F
  have Bet ?A ?B ?C
    using ⟨Bet A (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))⟩ by auto
  moreover have Bet ?A' ?B' ?C'
    using ⟨Bet D (Gradn D E (Suc n)) F⟩ by auto
  moreover have Cong ?A ?B ?A' ?B'
    by (metis Diff--Bet-Gradn-Suc cong-reverse-identity
      ⟨Cong A B D E⟩ ⟨Cong E F' B (Gradn A B (Suc n))⟩
      ⟨F' = Gradn D E (Suc n)⟩ l2-11-b not-cong-3412)
  moreover have Cong ?A ?C ?A' ?C'
    by (metis Diff--Bet-Gradn-Suc cong-reverse-identity
      ⟨Bet D E F⟩ ⟨Cong A B D E⟩
      ⟨Cong B (Gradn A B (Suc (Suc n))) E F⟩ l2-11-b)
  ultimately show ?thesis
    using l4-3-1 by blast
qed
hence Cong A B (Gradn D E (Suc n)) F
  using ⟨Cong A B (Gradn A B (Suc n)) (Gradn A B (Suc (Suc n)))⟩
    cong-transitivity by blast
hence Cong D E (Gradn D E (Suc n)) F
  using ⟨Cong A B D E⟩ cong-inner-transitivity by blast
hence Cong D E (Gradn D E ((Suc(Suc n))−1)) F
  by simp
ultimately
show ?thesis

```

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      using ⟨D ≠ E⟩ PreGrad-def by blast
    qed
  ultimately
  show ?thesis
    by (metis PreGrad-def LemSym-aux)
  qed
  hence F = Gradn D E (Suc (Suc n))
    by simp
  thus ?thesis by blast
  qed
  qed
}
  hence ∀ A B D E F. (Bet D E F ∧ Cong A B D E ∧
  Cong B (Gradn A B (Suc(Suc n))) E F) ⟶ F = Gradn D E (Suc(Suc n))
    by blast
}
  thus ?case
    using Suc.IH by fastforce
  qed
  thus ?thesis
    using assms(1) assms(2) assms(3) by blast
  qed

lemma bet-cong2-grad--grad2-aux:
  assumes n ≠ 0 and
    C = (Gradn A B n) and
    Bet D E F and
    Cong A B D E and
    Cong B C E F
  shows F = Gradn D E n
  proof -
    obtain k where n = Suc k
      using assms(1) not0-implies-Suc by blast
    thus ?thesis
      using assms(5) assms(2) assms(3) assms(4) bet-cong2-grad--grad2-aux-2 by blast
  qed

lemma bet-cong2-grad--grad2:
  assumes Grad A B C and
    Bet D E F and
    Cong A B D E and
    Cong B C E F
  shows Grad2 A B C D E F
  proof (cases A = B)
    case True
      thus ?thesis
        by (metis assms(1) assms(4) cong-diff-4 grad2-init grad-neq--neq12)
    next
      case False
        hence D ≠ E
          using assms(3) cong-diff by blast
        obtain n where n ≠ 0 ∧ C = (Gradn A B n)
          using Grad-def assms(1) by presburger
        have F = Gradn D E n
          using bet-cong2-grad--grad2-aux ⟨n ≠ 0 ∧ C = Gradn A B n⟩
            assms(2) assms(3) assms(4) by blast
        thus ?thesis
          using ⟨n ≠ 0 ∧ C = (Gradn A B n)⟩ Grad2-def by blast
  qed

lemma grad2--grad123:
  assumes Grad2 A B C D E F
  shows Grad A B C
  proof -
    obtain n where (n ≠ 0) ∧ (C = Gradn A B n) ∧ (F = Gradn D E n)
      using Grad2-def assms by presburger
  end

```

```

thus ?thesis
using Grad-def by blast
qed

lemma grad2--grad456:
assumes Grad2 A B C D E F
shows Grad D E F
proof –
obtain n where (n ≠ 0) ∧ (C = Gradn A B n) ∧ (F = Gradn D E n)
using Grad2-def assms by presburger
thus ?thesis
using Grad-def by blast
qed

lemma grad-sum-aux-R1:
assumes
  A C Le A D and
  Cong A D A E and
  Cong A C A E' and
  A Out E E'
shows Bet A E' E
by (meson Out-cases l5-6 assms(2) assms(3) assms(4) assms(1) l6-13-1)

lemma grad-sum-aux-0:
assumes A ≠ B and
  D = Gradn A B (Suc(Suc n)) and
  Cong A D A E and
  C = Gradn A B (Suc n) and
  Cong A C A E' and
  A Out E E'
shows Bet A E' E
proof –
{
assume Bet A E E'
have False
by (metis Gradn-aux-1-0 Gradn-uniq-aux-1
  Diff--Bet-Gradn-Gradn-Suc ‹Bet A E E'›
  assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
  between-cong-2 between-equality-2 cong-preserves-bet
  cong-transitivity l5-1 l6-6 not-cong-3412)
}
thus ?thesis
using Out-def assms(6) by auto
qed

lemma grad-sum-aux-1:
assumes A ≠ B and
  D = Gradn A B (Suc(Suc n)) and
  Bet A C E and
  Cong A D C E and
  F = Gradn A B (Suc n) and
  Bet A C E' and
  Cong A F C E' and
  C ≠ A
shows Bet A E' E
proof –
have A ≠ B ∧ D = Gradn A B (Suc(Suc n)) ∧
  Bet A C E ∧ Cong A D C E ∧
  F = Gradn A B (Suc n) ∧
  Bet A C E' ∧ Cong A F C E' ∧ A ≠ C  $\longrightarrow$ 
  Bet A E' E
proof (induction n)
case 0
{
fix A B C D E F

```

```

assume  $A \neq B$  and
   $D = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } 0))$  and
   $\text{Bet } A \ C \ E$  and
   $\text{Cong } A \ D \ C \ E$  and
   $F = \text{Gradn } A \ B \ (\text{Suc } 0)$  and
   $\text{Bet } A \ C \ E'$  and
   $\text{Cong } A \ F \ C \ E'$  and
   $A \neq C$ 
have  $F = B$ 
  by (simp add:  $\langle F = \text{Gradn } A \ B \ (\text{Suc } 0) \rangle$ )
have  $\text{Bet } C \ E' \ E$ 
proof –
  have  $C \text{ Out } E' \ E$ 
  proof –
    {
      assume  $C = E'$ 
      hence  $A = F$ 
      using  $\langle \text{Cong } A \ F \ C \ E' \rangle$  congr-identity by blast
      hence False
      using  $\langle A \neq B \rangle \langle F = B \rangle$  by fastforce
    }
  hence  $C \neq E'$ 
  by blast
moreover
  {
    assume  $C = E$ 
    hence  $A = D$ 
    using  $\langle \text{Cong } A \ D \ C \ E \rangle$  congr-identity by blast
    hence False
    by (metis Gradn-aux-1-0  $\langle A \neq B \rangle \langle D = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } 0)) \rangle$ )
  }
hence  $C \neq E$ 
by blast
moreover
have  $\text{Bet } E' \ C \ A$ 
  using  $\langle \text{Bet } A \ C \ E' \rangle$  Bet-cases by blast
moreover
have  $\text{Bet } E \ C \ A$ 
  using Bet-cases  $\langle \text{Bet } A \ C \ E \rangle$  by blast
ultimately
show ?thesis
  using  $\langle A \neq C \rangle$  l6-2 by metis
qed
moreover
have  $A \ F \text{ Le } A \ D$ 
  using  $\langle F = \text{Gradn } A \ B \ (\text{Suc } 0) \rangle \langle D = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } 0)) \rangle$ 
  Gradn-le-suc-1 by blast
hence  $C \ E' \text{ Le } C \ E$ 
  using l5-6  $\langle \text{Cong } A \ D \ C \ E \rangle \langle \text{Cong } A \ F \ C \ E' \rangle$  by blast
ultimately
show ?thesis
  using l6-13-1 by blast
qed
hence  $\text{Bet } A \ E' \ E$ 
  by (meson  $\langle \text{Bet } A \ C \ E \rangle$  between-exchange2)
}
thus ?case
  by force
next
case  $(\text{Suc } n)$ 
  {
    assume  $1: A \neq B \wedge D = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } n)) \wedge$ 
     $\text{Bet } A \ C \ E \wedge \text{Cong } A \ D \ C \ E \wedge$ 
     $F = \text{Gradn } A \ B \ (\text{Suc } n) \wedge$ 
     $\text{Bet } A \ C \ E' \wedge \text{Cong } A \ F \ C \ E' \wedge A \neq C \longrightarrow$ 
     $\text{Bet } A \ E' \ E$ 

```

```

{
  assume  $A \neq B$  and
     $D = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc}(\text{Suc } n)))$  and
     $\text{Bet } A \ C \ E$  and
     $\text{Cong } A \ D \ C \ E$  and
     $F = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } n))$  and
     $\text{Bet } A \ C \ E'$ 
     $\text{Cong } A \ F \ C \ E'$  and
     $A \neq C$ 
  obtain  $F'$  where  $F' = \text{Gradn } A \ B \ (\text{Suc } n)$ 
    by auto
  obtain  $E''$  where  $\text{Bet } A \ C \ E'' \wedge \text{Cong } A \ F' \ C \ E''$ 
    by (meson Cong-cases segment-construction)
  have  $\text{Bet } A \ E'' \ E'$ 
  proof -
    have  $A \neq B$ 
      by (simp add: assms(1))
    moreover
    have  $F = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } n))$ 
      by (simp add:  $\langle F = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n)) \rangle$ )
    moreover
    have  $\text{Bet } A \ C \ E'$ 
      by (simp add: assms(6))
    moreover
    have  $\text{Cong } A \ F \ C \ E'$ 
      by (simp add: assms(7))
    moreover
    have  $F' = \text{Gradn } A \ B \ (\text{Suc } n)$ 
      by (simp add:  $\langle F' = \text{Gradn } A \ B \ (\text{Suc } n) \rangle$ )
    moreover
    have  $\text{Bet } A \ C \ E''$ 
      by (simp add:  $\langle \text{Bet } A \ C \ E'' \wedge \text{Cong } A \ F' \ C \ E'' \rangle$ )
    moreover
    have  $\text{Cong } A \ F' \ C \ E''$ 
      by (simp add:  $\langle \text{Bet } A \ C \ E'' \wedge \text{Cong } A \ F' \ C \ E'' \rangle$ )
    moreover
    have  $A \neq C$ 
      using assms(8) by auto
    ultimately
    show ?thesis
  proof -
    {
      assume  $\text{Bet } C \ E'' \ E'$ 
      hence ?thesis
        using assms(6) between-exchange2 by blast
    }
    moreover
    {
      assume  $\text{Bet } C \ E' \ E''$ 
      hence ?thesis
        by (metis Diff--Bet-Gradn-Gradn-Suc Gradn-aux-1-0
          bet2--out cong-identity-inv cong-preserves-bet
           $\langle \text{Bet } A \ C \ E'' \wedge \text{Cong } A \ F' \ C \ E'' \rangle \langle F = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n)) \rangle$ 
           $\langle F' = \text{Gradn } A \ B \ (\text{Suc } n) \rangle$  assms(1) assms(7)
          calculation not-bet-distincts)
    }
    ultimately show ?thesis
      by (metis  $\langle \text{Bet } A \ C \ E'' \rangle$  assms(6) assms(8) l5-2)
  qed
qed
have  $C \ E'' \ Le \ C \ E'$ 
  using  $\langle \text{Bet } A \ C \ E'' \wedge \text{Cong } A \ F' \ C \ E'' \rangle \langle \text{Bet } A \ E'' \ E' \rangle$ 
    bet--le1213 between-exchange3 by blast
have  $A \ (\text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n))) \ Le \ A \ (\text{Gradn } A \ B \ (\text{Suc } (\text{Suc } (\text{Suc } n))))$ 
  using Gradn-Le-Gradn-Suc by blast
hence  $A \ F \ Le \ A \ D$ 

```



```

    using  $\langle F = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n)) \rangle$ 
     $\langle D = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } (\text{Suc } n))) \rangle$  by blast
  hence  $C \ E' \ Le \ C \ E$ 
    using assms(4) assms(7) l5-6 by blast
  have  $Bet \ A \ E' \ E$ 
  proof -
    {
      assume  $Bet \ A \ E \ E'$ 
      hence  $A \ E \ Le \ A \ E'$ 
        using bet--le1213 by auto
      hence  $C \ E \ Le \ C \ E'$ 
        using  $\langle Bet \ A \ E \ E' \rangle$  assms(3) bet--le1213 between-exchange3 by blast
      hence  $Cong \ C \ E \ C \ E'$ 
        by (simp add:  $\langle C \ E' \ Le \ C \ E \rangle$  le-anti-symmetry)
      hence  $Cong \ A \ D \ A \ F$ 
        by (meson assms(4) assms(7) cong-transitivity not-cong-3412)
      hence False
        by (metis Gradn-uniq-aux-1 Diff--Bet-Gradn-Gradn-Suc
           $\langle D = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } (\text{Suc } n))) \rangle$ 
           $\langle F = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n)) \rangle$ 
          assms(1) between-cong not-cong-3412)
    }
    thus ?thesis
      by (metis assms(3) assms(6) assms(8) l5-1)
  qed
}
}
thus ?case
  using Suc.IH by fastforce
qed
thus ?thesis
  using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
  assms(7) assms(8) by blast
qed

lemma grad-sum-aux-1A:
  assumes  $A \neq B$  and
     $C = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } 0))$ 
  shows  $B \ \text{Midpoint } A \ C$ 
proof -
  have  $C = (\text{Sym } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } 0)))$ 
    by (simp add: assms(2))
  have  $\dots = (\text{Sym } A \ B \ B)$ 
    by (simp add: assms(1))
  hence  $(\text{SOME } x::'p. \text{PreGrad } A \ B \ B \ x) = C$ 
    using assms(1) by (metis Gradn-aux-1-0 Sym.simps
       $\langle C = \text{Sym } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } 0)) \rangle$  assms(2))
  hence  $\text{PreGrad } A \ B \ B \ C$ 
    by (metis PreGrad-def Sym-Bet--Bet-Bet Sym-Bet--Cong
      Sym.elims assms(1) not-bet-distincts)
  thus ?thesis
    by (simp add: Midpoint-def PreGrad-def)
qed

lemma grad-sum-aux-2:
  assumes  $A \neq B$  and
     $D = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n))$  and
     $Bet \ A \ C \ E$  and
     $Cong \ A \ D \ C \ E$  and
     $F = \text{Gradn } A \ B \ (\text{Suc } n)$  and
     $Bet \ A \ C \ E'$  and
     $Cong \ A \ F \ C \ E'$  and
     $C \neq A$ 
  shows  $Cong \ A \ B \ E' \ E$ 
proof -
  have  $A \neq B \wedge D = \text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n)) \wedge Bet \ A \ C \ E \wedge Cong \ A \ D \ C \ E \wedge$ 

```

```

F = Gradn A B (Suc n) ∧ Bet A C E' ∧ Cong A F C E' ∧ A ≠ C →
Cong A B E' E
proof (induction n)
  case 0
  {
    fix A B C D E F
    assume A ≠ B and
      D = Gradn A B (Suc(Suc 0)) and
      Bet A C E and
      Cong A D C E and
      F = Gradn A B (Suc 0) and
      Bet A C E' and
      Cong A F C E' and
      A ≠ C
    have B Midpoint A D
      using grad-sum-aux-1A ⟨A ≠ B⟩ ⟨D = Gradn A B (Suc (Suc 0))⟩ by blast
    hence Bet A B D ∧ Cong A B B D
      using Midpoint-def by blast
    have B = F
      using One-nat-def Lem-Gradn-1 ⟨F = Gradn A B (Suc 0)⟩ by presburger
    hence Cong A B C E'
      by (simp add: ⟨Cong A F C E'⟩)
    have Cong B D C E'
      using ⟨B = F⟩ ⟨Bet A B D ∧ Cong A B B D⟩ ⟨Cong A B C E'⟩
      cong-inner-transitivity by blast
    have Cong B D E' E
      using cong-commutativity cong-diff-2 Tarski-neutral-dimensionless-axioms
      ⟨A ≠ B⟩ ⟨A ≠ C⟩ ⟨B = F⟩ ⟨Bet A B D ∧ Cong A B B D⟩ ⟨Bet A C E'⟩
      ⟨Bet A C E⟩ ⟨Cong A D C E⟩ ⟨Cong A F C E'⟩ bet-out between-symmetry
      l6-2 out-cong-cong by metis
    hence Cong C E' E' E
      using ⟨Cong B D C E'⟩ cong-inner-transitivity by blast
    hence Cong A B E' E
      using ⟨B = F⟩ ⟨Cong A F C E'⟩ cong-transitivity by blast
  }
  thus ?case
    by blast
next
  case (Suc n)
  have Cong A B E' E
  proof –
    have Bet A F D
      using Diff-Bet-Gradn-Gradn-Suc assms(1) assms(2) assms(5) by blast
    have Cong A B F D
      using Diff-Cong-Gradn-Suc-Gradn-Suc2 assms(1) assms(2)
      assms(5) by blast
    have Bet A E' E
      using grad-sum-aux-1 assms(1) assms(2) assms(3) assms(4)
      assms(5) assms(6) assms(7) assms(8) by blast
    {
      assume Bet C E E'
      hence Cong A B E' E
        by (metis ⟨Bet A E' E⟩ ⟨Bet A F D⟩ ⟨Cong A B F D⟩
          assms(1) assms(4) assms(6) assms(7) between-cong between-equality-2
          between-exchange3 cong-diff cong-transitivity not-cong-3412)
    }
    moreover
    {
      assume Bet C E' E
      hence Cong A B E' E
        using ⟨Bet A F D⟩ ⟨Cong A B F D⟩ assms(4) assms(7)
        cong-transitivity l4-3-1 by blast
    }
    ultimately show ?thesis
      by (metis assms(3) assms(6) assms(8) l5-2)
  qed

```

```

    thus ?case
    by blast
qed
thus ?thesis
using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) by blast
qed

lemma grad-sum-aux:
  assumes  $A \neq B$  and
     $C = \text{Gradn } A \ B \ (\text{Suc } n)$  and
     $D = \text{Gradn } A \ B \ (\text{Suc } m)$  and
     $\text{Bet } A \ C \ E$  and
     $\text{Cong } A \ D \ C \ E$ 
  shows  $E = \text{Gradn } A \ B \ ((\text{Suc } n) + (\text{Suc } m))$ 
proof -
  have  $\forall A \ B \ C \ D \ E. (A \neq B \wedge C = \text{Gradn } A \ B \ (\text{Suc } n) \wedge$ 
     $D = \text{Gradn } A \ B \ (\text{Suc } m) \wedge \text{Bet } A \ C \ E \wedge \text{Cong } A \ D \ C \ E) \longrightarrow$ 
     $E = \text{Gradn } A \ B \ ((\text{Suc } n) + (\text{Suc } m))$ 
  proof (induction m)
    case 0
    {
      fix  $A \ B \ C \ D \ E$ 
      assume  $A \neq B$  and
         $C = \text{Gradn } A \ B \ (\text{Suc } n)$  and
         $D = \text{Gradn } A \ B \ (\text{Suc } 0)$  and
         $\text{Bet } A \ C \ E$  and
         $\text{Cong } A \ D \ C \ E$ 
      have  $E = \text{Gradn } A \ B \ ((\text{Suc } n) + (\text{Suc } 0))$ 
      by (metis Nat.add-0-right Diff-Bet-Bet-Cong-Gradn-Suc
        Lem-Gradn-0 Diff--Bet-Gradn-Suc Cong-Gradn-Gradn-Suc
        cong-transitivity  $\langle A \neq B \rangle$   $\langle \text{Bet } A \ C \ E \rangle$   $\langle C = \text{Gradn } A \ B \ (\text{Suc } n) \rangle$ 
         $\langle \text{Cong } A \ D \ C \ E \rangle$   $\langle D = \text{Gradn } A \ B \ (\text{Suc } 0) \rangle$  add-Suc-shift between-exchange4)
    }
    thus ?case
    by force
  next
    case (Suc m)
    {
      assume 1:  $\forall A \ B \ C \ D \ E. (A \neq B \wedge C = \text{Gradn } A \ B \ (\text{Suc } n) \wedge$ 
         $D = \text{Gradn } A \ B \ (\text{Suc } m) \wedge \text{Bet } A \ C \ E \wedge \text{Cong } A \ D \ C \ E) \longrightarrow$ 
         $E = \text{Gradn } A \ B \ ((\text{Suc } n) + (\text{Suc } m))$ 
      {
        fix  $A \ B \ C \ D \ E$ 
        assume  $A \neq B$  and
           $C = \text{Gradn } A \ B \ (\text{Suc } n)$  and
           $D = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } m))$  and
           $\text{Bet } A \ C \ E$  and
           $\text{Cong } A \ D \ C \ E$ 
        obtain  $F$  where 2:  $F = \text{Gradn } A \ B \ (\text{Suc } m)$ 
        by simp
        obtain  $E'$  where 3:  $\text{Bet } A \ C \ E' \wedge \text{Cong } A \ F \ C \ E'$ 
        by (meson cong-4312 segment-construction)
        hence 4:  $E' = \text{Gradn } A \ B \ ((\text{Suc } n) + (\text{Suc } m))$ 
        using 1 2  $\langle A \neq B \rangle$   $\langle C = \text{Gradn } A \ B \ (\text{Suc } n) \rangle$  by blast
        have 5:  $\text{Bet } A \ B \ E$ 
        proof -
          have  $\text{Grad } A \ B \ C$ 
          using Grad-def  $\langle C = \text{Gradn } A \ B \ (\text{Suc } n) \rangle$  by blast
          hence  $\text{Bet } A \ B \ C$ 
          using grad--bet by blast
          thus ?thesis
          using  $\langle \text{Bet } A \ C \ E \rangle$  between-exchange4 by blast
        qed
        have 6:  $\text{Bet } A \ E' \ E$ 
        using grad-sum-aux-1  $\langle A \neq B \rangle$   $\langle D = \text{Gradn } A \ B \ (\text{Suc}(\text{Suc } m)) \rangle$ 
           $\langle \text{Bet } A \ C \ E \rangle$   $\langle \text{Cong } A \ D \ C \ E \rangle$  2 3 Gradn-aux-1-0
      }
    }
  qed

```

```

      ⟨C = Gradn A B (Suc n)⟩ by blast
    have Cong A B E' E
      using grad-sum-aux-2 2 3 Gradn-aux-1-0 ⟨A ≠ B⟩
      ⟨Bet A C E⟩ ⟨C = Gradn A B (Suc n)⟩ ⟨Cong A D C E⟩
      ⟨D = Gradn A B (Suc (Suc m))⟩ by blast
    hence E = Gradn A B (Suc(Suc n) + (Suc m))
      using Diff-Bet-Bet-Cong-Gradn-Suc 3 4 5 6
      by (metis ⟨A ≠ B⟩ add-Suc-right add-Suc-shift)
    hence E = Gradn A B ((Suc n) + (Suc(Suc m)))
      by simp
  }
  hence ∀ A B C D E. (A ≠ B ∧ C = Gradn A B (Suc n) ∧
D = Gradn A B (Suc(Suc m)) ∧ Bet A C E ∧ Cong A D C E) ⟶
E = Gradn A B ((Suc n) + (Suc(Suc m))) by blast
}
thus ?case
  using Suc.IH by fastforce
qed
thus ?thesis
  using assms(1) assms(2) assms(3) assms(4) assms(5) by blast
qed

lemma grad-sum:
  assumes Grad A B C and
    Grad A B D and
    Bet A C E and
    Cong A D C E
  shows Grad A B E
proof (cases A = B)
  case True
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) grad112--eq grad-stab by blast
next
  case False
  obtain n where 1: (n ≠ 0) ∧ (C = Gradn A B n)
    using Grad-def assms(1) by presburger
  obtain m where 2: (m ≠ 0) ∧ (D = Gradn A B m)
    using Grad-def assms(2) by presburger
  obtain k l where n = Suc k ∧ m = Suc l
    using 1 2 not0--implies-Suc by presburger
  hence E = Gradn A B (n + m)
    using False 1 2 grad-sum-aux assms(3) assms(4) by blast
  thus ?thesis
    by (meson Grad-def ⟨n ≠ 0 ∧ C = Gradn A B n⟩ add-is-0)
qed

lemma SymR-ex:
  assumes B Midpoint A C
  shows C = SymR A B
proof -
  have ∃ x. B Midpoint A x
    using assms(1) by blast
  thus ?thesis
    using assms(1) someI-ex
    by (metis SymR-uniq-aux SymR.elims)
qed

lemma SymR--midp:
  assumes C = SymR A B
  shows B Midpoint A C
  using SymR-ex assms symmetric-point-construction by blast

lemma SymR-uniq:
  assumes C = SymR A B and
    D = SymR A B
  shows C = D

```

```

by (simp add: assms(1) assms(2))

lemma GradExpn-1:
  shows GradExpn A A n = A
  by simp

lemma GradExpn-2:
  shows Bet A B (GradExpn A B (Suc n))
proof (cases A = B)
  case True
  thus ?thesis
    using between-trivial2 by blast
next
  case False
  show Bet A B (GradExpn A B (Suc n))
  proof(induction n)
    case 0
    have (GradExpn A B 1) = B
      using False by simp
    thus Bet A B (GradExpn A B (Suc 0))
      by (metis One-nat-def not-bet-distincts)
  next
    case (Suc n)
    {
      assume Bet A B (GradExpn A B (Suc n))
      have (Suc (Suc n))  $\neq$  0  $\wedge$  (Suc (Suc n))  $\neq$  1
        by presburger
      hence GradExpn A B (Suc(Suc n)) = SymR A (GradExpn A B (Suc n))
        using False by simp
      obtain C where (GradExpn A B (Suc n)) Midpoint A C
        using symmetric-point-construction by blast
      hence Bet A (GradExpn A B (Suc n)) C
        using Midpoint-def by blast
      have C = GradExpn A B (Suc(Suc n))
        using SymR-ex
         $\langle$ GradExpn A B (Suc (Suc n)) = SymR A (GradExpn A B (Suc n)) $\rangle$ 
         $\langle$ GradExpn A B (Suc n) Midpoint A C $\rangle$  by presburger
      hence Bet A B (GradExpn A B (Suc(Suc n)))
        using  $\langle$ Bet A (GradExpn A B (Suc n)) C $\rangle$ 
         $\langle$ Bet A B (GradExpn A B (Suc n)) $\rangle$  between-exchange4 by blast
    }
  thus ?case
    using Suc.IH by blast
qed
qed

lemma GradExpn-3:
  shows Bet A (GradExpn A B (Suc n)) (GradExpn A B (Suc(Suc n)))
proof (cases A = B)
  case True
  thus ?thesis
    using between-trivial2 by force
next
  case False
  show Bet A (GradExpn A B (Suc n)) (GradExpn A B (Suc(Suc n)))
  proof(induction n)
    case 0
    have (GradExpn A B 1) = B
      using False by simp
    thus Bet A (GradExpn A B (Suc 0)) (GradExpn A B (Suc(Suc 0)))
      using GradExpn-2 One-nat-def by presburger
  next
    case (Suc n)
    {
      assume Bet A (GradExpn A B (Suc n)) (GradExpn A B (Suc(Suc n)))
      have (Suc(Suc (Suc n)))  $\neq$  0  $\wedge$  (Suc(Suc (Suc n)))  $\neq$  1

```

```

    by presburger
  hence GradExpn A B (Suc(Suc(Suc n))) = SymR A (GradExpn A B (Suc(Suc n)))
    using False by simp
  obtain C where (GradExpn A B (Suc(Suc n))) Midpoint A C
    using symmetric-point-construction by blast
  hence Bet A (GradExpn A B (Suc(Suc n))) C
    using Midpoint-def by blast
  have C = GradExpn A B (Suc(Suc(Suc n)))
    using SymR-ex
      ⟨GradExpn A B (Suc(Suc (Suc n))) = SymR A (GradExpn A B (Suc(Suc n)))⟩
      ⟨GradExpn A B (Suc(Suc n)) Midpoint A C⟩ by presburger
  hence Bet A (GradExpn A B (Suc (Suc(n)))) (GradExpn A B (Suc(Suc(Suc n))))
    using ⟨Bet A (GradExpn A B (Suc (Suc n))) C⟩ by blast
}
thus ?case
  using Suc.IH by blast
qed
qed

lemma GradExpn-4:
  shows Cong A (GradExpn A B (Suc n)) (GradExpn A B (Suc n)) (GradExpn A B (Suc(Suc n)))
proof (cases A = B)
  case True
  thus ?thesis
    using GradExpn-1 cong-trivial-identity by presburger
next
  case False
  show Cong A (GradExpn A B (Suc n)) (GradExpn A B (Suc n)) (GradExpn A B (Suc(Suc n)))
  proof (induction n)
    case 0
    have GradExpn A B (Suc 0) = B
      using False by simp
    show Cong A (GradExpn A B (Suc 0)) (GradExpn A B (Suc 0)) (GradExpn A B (Suc(Suc 0)))
    proof -
      have (Suc(Suc 0)) ≠ 0 ∧ (Suc(Suc 0)) ≠ 1
        by presburger
      hence GradExpn A B (Suc(Suc 0)) = SymR A (GradExpn A B (Suc 0))
        using False by simp
      obtain C where (GradExpn A B (Suc 0)) Midpoint A C
        using symmetric-point-construction by blast
      hence Cong A (GradExpn A B (Suc 0)) (GradExpn A B (Suc 0)) C
        using Midpoint-def by blast
      have C = GradExpn A B (Suc(Suc 0))
        using SymR-ex ⟨GradExpn A B (Suc (Suc 0)) = SymR A (GradExpn A B (Suc 0))⟩
        ⟨GradExpn A B (Suc 0) Midpoint A C⟩ by presburger
      thus ?thesis
        using ⟨Cong A (GradExpn A B (Suc 0)) (GradExpn A B (Suc 0)) C⟩ by blast
    qed
  next
    case (Suc n)
    {
      assume Cong A (GradExpn A B (Suc n)) (GradExpn A B (Suc n)) (GradExpn A B (Suc(Suc n)))
      have (Suc(Suc(Suc n))) ≠ 0 ∧ (Suc(Suc(Suc n))) ≠ 1
        by presburger
      hence GradExpn A B (Suc(Suc(Suc n))) = SymR A (GradExpn A B (Suc(Suc n)))
        using False by simp
      obtain C where (GradExpn A B (Suc(Suc n))) Midpoint A C
        using symmetric-point-construction by blast
      hence Cong A (GradExpn A B (Suc(Suc n)) (GradExpn A B (Suc(Suc n))) C
        using Midpoint-def by blast
      have C = GradExpn A B (Suc(Suc(Suc n)))
        using SymR-ex
          ⟨GradExpn A B (Suc(Suc (Suc n))) = SymR A (GradExpn A B (Suc(Suc n)))⟩
          ⟨GradExpn A B (Suc(Suc n)) Midpoint A C⟩ by presburger
      hence Cong A (GradExpn A B (Suc(Suc n)))
        (GradExpn A B (Suc(Suc n))) (GradExpn A B (Suc(Suc(Suc n))))

```

```

    using ⟨Cong A (GradExpn A B (Suc (Suc n))) (GradExpn A B (Suc (Suc n))) C⟩
    by fastforce
  }
  thus ?case
    using Suc.IH by blast
qed
qed

```

```

lemma gradexp-init:
  shows GradExp A B B
proof -
  have B = GradExpn A B (Suc 0)
    by auto
  thus ?thesis
    using GradExp-def by blast
qed

```

```

lemma gradexp-stab-aux-0:
  assumes C = GradExpn A B 0 and
    Bet A C C' and
    Cong A C C C'
  shows C' = GradExpn A B 0
proof -
  obtain m where m = Suc 0
    by simp
  show ?thesis
  proof (cases A = B)
    case True
    thus ?thesis
      by (metis GradExpn-1 assms(1) assms(3) cong-reverse-identity)
  next
    case False
    hence A ≠ B
      by auto
    show ?thesis
    proof -
      have C = A
        using assms(1) by simp
      hence C = C'
        using assms(3) cong-reverse-identity by blast
      thus ?thesis
        using assms(1) by fastforce
    qed
  qed
qed

```

```

lemma gradexp-stab-aux-n:
  assumes C = GradExpn A B (Suc n) and
    Bet A C C' and
    Cong A C C C'
  shows C' = GradExpn A B (Suc(Suc n))
proof -
  obtain m where m = Suc(Suc n)
    by simp
  hence m - 1 = Suc n
    by simp
  show ?thesis
  proof (cases A = B)
    case True
    thus ?thesis
      by (metis GradExpn-1 assms(1) assms(3) cong-reverse-identity)
  next
    case False
    hence A ≠ B
      by auto
    show ?thesis

```

```

proof –
  have  $C$  Midpoint  $A$   $C'$ 
    using  $\text{assms}(2)$   $\text{assms}(3)$  midpoint-def by auto
  hence  $C' = \text{SymR } A \ C$ 
    using SymR-ex by blast
  hence  $C' = \text{SymR } A \ (\text{GradExpn } A \ B \ (m - 1))$ 
    using  $\langle m - 1 = \text{Suc } n \rangle$   $\text{assms}(1)$  by presburger
  hence  $C' = \text{GradExpn } A \ B \ m$ 
    by (simp add: False  $\langle m = \text{Suc } (\text{Suc } n) \rangle$ )
  thus ?thesis
    using  $\langle m = \text{Suc } (\text{Suc } n) \rangle$  by blast
qed
qed
qed

lemma gradexp-stab:
  assumes  $\text{GradExp } A \ B \ C$  and
     $\text{Bet } A \ C \ C'$  and
     $\text{Cong } A \ C \ C \ C'$ 
  shows  $\text{GradExp } A \ B \ C'$ 
proof –
  obtain  $n$  where  $C = \text{GradExpn } A \ B \ n$ 
    using GradExp-def  $\text{assms}(1)$  by blast
  show ?thesis
  proof (cases  $n = 0$ )
    case True
      thus ?thesis
        using  $\langle C = \text{GradExpn } A \ B \ n \rangle$   $\text{assms}(1)$   $\text{assms}(2)$   $\text{assms}(3)$ 
          gradexp-stab-aux-0 by blast
    next
      case False
        then obtain  $m$  where  $\text{Suc } m = n$ 
          using not0-implies-Suc by auto
        hence  $C = \text{GradExpn } A \ B \ (\text{Suc } m)$ 
          by (simp add:  $\langle C = \text{GradExpn } A \ B \ n \rangle$ )
        hence  $C' = \text{GradExpn } A \ B \ (\text{Suc}(\text{Suc } m))$ 
          using  $\text{assms}(2)$   $\text{assms}(3)$  gradexp-stab-aux-n by blast
        moreover
          have  $(\text{Suc } (\text{Suc } m)) \neq 0$ 
            by simp
          ultimately
            show ?thesis
              using GradExp-def by blast
        qed
      qed
    qed

lemma gradexp--grad-aux-1:
  shows  $\forall C. (C = (\text{GradExpn } A \ A \ (\text{Suc } n)) \longrightarrow \text{Grad } A \ A \ C)$ 
  by (simp add: grad-equiv-coq-1)

lemma gradexp--grad-aux:
  assumes  $A \neq B$ 
  shows  $\forall C. (C = (\text{GradExpn } A \ B \ (\text{Suc } n)) \longrightarrow \text{Grad } A \ B \ C)$ 
proof (induction  $n$ )
  case  $0$ 
  {
    fix  $C$ 
    assume  $C = (\text{GradExpn } A \ B \ (\text{Suc } 0))$ 
    hence  $C = B$ 
      by force
    hence  $\text{Grad } A \ B \ C$ 
      using grad-equiv-coq-1 by auto
  }
  hence  $(\forall C. (C = (\text{GradExpn } A \ B \ (\text{Suc } 0))) \longrightarrow \text{Grad } A \ B \ C)$ 
    by blast
  thus ?case

```



```

    by blast
next
case (Suc n)
{
  assume 1:  $\forall C. (C = (\text{GradExpn } A \ B \ (\text{Suc } n)) \longrightarrow \text{Grad } A \ B \ C)$ 
  {
    fix C'
    assume C' =  $(\text{GradExpn } A \ B \ ((\text{Suc } n) + 1))$ 
    have  $A \neq B \wedge ((\text{Suc } n) + 1) \neq 0 \wedge ((\text{Suc } n)+1) \neq 1$ 
      using assms by auto
    then
    have C' =  $(\text{SymR } A \ (\text{GradExpn } A \ B \ (\text{Suc } n)))$ 
      by (simp add:  $\langle C' = \text{GradExpn } A \ B \ ((\text{Suc } n)+1) \rangle$ )
    hence  $\text{Grad } A \ B \ (\text{GradExpn } A \ B \ (\text{Suc } n))$ 
      using 1 by blast
    have  $\text{Grad } A \ B \ C'$ 
    proof -
      have  $\text{Grad } A \ B \ (\text{GradExpn } A \ B \ (\text{Suc } n))$ 
        using  $\langle \text{Grad } A \ B \ (\text{GradExpn } A \ B \ (\text{Suc } n)) \rangle$  by blast
      moreover
      have  $\text{Bet } A \ (\text{GradExpn } A \ B \ (\text{Suc } n)) \ (\text{GradExpn } A \ B \ (\text{Suc}(\text{Suc } n)))$ 
        using GradExpn-3 by blast
      moreover
      have  $\text{Cong } A \ (\text{GradExpn } A \ B \ (\text{Suc } n)) \ (\text{GradExpn } A \ B \ (\text{Suc } n)) \ C'$ 
        using GradExpn-4  $\langle C' = \text{GradExpn } A \ B \ (\text{Suc } n + 1) \rangle$  by force
      ultimately
      show ?thesis
        using grad-sum by (metis Suc-eq-plus1  $\langle C' = \text{GradExpn } A \ B \ (\text{Suc } n + 1) \rangle$ )
    qed
  }
  hence  $\forall C. (C = (\text{GradExpn } A \ B \ ((\text{Suc } n) + 1)) \longrightarrow \text{Grad } A \ B \ C)$ 
    by blast
}
thus ?case
  using Suc.IH Suc-eq-plus1 by metis
qed

```

```

lemma gradexp--grad:
  assumes GradExp A B C
  shows Grad A B C
proof -
  obtain n where  $n \neq 0 \wedge C = \text{GradExpn } A \ B \ n$ 
    using GradExp-def assms by blast
  hence  $n \neq 0$ 
    by blast
  then obtain m where  $n = \text{Suc } m$ 
    using not0-implies-Suc by presburger
  thus ?thesis
    by (simp add:  $\langle n \neq 0 \wedge C = \text{GradExpn } A \ B \ n \rangle$  gradexp--grad-aux
      gradexp--grad-aux-1)
qed

```

```

lemma gradexp-le--reach:
  assumes GradExp A B B' and
    C D Le A B'
  shows Reach A B C D
  using Reach-def assms(1) assms(2) gradexp--grad by blast

```

```

lemma grad--ex-gradexp-le-aux-0:
  assumes A = B
  shows  $\exists D. \text{GradExp } A \ B \ D \wedge A \ (\text{Gradn } A \ B \ n) \ \text{Le } A \ D$ 
proof (induction n)
case 0
have  $\text{Gradn } A \ B \ 0 = A$ 
  by auto
moreover

```

```

have GradExp A B B
  by (simp add: gradexp-init)
moreover
have A A Le A B
  by (simp add: le-trivial)
ultimately
show  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B 0) \text{ Le } A D$ 
  by auto
next
case (Suc n)
{
  assume  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D$ 
  then obtain D where GradExp A B D  $\wedge A (\text{Gradn } A B n) \text{ Le } A D$ 
    by blast
  let ?n = Gradn A B n
  let ?sucn = Gradn A B (Suc n)
  have ?n = ?sucn
    using assms by force
  hence  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B (\text{Suc } n)) \text{ Le } A D$ 
    using  $\langle \exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D \rangle$  by presburger
}
thus ?case
  using Suc.IH by blast
qed

```

```

lemma grad--ex-gradexp-le-aux-1:
  assumes  $A \neq B$ 
  shows  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D$ 
proof (induction n)
  case 0
  have Gradn A B 0 = A
    by auto
  moreover
  have GradExp A B B
    by (simp add: gradexp-init)
  moreover
  have A A Le A B
    by (simp add: le-trivial)
  ultimately
  show  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B 0) \text{ Le } A D$ 
    by auto
next
case (Suc n)
{
  assume  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D$ 
  then obtain D where GradExp A B D  $\wedge A (\text{Gradn } A B n) \text{ Le } A D$ 
    by blast
  obtain D' where Bet A D D'  $\wedge \text{Cong } D D' A D$ 
    using segment-construction by blast
  have (Gradn A B n) (Gradn A B (Suc n)) Le D D'
    by (meson Cong-Gradn-Gradn-Suc  $\langle \text{Bet } A D D' \wedge \text{Cong } D D' A D \rangle$ 
       $\langle \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D \rangle$  grad--le gradexp--grad
      l5-6 le-right-comm not-cong-4312)
  hence GradExp A B D'  $\wedge A (\text{Gradn } A B (\text{Suc } n)) \text{ Le } A D'$ 
    by (meson Diff--Bet-Gradn-Gradn-Suc  $\langle \text{Bet } A D D' \wedge \text{Cong } D D' A D \rangle$ 
       $\langle \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D \rangle$  assms bet2-le2--le1346
      gradexp-stab not-cong-3412)
  hence  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B (\text{Suc } n)) \text{ Le } A D$ 
    by blast
}
thus ?case
  using Suc.IH by blast
qed

```

```

lemma grad--ex-gradexp-le-aux:
  shows  $\exists D. \text{GradExp } A B D \wedge A (\text{Gradn } A B n) \text{ Le } A D$ 

```

using *grad--ex-gradexp-le-aux-0 grad--ex-gradexp-le-aux-1* by *blast*

lemma *grad--ex-gradexp-le*:

assumes *Grad A B C*

shows $\exists D. \text{GradExp } A \ B \ D \wedge A \ C \ Le \ A \ D$

using *grad--ex-gradexp-le-aux Grad-def assms* by *auto*

lemma *reach--ex-gradexp-le*:

assumes *Reach A B C D*

shows $\exists B'. \text{GradExp } A \ B \ B' \wedge C \ D \ Le \ A \ B'$

by (*meson Reach-def le-transitivity assms grad--ex-gradexp-le*)

lemma *gradexp2-init*:

shows *GradExp2 A B B C D D*

proof –

have *B = GradExpn A B (Suc 0)*

by *auto*

moreover

have *D = GradExpn C D (Suc 0)*

by *auto*

ultimately

show *?thesis*

using *GradExp2-def* by *blast*

qed

lemma *GradExp2-stab*:

assumes *GradExp2 A B C D E F* and

Bet A C C' and

Cong A C C C' and

Bet D F F' and

Cong D F F F'

shows *GradExp2 A B C' D E F'*

proof –

obtain *n* where $(n \neq 0) \wedge (C = \text{GradExpn } A \ B \ n) \wedge (F = \text{GradExpn } D \ E \ n)$

using *GradExp2-def assms(1)* by *presburger*

hence $n \neq 0$

by *blast*

then obtain *m* where *Suc m = n*

using *not0-implies-Suc* by *auto*

hence *C = GradExpn A B (Suc m)*

using $\langle n \neq 0 \wedge C = \text{GradExpn } A \ B \ n \wedge F = \text{GradExpn } D \ E \ n \rangle$ by *blast*

hence *C' = GradExpn A B (Suc (Suc m))*

using *assms(2) assms(3) gradexp-stab-aux-n* by *blast*

moreover

have *F = GradExpn D E (Suc m)*

using $\langle \text{Suc } m = n \rangle \langle n \neq 0 \wedge C = \text{GradExpn } A \ B \ n \wedge F = \text{GradExpn } D \ E \ n \rangle$ by *blast*

hence *F' = GradExpn D E (Suc (Suc m))*

using *assms(4) assms(5) gradexp-stab-aux-n* by *blast*

moreover

have *Suc (Suc m) $\neq 0$*

by *simp*

ultimately

show *?thesis*

using *GradExp2-def* by *blast*

qed

lemma *gradexp2--gradexp123*:

assumes *GradExp2 A B C D E F*

shows *GradExp A B C*

proof –

obtain *n* where $n \neq 0 \wedge (C = \text{GradExpn } A \ B \ n) \wedge (F = \text{GradExpn } D \ E \ n)$

using *assms(1) GradExp2-def* by *blast*

thus *?thesis* using *GradExp-def* by *blast*

qed

lemma *gradexp2--gradexp456*:

```

assumes GradExp2 A B C D E F
shows GradExp D E F
proof –
  obtain n where  $n \neq 0 \wedge (C = \text{GradExpn } A \ B \ n) \wedge (F = \text{GradExpn } D \ E \ n)$ 
    using assms(1) GradExp2-def by blast
  thus ?thesis using GradExp-def by blast
qed

```

```

lemma MidR-uniq:
  assumes  $C = \text{MidR } A \ B$  and
     $D = \text{MidR } A \ B$ 
  shows  $C = D$ 
  by (simp add: assms(1) assms(2))

```

```

lemma MidR-ex-0:
  shows  $(\text{MidR } A \ B) \ \text{Midpoint } A \ B$ 
proof –
  have  $\text{MidR } A \ B = (\text{SOME } x. x \ \text{Midpoint } A \ B)$ 
    by simp
  thus ?thesis
    using someI-ex by (metis midpoint-existence)
qed

```

```

lemma MidR-ex-1:
  assumes  $C = (\text{MidR } A \ B)$ 
  shows  $C \ \text{Midpoint } A \ B$ 
  using assms MidR-ex-0 by blast

```

```

lemma MidR-ex-aux:
  assumes  $C \ \text{Midpoint } A \ B$ 
  shows  $C = (\text{SOME } x. x \ \text{Midpoint } A \ B)$ 
  by (metis assms MidR-uniq-aux someI)

```

```

lemma MidR-ex:
  assumes  $C \ \text{Midpoint } A \ B$ 
  shows  $C = (\text{MidR } A \ B)$ 
  using MidR-ex-0 MidR-uniq-aux assms by blast

```

```

lemma gradexpinv-init-aux:
  shows  $B = \text{GradExpInvn } A \ B \ 0$ 
  by simp

```

```

lemma gradexpinv-init:
  shows  $\text{GradExpInv } A \ B \ B$ 
  using gradexpinv-init-aux using GradExpInv-def by blast

```

```

lemma gradexpinv-stab-aux-0:
  assumes  $B = \text{GradExpInvn } A \ C \ 0$  and
     $\text{Bet } A \ B' \ B$  and
     $\text{Cong } A \ B' \ B' \ B$ 
  shows  $B' = \text{GradExpInvn } A \ C \ (\text{Suc } 0)$ 
proof –
  have  $B = C$ 
    using assms(1) by simp
  have  $B' \ \text{Midpoint } A \ B$ 
    by (simp add: assms(2) assms(3) midpoint-def)
  hence  $B' = \text{MidR } A \ B$ 
    using MidR-ex by blast
  show ?thesis
proof (cases  $A = C$ )
  case True
    hence  $B' = B$ 
      using  $\langle B = C \rangle$  assms(2) bet-neq12--neq by blast
    moreover
      have  $A = \text{GradExpInvn } A \ C \ (\text{Suc } 0)$ 
        using True by simp

```

```

ultimately
show ?thesis
  using True  $\langle B = C \rangle$  by force
next
case False
hence  $\text{MidR } A \ C = \text{GradExpInvn } A \ C \ (\text{Suc } 0)$ 
  by simp
thus ?thesis
  using  $\langle B = C \rangle \ \langle B' = \text{MidR } A \ B \rangle$  by blast
qed
qed

```

```

lemma gradexpinv-stab-aux-n:
  assumes Bet A B' B and
    Cong A B' B' B and
    B = GradExpInvn A C (Suc n)
  shows B' = GradExpInvn A C (Suc (Suc n))
proof -
  have Suc (Suc n)  $\neq 0$  by simp
  moreover
  have Suc (Suc n)  $\neq 1$  by simp
  moreover
  have B' Midpoint A B
    using Midpoint-def assms(1) assms(2) by blast
  hence B' = MidR A (GradExpInvn A C (Suc n))
    using MidR-ex assms(3) by blast
  ultimately
  show ?thesis
    by (metis GradExpInvn.simps One-nat-def assms(1) assms(3)
      diff-Suc-Suc l8-20-1-R1 l8-6 minus-nat.diff-0)
qed

```

```

lemma gradexpinv-stab:
  assumes Bet A B' B and
    Cong A B' B' B and
    GradExpInv A B C
  shows GradExpInv A B' C
proof -
  have GradExpInv A B C
    using GradExpInv-def assms(3) by auto
  then obtain n where B = GradExpInvn A C n
    using GradExpInv-def by blast
  have B' = GradExpInvn A C (Suc n)
  proof (cases n = 0)
    case True
    show ?thesis
      using True  $\langle B = \text{GradExpInvn } A \ C \ n \rangle$  assms(1) assms(2)
        gradexpinv-stab-aux-0 by blast
  next
    case False
    then obtain k where Suc k = n
      using not0-implies-Suc by auto
    hence B = GradExpInvn A C (Suc k)
      using  $\langle B = \text{GradExpInvn } A \ C \ n \rangle$  by blast
    hence B' = GradExpInvn A C (Suc (Suc k))
      using gradexpinv-stab-aux-n assms(1) assms(2) by blast
    show ?thesis
      using  $\langle B' = \text{GradExpInvn } A \ C \ (\text{Suc } (\text{Suc } k)) \rangle \ \langle \text{Suc } k = n \rangle$  by blast
  qed
  moreover
  have Suc n  $\neq 0$ 
    by simp
  ultimately
  show ?thesis
    using GradExpInv-def by blast
qed

```

```

lemma gradexp--gradexpinv-aux-1-0:
  assumes  $C = \text{GradExpn } A \ B \ (\text{Suc } 0)$ 
  shows  $B = \text{GradExpInvn } A \ C \ 0$ 
proof (cases  $A = B$ )
  case True
  hence  $C = A$ 
    by (simp add: assms)
  thus ?thesis
    by (simp add: True)
next
  case False
  hence  $C = B$ 
    using assms by simp
  thus ?thesis
    using False by simp
qed

lemma SymR-MidR:
  assumes  $A = \text{SymR } B \ C$ 
  shows  $C = \text{MidR } A \ B$ 
proof -
  have  $C \text{ Midpoint } B \ A$ 
    using assms SymR-ex symmetric-point-construction by blast
  thus ?thesis
    using MidR-ex-0 l7-17-bis by auto
qed

lemma MidR-comm:
  assumes  $C = \text{MidR } A \ B$ 
  shows  $C = \text{MidR } B \ A$ 
proof -
  have  $C \text{ Midpoint } B \ A$ 
    using MidR-ex-0 assms l7-2 by blast
  thus ?thesis
    by (meson MidR-ex  $\langle C \text{ Midpoint } B \ A \rangle$ )
qed

lemma MidR-SymR:
  assumes  $C = \text{MidR } A \ B$ 
  shows  $A = \text{SymR } B \ C$ 
proof -
  have  $C \text{ Midpoint } B \ A$ 
    using MidR-ex-0 assms l7-2 by blast
  thus ?thesis
    using SymR-ex by auto
qed

lemma MidR-AA:
  shows  $A = \text{MidR } A \ A$ 
  using l7-3-2 MidR-ex by blast

lemma MidR-AB:
  assumes  $A = \text{MidR } A \ B$ 
  shows  $A = B$ 
  by (metis MidR-AA MidR-SymR MidR-comm assms)

lemma gradexp--gradexpinv-aux-1-n-aa:
  shows  $\text{GradExpInvn } A' \ (\text{MidR } A' \ C') \ n = \text{GradExpInvn } A' \ C' \ (\text{Suc } n)$ 
proof (cases  $A' = C'$ )
  case True
  hence  $A' = \text{MidR } A' \ C'$ 
    using l7-3-2 MidR-ex by blast
  hence  $\text{GradExpInvn } A' \ (\text{MidR } A' \ C') \ n = A'$ 
    by simp
  moreover

```

```

have GradExpInvn A' C' (Suc n) = A'
  using True by simp
ultimately
show ?thesis
  by simp
next
case False
have GradExpInvn A' (MidR A' C') n = GradExpInvn A' C' (Suc n)
proof (induction n)
  case 0
  hence A' ≠ MidR A' C'
    using False MidR-AB by blast
  hence GradExpInvn A' (MidR A' C') 0 = (MidR A' C')
    by simp
  moreover
  have GradExpInvn A' C' 0 = C'
    using gradexpinv-init-aux by presburger
  hence GradExpInvn A' C' (Suc 0) = (MidR A' C')
    using False by simp
  ultimately
  have GradExpInvn A' (MidR A' C') 0 = GradExpInvn A' C' (Suc 0)
    by simp
  thus ?case
    by blast
next
case (Suc n)
{
  assume GradExpInvn A' (MidR A' C') n = GradExpInvn A' C' (Suc n)
  have GradExpInvn A' (MidR A' C') (Suc n)
    = (MidR A' (GradExpInvn A' (MidR A' C') ((Suc n) - 1)))
    by (metis GradExpInvn.simps MidR-AB Suc-neg-Zero diff-self-eq-0)
  have ... = MidR A' (GradExpInvn A' (MidR A' C') n)
    by simp
  have ... = MidR A' (GradExpInvn A' C' (Suc n))
    using Suc.IH by presburger
  have ... = MidR A' (GradExpInvn A' C' ((Suc (Suc n)) - 1))
    using diff-Suc-1 by presburger
  have ... = GradExpInvn A' C' (Suc (Suc n))
    using False by auto
  hence GradExpInvn A' (MidR A' C') (Suc n) = GradExpInvn A' C' (Suc (Suc n))
    using ⟨GradExpInvn A' (MidR A' C') (Suc n)
    = MidR A' (GradExpInvn A' (MidR A' C') (Suc n - 1))⟩
    ⟨MidR A' (GradExpInvn A' (MidR A' C') (Suc n - 1))
    = MidR A' (GradExpInvn A' (MidR A' C') n)⟩
    ⟨MidR A' (GradExpInvn A' (MidR A' C') n)
    = MidR A' (GradExpInvn A' C' (Suc n))⟩
    ⟨MidR A' (GradExpInvn A' C' (Suc n))
    = MidR A' (GradExpInvn A' C' (Suc (Suc n) - 1))⟩
    by presburger
}
thus ?case
  using Suc.IH by blast
qed
thus ?thesis by blast
qed

lemma gradexp--gradexpinv-aux-1-n-a:
  assumes ∀ A B C. (C = GradExpn A B (Suc n)) ⟶ B = GradExpInvn A C n
  shows ∀ A' B' C'. (C' = GradExpn A' B' (Suc (Suc n))) ⟶ B' = GradExpInvn A' C' (Suc n)
proof -
{
  fix A' B' C'
  assume C' = GradExpn A' B' (Suc (Suc n))
  have B' = GradExpInvn A' C' (Suc n)
  proof (cases A' = B')
    case True

```

```

{
  hence  $C' = A'$ 
    using True by (simp add:  $\langle C' = \text{GradExpn } A' B' (\text{Suc } (\text{Suc } n)) \rangle$ )
  hence  $A' = \text{GradExpInvn } A' C' (\text{Suc } n)$ 
    by simp
  hence  $B' = \text{GradExpInvn } A' C' (\text{Suc } n)$ 
    using True by simp
}
thus ?thesis
  using  $\langle C' = \text{GradExpn } A' B' (\text{Suc } (\text{Suc } n)) \rangle$  by blast
next
case False
{
  have  $\text{Suc } (\text{Suc } n) \neq 0$ 
    by simp
  moreover
  have  $\text{Suc } (\text{Suc } n) \neq 1$ 
    by simp
  moreover
  have  $C' = \text{SymR } A' (\text{GradExpn } A' B' ((\text{Suc } (\text{Suc } n)) - 1))$ 
    by (simp add: False  $\langle C' = \text{GradExpn } A' B' (\text{Suc } (\text{Suc } n)) \rangle$ )
  have  $(\text{Suc } (\text{Suc } n)) - 1 = \text{Suc } n$ 
    by simp
  let ?ssn1 =  $(\text{Suc } (\text{Suc } n)) - 1$ 
  let ?B1 =  $\text{GradExpn } A' B' ?ssn1$ 
  have  $?B1 = (\text{GradExpn } A' B' (\text{Suc } n))$ 
    by simp
  hence  $B' = (\text{GradExpInvn } A' ?B1 n)$ 
    using assms by blast
  have  $C' = \text{SymR } A' ?B1$ 
    using  $\langle C' = \text{SymR } A' (\text{GradExpn } A' B' (\text{Suc } (\text{Suc } n) - 1)) \rangle$  by blast
  {
    assume  $n = 0$ 
    hence  $B' = ?B1$ 
      by force
  }
  {
    assume  $n = 1$ 
    hence  $B' = \text{MidR } A' ?B1$ 
      by (metis False GradExpInvn.simps One-nat-def
         $\langle B' = \text{GradExpInvn } A' (\text{GradExpn } A' B' (\text{Suc } (\text{Suc } n) - 1)) n \rangle$ 
         $\langle \text{GradExpn } A' B' (\text{Suc } (\text{Suc } n) - 1) = \text{GradExpn } A' B' (\text{Suc } n) \rangle$ 
        n-not-Suc-n)
  }
  {
    assume  $n \neq 0 \wedge n \neq 1$ 
    hence  $B' = \text{MidR } A' (\text{GradExpInvn } A' ?B1 (n-1))$ 
      by (metis False GradExpInvn.simps
         $\langle B' = \text{GradExpInvn } A' (\text{GradExpn } A' B' (\text{Suc } (\text{Suc } n) - 1)) n \rangle$ )
  }
  {
    assume  $\text{Suc } n = 1$ 
    hence  $n = 0$ 
      by simp
    have  $C' = \text{SymR } A' ?B1$ 
      using  $\langle C' = \text{SymR } A' (\text{GradExpn } A' B' (\text{Suc } (\text{Suc } n) - 1)) \rangle$  by blast
    hence  $?B1 = \text{MidR } C' A'$ 
      using SymR-MidR by blast
    hence  $?B1 = \text{MidR } A' C'$ 
      using MidR-comm by blast
    hence  $B' = \text{MidR } A' C'$ 
      using  $\langle n = 0 \implies B' = \text{GradExpn } A' B' (\text{Suc } (\text{Suc } n) - 1) \rangle$   $\langle n = 0 \rangle$ 
      by force
  }
  moreover
  {

```



```

assume  $Suc\ n \neq 1$ 
hence  $B' = MidR\ A' (GradExpInvn\ A' ?B1\ (n-1))$ 
  by (metis One-nat-def
     $\langle n = 1 \implies B' = MidR\ A' (GradExpn\ A' B' (Suc\ (Suc\ n) - 1)) \rangle$ 
     $\langle n \neq 0 \wedge n \neq 1 \implies B'$ 
     $= MidR\ A' (GradExpInvn\ A' (GradExpn\ A' B' (Suc\ (Suc\ n) - 1))\ (n - 1)) \rangle$ 
    diff-Suc-1 gradexpinv-init-aux)
obtain  $k$  where  $Suc\ k = n$ 
  using  $\langle Suc\ n \neq 1 \rangle$  not0-implies-Suc by auto
have  $C' = SymR\ A' ?B1$ 
  using  $\langle C' = SymR\ A' (GradExpn\ A' B' (Suc\ (Suc\ n) - 1)) \rangle$  by blast
hence  $?B1 = MidR\ A' C'$ 
  using MidR-comm SymR-MidR by presburger
hence  $GradExpInvn\ A' ?B1\ k = (GradExpInvn\ A' C' (Suc\ k))$ 
  using gradexp--gradexpinv-aux-1-n-aa by presburger
hence  $GradExpInvn\ A' ?B1\ (n-1) = (GradExpInvn\ A' C' n)$ 
  using  $\langle Suc\ k = n \rangle$  by force
hence  $B' = MidR\ A' (GradExpInvn\ A' C' n)$ 
  using  $\langle B' = MidR\ A' (GradExpInvn\ A' (GradExpn\ A' B' (Suc\ (Suc\ n) - 1))\ (n - 1)) \rangle$ 
  by presburger
hence  $B' = MidR\ A' (GradExpInvn\ A' C' ((Suc\ n)-1))$ 
  using diff-Suc-1 by presburger
}
ultimately
have  $B' = GradExpInvn\ A' C' (Suc\ n)$ 
  by (metis GradExpInvn.elims GradExpn-2 Suc-neq-Zero
     $\langle C' = GradExpn\ A' B' (Suc\ (Suc\ n)) \rangle$  bet-neq12--neq)
}
hence  $(C' = GradExpn\ A' B' (Suc(Suc\ n))) \longrightarrow B' = GradExpInvn\ A' C' (Suc\ n)$  by blast
thus ?thesis
  using  $\langle C' = GradExpn\ A' B' (Suc\ (Suc\ n)) \rangle$  by blast
qed
}
thus ?thesis by blast
qed

```

lemma *gradexp--gradexpinv-aux-1-b:*

shows $\forall\ A\ B\ C. C = GradExpn\ A\ B\ (Suc\ n) \longrightarrow B = GradExpInvn\ A\ C\ n$

proof (*induction n*)

case 0

have $\forall\ A\ B\ C. C = GradExpn\ A\ B\ (Suc\ 0) \longrightarrow B = GradExpInvn\ A\ C\ 0$

by (*meson gradexp--gradexpinv-aux-1-0*)

thus *?case* **by** *blast*

next

case ($Suc\ n$)

{

assume $\forall\ A\ B\ C. C = GradExpn\ A\ B\ (Suc\ n) \longrightarrow B = GradExpInvn\ A\ C\ n$

{

fix $A\ B\ C$

assume $C = GradExpn\ A\ B\ (Suc(Suc\ n))$

have $B = GradExpInvn\ A\ C\ (Suc\ n)$

using $\langle C = GradExpn\ A\ B\ (Suc\ (Suc\ n)) \rangle$

$\langle \forall\ A\ B\ C. C = GradExpn\ A\ B\ (Suc\ n) \longrightarrow B = GradExpInvn\ A\ C\ n \rangle$

gradexp--gradexpinv-aux-1-n-a **by** *presburger*

}

hence $\forall\ A\ B\ C. C = GradExpn\ A\ B\ (Suc(Suc\ n)) \longrightarrow B = GradExpInvn\ A\ C\ (Suc\ n)$

by *blast*

}

thus *?case*

using *Suc.IH* **by** *fastforce*

qed

lemma *gradexp--gradexpinv-aux-1:*

assumes $C = GradExpn\ A\ B\ (Suc\ n)$

shows $B = GradExpInvn\ A\ C\ n$

using *assms gradexp--gradexpinv-aux-1-b* **by** *blast*

```

lemma gradexp--gradexpinv-aux:
  assumes GradExp A B C
  shows GradExpInv A B C
proof -
  obtain n where n ≠ 0 ∧ C = GradExpn A B n
    using assms GradExp-def by blast
  hence n ≠ 0
    by blast
  then obtain m where n = Suc m
    using not0-implies-Suc by presburger
  hence B = GradExpInvn A C m
    using gradexp--gradexpinv-aux-1
    using ⟨n ≠ 0 ∧ C = GradExpn A B n⟩ by blast
  thus ?thesis
    using GradExpInv-def ⟨n ≠ 0 ∧ C = GradExpn A B n⟩ by blast
qed

lemma gradexpinv--gradexp-aux-1-a-0:
  assumes B' = MidR A' (GradExpInvn A' C' (Suc 0))
  shows C' = SymR A' (GradExpn A' B' (Suc(Suc 0)))
proof (cases A' = C')
  case True
  hence B' = MidR A' C'
    by (simp add: assms)
  thus ?thesis
    by (metis GradExpn-1 MidR-AA MidR-SymR True)
next
  case False
  thus ?thesis
    by (metis GradExpn-2 GradExpn-4 Mid-cases SymR-MidR
      assms gradexp--gradexpinv-aux-1-0 gradexp--gradexpinv-aux-1-n-aa
      gradexpinv-init-aux gradexpinv-stab-aux-0 l7-9
      midpoint-def midpoint-existence)
qed

lemma sym-mid:
  shows SymR A (MidR A B) = B
  by (metis MidR-SymR MidR-comm)

lemma gradexpn-suc-suc:
  shows GradExpn A B (Suc n) = GradExpn A (MidR A B) (Suc(Suc n))
proof (cases A = B)
  case True
  thus ?thesis
    using GradExpn-1 MidR-AA by presburger
next
  case False
  hence A ≠ B
    by simp
  have GradExpn A B (Suc n) = GradExpn A (MidR A B) (Suc(Suc n))
  proof (induction n)
    case 0
    have GradExpn A (MidR A B) (Suc(Suc 0))
    = SymR A (GradExpn A (MidR A B) ((Suc(Suc 0)) - 1))
      using False MidR-AB
      by (metis diff-Suc-1 gradexp--gradexpinv-aux-1-b
        gradexp--gradexpinv-aux-1-n-aa gradexpinv-init-aux sym-mid)
    hence ... = SymR A (GradExpn A (MidR A B) (Suc 0))
      using diff-Suc-1 by presburger
    hence ... = SymR A (MidR A B)
      by simp
    hence ... = B
      using sym-mid by blast
    hence GradExpn A B (Suc 0) = GradExpn A (MidR A B) (Suc (Suc 0))
      by (metis gradexp--gradexpinv-aux-1 gradexp--gradexpinv-aux-1-n-aa

```

```

      gradexpinv--gradexp-aux-1-a-0 gradexpinv-init-aux)
    thus ?case
      by blast
  next
    case (Suc n)
    {
      assume  $\forall A' B'. \text{GradExpn } A' B' (\text{Suc } n) = \text{GradExpn } A' (\text{MidR } A' B') (\text{Suc}(\text{Suc } n))$ 
      {
        fix A B
        have  $\text{GradExpn } A (\text{MidR } A B) (\text{Suc}(\text{Suc}(\text{Suc } n)))$ 
          =  $\text{SymR } A (\text{GradExpn } A (\text{MidR } A B) (\text{Suc}(\text{Suc}(\text{Suc } n)) - 1))$ 
          using MidR-AA MidR-SymR by fastforce
        hence ... =  $\text{GradExpn } A B (\text{Suc}(\text{Suc } n))$ 
        by (metis GradExpn-3 GradExpn-4
           $\langle \forall A' B'. \text{GradExpn } A' B' (\text{Suc } n) = \text{GradExpn } A' (\text{MidR } A' B') (\text{Suc } (\text{Suc } n)) \rangle$ 
          gradexp-stab-aux-n)
        hence  $\text{GradExpn } A B (\text{Suc}(\text{Suc } n)) = \text{GradExpn } A (\text{MidR } A B) (\text{Suc}(\text{Suc}(\text{Suc } n)))$ 
        using  $\langle \text{GradExpn } A (\text{MidR } A B) (\text{Suc } (\text{Suc } (\text{Suc } n)))$ 
          =  $\text{SymR } A (\text{GradExpn } A (\text{MidR } A B) (\text{Suc } (\text{Suc } (\text{Suc } n)) - 1)) \rangle$ 
          by presburger
      }
      hence  $\forall A B. \text{GradExpn } A B (\text{Suc}(\text{Suc } n)) = \text{GradExpn } A (\text{MidR } A B) (\text{Suc}(\text{Suc}(\text{Suc } n)))$ 
      by blast
    }
  thus ?case
    using GradExpn-3 GradExpn-4 Suc.IH gradexp-stab-aux-n by blast
qed
thus ?thesis by blast
qed

```

lemma gradexpinv--gradexp-aux-1-a-n:

```

  assumes  $\forall A B C. (B = \text{MidR } A (\text{GradExpInvn } A C (\text{Suc } n))) \longrightarrow$ 
     $C = \text{SymR } A (\text{GradExpn } A B (\text{Suc}(\text{Suc } n)))$ 
  shows  $B' = \text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc}(\text{Suc } n))) \longrightarrow$ 
     $C' = \text{SymR } A' (\text{GradExpn } A' B' (\text{Suc}(\text{Suc}(\text{Suc } n))))$ 
  proof (cases  $A' = C'$ )
    case True
    thus ?thesis
      by (metis GradExpn-1 MidR-AA MidR-SymR gradexp--gradexpinv-aux-1)
  next
    case False
    {
      assume  $B' = \text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc}(\text{Suc } n)))$ 
      hence ... =  $\text{MidR } A' (\text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc}(\text{Suc } n) - 1)))$ 
      using False by force
      hence ... =  $\text{MidR } A' (\text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc } n)))$ 
      by simp
      let ?B1 =  $(\text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc } n)))$ 
      have  $C' = \text{SymR } A' (\text{GradExpn } A' ?B1 (\text{Suc}(\text{Suc } n)))$ 
      using assms by blast
      have  $\text{GradExpn } A' ?B1 (\text{Suc}(\text{Suc } n)) = \text{GradExpn } A' (\text{MidR } A' ?B1) (\text{Suc}(\text{Suc}(\text{Suc } n)))$ 
      using gradexpn-suc-suc by blast
      hence  $C' = \text{SymR } A' (\text{GradExpn } A' B' (\text{Suc}(\text{Suc}(\text{Suc } n))))$ 
      by (metis MidR-comm SymR-MidR  $\langle B' = \text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc } (\text{Suc } n))) \rangle$ 
         $\langle C' = \text{SymR } A' (\text{GradExpn } A' (\text{MidR } A' (\text{GradExpInvn } A' C' (\text{Suc } n))) (\text{Suc } (\text{Suc } n))) \rangle$ 
        gradexp--gradexpinv-aux-1 gradexp--gradexpinv-aux-1-n-aa)
    }
  thus ?thesis
    by blast
qed

```

lemma gradexpinv--gradexp-aux-1-a:

```

  shows  $\forall A B C. B = \text{MidR } A (\text{GradExpInvn } A C (\text{Suc } n)) \longrightarrow$ 
     $C = \text{SymR } A (\text{GradExpn } A B (\text{Suc}(\text{Suc } n)))$ 
  proof (induction n)
    case 0

```

```

thus ?case
  using gradexpinv--gradexp-aux-1-a-0 by blast
next
  case (Suc n)
  thus ?case
    using gradexpinv--gradexp-aux-1-a-n by blast
qed

lemma gradexpinv--gradexp-aux-1-n:
  assumes  $B = \text{GradExpInvn } A \ C \ n \longrightarrow C = \text{GradExpn } A \ B \ (\text{Suc } n)$ 
  shows  $B' = \text{GradExpInvn } A' \ C' \ (\text{Suc } n) \longrightarrow C' = \text{GradExpn } A' \ B' \ (\text{Suc}(\text{Suc } n))$ 
proof (cases  $A' = C'$ )
  case True
  thus ?thesis
    by force
next
  case False
  hence  $A' \neq C'$ 
    by blast
  {
    assume  $B' = \text{GradExpInvn } A' \ C' \ (\text{Suc } n)$ 
    have  $C' = \text{GradExpn } A' \ B' \ (\text{Suc}(\text{Suc } n))$ 
    proof (cases  $\text{Suc } n = 1$ )
    case True
    hence  $B' = \text{GradExpInvn } A' \ C' \ 1$ 
      using  $\langle B' = \text{GradExpInvn } A' \ C' \ (\text{Suc } n) \rangle$  by simp
    hence  $B' = \text{MidR } A' \ C'$ 
      using  $\langle A' \neq C' \rangle$  by simp
    hence  $C' = \text{SymR } A' \ B'$ 
      by (metis MidR-SymR MidR-comm)
    have  $A' \neq B'$ 
      using False MidR-AB  $\langle B' = \text{MidR } A' \ C' \rangle$  by blast
    have  $\text{Suc}(\text{Suc } n) \neq 0 \wedge \text{Suc } (\text{Suc } n) \neq 1$ 
      by simp
    hence  $\text{GradExpn } A' \ B' \ (\text{Suc}(\text{Suc } n)) = \text{SymR } A' \ (\text{GradExpn } A' \ B' \ (\text{Suc}(\text{Suc } n) - 1))$ 
      by (simp add:  $\langle A' \neq B' \rangle$ )
    hence  $\dots = \text{SymR } A' \ B'$ 
      by (simp add: True)
    thus ?thesis
      using  $\langle C' = \text{SymR } A' \ B' \rangle$ 
       $\langle \text{GradExpn } A' \ B' \ (\text{Suc } (\text{Suc } n)) = \text{SymR } A' \ (\text{GradExpn } A' \ B' \ (\text{Suc } (\text{Suc } n) - 1)) \rangle$ 
      by presburger
  }
next
  case False
  hence  $B' = (\text{MidR } A' \ (\text{GradExpInvn } A' \ C' \ ((\text{Suc } n) - 1)))$ 
    using  $\langle B' = \text{GradExpInvn } A' \ C' \ (\text{Suc } n) \rangle$ 
    by (simp add:  $\langle A' \neq C' \rangle$ )
  hence  $C' = \text{SymR } A' \ (\text{GradExpn } A' \ B' \ ((\text{Suc}(\text{Suc } n) - 1)))$ 
    using gradexpinv--gradexp-aux-1-a
    by (metis GradExpn.simps One-nat-def  $\langle A' \neq C' \rangle$ 
       $\langle B' = \text{GradExpInvn } A' \ C' \ (\text{Suc } n) \rangle$  gradexp--gradexpinv-aux-1
      nat.distinct(1) nat.inject)
  hence  $\dots = \text{GradExpn } A' \ B' \ (\text{Suc}(\text{Suc } n))$ 
    by (metis GradExpn.simps GradExpn-1 MidR-AB MidR-SymR
      One-nat-def  $\langle B' = \text{MidR } A' \ (\text{GradExpInvn } A' \ C' \ (\text{Suc } n - 1)) \rangle$ 
      nat.distinct(1) nat.inject)
  thus ?thesis
    using  $\langle C' = \text{SymR } A' \ (\text{GradExpn } A' \ B' \ (\text{Suc } (\text{Suc } n) - 1)) \rangle$  by blast
qed
}
thus ?thesis
  by blast
qed

```

```

lemma gradexpinv--gradexp-aux-1:
  shows  $B = \text{GradExpInvn } A \ C \ n \longrightarrow C = \text{GradExpn } A \ B \ (\text{Suc } n)$ 

```

```

proof (induction n)
  case 0
  thus ?case by force
next
  case (Suc n)
  thus ?case
    using gradexpinv--gradexp-aux-1-n by blast
qed

lemma gradexpinv--gradexp-aux:
  assumes GradExpInv A B C
  shows GradExp A B C
proof –
  obtain n where B = GradExpInvn A C n
    using GradExpInv-def assms by blast
  hence C = GradExpn A B (Suc n)
    using gradexpinv--gradexp-aux-1 by blast
  thus ?thesis
    using GradExp-def by blast
qed

lemma gradexp--gradexpinv:
  shows GradExp A B C  $\longleftrightarrow$  GradExpInv A B C
  using gradexp--gradexpinv-aux gradexpinv--gradexp-aux by blast

lemma reach--ex-gradexp-lt-aux:
  shows  $\forall A B C P Q R. ((A \neq B \wedge A B \text{ Le } P R \wedge R = \text{GradExpn } P Q (\text{Suc } n)) \longrightarrow$ 
     $(\exists C. \text{GradExp } A C B \wedge A C \text{ Lt } P Q))$ 
proof (induction n)
  case 0
  {
    fix A B C P Q R
    assume 1: A  $\neq$  B and 2: A B Le P R and 3: R = GradExpn P Q (Suc 0)
    obtain C where Bet A C B  $\wedge$  Cong A C C B
      by (meson midpoint-bet midpoint-cong midpoint-existence)
    have P  $\neq$  Q
      using 1 2 3 le-zero by force
    have R = Q
      by (simp add: 3)
    have GradExpInv A C B
    proof –
      have GradExpInv A B B
        using gradexpinv-init by blast
      thus ?thesis
        using  $\langle \text{Bet } A C B \wedge \text{Cong } A C C B \rangle$  gradexpinv-stab by blast
    qed
    hence GradExp A C B
      by (simp add: gradexpinv--gradexp-aux)
    moreover
    have A C Lt P Q
    proof –
      have A C Lt A B
        by (simp add: 1  $\langle \text{Bet } A C B \wedge \text{Cong } A C C B \rangle$  mid--lt midpoint-def)
      moreover
      have A B Le P Q
        using 2  $\langle R = Q \rangle$  by auto
      ultimately
      show ?thesis
        using le3456-lt--lt by blast
    qed
    ultimately
    have  $\exists C. \text{GradExp } A C B \wedge A C \text{ Lt } P Q$ 
      by blast
  }
thus ?case by blast

```

```

next
case (Suc n)
{
  assume H1:  $\forall A' B' C' P' Q' R'. (A' \neq B' \wedge A' B' \text{ Le } P' R' \wedge R' = \text{GradExpn } P' Q' (\text{Suc } n) \longrightarrow (\exists C'. \text{GradExp } A' C' B' \wedge A' C' \text{ Lt } P' Q'))$ 
  {
    fix A B C P Q R
    assume 1:  $A \neq B$  and
      2:  $A B \text{ Le } P R$  and
      3:  $R = \text{GradExpn } P Q (\text{Suc } (\text{Suc } n))$ 
    obtain M where M Midpoint A B
    using midpoint-existence by blast
    have  $P \neq R$ 
    using 1 2 le-diff by blast
    have  $M \neq A$ 
    using 1  $\langle M \text{ Midpoint } A B \rangle$  bet-cong-eq between-trivial2
      midpoint-cong by blast
    have  $M \neq B$ 
    using 1  $\langle M \text{ Midpoint } A B \rangle$  cong-identity midpoint-cong by blast
    have  $A M \text{ Le } P R$ 
    using 1 2  $\langle M \text{ Midpoint } A B \rangle$  le3456-lt--lt lt--le mid--lt by blast
    have  $R = \text{SymR } P (\text{GradExpn } P Q ((\text{Suc } (\text{Suc } n)) - 1))$ 
    using 3  $\langle P \neq R \rangle$  by auto
    hence ... =  $\text{SymR } P (\text{GradExpn } P Q (\text{Suc } n))$ 
    using diff-Suc-1 by presburger
    let ?R' =  $\text{GradExpn } P Q (\text{Suc } n)$ 
    have  $\exists C. (\text{GradExp } A C M \wedge A C \text{ Lt } P Q)$ 
    proof -
      have  $A M \text{ Le } P ?R'$ 
      using 2 3 GradExpn-3 GradExpn-4  $\langle M \text{ Midpoint } A B \rangle$ 
        le-mid2--le12 midpoint-def by blast
      moreover
      have  $M \neq A$ 
      using  $\langle M \neq A \rangle$  by force
      moreover
      have  $?R' = \text{GradExpn } P Q (\text{Suc } n)$ 
      by simp
      ultimately
      show ?thesis
      using H1 by force
    qed
    then obtain C where  $\text{GradExp } A C M \wedge A C \text{ Lt } P Q$ 
    by auto
    hence  $\exists C. \text{GradExp } A C B \wedge A C \text{ Lt } P Q$ 
    using  $\langle M \text{ Midpoint } A B \rangle$  gradexp-stab midpoint-bet
      midpoint-cong by blast
  }
  hence  $\forall A B C P Q R. A \neq B \wedge A B \text{ Le } P R \wedge R = \text{GradExpn } P Q (\text{Suc } (\text{Suc } n)) \longrightarrow (\exists C. \text{GradExp } A C B \wedge A C \text{ Lt } P Q)$ 
  by blast
}
thus ?case
using Suc.IH by presburger
qed

lemma reach--grad-min-1:
  assumes  $A \neq B$  and
    Bet A B C and
    A C Le A (Gradn A B (Suc 0))
  shows  $\exists D E. (\text{Bet } A D C \wedge \text{Grad } A B D \wedge E \neq C \wedge \text{Bet } A C E \wedge \text{Bet } A D E \wedge \text{Cong } A B D E)$ 
  proof (cases C = (Gradn A B (Suc 0)))
    case True
    let ?f = Gradn A B (Suc (Suc 0))
    have Grad A B B

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    by (simp add: grad-equiv-coq-1)
  moreover
  have ?f ≠ C
    by (metis Gradn-uniq-aux-1 True assms(1))
  moreover
  have Bet A C ?f
    using Bet-Gradn-Gradn-Suc True by blast
  moreover
  have Bet A B ?f
    by (meson Diff--Bet-Gradn-Suc assms(1))
  moreover
  have Cong A B B ?f
    by (metis Lem-Gradn-0 Cong-Gradn-Gradn-Suc True assms(2) between-cong)
  ultimately
  show ?thesis
    using assms(2) by blast
next
case False
let ?e = Gradn A B (Suc 0)
have Grad A B B
  by (simp add: grad-equiv-coq-1)
moreover
have ?e ≠ C
  using False by auto
moreover
have Bet A C ?e
  by (meson Gradn-aux-1-0 Diff--Bet-Gradn-Suc assms(1)
    assms(2) assms(3) bet-out l5-1 l6-13-1 l6-6)
moreover
have Bet A B ?e
  using Diff--Bet-Gradn-Suc not-bet-distincts by blast
moreover
have Cong A B B ?e
  by (metis (full-types) False Lem-Gradn-0 Cong-Gradn-Gradn-Suc
    assms(2) bet-neq23--neq between-cong-2 between-exchange3
    calculation(3) calculation(4) l5-1)
ultimately
show ?thesis
  using assms(2) by blast
qed

lemma reach--grad-min-n:
  assumes A ≠ B and
    Bet A B C and
    A C Le A (Gradn A B (Suc n)) →
  (∃ D E. (Bet A D C ∧ Grad A B D ∧ E ≠ C ∧ Bet A C E ∧ Bet A D E ∧ Cong A B D E))
  shows A C Le A (Gradn A B (Suc(Suc n))) →
  (∃ D E. (Bet A D C ∧ Grad A B D ∧ E ≠ C ∧ Bet A C E ∧ Bet A D E ∧ Cong A B D E))
proof -
{
  assume
    H1: A C Le A (Gradn A B (Suc(Suc n)))
  have ∃ D E. (Bet A D C ∧ Grad A B D ∧ E ≠ C ∧ Bet A C E ∧ Bet A D E ∧ Cong A B D E)
  proof (cases A C Le A (Gradn A B (Suc n)))
    case True
    hence A C Le A (Gradn A B (Suc n))
    by blast
    then obtain D E where
      Bet A D C ∧ Grad A B D ∧ E ≠ C ∧ Bet A C E ∧ Bet A D E ∧ Cong A B D E
    using assms(1) assms(3) by blast
    thus ?thesis
    by blast
  next
  case False
  hence A (Gradn A B (Suc n)) Lt A C
  by (meson nlt--le)

```

```

let ?f = Gradn A B (Suc (Suc n))
show ?thesis
proof (cases C = ?f)
case True
let ?e = Gradn A B (Suc (Suc (Suc n)))
show ?thesis
proof -
have Bet A C C
using not-bet-distincts by auto
moreover
have C = Gradn A B (Suc (Suc n))
using True by auto
hence Grad A B C
using Grad-def by blast
moreover
have ?e ≠ C
by (metis Gradn-uniq-aux-1 True assms(1))
moreover
have Bet A C ?e
using Bet-Gradn-Gradn-Suc True by blast
moreover
have Bet A B ?e
using Diff--Bet-Gradn-Suc not-bet-distincts by blast
moreover
have Cong A B C ?e
using Cong-Gradn-Gradn-Suc True by blast
ultimately
show ?thesis
using assms(2) by blast
qed
next
case False
hence A C Le A ?f
using H1 by blast
show ?thesis
proof -
let ?d = Gradn A B (Suc n)
let ?e = Gradn A B (Suc(Suc n))
have Bet A ?d C
by (meson Diff--Bet-Gradn-Suc
⟨A (Gradn A B (Suc n)) Lt A C⟩ assms(1) assms(2)
l5-1 l5-12-a lt--nle)
moreover
have Grad A B ?d
using Grad-def by blast
moreover
have ?e ≠ C
using False by auto
moreover
have Bet A C ?e
by (metis H1 Bet-Gradn-Gradn-Suc assms(1) bet--lt1213
calculation(1) calculation(2) grad-neq--neq13 l5-1 lt--nle)
moreover
have Bet A B ?e
using Diff--Bet-Gradn-Suc not-bet-distincts by blast
moreover
have Cong A B ?d ?e
using Cong-Gradn-Gradn-Suc by blast
ultimately
show ?thesis
using between-exchange4 by blast
qed
qed
qed
}
thus ?thesis

```


by *blast*
 qed

lemma *reach--grad-min-aux*:
 assumes $A \neq B$ and
 Bet A B C
 shows $(\text{Grad } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } n)) \wedge A \ C \ \text{Le } A \ (\text{Gradn } A \ B \ (\text{Suc } n)))$
 $\longrightarrow (\exists \ D \ E. (\text{Bet } A \ D \ C \wedge \text{Grad } A \ B \ D \wedge E \neq C \wedge \text{Bet } A \ C \ E \wedge \text{Bet } A \ D \ E \wedge \text{Cong } A \ B \ D \ E))$
proof (*induction* n)
 case 0
 show ?case
 using *assms*(1) *assms*(2) *reach--grad-min-1* by *blast*
next
 case (*Suc* n)
 have $\text{Grad } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } n)) \wedge \text{Grad } A \ B \ (\text{Gradn } A \ B \ (\text{Suc } (\text{Suc } n)))$
 using *Grad-def* by *blast*
 thus ?case
 using *assms*(1) *assms*(2) *reach--grad-min-n* *Suc.IH* by *blast*
 qed

lemma *reach--grad-min*:
 assumes $A \neq B$ and
 Bet A B C and
 Reach A B A C
 shows $\exists \ D \ E. (\text{Bet } A \ D \ C \wedge \text{Grad } A \ B \ D \wedge E \neq C \wedge \text{Bet } A \ C \ E \wedge \text{Bet } A \ D \ E \wedge \text{Cong } A \ B \ D \ E)$
proof –
 obtain D where $\text{Grad } A \ B \ D \wedge A \ C \ \text{Le } A \ D$
 by (*meson* *assms*(3) *gradexp--grad reach--ex-gradexp-le*)
 hence $A \ C \ \text{Le } A \ D$
 by *blast*
 have $A \ \text{Out } C \ D$
proof –
 have $A \ \text{Out } C \ B$
 using *assms*(1) *assms*(2) *bet-out l6-6* by *presburger*
 moreover
 have $\text{Grad } A \ B \ D$
 by (*simp* *add*: $\langle \text{Grad } A \ B \ D \wedge A \ C \ \text{Le } A \ D \rangle$)
 hence *Bet* A B D
 by (*simp* *add*: *grad--bet*)
 hence $A \ \text{Out } B \ D$
 using *assms*(1) *bet-out* by *force*
 ultimately
 show ?thesis
 using *l6-7* by *blast*
 qed
 hence *Bet* A C D
 using *l6-13-1* by (*simp* *add*: $\langle A \ C \ \text{Le } A \ D \rangle$)
 have $\text{Grad } A \ B \ D$
 by (*simp* *add*: $\langle \text{Grad } A \ B \ D \wedge A \ C \ \text{Le } A \ D \rangle$)
 then obtain n where $(n \neq 0) \wedge (D = \text{Gradn } A \ B \ n)$
 using *Grad-def* by *blast*
 hence $n \neq 0$
 by *simp*
 then obtain m where $\text{Suc } m = n$
 by (*metis* *not0-implies-Suc*)
 hence $D = \text{Gradn } A \ B \ (\text{Suc } m)$
 by (*simp* *add*: $\langle n \neq 0 \wedge D = \text{Gradn } A \ B \ n \rangle$)
 show ?thesis
 by (*metis* $\langle \text{Grad } A \ B \ D \wedge A \ C \ \text{Le } A \ D \rangle$
 $\langle \text{Suc } m = n \rangle \langle n \neq 0 \wedge D = \text{Gradn } A \ B \ n \rangle$
 assms(1) *assms*(2) *reach--grad-min-aux*)
 qed

lemma *reach--ex-gradexp-lt*:
 assumes $A \neq B$ and

```

  Reach P Q A B
shows  $\exists C. \text{GradExp } A \ C \ B \wedge A \ C \ Lt \ P \ Q$ 
proof -
  obtain R where GradExp P Q R  $\wedge A \ B \ Le \ P \ R$ 
  using assms(2) reach--ex-gradexp-le by blast
  then obtain n where  $n \neq 0 \wedge R = \text{GradExpn } P \ Q \ n$ 
  using GradExp-def by blast
  hence  $n \neq 0$ 
  by blast
  then obtain m where  $\text{Suc } m = n$ 
  using not0-implies-Suc by blast
  hence  $R = \text{GradExpn } P \ Q \ (\text{Suc } m)$ 
  by (simp add:  $\langle n \neq 0 \wedge R = \text{GradExpn } P \ Q \ n \rangle$ )
{
  fix B
  assume  $A \neq B$  and  $A \ B \ Le \ P \ R$ 
  hence  $\exists C. (\text{GradExp } A \ C \ B \wedge A \ C \ Lt \ P \ Q)$ 
  using  $\langle R = \text{GradExpn } P \ Q \ (\text{Suc } m) \rangle$  reach--ex-gradexp-lt-aux by force
}
thus ?thesis
  by (simp add:  $\langle \text{GradExp } P \ Q \ R \wedge A \ B \ Le \ P \ R \rangle$  assms(1))
qed

```

```

lemma t22-18-aux-0:
  assumes Bet A0 D1 A1 and
    Cong E0 E1 A1 D1 and
     $D = \text{Gradn } A0 \ D1 \ (\text{Suc } 0)$ 
  shows  $\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A)$ 
proof -
  have  $D = D1$ 
  by (simp add: assms(3))
  thus ?thesis
  using assms(1) assms(2) grad2-init by blast
qed

```

```

lemma t22-18-aux-n:
  assumes  $\forall A0 \ D1 \ A1 \ E0 \ E1 \ D.$ 
     $(\text{Bet } A0 \ D1 \ A1 \wedge \text{Cong } E0 \ E1 \ A1 \ D1 \wedge D = \text{Gradn } A0 \ D1 \ (\text{Suc } n)) \longrightarrow$ 
     $(\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A))$ 
  shows  $\forall A0 \ D1 \ A1 \ E0 \ E1 \ D.$ 
     $(\text{Bet } A0 \ D1 \ A1 \wedge \text{Cong } E0 \ E1 \ A1 \ D1 \wedge D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n))) \longrightarrow$ 
     $(\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A))$ 
proof -
  have  $\forall A0 \ D1 \ A1 \ E0 \ E1 \ D. (\text{Bet } A0 \ D1 \ A1 \wedge \text{Cong } E0 \ E1 \ A1 \ D1 \wedge$ 
     $D = \text{Gradn } A0 \ D1 \ (\text{Suc } n)) \longrightarrow$ 
     $(\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A))$ 
  using assms by blast
{
  fix A0 D1 A1 E0 E1 D
  assume 1: Bet A0 D1 A1 and
    2: Cong E0 E1 A1 D1 and
     $D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n))$ 
  let ?C = Gradn A0 D1 (Suc n)
  obtain A E where Grad2 A0 A1 A E0 E1 E and Cong E0 E A ?C and Bet A0 ?C A
  using 1 2 assms by blast
  obtain A' where Bet A0 A A' and Cong A A' A0 A1
  using segment-construction by blast
  obtain E' where Bet E0 E E' and Cong E E' E0 E1
  using segment-construction by blast
  have Grad2 A0 A1 A' E0 E1 E'
  using Grad2-stab  $\langle \text{Bet } A0 \ A \ A' \rangle \langle \text{Bet } E0 \ E \ E' \rangle \langle \text{Cong } A \ A' \ A0 \ A1 \rangle$ 
     $\langle \text{Cong } E \ E' \ E0 \ E1 \rangle \langle \text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \rangle$  cong-symmetry by blast
  moreover
  have  $E0 \ E1 \ Le \ A \ A'$ 

```

```

by (meson 1 2 ⟨Cong A A' A0 A1⟩ bet--le2313 l5-6
    le-left-comm not-cong-3412)
obtain D' where Bet A D' A' and Cong E0 E1 A D'
  using Le-def ⟨E0 E1 Le A A'⟩ by blast
have Cong E0 E' A' D
proof -
  have Cong E0 E' ?C D'
proof -
  have Bet ?C A D'
  by (metis between-exchange3 ⟨Bet A D' A'⟩
      ⟨Bet A0 (Gradn A0 D1 (Suc n)) A⟩ ⟨Bet A0 A A'⟩
      between-inner-transitivity)
moreover
have Cong E0 E ?C A
  using ⟨Cong E0 E A (Gradn A0 D1 (Suc n))⟩ not-cong-1243 by blast
moreover
have Cong E E' A D'
  by (metis cong-transitivity ⟨Cong E E' E0 E1⟩ ⟨Cong E0 E1 A D'⟩)
ultimately
show ?thesis
  using ⟨Bet E0 E E'⟩ l2-11 by blast
qed
moreover
have Cong ?C D' A' D
proof -
  have Bet ?C D' A'
  by (metis between-exchange3 ⟨Bet A D' A'⟩
      ⟨Bet A0 (Gradn A0 D1 (Suc n)) A⟩ ⟨Bet A0 A A'⟩ between-exchange2)
moreover
have Bet A' D ?C
proof -
  have Grad A0 A1 A
  using ⟨Grad2 A0 A1 A E0 E1 E⟩ grad2--grad123 by blast
hence Bet A0 A1 A
  by (simp add: grad--bet)
have Bet A0 D1 ?C
  using Diff--Bet-Gradn-Suc not-bet-distincts by blast
show ?thesis
proof (cases A0 = ?C)
case True
  thus ?thesis
  by (metis Gradn-aux-1-0 Lem-Gradn-id-n
      ⟨D = Gradn A0 D1 (Suc (Suc n))⟩ not-bet-distincts)
next
case False
hence A0 ≠ ?C
  by blast
show ?thesis
proof (cases D = ?C)
case True
  thus ?thesis
  using not-bet-distincts by blast
next
case False
hence D ≠ ?C
  by blast
show ?thesis
proof (cases A' = ?C)
case True
  thus ?thesis
  by (metis 1 Lem-Gradn-id-n cong-diff-3
      ⟨Bet A0 (Gradn A0 D1 (Suc n)) A⟩ ⟨Bet A0 A A'⟩
      ⟨Cong A A' A0 A1⟩ ⟨D = Gradn A0 D1 (Suc (Suc n))⟩
      between-equality-2 between-identity)
next
case False

```

```

hence  $A' \neq ?C$ 
  by blast
have  $?C \text{ Out } D \ A'$ 
proof -
  have  $D \neq ?C$ 
    using  $\langle D \neq \text{Gradn } A0 \ D1 \ (\text{Suc } n) \rangle$  by auto
  moreover
  have  $A' \neq ?C$ 
    using False by blast
  moreover
  have  $A0 \neq ?C$ 
    using  $\langle A0 \neq \text{Gradn } A0 \ D1 \ (\text{Suc } n) \rangle$  by auto
  moreover
  have  $\text{Bet } D \ ?C \ A0$ 
    using Bet-Gradn-Gradn-Suc
       $\langle D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n)) \rangle$  between-symmetry by blast
  ultimately
  show ?thesis
    by (meson  $\langle \text{Bet } A0 \ (\text{Gradn } A0 \ D1 \ (\text{Suc } n)) \ A \rangle \langle \text{Bet } A0 \ A \ A' \rangle$ 
        between-exchange4 between-symmetry l6-3-2)
qed
moreover
have  $?C \ D \ Le \ ?C \ A'$ 
proof -
  have  $?C \ D \ Le \ A \ A'$ 
    by (metis 1 Cong-Gradn-Gradn-Suc cong-symmetry
         $\langle \text{Cong } A \ A' \ A0 \ A1 \rangle \langle D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n)) \rangle$ 
        bet--le1213 l5-6)
  moreover
  have  $A \ A' \ Le \ ?C \ A'$ 
    by (metis between-exchange3
         $\langle \text{Bet } A0 \ (\text{Gradn } A0 \ D1 \ (\text{Suc } n)) \ A \rangle$ 
         $\langle \text{Bet } A0 \ A \ A' \rangle$  bet--le2313)
  ultimately
  show ?thesis
    using le-transitivity by blast
qed
ultimately
show ?thesis
  using between-symmetry l6-13-1 by blast
qed
qed
qed
moreover
have  $\text{Cong } ?C \ A' \ A' \ ?C$ 
  by (simp add: cong-pseudo-reflexivity)
moreover
have  $\text{Cong } D' \ A' \ D \ ?C$ 
proof -
  have  $\text{Cong } A0 \ D1 \ (\text{Gradn } A0 \ D1 \ (\text{Suc } n)) \ (\text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n)))$ 
    using Cong-Gradn-Gradn-Suc by blast
  hence  $\text{Cong } A0 \ D1 \ ?C \ D$ 
    using  $\langle D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n)) \rangle$  by blast
  hence  $\text{Cong } A0 \ D1 \ D \ ?C$ 
    using not-cong-1243 by blast
  have  $\text{Bet } A1 \ D1 \ A0$ 
    by (simp add: 1 between-symmetry)
  have  $\text{Cong } A1 \ D1 \ A \ D'$ 
    using 2  $\langle \text{Cong } E0 \ E1 \ A \ D' \rangle$  cong-inner-transitivity
    by blast
  have  $\text{Cong } A0 \ A1 \ A \ A'$ 
    using  $\langle \text{Cong } A \ A' \ A0 \ A1 \rangle$  cong-symmetry by presburger
  hence  $\text{Cong } A0 \ D1 \ D' \ A'$ 
    using  $\langle \text{Bet } A \ D' \ A' \rangle \langle \text{Bet } A1 \ D1 \ A0 \rangle \langle \text{Cong } A1 \ D1 \ A \ D' \rangle$ 
        l4-3-1 not-cong-2134 by blast

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```

    thus ?thesis
    using ⟨Cong A0 D1 D (Gradn A0 D1 (Suc n))⟩
    cong-inner-transitivity by blast
qed
ultimately
show ?thesis
    using l4-3 by blast
qed
ultimately
show ?thesis
    using Cong-cases cong-inner-transitivity by blast
qed
moreover
have Bet A0 D A'
proof -
    have Bet A0 ?C A'
    using ⟨Bet A0 (Gradn A0 D1 (Suc n)) A⟩ ⟨Bet A0 A A'⟩
    between-exchange4 by blast
    have Bet ?C D A'
    proof -
        have Grad A0 A1 A
        using ⟨Grad2 A0 A1 A E0 E1 E⟩ grad2--grad123 by blast
        hence Bet A0 A1 A
        by (simp add: grad--bet)
        have Bet A0 D1 ?C
        using Diff--Bet-Gradn-Suc not-bet-distincts by blast
        show ?thesis
        proof (cases A0 = ?C)
            case True
            thus ?thesis
            by (metis Gradn-aux-1-0 Lem-Gradn-id-n
                ⟨Bet A0 (Gradn A0 D1 (Suc n)) A'⟩
                ⟨D = Gradn A0 D1 (Suc (Suc n))⟩)
        next
            case False
            hence A0 ≠ ?C
            by blast
            show ?thesis
            proof (cases D = ?C)
                case True
                thus ?thesis
                using between-trivial2 by auto
            next
                case False
                hence D ≠ ?C
                by blast
                show ?thesis
                proof (cases A' = ?C)
                    case True
                    thus ?thesis
                    by (metis 1 Lem-Gradn-id-n cong-diff-3
                        ⟨Bet A0 (Gradn A0 D1 (Suc n)) A⟩ ⟨Bet A0 A A'⟩
                        ⟨Cong A A' A0 A1⟩ ⟨D = Gradn A0 D1 (Suc (Suc n))⟩
                        between-equality-2 between-identity)
                next
                    case False
                    hence A' ≠ ?C
                    by blast
                    have ?C Out D A'
                    proof -
                        have D ≠ ?C
                        using ⟨D ≠ Gradn A0 D1 (Suc n)⟩ by auto
                        moreover
                        have A' ≠ ?C
                        using False by blast
                        moreover

```

```

    have  $A0 \neq ?C$ 
      using  $\langle A0 \neq \text{Gradn } A0 \ D1 \ (\text{Suc } n) \rangle$  by auto
    moreover
    have  $\text{Bet } D \ ?C \ A0$ 
      using  $\text{Bet-Gradn-Gradn-Suc}$ 
       $\langle D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n)) \rangle$ 
      between-symmetry by blast
    ultimately
    show ?thesis
      by (meson  $\langle \text{Bet } A0 \ (\text{Gradn } A0 \ D1 \ (\text{Suc } n)) \ A' \rangle$ 
        between-symmetry l6-3-2)
  qed
  moreover
  have  $?C \ D \ Le \ ?C \ A'$ 
  proof -
    have  $?C \ D \ Le \ A \ A'$ 
      by (metis 1 Cong-Gradn-Gradn-Suc cong-symmetry
         $\langle \text{Cong } A \ A' \ A0 \ A1 \rangle \langle D = \text{Gradn } A0 \ D1 \ (\text{Suc } (\text{Suc } n)) \rangle$ 
        bet--le1213 l5-6)
    moreover
    have  $A \ A' \ Le \ ?C \ A'$ 
      by (metis between-exchange3  $\langle \text{Bet } A0 \ (\text{Gradn } A0 \ D1 \ (\text{Suc } n)) \ A \rangle$ 
         $\langle \text{Bet } A0 \ A \ A' \rangle$  bet--le2313)
    ultimately
    show ?thesis
      using le-transitivity by blast
  qed
  ultimately
  show ?thesis
    using l6-13-1 by blast
  qed
  qed
  qed
  qed
  thus ?thesis
    using  $\langle \text{Bet } A0 \ (\text{Gradn } A0 \ D1 \ (\text{Suc } n)) \ A' \rangle$  between-exchange2 by blast
  qed
  ultimately
  have  $\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A)$ 
    by blast
}
thus ?thesis by blast
qed

lemma t22-18-aux0:
  shows  $\forall A0 \ D1 \ A1 \ E0 \ E1 \ D.$ 
     $(\text{Bet } A0 \ D1 \ A1 \wedge \text{Cong } E0 \ E1 \ A1 \ D1 \wedge D = \text{Gradn } A0 \ D1 \ (\text{Suc } n)) \longrightarrow$ 
     $(\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A))$ 
  proof (induction n)
    case 0
    show ?case
      using t22-18-aux-0 by auto
  next
    case (Suc n)
    show ?case
      using t22-18-aux-n Suc.IH by blast
  qed

lemma t22-18-aux1:
  assumes  $\text{Bet } A0 \ D1 \ A1$  and
     $\text{Cong } E0 \ E1 \ A1 \ D1$  and
     $\text{Grad } A0 \ D1 \ D$ 
  shows  $\exists A \ E. (\text{Grad2 } A0 \ A1 \ A \ E0 \ E1 \ E \wedge \text{Cong } E0 \ E \ A \ D \wedge \text{Bet } A0 \ D \ A)$ 
  proof -
    obtain n where  $n \neq 0$  and  $D = \text{Gradn } A0 \ D1 \ n$ 

```

using *Grad-def assms(3)* by *blast*
 hence $n \neq 0$
 by *blast*
 then obtain m where $n = \text{Suc } m$
 using *not0-implies-Suc* by *presburger*
 hence $D = \text{Gradn } A0 \ D1 \ (\text{Suc } m)$
 by (*simp add: $\langle D = \text{Gradn } A0 \ D1 \ n \rangle$*)
 thus ?thesis
 using *assms(1) assms(2) t22-18-aux0* by *blast*
 qed

lemma *t22-18-aux2-0*:

assumes *Saccheri A0 B0 B1 A1* and
 $A = \text{Gradn } A0 \ A1 \ (\text{Suc } 0)$ and
 $E = \text{Gradn } B0 \ B1 \ (\text{Suc } 0)$ and
Saccheri A0 B0 B A
 shows $B0 \ B \ Le \ B0 \ E$

proof –

have *Per B0 A0 A1*
 using *Saccheri-def assms(1)* by *blast*
 have *Per A0 A1 B1*
 using *Saccheri-def assms(1)* by *blast*
 have *Cong A0 B0 B1 A1*
 using *Saccheri-def assms(1)* by *blast*
 have $A0 \ A1 \ OS \ B0 \ B1$
 using *Saccheri-def assms(1)* by *blast*
 have *Per B0 A0 A*
 using *Saccheri-def assms(4)* by *blast*
 have *Per A0 A B*
 using *Saccheri-def assms(4)* by *blast*
 have $A0 \ A \ OS \ B0 \ B$
 using *Saccheri-def assms(4)* by *blast*
 have *Cong A0 B0 B A*
 using *Saccheri-def assms(4)* by *blast*
 have $A = A1$
 by (*simp add: assms(2)*)
 have $E = B1$
 by (*simp add: assms(3)*)
 have *Saccheri A0 B0 B A1*
 using $\langle A = A1 \rangle$ *assms(4)* by *auto*
 hence *Cong A0 B0 B A1*
 using *Saccheri-def* by *blast*
 hence $B = B1$

proof –

have $A0 \neq A1$
 using $\langle A0 \ A1 \ OS \ B0 \ B1 \rangle$ *os-distincts* by *blast*
 have $A1 \ Out \ B \ B1$
 proof –
 have *Col A1 B B1*
 proof –
 have *Coplanar A0 B B1 A1*
 proof –
 have $\neg \text{Col } B0 \ A0 \ A1$
 by (*meson $\langle A0 \ A1 \ OS \ B0 \ B1 \rangle \text{ col123--nos not-col-permutation-2}$*)
 moreover
 have *Coplanar B0 A0 A1 B*
 using $\langle A = A1 \rangle$ *assms(4) coplanar-perm-7 sac--coplanar* by *blast*
 moreover
 have *Coplanar B0 A0 A1 B1*
 by (*meson $\langle A0 \ A1 \ OS \ B0 \ B1 \rangle \text{ ncoplanar-perm-8 os--coplanar}$*)
 ultimately
 show ?thesis
 by (*meson coplanar-pseudo-trans ncop-distincts*)
 qed
 moreover
 have $A0 \neq A1$

```

    using ⟨A0 ≠ A1⟩ by auto
  moreover
  have Per B A1 A0
    using ⟨A = A1⟩ ⟨Per A0 A B⟩ l8-2 by blast
  moreover
  have Per B1 A1 A0
    using Per-cases ⟨Per A0 A1 B1⟩ by blast
  ultimately
  show ?thesis
    using col-permutation-2 cop-per2--col by blast
qed
moreover
have A1 A0 OS B B1
proof -
  have A1 A0 OS B B0
    using Saccheri-def ⟨A = A1⟩ assms(4) invert-one-side
      one-side-symmetry by blast
  moreover
  have A1 A0 OS B0 B1
    by (simp add: ⟨A0 A1 OS B0 B1⟩ invert-one-side)
  ultimately
  show ?thesis
    using one-side-transitivity by blast
qed
ultimately
show ?thesis
  using col-one-side-out by auto
qed
moreover
have Cong A1 B A0 B0
  using Cong-cases ⟨Cong A0 B0 B A1⟩ by blast
moreover
have A1 Out B1 B1
  using calculation(1) out-diff2 out-trivial by auto
moreover
have Cong A1 B1 A0 B0
  using ⟨Cong A0 B0 B1 A1⟩ not-cong-3421 by blast
ultimately
show ?thesis
  using l6-11-uniqueness by blast
qed
thus ?thesis
  using ⟨E = B1⟩ local.le-cases by blast
qed

```

lemma t22-18-aux2-Sucn:

```

assumes ∀ A A0 A1 B B0 B1 E.
  Saccheri A0 B0 B1 A1 ∧ A = Gradn A0 A1 (Suc n) ∧
  E = Gradn B0 B1 (Suc n) ∧ Saccheri A0 B0 B A ⟶ B0 B Le B0 E
shows ∀ A A0 A1 B B0 B1 E.
  Saccheri A0 B0 B1 A1 ∧ A = Gradn A0 A1 (Suc(Suc n)) ∧
  E = Gradn B0 B1 (Suc(Suc n)) ∧ Saccheri A0 B0 B A ⟶ B0 B Le B0 E
proof -
{
  fix A A0 A1 B B0 B1 E
  assume Saccheri A0 B0 B1 A1 and
    A = Gradn A0 A1 (Suc(Suc n)) and
    E = Gradn B0 B1 (Suc(Suc n)) and
    Saccheri A0 B0 B A
  have A0 ≠ B0
    using ⟨Saccheri A0 B0 B1 A1⟩ sac-distincts by blast
  let ?A = Gradn A0 A1 (Suc n)
  let ?E = Gradn B0 B1 (Suc n)
  have Saccheri A0 B0 B ?A ⟶ B0 B Le B0 ?E
    using ⟨Saccheri A0 B0 B1 A1⟩ assms by blast
}

```



```

assume  $A0 = ?A$ 
have  $Bet\ A0\ A1\ ?A$ 
  using Diff--Bet-Gradn-Suc not-bet-distincts by blast
hence  $A1 = A0$ 
  by  $(metis\ \langle A0 = Gradn\ A0\ A1\ (Suc\ n) \rangle\ bet-neq32--neq)$ 
hence False
  using sac-distincts  $\langle Saccheri\ A0\ B0\ B1\ A1 \rangle$  by blast
}
hence  $A0 \neq ?A$ 
  by blast
have  $\exists\ B.\ Saccheri\ A0\ B0\ B\ ?A$ 
proof –
  have  $Grad\ A0\ A1\ ?A$ 
    using Grad-def by blast
  hence  $Bet\ A0\ A1\ ?A$ 
    by  $(simp\ add:\ grad--bet)$ 
  have  $\exists\ P.\ A0\ A1\ Perp\ P\ ?A \wedge A0\ A1\ OS\ B0\ P$ 
proof –
  have  $Grad\ A0\ A1\ ?A$ 
    using Grad-def by blast
  hence  $Bet\ A0\ A1\ ?A$ 
    by  $(simp\ add:\ grad--bet)$ 
  hence  $Col\ A0\ A1\ ?A$ 
    using Col-def by blast
  moreover
  have  $\neg\ Col\ A0\ A1\ B0$ 
    by  $(meson\ \langle Saccheri\ A0\ B0\ B1\ A1 \rangle\ Col-cases\ col-trivial-3\ l8-16-1\ sac--perp1214)$ 
  ultimately
  show ?thesis
    using l10-15 by blast
qed
then obtain  $P$  where  $A0\ A1\ Perp\ P\ ?A$  and  $A0\ A1\ OS\ B0\ P$ 
  by blast
hence  $P \neq ?A$ 
  using perp-not-eq-2 by blast
hence  $\exists\ B.\ ?A\ Out\ B\ P \wedge Cong\ ?A\ B\ A0\ B0$ 
  using l6-11-existence  $\langle A0 \neq B0 \rangle$  by simp
then obtain  $B'$  where  $?A\ Out\ B'\ P$  and  $Cong\ ?A\ B'\ A0\ B0$ 
  by blast
have  $Saccheri\ A0\ B0\ B'\ ?A$ 
proof –
  have  $Per\ B0\ A0\ ?A$ 
proof –
  have  $Per\ B0\ A0\ A1$ 
    by  $(meson\ perp-per-2\ \langle Saccheri\ A0\ B0\ B1\ A1 \rangle\ sac--perp1214)$ 
  moreover
  have  $Col\ A0\ A1\ ?A$ 
    using Col-def  $\langle Bet\ A0\ A1\ (Gradn\ A0\ A1\ (Suc\ n)) \rangle$  by blast
  ultimately
  show ?thesis
    using  $\langle Saccheri\ A0\ B0\ B1\ A1 \rangle\ per-col\ sac-distincts$  by blast
qed
moreover
have  $Per\ A0\ ?A\ B'$ 
proof –
  have  $?A\ A0\ Perp\ ?A\ P$ 
    using Perp-cases  $\langle A0\ A1\ Perp\ P\ (Gradn\ A0\ A1\ (Suc\ n)) \rangle$ 
     $\langle A0 \neq Gradn\ A0\ A1\ (Suc\ n) \rangle\ \langle Bet\ A0\ A1\ (Gradn\ A0\ A1\ (Suc\ n)) \rangle$ 
    bet-col perp-col1 by blast
  moreover
  have  $Col\ ?A\ P\ B'$ 
    using  $\langle Gradn\ A0\ A1\ (Suc\ n)\ Out\ B'\ P \rangle\ col-permutation-5\ out-col$  by blast
  ultimately
  show ?thesis
    by  $(meson\ Perp-cases\ col-trivial-3\ l8-16-1)$ 

```

```

qed
moreover
have Col A0 A1 ?A
  using Col-def ⟨Bet A0 A1 (Gradn A0 A1 (Suc n))⟩ by blast
hence A0 ?A OS B0 P
  using ⟨A0 A1 OS B0 P⟩ ⟨A0 ≠ Gradn A0 A1 (Suc n)⟩
    col-one-side by blast
hence ?A A0 OS B0 P
  using invert-one-side by blast
hence ?A A0 OS B0 B'
  using ⟨Gradn A0 A1 (Suc n) Out B' P⟩ l6-6
    out-out-one-side by blast
hence A0 ?A OS B0 B'
  using invert-one-side by blast
ultimately
show ?thesis
  using Saccheri-def ⟨Cong (Gradn A0 A1 (Suc n)) B' A0 B0⟩
    not-cong-4312 by blast
qed
thus ?thesis
  by blast
qed
then obtain B' where Saccheri A0 B0 B' ?A
  by blast
hence B0 B' Le B0 ?E
  using ⟨Saccheri A0 B0 B1 A1⟩ assms by blast
have Per B0 A0 ?A
  using ⟨Saccheri A0 B0 B' ?A⟩ Saccheri-def by blast
have Per A0 ?A B'
  using ⟨Saccheri A0 B0 B' ?A⟩ Saccheri-def by blast
have Cong A0 B0 B' ?A
  using ⟨Saccheri A0 B0 B' ?A⟩ Saccheri-def by blast
have A0 ?A OS B0 B'
  using ⟨Saccheri A0 B0 B' ?A⟩ Saccheri-def by blast
obtain C where Bet B0 B' C and Cong B' C ?E E
  using segment-construction by blast
have Cong B0 B1 B' B
proof -
  have Saccheri A0 B0 B1 A1
    by (simp add: ⟨Saccheri A0 B0 B1 A1⟩)
  moreover
  have Saccheri ?A B' B A
proof -
  have Saccheri A0 B0 B' ?A
    using ⟨Saccheri A0 B0 B' (Gradn A0 A1 (Suc n))⟩ by auto
  moreover
  have ?A ≠ A
    by (metis Gradn-uniq-aux-1 ⟨A = Gradn A0 A1 (Suc (Suc n))⟩
      ⟨A0 ≠ Gradn A0 A1 (Suc n)⟩ grad-rec-a-a)
  moreover
  have Coplanar A0 B0 ?A A
    by (metis Bet-Gradn-Gradn-Suc ncop--ncols
      ⟨A = Gradn A0 A1 (Suc (Suc n))⟩ bet-col)
  ultimately
  show ?thesis
    by (meson cop-sac2--sac ⟨Saccheri A0 B0 B A⟩)
qed
moreover
have Cong A0 B0 ?A B'
  using ⟨Cong A0 B0 B' (Gradn A0 A1 (Suc n))⟩
    cong-right-commutativity by blast
moreover
have Cong A0 A1 ?A A
  using Cong-Gradn-Gradn-Suc ⟨A = Gradn A0 A1 (Suc (Suc n))⟩ by blast
ultimately
show ?thesis

```

```

    using cong2-sac2--cong by blast
qed
have Cong B B' B' C
proof -
  have Cong B B' ?E E
    using Cong-Gradn-Gradn-Suc
      ⟨Cong B0 B1 B' B⟩ ⟨E = Gradn B0 B1 (Suc (Suc n))⟩
      cong-transitivity not-cong-3421 by blast
  moreover
  have Cong ?E E B' C
    using ⟨Cong B' C (Gradn B0 B1 (Suc n)) E⟩
      cong-inner-transitivity cong-reflexivity by blast
  ultimately
  show ?thesis
    by (meson cong-transitivity)
qed
hence B0 B Le B0 C
  by (meson ⟨Bet B0 B' C⟩ Cong-cases cong-reflexivity
      triangle-inequality-2)
moreover
have B0 C Le B0 E
proof -
  have Bet B0 B' C
    by (simp add: ⟨Bet B0 B' C⟩)
  moreover
  have Bet B0 ?E E
    using Bet-Gradn-Gradn-Suc ⟨E = Gradn B0 B1 (Suc (Suc n))⟩ by blast
  moreover
  have B0 B' Le B0 ?E
    using ⟨B0 B' Le B0 (Gradn B0 B1 (Suc n))⟩ by auto
  moreover
  have B' C Le ?E E
    using ⟨Cong B' C (Gradn B0 B1 (Suc n)) E⟩ cong--le by blast
  ultimately
  show ?thesis
    using ⟨Cong B' C (Gradn B0 B1 (Suc n)) E⟩
      bet2-le2--le1346 cong--le by blast
qed
ultimately
have B0 B Le B0 E
  using le-transitivity by blast
}
thus ?thesis
  by blast
qed

```

```

lemma t22-18-aux2-n:
  shows ∀ A A0 A1 B B0 B1 E.
    Saccheri A0 B0 B1 A1 ∧ A = Gradn A0 A1 (Suc n) ∧
    E = Gradn B0 B1 (Suc n) ∧ Saccheri A0 B0 B A
    → B0 B Le B0 E
proof (induction n)
  case 0
  show ?case
    using t22-18-aux2-0 by blast
next
  case (Suc n)
  show ?case
    using t22-18-aux2-Sucn Suc.IH by blast
qed

```

```

lemma t22-18-aux2:
  assumes Saccheri A0 B0 B1 A1 and
    Grad2 A0 A1 A B0 B1 E and
    Saccheri A0 B0 B A

```

```

shows  $B0\ B\ Le\ B0\ E$ 
proof -
  obtain  $n$  where  $n \neq 0$  and  $A = Gradn\ A0\ A1\ n$  and
     $E = Gradn\ B0\ B1\ n$ 
  using Grad2-def assms(2) by blast
  obtain  $m$  where  $n = Suc\ m$ 
  using  $\langle n \neq 0 \rangle$  not0-implies-Suc by blast
  hence  $A = Gradn\ A0\ A1\ (Suc\ m) \wedge E = Gradn\ B0\ B1\ (Suc\ m)$ 
  using  $\langle A = Gradn\ A0\ A1\ n \rangle \langle E = Gradn\ B0\ B1\ n \rangle$  by blast
  show ?thesis
  using t22-18-aux2-n
     $\langle A = Gradn\ A0\ A1\ (Suc\ m) \wedge E = Gradn\ B0\ B1\ (Suc\ m) \rangle$ 
    assms(1) assms(3) by blast
qed

lemma t22-18:
  assumes archimedes-axiom
  shows  $\forall\ A0\ B0\ B1\ A1. Saccheri\ A0\ B0\ B1\ A1 \longrightarrow \neg (B0\ B1\ Lt\ A1\ A0)$ 
proof -
  {
    fix  $A0\ B0\ B1\ A1$ 
    assume Saccheri  $A0\ B0\ B1\ A1$ 
    assume  $B0\ B1\ Lt\ A1\ A0$ 
    hence  $B0\ B1\ Le\ A1\ A0$ 
      using cong--nlt lt--le by blast
    then obtain  $D1$  where  $Bet\ A1\ D1\ A0$  and  $Cong\ B0\ B1\ A1\ D1$ 
      using Le-def by blast
    obtain  $C0$  where  $Bet\ A0\ D1\ C0$  and  $Cong\ D1\ C0\ A0\ B0$ 
      using segment-construction by fastforce
    obtain  $C$  where  $Bet\ A0\ C0\ C$  and  $Cong\ C0\ C\ A0\ B0$ 
      using segment-construction by fastforce
    {
      fix  $D$ 
      assume  $Grad\ A0\ D1\ D$ 
      have  $A0\ D\ Lt\ A0\ C$ 
      proof (cases  $A0 = D1$ )
        case True
        thus ?thesis
          using  $\langle B0\ B1\ Lt\ A1\ A0 \rangle \langle Cong\ B0\ B1\ A1\ D1 \rangle$  cong--nlt by blast
      next
        case False
        obtain  $A\ E$  where  $Grad2\ A0\ A1\ A\ B0\ B1\ E \wedge Cong\ B0\ E\ A\ D \wedge Bet\ A0\ D\ A$ 
          using t22-18-aux1 Bet-cases  $\langle Bet\ A1\ D1\ A0 \rangle$ 
           $\langle Cong\ B0\ B1\ A1\ D1 \rangle \langle Grad\ A0\ D1\ D \rangle$  by blast
        have  $Grad2\ A0\ A1\ A\ B0\ B1\ E$ 
          using  $\langle Grad2\ A0\ A1\ A\ B0\ B1\ E \wedge Cong\ B0\ E\ A\ D \wedge Bet\ A0\ D\ A \rangle$  by blast
        hence  $Grad\ A0\ A1\ A$ 
          using grad2--grad123 by blast
        hence  $Bet\ A0\ A1\ A$ 
          by (simp add: grad--bet)
        have  $Cong\ B0\ E\ A\ D$ 
          using  $\langle Grad2\ A0\ A1\ A\ B0\ B1\ E \wedge Cong\ B0\ E\ A\ D \wedge Bet\ A0\ D\ A \rangle$  by blast
        have  $Bet\ A0\ D\ A$ 
          using  $\langle Grad2\ A0\ A1\ A\ B0\ B1\ E \wedge Cong\ B0\ E\ A\ D \wedge Bet\ A0\ D\ A \rangle$  by blast
        have  $\exists\ P. A0\ A1\ Perp\ A\ P \wedge A0\ A1\ OS\ B0\ P$ 
        proof -
          have  $Col\ A0\ A1\ A$ 
            by (simp add: Col-def  $\langle Bet\ A0\ A1\ A \rangle$ )
          moreover
          have  $A0\ A1\ ParStrict\ B0\ B1$ 
            by (simp add:  $\langle Saccheri\ A0\ B0\ B1\ A1 \rangle$  sac--pars1423)
          hence  $\neg\ Col\ A0\ A1\ B0$ 
            using par-strict-not-col-1 by blast
          ultimately
          show ?thesis
            using Perp-cases l10-15 by blast
        qed
      }
  }

```

```

qed
then obtain  $P$  where  $A0\ A1\ \text{Perp}\ A\ P$  and  $A0\ A1\ OS\ B0\ P$ 
  by blast
have  $P \neq A$ 
  using  $\langle A0\ A1\ \text{Perp}\ A\ P \rangle$  perp-not-eq-2 by blast
have  $A0 \neq B0$ 
  using  $\langle A0\ A1\ OS\ B0\ P \rangle$  os-distincts by blast
then obtain  $B$  where  $A\ Out\ B\ P$  and  $Cong\ A\ B\ A0\ B0$ 
  using l6-11-existence  $\langle P \neq A \rangle$  by blast
have Saccheri  $A0\ B0\ B\ A$ 
proof -
  have  $\text{Per}\ B0\ A0\ A$ 
  proof -
    have  $A0 \neq A1$ 
      using  $\langle A0\ A1\ OS\ B0\ P \rangle$  os-distincts by blast
    moreover
    have  $\text{Per}\ B0\ A0\ A1$ 
      by (meson perp-per-2  $\langle \text{Saccheri}\ A0\ B0\ B1\ A1 \rangle$  sac--perp1214)
    moreover
    have  $Col\ A0\ A1\ A$ 
      by (simp add: Col-def  $\langle \text{Bet}\ A0\ A1\ A \rangle$ )
    ultimately
    show ?thesis
      using per-col by blast
  qed
  moreover
  have  $\text{Per}\ A0\ A\ B$ 
  proof -
    have  $A \neq B$ 
      using  $\langle A\ Out\ B\ P \rangle$  out-distinct by blast
    moreover
    have  $A\ A0\ \text{Perp}\ A\ P$ 
  proof -
    have  $A0 \neq A$ 
      using False  $\langle \text{Bet}\ A0\ D\ A \rangle$   $\langle \text{Grad}\ A0\ D1\ D \rangle$ 
      bet-neq32--neq grad-neq--neq13 by blast
    moreover
    have  $A0\ A1\ \text{Perp}\ P\ A$ 
      using Perp-cases  $\langle A0\ A1\ \text{Perp}\ A\ P \rangle$  by blast
    moreover
    have  $Col\ A0\ A1\ A$ 
      by (simp add: Col-def  $\langle \text{Bet}\ A0\ A1\ A \rangle$ )
    ultimately
    show ?thesis
      using  $\langle A0\ A1\ \text{Perp}\ A\ P \rangle$  col-trivial-3 perp-col2
      by presburger
  qed
  moreover
  have  $Col\ A\ P\ B$ 
    by (simp add:  $\langle A\ Out\ B\ P \rangle$  l6-6 out-col)
  ultimately
  show ?thesis
    by (meson perp-per-2 perp-col1)
qed
moreover
have  $Cong\ A0\ B0\ B\ A$ 
  using  $\langle Cong\ A\ B\ A0\ B0 \rangle$  not-cong-4312 by blast
moreover
have  $A\ A0\ OS\ B0\ B$ 
proof -
  have  $A\ A0\ OS\ B0\ P$ 
    using  $\langle A0\ A1\ OS\ B0\ P \rangle$   $\langle \text{Bet}\ A0\ A1\ A \rangle$  bet-col
    bet-neq32--neq col-one-side invert-one-side by blast
  moreover
  have  $A\ Out\ P\ B$ 
    using Out-cases  $\langle A\ Out\ B\ P \rangle$  by blast

```

ultimately
 show ?thesis
 using out-out-one-side by blast
 qed
 hence $A0 \ A \ OS \ B0 \ B$
 using invert-one-side by blast
 ultimately
 show ?thesis
 using Saccheri-def by blast
 qed
 have $B0 \ B \ Le \ B0 \ E$
 using $\langle Saccheri \ A0 \ B0 \ B1 \ A1 \rangle \langle Grad2 \ A0 \ A1 \ A \ B0 \ B1 \ E \rangle$
 $\langle Saccheri \ A0 \ B0 \ B \ A \rangle \ t22-18-aux2$ by blast
 have $B0 \ E \ Le \ A \ A0$
 using Le-def $\langle Bet \ A0 \ D \ A \rangle \langle Cong \ B0 \ E \ A \ D \rangle$
 between-symmetry by blast
 hence $B0 \ B \ Le \ A \ A0$
 using le-transitivity $\langle B0 \ B \ Le \ B0 \ E \rangle$ by blast
 then obtain Q where $Bet \ A \ Q \ A0$ and $Cong \ B0 \ B \ A \ Q$
 using Le-def by blast
 have $A0 \ D \ Le \ A0 \ Q$
 proof –
 have $Bet \ A0 \ Q \ A$
 using Bet-cases $\langle Bet \ A \ Q \ A0 \rangle$ by blast
 moreover
 have $Q \ A \ Le \ D \ A$
 proof –
 have $Cong \ B0 \ B \ Q \ A$
 using Cong-cases $\langle Cong \ B0 \ B \ A \ Q \rangle$ by blast
 moreover
 have $Cong \ B0 \ E \ D \ A$
 using Cong-cases $\langle Cong \ B0 \ E \ A \ D \rangle$ by blast
 ultimately
 show ?thesis
 using $\langle B0 \ B \ Le \ B0 \ E \rangle \ l5-6$ by blast
 qed
 moreover
 have $A0 \ A \ Le \ A0 \ A$
 using between-trivial2 l5-12-a by auto
 ultimately
 show ?thesis
 using $\langle Bet \ A0 \ D \ A \rangle \ bet2-le2--le1245$ by blast
 qed
 moreover
 have $A \neq A0$
 using False $\langle Bet \ A0 \ D \ A \rangle \langle Grad \ A0 \ D1 \ D \rangle$ between-identity
 grad-neq--neg13 by blast
 then obtain $B0'$ where $A0 \ Out \ B0' \ A \wedge Cong \ A0 \ B0' \ A0 \ B0$
 using $\langle A0 \neq B0 \rangle \ l6-11-existence$ by presburger
 have $A0 \ Out \ B0' \ A$
 using $\langle A0 \ Out \ B0' \ A \wedge Cong \ A0 \ B0' \ A0 \ B0 \rangle$ by blast
 have $Cong \ A0 \ B0' \ A0 \ B0$
 using $\langle A0 \ Out \ B0' \ A \wedge Cong \ A0 \ B0' \ A0 \ B0 \rangle$ by blast
 obtain B' where $Bet \ A0 \ B0' \ B'$ and $Cong \ B0' \ B' \ B0 \ B$
 using segment-construction by blast
 obtain A' where $Bet \ B0' \ B' \ A'$ and $Cong \ B' \ A' \ B \ A$
 using segment-construction by blast
 have $A0 \ A \ Le \ A0 \ A'$
 proof –
 obtain B'' where $Bet \ A0 \ B0' \ B''$ and $Cong \ B0' \ B'' \ B0 \ A$
 using segment-construction by blast
 have $A0 \ A \ Le \ A0 \ B''$
 by (meson $\langle Bet \ A0 \ B0' \ B'' \rangle \langle Cong \ A0 \ B0' \ A0 \ B0 \rangle$
 $\langle Cong \ B0' \ B'' \ B0 \ A \rangle \ cong--le3412 \ l2-11-b \ lt--le \ not-cong-3412$
 triangle-strict-inequality-2)
 moreover

have $A0\ B''\ Le\ A0\ A'$
proof –
 have $B0' \neq B'$
 using *sac-distincts*
 using $\langle Cong\ B0'\ B'\ B0\ B \rangle\ \langle Saccheri\ A0\ B0\ B\ A \rangle$
 cong-reverse-identity **by** *blast*
 hence $Bet\ A0\ B0'\ A'$
 using $\langle Bet\ B0'\ B'\ A' \rangle\ \langle Bet\ A0\ B0'\ B' \rangle$ *outer-transitivity-between*
 by *blast*
 moreover
 have $A0\ B0'\ Le\ A0\ B0'$
 using *local.le-cases* **by** *auto*
 moreover
 have $B0'\ B''\ Le\ B0'\ A'$
 proof –
 have $B0\ A\ Le\ B0'\ A'$
 proof –
 have $Cong\ B0\ B\ B0'\ B'$
 using $\langle Cong\ B0'\ B'\ B0\ B \rangle$ *not-cong-3412* **by** *blast*
 moreover
 have $Cong\ B\ A\ B'\ A'$
 using $\langle Cong\ B'\ A'\ B\ A \rangle$ *cong-symmetry* **by** *presburger*
 ultimately
 show *?thesis*
 using $\langle Bet\ B0'\ B'\ A' \rangle$ *triangle-inequality-2* **by** *blast*
 qed
 moreover
 have $Cong\ B0\ A\ B0'\ B''$
 using $\langle Cong\ B0'\ B''\ B0\ A \rangle$ *cong-symmetry* **by** *blast*
 moreover
 have $Cong\ B0'\ A'\ B0'\ A'$
 by (*simp add: cong-reflexivity*)
 ultimately
 show *?thesis*
 using *l5-6* **by** *blast*
 qed
 ultimately
 show *?thesis*
 using $\langle Bet\ A0\ B0'\ B'' \rangle$ *bet2-le2--le1346* **by** *blast*
qed
ultimately
show *?thesis*
 using *le-transitivity* **by** *blast*
qed
have $B0\ B\ Le\ A'\ B0'$
proof –
 have $B0'\ B'\ Le\ B0'\ A'$
 by (*simp add: $\langle Bet\ B0'\ B'\ A' \rangle$ l5-12-a*)
 moreover
 have $Cong\ B0'\ A'\ A'\ B0'$
 by (*simp add: cong-pseudo-reflexivity*)
 ultimately
 show *?thesis*
 using $\langle Cong\ B0'\ B'\ B0\ B \rangle$ *l5-6* **by** *blast*
qed
obtain Q' **where** $Bet\ A'\ Q'\ B0'$ **and** $Cong\ B0\ B\ A'\ Q'$
 using *Le-def* $\langle B0\ B\ Le\ A'\ B0' \rangle$ **by** *blast*
have $B0' \neq B'$
 using *sac-distincts* $\langle Cong\ B0'\ B'\ B0\ B \rangle\ \langle Saccheri\ A0\ B0\ B\ A \rangle$
 cong-reverse-identity **by** *blast*
hence $Bet\ A0\ B0'\ A'$
 using $\langle Bet\ A0\ B0'\ B' \rangle\ \langle Bet\ B0'\ B'\ A' \rangle$
 outer-transitivity-between **by** *blast*
have $A0\ Q\ Lt\ A0\ C$
proof –
 have $A0\ Q\ Le\ A0\ Q'$

```

proof -
  have Bet A0 Q' A'
    using ⟨Bet A' Q' B0'⟩ ⟨Bet A0 B0' A'⟩ bet3--bet
      between-trivial by blast
  moreover
  have Bet A0 Q A
    using Bet-cases ⟨Bet A Q A0⟩ by auto
  moreover
  have Q' A' Le Q A
    by (meson le-reflexivity ⟨Cong B0 B A Q⟩
      ⟨Cong B0 B A' Q'⟩ l5-6 le-comm)
  ultimately
  show ?thesis
    using ⟨A0 A Le A0 A'⟩ bet2-le2--le1245 by blast
qed
moreover
have A0 Q' Lt A0 C
proof -
  have Bet A0 D1 C
    using ⟨Bet A0 C0 C⟩ ⟨Bet A0 D1 C0⟩ between-exchange4 by blast
  hence D1 C Lt A0 C
    by (simp add: False bet--lt2313)
  moreover
  have Cong D1 C A0 Q'
proof -
  have Bet D1 C0 C
    using ⟨Bet A0 C0 C⟩ ⟨Bet A0 D1 C0⟩ between-exchange3 by blast
  moreover
  have Bet A0 B0' Q'
    using ⟨Bet A' Q' B0'⟩ ⟨Bet A0 B0' A'⟩ between-exchange3
      between-symmetry by blast
  moreover
  have Cong D1 C0 A0 B0'
    by (metis Cong-cases ⟨Cong A0 B0' A0 B0⟩
      ⟨Cong D1 C0 A0 B0⟩ cong-inner-transitivity)
  moreover
  have Cong B0' Q' A B
proof -
  have Cong B0' B' A' Q'
    using ⟨Cong B0 B A' Q'⟩ ⟨Cong B0' B' B0 B⟩
      cong-transitivity by blast
  have Cong B0' Q' A' B'
proof (cases Bet B0' Q' B')
  case True
  hence Bet A' B' Q'
    using Bet-perm ⟨Bet B0' B' A'⟩ between-exchange3 by blast
  moreover
  have Cong Q' B' B' Q'
    by (simp add: cong-pseudo-reflexivity)
  ultimately
  show ?thesis
    using ⟨Cong B0' B' A' Q'⟩ True l4-3 by blast
next
  case False
  hence Q' ≠ B0'
    using between-trivial2 by blast
  hence A' ≠ B0'
    using ⟨Bet A' Q' B0'⟩ between-identity by blast
  have B0' Out B' Q'
proof -
  have B0' Out B' A'
    using ⟨B0' ≠ B'⟩ ⟨Bet B0' B' A'⟩ bet-out by force
  moreover
  have B0' Out A' Q'
    by (simp add: ⟨Bet A' Q' B0'⟩ ⟨Q' ≠ B0'⟩ bet-out-1 l6-6)
  ultimately

```



```

    show ?thesis
      using l6-7 by blast
  qed
  hence  $Bet\ B0'\ B'\ Q'$ 
    using False Out-def by blast
  moreover
  have  $Bet\ A'\ Q'\ B'$ 
    using  $\langle Bet\ A'\ Q'\ B0' \rangle$  between-exchange3
      between-symmetry calculation by blast
  moreover
  have  $Cong\ B'\ Q'\ Q'\ B'$ 
    by (simp add: cong-pseudo-reflexivity)
  ultimately
  show ?thesis
    using  $\langle Cong\ B0'\ B'\ A'\ Q' \rangle$  l2-11-b by blast
  qed
  moreover
  have  $Cong\ A'\ B'\ A\ B$ 
    using  $\langle Cong\ B'\ A'\ B\ A \rangle$  not-cong-2143 by blast
  ultimately
  show ?thesis
    using cong-transitivity by blast
  qed
  hence  $Cong\ B0'\ Q'\ A0\ B0$ 
    using  $\langle Cong\ A\ B\ A0\ B0 \rangle$  cong-transitivity by blast
  hence  $Cong\ C0\ C\ B0'\ Q'$ 
    using cong-inner-transitivity  $\langle Cong\ C0\ C\ A0\ B0 \rangle$  cong-symmetry by blast
  ultimately
  show ?thesis
    using l2-11-b by blast
  qed
  moreover
  have  $Cong\ A0\ C\ A0\ C$ 
    by (simp add: cong-reflexivity)
  ultimately
  show ?thesis
    using cong2-lt--lt by blast
  qed
  ultimately
  show ?thesis
    by (meson le1234-lt--lt)
  qed
  ultimately
  show ?thesis
    using le1234-lt--lt by blast
  qed
}
hence  $\forall D. Grad\ A0\ D1\ D \longrightarrow A0\ D\ Lt\ A0\ C$ 
  by blast
have  $\neg Cong\ B0\ B1\ A1\ A0$ 
  using  $\langle B0\ B1\ Lt\ A1\ A0 \rangle$  cong--nlt lt--le by blast
have  $A0 \neq D1 \longrightarrow Reach\ A0\ D1\ A0\ C$ 
  using archimedes-axiom-def assms by blast
then obtain  $D$  where  $Grad\ A0\ D1\ D \wedge A0\ C\ Le\ A0\ D$ 
  using Reach-def  $\langle Cong\ B0\ B1\ A1\ D1 \rangle \neg Cong\ B0\ B1\ A1\ A0$  by blast
hence  $A0\ D\ Lt\ A0\ C$ 
  by (simp add:  $\langle \forall D. Grad\ A0\ D1\ D \longrightarrow A0\ D\ Lt\ A0\ C \rangle$ )
moreover
have  $A0\ C\ Lt\ A0\ C$ 
  using  $\langle Grad\ A0\ D1\ D \wedge A0\ C\ Le\ A0\ D \rangle$  calculation lt--nle by blast
ultimately
have False
  using nlt by auto
}
thus ?thesis
  by blast

```

qed

lemma *t22-19*:

assumes *archimedes-axiom*

shows $\forall A B C D. \text{Saccheri } A B C D \longrightarrow \neg \text{Obtuse } A B C$

proof –

```
{
  fix A B C D
  assume Saccheri A B C D
  hence  $\neg C B \text{ Lt } A D$ 
    using assms lt-comm t22-18 by blast
  moreover
  assume Obtuse A B C
  hence  $C B \text{ Lt } A D$ 
    using  $\langle \text{Saccheri } A B C D \rangle$  lt-left-comm lt-sac--obtuse-aux2 by blast
  ultimately
  have False
    by blast
}
```

thus ?thesis

by blast

qed

lemma *archi--obtuse-case-elimination*:

assumes *archimedes-axiom*

shows $\neg \text{hypothesis-of-obtuse-saccheri-quadrilaterals}$

proof –

have $\neg (\forall A B C D. \text{Saccheri } A B C D \longrightarrow \text{Obtuse } A B C)$

using assms ex-saccheri t22-19 by blast

thus ?thesis

using hypothesis-of-obtuse-saccheri-quadrilaterals-def by blast

qed

lemma *t22-23-aux*:

assumes $\neg \text{Col } A M N$ and

Per B C A and

$A \neq C$ and

M Midpoint A B and

Per M N A and

Col A C N and

M Midpoint N L

shows $\text{Bet } A N C \wedge \text{Lambert } N L B C \wedge \text{Cong } B L A N$

proof –

have $A \neq M$

using assms(1) col-trivial-1 by blast

have $N \neq M$

using assms(1) col-trivial-2 by blast

have $A \neq N$

using assms(1) col-trivial-3 by blast

have $L \neq N$

using $\langle N \neq M \rangle$ assms(7) l7-3 by blast

have $A \neq B$

using $\langle A \neq M \rangle$ assms(4) l7-3 by blast

hence $B \neq M$

using assms(4) is-midpoint-id-2 by blast

have $\text{Bet } A N C$

proof –

have $\text{Bet } A M B$

by (simp add: assms(4) midpoint-bet)

moreover

have $\text{Col } A N C$

using assms(6) not-col-permutation-5 by blast

moreover

have $\text{Per } A N M$

by (simp add: assms(5) l8-2)

moreover

```

have Per A C B
  by (simp add: assms(2) l8-2)
ultimately
show ?thesis
  by (metis per23-preserves-bet  $\langle A \neq N \rangle$  assms(3))
qed
moreover
have A M N CongA B M L
  using  $\langle A \neq M \rangle \langle N \neq M \rangle$  assms(4) assms(7) l7-3-2
  symmetry-preserves-conga by blast
hence Cong A N B L  $\wedge$  M A N CongA M B L  $\wedge$  M N A CongA M L B
  using per23-preserves-bet Cong-cases l11-49  $\langle A \neq N \rangle$  assms(4)
  assms(7) midpoint-cong by meson
have Lambert N L B C
proof -
{
  assume C = N
  have M = B
  proof -
    have  $\neg$  Col A M N
      by (simp add: assms(1))
    moreover
    have  $N \neq M$ 
      by (simp add:  $\langle N \neq M \rangle$ )
    moreover
    have Col A M M
      using not-col-distincts by blast
    moreover
    have Bet A M B
      by (simp add: assms(4) midpoint-bet)
    hence Col A M B
      by (simp add: Col-def)
    moreover
    have Col N M M
      by (simp add: col-trivial-2)
    moreover
    have Coplanar A B M N
      by (meson  $\langle A \neq M \rangle$  calculation(3) calculation(4)
        col--coplanar col-transitivity-1)
    hence Col N M B
      using  $\langle C = N \rangle \langle A \neq N \rangle$  assms(2) assms(5) col-permutation-3
      cop-per2--col by blast
    moreover
    show ?thesis
      using calculation(1) calculation(4) calculation(6) l6-16-1 by blast
  qed
  hence False
    using  $\langle B \neq M \rangle$  by auto
}
hence C  $\neq$  N
  by blast
moreover
have L  $\neq$  B
  using  $\langle A \neq N \rangle \langle Cong A N B L \wedge M A N CongA M B L \wedge M N A CongA M L B \rangle$ 
  cong-diff by blast
moreover
have B  $\neq$  C
  using  $\langle A \neq B \rangle$  assms(1) assms(4) assms(6) col-permutation-1
  l6-16-1 midpoint-col by blast
moreover
have N  $\neq$  L
  using  $\langle L \neq N \rangle$  by auto
moreover
have Per L N C
  by (metis per-col  $\langle A \neq N \rangle \langle N \neq M \rangle$  assms(5) assms(6) assms(7)
    col-per2--per l8-20-1-R1 midpoint-col not-col-permutation-1)

```

```

moreover
have  $Per\ N\ C\ B$ 
  by (metis per-col assms(2) assms(3) assms(6) col-permutation-4 l8-2)
moreover
have  $Per\ N\ L\ B$ 
proof –
  have  $Per\ M\ N\ A$ 
    by (simp add: assms(5))
  moreover
have  $M\ N\ A\ CongA\ N\ L\ B$ 
proof –
  have  $M\ N\ A\ CongA\ M\ L\ B$ 
    by (simp add:  $\langle Cong\ A\ N\ B\ L \wedge M\ A\ N\ CongA\ M\ B\ L \wedge M\ N\ A\ CongA\ M\ L\ B \rangle$ )
  moreover
have  $N\ Out\ M\ M$ 
    using  $\langle N \neq M \rangle$  out-trivial by auto
  moreover
have  $N\ Out\ A\ A$ 
    using  $\langle A \neq N \rangle$   $\langle Bet\ A\ N\ C \rangle$   $\langle C \neq N \rangle$  l6-3-2 by blast
  moreover
have  $M \neq L$ 
    using  $\langle N \neq L \rangle$  assms(7) is-midpoint-id-2 by blast
have  $Bet\ L\ M\ N$ 
    using assms(7) Bet-perm Midpoint-def by blast
hence  $L\ Out\ N\ M$ 
    by (simp add: Out-def  $\langle M \neq L \rangle$   $\langle N \neq L \rangle$ )
  moreover
have  $L\ Out\ B\ B$ 
    using  $\langle L \neq B \rangle$  out-trivial by auto
  ultimately
show ?thesis
    using l11-10 by blast
qed
ultimately
show ?thesis
    using l11-17 by blast
qed
moreover
have  $Coplanar\ N\ L\ B\ C$ 
proof –
  have  $Coplanar\ B\ C\ N\ M$ 
proof –
  have  $Coplanar\ C\ N\ A\ B$ 
    by (simp add:  $\langle Bet\ A\ N\ C \rangle$  bet--coplanar between-symmetry)
  moreover
have  $Bet\ A\ M\ B$ 
    by (simp add: assms(4) midpoint-bet)
hence  $Col\ B\ A\ M$ 
    using Col-def by auto
  ultimately
show ?thesis
    using  $\langle A \neq B \rangle$  col-cop--cop coplanar-perm-14
      coplanar-perm-7 by blast
qed
moreover
have  $Bet\ N\ M\ L$ 
    using assms(7) midpoint-bet by blast
hence  $Col\ N\ M\ L$ 
    by (simp add: Col-def)
  ultimately
show ?thesis
    using  $\langle N \neq M \rangle$  col-cop--cop ncoplanar-perm-16 by blast
qed
ultimately
show ?thesis
    using Lambert-def by blast

```

```

qed
moreover
have  $\text{Cong } A \ N \ B \ L$ 
  using  $\langle \text{Cong } A \ N \ B \ L \wedge M \ A \ N \ \text{Cong} \ A \ M \ B \ L \wedge M \ N \ A \ \text{Cong} \ A \ M \ L \ B \rangle$  by blast
hence  $\text{Cong } B \ L \ A \ N$ 
  by (simp add: cong-symmetry)
ultimately
show ?thesis
  by simp
qed

lemma t22-23:
  assumes  $\neg \text{hypothesis-of-obtuse-saccheri-quadrilaterals}$ 
  shows  $\forall \ A \ B \ C \ M \ N \ L.$ 
 $\neg \text{Col } A \ M \ N \wedge \text{Per } B \ C \ A \wedge A \neq C \wedge M \ \text{Midpoint } A \ B \wedge$ 
 $\text{Per } M \ N \ A \wedge \text{Col } A \ C \ N \wedge M \ \text{Midpoint } N \ L \longrightarrow$ 
 $(\text{Bet } A \ N \ C \wedge N \ C \ \text{Le } A \ N \wedge L \ N \ \text{Le } B \ C)$ 
proof -
{
  fix  $A \ B \ C \ M \ N \ L$ 
  assume  $\neg \text{Col } A \ M \ N$  and
     $\text{Per } B \ C \ A$  and
     $A \neq C$  and
     $M \ \text{Midpoint } A \ B$  and
     $\text{Per } M \ N \ A$  and
     $\text{Col } A \ C \ N$  and
     $M \ \text{Midpoint } N \ L$ 
  hence  $\text{Bet } A \ N \ C \wedge \text{Lambert } N \ L \ B \ C \wedge \text{Cong } B \ L \ A \ N$ 
    using t22-23-aux by blast
  have  $\text{Bet } A \ N \ C$ 
    using  $\langle \text{Bet } A \ N \ C \wedge \text{Lambert } N \ L \ B \ C \wedge \text{Cong } B \ L \ A \ N \rangle$  by blast
  moreover
  have  $\text{Lambert } N \ L \ B \ C$ 
    using  $\langle \text{Bet } A \ N \ C \wedge \text{Lambert } N \ L \ B \ C \wedge \text{Cong } B \ L \ A \ N \rangle$  by blast
  have  $\text{Cong } B \ L \ A \ N$ 
    using  $\langle \text{Bet } A \ N \ C \wedge \text{Lambert } N \ L \ B \ C \wedge \text{Cong } B \ L \ A \ N \rangle$  by blast
  have  $\neg \text{Obtuse } L \ B \ C$ 
    using  $\langle \text{Bet } A \ N \ C \wedge \text{Lambert } N \ L \ B \ C \wedge \text{Cong } B \ L \ A \ N \rangle$  assms
    lam-obtuse--oah-1 by blast
  have  $N \ L \ \text{OS } B \ C$ 
    using  $\langle \text{Lambert } N \ L \ B \ C \rangle$  by (simp add: lam--os)
  have  $\text{Lambert } N \ C \ B \ L$ 
    by (simp add: lam-perm  $\langle \text{Lambert } N \ L \ B \ C \rangle$ )
  hence  $N \ C \ \text{OS } B \ L$ 
    by (simp add: lam--os)
  have  $N \ C \ \text{Le } A \ N \wedge L \ N \ \text{Le } B \ C$ 
  proof (cases  $\text{Acute } L \ B \ C$ )
  case True
  have  $N \ C \ \text{Lt } A \ N$ 
  proof -
    have  $N \ C \ \text{Lt } B \ L$ 
      by (simp add: acute-lam--lt True
         $\langle \text{Bet } A \ N \ C \wedge \text{Lambert } N \ L \ B \ C \wedge \text{Cong } B \ L \ A \ N \rangle$  lt-right-comm)
    moreover
    have  $\text{Cong } N \ C \ N \ C$ 
      by (simp add: cong-reflexivity)
    moreover
    have  $\text{Cong } B \ L \ A \ N$ 
      by (simp add:  $\langle \text{Cong } B \ L \ A \ N \rangle$ )
    ultimately
    show ?thesis
      using cong2-lt--lt by blast
  qed
  moreover
  have  $N \ L \ \text{Lt } B \ C$ 
  proof -

```

```

have  $Per\ L\ N\ C$ 
  by (metis  $\langle Col\text{-}def\ \langle Bet\ A\ N\ C \rangle\ \langle M\ Midpoint\ N\ L \rangle$ 
     $\langle Per\ M\ N\ A \rangle\ \langle \neg\ Col\ A\ M\ N \rangle$  between-trivial col-per2--per
    l8-2 l8-5 midpoint-col)
moreover
have  $Per\ N\ C\ B$ 
  by (metis  $\langle A \neq C \rangle\ \langle Col\ A\ C\ N \rangle\ \langle Per\ B\ C\ A \rangle$ 
    col-permutation-4 l8-2 per-col)
moreover
have  $N\ C\ OS\ L\ B$ 
  by (simp add:  $\langle N\ C\ OS\ B\ L \rangle$  one-side-symmetry)
moreover
have  $L\ B\ C\ LtA\ N\ L\ B$ 
proof –
  have  $Acute\ L\ B\ C$ 
    by (simp add: True)
  moreover
  have  $Per\ N\ L\ B$ 
    using  $\langle Lambert\ N\ L\ B\ C \rangle$  Lambert-def by blast
  ultimately
  show ?thesis
    by (metis  $\langle Bet\ A\ N\ C \wedge Lambert\ N\ L\ B\ C \wedge Cong\ B\ L\ A\ N \rangle$ 
      acute-per--lta lam--os os-distincts)
qed
ultimately
show ?thesis
  by (simp add: lta-os-per2--lt)
qed
hence  $L\ N\ Lt\ B\ C$ 
  by (meson lt-left-comm)
ultimately
show ?thesis
  by (simp add: lt--le)
next
case False
hence  $Per\ L\ B\ C$ 
  by (metis  $\langle N\ C\ OS\ B\ L \rangle\ \langle N\ L\ OS\ B\ C \rangle\ \langle \neg\ Obtuse\ L\ B\ C \rangle$ 
    angle-partition os-distincts)
have  $Cong\ N\ C\ B\ L$ 
  using  $\langle Lambert\ N\ L\ B\ C \rangle\ \langle Per\ L\ B\ C \rangle$  cong-right-commutativity
  lam-per--cong by blast
hence  $Cong\ N\ C\ A\ N$ 
  using  $\langle Cong\ B\ L\ A\ N \rangle$  cong-transitivity by blast
moreover
have  $Cong\ L\ N\ B\ C$ 
  using  $\langle Lambert\ N\ C\ B\ L \rangle\ \langle Per\ L\ B\ C \rangle$  cong-commutativity
  l8-2 lam-per--cong by blast
ultimately
show ?thesis
  by (simp add: cong--le)
qed
ultimately
have  $Bet\ A\ N\ C \wedge N\ C\ Le\ A\ N \wedge L\ N\ Le\ B\ C$ 
  by blast
}
thus ?thesis
  by blast
qed

lemma t22-24-aux-0-a:
assumes
   $\neg\ Col\ A\ B0\ C0$  and
   $A\ C0\ Perp\ B0\ C0$  and
   $B = GradExpn\ A\ B0\ (Suc\ 0)$  and
   $E = GradExpn\ B0\ C0\ (Suc\ 0)$  and
   $A\ C0\ Perp\ B\ C$  and

```

```

  Col A C0 C
shows B0 E Le B C
proof -
  have A ≠ B0
    using assms(1) col-trivial-1 by auto
  hence B = B0
    using assms(3) by simp
  have B0 ≠ C0
    using assms(1) col-trivial-2 by fastforce
  hence C0 = E
    using assms(4) by simp
  thus ?thesis
    using ⟨B = B0⟩ assms(2) assms(6) assms(5) col-trivial-2
      l8-18-uniqueness le-reflexivity by blast
qed

```

lemma *t22-24-aux-0*:

```

shows ∀ A B0 C0 B C E.
  ¬ Col A B0 C0 ∧ A C0 Perp B0 C0 ∧
  (B = GradExpn A B0 (Suc 0)) ∧ (E = GradExpn B0 C0 (Suc 0)) ∧
  A C0 Perp B C ∧ Col A C0 C ⟶
  B0 E Le B C
using t22-24-aux-0-a by blast

```

lemma *t22-24-aux-suc*:

```

assumes ¬ hypothesis-of-obtuse-saccheri-quadrilaterals and
  ∀ A B0 C0 B C E.
  ¬ Col A B0 C0 ∧ A C0 Perp B0 C0 ∧
  (B = GradExpn A B0 (Suc n)) ∧ (E = GradExpn B0 C0 (Suc n)) ∧
  A C0 Perp B C ∧ Col A C0 C ⟶
  B0 E Le B C
shows
  ∀ A B0 C0 B C E.
  ¬ Col A B0 C0 ∧ A C0 Perp B0 C0 ∧
  (B = GradExpn A B0 (Suc(Suc n))) ∧ (E = GradExpn B0 C0 (Suc(Suc n))) ∧
  A C0 Perp B C ∧ Col A C0 C ⟶
  B0 E Le B C

```

proof -

```

{
  fix A B0 C0 B' C' E'
  assume ¬ Col A B0 C0 and
    A C0 Perp B0 C0 and
    B' = GradExpn A B0 (Suc(Suc n)) and
    E' = GradExpn B0 C0 (Suc(Suc n)) and
    A C0 Perp B' C' and
    Col A C0 C'
  let ?B = GradExpn A B0 (Suc n)
  have Cong A ?B ?B B'
    using GradExpn-4 ⟨B' = GradExpn A B0 (Suc (Suc n))⟩ by presburger
  have Bet A ?B B'
    using GradExpn-3 ⟨B' = GradExpn A B0 (Suc (Suc n))⟩ by blast
  have Bet A B0 ?B
    using GradExpn-2 by auto
  let ?E = GradExpn B0 C0 (Suc n)
  have Cong B0 ?E ?E E'
    using GradExpn-4 ⟨E' = GradExpn B0 C0 (Suc (Suc n))⟩ by presburger
  have Bet B0 ?E E'
    using GradExpn-3 ⟨E' = GradExpn B0 C0 (Suc (Suc n))⟩ by blast
  have ¬ Col A C0 ?B
    by (metis ⟨Bet A B0 (GradExpn A B0 (Suc n))⟩ ⟨¬ Col A B0 C0⟩
      bet-col bet-neq12--neq colx not-col-distincts not-col-permutation-5)
  then obtain C where Col A C0 C and A C0 Perp ?B C
    using l8-18-existence by blast
  have B0 ?E Le ?B C
    using ⟨A C0 Perp B0 C0⟩ ⟨A C0 Perp (GradExpn A B0 (Suc n)) C⟩

```

```

  ⟨Col A C0 C⟩ ⟨¬ Col A B0 C0⟩ assms(2) by blast
obtain D where ?B Midpoint C D
  using symmetric-point-construction by blast
have B0 E' Le D C
proof –
  have Bet B0 ?E E'
    using ⟨Bet B0 (GradExpn B0 C0 (Suc n)) E'⟩ by blast
  moreover
  have Bet D ?B C
    by (metis midpoint-bet ⟨GradExpn A B0 (Suc n) Midpoint C D⟩
      between-symmetry)
  moreover
  have B0 ?E Le D ?B
    by (meson ⟨B0 (GradExpn B0 C0 (Suc n)) Le (GradExpn A B0 (Suc n)) C⟩
      ⟨GradExpn A B0 (Suc n) Midpoint C D⟩ cong-pseudo-reflexivity
      l5-6 le-right-comm midpoint-cong)
  moreover
  have ?E E' Le ?B C
    by (meson ⟨Cong B0 (GradExpn B0 C0 (Suc n)) (GradExpn B0 C0 (Suc n)) E'⟩
      ⟨GradExpn A B0 (Suc n) Midpoint C D⟩ calculation(3)
      l5-6 l7-2 midpoint-cong)
  ultimately
  show ?thesis
    using bet2-le2--le1346 by blast
qed
moreover
{
  assume A = C
  have A = C0
  proof –
    have ¬ Col A C0 B0
      using ⟨¬ Col A B0 C0⟩ col-permutation-5 by blast
    moreover
    have Col A C0 A
      using col-trivial-3 by blast
    moreover
    have A C0 Perp B0 A
    proof –
      have A C0 Perp A ?B
        using ⟨A = C⟩ ⟨A C0 Perp (GradExpn A B0 (Suc n)) C⟩
          perp-right-comm by blast
      moreover
      have Col A ?B B0
        using ⟨Bet A B0 (GradExpn A B0 (Suc n))⟩ bet-col
          not-col-permutation-5 by blast
      ultimately
      show ?thesis
        using ⟨¬ Col A C0 B0⟩ not-col-distincts perp-col2-bis by blast
    qed
    moreover
    have Col A C0 C0
      by (simp add: col-trivial-2)
    ultimately
    show ?thesis
      using ⟨A C0 Perp B0 C0⟩ l8-18-uniqueness by blast
  qed
  hence False
    using ⟨¬ Col A B0 C0⟩ col-trivial-3 by blast
}
hence A ≠ C
  by auto
{
  assume A = C'
  have A = C0
  proof –
    have ¬ Col A C0 B0

```



```

    using  $\langle \neg \text{Col } A \ B0 \ C0 \rangle$  col-permutation-5 by blast
  moreover
  have  $\text{Col } A \ C0 \ A$ 
    using col-trivial-3 by blast
  moreover
  have  $A \ C0 \ \text{Perp } B0 \ A$ 
  proof -
    have  $A \ C0 \ \text{Perp } A \ B'$ 
      using  $\langle A = C' \rangle \langle A \ C0 \ \text{Perp } B' \ C' \rangle$  perp-right-comm by blast
    moreover
    have  $\text{Bet } A \ B0 \ B'$ 
      using  $\langle B' = \text{GradExpn } A \ B0 \ (\text{Suc}(\text{Suc } n)) \rangle$  GradExpn-2 by blast
    hence  $\text{Col } A \ B' \ B0$ 
      by (simp add: bet-col col-permutation-5)
    ultimately
    show ?thesis
      using  $\langle \neg \text{Col } A \ C0 \ B0 \rangle$  not-col-distincts perp-col2-bis by blast
  qed
  moreover
  have  $\text{Col } A \ C0 \ C0$ 
    by (simp add: col-trivial-2)
  ultimately
  show ?thesis
    using  $\langle A \ C0 \ \text{Perp } B0 \ C0 \rangle$  l8-18-uniqueness by blast
  qed
  hence False
    using  $\langle \neg \text{Col } A \ B0 \ C0 \rangle$  col-trivial-3 by blast
}
hence  $A \neq C'$ 
  by blast
have  $\text{Per } A \ C \ ?B$ 
  using  $\langle \text{Col } A \ C0 \ C \rangle \langle A \ C0 \ \text{Perp } (\text{GradExpn } A \ B0 \ (\text{Suc } n)) \ C \rangle$ 
    col-trivial-3 l8-16-1 l8-2 by blast
have  $D \ C \ \text{Le } B' \ C'$ 
  proof -
    have  $\neg \text{Col } A \ ?B \ C$ 
      using NCol-perm  $\langle A \neq C \rangle \langle \text{Col } A \ C0 \ C \rangle \langle \neg \text{Col } A \ C0 \ (\text{GradExpn } A \ B0 \ (\text{Suc } n)) \rangle$ 
        col-transitivity-1 by blast
    moreover
    have  $C' \ B' \ \text{Perp } A \ C0$ 
      using Perp-perm  $\langle A \ C0 \ \text{Perp } B' \ C' \rangle$  by blast
    hence  $C' \ B' \ \text{Perp } A \ C'$ 
      using  $\langle A \neq C' \rangle \langle \text{Col } A \ C0 \ C' \rangle$  perp-col1 by blast
    hence  $B' \ C' \ \text{Perp } C' \ A$ 
      using Perp-cases by blast
    hence  $\text{Per } B' \ C' \ A$ 
      using  $\langle C' \ B' \ \text{Perp } A \ C' \rangle$  perp-per-1 by blast
    moreover
    have  $?B \ \text{Midpoint } A \ B'$ 
      using  $\langle \text{Bet } A \ (\text{GradExpn } A \ B0 \ (\text{Suc } n)) \ B' \rangle$ 
         $\langle \text{Cong } A \ (\text{GradExpn } A \ B0 \ (\text{Suc } n)) \ (\text{GradExpn } A \ B0 \ (\text{Suc } n)) \ B' \rangle$ 
        midpoint-def by blast
    moreover
    have  $\text{Col } A \ C' \ C$ 
      by (metis  $\langle \text{Col } A \ C0 \ C' \rangle \langle \text{Col } A \ C0 \ C \rangle \langle \neg \text{Col } A \ B0 \ C0 \rangle$ 
        col-transitivity-1 not-col-distincts)
    ultimately
    show ?thesis using t22-23
      using  $\langle A \neq C' \rangle \langle \text{GradExpn } A \ B0 \ (\text{Suc } n) \ \text{Midpoint } C \ D \rangle$ 
         $\langle \text{Per } A \ C \ (\text{GradExpn } A \ B0 \ (\text{Suc } n)) \rangle$  assms(1) l8-2 by blast
  qed
  ultimately
  have  $B0 \ E' \ \text{Le } B' \ C'$ 
    using le-transitivity by blast
}
thus ?thesis

```

by blast
qed

lemma t22-24-aux-n:

assumes $\neg$ hypothesis-of-obtuse-saccheri-quadrilaterals

shows $\forall A B0 C0 B C E.$

$(\neg Col A B0 C0 \wedge A C0 Perp B0 C0 \wedge$
 $(B = GradExpn A B0 (Suc n)) \wedge (E = GradExpn B0 C0 (Suc n)) \wedge$
 $A C0 Perp B C \wedge Col A C0 C) \longrightarrow$
 $B0 E Le B C$

proof (induction n)

case 0

thus ?case

using t22-24-aux-0 by fastforce

next

case (Suc n)

thus ?case

using assms t22-24-aux-suc by presburger

qed

lemma t22-24-aux:

assumes $\neg$ hypothesis-of-obtuse-saccheri-quadrilaterals

shows $\forall A B0 B00 C0 B C E.$

$\neg Col A B0 C0 \wedge A C0 Perp B0 C0 \wedge B0 = B00 \wedge$
 $GradExp2 A B0 B B00 C0 E \wedge A C0 Perp B C \wedge Col A C0 C \longrightarrow$
 $B0 E Le B C$

proof -

{

fix A B0 B00 C0 B C E

assume $\neg Col A B0 C0$ and

A C0 Perp B0 C0 and

B0 = B00 and

GradExp2 A B0 B B00 C0 E and

A C0 Perp B C and

Col A C0 C

obtain n where $n \neq 0 \wedge (B = GradExpn A B0 n) \wedge (E = GradExpn B00 C0 n)$

using GradExp2-def $\langle GradExp2 A B0 B B00 C0 E \rangle$ by presburger

hence $n \neq 0$

by blast

then obtain m where $n = Suc m$

using not0-implies-Suc by blast

have $B = GradExpn A B0 (Suc m)$

by (simp add: $\langle n = Suc m \rangle \langle n \neq 0 \wedge B = GradExpn A B0 n \wedge E = GradExpn B00 C0 n \rangle$)

moreover

have $E = GradExpn B0 C0 (Suc m)$

by (simp add: $\langle B0 = B00 \rangle \langle n = Suc m \rangle$

$\langle n \neq 0 \wedge B = GradExpn A B0 n \wedge E = GradExpn B00 C0 n \rangle$)

ultimately

have $B0 E Le B C$

using $\langle A C0 Perp B C \rangle \langle A C0 Perp B0 C0 \rangle \langle Col A C0 C \rangle$

$\langle \neg Col A B0 C0 \rangle$ assms t22-24-aux-n by blast

}

thus ?thesis by blast

qed

lemma t22-24-aux1-0:

shows $\forall A B0 C0 E.$

$(\neg Col A B0 C0 \wedge A C0 Perp B0 C0 \wedge$

$E = GradExpn B0 C0 (Suc 0)) \longrightarrow$

$(\exists B C. GradExp2 A B0 B B0 C0 E \wedge A C0 Perp B C \wedge Col A C0 C)$

proof -

{

fix A B0 C0 E

assume $\neg Col A B0 C0$ and

```

  A C0 Perp B0 C0 and
  E = GradExpn B0 C0 (Suc 0)
have B0 ≠ C0
  using ⟨¬ Col A B0 C0⟩ not-col-distincts by blast
hence E = C0
  using ⟨E = GradExpn B0 C0 (Suc 0)⟩ by simp
have A ≠ B0
  using ⟨¬ Col A B0 C0⟩ col-trivial-1 by blast
hence B0 = GradExpn A B0 (Suc 0)
  by simp
have GradExp2 A B0 B0 B0 C0 E
  using ⟨E = C0⟩ gradexp2-init by auto
moreover
have Col A C0 C0
  by (simp add: col-trivial-2)
ultimately
have ∃ B C. GradExp2 A B0 B B0 C0 E ∧ A C0 Perp B C ∧ Col A C0 C
  using ⟨A C0 Perp B0 C0⟩ by blast
}
thus ?thesis
  by blast
qed

```

lemma t22-24-aux1-suc:

```

assumes ∀ A B0 C0 E.
  (¬ Col A B0 C0 ∧ A C0 Perp B0 C0 ∧
  E = GradExpn B0 C0 (Suc(n))) ⟶
  (∃ B C. GradExp2 A B0 B B0 C0 E ∧ A C0 Perp B C ∧ Col A C0 C)
shows ∀ A B0 C0 E.
  (¬ Col A B0 C0 ∧ A C0 Perp B0 C0 ∧
  E = GradExpn B0 C0 (Suc(Suc(n)))) ⟶
  (∃ B C. GradExp2 A B0 B B0 C0 E ∧ A C0 Perp B C ∧ Col A C0 C)
proof -
{
  fix A B0 C0 E'
  assume ¬ Col A B0 C0 and
    A C0 Perp B0 C0 and
    E' = GradExpn B0 C0 (Suc(Suc(n)))
  let ?E = GradExpn B0 C0 (Suc n)
  have Bet B0 ?E E'
    using GradExpn-3 ⟨E' = GradExpn B0 C0 (Suc (Suc n))⟩ by blast
  have Cong B0 ?E ?E E'
    using GradExpn-4 ⟨E' = GradExpn B0 C0 (Suc (Suc n))⟩ by blast
  obtain B C where GradExp2 A B0 B B0 C0 ?E ∧ A C0 Perp B C ∧ Col A C0 C
    using ⟨A C0 Perp B0 C0⟩ ⟨¬ Col A B0 C0⟩ assms by presburger
  have GradExp2 A B0 B B0 C0 ?E
    using ⟨GradExp2 A B0 B B0 C0 ?E ∧ A C0 Perp B C ∧ Col A C0 C⟩ by blast
  hence GradExp A B0 B
    by (simp add: gradexp2--gradexp123)
  hence Grad A B0 B
    by (simp add: gradexp--grad)
  hence Bet A B0 B
    by (simp add: grad--bet)
  have A C0 Perp B C
    using ⟨GradExp2 A B0 B B0 C0 ?E ∧ A C0 Perp B C ∧ Col A C0 C⟩ by blast
  have Col A C0 C
    using ⟨GradExp2 A B0 B B0 C0 ?E ∧ A C0 Perp B C ∧ Col A C0 C⟩ by blast
  obtain B' where Bet A B B' ∧ Cong B B' A B
    using segment-construction by presburger
  have Bet A B B'
    using ⟨Bet A B B' ∧ Cong B B' A B⟩ by blast
  have Cong B B' A B
    using ⟨Bet A B B' ∧ Cong B B' A B⟩ by blast
  have ¬ Col A C0 B'
    by (metis ⟨Bet A B0 B⟩ ⟨Bet A B B'⟩ ⟨¬ Col A B0 C0⟩
      bet-col bet-neq12--neq col-trivial-3 colx not-col-permutation-5)
}

```

```

then obtain C' where Col A C0 C'  $\wedge$  A C0 Perp B' C'
  using l8-18-existence by blast
moreover
have GradExp2 A B0 B' B0 C0 E'
proof -
  have GradExp2 A B0 B B0 C0 ?E
    using ⟨GradExp2 A B0 B B0 C0 (GradExpn B0 C0 (Suc n))⟩ by blast
  moreover
  have Bet A B B'
    by (simp add: ⟨Bet A B B'  $\wedge$  Cong B B' A B⟩)
  moreover
  have Cong A B B'
    using Cong-cases ⟨Cong B B' A B⟩ by blast
  moreover
  have Bet B0 ?E E'
    using ⟨Bet B0 (GradExpn B0 C0 (Suc n)) E'⟩ by auto
  moreover
  have Cong B0 ?E ?E E'
    using ⟨Cong B0 (GradExpn B0 C0 (Suc n)) (GradExpn B0 C0 (Suc n)) E'⟩
      by auto
  ultimately
  show ?thesis
    using GradExp2-stab by blast
qed
ultimately
have  $\exists B C. \text{GradExp2 } A B0 B B0 C0 E' \wedge A C0 \text{Perp } B C \wedge \text{Col } A C0 C$ 
  by blast
}
thus ?thesis
  by blast
qed

```

```

lemma t22-24-aux1-n:
shows  $\forall A B0 C0 E. (\neg \text{Col } A B0 C0 \wedge A C0 \text{Perp } B0 C0 \wedge E = \text{GradExpn } B0 C0 (\text{Suc}(n))) \longrightarrow (\exists B C. \text{GradExp2 } A B0 B B0 C0 E \wedge A C0 \text{Perp } B C \wedge \text{Col } A C0 C)$ 
proof (induction n)
case 0
thus ?case using t22-24-aux1-0 by blast
next
case (Suc n)
thus ?case using t22-24-aux1-suc by presburger
qed

```

```

lemma t22-24-aux1:
assumes  $\neg \text{Col } A B0 C0$  and
  A C0 Perp B0 C0 and
  GradExp B0 C0 E
shows  $\exists B C. \text{GradExp2 } A B0 B B0 C0 E \wedge A C0 \text{Perp } B C \wedge \text{Col } A C0 C$ 
proof -
obtain n where  $n \neq 0 \wedge E = \text{GradExpn } B0 C0 n$ 
  using GradExp-def assms(3) by presburger
hence  $n \neq 0$ 
  by simp
then obtain m where  $n = \text{Suc } m$ 
  using not0-implies-Suc by presburger
thus ?thesis
  using t22-24-aux1-n assms(1) assms(2)  $\langle n \neq 0 \wedge E = \text{GradExpn } B0 C0 n \rangle$  by blast
qed

```

```

lemma t22-24:
assumes archimedes-axiom
shows aristotle-s-axiom
proof -

```

```

{
  fix P Q D A B0
  assume  $\neg$  Col D A B0 and
    Acute D A B0
  obtain C0 where Col A D C0 and A D Perp B0 C0
  using Col-cases  $\langle \neg$  Col D A B0  $\rangle$  l8-18-existence by blast
  hence A  $\neq$  C0
  using  $\langle$  Acute D A B0  $\rangle$  acute-col-perp--out acute-sym
    col-trivial-3 l6-3-1 by blast
{
  assume Col A B0 C0
  hence Col C0 A B0
  using Col-cases by blast
  hence Col D A B0
  by (metis  $\langle$  A D Perp B0 C0  $\rangle$   $\langle$  Acute D A B0  $\rangle$ 
    acute-col-perp--out acute-sym col-trivial-3 l6-3-1
    not-col-permutation-1 perp-col1)
  hence False
  using  $\langle \neg$  Col D A B0  $\rangle$  by blast
}
}
hence  $\neg$  Col A B0 C0
  by blast
have A C0 Perp B0 C0
  using  $\langle$  Col A D C0  $\rangle$   $\langle$  A  $\neq$  C0  $\rangle$   $\langle$  A D Perp B0 C0  $\rangle$  perp-col by blast
have  $\exists$  X Y. A Out D X  $\wedge$  A Out B0 Y  $\wedge$  Per A X Y  $\wedge$  P Q Lt X Y
proof (cases P = Q)
  case True
  have Acute B0 A D
  by (simp add:  $\langle$  Acute D A B0  $\rangle$  acute-sym)
  hence A Out D C0
  using  $\langle$  Col A D C0  $\rangle$   $\langle$  A D Perp B0 C0  $\rangle$ 
    acute-col-perp--out l6-6 by blast
  moreover
  have A Out B0 B0
  by (metis  $\langle \neg$  Col D A B0  $\rangle$  col-trivial-2 out-trivial)
  moreover
  have Per A C0 B0
  by (simp add:  $\langle$  A C0 Perp B0 C0  $\rangle$  perp-comm perp-per-2)
  moreover
  have  $\neg$  Cong P P C0 B0
  using  $\langle$  A C0 Perp B0 C0  $\rangle$  cong-reverse-identity
    perp-not-eq-2 by blast
  hence P P Lt C0 B0
  by (metis cong-trivial-identity lt1123)
  ultimately
  show ?thesis
  using True by blast
next
case False
  obtain Q' where Bet P Q Q' and Cong Q Q' P Q
  using segment-construction by blast
  have B0  $\neq$  C0
  using  $\langle \neg$  Col A B0 C0  $\rangle$  col-trivial-2 by blast
  hence Reach B0 C0 P Q'
  using archimedes-axiom-def assms by blast
  then obtain E where GradExp B0 C0 E and P Q' Le B0 E
  using reach--ex-gradexp-le by blast
  obtain B C where GradExp2 A B0 B B0 C0 E  $\wedge$  A C0 Perp B C  $\wedge$  Col A C0 C
  using  $\langle$  A C0 Perp B0 C0  $\rangle$   $\langle$  GradExp B0 C0 E  $\rangle$   $\langle \neg$  Col A B0 C0  $\rangle$ 
    t22-24-aux1 by blast
  have GradExp2 A B0 B B0 C0 E
  using  $\langle$  GradExp2 A B0 B B0 C0 E  $\wedge$  A C0 Perp B C  $\wedge$  Col A C0 C  $\rangle$  by blast
  hence Grad A B0 B
  using gradexp2--gradexp123 gradexp--grad by blast
  hence Bet A B0 B
  by (simp add: grad--bet)

```

```

have A C0 Perp B C
  using ⟨GradExp2 A B0 B B0 C0 E ∧ A C0 Perp B C ∧ Col A C0 C⟩ by blast
have Col A C0 C
  using ⟨GradExp2 A B0 B B0 C0 E ∧ A C0 Perp B C ∧ Col A C0 C⟩ by blast
have A Out B0 B
  by (metis ⟨Bet A B0 B⟩ ⟨¬ Col D A B0⟩ bet-out col-trivial-2)
have Acute D A B
  by (metis ⟨Acute D A B0⟩ ⟨Bet A B0 B⟩ ⟨¬ Col D A B0⟩
    acute-col--out acute-out2--acute acute-trivial bet2--out
    bet-neq21--neq l5-1 not-col-distincts)
{
  assume A = C
  have Per D A B
  proof –
    have A C0 Perp B A
      using ⟨A = C⟩ ⟨A C0 Perp B C⟩ by auto
    moreover
    have Col A C0 D
      using Col-cases ⟨Col A D C0⟩ by blast
    ultimately
    show ?thesis
      by (meson col-trivial-3 l8-16-1 l8-2)
  qed
  hence False
    using ⟨Acute D A B⟩ acute-not-per by auto
}
hence A ≠ C
  by blast
have A Out D C
proof –
  have Acute B A D
    by (simp add: ⟨Acute D A B⟩ acute-sym)
  moreover
  have Col A D C
    using ⟨A ≠ C0⟩ ⟨Col A C0 C⟩ ⟨Col A D C0⟩ col-trivial-3
    colx by blast
  moreover
  have A D Perp B C
    by (metis ⟨A C0 Perp B C⟩ ⟨Col A D C0⟩ ⟨¬ Col D A B0⟩
      col-trivial-3 not-col-permutation-5 perp-col)
  ultimately
  show ?thesis
    using acute-col-perp--out l6-6 by blast
qed
moreover
have A Out B0 B
  by (simp add: ⟨A Out B0 B⟩)
moreover
have Per A C B
  using ⟨A C0 Perp B C⟩ ⟨Col A C0 C⟩ col-trivial-3
  l8-16-1 l8-2 by blast
moreover
have P Q Lt C B
proof –
  have P Q Lt P Q'
    by (metis False bet--lt1213 cong-diff-3
      ⟨Bet P Q Q'⟩ ⟨Cong Q Q' P Q⟩)
  moreover
  have B0 E Le B C
  proof –
    have ¬ hypothesis-of-obtuse-saccheri-quadrilaterals
      by (simp add: archi--obtuse-case-elimination assms)
    moreover
    have ¬ Col A B0 C0
      by (simp add: ⟨¬ Col A B0 C0⟩)
    moreover

```

```

    have A C0 Perp B0 C0
      by (simp add: ⟨A C0 Perp B0 C0⟩)
    moreover
    have GradExp2 A B0 B B0 C0 E
      by (simp add: ⟨GradExp2 A B0 B B0 C0 E⟩)
    moreover
    have A C0 Perp B C
      by (simp add: ⟨A C0 Perp B C⟩)
    moreover
    have Col A C0 C
      using ⟨Col A C0 C⟩ by auto
    ultimately
    show ?thesis
      using t22-24-aux by blast
  qed
  hence B0 E Le C B
    by (simp add: le-right-comm)
  hence P Q' Le C B
    using ⟨P Q' Le B0 E⟩ le-transitivity by blast
  ultimately
  show ?thesis
    using le3456-lt--lt by blast
  qed
  ultimately
  show ?thesis
    by blast
  qed
}
thus ?thesis
  by (simp add: aristotle-s-axiom-def)
qed

```

2.3 Equivalence Grad / GradI (inductive)

```

lemma GradI--GradI:
  shows GradI A B (Gradn A B 1)
proof -
  have B = Gradn A B 1
    by force
  thus ?thesis
    using gradi-init by auto
qed

lemma Gradn--GradI:
  shows GradI A B (Gradn A B (Suc n))
proof (induction n)
  show GradI A B (Gradn A B (Suc 0))
    by (simp add: gradi-init)
  {
    fix n
    assume GradI A B (Gradn A B (Suc n))
    have GradI A B (Gradn A B (Suc (Suc n)))
    proof(rule GradI.induct [where ?A=A and ?B=B])
      show GradI A B (Gradn A B (Suc (Suc n)))
        using Bet-Gradn-Gradn-Suc Cong-Gradn-Gradn-Suc GradI.simps
          ⟨GradI A B (Gradn A B (Suc n))⟩ by blast
      show GradI A B B
        by (simp add: gradi-init)
      show  $\bigwedge C C'. \text{GradI } A B C \implies \text{GradI } A B C \implies \text{Bet } A C C' \implies \text{Cong } A B C C' \implies \text{GradI } A B C'$ 
        using gradi-stab by blast
    qed
  }
  thus  $\bigwedge n. \text{GradI } A B (\text{Gradn } A B (\text{Suc } n)) \implies \text{GradI } A B (\text{Gradn } A B (\text{Suc } (\text{Suc } n)))$ 
    by blast
qed

```

```

lemma Grad--GradI:
  assumes Grad A B C
  shows GradI A B C
proof (cases B = C)
  case True
  thus ?thesis
    by (simp add: gradi-init)
next
  case False
  obtain n where n ≠ 0 and C = Gradn A B n
    using Grad-def assms by blast
  then obtain m where n = Suc m
    using not0-implies-Suc by blast
  hence C = Gradn A B (Suc m)
    using ⟨C = Gradn A B n⟩ by blast
  thus ?thesis
    using Gradn--GradI by blast
qed

lemma GradIAAB--not:
  assumes GradI A A B
  shows A = B
proof (rule GradI.induct [OF assms])
  show A = A
    by blast
  show  $\bigwedge C C'. \text{GradI } A A C \implies A = C \implies \text{Bet } A C C' \implies \text{Cong } A A C C' \implies A = C'$ 
    using between-cong by presburger
qed

lemma GradI--Grad:
  assumes GradI A B C
  shows Grad A B C
proof (cases A = B)
  case True
  hence A = C
    using GradIAAB--not assms by blast
  thus ?thesis
    using True grad-equiv-coq-1 by blast
next
  case False
  show ?thesis
  proof (rule GradI.induct [where ?A=A and ?B=B])
    show GradI A B C
      by (simp add: assms)
    show Grad A B B
      by (simp add: grad-equiv-coq-1)
    show  $\bigwedge C C'. \text{GradI } A B C \implies \text{Grad } A B C \implies \text{Bet } A C C' \implies \text{Cong } A B C C' \implies \text{Grad } A B C'$ 
      using grad-stab by blast
  qed
qed

theorem Grad-GradI:
  shows Grad A B C  $\longleftrightarrow$  GradI A B C
  using GradI--Grad Grad--GradI by blast

```

2.4 GradA

```

lemma grada-distincts:
  assumes GradA A B C D E F
  shows A ≠ B ∧ C ≠ B ∧ D ≠ E ∧ F ≠ E
proof (induction rule: GradA.induct [OF assms(1)])
  show  $\bigwedge D E F. A B C \text{ CongA } D E F \implies A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E$ 
    by (simp add: CongA-def)
  show  $\bigwedge D E F G H I. \text{GradA } A B C D E F \implies A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \implies$ 

```



```

    SAMS D E F A B C  $\implies$ 
    D E F A B C SumA G H I  $\implies$  A  $\neq$  B  $\wedge$  C  $\neq$  B  $\wedge$  G  $\neq$  H  $\wedge$  I  $\neq$  H
  using suma-distincts by blast
qed

lemma grada-ABC:
  assumes A  $\neq$  B and
    B  $\neq$  C
  shows GradA A B C A B C
proof -
  have A B C CongA A B C
    using assms(1) assms(2) conga-refl by force
  thus ?thesis
    using grada-init
    by force
qed

lemma gradaexp-distincts:
  assumes GradAExp A B C D E F
  shows A  $\neq$  B  $\wedge$  C  $\neq$  B  $\wedge$  D  $\neq$  E  $\wedge$  F  $\neq$  E
proof(induction rule: GradAExp.induct [OF assms(1)])
  show  $\bigwedge$  D E F. A B C CongA D E F  $\implies$  A  $\neq$  B  $\wedge$  C  $\neq$  B  $\wedge$  D  $\neq$  E  $\wedge$  F  $\neq$  E
    by (simp add: CongA-def)
  show  $\bigwedge$  D E F G H I.
    GradAExp A B C D E F  $\implies$ 
    A  $\neq$  B  $\wedge$  C  $\neq$  B  $\wedge$  D  $\neq$  E  $\wedge$  F  $\neq$  E  $\implies$ 
    SAMS D E F D E F  $\implies$  D E F D E F SumA G H I  $\implies$  A  $\neq$  B  $\wedge$  C  $\neq$  B  $\wedge$  G  $\neq$  H  $\wedge$  I  $\neq$  H
    using suma-distincts by blast
qed

lemma gradaexp-ABC:
  assumes A  $\neq$  B and
    B  $\neq$  C
  shows GradAExp A B C A B C
proof -
  have A B C CongA A B C
    using assms(1) assms(2) conga-refl by force
  thus ?thesis
    using gradaexp-init
    by force
qed

lemma conga2-grad--grada-aux1:
  assumes GradA A B C D E F and
    A B C CongA A' B' C'
  shows GradA A' B' C' D E F
proof (induction rule: GradA.cases [OF assms(1)])
{
  assume A B C CongA D E F
  hence GradA A' B' C' D E F
    using grada-init
    by (meson assms(2) not-conga not-conga-sym)
}
thus  $\bigwedge$  Da Ea Fa.
  D = Da  $\implies$ 
  E = Ea  $\implies$  F = Fa  $\implies$  A B C CongA Da Ea Fa  $\implies$  GradA A' B' C' D E F
  by blast
{
  fix Da Ea Fa
  assume 1: GradA A B C Da Ea Fa and
    2: SAMS Da Ea Fa A B C and
    3: Da Ea Fa A B C SumA D E F
  have GradA A' B' C' D E F
  proof (rule grada-stab)
    let ?D = Da
    let ?E = Ea

```

```

let ?F = Fa
show GradA A' B' C' ?D ?E ?F
proof (rule GradA.inducts)
  show GradA A B C Da Ea Fa
  by (simp add: 1)
  show  $\bigwedge D E F. A B C \text{ CongA } D E F \implies \text{GradA } A' B' C' D E F$ 
  by (meson conga-trans not-conga-sym assms(2) grada-init)
  {
    fix D E F G H I
    assume GradA A B C D E F and
      A: GradA A' B' C' D E F and
      B: SAMS D E F A B C and
      C: D E F A B C SumA G H I
    have GradA A' B' C' G H I
    proof (rule grada-stab)
      show GradA A' B' C' D E F
      by (simp add: A)
      show SAMS D E F A' B' C' using B
      by (meson C conga2-sams--sams assms(2) sams2-suma2--conga123)
      show D E F A' B' C' SumA G H I using C
      by (meson B conga3-suma--suma sams2-suma2--conga123 suma2--conga assms(2))
    qed
  }
  thus  $\bigwedge D E F G H I. \text{GradA } A B C D E F \implies \text{GradA } A' B' C' D E F \implies \text{SAMS } D E F A B C \implies D E F A B C \text{ SumA } G H I \implies \text{GradA } A' B' C' G H I$ 
  by blast
qed
show SAMS ?D ?E ?F A' B' C' using 2
by (meson 3 sams2-suma2--conga123 assms(2) conga2-sams--sams)
show ?D ?E ?F A' B' C' SumA D E F using 3
by (meson 2 sams2-suma2--conga123 assms(2) conga3-suma--suma suma2--conga)
qed
}
thus  $\bigwedge Da Ea Fa G H I. D = G \implies E = H \implies F = I \implies \text{GradA } A B C Da Ea Fa \implies \text{SAMS } Da Ea Fa A B C \implies Da Ea Fa A B C \text{ SumA } G H I \implies \text{GradA } A' B' C' D E F$ 
by blast
qed

lemma conga2-gradA--gradA-aux2:
  assumes GradA A B C D E F and
    D E F CongA D' E' F'
  shows GradA A B C D' E' F'
proof (induction rule: GradA.cases [OF assms(1)])
  {
    assume A B C CongA D E F
    hence A B C CongA D' E' F'
    by (metis conga-trans assms(2))
    hence GradA A B C D' E' F'
    by (simp add: grada-init)
  }
  thus  $\bigwedge Da Ea Fa. D = Da \implies E = Ea \implies F = Fa \implies A B C \text{ CongA } Da Ea Fa \implies \text{GradA } A B C D' E' F'$ 
  by blast
  {
    fix Da Ea Fa
    assume 1: GradA A B C Da Ea Fa and
      2: SAMS Da Ea Fa A B C and

```

```

3: Da Ea Fa A B C SumA D E F
have GradA A B C D' E' F'
proof (rule grada-stab)
  show GradA A B C Da Ea Fa
  by (simp add: 1)
  show SAMS Da Ea Fa A B C
  by (simp add: 2)
  show Da Ea Fa A B C SumA D' E' F'
  by (meson 2 3 sams2-suma2--conga123 assms(2)
    conga3-suma--suma sams2-suma2--conga456)
qed
}
thus  $\bigwedge Da Ea Fa G H I.$ 
 $D = G \implies$ 
 $E = H \implies$ 
 $F = I \implies$ 
 $GradA A B C Da Ea Fa \implies$ 
 $SAMS Da Ea Fa A B C \implies$ 
 $Da Ea Fa A B C SumA G H I \implies GradA A B C D' E' F'$ 
by blast
qed

lemma conga2-grad--grad:
assumes GradA A B C D E F and
  A B C CongA A' B' C' and
  D E F CongA D' E' F'
shows GradA A' B' C' D' E' F'
using assms(1) assms(2) assms(3)
  conga2-grad--grad-aux1 conga2-grad--grad-aux2 by blast

lemma grad--lea:
assumes GradA A B C D E F
shows A B C LeA D E F
proof (induction rule: GradA.cases [OF assms(1)])
  case (1 D E F)
  then show ?case
  by (simp add: conga--lea)
next
  case (2 D E F G H I)
  thus ?case
  by (metis sams-suma--lea456789)
qed

lemma grad-out--out:
assumes E Out D F and
  GradA A B C D E F
shows B Out A C
proof (induction rule: GradA.cases [OF assms(2)])
  case (1 D E F)
  then show ?case
  by (metis not-conga-sym assms(1) l11-21-a)
next
  case (2 D E F G H I)
  then show ?case
  by (metis sams-suma--lea456789 assms(1) out-lea--out)
qed

lemma grad2-sams-suma--grad-aux:
shows  $\forall A B C D E F G H I K L M.$ 
   $GradA A B C D E F \wedge GradA A B C G H I \wedge$ 
   $SAMS D E F G H I \wedge D E F G H I SumA K L M \longrightarrow GradA A B C K L M$ 
proof –
  {
    fix A B C D E F G' H' I' K L M
    assume K1: GradA A B C D E F and
    K2: GradA A B C G' H' I'
  }

```

```

have  $\forall K L M. (SAMS D E F G' H' I' \wedge D E F G' H' I' SumA K L M) \longrightarrow GradA A B C K L M$ 
proof(induction rule: GradA.induct [OF K2])
{
  fix  $Da Ea Fa$ 
  assume  $A B C CongA Da Ea Fa$ 
  {
    fix  $K L M$ 
    assume  $SAMS D E F Da Ea Fa$  and  $D E F Da Ea Fa SumA K L M$ 
    have  $SAMS D E F A B C$ 
    by (meson conga2-sams--sams not-conga-sym sams2-suma2--conga123
       $\langle A B C CongA Da Ea Fa \rangle \langle D E F Da Ea Fa SumA K L M \rangle \langle SAMS D E F Da Ea Fa \rangle$ )
    moreover have  $D E F A B C SumA K L M$ 
    by (meson conga3-suma--suma not-conga-sym sams2-suma2--conga123
       $\langle A B C CongA Da Ea Fa \rangle \langle D E F Da Ea Fa SumA K L M \rangle \langle SAMS D E F Da Ea Fa \rangle$ 
      suma2--conga)
    ultimately have  $GradA A B C K L M$ 
    using  $K1$  grada-stab by blast
  }
  hence  $\forall K L M. (SAMS D E F Da Ea Fa \wedge D E F Da Ea Fa SumA K L M) \longrightarrow GradA A B C K L M$ 
  by blast
}
thus  $\bigwedge Da Ea Fa.$ 
 $A B C CongA Da Ea Fa \implies$ 
 $\forall K L M. (SAMS D E F Da Ea Fa \wedge D E F Da Ea Fa SumA K L M) \longrightarrow GradA A B C K L M$ 
by blast
{
  fix  $G H I$ 
  {
    fix  $Da Ea Fa$ 
    assume  $GradA A B C Da Ea Fa$  and
       $SAMS Da Ea Fa A B C$  and
       $Da Ea Fa A B C SumA G H I$  and
       $P1: \forall K L M. SAMS D E F Da Ea Fa \wedge D E F Da Ea Fa SumA K L M \longrightarrow GradA A B C K L M$ 
    {
      fix  $K0 L0 M0$ 
      assume  $SAMS D E F G H I$  and
         $D E F G H I SumA K0 L0 M0$ 
      have  $Da \neq Ea$ 
      using  $\langle SAMS Da Ea Fa A B C \rangle$  sams-distincts by auto
      have  $Fa \neq Ea$ 
      using  $\langle GradA A B C Da Ea Fa \rangle$  grada-distincts by blast
      have  $D \neq E$ 
      using  $\langle SAMS D E F G H I \rangle$  sams-distincts by blast
      have  $E \neq F$ 
      using  $\langle SAMS D E F G H I \rangle$  sams-distincts by blast
      obtain  $K L M$  where  $D E F Da Ea Fa SumA K L M$ 
      using ex-suma  $\langle D \neq E \rangle \langle Da \neq Ea \rangle \langle E \neq F \rangle \langle Fa \neq Ea \rangle$  by presburger
      have  $SAMS D E F Da Ea Fa$ 
      proof (rule sams-lea2--sams [where ?A'=D and ?B'=E and ?C'=F and ?D'=G and ?E'=H and ?F'=I])
      show  $SAMS D E F G H I$ 
      using  $\langle SAMS D E F G H I \rangle$  by blast
      show  $D E F LeA D E F$ 
      using  $\langle D \neq E \rangle \langle E \neq F \rangle$  lea-refl by force
      show  $Da Ea Fa LeA G H I$ 
      proof (rule sams-suma--lea123789 [where ?D=A and ?E=B and ?F=C])
      show  $Da Ea Fa A B C SumA G H I$ 
      using  $\langle Da Ea Fa A B C SumA G H I \rangle$  by blast
      show  $SAMS Da Ea Fa A B C$ 
      by (simp add:  $\langle SAMS Da Ea Fa A B C \rangle$ )
    }
    qed
  }
  qed
  have  $GradA A B C K0 L0 M0$ 
  proof (rule grada-stab [where ?D = K and ?E = L and ?F = M])
  show  $GradA A B C K L M$ 
  using  $P1$   $\langle D E F Da Ea Fa SumA K L M \rangle \langle SAMS D E F Da Ea Fa \rangle$  by blast
}

```

```

show SAMS K L M A B C
  using sams-assoc-2 [where  $?A=D$  and  $?B=E$  and  $?C=F$  and
     $?D=Da$  and  $?E=Ea$  and  $?F=Fa$  and  $?D'=G$  and  $?E'=H$  and  $?F'=I$ ]
  using  $\langle D E F Da Ea Fa SumA K L M \rangle$   $\langle Da Ea Fa A B C SumA G H I \rangle$ 
     $\langle SAMS D E F Da Ea Fa \rangle$   $\langle SAMS D E F G H I \rangle$   $\langle SAMS Da Ea Fa A B C \rangle$  by blast
show K L M A B C SumA K0 L0 M0
  by (meson suma-assoc-2  $\langle D E F Da Ea Fa SumA K L M \rangle$ 
     $\langle D E F G H I SumA K0 L0 M0 \rangle$   $\langle Da Ea Fa A B C SumA G H I \rangle$   $\langle SAMS D E F Da Ea Fa \rangle$ 
     $\langle SAMS Da Ea Fa A B C \rangle$ )

qed
}
hence  $\forall K L M. SAMS D E F G H I \wedge D E F G H I SumA K L M \longrightarrow GradA A B C K L M$ 
by blast
}
hence  $\bigwedge Da Ea Fa. GradA A B C Da Ea Fa \wedge SAMS Da Ea Fa A B C \wedge$ 
   $Da Ea Fa A B C SumA G H I \wedge$ 
  ( $\forall K L M. SAMS D E F Da Ea Fa \wedge D E F Da Ea Fa SumA K L M \longrightarrow GradA A B C K L M$ )  $\longrightarrow$ 
  ( $\forall K L M. SAMS D E F G H I \wedge D E F G H I SumA K L M \longrightarrow GradA A B C K L M$ )
by blast
}
thus  $\bigwedge Da Ea Fa G H I.$ 
   $GradA A B C Da Ea Fa \implies$ 
  ( $\forall K L M. SAMS D E F Da Ea Fa \wedge D E F Da Ea Fa SumA K L M \longrightarrow GradA A B C K L M$ )  $\implies$ 
   $SAMS Da Ea Fa A B C \implies$ 
   $Da Ea Fa A B C SumA G H I \implies$ 
  ( $\forall K L M. SAMS D E F G H I \wedge D E F G H I SumA K L M \longrightarrow GradA A B C K L M$ )
by blast
qed
}
thus ?thesis by blast
qed

lemma grada2-sams-suma--grada:
assumes GradA A B C D E F and
  GradA A B C G H I and
  SAMS D E F G H I and
  D E F G H I SumA K L M
shows GradA A B C K L M
using assms(1) assms(2) assms(3) assms(4) grada2-sams-suma--grada-aux by blast

lemma gradaexp--grada:
assumes GradAExp A B C D E F
shows GradA A B C D E F
proof (rule GradAExp.induct [OF assms])
show  $\bigwedge D E F. A B C CongA D E F \implies GradA A B C D E F$ 
by (simp add: grada-init)
show  $\bigwedge D E F G H I.$ 
   $GradAExp A B C D E F \implies$ 
   $GradA A B C D E F \implies SAMS D E F D E F \implies D E F D E F SumA G H I \implies GradA A B C G H I$ 
using grada2-sams-suma--grada-aux by blast
qed

lemma acute-archi-aux:
assumes Per PO A B and
   $PO \neq A$  and
   $B \neq A$  and
   $C \neq D$  and
   $D \neq E$  and
  Bet A C D and
  Bet C D E and
  Bet D E B and
  C PO D CongA D PO E
shows C D Lt D E
proof –
have  $D \neq A$ 
using assms(4) assms(6) between-identity by blast

```

```

have C ≠ PO
  using assms(9) conga-diff1 by auto
have D ≠ PO
  using assms(9) conga-diff45 by blast
have ¬ Col PO A B
  by (metis assms(1) assms(2) assms(3) l8-8 not-col-permutation-2 per-col)
hence ¬ Col PO A D
  by (metis Col-def ⟨D ≠ A⟩ assms(4) assms(5) assms(6) assms(7) assms(8) l6-16-1)
have ¬ Col PO D E
  by (metis Col-def ⟨¬ Col PO A D⟩ assms(4) assms(5) assms(6) assms(7) l6-16-1)
then obtain P where A D PO CongA PO D P and PO D OS P E
  using ⟨¬ Col PO A D⟩ angle-construction-1 not-col-permutation-1 by blast
have Acute A D PO
  by (metis ⟨D ≠ A⟩ assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7)
    assms(8) bet-col1 between-trivial l11-43 outer-transitivity-between per-col)
hence Acute A D PO ⟷ A D PO LtA E D PO
  by (metis acute-chara-1 assms(4) assms(5) assms(6) assms(7) outer-transitivity-between2)
have A D PO LeA E D PO
  by (metis lta--lea outer-transitivity-between2 ⟨Acute A D PO⟩ acute-chara-1
    assms(4) assms(5) assms(6) assms(7))
hence PO D P LeA PO D E
  by (meson lea-right-comm lea-trans ⟨A D PO CongA PO D P⟩ conga--lea456123)
hence P InAngle PO D E
  using ⟨PO D OS P E⟩ lea-in-angle one-side-symmetry by presburger
have P ≠ D
  using ⟨PO D OS P E⟩ os-distincts by blast
obtain F where Bet PO F E and D Out F P
  using InAngle-def ⟨P InAngle PO D E⟩ ⟨¬ Col PO D E⟩ bet-col by auto
have A D PO CongA PO D F
  by (metis out2--conga ⟨A D PO CongA PO D P⟩ ⟨D Out F P⟩ ⟨D ≠ PO⟩
    bet-out-1 conga-trans not-bet-distincts)
have D Out A C
  by (simp add: assms(4) assms(6) bet-out-1 l6-6)
have D Out PO PO
  using ⟨D ≠ PO⟩ out-trivial by auto
have PO D C CongA A D PO
  by (simp add: out2--conga ⟨D Out A C⟩ ⟨D Out PO PO⟩ conga-left-comm)
have ¬ Col PO D F
  using ⟨D Out F P⟩ ⟨D Out PO PO⟩ ⟨PO D OS P E⟩ col-out2-col col-trivial-3 l9-19 by blast
have PO ≠ F
  using ⟨¬ Col PO D F⟩ col-trivial-3 by auto
have ¬ Col PO D C
  using ⟨¬ Col PO A D⟩ assms(4) assms(6) bet-col col2--eq col-permutation-5 by blast
have PO Out D D
  by (simp add: ⟨D ≠ PO⟩ out-trivial)
have PO Out C C
  by (simp add: ⟨C ≠ PO⟩ out-trivial)
have PO Out F E
  using ⟨Bet PO F E⟩ ⟨PO ≠ F⟩ bet-out by auto
hence D PO C CongA D PO F
  using l11-10 ⟨PO Out C C⟩ ⟨PO Out D D⟩ assms(9) conga-left-comm by blast
have PO D C CongA PO D F
  by (meson conga-trans ⟨A D PO CongA PO D F⟩ ⟨PO D C CongA A D PO⟩)
have Cong PO D PO D
  by (simp add: cong-reflexivity)
have Cong PO C PO F ∧ Cong D C D F ∧ PO C D CongA PO F D
  using l11-50-1 ⟨Cong PO D PO D⟩ ⟨D PO C CongA D PO F⟩ ⟨PO D C CongA PO D F⟩
    ⟨¬ Col PO D C⟩ by blast
hence Cong D F C D
  using not-cong-4312 by blast
moreover
{
  assume Col E D F
  {
    assume E = F
    have D ≠ C

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    using assms(4) by auto
  moreover have Per PO D C
    using ⟨E = F⟩ ⟨PO D C CongA PO D F⟩ assms(7) l11-18-2 by auto
  moreover have Col D C A
    using ⟨D Out A C⟩ not-col-permutation-5 out-col by blast
  ultimately have Per A D PO
    by (meson l11-17 ⟨PO D C CongA A D PO⟩)
  hence False
    using acute-not-per ⟨Acute A D PO⟩ by blast
}
moreover
{
  assume E ≠ F
  hence False
    by (metis col-permutation-1 ⟨Bet PO F E⟩ ⟨Col E D F⟩ ⟨¬ Col PO D F⟩ bet-col col2--eq)
}
ultimately have False
  by blast
}
hence ¬ Col E D F
  by blast
have E ≠ F
  using ⟨¬ Col E D F⟩ not-col-distincts by blast
have D E F LtA D F E
proof (rule lta-trans [where ?A1.0 = F and ?B1.0 = D and ?C1.0 = PO])
  have D PO E LtA PO D C ∧ D E PO LtA PO D C
    by (metis l11-41 not-col-permutation-1 one-side-not-col124
      ⟨∧thesis. (∧P. [A D PO CongA PO D P; PO D OS P E]) ⇒ thesis) ⇒ thesis)
      assms(4) assms(7) between-symmetry)
  show D E F LtA F D PO
proof -
  have D E PO CongA D E F
    by (metis bet-out-1 out2--conga ⟨Bet PO F E⟩ ⟨E ≠ F⟩ assms(5) out-trivial)
  moreover have PO D C CongA F D PO
    by (simp add: ⟨PO D C CongA PO D F⟩ conga-right-comm)
  ultimately show ?thesis
    by (simp add: conga-preserves-lta ⟨D PO E LtA PO D C ∧ D E PO LtA PO D C⟩)
qed
show F D PO LtA D F E
  by (metis bet-col col-lta--bet l11-41-aux not-col-permutation-2
    ⟨Bet PO F E⟩ ⟨D PO E LtA PO D C ∧ D E PO LtA PO D C⟩ ⟨E ≠ F⟩ ⟨PO D C CongA A D PO⟩
    ⟨PO D C CongA PO D F⟩
    ⟨∧thesis. (∧P. [A D PO CongA PO D P; PO D OS P E]) ⇒ thesis) ⇒ thesis)
    bet-conga--bet col123--nos ncol-conga-ncol)
qed
hence D F Lt D E
  using l11-44-2-b by blast
ultimately show ?thesis
  using cong2-lt--lt
  using cong-reflexivity by blast
qed

lemma acute-archi-aux1:
  assumes Per PO A0 B and
    B ≠ A0 and
    Bet A0 A1 B and
    Grada A0 PO A1 P Q R and
    A0 ≠ A1
  shows A0 PO B LeA P Q R ∨ (∃ A. Bet A0 A1 A ∧ Bet A0 A B ∧ P Q R CongA A0 PO A)
proof -
  have A0 ≠ PO
    using assms(4) grada-distincts by auto
  have A1 ≠ PO
    using assms(4) grada-distincts by auto
  have P ≠ Q
    using assms(4) grada-distincts by auto

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have R ≠ Q
  using assms(4) grada-distincts by auto
have ¬ Col PO A0 B
  by (metis per-not-col ⟨A0 ≠ PO⟩ assms(1) assms(2))
have PO ≠ B
  using assms(1) assms(2) per-distinct-1 by blast
{
  assume P Q R LeA A0 PO B
  {
    assume Col P Q R
    {
      assume Q Out P R
      hence PO Out A0 A1
        using grada-out--out [where ?D = P and ?E = Q and ?F = R] assms(4) by blast
      hence False
        using ⟨¬ Col PO A0 B⟩ assms(3) assms(4) assms(5) bet-col col-trivial-2
        colx out-col by blast
    }
    hence ¬ Q Out P R
      by blast
    hence Bet P Q R
      using ⟨Col P Q R⟩ not-out-bet by blast
    hence Bet A0 PO B
      using ⟨P Q R LeA A0 PO B⟩ bet-lea--bet [where ?A = P and ?B = Q and ?C = R]
      by blast
    hence False
      using ⟨¬ Col PO A0 B⟩ bet-col not-col-permutation-4 by blast
  }
  hence ¬ Col P Q R
    by blast
  then obtain C where P Q R CongA A0 PO C and A0 PO OS C B
    by (metis NCol-cases ⟨¬ Col PO A0 B⟩ angle-construction-1)
  have C InAngle A0 PO B
  proof (rule lea-in-angle)
    have A0 PO B CongA A0 PO B
      using ⟨A0 ≠ PO⟩ ⟨PO ≠ B⟩ conga-refl by auto
    thus A0 PO C LeA A0 PO B
      using l11-30 [where ?A=P and ?B=Q and ?C=R and ?D=A0 and ?E=PO and ?F=B]
      ⟨P Q R LeA A0 PO B⟩ ⟨P Q R CongA A0 PO C⟩ by blast
    show A0 PO OS B C
      by (simp add: ⟨A0 PO OS C B⟩ one-side-symmetry)
  qed
  have C ≠ PO
    using ⟨A0 PO OS C B⟩ os-distincts by blast
  obtain A where Bet A0 A B and A = PO ∨ PO Out A C
    using InAngle-def ⟨C InAngle A0 PO B⟩ by blast
  hence PO Out A C
    using Bet-cases Col-def ⟨¬ Col PO A0 B⟩ by blast
  have P Q R CongA A0 PO A
  proof (rule l11-10 [where ?A=P and ?C=R and ?D=A0 and ?F=C],
    insert ⟨P Q R CongA A0 PO C⟩ ⟨PO Out A C⟩)
    show Q Out P P
      by (simp add: ⟨P ≠ Q⟩ out-trivial)
    show Q Out R R
      using ⟨R ≠ Q⟩ out-trivial by auto
    show PO Out A0 A0
      by (simp add: ⟨A0 ≠ PO⟩ out-trivial)
  qed
  have Bet A0 A1 A
  proof (cases A0 = A1)
    case True
    thus ?thesis
      using assms(5) by auto
  next
  case False
  hence ¬ Col A0 PO A

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    by (metis Col-perm ⟨A0 PO OS C B⟩ ⟨PO Out A C⟩ l6-16-1
        one-side-not-col123 out-col out-distinct)
  have A1 InAngle A0 PO A
  proof (rule lea-in-angle)
    show A0 PO A1 LeA A0 PO A
    proof (rule l11-30 [where ?A=A0 and ?B=PO and ?C=A1 and ?D=P and ?E=Q and ?F=R])
      show A0 PO A1 LeA P Q R
      by (simp add: assms(4) grada--lea)
      show A0 PO A1 CongA A0 PO A1
      by (simp add: ⟨A0 ≠ PO⟩ ⟨A1 ≠ PO⟩ conga-refl)
      show P Q R CongA A0 PO A
      by (simp add: ⟨P Q R CongA A0 PO A⟩)
    qed
    show A0 PO OS A A1
    using False ⟨Bet A0 A B⟩ ⟨¬ Col A0 PO A⟩ assms(3) bet2--out
        not-col-distincts out-one-side by presburger
  qed
  obtain X where Bet A0 X A and X = PO ∨ PO Out X A1
  using InAngle-def ⟨A1 InAngle A0 PO A⟩ by blast
  hence PO Out X A1
  using ⟨¬ Col A0 PO A⟩ bet-col by blast
  have X = A1 ⟶ Bet A0 A1 A
  using ⟨Bet A0 X A⟩ by blast
  moreover
  have X ≠ A1 ⟶ Bet A0 A1 A
  by (meson ⟨Bet A0 A B⟩ ⟨Bet A0 X A⟩ ⟨PO Out X A1⟩ ⟨¬ Col A0 PO A⟩
      assms(3) bet-col bet-col1 col-permutation-2 colx out-col)
  ultimately
  show ?thesis
  by blast
qed
moreover have Bet A0 A B
  by (simp add: ⟨Bet A0 A B⟩)
ultimately have A0 PO B LeA P Q R ∨ (∃ A. Bet A0 A1 A ∧ Bet A0 A B ∧ P Q R CongA A0 PO A)
  using ⟨P Q R CongA A0 PO A⟩ by blast
}
thus ?thesis
  by (metis ⟨A0 ≠ PO⟩ ⟨P ≠ Q⟩ ⟨PO ≠ B⟩ ⟨R ≠ Q⟩ lea-total)
qed

```

lemma acute-archi-aux2-1-a:

assumes Per PO A0 B and

PO ≠ A0 and

B ≠ A0 and

Bet A0 A1 B and

A0 ≠ A1 and ¬ Col PO A0 B and ¬ Col A0 PO A1 and PO ≠ A1 and PO ≠ B

shows ∃ P Q R. (GradA A0 PO A1 P Q R ∧ (A0 PO B LeA P Q R ∨
 (∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ P Q R CongA A0 PO A' ∧ A0 A1 Le A0 A' ∧
 (∃ A. Bet A0 A A' ∧ A0 PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))

proof –

let ?P = A0

let ?Q = PO

let ?R = A1

have (GradA A0 PO A1 ?P ?Q ?R ∧ (A0 PO B LeA ?P ?Q ?R ∨
 (∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ ?P ?Q ?R CongA A0 PO A' ∧ A0 A1 Le A0 A' ∧
 (∃ A. Bet A0 A A' ∧ A0 PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))

by (metis assms(2) assms(4) assms(8) bet--le1213 conga-refl grada-init not-bet-distincts)

thus ?thesis by blast

qed

lemma acute-archi-aux2-1:

assumes Per PO A B and

PO ≠ A and

B ≠ A and

Bet A B0 B and

$A \neq B0$ and $\neg \text{Col } PO \ A \ B$ and $\neg \text{Col } A \ PO \ B0$ and $PO \neq B0$ and $PO \neq B$
shows $\exists \ P \ Q \ R. (\text{GradA } A \ PO \ B0 \ P \ Q \ R \wedge (A \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'. \text{Bet } A \ B0 \ A' \wedge \text{Bet } A \ A' \ B \wedge P \ Q \ R \text{CongA } A \ PO \ A' \wedge A \ B0 \text{Le } A \ A' \wedge$
 $(\exists \ A0. \text{Bet } A \ A0 \ A' \wedge A0 \ PO \ A' \text{CongA } A \ PO \ B0 \wedge A \ B0 \text{Le } A0 \ A'))))$
proof –
let $?A0 = A$
let $?A1 = B0$
have $\exists \ P \ Q \ R. (\text{GradA } ?A0 \ PO \ ?A1 \ P \ Q \ R \wedge (?A0 \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'. \text{Bet } ?A0 \ ?A1 \ A' \wedge \text{Bet } ?A0 \ A' \ B \wedge P \ Q \ R \text{CongA } ?A0 \ PO \ A' \wedge$
 $?A0 \ ?A1 \text{Le } ?A0 \ A' \wedge$
 $(\exists \ A. \text{Bet } ?A0 \ A \ A' \wedge ?A0 \ PO \ A' \text{CongA } ?A0 \ PO \ ?A1 \wedge ?A0 \ ?A1 \text{Le } A \ A'))))$
using *acute-archi-aux2-1-a* *assms(1)* *assms(2)* *assms(3)* *assms(4)* *assms(5)* *assms(6)* *assms(7)*
assms(8) *assms(9)* **by** *blast*
thus *?thesis*
by (*metis not-bet-distincts*)
qed

lemma *acute-archi-aux2-2*:
assumes *Per PO A0 B* and
 $PO \neq A0$ and
 $B \neq A0$ and
 $\text{Bet } A0 \ A1 \ B$ and
 $A0 \neq A1$ and
 $\text{Grad } A0 \ A1 \ C$ and
 $\text{Bet } A0 \ C \ C'$ and
 $\text{Cong } A0 \ A1 \ C \ C'$ and
 $\neg \text{Col } PO \ A0 \ B$ and
 $\neg \text{Col } A0 \ PO \ A1$ and
 $PO \neq A1$ and
 $PO \neq B$ and
 $\text{Per } PO \ A0 \ B \wedge PO \neq A0 \wedge$
 $B \neq A0 \wedge \text{Bet } A0 \ A1 \ B \wedge$
 $A0 \neq A1 \wedge$
 $\neg \text{Col } PO \ A0 \ B \wedge$
 $\neg \text{Col } A0 \ PO \ A1 \wedge$
 $PO \neq A1 \longrightarrow (\exists \ P \ Q \ R.$
 $(\text{GradA } A0 \ PO \ A1 \ P \ Q \ R \wedge$
 $(A0 \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'.$
 $\text{Bet } A0 \ A1 \ A' \wedge$
 $\text{Bet } A0 \ A' \ B \wedge$
 $P \ Q \ R \text{CongA } A0 \ PO \ A' \wedge$
 $A0 \ C \text{Le } A0 \ A' \wedge (\exists \ A. (\text{Bet } A0 \ A \ A' \wedge A \ PO \ A' \text{CongA } A0 \ PO \ A1 \wedge A0 \ A1 \text{Le } A \ A'))))))$
shows $\exists \ P \ Q \ R. (\text{GradA } A0 \ PO \ A1 \ P \ Q \ R \wedge$
 $(A0 \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'. \text{Bet } A0 \ A1 \ A' \wedge \text{Bet } A0 \ A' \ B \wedge P \ Q \ R \text{CongA } A0 \ PO \ A' \wedge A0 \ C' \text{Le } A0 \ A' \wedge$
 $(\exists \ A. \text{Bet } A0 \ A \ A' \wedge A \ PO \ A' \text{CongA } A0 \ PO \ A1 \wedge A0 \ A1 \text{Le } A \ A'))))$
proof –
have $\exists \ P \ Q \ R.$
 $(\text{GradA } A0 \ PO \ A1 \ P \ Q \ R \wedge (A0 \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'. \text{Bet } A0 \ A1 \ A' \wedge \text{Bet } A0 \ A' \ B \wedge P \ Q \ R \text{CongA } A0 \ PO \ A' \wedge$
 $A0 \ C' \text{Le } A0 \ A' \wedge (\exists \ A. (\text{Bet } A0 \ A \ A' \wedge A \ PO \ A' \text{CongA } A0 \ PO \ A1 \wedge A0 \ A1 \text{Le } A \ A'))))))$
using *assms(13)* *assms(1)* *assms(10)* *assms(11)* *assms(2)* *assms(3)* *assms(4)*
assms(5) *assms(9)* **by** *blast*
then obtain $P \ Q \ R$ **where** $\text{GradA } A0 \ PO \ A1 \ P \ Q \ R$ and
 $P2: A0 \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'. \text{Bet } A0 \ A1 \ A' \wedge \text{Bet } A0 \ A' \ B \wedge P \ Q \ R \text{CongA } A0 \ PO \ A' \wedge A0 \ C' \text{Le } A0 \ A' \wedge$
 $(\exists \ A. (\text{Bet } A0 \ A \ A' \wedge A \ PO \ A' \text{CongA } A0 \ PO \ A1 \wedge A0 \ A1 \text{Le } A \ A'))))$
by *blast*
{
assume $A0 \ PO \ B \text{LeA } P \ Q \ R$
hence $\exists \ P \ Q \ R. (\text{GradA } A0 \ PO \ A1 \ P \ Q \ R \wedge (A0 \ PO \ B \text{LeA } P \ Q \ R \vee$
 $(\exists \ A'. \text{Bet } A0 \ A1 \ A' \wedge \text{Bet } A0 \ A' \ B \wedge P \ Q \ R \text{CongA } A0 \ PO \ A' \wedge A0 \ C' \text{Le } A0 \ A' \wedge$
 $(\exists \ A. \text{Bet } A0 \ A \ A' \wedge A \ PO \ A' \text{CongA } A0 \ PO \ A1 \wedge A0 \ A1 \text{Le } A \ A'))))$
using $\langle \text{GradA } A0 \ PO \ A1 \ P \ Q \ R \rangle$ **by** *blast*
}

```

moreover
{
  assume  $\neg A0 PO B LeA P Q R$ 
  assume  $\exists A'$ .
     $Bet A0 A1 A' \wedge$ 
     $Bet A0 A' B \wedge$ 
     $P Q R CongA A0 PO A' \wedge$ 
     $A0 C Le A0 A' \wedge (\exists A. (Bet A0 A A' \wedge A PO A' CongA A0 PO A1 \wedge A0 A1 Le A A'))$ 
  then obtain  $A'$  where  $Bet A0 A1 A'$  and
     $Bet A0 A' B$  and
     $P Q R CongA A0 PO A'$  and
     $A0 C Le A0 A'$  and
     $P3: (\exists A. (Bet A0 A A' \wedge A PO A' CongA A0 PO A1 \wedge A0 A1 Le A A'))$ 
  by blast
  then obtain  $A$  where  $Bet A0 A A'$  and  $A PO A' CongA A0 PO A1$  and  $A0 A1 Le A A'$ 
  by blast
  have  $SAMS P Q R A0 PO A1$ 
  proof (rule sams-lea2--sams [where  $?A'=A0$  and  $?B'=PO$  and  $?C'=B$  and
     $?D'=A0$  and  $?E'=PO$  and  $?F'=B$ ])
  show  $SAMS A0 PO B A0 PO B$ 
    by (metis acute-chara-1 lea-right-comm lta--lea assms(1) assms(2) assms(3) l11-43
      point-construction-different sams-chara)
  show  $P Q R LeA A0 PO B$ 
  proof (rule l11-30 [where  $?A=A0$  and  $?B=PO$  and  $?C=A'$  and
     $?D=A0$  and  $?E=PO$  and  $?F=B$ ])
  have  $A' InAngle A0 PO B$ 
    by (metis InAngle-def Out-def  $\langle A PO A' CongA A0 PO A1 \rangle$ 
       $\langle Bet A0 A' B \rangle$  assms(12) assms(2) between-trivial conga-diff2)
  thus  $A0 PO A' LeA A0 PO B$ 
    using inangle--lea by force
  show  $A0 PO A' CongA P Q R$ 
    using  $\langle P Q R CongA A0 PO A' \rangle$  conga-sym-equiv by auto
  show  $A0 PO B CongA A0 PO B$ 
    using assms(12) assms(2) conga-refl by force
  qed
  have  $A1 InAngle A0 PO B$ 
    using InAngle-def assms(11) assms(12) assms(2) assms(4) out-trivial by auto
  thus  $A0 PO A1 LeA A0 PO B$ 
    by (simp add: inangle--lea)
  qed
  have  $A0 \neq A'$ 
    using  $\langle Bet A0 A1 A' \rangle$  assms(5) between-identity by blast
  have  $A \neq A'$ 
    using Le-def  $\langle A0 A1 Le A A' \rangle$  assms(5) between-identity cong-identity-inv by blast
  have  $PO \neq A$ 
    using  $\langle A PO A' CongA A0 PO A1 \rangle$  conga-diff1 by blast
  have  $P \neq Q$ 
    using  $\langle P Q R CongA A0 PO A' \rangle$  conga-diff1 by blast
  have  $PO \neq A'$ 
    using  $P3$  conga-diff2 by blast
  have  $Q \neq R$ 
    using  $\langle P Q R CongA A0 PO A' \rangle$  conga-diff2 by blast
  then obtain  $P' Q' R'$  where  $P Q R A0 PO A1 SumA P' Q' R'$ 
    using ex-suma  $\langle P \neq Q \rangle$  assms(11) assms(2) by fastforce
  have  $GradA A0 PO A1 P' Q' R'$ 
    using grada-stab [where  $?D=P$  and  $?E=Q$  and  $?F=R$ ]
     $\langle GradA A0 PO A1 P Q R \rangle \langle SAMS P Q R A0 PO A1 \rangle \langle P Q R A0 PO A1 SumA P' Q' R' \rangle$  by blast
  moreover
  have  $A0 PO B LeA P' Q' R' \vee (\exists A. Bet A0 A1 A \wedge Bet A0 A B \wedge P' Q' R' CongA A0 PO A)$ 
    using acute-archi-aux1 assms(1) assms(3) assms(4) assms(5) calculation by blast
  moreover
  {
    assume  $A0 PO B LeA P' Q' R'$ 
    hence  $GradA A0 PO A1 P' Q' R'$ 
    using calculation by auto
    hence  $\exists P Q R. (GradA A0 PO A1 P Q R \wedge$ 

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(A0 PO B LeA P Q R ∨
(∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ P Q R CongA A0 PO A' ∧ A0 C' Le A0 A' ∧
  (∃ A. Bet A0 A A' ∧ A PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))
  using ⟨A0 PO B LeA P' Q' R'⟩ by blast
}
moreover
{
  assume ∃ A. Bet A0 A1 A ∧ Bet A0 A B ∧ P' Q' R' CongA A0 PO A
  then obtain A'' where Bet A0 A1 A'' and Bet A0 A'' B and P' Q' R' CongA A0 PO A''
  by blast
  have ¬ Col A PO A'
  by (meson ⟨A ≠ A'⟩ ⟨Bet A0 A A'⟩ ⟨Bet A0 A' B⟩ ⟨Bet A0 A1 A'⟩ assms(10) assms(4)
    bet-col1 colx not-col-permutation-5)
  have ¬ Col A0 PO A'
  by (meson ⟨A0 ≠ A'⟩ ⟨Bet A0 A' B⟩ assms(10) assms(4) bet-col1 col-trivial-3 colx)
  {
    assume Col A' PO A''
    have A' ≠ A'' → False
    by (meson ⟨Bet A0 A1 A''⟩ ⟨Bet A0 A1 A'⟩ ⟨Col A' PO A''⟩ assms(10)
      bet-col col-permutation-5 colx)
  }
  moreover
  {
    assume A' = A''
    have A0 PO A' A0 PO A1 SumA A0 PO A'
    using conga3-suma--suma [where ?A=P and ?B=Q and ?C=R and
      ?D=A0 and ?E=PO and ?F=A1 and ?G=P' and ?H=Q' and ?I=R]
    using ⟨A' = A''⟩ ⟨P Q R A0 PO A1 SumA P' Q' R'⟩ ⟨P Q R CongA A0 PO A'⟩
      ⟨P' Q' R' CongA A0 PO A''⟩ assms(11) assms(2) conga-refl by force
    have ¬ Col A0 PO A''
    using ⟨A' = A''⟩ ⟨¬ Col A0 PO A'⟩ by auto
    have Bet A0 A' A''
    by (simp add: ⟨A' = A''⟩ between-trivial)
    have SAMS A0 PO A' A0 PO A1
    using conga2-sams--sams [where ?A=P and ?B=Q and ?C=R and
      ?D=A0 and ?E=PO and ?F=A1]
    using ⟨P Q R CongA A0 PO A'⟩ ⟨SAMS P Q R A0 PO A1⟩ assms(11) assms(2)
      conga-refl by presburger
    hence Col A0 PO A1
    using sams-suma--out546
    by (meson not-col-permutation-4 ⟨A0 PO A' A0 PO A1 SumA A0 PO A'⟩ out-col)
    hence False
    using assms(10) by blast
  }
  ultimately have False
  by blast
}
hence ¬ Col A' PO A''
by blast
have ¬ Col A0 PO A''
by (metis ⟨Bet A0 A'' B⟩ ⟨Bet A0 A1 A''⟩ assms(10) assms(4) bet-col1
  between-identity colx not-col-distincts)
have Bet A0 A' A''
proof (rule col-two-sides-bet [where ?B=PO])
  show Col A' A0 A''
  using ⟨Bet A0 A' B⟩ ⟨Bet A0 A'' B⟩ bet-col1 not-col-permutation-1 by blast
  have A' InAngle A0 PO A''
  proof (rule lea-in-angle)
    show A0 PO A' LeA A0 PO A''
    using l11-30 [where ?A=P and ?B=Q and ?C=R and
      ?D=P' and ?E=Q' and ?F=R]
    by (meson sams-suma--lea123789 ⟨P Q R A0 PO A1 SumA P' Q' R'⟩
      ⟨P Q R CongA A0 PO A'⟩ ⟨P' Q' R' CongA A0 PO A''⟩ ⟨SAMS P Q R A0 PO A1⟩)
    show A0 PO OS A'' A'
    by (metis ⟨A0 ≠ A'⟩ ⟨Bet A0 A' B⟩ ⟨Bet A0 A'' B⟩ ⟨Bet A0 A1 A''⟩
      ⟨¬ Col A0 PO A'⟩ assms(5) bet2--out between-identity out-one-side)
  qed

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thus  $A' PO TS A0 A''$ 
  by (simp add:  $\langle \neg Col A' PO A'' \rangle \langle \neg Col A0 PO A' \rangle$  in-angle-two-sides
    not-col-permutation-1 not-col-permutation-4)
qed
have  $A PO A' CongA A' PO A''$ 
proof (rule conga-trans [where  $?A=A0$  and  $?B'=PO$  and  $?C'=A1$ ])
  have  $PO \neq A''$ 
    using  $\langle \neg Col A' PO A'' \rangle$  not-col-distincts by blast
  have  $A' \neq A''$ 
    using  $\langle \neg Col A' PO A'' \rangle$  not-col-distincts by blast
  have  $\neg PO A' OS A0 A''$ 
    using  $\langle Bet A0 A' A'' \rangle$  col-trivial-3 one-side-chara by force
  show  $A PO A' CongA A0 PO A1$ 
    by (simp add:  $\langle A PO A' CongA A0 PO A1 \rangle$ )
  show  $A0 PO A1 CongA A' PO A''$ 
proof (rule sams2-suma2--conga456 [where  $?A=P$  and  $?B=Q$  and  $?C=R$ 
  and  $?G=P'$  and  $?H=Q'$  and  $?I=R'$ ])
  show  $SAMS P Q R A0 PO A1$ 
    using  $\langle SAMS P Q R A0 PO A1 \rangle$  by auto
  show  $SAMS P Q R A' PO A''$ 
proof (rule conga2-sams--sams [where  $?A=A0$  and  $?B=PO$  and  $?C=A'$  and
   $?D=A'$  and  $?E=PO$  and  $?F=A''$ ])
  show  $A0 PO A' CongA P Q R$ 
    using  $\langle P Q R CongA A0 PO A' \rangle$  conga-sym-equiv by blast
  show  $A' PO A'' CongA A' PO A''$ 
    using  $\langle PO \neq A'' \rangle \langle PO \neq A' \rangle$  conga-refl by auto
  show  $SAMS A0 PO A' A' PO A''$ 
    by (metis Col-cases bet-out bet-out-1 os2--sams out-one-side
       $\langle A' \neq A'' \rangle \langle A0 \neq A' \rangle \langle Bet A0 A' A'' \rangle \langle \neg Col A0 PO A'' \rangle$ )
qed
show  $P Q R A0 PO A1 SumA P' Q' R'$ 
  by (simp add:  $\langle P Q R A0 PO A1 SumA P' Q' R' \rangle$ )
show  $P Q R A' PO A'' SumA P' Q' R'$ 
proof (rule conga3-suma--suma [where  $?A=A0$  and  $?B=PO$  and  $?C=A'$  and
   $?D=A'$  and  $?E=PO$  and  $?F=A''$  and  $?G=A0$  and  $?H=PO$  and  $?I=A''$ ])
  show  $A0 PO A' A' PO A'' SumA A0 PO A''$ 
proof –
  have  $A' PO A'' CongA A' PO A''$ 
    using  $\langle PO \neq A'' \rangle \langle PO \neq A' \rangle$  conga-refl by auto
  moreover have  $\neg PO A' OS A0 A''$ 
    using  $\langle \neg PO A' OS A0 A'' \rangle$  by auto
  moreover have Coplanar  $A0 PO A' A''$ 
    using  $\langle Bet A0 A' A'' \rangle$  bet-col ncop--ncols by blast
  moreover have  $A0 PO A'' CongA A0 PO A''$ 
    using  $\langle PO \neq A'' \rangle$  assms(2) conga-refl by auto
  ultimately show ?thesis
    using SumA-def by blast
qed
show  $A0 PO A' CongA P Q R$ 
  using  $\langle P Q R CongA A0 PO A' \rangle$  conga-sym-equiv by auto
show  $A' PO A'' CongA A' PO A''$ 
  using  $\langle PO \neq A'' \rangle \langle PO \neq A' \rangle$  conga-refl by auto
show  $A0 PO A'' CongA P' Q' R'$ 
  using  $\langle P' Q' R' CongA A0 PO A'' \rangle$  conga-sym-equiv by auto
qed
qed
qed
have  $A A' Lt A' A''$ 
  using acute-archi-aux [where  $?PO=PO$  and  $?A=A0$  and  $?B=B$ ]
  by (metis ncol-conga-ncol not-col-distincts not-conga-sym
     $\langle A PO A' CongA A' PO A'' \rangle \langle A PO A' CongA A0 PO A1 \rangle \langle Bet A0 A A' \rangle$ 
     $\langle Bet A0 A' A'' \rangle \langle Bet A0 A'' B \rangle$  assms(1) assms(10) assms(3) between-exchange3)
hence  $A A' Le A' A''$ 
  using Lt-def by blast
hence  $A0 A1 Le A' A''$ 
  by (meson le-transitivity  $\langle A0 A1 Le A A' \rangle$ )

```

```

have Bet A0 A1 A''
  by (simp add: ⟨Bet A0 A1 A'⟩)
moreover have Bet A0 A'' B
  by (simp add: ⟨Bet A0 A'' B⟩)
moreover have P' Q' R' CongA A0 PO A''
  by (simp add: ⟨P' Q' R' CongA A0 PO A'⟩)
moreover have A0 C' Le A0 A''
  by (meson bet2-le2--le1346 ⟨A0 A1 Le A' A'⟩ ⟨A0 C Le A0 A'⟩
    ⟨Bet A0 A' A'⟩ assms(7) assms(8) cong-reflexivity l5-6)
moreover
have ∃ A. Bet A0 A A'' ∧ A PO A'' CongA A0 PO A1 ∧ A0 A1 Le A A''
proof -
  have Bet A0 A' A''
    using ⟨Bet A0 A' A'⟩ by fastforce
  moreover have A' PO A'' CongA A0 PO A1
    by (meson not-conga not-conga-sym ⟨A PO A' CongA A' PO A'⟩
      ⟨A PO A' CongA A0 PO A1⟩)
  moreover have A0 A1 Le A' A''
    using ⟨A0 A1 Le A' A'⟩ by auto
  ultimately show ?thesis by blast
qed
ultimately
have ∃ P Q R. (GradA A0 PO A1 P Q R ∧ (A0 PO B LeA P Q R ∨
  (∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ P Q R CongA A0 PO A' ∧ A0 C' Le A0 A' ∧
  (∃ A. Bet A0 A A' ∧ A PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))
  using ⟨GradA A0 PO A1 P' Q' R'⟩ by blast
}
ultimately
have ∃ P Q R. (GradA A0 PO A1 P Q R ∧ (A0 PO B LeA P Q R ∨
  (∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ P Q R CongA A0 PO A' ∧ A0 C' Le A0 A' ∧
  (∃ A. Bet A0 A A' ∧ A PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))
  using ⟨GradA A0 PO A1 P Q R⟩ by blast
}
ultimately
show ?thesis
  using P2 by blast
qed

lemma acute-archi-aux2:
  assumes Per PO A0 B and
    PO ≠ A0 and
    B ≠ A0 and
    Bet A0 A1 B and
    A0 ≠ A1 and
    Grad A0 A1 C
  shows ∃ P Q R. (GradA A0 PO A1 P Q R ∧ (A0 PO B LeA P Q R ∨
    (∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ P Q R CongA A0 PO A' ∧ A0 C Le A0 A' ∧
    (∃ A. Bet A0 A A' ∧ A PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))
proof -
  have ¬ Col PO A0 B
    by (metis assms(1) assms(2) assms(3) col-permutation-1 l8-8 per-col)
  have ¬ Col A0 PO A1
    by (metis ⟨¬ Col PO A0 B⟩ assms(4) assms(5) bet-col col-trivial-3 colx not-col-permutation-2)
  have PO ≠ A1
    using ⟨¬ Col A0 PO A1⟩ assms(4) bet-col1 by blast
  have PO ≠ B
    using assms(1) assms(3) per-distinct-1 by auto
  have GradI A0 A1 C
    by (simp add: Grad--GradI assms(6))
  let ?th = ∃ P Q R. (GradA A0 PO A1 P Q R ∧ (A0 PO B LeA P Q R ∨
    (∃ A'. Bet A0 A1 A' ∧ Bet A0 A' B ∧ P Q R CongA A0 PO A' ∧ A0 C Le A0 A' ∧
    (∃ A. Bet A0 A A' ∧ A PO A' CongA A0 PO A1 ∧ A0 A1 Le A A'))))
  have ?th
  proof (rule GradI.induct [where ?A=A0 and ?B=A1 and ?x=C])
    show GradI A0 A1 C
      using ⟨GradI A0 A1 C⟩ by blast
  end
end

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show  $\exists P Q R.$ 
   $\text{GradA } A0 PO A1 P Q R \wedge (A0 PO B \text{LeA } P Q R \vee$ 
     $(\exists A'. \text{Bet } A0 A1 A' \wedge \text{Bet } A0 A' B \wedge P Q R \text{CongA } A0 PO A' \wedge$ 
       $A0 A1 \text{Le } A0 A' \wedge (\exists A. \text{Bet } A0 A A' \wedge A PO A' \text{CongA } A0 PO A1 \wedge A0 A1 \text{Le } A A'))))$ 
by (metis bet-col assms(1) assms(2) assms(3) assms(4) assms(5) cong2-per2--cong-conga2
  cong-reflexivity grada-init le-reflexivity not-bet-distincts
  not-col-permutation-5 per-col)
{
  fix  $C0 C'$ 
  assume  $H1: \text{GradI } A0 A1 C0$  and
     $H2: \exists P Q R. \text{GradA } A0 PO A1 P Q R \wedge (A0 PO B \text{LeA } P Q R \vee$ 
       $(\exists A'. \text{Bet } A0 A1 A' \wedge \text{Bet } A0 A' B \wedge P Q R \text{CongA } A0 PO A' \wedge$ 
         $A0 C0 \text{Le } A0 A' \wedge (\exists A. \text{Bet } A0 A A' \wedge A PO A' \text{CongA } A0 PO A1 \wedge A0 A1 \text{Le } A A'))))$  and
     $H3: \text{Bet } A0 C0 C'$  and
     $H4: \text{Cong } A0 A1 C0 C'$ 
  have  $\exists P Q R. \text{GradA } A0 PO A1 P Q R \wedge (A0 PO B \text{LeA } P Q R \vee$ 
     $(\exists A'. \text{Bet } A0 A1 A' \wedge \text{Bet } A0 A' B \wedge P Q R \text{CongA } A0 PO A' \wedge$ 
       $A0 C' \text{Le } A0 A' \wedge (\exists A. \text{Bet } A0 A A' \wedge A PO A' \text{CongA } A0 PO A1 \wedge A0 A1 \text{Le } A A'))))$ 
  proof (rule acute-archi-aux2-2 [where ?C=C0])
    show  $\text{Per } PO A0 B$ 
      using assms(1) by auto
    show  $PO \neq A0$ 
      by (simp add: assms(2))
    show  $B \neq A0$ 
      by (simp add: assms(3))
    show  $\text{Bet } A0 A1 B$ 
      by (simp add: assms(4))
    show  $A0 \neq A1$ 
      by (simp add: assms(5))
    show  $\text{Grad } A0 A1 C0$ 
      by (simp add: Grad-GradI H1)
    show  $\text{Bet } A0 C0 C'$ 
      using  $H3$  by auto
    show  $\text{Cong } A0 A1 C0 C'$ 
      using  $H4$  by auto
    show  $\neg \text{Col } PO A0 B$ 
      using  $\langle \neg \text{Col } PO A0 B \rangle$  by auto
    show  $\neg \text{Col } A0 PO A1$ 
      by (simp add:  $\langle \neg \text{Col } A0 PO A1 \rangle$ )
    show  $PO \neq A1$ 
      by (simp add:  $\langle PO \neq A1 \rangle$ )
    show  $PO \neq B$ 
      using  $\langle PO \neq B \rangle$  by auto
    show  $\text{Per } PO A0 B \wedge PO \neq A0 \wedge B \neq A0 \wedge \text{Bet } A0 A1 B \wedge A0 \neq A1 \wedge$ 
       $\neg \text{Col } PO A0 B \wedge \neg \text{Col } A0 PO A1 \wedge PO \neq A1 \implies$ 
       $(\exists P Q R. \text{GradA } A0 PO A1 P Q R \wedge (A0 PO B \text{LeA } P Q R \vee$ 
         $(\exists A'. \text{Bet } A0 A1 A' \wedge \text{Bet } A0 A' B \wedge P Q R \text{CongA } A0 PO A' \wedge$ 
           $A0 C0 \text{Le } A0 A' \wedge (\exists A. \text{Bet } A0 A A' \wedge A PO A' \text{CongA } A0 PO A1 \wedge A0 A1 \text{Le } A A'))))$ 
      using  $H2$  by blast
  qed
}
thus  $\bigwedge C C'.$ 
   $\text{GradI } A0 A1 C \implies$ 
   $\exists P Q R. \text{GradA } A0 PO A1 P Q R \wedge (A0 PO B \text{LeA } P Q R \vee$ 
     $(\exists A'. \text{Bet } A0 A1 A' \wedge \text{Bet } A0 A' B \wedge P Q R \text{CongA } A0 PO A' \wedge$ 
       $A0 C \text{Le } A0 A' \wedge (\exists A. \text{Bet } A0 A A' \wedge A PO A' \text{CongA } A0 PO A1 \wedge A0 A1 \text{Le } A A')))) \implies$ 
   $\text{Bet } A0 C C' \implies$ 
   $\text{Cong } A0 A1 C C' \implies$ 
   $\exists P Q R.$ 
   $\text{GradA } A0 PO A1 P Q R \wedge$ 
   $(A0 PO B \text{LeA } P Q R \vee$ 
     $(\exists A'. \text{Bet } A0 A1 A' \wedge$ 
       $\text{Bet } A0 A' B \wedge$ 
       $P Q R \text{CongA } A0 PO A' \wedge$ 
       $A0 C' \text{Le } A0 A' \wedge (\exists A. \text{Bet } A0 A A' \wedge A PO A' \text{CongA } A0 PO A1 \wedge A0 A1 \text{Le } A A'))))$ 
by blast

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qed
thus ?thesis by blast
qed

lemma archi-in-acute-angles:
  assumes archimedes-axiom
  shows  $\forall A B C D E F. \neg \text{Col } A B C \wedge \text{Acute } D E F$ 
     $\longrightarrow (\exists P Q R. \text{GradA } A B C P Q R \wedge D E F \text{ LeA } P Q R)$ 
proof -
{
  fix A B C D E F
  assume  $\neg \text{Col } A B C$  and
    Acute D E F
  have  $A \neq B$ 
    using  $\langle \neg \text{Col } A B C \rangle$  col-trivial-1 by fastforce
  have  $C \neq B$ 
    using  $\langle \neg \text{Col } A B C \rangle$  col-trivial-2 by force
  have  $E \neq D$ 
    using  $\langle \text{Acute } D E F \rangle$  acute-distincts by blast
  have  $E \neq F$ 
    using  $\langle \text{Acute } D E F \rangle$  acute-distincts by blast
  have  $\exists P Q R. \text{GradA } A B C P Q R \wedge D E F \text{ LeA } P Q R$ 
  proof (cases Col D E F)
    case True
    thus ?thesis
      by (metis Col-def Out-cases bet-out bet-out-1 l11-31-1  $\langle A \neq B \rangle$ 
         $\langle \text{Acute } D E F \rangle \langle C \neq B \rangle \langle E \neq D \rangle \langle E \neq F \rangle$  acute-not-bet grada-ABC)
  next
    case False
    {
      assume D E F LeA A B C
      hence  $\exists P Q R. \text{GradA } A B C P Q R \wedge D E F \text{ LeA } P Q R$ 
        by (metis  $\langle A \neq B \rangle \langle C \neq B \rangle$  grada-ABC)
    }
  moreover
  {
    assume A B C LeA D E F
    obtain D0 where Col D E D0 and D E Perp F D0
      using False l8-18-existence by blast
    have E Out D0 D
    proof (rule acute-col-perp--out [where ?A=F])
      show Acute F E D
        using  $\langle \text{Acute } D E F \rangle$  acute-sym by blast
      show Col E D D0
        using  $\langle \text{Col } D E D0 \rangle$  not-col-permutation-4 by blast
      show E D Perp F D0
        using  $\langle D E \text{ Perp } F D0 \rangle$  perp-left-comm by blast
    qed
    have D0  $\neq$  F
      using  $\langle D E \text{ Perp } F D0 \rangle$  perp-not-eq-2 by blast
    have D0  $\neq$  E
      using  $\langle E \text{ Out } D0 D \rangle$  out-diff1 by auto
    have D E F CongA D0 E F
      by (metis acute-col-perp--out acute-sym out2--conga out-trivial
        perp-left-comm  $\langle \text{Acute } D E F \rangle \langle \text{Col } D E D0 \rangle \langle D E \text{ Perp } F D0 \rangle \langle E \neq F \rangle$ 
        not-col-permutation-4)
    have Acute D0 E F
      by (meson acute-conga--acute  $\langle \text{Acute } D E F \rangle \langle D E F \text{ CongA } D0 E F \rangle$ )
    have A B C LeA D0 E F
      by (meson conga--lea lea-trans  $\langle A B C \text{ LeA } D E F \rangle \langle D E F \text{ CongA } D0 E F \rangle$ )
    have  $\neg \text{Col } D E F$ 
      by (simp add: False)
    have Per E D0 F
      by (meson Per-cases l8-16-1  $\langle \text{Col } D E D0 \rangle \langle D E \text{ Perp } F D0 \rangle$  col-trivial-2)
    obtain D1' where A B C CongA D0 E D1' and D0 E OS D1' F
      by (metis Col-cases False angle-construction-1  $\langle \text{Col } D E D0 \rangle$ )
  }
}

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    ⟨D0 ≠ E⟩ ⟨¬ Col A B C⟩ col2--eq)
have D0 E F CongA D0 E F
  using ⟨D0 ≠ E⟩ ⟨E ≠ F⟩ conga-refl by auto
hence D0 E D1' LeA D0 E F
  using l11-30 [where ?A=A and ?B=B and ?C=C and ?D=D0 and ?E=E and ?F=F]
    ⟨A B C LeA D0 E F⟩ ⟨A B C CongA D0 E D1'⟩ by blast
have D0 E OS F D1'
  by (simp add: ⟨D0 E OS D1' F⟩ one-side-symmetry)
hence D1' InAngle D0 E F
  using lea-in-angle ⟨D0 E D1' LeA D0 E F⟩ by blast
then obtain D1 where Bet D0 D1 F and D1 = E ∨ E Out D1 D1'
  using InAngle-def by force
have D1 = E ⟶ (∃ P Q R. GradA A B C P Q R ∧ D E F LeA P Q R)
  using ⟨Acute D0 E F⟩ ⟨Bet D0 D1 F⟩ acute-not-bet by blast
moreover
{
  assume E Out D1 D1'
  have A B C CongA D0 E D1
  proof (rule l11-10 [where ?A=A and ?C=C and ?D=D0 and ?F=D1'],
    insert ⟨A B C CongA D0 E D1'⟩ ⟨E Out D1 D1'⟩)
    show B Out A A
      using ⟨A ≠ B⟩ out-trivial by auto
    show B Out C C
      by (simp add: ⟨C ≠ B⟩ out-trivial)
    show E Out D0 D0
      by (simp add: ⟨D0 ≠ E⟩ out-trivial)
  qed
  have ¬ Col D0 E D1'
    using ⟨D0 E OS D1' F⟩ col123--nos by force
  have D0 ≠ D1
    using Col-cases ⟨E Out D1 D1'⟩ ⟨¬ Col D0 E D1'⟩ out-col by blast
  obtain F' where Bet D0 F F' and Cong F F' D0 F
    using segment-construction by blast
  obtain G where Grad D0 D1 G and D0 F' Le D0 G
    using Reach-def ⟨D0 ≠ D1⟩ archimedes-axiom-def assms by blast
  have GradI D0 D1 G
    by (simp add: Grad--GradI ⟨Grad D0 D1 G⟩)
  have ∃ P Q R. (GradA D0 E D1 P Q R ∧ (D0 E F LeA P Q R ∨
    (∃ A'. Bet D0 D1 A' ∧ Bet D0 A' F ∧ P Q R CongA D0 E A' ∧
      D0 G Le D0 A' ∧ (∃ A. Bet D0 A A' ∧ A E A' CongA D0 E D1 ∧ D0 D1 Le A A'))))
    using acute-archi-aux2 ⟨Bet D0 D1 F⟩ ⟨D0 ≠ D1⟩ ⟨D0 ≠ E⟩ ⟨D0 ≠ F⟩
      ⟨Grad D0 D1 G⟩ ⟨Per E D0 F⟩ by blast
  then obtain P Q R where
    GradA D0 E D1 P Q R and
    D0 E F LeA P Q R ∨ (∃ A'. Bet D0 D1 A' ∧ Bet D0 A' F ∧
      P Q R CongA D0 E A' ∧ D0 G Le D0 A' ∧
      (∃ A. Bet D0 A A' ∧ A E A' CongA D0 E D1 ∧ D0 D1 Le A A')) by blast
  have D0 ≠ E
    using grada-distincts [where ?A=D0 and ?B=E and ?C=D1 and
      ?D=P and ?E=Q and ?F=R] ⟨GradA D0 E D1 P Q R⟩ by blast
  have D1 ≠ E
    using grada-distincts [where ?A=D0 and ?B=E and ?C=D1 and
      ?D=P and ?E=Q and ?F=R] ⟨GradA D0 E D1 P Q R⟩ by blast
  have P ≠ Q
    using grada-distincts [where ?A=D0 and ?B=E and ?C=D1 and
      ?D=P and ?E=Q and ?F=R] ⟨GradA D0 E D1 P Q R⟩ by blast
  have R ≠ Q
    using grada-distincts [where ?A=D0 and ?B=E and ?C=D1 and
      ?D=P and ?E=Q and ?F=R] ⟨GradA D0 E D1 P Q R⟩ by blast
  have GradA A B C P Q R
  proof (rule conga2-grad--grada [where ?A=D0 and ?B=E and ?C=D1
    and ?D=P and ?E=Q and ?F=R], insert ⟨GradA D0 E D1 P Q R⟩)
    show D0 E D1 CongA A B C
      using ⟨A B C CongA D0 E D1⟩ conga-sym-equiv by auto
    show P Q R CongA P Q R
      by (simp add: ⟨P ≠ Q⟩ ⟨R ≠ Q⟩ conga-refl)
  qed

```

```

qed
moreover
have  $D E F LeA P Q R$ 
proof -
{
  assume  $D0 E F LeA P Q R$ 
  have  $D E F LeA P Q R$ 
  proof (rule l11-30 [where ?A=D0 and ?B=E and ?C=F and
    ?D=P and ?E=Q and ?F=R])
    show  $D0 E F LeA P Q R$ 
    by (simp add:  $\langle D0 E F LeA P Q R \rangle$ )
    show  $D0 E F CongA D E F$ 
    using  $\langle D E F CongA D0 E F \rangle$  conga-sym-equiv by auto
    show  $P Q R CongA P Q R$ 
    using  $\langle P \neq Q \rangle \langle R \neq Q \rangle$  conga-refl by auto
  qed
}
moreover
{
  assume  $\exists A'. Bet D0 D1 A' \wedge Bet D0 A' F \wedge P Q R CongA D0 E A' \wedge$ 
 $D0 G Le D0 A' \wedge (\exists A. Bet D0 A A' \wedge A E A' CongA D0 E D1 \wedge D0 D1 Le A A')$ 
  then obtain  $A'$  where  $Bet D0 D1 A'$  and  $Bet D0 A' F$  and
 $P Q R CongA D0 E A'$  and  $D0 G Le D0 A'$  and
 $\exists A. Bet D0 A A' \wedge A E A' CongA D0 E D1 \wedge D0 D1 Le A A'$ 
  by blast
  have  $D0 A' Le D0 F$ 
  by (simp add:  $\langle Bet D0 A' F \rangle$  bet--le1213)
  hence  $D0 G Le D0 F$ 
  using  $\langle D0 G Le D0 A' \rangle$  le-transitivity by blast
  moreover have  $D0 F Lt D0 F'$ 
  using  $\langle Bet D0 F F' \rangle \langle Cong F F' D0 F \rangle \langle D0 \neq F \rangle$ 
  bet--lt1213 cong-diff-3 by presburger
  ultimately have  $D0 G Lt D0 F'$ 
  using le1234-lt--lt by blast
  hence False
  using  $\langle D0 F' Le D0 G \rangle$  le--nlt by auto
  hence  $D E F LeA P Q R$ 
  by blast
}
ultimately show ?thesis
using  $\langle D0 E F LeA P Q R \vee (\exists A'. Bet D0 D1 A' \wedge Bet D0 A' F \wedge$ 
 $P Q R CongA D0 E A' \wedge D0 G Le D0 A' \wedge$ 
 $(\exists A. Bet D0 A A' \wedge A E A' CongA D0 E D1 \wedge D0 D1 Le A A')) \rangle$  by blast
qed
ultimately have  $\exists P Q R. GradA A B C P Q R \wedge D E F LeA P Q R$ 
by blast
}
ultimately have  $\exists P Q R. GradA A B C P Q R \wedge D E F LeA P Q R$ 
using  $\langle D1 = E \vee E Out D1 D1' \rangle$  by blast
}
ultimately show ?thesis using lea-total
by (metis  $\langle A \neq B \rangle \langle C \neq B \rangle \langle E \neq D \rangle \langle E \neq F \rangle$ )
qed
}
thus ?thesis
by blast
qed

lemma angles-archi-aux:
  assumes  $GradA A B C D E F$  and
 $GradA A B C G H I$  and
 $\neg SAMS D E F G H I$ 
  shows  $\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C$ 
proof -
  have  $\neg SAMS D E F G H I \longrightarrow (\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C)$ 
  proof (rule GradA.induct [OF assms(2)])

```

show $\bigwedge Da Ea Fa$.

$A B C CongA Da Ea Fa \implies \neg SAMS D E F Da Ea Fa \longrightarrow$

$(\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C)$

by (*metis Tarski-neutral-dimensionless.conga-refl Tarski-neutral-dimensionless-axioms*
assms(1) conga2-sams--sams grada-distincts)

```
{
  fix Da Ea Fa G H I
  assume GradA A B C Da Ea Fa and
     $\neg SAMS D E F Da Ea Fa \longrightarrow (\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C)$  and
    SAMS Da Ea Fa A B C and
    Da Ea Fa A B C SumA G H I
  {
    assume  $\neg SAMS D E F G H I$ 
    {
      assume SAMS D E F Da Ea Fa
      have  $E \neq D$ 
      using  $\langle SAMS D E F Da Ea Fa \rangle$  sams-distincts by blast
      have  $E \neq F$ 
      using  $\langle SAMS D E F Da Ea Fa \rangle$  sams-distincts by blast
      have  $Ea \neq Da$ 
      using  $\langle SAMS D E F Da Ea Fa \rangle$  sams-distincts by blast
      have  $Ea \neq Fa$ 
      using  $\langle SAMS D E F Da Ea Fa \rangle$  sams-distincts by blast
      obtain P Q R where D E F Da Ea Fa SumA P Q R
      using ex-suma  $\langle E \neq D \rangle \langle E \neq F \rangle \langle Ea \neq Da \rangle \langle Ea \neq Fa \rangle$  by presburger
      have GradA A B C P Q R
      using grada2-sams--suma--grada [where ?D=D and ?E=E and ?F=F and
        ?G=Da and ?H=Ea and ?I=Fa]
        assms(1)  $\langle GradA A B C Da Ea Fa \rangle \langle SAMS D E F Da Ea Fa \rangle \langle D E F Da Ea Fa SumA P Q R \rangle$ 
      by blast
      moreover
      {
        assume SAMS P Q R A B C
        have SAMS D E F G H I
        using  $\langle SAMS D E F Da Ea Fa \rangle \langle SAMS Da Ea Fa A B C \rangle \langle D E F Da Ea Fa SumA P Q R \rangle$ 
           $\langle Da Ea Fa A B C SumA G H I \rangle \langle SAMS P Q R A B C \rangle$  sams-assoc-1 by blast
        hence False
        using  $\langle \neg SAMS D E F G H I \rangle$  by blast
      }
      ultimately have  $\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C$ 
      by blast
    }
    hence  $\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C$ 
    using  $\langle \neg SAMS D E F Da Ea Fa \longrightarrow (\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C) \rangle$ 
    by blast
  }
  hence  $\neg SAMS D E F G H I \longrightarrow (\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C)$ 
  by blast
}
thus  $\bigwedge Da Ea Fa G H I$ .
  GradA A B C Da Ea Fa  $\implies$ 
   $\neg SAMS D E F Da Ea Fa \longrightarrow (\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C) \implies$ 
  SAMS Da Ea Fa A B C  $\implies$  Da Ea Fa A B C SumA G H I  $\implies$ 
   $\neg SAMS D E F G H I \longrightarrow (\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C)$ 
  by blast
qed
thus ?thesis
  using assms(3) by blast
qed
```

lemma *angles-archi-aux1*:

assumes *archimedes-axiom*

shows $\forall A B C D E F$.

$\neg Col A B C \wedge \neg Bet D E F \longrightarrow$

$(\exists P Q R. GradA A B C P Q R \wedge (D E F LeA P Q R \vee \neg SAMS P Q R A B C))$

proof –

```

{
  fix A B C D E F
  assume  $\neg$  Col A B C and
     $\neg$  Bet D E F
  have  $D \neq E$ 
    using  $\langle \neg$  Bet D E F  $\rangle$  between-trivial2 by auto
  have  $F \neq E$ 
    using  $\langle \neg$  Bet D E F  $\rangle$  not-bet-distincts by blast
  obtain F1 where F1 InAngle D E F and F1 E D CongA F1 E F
    using angle-bisector  $\langle D \neq E \rangle \langle F \neq E \rangle$  by blast
  have  $F1 \neq E$ 
    using  $\langle F1 E D$  CongA F1 E F  $\rangle$  conga-distinct by auto
  have  $\neg$  E F1 OS D F
  proof (cases Col D E F1)
    case True
      thus ?thesis
        using NCol-perm col123--nos by blast
    next
      case False
      thus ?thesis
        by (meson  $\langle F1$  InAngle D E F  $\rangle$  col124--nos col-permutation-4 col-permutation-5
            in-angle-two-sides invert-one-side l9-9)
  qed
  have D E F1 D E F1 SumA D E F
    by (meson conga-refl  $\langle D \neq E \rangle \langle F \neq E \rangle \langle F1 E D$  CongA F1 E F  $\rangle$ 
         $\langle F1$  InAngle D E F  $\rangle \langle F1 \neq E \rangle$  conga3-suma--suma
        conga-left-comm inangle--suma not-conga-sym)
  have SAMS D E F1 D E F1
  proof -
    {
      assume Bet D E F1
      hence False
        using bet-in-angle-bet  $\langle F1$  InAngle D E F  $\rangle \langle \neg$  Bet D E F  $\rangle$  by blast
    }
    hence E Out D F1  $\vee \neg$  Bet D E F1
      by blast
    moreover
    have  $\exists J. F1 E J$  CongA D E F1  $\wedge \neg$  E F1 OS D J  $\wedge \neg$  D E TS F1 J  $\wedge$  Coplanar D E F1 J
    proof -
      have F1 E F CongA D E F1
        using  $\langle F1 E D$  CongA F1 E F  $\rangle$  conga-left-comm conga-sym-equiv by blast
      moreover have  $\neg$  E F1 OS D F
        by (simp add:  $\langle \neg$  E F1 OS D F  $\rangle$ )
      moreover have  $\neg$  D E TS F1 F
      proof (cases Col D E F1)
        case True
          thus ?thesis
            using TS-def col-permutation-2 by blast
        next
          case False
          thus ?thesis
            by (metis Col-cases TS-def  $\langle F1$  InAngle D E F  $\rangle$  in-angle-one-side l9-9)
      qed
      moreover have Coplanar D E F1 F
        by (meson inangle--coplanar  $\langle F1$  InAngle D E F  $\rangle$  coplanar-perm-8)
      ultimately show ?thesis
        by blast
    qed
    ultimately show ?thesis
      using SAMS-def  $\langle D \neq E \rangle$  by auto
  qed
  hence Acute D E F1
    by (metis nbet-sams-suma--acute  $\langle D E F1 D E F1$  SumA D E F  $\rangle \langle \neg$  Bet D E F  $\rangle$ )
  then obtain P1 Q1 R1 where GradA A B C P1 Q1 R1 and D E F1 LeA P1 Q1 R1
    using archi-in-acute-angles  $\langle \neg$  Col A B C  $\rangle$  assms by blast
  have  $P1 \neq Q1$ 

```

```

    using ⟨GradA A B C P1 Q1 R1⟩ grada-distincts by blast
have Q1 ≠ R1
    using ⟨GradA A B C P1 Q1 R1⟩ grada-distincts by blast
{
  assume SAMS P1 Q1 R1 P1 Q1 R1
  obtain P Q R where P1 Q1 R1 P1 Q1 R1 SumA P Q R
    using ex-suma ⟨P1 ≠ Q1⟩ ⟨Q1 ≠ R1⟩ by presburger
  have GradA A B C P Q R
    using grada2-sams-suma--grada [where ?D=P1 and ?E=Q1 and ?F=R1 and
      ?G=P1 and ?H=Q1 and ?I=R1] ⟨GradA A B C P1 Q1 R1⟩ ⟨GradA A B C P1 Q1 R1⟩
      ⟨SAMS P1 Q1 R1 P1 Q1 R1⟩ ⟨P1 Q1 R1 P1 Q1 R1 SumA P Q R⟩ by blast
  moreover have D E F LeA P Q R
    using sams-lea2-suma2--lea [where ?A=D and ?B=E and ?C=F1 and
      ?D=D and ?E=E and ?F=F1 and
      ?A'=P1 and ?B'=Q1 and ?C'=R1 and ?D'=P1 and ?E'=Q1 and ?F'=R1]
      ⟨D E F1 LeA P1 Q1 R1⟩ ⟨SAMS P1 Q1 R1 P1 Q1 R1⟩ ⟨D E F1 D E F1 SumA D E F⟩
      ⟨P1 Q1 R1 P1 Q1 R1 SumA P Q R⟩ by blast
  ultimately have ∃ P Q R. GradA A B C P Q R ∧ (D E F LeA P Q R ∨ ¬ SAMS P Q R A B C)
    by blast
}
moreover
{
  assume ¬ SAMS P1 Q1 R1 P1 Q1 R1
  hence ∃ P Q R. GradA A B C P Q R ∧ (D E F LeA P Q R ∨ ¬ SAMS P Q R A B C)
    using angles-archi-aux ⟨GradA A B C P1 Q1 R1⟩ calculation by blast
}
ultimately have ∃ P Q R. GradA A B C P Q R ∧ (D E F LeA P Q R ∨ ¬ SAMS P Q R A B C)
  by blast
}
thus ?thesis
  by blast
qed

```

lemma archi-in-angles:

assumes archimedes-axiom

shows $\forall A B C. \forall D :: 'p. \forall E :: 'p. \forall F :: 'p. (\neg \text{Col } A B C \wedge D \neq E \wedge F \neq E) \longrightarrow$
 $(\exists P Q R. \text{GradA } A B C P Q R \wedge (D E F \text{ LeA } P Q R \vee \neg \text{SAMS } P Q R A B C))$

proof –

```

{
  fix A B C
  fix D::'p
  fix E::'p
  fix F::'p
  assume ¬ Col A B C and
    D ≠ E and
    F ≠ E
  have ∃ P Q R. (GradA A B C P Q R ∧ (D E F LeA P Q R ∨ ¬ SAMS P Q R A B C))
    proof (cases Bet D E F)
      case True
      obtain A0 where Bet A B A0 and Cong B A0 A B
        using segment-construction by blast
      have A ≠ B
        using ⟨¬ Col A B C⟩ not-col-distincts by blast
      have C ≠ B
        using ⟨¬ Col A B C⟩ col-trivial-2 by auto
      have B ≠ A0
        using ⟨A ≠ B⟩ ⟨Cong B A0 A B⟩ cong-reverse-identity by blast
      have ¬ Col A0 B C
        by (meson ⟨B ≠ A0⟩ ⟨Bet A B A0⟩ ⟨¬ Col A B C⟩ bet-col col2--eq col-permutation-3)
      obtain P1 Q1 R1 where GradA A B C P1 Q1 R1 and
        C B A0 LeA P1 Q1 R1 ∨ ¬ SAMS P1 Q1 R1 A B C
        using angles-archi-aux1
        by (metis Col-def ⟨¬ Col A B C⟩ ⟨¬ Col A0 B C⟩ assms between-symmetry)
    }
  assume SAMS P1 Q1 R1 A B C

```

```

hence  $C B A0 LeA P1 Q1 R1$ 
  using  $\langle C B A0 LeA P1 Q1 R1 \vee \neg SAMS P1 Q1 R1 A B C \rangle$  by auto
have  $P1 \neq Q1$ 
  using  $\langle SAMS P1 Q1 R1 A B C \rangle$ 
  by (simp add: sams-distincts)
have  $R1 \neq Q1$ 
  using  $\langle C B A0 LeA P1 Q1 R1 \rangle$  lea-distincts by blast
obtain  $P Q R$  where  $P1 Q1 R1 A B C SumA P Q R$ 
  using ex-suma  $\langle A \neq B \rangle \langle C \neq B \rangle \langle P1 \neq Q1 \rangle \langle R1 \neq Q1 \rangle$  by presburger
have  $GradA A B C P Q R$ 
  using grada-stab [where  $?D=P1$  and  $?E=Q1$  and  $?F=R1$ ]
   $\langle GradA A B C P1 Q1 R1 \rangle \langle SAMS P1 Q1 R1 A B C \rangle \langle P1 Q1 R1 A B C SumA P Q R \rangle$  by auto
moreover
have  $P \neq Q$ 
  using calculation grada-distincts by blast
have  $R \neq Q$ 
  using calculation grada-distincts by blast
have  $A B A0 LeA P Q R$ 
proof (rule sams-lea2-suma2--lea [where
   $?A=A0$  and  $?B=B$  and  $?C=C$  and  $?D=A$  and  $?E=B$  and  $?F=C$ 
  and  $?A'=P1$  and  $?B'=Q1$  and  $?C'=R1$  and  $?D'=A$  and  $?E'=B$  and  $?F'=C$ ])
  show  $A0 B C LeA P1 Q1 R1$ 
    using  $\langle C B A0 LeA P1 Q1 R1 \rangle$  lea-left-comm by blast
  show  $A B C LeA A B C$ 
    using  $\langle A \neq B \rangle \langle C \neq B \rangle$  lea-refl by force
  show  $SAMS P1 Q1 R1 A B C$ 
    using  $\langle SAMS P1 Q1 R1 A B C \rangle$  by auto
  show  $A0 B C A B C SumA A B A0$ 
    by (metis Bet-cases  $\langle A \neq B \rangle \langle B \neq A0 \rangle \langle Bet A B A0 \rangle \langle C \neq B \rangle$ 
      bet--suma suma-middle-comm suma-right-comm)
  show  $P1 Q1 R1 A B C SumA P Q R$ 
    using  $\langle P1 Q1 R1 A B C SumA P Q R \rangle$  by auto
qed
hence  $Bet P Q R$ 
  using  $\langle Bet A B A0 \rangle$  bet-lea--bet by blast
hence  $D E F LeA P Q R$ 
  using l11-31-2  $\langle D \neq E \rangle \langle F \neq E \rangle \langle P \neq Q \rangle \langle R \neq Q \rangle$  by blast
ultimately have  $\exists P Q R. (GradA A B C P Q R \wedge (D E F LeA P Q R \vee \neg SAMS P Q R A B C))$ 
  by blast
}
moreover
{
  assume  $\neg SAMS P1 Q1 R1 A B C$ 
  hence  $\exists P Q R. (GradA A B C P Q R \wedge (D E F LeA P Q R \vee \neg SAMS P Q R A B C))$ 
    using  $\langle GradA A B C P1 Q1 R1 \rangle$  by blast
}
ultimately show ?thesis
  by blast
next
case False
thus ?thesis
  using  $\langle \neg Col A B C \rangle$  angles-archi-aux1 assms by blast
qed
}
thus ?thesis
  by blast
qed

```

```

lemma archi--grada-destruction:
  assumes archimedes-axiom
  shows  $\forall A B C. \neg Col A B C \longrightarrow$ 
     $(\exists P Q R. GradA A B C P Q R \wedge \neg SAMS P Q R A B C)$ 
proof –
{
  fix  $A B C$ 

```

```

assume  $\neg Col\ A\ B\ C$ 
obtain  $A0$  where  $Bet\ A\ B\ A0$  and  $Cong\ B\ A0\ A\ B$ 
  using segment-construction by blast
have  $A \neq B$ 
  using  $\langle \neg Col\ A\ B\ C \rangle$  col-trivial-1 by blast
have  $B \neq A0$ 
  using  $\langle A \neq B \rangle \langle Cong\ B\ A0\ A\ B \rangle$  cong-reverse-identity by blast
obtain  $P\ Q\ R$  where  $GradA\ A\ B\ C\ P\ Q\ R$  and  $A\ B\ A0\ LeA\ P\ Q\ R \vee \neg SAMS\ P\ Q\ R\ A\ B\ C$ 
  using archi-in-angles  $\langle A \neq B \rangle \langle B \neq A0 \rangle \langle \neg Col\ A\ B\ C \rangle$  assms by metis
{
  assume  $A\ B\ A0\ LeA\ P\ Q\ R$ 
  assume  $SAMS\ P\ Q\ R\ A\ B\ C$ 
  hence  $B\ Out\ A\ C \vee \neg Bet\ P\ Q\ R$ 
    using SAMS-def by blast
  have  $B\ Out\ A\ C \longrightarrow False$ 
    using Col-cases  $\langle \neg Col\ A\ B\ C \rangle$  out-col by blast
  moreover
  {
    assume  $\neg Bet\ P\ Q\ R$ 
    have  $Bet\ A\ B\ A0$ 
      by (simp add: Bet A B A0)
    hence False
      using bet-lea--bet  $\langle A\ B\ A0\ LeA\ P\ Q\ R \rangle \langle B\ Out\ A\ C \vee \neg Bet\ P\ Q\ R \rangle$  calculation by blast
  }
  hence  $\neg Bet\ P\ Q\ R \longrightarrow False$ 
    by blast
  ultimately have False
    using  $\langle B\ Out\ A\ C \vee \neg Bet\ P\ Q\ R \rangle$  by fastforce
}
hence  $A\ B\ A0\ LeA\ P\ Q\ R \longrightarrow \neg SAMS\ P\ Q\ R\ A\ B\ C$ 
  by blast
hence  $\neg SAMS\ P\ Q\ R\ A\ B\ C$ 
  using  $\langle A\ B\ A0\ LeA\ P\ Q\ R \vee \neg SAMS\ P\ Q\ R\ A\ B\ C \rangle$  by blast
hence  $\exists\ P\ Q\ R. GradA\ A\ B\ C\ P\ Q\ R \wedge \neg SAMS\ P\ Q\ R\ A\ B\ C$ 
  using  $\langle GradA\ A\ B\ C\ P\ Q\ R \rangle$  by blast
}
thus ?thesis
  by blast
qed

```

```

lemma gradaexp-destruction-aux:
  assumes  $GradA\ A\ B\ C\ P\ Q\ R$ 
  shows  $\exists\ S\ T\ U. GradAExp\ A\ B\ C\ S\ T\ U \wedge (Obtuse\ S\ T\ U \vee P\ Q\ R\ LeA\ S\ T\ U)$ 
proof (rule GradA.induct [OF assms(1)])
  show  $\bigwedge D\ E\ F. A\ B\ C\ CongA\ D\ E\ F \implies$ 
     $\exists\ S\ T\ U. GradAExp\ A\ B\ C\ S\ T\ U \wedge (Obtuse\ S\ T\ U \vee D\ E\ F\ LeA\ S\ T\ U)$ 
    by (metis conga--lea456123 conga-diff1 conga-diff2 gradaexp-ABC)
  {
    fix  $D\ E\ F\ G\ H\ I$ 
    assume  $GradA\ A\ B\ C\ D\ E\ F$  and
       $\exists\ S\ T\ U. GradAExp\ A\ B\ C\ S\ T\ U \wedge (Obtuse\ S\ T\ U \vee D\ E\ F\ LeA\ S\ T\ U)$  and
       $SAMS\ D\ E\ F\ A\ B\ C$  and
       $D\ E\ F\ A\ B\ C\ SumA\ G\ H\ I$ 
    then obtain  $P\ Q\ R$  where  $GradAExp\ A\ B\ C\ P\ Q\ R$  and  $Obtuse\ P\ Q\ R \vee D\ E\ F\ LeA\ P\ Q\ R$ 
      by blast
    have  $P \neq Q$ 
      using  $\langle GradAExp\ A\ B\ C\ P\ Q\ R \rangle$  gradaexp-distincts by blast
    have  $R \neq Q$ 
      using  $\langle GradAExp\ A\ B\ C\ P\ Q\ R \rangle$  gradaexp-distincts by blast
    {
      assume  $SAMS\ P\ Q\ R\ P\ Q\ R$ 
      {
        assume  $Obtuse\ P\ Q\ R$ 
        hence  $\exists\ S\ T\ U. GradAExp\ A\ B\ C\ S\ T\ U \wedge (Obtuse\ S\ T\ U \vee G\ H\ I\ LeA\ S\ T\ U)$ 
          using  $\langle GradAExp\ A\ B\ C\ P\ Q\ R \rangle$  by blast
      }
    }
  }

```

```

moreover
{
  assume  $D E F \text{ LeA } P Q R$ 
  obtain  $S T U$  where  $P Q R P Q R \text{ SumA } S T U$ 
  using  $ex\text{-}suma \langle P \neq Q \rangle \langle R \neq Q \rangle$  by presburger
  have  $\text{GradAExp } A B C S T U$ 
  proof (rule gradaexp-stab [where  $?D=P$  and  $?E=Q$  and  $?F=R$ ])
    show  $\text{GradAExp } A B C P Q R$ 
    by (simp add:  $\langle \text{GradAExp } A B C P Q R \rangle$ )
    show  $\text{SAMS } P Q R P Q R$ 
    by (simp add:  $\langle \text{SAMS } P Q R P Q R \rangle$ )
    show  $P Q R P Q R \text{ SumA } S T U$ 
    using  $\langle P Q R P Q R \text{ SumA } S T U \rangle$  by auto
  qed
  moreover
  have  $\text{GradA } A B C P Q R$ 
  using  $\langle \text{GradAExp } A B C P Q R \rangle$  gradaexp--grada by auto
  hence  $G H I \text{ LeA } S T U$ 
  by (meson  $\langle D E F A B C \text{ SumA } G H I \rangle \langle D E F \text{ LeA } P Q R \rangle \langle P Q R P Q R \text{ SumA } S T U \rangle$ 
     $\langle \text{SAMS } P Q R P Q R \rangle$  grada--lea sams-lea2-suma2--lea)
  ultimately have  $\exists S T U. \text{GradAExp } A B C S T U \wedge (\text{Obtuse } S T U \vee G H I \text{ LeA } S T U)$ 
  by blast
}
ultimately have  $\exists S T U. \text{GradAExp } A B C S T U \wedge (\text{Obtuse } S T U \vee G H I \text{ LeA } S T U)$ 
using  $\langle \text{Obtuse } P Q R \vee D E F \text{ LeA } P Q R \rangle$  by blast
}
moreover
{
  assume  $\neg \text{SAMS } P Q R P Q R$ 
  hence  $\text{Obtuse } P Q R$ 
  using  $\langle P \neq Q \rangle \langle R \neq Q \rangle$  nsams--obtuse by auto
  hence  $\exists S T U. \text{GradAExp } A B C S T U \wedge (\text{Obtuse } S T U \vee G H I \text{ LeA } S T U)$ 
  using  $\langle \text{GradAExp } A B C P Q R \rangle$  by blast
}
ultimately have  $\exists S T U. \text{GradAExp } A B C S T U \wedge (\text{Obtuse } S T U \vee G H I \text{ LeA } S T U)$ 
by blast
}
thus  $\bigwedge D E F G H I.$ 
 $\text{GradA } A B C D E F \implies$ 
 $\exists S T U. \text{GradAExp } A B C S T U \wedge (\text{Obtuse } S T U \vee D E F \text{ LeA } S T U) \implies$ 
 $\text{SAMS } D E F A B C \implies$ 
 $D E F A B C \text{ SumA } G H I \implies \exists S T U. \text{GradAExp } A B C S T U \wedge (\text{Obtuse } S T U \vee G H I \text{ LeA } S T U)$ 
by blast
qed

```

lemma *archi--gradaexp-destruction:*

```

assumes archimedes-axiom
shows  $\forall A B C. \neg \text{Col } A B C \longrightarrow (\exists P Q R. \text{GradAExp } A B C P Q R \wedge \text{Obtuse } P Q R)$ 
proof –
{
  fix  $A B C$ 
  assume  $\neg \text{Col } A B C$ 
  obtain  $D E F$  where  $\text{GradA } A B C D E F$  and  $\neg \text{SAMS } D E F A B C$ 
  using archi--grada-destruction  $\langle \neg \text{Col } A B C \rangle$  assms by blast
  obtain  $P Q R$  where  $\text{GradAExp } A B C P Q R$  and  $\text{Obtuse } P Q R \vee D E F \text{ LeA } P Q R$ 
  using  $\langle \text{GradA } A B C D E F \rangle$  gradaexp-destruction-aux by blast
  have  $P \neq Q$ 
  using  $\langle \text{GradAExp } A B C P Q R \rangle$  gradaexp-distincts by blast
  have  $R \neq Q$ 
  using  $\langle \text{GradAExp } A B C P Q R \rangle$  gradaexp-distincts by blast
  {
    assume  $D E F \text{ LeA } P Q R$ 
    {
      assume  $\text{SAMS } P Q R P Q R$ 
      have  $A B C \text{ LeA } P Q R$ 

```



```

    using ⟨GradAExp A B C P Q R⟩ grada--lea gradaexp--grada by blast
  hence SAMS D E F A B C
    using sams-lea2--sams [where ?A'=P and ?B'=Q and ?C'=R and
      ?D'=P and ?E'=Q and ?F'=R]
      ⟨SAMS P Q R P Q R⟩ ⟨D E F LeA P Q R⟩ by blast
  hence False
    using ⟨¬ SAMS D E F A B C⟩ by blast
}
hence ¬ SAMS P Q R P Q R
  by blast
hence Obtuse P Q R
  using ⟨P ≠ Q⟩ ⟨R ≠ Q⟩ nsams--obtuse by auto
}
hence ∃ P Q R. GradAExp A B C P Q R ∧ Obtuse P Q R
  using ⟨GradAExp A B C P Q R⟩ ⟨Obtuse P Q R ∨ D E F LeA P Q R⟩ by blast
}
thus ?thesis
  by blast
qed

```

2.5 Equivalence Greenberg - Aristotle

```

lemma aristotle--greenberg:
  assumes aristotle-s-axiom
  shows greenberg-s-axiom
proof -
{
  fix P Q R A B C
  assume ¬ Col A B C and
    Acute A B C and Q ≠ R and Per P Q R
  have ∃ S. P S Q LtA A B C ∧ Q Out S R
  proof (cases P = Q)
    case True
    have P R P LtA A B C
    proof -
      have P R P LeA A B C
      using True ⟨Q ≠ R⟩ ⟨¬ Col A B C⟩
        lea121345 not-col-distincts by presburger
    moreover
    have ¬ P R P CongA A B C
      using ⟨¬ Col A B C⟩ col-conga-col col-trivial-3 by blast
    ultimately
    show ?thesis
      by (simp add: LtA-def)
  qed
  moreover
  have P Out R R
    using True ⟨Q ≠ R⟩ out-trivial by auto
  ultimately
  show ?thesis
    using True by blast
}
next
case False
have ∃ X Y. B Out A X ∧ B Out C Y ∧ Per B X Y ∧ P Q Lt X Y
  using aristotle-s-axiom-def ⟨¬ Col A B C⟩
    ⟨Acute A B C⟩ assms by blast
then obtain X Y where B Out A X and B Out C Y and
  Per B X Y and P Q Lt X Y
  by blast
have P Q Le X Y
  by (simp add: ⟨P Q Lt X Y⟩ lt--le)
have ¬ Cong P Q X Y
  using ⟨P Q Lt X Y⟩ cong--nlt by blast
have X ≠ Y
  using ⟨P Q Lt X Y⟩ lt-diff by presburger
have P ≠ Q

```

using *False* by *blast*
 hence $\exists T. Q \text{ Out } T P \wedge \text{Cong } Q T X Y$
 using *l6-11-existence* $\langle X \neq Y \rangle$ by *simp*
 then obtain *T* where $Q \text{ Out } T P$ and $\text{Cong } Q T X Y$
 by *blast*
 have $X \neq B$
 using $\langle B \text{ Out } A X \rangle$ *out-diff2* by *blast*
 have $R \neq Q$
 using $\langle Q \neq R \rangle$ by *auto*
 hence $\exists S. Q \text{ Out } S R \wedge \text{Cong } Q S X B$
 using *l6-11-existence* $\langle R \neq Q \rangle \langle X \neq B \rangle$ by *simp*
 then obtain *S* where $Q \text{ Out } S R$ and $\text{Cong } Q S X B$
 by *blast*
 have $\text{Per } S Q P$
 proof –
 have $\text{Per } R Q P$
 using $\langle \text{Per } P Q R \rangle$ *l8-2* by *blast*
 moreover
 have $\text{Col } Q R S$
 using $\langle Q \text{ Out } S R \rangle$ *l6-6 out-col* by *blast*
 ultimately
 show *?thesis*
 using *l8-3* $\langle R \neq Q \rangle$ by *blast*
 qed
 have $\text{Per } T Q S$
 proof –
 have $\text{Per } P Q S$
 by (*simp add*: $\langle \text{Per } S Q P \rangle$ *l8-2*)
 moreover
 have $\text{Col } Q P T$
 by (*simp add*: $\langle Q \text{ Out } T P \rangle$ *l6-6 out-col*)
 ultimately
 show *?thesis*
 by (*metis False l8-2 per-col*)
 qed
 have $P \neq S$
 using $\langle \text{Per } S Q P \rangle \langle Q \text{ Out } S R \rangle$ *l6-3-1 l8-8* by *blast*
 have $T \neq S$
 using $\langle \text{Per } T Q S \rangle \langle Q \text{ Out } T P \rangle$ *l6-3-1 l8-8* by *blast*
 have $P S Q \text{ LtA } A B C$
 proof –
 have $P S Q \text{ CongA } P S Q$
 by (*metis* $\langle P \neq S \rangle \langle Q \text{ Out } S R \rangle$ *conga-refl out-diff1*)
 moreover
 have $T S Q \text{ CongA } A B C$
 proof –
 have $T S Q \text{ CongA } X B Y$
 by (*metis Per-cases cong-diff-3* $\langle \text{Cong } Q S X B \rangle$
 $\langle \text{Cong } Q T X Y \rangle \langle \text{Per } B X Y \rangle \langle \text{Per } T Q S \rangle \langle Q \text{ Out } T P \rangle$
 $\langle T \neq S \rangle \langle X \neq B \rangle$ *conga-right-comm l10-12*
l11-51 l6-3-1 not-cong-2143)
 moreover
 have $X B Y \text{ CongA } A B C$
 by (*simp add*: $\langle B \text{ Out } A X \rangle \langle B \text{ Out } C Y \rangle$ *out2--conga*)
 ultimately
 show *?thesis*
 using *not-conga* by *blast*
 qed
 moreover
 have $P S Q \text{ LtA } T S Q$
 proof –
 have $Q S P \text{ LeA } Q S T$
 proof –
 have $P \text{ InAngle } Q S T$
 proof –
 have $\text{Bet } Q P T$

```

proof -
  have  $Q \text{ Out } P \ T$ 
    by (simp add:  $\langle Q \text{ Out } T \ P \rangle$  l6-6)
  moreover
  have  $Q \ P \ Le \ Q \ T$ 
  proof -
    have  $Q \ P \ Le \ X \ Y$ 
      using Le-cases  $\langle P \ Q \ Le \ X \ Y \rangle$  by blast
    moreover
    have  $X \ Y \ Le \ Q \ T$ 
      by (simp add:  $\langle Cong \ Q \ T \ X \ Y \rangle$  cong--l6-12)
    ultimately
    show ?thesis
      using le-transitivity by blast
  qed
  ultimately
  show ?thesis
    by (simp add: l6-13-1)
  qed
  moreover
  have  $P = S \vee S \text{ Out } P \ P$ 
    using out-trivial by blast
  ultimately
  show ?thesis
    using InAngle-def Out-def  $\langle P \neq S \rangle \langle Q \text{ Out } S \ R \rangle \langle T \neq S \rangle$  by auto
  qed
  moreover
  have  $Q \ S \ T \ CongA \ Q \ S \ T$ 
    by (metis  $\langle Q \text{ Out } S \ R \rangle \langle T \neq S \rangle$  conga-refl l6-3-1)
  ultimately
  show ?thesis
    by (simp add: inangle--lea)
  qed
  hence  $P \ S \ Q \ LeA \ T \ S \ Q$ 
    by (simp add: lea-comm)
  moreover
  {
    assume  $P \ S \ Q \ CongA \ T \ S \ Q$ 
    have  $Cong \ Q \ P \ Q \ T \wedge Cong \ S \ P \ S \ T \wedge Q \ P \ S \ CongA \ Q \ T \ S$ 
    proof -
      {
        assume  $Col \ Q \ S \ P$ 
        hence  $Col \ S \ Q \ P$ 
          using Col-cases by blast
        hence  $S = Q \vee P = Q$ 
          by (simp add:  $\langle Per \ S \ Q \ P \rangle$  l8-9)
        hence False
          using False Out-def  $\langle Q \text{ Out } S \ R \rangle$  by blast
      }
    hence  $\neg Col \ Q \ S \ P$ 
      by blast
    moreover
    have  $S \ Q \ P \ CongA \ S \ Q \ T$ 
      using  $\langle Q \text{ Out } S \ R \rangle \langle Q \text{ Out } T \ P \rangle$  out2--conga
      out-diff1 out-trivial by force
    moreover
    have  $Q \ S \ P \ CongA \ Q \ S \ T$ 
      by (simp add:  $\langle P \ S \ Q \ CongA \ T \ S \ Q \rangle$  conga-comm)
    moreover
    have  $Cong \ Q \ S \ Q \ S$ 
      by (simp add: cong-reflexivity)
    ultimately
    show ?thesis
      by (metis l11-50-1)
  }
  qed
  have  $Cong \ P \ Q \ X \ Y$ 

```

```

proof –
  have Cong P Q T Q
    using Cong-cases
       $\langle \text{Cong } Q \ P \ Q \ T \wedge \text{Cong } S \ P \ S \ T \wedge Q \ P \ S \text{ CongA } Q \ T \ S \rangle$ 
    by blast
  moreover
  have Cong T Q X Y
    by (meson  $\langle \text{Cong } Q \ T \ X \ Y \rangle$  not-cong-2134)
  ultimately
  show ?thesis
    using cong-transitivity by blast
  qed
  hence False
    using  $\langle \neg \text{Cong } P \ Q \ X \ Y \rangle$  by auto
}
hence  $\neg P \ S \ Q \ \text{CongA } T \ S \ Q$ 
  by auto
ultimately
show ?thesis
  by (simp add: LtA-def)
qed
ultimately
show ?thesis
  using conga-preserves-lta by blast
qed
moreover
have Q Out S R
  by (simp add:  $\langle Q \ \text{Out } S \ R \rangle$ )
ultimately
show ?thesis
  by blast
qed
}
thus ?thesis
  by (simp add: greenberg-s-axiom-def)
qed

```

lemma *greenberg--aristotle:*

assumes *greenberg-s-axiom*

shows *aristotle-s-axiom*

proof –

```

{
  fix P Q A B C
  assume  $\neg \text{Col } A \ B \ C$  and Acute A B C
  obtain D' where A B Perp D' B and A B OS C D'
    using  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-2 l10-15 by blast
  have  $\exists X. \text{Col } A \ B \ X \wedge A \ B \ \text{Perp } C \ X$ 
    using  $\langle \neg \text{Col } A \ B \ C \rangle$  l8-18-existence by auto
  then obtain X where Col A B X and A B Perp C X
    by blast
  have  $\exists X \ Y. B \ \text{Out } A \ X \wedge B \ \text{Out } C \ Y \wedge \text{Per } B \ X \ Y \wedge P \ Q \ \text{Lt } X \ Y$ 
  proof (cases P = Q)
    case True
    have B Out A X
    proof –
      have Acute C B A
        by (simp add:  $\langle \text{Acute } A \ B \ C \rangle$  acute-sym)
      moreover
      have Col B A X
        using  $\langle \text{Col } A \ B \ X \rangle$  not-col-permutation-4 by blast
      moreover
      have B A Perp C X
        by (simp add: perp-left-comm  $\langle A \ B \ \text{Perp } C \ X \rangle$ )
      ultimately

```

```

  show ?thesis
  using acute-col-perp--out l6-6 by blast
qed
moreover
have B Out C C
  by (metis  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-2 out-trivial)
moreover
have Per B X C
proof (cases B = X)
  case True
  thus ?thesis
  using l8-20-1-R1 by force
next
  case False
  have Col A B B
  using col-trivial-2 by blast
  thus ?thesis
  by (metis  $\langle \text{Col } A \ B \ X \rangle$   $\langle A \ B \ \text{Perp } C \ X \rangle$  False perp-col2 perp-per-1)
qed
moreover
have X  $\neq$  C
  using  $\langle \text{Col } A \ B \ X \rangle$   $\langle \neg \text{Col } A \ B \ C \rangle$  by auto
hence P P Lt X C
  using lt1123 by simp
ultimately
show ?thesis
  using True by blast
next
  case False
  hence P  $\neq$  Q
  by simp
  have B  $\neq$  D'
  using  $\langle A \ B \ OS \ C \ D' \rangle$  os-distincts by blast
  then obtain P' where B Out D' P' and Cong B P' P Q
  using segment-construction-3  $\langle P \neq Q \rangle$  by blast
  have  $\exists C'. P' C' B \text{ LtA } A \ B \ C \wedge B \text{ Out } C' A$ 
  proof -
    have B  $\neq$  A
    using  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-1 by blast
    moreover
    have Per P' B A
    by (metis Perp-cases  $\langle A \ B \ \text{Perp } D' \ B \rangle$ 
       $\langle B \text{ Out } D' \ P' \rangle$  l8-20-1-R1 out-col
      perp-col perp-per-1)
    ultimately
    show ?thesis
    using  $\langle \neg \text{Col } A \ B \ C \rangle$   $\langle \text{Acute } A \ B \ C \rangle$ 
      greenberg-s-axiom-def assms by blast
  qed
  then obtain C' where P' C' B LtA A B C and B Out C' A
  by blast
  have  $\exists D''. B \ C' \ \text{Perp } D'' \ C' \wedge B \ C' \ OS \ C \ D''$ 
  proof -
    have Col B C' C'
    using col-trivial-2 by blast
    moreover
    have  $\neg \text{Col } B \ C' \ C$ 
    by (metis Out-def  $\langle B \text{ Out } C' \ A \rangle$   $\langle \neg \text{Col } A \ B \ C \rangle$ 
      col-transitivity-1 not-col-permutation-4 out-col)
    ultimately
    show ?thesis
    by (simp add: l10-15)
  qed
  then obtain D'' where B C' Perp D'' C' and B C' OS C D''
  by blast
  have C'  $\neq$  D''

```

```

    using ⟨B C' Perp D'' C'⟩ perp-not-eq-2 by blast
  then obtain P'' where C' Out D'' P'' and Cong C' P'' P Q
    using False segment-construction-3 by blast
  have B ≠ C
    using ⟨¬ Col A B C⟩ col-trivial-2 by blast
  have B ≠ P''
    by (metis Col-cases ⟨B C' OS C D''⟩
        ⟨C' Out D'' P''⟩ one-side-not-col124 out-col)
  then obtain Z where B Out C Z and Cong B Z B P''
    using segment-construction-3 ⟨B ≠ C⟩ by blast
  have ¬ Col A B Z
    by (metis Col-cases Out-def ⟨B Out C Z⟩
        ⟨¬ Col A B C⟩ l6-16-1 out-col)
  then obtain Z' where Col A B Z' and A B Perp Z Z'
    using l8-18-existence by blast
  have B Out A Z'
    by (metis ⟨A B Perp Z Z'⟩ ⟨Acute A B C⟩
        ⟨B Out C Z⟩ ⟨B Out C' A⟩ ⟨Col A B Z'⟩
        acute-col-perp--out acute-out2--acute
        acute-sym bet2--out l6-3-1 l6-6
        not-col-permutation-4 perp-left-comm)
  moreover
  have B Out C Z
    by (simp add: ⟨B Out C Z⟩)
  moreover
  have Per B Z' Z
  proof –
    have A B Perp Z' Z
      using Perp-perm ⟨A B Perp Z Z'⟩ by blast
    moreover
    have Z ≠ B
      using ⟨¬ Col A B Z⟩ col-trivial-2 by blast
    moreover
    have Col A B Z'
      by (simp add: ⟨Col A B Z'⟩)
    moreover
    have Col A B B
      by (simp add: col-trivial-2)
    ultimately
    show ?thesis
      by (meson l8-16-1 ⟨A B Perp Z Z'⟩ l8-2)
  qed
  moreover
  have C' P'' Lt Z' Z
  proof –
    have Per B Z' Z
      by (simp add: calculation(3))
    moreover
    have Per B C' P''
      by (metis out-distinct ⟨B C' Perp D'' C'⟩
          ⟨C' Out D'' P''⟩ out-col perp-col1
          perp-comm perp-per-2)
    moreover
    have B Z' Lt B C'
  proof –
    have B C' P' CongA A B P''
  proof –
    have B C' P' CongA C' B P''
  proof –
    have P' B C' CongA P'' C' B
  proof –
    have B P' Perp C' B
  proof –
    have B D' Perp C' B
  proof –
    have B A Perp B D'

```

```

    using Perp-perm ⟨A B Perp D' B⟩ by blast
  moreover
  have C' ≠ B
    using ⟨B Out C' A⟩ l6-3-1 by auto
  moreover
  have Col B A C'
    using Out-cases ⟨B Out C' A⟩ out-col by blast
  moreover
  have Col B A B
    by (simp add: col-trivial-3)
  ultimately
  show ?thesis
    using perp-col0 by blast
qed
moreover
have B ≠ P'
  using ⟨B Out D' P'⟩ out-distinct by blast
moreover
have Col B D' B
  by (simp add: col-trivial-3)
moreover
have Col B D' P'
  using ⟨B Out D' P'⟩ out-col by auto
ultimately
show ?thesis
  using perp-col by blast
qed
hence Per P' B C'
  using perp-per-1 by blast
moreover
have P' ≠ B
  using ⟨B Out D' P'⟩ out-diff2 by auto
moreover
have C' ≠ B
  using ⟨P' C' B LtA A B C⟩ lta-distincts by blast
moreover
have Per P'' C' B
  using ⟨Per B C' P''⟩ l8-2 by blast
moreover
have P'' ≠ C'
  using False ⟨Cong C' P'' P Q⟩ cong-diff-3 by blast
moreover
have B ≠ C'
  using calculation(3) by auto
ultimately
show ?thesis
  by (simp add: l11-16)
qed
moreover
have Cong B P' C' P''
  by (meson ⟨Cong B P' P Q⟩ ⟨Cong C' P'' P Q⟩
    cong-inner-transitivity cong-symmetry)
moreover
have Cong B C' C' B
  using cong-pseudo-reflexivity by blast
moreover
have P' ≠ C'
  using ⟨P' C' B LtA A B C⟩ lta-distincts by blast
ultimately
show ?thesis
  using l11-49 by blast
qed
moreover
have C' Out B B
  using ⟨P' C' B LtA A B C⟩
    lta-distincts out-trivial by blast

```

```

moreover
have  $C' \neq P'$ 
  using  $\langle P' C' B LtA A B C \rangle$  lta-distincts by blast
hence  $C' Out P' P'$ 
  using out-trivial by auto
moreover
have  $B Out A C'$ 
  using Out-cases  $\langle B Out C' A \rangle$  by auto
moreover
have  $B Out P'' P''$ 
  using  $\langle B \neq P'' \rangle$  out-trivial by auto
ultimately
show ?thesis
  using l11-10 by blast
qed
have  $P'' InAngle Z' B Z$ 
proof –
  have  $P'' InAngle A B C$ 
proof –
  have  $A B P'' LeA A B C$ 
proof –
  have  $B C' P' LeA A B C$ 
    by (meson  $\langle P' C' B LtA A B C \rangle$ 
      lea-left-comm lta--lea)
moreover
have  $B C' P' CongA A B P''$ 
  by (simp add:  $\langle B C' P' CongA A B P'' \rangle$ )
moreover
have  $A B C CongA A B C$ 
  using  $\langle Acute A B C \rangle$  acute-distincts
    conga-refl by blast
ultimately
show ?thesis
  using l11-30 by blast
qed
moreover
have  $A B OS C P''$ 
proof –
  have  $A B OS C D''$ 
proof –
  have  $A \neq B$ 
    using  $\langle \neg Col A B C \rangle$  not-col-distincts by blast
moreover
have  $Col B C' A$ 
  by (simp add:  $\langle B Out C' A \rangle$  out-col)
moreover
have  $Col B C' B$ 
  by (simp add: col-trivial-3)
ultimately
show ?thesis
  using  $\langle B C' OS C D'' \rangle$  col2-os--os by blast
qed
moreover
have  $A B OS D'' P''$ 
proof –
  have  $\neg Col B C' D''$ 
    using  $\langle B C' OS C D'' \rangle$  col124--nos by blast
hence  $\neg Col A B D''$ 
    using calculation col124--nos by blast
moreover
have  $Col A B C'$ 
  using  $\langle B Out C' A \rangle$  col-permutation-2 out-col by blast
moreover
have  $C' Out D'' P''$ 
  using  $\langle C' Out D'' P'' \rangle$  by auto
ultimately

```



```

    show ?thesis using out-one-side-1 by blast
qed
ultimately
show ?thesis
  using one-side-transitivity by blast
qed
ultimately
show ?thesis
  by (simp add: lea-in-angle)
qed
moreover
have B Out Z' A
  by (simp add: ⟨B Out A Z'⟩ l6-6)
moreover
have B Out Z C
  using Out-cases ⟨B Out C Z⟩ by blast
moreover
have B Out P'' P''
  using ⟨B ≠ P''⟩ out-trivial by auto
ultimately
show ?thesis
  using l11-25 by blast
qed
hence H3: Z' ≠ B ∧ Z ≠ B ∧ P'' ≠ B ∧
  (∃ X. Bet Z' X Z ∧ (X = B ∨ B Out X P''))
  using InAngle-def by blast
then consider T where Bet Z' T Z and T = B ∨ B Out T P''
  by blast

have ∃ T. Bet Z' T Z ∧ (Bet B T P'' ∨ Bet B P'' T)
proof -
  have T = B ⟶ (Bet B T P'' ∨ Bet B P'' T)
    using between-trivial2 by auto
  moreover
  {
    assume B Out T P''
    hence Bet B T P'' ∨ Bet B P'' T
      using Out-def by blast
  }
  ultimately
  show ?thesis
    by (metis Out-def
      ⟨∧thesis. (∧T. [Bet Z' T Z; T = B ∨ B Out T P'']) ⟹ thesis ⟹ thesis⟩
      between-trivial2)
qed
then obtain T where Bet Z' T Z and Bet B T P'' ∨ Bet B P'' T
  by blast
have Z Z' Par C' P''
proof -
  have Coplanar B C' Z C'
    by (meson ncop-distincts)
  moreover
  have Coplanar B C' Z P''
proof -
  have Coplanar B Z C C'
    using Out-cases ⟨B Out C Z⟩ out--coplanar by blast
  moreover
  have Coplanar C' P'' D'' B
    using Out-cases ⟨C' Out D'' P''⟩ out--coplanar by blast
  moreover
  have Coplanar C' B D'' C
    using ⟨B C' OS C D''⟩ ncoplanar-perm-7 os--coplanar by blast
  moreover have ¬ Col B C' D''
    using ⟨B C' OS C D''⟩ one-side-not-col124 by auto
  moreover have ¬ Col B C' C
    using ⟨B C' OS C D''⟩ col123--nos by blast

```

```

moreover have Coplanar  $B C' Z C$ 
  using calculation(1) ncoplanar-perm-3 by blast
moreover have Coplanar  $C' B P'' D''$ 
  using calculation(2) ncoplanar-perm-3 by blast
moreover have Coplanar  $C' C B D''$ 
  using calculation(3) ncoplanar-perm-3 by blast
ultimately
show ?thesis
  by (meson col-permutation-4 coplanar-perm-1 l9-30 ncop-distincts)
qed
moreover
have Coplanar  $B C' Z' C'$ 
  by (meson ncop-distincts)
moreover
have Coplanar  $B C' Z' P''$ 
  using  $\langle B \text{ Out } A Z' \rangle \langle B \text{ Out } C' A \rangle$  l6-7 out--coplanar by blast
moreover
have  $Z Z' \text{ Perp } B C'$ 
  by (metis Out-cases Perp-cases  $\langle A B \text{ Perp } Z Z' \rangle$ 
     $\langle B C' \text{ Perp } D'' C' \rangle \langle B \text{ Out } C' A \rangle$  out-col
    perp-col1 perp-not-eq-1)
moreover
have  $C' P'' \text{ Perp } B C'$ 
  by (metis False Perp-perm  $\langle B C' \text{ Perp } D'' C' \rangle$ 
     $\langle C' \text{ Out } D'' P'' \rangle \langle \text{Cong } C' P'' P Q \rangle$  cong-diff-3
    out-col perp-col1)
ultimately
show ?thesis
  using l12-9 by blast
qed
{
  assume  $T = Z$ 
  have  $A B P'' \text{ CongA } A B C$ 
  proof –
    have  $B \text{ Out } A A$ 
      using  $\langle B \text{ Out } C' A \rangle$  out-diff2 out-trivial by blast
    moreover
    have  $B \text{ Out } P'' C$ 
      using  $\langle B \text{ Out } C Z \rangle \langle \text{Bet } B T P'' \vee \text{Bet } B P'' T \rangle$ 
       $\langle \text{Cong } B Z B P'' \rangle \langle T = Z \rangle$  between-cong
      cong-symmetry l6-6 by blast
    ultimately
    show ?thesis
      by (meson l6-6 out2--conga)
  qed
  hence  $P' C' B \text{ CongA } A B C$ 
    by (meson conga-trans  $\langle B C' P' \text{ CongA } A B P'' \rangle$  conga-left-comm)
  hence False
    using  $\langle P' C' B \text{ LtA } A B C \rangle$  lta-not-conga by blast
}
hence  $T \neq Z$ 
  by auto
have  $T \neq Z'$ 
  by (metis Col-cases  $\langle A B \text{ OS } C D' \rangle \langle B C' P' \text{ CongA } A B P'' \rangle$ 
     $\langle B \text{ Out } A Z' \rangle \langle B \text{ Out } C' A \rangle \langle B \text{ Out } D' P' \rangle$ 
     $\langle \text{Bet } B T P'' \vee \text{Bet } B P'' T \rangle$  bet-col ncol-conga-ncol
    one-side-not-col123 one-side-not-col124 out-one-side)
{
  assume  $T = P''$ 
  have Acute  $B T Z$ 
    by (metis  $\langle B \neq P'' \rangle \langle \text{Cong } B Z B P'' \rangle \langle T = P'' \rangle$ 
       $\langle T \neq Z \rangle$  cong--acute not-cong-3412)
  have Obtuse  $B T Z$ 
  proof –
    have Acute  $B T Z'$ 
  proof –

```

```

have Per B Z' T
proof -
  have A B Perp T Z'
  proof -
    have Z Z' Perp A B
    by (meson Perp-perm ⟨A B Perp Z Z'⟩)
    moreover
    have Col Z Z' T
    by (simp add: Col-def ⟨Bet Z' T Z⟩)
    moreover
    have Col Z Z' Z'
    using not-col-distincts by auto
    ultimately
    show ?thesis
    using ⟨T ≠ Z'⟩ perp-col0 by blast
  qed
  moreover
  have Z' ≠ B
  by (simp add: H3)
  moreover
  have Col A B B
  by (simp add: col-trivial-2)
  ultimately
  show ?thesis
  by (meson ⟨Col A B Z'⟩ perp-col2 perp-per-1)
qed
hence Per B Z' T ∨ Obtuse B Z' T
by simp
thus ?thesis
by (metis acute-sym ⟨T ≠ Z'⟩ H3 l11-43)
qed
moreover
have B T Z' SuppA B T Z
proof -
  have Z' ≠ T
  using ⟨T ≠ Z'⟩ by auto
  moreover
  have T ≠ B
  using ⟨B ≠ P''⟩ ⟨T = P''⟩ by blast
  moreover
  have T ≠ Z
  by (simp add: ⟨T ≠ Z⟩)
  moreover
  have Bet Z' T Z
  using ⟨Bet Z' T Z⟩ by auto
  ultimately
  show ?thesis
  by (simp add: bet--suppa suppa-left-comm)
qed
ultimately
show ?thesis
using acute-suppa--obtuse by blast
qed
hence False
using ⟨B ≠ P''⟩ ⟨Cong B Z B P''⟩ ⟨T = P''⟩
⟨T ≠ Z⟩ acute--not-obtuse cong--acute
not-cong-3412 by blast
}
hence T ≠ P''
by blast
{
  assume Bet B P'' T
  have Obtuse Z T B
  proof -
    have Acute B T Z'
    proof -

```

```

have  $Z' \neq B$ 
  using  $H3$  by blast
moreover
have  $Z' \neq T$ 
  using  $\langle T \neq Z' \rangle$  by auto
moreover
have  $Per\ B\ Z'\ T \vee Obtuse\ B\ Z'\ T$ 
  by (metis  $Per$ -perm l8-3  $\langle Bet\ Z'\ T\ Z \rangle$   $\langle Per\ B\ Z'\ Z \rangle$ 
    bet-col bet-col1 bet-neg23--neg
    calculation(2) col-transitivity-1)
ultimately
show ?thesis
  by (meson acute-sym l11-43)
qed
moreover
have  $B\ T\ Z'\ SuppA\ Z\ T\ B$ 
proof -
  have  $Z' \neq T$ 
    using  $\langle T \neq Z' \rangle$  by fastforce
  moreover
  have  $T \neq B$ 
    using  $\langle B \neq P'' \rangle$   $\langle Bet\ B\ P''\ T \rangle$  bet-neg32--neg by blast
  ultimately
  show ?thesis
    by (simp add:  $\langle T \neq Z \rangle$   $\langle Bet\ Z'\ T\ Z \rangle$ 
        bet--suppa suppa-comm)
qed
ultimately
show ?thesis
  using acute-suppa--obtuse by blast
qed
have  $B\ Z\ T\ LtA\ B\ T\ Z$ 
proof -
  have  $Acute\ B\ Z\ T$ 
proof -
  have  $T \neq B$ 
    using  $\langle B \neq P'' \rangle$   $\langle Bet\ B\ P''\ T \rangle$  between-identity by blast
  moreover
  have  $Per\ B\ T\ Z \vee Obtuse\ B\ T\ Z$ 
    using  $\langle Obtuse\ Z\ T\ B \rangle$  obtuse-sym by blast
  ultimately
  show ?thesis
    by (simp add:  $\langle T \neq Z \rangle$   $\langle Obtuse\ Z\ T\ B \rangle$ 
        acute-sym l11-43-aux)
qed
moreover
have  $Obtuse\ B\ T\ Z$ 
  by (simp add:  $\langle Obtuse\ Z\ T\ B \rangle$  obtuse-sym)
ultimately
show ?thesis
  using acute-obtuse--lta by blast
qed
moreover
have  $B\ T\ Z\ LtA\ B\ Z\ T$ 
proof -
  have  $\neg Col\ Z\ B\ T$ 
    by (metis  $NCol$ -perm  $\langle Bet\ Z'\ T\ Z \rangle$   $\langle Col\ A\ B\ Z' \rangle$ 
         $\langle T \neq Z \rangle$   $H3$   $\langle \neg Col\ A\ B\ Z \rangle$  bet-out-1
        col-trivial-2 l6-21 out-col)
  moreover
  have  $B\ Z\ Lt\ B\ T$ 
    by (metis nlt--le  $\langle B \neq P'' \rangle$   $\langle Bet\ B\ P''\ T \rangle$ 
         $\langle Cong\ B\ Z\ B\ P'' \rangle$   $\langle T \neq P'' \rangle$  bet-out between-equality-2
        cong-reflexivity l5-6 l6-13-1 l6-6)
  ultimately
  show ?thesis

```

```

    using col-permutation-3 l11-44-2 by blast
qed
ultimately
have False
  using not-and-lta by blast
}
hence Bet B T P''
  using ⟨Bet B T P'' ∨ Bet B P'' T⟩ by auto
have C' P'' Par Z Z'
  by (simp add: ⟨Z Z' Par C' P''⟩ par-symmetry)
have Col Z Z' T
  using Col-def ⟨Bet Z' T Z⟩ by blast
have ¬ Col C' P'' T
  by (metis Col-cases H3 ⟨Bet B T P''⟩ ⟨Col A B Z'⟩
    ⟨Col Z Z' T⟩ ⟨T = P'' ⟹ False⟩ ⟨Z Z' Par C' P''⟩ ⟨¬ Col A B Z'⟩
    bet-col col2--eq not-strict-par2)
hence C' P'' ParStrict Z Z'
  using par-not-col-strict ⟨C' P'' Par Z Z'⟩
    ⟨Col Z Z' T⟩ by blast
have Z Z' OS C' P''
  by (simp add: ⟨C' P'' ParStrict Z Z'⟩ pars--os3412)
have P'' Out T B
  by (simp add: ⟨Bet B T P''⟩ ⟨T ≠ P''⟩ bet-out-1)
have B Out C' Z'
  using ⟨B Out A Z'⟩ ⟨B Out C' A⟩ l6-7 by blast
hence Bet B Z' C' ∨ Bet B C' Z'
  using Out-def by force
{
  assume Bet B Z' C'
  have Z' ≠ C'
    using ⟨Z Z' OS C' P''⟩ col123--nos col-trivial-2 by blast
  hence B Z' Lt B C'
    using bet--lt1213 by (simp add: ⟨Bet B Z' C'⟩)
}
moreover
{
  assume Bet B C' Z'
  have Z Z' TS B P''
  proof -
    have ¬ Col P'' Z Z'
      by (metis ⟨Col A B Z'⟩ ⟨Col Z Z' T⟩
        ⟨P'' InAngle Z' B Z⟩ ⟨P'' Out T B⟩ ⟨¬ Col A B Z'⟩
        ⟨¬ Col C' P'' T⟩ col-permutation-1 colx
        inangle-distincts not-col-distincts out-col)
    moreover
    have Col T Z Z'
      by (simp add: Col-def ⟨Bet Z' T Z⟩)
    moreover
    have Bet B T P''
      using ⟨Bet B T P''⟩ by blast
    ultimately
    show ?thesis
      by (metis ⟨Col A B Z'⟩ ⟨P'' InAngle Z' B Z⟩
        ⟨¬ Col A B Z'⟩ bet-col inangle-distincts l6-21 l9-18-R2
        not-col-distincts not-col-permutation-2
        not-col-permutation-5)
  qed
  moreover
  have Z Z' OS B C'
    by (metis ⟨Bet B C' Z'⟩ ⟨Z Z' OS C' P''⟩
      bet-out between-symmetry col-trivial-2 l9-19-R2
      one-side-not-col123 one-side-symmetry)
  ultimately
  have False
    using ⟨Z Z' OS C' P''⟩ l9-9 one-side-transitivity by blast
  hence B Z' Lt B C'

```

```

      by blast
    }
  thus ?thesis
    using ⟨Bet B Z' C' ∨ Bet B C' Z'⟩ calculation by fastforce
qed
ultimately
show ?thesis
  using ⟨Cong B Z B P''⟩ cong-lt-per2--lt-1 by blast
qed
hence P Q Lt Z' Z
  by (meson ⟨Cong C' P'' P Q⟩ cong--nlt le3456-lt--lt nle--lt)
ultimately
show ?thesis
  by blast
qed
}
thus ?thesis
  by (simp add: aristotle-s-axiom-def)
qed

```

```

theorem equiv-aristotle---greenberg:
  shows aristotle-s-axiom ⟷ greenberg-s-axiom
  using aristotle--greenberg greenberg--aristotle by blast

```

```

end
end

```

```

theory Tarski-Postulate-Parallels

```

```

imports
  Tarski-Neutral-Archimedes

```

```

begin

```

```

context Tarski-neutral-dimensionless

```

```

begin

```

3 Parallel's Postulate

3.1 Definitions

```

definition tarski-s-parallel-postulate ::

```

```

  bool

```

```

  (TarskiSParallelPostulate)

```

```

where

```

```

  tarski-s-parallel-postulate ≡

```

```

    ∀ A B C D T.

```

```

    Bet A D T ∧ Bet B D C ∧ A ≠ D

```

```

  ⟶

```

```

    (∃ X Y. Bet A B X ∧ Bet A C Y ∧ Bet X T Y)

```

```

definition euclid-5 ::

```

```

  bool (Euclid5)

```

```

where

```

```

  euclid-5 ≡

```

```

    ∀ P Q R S T U.

```

```

    BetS P T Q ∧ BetS R T S ∧ BetS Q U R ∧ ¬ Col P Q S ∧ Cong P T Q T ∧ Cong R T S T

```

```

  ⟶

```

```

    (∃ I. BetS S Q I ∧ BetS P U I)

```

```

definition euclid-s-parallel-postulate ::

```

```

  bool (EuclidSParallelPostulate)

```

where

euclid-s-parallel-postulate $\equiv$

$\forall A B C D P Q R.$

$B C OS A D \wedge SAMS A B C B C D \wedge A B C B C D Sum A P Q R \wedge \neg Bet P Q R$

$\longrightarrow$

$(\exists Y. B Out A Y \wedge C Out D Y)$

definition *playfair-s-postulate* ::

bool

(*PlayfairSPostulate*)

where

playfair-s-postulate $\equiv$

$\forall A1 A2 B1 B2 C1 C2 P.$

$A1 A2 Par B1 B2 \wedge Col P B1 B2 \wedge A1 A2 Par C1 C2 \wedge Col P C1 C2$

$\longrightarrow$

$Col C1 B1 B2 \wedge Col C2 B1 B2$

definition *decidability-of-intersection* ::

bool

(*DecidabilityIntersection*)

where

decidability-of-intersection $\equiv$

$\forall A B C D.$

$(\exists I. Col I A B \wedge Col I C D) \vee \neg (\exists I. Col I A B \wedge Col I C D)$

definition *alternate-interior-angles-postulate* ::

bool

(*AlternateInteriorAnglesPostulate*)

where

alternate-interior-angles-postulate $\equiv$

$\forall A B C D.$

$A C TS B D \wedge A B Par C D$

$\longrightarrow$

$B A C Cong A D C A$

definition *consecutive-interior-angles-postulate* ::

bool

(*ConsecutiveInteriorAnglesPostulate*)

where

consecutive-interior-angles-postulate $\equiv$

$\forall A B C D.$

$B C OS A D \wedge A B Par C D$

$\longrightarrow$

$A B C Supp A B C D$

definition *alternative-playfair-s-postulate* ::

bool

(*AlternativePlayfairSPostulate*)

where

alternative-playfair-s-postulate $\equiv$

$\forall A1 A2 B1 B2 C1 C2 P.$

$P Perp2 A1 A2 B1 B2 \wedge \neg Col A1 A2 P \wedge Col P B1 B2 \wedge$

$Coplanar A1 A2 B1 B2 \wedge A1 A2 Par C1 C2 \wedge Col P C1 C2$

$\longrightarrow$

$Col C1 B1 B2 \wedge Col C2 B1 B2$

definition *proclus-postulate* ::

bool

(*ProclusPostulate*)

where

proclus-postulate $\equiv$

$\forall A B C D P Q.$

$A B \text{ Par } C D \wedge \text{Col } A B P \wedge \neg \text{Col } A B Q \wedge \text{Coplanar } C D P Q$

$\longrightarrow$

$(\exists Y. \text{Col } P Q Y \wedge \text{Col } C D Y)$

definition *triangle-postulate* ::

bool

(TrianglePostulate)

where

triangle-postulate $\equiv$

$\forall A B C D E F.$

$A B C \text{ TriSum } A D E F$

$\longrightarrow$

$\text{Bet } D E F$

definition *bachmann-s-lotschnittaxiom* ::

bool

(BachmannsLotschnittaxiom)

where

bachmann-s-lotschnittaxiom $\equiv$

$\forall P Q R P1 R1.$

$(P \neq Q \wedge Q \neq R \wedge \text{Per } P Q R \wedge \text{Per } Q P P1 \wedge \text{Per } Q R R1 \wedge$
 $\text{Coplanar } P Q R P1 \wedge \text{Coplanar } P Q R R1$

$\longrightarrow$

$(\exists S. \text{Col } P P1 S \wedge \text{Col } R R1 S))$

definition *legendre-s-parallel-postulate* ::

bool

(LegendresParallelPostulate)

where

legendre-s-parallel-postulate $\equiv$

$\exists A B C.$

$\neg \text{Col } A B C \wedge$

$\text{Acute } A B C \wedge$

$(\forall T. T \text{ InAngle } A B C \longrightarrow (\exists X Y. B \text{ Out } A X \wedge B \text{ Out } C Y \wedge \text{Bet } X T Y))$

definition *weak-inverse-projection-postulate* ::

bool

(WeakInverseProjectionPostulate)

where

weak-inverse-projection-postulate $\equiv$

$\forall A B C D E F P Q.$

$((\text{Acute } A B C \wedge \text{Per } D E F \wedge A B C A B C \text{ Sum } A D E F \wedge$
 $B \text{ Out } A P \wedge P \neq Q \wedge \text{Per } B P Q \wedge \text{Coplanar } A B C Q)$

$\longrightarrow$

$(\exists Y. B \text{ Out } C Y \wedge \text{Col } P Q Y))$

definition *weak-triangle-circumscription-principle* ::

bool

(WeakTriangleCircumscriptionPrinciple)

where

weak-triangle-circumscription-principle $\equiv$

$\forall A B C A1 A2 B1 B2.$

$(\neg \text{Col } A B C \wedge \text{Per } A C B \wedge A1 A2 \text{ PerpBisect } B C \wedge$

$B1 B2 \text{ PerpBisect } A C \wedge \text{Coplanar } A B C A1 \wedge \text{Coplanar } A B C A2 \wedge$

$\text{Coplanar } A B C B1 \wedge \text{Coplanar } A B C B2$

$\longrightarrow$

$(\exists I. Col A1 A2 I \wedge Col B1 B2 I))$

definition *weak-tarski-s-parallel-postulate* ::

bool

$(WeakTarskiParallelPostulate)$

where

weak-tarski-s-parallel-postulate $\equiv$

$\forall A B C T.$

$(Per A B C \wedge T InAngle A B C$

$\longrightarrow$

$(\exists X Y. B Out A X \wedge B Out C Y \wedge Bet X T Y))$

definition *existential-playfair-s-postulate* ::

bool

$(ExistentialPlayfairPostulate)$

where

existential-playfair-s-postulate $\equiv$

$\exists A1 A2 P.$

$\neg Col A1 A2 P \wedge$

$(\forall B1 B2 C1 C2.$

$A1 A2 Par B1 B2 \wedge Col P B1 B2 \wedge A1 A2 Par C1 C2 \wedge Col P C1 C2$

$\longrightarrow$

$(Col C1 B1 B2 \wedge Col C2 B1 B2))$

definition *postulate-of-right-saccheri-quadrilaterals* ::

bool

$(PostulateRightSaccheriQuadrilaterals)$

where

postulate-of-right-saccheri-quadrilaterals $\equiv$

$\forall A B C D.$

$Saccheri A B C D$

$\longrightarrow$

$Per A B C$

definition *postulate-of-existence-of-a-right-saccheri-quadrilateral* ::

bool

$(PostulateExistenceRightSaccheriQuadrilateral)$

where

postulate-of-existence-of-a-right-saccheri-quadrilateral $\equiv$

$\exists A B C D.$

$Saccheri A B C D \wedge Per A B C$

definition *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights* ::

bool

$(PostulateExistenceTriangleAnglesSumTwoRights)$

where

postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights $\equiv$

$\exists A B C D E F.$

$\neg Col A B C \wedge A B C TriSum A D E F \wedge Bet D E F$

definition *inverse-projection-postulate* ::

bool

$(InverseProjectionPostulate)$

where

inverse-projection-postulate $\equiv$

$\forall A B C P Q.$

$Acute\ A\ B\ C \wedge B\ Out\ A\ P \wedge P \neq Q \wedge Per\ B\ P\ Q \wedge Coplanar\ A\ B\ C\ Q$
 $\longrightarrow$
 $(\exists\ Y. B\ Out\ C\ Y \wedge Col\ P\ Q\ Y)$

definition *alternative-proclus-postulate* ::

bool

(*AlternativeProclusPostulate*)

where

alternative-proclus-postulate $\equiv$

$\forall\ A\ B\ C\ D\ P\ Q.$

$P\ Perp2\ A\ B\ C\ D \wedge \neg\ Col\ C\ D\ P \wedge Coplanar\ A\ B\ C\ D \wedge Col\ A\ B\ P \wedge$

$\neg\ Col\ A\ B\ Q \wedge Coplanar\ C\ D\ P\ Q$

$\longrightarrow$

$(\exists\ Y. (Col\ P\ Q\ Y \wedge Col\ C\ D\ Y))$

definition *strong-parallel-postulate* ::

bool

(*StrongParallelPostulate*)

where

strong-parallel-postulate $\equiv$

$\forall\ P\ Q\ R\ S\ T\ U.$

$BetS\ P\ T\ Q \wedge BetS\ R\ T\ S \wedge \neg\ Col\ P\ R\ U \wedge Coplanar\ P\ Q\ R\ U \wedge$

$Cong\ P\ T\ Q\ T \wedge Cong\ R\ T\ S\ T$

$\longrightarrow$

$(\exists\ I. (Col\ S\ Q\ I \wedge Col\ P\ U\ I))$

definition *triangle-circumscription-principle* ::

bool

(*TriangleCircumscriptionPrinciple*)

where

triangle-circumscription-principle $\equiv$

$\forall\ A\ B\ C.$

$\neg\ Col\ A\ B\ C$

$\longrightarrow$

$(\exists\ D. Cong\ A\ D\ B\ D \wedge Cong\ A\ D\ C\ D \wedge Coplanar\ A\ B\ C\ D)$

definition *thales-converse-postulate* ::

bool

(*ThalesConversePostulate*) **where**

thales-converse-postulate $\equiv$

$\forall\ A\ B\ C\ M.$

$M\ Midpoint\ A\ B \wedge Per\ A\ C\ B$

$\longrightarrow$

$Cong\ M\ A\ M\ C$

definition *existential-thales-postulate* ::

bool

(*ExistentialThalesPostulate*) **where**

existential-thales-postulate $\equiv$

$\exists\ A\ B\ C\ M.$

$\neg\ Col\ A\ B\ C \wedge M\ Midpoint\ A\ B \wedge Cong\ M\ A\ M\ C \wedge Per\ A\ C\ B$

definition *thales-postulate* ::

bool

(*ThalesPostulate*) **where**

thales-postulate $\equiv$

$\forall\ A\ B\ C\ M.$

$M\ Midpoint\ A\ B \wedge Cong\ M\ A\ M\ C$

$\longrightarrow$

$Per\ A\ C\ B$

definition *posidonius-postulate* ::

bool

(*PosidoniusPostulate*) **where**

posidonius-postulate $\equiv$

$\exists\ A1\ A2\ B1\ B2.$

$\neg\ Col\ A1\ A2\ B1 \wedge B1 \neq B2 \wedge Coplanar\ A1\ A2\ B1\ B2 \wedge$
 $(\forall\ A3\ A4\ B3\ B4.$
 $Col\ A1\ A2\ A3 \wedge Col\ B1\ B2\ B3 \wedge A1\ A2\ Perp\ A3\ B3 \wedge$
 $Col\ A1\ A2\ A4 \wedge Col\ B1\ B2\ B4 \wedge A1\ A2\ Perp\ A4\ B4$
 $\longrightarrow$
 $Cong\ A3\ B3\ A4\ B4)$

definition *postulate-of-right-lambert-quadrilaterals* ::

bool

(*PostulateOfRightLambertQuadrilaterals*) **where**

postulate-of-right-lambert-quadrilaterals $\equiv$

$\forall\ A\ B\ C\ D.$

$Lambert\ A\ B\ C\ D$

$\longrightarrow$

$Per\ B\ C\ D$

definition *postulate-of-existence-of-a-right-lambert-quadrilateral* ::

bool

(*PostulateExistenceRightLambertQuadrilateral*) **where**

postulate-of-existence-of-a-right-lambert-quadrilateral $\equiv$

$\exists\ A\ B\ C\ D.$

$Lambert\ A\ B\ C\ D \wedge Per\ B\ C\ D$

definition *postulate-of-existence-of-similar-triangles* ::

bool

(*PostulateOfExistenceOfSimilarTriangles*) **where**

postulate-of-existence-of-similar-triangles $\equiv$

$\exists\ A\ B\ C\ D\ E\ F.$

$\neg\ Col\ A\ B\ C \wedge \neg\ Cong\ A\ B\ D\ E \wedge A\ B\ C\ Cong\ A\ D\ E\ F \wedge$
 $B\ C\ A\ Cong\ A\ E\ F\ D \wedge C\ A\ B\ Cong\ A\ F\ D\ E$

definition *midpoint-converse-postulate* ::

bool

(*MidpointConversePostulate*) **where**

midpoint-converse-postulate $\equiv$

$\forall\ A\ B\ C\ P\ Q.$

$\neg\ Col\ A\ B\ C \wedge P\ Midpoint\ B\ C \wedge A\ B\ Par\ Q\ P \wedge Col\ A\ C\ Q$

$\longrightarrow$

$Q\ Midpoint\ A\ C$

definition *postulate-of-transitivity-of-parallelism* ::

bool

(*PostulateOfTransitivityOfParallelism*) **where**

postulate-of-transitivity-of-parallelism $\equiv$

$\forall\ A1\ A2\ B1\ B2\ C1\ C2.$

$A1\ A2\ Par\ B1\ B2 \wedge B1\ B2\ Par\ C1\ C2$

$\longrightarrow$

$A1\ A2\ Par\ C1\ C2$

definition *perpendicular-transversal-postulate* ::

bool
(PerpendicularTransversalPostulate) **where**
perpendicular-transversal-postulate $\equiv$

$\forall A B C D P Q.$
 $A B \text{ Par } C D \wedge A B \text{ Perp } P Q \wedge \text{Coplanar } C D P Q$
 $\longrightarrow$
 $C D \text{ Perp } P Q$

definition *postulate-of-parallelism-of-perpendicular-transversals* ::
bool
(PostulateOfParallelismOfPerpendicularTransversals) **where**
postulate-of-parallelism-of-perpendicular-transversals $\equiv$

$\forall A1 A2 B1 B2 C1 C2 D1 D2.$
 $A1 A2 \text{ Par } B1 B2 \wedge A1 A2 \text{ Perp } C1 C2 \wedge B1 B2 \text{ Perp } D1 D2 \wedge$
 $\text{Coplanar } A1 A2 C1 D1 \wedge \text{Coplanar } A1 A2 C1 D2 \wedge$
 $\text{Coplanar } A1 A2 C2 D1 \wedge \text{Coplanar } A1 A2 C2 D2$
 $\longrightarrow$
 $C1 C2 \text{ Par } D1 D2$

definition *universal-posidonius-postulate* ::
bool
(UniversalPosidoniusPostulate) **where**
universal-posidonius-postulate $\equiv$

$\forall A1 A2 A3 A4 B1 B2 B3 B4.$
 $A1 A2 \text{ Par } B1 B2 \wedge \text{Col } A1 A2 A3 \wedge \text{Col } B1 B2 B3 \wedge A1 A2 \text{ Perp } A3 B3 \wedge$
 $\text{Col } A1 A2 A4 \wedge \text{Col } B1 B2 B4 \wedge A1 A2 \text{ Perp } A4 B4$
 $\longrightarrow$
 $\text{Cong } A3 B3 A4 B4$

definition *alternative-strong-parallel-postulate* ::
bool
(AlternativeStrongParallelPostulate) **where**
alternative-strong-parallel-postulate $\equiv$

$\forall A B C D P Q R.$
 $B C \text{ OS } A D \wedge A B C B C D \text{ SumA } P Q R \wedge \neg \text{Bet } P Q R$
 $\longrightarrow$
 $(\exists Y. \text{Col } B A Y \wedge \text{Col } C D Y)$

definition *Postulate01* :: *bool* **where**
Postulate01 $\equiv$ *tarski-s-parallel-postulate*

definition *Postulate02* :: *bool* **where**
Postulate02 $\equiv$ *playfair-s-postulate*

definition *Postulate03* :: *bool* **where**
Postulate03 $\equiv$ *triangle-postulate*

definition *Postulate04* :: *bool* **where**
Postulate04 $\equiv$ *bachmann-s-lotschnittaxiom*

definition *Postulate05* :: *bool* **where**
Postulate05 $\equiv$ *postulate-of-transitivity-of-parallelism*

definition *Postulate06* :: *bool* **where**
Postulate06 $\equiv$ *midpoint-converse-postulate*

definition *Postulate07* :: bool **where**
 Postulate07 $\equiv$ *alternate-interior-angles-postulate*

definition *Postulate08* :: bool **where**
 Postulate08 $\equiv$ *consecutive-interior-angles-postulate*

definition *Postulate09* :: bool **where**
 Postulate09 $\equiv$ *perpendicular-transversal-postulate*

definition *Postulate10* :: bool **where**
 Postulate10 $\equiv$ *postulate-of-parallelism-of-perpendicular-transversals*

definition *Postulate11* :: bool **where**
 Postulate11 $\equiv$ *universal-posidonius-postulate*

definition *Postulate12* :: bool **where**
 Postulate12 $\equiv$ *alternative-playfair-s-postulate*

definition *Postulate13* :: bool **where**
 Postulate13 $\equiv$ *proclus-postulate*

definition *Postulate14* :: bool **where**
 Postulate14 $\equiv$ *alternative-proclus-postulate*

definition *Postulate15* :: bool **where**
 Postulate15 $\equiv$ *triangle-circumscription-principle*

definition *Postulate16* :: bool **where**
 Postulate16 $\equiv$ *inverse-projection-postulate*

definition *Postulate17* :: bool **where**
 Postulate17 $\equiv$ *euclid-5*

definition *Postulate18* :: bool **where**
 Postulate18 $\equiv$ *strong-parallel-postulate*

definition *Postulate19* :: bool **where**
 Postulate19 $\equiv$ *alternative-strong-parallel-postulate*

definition *Postulate20* :: bool **where**
 Postulate20 $\equiv$ *euclid-s-parallel-postulate*

definition *Postulate21* :: bool **where**
 Postulate21 $\equiv$ *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*

definition *Postulate22* :: bool **where**
 Postulate22 $\equiv$ *posidonius-postulate*

definition *Postulate23* :: bool **where**
Postulate23 $\equiv$ *postulate-of-existence-of-similar-triangles*

definition *Postulate24* :: bool **where**
Postulate24 $\equiv$ *thales-postulate*

definition *Postulate25* :: bool **where**
Postulate25 $\equiv$ *thales-converse-postulate*

definition *Postulate26* :: bool **where**
Postulate26 $\equiv$ *existential-thales-postulate*

definition *Postulate27* :: bool **where**
Postulate27 $\equiv$ *postulate-of-right-saccheri-quadrilaterals*

definition *Postulate28* :: bool **where**
Postulate28 $\equiv$ *postulate-of-existence-of-a-right-saccheri-quadrilateral*

definition *Postulate29* :: bool **where**
Postulate29 $\equiv$ *postulate-of-right-lambert-quadrilaterals*

definition *Postulate30* :: bool **where**
Postulate30 $\equiv$ *postulate-of-existence-of-a-right-lambert-quadrilateral*

definition *Postulate31* :: bool **where**
Postulate31 $\equiv$ *weak-inverse-projection-postulate*

definition *Postulate32* :: bool **where**
Postulate32 $\equiv$ *weak-tarski-s-parallel-postulate*

definition *Postulate33* :: bool **where**
Postulate33 $\equiv$ *weak-triangle-circumscription-principle*

definition *Postulate34* :: bool **where**
Postulate34 $\equiv$ *legendre-s-parallel-postulate*

definition *Postulate35* :: bool **where**
Postulate35 $\equiv$ *existential-playfair-s-postulate*

3.2 Propositions

lemma *euclid-5--original-euclid*:
assumes *Euclid5*
shows *EuclidSParallelPostulate*
proof –
{
 fix *A B C D P Q R*
 assume *P1*: *B C OS A D* $\wedge$ *SAMS A B C B C D* $\wedge$ *A B C B C D SumA P Q R* $\wedge$ $\neg$ *Bet P Q R*
 obtain *M* **where** *P2*: *M Midpoint B C*
 using *midpoint-existence* **by** *auto*
 obtain *D'* **where** *P3*: *C Midpoint D D'*
 using *symmetric-point-construction* **by** *auto*
 obtain *E* **where** *P4*: *M Midpoint D' E*
 using *symmetric-point-construction* **by** *auto*

```

have P5:  $A \neq B$ 
  using P1 os-distincts by blast
have P6:  $B \neq C$ 
  using P1 os-distincts by blast
have P7:  $C \neq D$ 
  using P1 os-distincts by blast
have P10:  $M \neq B$ 
  using P2 P6 is-midpoint-id by auto
have P11:  $M \neq C$ 
  using P2 P6 is-midpoint-id-2 by auto
have P13:  $C \neq D'$ 
  using P3 P7 is-midpoint-id-2 by blast
have P16:  $\neg \text{Col } B \ C \ A$ 
  using one-side-not-col123 P1 by blast
have B C OS D A
  using P1 one-side-symmetry by blast
then have P17:  $\neg \text{Col } B \ C \ D$ 
  using one-side-not-col123 P1 by blast
then have P18:  $\neg \text{Col } M \ C \ D$ 
  using P2 Col-perm P11 col-transitivity-2 midpoint-col by blast
then have P19:  $\neg \text{Col } M \ C \ D'$ 
  by (metis P13 P3 Col-perm col-transitivity-2 midpoint-col)
then have P20:  $\neg \text{Col } D' \ C \ B$ 
  by (metis Col-perm P13 P17 P3 col-transitivity-2 midpoint-col)
then have P21:  $\neg \text{Col } M \ C \ E$ 
  by (metis P19 P4 bet-col col2--eq col-permutation-4 midpoint-bet midpoint-distinct-2)
have P22:  $M \ C \ D' \text{ CongA } M \ B \ E \wedge M \ D' \ C \text{ CongA } M \ E \ B$  using P13 l11-49
  by (metis Cong-cases P19 P2 P4 l11-51 l7-13-R1 l7-2 midpoint-cong not-col-distincts)
have P23:  $\text{Cong } C \ D' \ B \ E$ 
  using P11 P2 P4 l7-13-R1 l7-2 by blast
have P27:  $C \ B \ TS \ D \ D'$ 
  by (simp add: P13 P17 P3 bet--ts midpoint-bet not-col-permutation-4)
have P28:  $A \text{ InAngle } C \ B \ E$ 
proof -
  have C B A LeA C B E
  proof -
    have A B C LeA B C D'
    proof -
      have Bet D C D'
      by (simp add: P3 midpoint-bet)
      then show ?thesis using P1 P7 P13 sams-chara
      by (metis sams-left-comm sams-sym)
    qed
    moreover have A B C CongA C B A
      using P5 P6 conga-pseudo-refl by auto
    moreover have B C D' CongA C B E
      by (metis CongA-def Mid-cases P2 P22 P4 P6 symmetry-preserves-conga)
    ultimately show ?thesis
      using l11-30 by blast
  qed
  moreover have C B OS E A
  proof -
    have C B TS E D'
      using P2 P20 P4 l7-2 l9-2 mid-two-sides not-col-permutation-1 by blast
    moreover have C B TS A D'
      using P27  $\langle B \ C \ OS \ D \ A \rangle$  invert-two-sides l9-8-2 by blast
    ultimately show ?thesis
      using OS-def by blast
  qed
  ultimately show ?thesis
    using lea-in-angle by simp
qed
obtain A' where P30:  $\text{Bet } C \ A' \ E \wedge (A' = B \vee B \text{ Out } A' \ A)$  using P28 InAngle-def by auto
{
  assume A' = B
  then have Col D' C B

```

```

    by (metis Col-def P2 P21 P30 P6 col-transitivity-1 midpoint-col)
  then have False
    by (simp add: P20)
  then have  $\exists Y. B \text{ Out } A \ Y \wedge C \text{ Out } D \ Y$  by auto
}
{
  assume P31:  $B \text{ Out } A' \ A$ 
  have  $\exists I. \text{BetS } D' \ C \ I \wedge \text{BetS } B \ A' \ I$ 
  proof -
    have P32:  $\text{BetS } B \ M \ C$ 
      using BetS-def Midpoint-def P10 P11 P2 by auto
    moreover have  $\text{BetS } E \ M \ D'$ 
      using BetS-def Bet-cases P19 P21 P4 midpoint-bet not-col-distincts by fastforce
    moreover have  $\text{BetS } C \ A' \ E$ 
    proof -
      have P32A:  $C \neq A'$ 
        using P16 P31 out-col by auto
      {
        assume  $A' = E$ 
        then have P33:  $B \text{ Out } A \ E$ 
          using P31 l6-6 by blast
        then have  $A \ B \ C \ B \ C \ D \ \text{SumA } D' \ C \ D$ 
        proof -
          have  $D' \ C \ B \ \text{CongA } A \ B \ C$ 
          proof -
            have  $D' \ C \ M \ \text{CongA } E \ B \ M$ 
              by (simp add: P22 conga-comm)
            moreover have  $C \text{ Out } D' \ D'$ 
              using P13 out-trivial by auto
            moreover have  $C \text{ Out } B \ M$ 
              using BetSeq Out-cases P32 bet-out-1 by blast
            moreover have  $B \text{ Out } A \ E$ 
              using P33 by auto
            moreover have  $B \text{ Out } C \ M$ 
              using BetSeq Out-def P32 by blast
            ultimately show ?thesis
              using l11-10 by blast
          qed
        moreover have  $D' \ C \ B \ B \ C \ D \ \text{SumA } D' \ C \ D$ 
          by (simp add: P27 l9-2 ts--suma-1)
        moreover have  $B \ C \ D \ \text{CongA } B \ C \ D$ 
          using P6 P7 conga-refl by auto
        moreover have  $D' \ C \ D \ \text{CongA } D' \ C \ D$ 
          using P13 P7 conga-refl by presburger
        ultimately show ?thesis
          using conga3-suma--suma by blast
      }
    qed
    then have  $D' \ C \ D \ \text{CongA } P \ Q \ R$ 
      using P1 suma2--conga by auto
    then have  $\text{Bet } P \ Q \ R$ 
      using Bet-cases P3 bet-conga--bet midpoint-bet by blast
    then have False using P1 by simp
  }
  then have  $A' \neq E$  by auto
  then show ?thesis
    by (simp add: BetS-def P30 P32A)
  qed
  moreover have  $\neg \text{Col } B \ C \ D'$ 
    by (simp add: P20 not-col-permutation-3)
  moreover have  $\text{Cong } B \ M \ C \ M$ 
    using Midpoint-def P2 not-cong-1243 by blast
  moreover have  $\text{Cong } E \ M \ D' \ M$ 
    using Cong-perm Midpoint-def P4 by blast
  ultimately show ?thesis
    using euclid-5-def assms by blast
  qed

```



```

then obtain Y where P34: Bet D' C Y  $\wedge$  BetS B A' Y using BetSeq by blast
then have  $\exists Y. B \text{ Out } A Y \wedge C \text{ Out } D Y$ 
proof -
  have P35: B Out A Y
  by (metis BetSeq Out-def P31 P34 l6-7)
  moreover have C Out D Y
  proof -
    have D  $\neq$  C
    using P7 by auto
    moreover have Y  $\neq$  C
    using P16 P35 l6-6 out-col by blast
    moreover have D'  $\neq$  C
    using P13 by auto
    moreover have Bet D C D'
    by (simp add: P3 midpoint-bet)
    moreover have Bet Y C D'
    by (simp add: Bet-perm P34)
    ultimately show ?thesis
    using l6-2 by blast
  qed
  ultimately show ?thesis by auto
qed
}
then have  $\exists Y. B \text{ Out } A Y \wedge C \text{ Out } D Y$ 
using P30  $\langle A' = B \implies \exists Y. B \text{ Out } A Y \wedge C \text{ Out } D Y \rangle$  by blast
}
then show ?thesis using euclid-s-parallel-postulate-def by blast
qed

lemma tarski-s-euclid-implies-euclid-5:
  assumes TarskiSParallelPostulate
  shows Euclid5
proof -
  {
    fix P Q R S T U
    assume
      P1: BetS P T Q  $\wedge$  BetS R T S  $\wedge$  BetS Q U R  $\wedge$   $\neg$  Col P Q S  $\wedge$  Cong P T Q T  $\wedge$  Cong R T S T
    have P1A: BetS P T Q using P1 by simp
    have P1B: BetS R T S using P1 by simp
    have P1C: BetS Q U R using P1 by simp
    have P1D:  $\neg$  Col P Q S using P1 by simp
    have P1E: Cong P T Q T using P1 by simp
    have P1F: Cong R T S T using P1 by simp
    obtain V where P2: P Midpoint R V
    using symmetric-point-construction by auto
    have P3: Bet V P R
    using Mid-cases P2 midpoint-bet by blast
    then obtain W where P4: Bet P W Q  $\wedge$  Bet U W V using inner-pasch
    using BetSeq P1C by blast
    {
      assume P = W
      have P  $\neq$  V
      by (metis BetSeq Bet-perm Col-def Cong-perm Midpoint-def P1A P1B P1D P1E P1F
        P2 between-trivial is-midpoint-id-2 l7-9)
      have Col P Q S
      proof -
        have f1: Col V P R
        by (meson Col-def P3)
        have f2: Col U R Q
        by (simp add: BetSeq Col-def P1)
        have f3: Bet P T Q
        using BetSeq P1 by fastforce
        have f4: R = P  $\vee$  Col V P U
        by (metis (no-types) Col-def P4  $\langle P = W \rangle \langle P \neq V \rangle$  l6-16-1)
        have f5: Col Q P T
        using f3 by (meson Col-def)
      }
    }
  }

```

```

have f6: Col T Q P
  using f3 by (meson Col-def)
have f7: Col P T Q
  using f3 by (meson Col-def)
have f8: Col P Q P
  using Col-def P4  $\langle P = W \rangle$  by blast
have Col R T S
  by (meson BetSEq Col-def P1)
then have  $T = P \vee Q = P$ 
  using f8 f7 f6 f5 f4 f2 f1 by (metis (no-types) BetSEq P1  $\langle P \neq V \rangle$  colx l6-16-1)
then show ?thesis
  by (metis BetSEq P1)
qed
then have False
  by (simp add: P1D)
}
then have P5:  $P \neq W$  by auto
have Bet V W U
  using Bet-cases P4 by auto
then obtain X Y where P7:  $Bet P V X \wedge Bet P U Y \wedge Bet X Q Y$ 
  using assms(1) P1 P4 P5 tarski-s-parallel-postulate-def by blast
have Q S Par P R
proof -
  have  $Q \neq S$ 
    using P1D col-trivial-2 by auto
  moreover have T Midpoint Q P
    using BetSEq P1A P1E l7-2 midpoint-def not-cong-1243 by blast
  moreover have T Midpoint S R
    using BetSEq P1B P1F l7-2 midpoint-def not-cong-1243 by blast
  ultimately show ?thesis
    using l12-17 by auto
qed
then have P9:  $Q S ParStrict P R$ 
  using P1D Par-def par-strict-symmetry par-symmetry by blast
have P10:  $Q S TS P Y$ 
proof -
  have P10A:  $P \neq R$ 
    using P9 par-strict-distinct by auto
  then have P11:  $P \neq X$ 
    by (metis P2 P7 bet-neq12--neq midpoint-not-midpoint)
  have P12:  $\neg Col X Q S$ 
proof -
  have  $Q S ParStrict P R$ 
    by (simp add: P9)
  then have Col P R X
    by (metis P2 P3 P7 bet-col between-symmetry midpoint-not-midpoint
      not-col-permutation-4 outer-transitivity-between)
  then have  $P X ParStrict Q S$ 
    using P9 Par-strict-perm P11 par-strict-col-par-strict by blast
  then show ?thesis
    using par-strict-not-col-2 by auto
qed
{
  assume W1: Col Y Q S
  have W2:  $Q = Y$ 
    by (metis P12 P7 W1 bet-col bet-col1 colx)
  then have  $\neg Col Q P R$ 
    using P9 W1 par-not-col by auto
  then have W3:  $Q = U$ 
    by (smt BetS-def Col-def P1C P7 W2 col-transitivity-2)
  then have False
    using BetS-def P1C by auto
}
then have  $\neg Col Y Q S$  by auto
then have  $Q S TS X Y$ 
  by (metis P7 P12 bet--ts not-col-distincts not-col-permutation-1)

```

```

moreover have  $Q\ S\ OS\ X\ P$ 
proof –
  have  $P \neq V$ 
    using  $P10A\ P2\ is\ midpoint\ id\ 2$  by blast
  then have  $Q\ S\ ParStrict\ P\ X$ 
    by (meson Bet-perm P3 P7 P9 P11 bet-col not-col-permutation-4 par-strict-col-par-strict)
  then have  $Q\ S\ ParStrict\ X\ P$ 
    by (simp add: par-strict-right-comm)
  then show ?thesis
    by (simp add: l12-6)
qed
ultimately show ?thesis
  using  $l9-8-2$  by auto
qed
then obtain  $I$  where  $W4: Col\ I\ Q\ S \wedge Bet\ P\ I\ Y$ 
  using  $TS-def$  by blast
have  $\exists\ I. (Bet\ S\ Q\ I \wedge Bet\ S\ P\ U\ I)$ 
proof –
  have  $Bet\ S\ P\ U\ I$ 
proof –
  have  $P \neq Y$ 
    using  $P10\ not\ two\ sides\ id$  by auto
  have  $W4A: Bet\ P\ U\ I$ 
proof –
  have  $W5: Col\ P\ U\ I$ 
    using  $P7\ W4\ bet-col1$  by auto
  {
    assume  $W6: Bet\ U\ I\ P$ 
    have  $W7: Q\ S\ OS\ P\ U$ 
proof –
    have  $Q\ S\ OS\ R\ U$ 
proof –
    have  $\neg Col\ Q\ S\ R$ 
      using  $P9\ par\ strict\ not\ col\ 4$  by auto
    moreover have  $Q\ Out\ R\ U$ 
      using  $BetSEq\ Out-def\ P1C$  by blast
    ultimately show ?thesis
      by (simp add: out-one-side)
    qed
    moreover have  $Q\ S\ OS\ P\ R$ 
      by (simp add: P9 l12-6)
    ultimately show ?thesis
      using  $one-side-transitivity$  by blast
    qed
have  $W8: I\ Out\ P\ U \vee \neg Col\ Q\ S\ P$ 
      by (simp add: P1D not-col-permutation-1)
have False
proof –
    have  $I\ Out\ U\ P$ 
      using  $W4\ W6\ W7\ between-symmetry\ one-side-chara$  by blast
    then show ?thesis
      using  $W6\ not-bet-and-out$  by blast
    qed
  }
  {
    assume  $V1: Bet\ I\ P\ U$ 
    have  $P\ R\ OS\ I\ U$ 
proof –
    have  $P\ R\ OS\ I\ Q$ 
proof –
    {
      assume  $Q = I$ 
      then have  $Col\ P\ Q\ S$ 
        by (metis BetSEq Col-def P1C P7 P9 V1 W4 between-equality outer-transitivity-between par-not-col)
      then have False
    }
  }

```

```

    using P1D by blast
  }
  then have  $Q \neq I$  by blast
  moreover have  $P R \text{ ParStrict } Q S$ 
    using P9 par-strict-symmetry by blast
  moreover have  $\text{Col } Q S I$ 
    using Col-cases W4 by blast
  ultimately show ?thesis
    using one-side-symmetry par-strict-all-one-side by blast
qed
moreover have  $P R OS Q U$ 
proof -
  have  $Q S \text{ ParStrict } P R$ 
    using P9 by blast
  have  $R \text{ Out } Q U \wedge \neg \text{Col } P R Q$ 
    by (metis BetSEq Bet-cases Out-def P1C calculation col124--nos)
  then show ?thesis
    by (metis P7 V1 W4  $\langle \text{Bet } U I P \implies \text{False} \rangle$  between-equality
        col-permutation-2 not-bet-distincts out-col outer-transitivity-between)
qed
ultimately show ?thesis
  using one-side-transitivity by blast
qed
then have V2:  $P \text{ Out } I U$ 
  using P7 W4 bet2--out os-distincts by blast
then have  $\text{Col } P I U$ 
  using V1 not-bet-and-out by blast
then have False
  using V1 V2 not-bet-and-out by blast
}
then moreover have  $\neg (\text{Bet } U I P \vee \text{Bet } I P U)$ 
  using  $\langle \text{Bet } U I P \implies \text{False} \rangle$  by auto
ultimately show ?thesis
  using Col-def W5 by blast
qed
{
  assume  $P = U$ 
  then have  $\text{Col } P R Q$ 
    using BetSEq Col-def P1C by blast
  then have False
    using P9 par-strict-not-col-3 by blast
}
then have V6:  $P \neq U$  by auto
{
  assume  $U = I$ 
  have  $Q = U$ 
  proof -
    have f1:  $\text{BetS } Q I R$ 
      using P1C  $\langle U = I \rangle$  by blast
    then have f2:  $\text{Col } Q I R$ 
      using BetSEq Col-def by blast
    have f3:  $\text{Col } I R Q$ 
      using f1 by (simp add: BetSEq Col-def)
    { assume  $R \neq Q$ 
      moreover
      { assume  $(R \neq Q \wedge R \neq I) \wedge \neg \text{Col } I Q R$ 
        moreover
        { assume  $\exists p. (R \neq Q \wedge \neg \text{Col } I p I) \wedge \text{Col } Q I p$ 
          then have  $I = Q$ 
            using f1 by (metis (no-types) BetSEq Col-def col-transitivity-2) }
          ultimately have  $(\exists p \text{ pa. } ((\text{pa} \neq I \wedge \neg \text{Col } \text{pa } p R) \wedge \text{Col } Q I \text{pa}) \wedge \text{Col } I \text{pa } p) \vee I = Q$ 
            using f3 f2 by (metis (no-types) col-transitivity-2) }
          ultimately have  $(\exists p \text{ pa. } ((\text{pa} \neq I \wedge \neg \text{Col } \text{pa } p R) \wedge \text{Col } Q I \text{pa}) \wedge \text{Col } I \text{pa } p) \vee I = Q$ 
            using f1 by (metis (no-types) BetSEq P9 W4 col-transitivity-2 par-strict-not-col-4) }
        }
      then show ?thesis
        using f2 by (metis P9 W4  $\langle U = I \rangle$  col-transitivity-2 par-strict-not-col-4)
    }
  }
}
then show ?thesis
  using f2 by (metis P9 W4  $\langle U = I \rangle$  col-transitivity-2 par-strict-not-col-4)
}

```

```

qed
then have False
  using BetSEq P1C by blast
}
then have  $U \neq I$  by auto
then show ?thesis
  by (simp add: W4A V6 BetS-def)
qed
moreover have BetS S Q I
proof -
  have Q R TS S I
  proof -
    have Q R TS P I
    proof -
      have  $\neg$  Col P Q R
      using P9 col-permutation-5 par-strict-not-col-3 by blast
    moreover have  $\neg$  Col I Q R
    proof -
      {
        assume Col I Q R
        then have Col Q S R
        proof -
          have f1:  $\forall p \text{ pa pb. Col } p \text{ pa pb} \vee \neg \text{BetS } pb \text{ p pa}$ 
            by (meson BetSEq Col-def)
          then have f2: Col U I P
            using  $\langle \text{BetS } P \text{ U I} \rangle$  by blast
          have f3: Col I P U
            by (simp add: BetSEq Col-def  $\langle \text{BetS } P \text{ U I} \rangle$ )
          have f4:  $\forall p. (U = Q \vee \text{Col } Q \text{ p R}) \vee \neg \text{Col } Q \text{ U p}$ 
            by (metis BetSEq Col-def P1C col-transitivity-1)
          { assume  $P \neq Q$ 
            moreover
            { assume  $(P \neq Q \wedge U \neq Q) \wedge \text{Col } Q \text{ P Q}$ 
              then have  $(P \neq Q \wedge U \neq Q) \wedge \neg \text{Col } Q \text{ P R}$ 
                using Col-cases  $\langle \neg \text{Col } P \text{ Q R} \rangle$  by blast
            }
            moreover
            { assume  $\exists p. ((U \neq Q \wedge P \neq Q) \wedge \neg \text{Col } Q \text{ p P}) \wedge \text{Col } Q \text{ P p}$ 
              then have  $U \neq Q \wedge \neg \text{Col } Q \text{ P P}$ 
                by (metis col-transitivity-1)
              then have  $\neg \text{Col } U \text{ Q P}$ 
                using col-transitivity-2 by blast
            }
            ultimately have  $\neg \text{Col } U \text{ Q P} \vee I \neq Q$ 
              using f4 f3 by blast
          }
          ultimately have  $I \neq Q$ 
            using f2 f1 by (metis BetSEq P1C col-transitivity-1 col-transitivity-2)
        }
        then have  $I \neq Q$ 
          using BetSEq  $\langle \text{BetS } P \text{ U I} \rangle$  by blast
      }
      then show ?thesis
        by (simp add: W4  $\langle \text{Col } I \text{ Q R} \rangle$  col-transitivity-2)
    }
  }
qed
then have False
  using P9 par-strict-not-col-4 by blast
}
then show ?thesis by blast
qed
moreover have Col U Q R
  using BetSEq Bet-cases Col-def P1C by blast
moreover have Bet P U I
  by (simp add: BetSEq  $\langle \text{BetS } P \text{ U I} \rangle$ )
ultimately show ?thesis
  using TS-def by blast
qed
moreover have Q R OS P S
proof -
  have Q R Par P S
  proof -

```

```

    have  $Q \neq R$ 
      using BetSEq P1 by blast
    moreover have  $T \text{ Midpoint } Q P$ 
      using BetSEq Bet-cases P1A P1E cong-3421 midpoint-def by blast
    moreover have  $T \text{ Midpoint } R S$ 
      using BetSEq P1B P1F midpoint-def not-cong-1243 by blast
    ultimately show ?thesis
      using l12-17 by blast
  qed
  then have  $Q R \text{ ParStrict } P S$ 
    by (simp add: P1D Par-def not-col-permutation-4)
  then show ?thesis
    using l12-6 by blast
  qed
  ultimately show ?thesis
    using l9-8-2 by blast
  qed
  then show ?thesis
    by (metis BetS-def W4 col-two-sides-bet not-col-permutation-2 ts-distincts)
  qed
  ultimately show ?thesis
    by auto
  qed
}
then show ?thesis using euclid-5-def by blast
qed

lemma tarski-s-implies-euclid-s-parallel-postulate:
  assumes TarskiSParallelPostulate
  shows EuclidSParallelPostulate
  by (simp add: assms euclid-5--original-euclid tarski-s-euclid-implies-euclid-5)

theorem tarski-s-euclid-implies-playfair-s-postulate:
  assumes TarskiSParallelPostulate
  shows PlayfairSPostulate
proof -
{
  fix  $A1 A2 B1 B2 P C1 C2$ 
  assume  $P1: \neg \text{Col } P A1 A2 \wedge A1 A2 \text{ Par } B1 B2 \wedge \text{Col } P B1 B2 \wedge A1 A2 \text{ Par } C1 C2 \wedge \text{Col } P C1 C2$ 
  have  $P1A: \neg \text{Col } P A1 A2$ 
    by (simp add: P1)
  have  $P2: A1 A2 \text{ Par } B1 B2$ 
    by (simp add: P1)
  have  $P3: \text{Col } P B1 B2$ 
    by (simp add: P1)
  have  $P4: A1 A2 \text{ Par } C1 C2$ 
    by (simp add: P1)
  have  $P5: \text{Col } P C1 C2$ 
    by (simp add: P1)
  have  $P6: A1 A2 \text{ ParStrict } B1 B2$ 
  proof -
    have  $A1 A2 \text{ Par } B1 B2$ 
      by (simp add: P1)
    moreover have  $\text{Col } B1 B2 P$ 
      using P3 not-col-permutation-2 by blast
    moreover have  $\neg \text{Col } A1 A2 P$ 
      by (simp add: P1A not-col-permutation-1)
    ultimately show ?thesis
      using par-not-col-strict by auto
  qed
  have  $P7: A1 A2 \text{ ParStrict } C1 C2$ 
  proof -
    have  $A1 A2 \text{ Par } C1 C2$ 
      by (simp add: P1)
    moreover have  $\text{Col } C1 C2 P$ 
      using Col-cases P1 by blast

```

```

moreover have  $\neg \text{Col } A1 \ A2 \ P$ 
  by (simp add: P1A not-col-permutation-1)
ultimately show ?thesis
  using par-not-col-strict by auto
qed
{
  assume  $\neg \text{Col } C1 \ B1 \ B2 \vee \neg \text{Col } C2 \ B1 \ B2$ 
  have  $\exists \ C'. \text{Col } C1 \ C2 \ C' \wedge B1 \ B2 \ TS \ A1 \ C'$ 
  proof –
    have T2: Coplanar A1 A2 P A1
      using ncop-distincts by auto
    have T3: Coplanar A1 A2 B1 B2
      by (simp add: P1 par--coplanar)
    have T4: Coplanar A1 A2 C1 C2
      by (simp add: P7 pars--coplanar)
    have T5: Coplanar A1 A2 P B1
      using P1 col-trivial-2 ncop-distincts par--coplanar par-col2-par-bis by blast
    then have T6: Coplanar A1 A2 P B2
      using P3 T3 col-cop--cop by blast
    have T7: Coplanar A1 A2 P C1
      using P1 T4 col-cop--cop coplanar-perm-1 not-col-permutation-2 par-distincts by blast
    then have T8: Coplanar A1 A2 P C2
      using P5 T4 col-cop--cop by blast
    {
      assume  $\neg \text{Col } C1 \ B1 \ B2$ 
      moreover have  $C1 \neq C2$ 
        using P1 par-neq2 by auto
      moreover have  $\text{Col } B1 \ B2 \ P$ 
        using P1 not-col-permutation-2 by blast
      moreover have  $\text{Col } C1 \ C2 \ P$ 
        using Col-cases P5 by auto
      moreover have  $\neg \text{Col } B1 \ B2 \ C1$ 
        using Col-cases calculation(1) by auto
      moreover have  $\neg \text{Col } B1 \ B2 \ A1$ 
        using P6 par-strict-not-col-3 by auto
      moreover have  $\text{Coplanar } B1 \ B2 \ C1 \ A1$ 
        using Col-cases P1A T5 T2 T6 T7 coplanar-pseudo-trans by blast
      ultimately have  $\exists \ C'. \text{Col } C1 \ C2 \ C' \wedge B1 \ B2 \ TS \ A1 \ C'$ 
        using cop-not-par-other-side by blast
    }
    {
      assume  $\neg \text{Col } C2 \ B1 \ B2$ 
      moreover have  $C2 \neq C1$ 
        using P1 par-neq2 by blast
      moreover have  $\text{Col } B1 \ B2 \ P$ 
        using Col-cases P3 by auto
      moreover have  $\text{Col } C2 \ C1 \ P$ 
        using Col-cases P5 by auto
      moreover have  $\neg \text{Col } B1 \ B2 \ C2$ 
        by (simp add: calculation(1) not-col-permutation-1)
      moreover have  $\neg \text{Col } B1 \ B2 \ A1$ 
        using P6 par-strict-not-col-3 by auto
      moreover have  $\text{Coplanar } B1 \ B2 \ C2 \ A1$ 
        using Col-cases P1A T2 T5 T6 T8 coplanar-pseudo-trans by blast
      ultimately have  $\exists \ C'. \text{Col } C1 \ C2 \ C' \wedge B1 \ B2 \ TS \ A1 \ C'$  using cop-not-par-other-side
        by (meson not-col-permutation-4)
    }
  }
  then show ?thesis
    using  $\langle \neg \text{Col } C1 \ B1 \ B2 \implies \exists \ C'. \text{Col } C1 \ C2 \ C' \wedge B1 \ B2 \ TS \ A1 \ C' \rangle \langle \neg \text{Col } C1 \ B1 \ B2 \vee \neg \text{Col } C2 \ B1 \ B2 \rangle$ 
by blast
qed
then obtain  $C'$  where W1: Col C1 C2 C' ∧ B1 B2 TS A1 C' by auto
then have W2: ¬ Col A1 B1 B2
  using TS-def by blast
obtain  $B$  where W3: Col B B1 B2 ∧ Bet A1 B C'
  using TS-def W1 by blast

```

```

obtain  $C$  where  $W_4: P \text{ Midpoint } C' C$ 
  using symmetric-point-construction by blast
then have  $W_4A: \text{Bet } A1 B C' \wedge \text{Bet } C P C'$ 
  using Mid-cases  $W_3$  midpoint-bet by blast
then obtain  $D$  where  $W_5: \text{Bet } B D C \wedge \text{Bet } P D A1$  using inner-pasch by blast
have  $W_6: C' \neq P$ 
  using  $P_3$  TS-def  $W_1$  by blast
then have  $A1 A2 \text{ Par } C' P$ 
  by (meson  $P_1 W_1$  not-col-permutation-2 par-col2-par)
have  $W_9: A1 A2 \text{ ParStrict } C' P$ 
  using Col-cases  $P_5 P_7 W_1 W_6$  par-strict-col2-par-strict by blast
then have  $W_{10}: B \neq P$ 
  by (metis  $W_6 W_4A$  bet-out-1 out-col par-strict-not-col-3)
have  $W_{11}: P \neq C$ 
  using  $W_6 W_4$  is-midpoint-id-2 by blast
{
  assume  $P = D$ 
  then have False
    by (metis Col-def  $P_3 W_1 W_3 W_4A W_5 W_{10} W_{11}$  col-trivial-2 colx l9-18-R1)
}
then have  $P \neq D$  by auto
then obtain  $X Y$  where  $W_{12}: \text{Bet } P B X \wedge \text{Bet } P C Y \wedge \text{Bet } X A1 Y$ 
  using  $W_5$  assms tarski-s-parallel-postulate-def by blast
then have  $P \neq X$ 
  using  $W_{10}$  bet-neq12--neq by auto
then have  $A1 A2 \text{ ParStrict } P X$ 
  by (metis Col-cases  $P_3 P_6 W_{10} W_{12} W_3$  bet-col colx par-strict-col2-par-strict)
then have  $W_{15}: A1 A2 \text{ OS } P X$ 
  by (simp add: l12-6)
have  $P \neq Y$ 
  using  $W_{11} W_{12}$  between-identity by blast
then have  $A1 A2 \text{ ParStrict } P Y$ 
  by (metis Col-def  $W_{11} W_{12} W_4A W_9$  col-trivial-2 par-strict-col2-par-strict)
then have  $W_{16}: A1 A2 \text{ OS } P Y$ 
  using l12-6 by auto
have Col  $A1 X Y$ 
  by (simp add:  $W_{12}$  bet-col col-permutation-4)
then have  $A1 \text{ Out } X Y$  using col-one-side-out  $W_{15} W_{16}$ 
  using one-side-symmetry one-side-transitivity by blast
then have False
  using  $W_{12}$  not-bet-and-out by blast
}
then have Col  $C1 B1 B2 \wedge \text{Col } C2 B1 B2$ 
  by auto
}
{
  fix  $A1 A2 B1 B2 P C1 C2$ 
  assume  $P_1: \text{Col } P A1 A2 \wedge A1 A2 \text{ Par } B1 B2 \wedge \text{Col } P B1 B2 \wedge A1 A2 \text{ Par } C1 C2 \wedge \text{Col } P C1 C2$ 
  have Col  $C1 B1 B2$ 
    by (smt  $P_1$  l9-10 not-col-permutation-3 not-strict-par2 par-col2-par par-comm par-id-5 par-symmetry ts-distincts)
  moreover have Col  $C2 B1 B2$ 
    by (smt  $P_1$  l9-10 not-col-permutation-3 not-strict-par2 par-col2-par par-id-5 par-left-comm par-symmetry ts-distincts)
  ultimately have Col  $C1 B1 B2 \wedge \text{Col } C2 B1 B2$  by auto
}
then show ?thesis
  using playfair-s-postulate-def
  by (metis  $\langle \bigwedge P C_2 C_1 B_2 B_1 A_2 A_1. \neg \text{Col } P A_1 A_2 \wedge A_1 A_2 \text{ Par } B_1 B_2 \wedge \text{Col } P B_1 B_2 \wedge A_1 A_2 \text{ Par } C_1 C_2 \wedge \text{Col } P C_1 C_2 \implies \text{Col } C_1 B_1 B_2 \wedge \text{Col } C_2 B_1 B_2 \rangle$ )
qed

```

lemma *tarski-s-euclid-remove-degenerated-cases*:

```

assumes  $\forall A B C D T. A \neq B \wedge A \neq C \wedge A \neq D \wedge$ 
           $A \neq T \wedge B \neq C \wedge B \neq D \wedge B \neq T \wedge C \neq D \wedge$ 
           $C \neq T \wedge D \neq T \wedge \neg \text{Col } A B C \wedge \text{Bet } A D T \wedge$ 
           $\text{Bet } B D C \wedge \neg \text{Col } B C T \longrightarrow$ 
   $(\exists x y. (\text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y))$ 

```



```

shows  $\forall A B C D T. \text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D \longrightarrow$ 
       $(\exists x y. (\text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y))$ 
proof -
{
  fix A B C D T
  assume P1:  $\text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D$ 
  hence P2:  $\text{Bet } A D T$ 
    by auto
  have P3:  $\text{Bet } B D C$ 
    by (simp add: P1)
  have P4:  $A \neq D$ 
    using P1 by blast
  have  $(\exists x y. (\text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y))$ 
  proof cases
    assume A = B
    thus ?thesis
      using between-trivial between-trivial2 by blast
  next
    assume P5:  $A \neq B$ 
    thus ?thesis
  proof cases
    assume A = C
    thus ?thesis
      using between-trivial between-trivial2 by auto
  next
    assume P6:  $A \neq C$ 
    thus ?thesis
  proof cases
    assume A = T
    thus ?thesis
      using  $\langle \text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D \rangle$  between-identity by blast
  next
    assume P7:  $A \neq T$ 
    thus ?thesis
  proof cases
    assume B = C
    thus ?thesis
      using  $\langle \text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D \rangle$  between-identity
        between-trivial2 by blast
  next
    assume P8:  $B \neq C$ 
    thus ?thesis
  proof cases
    assume B = D
    thus ?thesis
      using  $\langle \text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D \rangle$  between-trivial2
        segment-construction by blast
  next
    assume P9:  $B \neq D$ 
    thus ?thesis
  proof cases
    assume B = T
    thus ?thesis
      using between-trivial between-trivial2 by blast
  next
    assume P10:  $B \neq T$ 
    thus ?thesis
  proof cases
    assume C = D
    thus ?thesis
      using  $\langle \text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D \rangle$  between-trivial by blast
  next
    assume P11:  $C \neq D$ 
    thus ?thesis
  proof cases
    assume C = T

```

```

    thus ?thesis
      using between-trivial by blast
next
assume P12:  $C \neq T$ 
thus ?thesis
proof cases
  assume  $D = T$ 
  thus ?thesis
    using  $\langle \text{Bet } A \ D \ T \wedge \text{Bet } B \ D \ C \wedge A \neq D \rangle$  between-trivial by blast
next
assume P13:  $D \neq T$ 
{
  assume Col A B C
  have Bet A B C  $\longrightarrow$  ?thesis
    by (metis  $\langle \text{Bet } A \ D \ T \rangle \langle \text{Bet } B \ D \ C \rangle$  between-equality
      between-exchange2 between-exchange3 between-exchange4
      between-trivial between-trivial2 l5-3)
  moreover
  have Bet B C A  $\longrightarrow$  ?thesis
    by (meson Bet-perm P1 between-exchange3 between-exchange4
      between-trivial)
  moreover
  have Bet C A B  $\longrightarrow$  ?thesis
    by (metis between-exchange3 Bet-perm P1
      between-exchange2 l5-3
      between-exchange4 between-trivial
      between-trivial2 l5-1)
  ultimately
  have ?thesis
    using Col-def  $\langle \text{Col } A \ B \ C \rangle$  by blast
}
moreover
{
  assume P14:  $\neg \text{Col } A \ B \ C$ 
  {
    assume Col B C T
    hence  $D = T$ 
      by (meson P1 P14 bet-col col-permutation-2
        col-permutation-5 colx)
    hence False
      by (simp add: P13)
  }
  hence ?thesis
    using assms P14 P13 P2 P3 P4 P5 P6 P7 P8
      P9 P10 P11 P12 by blast
}
ultimately
show ?thesis
  by blast
qed
qed
qed
qed
qed
qed
qed
qed
qed
}
thus ?thesis by auto
qed

```

```

lemma alternate-interior--consecutive-interior:
  assumes AlternateInteriorAnglesPostulate
  shows ConsecutiveInteriorAnglesPostulate
proof -
  {
    fix A B C D
    assume B C OS A D  $\wedge$  A B Par C D
    hence A  $\neq$  B
      using os-distincts by blast
    obtain A' where P3: Bet A B A'  $\wedge$  Cong B A' B A
      using segment-construction by blast
    have  $\neg$  Col B C A
      using  $\langle B C OS A D \wedge A B Par C D \rangle$  l9-19 not-col-distincts by blast
    hence B C TS A A'
      using P3  $\langle A \neq B \rangle$  bet--ts cong-diff-4 by blast
    hence B C TS A' D
      by (meson l9-8-2  $\langle B C OS A D \wedge A B Par C D \rangle$  l9-2)
    hence B C D CongA C B A'
      by (metis (full-types) P3 Par-cases l9-2
         $\langle B C OS A D \wedge A B Par C D \rangle$  alternate-interior-angles-postulate-def
        assms bet-col bet-col1 conga-comm invert-two-sides
        par-col2-par ts-distincts)
    hence A B C SuppA B C D
      using P3 SuppA-def  $\langle A \neq B \rangle$  by auto
  }
  thus ?thesis
    using consecutive-interior-angles-postulate-def by blast
qed

```

```

lemma alternate-interior--playfair-aux-1:
  assumes A1 A2 Par C1 C2 and
    Col P C1 C2 and
    Col P P1 P2 and
    P1 P2 Perp A1 A2 and
    Col Q A1 A2
  shows Coplanar P Q A1 C1
proof -
  have Coplanar A1 A2 C1 C1
    using ncop-distincts by blast
  moreover have Coplanar A1 A2 C1 A1
    using ncop-distincts by blast
  moreover have Coplanar A1 A2 C1 Q
    using Col-cases assms(5) ncop--ncols by blast
  moreover have Coplanar A1 A2 C1 P
    using assms(1) assms(2) col-cop--cop col-permutation-1
      par--coplanar par-neq2 by blast
  ultimately show ?thesis
    by (metis assms(1) assms(5) col-cop--cop col-permutation-1
      ncoplanar-perm-16 ncoplanar-perm-19 par-neq1)
qed

```

```

lemma alternate-interior--playfair-aux-2:
  assumes A1 A2 Par C1 C2 and
    Col P C1 C2 and
    Col Q A1 A2 and
    Col B1 B2 B3 and
     $\neg$  Col A1 A2 P and
    Col P B1 B2 and
    Coplanar A1 A2 B1 B2 and
    A1 A2 ParStrict B1 B2
  shows Coplanar P Q C1 B3
proof -
  have Coplanar A1 A2 P Q

```

```

    using assms(3) Col-cases ncop--ncols by blast
  moreover have Coplanar A1 A2 P B3
    using NCol-perm col2-cop--cop par-strict-neq2
      assms(4) assms(6) assms(7) assms(8) by blast
  moreover have Coplanar A1 A2 P P
    using ncop-distincts by blast
  moreover have Coplanar A1 A2 P C1
    using assms(1) assms(2)
    by (metis col-cop--cop coplanar-perm-1 not-col-permutation-2
      par--coplanar par-neq2)
  ultimately show ?thesis
    using assms(5) coplanar-pseudo-trans by presburger
qed

```

lemma *alternate-interior--playfair-aux:*

assumes *alternate-interior-angles-postulate*

shows $\forall A1 A2 B1 B2 C1 C2 P.$

$(P \text{ Perp2 } A1 A2 B1 B2 \wedge \neg \text{Col } A1 A2 P \wedge \text{Col } P B1 B2 \wedge$

$\text{Coplanar } A1 A2 B1 B2 \wedge$

$A1 A2 \text{ Par } C1 C2 \wedge \text{Col } P C1 C2) \longrightarrow$

$\text{Col } C1 B1 B2$

proof –

{

fix $A1 A2 B1 B2 C1 C2 P$

assume 1: $P \text{ Perp2 } A1 A2 B1 B2$ **and**

2: $\neg \text{Col } A1 A2 P$ **and**

3: $\text{Col } P B1 B2$ **and**

4: $\text{Coplanar } A1 A2 B1 B2$ **and**

5: $A1 A2 \text{ Par } C1 C2$ **and**

6: $\text{Col } P C1 C2$

have $\text{Col } C1 B1 B2$

proof *cases*

assume $P = C1$

thus ?thesis

using 3 **by** blast

next

assume $P \neq C1$

have $A1 A2 \text{ ParStrict } B1 B2$

using 1 2 3 4 *col-cop-perp2--pars-bis col-permutation-1* **by** blast

have $A1 A2 \text{ ParStrict } C1 C2$

using 2 5 6 *col-permutation-1 par-not-col-strict* **by** blast

obtain $P1 P2$ **where** $P1: \text{Col } P P1 P2 \wedge P1 P2 \text{ Perp } A1 A2 \wedge P1 P2 \text{ Perp } B1 B2$

using 1 *Perp2-def* **by** blast

hence $\text{Col } P P1 P2$

by blast

have $P1A: P1 P2 \text{ Perp } A1 A2$

using $P1$ **by** blast

then obtain Q **where** $P1B: \text{Col } Q P1 P2 \wedge \text{Col } Q A1 A2$

using *Perp-def* **by** (*meson NCol-cases perp-inter-perp-in-n*)

hence $P1C: \text{Col } Q P1 P2$

by blast

have $P1D: \text{Col } Q A1 A2$

using $P1B$ **by** blast

have $P1 P2 \text{ Perp } B1 B2$

using $P1$ **by** blast

then obtain P' **where** $P2: P' \text{ PerpAt } P1 P2 B1 B2$

using *perp-inter-perp-in-n* **by** blast

hence $P = P'$

using 3 $\langle \text{Col } P P1 P2 \rangle$ *l8-14-2-1b* **by** blast

have $P \neq Q$

using 2 *NCol-cases* $\langle \text{Col } Q P1 P2 \wedge \text{Col } Q A1 A2 \rangle$ **by** blast

hence $A1 A2 \text{ Perp } P Q$

using *perp-col0 P1A Perp-cases* $\langle \text{Col } P P1 P2 \rangle$

$\langle \text{Col } Q P1 P2 \wedge \text{Col } Q A1 A2 \rangle$ *not-col-permutation-3* **by** blast

have $B1 B2 \text{ Perp } P Q$

using *perp-col0*

```

  by (meson P1 <Col Q P1 P2 ∧ Col Q A1 A2> <P ≠ Q>
      col-permutation-3 perp-left-comm)
have ¬ Col Q C1 P
  using 6 <A1 A2 ParStrict C1 C2> <Col Q P1 P2 ∧ Col Q A1 A2>
      <P ≠ C1> col-trivial-2 colx par-not-col by blast
obtain B3 where P4: Col B1 B2 B3 ∧ B3 ≠ P
  using col-trivial-2 diff-col-ex by blast
hence P4A: Col B1 B2 B3
  by blast
have B3 ≠ P
  using P4 by blast
have Q ≠ C1
  using <¬ Col Q C1 P> col-trivial-1 by blast
have A1 ≠ A2
  using 2 col-trivial-1 by blast
have A2 ≠ P
  using 2 col-trivial-2 by blast
have A1 ≠ P
  using 2 col-trivial-3 by auto
have B1 ≠ B2
  using <P1 P2 Perp B1 B2> perp-distinct by blast
have A1 ≠ B1
  using <A1 A2 ParStrict B1 B2> col-trivial-3 par-strict-not-col-1 by blast
have A1 ≠ B2
  using <A1 A2 ParStrict B1 B2> col-trivial-1 col-trivial-3 par-not-col by blast
have A2 ≠ B1
  using <A1 A2 ParStrict B1 B2> col-trivial-2 par-strict-not-col-1 by blast
have A2 ≠ B2
  using <A1 A2 ParStrict B1 B2> col-trivial-2 par-strict-not-col-4 by blast
have A1 ≠ C1
  using <A1 A2 ParStrict C1 C2> not-par-strict-id by blast
have A1 ≠ C2
  using <A1 A2 ParStrict C1 C2> col-trivial-3 par-strict-not-col-4 by blast
have A2 ≠ C1
  using Par-strict-cases <A1 A2 ParStrict C1 C2> not-par-strict-id by blast
have A2 ≠ C2
  using Par-strict-cases <A1 A2 ParStrict C1 C2> not-par-strict-id by blast
have C1 ≠ C2
  using 5 par-distincts by blast
have Col P C1 B3
proof -
  have Per C1 P Q
  proof -
    have ∃ A3. Col A1 A2 A3 ∧ P Q TS C1 A3
  proof cases
    assume Col P Q A1
    have A2 ≠ A1
      using <A1 ≠ A2> by auto
    moreover
    have Col P Q Q
      by (simp add: col-trivial-2)
    moreover
    have Col A2 A1 Q
      by (simp add: P1D col-permutation-3)
    moreover
    have ¬ Col P Q A2
      using 2 <Col P Q A1> <P ≠ Q> col3 col-trivial-3 by blast
    moreover
    have ¬ Col P Q C1
      using NCol-cases <¬ Col Q C1 P> by blast
    moreover
    have Coplanar P Q A2 C1
      by (metis 5 6 P1D <Col P Q A1> <P ≠ C1>
          calculation(2) calculation(4) col-permutation-1 colx
          coplanar-perm-18 par--coplanar par-col-par)
  ultimately

```

```

have  $\exists Q0. Col\ A2\ A1\ Q0 \wedge P\ Q\ TS\ C1\ Q0$ 
  using cop-not-par-other-side by blast
thus ?thesis
  using col-permutation-4 by blast
next
assume  $\neg Col\ P\ Q\ A1$ 
have  $A1 \neq A2$ 
  by (simp add:  $\langle A1 \neq A2 \rangle$ )
moreover
have  $Col\ P\ Q\ Q$ 
  using col-trivial-2 by auto
moreover
have  $Col\ A1\ A2\ Q$ 
  by (simp add: P1D col-permutation-2)
moreover
{
  assume  $Col\ P'\ Q\ A1$ 
  hence  $Col\ A1\ P1\ P2$ 
    using  $\langle P = P' \rangle \langle \neg Col\ P\ Q\ A1 \rangle$  by auto
  hence  $A1 = Q$ 
    using  $\langle Col\ P'\ Q\ A1 \rangle \langle P = P' \rangle \langle \neg Col\ P\ Q\ A1 \rangle$  by auto
  hence False
    using  $\langle \neg Col\ P\ Q\ A1 \rangle$  col-trivial-2 by blast
}
hence  $\neg Col\ P\ Q\ A1$ 
  using  $\langle P = P' \rangle$  by blast
moreover
have  $\neg Col\ P\ Q\ C1$ 
  using NCol-cases  $\langle \neg Col\ Q\ C1\ P \rangle$  by blast
moreover
have Coplanar  $P\ Q\ A1\ C1$ 
  using alternate-interior--playfair-aux-1 5 6 P1 P1A P1D by blast
ultimately
show ?thesis
  using cop-not-par-other-side by blast
qed
then obtain  $A3$  where  $Col\ A1\ A2\ A3 \wedge P\ Q\ TS\ C1\ A3$ 
  by blast
hence  $\neg Col\ A3\ P\ Q$ 
  using TS-def by blast
have  $A3 \neq P$ 
  using 2  $\langle Col\ A1\ A2\ A3 \wedge P\ Q\ TS\ C1\ A3 \rangle$  by auto
have  $A3 \neq Q$ 
  using  $\langle \neg Col\ A3\ P\ Q \rangle$  col-trivial-3 by blast
have  $C1 \neq A3$ 
  using  $\langle Col\ A1\ A2\ A3 \wedge P\ Q\ TS\ C1\ A3 \rangle$  ts-distincts by blast
have  $Q\ A3\ Perp\ P\ Q$ 
proof -
  have  $A1\ A2\ Perp\ P\ Q$ 
    using  $\langle A1\ A2\ Perp\ P\ Q \rangle$  by blast
  moreover
  have  $Q \neq A3$ 
    using  $\langle \neg Col\ A3\ P\ Q \rangle$  col-trivial-3 by blast
  moreover
  have  $Col\ A1\ A2\ Q$ 
    using NCol-cases P1B by blast
  moreover
  have  $Col\ A1\ A2\ A3$ 
    using  $\langle Col\ A1\ A2\ A3 \wedge P\ Q\ TS\ C1\ A3 \rangle$  by blast
  ultimately
  show ?thesis
    using perp-col2 by blast
qed
hence  $Per\ A3\ Q\ P$ 
  by (simp add: perp-per-1)
moreover

```

```

have C1 P Q CongA A3 Q P
proof -
  have P Q TS C1 A3
    by (simp add: ⟨Col A1 A2 A3 ∧ P Q TS C1 A3⟩)
  moreover
  have P C1 Par Q A3
  proof -
    have P ≠ C1
      by (simp add: ⟨P ≠ C1⟩)
    moreover
    have Q ≠ A3
      using ⟨¬ Col A3 P Q⟩ col-trivial-3 by blast
    moreover
    have C1 C2 Par A1 A2
      by (simp add: 5 par-symmetry)
    moreover
    have Col C1 C2 P
      by (simp add: 6 col-permutation-1)
    moreover
    have Col C1 C2 C1
      by (simp add: col-trivial-3)
    moreover
    have Col A1 A2 Q
      using NCol-cases P1B by blast
    moreover
    have Col A1 A2 A3
      by (simp add: ⟨Col A1 A2 A3 ∧ P Q TS C1 A3⟩)
    ultimately
    show ?thesis
      using par-col4--par by blast
  qed
  ultimately
  show ?thesis
    using assms(1) alternate-interior-angles-postulate-def by blast
qed
hence A3 Q P CongA C1 P Q
  by (simp add: conga-sym)
ultimately
show ?thesis
  using l11-17 by blast
qed
hence P C1 Perp P Q
  using ⟨P ≠ C1⟩ ⟨P ≠ Q⟩ per-perp perp-left-comm by auto
moreover
have P B3 Perp P Q
proof -
  have B1 B2 Perp P Q
    by (simp add: ⟨B1 B2 Perp P Q⟩)
  moreover
  have P ≠ B3
    using ⟨B3 ≠ P⟩ by fastforce
  moreover
  have Col B1 B2 P
    using 3 not-col-permutation-2 by blast
  moreover
  have Col B1 B2 B3
    using P4A by auto
  ultimately
  show ?thesis
    using perp-col2 by blast
qed
moreover
have Coplanar P Q C1 B3
  using P1D P4A 2 3 4 5 6 alternate-interior--playfair-aux-2
    ⟨A1 A2 ParStrict B1 B2⟩ by blast
ultimately

```

```

    show ?thesis
    using cop-perp2--col by blast
qed
thus ?thesis
using 3 P4 colx not-col-permutation-1 by blast
qed
}
thus ?thesis by blast
qed

lemma alternate-interior--playfair-bis:
  assumes alternate-interior-angles-postulate
  shows alternative-playfair-s-postulate
proof -
{
  fix A1 A2 B1 B2 C1 C2 P
  assume 1: P Perp2 A1 A2 B1 B2 and
    2:  $\neg$  Col A1 A2 P and
    3: Col P B1 B2 and
    4: Coplanar A1 A2 B1 B2 and
    5: A1 A2 Par C1 C2 and
    6: Col P C1 C2
  have Col C1 B1 B2
    using alternate-interior--playfair-aux assms(1) 1 2 3 4 5 6
    by blast
  moreover
  have Col C2 B1 B2
    using alternate-interior--playfair-aux assms(1) 1 2 3 4 5 6
    by (meson col-permutation-5 par-right-comm)
  ultimately
  have Col C1 B1 B2  $\wedge$  Col C2 B1 B2
    by blast
}
thus ?thesis
using alternative-playfair-s-postulate-def by blast
qed

lemma alternate-interior--proclus-aux:
  assumes greenberg-s-axiom and
    alternate-interior-angles-postulate
  shows  $\forall A C D P Q. (P A \text{ ParStrict } C D \wedge C D \text{ Perp } P C \wedge$ 
 $P A \text{ OS } C Q \wedge P C \text{ OS } Q A \wedge P C \text{ OS } Q D)$ 
 $\longrightarrow$ 
 $(\exists Y. \text{Col } P Q Y \wedge \text{Col } C D Y)$ 
proof -
{
  fix A C D P Q
  assume 1: P A ParStrict C D and
    2: C D Perp P C and
    3: P A OS C Q and
    4: P C OS Q A and
    5: P C OS Q D
  obtain D' where P1: Bet D C D'  $\wedge$  Cong D C C D'
    using Cong-perm segment-construction by blast
  hence Bet D C D'
    by blast
  have Cong D C C D'
    using P1 by blast
  have C  $\neq$  D
    using 1 par-strict-neq2 by auto
  have C  $\neq$  D'
    using  $\langle C \neq D \rangle \langle \text{Cong } D C C D' \rangle$  cong-identity by blast
  have P  $\neq$  Q
    using 3 col124--nos col-trivial-3 by blast
  have P  $\neq$  A
    using 3 col124--nos not-col-distincts by blast

```



```

have  $\neg$  Col P A Q
  using 3 one-side-not-col124 by auto
have  $\neg$  Col P C Q
  using 4 one-side-not-col123 by auto
have  $\neg$  Col P C D
  using 5 one-side-not-col124 by blast
have A P Q LtA A P C
  by (meson 3 4 col-permutation-3 col-permutation-5
      inangle--lta l9-2 l9-9 one-side-not-col123 os-ts--inangle
      two-sides-cases)
have P C OS D A
  using 4 5 one-side-symmetry one-side-transitivity by blast
have Per A P C
proof -
  have Per P C D'
    using 2 P1 Perp-cases  $\langle C \neq D' \rangle$  bet-col bet-col1
    perp-col2-bis perp-per-2 by blast
  moreover
  have P C D' CongA A P C
proof -
  have P C TS D D'
    by (simp add:  $\langle \text{Bet } D \ C \ D' \rangle$   $\langle C \neq D' \rangle$   $\langle \neg \text{Col } P \ C \ D \rangle$ 
        bet--ts invert-two-sides not-col-permutation-4)
  hence P C TS A D'
    using  $\langle P \ C \ OS \ D \ A \rangle$  l9-8-2 by blast
  moreover
  have P A Par C D'
    by (meson 1 Par-def  $\langle \text{Bet } D \ C \ D' \rangle$   $\langle C \neq D' \rangle$  bet-col
        bet-col1 par-strict-col2-par-strict par-strict-right-comm)
  ultimately
  show ?thesis
    using alternate-interior-angles-postulate-def assms(2)
        conga-left-comm conga-sym by blast
qed
ultimately
show ?thesis
  using l11-17 by blast
qed
hence Acute A P Q
  using  $\langle A \ P \ Q \ LtA \ A \ P \ C \rangle$  Acute-def by blast
have  $\neg$  Col A P Q
  by (simp add:  $\langle \neg \text{Col } P \ A \ Q \rangle$  not-col-permutation-4)
have Per P C D
  using 2 Perp-perm perp-per-1 by blast
then obtain S where P2:  $P \ S \ C \ LtA \ A \ P \ Q \wedge C \text{ Out } S \ D$ 
  using  $\langle \text{Acute } A \ P \ Q \rangle$   $\langle \neg \text{Col } A \ P \ Q \rangle$   $\langle C \neq D \rangle$  assms(1)
      greenberg-s-axiom-def by blast
hence  $P \ S \ C \ LtA \ A \ P \ Q$ 
  by blast
have C Out S D
  using P2 by blast
have P C OS S D
  using  $\langle C \text{ Out } S \ D \rangle$   $\langle \neg \text{Col } P \ C \ D \rangle$  col-permutation-4
      invert-one-side out-one-side by blast
have Q InAngle C P S
proof -
  have P C OS S Q
    using 5  $\langle P \ C \ OS \ S \ D \rangle$  one-side-symmetry
        one-side-transitivity by blast
  moreover
  have P S OS C Q
proof -
  have P C OS S A
    using 4 calculation one-side-transitivity by blast
  hence P A ParStrict S C
    by (metis 1 Out-cases  $\langle C \text{ Out } S \ D \rangle$  os-distincts

```

```

    out-col par-strict-col-par-strict par-strict-comm
    par-strict-left-comm)
  hence  $P S TS C A$ 
    by (simp add:  $\langle P C OS S A \rangle$  l12-6 l9-31)
  moreover
  have  $A P S CongA C S P$ 
    using assms(2) alternate-interior-angles-postulate-def
    by (meson par-strict-par  $\langle P A ParStrict S C \rangle$  calculation l9-2)
  have  $A P S LtA A P Q$ 
    using conga-preserves-lta
    by (metis  $\langle A P S CongA C S P \rangle \langle P S C LtA A P Q \rangle$ 
       $\langle P \neq A \rangle \langle P \neq Q \rangle$  conga-right-comm conga-sym
      nltA or-lta2-conga)
  hence  $P_4: A P S LeA A P Q \wedge \neg A P S CongA A P Q$ 
    using lta-lea not-lta-and-conga by auto
  hence  $A P S LeA A P Q$ 
    by blast
  have  $\neg A P S CongA A P Q$ 
    using  $P_4$  by blast
  have  $P S TS Q A$ 
  proof -
    {
      assume  $Col P Q S$ 
      have  $P C OS Q S$ 
        by (simp add:  $\langle P C OS S Q \rangle$  one-side-symmetry)
      hence  $P Out Q S$ 
        using  $\langle Col P Q S \rangle$  col-one-side-out by blast
      have  $P Out A A$ 
        using  $\langle P \neq A \rangle$  out-trivial by auto
      hence False
        using out2--conga  $\langle P Out Q S \rangle \langle \neg A P S CongA A P Q \rangle$  by blast
    }
  hence  $\neg Col P Q S$ 
    by auto
  moreover
  have  $A P OS Q S$ 
    using 3  $\langle P C OS S A \rangle \langle P S TS C A \rangle$  invert-one-side
    one-side-symmetry one-side-transitivity
    os-ts1324--os by blast
  hence  $S InAngle Q P A$ 
    using l11-24 lea-in-angle  $\langle A P S LeA A P Q \rangle$  by blast
  ultimately
  show ?thesis
    by (metis NCol-cases TS-def  $\langle P S TS C A \rangle$  in-angle-two-sides)
  qed
  ultimately
  show ?thesis
    using l9-8-1 by blast
  qed
  ultimately
  show ?thesis
    using os2--inangle by blast
  qed
  then obtain  $Y$  where  $P_3: Bet C Y S \wedge (Y = P \vee P Out Y Q)$ 
    using InAngle-def by auto
  hence  $Bet C Y S$ 
    by blast
  have  $Y = P \vee P Out Y Q$ 
    using  $P_3$  by blast
  have  $P_4: Col P Q Y$ 
    using  $\langle Y = P \vee P Out Y Q \rangle$  col-permutation-5 col-trivial-3 out-col by blast
  have  $Col C D Y$ 
    by (meson Bet-cases Col-def Out-def  $\langle Bet C Y S \rangle$ 
       $\langle C Out S D \rangle$  bet-col1 between-exchange4)
  hence  $\exists Y. Col P Q Y \wedge Col C D Y$ 
    using  $P_4$  by blast

```

```

}
thus ?thesis by blast
qed

lemma alternate-interior--proclus:
  assumes greenberg-s-axiom and
    alternate-interior-angles-postulate
  shows proclus-postulate
proof -
{
  fix A B C D P Q
  assume 1: A B Par C D and
    2: Col A B P and
    3:  $\neg$  Col A B Q and
    4: Coplanar C D P Q
  have Col C D P  $\longrightarrow$  ( $\exists$  Y. Col P Q Y  $\wedge$  Col C D Y)
    using col-trivial-3 by blast
  moreover
  {
    assume  $\neg$  Col C D P
    hence A B ParStrict C D
      using 1 2 par-not-col-strict par-strict-symmetry par-symmetry by blast
    obtain C0 where P4: Col C D C0  $\wedge$  C D Perp P C0
      using l8-18-existence  $\langle \neg$  Col C D P  $\rangle$  by blast
    have Col P Q C0  $\longrightarrow$  ( $\exists$  Y. Col P Q Y  $\wedge$  Col C D Y)
      using  $\langle$  Col C D C0  $\wedge$  C D Perp P C0  $\rangle$  by blast
    moreover
    {
      assume  $\neg$  Col P Q C0
      have  $\exists$  Q1. Col Q P Q1  $\wedge$  A B OS C0 Q1
      proof -
        have Q  $\neq$  P
          using 2 3 by blast
        moreover
        have Col A B P
          by (simp add: 2)
        moreover
        have Col Q P P
          by (simp add: col-trivial-2)
        moreover
        have  $\neg$  Col A B Q
          by (simp add: 3)
        moreover
        have  $\neg$  Col C0 A B
      proof -
        have C D ParStrict A B
          by (simp add:  $\langle$  A B ParStrict C D  $\rangle$  par-strict-symmetry)
        moreover
        have Col C0 C D
          by (simp add:  $\langle$  Col C D C0  $\wedge$  C D Perp P C0  $\rangle$  col-permutation-2)
        ultimately
        show ?thesis
          using par-not-col by blast
      qed
    }
    hence  $\neg$  Col A B C0
      using col-permutation-2 by blast
    moreover
    have Coplanar A B Q C0
  proof -
    have Coplanar C D P A
      using 1 2 col-trivial-3 ncop-distincts
      par--coplanar par-col2-par par-symmetry by blast
    moreover
    have Coplanar C D P B
      using 1 2 Col-cases Par-cases calculation
      col-cop--cop par--coplanar by blast
  }
}

```

```

moreover
have Coplanar C D P C0
  using  $\langle \text{Col } C \ D \ C0 \wedge C \ D \ \text{Perp } P \ C0 \rangle \text{ ncop--ncols}$  by blast
ultimately
show ?thesis
  using  $\langle \neg \text{Col } C \ D \ P \rangle \ 4 \text{ coplanar-pseudo-trans}$  by blast
qed
ultimately
show ?thesis
  using cop-not-par-same-side by blast
qed
then obtain Q1 where P5: Col Q P Q1  $\wedge$  A B OS C0 Q1
  by blast
hence Col Q P Q1
  by blast
have A B OS C0 Q1
  using P5 by blast
have  $P \neq Q1$ 
  using 2 P5 one-side-not-col124 by blast
have  $\neg \text{Col } P \ C0 \ Q1$ 
  by (metis  $\langle \text{Col } Q \ P \ Q1 \rangle \langle P \neq Q1 \rangle \langle \neg \text{Col } P \ Q \ C0 \rangle$ 
    col-permutation-1 col-transitivity-1)
have  $\exists A1. \text{Col } A \ B \ A1 \wedge P \ C0 \ OS \ Q1 \ A1$ 
proof –
{
  assume Col P C0 A
  have ?thesis
  proof –
    have  $B \neq A$ 
      using 3 col-trivial-1 by blast
    moreover
    have Col P C0 P
      by (simp add: col-trivial-3)
    moreover
    have Col B A P
      by (simp add: 2 col-permutation-4)
    moreover
    have  $\neg \text{Col } P \ C0 \ B$ 
      by (metis P5  $\langle \text{Col } P \ C0 \ A \rangle \langle \neg \text{Col } P \ C0 \ Q1 \rangle$ 
        col-permutation-4 col-transitivity-1 not-col-distincts
        one-side-not-col123)
    moreover
    have  $\neg \text{Col } P \ C0 \ Q1$ 
      by (simp add:  $\langle \neg \text{Col } P \ C0 \ Q1 \rangle$ )
    moreover
    have Coplanar P C0 B Q1
      by (metis P5  $\langle \text{Col } P \ C0 \ A \rangle$  calculation(2) calculation(3)
        calculation(4) calculation(5) col-trivial-2 l6-21
        ncoplanar-perm-5 ts--coplanar two-sides-cases)
    ultimately
    show ?thesis
      using NCol-perm cop-not-par-same-side by blast
  qed
}
moreover
{
  assume  $\neg \text{Col } P \ C0 \ A$ 

  have ?thesis
  proof –
    have  $B \neq A$ 
      using 3 col-trivial-1 by blast
    moreover
    have Col P C0 P
      by (simp add: col-trivial-3)
    moreover

```

```

have Col A B P
  by (simp add: 2)
moreover
have  $\neg$  Col P C0 A
  by (simp add:  $\langle \neg$  Col P C0 A  $\rangle$ )
moreover
have  $\neg$  Col P C0 Q1
  using  $\langle \neg$  Col P C0 Q1  $\rangle$  by auto
moreover
have Coplanar P C0 A Q1
  by (metis 2 P5 calculation(4) calculation(5)
    col-one-side invert-one-side ncoplanar-perm-5 not-col-distincts
    ts--coplanar two-sides-cases)
ultimately
show ?thesis
  using cop-not-par-same-side by auto
qed
}
ultimately
show ?thesis
  by blast
qed
then obtain A1 where P6: Col A B A1  $\wedge$  P C0 OS Q1 A1
  by blast
hence Col A B A1
  by blast
have P C0 OS Q1 A1
  using P6 by blast
have  $\exists$  C1. Col C D C1  $\wedge$  P C0 OS Q1 C1
proof -
  have Coplanar C D P Q1
    using 4  $\langle$  Col Q P Q1  $\rangle$   $\langle \neg$  Col P Q C0  $\rangle$  col-cop--cop
    col-permutation-4 not-col-distincts by blast
  have P C0 Perp C D
    using Perp-perm  $\langle$  Col C D C0  $\wedge$  C D Perp P C0  $\rangle$  by blast
  moreover
  {
    assume  $\neg$  Col P C0 C
    have  $C \neq D$ 
      using 1 par-neq2 by auto
    moreover
    have Col P C0 C0
      by (simp add: col-trivial-2)
    moreover
    have Col C D C0
      using P4 by blast
    moreover
    have  $\neg$  Col P C0 C
      by (simp add:  $\langle \neg$  Col P C0 C  $\rangle$ )
    moreover
    have  $\neg$  Col P C0 Q1
      by (simp add:  $\langle \neg$  Col P C0 Q1  $\rangle$ )
    moreover
    have Coplanar P C0 C Q1
      using  $\langle$  Coplanar C D P Q1  $\rangle$  calculation(1) calculation(3)
      col2-cop--cop col-trivial-3 ncoplanar-perm-17
      ncoplanar-perm-18 by blast
    ultimately
    have ?thesis
      using cop-not-par-same-side by blast
  }
  moreover
  {
    assume  $\neg$  Col P C0 D
    have  $D \neq C$ 
      using 1 par-distinct by blast
  }

```

```

moreover
have  $Col\ P\ C0\ C0$ 
  by (simp add: col-trivial-2)
moreover
have  $Col\ D\ C\ C0$ 
  using  $P_4\ Col\text{-perm}$  by blast
moreover
have  $\neg\ Col\ P\ C0\ D$ 
  by (simp add:  $\langle\neg\ Col\ P\ C0\ D\rangle$ )
moreover
have  $\neg\ Col\ P\ C0\ Q1$ 
  by (simp add:  $\langle\neg\ Col\ P\ C0\ Q1\rangle$ )
moreover
have  $Coplanar\ P\ C0\ D\ Q1$ 
  using  $\langle Coplanar\ C\ D\ P\ Q1\rangle\ calculation(1)$ 
   $calculation(3)\ col\text{-cop--cop}\ ncoplanar\text{-perm-22}$ 
   $ncoplanar\text{-perm-5}$  by blast
ultimately
have ?thesis
  using  $NCol\text{-perm}\ cop\text{-not-par-same-side}$  by blast
}
ultimately
show ?thesis
  using  $perp\text{-not-col2}$  by blast
qed
then obtain  $C1$  where  $P7: Col\ C\ D\ C1 \wedge P\ C0\ OS\ Q1\ C1$ 
  by blast
hence  $Col\ C\ D\ C1$ 
  by blast
have  $P\ C0\ OS\ Q1\ C1$ 
  using  $P7$  by blast
have  $C0 \neq C1$ 
  using  $\langle P\ C0\ OS\ Q1\ C1\rangle\ col\text{-trivial-2}\ one\text{-side-not-col124}$  by blast
have  $P \neq A1$ 
  using  $\langle P\ C0\ OS\ Q1\ A1\rangle\ col\text{-trivial-3}\ one\text{-side-not-col124}$  by blast
have  $\exists\ Y. Col\ P\ Q\ Y \wedge Col\ C\ D\ Y$ 
proof –
  have  $P\ A1\ ParStrict\ C0\ C1$ 
proof –
  have  $P \neq A1$ 
    by (simp add:  $\langle P \neq A1\rangle$ )
  moreover
  have  $C0 \neq C1$ 
    using  $\langle C0 \neq C1\rangle$  by blast
  moreover
  have  $A\ B\ ParStrict\ C\ D$ 
    using  $\langle A\ B\ ParStrict\ C\ D\rangle$  by auto
  moreover
  have  $Col\ A\ B\ P$ 
    by (simp add: 2)
  moreover
  have  $Col\ A\ B\ A1$ 
    by (simp add:  $\langle Col\ A\ B\ A1\rangle$ )
  moreover
  have  $Col\ C\ D\ C0$ 
    using  $P_4$  by blast
  moreover
  have  $Col\ C\ D\ C1$ 
    by (simp add:  $\langle Col\ C\ D\ C1\rangle$ )
  moreover
  have  $C0\ C1\ Perp\ P\ C0$ 
proof –
  have  $C\ D\ Perp\ P\ C0$ 
    by (simp add:  $P_4$ )
  moreover
  have  $C0 \neq C1$ 

```

```

    by (simp add:  $\langle C0 \neq C1 \rangle$ )
  moreover
  have  $Col\ C\ D\ C0$ 
    by (simp add:  $\langle Col\ C\ D\ C0 \rangle$ )
  moreover
  have  $Col\ C\ D\ C1$ 
    by (simp add:  $\langle Col\ C\ D\ C1 \rangle$ )
  ultimately
  show ?thesis
    by (meson perp-col2)
qed
moreover
have  $P\ A1\ OS\ C0\ Q1$ 
  using 2  $P5\ P6\ calculation(1)\ col2-os--os$  by blast
ultimately
show ?thesis
  using par-strict-col4--par-strict  $P6$  by blast
qed
moreover
have  $C0\ C1\ Perp\ P\ C0$ 
proof -
  have  $C\ D\ Perp\ P\ C0$ 
    by (simp add:  $P4$ )
  moreover
  have  $C0 \neq C1$ 
    by (simp add:  $\langle C0 \neq C1 \rangle$ )
  moreover
  have  $Col\ C\ D\ C0$ 
    using  $P4$  by blast
  moreover
  have  $Col\ C\ D\ C1$ 
    by (simp add:  $\langle Col\ C\ D\ C1 \rangle$ )
  ultimately
  show ?thesis
    by (meson perp-col2)
qed
moreover
have  $P\ A1\ OS\ C0\ Q1$ 
  using 2  $P5\ P6\ \langle P \neq A1 \rangle\ col2-os--os$  by blast
moreover
have  $P\ C0\ OS\ Q1\ A1$ 
  by (simp add:  $\langle P\ C0\ OS\ Q1\ A1 \rangle$ )
moreover
have  $P\ C0\ OS\ Q1\ C1$ 
  by (simp add:  $\langle P\ C0\ OS\ Q1\ C1 \rangle$ )
ultimately
have  $\exists\ Y. Col\ P\ Q1\ Y \wedge Col\ C0\ C1\ Y$ 
  using alternate-interior--proclus-aux assms(1) assms(2)
  by blast
then obtain  $Y1$  where  $Col\ P\ Q1\ Y1$  and  $Col\ C0\ C1\ Y1$ 
  by blast
hence  $Col\ P\ Q1\ Y1$ 
  by blast
hence  $Col\ P\ Q\ Y1$ 
  using NCol-cases  $\langle Col\ Q\ P\ Q1 \rangle\ \langle P \neq Q1 \rangle\ col-transitivity-1$  by blast
moreover
have  $Col\ C\ D\ Y1$ 
  using  $P4\ \langle C0 \neq C1 \rangle\ \langle Col\ C\ D\ C1 \rangle\ \langle Col\ C0\ C1\ Y1 \rangle\ colx$  by blast
ultimately show ?thesis
  by blast
qed
}
ultimately
have  $\exists\ Y. Col\ P\ Q\ Y \wedge Col\ C\ D\ Y$ 
  by blast
}

```

```

ultimately
have  $\exists Y. \text{Col } P \ Q \ Y \wedge \text{Col } C \ D \ Y$ 
  by blast
}
thus ?thesis
  using proclus-postulate-def by blast
qed

```

lemma *alternate-interior--triangle*:

```

assumes alternate-interior-angles-postulate
shows triangle-postulate
proof -
{
  fix A B C D E F
  assume A B C TriSumA D E F
  have Bet D E F
  proof -
    have Col A B C  $\longrightarrow$  ?thesis
      using col-trisuma--bet by (simp add:  $\langle A \ B \ C \ \text{TriSumA } D \ E \ F \rangle$ )
    moreover
    {
      assume  $\neg \text{Col } A \ B \ C$ 
      have  $\exists B1. B \ C \ A \ \text{CongA } C \ B \ B1 \wedge C \ B \ \text{TS } B1 \ A$ 
      proof -
        have  $\neg \text{Col } B \ C \ A$ 
          by (simp add:  $\langle \neg \text{Col } A \ B \ C \rangle$  not-col-permutation-1)
        moreover
        have  $\neg \text{Col } C \ B \ A$ 
          by (simp add:  $\langle \neg \text{Col } A \ B \ C \rangle$  not-col-permutation-3)
        ultimately
        show ?thesis
          by (simp add: ex-conga-ts)
      qed
      then obtain B1 where P1:  $B \ C \ A \ \text{CongA } C \ B \ B1 \wedge C \ B \ \text{TS } B1 \ A$ 
        by blast
      hence  $B \ C \ A \ \text{CongA } C \ B \ B1$ 
        by blast
      have  $C \ B \ \text{TS } B1 \ A$ 
        using P1 by blast
      have  $A \ C \ \text{Par } B \ B1$ 
        using P1 conga-comm l12-21-b l9-2 par-left-comm by blast
      hence  $A \ C \ \text{ParStrict } B \ B1$ 
        using NCol-cases  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-3 par-not-col-strict by blast
      have  $\neg \text{Col } C \ B \ B1$ 
        using  $\langle A \ C \ \text{ParStrict } B \ B1 \rangle$  par-strict-not-col-2 by auto
      have  $\neg \text{Col } A \ B \ B1$ 
        using  $\langle A \ C \ \text{ParStrict } B \ B1 \rangle$  col-permutation-1 par-strict-not-col-3 by blast
      obtain B2 where P2:  $\text{Bet } B1 \ B \ B2 \wedge \text{Cong } B1 \ B \ B2$ 
        using Cong-perm segment-construction by blast
      hence  $\text{Bet } B1 \ B \ B2$ 
        by blast
      have  $\text{Cong } B1 \ B \ B2$ 
        using P2 by blast
      have  $A \neq B$ 
        using  $\langle \neg \text{Col } A \ B \ B1 \rangle$  col-trivial-1 by blast
      have  $B \neq B1$ 
        using  $\langle \neg \text{Col } C \ B \ B1 \rangle$  col-trivial-2 by blast
      have  $C \neq B$ 
        using  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-2 by auto
      have  $B2 \neq B1$ 
        using  $\langle B \neq B1 \rangle \langle \text{Bet } B1 \ B \ B2 \rangle$  between-identity by blast
      have  $B \neq B2$ 
        using  $\langle B2 \neq B1 \rangle \langle \text{Cong } B1 \ B \ B2 \rangle$  cong-identity by blast
      have  $B \ A \ \text{TS } B1 \ B2$ 
        using  $\langle B \neq B2 \rangle \langle \text{Bet } B1 \ B \ B2 \rangle \langle \neg \text{Col } A \ B \ B1 \rangle$  bet--ts col-permutation-4 by blast
      obtain D1 E1 F1 where P3:  $A \ B \ C \ B \ C \ A \ \text{SumA } D1 \ E1 \ F1 \wedge D1 \ E1 \ F1 \ C \ A \ B \ \text{SumA } D \ E \ F$ 

```



```

    using TriSumA-def ⟨A B C TriSumA D E F⟩ by blast
  hence A B C B C A SumA D1 E1 F1
    by blast
  have D1 E1 F1 C A B SumA D E F
    using P3 by blast
  have B1 B B2 CongA D E F
  proof -
    have D1 E1 F1 C A B SumA B1 B B2
  proof -
    have B1 B A A B B2 SumA B1 B B2
      by (simp add: ⟨B A TS B1 B2⟩ ts--suma-1)
    moreover
    have B1 B A CongA D1 E1 F1
  proof -
    have A B C B C A SumA B1 B A
  proof -
    have A B C C B B1 SumA A B B1
      by (simp add: ⟨C B TS B1 A⟩ l9-2 ts--suma)
    moreover
    have A B C CongA A B C
      by (simp add: ⟨A ≠ B⟩ ⟨C ≠ B⟩ conga-refl)
    moreover
    have C B B1 CongA B C A
      using ⟨B C A CongA C B B1⟩ not-conga-sym by blast
    moreover
    have A B B1 CongA B1 B A
      using ⟨A ≠ B⟩ ⟨B ≠ B1⟩ conga-pseudo-refl by auto
    ultimately
    show ?thesis
      using conga3-suma--suma by blast
  qed
  moreover
  have A B C B C A SumA D1 E1 F1
    by (simp add: ⟨A B C B C A SumA D1 E1 F1⟩)
  ultimately
  show ?thesis
    using suma2--conga by blast
  qed
  moreover
  have A B B2 CongA C A B
  proof -
    have B A TS B2 C
      by (metis Par-cases ⟨A C Par B B1⟩ ⟨B A TS B1 B2⟩ ⟨C B TS B1 A⟩
        invert-two-sides l9-2 l9-8-2 par-two-sides-two-sides ts-ts-os)
    moreover
    have B B2 Par A C
      by (metis Par-perm ⟨A C Par B B1⟩ ⟨B ≠ B2⟩ ⟨Bet B1 B B2⟩
        bet-col bet-col1 par-col2-par)
    ultimately
    show ?thesis
      using conga-left-comm alternate-interior-angles-postulate-def assms by blast
  qed
  moreover
  have B1 B B2 CongA B1 B B2
    using ⟨B ≠ B1⟩ ⟨B ≠ B2⟩ conga-refl by auto
  ultimately
  show ?thesis
    using conga3-suma--suma by blast
  qed
  moreover
  have D1 E1 F1 C A B SumA D E F
    by (simp add: P3)
  ultimately
  show ?thesis
    using suma2--conga by blast
  qed

```

```

    hence ?thesis
    using P2 bet-conga--bet by blast
  }
  ultimately
  show ?thesis
  by blast
qed
}
thus ?thesis
  using triangle-postulate-def by blast
qed

```

```

lemma bachmann-s-lotschnittaxiom-aux-R1:
  assumes bachmann-s-lotschnittaxiom
  shows  $\forall A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD.$ 
 $IAB \neq IAC \wedge IAB \neq IBD \wedge$ 
 $A1 A2 \text{ Perp } B1 B2 \wedge$ 
 $A1 A2 \text{ Perp } C1 C2 \wedge$ 
 $B1 B2 \text{ Perp } D1 D2 \wedge$ 
 $\text{Col } A1 A2 IAB \wedge$ 
 $\text{Col } B1 B2 IAB \wedge \text{Col } A1 A2 IAC \wedge$ 
 $\text{Col } C1 C2 IAC \wedge \text{Col } B1 B2 IBD \wedge \text{Col } D1 D2 IBD \wedge$ 
 $\text{Coplanar } IAB IAC IBD C1 \wedge$ 
 $\text{Coplanar } IAB IAC IBD C2 \wedge$ 
 $\text{Coplanar } IAB IAC IBD D1 \wedge \text{Coplanar } IAB IAC IBD D2 \longrightarrow$ 
 $(\exists I. \text{Col } C1 C2 I \wedge \text{Col } D1 D2 I)$ 
proof -
{
  fix A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD
  assume 1:  $IAB \neq IAC$  and
  2:  $IAB \neq IBD$  and
  3:  $A1 A2 \text{ Perp } B1 B2$  and
  4:  $A1 A2 \text{ Perp } C1 C2$  and
  5:  $B1 B2 \text{ Perp } D1 D2$  and
  6:  $\text{Col } A1 A2 IAB$  and
  7:  $\text{Col } B1 B2 IAB$  and
  8:  $\text{Col } A1 A2 IAC$  and
  9:  $\text{Col } C1 C2 IAC$  and
  10:  $\text{Col } B1 B2 IBD$  and
  11:  $\text{Col } D1 D2 IBD$  and
  12:  $\text{Coplanar } IAB IAC IBD C1$  and
  13:  $\text{Coplanar } IAB IAC IBD C2$  and
  14:  $\text{Coplanar } IAB IAC IBD D1$  and
  15:  $\text{Coplanar } IAB IAC IBD D2$ 
  have  $\text{Col } IAB IAC A1$ 
    using 4 6 8 col-transitivity-1 not-col-permutation-3 perp-distinct by blast
  have  $\text{Col } IAB IAC A2$ 
    using 4 6 8 col-permutation-3 col-transitivity-2 perp-distinct by blast
  have  $\text{Col } IAB IBD B1$ 
    using 10 5 7 col-transitivity-1 not-col-permutation-2 perp-not-eq-1 by blast
  have  $\text{Col } IAB IBD B2$ 
    using 10 5 7 Col-perm col-transitivity-2 perp-distinct by blast
  have  $\text{Coplanar } IAB IAC IBD A1$ 
    using  $\langle \text{Col } IAB IAC A1 \rangle \text{ncop--ncols}$  by blast
  have  $\text{Coplanar } IAB IAC IBD A2$ 
    using  $\langle \text{Col } IAB IAC A2 \rangle \text{ncop--ncols}$  by blast
  have  $\text{Coplanar } IAB IAC IBD B1$ 
    using  $\langle \text{Col } IAB IBD B1 \rangle \text{ncop--ncols}$  by blast
  have  $\text{Coplanar } IAB IAC IBD B2$ 
    using  $\langle \text{Col } IAB IBD B2 \rangle \text{ncop--ncols}$  by blast
  have  $IAB IAC \text{ Perp } IBD IAB$ 
    using perp-col4 1 2 3 6 7 8 10 by auto
  hence H1:  $\text{Per } IAC IAB IBD$ 
    using perp-per-1 1 l8-8 by blast
  have  $\neg \text{Col } IAC IAB IBD$ 
    using per-not-col 1 2  $\langle \text{Per } IAC IAB IBD \rangle$  by auto

```

```

have  $\neg$  Col A1 A2 IBD
  using 6 8 col3 par-neg1 4  $\langle \neg$  Col IAC IAB IBD  $\rangle$  perp-distinct by fastforce
have A1 A2 ParStrict D1 D2
proof -
  have A1 A2 Par D1 D2
  proof -
    have Coplanar B1 B2 A1 D1
    proof -
      have Coplanar IAB IAC IBD B1
        by (simp add:  $\langle$  Coplanar IAB IAC IBD B1  $\rangle$ )
      moreover
      have Coplanar IAB IAC IBD B2
        by (simp add:  $\langle$  Coplanar IAB IAC IBD B2  $\rangle$ )
      moreover
      have Coplanar IAB IAC IBD A1
        using  $\langle$  Coplanar IAB IAC IBD A1  $\rangle$  by force
      moreover
      have Coplanar IAB IAC IBD D1
        using 14 by blast
      ultimately
      show ?thesis
        by (meson  $\langle \neg$  Col IAC IAB IBD  $\rangle$  coplanar-perm-6 coplanar-pseudo-trans)
    qed
  moreover
  have Coplanar B1 B2 A1 D2
    using coplanar-pseudo-trans not-par-not-col par-id-1
    by (meson 15  $\langle$  Coplanar IAB IAC IBD A1  $\rangle$   $\langle$  Coplanar IAB IAC IBD B1  $\rangle$ 
       $\langle$  Coplanar IAB IAC IBD B2  $\rangle$   $\langle \neg$  Col IAC IAB IBD  $\rangle$  not-col-permutation-4)
  moreover
  have Coplanar B1 B2 A2 D1
  proof -
    have Coplanar IAB IAC IBD B1
      by (simp add:  $\langle$  Coplanar IAB IAC IBD B1  $\rangle$ )
    moreover
    have Coplanar IAB IAC IBD B2
      by (simp add:  $\langle$  Coplanar IAB IAC IBD B2  $\rangle$ )
    moreover
    have Coplanar IAB IAC IBD A2
      using  $\langle$  Coplanar IAB IAC IBD A2  $\rangle$  by blast
    moreover
    have Coplanar IAB IAC IBD D1
      using 14 by blast
    ultimately
    show ?thesis
      by (meson  $\langle \neg$  Col IAC IAB IBD  $\rangle$  coplanar-perm-6 coplanar-pseudo-trans)
    qed
  moreover
  have Coplanar B1 B2 A2 D2
  proof -
    have Coplanar IAB IAC IBD B1
      by (simp add:  $\langle$  Coplanar IAB IAC IBD B1  $\rangle$ )
    moreover
    have Coplanar IAB IAC IBD B2
      by (simp add:  $\langle$  Coplanar IAB IAC IBD B2  $\rangle$ )
    moreover
    have Coplanar IAB IAC IBD A2
      using  $\langle$  Coplanar IAB IAC IBD A2  $\rangle$  by blast
    moreover
    have Coplanar IAB IAC IBD D2
      using 15 by blast
    ultimately
    show ?thesis
      by (meson  $\langle \neg$  Col IAC IAB IBD  $\rangle$  coplanar-perm-6 coplanar-pseudo-trans)
    qed
  moreover
  have A1 A2 Perp B1 B2

```

```

    by (simp add: 3)
  moreover
  have  $D1 \ D2 \ \text{Perp} \ B1 \ B2$ 
    using 5 Perp-perm by blast
  ultimately
  show ?thesis
    using l12-9 by blast
qed
moreover
have  $Col \ D1 \ D2 \ IBD$ 
  by (simp add: 11)
moreover
have  $\neg \ Col \ A1 \ A2 \ IBD$ 
  using 6 8  $\langle \neg \ Col \ IAC \ IAB \ IBD \rangle$  calculation(1) col3 par-neq1 by blast
ultimately
show ?thesis
  using par-not-col-strict by blast
qed
have  $Q1: B1 \ B2 \ \text{ParStrict} \ C1 \ C2$ 
proof -
  have  $B1 \ B2 \ \text{Par} \ C1 \ C2$ 
  proof -
    have  $Coplanar \ A1 \ A2 \ B1 \ C1$ 
    proof -
      have  $Coplanar \ IAB \ IAC \ IBD \ A1$ 
        by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ A1 \rangle$ )
      moreover
      have  $Coplanar \ IAB \ IAC \ IBD \ A2$ 
        by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ A2 \rangle$ )
      moreover
      have  $Coplanar \ IAB \ IAC \ IBD \ B1$ 
        using  $\langle Coplanar \ IAB \ IAC \ IBD \ B1 \rangle$  by blast
      moreover
      have  $Coplanar \ IAB \ IAC \ IBD \ C1$ 
        using 12 by blast
      ultimately
      show ?thesis
        by (meson  $\langle \neg \ Col \ IAC \ IAB \ IBD \rangle$  coplanar-perm-6 coplanar-pseudo-trans)
    qed
    moreover
    have  $Coplanar \ A1 \ A2 \ B1 \ C2$ 
  proof -
    have  $Coplanar \ IAB \ IAC \ IBD \ A1$ 
      by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ A1 \rangle$ )
    moreover
    have  $Coplanar \ IAB \ IAC \ IBD \ A2$ 
      by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ A2 \rangle$ )
    moreover
    have  $Coplanar \ IAB \ IAC \ IBD \ B1$ 
      by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ B1 \rangle$ )
    moreover
    have  $Coplanar \ IAB \ IAC \ IBD \ C2$ 
      by (simp add: 13)
    ultimately
    show ?thesis
      by (meson  $\langle \neg \ Col \ IAC \ IAB \ IBD \rangle$  coplanar-perm-6 coplanar-pseudo-trans)
  qed
  moreover
  have  $Coplanar \ A1 \ A2 \ B2 \ C1$ 
  proof -
    have  $Coplanar \ IAB \ IAC \ IBD \ A1$ 
      by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ A1 \rangle$ )
    moreover
    have  $Coplanar \ IAB \ IAC \ IBD \ A2$ 
      by (simp add:  $\langle Coplanar \ IAB \ IAC \ IBD \ A2 \rangle$ )
    moreover

```

```

have Coplanar IAB IAC IBD B2
  by (simp add: ⟨Coplanar IAB IAC IBD B2⟩)
moreover
have Coplanar IAB IAC IBD C1
  using 12 by blast
ultimately
show ?thesis
  by (meson ⟨¬ Col IAC IAB IBD⟩ coplanar-perm-6 coplanar-pseudo-trans)
qed
moreover
have Coplanar A1 A2 B2 C2
proof -
  have Coplanar IAB IAC IBD A1
    by (simp add: ⟨Coplanar IAB IAC IBD A1⟩)
  moreover
  have Coplanar IAB IAC IBD A2
    by (simp add: ⟨Coplanar IAB IAC IBD A2⟩)
  moreover
  have Coplanar IAB IAC IBD B2
    using ⟨Coplanar IAB IAC IBD B2⟩ by blast
  moreover
  have Coplanar IAB IAC IBD C2
    using ⟨Coplanar IAB IAC IBD C2⟩ by blast
  ultimately
  show ?thesis
    by (meson ⟨¬ Col IAC IAB IBD⟩ coplanar-perm-6 coplanar-pseudo-trans)
qed
moreover
have B1 B2 Perp A1 A2
  using 3 Perp-perm by blast
moreover
have C1 C2 Perp A1 A2
  using 4 Perp-perm by blast
ultimately
show ?thesis
  using l12-9 by blast
qed
moreover
have Col C1 C2 IAC
  using 9 by auto
moreover
have ¬ Col B1 B2 IAC
  using 10 7 ⟨¬ Col IAC IAB IBD⟩ calculation(1) col3 par-neq1 by blast
ultimately
show ?thesis
  using par-not-col-strict by blast
qed
have IAC ≠ IAB
  using 1 by auto
have IAB ≠ IBD
  using 2 by auto
have Col C1 C2 IAC
  using 9 by auto
then obtain P1 where P1: C1 ≠ P1 ∧ C2 ≠ P1 ∧ IAC ≠ P1 ∧ Col C1 C2 P1
  using diff-col-ex3 by blast
hence C1 ≠ P1
  by blast
have C2 ≠ P1
  using P1 by blast
have IAC ≠ P1
  using P1 by blast
have Col C1 C2 P1
  using P1 by blast
have Col D1 D2 IBD
  by (simp add: 11)
then obtain R1 where P2: D1 ≠ R1 ∧ D2 ≠ R1 ∧ IBD ≠ R1 ∧ Col D1 D2 R1

```

```

    using diff-col-ex3 by blast
  hence  $D1 \neq R1$ 
    by blast
  have  $D2 \neq R1$ 
    using P2 by blast
  have  $IBD \neq R1$ 
    using P2 by blast
  have  $Col\ D1\ D2\ R1$ 
    using P2 by blast
  have  $IAC \neq IBD$ 
    using 8  $\langle \neg\ Col\ A1\ A2\ IBD \rangle$  by auto
  have  $D1 \neq D2$ 
    using  $\langle A1\ A2\ ParStrict\ D1\ D2 \rangle$  par-strict-distinct by blast
  have  $C1 \neq C2$ 
    using Q1 par-strict-neq2 by auto
  have P3:  $C1 \neq Q \wedge Q \neq D1 \wedge Per\ C1\ Q\ D1 \wedge Per\ Q\ C1\ C2 \wedge$ 
 $Per\ Q\ D1\ D2 \wedge Coplanar\ C1\ Q\ D1\ C2 \wedge Coplanar\ C1\ Q\ D1\ D2$ 
 $\longrightarrow$ 
  ( $\exists\ S.\ Col\ C1\ C2\ S \wedge Col\ D1\ D2\ S$ )
    using assms(1) bachmann-s-lotschnittaxiom-def by blast
  have  $\exists\ I.\ Col\ IAC\ P1\ I \wedge Col\ IBD\ R1\ I$ 
  proof -
    have  $IAC \neq IAB$ 
      using 1 by auto
    moreover
    have  $IAB \neq IBD$ 
      using 2 by auto
    moreover
    have  $Per\ IAC\ IAB\ IBD$ 
      using H1 by blast
    moreover
    have  $IAC\ IAB\ Perp\ P1\ IAC$ 
  proof -
    have  $IAC \neq IAB$ 
      using 1 by blast
    moreover
    have  $P1 \neq IAC$ 
      using P1 by blast
    ultimately
    show ?thesis
      using 8 6 P1 9 4 perp-col4 by blast
  qed
  hence  $Per\ IAB\ IAC\ P1$ 
    using perp-per-1 by blast
  moreover
  have  $IBD\ IAB\ Perp\ R1\ IBD$ 
  proof -
    have  $IBD \neq IAB$ 
      using 2 by auto
    moreover
    have  $R1 \neq IBD$ 
      using P2 by auto
    ultimately
    show ?thesis
      using 10 7 P2 5 11 perp-col4 by blast
  qed
  hence  $Per\ IAB\ IBD\ R1$ 
    using perp-per-1 by blast
  moreover
  have  $Coplanar\ IAC\ IAB\ IBD\ P1$ 
  proof -
    have  $C1 \neq C2$ 
      using Q1 par-strict-neq2 by auto
    moreover
    have  $Coplanar\ IAC\ IAB\ IBD\ C1$ 
      using 12 coplanar-perm-6 by blast

```

```

    moreover
    have Coplanar IAC IAB IBD C2
      using 13 coplanar-perm-6 by blast
    moreover
    have Col C1 C2 P1
      using P1 by blast
    ultimately
    show ?thesis
      using col-cop2--cop by blast
  qed
  moreover
  have Coplanar IAC IAB IBD R1
    using 14 15 P2  $\langle D1 \neq D2 \rangle$  col-cop2--cop coplanar-perm-6 by blast
  ultimately
  show ?thesis
    using assms(1) bachmann-s-lotschnittaxiom-def by blast
  qed
  then obtain I where P5: Col IAC P1 I  $\wedge$  Col IBD R1 I
    by auto
  hence Col IAC P1 I
    by blast
  have Col IBD R1 I
    using P5 by blast
  have Col C1 C2 I
    using 9  $\langle Col C1 C2 P1 \rangle \langle Col IAC P1 I \rangle \langle IAC \neq P1 \rangle$  colx by blast
  moreover
  have Col D1 D2 I
    using 11  $\langle Col D1 D2 R1 \rangle \langle Col IBD R1 I \rangle \langle IBD \neq R1 \rangle$  colx by blast
  ultimately
  have  $\exists I0. (Col C1 C2 I0 \wedge Col D1 D2 I0)$ 
    by blast
}
thus ?thesis by blast
qed

```

```

lemma bachmann-s-lotschnittaxiom-aux-R2:
  assumes  $\forall A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD.$ 
     $IAB \neq IAC \wedge IAB \neq IBD \wedge$ 
     $A1 A2 \text{ Perp } B1 B2 \wedge$ 
     $A1 A2 \text{ Perp } C1 C2 \wedge$ 
     $B1 B2 \text{ Perp } D1 D2 \wedge$ 
     $Col A1 A2 IAB \wedge$ 
     $Col B1 B2 IAB \wedge Col A1 A2 IAC \wedge$ 
     $Col C1 C2 IAC \wedge Col B1 B2 IBD \wedge Col D1 D2 IBD \wedge$ 
     $Coplanar IAB IAC IBD C1 \wedge$ 
     $Coplanar IAB IAC IBD C2 \wedge$ 
     $Coplanar IAB IAC IBD D1 \wedge Coplanar IAB IAC IBD D2 \longrightarrow$ 
     $(\exists I. Col C1 C2 I \wedge Col D1 D2 I)$ 
  shows bachmann-s-lotschnittaxiom
proof -
{
  fix P Q R P1 R1
  assume 1:  $P \neq Q$  and
    2:  $Q \neq R$  and
    3:  $Per P Q R$  and
    4:  $Per Q P P1$  and
    5:  $Per Q R R1$  and
    6:  $Coplanar P Q R P1$  and
    7:  $Coplanar P Q R R1$ 
  have  $\exists S. Col P P1 S \wedge Col R R1 S$ 
  proof cases
    assume  $P = P1$ 
    thus ?thesis
      using col-trivial-1 col-trivial-2 by blast
  next
    assume  $P \neq P1$ 

```

```

thus ?thesis
proof cases
  assume  $R = R1$ 
  thus ?thesis
    using col-trivial-1 col-trivial-2 by blast
next
  assume  $R \neq R1$ 
  have  $Q \neq P$ 
    using 1 by auto
  moreover
  have  $Q \neq R$ 
    by (simp add: 2)
  moreover
  have  $P \ Q \ \text{Perp} \ Q \ R$ 
    by (simp add: 1 2 3 per-perp)
  moreover
  have  $P \ Q \ \text{Perp} \ P \ P1$ 
    using 1 4  $\langle P \neq P1 \rangle$  per-perp perp-left-comm by auto
  moreover
  have  $Q \ R \ \text{Perp} \ R \ R1$ 
    by (simp add: 2 5  $\langle R \neq R1 \rangle$  per-perp)
  moreover
  have  $\text{Col} \ P \ Q \ Q$ 
    by (simp add: col-trivial-2)
  moreover
  have  $\text{Col} \ Q \ R \ Q$ 
    using col-trivial-3 by auto
  moreover
  have  $\text{Col} \ P \ Q \ P$ 
    by (simp add: col-trivial-3)
  moreover
  have  $\text{Col} \ P \ P1 \ P$ 
    by (simp add: col-trivial-3)
  moreover
  have  $\text{Col} \ Q \ R \ R$ 
    by (simp add: col-trivial-2)
  moreover
  have  $\text{Col} \ R \ R1 \ R$ 
    by (simp add: col-trivial-3)
  moreover
  have  $\text{Coplanar} \ Q \ P \ R \ P$ 
    using ncop-distincts by blast
  moreover
  have  $\text{Coplanar} \ Q \ P \ R \ P1$ 
    using 6 ncoplanar-perm-6 by blast
  moreover
  have  $\text{Coplanar} \ Q \ P \ R \ R$ 
    using ncop-distincts by blast
  moreover
  have  $\text{Coplanar} \ Q \ P \ R \ R1$ 
    using 7 ncoplanar-perm-6 by blast
  ultimately
  show ?thesis
    using assms(1) by blast
qed
qed
}
thus ?thesis
  using bachmann-s-lotschnittaxiom-def by blast
qed

lemma bachmann-s-lotschnittaxiom-aux:
  shows bachmann-s-lotschnittaxiom  $\longleftrightarrow$ 
 $(\forall \ A1 \ A2 \ B1 \ B2 \ C1 \ C2 \ D1 \ D2 \ IAB \ IAC \ IBD.$ 
 $IAB \neq IAC \wedge IAB \neq IBD \wedge$ 
 $A1 \ A2 \ \text{Perp} \ B1 \ B2 \wedge$ 

```


$A1 \ A2 \ \text{Perp} \ C1 \ C2 \wedge$
 $B1 \ B2 \ \text{Perp} \ D1 \ D2 \wedge$
 $\text{Col} \ A1 \ A2 \ IAB \wedge$
 $\text{Col} \ B1 \ B2 \ IAB \wedge \text{Col} \ A1 \ A2 \ IAC \wedge$
 $\text{Col} \ C1 \ C2 \ IAC \wedge \text{Col} \ B1 \ B2 \ IBD \wedge \text{Col} \ D1 \ D2 \ IBD \wedge$
 $\text{Coplanar} \ IAB \ IAC \ IBD \ C1 \wedge$
 $\text{Coplanar} \ IAB \ IAC \ IBD \ C2 \wedge$
 $\text{Coplanar} \ IAB \ IAC \ IBD \ D1 \wedge \text{Coplanar} \ IAB \ IAC \ IBD \ D2 \longrightarrow$
 $(\exists \ I. \ \text{Col} \ C1 \ C2 \ I \wedge \text{Col} \ D1 \ D2 \ I))$
using *bachmann-s-lotschnittaxiom-aux-R2 bachmann-s-lotschnittaxiom-aux-R1* **by** *blast*

lemma *bachmann-s-lotschnittaxiom--legendre-s-parallel-postulate:*

assumes *bachmann-s-lotschnittaxiom*

shows *legendre-s-parallel-postulate*

proof –

have $P0: \forall \ A1 \ A2 \ B1 \ B2 \ C1 \ C2 \ D1 \ D2 \ IAB \ IAC \ IBD.$

$IAB \neq IAC \wedge IAB \neq IBD \wedge$

$A1 \ A2 \ \text{Perp} \ B1 \ B2 \wedge$

$A1 \ A2 \ \text{Perp} \ C1 \ C2 \wedge$

$B1 \ B2 \ \text{Perp} \ D1 \ D2 \wedge$

$\text{Col} \ A1 \ A2 \ IAB \wedge$

$\text{Col} \ B1 \ B2 \ IAB \wedge \text{Col} \ A1 \ A2 \ IAC \wedge$

$\text{Col} \ C1 \ C2 \ IAC \wedge \text{Col} \ B1 \ B2 \ IBD \wedge \text{Col} \ D1 \ D2 \ IBD \wedge$

$\text{Coplanar} \ IAB \ IAC \ IBD \ C1 \wedge$

$\text{Coplanar} \ IAB \ IAC \ IBD \ C2 \wedge$

$\text{Coplanar} \ IAB \ IAC \ IBD \ D1 \wedge \text{Coplanar} \ IAB \ IAC \ IBD \ D2 \longrightarrow$

$(\exists \ I. \ \text{Col} \ C1 \ C2 \ I \wedge \text{Col} \ D1 \ D2 \ I)$

using *assms(1) bachmann-s-lotschnittaxiom-aux-R1* **by** *blast*

{

assume $(\exists \ A \ B \ C. (\neg \text{Col} \ A \ B \ C \wedge \text{Acute} \ A \ B \ C \wedge$

$(\forall \ P \ Q. \ B \ \text{Out} \ A \ P \wedge P \neq Q \wedge \text{Per} \ B \ P \ Q \wedge \text{Coplanar} \ A \ B \ C \ Q$

$\longrightarrow (\exists \ Y. \ B \ \text{Out} \ C \ Y \wedge \text{Col} \ P \ Q \ Y))))$

then obtain $A \ B \ C$ **where** $P1: (\neg \text{Col} \ A \ B \ C \wedge \text{Acute} \ A \ B \ C \wedge$

$(\forall \ P \ Q. \ B \ \text{Out} \ A \ P \wedge P \neq Q \wedge \text{Per} \ B \ P \ Q \wedge \text{Coplanar} \ A \ B \ C \ Q$

$\longrightarrow (\exists \ Y. \ B \ \text{Out} \ C \ Y \wedge \text{Col} \ P \ Q \ Y))))$

by *auto*

hence $\neg \text{Col} \ A \ B \ C$

by *auto*

have $\text{Acute} \ A \ B \ C$

using $P1$ **by** *blast*

have $Q1: \forall \ P \ Q. \ B \ \text{Out} \ A \ P \wedge P \neq Q \wedge \text{Per} \ B \ P \ Q \wedge \text{Coplanar} \ A \ B \ C \ Q$

$\longrightarrow (\exists \ Y. \ B \ \text{Out} \ C \ Y \wedge \text{Col} \ P \ Q \ Y)$

using $P1$ **by** *blast*

{

fix T

assume $T \ \text{InAngle} \ A \ B \ C$

{

assume $\text{Col} \ A \ B \ T$

have $\neg \text{Col} \ B \ C \ T$

by $(\text{metis} \ \langle \text{Col} \ A \ B \ T \rangle \ \langle T \ \text{InAngle} \ A \ B \ C \rangle \ \langle \neg \text{Col} \ A \ B \ C \rangle \ \text{col-trivial-3}$
 $\text{colx inangle-distincts not-col-permutation-2})$

then obtain Y **where** $P2: \text{Col} \ B \ C \ Y \wedge B \ C \ \text{Perp} \ T \ Y$

using *l8-18-existence* **by** *blast*

hence $\text{Col} \ B \ C \ Y$

by *auto*

have $B \ C \ \text{Perp} \ T \ Y$

using $P2$ **by** *blast*

have $B \ \text{Out} \ A \ T$

using $\text{Col-def NCol-perm} \ \langle \text{Col} \ A \ B \ T \rangle \ \langle T \ \text{InAngle} \ A \ B \ C \rangle \ \langle \neg \text{Col} \ A \ B \ C \rangle$
 col-in-angle-out **by** *blast*

moreover

have $T \ B \ A \ A \ B \ C \ \text{SumA} \ T \ B \ C$

proof –

have $A \ B \ C \ \text{CongA} \ A \ B \ C$

using $\langle \neg \text{Col} \ A \ B \ C \rangle \ \text{conga-refl not-col-distincts}$ **by** *fastforce*

moreover

```

have  $\neg B A OS T C$ 
  using  $\langle B Out A T \rangle$  one-side-not-col123 out-col by blast
moreover
have Coplanar  $T B A C$ 
  by (simp add:  $\langle Col A B T \rangle$  col--coplanar col-permutation-3)
moreover
have  $T B C CongA T B C$ 
  using  $\langle T InAngle A B C \rangle$  conga-refl inangle-distincts by auto
ultimately
show ?thesis
  using SumA-def by blast
qed
hence  $A B C CongA T B C$ 
  by (meson  $\langle Col A B T \rangle \langle T InAngle A B C \rangle \langle \neg Col A B C \rangle$  bet-col
    col-in-angle-out col-permutation-4 l6-6 out213-suma--conga)
hence Acute  $T B C$ 
  using  $\langle Acute A B C \rangle$  acute-conga--acute by blast
hence  $B Out C Y$ 
  using P2 acute-col-perp--out l6-6 by blast
moreover
have Bet  $T T Y$ 
  by (simp add: between-trivial2)
ultimately
have  $(\exists X Y. B Out A X \wedge B Out C Y \wedge Bet X T Y)$ 
  by blast
}
moreover
{
  assume  $\neg Col A B T$ 
  then obtain  $X$  where P3:  $Col A B X \wedge A B Perp T X$ 
    using l8-18-existence by blast
  hence  $Col A B X$ 
    by auto
  have  $A B Perp T X$ 
    using P3 by blast
  have  $B Out A X$ 
  proof -
    have Acute  $T B A$ 
    proof -
      have Acute  $A B C$ 
        by (simp add:  $\langle Acute A B C \rangle$ )
      moreover
      have  $A B T LeA A B C$ 
        by (simp add:  $\langle T InAngle A B C \rangle$  inangle--lea)
      hence  $T B A LeA A B C$ 
        using lea-left-comm by blast
      ultimately
      show ?thesis
        using acute-lea-acute by blast
    qed
  moreover
  have  $Col B A X$ 
    using  $\langle Col A B X \rangle$  not-col-permutation-4 by blast
  moreover
  have  $B A Perp T X$ 
    by (simp add:  $\langle A B Perp T X \rangle$  perp-left-comm)
  ultimately
  show ?thesis
    using acute-col-perp--out l6-6 by blast
  qed
  have  $\exists Y. B Out C Y \wedge Col X T Y$ 
  proof -
    have  $X \neq T$ 
      using P3  $\langle \neg Col A B T \rangle$  by blast
    moreover
    have  $Col A B B$ 

```

```

    by (simp add: col-trivial-2)
  hence Per B X T
    using  $\langle \text{Col } A \ B \ X \rangle \langle A \ B \ \text{Perp } T \ X \rangle$  Per-perm l8-16-1 by blast
  moreover
  have Coplanar A B C T
    using  $\langle T \ \text{InAngle } A \ B \ C \rangle$  coplanar-perm-9 inangle--coplanar by blast
  ultimately
  show ?thesis
    using  $\langle B \ \text{Out } A \ X \rangle$  Q1 by blast
qed
then obtain Y where P5:  $B \ \text{Out } C \ Y \wedge \text{Col } X \ T \ Y$ 
  by auto
hence  $B \ \text{Out } C \ Y$ 
  by auto
have  $\text{Col } X \ T \ Y$ 
  using P5 by blast
{
  assume  $\text{Bet } T \ Y \ X \vee \text{Bet } Y \ X \ T$ 
  have  $\text{Bet } X \ T \ Y$ 
  proof cases
    assume  $T = Y$ 
    thus ?thesis
      by (simp add: between-trivial)
  next
    assume  $T \neq Y$ 
    have  $\neg \text{Col } B \ C \ T$ 
      by (metis (full-types) P5  $\langle B \ \text{Out } A \ X \rangle \langle T \neq Y \rangle$ 
         $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-3 colx
        not-col-permutation-2 out-col out-distinct)
    {
      assume  $\text{Bet } T \ Y \ X$ 
      have  $C \ B \ \text{OS } T \ A$ 
        by (simp add:  $\langle T \ \text{InAngle } A \ B \ C \rangle \langle \neg \text{Col } A \ B \ C \rangle$ 
           $\langle \neg \text{Col } B \ C \ T \rangle$  in-angle-one-side
          l11-24 not-col-permutation-3)
      have  $C \ B \ \text{TS } T \ A$ 
      proof -
        have  $C \ B \ \text{TS } X \ T$ 
        proof -
          have  $\neg \text{Col } X \ C \ B$ 
            by (metis Out-def  $\langle B \ \text{Out } A \ X \rangle \langle \text{Col } A \ B \ X \rangle$ 
               $\langle \neg \text{Col } A \ B \ C \rangle$  col-permutation-2 col-transitivity-1)
          moreover have  $\neg \text{Col } T \ C \ B$ 
            using NCol-perm  $\langle \neg \text{Col } B \ C \ T \rangle$  by blast
          moreover have  $\exists Z. \text{Col } Z \ C \ B \wedge \text{Bet } X \ Z \ T$ 
            using Col-perm P5  $\langle \text{Bet } T \ Y \ X \rangle$  between-symmetry out-col by blast
          ultimately show ?thesis
            using TS-def by blast
        qed
      qed
    }
    moreover
    have  $C \ B \ \text{OS } X \ A$ 
      using  $\langle B \ \text{Out } A \ X \rangle \langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-3
      invert-one-side one-side-reflexivity
      os-out-os by blast
    ultimately
    show ?thesis
      using l9-2 l9-8-2 by blast
  qed
  hence  $\text{Bet } X \ T \ Y$ 
    using  $\langle C \ B \ \text{OS } T \ A \rangle$  l9-9 by blast
}
moreover
{
  assume  $\text{Bet } Y \ X \ T$ 
  have  $A \ B \ \text{OS } T \ C$ 
    by (simp add: NCol-perm  $\langle T \ \text{InAngle } A \ B \ C \rangle$ 

```

```

      <¬ Col A B C> <¬ Col A B T> in-angle-one-side)
have A B TS T C
proof -
  have A B TS Y T
  by (metis (full-types) <Bet Y X T> <Col A B X>
      <Col X T Y> <¬ Col A B T> between-equality between-trivial
      calculation col-permutation-5 colx l9-18 not-col-permutation-2)
  moreover
  have A B OS Y C
  using <A B OS T C> <B Out C Y> col124--nos
      invert-one-side l6-6 out-one-side by blast
  ultimately
  show ?thesis
  using l9-2 l9-8-2 by blast
qed
hence Bet X T Y
using <A B OS T C> l9-9 by blast
}
ultimately
show ?thesis
using <Bet T Y X ∨ Bet Y X T> by blast
qed
}
hence Bet X T Y
using Col-def <Col X T Y> by blast
hence ∃ X Y. B Out A X ∧ B Out C Y ∧ Bet X T Y
using <B Out A X> <B Out C Y> by blast
}
ultimately
have (∃ X Y. B Out A X ∧ B Out C Y ∧ Bet X T Y)
by blast
}
hence legendre-s-parallel-postulate
using legendre-s-parallel-postulate-def <Acute A B C>
    <¬ Col A B C> by blast
}
hence (∃ A B C. (¬ Col A B C ∧ Acute A B C ∧
    (∀ P Q. B Out A P ∧ P ≠ Q ∧ Per B P Q ∧ Coplanar A B C Q
    → (∃ Y. B Out C Y ∧ Col P Q Y)))) →
legendre-s-parallel-postulate
by blast
moreover
have ∃ A B C. (¬ Col A B C ∧
    Acute A B C ∧
    (∀ P Q. B Out A P ∧ P ≠ Q ∧
        Per B P Q ∧
        Coplanar A B C Q
    →
        (∃ Y. B Out C Y ∧ Col P Q Y)))
proof -
  obtain C E D9 where P6: ¬ (Bet C E D9 ∨ Bet E D9 C ∨ Bet D9 C E)
  using lower-dim by blast
  hence ¬ Col C E D9
  by (simp add: Col-def)
  then obtain B where P7: Col D9 E B ∧ D9 E Perp C B
  using l8-18-existence NCol-cases by blast
  hence Col D9 E B
  by auto
  have D9 E Perp C B
  using P7 by blast
  have ∃ F. Col D9 E F ∧ B ≠ F
  using <Col D9 E B> diff-col-ex3 by blast
  then obtain F where P8: Col D9 E F ∧ B ≠ F
  by blast
  hence Col D9 E F
  by auto

```

```

have B ≠ F
  using P8 by auto
then obtain A where P9: (Bet B F A ∨ Bet B A F) ∧ Cong B A B C
  using segment-construction-2 by presburger
hence Bet B F A ∨ Bet B A F
  by auto
have Cong B A B C
  using P9 by auto
have Col D9 E A
  using ⟨B ≠ F⟩ ⟨Bet B F A ∨ Bet B A F⟩ ⟨Col D9 E B⟩
    ⟨Col D9 E F⟩ bet-col1 between-trivial colx by blast
have C B Perp B A
proof —
  have D9 E Perp C B
    by (simp add: ⟨D9 E Perp C B⟩)
  moreover
  have B ≠ A
    by (metis NCol-cases ⟨Col D9 E A⟩ ⟨Cong B A B C⟩
      ⟨¬ Col C E D9⟩ cong-diff-3)
  moreover
  have Col D9 E B
    using ⟨Col D9 E B⟩ by blast
  moreover
  have Col D9 E A
    using ⟨Col D9 E A⟩ by auto
  ultimately
  show ?thesis
    using perp-col0 by blast
qed
hence B A Perp C B
  using Perp-perm by blast
have K1: ¬ Col B A C
  by (simp add: ⟨B A Perp C B⟩ perp-not-col)
obtain D where P8B: D Midpoint A C
  using midpoint-existence by blast
have ¬ Col A B D
  by (metis P8B ⟨¬ Col B A C⟩ col-transitivity-2
    midpoint-col midpoint-distinct-1
    not-col-permutation-1)
moreover
have B ≠ A
  using ⟨¬ Col B A C⟩ col-trivial-1 by blast
have A ≠ C
  using ⟨¬ Col B A C⟩ col-trivial-2 by blast
have B ≠ C
  using ⟨¬ Col B A C⟩ col-trivial-3 by blast
have D ≠ A
  using calculation not-col-distincts by blast
have D ≠ C
  using P8B ⟨A ≠ C⟩ midpoint-not-midpoint by blast
have Acute A B D
proof —
  have D ≠ B
    using calculation col-trivial-2 by blast
  have Per A B C
    using Perp-cases ⟨C B Perp B A⟩ perp-per-2 by blast
  moreover
  have A B D LtA A B C
  proof —
    have A B D LeA A B C
  proof —
    have D InAngle A B C
  proof —
    have Bet A D C
      using Midpoint-def P8B by blast
    moreover

```

```

have  $D = B \vee B$  Out  $D D$ 
  using out-trivial by blast
ultimately
show ?thesis
  using InAngle-def  $\langle B \neq A \rangle \langle B \neq C \rangle \langle D \neq B \rangle$  by auto
qed
moreover
have  $A B D$  CongA  $A B D$ 
  using  $\langle B \neq A \rangle \langle D \neq B \rangle$  conga-refl by auto
ultimately
show ?thesis
  by (simp add: inangle--lea)
qed
moreover
{
  assume  $A B D$  CongA  $A B C$ 
  hence  $Per A B D$ 
    using  $\langle Per A B C \rangle$  conga-sym l11-17 by blast
  have  $Per C B D$ 
    using  $P8B \langle B \neq A \rangle \langle D \neq C \rangle \langle Per A B C \rangle \langle Per A B D \rangle$ 
      col-per2--per l7-2 l8-2 l8-8 midpoint-col by blast
  have  $A B D C B D$  SumA  $A B C$ 
  proof -
    have  $D B C$  CongA  $C B D$ 
      using  $\langle B \neq C \rangle \langle D \neq B \rangle$  conga-pseudo-refl by auto
    moreover
    have  $\neg B D$  OS  $A C$ 
      by (meson  $P8B$  midpoint-bet col-trivial-3 one-side-chara)
    moreover
    have Coplanar  $A B D C$ 
  proof -
    have  $Bet A D C$ 
      using  $P8B$  by (simp add: midpoint-bet)
    thus ?thesis
      using bet-col ncop--ncols by blast
  qed
  moreover
  have  $A B C$  CongA  $A B C$ 
    using  $\langle B \neq A \rangle \langle B \neq C \rangle$  conga-refl by auto
  ultimately
  show ?thesis
    using SumA-def by blast
  qed
  hence  $Bet A B C$ 
    using per2-suma--bet  $\langle Per A B D \rangle \langle Per C B D \rangle$  by blast
  hence False
    using Col-def  $\langle \neg Col B A C \rangle$  between-symmetry by blast
}
hence  $\neg A B D$  CongA  $A B C$ 
  by auto
ultimately
show ?thesis
  by (simp add: LtA-def)
qed
ultimately
show ?thesis
  using Acute-def by blast
qed
moreover
{
  fix  $P Q$ 
  assume  $H1: B$  Out  $A P$  and
     $H2: P \neq Q$  and
     $H3: Per B P Q$  and
     $H4: Coplanar A B D Q$ 
  have  $B \neq P$ 

```

```

    using H1 Out-def by auto
  then obtain P' where H5: B Out P' C  $\wedge$  Cong B P' B P
    using  $\langle B \neq C \rangle$  l6-11-existence by fastforce
  hence H5A: B Out P' C
    by auto
  have H5B: Cong B P' B P
    using H5 by auto
  obtain Q' where H6: B C Perp Q' P'  $\wedge$  Coplanar B C A Q'
    using ex-perp-cop  $\langle B \neq C \rangle$  by blast
  hence H6A: B C Perp Q' P'
    by auto
  have H6B: Coplanar B C A Q'
    using H6 by blast
  have B A Perp Q P
    by (metis H1 H2 H3 NCol-cases  $\langle B \neq A \rangle$   $\langle B \neq P \rangle$  col-per-perp l8-2 out-col)
  have Coplanar B Q A C
  proof -
    have Coplanar B Q A D
      by (simp add: H4 coplanar-perm-10)
    moreover
    have Col A D A
      by (simp add: col-trivial-3)
    moreover
    have Col A D C
      using P8B by (simp add: Midpoint-def bet-col)
    ultimately
    show ?thesis
      using  $\langle D \neq A \rangle$  col2-cop--cop by blast
  qed
  have  $\exists I. Col P Q I \wedge Col P' Q' I$ 
  proof -
    have B  $\neq$  P'
      using  $\langle B Out P' C \rangle$  l6-3-1 by blast
    moreover
    have B  $\neq$  P
      using  $\langle B \neq P \rangle$  by auto
    moreover
    have B A Perp B C
      by (simp add:  $\langle B A Perp C B \rangle$  perp-right-comm)
    moreover
    have B A Perp P Q
      by (simp add:  $\langle B A Perp Q P \rangle$  perp-right-comm)
    moreover
    have B C Perp P' Q'
      by (simp add:  $\langle B C Perp Q' P' \rangle$  perp-right-comm)
    moreover
    have Col B A B
      by (simp add: col-trivial-3)
    moreover
    have Col B C B
      by (simp add: col-trivial-3)
    moreover
    have Col B A P
      by (simp add: H1 out-col)
    moreover
    have Col P Q P
      by (simp add: col-trivial-3)
    moreover
    have Col B C P'
      using Out-cases  $\langle B Out P' C \rangle$  out-col by blast
    moreover
    have Col P' Q' P'
      by (simp add: col-trivial-3)
    moreover
    have Coplanar B P P' P
      using ncop-distincts by blast

```

```

moreover
have Coplanar B P P' Q
proof –
  have Coplanar B A C Q
  by (simp add: <Coplanar B Q A C> coplanar-perm-3)
  moreover have Coplanar B A C P'
  using <Col B C P'> ncop--ncols by blast
  moreover have Coplanar B A C P
  using <Col B A P> ncop--ncols by blast
  moreover have Coplanar B A C B
  using ncop-distincts by blast
  ultimately show ?thesis
  using K1 coplanar-pseudo-trans by presburger
qed
moreover
have Coplanar B P P' P'
  using ncop-distincts by blast
moreover
have Coplanar B P P' Q'
  using H6B col2-cop--cop
  by (meson <B ≠ A> <B ≠ C> calculation(10) calculation(6)
    calculation(7) calculation(8) coplanar-perm-16
    coplanar-perm-21)
  ultimately
  show ?thesis
  using P0 by blast
qed
then obtain I where H7: Col P Q I ∧ Col P' Q' I
  by auto
hence Col P Q I
  by auto
have Col P' Q' I
  using H7 by blast
have B ≠ D
  using <¬ Col A B D> col-trivial-2 by blast
hence B C ParStrict P I
proof –
  have P ≠ I
  proof –
    {
      assume P = I
      {
        assume Col B A C
        hence ¬ B A Perp C B
        using K1 by blast
        hence False
        using perp-not-col using <B A Perp C B> by blast
      }
    }
  moreover
  have Col B A C
  proof –
    have A B Par P' Q'
    proof –
      have Coplanar B C A P'
      using H5A ncop--ncols ncoplanar-perm-3 out-col by blast
      moreover
      have Coplanar B C A Q'
      by (simp add: H6B)
      moreover
      have Coplanar B C B P'
      using ncop-distincts by blast
      moreover
      have Coplanar B C B Q'
      using ncop-distincts by blast
      moreover
      have A B Perp B C

```



```

    using Perp-perm  $\langle B \ A \ \text{Perp} \ C \ B \rangle$  by blast
  moreover
  have  $P' \ Q' \ \text{Perp} \ B \ C$ 
    using H6A Perp-perm by blast
  ultimately
  show ?thesis
    using l12-9 by blast
qed
moreover
have  $\text{Col} \ A \ B \ P$ 
  using Col-perm H1 out-col by blast
moreover
have  $\text{Col} \ P' \ Q' \ P$ 
  by (simp add:  $\langle \text{Col} \ P' \ Q' \ I \rangle \ \langle P = I \rangle$ )
ultimately
show ?thesis
  using not-strict-par
  by (metis H5A col-trivial-2 col-trivial-3
    colx out-col out-distinct)
qed
ultimately
have False
  by blast
}
thus ?thesis
  by auto
qed
moreover
have  $B \ C \ \text{ParStrict} \ P \ Q$ 
proof -
  have  $B \ C \ \text{Par} \ P \ Q$ 
  proof -
    have Coplanar  $A \ B \ B \ P$ 
      using ncop-distincts by blast
    moreover
    have Coplanar  $A \ B \ B \ Q$ 
      using ncop-distincts by blast
    moreover
    have Coplanar  $A \ B \ C \ P$ 
      using H1 ncop--ncols ncoplanar-perm-14 out-col by blast
    moreover
    have Coplanar  $A \ B \ C \ Q$ 
      using  $\langle \text{Coplanar} \ B \ Q \ A \ C \rangle$  ncoplanar-perm-10 by blast
    moreover
    have  $B \ C \ \text{Perp} \ A \ B$ 
      by (simp add:  $\langle C \ B \ \text{Perp} \ B \ A \rangle$  perp-comm)
    moreover
    have  $P \ Q \ \text{Perp} \ A \ B$ 
      using Perp-perm  $\langle B \ A \ \text{Perp} \ Q \ P \rangle$  by blast
    ultimately
    show ?thesis
      using l12-9 by blast
  qed
  moreover
  have  $\text{Col} \ P \ Q \ P$ 
    using col-trivial-3 by auto
  moreover
  have  $\neg \ \text{Col} \ B \ C \ P$ 
    using H1  $\langle B \neq P \rangle \ \langle \neg \ \text{Col} \ B \ A \ C \rangle$  col-permutation-5
    col-trivial-3 colx out-col by blast
  ultimately
  show ?thesis
    using par-not-col-strict by blast
qed
ultimately
show ?thesis

```

```

    using ⟨Col P Q I⟩ par-strict-col-par-strict by blast
qed
hence B C OS P I
  by (simp add: l12-6)
have Col D B I
proof cases
  assume D = I
  thus ?thesis
    using col-trivial-3 by blast
next
  assume D ≠ I
  have Coplanar A B C I
  proof -
    have P ≠ Q
      by (simp add: H2)
    moreover
    have Coplanar A B C P
      using H1 ncoplanar-perm-7 out--coplanar by blast
    moreover
    have Coplanar A B C Q
      using ⟨Coplanar B Q A C⟩ ncoplanar-perm-10 by blast
    moreover
    have Col P Q I
      by (simp add: ⟨Col P Q I⟩)
    ultimately
    show ?thesis
      using col-cop2--cop by blast
  qed
  moreover
  have B D PerpBisect A C
    by (simp add: P8B ⟨A ≠ C⟩ ⟨B ≠ D⟩ ⟨Cong B A B C⟩
      cong-commutativity cong-mid-perp-bisect)
  hence D B Perp A C
    using Perp-perm perp-bisect-perp by blast
  moreover
  have D I PerpBisect A C
  proof -
    have D ≠ I
      by (simp add: ⟨D ≠ I⟩)
    moreover
    have A ≠ C
      using ⟨A ≠ C⟩ by blast
    moreover
    have Cong A I C I
  proof -
    have P I Perp B P
  proof -
    have A B Perp P I
  proof -
    have P Q Perp A B
      using Perp-perm ⟨B A Perp Q P⟩ by blast
    moreover
    {
      assume P = I
      have B A Par P' Q'
    proof -
      have Coplanar B C B P'
        using ncop-distincts by blast
      moreover
      have Coplanar B C B Q'
        using ncop-distincts by blast
      moreover
      have Coplanar B C A P'
        using H5A coplanar-perm-4 ncop--ncols out-col by blast
      moreover
      have Coplanar B C A Q'

```

```

    using H6B by blast
  moreover
  have B A Perp B C
    by (simp add: ⟨B A Perp C B⟩ perp-right-comm)
  moreover
  have P' Q' Perp B C
    using H6A Perp-perm by blast
  ultimately
  show ?thesis using l12-9 by blast
qed
have Col B A P
  by (simp add: H1 out-col)
have Col P' Q' P
  by (simp add: ⟨Col P' Q' I⟩ ⟨P = I⟩)
hence Col B A C
  using not-strict-par
  by (metis H5A ⟨B A Par P' Q'⟩ ⟨Col B A P⟩
    col-trivial-3 colx out-col out-distinct)
hence False
  using K1 by blast
}
hence P ≠ I
  by auto
moreover
have Col P Q P
  by (simp add: col-trivial-3)
moreover
have Col P Q I
  by (simp add: ⟨Col P Q I⟩)
ultimately
show ?thesis
  using perp-col0 by blast
qed
moreover
have B ≠ P
  by (simp add: ⟨B ≠ P⟩)
moreover
have Col A B B
  using not-col-distincts by blast
moreover
have Col A B P
  using Col-perm H1 out-col by blast
ultimately
show ?thesis
  using perp-col0 by blast
qed
have P' I Perp B P'
proof -
  have B C Perp P' I
  proof -
    have P' Q' Perp B C
      using H6A Perp-perm by blast
  moreover
  {
    assume P' = I
    have B C Par P Q
    proof -
      have Coplanar B A B P
        using ncop-distincts by blast
      moreover
      have Coplanar B A B Q
        using ncop-distincts by blast
      moreover
      have Coplanar B A C P
        using H1 ncop--ncols out-col by blast
      moreover

```

```

have Coplanar B A C Q
  by (simp add: ⟨Coplanar B Q A C⟩ coplanar-perm-3)
moreover
have B C Perp B A
  by (simp add: perp-left-comm ⟨C B Perp B A⟩)
moreover
have P Q Perp B A
  using Perp-cases ⟨B A Perp Q P⟩ by blast
ultimately
show ?thesis using l12-9 by blast
qed
have Col B C P'
  using H5A col-permutation-5 out-col by blast
have Col P Q P'
  by (simp add: ⟨Col P Q I⟩ ⟨P' = I⟩)
hence Col B A C
  using not-strict-par not-par-inter-uniqueness
  by (metis H1 ⟨B C Par P Q⟩ ⟨B ≠ A⟩ ⟨B ≠ P⟩
    ⟨Col B C P'⟩ col-trivial-3 out-col)
hence False
  using K1 by blast
}
hence P' ≠ I
  by auto
moreover
have Col P' Q' P'
  by (simp add: col-trivial-3)
moreover
have Col P' Q' I
  by (simp add: ⟨Col P' Q' I⟩)
ultimately
show ?thesis
  using perp-col0 by blast
qed
moreover
have B ≠ P'
  using H5A out-diff1 by blast
moreover
have Col B C B
  using not-col-distincts by blast
moreover
have Col B C P'
  using Col-perm H5A out-col by blast
ultimately
show ?thesis
  using perp-col0 by blast
qed
have P' ≠ I
  using ⟨P' I Perp B P'⟩ perp-distinct by blast
have P ≠ I
  using ⟨P I Perp B P'⟩ perp-distinct by auto
have B P Lt B I
  using H2 H3 ⟨B ≠ P'⟩ ⟨Col P Q I⟩ ⟨P ≠ I⟩ per-col per-lt by presburger
have A B I CongA C B I
proof -
  have P B I CongA P' B I
proof -
  have I P B CongA I P' B
    using H5B ⟨B ≠ P'⟩ ⟨P I Perp B P'⟩ ⟨P ≠ I⟩
    ⟨P' I Perp B P'⟩ ⟨P' ≠ I⟩ cong-diff-3 l11-16
    perp-per-1 by blast
  show ?thesis
    using H5B ⟨B ≠ P'⟩ ⟨P I Perp B P'⟩ ⟨P ≠ I⟩
    ⟨P' I Perp B P'⟩ cong2-per2--cong-conga2 cong-4321
    cong-reflexivity perp-per-1 by auto
qed

```

```

moreover
have  $B \text{ Out } A \ P$ 
  by (simp add: H1)
moreover
have  $B \neq I$ 
  using  $\langle B \ C \ OS \ P \ I \rangle \text{ os-distincts}$  by blast
hence  $B \text{ Out } I \ I$ 
  using out-trivial by auto
moreover
have  $B \text{ Out } C \ P'$ 
  by (simp add: H5A l6-6)
ultimately
show ?thesis using l11-10 by blast
qed
thus ?thesis
  using  $\langle Cong \ B \ A \ B \ C \rangle \text{ cong2-conga-cong}$ 
    cong-reflexivity not-cong-2143 by blast
qed
moreover
have  $D \text{ Midpoint } A \ C$ 
  by (simp add: P8B)
ultimately
show ?thesis
  using cong-mid-perp-bisect perp-bisect-sym-1 by auto
qed
ultimately
show ?thesis
  using cop-perp2--col ncoplanar-perm-2 perp-bisect-perp by blast
qed
moreover
have  $\neg \text{Bet } D \ B \ I$ 
proof –
{
  assume  $\text{Bet } D \ B \ I$ 
  have  $B \ C \ TS \ P \ I$ 
  proof –
    have  $\neg \text{Col } B \ C \ D$ 
      by (metis NCol-cases P8B midpoint-out  $\langle A \neq C \rangle$ 
         $\langle C \ B \ \text{Perp } B \ A \rangle \langle D \neq C \rangle \text{ l6-16-1 out-col perp-not-col2}$ )
    have  $B \ C \ TS \ D \ I$ 
    proof –
      have  $B \ C \ OS \ I \ P$ 
        by (simp add:  $\langle B \ C \ OS \ P \ I \rangle \text{ one-side-symmetry}$ )
      hence  $\neg \text{Col } B \ C \ I$ 
        using col123--nos by blast
      have  $\text{Bet } D \ B \ I$ 
        by (simp add:  $\langle \text{Bet } D \ B \ I \rangle$ )
      thus ?thesis
        using  $\langle \neg \text{Col } B \ C \ D \rangle \langle \neg \text{Col } B \ C \ I \rangle \text{ bet--ts}$ 
          not-col-distincts by auto
    qed
  moreover
  have  $B \ C \ OS \ D \ A$ 
  proof –
    have  $\text{Col } B \ C \ C$ 
      by (simp add: col-trivial-2)
    moreover
    have  $\text{Col } D \ A \ C$ 
      by (simp add: P8B midpoint-col)
    moreover
    have  $C \text{ Out } D \ A$ 
      using P8B by (simp add:  $\langle D \neq C \rangle \text{ bet-out-1 midpoint-bet}$ )
    ultimately
    show ?thesis
      using  $\langle \neg \text{Col } B \ C \ D \rangle \text{ l9-19-R2}$  by blast
  qed

```

```

moreover
have  $B\ C\ OS\ A\ P$ 
proof –
  have  $Col\ B\ C\ C$ 
    by (simp add: col-trivial-2)
  moreover
  have  $Col\ D\ A\ C$ 
    by (simp add: P8B midpoint-col)
  moreover
  have  $C\ Out\ D\ A$ 
    using  $P8B$ 
    by (simp add:  $\langle D \neq C \rangle$  bet-out-1 midpoint-bet)
  ultimately
  show ?thesis
    using  $\langle \neg\ Col\ B\ C\ D \rangle\ H1\ K1\ col-trivial-3$ 
      not-col-permutation-5 out-one-side-1 by blast
  qed
  ultimately
  show ?thesis
    using l9-8-2 by blast
  qed
  hence  $\neg\ Bet\ D\ B\ I$ 
    using  $\langle B\ C\ OS\ P\ I \rangle\ l9-9$  by blast
  hence False
    by (simp add:  $\langle Bet\ D\ B\ I \rangle$ )
}
thus ?thesis
  by auto
qed
ultimately
have  $\exists\ Y.\ B\ Out\ D\ Y \wedge Col\ P\ Q\ Y$ 
  using l6-4-2  $\langle Col\ P\ Q\ I \rangle$  by blast
}
ultimately
show ?thesis
  by blast
qed
ultimately
show ?thesis by blast
qed

```

lemma *bachmann-s-lotschnittaxiom--weak-inverse-projection-postulate:*

assumes *bachmann-s-lotschnittaxiom*

shows *weak-inverse-projection-postulate*

proof –

have *lotschnitt*: $\forall\ A1\ A2\ B1\ B2\ C1\ C2\ D1\ D2\ IAB\ IAC\ IBD.$
 $IAB \neq IAC \wedge IAB \neq IBD \wedge$
 $A1\ A2\ Perp\ B1\ B2 \wedge A1\ A2\ Perp\ C1\ C2 \wedge$
 $B1\ B2\ Perp\ D1\ D2 \wedge Col\ A1\ A2\ IAB \wedge$
 $Col\ B1\ B2\ IAB \wedge Col\ A1\ A2\ IAC \wedge$
 $Col\ C1\ C2\ IAC \wedge Col\ B1\ B2\ IBD \wedge$
 $Col\ D1\ D2\ IBD \wedge Coplanar\ IAB\ IAC\ IBD\ C1 \wedge$
 $Coplanar\ IAB\ IAC\ IBD\ C2 \wedge$
 $Coplanar\ IAB\ IAC\ IBD\ D1 \wedge$
 $Coplanar\ IAB\ IAC\ IBD\ D2$

→

$(\exists\ I.\ Col\ C1\ C2\ I \wedge Col\ D1\ D2\ I)$

using *assms bachmann-s-lotschnittaxiom-aux-R1* **by** auto

{

fix $A\ B\ C\ D\ E\ F\ P\ Q$

assume $1 : Acute\ A\ B\ C$ **and**

$2 : Per\ D\ E\ F$ **and**

$3 : A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F$ **and**

$4 : B\ Out\ A\ P$ **and**

$5 : P \neq Q$ **and**

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6: Per B P Q and
7: Coplanar A B C Q
have A ≠ B
  using 4 l6-3-1 by blast
have P ≠ B
  using 4 l6-3-1 by blast
have B ≠ C
  using 1 acute-distincts by blast
have D ≠ E
  using 3 suma-distincts by blast
have E ≠ F
  using 3 suma-distincts by blast
{
  assume Col A B C
  hence Col D E F using col2-suma--col
    using 3 by blast
  hence False
    using 2 ⟨D ≠ E⟩ ⟨E ≠ F⟩ per-col-eq by blast
}
hence ¬ Col A B C
  by auto
have ¬ Col B C P
  by (metis 4 ⟨P ≠ B⟩ ⟨¬ Col A B C⟩ col-permutation-5
    col-trivial-2 col-trivial-3 col: out-col)
obtain P' where P2: P P' ReflectL B C
  using l10-6-existence-spec by (metis col--refl col-trivial-2)
hence P P' Reflect B C
  by (simp add: ⟨B ≠ C⟩ is-image-is-image-spec)
have ¬ Col B C P'
  using P2 ⟨¬ Col B C P⟩ col-image-spec--eq l10-4-spec by blast
obtain Q' where P3: Q Q' ReflectL B C
  using l10-6-existence-spec by (metis image-spec-triv)
hence Q Q' Reflect B C
  by (simp add: ⟨B ≠ C⟩ is-image-is-image-spec)
have B ≠ P'
  using ⟨¬ Col B C P'⟩ col-trivial-3 by blast
have P' ≠ Q'
  using 5 P2 P3 l10-2-uniqueness-spec by blast
have C B P' CongA A B C
proof -
  have C B P' CongA P B C
    by (metis P2 ⟨B ≠ C⟩ ⟨P ≠ B⟩ conga-left-comm
      is-image-spec-rev not-conga-sym reflectl--conga)
  moreover
  have B Out C C
    using ⟨B ≠ C⟩ out-trivial by auto
  moreover
  have B Out P' P'
    using ⟨B ≠ P'⟩ out-trivial by auto
  ultimately
  show ?thesis
    using 4 l11-10 by blast
qed
obtain X where X Midpoint P' P ∧ Col B C X
  using P2 ReflectL-def by blast
hence ∃ T. Col T B C ∧ Bet P T P'
  using Mid-cases col-permutation-2 midpoint-bet by blast
hence B C TS P P'
  using TS-def ⟨¬ Col B C P'⟩ ⟨¬ Col B C P⟩ col-permutation-1 by blast
have B Out P A
  by (simp add: 4 l6-6)
have Coplanar P' C A B
proof -
  have Coplanar P' C P B
    using ⟨B C TS P P'⟩ ncoplanar-perm-21 ts--coplanar by blast
  moreover

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have P ≠ B
  by (simp add: ⟨P ≠ B⟩)
moreover
have Col P B A
  using 4 col-permutation-2 out-col by blast
moreover
have Col P B B
  by (simp add: col-trivial-2)
ultimately
show ?thesis
  using col2-cop--cop by blast
qed
have Coplanar B P P' Q
proof -
  have Coplanar C A B B
    using ncop-distincts by blast
  moreover have Coplanar C A B P
    using 4 coplanar-perm-20 out--coplanar by blast
  moreover have Coplanar C A B P'
    using ⟨Coplanar P' C A B⟩ ncoplanar-perm-18 by blast
  moreover have Coplanar C A B Q
    using 7 ncoplanar-perm-8 by blast
  ultimately show ?thesis
    using ⟨¬ Col A B C⟩ NCol-cases coplanar-pseudo-trans by blast
qed
have A B C A B C SumA A B P'
proof -
  have A B C C B P' SumA A B P'
  proof -
    have Col B B C
      by (simp add: col-trivial-1)
    hence ¬ B C OS A P'
      using l9-5 l9-9 ⟨B Out P A⟩ ⟨B C TS P P'⟩ by blast
    moreover
    have Coplanar A B C P'
      by (simp add: ⟨Coplanar P' C A B⟩ coplanar-perm-17)
    ultimately
    show ?thesis
      using conga-refl
      by (metis SumA-def ⟨A ≠ B⟩ ⟨B ≠ C⟩ ⟨B ≠ P'⟩)
  qed
  moreover
  have A B C CongA A B C
    using ⟨A ≠ B⟩ ⟨B ≠ C⟩ conga-refl by auto
  moreover
  have C B P' CongA A B C
    by (simp add: ⟨C B P' CongA A B C⟩)
  moreover
  have A B P' CongA A B P'
    using calculation(1) suma2--conga by blast
  ultimately
  show ?thesis
    using conga3-suma--suma by blast
qed
hence D E F CongA A B P'
  using suma2--conga 3 by blast
hence Per A B P'
  using l11-17 2 by blast
have ¬ Col A B P'
  using perp-not-col ⟨A ≠ B⟩ ⟨B ≠ P'⟩ ⟨Per A B P'⟩
  perp-col-eq by blast
have ¬ Col B P' P
  by (meson ⟨B Out P A⟩ ⟨P ≠ B⟩ ⟨¬ Col A B P'⟩
    col-trivial-3 colx not-col-permutation-1 out-col)
have A B Perp B P'
  by (simp add: ⟨A ≠ B⟩ ⟨B ≠ P'⟩ ⟨Per A B P'⟩ per-perp)

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have Q1: A B Perp P Q
  by (metis 4 5 6 Col-perm Per-cases col-per-perp
    ⟨A ≠ B⟩ ⟨P ≠ B⟩ out-col perp-comm)
have B B ReflectL B C
  by (simp add: col-image-spec col-trivial-3)
hence B P' Perp P' Q'
  using 6 P2 P3 ⟨B ≠ P'⟩ ⟨P' ≠ Q'⟩ image-spec-preserves-per
    per-perp by auto
have ¬ Col P' P Q
proof -
  have B P' Par P Q
  proof -
    have Coplanar A B B P
      using ncop-distincts by blast
    moreover
    have Coplanar A B B Q
      using ncop-distincts by blast
    moreover
    have Coplanar A B P' P
      by (meson ⟨B Out P A⟩ ncop--ncols
        ncoplanar-perm-9 out-col)
    moreover
    have Coplanar A B P' Q
      using ⟨B Out P A⟩ ⟨Coplanar B P P' Q⟩
        ⟨P ≠ B⟩ col-cop--cop ncoplanar-perm-17 out-col
        coplanar-perm-16 by blast
    moreover
    have B P' Perp A B
      using Perp-perm ⟨A B Perp B P'⟩ by blast
    moreover
    have P Q Perp A B
      using Perp-perm ⟨A B Perp P Q⟩ by blast
    ultimately
    show ?thesis
      using l12-9 by blast
  qed
hence B P' ParStrict P Q
  using Par-def ⟨¬ Col B P' P⟩ not-col-permutation-2
    par-strict-symmetry par-symmetry by blast
moreover
have Col P' B P'
  by (simp add: col-trivial-3)
ultimately
show ?thesis
  using par-strict-not-col-2 by blast
qed
obtain I where P5: Col I A B ∧ Col I P Q
  using Q1 NCol-cases perp-inter-perp-in-n by blast
hence Col I A B
  by simp
have Col I P Q
  using P5 by simp
have ∃ Y. Col P' Q' Y ∧ Col P Q Y
proof -
  have B ≠ P
    using ⟨P ≠ B⟩ by blast
  moreover
  have B ≠ P'
    by (simp add: ⟨B ≠ P'⟩)
  moreover
  have A B Perp B P'
    by (simp add: ⟨A B Perp B P'⟩)
  moreover
  have A B Perp P Q
    by (simp add: Q1)
  moreover

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have B P' Perp P' Q'
  by (simp add: ⟨B P' Perp P' Q'⟩)
moreover
have Col A B B
  by (simp add: col-trivial-2)
moreover
have Col B P' B
  by (simp add: col-trivial-3)
moreover
have Col A B P
  using 4 not-col-permutation-4 out-col by blast
moreover
have Col P Q P
  by (simp add: col-trivial-3)
moreover
have Col B P' P'
  by (simp add: col-trivial-2)
moreover
have Col P' Q' P'
  by (simp add: col-trivial-3)
moreover
have Coplanar B P P' P
  using ncop-distincts by blast
moreover
have Coplanar B P P' Q
  using ⟨Coplanar B P P' Q⟩ by blast
moreover
have Coplanar B P P' P'
  using ncop-distincts by blast
moreover
have Coplanar B P P' Q'
proof -
{
  assume Col B C Q
  have Q = Q'
    using P3 ⟨Col B C Q⟩ col-image-spec--eq by blast
  hence Coplanar B P P' Q'
    using calculation(13) by blast
}
moreover
{
  assume ¬ Col B C Q
  moreover
  have Coplanar B C Q B
    using ncop-distincts by blast
  moreover
  have Coplanar B C Q P
    using 7 ⟨A ≠ B⟩ ⟨Col A B B⟩ ⟨Col A B P⟩
      col2-cop--cop ncoplanar-perm-14
      ncoplanar-perm-17 by blast
  moreover
  have Coplanar B C Q P'
    by (metis ⟨∃ T. Col T B C ∧ Bet P T P'⟩
      ⟨¬ Col B C P⟩ bet-col calculation(3) col-cop2--cop
      col-permutation-1 ncop--ncols)
  moreover
  have Coplanar B C Q Q'
    using P3 ncoplanar-perm-16 reflectl--coplanar by blast
  ultimately
  have ?thesis
    using coplanar-pseudo-trans by blast
}
ultimately
show ?thesis by auto
qed
ultimately

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  show ?thesis using lotschnitt by blast
qed
then obtain Y where P7: Col P' Q' Y  $\wedge$  Col P Q Y
  by auto
hence Col P' Q' Y
  by simp
have Col P Q Y
  using P7 by simp
have B Out C Y
proof -
  have Col B C Y
  proof -
    have P P' Reflect B C
      by (simp add:  $\langle P P' \text{ Reflect } B C \rangle$ )
    moreover
    have Q Q' Reflect B C
      by (simp add:  $\langle Q Q' \text{ Reflect } B C \rangle$ )
    moreover
    have  $\neg$  Col P Q P'
      using NCol-perm  $\langle \neg \text{ Col } P' P Q \rangle$  by blast
    moreover
    have Col P Q Y
      using  $\langle \text{ Col } P Q Y \rangle$  by blast
    moreover
    have Col P' Q' Y
      using  $\langle \text{ Col } P' Q' Y \rangle$  by blast
    ultimately
    show ?thesis
      using intersection-with-image-gen not-col-permutation-2 by blast
  qed
moreover
have B A OS C Y
proof -
  have A B OS C P'
  proof -
    have Coplanar A B C P'
      by (simp add:  $\langle \text{ Coplanar } P' C A B \rangle$  coplanar-perm-17)
    moreover
    have  $\neg$  Col C A B
      by (simp add:  $\langle \neg \text{ Col } A B C \rangle$  not-col-permutation-2)
    moreover
    have  $\neg$  Col P' A B
      using Col-cases  $\langle \neg \text{ Col } A B P' \rangle$  by blast
    moreover
    have  $\neg$  A B TS C P'
  proof -
    have SAMS A B C A B C
      by (simp add: 1 acute--sams)
    moreover
    have C B P' CongA A B C
      using  $\langle C B P' \text{ CongA } A B C \rangle$  by blast
    moreover
    have  $\neg$  B C OS A P'
  proof -
    have B C TS P P'
      using  $\langle B C TS P P' \rangle$  by blast
    moreover
    have Col B B C
      by (simp add: col-trivial-1)
    moreover
    have B Out P A
      by (simp add:  $\langle B \text{ Out } P A \rangle$ )
    ultimately
    show ?thesis
      using l9-5 l9-9 by blast
  qed

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ultimately
show ?thesis
  using conga-sams-nos--nts by blast
qed
ultimately
show ?thesis
  using cop-nts--os by blast
qed
moreover
have A B OS P' Y
proof -
  have P' ≠ Y
    using ⟨Col B C Y⟩ ⟨¬ Col B C P'⟩ by blast
  moreover
  have A B ParStrict P' Q'
  proof -
    have A B Par P' Q'
    proof -
      have Coplanar B P' A P'
        using ncop-distincts by blast
      moreover
      have Coplanar B P' A Q'
      proof -
        {
          assume Col B C Q
          hence Q = Q'
            using P3 col-image-spec--eq by blast
          have ?thesis
            using P3 ⟨Col B C Q⟩ ⟨B ≠ C⟩ ⟨Coplanar P' C A B⟩
              ⟨Q = Q'⟩ col2-cop--cop col-trivial-3 coplanar-perm-13
              ncoplanar-perm-14 by blast
        }
      moreover
      {
        assume ¬ Col B C Q
        have Coplanar B C Q B
          using ncop-distincts by blast
        moreover
        have Coplanar B C Q P'
          using 7 ⟨Coplanar P' C A B⟩ ⟨¬ Col A B C⟩
            coplanar-trans-1 ncoplanar-perm-22 by blast
        moreover
        have Coplanar B C Q A
          by (meson 7 ncoplanar-perm-18)
        moreover
        have Coplanar B C Q Q'
        proof -
          have Coplanar A B C B
            using ncop-distincts by blast
          moreover have Coplanar A B C C
            using ncop-distincts by blast
          moreover have Coplanar A B C Q'
            by (meson ⟨B ≠ C⟩ ⟨Col B C Y⟩ ⟨Col P' Q' Y⟩
              ⟨Coplanar P' C A B⟩ ⟨P' ≠ Y⟩ calculation(1)
              calculation(2) col-cop2--cop
              col-permutation-5 coplanar-perm-17)
          ultimately show ?thesis
            using ⟨¬ Col A B C⟩ 7 coplanar-trans-1 by blast
        qed
      }
      ultimately
      have Coplanar B P' A Q'
        using ⟨¬ Col B C Q⟩ coplanar-pseudo-trans by blast
    }
  ultimately
  show ?thesis
    by auto

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qed
moreover
have Coplanar B P' B P'
  using ncop-distincts by blast
moreover
have Coplanar B P' B Q'
  using ncop-distincts by blast
moreover
have A B Perp B P'
  by (simp add: ⟨A B Perp B P'⟩)
moreover
have P' Q' Perp B P'
  using Perp-perm ⟨B P' Perp P' Q'⟩ by blast
ultimately
show ?thesis
  using l12-9 by blast
qed
moreover
have Col P' Q' P'
  by (simp add: col-trivial-3)
moreover
have ¬ Col A B P'
  by (simp add: ⟨¬ Col A B P'⟩)
ultimately
show ?thesis
  using par-not-col-strict by blast
qed
moreover
have Col P' Q' Y
  by (simp add: ⟨Col P' Q' Y⟩)
ultimately
show ?thesis
  using par-strict-all-one-side by blast
qed
ultimately
show ?thesis
  using invert-one-side one-side-transitivity by blast
qed
ultimately
show ?thesis
  using col-one-side-out by blast
qed
hence ∃ Y. B Out C Y ∧ Col P Q Y
  using P7 by auto
}
thus ?thesis
  using weak-inverse-projection-postulate-def by blast
qed

lemma bachmann-s-lotschnittaxiom--weak-triangle-circumscription-principle:
  assumes bachmann-s-lotschnittaxiom
  shows weak-triangle-circumscription-principle
proof -
  have P1: ∀ A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD.
    IAB ≠ IAC ∧ IAB ≠ IBD ∧ A1 A2 Perp B1 B2 ∧
    A1 A2 Perp C1 C2 ∧ B1 B2 Perp D1 D2 ∧
    Col A1 A2 IAB ∧ Col B1 B2 IAB ∧
    Col A1 A2 IAC ∧ Col C1 C2 IAC ∧
    Col B1 B2 IBD ∧ Col D1 D2 IBD ∧
    Coplanar IAB IAC IBD C1 ∧ Coplanar IAB IAC IBD C2 ∧
    Coplanar IAB IAC IBD D1 ∧ Coplanar IAB IAC IBD D2
  →
    (∃ I. Col C1 C2 I ∧ Col D1 D2 I)
  using assms bachmann-s-lotschnittaxiom-aux by auto
moreover
{

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fix A B C A1 A2 B1 B2
assume 1:  $\neg \text{Col } A B C$  and
  2:  $\text{Per } A C B$  and
  3:  $A1 A2 \text{ PerpBisect } B C$  and
  4:  $B1 B2 \text{ PerpBisect } A C$  and
  5:  $\text{Coplanar } A B C A1$  and
  6:  $\text{Coplanar } A B C A2$  and
  7:  $\text{Coplanar } A B C B1$  and
  8:  $\text{Coplanar } A B C B2$ 
obtain A3 where 9:  $A1 \neq A2 \wedge \text{Col } A3 A1 A2 \wedge \text{Col } A3 B C$ 
  using 3 perp-bisect-perp Perp-def PerpAt-def by blast
obtain B3 where 10:  $B1 \neq B2 \wedge \text{Col } B3 B1 B2 \wedge \text{Col } B3 A C$ 
  using 4 perp-bisect-perp Perp-def PerpAt-def by blast
have A  $\neq B$ 
  using 1 col-trivial-1 by blast
have B  $\neq C$ 
  using 1 col-trivial-2 by auto
have A  $\neq C$ 
  using 1 col-trivial-3 by blast
have C  $\neq A3$ 
proof -
  obtain C' where 11:  $C' \text{ Midpoint } C B \wedge \text{Col } A1 A2 C'$ 
    by (metis (full-types) 3 9  $\langle B \neq C \rangle$  l4-17
        l7-20-bis not-col-permutation-2 not-cong-4321
        perp-bisect-cong-1 perp-bisect-cong-2)
  have C'  $\neq C$ 
    using 11  $\langle B \neq C \rangle$  cong-reverse-identity midpoint-cong by blast
  have C'  $\neq B$ 
    using 11  $\langle B \neq C \rangle$  is-midpoint-id-2 by blast
  {
    assume C = A3
    {
      assume  $\neg \text{Col } A1 A2 B$ 
      have C = C'
        by (metis 11 9  $\langle C = A3 \rangle \langle \neg \text{Col } A1 A2 B \rangle$ 
            colx midpoint-col not-col-permutation-2)
    }
    moreover
    {
      assume  $\neg \text{Col } A1 A2 C$ 
      have C = C'
        using 9  $\langle C = A3 \rangle \langle \neg \text{Col } A1 A2 C \rangle$  col-permutation-1 by blast
    }
    ultimately
    have C = C'
      using 3 perp-not-col2 perp-bisect-perp by blast
    hence False
      using  $\langle C' \neq C \rangle$  by blast
  }
  thus ?thesis by auto
qed
moreover
have C  $\neq B3$ 
proof -
  obtain C' where 12:  $C' \text{ Midpoint } C A \wedge \text{Col } B1 B2 C'$ 
    by (metis (full-types) 10 4  $\langle A \neq C \rangle$  l4-17
        l7-20-bis not-col-permutation-2 not-cong-4321
        perp-bisect-cong-1 perp-bisect-cong-2)
  have C'  $\neq C$ 
    by (metis 12  $\langle A \neq C \rangle$  midpoint-distinct-1)
  have C'  $\neq A$ 
    using 12  $\langle A \neq C \rangle$  is-midpoint-id-2 by force
  {
    assume C = B3
    {
      assume  $\neg \text{Col } B1 B2 A$ 

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    have  $C = C'$ 
      by (metis 10 12  $\langle C = B3 \rangle \langle \neg \text{Col } B1 \ B2 \ A \rangle$ 
          colx midpoint-col not-col-permutation-2)
  }
  moreover
  {
    assume  $\neg \text{Col } B1 \ B2 \ C$ 
    have  $C = C'$ 
      using 10  $\langle C = B3 \rangle \langle \neg \text{Col } B1 \ B2 \ C \rangle$  not-col-permutation-2 by blast
  }
  ultimately
  have  $C = C'$ 
    using 4 perp-not-col2 perp-bisect-perp by blast
  hence False
    using  $\langle C' \neq C \rangle$  by blast
  }
  thus ?thesis by auto
qed
moreover
have  $B \ C \ \text{Perp} \ A \ C$ 
  by (metis 2 Perp-perm  $\langle A \neq C \rangle \langle B \neq C \rangle$  per-perp)
moreover
have  $B \ C \ \text{Perp} \ A1 \ A2$ 
  using 3 Perp-perm perp-bisect-perp by blast
moreover
have  $A \ C \ \text{Perp} \ B1 \ B2$ 
  using 4 Perp-perm perp-bisect-perp by blast
moreover
have  $\text{Col } B \ C \ C$ 
  by (simp add: col-trivial-2)
moreover
have  $\text{Col } A \ C \ C$ 
  by (simp add: col-trivial-2)
moreover
have  $\text{Col } B \ C \ A3$ 
  using 9 not-col-permutation-2 by blast
moreover
have  $\text{Col } A1 \ A2 \ A3$ 
  using 9 NCol-cases by blast
moreover
have  $\text{Col } A \ C \ B3$ 
  using 10 Col-perm by blast
moreover
have  $\text{Col } B1 \ B2 \ B3$ 
  using 10 Col-cases by blast
moreover
have Coplanar  $C \ A3 \ B3 \ A1$ 
proof -
  have Coplanar  $A \ B \ C \ C$ 
    using ncop-distincts by blast
  moreover have Coplanar  $A \ B \ C \ A3$ 
    using  $\langle \text{Col } B \ C \ A3 \rangle$  ncop--ncols by blast
  moreover have Coplanar  $A \ B \ C \ B3$ 
    using  $\langle \text{Col } A \ C \ B3 \rangle$  ncop--ncols by auto
  ultimately show ?thesis
    using 1 5 coplanar-pseudo-trans by blast
qed
moreover
have Coplanar  $C \ A3 \ B3 \ A2$ 
proof -
  have Coplanar  $A \ B \ C \ C$ 
    using ncop-distincts by blast
  moreover have Coplanar  $A \ B \ C \ A3$ 
    using  $\langle \text{Col } B \ C \ A3 \rangle$  ncop--ncols by blast
  moreover have Coplanar  $A \ B \ C \ B3$ 
    by (meson  $\langle \text{Col } A \ C \ B3 \rangle$  ncop--ncols)

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```

    ultimately show ?thesis
    using 1 6 coplanar-pseudo-trans by blast
qed
moreover
have Coplanar C A3 B3 B1
proof -
  have Coplanar A B C C
    using ncop-distincts by blast
  moreover have Coplanar A B C A3
    using ⟨Col B C A3⟩ ncop--ncols by blast
  moreover have Coplanar A B C B3
    using ⟨Col A C B3⟩ ncop--ncols by blast
  ultimately show ?thesis
    using 1 7 coplanar-pseudo-trans by blast
qed
moreover
have Coplanar C A3 B3 B2
proof -
  have Coplanar A B C C
    using ncop-distincts by blast
  moreover have Coplanar A B C A3
    using ⟨Col B C A3⟩ ncop--ncols by blast
  moreover have Coplanar A B C B3
    using ⟨Col A C B3⟩ ncop--ncols by blast
  ultimately show ?thesis
    using 1 coplanar-pseudo-trans 8 by blast
qed
ultimately
have ∃ I. Col A1 A2 I ∧ Col B1 B2 I
  using P1 by blast
}
thus ?thesis using P1 weak-triangle-circumscription-principle-def by auto
qed

```

```

lemma consecutive-interior--alternate-interior:
  assumes consecutive-interior-angles-postulate
  shows alternate-interior-angles-postulate
proof -
  {
    fix A B C D
    assume 1: A C TS B D and
      2: A B Par C D
    obtain D' where 3: Bet D C D' ∧ Cong C D' C D
      using segment-construction by blast
    have A C TS D' D
      by (metis 1 3 bet--ts bet-col between-trivial
        cong-reverse-identity invert-two-sides l9-18 l9-2)
    hence A C OS B D'
      using 1 l9-8-1 by blast
    hence B A C SuppA A C D'
      by (metis 2 3 assms bet-col bet-col1
        consecutive-interior-angles-postulate-def
        os-distincts par-col2-par par-comm)
    moreover
    have D C A SuppA A C D'
      by (metis 2 3 ⟨A C OS B D'⟩ bet--suppa os-distincts par-neq2)
    ultimately
    have B A C CongA D C A
      using suppa2--conga123 by blast
  }
  thus ?thesis
    by (simp add: alternate-interior-angles-postulate-def)
qed

```

```

lemma existential-playfair--rah-1:
  postulate-of-right-saccheri-quadrilaterals ⟷

```


hypothesis-of-right-saccheri-quadrilaterals

proof –

```

{
  assume postulate-of-right-saccheri-quadrilaterals
  {
    fix A B C D
    assume Saccheri A B C D
    have Per A B C
      using  $\langle \text{PostulateRightSaccheriQuadrilaterals} \rangle$ 
       $\langle \text{Saccheri } A \ B \ C \ D \rangle$  postulate-of-right-saccheri-quadrilaterals-def
      by blast
  }
  hence hypothesis-of-right-saccheri-quadrilaterals
    by (simp add: hypothesis-of-right-saccheri-quadrilaterals-def)
}
moreover
{
  assume hypothesis-of-right-saccheri-quadrilaterals
  {
    fix A B C D
    assume Saccheri A B C D
    have Per A B C
      using  $\langle \text{HypothesisRightSaccheriQuadrilaterals} \rangle$ 
       $\langle \text{Saccheri } A \ B \ C \ D \rangle$  hypothesis-of-right-saccheri-quadrilaterals-def
      by blast
  }
  hence postulate-of-right-saccheri-quadrilaterals
    by (simp add: postulate-of-right-saccheri-quadrilaterals-def)
}
ultimately
show ?thesis by auto
qed

```

lemma *existential-playfair--rah:*

assumes *existential-playfair-s-postulate*

shows *postulate-of-right-saccheri-quadrilaterals*

proof –

```

obtain A1 A2 P where  $1: \neg \text{Col } A1 \ A2 \ P \wedge$ 
   $(\forall \ B1 \ B2 \ C1 \ C2.$ 
     $(A1 \ A2 \ \text{Par } B1 \ B2 \wedge \text{Col } P \ B1 \ B2 \wedge$ 
       $A1 \ A2 \ \text{Par } C1 \ C2 \wedge \text{Col } P \ C1 \ C2) \longrightarrow$ 
       $(\text{Col } C1 \ B1 \ B2 \wedge \text{Col } C2 \ B1 \ B2))$ 
  using assms existential-playfair-s-postulate-def by blast
have  $\neg \text{Col } A1 \ A2 \ P$ 
  using 1 by blast
have  $\forall \ B1 \ B2 \ C1 \ C2.$ 
   $(A1 \ A2 \ \text{Par } B1 \ B2 \wedge \text{Col } P \ B1 \ B2 \wedge$ 
     $A1 \ A2 \ \text{Par } C1 \ C2 \wedge \text{Col } P \ C1 \ C2) \longrightarrow$ 
     $(\text{Col } C1 \ B1 \ B2 \wedge \text{Col } C2 \ B1 \ B2)$ 
  using 1 by blast
obtain Q where  $\text{Col } A1 \ A2 \ Q \wedge A1 \ A2 \ \text{Perp } P \ Q$ 
  using  $\langle \neg \text{Col } A1 \ A2 \ P \rangle$  l8-18-existence by presburger
have  $\exists \ A3. \text{Col } A1 \ A2 \ A3 \wedge A3 \neq Q$ 
  by (metis col-trivial-2 diff-col-ex)
then obtain A3 where  $\text{Col } A1 \ A2 \ A3 \wedge A3 \neq Q$ 
  by auto
have  $P \neq Q$ 
  using  $\langle \text{Col } A1 \ A2 \ Q \wedge A1 \ A2 \ \text{Perp } P \ Q \rangle \langle \neg \text{Col } A1 \ A2 \ P \rangle$  by auto
then obtain R where  $P \ Q \ \text{Perp } R \ P \wedge \text{Coplanar } P \ Q \ A3 \ R$ 
  using ex-perp-cop by presburger
have  $P \ Q \ \text{Perp } R \ P$ 
  by (simp add:  $\langle P \ Q \ \text{Perp } R \ P \wedge \text{Coplanar } P \ Q \ A3 \ R \rangle$ )
have Coplanar P Q A3 R
  by (simp add:  $\langle P \ Q \ \text{Perp } R \ P \wedge \text{Coplanar } P \ Q \ A3 \ R \rangle$ )
have  $A1 \neq A2$ 
  using  $\langle \neg \text{Col } A1 \ A2 \ P \rangle$  col-trivial-1 by blast

```

```

have  $Q \neq P$ 
  using  $\langle P \neq Q \rangle$  by auto
have  $R \neq P$ 
  using  $\langle P \ Q \ \text{Perp} \ R \ P \rangle$  perp-distinct by auto
have  $A1 \ A2 \ \text{Par} \ P \ R$ 
proof -
  have Coplanar  $P \ Q \ A1 \ P$ 
    using ncop-distincts by blast
  moreover
  have Coplanar  $P \ R \ Q \ A1$ 
proof -
  have Coplanar  $P \ R \ Q \ A3$ 
    by (simp add:  $\langle P \ Q \ \text{Perp} \ R \ P \wedge \text{Coplanar} \ P \ Q \ A3 \ R \rangle$  coplanar-perm-4)
  moreover
  have  $Q \neq A3$ 
    using  $\langle \text{Col} \ A1 \ A2 \ A3 \wedge A3 \neq Q \rangle$  by auto
  moreover
  have Col  $Q \ A3 \ A1$ 
    by (meson  $\langle A1 \neq A2 \rangle \langle \text{Col} \ A1 \ A2 \ A3 \wedge A3 \neq Q \rangle$ 
       $\langle \text{Col} \ A1 \ A2 \ Q \wedge A1 \ A2 \ \text{Perp} \ P \ Q \rangle$  col3 col-trivial-3)
  ultimately
  show ?thesis
    using col-cop--cop by blast
qed
hence Coplanar  $P \ Q \ A1 \ R$ 
  using ncoplanar-perm-4 by blast
moreover
have Coplanar  $P \ Q \ A2 \ P$ 
  using ncop-distincts by blast
moreover
have Coplanar  $P \ R \ Q \ A2$ 
proof -
  have Coplanar  $P \ R \ Q \ A3$ 
    by (simp add:  $\langle P \ Q \ \text{Perp} \ R \ P \wedge \text{Coplanar} \ P \ Q \ A3 \ R \rangle$  coplanar-perm-4)
  moreover
  have  $Q \neq A3$ 
    using  $\langle \text{Col} \ A1 \ A2 \ A3 \wedge A3 \neq Q \rangle$  by auto
  moreover
  have Col  $Q \ A3 \ A2$ 
    by (meson  $\langle A1 \neq A2 \rangle \langle \text{Col} \ A1 \ A2 \ A3 \wedge A3 \neq Q \rangle$ 
       $\langle \text{Col} \ A1 \ A2 \ Q \wedge A1 \ A2 \ \text{Perp} \ P \ Q \rangle$  col3 col-trivial-2)
  ultimately
  show ?thesis
    using col-cop--cop by blast
qed
hence Coplanar  $P \ Q \ A2 \ R$ 
  by (simp add: coplanar-perm-3)
moreover
have  $A1 \ A2 \ \text{Perp} \ P \ Q$ 
  by (simp add:  $\langle \text{Col} \ A1 \ A2 \ Q \wedge A1 \ A2 \ \text{Perp} \ P \ Q \rangle$ )
moreover
have  $P \ R \ \text{Perp} \ P \ Q$ 
  using Perp-perm  $\langle P \ Q \ \text{Perp} \ R \ P \wedge \text{Coplanar} \ P \ Q \ A3 \ R \rangle$  by blast
ultimately
show ?thesis
  using l12-9 by blast
qed
have Coplanar  $A1 \ A2 \ P \ R$ 
  by (simp add:  $\langle A1 \ A2 \ \text{Par} \ P \ R \rangle$  par--coplanar)
have  $A1 \ A2 \ \text{ParStrict} \ P \ R$ 
  using  $\langle A1 \ A2 \ \text{Par} \ P \ R \rangle \langle \neg \text{Col} \ A1 \ A2 \ P \rangle$  col-trivial-3
  par-not-col-strict by blast
have  $\neg \text{Col} \ A1 \ A2 \ R$ 
  using  $\langle A1 \ A2 \ \text{ParStrict} \ P \ R \rangle$  par-strict-not-col-4 by auto
obtain  $S$  where  $\text{Col} \ A1 \ A2 \ S \wedge A1 \ A2 \ \text{Perp} \ R \ S$ 
  using  $\langle \neg \text{Col} \ A1 \ A2 \ R \rangle$  l8-18-existence by blast

```

```

hence  $Col\ A1\ A2\ S$ 
  by auto
have  $A1\ A2\ Perp\ R\ S$ 
  by (simp add:  $\langle Col\ A1\ A2\ S \wedge A1\ A2\ Perp\ R\ S \rangle$ )
have  $P\ Q\ Par\ R\ S$ 
proof –
  have  $Coplanar\ A1\ A2\ P\ R$ 
    by (simp add:  $\langle Coplanar\ A1\ A2\ P\ R \rangle$ )
  moreover
  have  $Coplanar\ A1\ A2\ P\ S$ 
    using  $\langle Col\ A1\ A2\ S \wedge A1\ A2\ Perp\ R\ S \rangle$  ncop--ncols by blast
  moreover
  have  $Coplanar\ A1\ A2\ Q\ R$ 
    using  $\langle Col\ A1\ A2\ Q \wedge A1\ A2\ Perp\ P\ Q \rangle$  ncop--ncols by blast
  moreover
  have  $Coplanar\ A1\ A2\ Q\ S$ 
    by (meson  $\langle \neg\ Col\ A1\ A2\ P \rangle \langle \neg\ Col\ A1\ A2\ R \rangle$  calculation(1)
      calculation(2) calculation(3) l9-30 ncop-distincts)
  moreover
  have  $P\ Q\ Perp\ A1\ A2$ 
    using Perp-perm  $\langle Col\ A1\ A2\ Q \wedge A1\ A2\ Perp\ P\ Q \rangle$  by blast
  moreover
  have  $R\ S\ Perp\ A1\ A2$ 
    using Perp-perm  $\langle Col\ A1\ A2\ S \wedge A1\ A2\ Perp\ R\ S \rangle$  by blast
  ultimately
  show ?thesis
    using l12-9 by blast
qed
have  $\neg\ Col\ P\ Q\ R$ 
  by (simp add:  $\langle P\ Q\ Perp\ R\ P \wedge Coplanar\ P\ Q\ A3\ R \rangle$  perp-not-col)
hence  $P\ Q\ ParStrict\ R\ S$ 
  using  $\langle P\ Q\ Par\ R\ S \rangle$  col-trivial-3 par-not-col-strict by blast
hence  $\neg\ Col\ R\ S\ P$ 
  by (meson par-strict-not-col-3)
then obtain  $R'$  where  $Col\ R\ S\ R' \wedge R\ S\ Perp\ P\ R'$ 
  using l8-18-existence by blast
hence  $Col\ R\ S\ R'$ 
  by auto
have  $R\ S\ Perp\ P\ R'$ 
  by (simp add:  $\langle Col\ R\ S\ R' \wedge R\ S\ Perp\ P\ R' \rangle$ )
have  $A1\ A2\ Par\ P\ R'$ 
proof –
  have  $Coplanar\ R\ P\ A1\ A2$ 
    using  $\langle Coplanar\ A1\ A2\ P\ R \rangle$  ncoplanar-perm-17 by blast
  hence  $Coplanar\ R\ S\ A1\ P$ 
    using  $\langle Col\ A1\ A2\ S \rangle \langle A1 \neq A2 \rangle$  col-cop--cop coplanar-perm-5 by blast
  moreover
  have  $Coplanar\ R\ S\ A1\ R'$ 
    using  $\langle Col\ R\ S\ R' \rangle$  ncop--ncols by blast
  moreover
  have  $Coplanar\ R\ S\ A2\ P$ 
    by (metis  $\langle Col\ A1\ A2\ S \rangle \langle Coplanar\ R\ P\ A1\ A2 \rangle$ 
      calculation(1) col-cop--cop coplanar-perm-4
      not-col-permutation-1)
  moreover
  have  $Coplanar\ R\ S\ A2\ R'$ 
    using  $\langle Col\ R\ S\ R' \rangle$  ncop--ncols by blast
  moreover
  have  $A1\ A2\ Perp\ R\ S$ 
    by (simp add:  $\langle Col\ A1\ A2\ S \wedge A1\ A2\ Perp\ R\ S \rangle$ )
  moreover
  have  $P\ R'\ Perp\ R\ S$ 
    using Perp-perm  $\langle R\ S\ Perp\ P\ R' \rangle$  by blast
  ultimately
  show ?thesis
    using l12-9 by blast

```

```

qed
have Col R' P R
  using 1 ⟨A1 A2 Par P R'⟩ ⟨A1 A2 Par P R⟩ col-trivial-1 by blast
hence R = R'
  using l6-21
  by (metis Perp-perm ⟨Col R S R' ∧ R S Perp P R'⟩
      perp-col perp-not-col)
have P Q ParStrict R S
  by (simp add: ⟨P Q ParStrict R S⟩)
have Per Q S R
proof -
  have Q S Perp R S
  proof -
    have Q ≠ S
      using ⟨P Q ParStrict R S⟩ col-trivial-2
      par-strict-not-col-4 by blast
    moreover
    have Col A1 A2 Q
      using ⟨Col A1 A2 Q ∧ A1 A2 Perp P Q⟩ by auto
    ultimately
    show ?thesis
      using ⟨A1 A2 Perp R S⟩ ⟨Col A1 A2 S⟩ perp-col2 by blast
  qed
  moreover
  have Col S Q S
    using col-trivial-3 by blast
  moreover
  have Col S R S
    by (simp add: col-trivial-3)
  ultimately
  show ?thesis
    using perp-comm perp-per-2 by blast
qed
moreover
have Lambert P Q S R
proof -
  have Q ≠ S
    using ⟨P Q ParStrict R S⟩ col-trivial-3
    par-strict-not-col-2 by blast
  moreover
  have S ≠ R
    using ⟨¬ Col R S P⟩ col-trivial-1 by blast
  moreover
  have Per Q P R
    by (simp add: ⟨P Q Perp R P⟩ perp-per-1)
  moreover
  have Per P R S
    using ⟨R = R'⟩ ⟨R S Perp P R'⟩ l8-2 perp-per-1 by blast
  moreover
  have Per P Q S
  proof -
    have A1 A2 Perp P Q
      by (simp add: ⟨Col A1 A2 Q ∧ A1 A2 Perp P Q⟩)
    moreover
    have Col A1 A2 Q
      using ⟨Col A1 A2 Q ∧ A1 A2 Perp P Q⟩ by auto
    ultimately
    show ?thesis
      using ⟨Q ≠ S⟩ ⟨Col A1 A2 S⟩ l8-16-1 by blast
  qed
  moreover
  have Coplanar P Q S R
    using ⟨P Q Par R S⟩ coplanar-perm-1 par--coplanar by blast
  ultimately
  show ?thesis
    using Lambert-def ⟨Q ≠ P⟩ ⟨R ≠ P⟩ by blast

```

```

qed
ultimately
have hypothesis-of-right-saccheri-quadrilaterals
  using lam-per--rah-1 by blast
show ?thesis
  by (simp add: ⟨HypothesisRightSaccheriQuadrilaterals⟩
      existential-playfair--rah-1)
qed

lemma existential-saccheri--rah:
  assumes postulate-of-existence-of-a-right-saccheri-quadrilateral
  shows postulate-of-right-saccheri-quadrilaterals
  using per-sac--rah assms existential-playfair--rah-1
    postulate-of-existence-of-a-right-saccheri-quadrilateral-def by blast

lemma existential-triangle--rah:
  assumes postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights
  shows postulate-of-right-saccheri-quadrilaterals
  using t22-14--rah assms
    existential-playfair--rah-1
    postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-def
  by blast

lemma inverse-projection-postulate--proclus-bis:
  assumes inverse-projection-postulate
  shows alternative-proclus-postulate
proof -
{
  fix A B C D P Q
  assume 1: P Perp2 A B C D ∧ ¬ Col C D P ∧ Coplanar A B C D ∧
    Col A B P ∧ ¬ Col A B Q ∧ Coplanar C D P Q
  hence P Perp2 A B C D
    by auto
  have ¬ Col C D P
    using 1 by auto
  have Coplanar A B C D
    using 1 by auto
  have Col A B P
    using 1 by auto
  have ¬ Col A B Q
    using 1 by auto
  have Coplanar C D P Q
    using 1 by auto
  have C D ParStrict A B
  proof -
    have Coplanar C D A B
      by (simp add: ⟨Coplanar A B C D⟩ coplanar-perm-16)
    moreover
    have P Perp2 C D A B
      using 1 perp2-sym by blast
    ultimately
    show ?thesis
      using ⟨Col A B P⟩ ⟨¬ Col C D P⟩ col-cop-perp2--pars-bis by blast
  qed
  obtain P1 P2 where 2: Col P P1 P2 ∧ P1 P2 Perp A B ∧ P1 P2 Perp C D
    using Perp2-def ⟨P Perp2 A B C D⟩ by blast
  have Col P P1 P2
    using 2 by blast
  have P1 P2 Perp A B
    using 2 by blast
  have P1 P2 Perp C D
    using 2 by blast
  then obtain C0 where C0 PerpAt P1 P2 C D
    using perp-inter-perp-in-n by blast
  hence Col C0 P1 P2 ∧ Col C0 C D
    using Col-cases perp-in-col by blast
}

```

```

hence  $Col\ C0\ P1\ P2$ 
  by blast
have  $Col\ C0\ C\ D$ 
  using  $\langle Col\ C0\ P1\ P2 \wedge Col\ C0\ C\ D \rangle$  by auto
obtain  $P'$  where  $P' \perp P1\ P2\ A\ B$ 
  using  $\langle P1\ P2 \perp P1\ P2\ A\ B \rangle$  perp-inter-perp-in-n by blast
have  $Col\ P\ A\ B$ 
  using Col-perm  $\langle Col\ A\ B\ P \rangle$  by blast
hence  $P = P'$ 
  using  $\langle Col\ P\ P1\ P2 \rangle \langle P' \perp P1\ P2\ A\ B \rangle$  l8-14-2-1b by blast
have  $P \neq C0$ 
proof –
  {
    assume  $P = C0$ 
    hence  $Col\ P\ C\ D$ 
    by (simp add:  $\langle Col\ C0\ C\ D \rangle$ )
    hence False
    using  $\langle \neg Col\ C\ D\ P \rangle$  col-permutation-1 by blast
  }
thus ?thesis
by auto
qed
have  $A\ B \perp P\ C0$ 
proof –
  have  $Col\ P1\ P2\ P$ 
  by (simp add:  $\langle Col\ P\ P1\ P2 \rangle$  col-permutation-1)
  moreover
  have  $Col\ P1\ P2\ C0$ 
  by (simp add:  $\langle Col\ C0\ P1\ P2 \rangle$  col-permutation-1)
  ultimately
  show ?thesis
  using  $\langle P \neq C0 \rangle \langle P1\ P2 \perp P1\ P2\ A\ B \rangle$  perp-col0 by blast
qed
have  $C\ D \perp P\ C0$ 
proof –
  have  $Col\ P1\ P2\ P$ 
  by (simp add:  $\langle Col\ P\ P1\ P2 \rangle$  col-permutation-1)
  moreover
  have  $Col\ P1\ P2\ C0$ 
  by (simp add:  $\langle Col\ C0\ P1\ P2 \rangle$  col-permutation-1)
  ultimately
  show ?thesis
  using  $\langle P \neq C0 \rangle \langle P1\ P2 \perp P1\ P2\ C\ D \rangle$  perp-col0 by blast
qed
have  $P \neq Q$ 
  using  $\langle Col\ A\ B\ P \rangle \langle \neg Col\ A\ B\ Q \rangle$  by auto
have  $\exists Y. (Col\ P\ Q\ Y \wedge Col\ C\ D\ Y)$ 
proof (cases  $Col\ P\ Q\ C0$ )
  case True
  thus ?thesis
  using  $\langle Col\ C0\ C\ D \rangle$  col-permutation-1 by blast
next
  case False
  have  $\neg Col\ C0\ A\ B$ 
  using  $\langle C\ D \text{ ParStrict } A\ B \rangle \langle Col\ C0\ C\ D \rangle$  par-not-col by blast
  have  $\exists Q0. Col\ Q\ P\ Q0 \wedge A\ B\ OS\ C0\ Q0$ 
proof –
  have  $Q \neq P$ 
  using  $\langle P \neq Q \rangle$  by auto
  moreover
  have  $Col\ Q\ P\ P$ 
  by (simp add: col-trivial-2)
  moreover
  have  $\neg Col\ A\ B\ C0$ 
  using NCol-perm  $\langle \neg Col\ C0\ A\ B \rangle$  by blast
  moreover

```

```

have Coplanar A B Q C0
proof -
  have Coplanar C D P A
  by (metis ⟨Col A B P⟩ ⟨Coplanar A B C D⟩ ⟨¬ Col C0 A B⟩
    col-cop--cop coplanar-perm-1 coplanar-perm-16
    not-col-distincts)
  moreover have Coplanar C D P B
  using ⟨Col P A B⟩ ⟨Coplanar A B C D⟩ calculation
    col-cop--cop coplanar-perm-16 by blast
  moreover have Coplanar C D P C0
  using Col-cases ⟨Col C0 C D⟩ ncop--ncols by blast
  ultimately show ?thesis
  using ⟨¬ Col C D P⟩ ⟨Coplanar C D P Q⟩
    coplanar-pseudo-trans by blast
qed
ultimately
show ?thesis
  using ⟨¬ Col A B Q⟩ ⟨Col A B P⟩ cop-not-par-same-side by blast
qed
then obtain Q0 where Col Q P Q0 ∧ A B OS C0 Q0
  by auto
hence Col Q P Q0
  by auto
have A B OS C0 Q0
  by (simp add: ⟨Col Q P Q0 ∧ A B OS C0 Q0⟩)
have ¬ Col A B Q0
  using ⟨Col Q P Q0 ∧ A B OS C0 Q0⟩ one-side-not-col124 by blast
{
  assume P = Q0
  hence Col A B P
  using ⟨Col A B P⟩ by auto
  hence False
  by (simp add: ⟨P = Q0⟩ ⟨¬ Col A B Q0⟩)
}
hence P ≠ Q0
  by auto
have ¬ Col P C0 Q0
  by (metis False ⟨Col Q P Q0⟩ ⟨P ≠ Q0⟩ col-transitivity-1
    not-col-permutation-2)
have ∃ C1. Col C D C1 ∧ C1 ≠ C0
  by (metis col-trivial-3 diff-col-ex)
then obtain C1 where Col C D C1 ∧ C1 ≠ C0
  by auto
hence Col C D C1
  by auto
have C1 ≠ C0
  by (simp add: ⟨Col C D C1 ∧ C1 ≠ C0⟩)
have ∃ A0. Col A B A0 ∧ P C0 OS Q0 A0
proof (cases Col P C0 A)
case True
  have ¬ Col P C0 B
  using True ⟨P ≠ C0⟩ ⟨¬ Col C0 A B⟩ col-transitivity-2 by blast
  have ∃ Q. Col B A Q ∧ P C0 OS Q0 Q
  proof -
    have B ≠ A
    using ⟨¬ Col C0 A B⟩ col-trivial-2 by auto
    moreover
    have Col P C0 P
    by (simp add: col-trivial-3)
    moreover
    have Col B A P
    using Col-cases ⟨Col A B P⟩ by blast
    moreover
    have ¬ Col P C0 B
    by (simp add: ⟨¬ Col P C0 B⟩)
    moreover

```

```

have  $\neg \text{Col } P \ C0 \ Q0$ 
  by (simp add:  $\langle \neg \text{Col } P \ C0 \ Q0 \rangle$ )
moreover
have  $\text{Coplanar } P \ C0 \ B \ Q0$ 
  by (metis True  $\langle \text{Col } P \ A \ B \rangle \langle \text{Col } Q \ P \ Q0 \wedge A \ B \ OS \ C0 \ Q0 \rangle$ 
      calculation(2) calculation(4) colx
      coplanar-perm-2 os--coplanar)
ultimately
show ?thesis
  using cop-not-par-same-side by blast
qed
thus ?thesis
  using not-col-permutation-4 by blast
next
case False
have  $\text{Coplanar } P \ C0 \ A \ Q0$ 
  by (metis False  $\langle \text{Col } A \ B \ P \rangle \langle \text{Col } Q \ P \ Q0 \wedge A \ B \ OS \ C0 \ Q0 \rangle$ 
      col2-os--os col-trivial-3 coplanar-perm-2 os--coplanar)
thus ?thesis
  by (metis False  $\langle \text{Col } A \ B \ P \rangle \langle \neg \text{Col } C0 \ A \ B \rangle \langle \neg \text{Col } P \ C0 \ Q0 \rangle$ 
      col-trivial-2 col-trivial-3 cop-not-par-same-side)
qed
then obtain  $A0$  where  $\text{Col } A \ B \ A0 \wedge P \ C0 \ OS \ Q0 \ A0$ 
  by auto
hence  $\text{Col } A \ B \ A0$ 
  by auto
have  $P \ C0 \ OS \ Q0 \ A0$ 
  by (simp add:  $\langle \text{Col } A \ B \ A0 \wedge P \ C0 \ OS \ Q0 \ A0 \rangle$ )
have  $\neg \text{Col } P \ C0 \ A0$ 
  using  $\langle P \ C0 \ OS \ Q0 \ A0 \rangle$  one-side-not-col124 by blast
have  $\exists Y. P \text{ Out } Q0 \ Y \wedge \text{Col } C0 \ C1 \ Y$ 
proof -
  have  $\text{Acute } C0 \ P \ Q0$ 
  proof -
    have  $A \ B \ \text{Perp } C0 \ P$ 
      by (simp add:  $\langle A \ B \ \text{Perp } P \ C0 \rangle$  perp-right-comm)
    hence  $\text{Per } C0 \ P \ A0$ 
      using  $\langle \text{Col } A \ B \ P \rangle \langle \text{Col } A \ B \ A0 \rangle$  l8-16-1 by blast
    moreover
    have  $C0 \ P \ Q0 \ \text{LtA } C0 \ P \ A0$ 
    proof -
      have  $P \ A0 \ OS \ C0 \ Q0$ 
        by (metis  $\langle \text{Col } A \ B \ A0 \wedge P \ C0 \ OS \ Q0 \ A0 \rangle \langle \text{Col } A \ B \ P \rangle$ 
             $\langle \text{Col } Q \ P \ Q0 \wedge A \ B \ OS \ C0 \ Q0 \rangle \langle \neg \text{Col } P \ C0 \ A0 \rangle$ 
            col2-os--os col-trivial-3)
      hence  $Q0 \ \text{InAngle } C0 \ P \ A0$ 
        by (simp add:  $\langle P \ C0 \ OS \ Q0 \ A0 \rangle$  one-side-symmetry os2--inangle)
      hence  $C0 \ P \ Q0 \ \text{LeA } C0 \ P \ A0$ 
        using inangle--lea by force
    moreover
    {
      assume  $C0 \ P \ Q0 \ \text{CongA } C0 \ P \ A0$ 
      have  $C0 \ P \ OS \ Q0 \ A0$ 
        by (meson  $\langle P \ C0 \ OS \ Q0 \ A0 \rangle$  invert-one-side)
      hence  $P \ \text{Out } Q0 \ A0$ 
        using conga-os--out
        using  $\langle C0 \ P \ Q0 \ \text{CongA } C0 \ P \ A0 \rangle$  by blast
      hence  $\text{Col } A \ B \ Q0$ 
        using  $\langle P \ A0 \ OS \ C0 \ Q0 \rangle$  os-distincts out-out-one-side by blast
      hence False
        using  $\langle \neg \text{Col } A \ B \ Q0 \rangle$  by auto
    }
  hence  $\neg C0 \ P \ Q0 \ \text{CongA } C0 \ P \ A0$ 
    by auto
  ultimately
  show ?thesis

```



```

    using LtA-def by blast
qed
ultimately
show ?thesis
    using Acute-def by blast
qed
moreover
have P Out C0 C0
    using ⟨P ≠ C0⟩ out-trivial by auto
moreover
have C0 ≠ C1
    using ⟨C1 ≠ C0⟩ by blast
moreover
have Col C D C0
    by (simp add: ⟨Col C0 C D⟩ col-permutation-1)
hence Per P C0 C1
    using l8-16-1 ⟨C D Perp P C0⟩ ⟨Col C D C1⟩ by blast
moreover
have Coplanar C0 P Q0 C1
proof -
    have Coplanar C D P C0
        using ⟨Col C D C0⟩ ncop--ncols by blast
    moreover have Coplanar C D P P
        using ncop-distincts by auto
    moreover have Coplanar C D P Q0
        by (metis ⟨Col Q P Q0⟩ ⟨Coplanar C D P Q⟩
            ⟨P ≠ Q⟩ calculation(2) col-cop2--cop)
    moreover have Coplanar C D P C1
        using ⟨Col C D C1⟩ ncop--ncols by blast
    ultimately show ?thesis
        using ⟨¬ Col C D P⟩ coplanar-pseudo-trans by blast
qed
ultimately
show ?thesis
    using assms inverse-projection-postulate-def by blast
qed
then obtain Y where P Out Q0 Y and Col C0 C1 Y
    by blast
have Col P Q Y
    by (meson ⟨Col Q P Q0⟩ ⟨P Out Q0 Y⟩ ⟨P ≠ Q0⟩
        col-permutation-1 l6-16-1 out-col)
have Col C D Y
    by (metis ⟨Col C D C1 ∧ C1 ≠ C0⟩ ⟨Col C0 C D⟩
        ⟨Col C0 C1 Y⟩ colx not-col-permutation-2)
thus ?thesis
    using ⟨Col P Q Y⟩ by blast
qed
}
thus ?thesis
    using alternative-proclus-postulate-def by fastforce
qed

```

lemma strong-parallel-postulate-implies-inter-dec:

```

shows decidability-of-intersection
proof -
{
    fix P Q S U
    have (∃ I. Col I S Q ∧ Col I P U) ∨ ¬ (∃ I. Col I S Q ∧ Col I P U)
        by blast
}
thus ?thesis
    using decidability-of-intersection-def by blast
qed

```

```

lemma impossible-case-1:
  assumes Bet A B x and
    Bet C y A and
    B C ParStrict x y
  shows False
proof -
  have Bet x B A
    using Bet-cases assms(1) by blast
  then obtain I where Bet B I C  $\wedge$  Bet y I x
    using assms(2) inner-pasch by presburger
  thus ?thesis
    by (meson assms(3) bet-col col-permutation-1 col-permutation-4 par-not-col)
qed

```

```

lemma impossible-case-2:
  assumes A  $\neq$  D and
    B  $\neq$  D and

     $\neg$  Col A B C and
    Col A B x and
    Bet A D T and

    Bet B D C and
    Bet y A C and
    Bet x T y
  shows False
proof -
  {
    assume A = y
    hence Col A B C
      by (metis NCol-perm assms(2) assms(4) assms(5) assms(6)
        assms(8) bet-col between-identity col-trivial-2 colx)
    hence False
      using assms(3) by auto
  }
  hence A  $\neq$  y
    by blast
  have A B TS y T
  proof -
    have A B TS C y
      by (simp add:  $\langle A \neq y \rangle$  assms(7) assms(3) bet-ts between-symmetry)
    moreover
    have A B OS C T
      using bet-out calculation invert-one-side l9-8-1
        one-side-not-col124 one-side-symmetry out-one-side
        out-out-one-side by (metis assms(1) assms(2) assms(5) assms(6))
    ultimately
    show ?thesis
      using l9-2 l9-8-2 by blast
  qed
  hence A B OS T x
    using Bet-cases assms(8) bet-ts--os by blast
  thus ?thesis
    using assms(4) col124--nos by auto
qed

```

```

lemma impossible-case-3:
  assumes
    Bet A D T and
     $\neg$  Col B C T and
    Bet B D C and
    Bet B x A and
    Bet x T y and
    B C ParStrict x y
  shows False

```

```

proof –
  have  $B\ C\ OS\ x\ y$ 
    using  $assms(6)\ l12-6$  by blast
  hence  $B\ C\ TS\ x\ T$ 
    by (metis (full-types) Col-cases  $\langle B\ C\ OS\ x\ y \rangle$ 
       $assms(1)\ assms(2)\ assms(3)\ assms(4)\ bet-col\ bet-out$ 
      between-trivial l9-18 l9-8-2 one-side-not-col124
      one-side-symmetry out-out-one-side)
  hence  $B\ C\ TS\ x\ y$ 
    by (meson  $assms(5)\ bet-ts--ts$ )
  thus ?thesis
    using  $\langle B\ C\ OS\ x\ y \rangle\ l9-9$  by blast
qed

lemma impossible-case-4-1:
  assumes  $A \neq D$  and
     $C \neq D$  and
     $\neg Col\ A\ B\ C$  and
     $Col\ A\ C\ y$  and
     $Bet\ A\ D\ T$  and

     $Bet\ B\ D\ C$  and
     $A\ Out\ B\ x$  and
     $Bet\ T\ y\ x$ 
  shows False
proof –
  have  $A\ C\ OS\ B\ D$ 
    by (metis  $assms(2)\ assms(3)\ assms(6)\ bet-col$ 
      between-equality-2 col-permutation-4 col-trivial-2
      cop-nos--ts l6-21 l9-18 ncop--ncols not-col-permutation-5)
  hence  $A\ C\ OS\ T\ x$ 
    by (metis  $assms(1)\ assms(5)\ assms(7)\ bet-out$ 
      col-trivial-3 one-side-symmetry os-out-os)
  hence  $A\ C\ OS\ T\ y$ 
    using  $assms(8)\ l9-17$  by blast
  thus ?thesis
    using  $assms(4)\ one-side-not-col124$  by auto
qed

lemma impossible-case-4-2:
  assumes  $\neg Col\ A\ B\ C$  and
     $Col\ A\ C\ y$  and
     $Bet\ A\ D\ T$  and
     $\neg Col\ B\ C\ T$  and
     $Bet\ B\ D\ C$  and
     $Bet\ B\ A\ x$  and
     $Bet\ T\ y\ x$  and
     $B\ C\ ParStrict\ x\ y$ 
  shows False
proof –
  have  $B\ C\ TS\ A\ T$ 
    by (metis Col-cases TS-def  $assms(1)\ assms(3)\ assms(4)$ 
       $assms(5)\ bet-out-1\ not-col-distincts\ out-col$ )
  have  $B\ C\ OS\ A\ x$ 
    by (metis  $assms(1)\ assms(6)\ bet-out\ not-col-distincts$ 
      one-side-reflexivity out-out-one-side)
  hence  $B\ C\ TS\ x\ T$ 
    using  $\langle B\ C\ TS\ A\ T \rangle\ l9-8-2$  by blast
  hence  $T \neq x$ 
    using ts-distincts by blast
  have  $B\ C\ ParStrict\ x\ T$ 
    by (metis  $\langle B\ C\ TS\ x\ T \rangle\ assms(7)\ assms(8)\ bet-col$ 
      between-symmetry par-strict-col-par-strict ts-distincts)
  thus ?thesis
    by (meson  $\langle B\ C\ TS\ x\ T \rangle\ l12-6\ l9-9$ )
qed

```

lemma *impossible-case-4*:
assumes $A \neq D$ **and**

$C \neq D$ **and**
 $D \neq T$ **and**
 $\neg \text{Col } A \ B \ C$ **and**
 $\text{Col } A \ C \ y$ **and**
 $\text{Bet } A \ D \ T$ **and**
 $\neg \text{Col } B \ C \ T$ **and**
 $\text{Bet } B \ D \ C$ **and**
 $\text{Col } A \ B \ x$ **and**
 $\text{Bet } T \ y \ x$ **and**
 $B \ C \ \text{ParStrict } x \ y$

shows *False*

proof –

have $\text{Bet } A \ B \ x \vee A \ \text{Out } B \ x \vee B \ \text{Out } x \ A$
using *assms(9) l6-6 or-bet-out* **by** *blast*

moreover

have $\neg \text{Bet } A \ B \ x$
by (*metis assms(1) assms(10) assms(2) assms(4) assms(5)*
assms(6) assms(8) bet-out impossible-case-4-1 not-col-distincts)

moreover

have $\neg A \ \text{Out } B \ x$
using *assms(1) assms(10) assms(2) assms(4) assms(5)*
assms(6) assms(8) impossible-case-4-1 **by** *blast*

moreover

have $\neg B \ \text{Out } x \ A$
using *calculation(3) impossible-case-4-2*
by (*meson assms(9) assms(10) assms(11) assms(4)*
assms(5) assms(6) assms(7) assms(8)
col-permutation-4 or-bet-out)

ultimately

show *?thesis* **by** *blast*

qed

lemma *impossible-two-sides-not-col*:
assumes $A \neq D$ **and**

$C \neq D$ **and**

$\neg \text{Col } A \ B \ C$ **and**
 $\text{Bet } A \ D \ T$ **and**

$\text{Bet } B \ D \ C$ **and**
 $\text{Bet } B \ Y \ T$

shows $\neg \text{Col } A \ C \ Y$

proof –

have $A \ C \ \text{OS } B \ D$
by (*metis Col-perm assms(2) assms(3) assms(5)*
bet-col between-equality-2 invert-one-side l6-4-1
out-one-side out-to-bet)

have $A \ C \ \text{OS } D \ T$
by (*metis $\langle A \ C \ \text{OS } B \ D \rangle$ assms(1) assms(4) bet-out*
col124--nos out-one-side)

hence $A \ C \ \text{OS } B \ T$
using *$\langle A \ C \ \text{OS } B \ D \rangle$ one-side-transitivity* **by** *blast*

hence $A \ C \ \text{OS } B \ Y$
using *assms(6) l9-17* **by** *blast*

thus *?thesis*
using *one-side-not-col124* **by** *auto*

qed

lemma *triangle-circumscription-implies-tarski-s-euclid-aux1*:
assumes *triangle-circumscription-principle* **and**

```

B ≠ D and
C ≠ D and
D ≠ T and
T ≠ X and
¬ Col A B C and
Col A B M1 and
Bet A D T and
¬ Col B C T and
Bet B D C and
Col T Y Z and
Bet Y T X and
Bet Y M1 Z1 and
Cong Y T T X and
Cong Y M1 M1 Z1 and
B C Perp T Z and
A B Perp Y Z1
shows ∃ x. Col A B x ∧ B C ParStrict x T ∧ Cong X x Y x
proof –
  have A ≠ D
    using Bet-cases Col-def assms(10) assms(6) by blast
  have B ≠ C
    using assms(6) col-trivial-2 by blast
  have A ≠ B
    using assms(6) col-trivial-1 by auto
  have A ≠ C
    using assms(6) col-trivial-3 by blast
  have X ≠ Y
    using assms(12) assms(5) between-identity by blast
  have Y ≠ T
    using assms(14) assms(5) cong-reverse-identity by blast
  have Y ≠ Z1
    using assms(17) perp-not-eq-2 by auto
  have T ≠ Z
    using assms(16) perp-not-eq-2 by blast
  have Coplanar B C T A
    by (metis assms(10) assms(8) bet--coplanar bet-cop--tsp
      coplanar-perm-1 coplanar-perm-5 tsp-distincts)
  have Coplanar B C T B
    using ncop-distincts by blast
  have Coplanar B C T C
    using ncop-distincts by blast
  have Coplanar B C T T
    using ncop-distincts by auto
  have Coplanar B C T Z
    by (simp add: assms(16) perp--coplanar)
  have Coplanar B C T Y
    by (meson ⟨Coplanar B C T Z⟩ ⟨T ≠ Z⟩ assms(11)
      col-cop--cop not-col-permutation-5)
  have Coplanar B C T X
    using ⟨Coplanar B C T T⟩ ⟨Coplanar B C T Y⟩ ⟨Y ≠ T⟩
      assms(12) bet-col col-cop2--cop by blast
  {
    assume Col X Y Z1
    hence Col T Z Z1
      by (metis Col-perm ⟨Y ≠ T⟩ assms(11) assms(12) bet-col
        between-identity col-transitivity-1)
    have Coplanar B C Y Z1
      by (metis Col-cases ⟨Col T Z Z1⟩ ⟨Coplanar B C T Y⟩
        ⟨T ≠ Z⟩ ⟨Y ≠ T⟩ assms(11) col-cop--cop col-trivial-3
        colx coplanar-perm-1)
    have B A Par B C
    proof –
      have Coplanar Y Z1 A C
    proof –
      have Coplanar B C T Z1
        using ⟨Col T Z Z1⟩ ⟨Coplanar B C T Z⟩ ⟨T ≠ Z⟩ col-cop--cop by blast

```

```

    moreover have Coplanar B C T C
      using ⟨Coplanar B C T C⟩ by auto
    ultimately show ?thesis
      using assms(9) ⟨Coplanar B C T A⟩
        ⟨Coplanar B C T Y⟩ coplanar-pseudo-trans by presburger
  qed
  moreover
  have B C Perp Y Z1
  proof –
    have T Z Perp B C
      using Perp-perm assms(16) by blast
    moreover
    have Col T Z Y
      using Col-cases assms(11) by blast
    ultimately
    show ?thesis
      using ⟨Y ≠ Z1⟩ ⟨Col T Z Z1⟩ perp-col0 by blast
  qed
  ultimately
  show ?thesis
    by (metis ⟨A ≠ B⟩ ⟨B ≠ C⟩ assms(17) cop-perp2--col
      not-par-not-col perp-comm perp-right-comm)
  qed
  hence False
    using assms(6) par-id-1 by auto
}
hence ¬ Col X Y Z1
  by blast
obtain x where Cong X x Y x ∧ Cong X x Z1 x ∧ Coplanar X Y Z1 x
  using triangle-circumscription-principle-def
    ⟨¬ Col X Y Z1⟩ assms(1) by blast
have Cong X x Y x
  by (simp add: ⟨Cong X x Y x ∧ Cong X x Z1 x ∧ Coplanar X Y Z1 x⟩)
have Cong X x Z1 x
  by (simp add: ⟨Cong X x Y x ∧ Cong X x Z1 x ∧ Coplanar X Y Z1 x⟩)
have Coplanar X Y Z1 x
  by (simp add: ⟨Cong X x Y x ∧ Cong X x Z1 x ∧ Coplanar X Y Z1 x⟩)
have Y ≠ M1
  using ⟨Y ≠ Z1⟩ assms(15) cong-reverse-identity by blast
have Coplanar B C T Z1
  by (meson ⟨A ≠ B⟩ ⟨Coplanar B C T A⟩ ⟨Coplanar B C T B⟩
    ⟨Coplanar B C T Y⟩ ⟨Y ≠ M1⟩ assms(13) assms(7)
    bet-col col-cop2--cop)
have Coplanar B C T x
  by (meson ⟨Coplanar B C T X⟩ ⟨Coplanar B C T Y⟩
    ⟨Coplanar B C T Z1⟩ ⟨Coplanar X Y Z1 x⟩ ⟨¬ Col X Y Z1⟩
    l9-30 ncop-distincts)
have Col A B x
  proof –
    have Coplanar A x Y Z1
      using assms(9) ⟨Coplanar B C T A⟩ ⟨Coplanar B C T Y⟩
        ⟨Coplanar B C T Z1⟩ ⟨Coplanar B C T x⟩
        coplanar-pseudo-trans by blast
    moreover
    have Coplanar B x Y Z1
      using assms(9)
      by (meson ⟨Coplanar B C T B⟩ ⟨Coplanar B C T Y⟩
        ⟨Coplanar B C T Z1⟩ ⟨Coplanar B C T x⟩ coplanar-pseudo-trans)
    moreover
    have Cong x Y x Z1
      proof –
        have Cong x Y X x
          using ⟨Cong X x Y x⟩ not-cong-3421 by blast
        moreover
        have Cong X x x Z1
          using ⟨Cong X x Z1 x⟩ not-cong-1243 by blast

```

```

ultimately
show ?thesis
  by (meson cong-transitivity)
qed
moreover
have A B PerpBisect Y Z1
proof -
  have M1 Midpoint Y Z1
    by (simp add: Midpoint-def assms(13) assms(15))
  hence (∃ X. X Midpoint Y Z1 ∧ Col A B X) ∧ (A B Perp Y Z1 ∨ Y = Z1)
    using assms(7) assms(17) by blast
  hence Y Z1 ReflectL A B
    using ReflectL-def l10-4-spec by presburger
  thus ?thesis
    using ⟨Y ≠ Z1⟩ Perp-bisect-def by auto
qed
ultimately
show ?thesis
  using cong-cop2-perp-bisect-col by blast
qed
have B C ParStrict x T
proof -
  have B C Par x T
  proof -
    have Coplanar X Y B x
      by (meson ⟨Coplanar B C T B⟩ ⟨Coplanar B C T X⟩
        ⟨Coplanar B C T Y⟩ ⟨Coplanar B C T x⟩ assms(9)
        coplanar-pseudo-trans)
    moreover
    have Coplanar X Y B T
      by (meson ⟨Coplanar B C T B⟩ ⟨Coplanar B C T T⟩
        ⟨Coplanar B C T X⟩ ⟨Coplanar B C T Y⟩ assms(9)
        coplanar-pseudo-trans)
    moreover
    have Coplanar X Y C x
      by (meson ⟨Coplanar B C T C⟩ ⟨Coplanar B C T X⟩
        ⟨Coplanar B C T Y⟩ ⟨Coplanar B C T x⟩ assms(9)
        coplanar-pseudo-trans)
    moreover
    have Coplanar X Y C T
      by (meson ⟨Coplanar B C T C⟩ ⟨Coplanar B C T T⟩
        ⟨Coplanar B C T X⟩ ⟨Coplanar B C T Y⟩ assms(9)
        coplanar-pseudo-trans)
    moreover
    have B C Perp X Y
  proof -
    have T Z Perp B C
      using Perp-cases assms(16) by blast
    moreover
    have Col T Z X
      by (meson ⟨Y ≠ T⟩ assms(11) assms(12) bet-col
        bet-col1 col3 not-col-permutation-4)
    moreover
    have Col T Z Y
      by (simp add: assms(11) col-permutation-5)
    ultimately
    show ?thesis
      using ⟨X ≠ Y⟩ perp-col0 by blast
  qed
  moreover
  have x T PerpBisect X Y
  proof -
    have x ≠ T
      using ⟨A ≠ D⟩ ⟨Col A B x⟩ assms(10) assms(2)
        assms(6) assms(8) between-trivial2
        impossible-case-2 by blast
  qed

```

```

moreover
have Coplanar  $x$   $T$   $X$   $Y$ 
  by (meson assms(12) bet-col ncop--ncol ncoplanar-perm-21)
moreover
have Cong  $X$   $T$   $Y$   $T$ 
  by (meson assms(14) not-cong-3421)
ultimately
show ?thesis
  using  $\langle \text{Cong } X \ x \ Y \ x \rangle \ \langle X \neq Y \rangle$  cong-cop-perp-bisect by presburger
qed
hence  $x$   $T$  Perp  $X$   $Y$ 
  by (simp add: perp-bisect-perp)
ultimately
show ?thesis
  using l12-9 by blast
qed
moreover
have Col  $x$   $T$   $T$ 
  by (simp add: col-trivial-2)
moreover
have  $\neg$  Col  $B$   $C$   $T$ 
  by (simp add: assms(9))
ultimately
show ?thesis
  using par-not-col-strict by blast
qed
thus ?thesis
  using  $\langle \text{Col } A \ B \ x \rangle \ \langle \text{Cong } X \ x \ Y \ x \rangle$  by blast
qed

```

lemma *triangle-circumscription-implies-tarski-s-euclid-aux:*

```

assumes triangle-circumscription-principle and
   $B \neq D$  and
   $C \neq D$  and
   $D \neq T$  and
   $T \neq X$  and
   $\neg$  Col  $A$   $B$   $C$  and
  Col  $A$   $B$   $M1$  and
  Col  $A$   $C$   $M2$  and
  Bet  $A$   $D$   $T$  and
   $\neg$  Col  $B$   $C$   $T$  and
  Bet  $B$   $D$   $C$  and
  Col  $T$   $Y$   $Z$  and
  Bet  $Y$   $T$   $X$  and
  Bet  $Y$   $M1$   $Z1$  and
  Bet  $Y$   $M2$   $Z2$  and
  Cong  $Y$   $T$   $T$   $X$  and
  Cong  $Y$   $M1$   $M1$   $Z1$  and
  Cong  $Y$   $M2$   $M2$   $Z2$  and
   $B$   $C$  Perp  $T$   $Z$  and
   $A$   $B$  Perp  $Y$   $Z1$  and
   $A$   $C$  Perp  $Y$   $Z2$ 
shows  $\exists \ x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
proof –
have  $\exists x. \text{Col } A \ B \ x \wedge B \ C \text{ParStrict } x \ T \wedge \text{Cong } X \ x \ Y \ x$ 
  using triangle-circumscription-implies-tarski-s-euclid-aux1
    assms(1) assms(2) assms(3) assms(4) assms(5)
    assms(6) assms(7) assms(9) assms(10)
    assms(11) assms(12) assms(13) assms(14)
    assms(16) assms(17) assms(19) assms(20)
  by blast
then obtain  $x$  where  $\text{Col } A \ B \ x \wedge B \ C \text{ParStrict } x \ T \wedge \text{Cong } X \ x \ Y \ x$ 
  by auto
hence  $\text{Col } A \ B \ x$ 
  by auto
have  $B \ C \text{ParStrict } x \ T$ 

```



```

  by (simp add: ⟨Col A B x ∧ B C ParStrict x T ∧ Cong X x Y x⟩)
have Cong X x Y x
  by (simp add: ⟨Col A B x ∧ B C ParStrict x T ∧ Cong X x Y x⟩)
have ∃ y. Col A C y ∧ C B ParStrict y T ∧ Cong X y Y y
proof -
  have ¬ Col A C B
    using assms(6) col-permutation-5 by blast
  moreover
  have Col A C M2
    by (simp add: assms(8))
  moreover
  have Bet A D T
    by (simp add: assms(9))
  moreover
  have ¬ Col C B T
    using NCol-cases assms(10) by blast
  moreover
  have Bet C D B
    using Bet-cases assms(11) by blast
  moreover
  have C B Perp T Z
    by (simp add: assms(19) perp-left-comm)
  ultimately
  show ?thesis
    using triangle-circumscription-implies-tarski-s-euclid-aux1
      assms(1) assms(2) assms(3) assms(4) assms(5)
      assms(8) assms(9) assms(12) assms(13) assms(15)
      assms(16) assms(18) assms(21) by blast
qed
then obtain y where Col A C y ∧ C B ParStrict y T ∧ Cong X y Y y
  by auto
hence Col A C y
  by auto
have C B ParStrict y T
  by (simp add: ⟨Col A C y ∧ C B ParStrict y T ∧ Cong X y Y y⟩)
have Cong X y Y y
  by (simp add: ⟨Col A C y ∧ C B ParStrict y T ∧ Cong X y Y y⟩)
have Col x T y
proof (cases x = T)
  case True
  thus ?thesis
    by (simp add: col-trivial-1)
next
  case False
  thus ?thesis
  proof (cases y = T)
    case True
    thus ?thesis
      using col-trivial-2 by auto
  next
    case False
    thus ?thesis
    proof -
      have X ≠ Y
        using assms(13) assms(5) between-identity by blast
      have Y ≠ T
        using assms(16) assms(5) cong-reverse-identity by blast
      have T ≠ Z
        using assms(19) perp-not-eq-2 by auto
      have Coplanar X Y x y
      proof -
        have Coplanar B C T Y
        proof -
          have Coplanar B C T Z
            using assms(19) perp--coplanar by auto
          moreover

```

```

    have Col T Z Y
      by (simp add: assms(12) col-permutation-5)
    ultimately
    show ?thesis
      using  $\langle T \neq Z \rangle$  col-cop--cop by blast
  qed
  moreover
  have Coplanar B C T X
    by (meson  $\langle \text{Coplanar } B \ C \ T \ Y \rangle \langle Y \neq T \rangle$  assms(13)
      bet-col bet-col1 col2-cop--cop ncoplanar-perm-7)
  moreover
  have Coplanar B C T x
    by (meson  $\langle B \ C \ \text{ParStrict } x \ T \rangle$  par-strict-right-comm
      pars--coplanar)
  moreover
  have Coplanar B C T y
    by (simp add:  $\langle C \ B \ \text{ParStrict } y \ T \rangle$  par-strict-comm
      pars--coplanar)
  ultimately
  show ?thesis
    by (meson assms(10) coplanar-pseudo-trans)
  qed
  moreover
  have T x PerpBisect X Y
  proof -
    have T  $\neq$  x
      using  $\langle B \ C \ \text{ParStrict } x \ T \rangle$  par-strict-distinct by blast
    moreover
    have Coplanar T x X Y
      by (meson assms(13) assms(16) midpoint--coplanar
        midpoint-def ncoplanar-perm-5)
    moreover
    have Cong X T Y T
      using Cong-perm assms(16) by blast
    ultimately
    show ?thesis
      using  $\langle X \neq Y \rangle \langle \text{Cong } X \ x \ Y \ x \rangle$  cong-cop-perp-bisect by auto
  qed
  hence T x Perp X Y
    using perp-bisect-perp by auto
  moreover
  have T y PerpBisect X Y
  proof -
    have T  $\neq$  y
      using False by auto
    moreover
    have Coplanar T y X Y
      by (meson assms(13) bet--coplanar between-symmetry
        ncoplanar-perm-13)
    moreover
    have Cong X T Y T
      using assms(16) cong-4312 by blast
    ultimately
    show ?thesis
      by (simp add:  $\langle \text{Cong } X \ y \ Y \ y \rangle \langle X \neq Y \rangle$  cong-cop-perp-bisect)
  qed
  hence T y Perp X Y
    by (simp add: perp-bisect-perp)
  ultimately
  show ?thesis
    using col-permutation-4 cop-perp2--col by blast
  qed
  qed
  have B C ParStrict x y
  proof -

```

```

{
  assume  $x = y$ 
  hence  $A \neq C$ 
  using assms(6) col-trivial-3 by force
  hence  $A = x$ 
  using NCol-perm  $\langle \text{Col } A \ B \ x \rangle \langle \text{Col } A \ C \ y \rangle \langle x = y \rangle$  assms(6)
    col-trivial-3 colx by blast
  hence  $\exists X0. \text{Col } X0 \ B \ C \wedge \text{Col } X0 \ x \ T$ 
  using Bet-cases Col-def assms(11) assms(9) by blast
  hence False
  using  $\langle B \ C \ \text{ParStrict } x \ T \rangle$  par-not-col by blast
}
moreover have  $B \ C \ \text{ParStrict } x \ T$ 
  using  $\langle B \ C \ \text{ParStrict } x \ T \rangle$  by blast
moreover have  $\text{Col } x \ T \ y$ 
  by (simp add:  $\langle \text{Col } x \ T \ y \rangle$ )
ultimately show ?thesis
  using par-strict-col-par-strict by blast
qed
have  $A \neq D$ 
  using Col-cases assms(11) assms(6) bet-col by blast
{
  assume  $\text{Bet } x \ T \ y$ 
  {
    assume  $\text{Bet } A \ B \ x$ 
    hence  $\exists x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
    using Col-def  $\langle A \neq D \rangle \langle B \ C \ \text{ParStrict } x \ y \rangle \langle \text{Bet } x \ T \ y \rangle$ 
       $\langle \text{Col } A \ B \ x \rangle \langle \text{Col } A \ C \ y \rangle$  assms(11) assms(2) assms(6) assms(9)
      impossible-case-1 impossible-case-2 by blast
  }
  moreover
  {
    assume  $\text{Bet } B \ x \ A$ 
    hence  $\exists x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
    using  $\langle B \ C \ \text{ParStrict } x \ y \rangle \langle \text{Bet } x \ T \ y \rangle$  assms(10)
      assms(11) assms(9) impossible-case-3 by blast
  }
  moreover
  {
    assume  $\text{Bet } x \ A \ B$ 
    hence  $\exists x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
    by (meson  $\langle A \neq D \rangle \langle \text{Bet } x \ T \ y \rangle \langle \text{Col } A \ C \ y \rangle$  assms(11)
      assms(3) assms(6) assms(9) between-symmetry
      col-permutation-5 impossible-case-2)
  }
  ultimately
  have  $\exists x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
  using Col-def  $\langle \text{Col } A \ B \ x \rangle$  by blast
}
moreover
{
  assume  $\text{Bet } T \ y \ x$ 
  {
    assume  $\text{Bet } A \ B \ x$ 
    hence  $\exists x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
    using  $\langle A \neq D \rangle \langle B \ C \ \text{ParStrict } x \ y \rangle \langle \text{Bet } T \ y \ x \rangle$ 
       $\langle \text{Col } A \ B \ x \rangle \langle \text{Col } A \ C \ y \rangle$  assms(10) assms(11) assms(3)
      assms(4) assms(6) assms(9) impossible-case-4 by blast
  }
  moreover
  {
    assume  $\text{Bet } B \ x \ A$ 
    hence  $\exists x \ y. \text{Bet } A \ B \ x \wedge \text{Bet } A \ C \ y \wedge \text{Bet } x \ T \ y$ 
    using  $\langle A \neq D \rangle \langle B \ C \ \text{ParStrict } x \ y \rangle \langle \text{Bet } T \ y \ x \rangle \langle \text{Col } A \ B \ x \rangle$ 
       $\langle \text{Col } A \ C \ y \rangle$  assms(10) assms(11) assms(3) assms(4) assms(6)
      assms(9) impossible-case-4 by blast
  }
}

```

```

}
moreover
{
  assume Bet x A B
  hence  $\exists x y. \text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y$ 
  using  $\langle A \neq D \rangle \langle B C \text{ ParStrict } x y \rangle \langle \text{Bet } T y x \rangle$ 
     $\langle \text{Col } A B x \rangle \langle \text{Col } A C y \rangle \text{assms}(10) \text{assms}(11) \text{assms}(3) \text{assms}(4)$ 
     $\text{assms}(6) \text{assms}(9) \text{impossible-case-4}$  by blast
}
ultimately
have  $\exists x y. \text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y$ 
  using Col-def  $\langle \text{Col } A B x \rangle$  by blast
}
moreover
{
  assume Bet y x T
  {
    assume Bet A B x
    hence  $\exists x y. \text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y$ 
    by (meson  $\langle A \neq D \rangle \langle B C \text{ ParStrict } x y \rangle \langle \text{Bet } y x T \rangle$ 
       $\langle \text{Col } A B x \rangle \langle \text{Col } A C y \rangle \text{assms}(10) \text{assms}(11) \text{assms}(2)$ 
       $\text{assms}(4) \text{assms}(6) \text{assms}(9) \text{between-symmetry}$ 
      col-permutation-5 impossible-case-4
      not-col-permutation-3 par-strict-comm)
  }
  moreover
  {
    assume Bet B x A
    hence  $\exists x y. \text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y$ 
    using  $\langle B C \text{ ParStrict } x T \rangle \text{assms}(10) \text{assms}(11)$ 
       $\text{assms}(9) \text{between-trivial impossible-case-3}$  by blast
  }
  moreover
  {
    assume Bet x A B
    hence  $\exists x y. \text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y$ 
    by (meson Par-strict-cases  $\langle A \neq D \rangle \langle B C \text{ ParStrict } x y \rangle$ 
       $\langle \text{Bet } y x T \rangle \langle \text{Col } A B x \rangle \langle \text{Col } A C y \rangle \text{assms}(10) \text{assms}(11)$ 
       $\text{assms}(2) \text{assms}(4) \text{assms}(6) \text{assms}(9) \text{between-symmetry}$ 
      col-permutation-5 impossible-case-4 not-col-permutation-2)
  }
  ultimately
  have  $\exists x y. \text{Bet } A B x \wedge \text{Bet } A C y \wedge \text{Bet } x T y$ 
    using Col-def  $\langle \text{Col } A B x \rangle$  by blast
}
ultimately
show ?thesis
  using Col-def  $\langle \text{Col } x T y \rangle$  by blast
qed

```

lemma *triangle-circumscription-implies-tarski-s-euclid*:

assumes *triangle-circumscription-principle*

shows *tarski-s-parallel-postulate*

proof –

```

{
  fix A B C D T
  assume Bet A D T  $\wedge$  Bet B D C  $\wedge$  A  $\neq$  D
  have  $\exists X Y. \text{Bet } A B X \wedge \text{Bet } A C Y \wedge \text{Bet } X T Y$ 
  proof –
    {
      fix A' B' C' D' T'
      assume H1:  $A' \neq B' \wedge A' \neq C' \wedge A' \neq D' \wedge A' \neq T' \wedge$ 
         $B' \neq C' \wedge B' \neq D' \wedge B' \neq T' \wedge C' \neq D' \wedge$ 
         $C' \neq T' \wedge D' \neq T' \wedge \neg \text{Col } A' B' C' \wedge$ 
         $\text{Bet } A' D' T' \wedge \text{Bet } B' D' C' \wedge \neg \text{Col } B' C' T'$ 
    }
  }

```

```

have  $A' \neq B'$ 
  using  $H1$  by blast
have  $A' \neq C'$ 
  using  $H1$  by blast
have  $A' \neq D'$ 
  using  $H1$  by blast
have  $B' \neq C'$ 
  using  $H1$  by blast
have  $B' \neq D'$ 
  using  $H1$  by blast
have  $C' \neq D'$ 
  using  $H1$  by blast
have  $D' \neq T'$ 
  using  $H1$  by blast
have  $\neg \text{Col } A' B' C'$ 
  using  $H1$  by blast
have  $\text{Bet } A' D' T'$ 
  using  $H1$  by blast
have  $\text{Bet } B' D' C'$ 
  using  $H1$  by blast
have  $\neg \text{Col } B' C' T'$ 
  using  $H1$  by blast
obtain  $Y'$  where  $\text{Col } B' C' Y' \wedge B' C' \text{Perp } T' Y'$ 
  using  $\langle \neg \text{Col } B' C' T' \rangle$  l8-18-existence by presburger
{
  fix  $B'' C''$ 
  assume  $H2: A' \neq B'' \wedge A' \neq C'' \wedge B'' \neq C'' \wedge$ 
     $B'' \neq D' \wedge C'' \neq D' \wedge \neg \text{Col } A' B'' C'' \wedge$ 
     $\text{Bet } B'' D' C'' \wedge \neg \text{Col } B'' C'' T' \wedge$ 
     $\text{Col } B'' C'' Y' \wedge B'' C'' \text{Perp } T' Y'$ 
  {
    fix  $B''' C'''$ 
    assume  $A1: A' \neq C''' \wedge B''' \neq C''' \wedge B''' \neq D' \wedge$ 
       $C''' \neq D' \wedge \neg \text{Col } A' B''' C''' \wedge$ 
       $\text{Bet } B''' D' C''' \wedge \neg \text{Col } B''' C''' T' \wedge$ 
       $\text{Col } B''' C''' Y' \wedge B''' C''' \text{Perp } T' Y' \wedge$ 
       $B''' \neq Y'$ 
    have  $\exists x y. (\text{Bet } A' B''' x \wedge \text{Bet } A' C''' y \wedge \text{Bet } x T' y)$ 
    proof -
      {
        assume  $C''' = Y'$ 
        have  $C''' \neq T'$ 
          using  $A1$  col-trivial-2 by blast
        obtain  $Y1$  where  $Y1$  Midpoint  $C''' T'$ 
          using midpoint-existence by fastforce
        have  $\neg \text{Col } A' B''' C'''$ 
          using  $A1$  by blast
      }
      have  $A' \neq C''' \wedge B''' \neq C''' \wedge B''' \neq D' \wedge$ 
         $C''' \neq D' \wedge \neg \text{Col } A' B''' C''' \wedge$ 
         $\text{Bet } B''' D' C''' \wedge \neg \text{Col } B''' C''' T' \wedge$ 
         $\text{Col } B''' C''' Y' \wedge B''' C''' \text{Perp } T' Y' \wedge$ 
         $B''' \neq Y'$  using  $A1$  by blast
      {
        assume  $Y1 = A'$ 
        have  $C''' \neq D'$  using  $A1$  by blast
        have  $\text{Bet } B''' D' C'''$  using  $A1$  by blast
        have  $\text{Col } A' C''' T'$ 
          using  $A1$ 
          using  $\langle Y1 = A' \rangle \langle Y1 \text{ Midpoint } C''' T' \rangle$  midpoint-col by blast
        hence  $\text{Col } A' B''' C'''$ 
          using  $\langle A' \neq D' \rangle \langle \text{Bet } A' D' T' \rangle \langle \text{Bet } B''' D' C''' \rangle$ 
           $\langle C''' \neq D' \rangle$  between-trivial
          impossible-two-sides-not-col by blast
        hence False
          using  $\langle \neg \text{Col } A' B''' C''' \rangle$  by blast
      }
    }
  }

```

```

}
hence  $Y1 \neq A'$ 
  by blast
have  $C''' \neq Y1$ 
  using  $\langle C''' \neq T' \rangle \langle Y1 \text{ Midpoint } C''' T' \rangle$ 
    midpoint-distinct-1 by blast
have  $B''' \neq Y1$ 
  by (metis A1 midpoint-col  $\langle Y1 \text{ Midpoint } C''' T' \rangle$ )
have  $\text{Bet } C''' Y1 T'$ 
  by (simp add:  $\langle Y1 \text{ Midpoint } C''' T' \rangle$  midpoint-bet)
have  $\text{Cong } C''' Y1 Y1 T'$ 
  by (simp add:  $\langle Y1 \text{ Midpoint } C''' T' \rangle$  midpoint-cong)
have  $\neg \text{Col } A' B''' Y1$ 
  using impossible-two-sides-not-col A1
  by (meson  $\langle A' \neq D' \rangle \langle \text{Bet } A' D' T' \rangle$ 
     $\langle \text{Bet } C''' Y1 T' \rangle$  between-symmetry
    not-col-permutation-5)
obtain X where  $T' \text{ Midpoint } Y1 X$ 
  using symmetric-point-construction by auto
hence  $X \neq Y1$ 
  by (metis  $\langle C''' \neq Y1 \rangle \langle Y1 \text{ Midpoint } C''' T' \rangle$  l7-9-bis l8-20-2)
hence  $T' \neq X$ 
  using  $\langle T' \text{ Midpoint } Y1 X \rangle$  is-midpoint-id-2 by blast
have  $\text{Bet } Y1 T' X$ 
  by (simp add:  $\langle T' \text{ Midpoint } Y1 X \rangle$  midpoint-bet)
have  $\text{Cong } Y1 T' T' X$ 
  by (simp add:  $\langle T' \text{ Midpoint } Y1 X \rangle$  midpoint-cong)
obtain Z1 where  $Z1 Y1 \text{ Reflect } A' B'''$ 
  using l10-2-existence by blast
hence  $Z1 Y1 \text{ ReflectL } A' B'''$ 
  using  $\langle \neg \text{Col } A' B''' Y1 \rangle$  is-image-is-image-spec
    not-col-distincts by blast
then obtain M1 where HH1:  $\text{Bet } Y1 M1 Z1 \wedge \text{Cong } Y1 M1 M1 Z1 \wedge$ 
     $\text{Col } A' B''' M1 \wedge$ 
     $(A' B''' \text{ Perp } Y1 Z1 \vee Y1 = Z1)$ 
  using ReflectL-def midpoint-bet midpoint-cong by blast
obtain Z2 where  $Z2 Y1 \text{ Reflect } A' C'''$ 
  using l10-2-existence by blast
hence  $Z2 Y1 \text{ ReflectL } A' C'''$ 
  using  $\langle \neg \text{Col } A' B''' C''' \rangle$  is-image-is-image-spec
    not-col-distincts by blast
then obtain M2 where HH2:  $\text{Bet } Y1 M2 Z2 \wedge \text{Cong } Y1 M2 M2 Z2 \wedge$ 
     $\text{Col } A' C''' M2 \wedge$ 
     $(A' C''' \text{ Perp } Y1 Z2 \vee Y1 = Z2)$ 
  using ReflectL-def midpoint-bet midpoint-cong by blast
have  $\text{Bet } Y1 M2 Z2$ 
  using HH2 by blast
have  $\text{Cong } Y1 M2 M2 Z2$ 
  using HH2 by blast
have  $\text{Col } A' C''' M2$ 
  using HH2 by blast
have  $A' C''' \text{ Perp } Y1 Z2 \vee Y1 = Z2$ 
  using HH2 by blast

{
  assume  $Y1 = Z2$ 
  hence  $Y1 = M2$ 
    using HH2 between-identity by blast
  hence  $\text{Col } A' C''' Y1$ 
    using HH2 by blast
  have  $\text{Col } A' B''' C'''$ 
  proof -
    have  $\text{Bet } A' D' T'$ 
      using  $\langle \text{Bet } A' D' T' \rangle$  by auto
    moreover have  $\text{Bet } C''' Y1 T'$ 
      using  $\langle \text{Bet } C''' Y1 T' \rangle$  by blast

```

```

moreover have  $C''' \neq Y1$ 
  using  $\langle C''' \neq Y1 \rangle$  by auto
moreover have  $Col\ A'\ C''' Y1$ 
  using  $\langle Col\ A'\ C''' Y1 \rangle$  by blast
moreover
{
  assume  $\neg Col\ B''' C''' Y1$ 
  have  $A' \neq C'''$ 
    using  $A1$  by blast
  have  $B''' \neq C'''$ 
    using  $A1$  by blast
  have  $C''' \neq D'$ 
    using  $A1$  by blast
  have  $Bet\ B''' D' C'''$ 
    using  $A1$  by blast
  have  $\neg Col\ B''' C''' T'$ 
    using  $A1$  by blast
  have  $B''' \neq Y'$ 
    using  $A1$  by blast
  hence  $A' = D'$ 
    by (metis (no-types, lifting) NCol-perm
      calculation(3) col-transitivity-1
       $\langle A' \neq C''' \rangle$   $\langle B''' \neq C''' \rangle$   $\langle C''' \neq D' \rangle$ 
       $\langle Bet\ B''' D' C''' \rangle$   $\langle \neg Col\ B''' C''' T' \rangle$ 
      bet-col between-identity calculation(1)
      calculation(2) not-col-distincts
      calculation(4))
  hence False
    using  $\langle A' \neq D' \rangle$  by blast
}
hence  $Col\ B''' C''' Y1$ 
  by blast
ultimately show ?thesis using l6-16-1
  using col-permutation-2 by blast
qed
hence False
  using  $A1$  by blast
hence  $\exists x\ y. (Bet\ A'\ B''' x \wedge Bet\ A'\ C''' y \wedge Bet\ x\ T'\ y)$ 
  by auto
}
moreover
{
  assume  $A'\ C''' Perp\ Y1\ Z2$ 
  hence  $\exists x\ y. (Bet\ A'\ B''' x \wedge Bet\ A'\ C''' y \wedge Bet\ x\ T'\ y)$ 
  proof -
    have  $B''' \neq D'$ 
      using  $A1$  by blast
    moreover
    have  $C''' \neq D'$ 
      using  $A1$  by blast
    moreover
    have  $D' \neq T'$ 
      using  $\langle D' \neq T' \rangle$  by auto
    moreover
    have  $T' \neq X$ 
      by (simp add:  $\langle T' \neq X \rangle$ )
    moreover
    have  $\neg Col\ A'\ B''' C'''$ 
      using  $A1$  by blast
    moreover
    have  $Col\ A'\ B''' M1$ 
      using HH1 by blast
    moreover
    have  $Col\ A'\ C''' M2$ 
      using HH2 by blast
    moreover

```

```

have Bet A' D' T'
  by (simp add: ⟨Bet A' D' T'⟩)
moreover
have ¬ Col B''' C''' T'
  using A1 by blast
moreover
have Bet B''' D' C'''
  using A1 by blast
moreover
have Col T' Y1 C'''
  by (simp add: ⟨Bet C''' Y1 T'⟩ bet-col between-symmetry)
moreover
have Bet Y1 T' X
  by (simp add: ⟨Bet Y1 T' X⟩)
moreover
have Bet Y1 M1 Z1
  using HH1 by blast
moreover
have Bet Y1 M2 Z2
  using HH2 by blast
moreover
have Cong Y1 T' T' X
  by (simp add: ⟨Cong Y1 T' T' X⟩)
moreover
have Cong Y1 M1 M1 Z1
  using HH1 by blast
moreover
have Cong Y1 M2 M2 Z2
  using HH2 by blast
moreover
have B''' C''' Perp T' C'''
  using A1 ⟨C''' = Y'⟩ by blast
moreover
have A' B''' Perp Y1 Z1
  using HH1 ⟨¬ Col A' B''' Y1⟩ bet-neq12--neq by blast
moreover
have A' C''' Perp Y1 Z2
  by (simp add: ⟨A' C''' Perp Y1 Z2⟩)
ultimately
show ?thesis
  using assms(1)
    triangle-circumscription-implies-tarski-s-euclid-aux
  by blast
qed
}
ultimately
have ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)
  using HH2 by blast
}
}
moreover
{
  assume C''' ≠ Y'
  obtain X where T' Midpoint Y' X
    using symmetric-point-construction by blast
  have T' ≠ Y'
    using A1 by blast
  hence X ≠ Y'
    using ⟨T' Midpoint Y' X⟩ l7-3 by blast
  have T' ≠ X
    using ⟨T' Midpoint Y' X⟩ ⟨T' ≠ Y'⟩ is-midpoint-id-2 by blast
  have Bet Y' T' X
    by (simp add: ⟨T' Midpoint Y' X⟩ midpoint-bet)
  have Cong Y' T' T' X
    by (simp add: ⟨T' Midpoint Y' X⟩ midpoint-cong)
  obtain Z1 where Z1 Y' Reflect A' B'''

```



```

    using l10-2-existence by blast
have  $\neg (A' = B''' \wedge A' \text{ Midpoint } Y' Z1)$ 
    using A1 not-col-distincts by blast
hence  $Z1 \ Y' \ \text{ReflectL } A' \ B'''$ 
    using Reflect-def  $\langle Z1 \ Y' \ \text{Reflect } A' \ B''' \rangle$  by force
then obtain M1 where HH3:  $\text{Bet } Y' \ M1 \ Z1 \wedge \text{Cong } Y' \ M1 \ M1 \ Z1 \wedge$ 
                         $\text{Col } A' \ B''' \ M1 \wedge$ 
                         $(A' \ B''' \ \text{Perp } Y' \ Z1 \vee Y' = Z1)$ 
    using ReflectL-def midpoint-bet midpoint-cong by blast
obtain Z2 where  $Z2 \ Y' \ \text{Reflect } A' \ C'''$ 
    using l10-2-existence by blast
have  $\neg (A' = C''' \wedge A' \text{ Midpoint } Y' Z2)$ 
    using A1 not-col-distincts by blast
hence  $Z2 \ Y' \ \text{ReflectL } A' \ C'''$ 
    using Reflect-def  $\langle Z2 \ Y' \ \text{Reflect } A' \ C''' \rangle$  by force
then obtain M2 where HH4:  $\text{Bet } Y' \ M2 \ Z2 \wedge \text{Cong } Y' \ M2 \ M2 \ Z2 \wedge$ 
                         $\text{Col } A' \ C''' \ M2 \wedge$ 
                         $(A' \ C''' \ \text{Perp } Y' \ Z2 \vee Y' = Z2)$ 
    using ReflectL-def midpoint-bet midpoint-cong by blast
have  $\neg \text{Col } A' \ B''' \ Y'$ 
proof -
{
  assume  $\text{Col } A' \ B''' \ Y'$ 
  have False
  proof -
    have  $\neg \text{Col } A' \ B''' \ C'''$ 
      using A1 by blast
    moreover
    have  $C''' \neq B'''$ 
      using A1 by blast
    moreover
    have  $\text{Col } A' \ B''' \ B'''$ 
      by (simp add: col-trivial-2)
    moreover
    have  $\text{Col } A' \ B''' \ Y'$ 
      using  $\langle \text{Col } A' \ B''' \ Y' \rangle$  by blast
    moreover
    have  $\text{Col } C''' \ B''' \ B'''$ 
      by (simp add: col-trivial-2)
    moreover
    have  $\text{Col } C''' \ B''' \ Y'$ 
      using A1 Col-cases by blast
    ultimately
    show ?thesis
      using A1 l6-21 by blast
  qed
}
thus ?thesis
  by blast qed
have  $\neg \text{Col } A' \ C''' \ Y'$ 
proof -
{
  assume  $\text{Col } A' \ C''' \ Y'$ 
  have False
  proof -
    have  $\neg \text{Col } A' \ C''' \ B'''$ 
      using A1 Col-cases by blast
    moreover
    have  $B''' \neq C'''$ 
      using A1 by blast
    moreover
    have  $\text{Col } A' \ C''' \ C'''$ 
      by (simp add: col-trivial-2)
    moreover
    have  $\text{Col } A' \ C''' \ Y'$ 
      using  $\langle \text{Col } A' \ C''' \ Y' \rangle$  by blast
  }
}

```

```

    moreover
    have Col B''' C''' C'''
      by (simp add: col-trivial-2)
    moreover
    have Col B''' C''' Y'
      using A1 by blast
    ultimately
    show ?thesis
      using A1 l6-21 by (meson ⟨C''' ≠ Y'⟩)
  qed
}
thus ?thesis
  by blast
qed
have ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)
proof-
{
  assume A' B''' Perp Y' Z1 ∧ A' C''' Perp Y' Z2
  have ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)
  proof -
    have B''' ≠ D'
      using A1 by blast
    moreover
    have C''' ≠ D'
      using A1 by blast
    moreover
    have D' ≠ T'
      by (simp add: ⟨D' ≠ T'⟩)
    moreover
    have T' ≠ X
      by (simp add: ⟨T' ≠ X⟩)
    moreover
    have ¬ Col A' B''' C'''
      using A1 by blast
    moreover
    have Col A' B''' M1
      using HH3 by blast
    moreover
    have Col A' C''' M2
      using HH4 by blast
    moreover
    have Bet A' D' T'
      using ⟨Bet A' D' T'⟩ by blast
    moreover
    have ¬ Col B''' C''' T'
      using A1 by blast
    moreover
    have Bet B''' D' C'''
      using A1 by blast
    moreover
    have Col T' Y' Y'
      using col-trivial-2 by presburger
    moreover
    have Bet Y' T' X
      using ⟨Bet Y' T' X⟩ by blast
    moreover
    have Bet Y' M1 Z1
      using HH3 by blast
    moreover
    have Bet Y' M2 Z2
      using HH4 by blast
    moreover
    have Cong Y' T' T' X
      using ⟨Cong Y' T' T' X⟩ by blast
    moreover
    have Cong Y' M1 M1 Z1

```

```

    using HH3 by blast
  moreover
  have Cong Y' M2 M2 Z2
    using HH4 by blast
  moreover
  have B''' C''' Perp T' Y'
    using A1 by blast
  moreover
  have A' B''' Perp Y' Z1
    using ⟨A' B''' Perp Y' Z1 ∧ A' C''' Perp Y' Z2⟩ by blast
  moreover
  have A' C''' Perp Y' Z2
    using ⟨A' B''' Perp Y' Z1 ∧ A' C''' Perp Y' Z2⟩ by blast
  ultimately
  show ?thesis
    using assms(1)
      triangle-circumscription-implies-tarski-s-euclid-aux
    by blast
qed
}
moreover
{
  assume Y' = Z1
  hence ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)
    using HH3 ⟨¬ Col A' B''' Y'⟩ bet-neq12--neq by blast
}
moreover
{
  assume Y' = Z2
  hence ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)
    using ⟨Z2 Y' Reflect A' C'⟩ ⟨¬ Col A' C''' Y'⟩
      col-permutation-1 l10-8 by blast
}
ultimately
show ?thesis
  using HH3 HH4 by blast
qed
}
ultimately
show ?thesis
  by blast
qed
}
hence Haux: ∀ B''' C'''. (A' ≠ C''' ∧ B''' ≠ C''' ∧
  B''' ≠ D' ∧ C''' ≠ D' ∧
  ¬ Col A' B''' C''' ∧
  Bet B''' D' C''' ∧
  ¬ Col B''' C''' T' ∧
  Col B''' C''' Y' ∧
  B''' C''' Perp T' Y' ∧ B''' ≠ Y')
  →
  (∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y))
  by auto
moreover
{
  fix B''' C'''
  assume A2: A' ≠ C''' ∧ B''' ≠ C''' ∧ B''' ≠ D' ∧
    C''' ≠ D' ∧ ¬ Col A' B''' C''' ∧
    Bet B''' D' C''' ∧ ¬ Col B''' C''' T' ∧
    Col B''' C''' Y' ∧ B''' C''' Perp T' Y' ∧
    B''' = Y'
  have ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)
  proof -
    {
      assume C''' = Y'
      hence ∃ x y. (Bet A' B''' x ∧ Bet A' C''' y ∧ Bet x T' y)

```

```

    using A2 by fastforce
  }
  moreover
  {
    assume  $C''' \neq Y'$ 
    have  $\exists x y. (Bet\ A'\ B'''\ x \wedge Bet\ A'\ C''' y \wedge Bet\ x\ T'\ y)$ 
    proof -
      have  $A' \neq C'''$ 
        using A2 by blast
      moreover
      have  $A' \neq B'''$ 
        using A2 not-col-distincts by blast
      moreover
      have  $C''' \neq B'''$ 
        using A2 by blast
      moreover
      have  $C''' \neq D'$ 
        using A2 by blast
      moreover
      have  $B''' \neq D'$ 
        using A2 by blast
      moreover
      have  $\neg Col\ A'\ C''' B'''$ 
        using A2 Col-cases by blast
      moreover
      have  $Bet\ C''' D' B'''$ 
        using A2 between-symmetry by blast
      moreover
      have  $\neg Col\ C''' B''' T'$ 
        using A2 Col-cases by blast
      moreover
      have  $Col\ C''' B''' Y'$ 
        using A2 Col-cases by blast
      moreover
      have  $C''' B''' Perp\ T' Y'$ 
        using A2 by (metis col-trivial-2 perp-col2)
      moreover
      have  $C''' \neq Y'$ 
        using A2 by blast
      ultimately
      have  $(\exists x y. (Bet\ A'\ C''' x \wedge Bet\ A'\ B''' y \wedge Bet\ x\ T' y))$ 
        using Haux by blast
      thus ?thesis
        using Bet-cases by blast
    qed
  }
  ultimately
  show ?thesis
    by auto
  qed
}
ultimately
have  $\exists x y. (Bet\ A'\ B'' x \wedge Bet\ A'\ C'' y \wedge Bet\ x\ T' y)$ 
  using H2 by blast
}
hence  $\exists x y. (Bet\ A'\ B' x \wedge Bet\ A'\ C' y \wedge Bet\ x\ T' y)$ 
  using H1  $\langle Col\ B'\ C'\ Y' \wedge B'\ C'\ Perp\ T' Y' \rangle$  by presburger
}
hence  $\forall A\ B\ C\ D\ T. Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D \longrightarrow$ 
   $(\exists x y. (Bet\ A\ B\ x \wedge Bet\ A\ C\ y \wedge Bet\ x\ T\ y))$ 
  using tarski-s-euclid-remove-degenerated-cases by blast
thus ?thesis
  using  $\langle Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D \rangle$  by blast
qed
}
thus ?thesis

```

using *tarski-s-parallel-postulate-def* by *blast*
qed

lemma *thales-converse-postulate--thales-existence*:
 assumes *thales-converse-postulate*
 shows *existential-thales-postulate*
 proof –
 obtain *A B C* where $\neg (Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$
 using *lower-dim* by *blast*
 hence $\neg Col\ C\ A\ B$
 using *Col-def* by *auto*
 have *Col C A C*
 by (*simp add: col-trivial-3*)
 then obtain *B'* where $C\ A\ Perp\ B'\ C \wedge C\ A\ OS\ B\ B'$
 using *l10-15* $\langle \neg Col\ C\ A\ B \rangle$ by *blast*
 have $C\ A\ Perp\ B'\ C$
 by (*simp add: $\langle C\ A\ Perp\ B'\ C \wedge C\ A\ OS\ B\ B' \rangle$*)
 have $C\ A\ OS\ B\ B'$
 by (*simp add: $\langle C\ A\ Perp\ B'\ C \wedge C\ A\ OS\ B\ B' \rangle$*)
 have $\neg Col\ C\ A\ B'$
 by (*simp add: $\langle C\ A\ Perp\ B'\ C \wedge C\ A\ OS\ B\ B' \rangle$ perp-not-col*)
 hence $\neg Col\ A\ B'\ C$
 using *Col-cases* by *blast*
 moreover
 have *Per A C B'*
 by (*simp add: $\langle C\ A\ Perp\ B'\ C \rangle$ perp-per-1*)
 moreover
 obtain *M* where *M* *Midpoint A B'*
 using *midpoint-existence* by *blast*
 moreover
 have *Cong M A M C*
 using *assms thales-converse-postulate-def calculation(2) calculation(3)* by *blast*
 ultimately
 show *?thesis*
 using *existential-thales-postulate-def* by *blast*
 qed

lemma *thales-converse-postulate--weak-triangle-circumscription-principle*:
 assumes *thales-converse-postulate*
 shows *weak-triangle-circumscription-principle*
 proof –
 {
 fix *A B C A1 A2 B1 B2*
 assume 1: $\neg Col\ A\ B\ C$ and
 2: *Per A C B* and
 3: *A1 A2 PerpBisect B C* and
 4: *B1 B2 PerpBisect A C* and
 5: *Coplanar A B C A1* and
 6: *Coplanar A B C A2* and
 7: *Coplanar A B C B1* and
 8: *Coplanar A B C B2*
 obtain *M* where *M* *Midpoint A B*
 using *midpoint-existence* by *blast*
 hence *Cong M A M C*
 using *thales-converse-postulate-def 2 assms* by *blast*
 have *Bet A M B*
 by (*simp add: $\langle M\ Midpoint\ A\ B \rangle$ midpoint-bet*)
 have $A \neq B$
 using 1 *col-trivial-1* by *auto*
 have *Col A1 A2 M*
 proof –

```

have Coplanar A1 M B C
proof -
  have Coplanar A1 C A B
    using 5 ncoplanar-perm-17 by blast
  moreover
  have Col A B M
    using ⟨Bet A M B⟩ bet-col not-col-permutation-5 by blast
  moreover
  have Col A B B
    using col-trivial-2 by blast
  ultimately
  show ?thesis
    by (metis ⟨A ≠ B⟩ col-cop--cop col-permutation-4
      coplanar-perm-1 coplanar-perm-5)
qed
moreover
have Coplanar A2 M B C
proof -
  have Coplanar A2 C A B
    using 6 ncoplanar-perm-17 by blast
  moreover
  have Col A B M
    using ⟨Bet A M B⟩ bet-col not-col-permutation-5 by blast
  moreover
  have Col A B B
    using col-trivial-2 by blast
  ultimately
  show ?thesis
    by (metis ⟨A ≠ B⟩ col-cop--cop col-permutation-4
      coplanar-perm-1 coplanar-perm-5)
qed
moreover
have Cong M B M C
  by (meson ⟨Cong M A M C⟩ ⟨M Midpoint A B⟩
    cong-inner-transitivity cong-left-commutativity midpoint-cong)
ultimately
show ?thesis
  using 3 cong-cop2-perp-bisect-col by blast
qed
moreover
have Col B1 B2 M
proof -
  have Coplanar B1 M A C
  proof -
    have Coplanar B1 C A B
      using 7 ncoplanar-perm-17 by blast
    moreover
    have Col A B M
      using ⟨Bet A M B⟩ bet-col not-col-permutation-5 by blast
    moreover
    have Col A B A
      by (simp add: col-trivial-3)
    ultimately
    show ?thesis
      by (metis ⟨A ≠ B⟩ col-cop--cop coplanar-perm-5)
  qed
  moreover
  have Coplanar B2 M A C
  proof -
    have Coplanar B2 C A B
      using 8 ncoplanar-perm-17 by blast
    moreover
    have Col A B M
      using ⟨Bet A M B⟩ bet-col not-col-permutation-5 by blast
    moreover
    have Col A B A

```

```

      by (simp add: col-trivial-3)
    ultimately
    show ?thesis
      by (metis ‹A ≠ B› col-cop--cop coplanar-perm-5)
  qed
  ultimately
  show ?thesis
    using ‹Cong M A M C› 4 cong-cop2-perp-bisect-col by blast
  qed
  ultimately
  have ∃ I. Col A1 A2 I ∧ Col B1 B2 I
    by blast
}
thus ?thesis
  using weak-triangle-circumscription-principle-def by blast
qed

```

```

lemma thales-existence--rah:
  assumes existential-thales-postulate
  shows postulate-of-right-saccheri-quadrilaterals
proof -
{
  fix A B C D
  assume Saccheri A B C D
  hence Per A B C
    using t22-17--rah assms existential-thales-postulate-def
    existential-playfair--rah-1
    postulate-of-right-saccheri-quadrilaterals-def by fastforce
}
thus ?thesis
  using postulate-of-right-saccheri-quadrilaterals-def by blast
qed

```

```

lemma thales-postulate--thales-converse-postulate:
  assumes thales-postulate
  shows thales-converse-postulate
proof -
{
  fix A B C M
  assume M Midpoint A B and Per A C B
  hence Bet A M B
    by (simp add: midpoint-bet)
  have Cong M A M C
  proof (cases Col A B C)
    case True
    thus ?thesis
      using ‹M Midpoint A B› ‹Per A C B› cong-left-commutativity
      cong-reflexivity l8-9 midpoint-cong not-col-permutation-5 by blast
  next
    case False
    hence ¬ Col A B C
      by simp
    have A ≠ B
      using False col-trivial-1 by auto
    have C ≠ B
      using False col-trivial-2 by blast
    have A ≠ M
      using False ‹M Midpoint A B› is-midpoint-id not-col-distincts by blast
    have A ≠ C
      using False col-trivial-3 by blast
    have M ≠ C
      using Col-def False ‹M Midpoint A B› l7-2 midpoint-bet by blast
  }
}

```

```

then obtain  $C'$  where  $M \text{ Out } C C'$  and  $\text{Cong } M C' M A$ 
  using segment-construction-3  $\langle A \neq M \rangle$  by metis
hence  $\text{Cong } M A M C'$ 
  by (meson cong-symmetry)
show ?thesis
proof (cases  $C = C'$ )
  case True
    thus ?thesis
      using  $\langle \text{Cong } M A M C' \rangle$  by auto
  next
  case False
    hence  $C \neq C'$  by simp
    {
      assume  $\neg \text{Cong } M A M C$ 
      have  $M \neq C'$ 
        using  $\langle A \neq M \rangle \langle \text{Cong } M C' M A \rangle$  cong-reverse-identity by blast
      {
        assume  $\text{Col } A B C'$ 
        hence  $\neg \text{Per } A C B$ 
          by (metis Col-cases midpoint-col
             $\langle M \text{ Midpoint } A B \rangle \langle M \text{ Out } C C' \rangle \langle M \neq C' \rangle$ 
             $\langle \neg \text{Col } A B C \rangle$  colx out-col)
        hence False
          by (simp add:  $\langle \text{Per } A C B \rangle$ )
      }
      hence  $\neg \text{Col } A B C'$ 
        by auto
    }
    {
      assume  $\text{Col } A C C'$ 
      hence  $\text{Col } A C B$ 
        by (meson False  $\langle A \neq M \rangle \langle \text{Bet } A M B \rangle \langle M \text{ Out } C C' \rangle$ 
          bet-col l6-16-1 not-col-permutation-3
          not-col-permutation-5 out-col)
      hence False
        using  $\langle \neg \text{Col } A B C \rangle$  not-col-permutation-5 by blast
    }
    hence  $\neg \text{Col } A C C'$ 
      by auto
    {
      assume  $\text{Col } B C C'$ 
      hence  $\neg \text{Per } A C B$ 
        by (metis Col-cases False midpoint-col
           $\langle M \text{ Midpoint } A B \rangle \langle M \text{ Out } C C' \rangle \langle \neg \text{Col } A B C \rangle$ 
          col-trivial-2 l6-21 midpoint-distinct-2
          out-col)
      hence False
        by (simp add:  $\langle \text{Per } A C B \rangle$ )
    }
    hence  $\neg \text{Col } B C C'$ 
      by auto
    have  $B \neq C'$ 
      using  $\langle \neg \text{Col } A B C' \rangle$  col-trivial-2 by auto
    have  $A \neq C'$ 
      using  $\langle \neg \text{Col } A B C' \rangle$  col-trivial-3 by blast
    have  $\text{Per } A C' B$ 
      using  $\langle \text{Cong } M A M C' \rangle \langle M \text{ Midpoint } A B \rangle$ 
        assms thales-postulate-def by blast
    hence  $A C B \text{ Cong } A C' B$ 
      using  $\langle \text{Per } A C B \rangle \langle A \neq C' \rangle \langle A \neq C \rangle$ 
         $\langle B \neq C' \rangle \langle C \neq B \rangle$  l11-16 by force
    have  $\text{Col } A B M$ 
      using Col-perm  $\langle \text{Bet } A M B \rangle$  bet-col by blast
    hence  $A B \text{ OS } C C'$ 
      using  $\langle \neg \text{Col } A B C \rangle \langle M \text{ Out } C C' \rangle$  l9-19-R2 by blast
    {
      assume  $\text{Bet } M C C'$ 

```



```

have A C' B LtA A C B
proof -
  have C' A OS B C
  proof -
    have C' A OS B M
    by (metis NCol-cases out-one-side
      ⟨A ≠ B⟩ ⟨A ≠ M⟩ ⟨Bet A M B⟩ ⟨¬ Col A B C'⟩
      bet2--out invert-one-side l5-1)
    moreover
    have C' A OS M C
    by (metis False bet-out-1 out-one-side
      ⟨Bet M C C'⟩ ⟨¬ Col A C C'⟩
      not-col-permutation-2 one-side-symmetry)
    ultimately
    show ?thesis
    using one-side-transitivity by blast
  qed
  moreover
  have A B OS C' C
  by (simp add: ⟨A B OS C C'⟩ one-side-symmetry)
  moreover
  have C' B OS A C
  proof -
    have C' B OS A M
    by (meson out-one-side ⟨A ≠ B⟩ ⟨M Midpoint A B⟩
      ⟨¬ Col A B C'⟩ invert-one-side midpoint-out-1
      not-col-permutation-1)
    moreover
    have C' B OS M C
    by (meson False ⟨Bet M C C'⟩ ⟨¬ Col B C C'⟩
      bet-out-1 l6-6 not-col-permutation-2
      out-one-side)
    ultimately
    show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
  by (simp add: os3--lta)
}
moreover
{
  assume Bet M C' C
  have A C B LtA A C' B
  proof -
    have C A OS B C'
    proof -
      have C A OS B M
      by (metis ⟨A ≠ M⟩ ⟨Bet A M B⟩ ⟨¬ Col A B C⟩
        bet2--out col-trivial-3 invert-one-side l5-1
        not-col-permutation-5 out-one-side-1)
      moreover
      have C A OS M C'
      by (metis ⟨Bet M C C' ⟹ False⟩ ⟨M Out C C'⟩
        calculation col-trivial-3 l9-19-R2
        one-side-not-col124 or-bet-out out-col)
      ultimately
      show ?thesis
      using one-side-transitivity by blast
    qed
    moreover
    have C B OS A C'

```

```

proof –
  have  $C B OS A M$ 
    by (meson out-one-side  $\langle A \neq B \rangle \langle M \text{ Midpoint } A B \rangle$ 
       $\langle \neg Col A B C \rangle$  invert-one-side midpoint-out-1
      not-col-permutation-1)
  moreover
  have  $C B OS M C'$ 
    by (meson  $\langle Bet M C C' \implies False \rangle \langle M Out C C' \rangle$ 
      calculation col-trivial-3 l6-4-2 l9-19-R2
      one-side-not-col124 out-col)
  ultimately
  show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
    by (simp add:  $\langle A B OS C C' \rangle$  os3--lta)
  qed
  hence False
    using  $\langle A C B CongA A C' B \rangle$ 
    by (meson lta-not-conga)
}
ultimately
have False
  using Out-def  $\langle M Out C C' \rangle$  by presburger
}
thus ?thesis
  by auto
qed
qed
}
thus ?thesis
  using thales-converse-postulate-def by blast
qed

lemma triangle--existential-triangle:
  assumes triangle-postulate
  shows postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights
proof –
  obtain  $A B C$  where  $\neg (Bet A B C \vee Bet B C A \vee Bet C A B)$ 
    using lower-dim-ex by blast
  hence  $\neg Col A B C$ 
    by (simp add: Col-def)
  moreover
  have  $A \neq B$ 
    using calculation col-trivial-1 by blast
  have  $B \neq C$ 
    using calculation col-trivial-2 by blast
  have  $A \neq C$ 
    using calculation col-trivial-3 by blast
  obtain  $D E F$  where  $A B C TriSumA D E F$ 
    using ex-trisuma  $\langle A \neq B \rangle \langle B \neq C \rangle \langle A \neq C \rangle$  by blast
  have  $Bet D E F$ 
    using triangle-postulate-def assms  $\langle A B C TriSumA D E F \rangle$  by blast
  hence  $\exists A B C D E F. (\neg Col A B C \wedge A B C TriSumA D E F \wedge Bet D E F)$ 
    using  $\langle A B C TriSumA D E F \rangle \langle \neg Col A B C \rangle$  by blast
  thus ?thesis
    using postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-def
    by presburger
qed

lemma legendre-aux-tr:
  assumes greenberg-s-axiom and
    triangle-postulate
  shows  $(\forall A B C P Q. \neg($ 

```

```

  Q A Perp P Q ∧ P B Perp P Q ∧ Q A ParStrict P B ∧
  Q A ParStrict P C ∧ P B OS Q C ∧ P Q OS C A ∧ P Q OS C B))
proof -
{
  fix A B C P Q
  assume Q A Perp P Q and
    P B Perp P Q and
    Q A ParStrict P B and
    Q A ParStrict P C and
    P B OS Q C and
    P Q OS C A and
    P Q OS C B
  obtain B' where P Midpoint B B'
    using symmetric-point-construction by blast
  have C InAngle Q P B
    by (meson ⟨P B OS Q C⟩ ⟨P Q OS C B⟩ col123--nos l11-24
      l9-2 l9-9-bis os-ts--inangle two-sides-cases)
  have ¬ Col P B C
    using ⟨P B OS Q C⟩ col124--nos by blast
  have ¬ Col P Q C
    by (meson ⟨P Q OS C A⟩ col123--nos)
  have ¬ Col P Q A
    using ⟨P Q OS C A⟩ col124--nos by blast
  have B P C LtA B P Q
  proof -
    have ¬ Col C P Q
      by (simp add: ⟨¬ Col P Q C⟩ not-col-permutation-2)
    moreover
    have C InAngle B P Q
      by (simp add: ⟨C InAngle Q P B⟩ l11-24)
    ultimately
    show ?thesis
      by (simp add: inangle--lta)
  qed
  have Per B P Q
    by (simp add: ⟨P B Perp P Q⟩ perp-per-2)
  hence Acute B P C
    using ⟨B P C LtA B P Q⟩ Acute-def by blast
  have P ≠ Q
    using ⟨¬ Col P Q C⟩ col-trivial-1 by auto
  have A ≠ Q
    using ⟨¬ Col P Q A⟩ col-trivial-2 by fastforce
  have P ≠ A
    using ⟨¬ Col P Q A⟩ col-trivial-3 by auto
  have P ≠ C
    using ⟨¬ Col P Q C⟩ col-trivial-3 by blast
  have P ≠ B
    using ⟨¬ Col P B C⟩ col-trivial-1 by auto
  have B ≠ B'
    using ⟨P Midpoint B B'⟩ ⟨P ≠ B⟩ l8-20-2 by blast
  hence P ≠ B'
    using ⟨P Midpoint B B'⟩ midpoint-not-midpoint by blast
  have ∃ R. (P R Q LtA B P C ∧ Q Out R A)
  proof -
    have ¬ Col B P C
      by (simp add: ⟨¬ Col P B C⟩ not-col-permutation-4)
    moreover
    have Q ≠ A
      using ⟨A ≠ Q⟩ by auto
    moreover
    have Per P Q A
      by (meson ⟨Q A Perp P Q⟩ l8-2 perp-per-1)
    ultimately
    show ?thesis
      using ⟨Acute B P C⟩ greenberg-s-axiom-def assms by blast
  qed
}

```

```

then obtain  $R$  where  $P R Q$  LtA  $B P C$  and  $Q$  Out  $R A$ 
  by blast
have  $R \neq Q$ 
  using  $\langle Q$  Out  $R A \rangle$  l6-3-1 by auto
have  $R \neq P$ 
  using  $\langle P R Q$  LtA  $B P C \rangle$  lta-distincts by blast
have  $P Q$  OS  $R A$ 
  by (meson Out-cases  $\langle Q$  Out  $R A \rangle$   $\langle \neg$  Col  $P Q A \rangle$  col-trivial-2
    one-side-symmetry out-one-side-1)
have  $\neg$  Col  $Q P R \vee \neg$  Col  $Q P A$ 
  using  $\langle \neg$  Col  $P Q A \rangle$  col-permutation-4 by blast
hence  $P C$  OS  $Q R$ 
  by (meson  $\langle Q$  Out  $R A \rangle$   $\langle Q A$  ParStrict  $P C \rangle$  col-permutation-5
    out-col par-strict-one-side par-strict-symmetry)
obtain  $D E F$  where  $B' P R P R Q$  SumA  $D E F$ 
  using ex-suma  $\langle P \neq B' \rangle$   $\langle R \neq P \rangle$   $\langle R \neq Q \rangle$  by presburger
have  $R Q P Q P R$  SumA  $B' P R$ 
proof -
  have  $B' P Q Q P R$  SumA  $B' P R$ 
  proof -
    have  $Q P R$  CongA  $Q P R$ 
      using  $\langle P \neq Q \rangle$   $\langle R \neq P \rangle$  conga-refl by auto
    moreover
    have  $P Q$  TS  $R B'$ 
    proof -
      have  $P Q$  TS  $B B'$ 
      proof -
        have  $\neg$  Col  $P Q B$ 
          using  $\langle P Q$  OS  $C B \rangle$  col124--nos by blast
        moreover
        have Bet  $B P B'$ 
          using  $\langle P$  Midpoint  $B B' \rangle$  midpoint-bet by blast
        ultimately
        show ?thesis
          by (simp add:  $\langle P \neq B' \rangle$  bet--ts)
      qed
    moreover
    have  $P Q$  OS  $B R$ 
      by (meson  $\langle P Q$  OS  $C A \rangle$   $\langle P Q$  OS  $C B \rangle$   $\langle P Q$  OS  $R A \rangle$ 
        one-side-symmetry one-side-transitivity)
    ultimately
    show ?thesis
      using l9-8-2 by blast
  qed
  hence  $\neg$   $P Q$  OS  $B' R$ 
    by (meson l9-2 l9-9-bis)
  moreover
  have Coplanar  $B' P Q R$ 
    by (meson  $\langle P Q$  TS  $R B' \rangle$  coplanar-perm-18 ts--coplanar)
  moreover
  have  $B' P R$  CongA  $B' P R$ 
    using  $\langle P \neq B' \rangle$   $\langle R \neq P \rangle$  conga-refl by auto
  ultimately
  show ?thesis
    using SumA-def by blast
qed
moreover
have  $B' P Q$  CongA  $R Q P$ 
proof -
  have Per  $B' P Q$ 
    by (meson  $\langle P$  Midpoint  $B B' \rangle$   $\langle Per B P Q \rangle$  l8-2 l8-4)
  moreover
  have Per  $P Q R$ 
  proof -
    have Per  $P Q A$ 
      using  $\langle Q A$  Perp  $P Q \rangle$  l8-2 perp-per-1 by blast

```

```

moreover
have  $Col\ Q\ A\ R$ 
  by (metis Col-def Out-def  $\langle Q\ Out\ R\ A \rangle$  col-permutation-3)
ultimately
show ?thesis
  by (metis  $\langle A \neq Q \rangle$  per-col)
qed
hence  $Per\ R\ Q\ P$ 
  by (meson l8-2)
ultimately
show ?thesis
  using  $\langle P \neq B' \rangle \langle P \neq Q \rangle \langle R \neq Q \rangle$  l11-16 by blast
qed
moreover
have  $Q\ P\ R\ CongA\ Q\ P\ R$ 
  using  $\langle P \neq Q \rangle \langle R \neq P \rangle$  conga-refl by force
moreover
have  $B'\ P\ R\ CongA\ B'\ P\ R$ 
  using  $\langle P \neq B' \rangle \langle R \neq P \rangle$  conga-refl by auto
ultimately
show ?thesis
  by (meson conga3-suma--suma)
qed
hence  $R\ Q\ P\ TriSumA\ D\ E\ F$ 
  using TriSumA-def  $\langle B'\ P\ R\ P\ R\ Q\ SumA\ D\ E\ F \rangle$  by blast
obtain  $I\ J\ K$  where  $B'\ P\ R\ C\ P\ B\ SumA\ I\ J\ K$ 
  using ex-suma  $\langle P \neq B' \rangle \langle R \neq P \rangle \langle P \neq C \rangle \langle P \neq B \rangle$  by presburger
have  $\neg\ Col\ R\ P\ B'$ 
proof –
  have  $Bet\ B\ P\ B'$ 
    using  $\langle P\ Midpoint\ B\ B' \rangle$  midpoint-bet by blast
  hence  $Col\ P\ B\ B'$ 
    using bet-col not-col-permutation-4 by blast
  hence  $Q\ A\ ParStrict\ P\ B'$ 
    using  $\langle Q\ A\ ParStrict\ P\ B \rangle \langle P \neq B' \rangle$  par-strict-col-par-strict by blast
moreover
have  $Col\ R\ Q\ A$ 
  by (metis Col-def Out-def  $\langle Q\ Out\ R\ A \rangle$  col-permutation-3)
ultimately
show ?thesis
  using par-not-col by blast
qed
have  $B'\ P\ R\ R\ P\ B\ SumA\ B'\ P\ B$ 
  by (metis Bet-cases  $\langle P\ Midpoint\ B\ B' \rangle \langle P \neq B' \rangle \langle P \neq B \rangle$ 
     $\langle R \neq P \rangle$  bet--suma midpoint-bet)
have  $SAMS\ B'\ P\ R\ R\ P\ B$ 
  by (meson Mid-cases  $\langle B'\ P\ R\ R\ P\ B\ SumA\ B'\ P\ B \rangle$ 
     $\langle P\ Midpoint\ B\ B' \rangle$  bet-suma--sams midpoint-bet)
have  $C\ P\ B\ LeA\ R\ P\ B$ 
proof –
  have  $P\ C\ TS\ Q\ B$ 
    by (simp add:  $\langle P\ B\ OS\ Q\ C \rangle \langle P\ Q\ OS\ C\ B \rangle$  l9-31 one-side-symmetry)
  hence  $P\ C\ TS\ B\ R$ 
    using  $\langle P\ C\ OS\ Q\ R \rangle$  l9-2 l9-8-2 by blast
moreover
have  $P\ B\ OS\ R\ Q$ 
  by (metis Out-def Par-strict-cases  $\langle Q\ A\ ParStrict\ P\ B \rangle$ 
     $\langle Q\ Out\ R\ A \rangle$  bet-col l9-17 one-side-symmetry
    par-strict-one-side pars--os3412)
hence  $P\ B\ OS\ R\ C$ 
  using  $\langle P\ B\ OS\ Q\ C \rangle$  one-side-transitivity by blast
ultimately
show ?thesis
  by (simp add: inangle--lea lea-comm os-ts--inangle)
qed
have  $D\ E\ F\ LtA\ B'\ P\ B$ 

```

```

proof –
  have  $D E F LtA I J K$ 
proof –
  have  $B' P R LeA B' P R$ 
    using  $\langle P \neq B' \rangle \langle R \neq P \rangle$  lea-refl by auto
  moreover
  have  $P R Q LtA C P B$ 
    by (simp add:  $\langle P R Q LtA B P C \rangle$  lta-right-comm)
  moreover
  have  $SAMS B' P R C P B$ 
    by (meson  $\langle C P B LeA R P B \rangle \langle SAMS B' P R R P B \rangle$ 
      calculation(1) sams-lea2--sams)
  ultimately
  show ?thesis
    by (meson  $\langle B' P R P R Q SumA D E F \rangle$ 
       $\langle B' P R C P B SumA I J K \rangle$  sams-lea-lta456-suma2--lta)
qed
moreover
have  $I J K LeA B' P B$ 
  using  $\langle B' P R C P B SumA I J K \rangle$ 
     $\langle B' P R R P B SumA B' P B \rangle \langle C P B LeA R P B \rangle$ 
     $\langle SAMS B' P R R P B \rangle$  sams-lea456-suma2--lea by blast
ultimately
show ?thesis
  using lea456789-lta--lta by blast
qed
have Bet D E F
  using  $\langle R Q P TriSumA D E F \rangle$  triangle-postulate-def assms(2) by blast
hence  $D E F CongA B' P B$ 
  by (metis  $\langle D E F LtA B' P B \rangle \langle P Midpoint B B' \rangle$ 
    conga-line conga-right-comm lta-distincts
    midpoint-bet)
hence False
  using  $\langle D E F LtA B' P B \rangle$  lta-not-conga by blast
}
thus ?thesis by blast
qed

```

```

lemma legendre-aux1-tr:
  assumes greenberg-s-axiom and
    triangle-postulate
  shows  $\forall A1 A2 B1 B2 C1 C2 P.$ 
    ( $P Perp2 A1 A2 B1 B2 \wedge \neg Col A1 A2 P \wedge$ 
       $Col P B1 B2 \wedge Coplanar A1 A2 B1 B2 \wedge$ 
       $A1 A2 Par C1 C2 \wedge Col P C1 C2 \wedge$ 
       $\neg B1 B2 TS A1 C1$ )
     $\longrightarrow$ 
       $Col C1 B1 B2$ 
proof –
  {
    fix  $A1 A2 B1 B2 C1 C2 P$ 
    assume  $P Perp2 A1 A2 B1 B2$  and
       $\neg Col A1 A2 P$  and
       $Col P B1 B2$  and
       $Coplanar A1 A2 B1 B2$  and
       $A1 A2 Par C1 C2$  and
       $Col P C1 C2$  and
       $\neg B1 B2 TS A1 C1$ 
    have  $Col B1 B2 P$ 
      using Col-cases  $\langle Col P B1 B2 \rangle$  by blast
    hence  $A1 A2 ParStrict B1 B2$ 
      using  $\langle P Perp2 A1 A2 B1 B2 \rangle \langle Coplanar A1 A2 B1 B2 \rangle$ 
       $\langle \neg Col A1 A2 P \rangle \langle Coplanar A1 A2 B1 B2 \rangle \langle P Perp2 A1 A2 B1 B2 \rangle$ 

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    col-cop-perp2--pars-bis by blast
have A1 A2 ParStrict C1 C2
  using Col-cases ⟨A1 A2 Par C1 C2⟩ ⟨Col P C1 C2⟩
    ⟨¬ Col A1 A2 P⟩ par-not-col-strict by blast
{
  assume ¬ Col C1 B1 B2
  have P ≠ C1
    using ⟨Col P B1 B2⟩ ⟨¬ Col C1 B1 B2⟩ by auto
  obtain P1 P2 where Col P P1 P2 and
    P1 P2 Perp A1 A2 and
    P1 P2 Perp B1 B2
    using ⟨P Perp2 A1 A2 B1 B2⟩ Perp2-def by blast
  obtain Q where Col Q P1 P2 and Col Q A1 A2
    using ⟨P1 P2 Perp A1 A2⟩ col-permutation-2
      perp-inter-perp-in-n by blast
  have P ≠ Q
    using NCol-perm ⟨Col Q A1 A2⟩ ⟨¬ Col A1 A2 P⟩ by blast
  have A1 A2 Perp P Q
    by (meson ⟨Col P P1 P2⟩ ⟨Col Q P1 P2⟩ ⟨P ≠ Q⟩
      ⟨P1 P2 Perp A1 A2⟩ col-permutation-1 perp-col0)
  have B1 B2 Perp P Q
    by (meson ⟨Col P P1 P2⟩ ⟨Col Q P1 P2⟩ ⟨P ≠ Q⟩
      ⟨P1 P2 Perp B1 B2⟩ col-permutation-3 perp-col0
      perp-left-comm)
  have P ≠ Q
    using ⟨P ≠ Q⟩ by auto
  have C1 ≠ B1
    using ⟨¬ Col C1 B1 B2⟩ col-trivial-1 by blast
  have B1 ≠ B2
    using ⟨¬ Col C1 B1 B2⟩ not-col-distincts by blast
  have C1 ≠ B2
    using ⟨¬ Col C1 B1 B2⟩ not-col-distincts by blast
  have A1 ≠ A2
    using ⟨¬ Col A1 A2 P⟩ col-trivial-1 by auto
  have P ≠ A2
    using ⟨¬ Col A1 A2 P⟩ col-trivial-2 by blast
  have P ≠ A1
    using ⟨¬ Col A1 A2 P⟩ col-trivial-3 by blast
  have A1 ≠ B1
    using ⟨A1 A2 ParStrict B1 B2⟩ not-par-strict-id by blast
  have A1 ≠ B2
    using ⟨A1 A2 ParStrict B1 B2⟩ col-trivial-2
      par-strict-not-col-3 by blast
  have A2 ≠ B1
    using ⟨A1 A2 ParStrict B1 B2⟩ not-col-distincts
      par-strict-not-col-2 by blast
  have A2 ≠ B2
    using ⟨A1 A2 ParStrict B1 B2⟩ not-col-distincts
      par-strict-not-col-4 by blast
  have A1 ≠ C1
    using ⟨A1 A2 ParStrict C1 C2⟩ not-par-strict-id by blast
  have A1 ≠ C2
    using ⟨A1 A2 ParStrict C1 C2⟩ col-trivial-3
      par-strict-not-col-4 by blast
  have A2 ≠ C1
    by (metis ⟨A1 A2 ParStrict C1 C2⟩ ⟨Col P C1 C2⟩
      col3 col-trivial-2 par-strict-not-col-2)
  have A2 ≠ C2
    using ⟨A1 A2 ParStrict C1 C2⟩ col-trivial-2
      par-strict-not-col-4 by blast
  have C2 ≠ C1
    using ⟨A1 A2 Par C1 C2⟩ par-neq2 by blast
  have B1 B2 OS Q C1
  proof –
    have B1 B2 OS Q A1
    proof (cases A1 = Q)

```

```

case True
thus ?thesis
  using ⟨A1 A2 ParStrict B1 B2⟩ not-col-distincts
    par-strict-one-side par-strict-symmetry by blast
next
case False
have B1 B2 ParStrict A1 A2
  using ⟨A1 A2 ParStrict B1 B2⟩ par-strict-symmetry by blast
moreover
have Col A1 A2 Q
  using ⟨Col Q A1 A2⟩ not-col-permutation-2 by blast
ultimately
show ?thesis
  using one-side-symmetry par-strict-one-side by blast
qed
moreover
have B1 B2 OS A1 C1
proof -
  have Coplanar B1 B2 A1 C1
  proof -
    have ¬ Col A1 A2 P
    by (simp add: ⟨¬ Col A1 A2 P⟩)
    moreover
    have Coplanar A1 A2 P B1
    by (meson ⟨B1 ≠ B2⟩ ⟨Col B1 B2 P⟩
      ⟨Coplanar A1 A2 B1 B2⟩ col2-cop--cop col-trivial-3)
    moreover
    have Coplanar A1 A2 P B2
    using ⟨B1 ≠ B2⟩ ⟨Col B1 B2 P⟩ ⟨Coplanar A1 A2 B1 B2⟩
      col2-cop--cop col-trivial-2 by blast
    moreover
    have Coplanar A1 A2 P A1
    using ncop-distincts by blast
    moreover
    have Coplanar A1 A2 P C1
    using ⟨A1 A2 Par C1 C2⟩ ⟨C2 ≠ C1⟩ ⟨Col P C1 C2⟩
      col-cop--cop coplanar-perm-1 not-col-permutation-2
      par--coplanar by blast
    ultimately
    show ?thesis
    by (meson coplanar-pseudo-trans)
  qed
  moreover
  have ¬ Col A1 B1 B2
  using ⟨A1 A2 ParStrict B1 B2⟩ not-col-permutation-2
    par-strict-not-col-3 by blast
  ultimately
  show ?thesis
  using ⟨¬ Col C1 B1 B2⟩ ⟨¬ B1 B2 TS A1 C1⟩ cop-nts--os by blast
qed
ultimately
show ?thesis
  using one-side-transitivity by blast
qed
have ¬ Col Q C1 P
  by (metis Col-cases ⟨A1 A2 ParStrict C1 C2⟩
    ⟨Col P C1 C2⟩ ⟨Col Q A1 A2⟩ ⟨P ≠ C1⟩
    col-transitivity-2 par-not-col)
have ¬ Col B1 B2 Q
  using ⟨B1 B2 OS Q C1⟩ col123--nos by blast
have ∃ B3. Col B1 B2 B3 ∧ P Q OS C1 B3
proof (cases Col P Q B1)
case True
hence P = B1
  by (meson ⟨Col P B1 B2⟩ ⟨¬ Col B1 B2 Q⟩
    col-transitivity-2 not-col-permutation-5)

```



```

have  $\exists B3. \text{Col } B2 P B3 \wedge P Q OS C1 B3$ 
proof -
  have  $B2 \neq P$ 
    using  $\langle B1 \neq B2 \rangle \langle P = B1 \rangle$  by blast
  moreover
  have  $\text{Col } P Q P$ 
    using  $\text{True } \langle P = B1 \rangle$  by auto
  moreover
  have  $\text{Col } B2 P P$ 
    using not-col-distincts by auto
  moreover
  have  $\neg \text{Col } P Q B2$ 
    using Col-cases  $\langle P = B1 \rangle \langle \neg \text{Col } B1 B2 Q \rangle$  by blast
  moreover
  have  $\neg \text{Col } P Q C1$ 
    using Col-cases  $\langle \neg \text{Col } Q C1 P \rangle$  by blast
  moreover
  have Coplanar  $P Q B2 C1$ 
    using  $\langle B1 B2 OS Q C1 \rangle \langle P = B1 \rangle$  coplanar-perm-2
      os--coplanar by blast
  ultimately
  show ?thesis
    using cop-not-par-same-side by blast
qed
then obtain  $B3$  where  $\text{Col } B2 P B3$  and  $P Q OS C1 B3$ 
  by blast
thus ?thesis
  using  $\langle P = B1 \rangle$  not-col-permutation-4 by blast
next
case False
have  $\text{Col } P Q P$ 
  by (meson not-col-distincts)
moreover
have Coplanar  $P Q B1 C1$ 
proof -
  have  $\neg \text{Col } B2 Q B1$ 
    using Col-cases  $\langle \neg \text{Col } B1 B2 Q \rangle$  by blast
  moreover
  have Coplanar  $B2 Q B1 P$ 
    using NCol-perm  $\langle \text{Col } P B1 B2 \rangle$  ncop--ncols by blast
  moreover
  have Coplanar  $B2 Q B1 C1$ 
    using  $\langle B1 B2 OS Q C1 \rangle$  ncoplanar-perm-12
      os--coplanar by blast
  ultimately
  show ?thesis
    by (meson coplanar-trans-1 ncoplanar-perm-8)
qed
ultimately
show ?thesis
  using  $\langle B1 \neq B2 \rangle \langle \text{Col } B1 B2 P \rangle$  False
     $\langle \neg \text{Col } Q C1 P \rangle$  cop-not-par-same-side
    Col-cases by blast
qed
then obtain  $B3$  where  $\text{Col } B1 B2 B3$  and  $P Q OS C1 B3$ 
  by blast
have  $\neg \text{Col } P Q B3$ 
  using  $\langle P Q OS C1 B3 \rangle$  one-side-not-col124 by auto
have Coplanar  $A1 A2 C1 P$ 
  using Col-cases  $\langle A1 A2 \text{Par } C1 C2 \rangle \langle C2 \neq C1 \rangle$ 
     $\langle \text{Col } P C1 C2 \rangle$  col-cop--cop par--coplanar by blast
have  $\exists A3. \text{Col } A1 A2 A3 \wedge P Q OS C1 A3$ 
proof (cases  $\text{Col } P Q A1$ )
  case True
  have  $Q = A1$ 
    by (meson True  $\langle \text{Col } Q A1 A2 \rangle \langle \neg \text{Col } A1 A2 P \rangle$ )

```

```

      col-permutation-1 col-trivial-2 colx)
have  $\exists A3. \text{Col } A2 \ Q \ A3 \wedge P \ Q \ OS \ C1 \ A3$ 
proof -
  have  $A2 \neq Q$ 
    using  $\langle A1 \neq A2 \rangle \langle Q = A1 \rangle$  by auto
  moreover
  have  $\text{Col } P \ Q \ Q$ 
    by (simp add: col-trivial-2)
  moreover
  have  $\text{Col } A2 \ Q \ Q$ 
    by (simp add: col-trivial-2)
  moreover
  have  $\neg \text{Col } P \ Q \ A2$ 
    using Col-cases  $\langle Q = A1 \rangle \langle \neg \text{Col } A1 \ A2 \ P \rangle$  by blast
  moreover
  have  $\neg \text{Col } P \ Q \ C1$ 
    using Col-cases  $\langle \neg \text{Col } Q \ C1 \ P \rangle$  by blast
  moreover
  have Coplanar  $P \ Q \ A2 \ C1$ 
    using  $\langle \text{Coplanar } A1 \ A2 \ C1 \ P \rangle \langle Q = A1 \rangle$  ncoplanar-perm-9 by blast
  ultimately
  show ?thesis
    using cop-not-par-same-side by blast
qed
then obtain  $A3$  where  $\text{Col } A2 \ Q \ A3$  and  $P \ Q \ OS \ C1 \ A3$ 
  by blast
thus ?thesis
  using  $\langle Q = A1 \rangle$  not-col-permutation-4 by blast
next
case False
have  $\text{Col } P \ Q \ Q$ 
  by (simp add: col-trivial-2)
moreover
have  $\text{Col } A1 \ A2 \ Q$ 
  using NCol-cases  $\langle \text{Col } Q \ A1 \ A2 \rangle$  by blast
moreover
have  $\neg \text{Col } P \ Q \ C1$ 
  using Col-cases  $\langle \neg \text{Col } Q \ C1 \ P \rangle$  by blast
moreover
have Coplanar  $P \ Q \ A1 \ C1$ 
  using  $\langle A1 \neq A2 \rangle \langle \text{Coplanar } A1 \ A2 \ C1 \ P \rangle$ 
    calculation(2) col-cop--cop coplanar-perm-16
    ncoplanar-perm-19 by blast
ultimately
show ?thesis
  using False  $\langle A1 \neq A2 \rangle$  cop-not-par-same-side by blast
qed
then obtain  $A3$  where  $\text{Col } A1 \ A2 \ A3$  and  $P \ Q \ OS \ C1 \ A3$ 
  by blast
have  $Q \neq B3$ 
  using  $\langle \text{Col } B1 \ B2 \ B3 \rangle \langle \neg \text{Col } B1 \ B2 \ Q \rangle$  by blast
have  $P \neq B3$ 
  using  $\langle \neg \text{Col } P \ Q \ B3 \rangle$  col-trivial-3 by auto
have  $B2 \neq Q$ 
  using  $\langle \neg \text{Col } B1 \ B2 \ Q \rangle$  col-trivial-2 by auto
have  $Q \neq B1$ 
  using  $\langle \neg \text{Col } B1 \ B2 \ Q \rangle$  col-trivial-3 by blast
have  $Q \neq C1$ 
  using  $\langle \neg \text{Col } Q \ C1 \ P \rangle$  not-col-distincts by blast
have  $P \neq A3$ 
  using  $\langle \text{Col } A1 \ A2 \ A3 \rangle \langle \neg \text{Col } A1 \ A2 \ P \rangle$  by auto
have  $Q \neq A3$ 
  using  $\langle P \ Q \ OS \ C1 \ A3 \rangle$  os-distincts by blast
have False
proof -
  have  $H1: Q \ A3 \text{ Perp } P \ Q \wedge P \ B3 \text{ Perp } P \ Q \wedge Q \ A3 \text{ ParStrict } P \ B3 \wedge$ 

```

```

      Q A3 ParStrict P C1 ∧ P B3 OS Q C1 ∧
      P Q OS C1 A3 ∧ P Q OS C1 B3
proof –
  have Col A1 A2 Q
    using Col-cases ⟨Col Q A1 A2⟩ by blast
  hence Q A3 Perp P Q
    using ⟨A1 A2 Perp P Q⟩ ⟨Q ≠ A3⟩ ⟨Col A1 A2 A3⟩ perp-col2 by blast
  moreover
  have Col B1 B2 P
    by (simp add: ⟨Col B1 B2 P⟩)
  hence P B3 Perp P Q
    using perp-col2 ⟨B1 B2 Perp P Q⟩ ⟨Col B1 B2 B3⟩ ⟨P ≠ B3⟩ by blast
  moreover
  have Q A3 ParStrict P B3
    using ⟨Q ≠ A3⟩ ⟨P ≠ B3⟩ ⟨A1 A2 ParStrict B1 B2⟩
      ⟨Col A1 A2 Q⟩ ⟨Col A1 A2 A3⟩ ⟨Col B1 B2 P⟩
      ⟨Col B1 B2 B3⟩ par-strict-col4--par-strict by blast
  moreover
  have Q A3 ParStrict P C1
proof –
  have Col C1 C2 P
    using Col-cases ⟨Col P C1 C2⟩ by auto
  moreover
  have Col C1 C2 C1
    using not-col-distincts by blast
  ultimately
  show ?thesis
    using ⟨Q ≠ A3⟩ ⟨P ≠ C1⟩ ⟨A1 A2 ParStrict C1 C2⟩
      ⟨Col A1 A2 Q⟩ ⟨Col A1 A2 A3⟩
      par-strict-col4--par-strict by blast
qed
moreover
have P B3 OS Q C1
  using ⟨P ≠ B3⟩ ⟨Col B1 B2 P⟩ ⟨Col B1 B2 B3⟩
    ⟨B1 B2 OS Q C1⟩ col2-os--os by blast
ultimately
show ?thesis by (simp add: ⟨P Q OS C1 A3⟩ ⟨P Q OS C1 B3⟩)
qed
thus ?thesis
  using legendre-aux-tr assms(1) assms(2) by blast
qed
}
hence Col C1 B1 B2
  by auto
}
thus ?thesis by blast
qed

```

lemma *legendre-aux2-tr*:

assumes *greenberg-s-axiom* **and**
triangle-postulate

shows $\forall A1 A2 B1 B2 C1 C2 P.$

$(P \text{ Perp2 } A1 A2 B1 B2 \wedge \neg \text{Col } A1 A2 P \wedge$
 $\text{Col } P B1 B2 \wedge \text{Coplanar } A1 A2 B1 B2 \wedge$
 $A1 A2 \text{ Par } C1 C2 \wedge \text{Col } P C1 C2)$

$\longrightarrow$

$\text{Col } C1 B1 B2$

proof –

```

{
  fix A1 A2 B1 B2 C1 C2 P
  assume 1: P Perp2 A1 A2 B1 B2 and
    2:  $\neg \text{Col } A1 A2 P$  and
    3:  $\text{Col } P B1 B2$  and
    4:  $\text{Coplanar } A1 A2 B1 B2$  and

```

```

5: A1 A2 Par C1 C2 and
6: Col P C1 C2
have A1 A2 ParStrict B1 B2
using Col-cases ⟨Col P B1 B2⟩ ⟨Coplanar A1 A2 B1 B2⟩
  ⟨P Perp2 A1 A2 B1 B2⟩ ⟨¬ Col A1 A2 P⟩
  col-cop-perp2--pars-bis by blast
{
  assume B1 B2 TS A1 C1
  hence ¬ Col C1 B1 B2
    using TS-def by blast
  have C1 ≠ P
    using 3 ⟨¬ Col C1 B1 B2⟩ by auto
  obtain C3 where P Midpoint C1 C3
    using symmetric-point-construction by blast
  hence C1 ≠ C3
    by (metis ⟨C1 ≠ P⟩ midpoint-distinct-2)
  have ¬ Col C3 B1 B2
    by (metis 3 ⟨P Midpoint C1 C3⟩ ⟨¬ Col C1 B1 B2⟩
      col-trivial-2 l6-21 midpoint-col midpoint-distinct-2
      not-col-permutation-3 not-col-permutation-4)
  moreover
  have Col C3 B1 B2
  proof -
    have P Perp2 A1 A2 B1 B2
      by (simp add: 1)
    moreover
    have ¬ Col A1 A2 P
      by (simp add: 2)
    moreover
    have Col P B1 B2
      by (simp add: 3)
    moreover
    have Coplanar A1 A2 B1 B2
      by (simp add: 4)
    moreover
    have A1 A2 Par C1 P
      by (metis 5 6 ⟨C1 ≠ P⟩ col-trivial-2
        par-col2-par-bis par-right-comm)
    hence A1 A2 Par C1 C3
      using Col-def Midpoint-def ⟨C1 ≠ C3⟩
      ⟨P Midpoint C1 C3⟩ par-col-par by blast
    hence A1 A2 Par C3 C1
      using Par-perm by blast
    moreover
    have Col P C3 C1
      using ⟨P Midpoint C1 C3⟩
      by (meson Mid-cases midpoint-col)
    moreover
    have Bet C3 P C1
      using ⟨P Midpoint C1 C3⟩
      by (metis midpoint-bet between-symmetry)
    hence B1 B2 TS C3 C1
      using 3 TS-def ⟨¬ Col C1 B1 B2⟩ ⟨¬ Col C3 B1 B2⟩ by blast
    hence B1 B2 OS A1 C3
      using ⟨B1 B2 TS A1 C1⟩ l9-8-1 by blast
    hence ¬ B1 B2 TS A1 C3
      using l9-9-bis by auto
    ultimately
    show ?thesis
      using legendre-aux1-tr greenberg-s-axiom-def
      triangle-postulate-def assms(1) assms(2) by blast
  qed
  ultimately
  have False
    using ⟨¬ Col C3 B1 B2⟩ ⟨Col C3 B1 B2⟩ by blast
}

```

```

  hence  $\neg B1\ B2\ TS\ A1\ C1$ 
    by blast
  hence  $Col\ C1\ B1\ B2$ 
    using legendre-aux1-tr greenberg-s-axiom-def
      assms(1) assms(2) 1 2 3 4 5 6 by blast
}
thus ?thesis
  by blast
qed

lemma triangle--playfair-bis:
  assumes greenberg-s-axiom and
    triangle-postulate
  shows alternative-playfair-s-postulate
proof -
{
  fix  $A1\ A2\ B1\ B2\ C1\ C2\ P$ 
  assume 1:  $P\ Perp2\ A1\ A2\ B1\ B2$  and
    2:  $\neg Col\ A1\ A2\ P$  and
    3:  $Col\ P\ B1\ B2$  and
    4:  $Coplanar\ A1\ A2\ B1\ B2$  and
    5:  $A1\ A2\ Par\ C1\ C2$  and
    6:  $Col\ P\ C1\ C2$ 
  have  $Col\ C1\ B1\ B2$ 
    using 1 2 3 4 5 6 legendre-aux2-tr
      greenberg-s-axiom-def assms(1) assms(2)
    by blast
  moreover
  have  $Col\ C2\ B1\ B2$ 
  proof -
    have  $A1\ A2\ Par\ C2\ C1$ 
      by (simp add: 5 par-right-comm)
    moreover
    have  $Col\ P\ C2\ C1$ 
      by (simp add: 6 col-permutation-5)
    ultimately
    show ?thesis
      using 1 2 3 4 legendre-aux2-tr greenberg-s-axiom-def
        assms(1) assms(2) by blast
  qed
  ultimately
  have  $Col\ C1\ B1\ B2 \wedge Col\ C2\ B1\ B2$ 
    by simp
}
thus ?thesis
  using alternative-playfair-s-postulate-def by blast
qed

lemma rah--existential-saccheri:
  assumes postulate-of-right-saccheri-quadrilaterals
  shows postulate-of-existence-of-a-right-saccheri-quadrilateral
proof -
  obtain  $A\ B\ C\ D$  where Saccheri  $A\ B\ C\ D$ 
    using ex-saccheri by blast
  moreover
  have  $Per\ A\ B\ C$ 
    using assms
      calculation postulate-of-right-saccheri-quadrilaterals-def
    by blast
  ultimately
  show ?thesis
    using postulate-of-existence-of-a-right-saccheri-quadrilateral-def
      by blast
qed

lemma rah--posidonius-aux:

```

```

assumes postulate-of-right-saccheri-quadrilaterals
shows  $\forall A1 A2 A3 B1 B2 B3.$ 
  (Per  $A2 A1 B1 \wedge$  Per  $A1 A2 B2 \wedge$ 
   Cong  $A1 B1 A2 B2 \wedge A1 A2 OS B1 B2 \wedge$ 
   Col  $A1 A2 A3 \wedge Col B1 B2 B3 \wedge$ 
    $A1 A2 \text{Perp} A3 B3$ )
 $\longrightarrow$ 
  Cong  $A3 B3 A1 B1$ 
proof –
{
  fix  $A1 A2 A3 B1 B2 B3$ 
  assume 1: Per  $A2 A1 B1$  and
    2: Per  $A1 A2 B2$  and
    3: Cong  $A1 B1 A2 B2$  and
    4: A1 A2 OS B1 B2 and
    5: Col  $A1 A2 A3$  and
    6: Col  $B1 B2 B3$  and
    7:  $A1 A2 \text{Perp} A3 B3$ 
  have  $A1 \neq B1$ 
    using  $\langle A1 A2 OS B1 B2 \rangle \langle Cong A1 B1 A2 B2 \rangle$ 
      col-trivial-2 cong-reverse-identity
      one-side-not-col124 by blast
  have  $B2 \neq B1$ 
    using  $\langle A1 A2 \text{Perp} A3 B3 \rangle \langle Cong A1 B1 A2 B2 \rangle$ 
       $\langle \text{Per } A1 A2 B2 \rangle \langle \text{Per } A2 A1 B1 \rangle$  diff-per-diff
      perp-not-eq-1 by blast
  have  $A2 \neq B2$ 
    using  $\langle A1 \neq B1 \rangle \langle Cong A1 B1 A2 B2 \rangle$  cong-identity-inv by blast
  have  $A1 \neq A2$ 
    using  $\langle A1 A2 OS B1 B2 \rangle$  os-distincts by auto
  have  $A1 \neq B2$ 
    using  $\langle A2 \neq B2 \rangle \langle \text{Per } A1 A2 B2 \rangle$  per-distinct-1 by blast
  have  $A2 \neq B1$ 
    using  $\langle A1 \neq A2 \rangle \langle \text{Per } A2 A1 B1 \rangle$  per-distinct-1 by blast
  have Cong  $A3 B3 A1 B1$ 
  proof (cases  $A1 = A3$ )
    case True
      hence  $A1 = A3$ 
      by blast
      have  $B1 = B3$ 
      proof –
        have  $\neg Col B1 B2 A1$ 
          using 1 2  $\langle A1 \neq A2 \rangle \langle A1 \neq B1 \rangle \langle A2 \neq B2 \rangle$ 
            col-permutation-5 per-not-colp by blast
        moreover
          have Col  $B1 B2 B1$ 
            using not-col-distincts by blast
        moreover
          have Col  $A1 B1 B1$ 
            using not-col-distincts by blast
        moreover
          have Col  $A1 B1 B3$ 
        proof –
          {
            assume  $\neg Col A1 B1 B3$ 
            have Per  $A2 A1 B3$ 
              using perp-per-2 7  $\langle A1 = A3 \rangle$  by blast
            hence  $\neg Col A1 A2 B3$ 
              by (meson 5 7 l8-16-1)
            have Per  $A1 A2 B3$ 
            proof –
              have  $f2$ : Per  $B2 A2 A1$ 
                by (simp add: 2 l8-2)
              have  $f3$ : Per  $B1 A1 A2$ 
                by (simp add: 1 l8-2)
              have Col  $B3 B1 B2$ 
          }
        }
    }

```

```

    by (simp add: 6 col-permutation-2)
  then show ?thesis
    by (metis col-per2--per l8-2 l8-7 not-col-distincts
      ⟨Per A2 A1 B3⟩ ⟨¬ Col A1 B1 B3⟩
      ⟨Per B2 A2 A1⟩ ⟨Per B1 A1 A2⟩)
  qed
  hence A1 = A2
    using ⟨Per A2 A1 B3⟩ l8-2 l8-7 by blast
  hence False
    by (simp add: ⟨A1 ≠ A2⟩)
}
thus ?thesis
  by blast
qed
ultimately
show ?thesis
  using 6 ⟨A1 ≠ B1⟩ l6-21 by blast
qed
thus ?thesis
  using True cong-reflexivity by auto
next
case False
have A3 ≠ B3
  using 7 perp-not-eq-2 by auto
then obtain B'3 where A3 Out B3 B'3 and Cong A3 B'3 A1 B1
  using segment-construction-3 ⟨A1 ≠ B1⟩ by blast
hence A3 ≠ B'3
  using out-distinct by blast
have B3 = B'3
proof -
  have Saccheri A1 B1 B2 A2
    by (meson 1 2 3 4 Per-cases Saccheri-def not-cong-1243)
  hence A1 A2 ParStrict B1 B2
    using sac--pars1423 by auto
  hence B1 B2 ParStrict A1 A2
    by (simp add: par-strict-symmetry)
  hence B1 B2 ParStrict A1 A3
    using 5 False par-strict-col-par-strict by blast
  have ¬ Col B1 B2 A3
    using ⟨B1 B2 ParStrict A1 A3⟩ par-strict-not-col-4 by auto
  moreover
  have A3 ≠ B3
    using ⟨A3 ≠ B3⟩ by auto
  moreover
  have Col B1 B2 B3
    using 6 by auto
  moreover
  have Col B2 B'3 B1
  proof -
    have Coplanar A1 B2 B'3 B1
    proof -
      have ¬ Col A3 B1 B2
        using NCol-cases calculation(1) by blast
      moreover
      have Coplanar A3 B1 B2 A1
        using ⟨B1 B2 ParStrict A1 A3⟩ ncoplanar-perm-9
          pars--coplanar by blast
      moreover
      have Coplanar A3 B1 B2 B'3
        using 6 Coplanar-def Out-cases ⟨A3 Out B3 B'3⟩
          out-col by blast
      ultimately
      show ?thesis
        by (meson coplanar-trans-1 ncoplanar-perm-20)
    qed
  moreover

```

```

have Per B2 B1 A1
  using ⟨Saccheri A1 B1 B2 A2⟩ Per-perm assms
  postulate-of-right-saccheri-quadrilaterals-def by blast
moreover
have Saccheri A1 B1 B'3 A3
proof -
{
  assume B1 = B3
  have A1 = A3
  proof -
    have ¬ Col A1 A2 B1
      using 4 col123--nos by blast
    moreover
    have Col A1 A2 A1
      by (simp add: col-trivial-3)
    moreover
    have A1 A2 Perp B1 A1
      using 1 ⟨A1 ≠ A2⟩ ⟨A1 ≠ B1⟩ per-perp
      perp-comm by presburger
    moreover
    have A1 A2 Perp B1 A3
      by (simp add: 7 ⟨B1 = B3⟩ perp-right-comm)
    ultimately
    show ?thesis
      using 5 l8-18-uniqueness by blast
  qed
  hence False
    using False by auto
}
hence B1 ≠ B3
  by blast
have Per A1 A3 B'3
  by (metis 5 7 ⟨A3 Out B3 B'3⟩ col-trivial-3
    l8-16-1 l8-2 l8-3 out-col perp-right-comm)
moreover
have Per B1 A1 A3
  using 1 5 ⟨A1 ≠ A2⟩ l8-2 l8-3 by blast
moreover
have Cong A1 B1 B'3 A3
  using Cong-cases ⟨Cong A3 B'3 A1 B1⟩ by blast
moreover
have A1 A3 OS B1 B'3
proof -
  have A1 A3 OS B1 B3
    by (meson 5 6 False ⟨B1 B2 ParStrict A1 A2⟩
      ⟨B1 ≠ B3⟩ col-trivial-3 par-strict-col4--par-strict
      pars--os3412)
  moreover
  have A1 A3 OS B3 B'3
    using ⟨A3 Out B3 B'3⟩ calculation col-trivial-2
    one-side-not-col124 out-one-side-1 by blast
  ultimately
  show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
    using Saccheri-def by blast
  qed
hence Per B'3 B1 A1
  using Per-perm assms
  postulate-of-right-saccheri-quadrilaterals-def by blast
ultimately
show ?thesis
  using ⟨A1 ≠ B1⟩ cop-per2--col by blast
qed

```



```

    hence Col B1 B2 B'3
      using Col-cases by blast
    moreover
    have Col A3 B3 B3
      by (metis not-col-distincts)
    moreover
    have Col A3 B3 B'3
      using  $\langle A3 \text{ Out } B3 \text{ B'3} \rangle$  out-col by auto
    ultimately
    show ?thesis
      using l6-21 by blast
  qed
  thus ?thesis
    using  $\langle \text{Cong } A3 \text{ B'3 } A1 \text{ B1} \rangle$  by auto
  qed
}
thus ?thesis by blast
qed

```

```

lemma rah--posidonius:
  assumes postulate-of-right-saccheri-quadrilaterals
  shows posidonius-postulate
proof -
  obtain A1 B1 B2 A2 where Saccheri A1 B1 B2 A2
    using ex-saccheri by presburger
  hence H1: Per B1 A1 A2  $\wedge$  Per A1 A2 B2  $\wedge$ 
        Cong A1 B1 B2 A2  $\wedge$  A1 A2 OS B1 B2
    using Saccheri-def by blast
  hence Per B1 A1 A2 by blast
  have Per A1 A2 B2
    by (simp add: H1)
  have Cong A1 B1 B2 A2
    by (simp add: H1)
  have A1 A2 OS B1 B2
    by (simp add: H1)
  have  $\neg \text{Col } A1 \text{ A2 B1}$ 
    using  $\langle A1 \text{ A2 OS B1 B2} \rangle$  col123--nos by blast
  moreover
  have B1  $\neq$  B2
    using  $\langle \text{Saccheri } A1 \text{ B1 B2 A2} \rangle$  sac-distincts by blast
  moreover
  have Coplanar A1 A2 B1 B2
    using  $\langle \text{Saccheri } A1 \text{ B1 B2 A2} \rangle$  ncoplanar-perm-3
      sac--coplanar by blast
  moreover
  {
    fix A3 A4 B3 B4
    assume Col A1 A2 A3 and
           Col B1 B2 B3 and
           A1 A2 Perp A3 B3 and
           Col A1 A2 A4 and
           Col B1 B2 B4 and
           A1 A2 Perp A4 B4
    have  $(\text{Per } A2 \text{ A1 B1} \wedge \text{Per } A1 \text{ A2 B2} \wedge \text{Cong } A1 \text{ B1 A2 B2} \wedge$ 
           $\text{A1 A2 OS B1 B2} \wedge \text{Col } A1 \text{ A2 A3} \wedge \text{Col } B1 \text{ B2 B3} \wedge$ 
           $\text{A1 A2 Perp A3 B3}) \longrightarrow \text{Cong } A3 \text{ B3 A1 B1}$ 
      using rah--posidonius-aux assms by blast
    hence Cong A3 B3 A1 B1
      using Cong-cases  $\langle A1 \text{ A2 OS B1 B2} \rangle$   $\langle A1 \text{ A2 Perp A3 B3} \rangle$ 
         $\langle \text{Col } A1 \text{ A2 A3} \rangle$   $\langle \text{Col } B1 \text{ B2 B3} \rangle$   $\langle \text{Cong } A1 \text{ B1 B2 A2} \rangle$ 
         $\langle \text{Per } A1 \text{ A2 B2} \rangle$   $\langle \text{Per } B1 \text{ A1 A2} \rangle$  l8-2 by blast
    hence Cong A1 B1 A3 B3
      using not-cong-3412 by blast
  }
  moreover

```

```

have (Per A2 A1 B1  $\wedge$  Per A1 A2 B2  $\wedge$  Cong A1 B1 A2 B2  $\wedge$ 
  A1 A2 OS B1 B2  $\wedge$  Col A1 A2 A4  $\wedge$  Col B1 B2 B4  $\wedge$ 
  A1 A2 Perp A4 B4)  $\longrightarrow$  Cong A4 B4 A1 B1
using rah--posidonius-aux assms by blast
hence Cong A4 B4 A1 B1
using Cong-cases  $\langle$ A1 A2 OS B1 B2 $\rangle$   $\langle$ A1 A2 Perp A4 B4 $\rangle$ 
 $\langle$ Col A1 A2 A4 $\rangle$   $\langle$ Col B1 B2 B4 $\rangle$   $\langle$ Cong A1 B1 B2 A2 $\rangle$ 
 $\langle$ Per A1 A2 B2 $\rangle$   $\langle$ Per B1 A1 A2 $\rangle$  l8-2 by blast
hence Cong A1 B1 A4 B4
by (simp add: cong-symmetry)
ultimately
have Cong A3 B3 A4 B4
using cong-inner-transitivity by blast
}
ultimately
show ?thesis
using posidonius-postulate-def by blast
qed

```

```

lemma rah--rectangle-principle:
  assumes postulate-of-right-saccheri-quadrilaterals
  shows postulate-of-right-lambert-quadrilaterals
proof -
{
  fix A B C D
  assume Lambert A B C D
  hence Per B C D
    using lam-per--rah assms existential-playfair--rah-1 by blast
}
thus ?thesis
  by (simp add: postulate-of-right-lambert-quadrilaterals-def)
qed

```

```

lemma rah--similar:
  assumes postulate-of-right-saccheri-quadrilaterals
  shows postulate-of-existence-of-similar-triangles
proof -
  obtain A B0 C where  $\neg$  (Bet A B0 C  $\vee$  Bet B0 C A  $\vee$  Bet C A B0)
    using lower-dim-ex by blast
  hence  $\neg$  Col A B0 C
    by (simp add: Col-def)
  then obtain B where C A Perp B C and C A OS B0 B
    using l10-15 by (metis Col-cases col-trivial-3)
  hence  $\neg$  Col C A B
    using col124--nos by blast
  have Per A C B
    by (simp add:  $\langle$ C A Perp B C $\rangle$  perp-per-1)
  obtain B' where B Midpoint A B'
    using symmetric-point-construction by blast
  have C  $\neq$  A
    using  $\langle$  $\neg$  Col A B0 C $\rangle$  col-trivial-3 by blast
  have B  $\neq$  A
    using  $\langle$  $\neg$  Col C A B $\rangle$  col-trivial-2 by blast
  have C  $\neq$  B
    using  $\langle$  $\neg$  Col C A B $\rangle$  col-trivial-3 by auto
  have B0  $\neq$  A
    using  $\langle$  $\neg$  Col A B0 C $\rangle$  col-trivial-1 by force
  have B'  $\neq$  A
    using  $\langle$ B Midpoint A B' $\rangle$   $\langle$ B  $\neq$  A $\rangle$  l7-3 by blast
  have B  $\neq$  B'
    by (metis  $\langle$ B Midpoint A B' $\rangle$   $\langle$ B'  $\neq$  A $\rangle$  midpoint-not-midpoint)

```

have $\neg \text{Col } A \ C \ B'$
using $\langle B \text{ Midpoint } A \ B' \rangle \langle B' \neq A \rangle \langle \neg \text{Col } C \ A \ B \rangle$
col2--eq midpoint-col not-col-permutation-4 **by** *blast*
hence $\neg \text{Col } B \ B' \ C$
by (*metis* $\langle B \text{ Midpoint } A \ B' \rangle \langle B \neq B' \rangle$ *col2--eq*
col-permutation-1 midpoint-col)
obtain C' **where** $\text{Col } A \ C \ C'$ **and** $A \ C \text{ Perp } B' \ C'$
using $\langle \neg \text{Col } A \ C \ B' \rangle$ *l8-18-existence* **by** *blast*
have $\neg \text{Col } A \ B \ C$
using $\langle \neg \text{Col } C \ A \ B \rangle$ *col-permutation-2* **by** *blast*
moreover
have $\neg \text{Cong } A \ B \ A \ B'$
by (*metis* *midpoint-bet* $\langle B \text{ Midpoint } A \ B' \rangle$
 $\langle B \neq B' \rangle$ *between-cong*)
moreover
have $B \ C \text{ ParStrict } B' \ C'$
proof –
have $B \ C \text{ Par } B' \ C'$
proof –
have $\text{Bet } A \ B \ B'$
by (*simp* *add:* $\langle B \text{ Midpoint } A \ B' \rangle$ *midpoint-bet*)
hence $\text{Coplanar } A \ C \ B \ B'$
using *bet-col ncop--ncols* **by** *blast*
moreover
have $\text{Coplanar } A \ C \ B \ C'$
by (*meson* $\langle \text{Col } A \ C \ C' \rangle$ *ncop--ncols*)
moreover
have $\text{Coplanar } A \ C \ C \ B'$
using *ncop-distincts* **by** *blast*
moreover
have $\text{Coplanar } A \ C \ C \ C'$
using *ncop-distincts* **by** *blast*
moreover
have $B \ C \text{ Perp } A \ C$
using *Perp-perm* $\langle C \ A \text{ Perp } B \ C \rangle$ **by** *blast*
moreover
have $B' \ C' \text{ Perp } A \ C$
using *Perp-perm* $\langle A \ C \text{ Perp } B' \ C' \rangle$ **by** *blast*
ultimately
show *?thesis*
using *l12-9* **by** *blast*
qed
moreover
have $\text{Col } B' \ C' \ B'$
using *not-col-distincts* **by** *blast*
moreover
have $\neg \text{Col } B \ C \ B'$
using $\langle \neg \text{Col } B \ B' \ C \rangle$ *not-col-permutation-5* **by** *blast*
ultimately
show *?thesis*
using *par-not-col-strict* **by** *auto*
qed
have $\neg \text{Col } B \ C \ C'$
by (*meson* $\langle B \ C \text{ ParStrict } B' \ C' \rangle$ *not-col-permutation-1*
par-strict-not-col-2 par-strict-symmetry)
hence $C \neq C'$
using *col-trivial-2* **by** *auto*
have $B' \neq C'$
using $\langle B \ C \text{ ParStrict } B' \ C' \rangle$ *col-trivial-1 col-trivial-3*
one-side-not-col124 par-strict-one-side **by** *blast*
have $\text{Bet } A \ C \ C'$
proof –
have $\text{Col } C \ A \ C'$
by (*simp* *add:* $\langle \text{Col } A \ C \ C' \rangle$ *col-permutation-4*)
moreover
have $C \ B \text{ TS } C' \ A$

```

proof -
  have  $C B TS B' A$ 
proof -
  have  $\neg Col B' C B$ 
    using  $\langle \neg Col B B' C \rangle$  col-permutation-2 by blast
  moreover
  have  $\neg Col A C B$ 
    using  $\langle \neg Col A B C \rangle$  col-permutation-5 by blast
  moreover
  have  $Col B C B$ 
    using not-col-distincts by blast
  moreover
  have  $Bet B' B A$ 
    using  $\langle B \text{ Midpoint } A B' \rangle$  between-symmetry
    midpoint-bet by blast
  ultimately
  show ?thesis
    using TS-def by blast
qed
moreover
have  $C B OS B' C'$ 
  using Par-strict-perm  $\langle B C \text{ ParStrict } B' C' \rangle$ 
  pars--os3412 by blast
ultimately
show ?thesis
  using l9-8-2 by blast
qed
hence  $C B TS A C'$ 
  by (simp add: l9-2)
ultimately
show ?thesis
  using col-two-sides-bet by blast
qed
have  $A \neq C'$ 
  using  $\langle Bet A C C' \rangle \langle C \neq C' \rangle$  between-equality-2
  between-trivial2 by blast
have  $C' B' Perp A C'$ 
proof -
  have  $A \neq C'$ 
    using  $\langle A \neq C' \rangle$  by auto
  moreover
  have  $C' B' Perp A C$ 
    using Perp-perm  $\langle A C \text{ Perp } B' C' \rangle$  by blast
  ultimately
  show ?thesis
    using  $\langle Col A C C' \rangle$  perp-col1 by blast
qed
hence  $Per B' C' A$ 
  using perp-per-1 by blast
have  $A B C CongA A B' C'$ 
proof -
  have  $SAMS C A B A B C$ 
    using  $\langle B \neq A \rangle \langle C \neq A \rangle \langle C \neq B \rangle$  sams123231 by auto
  moreover
  have  $SAMS C A B A B' C'$ 
proof -
  have  $C' A B' CongA C A B$ 
    by (metis midpoint-bet out2--conga
       $\langle B \text{ Midpoint } A B' \rangle \langle B \neq A \rangle \langle Bet A C C' \rangle$ 
       $\langle C \neq A \rangle$  bet-out)
  moreover
  have  $A B' C' CongA A B' C'$ 
    using  $\langle B' \neq A \rangle \langle B' \neq C' \rangle$  lea-asym lea-refl by auto
  moreover
  have  $SAMS C' A B' A B' C'$ 
    using  $\langle A \neq C' \rangle \langle B' \neq A \rangle \langle B' \neq C' \rangle$  sams123231 by force

```

ultimately
 show ?thesis
 using conga2-sams--sams by blast
 qed
 moreover
 have $C \ A \ B \ A \ B \ C \ \text{Sum} \ A \ B \ C \ A$
 proof –
 have $\text{Per} \ B \ C \ A$
 using $\text{Per-perm} \ \langle \text{Per} \ A \ C \ B \rangle$ by blast
 moreover
 have hypothesis-of-right-saccheri-quadrilaterals
 using assms existential-playfair--rah-1 by blast
 ultimately
 show ?thesis
 using $\langle C \neq A \rangle \ \langle C \neq B \rangle \ t22-12\text{--}rah-2$ by presburger
 qed
 moreover
 have $C \ A \ B \ A \ B' \ C' \ \text{Sum} \ A \ B \ C \ A$
 proof –
 have hypothesis-of-right-saccheri-quadrilaterals
 using assms existential-playfair--rah-1 by auto
 hence $C' \ A \ B' \ A \ B' \ C' \ \text{Sum} \ A \ B' \ C' \ A$
 using $\langle A \neq C' \rangle \ \langle B' \neq C' \rangle \ \langle \text{Per} \ B' \ C' \ A \rangle$
 t22-12--rah-2 by presburger
 moreover
 have $C' \ A \ B' \ \text{Cong} \ A \ C \ A \ B$
 by (metis midpoint-bet out2--conga
 $\langle B \ \text{Midpoint} \ A \ B' \rangle \ \langle B \neq A \rangle \ \langle \text{Bet} \ A \ C \ C' \rangle$
 $\langle C \neq A \rangle \ \text{bet-out}$)
 moreover
 have $A \ B' \ C' \ \text{Cong} \ A \ A \ B' \ C'$
 using $\langle B' \neq A \rangle \ \langle B' \neq C' \rangle \ \text{conga-refl}$ by auto
 moreover
 have $B' \ C' \ A \ \text{Cong} \ A \ B \ C \ A$
 by (metis $\langle A \ C \ \text{Perp} \ B' \ C' \rangle \ \langle A \neq C' \rangle$
 $\langle C \ A \ \text{Perp} \ B \ C \rangle \ \langle C \neq A \rangle \ \langle C \neq B \rangle \ \langle C \neq C' \rangle$
 $\langle \text{Col} \ A \ C \ C' \rangle \ \text{col2--eq} \ \text{conga-right-comm} \ l11-16$
l8-16-1 perp-per-1)
 ultimately
 show ?thesis
 by (meson conga3-suma--suma)
 qed
 ultimately
 show ?thesis
 using sams2-suma2--conga456 by blast
 qed
 moreover
 have $B \ C \ A \ \text{Cong} \ A \ B' \ C' \ A$
 by (metis $\langle A \ C \ \text{Perp} \ B' \ C' \rangle \ \langle A \neq C' \rangle$
 $\langle C \ A \ \text{Perp} \ B \ C \rangle \ \langle C \neq A \rangle \ \langle C \neq B \rangle \ \langle C \neq C' \rangle$
 $\langle \text{Col} \ A \ C \ C' \rangle \ \text{conga-left-comm} \ l11-16 \ l6-16-1$
l8-16-1 perp-per-1)
 moreover
 have $C \ A \ B \ \text{Cong} \ A \ C' \ A \ B'$
 proof –
 have $A \ \text{Out} \ C' \ C$
 by (simp add: $\langle \text{Bet} \ A \ C \ C' \rangle \ \langle C \neq A \rangle \ \text{bet-out} \ l6-6$)
 moreover
 have $A \ \text{Out} \ B' \ B$
 by (simp add: $\langle B \ \text{Midpoint} \ A \ B' \rangle \ \langle B' \neq A \rangle \ l7-2 \ \text{midpoint-out-1}$)
 ultimately
 show ?thesis
 by (simp add: out2--conga)
 qed
 ultimately
 show ?thesis

using *postulate-of-existence-of-similar-triangles-def* by blast
qed

lemma *rah--thales-postulate*:

assumes *postulate-of-right-saccheri-quadrilaterals*
shows *thales-postulate*

proof –

```
{
  fix A B C M
  assume M Midpoint A B and Cong M A M C
  have Per A C B
  proof (cases Col A B C)
    case True
    show ?thesis
    proof (cases A = B)
      case True
      thus ?thesis
        using ⟨Cong M A M C⟩ ⟨M Midpoint A B⟩
        cong-reverse-identity l8-20-2 l8-5 by blast
    next
    case False
    hence A ≠ B
      by auto
    moreover
    have Col A M B
      using ⟨M Midpoint A B⟩ midpoint-col
      not-col-permutation-4 by blast
    hence Col A M C
      by (meson True calculation col-trivial-3 colx)
    ultimately
    show ?thesis
      by (metis ⟨Cong M A M C⟩ ⟨M Midpoint A B⟩
        cong-col-mid cong-left-commutativity l8-2 l8-5
        symmetric-point-uniqueness)
  qed
next
case False
thus ?thesis
  using t22-17--rah ⟨Cong M A M C⟩ ⟨M Midpoint A B⟩
  assms existential-playfair--rah-1 by blast
qed
}
thus ?thesis
  using thales-postulate-def by blast
qed
```

lemma *rah--triangle*:

assumes *postulate-of-right-saccheri-quadrilaterals*
shows *triangle-postulate*

proof –

```
{
  fix A B C D E F
  assume A B C TriSumA D E F
  hence Bet D E F
    using assms existential-playfair--rah-1 t22-14--bet by blast
}
thus ?thesis
  using triangle-postulate-def by blast
qed
```

lemma *rectangle-principle--rectangle-existence*:

assumes *postulate-of-right-lambert-quadrilaterals*
shows *postulate-of-existence-of-a-right-lambert-quadrilateral*

proof –

```

obtain  $A B C D$  where  $Saccheri A B C D$ 
  using  $ex-saccheri$  by  $blast$ 
moreover
obtain  $M$  where  $M$   $Midpoint B C$ 
  using  $midpoint-existence$  by  $blast$ 
moreover
obtain  $N$  where  $N$   $Midpoint A D$ 
  using  $midpoint-existence$  by  $blast$ 
ultimately
have  $Lambert N M B A$ 
  by ( $meson mid2-sac--lam6521$ )
moreover
have  $Per M B A$ 
  using  $postulate-of-right-lambert-quadrilaterals-def$ 
     $assms \langle Lambert N M B A \rangle$  by  $blast$ 
ultimately
show  $?thesis$ 
  using  $postulate-of-existence-of-a-right-lambert-quadrilateral-def$ 
    by  $blast$ 
qed

```

```

lemma  $similar--rah-aux$ :
  assumes  $\neg Col A B C$  and
     $A B C Cong A D E F$  and
     $B C A Cong A E F D$  and
     $C A B Cong A F D E$  and
     $B C A Le A A B C$  and
     $D E Lt A B$ 
  shows  $postulate-of-right-saccheri-quadrilaterals$ 
proof –
  have  $A \neq B$ 
    using  $assms(1)$   $col-trivial-1$  by  $auto$ 
  have  $C \neq B$ 
    using  $assms(1)$   $col-trivial-2$  by  $blast$ 
  have  $A \neq C$ 
    using  $assms(1)$   $col-trivial-3$  by  $blast$ 
  have  $F \neq E$ 
    using  $assms(2)$   $conga-distinct$  by  $presburger$ 
  have  $D \neq E$ 
    using  $assms(4)$   $conga-diff56$  by  $blast$ 
  then obtain  $G$  where  $A Out B G$  and  $Cong A G D E$ 
    using  $segment-construction-3 \langle A \neq B \rangle$  by  $blast$ 
  have  $D \neq F$ 
    using  $assms(4)$   $conga-distinct$  by  $blast$ 
  then obtain  $H$  where  $A Out C H$  and  $Cong A H D F$ 
    using  $segment-construction-3 \langle A \neq C \rangle$  by  $blast$ 
  have  $A G Lt A B$ 
    by ( $meson \langle Cong A G D E \rangle assms(6) cong2-lt--lt$ 
       $cong-reflexivity cong-symmetry$ )
  have  $Bet A G B$ 
proof –
  have  $A Out G B$ 
    by ( $simp add: \langle A Out B G \rangle l6-6$ )
  moreover
  have  $A G Le A B$ 
    by ( $simp add: \langle A G Lt A B \rangle lt--le$ )
  ultimately
  show  $?thesis$ 
    by ( $meson l6-13-1$ )
qed
have  $B \neq G$ 
  using  $\langle A G Lt A B \rangle not-and-lt$  by  $blast$ 
have  $C A B Cong A H A G$ 
  by ( $meson Out-cases \langle A Out B G \rangle \langle A Out C H \rangle out2--conga$ )

```

```

have D F E CongA A H G  $\wedge$  D E F CongA A G H
proof -
  have F D E CongA H A G
    by (meson  $\langle C A B \text{ CongA } H A G \rangle$  assms(4)
        not-conga not-conga-sym)
  moreover
  have Cong D F A H
    by (meson  $\langle \text{Cong } A H D F \rangle$  not-cong-3412)
  moreover
  have Cong D E A G
    by (simp add:  $\langle \text{Cong } A G D E \rangle$  cong-symmetry)
  ultimately
  show ?thesis
    using  $\langle F \neq E \rangle$  l11-49 by blast
qed
have D E F CongA A G H
  using  $\langle D F E \text{ CongA } A H G \wedge D E F \text{ CongA } A G H \rangle$  by auto
hence A B C CongA A G H
  using assms(2) conga-trans by blast
have E F D CongA G H A
  by (simp add:  $\langle D F E \text{ CongA } A H G \wedge D E F \text{ CongA } A G H \rangle$ 
      conga-comm)
hence B C A CongA G H A
  using assms(3) conga-trans by blast
have  $\neg \text{Col } A G H$ 
  by (meson Out-cases  $\langle A \text{ Out } B G \rangle \langle A \text{ Out } C H \rangle$  assms(1)
      col-out2-col not-col-permutation-2)
have  $\neg \text{Col } B G H$ 
  using  $\langle B \neq G \rangle \langle \text{Bet } A G B \rangle \langle \neg \text{Col } A G H \rangle$  bet-col
      col2--eq col-permutation-3 by blast
have G H ParStrict B C
proof -
  have B C Par G H
  proof -
    have A B OS C H
      by (simp add:  $\langle A \text{ Out } C H \rangle$  assms(1) out-one-side)
    moreover
    have C B A CongA H G A
      by (simp add:  $\langle A B C \text{ CongA } A G H \rangle$  conga-comm)
    ultimately
    show ?thesis
      using  $\langle A \text{ Out } B G \rangle$  l12-22-b by auto
  qed
hence G H Par B C
  using par-symmetry by blast
moreover
have Col B C B
  by (simp add: col-trivial-3)
moreover
have  $\neg \text{Col } G H B$ 
  by (simp add:  $\langle \neg \text{Col } B G H \rangle$  not-col-permutation-1)
ultimately
show ?thesis
  using par-not-col-strict by blast
qed
have  $\neg \text{Col } G H C$ 
  by (meson  $\langle G H \text{ ParStrict } B C \rangle$  l12-6 one-side-not-col124)
have  $\neg \text{Col } B C G$ 
  by (meson  $\langle A \text{ Out } B G \rangle \langle B \neq G \rangle$  assms(1) col-trivial-2
      colx not-col-permutation-5 out-col)
have  $\neg \text{Col } H B C$ 
  by (metis  $\langle A \text{ Out } C H \rangle \langle \neg \text{Col } G H C \rangle$  assms(1)
      col-trivial-3 colx not-col-permutation-2 out-col)
have  $H \neq C$ 
  using  $\langle \neg \text{Col } H B C \rangle$  col-trivial-3 by auto
have  $G \neq C$ 

```



```

  using  $\langle \neg \text{Col } B \ C \ G \rangle$  col-trivial-2 by auto
have  $H \neq G$ 
  using  $\langle \neg \text{Col } A \ G \ H \rangle$  col-trivial-2 by blast
have  $A \neq G$ 
  using  $\langle \neg \text{Col } A \ G \ H \rangle$  col-trivial-1 by auto
have  $H \neq A$ 
  using  $\langle \neg \text{Col } A \ G \ H \rangle$  col-trivial-3 by blast
have  $C \text{ Out } H \ A$ 
proof –
  have  $\text{Col } C \ H \ A$ 
    using  $\langle A \text{ Out } C \ H \rangle$  col-permutation-1 out-col by blast
  moreover
  have  $B \ C \ OS \ H \ A$ 
proof –
  have  $B \ C \ OS \ H \ G$ 
    using  $\langle G \ H \ \text{ParStrict } B \ C \rangle$  one-side-symmetry
    pars--os3412 by blast
  moreover
  have  $B \ C \ OS \ G \ A$ 
    using  $\langle B \neq G \rangle \langle \text{Bet } A \ G \ B \rangle \langle \neg \text{Col } B \ C \ G \rangle$ 
    bet-out-1 out-one-side by presburger
  ultimately
  show ?thesis
    using one-side-transitivity by blast
qed
hence  $C \ B \ OS \ H \ A$ 
  by (simp add: invert-one-side)
ultimately
show ?thesis
  using col-one-side-out by force
qed
hence  $\text{Bet } A \ H \ C$ 
  by (meson  $\langle A \text{ Out } C \ H \rangle$  between-symmetry out2--bet)
have  $\text{SAMS } B \ G \ H \ H \ C \ B$ 
proof –
  have  $\text{Bet } B \ G \ A$ 
    using Bet-cases  $\langle \text{Bet } A \ G \ B \rangle$  by blast
  moreover
  have  $H \ C \ B \ \text{LeA } H \ G \ A$ 
proof –
  have  $B \ C \ A \ \text{LeA } A \ B \ C$ 
    by (simp add: assms(5))
  moreover
  have  $B \ C \ A \ \text{CongA } H \ C \ B$ 
    by (metis CongA-def  $\langle B \ C \ A \ \text{CongA } G \ H \ A \rangle$ 
     $\langle \text{Bet } A \ H \ C \rangle \langle H \neq C \rangle$  bet2--out bet-out-1
    conga-left-comm out2--conga)
  moreover
  have  $A \ B \ C \ \text{CongA } H \ G \ A$ 
    using  $\langle A \ B \ C \ \text{CongA } A \ G \ H \rangle$  conga-right-comm by blast
  ultimately
  show ?thesis
    using l11-30 by blast
qed
ultimately
show ?thesis
  using  $\langle A \neq G \rangle \langle B \neq G \rangle$  sams-chara by blast
qed
have  $A \ G \ H \ \text{CongA } G \ B \ C$ 
  by (metis out2--conga  $\langle A \ B \ C \ \text{CongA } A \ G \ H \rangle$ 
   $\langle B \neq G \rangle \langle \text{Bet } A \ G \ B \rangle \langle C \neq B \rangle$ 
  bet-out-1 conga-sym-equiv
  conga-trans out-trivial)
have  $G \ H \ A \ \text{CongA } B \ C \ H$ 
  by (metis Out-def  $\langle B \ C \ A \ \text{CongA } G \ H \ A \rangle \langle C \text{ Out } H \ A \rangle$ 
   $\langle C \neq B \rangle$  not-bet-distincts not-conga)

```

```

    not-conga-sym out2--conga)
have SAMS B G H C B G
proof -
  have B G H CongA B G H
    using ⟨B ≠ G⟩ ⟨H ≠ G⟩ conga-refl by blast
  moreover
  have H G A CongA C B G
    by (simp add: ⟨A G H CongA G B C⟩ conga-comm)
  moreover
  have SAMS B G H H G A
    by (metis ⟨A ≠ G⟩ ⟨B ≠ G⟩ ⟨Bet A G B⟩ ⟨H ≠ G⟩
      bet--sams between-symmetry)
  ultimately
  show ?thesis
    by (meson conga2-sams--sams)
qed
have SAMS C H G B C H
proof -
  have C H G CongA C H G
    using ⟨H ≠ C⟩ ⟨H ≠ G⟩ conga-refl by auto
  moreover
  have SAMS C H G G H A
    using ⟨Bet A H C⟩ ⟨H ≠ A⟩ ⟨H ≠ C⟩ ⟨H ≠ G⟩
      bet--sams between-symmetry by presburger
  ultimately
  show ?thesis
    by (meson ⟨G H A CongA B C H⟩ conga2-sams--sams)
qed
obtain I J K where B C G C G B SumA I J K
  using ex-suma ⟨B ≠ G⟩ ⟨C ≠ B⟩ ⟨G ≠ C⟩ by presburger
have C ≠ H
  using ⟨H ≠ C⟩ by auto
then obtain L M N where H C G C G H SumA L M N
  using ex-suma ⟨H ≠ G⟩ ⟨G ≠ C⟩ by presburger
have L ≠ M
  using ⟨H C G C G H SumA L M N⟩ suma-distincts by presburger
have N ≠ M
  using ⟨H C G C G H SumA L M N⟩ suma-distincts by blast
have I ≠ J
  using ⟨B C G C G B SumA I J K⟩ suma-distincts by blast
have J ≠ K
  using ⟨B C G C G B SumA I J K⟩ suma-distincts by blast
have I ≠ K
  by (metis Par-strict-cases par-strict-not-col-4
    ⟨B C G C G B SumA I J K⟩ ⟨G H ParStrict B C⟩
    col-trivial-3 ncol-suma--ncol)
have B ≠ C
  using ⟨C ≠ B⟩ by auto
have G ≠ B
  using ⟨B ≠ G⟩ by auto
then obtain O0 P Q where I J K G B C SumA O0 P Q
  using ex-suma ⟨I ≠ J⟩ ⟨J ≠ K⟩ ⟨B ≠ C⟩ by blast
have G ≠ H
  using ⟨H ≠ G⟩ by auto
have M ≠ N
  using ⟨N ≠ M⟩ by auto
then obtain R S T where L M N G H C SumA R S T
  using ex-suma ⟨L ≠ M⟩ ⟨M ≠ N⟩ ⟨G ≠ H⟩ ⟨H ≠ C⟩ by blast
obtain U V W where I J K L M N SumA U V W
  using ex-suma ⟨I ≠ J⟩ ⟨J ≠ K⟩ ⟨L ≠ M⟩ ⟨M ≠ N⟩ by blast
have R ≠ S
  using ⟨L M N G H C SumA R S T⟩ suma-distincts by blast
have S ≠ T
  using ⟨L M N G H C SumA R S T⟩ suma-distincts by blast
have O0 ≠ P
  using ⟨I J K G B C SumA O0 P Q⟩ suma-distincts by blast

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have Q ≠ P
  using ⟨I J K G B C SumA O0 P Q⟩ suma-distincts by blast
have SAMS I J K L M N ∧ H G B B C H SumA U V W
proof —
  have G C TS B H
    by (metis Col-cases ⟨Bet A G B⟩ ⟨C Out H A⟩
      ⟨G H ParStrict B C⟩ ⟨¬ Col G H C⟩ bet-out l12-6
      l9-19-R2 l9-9 out-col par-one-or-two-sides
      two-sides-cases)
  have SAMS H G B B C G
  proof —
    have SAMS H G B B C H
      by (simp add: ⟨SAMS B G H H C B⟩ sams-comm)
    moreover
    have H G B LeA H G B
      using ⟨G ≠ B⟩ ⟨H ≠ G⟩ lea-refl by force
    moreover
    have G InAngle B C H
      by (meson Par-strict-cases ⟨G C TS B H⟩
        ⟨G H ParStrict B C⟩ invert-two-sides
        l12-6 os-ts--inangle)
    hence B C G LeA B C H
      by (simp add: inangle--lea)
    ultimately
    show ?thesis
      by (meson sams-lea2--sams)
  qed
have C ≠ G
  using ⟨G ≠ C⟩ by auto
then obtain X Y Z where B C G H G B SumA X Y Z
  using ex-suma ⟨B ≠ C⟩ ⟨H ≠ G⟩ ⟨G ≠ B⟩ by blast
have B G C C G H SumA H G B
  by (meson ⟨G C TS B H⟩ suma-right-comm ts--suma-1)
have B G OS C H
  by (meson Col-perm ⟨A Out B G⟩ ⟨A Out C H⟩
    ⟨¬ Col B C G⟩ one-side-reflexivity one-side-symmetry
    os-out-os out-col)
hence SAMS B G C C G H
  by (simp add: ⟨G C TS B H⟩ os-ts--sams)
have SAMS I J K C G H
proof —
  have SAMS B C G C G B
    using ⟨C ≠ B⟩ ⟨G ≠ B⟩ ⟨G ≠ C⟩ sams123231 by auto
  moreover
  have SAMS C G B C G H
    by (simp add: ⟨SAMS B G C C G H⟩ sams-left-comm)
  moreover
  have C G B C G H SumA H G B
    using ⟨B G C C G H SumA H G B⟩ suma-left-comm by blast
  moreover
  have SAMS B C G H G B
    by (simp add: ⟨SAMS H G B B C G⟩ sams-sym)
  ultimately
  show ?thesis
    using ⟨B C G C G B SumA I J K⟩ sams-assoc by blast
  qed
have I J K C G H SumA X Y Z
proof —
  have SAMS B C G C G B
    using ⟨C ≠ B⟩ ⟨G ≠ B⟩ ⟨G ≠ C⟩ sams123231 by auto
  moreover
  have SAMS C G B C G H
    by (simp add: ⟨SAMS B G C C G H⟩ sams-left-comm)
  moreover
  have C G B C G H SumA H G B
    using ⟨B G C C G H SumA H G B⟩ suma-left-comm by blast

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ultimately
show *?thesis*
using $\langle B \ C \ G \ C \ G \ B \ SumA \ I \ J \ K \rangle \langle B \ C \ G \ H \ G \ B \ SumA \ X \ Y \ Z \rangle$
suma-assoc **by** *blast*
qed
have $SAMS \ B \ C \ G \ H \ C \ G$
by (*meson* $\langle G \ C \ TS \ B \ H \rangle \langle G \ H \ ParStrict \ B \ C \rangle$
invert-two-sides os-ts--sams sams-right-comm)
have $B \ C \ G \ H \ C \ G \ SumA \ H \ C \ B$
by (*meson* $\langle G \ C \ TS \ B \ H \rangle$ *suma-middle-comm*
suma-right-comm ts--suma)
have $SAMS \ I \ J \ K \ L \ M \ N$
proof –
have $SAMS \ X \ Y \ Z \ H \ C \ G$
proof –
have $H \ G \ B \ B \ C \ G \ SumA \ X \ Y \ Z$
by (*simp add:* $\langle B \ C \ G \ H \ G \ B \ SumA \ X \ Y \ Z \rangle$ *suma-sym*)
moreover
have $SAMS \ H \ G \ B \ H \ C \ B$
by (*simp add:* $\langle SAMS \ B \ G \ H \ H \ C \ B \rangle$ *sams-left-comm*)
ultimately
show *?thesis*
by (*meson* $\langle SAMS \ H \ G \ B \ B \ C \ G \rangle \langle SAMS \ B \ C \ G \ H \ C \ G \rangle$
 $\langle B \ C \ G \ H \ C \ G \ SumA \ H \ C \ B \rangle$ *sams-assoc*)
qed
have $SAMS \ C \ G \ H \ H \ C \ G$
by (*simp add:* $\langle G \neq C \rangle \langle H \neq C \rangle \langle H \neq G \rangle$ *sams123231 sams-comm*)
moreover
have $C \ G \ H \ H \ C \ G \ SumA \ L \ M \ N$
by (*meson* $\langle H \ C \ G \ C \ G \ H \ SumA \ L \ M \ N \rangle$ *suma-sym*)
ultimately
show *?thesis*
using $\langle SAMS \ I \ J \ K \ C \ G \ H \rangle \langle I \ J \ K \ C \ G \ H \ SumA \ X \ Y \ Z \rangle$
 $\langle SAMS \ X \ Y \ Z \ H \ C \ G \rangle$ *sams-assoc-1* **by** *blast*
qed
moreover
have $H \ G \ B \ B \ C \ H \ SumA \ U \ V \ W$
proof –
have $X \ Y \ Z \ H \ C \ G \ SumA \ U \ V \ W$
proof –
have $SAMS \ C \ G \ H \ H \ C \ G$
by (*simp add:* $\langle G \neq C \rangle \langle H \neq C \rangle \langle H \neq G \rangle$
sams123231 sams-comm)
moreover
have $C \ G \ H \ H \ C \ G \ SumA \ L \ M \ N$
by (*simp add:* $\langle H \ C \ G \ C \ G \ H \ SumA \ L \ M \ N \rangle$ *suma-sym*)
ultimately
show *?thesis*
using $\langle SAMS \ I \ J \ K \ C \ G \ H \rangle \langle I \ J \ K \ C \ G \ H \ SumA \ X \ Y \ Z \rangle$
 $\langle I \ J \ K \ L \ M \ N \ SumA \ U \ V \ W \rangle$ *suma-assoc* **by** *blast*
qed
moreover
have $H \ G \ B \ B \ C \ G \ SumA \ X \ Y \ Z$
by (*simp add:* $\langle B \ C \ G \ H \ G \ B \ SumA \ X \ Y \ Z \rangle$ *suma-sym*)
moreover
have $B \ C \ G \ H \ C \ G \ SumA \ B \ C \ H$
by (*simp add:* $\langle B \ C \ G \ H \ C \ G \ SumA \ H \ C \ B \rangle$ *suma-right-comm*)
ultimately
show *?thesis*
using $\langle SAMS \ H \ G \ B \ B \ C \ G \rangle \langle SAMS \ B \ C \ G \ H \ C \ G \rangle$
suma-assoc **by** *blast*
qed
ultimately
show *?thesis* **by** *blast*
qed
hence $SAMS \ I \ J \ K \ L \ M \ N$

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by simp
have H G B B C H SumA U V W
using ⟨SAMS I J K L M N ∧ H G B B C H SumA U V W⟩ by simp
{
  fix A' B' C' D'
  assume Saccheri A' B' C' D'
  {
    assume hypothesis-of-acute-saccheri-quadrilaterals
    have U V W LtA U V W
    proof -
      have ¬ Col C G B
      using Col-cases ⟨¬ Col B C G⟩ by blast
      hence SAMS I J K G B C ∧ ¬ Bet O0 P Q
      using ⟨B C G C G B SumA I J K⟩ ⟨I J K G B C SumA O0 P Q⟩
      t22-14--sams-nbet
      ⟨hypothesis-of-acute-saccheri-quadrilaterals⟩ by blast
      have I J K LtA H G B
      proof -
        have G B C LeA G B C
        by (simp add: ⟨C ≠ B⟩ ⟨G ≠ B⟩ lea-refl)
        moreover
        have O0 P Q LtA A G B
        proof -
          have O0 P Q LeA A G B
          using ⟨A ≠ G⟩ ⟨Bet A G B⟩ ⟨G ≠ B⟩ ⟨O0 ≠ P⟩
          ⟨Q ≠ P⟩ l11-31-2 by force
          moreover
          {
            assume O0 P Q CongA A G B
            hence Bet O0 P Q
            using ⟨Bet A G B⟩ bet-lea--bet conga--lea456123 by blast
            hence False
            using ⟨SAMS I J K G B C ∧ ¬ Bet O0 P Q⟩ by blast
          }
          hence ¬ O0 P Q CongA A G B
          by auto
          ultimately
          show ?thesis
          by (simp add: LtA-def)
        qed
        moreover
        have SAMS I J K G B C
        by (simp add: ⟨SAMS I J K G B C ∧ ¬ Bet O0 P Q⟩)
        moreover
        have I J K G B C SumA O0 P Q
        by (simp add: ⟨I J K G B C SumA O0 P Q⟩)
        moreover
        have H G B G B C SumA A G B
        proof -
          have B G H H G A SumA A G B
          by (metis ⟨A Out B G⟩ ⟨Bet A G B⟩ ⟨G ≠ B⟩
            ⟨H ≠ G⟩ bet--suma between-symmetry l6-3-1
            suma-right-comm)
          moreover
          have B G H CongA H G B
          using ⟨G ≠ B⟩ ⟨H ≠ G⟩ conga-pseudo-refl by auto
          moreover
          have H G A CongA G B C
          by (simp add: ⟨A G H CongA G B C⟩ conga-left-comm)
          moreover
          have A G B CongA A G B
          using ⟨A ≠ G⟩ ⟨G ≠ B⟩ conga-refl by auto
          ultimately
          show ?thesis
          by (meson conga3-suma--suma)
        qed
      }
    }
  }

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ultimately
show ?thesis
  by (meson sams-lea-lta789-suma2--lta123)
qed
moreover
have  $\neg \text{Col } C \ G \ H$ 
  by (meson Col-perm  $\langle \neg \text{Col } G \ H \ C \rangle$ )
hence  $\text{SAMS } L \ M \ N \ G \ H \ C \wedge \neg \text{Bet } R \ S \ T$ 
  using  $\langle H \ C \ G \ C \ G \ H \text{ SumA } L \ M \ N \rangle \langle L \ M \ N \ G \ H \ C \text{ SumA } R \ S \ T \rangle$ 
  t22-14--sams-nbet  $\langle \text{hypothesis-of-acute-saccheri-quadrilaterals} \rangle$ 
  by blast
have  $L \ M \ N \text{ LtA } B \ C \ H$ 
proof -
  have  $G \ H \ C \text{ LeA } G \ H \ C$ 
  using  $\langle H \neq C \rangle \langle H \neq G \rangle$  lea-refl by auto
  moreover
  have  $R \ S \ T \text{ LtA } A \ H \ C$ 
  proof -
    have  $R \ S \ T \text{ LeA } A \ H \ C$ 
    using  $\langle \text{Bet } A \ H \ C \rangle \langle H \neq A \rangle \langle H \neq C \rangle \langle R \neq S \rangle$ 
     $\langle S \neq T \rangle$  l11-31-2 by auto
    moreover
    {
      assume  $R \ S \ T \text{ CongA } A \ H \ C$ 
      have  $\text{Bet } R \ S \ T$ 
      proof -
        have  $\text{Bet } A \ H \ C$ 
        by (simp add:  $\langle \text{Bet } A \ H \ C \rangle$ )
        moreover
        have  $A \ H \ C \text{ CongA } R \ S \ T$ 
        using  $\langle R \ S \ T \text{ CongA } A \ H \ C \rangle$  conga-sym-equiv by blast
        moreover
        have  $B \ C \ H \ G \ H \ C \text{ SumA } A \ H \ C$ 
        proof -
          have  $A \ H \ G \ G \ H \ C \text{ SumA } A \ H \ C$ 
          using  $\langle H \neq A \rangle \langle H \neq C \rangle \langle H \neq G \rangle$ 
          bet--suma calculation(1) by force
          moreover
          have  $A \ H \ G \text{ CongA } B \ C \ H$ 
          by (simp add:  $\langle G \ H \ A \text{ CongA } B \ C \ H \rangle$  conga-left-comm)
          moreover
          have  $G \ H \ C \text{ CongA } G \ H \ C$ 
          using  $\langle H \neq C \rangle \langle H \neq G \rangle$  conga-refl by auto
          moreover
          have  $A \ H \ C \text{ CongA } A \ H \ C$ 
          using  $\langle H \neq A \rangle \langle H \neq C \rangle$  conga-refl by auto
          ultimately
          show ?thesis
          by (meson conga3-suma--suma)
        qed
        ultimately
        show ?thesis
        using bet-conga--bet by blast
      qed
    }
  hence  $\neg R \ S \ T \text{ CongA } A \ H \ C$ 
  by auto
  ultimately
  show ?thesis
  using LtA-def by blast
qed
moreover
have  $\text{SAMS } L \ M \ N \ G \ H \ C$ 
  by (simp add:  $\langle \text{SAMS } L \ M \ N \ G \ H \ C \wedge \neg \text{Bet } R \ S \ T \rangle$ )

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moreover
have  $L\ M\ N\ G\ H\ C\ SumA\ R\ S\ T$ 
  using  $\langle L\ M\ N\ G\ H\ C\ SumA\ R\ S\ T \rangle$  by auto
moreover
have  $B\ C\ H\ G\ H\ C\ SumA\ A\ H\ C$ 
proof –
  have  $A\ H\ G\ G\ H\ C\ SumA\ A\ H\ C$ 
    using  $\langle H \neq A \rangle \langle H \neq C \rangle \langle H \neq G \rangle$ 
      bet--suma calculation(1)  $\langle Bet\ A\ H\ C \rangle$  by auto
  moreover
  have  $A\ H\ G\ CongA\ B\ C\ H$ 
    by (simp add:  $\langle G\ H\ A\ CongA\ B\ C\ H \rangle$  conga-left-comm)
  moreover
  have  $G\ H\ C\ CongA\ G\ H\ C$ 
    using  $\langle H \neq C \rangle \langle H \neq G \rangle$  conga-refl by auto
  moreover
  have  $A\ H\ C\ CongA\ A\ H\ C$ 
    using  $\langle H \neq A \rangle \langle H \neq C \rangle$  conga-refl by auto
  ultimately
  show ?thesis
    by (meson conga3--suma--suma)
qed
ultimately
show ?thesis
  by (meson sams-lea-lta789--suma2--lta123)
qed
moreover
have  $SAMS\ H\ G\ B\ B\ C\ H$ 
  by (simp add:  $\langle SAMS\ B\ G\ H\ H\ C\ B \rangle$  sams-comm)
ultimately
show ?thesis
  by (meson  $\langle I\ J\ K\ L\ M\ N\ SumA\ U\ V\ W \rangle$ 
     $\langle H\ G\ B\ B\ C\ H\ SumA\ U\ V\ W \rangle$  lta--lea
    sams-lea-lta456--suma2--lta)
qed
hence False
  by (simp add: nltA)
hence  $Per\ A'\ B'\ C'$ 
  by simp
}
moreover
{
  assume hypothesis-of-right-saccheri-quadrilaterals
  hence  $Per\ A'\ B'\ C'$ 
    using  $\langle Saccheri\ A'\ B'\ C'\ D' \rangle$ 
      hypothesis-of-right-saccheri-quadrilaterals-def
    by blast
}
moreover
{
  assume hypothesis-of-obtuse-saccheri-quadrilaterals
  have  $U\ V\ W\ LtA\ U\ V\ W$ 
proof –
  have  $H\ G\ B\ LtA\ I\ J\ K$ 
proof –
  {
    assume  $I\ J\ K\ LeA\ H\ G\ B$ 
    have False
    proof –
    have  $\neg\ Col\ C\ G\ B$ 
      using  $\langle \neg\ Col\ B\ C\ G \rangle$  col-permutation-2 by blast
    moreover
    have  $B\ C\ G\ C\ G\ B\ SumA\ I\ J\ K$ 
      by (simp add:  $\langle B\ C\ G\ C\ G\ B\ SumA\ I\ J\ K \rangle$ )
    moreover
    have  $SAMS\ I\ J\ K\ G\ B\ C$ 

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proof -
  have  $SAMS\ H\ G\ B\ G\ B\ C$ 
    by (simp add:  $\langle SAMS\ B\ G\ H\ C\ B\ G \rangle$  sams-comm)
  moreover
  have  $G\ B\ C\ LeA\ G\ B\ C$ 
    using  $\langle C \neq B \rangle \langle G \neq B \rangle$  lea-refl by blast
  ultimately
  show ?thesis
    using  $\langle I\ J\ K\ LeA\ H\ G\ B \rangle$  sams-lea2--sams by blast
qed
ultimately show ?thesis
  using t22-14--nsams
     $\langle HypothesisObtuseSaccheriQuadrilaterals \rangle$ 
  by blast
qed
}
hence  $\neg\ I\ J\ K\ LeA\ H\ G\ B$ 
  by auto
thus ?thesis
  using  $\langle G \neq B \rangle \langle H \neq G \rangle \langle I \neq J \rangle \langle J \neq K \rangle$  nlea--lta by blast
qed
moreover
{
  assume  $L\ M\ N\ LeA\ B\ C\ H$ 
  have False
  proof -
    have  $\neg\ Col\ C\ G\ H$ 
      using  $\langle \neg\ Col\ G\ H\ C \rangle$  col-permutation-1 by blast
    moreover
    have  $H\ C\ G\ C\ G\ H\ SumA\ L\ M\ N$ 
      by (simp add:  $\langle H\ C\ G\ C\ G\ H\ SumA\ L\ M\ N \rangle$ )
    moreover
    have  $SAMS\ L\ M\ N\ G\ H\ C$ 
    proof -
      have  $SAMS\ B\ C\ H\ G\ H\ C$ 
        by (meson  $\langle SAMS\ C\ H\ G\ B\ C\ H \rangle$  sams-right-comm sams-sym)
      moreover
      have  $G\ H\ C\ LeA\ G\ H\ C$ 
        using  $\langle H \neq C \rangle \langle H \neq G \rangle$  lea-refl by auto
      ultimately
      show ?thesis
        using  $\langle L\ M\ N\ LeA\ B\ C\ H \rangle$  sams-lea2--sams by blast
    qed
    ultimately
    show ?thesis
      using  $\langle HypothesisObtuseSaccheriQuadrilaterals \rangle$ 
        t22-14--nsams by blast
    qed
  }
  hence  $\neg\ L\ M\ N\ LeA\ B\ C\ H$ 
    by auto
  hence  $B\ C\ H\ LtA\ L\ M\ N$ 
    using  $\langle C \neq B \rangle \langle H \neq C \rangle \langle L \neq M \rangle \langle N \neq M \rangle$  nlta--lea by blast
  ultimately
  show ?thesis
    by (meson  $\langle SAMS\ I\ J\ K\ L\ M\ N \rangle \langle H\ G\ B\ B\ C\ H\ SumA\ U\ V\ W \rangle$ 
       $\langle I\ J\ K\ L\ M\ N\ SumA\ U\ V\ W \rangle$  lta--lea
      sams-lea-lta456-suma2--lta)
qed
hence  $Per\ A'\ B'\ C'$ 
  by (simp add: nlta)
}
ultimately
have  $Per\ A'\ B'\ C'$ 
  using saccheri-s-three-hypotheses by blast
}

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thus ?thesis
  using postulate-of-right-saccheri-quadrilaterals-def by blast
qed

lemma similar--rah:
  assumes postulate-of-existence-of-similar-triangles
  shows postulate-of-right-saccheri-quadrilaterals
proof -
  obtain A B C D E F where
     $\neg$  Col A B C and
     $\neg$  Cong A B D E and
    A B C CongA D E F and
    B C A CongA E F D and
    C A B CongA F D E
  using assms postulate-of-existence-of-similar-triangles-def by blast
  have A  $\neq$  B
  using  $\langle \neg$  Col A B C  $\rangle$  col-trivial-1 by blast
  have C  $\neq$  B
  using  $\langle \neg$  Col A B C  $\rangle$  col-trivial-2 by force
  have A  $\neq$  C
  using  $\langle \neg$  Col A B C  $\rangle$  col-trivial-3 by blast
  have D  $\neq$  F
  using  $\langle$  B C A CongA E F D  $\rangle$  conga-diff56 by blast
  have E  $\neq$  F
  using  $\langle$  A B C CongA D E F  $\rangle$  conga-diff56 by blast
  have D  $\neq$  E
  using  $\langle$  C A B CongA F D E  $\rangle$  conga-distinct by blast
  {
    assume B C A LeA A B C
    {
      assume D E Le A B
      hence D E Lt A B
      using  $\langle \neg$  Cong A B D E  $\rangle$  le-anti-symmetry nle--lt by blast
      hence postulate-of-right-saccheri-quadrilaterals
      using  $\langle \neg$  Col A B C  $\rangle$   $\langle$  A B C CongA D E F  $\rangle$ 
         $\langle$  B C A CongA E F D  $\rangle$   $\langle$  C A B CongA F D E  $\rangle$   $\langle$  B C A LeA A B C  $\rangle$ 
        similar--rah-aux by blast
    }
  }
  moreover
  {
    assume A B Le D E
    hence A B Lt D E
    by (simp add: Lt-def  $\langle \neg$  Cong A B D E  $\rangle$ )
    moreover
    have  $\neg$  Col D E F
    using  $\langle \neg$  Col A B C  $\rangle$   $\langle$  A B C CongA D E F  $\rangle$  ncol-conga-ncol by blast
    moreover
    have E F D LeA D E F
    using  $\langle$  B C A LeA A B C  $\rangle$   $\langle$  B C A CongA E F D  $\rangle$ 
       $\langle$  A B C CongA D E F  $\rangle$  l11-30 by blast
    moreover
    have D E F CongA A B C
    using  $\langle$  A B C CongA D E F  $\rangle$  conga-sym by blast
    moreover
    have E F D CongA B C A
    by (simp add:  $\langle$  B C A CongA E F D  $\rangle$  conga-sym)
    moreover
    have F D E CongA C A B
    by (simp add:  $\langle$  C A B CongA F D E  $\rangle$  conga-sym)
    ultimately
    have postulate-of-right-saccheri-quadrilaterals
    using similar--rah-aux by blast
  }
  ultimately
  have postulate-of-right-saccheri-quadrilaterals
  using local.le-cases by blast

```

```

}
moreover
{
  assume  $A B C \text{ Le } A B C A$ 
  {
    assume  $D F \text{ Le } A C$ 
    {
      assume  $\text{Cong } D F A C$ 
      have  $\text{False}$ 
      proof –
        have  $\neg \text{Col } A C B$ 
        using  $\text{Col-cases } \langle \neg \text{Col } A B C \rangle$  by blast
        moreover
        have  $A C B \text{ Cong } A D F E$ 
        using  $\langle B C A \text{ Cong } A E F D \rangle$  conga-comm by blast
        moreover
        have  $\text{Cong } A C D F$ 
        using  $\text{Cong-cases } \langle \text{Cong } D F A C \rangle$  by blast
        ultimately
        show ?thesis
        using  $\langle C A B \text{ Cong } A F D E \rangle \langle \neg \text{Cong } A B D E \rangle$ 
        l11-50-1 by blast
      qed
    }
    hence  $D F \text{ Lt } A C$ 
    using  $\text{Lt-def } \langle D F \text{ Le } A C \rangle$  by blast
    moreover
    have  $\neg \text{Col } A C B$ 
    using  $\text{Col-cases } \langle \neg \text{Col } A B C \rangle$  by blast
    moreover
    have  $A C B \text{ Cong } A D F E$ 
    by (simp add:  $\langle B C A \text{ Cong } A E F D \rangle$  conga-comm)
    moreover
    have  $C B A \text{ Cong } A F E D$ 
    by (simp add:  $\langle A B C \text{ Cong } A D E F \rangle$  conga-comm)
    moreover
    have  $B A C \text{ Cong } A E D F$ 
    by (meson  $\langle C A B \text{ Cong } A F D E \rangle$  conga-comm)
    moreover
    have  $C B A \text{ Le } A A C B$ 
    by (simp add:  $\langle A B C \text{ Le } A B C A \rangle$  lea-comm)
    ultimately
    have  $\text{postulate-of-right-saccheri-quadrilaterals}$ 
    using  $\text{similar--rah-aux}$  by blast
  }
moreover
{
  assume  $A C \text{ Le } D F$ 
  have  $\neg \text{Col } D F E$ 
  proof –
    have  $\neg \text{Col } A C B$ 
    using  $\text{Col-cases } \langle \neg \text{Col } A B C \rangle$  by blast
    moreover
    have  $A C B \text{ Cong } A D F E$ 
    using  $\langle B C A \text{ Cong } A E F D \rangle$  conga-comm by blast
    ultimately
    show ?thesis
    using  $\text{ncol-conga-ncol}$  by blast
  qed
  moreover
  have  $D F E \text{ Cong } A A C B$ 
  by (meson  $\langle B C A \text{ Cong } A E F D \rangle$  conga-comm conga-sym)
  moreover
  have  $F E D \text{ Cong } A C B A$ 
  by (meson  $\langle A B C \text{ Cong } A D E F \rangle$  conga-comm conga-sym)
  moreover

```

```

have E D F CongA B A C
  by (meson ⟨C A B CongA F D E⟩ conga-comm conga-sym)
moreover
have F E D LeA D F E
proof -
  have A B C LeA B C A
    by (simp add: ⟨A B C LeA B C A⟩)
  moreover
  have A B C CongA F E D
    by (simp add: ⟨A B C CongA D E F⟩ conga-right-comm)
  moreover
  have B C A CongA D F E
    by (simp add: ⟨B C A CongA E F D⟩ conga-right-comm)
  ultimately
  show ?thesis
    using l11-30 by blast
qed
moreover
{
  assume Cong A C D F
  have ¬ Col A C B
    using Col-cases ⟨¬ Col A B C⟩ by auto
  moreover
  have A C B CongA D F E
    by (simp add: ⟨B C A CongA E F D⟩ conga-comm)
  ultimately
  have False
    using ⟨Cong A C D F⟩ ⟨¬ Cong A B D E⟩
      ⟨C A B CongA F D E⟩ l11-50-1 by blast
}
hence A C Lt D F
  using Lt-def ⟨A C Le D F⟩ by blast
ultimately
have postulate-of-right-saccheri-quadrilaterals
  using similar--rah-aux by blast
}
ultimately
have postulate-of-right-saccheri-quadrilaterals
  using local.le-cases by blast
}
ultimately
show ?thesis
  by (metis ⟨A ≠ B⟩ ⟨A ≠ C⟩ ⟨C ≠ B⟩ lea-total)
qed

```

lemma impossible-case-5:

shows $\forall P Q R S U I.$

$\neg (BetS Q U R \wedge \neg Col P Q S \wedge$
 $\neg Col P R U \wedge P R Par Q S \wedge$
 $Bet S Q I \wedge Bet U I P)$

proof -

```

{
  fix P Q R S U I
  assume 1: BetS Q U R and
    2: ¬ Col P Q S and
    3: ¬ Col P R U and
    4: P R Par Q S and
    5: Bet S Q I and
    6: Bet U I P
  have Bet Q U R
    by (simp add: 1 BetSEq)
  have Q ≠ U
    using 1 BetSEq by blast
  have Q ≠ R
    using 1 BetSEq by blast
  have R ≠ U

```

```

    using 1 BetSEq by blast
  have Q S Par P R
    by (simp add: 4 par-symmetry)
  have Q S ParStrict P R
    using 2 4 Par-def par-strict-symmetry by blast
  have Q S OS R U
    by (metis 4 ⟨Bet Q U R⟩ ⟨Q S ParStrict P R⟩
      ⟨Q ≠ U⟩ bet2--out col-trivial-2 invert-one-side
      l5-1 l9-19-R2 not-strict-par par-right-comm
      par-strict-not-col-3)
  have False
  proof -
    have Q S TS P U
      by (metis 2 5 6 NCol-perm ⟨Q S OS R U⟩
        bet-col between-symmetry col124--nos l9-18)
    moreover
    have Q S OS P U
      by (metis ⟨Bet Q U R⟩ ⟨Q S ParStrict P R⟩
        ⟨Q ≠ R⟩ ⟨Q ≠ U⟩ bet2--out l12-6 l5-1
        out-out-one-side)
    ultimately
    show ?thesis
      using l9-9-bis by blast
  qed
}
thus ?thesis by blast
qed

lemma impossible-case-6:
  shows  $\forall P Q R S U I.$ 
     $\neg (BetS Q U R \wedge \neg Col P Q S \wedge$ 
       $P S Par Q R \wedge Bet S Q I \wedge Bet I P U)$ 
  proof -
    {
      fix P Q R S U I
      assume 1: BetS Q U R and
      2:  $\neg Col P Q S$  and
      3: P S Par Q R and
      4: Bet S Q I and
      5: Bet I P U
      have Bet Q U R
        by (simp add: 1 BetSEq)
      have Q ≠ U
        using 1 BetSEq by blast
      have Q ≠ R
        using 1 BetSEq by blast
      have R ≠ U
        using 1 BetSEq by blast
      have Bet U P I
        using 5 Bet-cases by blast
      then obtain J where Bet Q J U and Bet P J S
        using 4 inner-pasch by blast
      have P S ParStrict Q U
      proof -
        have P S ParStrict Q R
          using 2 3 Col-cases col-trivial-3 par-not-col-strict by blast
        moreover
        have Col Q R U
          by (simp add: ⟨Bet Q U R⟩ bet-col col-permutation-5)
        ultimately
        show ?thesis
          by (meson ⟨Q ≠ U⟩ par-strict-col-par-strict)
      qed
      hence False
        using 4 5 impossible-case-1 par-strict-right-comm by blast
    }
  
```

```

thus ?thesis
by blast
qed

lemma impossible-case-7:
shows  $\forall P Q R S U I.$ 
   $\neg (BetS\ Q\ U\ R \wedge \neg Col\ P\ Q\ S \wedge$ 
     $\neg Col\ P\ R\ U \wedge P\ R\ Par\ Q\ S \wedge$ 
     $P\ S\ Par\ Q\ R \wedge Col\ P\ U\ I \wedge Bet\ Q\ I\ S)$ 

proof –
{
  fix P Q R S U I
  assume 1:BetS Q U R and
    2:  $\neg Col\ P\ Q\ S$  and
    3:  $\neg Col\ P\ R\ U$  and
    4: P R Par Q S and
    5: P S Par Q R and
    6: Col P U I and
    7: Bet Q I S
  have Bet Q U R
    by (simp add: 1 BetSEq)
  have Q  $\neq$  U
    using 1 BetSEq by blast
  have Q  $\neq$  R
    using 1 BetSEq by blast
  have R  $\neq$  U
    using 1 BetSEq by blast
  have Bet R U Q
    using Bet-cases  $\langle Bet\ Q\ U\ R \rangle$  by auto
  have P U OS Q S
  proof –
    have P U TS Q R
      by (metis 3  $\langle Bet\ Q\ U\ R \rangle$   $\langle Q \neq U \rangle$  bet-col
        col-permutation-2 col-transitivity-1
        col-trivial-2 l9-18-R2)
    moreover
    have P U TS S R
    proof –
      have P S OS R Q
      proof –
        have P S Par R Q
          by (simp add: 5 par-right-comm)
        moreover
        have Col R Q Q
          by (simp add: col-trivial-2)
        moreover
        have  $\neg Col\ P\ S\ Q$ 
          by (simp add: 2 not-col-permutation-5)
        ultimately
        show ?thesis
          using par-not-col-strict par-strict-symmetry
            pars--os3412 by blast
      qed
    hence P S OS U R
      using  $\langle Bet\ R\ U\ Q \rangle$  l9-17 one-side-symmetry by blast
    moreover
    have P R OS U S
    proof –
      have P R OS U Q
        by (metis 3  $\langle Bet\ R\ U\ Q \rangle$  bet-out col-trivial-2 l9-19-R2)
      moreover
      have P R OS Q S
        by (meson 2 4 col-trivial-3
          not-col-permutation-1 par-not-col-strict
          par-symmetry pars--os3412)
      ultimately

```

```

    show ?thesis
    using one-side-transitivity by blast
qed
ultimately
show ?thesis
  by (simp add: l9-31)
qed
ultimately
show ?thesis
  using l9-8-1 by blast
qed
hence  $P \cup OS \subseteq Q \cap I$ 
  using 7 l9-17 by blast
hence False
  using 6 one-side-not-col124 by auto
}
thus ?thesis by blast
qed

lemma impossible-case-8:
shows  $\forall P Q R S U I.$ 
   $\neg (BetS \ Q \ U \ R \wedge \neg Col \ P \ Q \ S \wedge$ 
     $P \ R \ Par \ Q \ S \wedge P \ S \ Par \ Q \ R \wedge$ 
     $Col \ P \ U \ I \wedge Bet \ I \ S \ Q)$ 
proof -
{
  fix  $P \ Q \ R \ S \ U \ I$ 
  assume 1:  $BetS \ Q \ U \ R$  and
    2:  $\neg Col \ P \ Q \ S$  and
    3:  $P \ R \ Par \ Q \ S$  and
    4:  $P \ S \ Par \ Q \ R$  and
    5:  $Col \ P \ U \ I$  and
    6:  $Bet \ I \ S \ Q$ 
  have  $Bet \ Q \ U \ R$ 
    by (simp add: 1 BetSEq)
  have  $Q \neq U$ 
    using 1 BetSEq by blast
  have  $Q \neq R$ 
    using 1 BetSEq by blast
  have  $R \neq U$ 
    using 1 BetSEq by blast
  have  $P \ S \ ParStrict \ Q \ U$ 
  proof -
    have  $P \ S \ ParStrict \ Q \ R$ 
      using 2 4 Col-cases col-trivial-3
      par-not-col-strict by blast
    moreover
    have  $Col \ Q \ R \ U$ 
      using Col-perm  $\langle Bet \ Q \ U \ R \rangle$  bet-col by blast
    ultimately
    show ?thesis
      by (meson  $\langle Q \neq U \rangle$  par-strict-col-par-strict)
  qed
  have  $Bet \ Q \ S \ I$ 
    using 6 Bet-cases by auto
  {
    assume  $Bet \ P \ U \ I$ 
    hence False
      by (meson Par-strict-cases  $\langle Bet \ Q \ S \ I \rangle$ 
         $\langle P \ S \ ParStrict \ Q \ U \rangle$  between-symmetry impossible-case-1)
  }
  moreover
  {
    assume  $Bet \ U \ I \ P$ 
    then obtain  $J$  where  $Bet \ U \ J \ Q$  and  $Bet \ P \ S \ J$ 
      using  $\langle Bet \ Q \ S \ I \rangle$  outer-pasch by blast
  }
}

```

```

    hence False
    by (metis ‹Bet Q U R› ‹P S ParStrict Q U›
        bet-col bet-col1 between-symmetry l12-6 l5-3
        l9-19 not-bet-and-out)
  }
  moreover
  {
    assume Bet I P U
    then obtain J where Bet Q J I ∧ Bet R P J
    using ‹Bet Q U R› outer-pasch by blast
    have P R ParStrict Q I
    using 2 3 Par-def ‹Bet Q S I› bet-col
        bet-neq32--neq col2--eq
        par-strict-col2-par-strict by blast
    hence False
    by (meson 3 ‹Bet Q J I ∧ Bet R P J›
        ‹Bet Q S I› bet-col bet-col1 inter-uniqueness-not-par
        par-left-comm par-strict-comm par-strict-not-col-4)
  }
  ultimately
  have False
  using 5 Col-def by blast
}
thus ?thesis
by blast
qed

```

lemma *strong-parallel-postulate-implies-tarski-s-euclid-aux:*

```

assumes strong-parallel-postulate and
  A ≠ B and
  A ≠ D and
  A ≠ T and
  B ≠ D and
  B ≠ T and
  D ≠ T and
  ¬ Col A B T and
  Bet A D T
shows ∃ B' B'' MB X.
  (Bet A B X ∧ T X ParStrict B D ∧
   BetS B MB T ∧ BetS B' MB B'' ∧
   Cong B MB T MB ∧ Cong B' MB B'' MB ∧
   Col B B' D ∧ Bet B'' T X ∧
   B ≠ B' ∧ B'' ≠ T)

```

proof –

```

obtain B' where B Midpoint D B'
using symmetric-point-construction by blast
have D ≠ B'
using ‹B Midpoint D B'› assms(5) l7-3 by blast
have B ≠ B'
using ‹B Midpoint D B'› ‹D ≠ B'› midpoint-not-midpoint by blast
moreover
have Bet D B B'
by (simp add: ‹B Midpoint D B'› midpoint-bet)
have Cong D B B B'
using ‹B Midpoint D B'› midpoint-cong by auto
have Bet T D A
using Bet-cases assms(9) by blast
have Bet B' B D
using Bet-cases ‹Bet D B B'› by blast
obtain MB where MB Midpoint B T
using midpoint-existence by blast
hence BetS B MB T
by (metis BetS-def assms(6) midpoint-bet midpoint-distinct-1)
moreover
have Cong B MB T MB
using Midpoint-def ‹MB Midpoint B T› not-cong-1243 by blast

```

```

moreover
have  $MB \neq B$ 
  using  $\langle MB \text{ Midpoint } B \ T \rangle$   $assms(6)$  is-midpoint-id by blast
have  $MB \neq T$ 
  using BetSEq calculation(2) by blast
obtain  $B''$  where  $MB \text{ Midpoint } B' \ B''$ 
  using symmetric-point-construction by blast
have  $MB \neq B'$ 
  using  $\langle Bet \ D \ B \ B' \rangle$   $\langle MB \text{ Midpoint } B \ T \rangle$   $assms(8)$   $assms(9)$ 
    bet-col between-exchange2 calculation(1) midpoint-bet
    outer-transitivity-between by blast
hence  $BetS \ B' \ MB \ B''$ 
  by (metis BetS-def  $\langle MB \text{ Midpoint } B' \ B'' \rangle$  midpoint-bet
    midpoint-not-midpoint)
hence  $B' \neq B''$ 
  using BetSEq by blast
have  $MB \neq B''$ 
  using BetSEq  $\langle BetS \ B' \ MB \ B'' \rangle$  by blast
have  $Cong \ B' \ MB \ B'' \ MB$ 
  using Midpoint-def  $\langle MB \text{ Midpoint } B' \ B'' \rangle$ 
    cong-right-commutativity by blast
moreover
have  $\neg \text{Col } B \ T \ B'$ 
  by (metis Inter-def  $\langle Bet \ D \ B \ B' \rangle$   $assms(7)$ 
     $assms(8)$   $assms(9)$  bet-col calculation(1)
    col-trivial-3 inter-trivial
    l6-21 not-col-permutation-5)
obtain  $B'''$  where  $Bet \ T \ B''' \ B'$  and  $Bet \ A \ B \ B'''$ 
  using  $\langle Bet \ B' \ B \ D \rangle$   $\langle Bet \ T \ D \ A \rangle$  outer-pasch by blast
have  $B' \neq B'''$ 
  by (metis  $\langle Bet \ A \ B \ B''' \rangle$   $\langle Bet \ B' \ B \ D \rangle$   $\langle \neg \text{Col } B \ T \ B' \rangle$ 
     $assms(3)$   $assms(9)$  bet-col bet-out-1
    col-permutation-1 colx out-col)
have  $B''' \neq T$ 
  using  $\langle Bet \ A \ B \ B''' \rangle$   $assms(8)$  bet-col by blast
hence  $BetS \ T \ B''' \ B'$ 
  using BetS-def  $\langle B' \neq B''' \rangle$   $\langle Bet \ T \ B''' \ B' \rangle$  by presburger
have  $\neg \text{Col } B \ B' \ B'''$ 
  by (meson Col-perm  $\langle B' \neq B''' \rangle$   $\langle Bet \ T \ B''' \ B' \rangle$ 
     $\langle \neg \text{Col } B \ T \ B' \rangle$  bet-col col2--eq)
have  $Coplanar \ B \ T \ B' \ B'''$ 
  by (meson  $\langle Bet \ T \ B''' \ B' \rangle$  bet-col col--coplanar ncoplanar-perm-11)
hence  $\exists X. (\text{Col } B'' \ T \ X \wedge \text{Col } B \ B''' \ X)$ 
  using  $assms(1)$  calculation(2)  $\langle BetS \ B' \ MB \ B'' \rangle$ 
     $\langle \neg \text{Col } B \ B' \ B''' \rangle$   $\langle Cong \ B \ MB \ T \ MB \rangle$   $\langle Cong \ B' \ MB \ B'' \ MB \rangle$ 
    strong-parallel-postulate-def by blast
then obtain  $X$  where  $\text{Col } B'' \ T \ X$  and  $\text{Col } B \ B''' \ X$ 
  by blast
have  $B \ B' \ Par \ T \ B''$ 
  using l12-17 calculation(1)  $\langle MB \text{ Midpoint } B \ T \rangle$ 
     $\langle MB \text{ Midpoint } B' \ B'' \rangle$  by blast
have  $B \neq B''$ 
  using  $\langle B \ B' \ Par \ T \ B'' \rangle$   $\langle \neg \text{Col } B \ T \ B' \rangle$  par-id-3
    par-right-comm by blast
hence  $B \ B'' \ Par \ T \ B'$ 
  using l12-17
  by (meson  $\langle MB \text{ Midpoint } B \ T \rangle$   $\langle MB \text{ Midpoint } B' \ B'' \rangle$  mid-par-cong2)
{
  assume  $Bet \ B'' \ T \ X$ 
  {
    assume  $Bet \ B \ B''' \ X$ 
    hence  $BetS \ B'' \ T \ X$ 
    by (metis BetS-def  $\langle B \ B' \ Par \ T \ B'' \rangle$   $\langle Bet \ A \ B \ B''' \rangle$ 
       $\langle Bet \ B' \ B \ D \rangle$   $\langle Bet \ B'' \ T \ X \rangle$   $\langle \neg \text{Col } B \ B' \ B''' \rangle$   $assms(8)$ 
      bet-col bet-col1 not-col-permutation-4
      outer-transitivity-between par-distincts)
  }
}

```



```

have BetS B B''' X
  by (metis BetS-def ⟨B B' Par T B''⟩ ⟨Bet B B''' X⟩
    ⟨Bet T B''' B'⟩ ⟨BetS B'' T X⟩ ⟨¬ Col B B' B'''⟩
    ⟨¬ Col B T B'⟩ bet-col col-permutation-5 not-col-distincts
    not-strict-par outer-transitivity-between par-right-comm)
have B'' ≠ T
  using BetSeq ⟨BetS B'' T X⟩ by blast
have T ≠ X
  using BetSeq ⟨BetS B'' T X⟩ by blast
have ?thesis
proof -
  have Bet A B X
    using BetSeq ⟨Bet A B B'''⟩ ⟨BetS B B''' X⟩
    outer-transitivity-between by blast
  moreover
  have T X ParStrict B D
  proof -
    have T X ParStrict B B'
  proof -
    have B B' ParStrict T B''
      by (metis Par-def ⟨B B' Par T B''⟩ ⟨¬ Col B T B'⟩
        not-col-distincts par-col2-par-bis par-id-3)
    moreover
    have Col T B'' X
      using Col-cases ⟨Col B'' T X⟩ by blast
    ultimately
    show ?thesis
      by (meson ⟨T ≠ X⟩ par-strict-col-par-strict
        par-strict-symmetry)
  qed
  moreover
  have Col B B' D
    by (simp add: Col-def ⟨Bet D B B'⟩)
  ultimately
  show ?thesis
    using assms(5) par-strict-col-par-strict by blast
  qed
  moreover
  have Col B B' D
    by (simp add: Col-def ⟨Bet D B B'⟩)
  ultimately
  show ?thesis
    using ⟨Bet B'' T X⟩ ⟨Cong B' MB B'' MB⟩ ⟨B ≠ B'⟩
    ⟨B'' ≠ T⟩ ⟨BetS B MB T⟩ ⟨BetS B' MB B''⟩
    ⟨Cong B MB T MB⟩ by blast
  qed
}
moreover
{
  assume Bet B''' X B
  have ¬ Col B T B''
    by (meson ⟨B B' Par T B''⟩ ⟨¬ Col B T B'⟩ assms(6)
      col-trivial-2 par-col2-par-bis par-id-3)
  hence False
    using impossible-case-5 ⟨BetS T B''' B'⟩ ⟨¬ Col B B' B'''⟩
    ⟨B B' Par T B''⟩ ⟨Bet B'' T X⟩ ⟨Bet B''' X B⟩ by blast
  hence ?thesis
    by blast
}
moreover
{
  assume Bet X B B'''
  have ¬ Col B T B''
    using ⟨B B' Par T B''⟩ ⟨¬ Col B T B'⟩ assms(6)
    col-trivial-2 par-col2-par-bis par-id-3 by blast
  hence ?thesis

```

```

    using impossible-case-6  $\langle \text{BetS } T B''' B' \rangle \langle B B'' \text{ Par } T B' \rangle$ 
       $\langle \text{Bet } B'' T X \rangle \langle \text{Bet } X B B''' \rangle$  by blast
  }
  ultimately
  have ?thesis
    using Bet-perm  $\langle \text{Col } B B''' X \rangle$  third-point by blast
}
moreover
{
  assume Bet T X B''
  have  $\neg \text{Col } B T B''$ 
    by (meson  $\langle B B' \text{ Par } T B'' \rangle \langle \neg \text{Col } B T B' \rangle$  assms(6)
      col-trivial-2 par-col2-par-bis par-id-3)
  hence ?thesis
    using impossible-case-7  $\langle \text{BetS } T B''' B' \rangle \langle \neg \text{Col } B B' B''' \rangle$ 
       $\langle B B' \text{ Par } T B'' \rangle \langle B B'' \text{ Par } T B' \rangle \langle \text{Col } B B''' X \rangle$ 
       $\langle \text{Bet } T X B'' \rangle$  by blast
}
moreover
{
  assume Bet X B'' T
  have  $\neg \text{Col } B T B''$ 
    by (meson  $\langle B B' \text{ Par } T B'' \rangle \langle \neg \text{Col } B T B' \rangle$  assms(6)
      col-trivial-2 par-col2-par-bis par-id-3)
  hence ?thesis
    using impossible-case-8  $\langle \text{BetS } T B''' B' \rangle \langle B B' \text{ Par } T B'' \rangle$ 
       $\langle B B'' \text{ Par } T B' \rangle \langle \text{Bet } X B'' T \rangle \langle \text{Col } B B''' X \rangle$  by blast
}
ultimately
show ?thesis
  using Bet-perm  $\langle \text{Col } B'' T X \rangle$  third-point by blast
qed

```

lemma strong-parallel-postulate-implies-tarski-s-euclid :

assumes strong-parallel-postulate

shows tarski-s-parallel-postulate

proof –

```

{
  fix A' B' C' D' T'
  assume 1: Bet A' D' T' and
    2: Bet B' D' C' and
    3: A'  $\neq$  D'
  {
    fix A B C D T
    assume H1: A  $\neq$  B and
      H2: A  $\neq$  C and
      H3: A  $\neq$  D and
      H4: A  $\neq$  T and
      H5: B  $\neq$  C and
      H6: B  $\neq$  D and
      H7: B  $\neq$  T and
      H8: C  $\neq$  D and
      H9: C  $\neq$  T and
      H10: D  $\neq$  T and
      H11:  $\neg \text{Col } A B C$  and
      H12: Bet A D T and
      H13: Bet B D C and
      H14:  $\neg \text{Col } B C T$ 
    have  $\neg \text{Col } A B T$ 
      by (meson H11 H12 H13 H3 H6 between-trivial2
        impossible-case-2)
    moreover
    have  $\neg \text{Col } A C T$ 
      using H11 H12 H13 H3 H8 between-trivial
        impossible-two-sides-not-col by blast
    ultimately

```

```

have  $\exists B' B'' MB X$ .
  (Bet A B X  $\wedge$  T X ParStrict B D  $\wedge$ 
   BetS B MB T  $\wedge$  BetS B' MB B''  $\wedge$ 
   Cong B MB T MB  $\wedge$  Cong B' MB B'' MB  $\wedge$ 
   Col B B' D  $\wedge$  Bet B'' T X  $\wedge$ 
    $B \neq B' \wedge B'' \neq T$ )
  using assms strong-parallel-postulate-implies-tarski-s-euclid-aux
    H1 H2 H3 H4 H5 H6 H7 H8 H9 H10 H11 H12 H13 H14 by blast
then obtain  $B' B'' MB X$  where Bet A B X and
  T X ParStrict B D and
  BetS B MB T and
  BetS B' MB B'' and
  Cong B MB T MB and
  Cong B' MB B'' MB and
  Col B B' D and
  Bet B'' T X and
   $B \neq B'$  and
   $B'' \neq T$ 
  by auto
have  $\neg Col A B T$ 
  by (meson H11 H12 H13 H3 H6 between-trivial2 impossible-case-2)
moreover
have  $\neg Col A C T$ 
  using H11 H12 H13 H3 H8 between-trivial
    impossible-two-sides-not-col by blast
ultimately
have  $\exists B' B'' MB X$ .
  (Bet A C X  $\wedge$  T X ParStrict C D  $\wedge$ 
   BetS C MB T  $\wedge$  BetS B' MB B''  $\wedge$ 
   Cong C MB T MB  $\wedge$  Cong B' MB B'' MB  $\wedge$ 
   Col C B' D  $\wedge$  Bet B'' T X  $\wedge$ 
    $C \neq B' \wedge B'' \neq T$ )
  using assms strong-parallel-postulate-implies-tarski-s-euclid-aux
    H1 H2 H3 H4 H5 H6 H7 H8 H9 H10 H11 H12 H13 H14 by blast
then obtain  $B9' B9'' MB9 Y$  where Bet A C Y and
  T Y ParStrict C D and
  BetS C MB9 T and
  BetS B9' MB9 B9'' and
  Cong C MB9 T MB9 and
  Cong B9' MB9 B9'' MB9 and
  Col C B9' D and
  Bet B9'' T Y and
   $C \neq B9'$  and
   $B9'' \neq T$ 
  by auto
have Bet X T Y
proof (cases Col X T Y)
  case True
    have  $\neg Col A B C$ 
    by (simp add: H11)
  {
    assume Col A X Y
    hence False
    by (metis H1 H11 <Bet A B X> <Bet A C Y>
      bet-col bet-neq12--neq
      col-trivial-3
      not-par-inter-uniqueness par-id)
  }
hence  $\neg Col A X Y$ 
by blast
have Bet Y C A
  using <Bet A C Y> between-symmetry by blast
then obtain  $U$  where Bet Y U B and Bet A D U
  using H13 outer-pasch by blast
have Bet X B A
  by (meson Bet-perm <Bet A B X>)

```

```

then obtain V where Bet X V Y and Bet A U V
  using ⟨Bet Y U B⟩ outer-pasch by blast
have T = V
proof -
  have ¬ Col X Y A
    using Col-cases ⟨¬ Col A X Y⟩ by blast
  moreover
  have A ≠ D
    by (simp add: H3)
  moreover
  have Col X Y T
    using True not-col-permutation-5 by blast
  moreover
  have Col X Y V
    using ⟨Bet X V Y⟩ bet-col col-permutation-5 by blast
  moreover
  have Col A D T
    by (simp add: Col-def H12)
  moreover
  have Col A D V
    by (meson ⟨Bet A D U⟩ ⟨Bet A U V⟩ bet-col
      between-exchange4)
  ultimately
  show ?thesis
    using l6-21 by blast
qed
thus ?thesis
  by (simp add: ⟨Bet X V Y⟩)
next
case False
hence ¬ Col X T Y
  by simp
have ¬ Col T B'' Y
  by (meson False ⟨B'' ≠ T⟩ ⟨Bet B'' T X⟩ bet-col
    col-transitivity-2 not-col-permutation-4)
have T Y ParStrict C B
  by (metis H13 H5 ⟨T Y ParStrict C D⟩ bet-col
    between-symmetry col-trivial-3 par-strict-col2-par-strict)
have T X ParStrict B C
  by (meson H13 H5 ⟨T X ParStrict B D⟩ bet-col
    par-strict-col-par-strict)
have X ≠ T
  using False col-trivial-1 by auto
have Y ≠ T
  using False col-trivial-2 by blast
have X ≠ Y
  using False col-trivial-3 by force
have Coplanar T B B'' Y
proof -
  have Coplanar B Y T X
  proof -
    have ¬ Col C B T
      using Col-cases H14 by blast
    moreover
    have Coplanar C B T X
      using ⟨T X ParStrict B C⟩ ncoplanar-perm-17
        pars--coplanar by blast
    moreover
    have Coplanar C B T Y
      using ⟨T Y ParStrict C B⟩ coplanar-perm-16
        pars--coplanar by blast
    ultimately
    show ?thesis
      by (meson coplanar-trans-1 ncoplanar-perm-3)
  qed
qed
moreover

```

```

have Col T X B''
  by (simp add: Col-def ⟨Bet B'' T X⟩)
ultimately
show ?thesis
  using ⟨X ≠ T⟩ col-cop--cop coplanar-perm-13 by blast
qed
have ∃ I. Col B' B I ∧ Col T Y I
proof -
  have BetS T MB B
    by (metis BetS-def ⟨BetS B MB T⟩ between-symmetry)
  moreover
  have BetS B'' MB B'
    by (metis BetS-def Bet-cases ⟨BetS B' MB B''⟩)
  moreover
  have Cong T MB B MB
    by (meson ⟨Cong B MB T MB⟩ not-cong-3412)
  moreover
  have Cong B'' MB B' MB
    by (simp add: ⟨Cong B' MB B'' MB⟩ cong-symmetry)
  ultimately
  show ?thesis
    using assms ⟨¬ Col T B'' Y⟩ ⟨Coplanar T B B'' Y⟩
      strong-parallel-postulate-def by blast
qed
then obtain I where Col B' B I and Col T Y I
  by auto
thus ?thesis
  by (metis H13 H6 ⟨B ≠ B'⟩ ⟨Col B B' D⟩
    ⟨T Y ParStrict C D⟩ bet-col col3 col-trivial-2
    not-col-permutation-4 not-strict-par
    par-strict-not-col-4 par-strict-par)
qed
hence ∃ x y. (Bet A B x ∧ Bet A C y ∧ Bet x T y)
  using ⟨Bet A B X⟩ ⟨Bet A C Y⟩ by auto
}
hence ∃ X Y. Bet A' B' X ∧ Bet A' C' Y ∧ Bet X T' Y
  using 1 2 3 tarski-s-euclid-remove-degenerated-cases by blast
}
thus ?thesis
  using tarski-s-parallel-postulate-def by blast
qed

```

```

lemma midpoint-converse-postulate-implies-playfair:
  assumes midpoint-converse-postulate
  shows playfair-s-postulate
proof -
  {
    fix A1 A2 B1 B2 C1 C2 P
    assume A1 A2 Par B1 B2 and
      Col P B1 B2 and
      A1 A2 Par C1 C2 and
      Col P C1 C2
    {
      assume A1 A2 ParStrict B1 B2
      {
        assume A1 A2 ParStrict C1 C2
        hence P ≠ A1
          using Col-cases ⟨Col P C1 C2⟩ par-strict-not-col-3 by blast
        obtain X where P Midpoint A1 X
          using symmetric-point-construction by blast
      }
      fix B1' B2'
      assume Col P B1' B2' and

```

```

  A1 A2 ParStrict B1' B2' and
  P ≠ B1'
moreover
{
  fix B1'' B2''
  assume Col P B1'' B2'' and
    A1 A2 ParStrict B1'' B2'' and
    P ≠ B1''
  have A1 ≠ X
    using ⟨P Midpoint A1 X⟩ ⟨P ≠ A1⟩ l7-3 by blast
  have P ≠ X
    using ⟨A1 ≠ X⟩ ⟨P Midpoint A1 X⟩ midpoint-distinct-1 by blast
  have A1 ≠ A2
    using ⟨A1 A2 ParStrict B1' B2'⟩ par-strict-neq1 by blast
  have A1 ≠ B1''
    using ⟨A1 A2 ParStrict B1'' B2''⟩
      not-par-strict-id by fastforce
  have A1 ≠ B2''
    using ⟨A1 A2 ParStrict B1'' B2''⟩ ⟨Col P B1'' B2''⟩
      ⟨P ≠ B1''⟩ col-transitivity-2 par-strict-not-col-3 by blast
  have A2 ≠ B1''
    using ⟨A1 A2 ParStrict B1'' B2''⟩ not-par-strict-id
      par-strict-left-comm by blast
  have A2 ≠ B2''
    using Par-strict-cases ⟨A1 A2 ParStrict B1'' B2''⟩
      not-par-strict-id by blast
  have B2'' ≠ B1''
    using ⟨A1 A2 ParStrict B1'' B2''⟩ col-trivial-1
      par-strict-not-col-3 by blast
  have ∃ B3. Col B1'' P B3 ∧ (BetS X B3 A2 ∨ BetS A1 B3 A2)
  proof –
    have Coplanar X A1 A2 B1''
    proof –
      have Coplanar A2 B1'' A1 P
        using NCol-perm ⟨A1 A2 ParStrict B1'' B2''⟩
          ⟨B2'' ≠ B1''⟩ ⟨Col P B1'' B2''⟩ col-cop--cop
          ncoplanar-perm-12 pars--coplanar by blast
      moreover
      have Col A1 P X
        using Midpoint-def ⟨P Midpoint A1 X⟩
          bet-col by blast
      ultimately
      show ?thesis
        using ⟨P ≠ A1⟩ col-cop--cop
          ncoplanar-perm-17 by blast
    qed
  moreover
  have ¬ Col A2 P B1''
    by (metis Par-strict-cases ⟨A1 A2 ParStrict B1'' B2''⟩
      ⟨Col P B1'' B2''⟩ ⟨P ≠ B1''⟩ col3 col-trivial-2
      not-col-permutation-2 par-strict-not-col-3)
  moreover
  {
    assume Col X A1 B1''
    have Bet A1 P X
      using ⟨P Midpoint A1 X⟩ midpoint-bet by blast
    hence Col A1 P X
      using Col-def by blast
    hence Col A1 P B1''
      using ⟨Col X A1 B1''⟩
      by (metis ⟨A1 ≠ X⟩ col-permutation-2
        col-trivial-2 l6-21)
    moreover
    have ¬ Col A1 P B1''
      by (metis ⟨A1 A2 ParStrict B1'' B2''⟩
        ⟨Col P B1'' B2''⟩ ⟨P ≠ B1''⟩ col3 col-trivial-2

```

```

      col-trivial-3 par-strict-not-col-3)
ultimately
have False
by simp
}
hence  $\neg \text{Col } X \ A1 \ B1''$ 
by blast
moreover
have  $\text{BetS } X \ P \ A1$ 
using  $\text{BetS-def Midpoint-def } \langle P \ \text{Midpoint } A1 \ X \rangle$ 
 $\langle P \neq A1 \rangle \langle P \neq X \rangle$  between-symmetry by presburger
ultimately
show ?thesis
using hilbert-s-version-of-pasch by blast
qed
then obtain  $B3$  where  $\text{Col } B1'' \ P \ B3$  and  $\text{BetS } X \ B3 \ A2 \vee \text{BetS } A1 \ B3 \ A2$ 
by blast
have  $\text{Col } B1'' \ B2'' \ B3$ 
by (meson  $\langle \text{Col } B1'' \ P \ B3 \rangle \langle \text{Col } P \ B1'' \ B2'' \rangle$ 
 $\langle P \neq B1'' \rangle$  col-transitivity-2 not-col-permutation-4)
moreover
{
  assume  $\text{BetS } A1 \ B3 \ A2$ 
  have  $\text{Col } B3 \ A1 \ A2$ 
  using  $\text{BetSEq Col-def } \langle \text{BetS } A1 \ B3 \ A2 \rangle$ 
  col-permutation-3 by blast
  moreover
  have  $\text{Col } B3 \ B1'' \ B2''$ 
  using  $\text{Col-perm } \langle \text{Col } B1'' \ B2'' \ B3 \rangle$  by blast
  ultimately
  have False
  using  $\langle A1 \ A2 \ \text{ParStrict } B1'' \ B2'' \rangle$  par-not-col by blast
}
hence  $\text{BetS } A2 \ B3 \ X$ 
by (metis  $\text{BetS-def } \langle \text{BetS } X \ B3 \ A2 \vee \text{BetS } A1 \ B3 \ A2 \rangle$ 
between-symmetry)
moreover
have  $A1 \ A2 \ \text{ParStrict } P \ B3$ 
proof -
  have  $\text{Col } B1'' \ B2'' \ P$ 
  by (simp add:  $\langle \text{Col } P \ B1'' \ B2'' \rangle$  col-permutation-1)
  moreover
  {
    assume  $P = B3$ 
    have  $\text{Bet } A2 \ B3 \ X$ 
    using  $\text{BetSEq } \langle \text{BetS } A2 \ B3 \ X \rangle$  by blast
    hence  $\text{Col } X \ P \ A2$ 
    using  $\text{BetSEq } \langle \text{BetS } A1 \ B3 \ A2 \implies \text{False} \rangle$ 
     $\langle \text{BetS } X \ B3 \ A2 \vee \text{BetS } A1 \ B3 \ A2 \rangle \langle P = B3 \rangle$  bet-col by blast
    have  $A2 \neq B3$ 
    using  $\text{BetSEq } \langle \text{BetS } A2 \ B3 \ X \rangle$  by auto
    have  $B3 \neq X$ 
    using  $\text{BetSEq } \langle \text{BetS } A2 \ B3 \ X \rangle$  by auto
    hence  $\text{Col } P \ A1 \ A2$ 
  }
  proof -
    have  $\text{Col } X \ P \ A2$ 
    by (simp add:  $\langle \text{Col } X \ P \ A2 \rangle$ )
    moreover
    have  $\text{Bet } A1 \ P \ X$ 
    by (simp add:  $\langle P \ \text{Midpoint } A1 \ X \rangle$  midpoint-bet)
    hence  $\text{Col } X \ P \ A1$ 
    using  $\text{Bet-cases Col-def}$  by blast
    ultimately
    show ?thesis
    using  $\langle A1 \ A2 \ \text{ParStrict } B1'' \ B2'' \rangle$ 
    by (metis  $\langle P \neq X \rangle$  col-transitivity-2)
  
```

```

    qed
    hence False
    using ⟨Col P B1'' B2''⟩ ⟨A1 A2 ParStrict B1'' B2''⟩
      par-not-col by blast
  }
  hence P ≠ B3
  by blast
  ultimately
  show ?thesis
  using ⟨Col B1'' B2'' B3⟩ ⟨A1 A2 ParStrict B1'' B2''⟩
    par-strict-col2-par-strict by blast
  qed
  ultimately
  have ∃ B3. Col B1'' B2'' B3 ∧ BetS A2 B3 X ∧ A1 A2 ParStrict P B3
  by blast
}
ultimately
have ∃ B3. Col B1' B2' B3 ∧ BetS A2 B3 X ∧ A1 A2 ParStrict P B3
  using ⟨A1 A2 ParStrict B1' B2'⟩ ⟨Col P B1' B2'⟩ by blast
}
moreover
{
  fix B1' B2'
  assume Col P B1' B2' and
    A1 A2 ParStrict B1' B2' and
    P = B1'
  have P ≠ B2'
  using ⟨P = B1'⟩ ⟨A1 A2 ParStrict B1' B2'⟩
    par-strict-distinct by presburger
  moreover
  have Col P B2' B1'
  using ⟨P = B1'⟩ col-trivial-3 by blast
  moreover
  have A1 A2 ParStrict B2' B1'
  by (simp add: ⟨A1 A2 ParStrict B1' B2'⟩
    par-strict-right-comm)
  ultimately
  have ∃ B3. Col B1' B2' B3 ∧ BetS A2 B3 X ∧ A1 A2 ParStrict P B3
  using Col-perm
    ⟨∧ B2' B1'. [⟨Col P B1' B2'⟩; A1 A2 ParStrict B1' B2'; P ≠ B1']⟩
    ⇒ ∃ B3. Col B1' B2' B3 ∧ BetS A2 B3 X ∧ A1 A2 ParStrict P B3⟩
  by blast
}
ultimately
have H1: ∀ B1' B2'. (Col P B1' B2' ∧ A1 A2 ParStrict B1' B2')
  →
  (∃ B3. Col B1' B2' B3 ∧ BetS A2 B3 X ∧ A1 A2 ParStrict P B3)
  by blast
then obtain B3 where Col B1 B2 B3 and BetS A2 B3 X and
  A1 A2 ParStrict P B3
  using ⟨A1 A2 ParStrict B1 B2⟩ ⟨Col P B1 B2⟩ by blast
obtain C3 where Col C1 C2 C3 and BetS A2 C3 X and
  A1 A2 ParStrict P C3
  using ⟨A1 A2 ParStrict C1 C2⟩ ⟨Col P C1 C2⟩ H1 by blast
have Col A2 X B3
  by (meson BetSEq Bet-cases Col-def ⟨BetS A2 B3 X⟩)
have Col A2 X C3
  by (meson BetSEq Bet-cases Col-def ⟨BetS A2 C3 X⟩)
have ¬ Col A1 A2 X
  by (metis BetSEq ⟨A1 A2 ParStrict P C3⟩
    ⟨BetS A2 B3 X⟩ ⟨Col A2 X C3⟩ col124--nos col3
    col-trivial-3 l12-6 not-col-permutation-2)
have B3 = C3
proof -
  have B3 Midpoint A2 X
proof -

```



```

have  $\neg \text{Col } A2 \ A1 \ X$ 
  by (meson Col-cases  $\langle \neg \text{Col } A1 \ A2 \ X \rangle$ )
moreover
have  $A2 \ A1 \ \text{Par } B3 \ P$ 
  by (simp add:  $\langle A1 \ A2 \ \text{ParStrict } P \ B3 \rangle$ 
      par-strict-comm par-strict-par)
ultimately
show ?thesis
  using assms  $\langle P \ \text{Midpoint } A1 \ X \rangle \langle \text{Col } A2 \ X \ B3 \rangle$ 
    midpoint-converse-postulate-def by blast
qed
moreover
have  $C3 \ \text{Midpoint } A2 \ X$ 
proof -
  have  $\neg \text{Col } A2 \ A1 \ X$ 
    by (meson Col-cases  $\langle \neg \text{Col } A1 \ A2 \ X \rangle$ )
  moreover
  have  $A2 \ A1 \ \text{Par } C3 \ P$ 
    by (simp add:  $\langle A1 \ A2 \ \text{ParStrict } P \ C3 \rangle$ 
        par-strict-comm par-strict-par)
  ultimately
  show ?thesis
    using assms  $\langle P \ \text{Midpoint } A1 \ X \rangle \langle \text{Col } A2 \ X \ C3 \rangle$ 
      midpoint-converse-postulate-def by blast
qed
ultimately
show ?thesis
  by (meson Mid-cases l7-17-bis)
qed
have  $\text{Col } C1 \ B1 \ B2$ 
proof -
  have  $\text{Col } B1 \ B2 \ C3$ 
    using  $\langle B3 = C3 \rangle \langle \text{Col } B1 \ B2 \ B3 \rangle$  by auto
  moreover
  have  $\text{Col } C1 \ C2 \ C3$ 
    using  $\langle \text{Col } C1 \ C2 \ C3 \rangle$  by auto
  moreover
  have  $\text{Col } P \ C1 \ C2$ 
    by (simp add:  $\langle \text{Col } P \ C1 \ C2 \rangle$ )
  moreover
  have  $\text{Col } P \ B1 \ B2$ 
    by (simp add:  $\langle \text{Col } P \ B1 \ B2 \rangle$ )
  ultimately
  show ?thesis
    by (metis  $\langle A1 \ A2 \ \text{ParStrict } C1 \ C2 \rangle \langle A1 \ A2 \ \text{ParStrict } P \ C3 \rangle$ 
        l6-21 not-col-permutation-2 par-strict-neq2)
qed
moreover
have  $\text{Col } C2 \ B1 \ B2$ 
proof -
  have  $\text{Col } B1 \ B2 \ C3$ 
    using  $\langle B3 = C3 \rangle \langle \text{Col } B1 \ B2 \ B3 \rangle$  by auto
  moreover
  have  $\text{Col } C1 \ C2 \ C3$ 
    using  $\langle \text{Col } C1 \ C2 \ C3 \rangle$  by auto
  moreover
  have  $\text{Col } P \ C1 \ C2$ 
    by (simp add:  $\langle \text{Col } P \ C1 \ C2 \rangle$ )
  moreover
  have  $\text{Col } P \ B1 \ B2$ 
    by (simp add:  $\langle \text{Col } P \ B1 \ B2 \rangle$ )
  ultimately
  show ?thesis
    by (metis  $\langle A1 \ A2 \ \text{ParStrict } P \ C3 \rangle \langle \text{Col } C1 \ B1 \ B2 \rangle$ 
        l6-21 not-col-permutation-2 not-col-permutation-3
        par-strict-neq2)

```

```

qed
ultimately
have Col C1 B1 B2  $\wedge$  Col C2 B1 B2
  by simp
}
moreover
have (A1  $\neq$  A2  $\wedge$  B1  $\neq$  B2  $\wedge$  Col A1 B1 B2  $\wedge$  Col A2 B1 B2)
   $\longrightarrow$ 
    Col C1 B1 B2  $\wedge$  Col C2 B1 B2
  using NCol-perm  $\langle$ A1 A2 ParStrict B1 B2 $\rangle$ 
    par-strict-not-col-3 by blast
{
  assume A1  $\neq$  A2 and
    B1  $\neq$  B2 and
    Col A1 B1 B2 and
    Col A2 B1 B2
  hence Col C1 B1 B2  $\wedge$  Col C2 B1 B2
    using NCol-perm  $\langle$ A1 A2 ParStrict B1 B2 $\rangle$ 
      par-strict-not-col-3 by blast
}
ultimately
have Col C1 B1 B2  $\wedge$  Col C2 B1 B2
  by (meson  $\langle$ A1 A2 Par C1 C2 $\rangle$   $\langle$ A1 A2 ParStrict B1 B2 $\rangle$ 
     $\langle$ Col P B1 B2 $\rangle$   $\langle$ Col P C1 C2 $\rangle$  col-permutation-1
    par-not-col par-not-col-strict)
}
moreover
{
  assume A1  $\neq$  A2 and B1  $\neq$  B2 and
    Col A1 B1 B2 and Col A2 B1 B2
  hence  $\neg$  A1 A2 ParStrict B1 B2
    using par-strict-not-col-2 by blast
  have Col B1 B2 C1  $\wedge$  Col B1 B2 C2
  proof -
    have B1 B2 Par C1 C2
      using NCol-perm  $\langle$ A1 A2 Par C1 C2 $\rangle$   $\langle$ B1  $\neq$  B2 $\rangle$ 
         $\langle$ Col A1 B1 B2 $\rangle$   $\langle$ Col A2 B1 B2 $\rangle$ 
        par-col2-par-bis par-symmetry by blast
    moreover have Col B1 B2 A1
      using  $\langle$ Col A1 B1 B2 $\rangle$ 
        not-col-permutation-2 by blast
    moreover have Col C1 C2 A1
      using Par-cases  $\langle$ Col P B1 B2 $\rangle$ 
         $\langle$ Col P C1 C2 $\rangle$  calculation(1) calculation(2)
        col-not-col-not-par col-permutation-1 by blast
    ultimately show ?thesis
      using inter-uniqueness-not-par not-strict-par2 by blast
  qed
  hence Col C1 B1 B2  $\wedge$  Col C2 B1 B2
    using col-permutation-2 by blast
}
ultimately
have Col C1 B1 B2  $\wedge$  Col C2 B1 B2
  using Par-def  $\langle$ A1 A2 Par B1 B2 $\rangle$  by presburger
}
thus ?thesis
  using playfair-s-postulate-def by blast
qed

```

```

lemma playfair-implies-par-trans:
  assumes playfair-s-postulate
  shows postulate-of-transitivity-of-parallelism
proof -
{
  fix A1 A2 B1 B2 C1 C2
  assume A1 A2 Par B1 B2 and

```

```

  B1 B2 Par C1 C2
have B1 ≠ B2
  using ⟨A1 A2 Par B1 B2⟩ par-distincts by blast
have A1 ≠ A2
  using ⟨A1 A2 Par B1 B2⟩ par-distincts by auto
have C1 ≠ C2
  using ⟨B1 B2 Par C1 C2⟩ par-distincts by blast
have A1 A2 Par C1 C2
proof (cases Coplanar A1 A2 C1 B1)
  case True
  show ?thesis
  proof (cases Col A1 A2 C1)
    case True
    hence Col A1 A2 C2
    proof –
      have B1 B2 Par A1 A2
        by (simp add: ⟨A1 A2 Par B1 B2⟩ par-symmetry)
      moreover have Col C1 A1 A2
        using True not-col-permutation-1 by blast
      moreover have Col C1 C1 C2
        by (simp add: col-trivial-1)
      ultimately show ?thesis
        using playfair-s-postulate-def ⟨B1 B2 Par C1 C2⟩
          assms col-permutation-1 by blast
    qed
    have B1 B2 Par C1 C2
      by (simp add: ⟨B1 B2 Par C1 C2⟩)
    moreover
    have Col C1 C1 C2
      by (simp add: col-trivial-1)
    moreover
    have Col A2 C1 C2
      using Par-cases True ⟨A1 A2 Par B1 B2⟩ assms
        calculation(1) calculation(2) not-col-permutation-3
        playfair-s-postulate-def by blast
    moreover
    have B1 B2 Par A1 A2
      using Par-cases ⟨A1 A2 Par B1 B2⟩ by blast
    moreover
    have Col A1 A2 C1
      using True by simp
    hence Col A1 C1 C2
      using calculation(3)
      by (metis ⟨Col A1 A2 C2⟩ col-transitivity-1)
    ultimately
    show ?thesis
      using Par-def ⟨A1 ≠ A2⟩ ⟨C1 ≠ C2⟩ by blast
  next
  case False
  hence ¬ Col A1 A2 C1
    by simp
  have A1 A2 ParStrict C1 C2
  proof –
    have Coplanar A1 A2 C1 C2
    proof –
      have C1 C2 Par B1 B2
        by (simp add: ⟨B1 B2 Par C1 C2⟩ par-symmetry)
      moreover
      {
        assume A1 A2 Par B1 B2 and C1 C2 ParStrict B1 B2
        hence Coplanar C1 C2 B1 B2
          using ParStrict-def by blast
        have Coplanar A1 A2 C1 C2
        proof –
          {
            assume A1 A2 ParStrict B1 B2

```

```

hence Coplanar A1 A2 B1 B2
  using pars--coplanar by auto
moreover
have Coplanar C1 C2 B1 B2
  by (simp add: ⟨Coplanar C1 C2 B1 B2⟩)
moreover
have Coplanar A1 A2 C1 B1
  by (simp add: True)
ultimately
have Coplanar A1 A2 C1 B2
proof -
  have ¬ Col A2 B1 B2
    using ⟨A1 A2 ParStrict B1 B2⟩ par-strict-not-col-2 by auto
  moreover have Coplanar A2 B1 B2 A1
    using ⟨Coplanar A1 A2 B1 B2⟩ ncoplanar-perm-18 by blast
  moreover have Coplanar A2 B1 B2 A2
    using ncop-distincts by blast
  moreover have Coplanar A2 B1 B2 C1
    by (meson True ⟨A1 A2 ParStrict B1 B2⟩
        calculation(2) coplanar-perm-1 l9-30
        ncop-distincts par-strict-not-col-3
        par-strict-symmetry)
  moreover have Coplanar A2 B1 B2 B2
    using ncop-distincts by blast
  ultimately show ?thesis
    using coplanar-pseudo-trans by blast
qed
have Coplanar A1 A2 C1 C1
  using ncop-distincts by blast
have Coplanar B1 B2 C1 A1
  by (meson False True ⟨Coplanar A1 A2 C1 B2⟩
      coplanar-pseudo-trans ncop-distincts)
moreover
have Coplanar B1 B2 C1 A2
  by (meson False True ⟨Coplanar A1 A2 C1 B2⟩
      coplanar-trans-1 ncoplanar-perm-22)
moreover
have Coplanar B1 B2 C1 C1
  using ncop-distincts by blast
moreover
have Coplanar B1 B2 C1 C2
  by (simp add: ⟨Coplanar C1 C2 B1 B2⟩
      coplanar-perm-16)
ultimately
have Coplanar A1 A2 C1 C2
  by (meson NCol-perm ⟨C1 C2 ParStrict B1 B2⟩
      coplanar-pseudo-trans not-col-distincts
      par-not-col)
}
moreover
{
  assume A1 ≠ A2 ∧ B1 ≠ B2 ∧ Col A1 B1 B2 ∧ Col A2 B1 B2
  hence Coplanar A1 A2 C1 C2
    by (meson ⟨C1 C2 Par B1 B2⟩ col2-cop--cop
        col-permutation-1 ncoplanar-perm-16 par--coplanar)
}
ultimately
show ?thesis
  by (metis Par-cases ⟨A1 A2 Par B1 B2⟩ ⟨A1 ≠ A2⟩
      ⟨B1 ≠ B2⟩ col-permutation-2 not-col-distincts
      par-not-col-strict par-strict-symmetry)
qed
}
moreover
{
  assume A1 A2 Par B1 B2 and C1 ≠ C2 and

```

```

    B1 ≠ B2 and Col C1 B1 B2 and Col A2 B1 B2
  hence Coplanar A1 A2 C1 C2
  by (meson col-permutation-1 col-trivial-3
      ncop--ncols par-not-col par-not-col-strict)
}
ultimately
show ?thesis
  by (metis ⟨A1 A2 Par B1 B2⟩ ⟨B1 ≠ B2⟩
      all-one-side-par-strict col2-cop--cop not-strict-par1
      not-strict-par2 par--coplanar par-not-col-strict
      par-symmetry)
qed
moreover
{
  assume ∃ X. Col X A1 A2 ∧ Col X C1 C2
  then obtain X where Col X A1 A2 and Col X C1 C2
  by blast
  have B1 B2 Par A1 A2
  using ⟨A1 A2 Par B1 B2⟩ par-symmetry by blast
  hence False
  using ⟨B1 B2 Par C1 C2⟩ ⟨Col X A1 A2⟩
      ⟨Col X C1 C2⟩ False assms col-permutation-1
      playfair-s-postulate-def by blast
}
ultimately
show ?thesis
  using ParStrict-def by blast
qed
thus ?thesis
  by (simp add: Par-def)
qed
next
case False
hence A1 A2 ParStrict B1 B2
  using ⟨A1 A2 Par B1 B2⟩ col-trivial-3
      ncop--ncols par-not-col-strict by blast
have B1 B2 ParStrict C1 C2
proof -
  have Col C1 C2 C1
  by (simp add: col-trivial-3)
  moreover
  have ¬ Col B1 B2 C1
  by (meson False ⟨A1 A2 Par B1 B2⟩ ⟨B1 ≠ B2⟩
      col-cop--cop ncoplanar-perm-1 par--coplanar)
  ultimately
  show ?thesis
  using ⟨B1 B2 Par C1 C2⟩ par-not-col-strict by blast
qed
have ∃ C'. Coplanar A1 A2 C1 C' ∧ Coplanar B1 B2 C1 C' ∧ C1 ≠ C'
proof -
  have Coplanar A1 A2 C1 C1
  using ncop-distincts by blast
  moreover
  have A1 A2 C1 OSP B1 B2
  by (meson False ⟨A1 A2 ParStrict B1 B2⟩
      col-trivial-2 cop2-os--osp
      ncop-distincts par-strict-one-side)
  ultimately
  show ?thesis
  by (simp add: cop-osp--ex-cop2)
qed
then obtain C' where Coplanar A1 A2 C1 C' and Coplanar B1 B2 C1 C'
and C1 ≠ C'
by auto
{
  fix X

```

```

assume Coplanar A1 A2 B1 X and
  Col X C1 C'
have Col X A1 A2
proof –
  have  $\neg$  Coplanar A1 A2 C1 B1
    by (simp add: False)
  moreover
  have  $\neg$  Col A1 A2 B1
    using calculation ncop--ncols by auto
  moreover
  have Coplanar A1 A2 B1 B1
    using ncop-distincts by blast
  moreover
  have Coplanar A1 A2 C1 X
    using  $\langle C1 \neq C' \rangle \langle Col X C1 C' \rangle \langle Coplanar A1 A2 C1 C' \rangle$ 
      col-cop--cop col-permutation-1 by blast
  moreover
  have Coplanar A1 A2 C1 A1
    using ncop-distincts by blast
  moreover
  have Coplanar A1 A2 C1 A2
    using ncop-distincts by blast
  moreover
  have Coplanar A1 A2 B1 X
    by (simp add:  $\langle Coplanar A1 A2 B1 X \rangle$ )
  moreover
  have Coplanar A1 A2 B1 A1
    using ncop-distincts by blast
  moreover
  have Coplanar A1 A2 B1 A2
    using ncop-distincts by blast
  ultimately
  show ?thesis
    using l9-30 by blast
qed
moreover
have Col X B1 B2
proof –
  have  $\neg$  Coplanar A1 A2 B1 C1
    using False ncoplanar-perm-1 by blast
  moreover
  have  $\neg$  Col B1 B2 C1
    using  $\langle B1 B2 ParStrict C1 C2 \rangle$  col123--nos l12-6 by blast
  moreover
  have Coplanar B1 B2 C1 C1
    using ncop-distincts by blast
  moreover
  have Coplanar A1 A2 B1 X
    by (simp add:  $\langle Coplanar A1 A2 B1 X \rangle$ )
  moreover
  have Coplanar A1 A2 B1 B1
    using ncop-distincts by blast
  moreover
  have Coplanar A1 A2 B1 B2
    using ParStrict-def  $\langle A1 A2 ParStrict B1 B2 \rangle$  by blast
  moreover
  have Coplanar B1 B2 C1 X
    by (meson  $\langle C1 \neq C' \rangle \langle Col X C1 C' \rangle$ 
       $\langle Coplanar B1 B2 C1 C' \rangle$  col-cop--cop
      not-col-permutation-1)
  moreover
  have Coplanar B1 B2 C1 B1
    using ncop-distincts by blast
  moreover
  have Coplanar B1 B2 C1 B2
    using ncop-distincts by blast

```

```

    ultimately
    show ?thesis
    using l9-30 by blast
qed
ultimately
have False
    using ⟨A1 A2 ParStrict B1 B2⟩ par-not-col by blast
}
have A1 A2 ParStrict C1 C2
proof -
{
    assume  $\exists X. \text{Col } X \ A1 \ A2 \wedge \text{Col } X \ C1 \ C'$ 
    then obtain X where  $\text{Col } X \ A1 \ A2$  and  $\text{Col } X \ C1 \ C'$ 
    by auto
    have Coplanar A1 A2 B1 X
    using ⟨Col X A1 A2⟩ col-permutation-1 ncop--ncols by blast
    hence False
    using ⟨Col X C1 C'⟩
    ⟨ $\bigwedge X. [\text{Coplanar } A1 \ A2 \ B1 \ X; \text{Col } X \ C1 \ C'] \implies \text{False}$ ⟩
    by blast
}
hence A1 A2 ParStrict C1 C'
    using ParStrict-def ⟨Coplanar A1 A2 C1 C'⟩ by blast
moreover
have B1 B2 ParStrict C1 C'
proof -
    have B1 B2 Par C1 C'
    using ⟨A1 A2 Par B1 B2⟩ ⟨B1  $\neq$  B2⟩ ⟨Coplanar B1 B2 C1 C'⟩
    assms calculation col-cop--cop col-permutation-1
    coplanar-perm-1 par--coplanar par-strict-par
    playfair-s-postulate-def
    by (meson False cop-npars--inter-exists)
    thus ?thesis
    using Par-def
    ⟨ $\bigwedge X. [\text{Coplanar } A1 \ A2 \ B1 \ X; \text{Col } X \ C1 \ C'] \implies \text{False}$ ⟩
    ncop-distincts by blast
qed
hence Col C1 C' C2
    using ⟨B1 B2 Par C1 C2⟩ assms col-permutation-1
    col-trivial-1 par-strict-par
    playfair-s-postulate-def by blast
ultimately
show ?thesis
    using ⟨C1  $\neq$  C2⟩ par-strict-col-par-strict by blast
qed
thus ?thesis
    by (simp add: Par-def)
qed
}
thus ?thesis
    using postulate-of-transitivity-of-parallelism-def by blast
qed

```

```

lemma par-trans-implies-playfair:
  assumes postulate-of-transitivity-of-parallelism
  shows playfair-s-postulate
proof -
{
  fix A1 A2 B1 B2 C1 C2 P
  assume A1 A2 Par B1 B2 and
    Col P B1 B2 and
    A1 A2 Par C1 C2 and
    Col P C1 C2
  have C1 C2 Par B1 B2
  using ⟨A1 A2 Par B1 B2⟩ ⟨A1 A2 Par C1 C2⟩ assms
    par-symmetry postulate-of-transitivity-of-parallelism-def by blast
}

```

```

hence  $\text{Col } C1 \ B1 \ B2 \wedge \text{Col } C2 \ B1 \ B2$ 
using  $\text{Par-def } \langle \text{Col } P \ B1 \ B2 \rangle \ \langle \text{Col } P \ C1 \ C2 \rangle$ 
       $\text{par-not-col}$  by blast
}
thus ?thesis
using playfair-s-postulate-def by blast
qed

lemma par-perp-perp-implies-par-perp-2-par:
assumes perpendicular-transversal-postulate
shows postulate-of-parallelism-of-perpendicular-transversals
proof -
{
  fix  $A1 \ A2 \ B1 \ B2 \ C1 \ C2 \ D1 \ D2$ 
  assume  $A1 \ A2 \ \text{Par } B1 \ B2$  and  $A1 \ A2 \ \text{Perp } C1 \ C2$  and  $B1 \ B2 \ \text{Perp } D1 \ D2$  and
     $\text{Coplanar } A1 \ A2 \ C1 \ D1$  and  $\text{Coplanar } A1 \ A2 \ C1 \ D2$  and
     $\text{Coplanar } A1 \ A2 \ C2 \ D1$  and  $\text{Coplanar } A1 \ A2 \ C2 \ D2$ 
  have  $C1 \ C2 \ \text{Perp } A1 \ A2$ 
    using  $\text{Perp-perm } \langle A1 \ A2 \ \text{Perp } C1 \ C2 \rangle$  by blast
  moreover
  have  $D1 \ D2 \ \text{Perp } A1 \ A2$ 
  proof -
    have  $B1 \ B2 \ \text{Par } A1 \ A2$ 
      using  $\text{Par-cases } \langle A1 \ A2 \ \text{Par } B1 \ B2 \rangle$  by blast
    moreover
    have  $B1 \ B2 \ \text{Perp } D1 \ D2$ 
      by (simp add:  $\langle B1 \ B2 \ \text{Perp } D1 \ D2 \rangle$ )
    moreover

    have  $\text{Coplanar } A1 \ A2 \ D1 \ D2$ 
    proof -
      {
        assume  $\neg \text{Col } A1 \ A2 \ C1$ 
        hence  $\text{Coplanar } A1 \ A2 \ D1 \ D2$ 
          by (meson  $\langle \text{Coplanar } A1 \ A2 \ C1 \ D1 \rangle \ \langle \text{Coplanar } A1 \ A2 \ C1 \ D2 \rangle$ 
            coplanar-pseudo-trans ncop-distincts)
        }
      moreover
      {
        assume  $\neg \text{Col } A1 \ A2 \ C2$ 
        hence  $\text{Coplanar } A1 \ A2 \ D1 \ D2$ 
          by (meson  $\langle \text{Coplanar } A1 \ A2 \ C2 \ D1 \rangle \ \langle \text{Coplanar } A1 \ A2 \ C2 \ D2 \rangle$ 
            coplanar-perm-12 coplanar-trans-1 not-col-permutation-2)
        }
      ultimately
      show ?thesis
        using  $\text{perp-not-col2 } \langle A1 \ A2 \ \text{Perp } C1 \ C2 \rangle$  by blast
    qed
    ultimately
    show ?thesis
      using  $\text{Perp-perm assms perpendicular-transversal-postulate-def}$  by blast
    qed
    ultimately
    have  $C1 \ C2 \ \text{Par } D1 \ D2$ 
      using l12-9  $\langle \text{Coplanar } A1 \ A2 \ C1 \ D1 \rangle \ \langle \text{Coplanar } A1 \ A2 \ C1 \ D2 \rangle$ 
         $\langle \text{Coplanar } A1 \ A2 \ C2 \ D1 \rangle \ \langle \text{Coplanar } A1 \ A2 \ C2 \ D2 \rangle$ 
      by blast
  }
thus ?thesis
using postulate-of-parallelism-of-perpendicular-transversals-def by blast
qed

lemma par-perp-2-par-implies-par-perp-perp:
assumes postulate-of-parallelism-of-perpendicular-transversals
shows perpendicular-transversal-postulate
proof -

```



```

{
  fix A B C D P Q
  assume A B Par C D and
    A B Perp P Q and
    Coplanar C D P Q
  {
    assume A B ParStrict C D
    obtain X where X PerpAt A B P Q
    using Perp-def ⟨A B Perp P Q⟩ by blast
    have C D Perp P Q
    proof (cases Col C D X)
      case True
      have Col X A B
      using ⟨X PerpAt A B P Q⟩ Col-cases perp-in-col by blast
    moreover
    have Col X C D
    using Col-cases True by blast
    ultimately
    have ∃ X0. Col X0 A B ∧ Col X0 C D
    by blast
    hence False
    using ⟨A B ParStrict C D⟩ par-not-col by blast
    thus ?thesis
    by blast
  next
  case False
  then obtain Y where Col C D Y and C D Perp X Y
  using False l8-18-existence by blast
  have A ≠ B
  using ⟨A B Par C D⟩ par-distinct by auto
  have P ≠ Q
  using ⟨A B Perp P Q⟩ perp-not-eq-2 by auto
  have P Q Par X Y
  proof -
    have Coplanar C D X A
    proof -
      have Coplanar C D A B
      using Par-cases ⟨A B Par C D⟩ par--coplanar by blast
    moreover
    have Col X A B
    using ⟨X PerpAt A B P Q⟩ Col-cases perp-in-col by blast
    hence Col A B X
    using Col-cases by blast
    moreover
    have Col A B A
    by (simp add: col-trivial-3)
    ultimately
    show ?thesis
    using ⟨A ≠ B⟩ col2-cop--cop by blast
  qed
  have Coplanar C D X B
  proof -
    have Coplanar C D A B
    using Par-perm ⟨A B Par C D⟩ par--coplanar by blast
    moreover
    have Col X A B
    using ⟨X PerpAt A B P Q⟩ Col-cases perp-in-col by blast
    hence Col A B X
    using Col-cases by blast
    moreover
    have Col A B B
    by (simp add: col-trivial-2)
    ultimately
    show ?thesis
    using ⟨A ≠ B⟩ col2-cop--cop by blast
  qed
}

```

```

have Coplanar C D X P
  by (meson ⟨Coplanar C D P Q⟩ ⟨P ≠ Q⟩
      ⟨X PerpAt A B P Q⟩ col-cop--cop ncoplanar-perm-6
      ncoplanar-perm-7 perp-in-col)
have Coplanar C D X Q
proof —
  have Coplanar C D P Q
    by (simp add: ⟨Coplanar C D P Q⟩)
  moreover
  have Col P Q X
    using ⟨X PerpAt A B P Q⟩ perp-in-col by blast
  moreover
  have Col P Q Q
    using col-trivial-2 by auto
  ultimately
  show ?thesis
    using ⟨P ≠ Q⟩ col2-cop--cop by blast
qed
have Coplanar C D X Y
  by (simp add: ⟨C D Perp X Y⟩ perp--coplanar)
have Coplanar A B P X
  using ⟨X PerpAt A B P Q⟩ col--coplanar
  coplanar-perm-1 perp-in-col by blast
moreover
have Coplanar A B P Y
  by (meson False ⟨Coplanar C D X A⟩ ⟨Coplanar C D X B⟩
      ⟨Coplanar C D X P⟩ ⟨Coplanar C D X Y⟩
      coplanar-pseudo-trans)
moreover
have Coplanar A B Q X
  by (meson ⟨X PerpAt A B P Q⟩ ncop--ncol
      ncoplanar-perm-1 perp-in-col)
moreover
have Coplanar A B Q Y
  by (meson ⟨Coplanar C D X A⟩ ⟨Coplanar C D X B⟩
      ⟨Coplanar C D X Q⟩ ⟨Coplanar C D X Y⟩ False
      coplanar-pseudo-trans)
ultimately
show ?thesis
  using ⟨C D Perp X Y⟩ ⟨A B Perp P Q⟩ ⟨A B Par C D⟩
  postulate-of-parallelism-of-perpendicular-transversals-def
  assms by blast
qed
{
  assume P ≠ Q ∧ X ≠ Y ∧ Col P X Y ∧ Col Q X Y
  hence C D Perp P Q
    by (meson ⟨C D Perp X Y⟩ not-col-permutation-2 perp-col2-bis)
}
moreover
{
  assume P Q ParStrict X Y
  have Col X P Q
    using ⟨X PerpAt A B P Q⟩ Col-cases perp-in-col by blast
  moreover
  have Col X X Y
    by (simp add: col-trivial-1)
  ultimately
  have False
    using ⟨P Q ParStrict X Y⟩ par-not-col by blast
  hence C D Perp P Q
    by blast
}
ultimately
show ?thesis
  using ⟨P Q Par X Y⟩ Par-def by blast
qed

```

```

}
hence C D Perp P Q
  by (metis Par-perm Par-strict-perm par-neq2
    par-not-col-strict perp-col2 perp-inter-perp-in
    ⟨A B Par C D⟩ ⟨A B Perp P Q⟩ not-strict-par2)
}
thus ?thesis
  using perpendicular-transversal-postulate-def by blast
qed

lemma par-perp-perp-implies-playfair:
  assumes perpendicular-transversal-postulate
  shows playfair-s-postulate
proof -
{
  fix A1 A2 B1 B2 C1 C2 P
  assume A1 A2 Par B1 B2 and
    Col P B1 B2 and
    A1 A2 Par C1 C2 and
    Col P C1 C2
  {
    assume H1:  $\forall C1 C2. P \neq C1 \wedge A1 A2 Par C1 C2 \wedge Col P C1 C2$ 
       $\longrightarrow$ 
       $(Col C1 B1 B2 \wedge Col C2 B1 B2)$ 
    fix C1 C2
    assume A1 A2 Par C1 C2 and Col P C1 C2
    have Col C1 B1 B2  $\wedge$  Col C2 B1 B2
    proof (cases P = C1)
      case True
      hence P = C1
      by simp
      show ?thesis
      proof (cases P = C2)
        case True
        thus ?thesis
          using ⟨Col P B1 B2⟩ ⟨P = C1⟩ by auto
      next
        case False
        hence P  $\neq$  C2
        by simp
        show ?thesis
          by (metis ⟨A1 A2 Par C1 C2⟩ ⟨Col P B1 B2⟩
            ⟨Col P C1 C2⟩ H1 not-col-permutation-5
            par-right-comm)
      qed
    next
      case False
      hence P  $\neq$  C1
      by simp
      thus ?thesis
        using ⟨A1 A2 Par C1 C2⟩ ⟨Col P C1 C2⟩ H1 by blast
    qed
  }
}
{
  fix C1 C2
  assume P  $\neq$  C1 and
    A1 A2 Par C1 C2 and
    Col P C1 C2
  have Col C1 B1 B2  $\wedge$  Col C2 B1 B2
  proof (cases Col A1 A2 P)
    case True
    have Col C1 B1 B2
      using ⟨A1 A2 Par B1 B2⟩ ⟨A1 A2 Par C1 C2⟩
        ⟨Col P B1 B2⟩ ⟨Col P C1 C2⟩ col3 not-strict-par1
        not-strict-par2 par-neq1
    by (meson True not-col-permutation-2)
  }
}

```

```

moreover
have  $Col\ C2\ B1\ B2$ 
  using  $\langle A1\ A2\ Par\ B1\ B2 \rangle\ \langle A1\ A2\ Par\ C1\ C2 \rangle$ 
     $\langle Col\ P\ B1\ B2 \rangle\ \langle Col\ P\ C1\ C2 \rangle$ 
     $col3\ not\ strict\ par1\ not\ strict\ par2\ par\ neq1$ 
  by (metis  $\langle P \neq C1 \rangle$  calculation
     $col\ permutation\ 1\ col\ trivial\ 2$ )
ultimately show ?thesis
  by blast
next
case False
hence  $\neg\ Col\ A1\ A2\ P$ 
  by simp
then obtain  $I$  where  $Col\ A1\ A2\ I$  and  $A1\ A2\ Perp\ P\ I$ 
  using l8-18-existence by blast
have  $Coplanar\ B1\ B2\ P\ I$ 
  using  $\langle Col\ P\ B1\ B2 \rangle\ col\ permutation\ 1\ ncop\ --ncol$  by blast
have  $Coplanar\ C1\ C2\ P\ I$ 
  by (meson  $\langle Col\ P\ C1\ C2 \rangle\ col\ permutation\ 1\ ncop\ --ncol$ )
moreover
have  $A1\ A2\ Par\ B1\ B2$ 
  using  $\langle A1\ A2\ Par\ B1\ B2 \rangle$  by auto
ultimately
have  $B1\ B2\ Perp\ P\ I$ 
  using perpendicular-transversal-postulate-def assms
     $\langle A1\ A2\ Perp\ P\ I \rangle\ \langle Coplanar\ B1\ B2\ P\ I \rangle$  by blast
have  $C1\ C2\ Perp\ P\ I$ 
  using perpendicular-transversal-postulate-def assms
     $\langle A1\ A2\ Perp\ P\ I \rangle\ \langle Coplanar\ C1\ C2\ P\ I \rangle\ \langle A1\ A2\ Par\ C1\ C2 \rangle$  by blast
have  $Coplanar\ A1\ A2\ P\ B1$ 
  using  $\langle A1\ A2\ Par\ B1\ B2 \rangle\ \langle Col\ P\ B1\ B2 \rangle\ col2\ cop\ --cop$ 
     $col\ permutation\ 1\ col\ trivial\ 3\ par\ --coplanar$ 
     $par\ neq2$  by blast
have  $Coplanar\ A1\ A2\ P\ B2$ 
  using  $\langle A1\ A2\ Par\ B1\ B2 \rangle\ \langle Col\ P\ B1\ B2 \rangle\ col2\ cop\ --cop$ 
     $col\ permutation\ 1\ col\ trivial\ 3\ par\ --coplanar$ 
     $par\ neq2$  by blast
have  $Coplanar\ A1\ A2\ P\ C1$ 
  using  $\langle A1\ A2\ Par\ C1\ C2 \rangle\ \langle Col\ P\ C1\ C2 \rangle\ col\ trivial\ 2$ 
     $ncop\ distincts\ par\ --coplanar$ 
     $par\ col2\ par\ bis$  by blast
have  $Coplanar\ A1\ A2\ P\ I$ 
  by (simp add:  $\langle A1\ A2\ Perp\ P\ I \rangle\ perp\ --coplanar$ )
have  $P\ C1\ Perp\ P\ I$ 
proof –
  have  $C1 \neq P$ 
  using  $\langle P \neq C1 \rangle$  by blast
moreover
have  $C1\ C2\ Perp\ P\ I$ 
  by (simp add:  $\langle C1\ C2\ Perp\ P\ I \rangle$ )
moreover
have  $Col\ C1\ C2\ P$ 
  using Col-cases  $\langle Col\ P\ C1\ C2 \rangle$  by blast
ultimately
show ?thesis
  by (meson perp-col perp-left-comm)
qed
have  $Col\ P\ C1\ B1$ 
proof (cases  $P = B1$ )
  case True
  thus ?thesis
  by (simp add:  $col\ trivial\ 3$ )
next
case False
show ?thesis
proof –

```

```

have Coplanar P I C1 B1
proof -
  have Coplanar A1 A2 P P
    using ncop-distincts by blast
  thus ?thesis
    by (meson <¬ Coplanar A1 A2 P> <Coplanar A1 A2 P B1>
      <Coplanar A1 A2 P C1> <Coplanar A1 A2 P I>
      coplanar-pseudo-trans)
qed
moreover
have P B1 Perp P I
  by (metis False Perp-perm <B1 B2 Perp P I> <Col P B1 B2>
    col-permutation-1 perp-col)
ultimately
show ?thesis
  using <P C1 Perp P I> cop-perp2--col by blast
qed
qed
moreover
have Col P C1 B2
proof (cases P = B2)
  case True
  thus ?thesis
    by (simp add: col-trivial-3)
next
case False
show ?thesis
proof -
  have Coplanar P I C1 B2
  proof -
    have Coplanar A1 A2 P P
      using ncop-distincts by blast
    thus ?thesis
      by (meson <¬ Coplanar A1 A2 P> <Coplanar A1 A2 P B2>
        <Coplanar A1 A2 P C1> <Coplanar A1 A2 P I>
        coplanar-pseudo-trans)
  qed
  moreover
  have P B2 Perp P I
    by (metis False Perp-perm <B1 B2 Perp P I>
      <Col P B1 B2> col-permutation-1 perp-col)
  ultimately
  show ?thesis
    using <P C1 Perp P I> cop-perp2--col by blast
qed
qed
ultimately
show ?thesis
  by (metis <Col P C1 C2> <P ≠ C1> col-transitivity-2)
qed
}
hence Col C1 B1 B2 ∧ Col C2 B1 B2
  using <A1 A2 Par C1 C2> <Col P C1 C2>
  <∧ C2a C1a. [∀ C1 C2. P ≠ C1 ∧ A1 A2 Par C1 C2 ∧ Col P C1 C2]
  → Col C1 B1 B2 ∧ Col C2 B1 B2; A1 A2 Par C1a C2a; Col P C1a C2a]
  ⇒ Col C1a B1 B2 ∧ Col C2a B1 B2>
  by blast
}
thus ?thesis
  using playfair-s-postulate-def by blast
qed

lemma playfair--universal-posidonius-postulate:
  assumes playfair-s-postulate
  shows universal-posidonius-postulate
proof -

```

```

{
  fix A1 A2 A3 A4 B1 B2 B3 B4
  assume A1 A2 Par B1 B2 and
    Col A1 A2 A3 and Col B1 B2 B3 and
    A1 A2 Perp A3 B3 and
    Col A1 A2 A4 and Col B1 B2 B4 and A1 A2 Perp A4 B4
  {
    assume A1 A2 ParStrict B1 B2
    hence  $\neg$  Col A1 A2 B1
      by (meson par-strict-not-col-1)
    then obtain A1' where Col A1 A2 A1' and A1 A2 Perp B1 A1'
      using l8-18-existence by blast
    {
      fix A3 B3
      assume Col A1 A2 A3 and
        Col B1 B2 B3 and
        A1 A2 Perp A3 B3
      have B3  $\neq$  A3
        using  $\langle A1 A2 Perp A3 B3 \rangle \langle Col A1 A2 A3 \rangle$  perp-not-col2 by blast
      then obtain B3' where Bet A3 B3 B3'  $\vee$  Bet A3 B3' B3 and Cong A3 B3' A1' B1
        using segment-construction-2 by blast
      {
        assume A1' = A3
        have Col A1' B1 B3
        proof -
          have Coplanar A1 A2 B1 B3
            using  $\langle A1 A2 Par B1 B2 \rangle \langle Col B1 B2 B3 \rangle$  col2-cop--cop
              col-trivial-3 par--coplanar par-neq2 by blast
          moreover
          have A1' B1 Perp A1 A2
            using Perp-perm  $\langle A1 A2 Perp B1 A1' \rangle$  by blast
          moreover
          have A1' B3 Perp A1 A2
            using Perp-perm  $\langle A1' = A3 \rangle \langle A1 A2 Perp A3 B3 \rangle$  by blast
          ultimately
          show ?thesis
            using cop-perp2--col by blast
        qed
        have B1 = B3
        proof -
          have  $\neg$  Col B1 B2 A1'
          proof (cases A1 = A1')
            case True
            thus ?thesis
              using  $\langle A1 A2 Par B1 B2 \rangle \langle Col A1 A2 A1' \rangle$ 
                 $\langle \neg Col A1 A2 B1 \rangle$  not-strict-par1 by blast
          next
            case False
            thus ?thesis
              using  $\langle A1' = A3 \rangle \langle A1 A2 Par B1 B2 \rangle \langle Col A1 A2 A3 \rangle \langle \neg Col A1 A2 B1 \rangle$ 
                not-strict-par by blast
          qed
          moreover
          have Col B1 B2 B1
            by (simp add: col-trivial-3)
          moreover
          have Col A1' B1 B1
            using col-trivial-2 by auto
          ultimately
          show ?thesis
            by (meson  $\langle Col B1 B2 B3 \rangle \langle Col A1' B1 B3 \rangle$ 
              col-permutation-4 l6-16-1)
        qed
        hence Cong A3 B3 A1' B1
          using cong-reflexivity by (simp add:  $\langle A1' = A3 \rangle$ )
      }
    }
  }
}

```

```

moreover
{
  assume  $A1' \neq A3$ 
  have Saccheri  $A1' B1 B3' A3$ 
  proof -
    have  $A1' B1 \text{ Perp } A3 A1'$ 
      using  $\langle A1' \neq A3 \rangle$  Perp-cases  $\langle A1 A2 \text{ Perp } B1 A1' \rangle$ 
         $\langle \text{Col } A1 A2 A1' \rangle$   $\langle \text{Col } A1 A2 A3 \rangle$  perp-col2-bis by blast
    hence  $\text{Per } B1 A1' A3$ 
      using perp-per-1 by blast
    moreover
    have  $A3 A1' \text{ Perp } B3' A3$ 
    proof -
      have  $A1 A2 \text{ Perp } A3 B3$ 
        by (simp add:  $\langle A1 A2 \text{ Perp } A3 B3 \rangle$ )
      hence  $A3 B3 \text{ Perp } A3 A1'$ 
        using perp-col0  $\langle A1' \neq A3 \rangle$   $\langle \text{Col } A1 A2 A3 \rangle$ 
           $\langle \text{Col } A1 A2 A1' \rangle$  by presburger
      moreover
      have  $\text{Col } A3 B3 B3'$ 
        using Col-def  $\langle \text{Bet } A3 B3 B3' \vee \text{Bet } A3 B3' B3 \rangle$ 
          col-permutation-4 by blast
      moreover
      have  $\text{Col } A3 B3 A3$ 
        by (simp add: col-trivial-3)
      ultimately
      show ?thesis
        by (metis  $\langle A1' B1 \text{ Perp } A3 A1' \rangle$   $\langle \text{Cong } A3 B3' A1' B1 \rangle$ 
          cong-diff-3 perp-col0 perp-not-eq-1)
    qed
  hence  $\text{Per } A1' A3 B3'$ 
    using perp-per-1 by blast
  moreover
  have  $\text{Cong } A1' B1 B3' A3$ 
    using  $\langle \text{Cong } A3 B3' A1' B1 \rangle$  not-cong-4312 by blast
  moreover
  have  $A1' A3 \text{ OS } B1 B3'$ 
  proof -
    have  $A1' A3 \text{ OS } B1 B3$ 
    proof (cases  $B1 = B3$ )
      case True
      thus ?thesis
        by (metis Col-cases  $\langle \neg \text{Col } A1 A2 B1 \rangle$   $\langle A1' \neq A3 \rangle$ 
           $\langle \text{Col } A1 A2 A1' \rangle$   $\langle \text{Col } A1 A2 A3 \rangle$  colx
          one-side-reflexivity)
    next
      case False
      have  $\text{Col } A1 A2 A1'$ 
        using  $\langle \text{Col } A1 A2 A1' \rangle$  by auto
      moreover
      have  $\text{Col } A1 A2 A3$ 
        by (simp add:  $\langle \text{Col } A1 A2 A3 \rangle$ )
      moreover
      have  $A1 A2 \text{ OS } B1 B3$ 
        by (meson  $\langle A1 A2 \text{ ParStrict } B1 B2 \rangle$   $\langle \text{Col } B1 B2 B3 \rangle$ 
          par-strict-one-side)
      ultimately
      show ?thesis
        using col2-os--os  $\langle A1' \neq A3 \rangle$  by blast
    qed
  moreover
  have  $A1' A3 \text{ OS } B3 B3'$ 
    by (metis  $\langle \text{Bet } A3 B3 B3' \vee \text{Bet } A3 B3' B3 \rangle$ 
       $\langle \text{Cong } A1' B1 B3' A3 \rangle$  bet-out calculation col124--nos
      cong-identity invert-one-side l6-6 os-distincts
      out-one-side)

```

```

ultimately
show ?thesis
  using one-side-transitivity by blast
qed
ultimately
show ?thesis
  using Saccheri-def by blast
qed
hence A1' A3 ParStrict B1 B3'
  using sac--pars1423 by blast
have Col B3' B1 B2
proof -
  have A1 A2 Par B1 B2
    using ⟨A1 A2 Par B1 B2⟩ by auto
  moreover
  have Col B1 B1 B2
    by (simp add: col-trivial-1)
  moreover
  have A1 A2 Par B1 B3'
proof -
  have A1 ≠ A2
    using ⟨¬ Col A1 A2 B1⟩ col-trivial-1 by blast
  moreover
  have B1 B3' Par A1' A3
    using Par-strict-perm ⟨A1' A3 ParStrict B1 B3'⟩
      par-strict-par by blast
  moreover
  have Col A1' A3 A1
    by (meson ⟨Col A1 A2 A1'⟩ ⟨Col A1 A2 A3⟩
      calculation(1) col-transitivity-1
      not-col-permutation-3)
  moreover
  have Col A1' A3 A2
    by (meson ⟨Col A1 A2 A1'⟩ ⟨Col A1 A2 A3⟩
      calculation(1) col-transitivity-2
      not-col-permutation-2)
  ultimately
  show ?thesis
    by (meson par-col2-par par-symmetry)
qed
moreover
have Col B1 B1 B3'
  by (simp add: col-trivial-1)
ultimately
show ?thesis
  using assms playfair-s-postulate-def by blast
qed
have B3 = B3'
proof -
  have ¬ Col B1 B3' A3
    by (metis ⟨A1 A2 Par B1 B2⟩ ⟨A1' A3 ParStrict B1 B3'⟩
      ⟨Col A1 A2 A3⟩ ⟨Col B3' B1 B2⟩ ⟨¬ Col A1 A2 B1⟩
      col-transitivity-2 not-col-permutation-4
      not-strict-par par-strict-neq2)
  moreover
  have A3 ≠ B3
    using ⟨B3 ≠ A3⟩ by auto
  moreover
  have Col B1 B3' B3
    by (metis ⟨A1 A2 ParStrict B1 B2⟩ ⟨Col B1 B2 B3⟩
      ⟨Col B3' B1 B2⟩ colx not-col-distincts
      par-strict-neq2)
  moreover
  have Col B1 B3' B3'
    by (simp add: col-trivial-2)
  moreover

```



```

    have Col A3 B3 B3
      by (simp add: col-trivial-2)
    moreover
    have Col A3 B3 B3'
      using ⟨Bet A3 B3 B3' ∨ Bet A3 B3' B3⟩
        bet-col1 between-trivial by blast
    ultimately
    show ?thesis
      using l6-21 by blast
  qed
  hence Cong A3 B3 A1' B1
    using ⟨Cong A3 B3' A1' B1⟩ by auto
}
ultimately
have Cong A3 B3 A1' B1
  by blast
}
moreover
have Cong A3 B3 A1' B1
  using calculation(1)
  by (simp add: ⟨A1 A2 Perp A3 B3⟩ ⟨Col A1 A2 A3⟩
    ⟨Col B1 B2 B3⟩)
moreover
have Cong A1' B1 A4 B4
  using calculation(1) Cong-perm ⟨A1 A2 Perp A4 B4⟩
    ⟨Col A1 A2 A4⟩ ⟨Col B1 B2 B4⟩ by blast
ultimately
have Cong A3 B3 A4 B4
  using cong-transitivity by blast
}
moreover
{
  assume A1 ≠ A2 and B1 ≠ B2 and Col A1 B1 B2 and Col A2 B1 B2
  hence Cong A3 B3 A4 B4
    using ⟨A1 A2 Par B1 B2⟩ ⟨A1 A2 Perp A3 B3⟩
      ⟨Col A1 A2 A3⟩ ⟨Col B1 B2 B3⟩ calculation
      par-not-col-strict perp-not-col2 by blast
}
ultimately
have Cong A3 B3 A4 B4
  using Par-def ⟨A1 A2 Par B1 B2⟩ by presburger
}
thus ?thesis
  using universal-posidonius-postulate-def by blast
qed

```

lemma *weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom-aux:*

assumes *weak-inverse-projection-postulate*

shows $\forall A1 A2 B1 B2 C1 C2 Q P R M.$

$(A1 A2 Perp B1 B2 \wedge A1 A2 Perp C1 C2 \wedge$
 $Col A1 A2 Q \wedge Col B1 B2 Q \wedge$
 $Col A1 A2 P \wedge Col C1 C2 P \wedge$
 $Col B1 B2 R \wedge Coplanar Q P R C1 \wedge$
 $Coplanar Q P R C2 \wedge \neg Col Q P R \wedge$
 $M InAngle P Q R \wedge M Q P CongA M Q R)$

→

$(B1 B2 ParStrict C1 C2 \wedge$
 $(\exists S. Q Out M S \wedge Col C1 C2 S))$

proof –

```

{
  fix A1 A2 B1 B2 C1 C2 Q P R M
  assume A1 A2 Perp B1 B2 and
    A1 A2 Perp C1 C2 and
    Col A1 A2 Q and

```

```

Col B1 B2 Q and
Col A1 A2 P and
Col C1 C2 P and
Col B1 B2 R and
Coplanar Q P R C1 and
Coplanar Q P R C2 and
¬ Col Q P R and
M InAngle P Q R and
M Q P CongA M Q R
have Q ≠ P
  using ⟨¬ Col Q P R⟩ col-trivial-1 by auto
have Q ≠ R
  using ⟨¬ Col Q P R⟩ col-trivial-3 by auto
have A1 ≠ A2
  using ⟨A1 A2 Perp C1 C2⟩ perp-distinct by auto
have C1 ≠ C2
  using ⟨A1 A2 Perp C1 C2⟩ perp-distinct by auto
have B1 ≠ B2
  using ⟨A1 A2 Perp B1 B2⟩ perp-distinct by auto
have Q ≠ M
  using ⟨M InAngle P Q R⟩ inangle-distincts by blast
have ¬ Col A1 A2 R
  using ⟨A1 A2 Perp B1 B2⟩ ⟨Col A1 A2 P⟩ ⟨Col A1 A2 Q⟩
    ⟨¬ Col Q P R⟩ col3 perp-not-eq-1 by blast
have ¬ Col B1 B2 P
  using ⟨B1 ≠ B2⟩ ⟨Col B1 B2 Q⟩ ⟨Col B1 B2 R⟩
    ⟨¬ Col Q P R⟩ col3 by blast
have B1 B2 ParStrict C1 C2
proof —
  have B1 B2 Par C1 C2
proof —
  have Col A1 P Q ∧ Col A2 P Q ∧ Col B1 R Q ∧ Col B2 R Q
    by (meson ⟨A1 ≠ A2⟩ ⟨B1 ≠ B2⟩ ⟨Col A1 A2 P⟩
      ⟨Col A1 A2 Q⟩ ⟨Col B1 B2 Q⟩ ⟨Col B1 B2 R⟩
      col-transitivity-1 col-transitivity-2)
  have Coplanar Q P R A1
    by (meson ⟨Col A1 P Q ∧ Col A2 P Q ∧ Col B1 R Q ∧ Col B2 R Q⟩
      ncop--ncols not-col-permutation-3)
  have Coplanar Q P R A2
    using ⟨Col A1 P Q ∧ Col A2 P Q ∧ Col B1 R Q ∧ Col B2 R Q⟩
      ncop--ncols not-col-permutation-3 by blast
  have Coplanar Q P R B1
    using ⟨Col A1 P Q ∧ Col A2 P Q ∧ Col B1 R Q ∧ Col B2 R Q⟩
      ncop--ncols not-col-permutation-3 by blast
  have Coplanar Q P R B2
    using NCol-perm ⟨Col A1 P Q ∧ Col A2 P Q ∧ Col B1 R Q ∧ Col B2 R Q⟩
      ncop--ncols by blast
  have Coplanar A1 A2 B1 C1
    by (meson ⟨Coplanar Q P R A1⟩ ⟨Coplanar Q P R A2⟩
      ⟨Coplanar Q P R B1⟩ ⟨Coplanar Q P R C1⟩ ⟨¬ Col Q P R⟩
      coplanar-pseudo-trans)
  moreover
  have Coplanar A1 A2 B1 C2
    by (meson ⟨Coplanar Q P R A1⟩ ⟨Coplanar Q P R A2⟩
      ⟨Coplanar Q P R B1⟩ ⟨Coplanar Q P R C2⟩ ⟨¬ Col Q P R⟩
      coplanar-pseudo-trans)
  moreover
  have Coplanar A1 A2 B2 C1
    by (meson ⟨Coplanar Q P R A1⟩ ⟨Coplanar Q P R A2⟩
      ⟨Coplanar Q P R B2⟩ ⟨Coplanar Q P R C1⟩ ⟨¬ Col Q P R⟩
      coplanar-pseudo-trans)
  moreover
  have Coplanar A1 A2 B2 C2
    by (meson ⟨Coplanar Q P R A1⟩ ⟨Coplanar Q P R A2⟩
      ⟨Coplanar Q P R B2⟩ ⟨Coplanar Q P R C2⟩ ⟨¬ Col Q P R⟩
      coplanar-pseudo-trans)

```

```

moreover
have  $B1\ B2\ Perp\ A1\ A2$ 
  using  $Perp\text{-}perm\ \langle A1\ A2\ Perp\ B1\ B2 \rangle$  by blast
moreover
have  $C1\ C2\ Perp\ A1\ A2$ 
  using  $Perp\text{-}perm\ \langle A1\ A2\ Perp\ C1\ C2 \rangle$  by blast
ultimately
show ?thesis
  by (meson l12-9)
qed
moreover
have  $Col\ C1\ C2\ P$ 
  by (simp add:  $\langle Col\ C1\ C2\ P \rangle$ )
moreover
have  $\neg\ Col\ B1\ B2\ P$ 
  using  $\langle \neg\ Col\ B1\ B2\ P \rangle$  by auto
ultimately
show ?thesis
  using par-not-col-strict by blast
qed
moreover
have  $\neg\ Col\ Q\ C1\ C2$ 
  using  $\langle Col\ B1\ B2\ Q \rangle$  calculation
  not-col-permutation-1 par-not-col by blast
have  $Per\ P\ Q\ R$ 
proof –
  have  $B1\ B2\ Perp\ A1\ A2$ 
    using  $Perp\text{-}perm\ \langle A1\ A2\ Perp\ B1\ B2 \rangle$  by blast
  hence  $A1\ A2\ Perp\ Q\ R$ 
    using  $\langle Q \neq R \rangle\ \langle Col\ B1\ B2\ R \rangle\ \langle Col\ B1\ B2\ Q \rangle$  perp-col0 by blast
  moreover
  have  $Col\ A1\ A2\ Q$ 
    using  $\langle Col\ A1\ A2\ Q \rangle$  by auto
  moreover
  have  $Col\ A1\ A2\ P$ 
    by (simp add:  $\langle Col\ A1\ A2\ P \rangle$ )
  ultimately
  show ?thesis
    by (meson  $\langle Q \neq P \rangle$  perp-col2 perp-per-2)
qed
have  $P\ Q\ M\ P\ Q\ M\ SumA\ P\ Q\ R$ 
proof –
  have  $P\ Q\ M\ M\ Q\ R\ SumA\ P\ Q\ R$ 
    by (simp add:  $\langle M\ InAngle\ P\ Q\ R \rangle$  inangle--suma)
  moreover
  have  $P\ Q\ M\ CongA\ P\ Q\ M$ 
    using  $\langle Q \neq M \rangle\ \langle Q \neq P \rangle$  conga-refl by auto
  moreover
  have  $M\ Q\ R\ CongA\ P\ Q\ M$ 
    by (meson  $\langle M\ Q\ P\ CongA\ M\ Q\ R \rangle$  conga-right-comm conga-sym)
  moreover
  have  $P\ Q\ R\ CongA\ P\ Q\ R$ 
    using  $\langle Q \neq P \rangle\ \langle Q \neq R \rangle$  conga-refl by auto
  ultimately
  show ?thesis
    by (meson conga3-suma--suma)
qed
have  $Acute\ P\ Q\ M$ 
proof –
  have  $\neg\ Bet\ P\ Q\ R$ 
    using  $Col\text{-}cases\ \langle \neg\ Col\ Q\ P\ R \rangle$  bet-col by blast
  moreover
  have  $SAMS\ P\ Q\ M\ P\ Q\ M$ 
    by (meson  $\langle M\ InAngle\ P\ Q\ R \rangle\ \langle M\ Q\ P\ CongA\ M\ Q\ R \rangle$ 
      conga2-sams--sams conga-left-comm conga-sym
      conga-trans inangle--sams)

```

```

moreover
have  $P \ Q \ M \ P \ Q \ M \ \text{SumA} \ P \ Q \ R$ 
  by (simp add:  $\langle P \ Q \ M \ P \ Q \ M \ \text{SumA} \ P \ Q \ R \rangle$ )
ultimately
show ?thesis
  using nbt-sams-suma--acute by blast
qed
have  $\exists \ C3. \ \text{Col} \ C1 \ C2 \ C3 \wedge P \ Q \ OS \ R \ C3$ 
proof –
  obtain  $C0$  where  $C1 \neq C0$  and  $C2 \neq C0$  and
     $P \neq C0$  and  $\text{Col} \ C1 \ C2 \ C0$ 
    using  $\langle \text{Col} \ C1 \ C2 \ P \rangle \ \text{diff-col-ex3}$  by blast
  have  $\exists \ C3. \ \text{Col} \ C0 \ P \ C3 \wedge P \ Q \ OS \ R \ C3$ 
  proof –
    have  $C0 \neq P$ 
      using  $\langle P \neq C0 \rangle$  by auto
    moreover
    have  $\text{Col} \ P \ Q \ P$ 
      using col-trivial-3 by blast
    moreover
    have  $\text{Col} \ C0 \ P \ P$ 
      by (simp add: col-trivial-2)
    moreover
    {
      assume  $\text{Col} \ P \ Q \ C0$ 
      hence  $\text{Col} \ C1 \ C2 \ Q$ 
      by (metis (full-types)  $\langle \text{Col} \ C1 \ C2 \ C0 \rangle \langle \text{Col} \ C1 \ C2 \ P \rangle$ 
        calculation(1) col-permutation-4 col-trivial-2 l6-21)
      hence False
      using  $\langle \neg \ \text{Col} \ Q \ C1 \ C2 \rangle \ \text{not-col-permutation-1}$  by blast
    }
    moreover
    have  $\neg \ \text{Col} \ P \ Q \ R$ 
      using Col-cases  $\langle \neg \ \text{Col} \ Q \ P \ R \rangle$  by blast
    moreover
    have Coplanar  $P \ Q \ R \ C0$ 
    proof –
      have Coplanar  $P \ Q \ R \ C1$ 
        using  $\langle \text{Coplanar} \ Q \ P \ R \ C1 \rangle \ \text{ncoplanar-perm-6}$  by blast
      moreover
      have Coplanar  $P \ Q \ R \ C2$ 
        by (simp add:  $\langle \text{Coplanar} \ Q \ P \ R \ C2 \rangle \ \text{coplanar-perm-6}$ )
      ultimately
      show ?thesis
        using  $\langle C1 \neq C2 \rangle \langle \text{Col} \ C1 \ C2 \ C0 \rangle \ \text{col-cop2--cop}$  by blast
    qed
    hence Coplanar  $P \ Q \ C0 \ R$ 
      by (simp add: coplanar-perm-1)
    ultimately
    show ?thesis
      using cop-not-par-same-side by blast
    qed
  then obtain  $C3$  where  $\text{Col} \ C0 \ P \ C3$  and  $P \ Q \ OS \ R \ C3$ 
    by blast
  thus ?thesis
    by (metis  $\langle \text{Col} \ C1 \ C2 \ C0 \rangle \langle \text{Col} \ C1 \ C2 \ P \rangle \langle P \neq C0 \rangle \ \text{colx}$ )
qed
then obtain  $C3$  where  $\text{Col} \ C1 \ C2 \ C3$  and  $P \ Q \ OS \ R \ C3$ 
  by blast
have  $\exists \ S. \ Q \ \text{Out} \ M \ S \wedge \text{Col} \ P \ C3 \ S$ 
proof –
  have  $Q \ \text{Out} \ P \ P$ 
    using  $\langle Q \neq P \rangle \ \text{out-trivial}$  by auto
  moreover
  have  $P \neq C3$ 
    by (metis os-distincts  $\langle P \ Q \ OS \ R \ C3 \rangle$ )

```

```

moreover
have  $Per\ Q\ P\ C3$ 
proof –
  have  $A1\ A2\ Perp\ C1\ C2$ 
    by (simp add: <A1 A2 Perp C1 C2>)
  then obtain  $P'$  where  $P'\ PerpAt\ A1\ A2\ C1\ C2$ 
    using perp-inter-perp-in-n by blast
  hence  $Col\ P'\ A1\ A2 \wedge Col\ P'\ C1\ C2 \wedge$ 
     $(\forall\ U\ V. (Col\ U\ A1\ A2 \wedge Col\ V\ C1\ C2) \longrightarrow Per\ U\ P'\ V)$ 
    using PerpAt-def by blast
  hence  $\forall\ U\ V. (Col\ U\ A1\ A2 \wedge Col\ V\ C1\ C2) \longrightarrow Per\ U\ P'\ V$ 
    by blast
  hence  $P = P'$ 
    by (metis Col-cases <Col A1 A2 P> <Col C1 C2 P>
      <P' PerpAt A1 A2 C1 C2> l8-14-2-1b)
  moreover
  have  $Col\ Q\ A1\ A2$ 
    using NCol-cases <Col A1 A2 Q> by blast
  moreover
  have  $Col\ C3\ C1\ C2$ 
    by (simp add: <Col C1 C2 C3> col-permutation-2)
  ultimately
  show ?thesis
    using  $\forall\ U\ V. Col\ U\ A1\ A2 \wedge Col\ V\ C1\ C2 \longrightarrow Per\ U\ P'\ V$  by blast
qed
moreover
have  $Coplanar\ P\ Q\ M\ C3$ 
proof –
  have  $Coplanar\ P\ Q\ R\ P$ 
    using ncop-distincts by blast
  moreover
  have  $Coplanar\ P\ Q\ R\ Q$ 
    using ncop-distincts by blast
  moreover
  have  $Coplanar\ P\ Q\ R\ M$ 
    by (meson <M InAngle P Q R> inangle--coplanar
      ncoplanar-perm-18)
  moreover
  have  $Coplanar\ P\ Q\ R\ C3$ 
    using  $\langle P\ Q\ OS\ R\ C3 \rangle$  os--coplanar by auto
  ultimately
  show ?thesis
    using  $Col\text{-}perm\ \neg\ Col\ Q\ P\ R$  coplanar-pseudo-trans by blast
qed
ultimately
show ?thesis
  using  $\langle Acute\ P\ Q\ M \rangle\ \langle Per\ P\ Q\ R \rangle\ \langle P\ Q\ M\ P\ Q\ M\ SumA\ P\ Q\ R \rangle$ 
    assms weak-inverse-projection-postulate-def by blast
qed
then obtain  $S$  where  $Q\ Out\ M\ S$  and  $Col\ P\ C3\ S$ 
  by blast
hence  $\exists\ S. Q\ Out\ M\ S \wedge Col\ C1\ C2\ S$ 
  by (metis <Col C1 C2 C3> <P Q OS R C3> <Col C1 C2 P>
    colx not-col-distincts one-side-not-col124)
  ultimately
have  $B1\ B2\ ParStrict\ C1\ C2 \wedge (\exists\ S. Q\ Out\ M\ S \wedge Col\ C1\ C2\ S)$ 
  by blast
}
thus ?thesis
  by blast
qed

lemma weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom:
  assumes weak-inverse-projection-postulate
  shows bachmann-s-lotschnittaxiom
proof –

```

```

{
  fix A1 A2 B1 B2 C1 C2 D1 D2 Q P R
  assume Q ≠ P and
    Q ≠ R and
    A1 A2 Perp B1 B2 and
    A1 A2 Perp C1 C2 and
    B1 B2 Perp D1 D2 and
    Col A1 A2 Q and
    Col B1 B2 Q and
    Col A1 A2 P and
    Col C1 C2 P and
    Col B1 B2 R and
    Col D1 D2 R and
    Coplanar Q P R C1 and
    Coplanar Q P R C2 and
    Coplanar Q P R D1 and
    Coplanar Q P R D2
  have Q P Perp R Q
    using ⟨Q ≠ P⟩ ⟨Q ≠ R⟩ ⟨Col A1 A2 Q⟩ ⟨Col A1 A2 P⟩
      ⟨Col B1 B2 R⟩ ⟨Col B1 B2 Q⟩ ⟨A1 A2 Perp B1 B2⟩
    perp-col4 by auto
  hence ¬ Col P Q R
    by (meson NCol-cases perp-not-col)
  have P ≠ Q
    using ⟨Q ≠ P⟩ by auto
  have R ≠ Q
    using ⟨Q ≠ R⟩ by blast
  obtain M where (M InAngle P Q R) and (M Q P CongA M Q R)
    using angle-bisector ⟨P ≠ Q⟩ ⟨R ≠ Q⟩ by blast
  have B1 ≠ B2
    using ⟨B1 B2 Perp D1 D2⟩ col-trivial-1 perp-not-col2 by blast
  have M ≠ Q
    using ⟨M InAngle P Q R⟩ inangle-distincts by presburger
  have P Q M P Q M SumA P Q R
  proof –
    have P Q M M Q R SumA P Q R
      by (simp add: ⟨M InAngle P Q R⟩ inangle--suma)
    moreover
    have P Q M CongA P Q M
      using ⟨M ≠ Q⟩ ⟨Q ≠ P⟩ conga-refl by auto
    moreover
    have M Q R CongA P Q M
      by (meson ⟨M Q P CongA M Q R⟩ conga-right-comm conga-sym)
    moreover
    have P Q R CongA P Q R
      using ⟨Q ≠ P⟩ ⟨R ≠ Q⟩ conga-refl by auto
    ultimately
    show ?thesis
      by (meson conga3-suma--suma)
  qed
  have Acute P Q M
  proof –
    have ¬ Bet P Q R
      using ⟨¬ Col P Q R⟩ bet-col by force
    moreover
    have SAMS P Q M P Q M
      by (meson ⟨M InAngle P Q R⟩ ⟨M Q P CongA M Q R⟩
        conga2-sams--sams conga-right-comm conga-trans
        inangle--sams not-conga-sym)
    ultimately
    show ?thesis
      by (meson ⟨P Q M P Q M SumA P Q R⟩ nbet-sams-suma--acute)
  qed
  have B1 B2 ParStrict C1 C2 ∧ (∃ S. Q Out M S ∧ Col C1 C2 S)
  proof –
    have A1 A2 Perp B1 B2

```

```

    using ⟨A1 A2 Perp B1 B2⟩ by auto
  moreover
  have A1 A2 Perp C1 C2
    by (simp add: ⟨A1 A2 Perp C1 C2⟩)
  moreover
  have Col A1 A2 Q
    by (simp add: ⟨Col A1 A2 Q⟩)
  moreover
  have Col B1 B2 Q
    by (simp add: ⟨Col B1 B2 Q⟩)
  moreover
  have Col A1 A2 P
    by (simp add: ⟨Col A1 A2 P⟩)
  moreover
  have Col C1 C2 P
    by (simp add: ⟨Col C1 C2 P⟩)
  moreover
  have Col B1 B2 R
    using ⟨Col B1 B2 R⟩ by auto
  moreover
  have Coplanar Q P R C1
    by (simp add: ⟨Coplanar Q P R C1⟩)
  moreover
  have Coplanar Q P R C2
    using ⟨Coplanar Q P R C2⟩ by auto
  moreover
  have ¬ Col Q P R
    using Col-cases ⟨¬ Col P Q R⟩ by blast
  moreover
  have M InAngle P Q R
    by (simp add: ⟨M InAngle P Q R⟩)
  moreover
  have M Q P CongA M Q R
    using ⟨M Q P CongA M Q R⟩ by blast
  ultimately
  show ?thesis
    using assms
      weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom-aux
    by blast
qed
then obtain S where Q Out M S and Col C1 C2 S
  by blast
have B1 B2 ParStrict C1 C2
  using ⟨B1 B2 ParStrict C1 C2 ∧ (∃ S. Q Out M S ∧ Col C1 C2 S)⟩ by blast
have A1 A2 ParStrict D1 D2 ∧ (∃ T. Q Out M T ∧ Col D1 D2 T)
proof -
  have B1 B2 Perp A1 A2
    using Perp-perm ⟨A1 A2 Perp B1 B2⟩ by blast
  moreover
  have B1 B2 Perp D1 D2
    by (simp add: ⟨B1 B2 Perp D1 D2⟩)
  moreover
  have Col B1 B2 Q
    by (simp add: ⟨Col B1 B2 Q⟩)
  moreover
  have Col A1 A2 Q
    using ⟨Col A1 A2 Q⟩ by blast
  moreover
  have Col B1 B2 R
    by (simp add: ⟨Col B1 B2 R⟩)
  moreover
  have Col D1 D2 R
    by (simp add: ⟨Col D1 D2 R⟩)
  moreover
  have Col A1 A2 P
    using ⟨Col A1 A2 P⟩ by auto

```

```

moreover
have Coplanar Q R P D1
  by (simp add: <Coplanar Q P R D1> coplanar-perm-2)
moreover
have Coplanar Q R P D2
  using <Coplanar Q P R D2> coplanar-perm-2 by blast
moreover
have  $\neg$  Col Q R P
  using Col-cases < $\neg$  Col P Q R> by blast
moreover
have M InAngle R Q P
  by (simp add: <M InAngle P Q R> l11-24)
moreover
have M Q R CongA M Q P
  by (simp add: <M Q P CongA M Q R> conga-sym-equiv)
ultimately
show ?thesis
  using assms
    weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom-aux
  by blast
qed
then obtain T where Q Out M T and Col D1 D2 T
  by blast
have A1 A2 ParStrict D1 D2
  using <A1 A2 ParStrict D1 D2  $\wedge$  ( $\exists$  T. Q Out M T  $\wedge$  Col D1 D2 T)> by blast
have  $\exists$  I. Col C1 C2 I  $\wedge$  Col D1 D2 I
proof (cases Col C1 C2 T)
  case True
    thus ?thesis
    using <Col D1 D2 T> by auto
next
  case False
    hence  $\neg$  Col C1 C2 T
    by blast
    show ?thesis
    proof (cases Col D1 D2 S)
      case True
        thus ?thesis
        using <Col C1 C2 S> by auto
      next
        case False
          hence  $\neg$  Col D1 D2 S
          by auto
          have Q Out S T
          using Out-cases <Q Out M S> <Q Out M T> l6-7 by blast
          have  $S \neq Q$ 
          using <Q Out M S> l6-3-1 by blast
          have  $T \neq Q$ 
          using <Q Out M T> l6-3-1 by blast
          {
            assume Bet Q S T
            have C1 C2 TS R T
            proof –
              have C1 C2 TS Q T
              by (metis TS-def <Bet Q S T> <Col C1 C2 S>
                <S  $\neq$  Q> < $\neg$  Col C1 C2 T> bet-col colx
                not-col-permutation-2)
              moreover
              have C1 C2 ParStrict Q R
              by (metis Par-strict-cases <B1 B2 ParStrict C1 C2>
                <Col B1 B2 Q> <Col B1 B2 R> <R  $\neq$  Q>
                par-strict-col2-par-strict)
              hence C1 C2 OS Q R
              using l12-6 by blast
              ultimately
              show ?thesis
          }
    }

```



```

    using l9-8-2 by blast
qed
then obtain I where Col I C1 C2 and Bet R I T
  using TS-def by blast
have T ≠ R
  using ⟨C1 C2 TS R T⟩ ts-distincts by blast
hence ∃ I. Col C1 C2 I ∧ Col D1 D2 I
  by (meson ⟨Bet R I T⟩ ⟨Col D1 D2 R⟩ ⟨Col D1 D2 T⟩
    ⟨Col I C1 C2⟩ bet-col colx not-col-permutation-2)
}
moreover
{
  assume Bet Q T S
  have D1 D2 TS P S
  proof –
    have D1 D2 TS Q S
      by (metis Col-cases False ⟨Bet Q T S⟩ ⟨Col D1 D2 T⟩
        ⟨T ≠ Q⟩ bet-col colx l9-18)
    moreover
    have D1 D2 ParStrict A1 A2
      by (meson ⟨A1 A2 ParStrict D1 D2⟩ par-strict-symmetry)
    hence D1 D2 ParStrict Q P
      using ⟨Q ≠ P⟩ ⟨Col A1 A2 Q⟩ ⟨Col A1 A2 P⟩
        par-strict-col2-par-strict by blast
    hence D1 D2 OS Q P
      by (simp add: l12-6)
    ultimately
    show ?thesis
      using l9-8-2 by blast
  qed
then obtain I where Col I D1 D2 and Bet P I S
  using TS-def by blast
hence Col D1 D2 I
  using Col-cases by blast
have P ≠ S
  using ⟨D1 D2 TS P S⟩ not-two-sides-id by blast
hence ∃ I. Col C1 C2 I ∧ Col D1 D2 I
  by (metis (full-types) ⟨Col C1 C2 P⟩ ⟨Col C1 C2 S⟩
    ⟨∧thesis. (∧I. [Col I D1 D2; Bet P I S] ⇒ thesis) ⇒ thesis⟩
    bet-col col-trivial-2 l6-21 not-col-permutation-2)
}
ultimately
show ?thesis
  using Out-def ⟨Q Out S T⟩ by force
qed
qed
}
thus ?thesis
  using bachmann-s-lotschnittaxiom-aux-R2 by blast
qed

lemma weak-triangle-circumscription-principle--bachmann-s-lotschnittaxiom:
  assumes weak-triangle-circumscription-principle
  shows bachmann-s-lotschnittaxiom
proof –
{
  fix A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD
  assume IAB ≠ IAC and IAB ≠ IBD and
    A1 A2 Perp B1 B2 and
    A1 A2 Perp C1 C2 and
    B1 B2 Perp D1 D2 and
    Col A1 A2 IAB and
    Col B1 B2 IAB and
    Col A1 A2 IAC and
    Col C1 C2 IAC and
    Col B1 B2 IBD and

```

Col D1 D2 IBD and
Coplanar IAB IAC IBD C1 and
Coplanar IAB IAC IBD C2 and
Coplanar IAB IAC IBD D1 and
Coplanar IAB IAC IBD D2
obtain *E where IAC Midpoint IAB E*
 using *symmetric-point-construction* **by** *blast*
hence *Bet IAB IAC E*
 using *midpoint-bet* **by** *auto*
hence *Col IAB IAC E*
 using *Col-def* **by** *blast*
obtain *F where IBD Midpoint IAB F*
 using *symmetric-point-construction* **by** *blast*
hence *Bet IAB IBD F*
 using *midpoint-bet* **by** *auto*
hence *Col IAB IBD F*
 using *Col-def* **by** *blast*
have *IAB $\neq$ F*
 using $\langle \text{Bet IAB IBD F} \rangle \langle \text{IAB} \neq \text{IBD} \rangle$ *between-identity* **by** *blast*
have *IAB $\neq$ E*
 using $\langle \text{IAB} \neq \text{IAC} \rangle \langle \text{IAC Midpoint IAB E} \rangle$ *l8-20-2* **by** *blast*
have *E IAB Perp IAB F*
proof –
 have *B1 B2 Perp E IAB*
 proof –
 have *A1 A2 Perp B1 B2*
 using $\langle \text{A1 A2 Perp B1 B2} \rangle$ **by** *blast*
 moreover
 have *E $\neq$ IAB*
 using $\langle \text{IAB} \neq \text{E} \rangle$ **by** *blast*
 moreover
 have *Col A1 A2 E*
 using $\langle \text{Col A1 A2 IAB} \rangle \langle \text{Col A1 A2 IAC} \rangle \langle \text{Col IAB IAC E} \rangle$
 $\langle \text{IAB} \neq \text{IAC} \rangle$ *colx* **by** *blast*
 moreover
 have *Col A1 A2 IAB*
 by (*simp add:* $\langle \text{Col A1 A2 IAB} \rangle$)
 ultimately
 show *?thesis*
 using *perp-col0* **by** *blast*
 qed
 moreover
 have *IAB $\neq$ F*
 using $\langle \text{IAB} \neq \text{F} \rangle$ **by** *auto*
 moreover
 have *Col B1 B2 IAB*
 by (*simp add:* $\langle \text{Col B1 B2 IAB} \rangle$)
 moreover
 have *Col B1 B2 F*
 using $\langle \text{Col B1 B2 IBD} \rangle \langle \text{Col IAB IBD F} \rangle \langle \text{IAB} \neq \text{IBD} \rangle$
 calculation(3) *colx* **by** *blast*
 ultimately
 show *?thesis*
 using *perp-col0* **by** *blast*
qed
have *Per IAC IAB IBD*
proof –
 have *IAB $\neq$ F*
 by (*simp add:* $\langle \text{IAB} \neq \text{F} \rangle$)
 moreover
 have *Per IAC IAB F*
proof –
 have *IAB $\neq$ E*
 using $\langle \text{IAB} \neq \text{E} \rangle$ **by** *blast*
 moreover
 have *Per F IAB E*

```

    using Perp-perm ⟨E IAB Perp IAB F⟩ perp-per-2 by blast
  moreover
  have Col IAB E IAC
    using ⟨Col IAB IAC E⟩ not-col-permutation-5 by blast
  ultimately
  show ?thesis
    using l8-2 per-col by blast
qed
moreover
have Col IAB F IBD
  using ⟨Col IAB IBD F⟩ not-col-permutation-5 by blast
ultimately
show ?thesis
  using per-col by blast
qed
hence ¬ Col IAC IAB IBD
  using ⟨IAB ≠ IAC⟩ ⟨IAB ≠ IBD⟩ l8-9 by blast
have ∃ I. Col C1 C2 I ∧ Col D1 D2 I
proof -
  have ¬ Col E F IAB
    by (metis ⟨Col IAB IAC E⟩ ⟨Col IAB IBD F⟩ ⟨IAB ≠ E⟩
      ⟨IAB ≠ F⟩ ⟨¬ Col IAC IAB IBD⟩ col-permutation-1
      col-transitivity-2)
  moreover
  have E IAB Perp F IAB
    by (simp add: ⟨E IAB Perp IAB F⟩ perp-right-comm)
  hence Per E IAB F
    using Perp-cases perp-per-2 by blast
  moreover
  have F IAB ReflectL D1 D2
  proof -
    have IBD Midpoint IAB F
      by (simp add: ⟨IBD Midpoint IAB F⟩)
    moreover
    have Col D1 D2 IBD
      by (simp add: ⟨Col D1 D2 IBD⟩)
    moreover
    have D1 D2 Perp IAB F
    proof -
      have B1 B2 Perp D1 D2
        using ⟨B1 B2 Perp D1 D2⟩ by blast
      moreover
      have IAB ≠ F
        using ⟨IAB ≠ F⟩ by auto
      moreover
      have Col B1 B2 IAB
        by (simp add: ⟨Col B1 B2 IAB⟩)
      moreover
      have Col B1 B2 F
        using ⟨Col B1 B2 IBD⟩ ⟨Col IAB IBD F⟩ ⟨IAB ≠ IBD⟩
          calculation(3) colx by blast
      ultimately
      show ?thesis
        using perp-col0 by blast
    qed
  hence (D1 D2 Perp IAB F) ∨ (IAB = F)
    by blast
  ultimately
  show ?thesis
    using ReflectL-def by blast
qed
hence D1 D2 PerpBisect F IAB
  using Perp-bisect-def ⟨IAB ≠ F⟩ by blast
moreover
have E IAB ReflectL C1 C2
proof -

```

```

have IAC Midpoint IAB E
  by (simp add: ⟨IAC Midpoint IAB E⟩)
moreover
have Col C1 C2 IAC
  by (simp add: ⟨Col C1 C2 IAC⟩)
moreover
have C1 C2 Perp IAB E
proof -
  have A1 A2 Perp C1 C2
    by (simp add: ⟨A1 A2 Perp C1 C2⟩)
  moreover
  have IAB ≠ E
    using ⟨IAB ≠ E⟩ by auto
  moreover
  have Col A1 A2 IAB
    by (simp add: ⟨Col A1 A2 IAB⟩)
  moreover
  have Col A1 A2 E
    using ⟨Col A1 A2 IAC⟩ ⟨Col IAB IAC E⟩ ⟨IAB ≠ IAC⟩
      calculation(3) colx by blast
  ultimately
  show ?thesis
    using perp-col0 by blast
qed
hence (C1 C2 Perp IAB E) ∨ (IAB = E)
  by blast
ultimately
show ?thesis
  using ReflectL-def by blast
qed
hence C1 C2 PerpBisect E IAB
  using Perp-bisect-def ⟨IAB ≠ E⟩ by blast
moreover
have Coplanar E F IAB D1
proof -
  have Coplanar IAB IAC IBD E
    using ⟨Col IAB IAC E⟩ ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD F
    using ⟨Col IAB IBD F⟩ ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD IAB
    using ⟨Coplanar IAB IAC IBD D1⟩ ncop--distincts by blast
  ultimately
  show ?thesis
    by (meson ⟨Coplanar IAB IAC IBD D1⟩ ⟨¬ Col IAC IAB IBD⟩
      col-permutation-4 coplanar-pseudo-trans)
qed
moreover
have Coplanar E F IAB D2
proof -
  have Coplanar IAB IAC IBD E
    using ⟨Col IAB IAC E⟩ ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD F
    using ⟨Col IAB IBD F⟩ ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD IAB
    using ncop--distincts by blast
  moreover
  have Coplanar IAB IAC IBD D2
    using ⟨Coplanar IAB IAC IBD D2⟩ by blast
  ultimately
  show ?thesis
    using NCol-perm ⟨¬ Col IAC IAB IBD⟩ coplanar-pseudo-trans by blast
qed

```

```

moreover
have Coplanar E F IAB C1
proof –
  have Coplanar IAB IAC IBD E
    using  $\langle \text{Col IAB IAC E} \rangle$  ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD F
    using  $\langle \text{Col IAB IBD F} \rangle$  ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD IAB
    using ncop-distincts by blast
  ultimately
  show ?thesis
    using Col-perm  $\langle \text{Coplanar IAB IAC IBD C1} \rangle$ 
       $\langle \neg \text{Col IAC IAB IBD} \rangle$  coplanar-pseudo-trans by blast
qed
moreover
have Coplanar E F IAB C2
proof –
  have Coplanar IAB IAC IBD E
    using  $\langle \text{Col IAB IAC E} \rangle$  ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD F
    using  $\langle \text{Col IAB IBD F} \rangle$  ncop--ncols by blast
  moreover
  have Coplanar IAB IAC IBD IAB
    using ncop-distincts by blast
  moreover
  have Coplanar IAB IAC IBD C2
    using  $\langle \text{Coplanar IAB IAC IBD C2} \rangle$  by auto
  ultimately
  show ?thesis
    using NCol-cases  $\langle \neg \text{Col IAC IAB IBD} \rangle$ 
      coplanar-pseudo-trans by blast
qed
ultimately
show ?thesis
  using assms weak-triangle-circumscription-principle-def by blast
qed
}
thus ?thesis
  using bachmann-s-lotschnittaxiom-aux-R2 by blast
qed

lemma universal-posidonius-postulate--perpendicular-transversal-postulate-aux-lem:
fixes A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD
assumes
  IAB  $\neq$  IAC and
  IAB  $\neq$  IBD and
  A1 A2 Perp B1 B2 and
  A1 A2 Perp C1 C2 and
  B1 B2 Perp D1 D2 and
  Col A1 A2 IAB and
  Col B1 B2 IAB and
  Col A1 A2 IAC and
  Col C1 C2 IAC and
  Col B1 B2 IBD and
  Col D1 D2 IBD and
  Coplanar IAB IAC IBD C1 and
  Coplanar IAB IAC IBD C2 and
  Coplanar IAB IAC IBD D1 and
  Coplanar IAB IAC IBD D2 and
  postulate-of-right-saccheri-quadrilaterals
shows  $\exists I. \text{Col } C1 \ C2 \ I \wedge \text{Col } D1 \ D2 \ I$ 
proof –
  have thales-postulate

```

using *assms*(16) *rah--thales-postulate* by *blast*
 hence *thales-converse-postulate*
 using *thales-postulate--thales-converse-postulate* by *blast*
 hence *weak-triangle-circumscription-principle*
 using *thales-converse-postulate--weak-triangle-circumscription-principle*
 by *blast*
 hence *bachmann-s-lotschnittaxiom*
 using *weak-triangle-circumscription-principle--bachmann-s-lotschnittaxiom*
 by *blast*
 thus *?thesis*
 using *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5)
 assms(6) *assms*(7) *assms*(8) *assms*(9)
 assms(10) *assms*(11) *assms*(12)
 assms(13) *assms*(14) *assms*(15)
 bachmann-s-lotschnittaxiom-aux-R1 by *blast*
 qed

lemma *universal-posidonius-postulate--perpendicular-transversal-postulate-aux*:

assumes *universal-posidonius-postulate*

shows $\forall E F G H R P.$

$E G \text{ Perp } R P \wedge \text{Coplanar } F H P R \wedge \text{Col } E G R \wedge \text{Saccheri } E F H G \longrightarrow$
 $F H \text{ Perp } P R$

proof –

{
 fix $E F G H R P$
 assume $E G \text{ Perp } R P$ and $\text{Coplanar } F H P R$ and
 $\text{Col } E G R$ and $\text{Saccheri } E F H G$
 have $\text{Per } F E G$
 using $\langle \text{Saccheri } E F H G \rangle \text{ perp-per-2 sac--perp1214}$ by *blast*
 have $E G \text{ Perp } E F$
 using $\text{Perp-cases } \langle \text{Saccheri } E F H G \rangle \text{ sac--perp1214}$ by *blast*
 have $E G \text{ ParStrict } F H$
 by (*simp add*: $\langle \text{Saccheri } E F H G \rangle \text{ sac--pars1423}$)
 hence $E \neq G$
 using *par-strict-distinct* by *presburger*
 have $E \neq F$
 using $\langle E G \text{ Perp } E F \rangle \text{ perp-distinct}$ by *blast*
 have $P \neq R$
 using $\langle E G \text{ Perp } R P \rangle \text{ perp-distinct}$ by *blast*
 have $E \neq H$
 using $\langle \text{Saccheri } E F H G \rangle \text{ sac-distincts}$ by *blast*
 have $F \neq H$
 using $\langle E G \text{ ParStrict } F H \rangle \text{ par-strict-distinct}$ by *auto*
 have *postulate-of-right-saccheri-quadrilaterals*
 proof –
 obtain $M1$ where $M1 \text{ Midpoint } E G$
 using *midpoint-existence* by *presburger*
 hence $\text{Bet } E M1 G$
 by (*simp add*: *midpoint-bet*)
 obtain $M2$ where $M2 \text{ Midpoint } F H$
 using *midpoint-existence* by *presburger*
 hence $\text{Bet } F M2 H$
 by (*simp add*: *midpoint-bet*)
 have *Lambert* $M1 M2 F E$
 using *mid2-sac--lam6521* $\langle M1 \text{ Midpoint } E G \rangle \langle M2 \text{ Midpoint } F H \rangle$
 $\langle \text{Saccheri } E F H G \rangle$ by *blast*
 have $M1 \neq M2$
 using $\langle \text{Lambert } M1 M2 F E \rangle \text{ Lambert-def}$ by *blast*
 have $F \neq M2$
 using $\langle \text{Lambert } M1 M2 F E \rangle \text{ Lambert-def}$ by *blast*
 have $F \neq E$
 using $\langle \text{Lambert } M1 M2 F E \rangle \text{ Lambert-def}$ by *blast*
 have $M1 \neq E$
 using $\langle \text{Lambert } M1 M2 F E \rangle \text{ Lambert-def}$ by *blast*
 have $\text{Per } M2 M1 E$
 using $\langle \text{Lambert } M1 M2 F E \rangle \text{ Lambert-def}$ by *blast*
 }

```

have Per M1 E F
  using ⟨Lambert M1 M2 F E⟩ Lambert-def by blast
have Per M1 M2 F
  using ⟨Lambert M1 M2 F E⟩ Lambert-def by blast
have Coplanar M1 M2 F E
  using ⟨Lambert M1 M2 F E⟩ Lambert-def by blast
have Saccheri M1 M2 F E
proof –
  have Cong F E M1 M2
  proof –
    have E G Par F H
      using ⟨Saccheri E F H G⟩ sac--par1423 by blast
    moreover
    have Col E G E
      by (simp add: col-trivial-3)
    moreover
    have Col F H F
      by (simp add: col-trivial-3)
    moreover
    have E G Perp E F
      by (simp add: ⟨E G Perp E F⟩)
    moreover
    have Col E G M1
      using ⟨Bet E M1 G⟩ bet-col not-col-permutation-5 by blast
    moreover
    have Col F H M2
      using ⟨Bet F M2 H⟩ bet-col col-permutation-5 by blast
    moreover
    have E G Perp M1 M2
  proof –
    have E M1 Perp M1 M2
      by (metis l8-2 ⟨M1 ≠ E⟩ ⟨M1 ≠ M2⟩ ⟨Per M2 M1 E⟩ per-perp)
    moreover
    have Col E M1 E
      by (simp add: col-trivial-3)
    moreover
    have Col E M1 G
      by (simp add: Col-def ⟨Bet E M1 G⟩)
    ultimately
    show ?thesis
      using ⟨E ≠ G⟩ perp-col2 by blast
  qed
  ultimately
  show ?thesis
    using assms universal-posidonius-postulate-def
      cong-left-commutativity by blast
  qed
hence Cong M1 M2 F E
  using cong-symmetry by blast
moreover
have M1 E ParStrict M2 F
proof –
  have M1 ≠ E
    by (simp add: ⟨M1 ≠ E⟩)
  moreover have M2 ≠ F
    using ⟨F ≠ M2⟩ by blast
  moreover have Col E G M1
    using ⟨Bet E M1 G⟩ bet-col not-col-permutation-5 by blast
  moreover
  have Col E G E
    by (simp add: col-trivial-3)
  moreover
  have Col F H M2
    using ⟨Bet F M2 H⟩ bet-col col-permutation-5 by blast
  moreover
  have Col F H F

```

```

    by (simp add: col-trivial-3)
  moreover have  $E \ G \ \text{ParStrict} \ F \ H$ 
    by (simp add:  $\langle E \ G \ \text{ParStrict} \ F \ H \rangle$ )
  ultimately
  show ?thesis
    using par-strict-col4--par-strict by blast
qed
hence  $M1 \ E \ OS \ M2 \ F$ 
  using l12-6 by auto
ultimately
show ?thesis
  using  $\langle \text{Per} \ M2 \ M1 \ E \rangle \ \langle \text{Per} \ M1 \ E \ F \rangle$ 
  by (simp add: Saccheri-def)
qed
thus ?thesis
  using  $\langle \text{Per} \ M1 \ M2 \ F \rangle$  per-sac--rah
  existential-playfair--rah-1 by blast
qed
have  $F \ H \ \text{Perp} \ P \ R$ 
proof (cases  $E = R$ )
case True
hence  $E \ P \ \text{Perp} \ F \ H$ 
proof -
  have  $Col \ F \ P \ E$ 
  proof -
    have Coplanar  $G \ F \ P \ E$ 
    proof -
      have Coplanar  $H \ R \ F \ G$ 
      using True  $\langle E \ G \ \text{ParStrict} \ F \ H \rangle$  ncoplanar-perm-11
      pars--coplanar by blast
    moreover have Coplanar  $H \ R \ F \ F$ 
      using ncop-distincts by blast
    moreover have Coplanar  $H \ R \ F \ P$ 
      using  $\langle \text{Coplanar} \ F \ H \ P \ R \rangle$  ncoplanar-perm-13 by blast
    moreover have Coplanar  $H \ R \ F \ E$ 
      using True ncop-distincts by blast
    ultimately show ?thesis
      using ParStrict-def True  $\langle E \ G \ \text{ParStrict} \ F \ H \rangle$ 
      col-trivial-1 coplanar-pseudo-trans
      not-col-permutation-2 by blast
    qed
  moreover
  have  $\text{Per} \ P \ E \ G$ 
    using True  $\langle E \ G \ \text{Perp} \ R \ P \rangle$  l8-2 perp-per-2 by blast
  ultimately
  show ?thesis
    using  $\langle E \neq G \rangle \ \langle \text{Per} \ F \ E \ G \rangle$  cop-per2--col by blast
  qed
hence  $Col \ E \ F \ P$ 
  by (simp add: col-permutation-2)
moreover
have  $\text{Per} \ E \ F \ H$ 
  using  $\langle \text{Saccheri} \ E \ F \ H \ G \rangle$ 
   $\langle \text{postulate-of-right-saccheri-quadrilaterals} \rangle$ 
  postulate-of-right-saccheri-quadrilaterals-def by blast
hence  $E \ F \ \text{Perp} \ F \ H$ 
  by (simp add:  $\langle E \neq F \rangle \ \langle F \neq H \rangle$  per-perp)
ultimately
show ?thesis
  using True  $\langle P \neq R \rangle$  col-trivial-3 perp-col2 by metis
qed
thus ?thesis
  using Perp-perm True by blast
next
case False
have  $\neg \text{Col} \ F \ H \ R$ 

```



```

  by (meson ⟨Col E G R⟩ ⟨E G ParStrict F H⟩
      not-col-permutation-1 par-not-col)
have ∃ I. Col P R I ∧ Col F H I
proof -
  have E ≠ F
    by (simp add: ⟨E ≠ F⟩)
  moreover
  have E G Perp E F
    by (simp add: ⟨E G Perp E F⟩)
  moreover
  have E G Perp P R
    using Perp-cases ⟨E G Perp R P⟩ by blast
  moreover
  have Per E F H
    using ⟨Saccheri E F H G⟩
      ⟨postulate-of-right-saccheri-quadrilaterals⟩
      postulate-of-right-saccheri-quadrilaterals-def by blast
  hence E F Perp F H
    by (simp add: ⟨E ≠ F⟩ ⟨F ≠ H⟩ per-perp)
  moreover
  have Col E G E
    by (simp add: col-trivial-3)
  moreover
  have Col E F E
    by (simp add: col-trivial-3)
  moreover
  have Col E G R
    by (simp add: ⟨Col E G R⟩)
  moreover
  have Col P R R
    by (simp add: col-trivial-2)
  moreover
  have Col E F F
    using col-trivial-2 by auto
  moreover
  have Col F H F
    by (simp add: col-trivial-3)
  moreover
  have Coplanar E R F P
  proof -
    have Coplanar F H R P
      using ⟨Coplanar F H P R⟩ coplanar-perm-1 by blast
    moreover
    have Coplanar F H R E
      by (metis False ⟨Col E G E⟩ ⟨Col E G R⟩
          ⟨E G ParStrict F H⟩ par-strict-col2-par-strict
          par-strict-symmetry pars--coplanar)
    moreover
    have Coplanar F H R F
      using ncop-distincts by blast
    moreover
    have Coplanar F H R R
      using ncop-distincts by blast
    ultimately
    show ?thesis
      by (meson ⟨¬ Col F H R⟩ coplanar-pseudo-trans)
  qed
  moreover
  have Coplanar E R F R
    using ncop-distincts by blast
  moreover
  have Coplanar E R F F
    using ncop-distincts by blast
  moreover
  have Coplanar F H E R
    by (meson False ⟨Col E G R⟩ ⟨E G ParStrict F H⟩

```

$col\text{-}trivial\text{-}3$ $par\text{-}strict\text{-}col2\text{-}par\text{-}strict$
 $par\text{-}strict\text{-}symmetry$ $pars\text{-}coplanar$)
hence $Coplanar\ E\ R\ F\ H$
by ($simp\ add$: $coplanar\text{-}perm\text{-}16$)
ultimately
show $?thesis$
using $False$ $\langle postulate\text{-}of\text{-}right\text{-}saccheri\text{-}quadrilaterals \rangle$
 $universal\text{-}posidonius\text{-}postulate\text{-}perpendicular\text{-}transversal\text{-}postulate\text{-}aux\text{-}lem$
by $blast$
qed
then obtain S **where** $Col\ P\ R\ S$ **and** $Col\ F\ H\ S$
by $blast$
have $S \neq R$
using $\langle Col\ F\ H\ S \rangle \langle \neg\ Col\ F\ H\ R \rangle$ **by** $auto$
have $Saccheri\ E\ F\ S\ R$
proof –
have $Per\ F\ E\ R$
proof –
have $E \neq G$
by ($simp\ add$: $\langle E \neq G \rangle$)
moreover
have $Per\ F\ E\ G$
using $\langle Saccheri\ E\ F\ H\ G \rangle$ $perp\text{-}per\text{-}2$ $sac\text{-}perp1214$ **by** $blast$
ultimately
show $?thesis$
using $\langle Col\ E\ G\ R \rangle$ $per\text{-}col$ **by** $blast$
qed
moreover
have $R\ E\ Perp\ S\ R$
proof –
have $Col\ E\ G\ E$
by ($simp\ add$: $col\text{-}trivial\text{-}3$)
moreover
have $Col\ R\ P\ S$
using $Col\text{-}cases\ \langle Col\ P\ R\ S \rangle$ **by** $blast$
moreover
have $Col\ R\ P\ R$
by ($simp\ add$: $col\text{-}trivial\text{-}3$)
ultimately
show $?thesis$
using $False\ \langle S \neq R \rangle \langle E\ G\ Perp\ R\ P \rangle$
 $\langle Col\ E\ G\ R \rangle$ $perp\text{-}col4$ **by** $auto$
qed
hence $Per\ E\ R\ S$
using $perp\text{-}per\text{-}1$ **by** $force$
moreover
have $Cong\ E\ F\ R\ S$
proof –
have $E\ G\ Par\ F\ H$
using $\langle E\ G\ ParStrict\ F\ H \rangle$ $par\text{-}strict\text{-}par$ **by** $auto$
moreover
have $Col\ E\ G\ E$
by ($simp\ add$: $col\text{-}trivial\text{-}3$)
moreover
have $Col\ F\ H\ F$
by ($simp\ add$: $col\text{-}trivial\text{-}3$)
moreover
have $E\ G\ Perp\ R\ S$
proof –
have $R\ P\ Perp\ E\ G$
using $Perp\text{-}perm\ \langle E\ G\ Perp\ R\ P \rangle$ **by** $blast$
moreover
have $Col\ R\ P\ S$
using $\langle Col\ P\ R\ S \rangle$ $not\text{-}col\text{-}permutation\text{-}4$ **by** $blast$
ultimately
show $?thesis$

```

    using ⟨E G Perp R P⟩ ⟨S ≠ R⟩ perp-col1 by auto
qed
ultimately
show ?thesis
    using ⟨E G Perp E F⟩ ⟨Col E G R⟩ ⟨Col F H S⟩
    assms universal-posidonius-postulate-def by blast
qed
hence Cong E F S R
    using not-cong-1243 by blast
moreover
have E R OS F S
proof -
{
    assume F = S
    hence ¬ Col E G F
    by (meson ⟨Col F H S⟩ ⟨E G ParStrict F H⟩
        not-col-permutation-1 par-not-col)
    have Col E P R
    proof -
        have F E Perp E G
        using Perp-cases ⟨E G Perp E F⟩ by blast
    moreover
    have P R Perp E G
    using Perp-perm ⟨E G Perp R P⟩ by blast
    moreover
    have Col F P R
    by (simp add: ⟨Col P R S⟩ ⟨F = S⟩ col-permutation-2)
    moreover
    have Coplanar E G E P
    using ncop-distincts by blast
    moreover
    have Coplanar E G E R
    using ncop-distincts by blast
    ultimately
    show ?thesis
    using col-cop2-perp2--col by blast
    qed
    hence False
    by (metis False ⟨Col E G R⟩ ⟨E G Perp R P⟩
        col-trivial-3 l6-21 not-col-permutation-2
        perp-not-col2)
}
moreover
have E R ParStrict F H
    by (meson False ⟨Col E G R⟩ ⟨E G ParStrict F H⟩
        ⟨F ≠ H⟩ col-trivial-2 col-trivial-3
        par-strict-col4--par-strict)
ultimately
show ?thesis
    using ⟨Col F H S⟩ par-strict-one-side by blast
qed
ultimately
show ?thesis
    using Saccheri-def by blast
qed
have F ≠ S
    using ⟨Saccheri E F S R⟩ sac-distincts by blast
have F S Perp P R
proof -
    have R ≠ P
    using ⟨P ≠ R⟩ by auto
    moreover
    have Saccheri R S F E
    by (simp add: ⟨Saccheri E F S R⟩ sac-perm)
    hence Per R S F
    using ⟨postulate-of-right-saccheri-quadrilaterals⟩

```

```

    postulate-of-right-saccheri-quadrilaterals-def by blast
  hence  $R \ S \ \text{Perp} \ S \ F$ 
    using  $\langle F \neq S \rangle \ \langle S \neq R \rangle \ \text{per-perp}$  by auto
  moreover
  have  $\text{Col} \ R \ S \ P$ 
    using  $\langle \text{Col} \ P \ R \ S \rangle \ \text{not-col-permutation-2}$  by blast
  ultimately
  show ?thesis
    using  $\text{Perp-perm} \ \text{perp-col}$  by blast
qed
moreover
have  $\text{Col} \ F \ S \ H$ 
  using  $\text{Col-cases} \ \langle \text{Col} \ F \ H \ S \rangle$  by blast
ultimately
show ?thesis
  using  $\langle F \neq H \rangle \ \text{perp-col}$  by blast
qed
}
thus ?thesis
  by blast
qed

lemma universal-posidonius-postulate--perpendicular-transversal-postulate:
  assumes universal-posidonius-postulate
  shows perpendicular-transversal-postulate
proof -
{
  fix  $A \ B \ C \ D \ P \ Q$ 
  assume  $A \ B \ \text{Par} \ C \ D$  and
     $A \ B \ \text{Perp} \ P \ Q$  and
     $\text{Coplanar} \ C \ D \ P \ Q$ 
{
  fix  $P \ Q$ 
  assume  $A \ B \ \text{Perp} \ P \ Q$  and
     $\text{Coplanar} \ C \ D \ P \ Q$  and
     $\neg \text{Col} \ A \ B \ P$ 
  have  $A \ B \ \text{Par} \ C \ D$ 
    using  $\langle A \ B \ \text{Par} \ C \ D \rangle$  by auto
  hence  $A \ B \ \text{ParStrict} \ C \ D \vee (A \neq B \wedge C \neq D \wedge \text{Col} \ A \ C \ D \wedge \text{Col} \ B \ C \ D)$ 
    by (simp add: Par-def)
  moreover
  {
    assume  $A \ B \ \text{ParStrict} \ C \ D$ 
    obtain  $R$  where  $\text{Col} \ A \ B \ R$  and  $\text{Col} \ P \ Q \ R$ 
      using  $\langle A \ B \ \text{Perp} \ P \ Q \rangle \ \text{perp-inter-exists}$  by blast
    have  $\neg \text{Col} \ A \ B \ C$ 
      using  $\langle A \ B \ \text{ParStrict} \ C \ D \rangle \ \text{l12-6 one-side-not-col123}$  by blast
    then obtain  $E$  where  $\text{Col} \ A \ B \ E$  and  $A \ B \ \text{Perp} \ C \ E$ 
      using l8-18-existence by blast
    have  $\neg \text{Col} \ A \ B \ D$ 
      using  $\langle A \ B \ \text{ParStrict} \ C \ D \rangle \ \text{par-strict-not-col-4}$  by auto
    then obtain  $G$  where  $\text{Col} \ A \ B \ G$  and  $A \ B \ \text{Perp} \ D \ G$ 
      using l8-18-existence by blast
    {
      assume  $E = G$ 
      hence  $\text{Col} \ C \ D \ E$ 
      proof -
        have  $E \ C \ \text{Perp} \ A \ B$ 
          using  $\text{Perp-perm} \ \langle A \ B \ \text{Perp} \ C \ E \rangle$  by blast
        moreover
        have  $D \ E \ \text{Perp} \ A \ B$ 
          using  $\text{Perp-cases} \ \langle A \ B \ \text{Perp} \ D \ G \rangle \ \langle E = G \rangle$  by blast
        moreover
        have  $\text{Col} \ E \ D \ E$ 
          by (simp add: col-trivial-3)
        moreover

```

```

have Coplanar A B C D
  by (simp add: ⟨A B ParStrict C D⟩ pars--coplanar)
moreover
have Coplanar A B C E
  using ⟨Col A B E⟩ ncop--ncols by blast
ultimately
show ?thesis
  using col-cop2-perp2--col by blast
qed
hence Col E A B ∧ Col E C D
  using Col-cases ⟨Col A B E⟩ by blast
hence False
  using ⟨A B Par C D⟩ ⟨Col A B G⟩ ⟨Col C D E⟩ ⟨E = G⟩
    ⟨¬ Col A B C⟩ not-strict-par by blast
}
have Saccheri E C D G
proof -
  have A B Perp E C
    by (simp add: ⟨A B Perp C E⟩ perp-right-comm)
  hence Per C E G
    using ⟨Col A B E⟩ ⟨Col A B G⟩ ⟨A B Perp C E⟩
      l8-16-1 by blast
  moreover
  have Per E G D
    using ⟨A B Perp D G⟩ ⟨Col A B G⟩ ⟨Col A B E⟩
      Per-perm l8-16-1 by blast
  moreover
  have Cong E C D G
proof -
  have Col C D C
    by (simp add: col-trivial-3)
  moreover
  have Col C D D
    by (simp add: col-trivial-2)
  moreover
  have A B Perp G D
    by (simp add: ⟨A B Perp D G⟩ perp-right-comm)
  ultimately
  show ?thesis
    using assms ⟨A B Par C D⟩ ⟨Col A B E⟩ ⟨A B Perp E C⟩
      ⟨Col A B G⟩ not-cong-1243
      universal-posidonius-postulate-def by blast
qed
moreover
have C D ParStrict A B
  using ⟨A B ParStrict C D⟩ par-strict-symmetry by blast
hence C D ParStrict E G
  using ⟨Col A B E⟩ ⟨Col A B G⟩ ⟨E = G ⟹ False⟩
    par-strict-col2-par-strict by blast
hence E G OS C D
  by (simp add: pars--os3412)
ultimately
show ?thesis
  using Saccheri-def by blast
qed
have P ≠ Q
  using ⟨A B Perp P Q⟩ perp-distinct by auto
have P ≠ R
  using ⟨Col A B R⟩ ⟨¬ Col A B P⟩ by auto
have A ≠ B
  using ⟨¬ Col A B C⟩ col-trivial-1 by auto
have C D Perp P R
proof -
  have A B Perp R P
    by (meson ⟨A B Perp P Q⟩ ⟨Col P Q R⟩ ⟨P ≠ R⟩
      perp-col1 perp-right-comm)

```

```

hence  $E \ G \ \text{Perp} \ R \ P$ 
  using  $\langle \text{Col } A \ B \ E \rangle \ \langle \text{Col } A \ B \ G \rangle \ \langle E = G \implies \text{False} \rangle$ 
    perp-col2 by blast
moreover
have  $\text{Coplanar } C \ D \ P \ R$ 
  using  $\langle \text{Coplanar } C \ D \ P \ Q \rangle \ \langle \text{Col } P \ Q \ R \rangle \ \langle P \neq Q \rangle$ 
    col-cop--cop by blast
moreover
have  $\text{Col } E \ G \ R$ 
  using  $\langle \text{Col } A \ B \ R \rangle \ \langle \text{Col } A \ B \ E \rangle \ \langle \text{Col } A \ B \ G \rangle$ 
     $\langle A \neq B \rangle$  col3 by blast
ultimately
show ?thesis
  using assms
    universal-posidonius-postulate--perpendicular-transversal-postulate-aur
     $\langle \text{Saccheri } E \ C \ D \ G \rangle$  by blast
qed
hence  $P \ R \ \text{Perp} \ C \ D$ 
  using Perp-perm by blast
hence  $C \ D \ \text{Perp} \ P \ Q$ 
  using NCol-perm Perp-cases  $\langle \text{Col } P \ Q \ R \rangle \ \langle P \neq Q \rangle$ 
    perp-col1 by blast
}
moreover
have  $(A \neq B \wedge C \neq D \wedge \text{Col } A \ C \ D \wedge \text{Col } B \ C \ D) \longrightarrow C \ D \ \text{Perp} \ P \ Q$ 
  by (meson  $\langle A \ B \ \text{Par} \ C \ D \rangle \ \langle A \ B \ \text{Perp} \ P \ Q \rangle$  not-col-distincts
    not-col-permutation-2 not-strict-par perp-col2)
ultimately
have  $C \ D \ \text{Perp} \ P \ Q$ 
  by blast
}
moreover
have  $\neg \text{Col } A \ B \ P \longrightarrow C \ D \ \text{Perp} \ P \ Q$ 
  by (simp add:  $\langle A \ B \ \text{Perp} \ P \ Q \rangle \ \langle \text{Coplanar } C \ D \ P \ Q \rangle$  calculation)
moreover
have  $\neg \text{Col } A \ B \ Q \longrightarrow C \ D \ \text{Perp} \ P \ Q$ 
  using  $\langle A \ B \ \text{Perp} \ P \ Q \rangle \ \langle \text{Coplanar } C \ D \ P \ Q \rangle$  calculation(1)
    ncoplanar-perm-1 perp-right-comm by blast
ultimately
have  $C \ D \ \text{Perp} \ P \ Q$ 
  using  $\langle A \ B \ \text{Perp} \ P \ Q \rangle \ \langle \text{Coplanar } C \ D \ P \ Q \rangle$  perp-not-col2 by blast
}
thus ?thesis
  using perpendicular-transversal-postulate-def by blast
qed

```

lemma *playfair--alternate-interior:*
assumes *playfair-s-postulate*
shows *alternate-interior-angles-postulate*

```

proof –
{
  fix  $A \ B \ C \ D$ 
  assume  $A \ C \ \text{TS} \ B \ D$  and
     $A \ B \ \text{Par} \ C \ D$ 
  have  $\neg \text{Col } A \ B \ C$ 
    using TS-def  $\langle A \ C \ \text{TS} \ B \ D \rangle$  not-col-permutation-4 by blast
  hence  $\neg \text{Col } B \ A \ C \wedge \neg \text{Col } A \ C \ B$ 
    using Col-cases by blast
  then obtain  $D'$  where  $B \ A \ C \ \text{Cong} \ A \ C \ D'$  and  $A \ C \ \text{TS} \ D' \ B$ 
    using ex-conga-ts by blast
  have  $B \ A \ C \ \text{Cong} \ A \ D' \ C \ A$ 
    by (simp add:  $\langle B \ A \ C \ \text{Cong} \ A \ C \ D' \rangle$  conga-right-comm)
  moreover
  have  $D' \ C \ A \ \text{Cong} \ A \ D \ C \ A$ 
  proof –
    have  $C \ \text{Out} \ D \ D'$ 

```

```

proof -
  have  $Col\ C\ C\ D \wedge Col\ D'\ C\ D$ 
proof -
  have  $A\ B\ Par\ C\ D$ 
  using  $\langle A\ B\ Par\ C\ D \rangle$  by auto
  moreover
  have  $Col\ C\ C\ D$ 
  by (simp add: col-trivial-1)
  moreover
  have  $A\ B\ Par\ C\ D'$ 
proof -
  have  $A\ C\ TS\ B\ D'$ 
  by (meson  $\langle A\ C\ TS\ D'\ B \rangle$  l9-2)
  moreover
  have  $B\ A\ C\ CongA\ D'\ C\ A$ 
  by (simp add:  $\langle B\ A\ C\ CongA\ D'\ C\ A \rangle$ )
  ultimately
  show ?thesis
  using l12-21-b by blast
qed
moreover
have  $Col\ C\ C\ D'$ 
using col-trivial-1 by blast
ultimately
show ?thesis
using assms playfair-s-postulate-def by blast
qed
hence  $Col\ C\ D\ D'$ 
using not-col-permutation-1 by blast
moreover
have  $A\ C\ OS\ D\ D'$ 
using  $\langle A\ C\ TS\ B\ D \rangle\ \langle A\ C\ TS\ D'\ B \rangle$  l9-2 l9-8-1 by blast
hence  $C\ A\ OS\ D\ D'$ 
by (simp add: invert-one-side)
ultimately
show ?thesis
using col-one-side-out by blast
qed
moreover
have  $C\ Out\ A\ A$ 
using  $\langle \neg\ Col\ B\ A\ C \wedge \neg\ Col\ A\ C\ B \rangle$ 
not-col-distincts out-trivial by blast
ultimately
show ?thesis
by (simp add: out2--conga)
qed
ultimately
have  $B\ A\ C\ CongA\ D\ C\ A$ 
by (meson conga-trans)
}
thus ?thesis
using alternate-interior-angles-postulate-def by presburger
qed

lemma playfair-bis--playfair:
  assumes alternative-playfair-s-postulate
  shows playfair-s-postulate
proof -
  {
    fix  $A1\ A2\ B1\ B2\ C1\ C2\ P$ 
    assume  $A1\ A2\ Par\ B1\ B2$  and
       $Col\ P\ B1\ B2$  and
       $A1\ A2\ Par\ C1\ C2$  and
       $Col\ P\ C1\ C2$ 
    have  $Col\ C1\ B1\ B2 \wedge Col\ C2\ B1\ B2$ 
    proof (cases  $Col\ A1\ A2\ P$ )

```

```

case True
{
  assume H1: (A1 ≠ A2 ∧ B1 ≠ B2 ∧ Col A1 B1 B2 ∧ Col A2 B1 B2) →
(Col C1 B1 B2 ∧ Col C2 B1 B2)
  have ?thesis
    by (metis Par-def True ⟨A1 A2 Par B1 B2⟩ H1
      ⟨Col P B1 B2⟩ col-permutation-2 par-not-col)
}
moreover
have A1 A2 ParStrict C1 C2 → ?thesis
  by (metis True ⟨A1 A2 Par C1 C2⟩ ⟨Col P C1 C2⟩
    inter-uniqueness-not-par not-col-permutation-2
    par-strict-not-col-1)
ultimately
show ?thesis using ⟨A1 A2 Par C1 C2⟩
  by (metis True ⟨Col P C1 C2⟩ col-permutation-2 colx not-strict-par)
next
case False
have A1 ≠ A2
  using False not-col-distincts by presburger
then obtain X where P X Perp A1 A2
  using perp-exists by blast
{
  assume H2: ∀ A1 A2. (A1 A2 Par B1 B2 ∧
    A1 A2 Par C1 C2 ∧ ¬ Col A1 A2 P ∧
    P X Perp A1 A2 ∧ ¬ Col P X A1)
    →
    (Col C1 B1 B2 ∧ Col C2 B1 B2)
  fix A1 A2
  assume A1 A2 Par B1 B2 and
    A1 A2 Par C1 C2 and
    ¬ Col A1 A2 P and
    P X Perp A1 A2
  {
    assume Col P X A1
    have Col C1 B1 B2 ∧ Col C2 B1 B2
      using ⟨A1 A2 Par B1 B2⟩ ⟨A1 A2 Par C1 C2⟩
      ⟨¬ Col A1 A2 P⟩ ⟨P X Perp A1 A2⟩
    by (metis Col-perm Perp-cases H2
      l8-16-1 par-left-comm)
  }
  moreover
  {
    assume Col P X A2
    have Col C1 B1 B2 ∧ Col C2 B1 B2
      using ⟨A1 A2 Par B1 B2⟩ ⟨A1 A2 Par C1 C2⟩
      ⟨Col P X A1 ⇒ Col C1 B1 B2 ∧ Col C2 B1 B2⟩
      ⟨P X Perp A1 A2⟩ H2 ⟨¬ Col A1 A2 P⟩ by blast
  }
  ultimately
  have Col C1 B1 B2 ∧ Col C2 B1 B2
    using ⟨A1 A2 Par B1 B2⟩ ⟨A1 A2 Par C1 C2⟩ ⟨P X Perp A1 A2⟩
    H2 ⟨¬ Col A1 A2 P⟩ by blast
}
{
  fix A1' A2'
  assume A1' A2' Par B1 B2 and
    A1' A2' Par C1 C2 and
    ¬ Col A1' A2' P and
    P X Perp A1' A2' and
    ¬ Col P X A1'
  have Coplanar P X A1' A2'
    by (simp add: ⟨P X Perp A1' A2'⟩ perp--coplanar)
  have P ≠ X
    using ⟨¬ Col P X A1'⟩ col-trivial-1 by force
  then obtain D where P X Perp D P and Coplanar P X A1' D

```



```

    using ex-perp-cop by blast
  hence  $D \neq P$ 
    using perp-not-eq-2 by blast
  have  $P \text{ Perp2 } A1' A2' D P$ 
  proof -
    have  $\text{Col } P X P$ 
      by (simp add: col-trivial-3)
    moreover
    have  $X P \text{ Perp } A1' A2'$ 
      using Perp-cases  $\langle P X \text{ Perp } A1' A2' \rangle$  by blast
    moreover
    have  $X P \text{ Perp } P D$ 
      using Perp-cases  $\langle P X \text{ Perp } D P \rangle$  by blast
    ultimately
    show ?thesis
      using Perp2-def perp-right-comm by blast
  qed
  have  $\text{Col } B1 P D \wedge \text{Col } B2 P D$ 
  proof -
    have  $\text{Col } P P D$ 
      using col-trivial-1 by auto
    moreover
    have  $\text{Coplanar } A1' A2' P D$ 
      by (meson  $\langle \text{Coplanar } P X A1' A2' \rangle \langle \text{Coplanar } P X A1' D \rangle$ 
         $\langle \neg \text{Col } P X A1' \rangle$  coplanar-perm-16 l9-30 ncop-distincts)
    ultimately
    show ?thesis
      using  $\langle P \text{ Perp2 } A1' A2' D P \rangle \langle \neg \text{Col } A1' A2' P \rangle$ 
         $\langle A1' A2' \text{ Par } B1 B2 \rangle \langle \text{Col } P B1 B2 \rangle$ 
        alternative-playfair-s-postulate-def assms
        coplanar-perm-1 not-col-permutation-5 by blast
  qed
  moreover
  have  $\text{Col } C1 P D \wedge \text{Col } C2 P D$ 
  proof -
    have  $\text{Col } P P D$ 
      using col-trivial-1 by auto
    moreover
    have  $\text{Coplanar } A1' A2' P D$ 
      by (meson  $\langle \text{Coplanar } P X A1' A2' \rangle \langle \text{Coplanar } P X A1' D \rangle$ 
         $\langle \neg \text{Col } P X A1' \rangle$  coplanar-perm-16 l9-30 ncop-distincts)
    ultimately
    show ?thesis
      using  $\langle P \text{ Perp2 } A1' A2' D P \rangle \langle \neg \text{Col } A1' A2' P \rangle$ 
         $\langle A1' A2' \text{ Par } C1 C2 \rangle \langle \text{Col } P C1 C2 \rangle$ 
        alternative-playfair-s-postulate-def assms
        coplanar-perm-1 not-col-permutation-5 by blast
  qed
  ultimately
  have  $\text{Col } P D C1 \wedge \text{Col } P D C2 \wedge \text{Col } P D B1 \wedge \text{Col } P D B2$ 
    using Col-cases by blast
  hence  $\text{Col } C1 B1 B2 \wedge \text{Col } C2 B1 B2$ 
    by (metis  $\langle D \neq P \rangle$  col3)
}
thus ?thesis
  using False  $\langle A1 A2 \text{ Par } B1 B2 \rangle \langle A1 A2 \text{ Par } C1 C2 \rangle$ 
     $\langle P X \text{ Perp } A1 A2 \rangle$ 
     $\langle \bigwedge A2a A1a. [\forall A1 A2. A1 A2 \text{ Par } B1 B2 \wedge A1 A2 \text{ Par } C1 C2 \wedge$ 
       $\neg \text{Col } A1 A2 P \wedge P X \text{ Perp } A1 A2 \wedge \neg \text{Col } P X A1$ 
       $\longrightarrow \text{Col } C1 B1 B2 \wedge \text{Col } C2 B1 B2; A1a A2a \text{ Par } B1 B2;$ 
       $A1a A2a \text{ Par } C1 C2; \neg \text{Col } A1a A2a P; P X \text{ Perp } A1a A2a]$ 
     $\implies \text{Col } C1 B1 B2 \wedge \text{Col } C2 B1 B2 \rangle$  by blast
  qed
}
thus ?thesis
  using playfair-s-postulate-def by blast

```

qed

lemma *playfair-s-postulate-implies-midpoint-converse-postulate*:

assumes *playfair-s-postulate*

shows *midpoint-converse-postulate*

proof –

```

{
  fix A B C P Q
  assume  $\neg$  Col A B C and
    P Midpoint B C and
    A B Par Q P and
    Col A C Q
  obtain X where X Midpoint A C
    using midpoint-existence by blast
  hence Bet A X C
    by (simp add: midpoint-bet)
  have Cong A X X C
    using  $\langle X \text{ Midpoint } A \ C \rangle$  midpoint-cong by blast
  have A B ParStrict X P
    using triangle-mid-par  $\langle P \text{ Midpoint } B \ C \rangle$   $\langle X \text{ Midpoint } A \ C \rangle$ 
       $\langle \neg \text{ Col } A \ B \ C \rangle$  by blast
  have X = Q
  proof –
    have Col A C X
      by (meson  $\langle \text{Bet } A \ X \ C \rangle$  bet-col1 between-trivial)
    moreover
    have Col P Q X
      using  $\langle A \ B \ \text{Par } Q \ P \rangle$   $\langle A \ B \ \text{ParStrict } X \ P \rangle$  assms
        col-trivial-3 not-col-permutation-3 par-strict-par
        playfair-s-postulate-def by blast
    moreover
    have Col P Q Q
      by (simp add: col-trivial-2)
    ultimately
    show ?thesis
      using l6-21
      by (metis  $\langle A \ B \ \text{Par } Q \ P \rangle$   $\langle \text{Col } A \ C \ Q \rangle$   $\langle \neg \text{ Col } A \ B \ C \rangle$ 
        not-col-distincts par-col2-par-bis par-id)
  qed
  hence Q Midpoint A C
    using  $\langle X \text{ Midpoint } A \ C \rangle$  by auto
}
thus ?thesis
  using midpoint-converse-postulate-def by blast
qed

```

lemma *inter-dec-plus-par-perp-perp-imp-ly-triangle-circumscription*:

assumes

perpendicular-transversal-postulate

shows *triangle-circumscription-principle*

proof –

```

{
  fix A B C
  assume  $\neg$  Col A B C
  hence A  $\neq$  B
    using col-trivial-1 by blast
  then obtain C1 C2 where C1 C2 PerpBisect A B and
    Coplanar A B C C1 and Coplanar A B C C2
    using perp-bisect-existence-cop by blast
  have A  $\neq$  C
    using  $\langle \neg \text{ Col } A \ B \ C \rangle$  col-trivial-3 by blast
  then obtain B1 B2 where B1 B2 PerpBisect A C and
    Coplanar A C B B1 and Coplanar A C B B2
    using perp-bisect-existence-cop by blast
  {
    assume  $\exists I. \text{ Col } I \ B1 \ B2 \wedge \text{ Col } I \ C1 \ C2$ 

```

```

then obtain CC where Col CC B1 B2 and Col CC C1 C2
  by auto
have Cong A CC B CC
proof (cases CC = C1)
  case True
  thus ?thesis
    using ⟨C1 C2 PerpBisect A B⟩ perp-bisect-cong2 by blast
next
  case False
  hence CC ≠ C1
  by auto
  have C1 C2 PerpBisectBis A B
    using ⟨C1 C2 PerpBisect A B⟩ perp-bisect-equiv-defA by blast
  then obtain I where I PerpAt C1 C2 A B and I Midpoint A B
    using Perp-bisect-bis-def by blast
  have I PerpAt C1 CC A B
    by (metis False perp-in-sym ⟨Col CC C1 C2⟩
      ⟨I PerpAt C1 C2 A B⟩ not-col-permutation-2
      perp-in-col-perp-in)
  hence C1 CC PerpBisectBis A B
    using Perp-bisect-bis-def ⟨I Midpoint A B⟩ by blast
  hence C1 CC PerpBisect A B
    by (simp add: perp-bisect-equiv-defB)
  hence CC C1 PerpBisect A B
    by (simp add: perp-bisect-sym-1)
  thus ?thesis
    by (simp add: perp-bisect-cong-1)
qed
moreover
have Cong A CC C CC
proof (cases CC = B1)
  case True
  thus ?thesis
    using ⟨B1 B2 PerpBisect A C⟩ perp-bisect-cong-1 by auto
next
  case False
  hence CC ≠ B1
  by auto
  have B1 B2 PerpBisectBis A C
    using ⟨B1 B2 PerpBisect A C⟩ perp-bisect-equiv-defA by blast
  then obtain I where I PerpAt B1 B2 A C and I Midpoint A C
    using Perp-bisect-bis-def by blast
  have I PerpAt B1 CC A C
    by (metis False perp-in-sym ⟨Col CC B1 B2⟩
      ⟨I PerpAt B1 B2 A C⟩ not-col-permutation-2
      perp-in-col-perp-in)
  hence B1 CC PerpBisectBis A C
    using Perp-bisect-bis-def ⟨I Midpoint A C⟩ by blast
  hence B1 CC PerpBisect A C
    by (simp add: perp-bisect-equiv-defB)
  hence CC B1 PerpBisect A C
    by (simp add: perp-bisect-sym-1)
  thus ?thesis
    by (simp add: perp-bisect-cong-1)
qed
moreover
have C1 C2 Perp A B ∨ B = A
  using ⟨C1 C2 PerpBisect A B⟩ perp-bisect-perp by blast
have Coplanar A B C CC
proof (cases A = B)
  case True
  thus ?thesis
    using ⟨A ≠ B⟩ by auto
next
  case False
  hence C1 C2 Perp A B

```

```

    using ⟨C1 C2 Perp A B ∨ B = A⟩ by auto
  hence Coplanar C1 C2 A B
    using perp--coplanar by simp
  have C1 ≠ C2
    using ⟨C1 C2 PerpBisect A B⟩ col-trivial-1
      perp-bisect-perp perp-not-col2 by blast
  moreover
  have Coplanar A B C C1
    using ⟨Coplanar A B C C1⟩ by auto
  moreover
  have Coplanar A B C C2
    by (simp add: ⟨Coplanar A B C C2⟩)
  ultimately
  show ?thesis
    by (meson Col-cases ⟨Col CC C1 C2⟩ col-cop2--cop)
qed
ultimately
have ∃ CC. Cong A CC B CC ∧ Cong A CC C CC ∧ Coplanar A B C CC
  by auto
}
moreover
{
  assume ¬ (∃ I. Col I B1 B2 ∧ Col I C1 C2)
  have Coplanar B1 B2 C1 C2
  proof -
    have Coplanar A B C B1
      using ⟨Coplanar A C B B1⟩ ncoplanar-perm-2 by blast
    moreover
    have Coplanar A B C B2
      using ⟨Coplanar A C B B2⟩ coplanar-perm-2 by blast
    moreover
    have Coplanar A B C C1
      using ⟨Coplanar A B C C1⟩ by auto
    moreover
    have Coplanar A B C C2
      by (simp add: ⟨Coplanar A B C C2⟩)
    ultimately
    show ?thesis
      by (meson ⟨¬ Col A B C⟩ coplanar-pseudo-trans)
  qed
  hence B1 B2 ParStrict C1 C2
    using ⟨⊢ I. Col I B1 B2 ∧ Col I C1 C2⟩
      by (simp add: ParStrict-def)
  hence B1 B2 Par C1 C2
    by (simp add: Par-def)
  have C1 C2 Perp A B
    using ⟨C1 C2 PerpBisect A B⟩ perp-bisect-perp by auto
  have B1 B2 Perp A C
    using ⟨B1 B2 PerpBisect A C⟩ perp-bisect-perp by auto
  hence Coplanar C1 C2 A C ⟶ C1 C2 Perp A C
    using ⟨B1 B2 Par C1 C2⟩ assms
      perpendicular-transversal-postulate-def by simp
  have A B Par A C
  proof -
    have Coplanar C1 C2 A A
      using ncop-distincts by blast
    moreover
    have Coplanar C1 C2 A C
      by (meson ⟨Coplanar A B C C1⟩ ⟨Coplanar A B C C2⟩
        ⟨¬ Col A B C⟩ coplanar-pseudo-trans
          ncop-distincts)
    moreover
    have Coplanar C1 C2 A B
      using ⟨C1 C2 PerpBisect A B⟩ perp--coplanar
        perp-bisect-perp by blast
    hence Coplanar C1 C2 B A

```

```

    by (simp add: coplanar-perm-1)
  moreover
  have Coplanar C1 C2 B C
    by (meson ⟨Coplanar A B C C1⟩ ⟨Coplanar A B C C2⟩
      ⟨¬ Col A B C⟩ coplanar-pseudo-trans ncop-distincts)
  moreover
  have A B Perp C1 C2
    using Perp-cases ⟨C1 C2 Perp A B⟩ by blast
  moreover
  have A C Perp C1 C2
    using Perp-perm ⟨Coplanar C1 C2 A C ⟶ C1 C2 Perp A C⟩
      calculation(2) by blast
  ultimately
  show ?thesis
    by (simp add: l12-9)
qed
hence Col A B C
  by (simp add: par-id)
hence False
  using ⟨¬ Col A B C⟩ by blast
hence ∃ CC. Cong A CC B CC ∧ Cong A CC C CC ∧ Coplanar A B C CC
  by simp
}
ultimately
have ∃ CC. Cong A CC B CC ∧ Cong A CC C CC ∧ Coplanar A B C CC
  by auto
}
thus ?thesis
  using triangle-circumscription-principle-def by blast
qed

```

```

lemma original-euclid--original-spp:
  assumes euclid-s-parallel-postulate
  shows alternative-strong-parallel-postulate
proof -
{
  fix A B C D P Q R
  assume B C OS A D and
    A B C B C D SumA P Q R and
    ¬ Bet P Q R
  obtain A' where B Midpoint A A'
    using symmetric-point-construction by blast
  hence Bet A B A'
    by (simp add: midpoint-bet)
  hence Col A B A'
    by (simp add: bet-col)
  obtain D' where C Midpoint D D'
    using symmetric-point-construction by blast
  hence Bet D C D'
    by (simp add: midpoint-bet)
  hence Col D C D'
    by (simp add: bet-col)
  have A ≠ B
    using ⟨B C OS A D⟩ os-distincts by blast
  have C ≠ B
    using ⟨A B C B C D SumA P Q R⟩ suma-distincts by blast
  have C ≠ D
    using ⟨B C OS A D⟩ os-distincts by blast
  have D ≠ D'
    using ⟨Bet D C D'⟩ ⟨C ≠ D⟩ bet-neq12--neq by blast
  hence C ≠ D'
    using ⟨C Midpoint D D'⟩ midpoint-distinct-1 by blast
  have A' ≠ B
    using ⟨A ≠ B⟩ ⟨B Midpoint A A'⟩ midpoint-not-midpoint by blast
}

```

```

{
  assume  $B C D \text{ LeA } C B A'$ 
  have  $\text{SAMS } A B C B C D$ 
  using  $\langle \text{Midpoint-def } \langle A \neq B \rangle \langle A' \neq B \rangle \langle B C D \text{ LeA } C B A' \rangle$ 
     $\langle B \text{ Midpoint } A A' \rangle \text{ sams-chara} \text{ by blast}$ 
  then obtain  $Y$  where  $B \text{ Out } A Y \wedge C \text{ Out } D Y$ 
  using  $\langle B C \text{ OS } A D \rangle \langle A B C B C D \text{ SumA } P Q R \rangle \langle \neg \text{Bet } P Q R \rangle$ 
     $\text{assms euclid-s-parallel-postulate-def} \text{ by blast}$ 
  hence  $\exists Y. \text{Col } B A Y \wedge \text{Col } C D Y$ 
  using  $\text{out-col} \text{ by blast}$ 
}
moreover
{
  assume  $C B A' \text{ LeA } B C D$ 
  have  $\text{SAMS } D' C B C B A'$ 
  proof –
    have  $\text{Bet } D' C D$ 
    by ( $\text{simp add: } \langle C \text{ Midpoint } D D' \rangle \text{ l7-2 midpoint-bet}$ )
  moreover
  have  $C B A' \text{ LeA } B C D$ 
  by ( $\text{simp add: } \langle C B A' \text{ LeA } B C D \rangle$ )
  ultimately
  show ?thesis
  by ( $\text{metis } \langle C \neq D' \rangle \langle C \neq D \rangle \text{ sams-chara}$ )
qed
have  $B \neq C$ 
using  $\langle C \neq B \rangle \text{ by auto}$ 
have  $D' \neq C$ 
using  $\langle C \neq D' \rangle \text{ by auto}$ 
obtain  $P' Q' R'$  where  $A' B C B C D' \text{ SumA } P' Q' R'$ 
using  $\text{ex-suma } \langle A' \neq B \rangle \langle B \neq C \rangle \langle C \neq D' \rangle \text{ by blast}$ 
have  $B C \text{ OS } A' D'$ 
proof –
  have  $\neg \text{Col } B C A$ 
  using  $\langle B C \text{ OS } A D \rangle \text{ col123--nos} \text{ by blast}$ 
  have  $\neg \text{Col } B C D$ 
  using  $\langle B C \text{ OS } A D \rangle \text{ col124--nos} \text{ by blast}$ 
  have  $B C \text{ TS } A A'$ 
  using  $\langle A' \neq B \rangle \langle \text{Bet } A B A' \rangle \langle \neg \text{Col } B C A \rangle \text{ bet--ts} \text{ by force}$ 
  hence  $B C \text{ TS } A' D$ 
  using  $\langle B C \text{ OS } A D \rangle \text{ l9-2 l9-8-2} \text{ by blast}$ 
  moreover
  have  $B C \text{ TS } D D'$ 
  using  $\langle \text{Bet } D C D' \rangle \langle D' \neq C \rangle \langle \neg \text{Col } B C D \rangle \text{ bet--ts}$ 
     $\text{invert-two-sides not-col-permutation-4} \text{ by presburger}$ 
  ultimately
  show ?thesis
  by ( $\text{meson l9-2 l9-8-1}$ )
qed
moreover
{
  assume  $\text{Bet } P' Q' R'$ 
  hence  $A' B C \text{ SuppA } B C D'$ 
  using  $\text{bet-suma--suppa } \langle A' B C B C D' \text{ SumA } P' Q' R' \rangle \text{ by auto}$ 
  have  $A B C \text{ SuppA } B C D$ 
  proof –
    have  $B C D' \text{ CongA } A B C$ 
    proof –
      have  $A' B C \text{ SuppA } B C D'$ 
      by ( $\text{simp add: } \langle A' B C \text{ SuppA } B C D' \rangle$ )
    moreover
    have  $A' B C \text{ SuppA } A B C$ 
    by ( $\text{metis between-symmetry } \langle A \neq B \rangle \langle A' \neq B \rangle$ 
       $\langle \text{Bet } A B A' \rangle \langle C \neq B \rangle \text{ bet--suppa suppa-right-comm}$ )
    ultimately
    show ?thesis
  }
}

```

```

    by (simp add: suppa2--conga456)
  qed
  moreover
  have  $A' B C \text{ CongA } B C D$ 
  proof -
    have  $A' B C \text{ SuppA } B C D'$ 
      using  $\langle A' B C \text{ SuppA } B C D' \rangle$  by auto
    moreover
    have  $B C D \text{ SuppA } B C D'$ 
      using  $\langle \text{Bet } D C D' \rangle \langle C \neq B \rangle \langle C \neq D \rangle \langle D' \neq C \rangle$ 
      bet--suppa suppa-left-comm by presburger
    ultimately
    show ?thesis
      by (simp add: suppa2--conga123)
  qed
  moreover
  have  $B C D' \text{ SuppA } A' B C$ 
    by (simp add:  $\langle A' B C \text{ SuppA } B C D' \rangle$  suppa-sym)
  ultimately
  show ?thesis
    by (meson conga2-suppa--suppa)
  qed
  hence  $\text{Bet } P Q R$ 
    using  $\langle A B C B C D \text{ SumA } P Q R \rangle$  suma-suppa--bet by auto
  hence False
    using  $\langle \neg \text{Bet } P Q R \rangle$  by auto
}
ultimately
have  $\exists Y. B \text{ Out } A' Y \wedge C \text{ Out } D' Y$ 
  using  $\langle A' B C B C D' \text{ SumA } P' Q' R' \rangle$  assms
  euclid-s-parallel-postulate-def  $\langle \text{SAMS } D' C B C B A' \rangle$ 
  sams-comm sams-sym by blast
then obtain  $Y$  where  $B \text{ Out } A' Y$  and  $C \text{ Out } D' Y$ 
  by auto
have  $\text{Col } B A' Y$ 
  using  $\langle B \text{ Out } A' Y \rangle$  out-col by auto
hence  $\text{Col } B A Y$ 
  by (meson  $\langle A' \neq B \rangle \langle \text{Col } A B A' \rangle$  col-permutation-4
  col-trivial-2 colx)
moreover
have  $\text{Col } C D' Y$ 
  using  $\langle C \text{ Out } D' Y \rangle$  out-col by auto
hence  $\text{Col } C D Y$ 
  by (meson  $\langle \text{Col } D C D' \rangle \langle D' \neq C \rangle$  col-trivial-2
  colx not-col-permutation-4)
ultimately
have  $\exists Y. \text{Col } B A Y \wedge \text{Col } C D Y$ 
  by auto
}
ultimately
have  $\exists Y. \text{Col } B A Y \wedge \text{Col } C D Y$ 
  using lea-total by (metis  $\langle A' \neq B \rangle \langle C \neq B \rangle \langle C \neq D \rangle$ )
}
thus ?thesis
  using alternative-strong-parallel-postulate-def by blast
qed

lemma original-spp--inverse-projection-postulate:
  assumes alternative-strong-parallel-postulate
  shows inverse-projection-postulate
proof -
  {
    fix  $A B C P Q$ 
    assume Acute  $A B C$  and
       $B \text{ Out } A P$  and
       $P \neq Q$  and

```

```

  Per B P Q and
  Coplanar A B C Q
have B ≠ A
  using ⟨Acute A B C⟩ acute-distincts by blast
have B ≠ P
  using ⟨B Out A P⟩ l6-3-1 by blast
have B ≠ C
  using ⟨Acute A B C⟩ acute-distincts by blast
have Col B A P
  using ⟨B Out A P⟩ out-col by blast
have ∃ Y. B Out C Y ∧ Col P Q Y
proof (cases Col A B C)
  case True
  {
    assume Bet C B A
    obtain x y z where Per x y z and A B C LtA x y z
      using ⟨Acute A B C⟩ ⟨Bet C B A⟩ acute-not-bet
        between-symmetry by blast
    have ¬ (A B C LtA x y z ∧ x y z LtA A B C)
      by (simp add: not-and-lta)
    have A B C LtA x y z
      using ⟨A B C LtA x y z⟩ by auto
    moreover
    have x y z LtA A B C
      using ⟨Acute A B C⟩ ⟨Bet C B A⟩ acute-not-bet
        between-symmetry by blast
    ultimately
    have False
      using ⟨¬ (A B C LtA x y z ∧ x y z LtA A B C)⟩ by blast
  }
hence ¬ Bet C B A
  by auto
hence B Out C A
  by (simp add: True col-permutation-3 l6-4-2)
hence B Out C P
  using l6-7 ⟨B Out A P⟩ by blast
moreover
have Col P Q P
  by (simp add: col-trivial-3)
ultimately
show ?thesis
  by auto
next
case False
have ¬ Col B P Q
  using ⟨B ≠ P⟩ ⟨P ≠ Q⟩ ⟨Per B P Q⟩ l8-9 by blast
have ∃ Q0. Col Q P Q0 ∧ A B OS C Q0
proof –
  have Q ≠ P
    using ⟨P ≠ Q⟩ by auto
  moreover
  have Col A B P
    using Col-cases ⟨Col B A P⟩ by blast
  moreover
  have Col Q P P
    using col-trivial-2 by blast
  moreover
  have ¬ Col A B Q
    by (metis Out-def ⟨B Out A P⟩ ⟨¬ Col B P Q⟩
      calculation(2) col-transitivity-2)
  moreover
  have ¬ Col A B C
    by (simp add: False)
  moreover
  have Coplanar A B Q C
    using ⟨Coplanar A B C Q⟩ coplanar-perm-1 by blast

```



```

ultimately
show ?thesis
  by (simp add: cop-not-par-same-side)
qed
then obtain Q0 where Col Q P Q0 and A B OS C Q0
  by auto
have  $\neg$  Col A B Q0
  using  $\langle A B OS C Q0 \rangle$  one-side-not-col124 by blast
have  $P \neq Q0$ 
  using NCol-cases  $\langle Col B A P \rangle \langle \neg Col A B Q0 \rangle$  by blast
then obtain D E F where C B P B P Q0 SumA D E F
  using ex-suma  $\langle B \neq C \rangle \langle B \neq P \rangle$  by presburger
have  $\exists Y. Col B C Y \wedge Col P Q0 Y$ 
proof -
  have B P OS C Q0
    using  $\langle A B OS C Q0 \rangle \langle B \neq P \rangle \langle Col B A P \rangle$ 
    col-one-side invert-one-side by blast
  moreover
  have C B P B P Q0 SumA D E F
    by (simp add:  $\langle C B P B P Q0 SumA D E F \rangle$ )
  moreover
  {
    assume Bet D E F
    have Per B P Q0
      using  $\langle Col Q P Q0 \rangle \langle P \neq Q \rangle \langle Per B P Q \rangle$ 
      not-col-permutation-4 per-col by blast
    hence Per C B P
      using  $\langle Bet D E F \rangle \langle C B P B P Q0 SumA D E F \rangle$ 
      bet-per-suma--per123 by blast
    hence A B C LtA C B P
      using  $\langle Acute A B C \rangle$  acute-per--lta  $\langle B \neq C \rangle \langle B \neq P \rangle$  by simp
    hence A B C LeA C B P  $\wedge \neg A B C CongA C B P$ 
      by (simp add: lta--lea lta-not-conga)
    moreover
    have A B C CongA P B C
    proof -
      have B Out P A
        using  $\langle B Out A P \rangle$  l6-6 by blast
      moreover
      have B Out C C
        using  $\langle B \neq C \rangle$  out-trivial by fastforce
      ultimately
      show ?thesis
        by (simp add: out2--conga)
    qed
    hence A B C CongA C B P
      by (simp add: conga-right-comm)
    ultimately
    have False
      by blast
  }
ultimately
show ?thesis
  using alternative-strong-parallel-postulate-def assms by blast
qed
then obtain Y where Col B C Y  $\wedge$  Col P Q0 Y
  by auto
have Col A B B
  by (simp add: col-trivial-2)
hence  $\exists B0. A B Perp B0 B \wedge A B OS C B0$ 
  by (simp add: False l10-15)
then obtain B0 where A B Perp B0 B and A B OS C B0
  by auto
have  $\neg$  Col A B B0
  using  $\langle A B OS C B0 \rangle$  col124--nos by blast
have  $\neg$  Col B C P

```

```

by (meson False  $\langle B \neq P \rangle \langle Col\ B\ A\ P \rangle col2\text{--}eq\ col\text{--}permutation\text{--}4$ )
have  $B0 \neq B$ 
using  $\langle Col\ A\ B\ B \rangle \langle \neg\ Col\ A\ B\ B0 \rangle$  by auto
have  $P \neq Y$ 
using  $\langle Col\ B\ C\ Y \wedge Col\ P\ Q0\ Y \rangle \langle \neg\ Col\ B\ C\ P \rangle$  by blast
have  $B\ B0\ OS\ C\ Y$ 
proof -
  have  $B\ B0\ OS\ C\ P$ 
  proof -
    have  $B0\ B\ OS\ C\ A$ 
    proof -
      have  $\neg\ Col\ B0\ B\ A$ 
      using  $\langle \neg\ Col\ A\ B\ B0 \rangle not\text{--}col\text{--}permutation\text{--}3$  by blast
    moreover
    {
      assume  $Col\ B\ B0\ C$ 
      hence  $A\ B\ C\ LtA\ A\ B\ C$ 
      by (metis Perp-cases  $\langle A\ B\ Perp\ B0\ B \rangle \langle$ 
Acute  $A\ B\ C \rangle \langle B \neq C \rangle acute\text{--}per\text{--}lta$ 
col-trivial-3 l8-16-1)
      hence False
      using not-and-lta by blast
    }
  moreover
  have  $A\ B\ C\ LeA\ A\ B\ B0$ 
  using  $\langle A\ B\ Perp\ B0\ B \rangle \langle Acute\ A\ B\ C \rangle \langle B \neq A \rangle$ 
 $\langle B0 \neq B \rangle acute\text{--}per\text{--}lta\ lta\text{--}lea\ perp\text{--}comm$ 
perp-per-2 by presburger
  hence  $C\ InAngle\ A\ B\ B0$ 
  by (simp add:  $\langle A\ B\ OS\ C\ B0 \rangle lea\text{--}in\text{--}angle$ 
one-side-symmetry)
  hence  $C\ InAngle\ B0\ B\ A$ 
  using l11-24 by blast
  ultimately
  show ?thesis
  using in-angle-one-side by blast
qed
have  $B\ B0\ OS\ C\ A$ 
using invert-one-side by blast
moreover
have  $B\ B0\ OS\ A\ P$ 
using  $\langle B\ Out\ A\ P \rangle calculation\ col124\text{--}nos$ 
col-trivial-3 l9-19-R2 by blast
ultimately
show ?thesis
using one-side-transitivity by blast
qed
moreover
have  $B\ B0\ Par\ P\ Y$ 
proof -
  have Coplanar  $A\ B\ B\ P$ 
  using ncop-distincts by blast
  moreover
  have Coplanar  $A\ B\ B\ Y$ 
  using ncop-distincts by blast
  moreover
  have Coplanar  $A\ B\ B0\ P$ 
  by (meson  $\langle Col\ B\ A\ P \rangle col\text{--}permutation\text{--}4\ ncop\text{--}ncols$ )
  moreover
  have Coplanar  $A\ B\ B0\ Y$ 
  proof -
    have  $\neg\ Col\ C\ A\ B$ 
    using Col-cases False by blast
  moreover
  have Coplanar  $C\ A\ B\ B0$ 
  proof -

```

```

have Coplanar A B B0 C
  by (meson ⟨A B OS C B0⟩ cop2-os--osp
      ncop-distincts osp-distincts)
moreover
have Coplanar A B B0 A
  using ncop-distincts by auto
moreover
have Coplanar A B B0 B
  using ncop-distincts by blast
moreover
have Coplanar A B B0 B0
  using ncop-distincts by blast
ultimately
show ?thesis
  using coplanar-perm-18 by blast
qed
moreover
have Coplanar C A B Y
  using ⟨Col B C Y ∧ Col P Q0 Y⟩
      col-permutation-4 ncop--ncols by blast
ultimately
show ?thesis
  using coplanar-trans-1 by blast
qed
moreover
have B B0 Perp A B
  using Perp-cases ⟨A B Perp B0 B⟩ by blast
moreover
have B P Perp P Y
proof —
  have B P Perp P Q
    using ⟨B ≠ P⟩ ⟨P ≠ Q⟩ ⟨Per B P Q⟩ per-perp by blast
  moreover
  have Col P Q Y
    by (meson ⟨Col B C Y ∧ Col P Q0 Y⟩ ⟨Col Q P Q0⟩
        ⟨P ≠ Q0⟩ col-trivial-2 colx not-col-permutation-4)
  ultimately
  show ?thesis
    using ⟨P ≠ Y⟩ perp-col1 by blast
qed
hence P Y Perp B P
  using Perp-perm by blast
hence P Y Perp B A
  using ⟨B Out A P⟩ ⟨B ≠ A⟩ col-permutation-5
      out-col perp-col1 by blast
hence P Y Perp A B
  by (simp add: perp-right-comm)
ultimately
show ?thesis
  using l12-9 by blast
qed
hence B B0 ParStrict P Y
  using calculation not-col-distincts
      one-side-not-col124 par-not-col-strict by blast
hence B B0 OS P Y
  using l12-6 by blast
ultimately
show ?thesis
  using one-side-transitivity by blast
qed
hence B Out C Y
  using ⟨Col B C Y ∧ Col P Q0 Y⟩
      col-one-side-out by blast
moreover
have Col P Q Y
  by (meson ⟨Col B C Y ∧ Col P Q0 Y⟩ ⟨Col Q P Q0⟩

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      ⟨P ≠ Q0⟩ col-trivial-2 colx not-col-permutation-4)
    ultimately
    show ?thesis
      by auto
  qed
}
thus ?thesis
  using inverse-projection-postulate-def by auto
qed

lemma proclus-bis--proclus:
  assumes alternative-proclus-postulate
  shows proclus-postulate
proof -
{
  fix A B C D P Q
  assume A B Par C D and
    Col A B P and
    ¬ Col A B Q and
    Coplanar C D P Q
  have ∃ Y. Col P Q Y ∧ Col C D Y
  proof (cases Col C D P)
    case True
    thus ?thesis
      using col-trivial-3 by blast
  next
    case False
    hence ¬ Col C D P
      by simp
    have C D Par A B
      by (simp add: ⟨A B Par C D⟩ par-symmetry)
    hence C D ParStrict A B
      using False ⟨Col A B P⟩ par-not-col-strict by blast
    obtain C0 where Col C D C0 and C D Perp P C0
      using False l8-18-existence by blast
    have ¬ Col C0 A B
      by (meson ⟨C D ParStrict A B⟩ ⟨Col C D C0⟩
        col-permutation-2 par-not-col)
    have ∃ A0. Col A B A0 ∧ ¬ Col C0 P A0
  proof (cases Col C0 P A)
    case True
    thus ?thesis
      by (metis (full-types) ⟨C D ParStrict A B⟩
        ⟨Col A B P⟩ ⟨Col C D C0⟩ col-permutation-5 col-trivial-2
        not-col-permutation-1 par-not-col par-strict-col-par-strict)
  next
    case False
    thus ?thesis
      using col-trivial-3 by blast
  qed
  then obtain A0 where Col A B A0 and ¬ Col C0 P A0
    by blast
  have Col C0 P P
    by (simp add: col-trivial-2)
  then obtain A' where C0 P Perp A' P and C0 P OS A0 A'
    using ⟨¬ Col C0 P A0⟩ l10-15 by blast
  have P ≠ A0
    using ⟨Col C0 P P⟩ ⟨¬ Col C0 P A0⟩ by blast
  hence Coplanar C D P A0
    by (meson ⟨C D ParStrict A B⟩ ⟨Col A B A0⟩
      ⟨Col A B P⟩ par-strict-col2-par-strict
      pars--coplanar)
  show ?thesis
proof (cases Col A0 P A')
  case True
  show ?thesis

```

```

proof –
  have  $P \text{ Perp2 } A0 \ P \ C \ D$ 
  proof –
    have  $Col \ P \ C0 \ P$ 
    by (simp add: col-trivial-3)
    moreover
    have  $C0 \ P \ Perp \ A0 \ P$ 
    by (metis NCol-perm Perp-perm True
       $\langle C0 \ P \ Perp \ A' \ P \rangle \langle P \neq A0 \rangle$  perp-col)
    moreover
    have  $C0 \ P \ Perp \ C \ D$ 
    using Perp-perm  $\langle C \ D \ Perp \ P \ C0 \rangle$  by blast
    ultimately
    show ?thesis
    using Perp2-def by blast
  qed
  moreover
  have  $\neg Col \ C \ D \ P$ 
  using False by auto
  moreover
  have Coplanar  $A0 \ P \ C \ D$ 
  by (simp add:  $\langle Coplanar \ C \ D \ P \ A0 \rangle$  coplanar-perm-22)
  moreover
  have  $Col \ A0 \ P \ P$ 
  by (simp add: col-trivial-2)
  moreover
  have  $\neg Col \ A0 \ P \ Q$ 
  by (metis  $\langle Col \ A \ B \ A0 \rangle \langle Col \ A \ B \ P \rangle$ 
     $\langle P \neq A0 \rangle \langle \neg Col \ A \ B \ Q \rangle$  colx)
  ultimately
  show ?thesis
  using  $\langle Coplanar \ C \ D \ P \ Q \rangle$  assms
    alternative-proclus-postulate-def by simp
qed
next
case False
hence  $\neg Col \ A0 \ P \ A'$ 
by simp
have  $\exists Y. Col \ P \ A0 \ Y \wedge Col \ C \ D \ Y$ 
proof –
  have  $P \text{ Perp2 } A' \ P \ C \ D$ 
  proof –
    have  $Col \ P \ C0 \ P$ 
    by (simp add: col-trivial-3)
    moreover
    have  $C0 \ P \ Perp \ C \ D$ 
    using Perp-cases  $\langle C \ D \ Perp \ P \ C0 \rangle$  by auto
    ultimately
    show ?thesis
    using  $\langle C0 \ P \ Perp \ A' \ P \rangle$  Perp2-def by blast
  qed
  moreover
  have  $\neg Col \ C \ D \ P$ 
  by (simp add:  $\langle \neg Col \ C \ D \ P \rangle$ )
  moreover
  have Coplanar  $C \ D \ P \ A0$ 
  by (meson  $\langle C \ D \ ParStrict \ A \ B \rangle \langle Col \ A \ B \ A0 \rangle$ 
     $\langle Col \ A \ B \ P \rangle \langle P \neq A0 \rangle$  par-strict-col2-par-strict
    pars--coplanar)
  have  $C0 \ P \ OS \ A0 \ A'$ 
  by (simp add:  $\langle C0 \ P \ OS \ A0 \ A' \rangle$ )
  hence Coplanar  $C0 \ P \ A0 \ A'$ 
  by (simp add: os--coplanar)
  hence Coplanar  $A' \ P \ C \ D$ 
  proof –
    have Coplanar  $C0 \ P \ A0 \ A'$ 

```

```

    by (simp add: ⟨Coplanar C0 P A0 A'⟩)
  moreover have Coplanar C0 P A0 P
    using ncop-distincts by blast
  moreover have Coplanar C0 P A0 C
    by (metis ⟨Col C D C0⟩ ⟨Coplanar C D P A0⟩
      ⟨¬ Col C D P⟩ col-cop--cop coplanar-perm-20
      coplanar-perm-22 not-col-distincts)
  moreover have Coplanar C0 P A0 D
    by (metis ⟨Col C D C0⟩ ⟨Coplanar C D P A0⟩
      calculation(3) col-cop2--cop coplanar-perm-21
      coplanar-perm-22 ncop-distincts)
  ultimately show ?thesis
    using ⟨¬ Col C0 P A0⟩ coplanar-pseudo-trans by blast
qed
moreover
have Col A' P P
  by (simp add: col-trivial-2)
moreover
have ¬ Col A' P A0
  by (simp add: False not-col-permutation-3)
ultimately
show ?thesis
  using ⟨Coplanar C D P A0⟩ assms
  alternative-proclus-postulate-def by simp
qed
then obtain Y where Col P A0 Y and Col C D Y
  by auto
have C D ParStrict A B
  by (simp add: ⟨C D ParStrict A B⟩)
moreover
have Col Y C D
  using Col-cases ⟨Col C D Y⟩ by blast
moreover
have Col Y A B
  by (meson ⟨Col A B A0⟩ ⟨Col A B P⟩ ⟨Col P A0 Y⟩
    ⟨P ≠ A0⟩ colx not-col-permutation-1)
ultimately
have False
  using par-not-col by simp
thus ?thesis
  by simp
qed
qed
}
thus ?thesis
  using proclus-postulate-def by blast
qed

```

lemma proclus-s-postulate-implies-strong-parallel-postulate:

assumes proclus-postulate

shows strong-parallel-postulate

proof –

```

{
  fix P Q R S T U
  assume BetS P T Q and
    BetS R T S and
    ¬ Col P R U and
    Coplanar P Q R U and
    Cong P T Q T and
    Cong R T S T
  have ∃ I. Col S Q I ∧ Col P U I
  proof (cases Col P Q R)
    case True
    have Col S Q P
    by (metis (full-types) BetS-def Col-cases
      True ⟨BetS P T Q⟩ ⟨BetS R T S⟩ bet-col colx)
  }

```

```

moreover
have Col P U P
  by (simp add: col-trivial-3)
ultimately
show ?thesis
  by auto
next
case False
show ?thesis
proof –
  have P R Par Q S
  proof –
    have T Midpoint P Q
    by (simp add: BetSEq midpoint-def  $\langle \text{BetS } P \ T \ Q \rangle$ 
       $\langle \text{Cong } P \ T \ Q \ T \rangle$  cong-right-commutativity)
    moreover
    have T Midpoint R S
    by (meson BetSEq midpoint-def  $\langle \text{BetS } R \ T \ S \rangle$ 
       $\langle \text{Cong } R \ T \ S \ T \rangle$  not-cong-1243)
    ultimately
    show ?thesis
    using False not-col-distincts sym-par by blast
  qed
moreover
have Col P R P
  by (simp add: col-trivial-3)
moreover
have Coplanar P Q R S
  using calculation(1) coplanar-perm-2 par--coplanar by blast
hence Coplanar Q S P U
  by (meson False  $\langle \text{Coplanar } P \ Q \ R \ U \rangle$  col-permutation-1
    coplanar-trans-1 ncoplanar-perm-8)
ultimately
show ?thesis
  using  $\langle \neg \text{Col } P \ R \ U \rangle$  assms proclus-postulate-def
    not-col-permutation-4 by blast
qed
qed
}
thus ?thesis
using strong-parallel-postulate-def by blast
qed

```

```

lemma rectangle-existence--rah:
assumes postulate-of-existence-of-a-right-lambert-quadrilateral
shows postulate-of-right-saccheri-quadrilaterals
proof –
{
  fix A B C D
  assume Saccheri A B C D
  hence Per A B C
  using assms lam-per--rah
    hypothesis-of-right-saccheri-quadrilaterals-def
    postulate-of-existence-of-a-right-lambert-quadrilateral-def
  by blast
}
thus ?thesis
by (simp add: postulate-of-right-saccheri-quadrilaterals-def)
qed

```

```

lemma posidonius-postulate--rah:
assumes posidonius-postulate
shows postulate-of-right-saccheri-quadrilaterals
proof –
have  $\exists \ A1 \ A2 \ B1 \ B2.$ 
   $\text{Per } A2 \ A1 \ B1 \ \wedge \ \text{Per } A1 \ A2 \ B2 \ \wedge$ 

```

$Cong\ A1\ B1\ A2\ B2 \wedge A1\ A2\ OS\ B1\ B2 \wedge$
 $(\forall\ A3\ B3.$
 $(Col\ A1\ A2\ A3 \wedge Col\ B1\ B2\ B3 \wedge$
 $A1\ A2\ Perp\ A3\ B3)$
 $\longrightarrow Cong\ A3\ B3\ A1\ B1)$

proof—

obtain $A1'\ A2'\ B1\ B2$ **where**
 $\neg\ Col\ A1'\ A2'\ B1$ **and** $B1 \neq B2$ **and**
 $Coplanar\ A1'\ A2'\ B1\ B2$ **and**
 $H1: \forall\ A3\ A4\ B3\ B4.$
 $(Col\ A1'\ A2'\ A3 \wedge Col\ B1\ B2\ B3 \wedge A1'\ A2'\ Perp\ A3\ B3 \wedge$
 $Col\ A1'\ A2'\ A4 \wedge Col\ B1\ B2\ B4 \wedge A1'\ A2'\ Perp\ A4\ B4) \longrightarrow$
 $Cong\ A3\ B3\ A4\ B4$

using *assms posidonius-postulate-def* **by** *blast*

obtain $A1$ **where** $Col\ A1'\ A2'\ A1$ **and** $A1'\ A2'\ Perp\ B1\ A1$

using $\langle \neg\ Col\ A1'\ A2'\ B1 \rangle$ *l8-18-existence* **by** *blast*

have $A1' \neq A2'$

using $\langle \neg\ Col\ A1'\ A2'\ B1 \rangle$ *col-trivial-1* **by** *blast*

have $A1 \neq B1$

using $\langle A1'\ A2'\ Perp\ B1\ A1 \rangle$ *perp-not-eq-2* **by** *blast*

have $A1'\ A2'\ ParStrict\ B1\ B2$

proof —

{
assume $\exists\ I. Col\ I\ A1'\ A2' \wedge Col\ I\ B1\ B2$
then obtain I **where** $Col\ I\ A1'\ A2'$ **and** $Col\ I\ B1\ B2$
by *blast*
have $B1 \neq I$
using $\langle Col\ I\ A1'\ A2' \rangle$ $\langle \neg\ Col\ A1'\ A2'\ B1 \rangle$
not-col-permutation-2 **by** *blast*
obtain $B3$ **where** $B3\ Midpoint\ B1\ I$
using *midpoint-existence* **by** *presburger*
hence $B1 \neq B3$
using $\langle B1 \neq I \rangle$ *l7-3-2 l7-9-bis* **by** *blast*
have $Bet\ B1\ B3\ I$
by (*metis midpoint-bet* $\langle B3\ Midpoint\ B1\ I \rangle$)
hence $Col\ B1\ B3\ I$
using *bet-col* **by** *auto*
{
assume $Col\ A1'\ A2'\ B3$
have $Col\ B1\ B3\ A1'$
by (*metis* $\langle A1' \neq A2' \rangle$ $\langle B3\ Midpoint\ B1\ I \rangle$
 $\langle Col\ A1'\ A2'\ B3 \rangle$ $\langle Col\ B1\ B3\ I \rangle$ $\langle Col\ I\ A1'\ A2' \rangle$ *col2--eq*
col-permutation-4 midpoint-distinct-1 not-col-distincts)
moreover
have $Col\ B1\ B3\ A2'$
by (*metis* $\langle A1' \neq A2' \rangle$ $\langle B3\ Midpoint\ B1\ I \rangle$
 $\langle Col\ A1'\ A2'\ B3 \rangle$ $\langle Col\ B1\ B3\ I \rangle$ $\langle Col\ I\ A1'\ A2' \rangle$ *col2--eq*
col-permutation-4 midpoint-distinct-1 not-col-distincts)
moreover
have $Col\ B1\ B3\ B1$
by (*simp add: col-trivial-3*)
ultimately
have $Col\ A1'\ A2'\ B1$
using $\langle B1 \neq B3 \rangle$ *col3* **by** *blast*
hence *False*
using $\langle \neg\ Col\ A1'\ A2'\ B1 \rangle$ **by** *auto*
}

then obtain $A3$ **where** $Col\ A1'\ A2'\ A3$ **and** $A1'\ A2'\ Perp\ B3\ A3$

using *l8-18-existence* **by** *blast*

have $Cong\ A1\ B1\ A3\ B3$

proof —

have $Col\ A1'\ A2'\ A1$

by (*simp add:* $\langle Col\ A1'\ A2'\ A1 \rangle$)

moreover

have $Col\ B1\ B2\ B1$

using *col-trivial-3* **by** *blast*


```

moreover
have  $A1' A2' \text{ Perp } A1 B1$ 
  using  $\text{Perp-cases } \langle A1' A2' \text{ Perp } B1 A1 \rangle$  by blast
moreover
have  $\text{Col } A1' A2' A3$ 
  by (simp add: Col A1' A2' A3)
moreover
have  $\text{Col } B1 B2 B3$ 
  by (metis B1 ≠ I Col B1 B3 I Col I B1 B2
    col-permutation-3 col-permutation-5 col-transitivity-2)
moreover
have  $A1' A2' \text{ Perp } A3 B3$ 
  using  $\text{Perp-perm } \langle A1' A2' \text{ Perp } B3 A3 \rangle$  by blast
ultimately
show ?thesis
  using H1 by blast
qed
have  $\text{Cong } B1 I B3 I$ 
proof –
{
  assume  $A3 = I$ 
  have  $A3 B3 \text{ Perp } A1' A2'$ 
    using  $\text{Perp-cases } \langle A1' A2' \text{ Perp } B3 A3 \rangle$  by blast
  hence  $A1' A2' \text{ Perp } B1 A3$ 
    by (metis between-symmetry A3 = I B1 ≠ B3
      B1 ≠ I Bet B1 B3 I bet-col bet-col1
      col3 perp-col0)
  hence  $A1 = A3$ 
    using  $\langle A1' A2' \text{ Perp } B1 A1 \rangle \langle \text{Col } A1' A2' A1 \rangle$ 
       $\langle \text{Col } A1' A2' A3 \rangle$  l8-18-uniqueness
    by blast
  have  $B3 \neq B1$ 
    using  $\langle B1 \neq B3 \rangle$  by blast
  hence False
    using  $\langle A1 = A3 \rangle \langle A3 = I \rangle \langle \text{Bet } B1 B3 I \rangle$ 
       $\langle \text{Cong } A1 B1 A3 B3 \rangle$  between-cong between-symmetry
      cong-inner-transitivity cong-reflexivity by blast
}
hence  $A3 \neq I$ 
  by auto
have  $B3 \neq I$ 
  using  $\langle B1 \neq I \rangle \langle B3 \text{ Midpoint } B1 I \rangle$ 
    midpoint-distinct-1 by blast
{
  assume  $A1 = I$ 
  hence  $A3 = A1$ 
  proof –
    have  $\neg \text{Col } A1' A2' B3$ 
      using  $\langle \text{Col } A1' A2' B3 \implies \text{False} \rangle$  by auto
    moreover
    have  $A1' A2' \text{ Perp } B3 A1$ 
    proof –
      have  $\text{Col } B1 A1 B3$ 
        using  $\langle A1 = I \rangle \langle \text{Col } B1 B3 I \rangle$ 
          col-permutation-5 by blast
      moreover
      have  $\text{Col } B1 A1 A1$ 
        by (simp add: col-trivial-2)
      ultimately
      show ?thesis
        using  $\langle A1' A2' \text{ Perp } B1 A1 \rangle \langle \text{Col } A1' A2' A1 \rangle$ 
           $\langle \neg \text{Col } A1' A2' B3 \rangle$  perp-col2-bis by blast
    }
qed
ultimately
show ?thesis
  using  $\langle \text{Col } A1' A2' B3 \implies \text{False} \rangle$ 

```

```

      ⟨Col A1' A2' A3⟩ ⟨A1' A2' Perp B3 A3⟩ ⟨Col A1' A2' A1⟩
      l8-18-uniqueness by blast
qed
hence False
  using ⟨A1 = I⟩ ⟨A3 ≠ I⟩ by auto
}
hence A1 ≠ I
  by blast
{
  assume Col B1 A1 I
  hence A3 = I
    by (metis ⟨A1 = I ⟹ False⟩ ⟨Col A1' A2' A1⟩
      ⟨Col I A1' A2'⟩ ⟨¬ Col A1' A2' B1⟩
      l6-21 not-col-distincts)
  hence False
    using ⟨A3 ≠ I⟩ by blast
}
moreover
have Bet I A3 A1
proof -
  have A3 B3 TS I A1
  proof -
    have ¬ Col A3 B3 B1
      by (meson ⟨A3 = I ⟹ False⟩ ⟨B1 ≠ B3⟩
        ⟨Col A1' A2' A3⟩ ⟨Col B1 B3 I⟩ ⟨Col I A1' A2'⟩
        ⟨¬ Col A1' A2' B1⟩ col-permutation-3
        col-permutation-4 l6-21)
    have A3 B3 TS B1 I
      using Col-perm ⟨B3 ≠ I⟩ ⟨Bet B1 B3 I⟩
        ⟨¬ Col A3 B3 B1⟩ bet--ts
        invert-two-sides by blast
  moreover
  have A3 B3 OS B1 A1
  proof -
    have A3 B3 Par B1 A1
    proof -
      have Coplanar A1' A2' A3 B1
        by (simp add: ⟨Col A1' A2' A3⟩ col--coplanar)
      moreover
      have Coplanar A1' A2' A3 A1
        using ⟨Col A1' A2' A3⟩ ncop--ncols by blast
      moreover
      have Coplanar A1' A2' B3 B1
        using Coplanar-def ⟨Col B1 B3 I⟩
          ⟨Col I A1' A2'⟩ col-permutation-4
          not-col-permutation-2 by blast
      moreover
      have Coplanar A1' A2' B3 A1
        using ⟨Col A1' A2' A1⟩ ncop--ncols by blast
      moreover
      have A3 B3 Perp A1' A2'
        using Perp-perm ⟨A1' A2' Perp B3 A3⟩ by blast
      moreover
      have B1 A1 Perp A1' A2'
        using Perp-perm ⟨A1' A2' Perp B1 A1⟩ by blast
      ultimately
      show ?thesis
        using l12-9 by blast
    qed
  moreover
  have Col B1 A1 B1
    by (simp add: col-trivial-3)
  ultimately
  show ?thesis
    by (meson ⟨¬ Col A3 B3 B1⟩ col-trivial-2
      par-not-col-strict par-strict-one-side)

```

```

    qed
    ultimately
    show ?thesis
      using l9-2 l9-8-2 by blast
  qed
  moreover
  have Col A1' A2' I
    using ⟨Col I A1' A2'⟩ not-col-permutation-2 by blast
  hence Col A3 I A1
    using ⟨Col A1' A2' A3⟩ ⟨Col A1' A2' A1⟩
      ⟨A1' ≠ A2'⟩ col3 by blast
  ultimately
  show ?thesis
    using col-two-sides-bet by blast
  qed
  hence I Out A3 A1
    by (simp add: ⟨A3 ≠ I⟩ bet-out)
  hence A1 I B1 CongA A3 I B3
    by (simp add: ⟨B3 ≠ I⟩ ⟨Bet B1 B3 I⟩ bet-out-1 out2--conga)
  moreover
  have B1 A1 I CongA B3 A3 I
  proof -
    have Per B1 A1 I
    proof -
      have A1' A2' Perp A1 B1
        using Perp-cases ⟨A1' A2' Perp B1 A1⟩ by blast
      moreover
      have Col A1' A2' I
        using ⟨Col I A1' A2'⟩ not-col-permutation-2 by blast
      ultimately
      show ?thesis
        using Perp-cases ⟨Col A1' A2' A1⟩ l8-16-1 by blast
    qed
    moreover
    have Col A1' A2' I
      using Col-perm ⟨Col I A1' A2'⟩ by blast
    hence Per B3 A3 I
      using ⟨Col A1' A2' A3⟩ ⟨A1' A2' Perp B3 A3⟩ l8-16-1 by blast
    ultimately
    show ?thesis
      by (metis ⟨A1 ≠ B1⟩ ⟨A1 ≠ I⟩ ⟨A3 ≠ I⟩
        ⟨Cong A1 B1 A3 B3⟩ cong-diff l11-16)
  qed
  moreover
  have Cong B1 A1 B3 A3
    using ⟨Cong A1 B1 A3 B3⟩ not-cong-2143 by blast
  ultimately
  show ?thesis
    using l11-50-2 by blast
  qed
  have B3 ≠ B1
    using ⟨B1 ≠ B3⟩ by auto
  hence False
    by (metis ⟨Bet B1 B3 I⟩ ⟨Col B1 B3 I⟩
      ⟨Cong B1 I B3 I⟩ between-equality col-cong2-bet2
      col-transitivity-1 not-cong-1243)
}
thus ?thesis
  using ParStrict-def ⟨Coplanar A1' A2' B1 B2⟩ by blast
qed
hence ¬ Col A1' A2' B2
  using col124--nos l12-6 by blast
then obtain A2 where Col A1' A2' A2 and A1' A2' Perp B2 A2
  using l8-18-existence by blast
{
  assume A1 = A2

```

```

have A2' A1' ParStrict B2 B1
  using ⟨A1' A2' ParStrict B1 B2⟩ par-strict-comm by blast
hence ¬ Col A2 B2 B1
  using ⟨Col A1' A2' A2⟩ not-col-permutation-3 par-not-col by blast
moreover
have Col A1 B1 B2
  using ⟨A1 = A2⟩ Perp-cases ⟨A1' A2' Perp B1 A1⟩
    ⟨A1' A2' Perp B2 A2⟩ cop-perp2--col
    ⟨Coplanar A1' A2' B1 B2⟩ by blast
ultimately
have False
  using ⟨A1 = A2⟩ ⟨¬ Col A2 B2 B1⟩ not-col-permutation-5 by blast
}
hence A1 ≠ A2
  by blast
have A1' A2' Perp A1 B1
  using Perp-cases ⟨A1' A2' Perp B1 A1⟩ by blast
have A1' A2' Perp A2 B2
  by (simp add: ⟨A1' A2' Perp B2 A2⟩ perp-right-comm)
have Per A2 A1 B1
  by (meson ⟨Col A1' A2' A1⟩ ⟨Col A1' A2' A2⟩
    ⟨A1' A2' Perp A1 B1⟩ ⟨A1 ≠ A2⟩ perp-col2 perp-per-2)
moreover
have Per A1 A2 B2
  by (metis ⟨Col A1' A2' A1⟩ ⟨Col A1' A2' A2⟩
    ⟨A1' A2' Perp A2 B2⟩ ⟨A1 ≠ A2⟩
    perp-col2 perp-per-2)
moreover
have Cong A1 B1 A2 B2
proof -
  have Col B1 B2 B1
    by (simp add: col-trivial-3)
  moreover
  have Col B1 B2 B2
    by (simp add: col-trivial-2)
  ultimately
  show ?thesis
    using ⟨Col A1' A2' A1⟩ ⟨Col A1' A2' A2⟩
      ⟨A1' A2' Perp A1 B1⟩ ⟨A1' A2' Perp A2 B2⟩ H1 by blast
qed
moreover
have A1' A2' OS B1 B2
  using ⟨A1' A2' ParStrict B1 B2⟩ l12-6 by force
hence A1 A2 OS B1 B2
  using ⟨Col A1' A2' A1⟩ ⟨Col A1' A2' A2⟩ ⟨A1 ≠ A2⟩ col2-os--os by blast
moreover
{
  fix A3 B3
  assume Col A1 A2 A3 and
    Col B1 B2 B3 and
    A1 A2 Perp A3 B3
  have Col A1' A2' A3
    using ⟨A1 ≠ A2⟩ ⟨Col A1 A2 A3⟩ ⟨Col A1' A2' A1⟩
      ⟨Col A1' A2' A2⟩ colx by blast
  moreover
  have A1' A2' Perp A3 B3
  proof -
    have Col A1 A2 A1'
      by (metis ⟨A1' ≠ A2'⟩ ⟨Col A1' A2' A1⟩
        ⟨Col A1' A2' A2⟩ col-transitivity-2
        not-col-permutation-2)
    moreover
    have Col A1 A2 A2'
      by (meson ⟨A1' ≠ A2'⟩ ⟨Col A1' A2' A1⟩
        ⟨Col A1' A2' A2⟩ col-permutation-1
        col-transitivity-2)

```

```

    ultimately
    show ?thesis
    using ⟨A1 A2 Perp A3 B3⟩ ⟨A1' ≠ A2'⟩ perp-col2 by blast
qed
moreover
have Col B1 B2 B1
  by (simp add: col-trivial-3)
ultimately
have Cong A3 B3 A1 B1
  using ⟨Col B1 B2 B3⟩ ⟨Col A1' A2' A1⟩
  ⟨A1' A2' Perp A1 B1⟩ H1 by blast
}
ultimately
show ?thesis
  by blast
qed
then obtain A D B C where Per D A B and
  Per A D C and Cong A B D C and A D OS B C and
  H2: ∀ A3 B3. (Col A D A3 ∧ Col B C B3 ∧ A D Perp A3 B3) ⟶ Cong A3 B3 A B
  by blast
have Saccheri A B C D
  using Saccheri-def ⟨A D OS B C⟩ ⟨Cong A B D C⟩
  ⟨Per A D C⟩ ⟨Per D A B⟩ l8-2 not-cong-1243 by blast
have Per A B C
proof -
  obtain M where M Midpoint B C
    using midpoint-existence by blast
  hence Bet B M C
    using midpoint-bet by auto
  moreover
  obtain N where N Midpoint A D
    using midpoint-existence by blast
  hence Bet A N D
    by (simp add: midpoint-bet)
  moreover
  have A D Perp M N
    using mid2-sac--perp-lower ⟨M Midpoint B C⟩
    ⟨N Midpoint A D⟩ ⟨Saccheri A B C D⟩ by blast
  have A ≠ N
    using ⟨A D Perp M N⟩ ⟨N Midpoint A D⟩
    is-midpoint-id perp-distinct by blast
  moreover
  have N ≠ D
    by (metis ⟨N Midpoint A D⟩ calculation(3) midpoint-distinct-2)
  moreover
  have Per M N A
proof -
  have M N Perp A D
    using Perp-cases ⟨A D Perp M N⟩ by blast
  moreover
  have Col A D N
    using ⟨Bet A N D⟩ bet-col col-permutation-5 by blast
  ultimately
  show ?thesis
    by (meson Perp-cases col-trivial-3 l8-16-1)
qed
moreover
have Cong M N A B
proof -
  have Col A D N
    using bet-col calculation(2) not-col-permutation-5 by blast
  moreover
  have Col B C M
    by (meson ⟨Bet B M C⟩ bet-col not-col-permutation-5)
  ultimately
  show ?thesis

```

```

    using Cong-perm ⟨A D Perp M N⟩
    H2 perp-right-comm by blast
qed
ultimately
show ?thesis
  using ⟨Saccheri A B C D⟩ t22-7--per by blast
qed
thus ?thesis
  using ⟨Saccheri A B C D⟩ existential-saccheri--rah
  postulate-of-existence-of-a-right-saccheri-quadrilateral-def
  by blast
qed

lemma playfair--existential-playfair:
  assumes playfair-s-postulate
  shows existential-playfair-s-postulate
proof -
  obtain PA PB PC where ¬ Bet PA PB PC and ¬ Bet PB PC PA and ¬ Bet PC PA PB
  using lower-dim-ex by blast
  hence ¬ Col PA PB PC
  using Col-def by presburger
  moreover
  {
    fix B1 B2 C1 C2
    assume PA PB Par B1 B2 and Col PC B1 B2 and
      PA PB Par C1 C2 and
      Col PC C1 C2
    have Col C1 B1 B2 ∧ Col C2 B1 B2
    using assms ⟨Col PC B1 B2⟩ ⟨Col PC C1 C2⟩ ⟨PA PB Par B1 B2⟩
      ⟨PA PB Par C1 C2⟩ playfair-s-postulate-def by blast
  }
  ultimately show ?thesis
  using existential-playfair-s-postulate-def by blast
qed

```

3.3 Equivalences

```

lemma proclus--aristotle:
  assumes proclus-postulate
  shows aristotle-s-axiom
proof -
  {
    fix P Q A B C
    assume ¬ Col A B C and
      Acute A B C
    have (Col B A B ∧ ¬ Col B A C) ⟶ (∃ Q. B A Perp Q B ∧ B A OS C Q)
    using l10-15 by blast
    then obtain D0 where B A Perp D0 B and B A OS C D0
    using Col-cases ⟨¬ Col A B C⟩ col-trivial-3 by blast
    obtain D where Bet B D0 D and Cong D0 D P Q
    using segment-construction by blast
    have ¬ Col B A D0
    using ⟨B A OS C D0⟩ one-side-not-col124 by auto
    have A ≠ B
    using ⟨¬ Col B A D0⟩ col-trivial-1 by blast
    have D0 ≠ B
    using ⟨¬ Col B A D0⟩ col-trivial-3 by auto
    have B ≠ D
    using ⟨Bet B D0 D⟩ ⟨D0 ≠ B⟩ between-identity by blast
    have ¬ Col D B A
    by (metis ⟨B ≠ D⟩ ⟨Bet B D0 D⟩ ⟨¬ Col B A D0⟩
      bet-col col-transitivity-1 not-col-permutation-1)
    have (Col D B D ∧ ¬ Col D B A) ⟶ (∃ Q. D B Perp Q D ∧ D B OS A Q)
    using l10-15 by blast
    then obtain Y0 where D B Perp Y0 D and D B OS A Y0
    using ⟨¬ Col D B A⟩ col-trivial-3 by blast
  }

```

```

have B A Perp B D
  by (metis Perp-perm os-distincts ⟨B A OS C D0⟩
      ⟨B A Perp D0 B⟩ ⟨Bet B D0 D⟩ bet-col1
      bet-neq21--neg l5-1 perp-col1)
have B A Par Y0 D
proof —
  have Coplanar D B B Y0
    using ncop-distincts by auto
  moreover
  have Coplanar D B B D
    using ncop-distincts by blast
  moreover
  have Coplanar D B A Y0
    by (meson ⟨D B OS A Y0⟩ inangle--coplanar ncop--ncols
        ncoplanar-perm-14 ncoplanar-perm-20 one-side-symmetry
        os-ts--inangle two-sides-cases)
  moreover
  have Coplanar D B A D
    using ncop-distincts by blast
  moreover
  have B A Perp D B
    by (simp add: Perp-perm ⟨B A Perp B D⟩)
  moreover
  have Y0 D Perp D B
    using Perp-perm ⟨D B Perp Y0 D⟩ by blast
  ultimately
  show ?thesis
    using l12-9 by blast
qed
have ∃ Y. Col B C Y ∧ Col Y0 D Y
proof —
  have (B A Par Y0 D ∧ Col B A B ∧
      ¬ Col B A C ∧ Coplanar Y0 D B C)
    →
    (∃ Y. Col B C Y ∧ Col Y0 D Y)
    using assms proclus-postulate-def by blast
  moreover
  have Coplanar Y0 D B C
proof —
  have Coplanar A D B Y0
    using ⟨B A Par Y0 D⟩ ncoplanar-perm-13
    par--coplanar by blast
  moreover have Coplanar A D B D
    using ncop-distincts by blast
  moreover have Coplanar A D B B
    using ncop-distincts by blast
  moreover have Coplanar A D B C
proof —
  have Coplanar A B C D0
    using ⟨B A OS C D0⟩ coplanar-perm-6
    os--coplanar by blast
  thus ?thesis
    using ⟨Bet B D0 D⟩ ⟨D0 ≠ B⟩ bet-col
    col-cop--cop coplanar-perm-2 coplanar-perm-5 by blast
qed
ultimately show ?thesis
  using ⟨¬ Col D B A⟩ col-permutation-1
  coplanar-pseudo-trans by blast
qed
moreover
have B A Par Y0 D
  using ⟨B A Par Y0 D⟩ by auto
moreover
have Col B A B
  by (simp add: col-trivial-3)
moreover

```

```

have  $\neg \text{Col } B \ A \ C$ 
  using Col-cases  $\langle \neg \text{Col } A \ B \ C \rangle$  by blast
ultimately
show ?thesis
  using Col-cases  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-3 by blast
qed
then obtain  $Y$  where  $\text{Col } B \ C \ Y$  and  $\text{Col } Y \ 0 \ D \ Y$ 
  by blast
have  $B \ A \ \text{ParStrict } Y \ 0 \ D$ 
  using Col-perm Par-def  $\langle B \ A \ \text{Par } Y \ 0 \ D \rangle$ 
   $\langle D \ B \ \text{OS } A \ Y \ 0 \rangle$  col124--nos by blast
have  $\neg \text{Col } Y \ B \ A$ 
  by (meson  $\langle B \ A \ \text{ParStrict } Y \ 0 \ D \rangle$   $\langle \text{Col } Y \ 0 \ D \ Y \rangle$ 
  col-permutation-2 par-not-col)
have  $\neg \text{Col } A \ B \ Y \longrightarrow (\exists X. \text{Col } A \ B \ X \wedge A \ B \ \text{Perp } Y \ X)$ 
  using l8-18-existence by blast
then obtain  $X$  where  $\text{Col } A \ B \ X$  and  $A \ B \ \text{Perp } Y \ X$ 
  using Col-cases  $\langle \neg \text{Col } Y \ B \ A \rangle$  by blast
have  $B \ A \ \text{OS } C \ D$ 
  by (meson  $\langle B \ A \ \text{OS } C \ D \ 0 \rangle$   $\langle \text{Bet } B \ D \ 0 \ D \rangle$   $\langle D \ 0 \neq B \rangle$ 
  bet-out out-out-one-side)
have  $A \ B \ C \ \text{LtA } A \ B \ D$ 
  by (metis  $\langle A \neq B \rangle$   $\langle \text{Acute } A \ B \ C \rangle$   $\langle B \ A \ \text{Perp } D \ 0 \ B \rangle$ 
   $\langle B \neq D \rangle$   $\langle \text{Bet } B \ D \ 0 \ D \rangle$   $\langle D \ 0 \neq B \rangle$  acute-per--lta
  bet-col per-col perp-per-1)
{
  assume  $\text{Col } B \ C \ D$ 
  {
    assume  $\text{Bet } C \ B \ D$ 
    hence  $B \ A \ \text{TS } C \ D$ 
      using Col-cases  $\langle B \neq D \rangle$   $\langle \neg \text{Col } A \ B \ C \rangle$  bet--ts by blast
    hence False
      using  $\langle B \ A \ \text{OS } C \ D \rangle$  l9-9 by blast
  }
  moreover
  {
    assume  $\neg \text{Bet } C \ B \ D$ 
    hence  $\neg A \ B \ C \ \text{CongA } A \ B \ D$ 
      by (metis  $\langle A \neq B \rangle$   $\langle \text{Acute } A \ B \ C \rangle$   $\langle B \ A \ \text{Perp } D \ 0 \ B \rangle$ 
       $\langle B \neq D \rangle$   $\langle \text{Bet } B \ D \ 0 \ D \rangle$   $\langle D \ 0 \neq B \rangle$  acute-per--lta
      bet-col lta-not-conga per-col perp-per-1)
    moreover
    have  $A \ B \ C \ \text{CongA } A \ B \ D$ 
    proof –
      have  $B \ \text{Out } A \ A$ 
        by (simp add:  $\langle A \neq B \rangle$  out-trivial)
      moreover
      have  $B \ \text{Out } D \ C$ 
        using  $\langle B \ A \ \text{OS } C \ D \rangle$   $\langle \text{Col } B \ C \ D \rangle$ 
        col-one-side-out l6-6 by blast
      ultimately
      show ?thesis
        by (simp add: out2--conga)
    qed
    ultimately
    have False
      by blast
  }
  ultimately
  have False
    by blast
}
hence  $\neg \text{Col } B \ C \ D$ 
  by auto
have  $Y \neq B$ 
  using  $\langle \neg \text{Col } Y \ B \ A \rangle$  col-trivial-1 by blast

```



```

have  $Y \neq X$ 
  using Col-cases  $\langle \text{Col } A \ B \ X \rangle \langle \neg \text{Col } Y \ B \ A \rangle$  by blast
have  $Y0 \neq D$ 
  using  $\langle D \ B \ OS \ A \ Y0 \rangle$  col-trivial-3 one-side-not-col124 by blast
have  $Y \neq D$ 
  using  $\langle \text{Col } B \ C \ Y \rangle \langle \neg \text{Col } B \ C \ D \rangle$  by auto
have  $B \ A \ OS \ C \ D$ 
  using  $\langle B \ A \ OS \ C \ D \rangle$  by auto
have  $B \ D \ ParStrict \ X \ Y$ 
proof –
  have  $B \ D \ Par \ X \ Y$ 
  proof –
    have Coplanar  $B \ A \ B \ X$ 
      using ncop-distincts by blast
    moreover
    have Coplanar  $B \ A \ B \ Y$ 
      using ncop-distincts by auto
    moreover
    have Coplanar  $B \ A \ D \ X$ 
      using NCol-perm  $\langle \text{Col } A \ B \ X \rangle$  ncop--ncols by blast
    moreover
    have Coplanar  $B \ A \ D \ Y$ 
      using  $\langle B \ A \ ParStrict \ Y0 \ D \rangle \langle \text{Col } Y0 \ D \ Y \rangle$ 
       $\langle Y0 \neq D \rangle$  col2-cop--cop col-trivial-2
      pars--coplanar by blast
    moreover
    have  $B \ D \ Perp \ B \ A$ 
      using Perp-perm  $\langle B \ A \ Perp \ B \ D \rangle$  by blast
    moreover
    have  $X \ Y \ Perp \ B \ A$ 
      using Perp-perm  $\langle A \ B \ Perp \ Y \ X \rangle$  by blast
    ultimately
    show ?thesis
      using l12-9 by blast
  qed
  moreover
  have Col  $X \ Y \ Y$ 
    by (simp add: col-trivial-2)
  moreover
  have  $\neg \text{Col } B \ D \ Y$ 
    by (meson  $\langle \text{Col } B \ C \ Y \rangle \langle Y \neq B \rangle \langle \neg \text{Col } B \ C \ D \rangle$ 
      col-transitivity-2 not-col-permutation-2)
  ultimately
  show ?thesis
    using par-not-col-strict by blast
qed
have C InAngle  $A \ B \ D$ 
proof –
  have  $A \ B \ C \ LeA \ A \ B \ D$ 
    by (simp add:  $\langle A \ B \ C \ LtA \ A \ B \ D \rangle$  lta--lea)
  moreover
  have  $A \ B \ OS \ D \ C$ 
    using  $\langle B \ A \ OS \ C \ D \rangle$  invert-one-side
    one-side-symmetry by blast
  ultimately
  show ?thesis
    by (simp add: lea-in-angle)
qed
have  $B \ A \ OS \ D \ Y$ 
  using  $\langle B \ A \ ParStrict \ Y0 \ D \rangle \langle \text{Col } Y0 \ D \ Y \rangle \langle Y \neq D \rangle$ 
  col-trivial-2 l12-6 par-strict-col2-par-strict by presburger
hence  $B \ A \ OS \ C \ Y$ 
  using  $\langle B \ A \ OS \ C \ D \rangle$  one-side-transitivity by blast
have  $B \ Out \ A \ X$ 
proof –
  have Col  $B \ A \ X$ 

```

```

    using Col-perm  $\langle \text{Col } A \ B \ X \rangle$  by blast
  moreover
  have  $B \ D \ OS \ X \ A$ 
  proof -
    have  $B \ D \ OS \ X \ Y$ 
      using  $\langle B \ D \ ParStrict \ X \ Y \rangle$  l12-6 by auto
    moreover
    have  $B \ D \ OS \ Y \ A$ 
      by (meson  $\langle B \ A \ OS \ C \ D \rangle \langle B \ A \ OS \ C \ Y \rangle$ 
         $\langle C \ InAngle \ A \ B \ D \rangle \langle \text{Col } B \ C \ Y \rangle \langle \neg \text{Col } A \ B \ C \rangle$ 
         $\langle \neg \text{Col } B \ C \ D \rangle$  col-one-side-out col-permutation-5
        in-angle-two-sides invert-two-sides
        not-col-permutation-4 one-side-symmetry
        os-ts1324--os out-out-one-side)
    ultimately
    show ?thesis
      using one-side-transitivity by blast
  qed
  hence  $B \ D \ OS \ A \ X$ 
    using one-side-symmetry by blast
  ultimately
  show ?thesis
    using col-one-side-out by blast
  qed
  moreover
  have  $B \ Out \ C \ Y$ 
  proof -
    have  $D \neq Y$ 
      using  $\langle Y \neq D \rangle$  by auto
    moreover
    have  $B \ A \ ParStrict \ D \ Y0$ 
      using  $\langle B \ A \ ParStrict \ Y0 \ D \rangle$  par-strict-right-comm by blast
    moreover
    have  $\text{Col } D \ Y0 \ Y$ 
      using Col-perm  $\langle \text{Col } Y0 \ D \ Y \rangle$  by blast
    ultimately
    show ?thesis
      using  $\langle B \ A \ OS \ C \ Y \rangle \langle \text{Col } B \ C \ Y \rangle$  col-one-side-out by blast
  qed
  moreover
  have  $\neg \text{Col } B \ D \ X$ 
    by (metis  $\langle \text{Col } A \ B \ X \rangle \langle \neg \text{Col } D \ B \ A \rangle$ 
      calculation(1) col-trivial-2 colx
      not-col-permutation-3 not-col-permutation-5 out-diff2)
  hence  $\text{Per } B \ X \ Y$ 
  proof -
    have  $B \neq X$ 
      using  $\langle \neg \text{Col } B \ D \ X \rangle$  col-trivial-3 by auto
    moreover
    have  $B \ A \ Perp \ Y \ X$ 
      using Perp-perm  $\langle A \ B \ Perp \ Y \ X \rangle$  by blast
    moreover
    have  $\text{Col } B \ A \ X$ 
      by (simp add:  $\langle B \ Out \ A \ X \rangle$  out-col)
    ultimately
    show ?thesis
      using Per-perm col-trivial-3 l8-16-1 by blast
  qed
  moreover
  have  $X \neq B$ 
    using  $\langle \neg \text{Col } B \ D \ X \rangle$  col-trivial-3 by auto
  have  $\text{Cong } B \ D \ X \ Y$ 
  proof -
    have  $(\neg \text{Col } B \ Y \ D \wedge Y \ D \ B \ CongA \ B \ X \ Y \wedge$ 
       $B \ Y \ D \ CongA \ Y \ B \ X \wedge \text{Cong } B \ Y \ Y \ B)$ 
       $\longrightarrow$ 

```

```

      (Cong B D Y X ∧ Cong Y D B X ∧ D B Y CongA X Y B)
    using l11-50-2 by blast
  moreover
  have ¬ Col B Y D
  by (metis ‹Col Y0 D Y› ‹D B OS A Y0› ‹Y ≠ D›
      col124--nos col-transitivity-1
      not-col-permutation-2)
  moreover
  have Per Y D B
  proof –
    have D Y0 Perp D B
    using Perp-perm ‹D B Perp Y0 D› by blast
  moreover
  have Col D Y0 Y
  using Col-perm ‹Col Y0 D Y› by blast
  ultimately
  show ?thesis
  using ‹Y ≠ D› perp-col perp-per-2 by presburger
qed
hence Y D B CongA B X Y
  using ‹Per B X Y› ‹B ≠ D› ‹X ≠ B› ‹Y ≠ D›
  ‹Y ≠ X› l11-16 by blast
moreover
have D Y B CongA X B Y
proof –
  have Y B TS X D
  proof –
    have Y B TS A D
    proof –
      have Col C B Y
      using Col-perm ‹Col B C Y› by blast
    moreover
    have Col C B B
    using col-trivial-2 by auto
  moreover
  have C B TS A D
  using Col-cases ‹B A OS C D›
  ‹C InAngle A B D› ‹Col B C D ⇒ False›
  col123--nos in-angle-two-sides by blast
  ultimately
  show ?thesis
  using ‹Y ≠ B› col-preserves-two-sides by blast
qed
moreover
have ¬ Col B Y A ∨ ¬ Col B Y X
  using Col-cases ‹¬ Col Y B A› by blast
hence Y B OS A X
  by (meson ‹B Out A X› ‹Y B TS A D›
      invert-one-side l9-8-1 out-out-one-side)
  ultimately
  show ?thesis
  using l9-8-2 by blast
qed
hence Y B TS D X
  using l9-2 by blast
moreover
have D Y0 Par B A
  using Par-perm ‹B A Par Y0 D› by blast
hence D Y Par B A
  by (metis ‹Col Y0 D Y› ‹Y ≠ D›
      not-col-permutation-4 par-col-par par-symmetry)
hence D Y Par B X
  by (metis ‹Col A B X› ‹X ≠ B›
      col-permutation-4 par-col-par)
hence Y D Par B X
  using Par-perm by blast

```

```

moreover
have alternate-interior-angles-postulate
  by (simp add: assms
    playfair--alternate-interior
    proclus-s-postulate-implies-strong-parallel-postulate
    strong-parallel-postulate-implies-tarski-s-euclid
    tarski-s-euclid-implies-playfair-s-postulate)
ultimately
show ?thesis
  using alternate-interior-angles-postulate-def by blast
qed
hence  $B\ Y\ D\ \text{Cong}\ A\ Y\ B\ X$ 
  by (simp add: conga-comm)
moreover
have  $\text{Cong}\ B\ Y\ Y\ B$ 
  by (simp add: cong-pseudo-reflexivity)
ultimately
show ?thesis
  using not-cong-1243 by blast
qed
have  $P\ Q\ Lt\ X\ Y$ 
proof –
  have  $P\ Q\ Le\ B\ D$ 
    by (meson  $\langle \text{Bet}\ B\ D0\ D \rangle\ \langle \text{Cong}\ D0\ D\ P\ Q \rangle$ 
      bet--le2313 cong--le3412 le-transitivity)
  hence  $P\ Q\ Le\ X\ Y$ 
    using  $\langle \text{Bet}\ B\ D0\ D \rangle\ \langle \text{Cong}\ B\ D\ X\ Y \rangle\ \langle \text{Cong}\ D0\ D\ P\ Q \rangle$ 
    bet--le2313 l5-6 by blast
  moreover
  have  $\neg\ \text{Cong}\ P\ Q\ X\ Y$ 
    by (metis  $\langle \text{Bet}\ B\ D0\ D \rangle\ \langle \text{Cong}\ B\ D\ X\ Y \rangle$ 
       $\langle \text{Cong}\ D0\ D\ P\ Q \rangle\ \langle D0 \neq B \rangle$  bet-le-eq
      cong--le cong-symmetry l5-6)
  ultimately
  show ?thesis
    using Lt-def by blast
qed
ultimately
have  $\exists\ X\ Y.\ B\ \text{Out}\ A\ X \wedge B\ \text{Out}\ C\ Y \wedge \text{Per}\ B\ X\ Y \wedge P\ Q\ Lt\ X\ Y$ 
  by blast
}
thus ?thesis
  using aristotle-s-axiom-def by blast
qed

```

```

lemma aristotle--obtuse-case-elimination:
assumes aristotle-s-axiom
shows  $\neg\ \text{hypothesis-of-obtuse-saccheri-quadrilaterals}$ 
proof –
{
  assume hypothesis-of-obtuse-saccheri-quadrilaterals
  obtain  $Q'\ C'\ P\ Q$  where Lambert  $Q'\ C'\ P\ Q$ 
    using ex-lambert by blast
  have Obtuse  $C'\ P\ Q$ 
    using  $\langle \text{HypothesisObtuseSaccheriQuadrilaterals} \rangle$ 
     $\langle \text{Lambert}\ Q'\ C'\ P\ Q \rangle$  lam-obtuse--oah by blast
  have  $Q'\ Q\ \text{ParStrict}\ C'\ P$ 
    using lam--pars1423  $\langle \text{Lambert}\ Q'\ C'\ P\ Q \rangle$  by force
  have  $\exists\ A'.\ P\ Q\ \text{Perp}\ A'\ P \wedge P\ Q\ OS\ C'\ A'$ 
  proof –
    have  $Col\ P\ Q\ P$ 
      by (simp add: col-trivial-3)
    moreover
    have  $\neg\ Col\ P\ Q\ C'$ 

```

```

    by (meson Par-strict-cases ⟨Q' Q ParStrict C' P⟩
        not-col-distincts not-col-permutation-4 par-not-col)
ultimately
show ?thesis
using l10-15 by force
qed
then obtain A' where P Q Perp A' P and P Q OS C' A'
    by blast
have Q P A' LtA C' P Q
proof -
    have Obtuse C' P Q
        by (simp add: ⟨Obtuse C' P Q⟩)
    moreover
    have Q ≠ P
        using calculation obtuse-distincts by blast
    moreover
    have P ≠ A'
        using ⟨P Q Perp A' P⟩ perp-not-eq-2 by blast
    moreover
    have Per Q P A'
        using ⟨P Q Perp A' P⟩ perp-per-1 by auto
    ultimately
    show ?thesis
        by (simp add: obtuse-per--lta)
qed
have Q P A' LeA C' P Q
    by (simp add: ⟨Q P A' LtA C' P Q⟩ lta--lea)
have ¬ Q P A' CongA C' P Q
    by (simp add: ⟨Q P A' LtA C' P Q⟩ lta-not-conga)
have A' InAngle Q P C'
proof -
    have Q P A' LeA Q P C'
        by (simp add: ⟨Q P A' LeA C' P Q⟩ lea-right-comm)
    moreover
    have Q P OS C' A'
        by (simp add: ⟨P Q OS C' A'⟩ invert-one-side)
    ultimately
    show ?thesis
        by (simp add: lea-in-angle)
qed
obtain C where Bet C' P C and Cong P C C' P
    by (metis between-inner-transitivity
        outer-transitivity-between
        point-construction-different segment-construction-2)
obtain A where Bet A' P A and Cong P A A' P
    by (metis between-inner-transitivity
        outer-transitivity-between point-construction-different
        segment-construction-2)
have C InAngle A P Q
proof -
    {
        assume A = P
        hence A' = A
            using ⟨Cong P A A' P⟩ cong-diff-3 by blast
        hence A' = P
            by (simp add: ⟨A = P⟩)
        hence False
            using ⟨P Q Perp A' P⟩ perp-not-eq-2 by blast
    }
    hence A ≠ P
        by auto
    moreover
    have Bet A' P A
        by (simp add: ⟨Bet A' P A⟩)
    moreover
    have Q InAngle A' P C

```

```

proof -
{
  assume  $C = P$ 
  hence  $C = C'$ 
  using  $\langle \text{Cong } P \ C \ C' \ P \rangle$  cong-diff-4 by blast
  hence  $C' = P$ 
  using  $\langle C = P \rangle$  by blast
  hence False
  using  $\langle \text{Obtuse } C' \ P \ Q \rangle$  obtuse-distincts by blast
}
hence  $C \neq P$ 
  by blast
moreover
have  $A' \text{ InAngle } C' \ P \ Q$ 
  using  $\langle A' \text{ InAngle } Q \ P \ C' \rangle$  l11-24 by blast
ultimately
show ?thesis
  by (meson  $\langle \text{Bet } C' \ P \ C \rangle$  in-angle-reverse l11-24)
qed
ultimately
show ?thesis
  using in-angle-reverse by blast
qed
{
  assume  $\text{Col } P \ C' \ A'$ 
  have  $P \text{ Out } Q \ Q$ 
  using  $\langle C \text{ InAngle } A \ P \ Q \rangle$  inangle-distincts out-trivial by blast
  moreover
  have  $P \text{ Out } C' \ A'$ 
  using  $\langle \text{Col } P \ C' \ A' \rangle \langle P \ Q \text{ OS } C' \ A' \rangle$  col-one-side-out by auto
  ultimately
  have False
  using conga-right-comm by (meson  $\langle \neg Q \ P \ A' \text{ CongA } C' \ P \ Q \rangle$  out2--conga)
}
hence  $\neg \text{Col } P \ C' \ A'$ 
  by blast
have  $\neg \text{Col } P \ C \ A$ 
  by (metis  $\langle \text{Bet } A' \ P \ A \rangle \langle \text{Bet } C' \ P \ C \rangle$ 
 $\langle \text{Col } P \ C' \ A' \implies \text{False} \rangle \langle \text{Cong } P \ A \ A' \ P \rangle \langle \text{Cong } P \ C \ C' \ P \rangle$ 
bet-col bet-col1 col-permutation-4
col-permutation-5 colx cong-diff-4)
have  $\neg \text{Col } P \ Q \ C$ 
  by (metis (no-types, lifting)  $\langle \text{Bet } C' \ P \ C \rangle$ 
 $\langle \text{Cong } P \ C \ C' \ P \rangle \langle P \ Q \text{ OS } C' \ A' \rangle$  bet-col bet-cong-eq
col-permutation-1 col-transitivity-1
one-side-not-col123)
have  $\text{Per } A \ P \ Q$ 
  by (metis Perp-cases  $\langle \text{Bet } A' \ P \ A \rangle \langle P \ Q \text{ Perp } A' \ P \rangle$ 
bet-col bet-col1 l8-20-1-R1 perp-col2 perp-per-2)
have  $A \ P \text{ OS } C \ Q$ 
  by (metis  $\langle \text{Bet } A' \ P \ A \rangle \langle C \text{ InAngle } A \ P \ Q \rangle$ 
 $\langle P \ Q \text{ OS } C' \ A' \rangle \langle \neg \text{Col } P \ C \ A \rangle$  bet-col col124--nos
col-permutation-5 colx in-angle-one-side
not-col-distincts)
have  $\exists X \ Y. P \text{ Out } A \ X \wedge P \text{ Out } C \ Y \wedge \text{Per } P \ X \ Y \wedge P \ Q \text{ Lt } X \ Y$ 
proof -
  have  $\neg \text{Col } A \ P \ C$ 
  using  $\langle \neg \text{Col } P \ C \ A \rangle$  not-col-permutation-2 by blast
  moreover
  have  $A \ P \ C \text{ LtA } A \ P \ Q$ 
proof -
  have  $A \ P \ C \text{ LeA } A \ P \ Q$ 
  by (simp add:  $\langle C \text{ InAngle } A \ P \ Q \rangle$  inangle--lea)
  moreover
  {
    assume  $A \ P \ C \text{ CongA } A \ P \ Q$ 

```

```

have A P OS C Q
  by (simp add: ⟨A P OS C Q⟩)
hence P Out C Q
  using conga-os--out ⟨A P C CongA A P Q⟩ by blast
hence Col P Q C
  by (meson not-col-permutation-5 out-col)
hence False
  using ⟨¬ Col P Q C⟩ by auto
}
hence ¬ A P C CongA A P Q
  by blast
ultimately
show ?thesis
  by (simp add: LtA-def)
qed
hence Acute A P C
  using ⟨Per A P Q⟩ Acute-def by blast
ultimately
show ?thesis
  using aristotle-s-axiom-def assms by blast
qed
then obtain X Y where P Out A X and P Out C Y
  and Per P X Y and P Q Lt X Y
  by blast
have ¬ Col P Q Y
  by (metis Out-def ⟨P Out C Y⟩ ⟨¬ Col P Q C⟩
    colx not-col-distincts out-col)
then obtain Z where Col P Q Z and P Q Perp Y Z
  using l8-18-existence by blast
have X Y Lt P Q
proof -
  have X Y Lt P Z
  proof -
    {
      assume P = Z
      have P ≠ Y
        using ⟨¬ Col P Q Y⟩ not-col-distincts by blast
      moreover
      have Per Q P Y
        using ⟨P = Z⟩ ⟨P Q Perp Y Z⟩ perp-per-1 by auto
      moreover
      have Col P Y C
        using ⟨P Out C Y⟩ l6-6 out-col by blast
      ultimately
      have Per Q P C
        using per-col by blast
      have Col P C A
      proof -
        have Coplanar P Q C A
          by (meson ⟨C InAngle A P Q⟩
            inangle--coplanar ncoplanar-perm-16)
        moreover
        have P C Perp P Q
          using ⟨Per Q P C⟩
          by (metis Perp-perm os-distincts out-distinct
            ⟨P Out C Y⟩ ⟨P Q OS C' A'⟩ per-perp)
        moreover
        have P A Perp P Q
          by (metis Perp-cases bet-col1 l5-1
            ⟨Bet A' P A⟩ ⟨Cong P A A' P⟩ ⟨P Q Perp A' P⟩
            bet-cong-eq perp-col2 perp-distinct)
        ultimately
        show ?thesis
          using cop-perp2--col by blast
      qed
    }
  qed
hence False

```

```

    using  $\langle \neg \text{Col } P \ C \ A \rangle$  by auto
  }
hence  $P \neq Z$ 
  by blast
have  $\text{Lambert } P \ X \ Y \ Z$ 
proof -
  have  $P \neq X$ 
    using  $\langle P \ \text{Out } A \ X \rangle$  out-diff2 by blast
  moreover
  have  $X \neq Y$ 
    using  $\langle P \ Q \ Lt \ X \ Y \rangle$  lt-diff by blast
  moreover
  have  $Y \neq Z$ 
    using  $\langle \text{Col } P \ Q \ Z \rangle \langle \neg \text{Col } P \ Q \ Y \rangle$  by blast
  moreover
  have  $P \neq Z$ 
    using  $\langle P \neq Z \rangle$  by blast
  moreover
  have  $\text{Per } X \ P \ Z$ 
proof -
  have  $P \neq Q$ 
    using  $\langle \neg \text{Col } P \ Q \ Y \rangle$  col-trivial-1 by force
  moreover
  have  $\text{Per } X \ P \ Q$ 
    by (metis out-distinct  $\langle P \ \text{Out } A \ X \rangle \langle \text{Per } A \ P \ Q \rangle$  l8-3 out-col)
  ultimately
  show ?thesis
    using  $\langle \text{Col } P \ Q \ Z \rangle$  per-col by blast
qed
moreover
have  $\text{Per } P \ Z \ Y$ 
proof -
  have  $P \neq Z$ 
    using calculation(4) by blast
  moreover
  have  $P \ Q \ \text{Perp } Y \ Z$ 
    using  $\langle P \ Q \ \text{Perp } Y \ Z \rangle$  by auto
  ultimately
  show ?thesis
    by (meson  $\langle \text{Col } P \ Q \ Z \rangle$  perp-col perp-comm perp-per-2)
qed
moreover
have  $Y \ \text{InAngle } X \ P \ Q$ 
proof -
  have  $P \ \text{Out } X \ A$ 
    using Out-cases  $\langle P \ \text{Out } A \ X \rangle$  by blast
  moreover
  have  $P \ \text{Out } Q \ Q$ 
    using  $\langle A' \ \text{InAngle } Q \ P \ C' \rangle$  inangle-distincts out-trivial by blast
  moreover
  have  $P \ \text{Out } Y \ C$ 
    by (simp add:  $\langle P \ \text{Out } C \ Y \rangle$  l6-6)
  ultimately
  show ?thesis
    using  $\langle C \ \text{InAngle } A \ P \ Q \rangle$  l11-25 by blast
qed
hence  $\text{Coplanar } P \ X \ Y \ Z$ 
  using  $\langle \text{Col } P \ Q \ Z \rangle \langle \neg \text{Col } P \ Q \ Y \rangle$  col-cop--cop
    inangle--coplanar ncoplanar-perm-14
    not-col-distincts by blast
ultimately
show ?thesis
  using Lambert-def  $\langle \text{Per } P \ X \ Y \rangle$  by blast
qed
hence  $\text{Obtuse } X \ Y \ Z$ 
  using  $\langle \text{HypothesisObtuseSaccheriQuadrilaterals} \rangle$ 

```



```

    lam-obtuse--oah-2 by blast
  thus ?thesis
    by (simp add: ⟨Lambert P X Y Z⟩ lam-obtuse--lt)
qed
moreover
have P Z Lt P Q
proof -
  have Q Out P Z
  proof -
    have Col Q P Z
    using Col-cases ⟨Col P Q Z⟩ by auto
  moreover
  have Q Q' OS P Z
  proof -
    have Q Q' ParStrict P Y
    by (metis ⟨Bet C' P C⟩ ⟨P Out C Y⟩
      ⟨Q' Q ParStrict C' P⟩ ⟨¬ Col P Q C⟩
      bet-col not-col-distincts out-col
      out-diff2 par-strict-col2-par-strict
      par-strict-left-comm)
  hence Q Q' OS P Y
  using l12-6 by blast
  moreover
  have Q Q' OS Y Z
  proof -
    have Q Q' Par Y Z
    proof -
      have Coplanar P Q Q Y
      using ncop-distincts by blast
    moreover
    have Coplanar P Q Q Z
    using ncop-distincts by blast
  moreover
  have Coplanar P Q Q' Y
  by (meson ⟨Q Q' OS P Y⟩ cop2-os--osp
    ncop-distincts one-side-symmetry osp-distincts)
  moreover
  have Coplanar P Q Q' Z
  using ⟨Col P Q Z⟩ ncop--ncols by blast
  moreover
  have Lambert Q' C' P Q
  by (simp add: ⟨Lambert Q' C' P Q⟩)
  hence Per C' Q' Q ∧ Per Q' Q P ∧ Per Q' C' P
  using Lambert-def by blast
  hence Q Q' Perp P Q
  by (metis os-distincts ⟨Lambert Q' C' P Q⟩
    ⟨P Q OS C' A'⟩ lam--os per-perp perp-comm)
  moreover
  have Y Z Perp P Q
  by (meson Perp-perm ⟨P Q Perp Y Z⟩)
  ultimately
  show ?thesis
  using l12-9 by blast
qed
moreover
have Col Y Z Y
  by (simp add: col-trivial-3)
moreover
have ¬ Col Q Q' Y
proof -
  have P C' ParStrict Q Q'
  using Par-strict-perm ⟨Q' Q ParStrict C' P⟩ by blast
  moreover
  have Col Y P C
  using ⟨P Out C Y⟩ not-col-permutation-1 out-col by blast
  ultimately

```

```

    show ?thesis
    using ⟨Q Q' OS P Y⟩ one-side-not-col124 by blast
qed
ultimately
show ?thesis
  by (meson l12-6 par-not-col-strict)
qed
ultimately
show ?thesis
  using one-side-transitivity by blast
qed
ultimately
show ?thesis
  using col-one-side-out by blast
qed
have P Out Z Q
proof -
  have Col P Z Q
    using NCol-perm ⟨Col P Q Z⟩ by blast
  moreover
  have P A OS Z Q
  proof -
    have P A OS Z Y
    proof -
      have P A Par Z Y
    proof -
      have Coplanar P Q P Z
        using ncop-distincts by blast
      moreover
      have Coplanar P Q P Y
        using ncop-distincts by blast
      moreover
      have Coplanar P Q A Z
        using ⟨Col P Q Z⟩ ncop--ncols by blast
      moreover
      have Coplanar P Q A Y
    proof -
      have Coplanar Q A P C
        by (meson ⟨C InAngle A P Q⟩ inangle--coplanar ncoplanar-perm-21)
      moreover
      have P ≠ C
        using ⟨¬ Col P C A⟩ not-col-distincts by blast
      ultimately
      show ?thesis
        by (meson ⟨P Out C Y⟩ col-cop--cop
          ncoplanar-perm-8 out-col)
    qed
  qed
  moreover
  have P A Perp P Q
    by (metis Perp-cases ⟨Bet A' P A⟩
      ⟨Cong P A A' P⟩ ⟨P Q Perp A' P⟩ bet-col
      bet-col1 cong-diff-3 perp-col2)
  moreover
  have Z Y Perp P Q
    using Perp-perm ⟨P Q Perp Y Z⟩ by blast
  ultimately
  show ?thesis
    using l12-9 by blast
qed
moreover
have Col Z Y Y
  using col-trivial-2 by auto
moreover
have ¬ Col P A Y
  by (metis Out-def ⟨P Out C Y⟩ ⟨¬ Col P C A⟩
    colx not-col-distincts

```

```

      not-col-permutation-5 out-col)
    ultimately
    show ?thesis
      using l12-6 par-not-col-strict by blast
  qed
  moreover
  have P A OS Y Q
    by (meson ⟨A P OS C Q⟩ ⟨P Out C Y⟩
        invert-one-side not-col-distincts os-out-os)
  ultimately
  show ?thesis
    using one-side-transitivity by blast
  qed
  ultimately
  show ?thesis
    using col-one-side-out by blast
  qed
  hence Bet P Z Q
    using ⟨Q Out P Z⟩ out2--bet by blast
  thus ?thesis
    by (metis ⟨P Q Lt X Y⟩ bet--lt1213 calculation lt-transitivity)
  qed
  ultimately
  show ?thesis
    by (meson lt-transitivity)
  qed
  hence False
    using ⟨P Q Lt X Y⟩ not-and-lt by blast
}
thus ?thesis
  by auto
qed

```

```

lemma aristotle--acute-or-right:
  assumes aristotle-s-axiom
  shows hypothesis-of-acute-saccheri-quadrilaterals
    ∨
    hypothesis-of-right-saccheri-quadrilaterals
proof -
{
  assume hypothesis-of-obtuse-saccheri-quadrilaterals
  have hypothesis-of-acute-saccheri-quadrilaterals ∨
    hypothesis-of-right-saccheri-quadrilaterals
    using aristotle--obtuse-case-elimination assms
    saccheri-s-three-hypotheses by blast
}
thus ?thesis
  using saccheri-s-three-hypotheses
    aristotle--obtuse-case-elimination assms by blast
qed

```

```

lemma Axiom1ProofIsabelleHOL:
  shows Axiom1
  using Axiom1-def by blast

```

```

theorem equivalent-postulates-without-decidability-of-intersection-of-lines:
  shows (Postulate07 ⟷ Postulate12) ∧
    (Postulate12 ⟷ Postulate08) ∧
    (Postulate08 ⟷ Postulate06) ∧
    (Postulate06 ⟷ Postulate09) ∧
    (Postulate09 ⟷ Postulate02) ∧
    (Postulate02 ⟷ Postulate11) ∧
    (Postulate11 ⟷ Postulate10) ∧
    (Postulate10 ⟷ Postulate05)
proof -

```

```

have Postulate09  $\longleftrightarrow$  Postulate10
proof -
  have Postulate09  $\longrightarrow$  Postulate10
    using Postulate09-def Postulate10-def
      par-perp-perp-implies-par-perp-2-par by auto
  moreover
  have Postulate10  $\longrightarrow$  Postulate09
    using Postulate09-def Postulate10-def
      par-perp-2-par-implies-par-perp-perp by blast
  ultimately
  show ?thesis
    by blast
qed
moreover
have Postulate09  $\longrightarrow$  Postulate02
  by (simp add: Postulate02-def Postulate09-def
    par-perp-perp-implies-playfair)
moreover
have Postulate02  $\longrightarrow$  Postulate11
  using Postulate02-def Postulate11-def
    playfair--universal-posidonius-postulate by blast
moreover
have Postulate11  $\longrightarrow$  Postulate09
  by (simp add: Postulate09-def Postulate11-def
    universal-posidonius-postulate--perpendicular-transversal-postulate)
moreover
have Postulate02  $\longleftrightarrow$  Postulate05
proof -
  have Postulate02  $\longrightarrow$  Postulate05
    using Postulate02-def Postulate05-def
      playfair-implies-par-trans by auto
  moreover
  have Postulate05  $\longrightarrow$  Postulate02
    using Postulate02-def Postulate05-def
      par-trans-implies-playfair by blast
  ultimately
  show ?thesis
    by blast
qed
moreover
have Postulate02  $\longrightarrow$  Postulate07
  using Postulate07-def Postulate11-def calculation(3)
    par-perp-perp-implies-playfair playfair--alternate-interior
    universal-posidonius-postulate--perpendicular-transversal-postulate by blast
moreover
have Postulate07  $\longrightarrow$  Postulate12
  using Postulate07-def Postulate12-def
    alternate-interior--playfair-bis by blast
moreover
have Postulate12  $\longrightarrow$  Postulate02
  using Postulate11-def Postulate12-def calculation(2)
    calculation(4) playfair--universal-posidonius-postulate
    playfair-bis--playfair by blast
moreover
have Postulate07  $\longleftrightarrow$  Postulate08
proof -
  have Postulate07  $\longrightarrow$  Postulate08
    using Postulate07-def Postulate08-def
      alternate-interior--consecutive-interior by auto
  moreover
  have Postulate08  $\longrightarrow$  Postulate07
    using Postulate07-def Postulate08-def
      consecutive-interior--alternate-interior by blast
  ultimately
  show ?thesis
    by blast

```

```

qed
moreover
have Postulate02  $\longleftrightarrow$  Postulate06
proof -
  have Postulate02  $\longrightarrow$  Postulate06
    using Postulate02-def Postulate06-def
    playfair-s-postulate-implies-midpoint-converse-postulate by blast
  moreover
  have Postulate06  $\longrightarrow$  Postulate02
    using Postulate02-def Postulate06-def
    midpoint-converse-postulate-implies-playfair by auto
  ultimately
  show ?thesis
    by blast
qed
ultimately
show ?thesis
  by blast
qed

```

```

lemma Cycle-1:
  shows (Postulate01  $\longrightarrow$  Postulate02)  $\wedge$ 
        (Postulate09  $\longrightarrow$  Postulate15)  $\wedge$ 
        (Postulate15  $\longrightarrow$  Postulate01)
proof -
  have Postulate01  $\longrightarrow$  Postulate02
    by (simp add: Postulate01-def Postulate02-def
        tarski-s-euclid-implies-playfair-s-postulate)
  moreover
  have Postulate09  $\longrightarrow$  Postulate15
    using inter-dec-plus-par-perp-perp-imp-ly-triangle-circumscription
        Postulate09-def Postulate15-def by fastforce
  moreover
  have Postulate15  $\longrightarrow$  Postulate01
    by (simp add: Postulate01-def Postulate15-def
        triangle-circumscription-implies-tarski-s-euclid)
  ultimately
  show ?thesis
    by auto
qed

```

```

lemma Cycle-2:
  shows (Postulate01  $\longleftrightarrow$  Postulate13)  $\wedge$ 
        (Postulate13  $\longleftrightarrow$  Postulate14)  $\wedge$ 
        (Postulate14  $\longleftrightarrow$  Postulate16)  $\wedge$ 
        (Postulate16  $\longleftrightarrow$  Postulate17)  $\wedge$ 
        (Postulate17  $\longleftrightarrow$  Postulate18)  $\wedge$ 
        (Postulate18  $\longleftrightarrow$  Postulate19)  $\wedge$ 
        (Postulate19  $\longleftrightarrow$  Postulate20)  $\wedge$ 
        (Postulate20  $\longleftrightarrow$  Postulate01)
proof -
  have Postulate01  $\longrightarrow$  Postulate17
    using Postulate01-def Postulate17-def
    tarski-s-euclid-implies-euclid-5 by blast
  moreover
  have Postulate17  $\longrightarrow$  Postulate20
    by (simp add: Postulate17-def Postulate20-def
        euclid-5--original-euclid)
  moreover
  have Postulate20  $\longrightarrow$  Postulate19
    using Postulate19-def Postulate20-def
    original-euclid--original-spp by blast
  moreover
  have Postulate19  $\longrightarrow$  Postulate16
    by (simp add: Postulate16-def Postulate19-def
        original-spp--inverse-projection-postulate)

```

```

moreover
have Postulate16  $\longrightarrow$  Postulate14
  using Postulate14-def Postulate16-def
    inverse-projection-postulate--proclus-bis by blast
moreover
have Postulate14  $\longrightarrow$  Postulate13
  by (simp add: Postulate13-def Postulate14-def
    proclus-bis--proclus)
moreover
have Postulate13  $\longrightarrow$  Postulate18
  by (simp add: Postulate13-def Postulate18-def
    proclus-s-postulate-implies-strong-parallel-postulate)
moreover
have Postulate18  $\longrightarrow$  Postulate01
  using Postulate01-def Postulate18-def
    strong-parallel-postulate-implies-tarski-s-euclid by blast
ultimately
show ?thesis
  by auto
qed

```

lemma *Cycle-3:*

```

shows (Postulate03  $\longleftrightarrow$  Postulate21)  $\wedge$ 
  (Postulate21  $\longleftrightarrow$  Postulate22)  $\wedge$ 
  (Postulate22  $\longleftrightarrow$  Postulate23)  $\wedge$ 
  (Postulate23  $\longleftrightarrow$  Postulate24)  $\wedge$ 
  (Postulate24  $\longleftrightarrow$  Postulate25)  $\wedge$ 
  (Postulate25  $\longleftrightarrow$  Postulate26)  $\wedge$ 
  (Postulate26  $\longleftrightarrow$  Postulate27)  $\wedge$ 
  (Postulate27  $\longleftrightarrow$  Postulate28)  $\wedge$ 
  (Postulate28  $\longleftrightarrow$  Postulate29)  $\wedge$ 
  (Postulate29  $\longleftrightarrow$  Postulate30)  $\wedge$ 
  (Postulate30  $\longleftrightarrow$  Postulate03)
proof –
  have Postulate03  $\longrightarrow$  Postulate21
    using Postulate03-def Postulate21-def
      triangle--existential-triangle by blast
  moreover
  have Postulate21  $\longrightarrow$  Postulate27
    using Postulate21-def Postulate27-def
      existential-triangle--rah by fastforce
  moreover
  have Postulate27  $\longrightarrow$  Postulate03
    by (simp add: Postulate03-def
      Postulate27-def rah--triangle)
  moreover
  have Postulate27  $\longleftrightarrow$  Postulate28
proof –
  have Postulate27  $\longrightarrow$  Postulate28
    by (simp add: Postulate27-def Postulate28-def
      rah--existential-saccheri)
  moreover
  have Postulate28  $\longrightarrow$  Postulate27
    by (simp add: Postulate27-def Postulate28-def
      existential-saccheri--rah)
  ultimately show ?thesis
    by auto
qed
  moreover
  have Postulate27  $\longrightarrow$  Postulate29
    using Postulate27-def Postulate29-def
      rah--rectangle-principle by fastforce
  moreover
  have Postulate29  $\longrightarrow$  Postulate30
    using Postulate29-def Postulate30-def
      rectangle-principle--rectangle-existence by blast

```

```

moreover
have Postulate30  $\longrightarrow$  Postulate27
  by (simp add: Postulate27-def Postulate30-def
    rectangle-existence--rah)
moreover
have Postulate27  $\longleftrightarrow$  Postulate22
proof –
  have Postulate27  $\longrightarrow$  Postulate22
    using Postulate22-def Postulate27-def
      rah--posidonius by blast
  moreover
  have Postulate22  $\longrightarrow$  Postulate27
    using Postulate22-def Postulate27-def
      posidonius-postulate--rah by blast
  ultimately
  show ?thesis
    by auto
qed
moreover
have Postulate27  $\longrightarrow$  Postulate24
  using Postulate24-def Postulate27-def
    rah--thales-postulate by presburger
moreover
have Postulate24  $\longrightarrow$  Postulate25
  by (simp add: Postulate24-def Postulate25-def
    thales-postulate--thales-converse-postulate)
moreover
have Postulate25  $\longrightarrow$  Postulate26
  using Postulate25-def Postulate26-def
    thales-converse-postulate--thales-existence by fastforce
moreover
have Postulate26  $\longrightarrow$  Postulate27
  using Postulate26-def Postulate30-def calculation(7)
    rah--rectangle-principle
    rectangle-principle--rectangle-existence thales-existence--rah by blast
moreover
have Postulate27  $\longleftrightarrow$  Postulate23
proof –
  have Postulate27  $\longrightarrow$  Postulate23
    using Postulate23-def Postulate26-def calculation(10)
      calculation(11) calculation(9) rah--similar
      thales-existence--rah by blast
  moreover
  have Postulate23  $\longrightarrow$  Postulate27
    using Postulate22-def Postulate23-def
       $\langle \textit{Postulate27} = \textit{Postulate22} \rangle$  rah--posidonius
      similar--rah by blast
  ultimately
  show ?thesis
    by auto
qed
ultimately
show ?thesis by blast
qed

```

```

lemma InterCycle1:
assumes greenberg-s-axiom and Postulate03
shows Postulate12
using greenberg-s-axiom-def Postulate03-def Postulate12-def
  assms(1) assms(2) triangle--playfair-bis by blast

```

```

lemma InterCycle2:
assumes Postulate07
shows Postulate03
using Postulate03-def Postulate07-def
  alternate-interior--triangle assms by blast

```

```

lemma InterCycle3:
  assumes Postulate01
  shows Postulate02
  using Postulate01-def Postulate02-def assms
    tarski-s-euclid-implies-playfair-s-postulate by blast

lemma InterCycle4:
  assumes Postulate09
  shows Postulate15
  using Postulate09-def Postulate15-def assms
    inter-dec-plus-par-perp-perp-impl-triangle-circumscription
  by auto

lemma InterAx1-R1:
  assumes Postulate13
  shows aristotle-s-axiom
  using proclus--aristotle aristotle-s-axiom-def
    Postulate13-def assms by blast

lemma InterAx1:
  assumes Postulate01
  shows aristotle-s-axiom
  using InterAx1-R1 Postulate01-def Postulate13-def
    assms euclid-5--original-euclid
    inverse-projection-postulate--proclus-bis
    original-euclid--original-spp original-spp--inverse-projection-postulate
    proclus-bis--proclus
    tarski-s-euclid-implies-euclid-5 by blast

lemma InterAx3:
  assumes greenberg-s-axiom and Postulate02
  shows Postulate01
proof -
  have Postulate03
  using Postulate02-def Postulate03-def alternate-interior--triangle
    assms(2) playfair--alternate-interior by blast
  hence Postulate12
  using InterCycle1 assms(1) by auto
  thus ?thesis
  using greenberg-s-axiom-def Postulate01-def Postulate07-def
    alternate-interior--proclus assms(1)
    equivalent-postulates-without-decidability-of-intersection-of-lines
    proclus-s-postulate-implies-strong-parallel-postulate
    strong-parallel-postulate-implies-tarski-s-euclid by blast
qed

lemma InterAx4:
  assumes Postulate01
  shows Axiom1
  by (simp add: Axiom1ProofIsabelleHOL)

lemma InterAx5:
  assumes Postulate02
  shows Postulate01
proof -
  have Postulate15
  using InterCycle4 Postulate02-def Postulate09-def
    assms playfair--universal-posidonius-postulate
    universal-posidonius-postulate--perpendicular-transversal-postulate
  by blast
  thus ?thesis
  using Postulate01-def Postulate15-def
    triangle-circumscription-implies-tarski-s-euclid by blast
qed

```



```

lemma PPR-Theorem-4-bis:
  assumes Postulate01
  shows greenberg-s-axiom
  using aristotle-s-axiom-def greenberg-s-axiom-def InterAx1 InterAx5
    aristotle--greenberg assms by blast

lemma PPR-Proposition-6:
  assumes archimedes-axiom
  shows aristotle-s-axiom
  using t22-24 archimedes-axiom-def aristotle-s-axiom-def assms by blast

lemma InterCycle1bis:
  assumes Postulate01
  shows Postulate03  $\longrightarrow$  Postulate12
  using Cycle-1 assms
    equivalent-postulates-without-decidability-of-intersection-of-lines by blast

lemma weak-inverse-projection-postulate--weak-tarski-s-parallel-postulate:
  assumes weak-inverse-projection-postulate
  shows weak-tarski-s-parallel-postulate
proof -
  {
    fix A B C P T
    assume Per A B C and
      T InAngle A B C and
      P  $\neq$  T and
      P B A CongA P B C and
      Per B P T and
      Coplanar A B C P
    have B  $\neq$  P
      using  $\langle P B A \text{ CongA } P B C \rangle$  conga-diff45 by blast
    have B  $\neq$  A
      using  $\langle P B A \text{ CongA } P B C \rangle$  conga-diff2 by blast
    have B  $\neq$  C
      using  $\langle T \text{ InAngle } A B C \rangle$  inangle-distincts by blast
    have P InAngle A B C
      by (meson conga-inangle-per2--inangle
         $\langle \text{Coplanar } A B C P \rangle \langle P B A \text{ CongA } P B C \rangle \langle \text{Per } A B C \rangle$ 
         $\langle \text{Per } B P T \rangle \langle T \text{ InAngle } A B C \rangle$ )
    have P B A P B A SumA A B C
    proof -
      have A B P P B C SumA A B C
        by (simp add: inangle--suma  $\langle P \text{ InAngle } A B C \rangle$ )
      moreover have A B P CongA P B A
        using  $\langle B \neq A \rangle \langle B \neq P \rangle$  conga-pseudo-refl by auto
      moreover have P B C CongA P B A
        using  $\langle P B A \text{ CongA } P B C \rangle$  conga-sym-equiv by force
      moreover have A B C CongA A B C
        using  $\langle B \neq A \rangle \langle B \neq C \rangle$  conga-refl by auto
      ultimately show ?thesis
        using conga3-suma--suma by blast
    qed
    have Acute P B A
      by (meson conga-inangle-per--acute  $\langle P B A \text{ CongA } P B C \rangle$ 
         $\langle P \text{ InAngle } A B C \rangle \langle \text{Per } A B C \rangle$  acute-sym)
    have B Out P P
      using  $\langle B \neq P \rangle$  out-trivial by auto
    {
      assume Col A B C
      hence  $\neg \text{Per } A B C$ 
        using  $\langle B \neq A \rangle \langle B \neq C \rangle$  per-not-col by force
      hence False
        using  $\langle \text{Per } A B C \rangle$  by blast
    }
  }
  hence  $\neg \text{Col } A B C$ 
    by blast

```

```

have Coplanar P B A T
  by (metis Col-cases ⟨Coplanar A B C P⟩ ⟨T InAngle A B C⟩ ⟨¬ Col A B C⟩
    coplanar-perm-12 coplanar-perm-21 coplanar-trans-1 inangle--coplanar)
hence ∃ X. B Out A X ∧ Col P T X
  using ⟨Acute P B A⟩ ⟨Per A B C⟩ ⟨P B A P B A SumA A B C⟩ ⟨B Out P P⟩
    ⟨P ≠ T⟩ ⟨Per B P T⟩
    assms weak-inverse-projection-postulate-def
  by blast
then obtain X where B Out A X and Col P T X
  by blast
have ∃ Y. B Out C Y ∧ Col P T Y
proof -
  have Acute P B C
    using ⟨Acute P B A⟩ ⟨P B A CongA P B C⟩ acute-conga--acute by blast
  moreover have Per C B A
    using Per-perm ⟨Per A B C⟩ by blast
  moreover have P B C P B C SumA C B A
    by (metis Tarski-neutral-dimensionless.conga3-suma--suma
      Tarski-neutral-dimensionless-axioms ⟨B ≠ A⟩ ⟨B ≠ C⟩
      ⟨P B A CongA P B C⟩ ⟨P B A P B A SumA A B C⟩ conga-pseudo-refl)
  moreover have Coplanar P B C T
    by (meson ⟨Col A B C ⟹ False⟩ ⟨Coplanar A B C P⟩
      ⟨T InAngle A B C⟩ coplanar-pseudo-trans inangle--coplanar
      ncop-distincts ncoplanar-perm-18)
  ultimately show ?thesis
    using ⟨B Out P P⟩ ⟨P ≠ T⟩ ⟨Per B P T⟩
      assms weak-inverse-projection-postulate-def
    by blast
qed
then obtain Y where B Out C Y and Col P T Y
  by blast
hence ∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y)
  using ⟨B Out A X⟩ ⟨Col P T X⟩ not-col-permutation-4 by blast
}
moreover
{
  assume LEMME: ∀ A B C P T. Per A B C ∧ T InAngle A B C ∧
    P ≠ T ∧ P B A CongA P B C ∧ Per B P T ∧
    Coplanar A B C P
    ⟹
    (∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y))
  {
    fix A B C T
    assume Per A B C and T InAngle A B C
    obtain P0 where P0 InAngle A B C and P0 B A CongA P0 B C
      using ⟨T InAngle A B C⟩ angle-bisector inangle-distincts by blast
    have A ≠ B
      using ⟨P0 B A CongA P0 B C⟩ conga-diff2 by auto
    have C ≠ B
      using ⟨T InAngle A B C⟩ inangle-distincts by blast
    have T ≠ B
      using ⟨T InAngle A B C⟩ inangle-distincts by blast
    have P0 ≠ B
      using ⟨P0 B A CongA P0 B C⟩ conga-diff45 by blast
    {
      assume Col A B C
      hence ¬ Per A B C
        using ⟨A ≠ B⟩ ⟨C ≠ B⟩ per-not-col by force
      hence False
        by (simp add: ⟨Per A B C⟩)
    }
    hence ¬ Col A B C
      by blast
    have ¬ Col P0 B A
    proof -
      have P0 B A P0 B A SumA A B C

```

```

proof -
  have A B P0 P0 B C SumA A B C
    using ⟨P0 InAngle A B C⟩ inangle--suma by force
  moreover have A B P0 CongA P0 B A
    using ⟨A ≠ B⟩ ⟨P0 ≠ B⟩ conga-pseudo-refl by auto
  moreover have P0 B C CongA P0 B A
    using ⟨P0 B A CongA P0 B C⟩ not-conga-sym by blast
  moreover have A B C CongA A B C
    by (simp add: ⟨A ≠ B⟩ ⟨C ≠ B⟩ conga-refl)
  ultimately show ?thesis
    using conga3-suma--suma by blast
qed
thus ?thesis
  using ⟨¬ Col A B C⟩ col2-suma--col by blast
qed
{
  assume Col B P0 T
  have ∃ T'. T' InAngle A B C ∧ B T Perp T T'
  proof -
    obtain T0 where B P0 Perp T0 T and B P0 OS A T0
      using Col-cases ⟨Col B P0 T⟩ ⟨¬ Col P0 B A⟩ l10-15 by blast
    have ∃ T'. T' InAngle A B C ∧ T ≠ T' ∧ Col T T0 T'
    proof -
      have ¬ B Out A C
        using Col-cases ⟨¬ Col A B C⟩ out-col by blast
      moreover have T InAngle A B C
        by (simp add: ⟨T InAngle A B C⟩)
      moreover have Coplanar A B C T0
      proof -
        have Coplanar P0 B A A
          using ncop-distincts by auto
        moreover have Coplanar P0 B A B
          using ncop-distincts by auto
        moreover have Coplanar P0 B A C
          using ⟨P0 InAngle A B C⟩ inangle--coplanar ncoplanar-perm-2 by blast
        moreover have Coplanar P0 B A T0
          by (simp add: ⟨B P0 OS A T0⟩ invert-one-side os--coplanar)
        ultimately show ?thesis
          using ⟨¬ Col P0 B A⟩ coplanar-perm-6 coplanar-trans-1 by blast
      qed
      ultimately show ?thesis
        using cop-inangle--ex-col-inangle by blast
    qed
    then obtain T' where T' InAngle A B C and
      T ≠ T' and Col T T0 T' by blast
    have B P0 Perp T T'
      by (metis ⟨B P0 Perp T0 T⟩ ⟨Col T T0 T'⟩ ⟨T ≠ T'⟩
        not-col-distincts perp-col2-bis)
    hence B T Perp T T'
      by (metis perp-col ⟨Col B P0 T⟩ ⟨T ≠ B⟩)
    thus ?thesis
      using ⟨T' InAngle A B C⟩ by blast
  qed
  then obtain T' where T' InAngle A B C and B T Perp T T'
    by blast
  have T ≠ T'
    using ⟨B T Perp T T'⟩ perp-not-eq-2 by auto
  have ∃ X Y. B Out A X ∧ Col T' T X ∧ B Out C Y ∧ Col T' T Y
  proof -
    have T B A CongA T B C
      using ⟨T ≠ B⟩ ⟨P0 B A CongA P0 B C⟩ ⟨Col B P0 T⟩ col-conga--conga by blast
    moreover have Per B T T'
      using Perp-cases ⟨B T Perp T T'⟩ perp-per-1 by blast
    moreover have Coplanar A B C T
      using ⟨T InAngle A B C⟩ coplanar-perm-9 inangle--coplanar by blast
    ultimately show ?thesis

```

```

    using LEMME ⟨Per A B C⟩ ⟨T' InAngle A B C⟩ ⟨T ≠ T'⟩ by blast
qed
then obtain X Y where B Out A X and Col T' T X and B Out C Y and Col T' T Y
by blast
hence ∃ X Y. B Out A X ∧ B Out C Y ∧ Col X T Y
by (metis Col-cases ⟨T ≠ T'⟩ col-transitivity-2)
}
moreover
{
  assume ¬ Col B P0 T
  obtain P where Col B P0 P and B P0 Perp T P
  using ⟨¬ Col B P0 T⟩ l8-18-existence by blast
  have B Out P P0
  proof -
    have Acute T B P0
    using ⟨Per A B C⟩ ⟨T InAngle A B C⟩ ⟨P0 B A CongA P0 B C⟩
    ⟨P0 InAngle A B C⟩ acute-sym conga-inangle2-per--acute by blast
    thus ?thesis
    by (meson ⟨B P0 Perp T P⟩ ⟨Col B P0 P⟩ acute-col-perp--out)
  qed
  have P ≠ T
  using ⟨Col B P0 P⟩ ⟨¬ Col B P0 T⟩ by blast
  have P ≠ B
  using ⟨B Out P P0⟩ l6-3-1 by auto
  have ∃ X Y. B Out A X ∧ B Out C Y ∧ Col X T Y
  proof -
    have ∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y)
    proof -
      have P B A CongA P B C
      by (metis col-conga--conga ⟨Col B P0 P⟩ ⟨P ≠ B⟩ ⟨P0 B A CongA P0 B C⟩)
      moreover have Per B P T
      proof -
        have B ≠ P
        using ⟨P ≠ B⟩ by auto
        moreover have B P0 Perp T P
        by (simp add: ⟨B P0 Perp T P⟩)
        moreover have Col B P0 P
        by (simp add: ⟨Col B P0 P⟩)
        ultimately show ?thesis
        using col-trivial-3 l8-16-1 l8-2 by blast
      qed
      moreover have Coplanar A B C P
      using ⟨Col B P0 P⟩ ⟨P0 InAngle A B C⟩ ⟨P0 ≠ B⟩ col-cop--cop
      coplanar-perm-11 inangle--coplanar ncoplanar-perm-2 by blast
      ultimately show ?thesis
      using ⟨Per A B C⟩ ⟨T InAngle A B C⟩ ⟨P ≠ T⟩ LEMME by blast
    qed
    then obtain X Y where B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y by blast
    thus ?thesis
    using ⟨P ≠ T⟩ col2--eq col-permutation-2 by blast
  qed
}
ultimately
have ∃ X Y. B Out A X ∧ B Out C Y ∧ Col X T Y
by blast
then obtain X Y where B Out A X and B Out C Y and Col X T Y
by blast
have Y ≠ B
using Out-def ⟨B Out C Y⟩ by auto
have X ≠ B
using Out-def ⟨B Out A X⟩ by auto
have ¬ Col A B Y
using Out-cases ⟨A ≠ B⟩ ⟨B Out C Y⟩ ⟨¬ Col A B C⟩ col-out2-col out-trivial by blast
have X ≠ Y
using ⟨B Out A X⟩ ⟨¬ Col A B Y⟩ not-col-permutation-4 out-col by blast
have Bet X T Y

```

```

proof -
{
  assume  $T = X$  and  $T = Y$ 
  have  $Bet\ X\ T\ Y$ 
  using  $\langle T = X \rangle \langle T = Y \rangle \langle X \neq Y \rangle$  by auto
}
moreover
{
  assume  $T \neq X$  and  $T \neq Y$ 
  have  $B\ A\ OS\ T\ C$ 
  by (metis Col-cases  $\langle B\ Out\ A\ X \rangle \langle Col\ X\ T\ Y \rangle \langle T\ InAngle\ A\ B\ C \rangle \langle T \neq X \rangle$ 
     $\langle \neg Col\ A\ B\ C \rangle \langle \neg Col\ A\ B\ Y \rangle$  colx in-angle-one-side invert-one-side out-col)
  hence  $B\ A\ OS\ T\ Y$ 
  using  $\langle B\ Out\ C\ Y \rangle$  out-out-one-side by blast
  hence  $X\ B\ OS\ T\ Y$ 
  by (meson  $\langle B\ Out\ A\ X \rangle \langle X \neq B \rangle$  col2-os--os not-col-distincts out-col)
  hence  $X\ Out\ T\ Y$ 
  using  $\langle Col\ X\ T\ Y \rangle$  col-one-side-out by blast
  have  $B\ C\ OS\ T\ A$ 
  proof -
    have  $\neg Col\ C\ B\ A$ 
    using  $\langle \neg Col\ A\ B\ C \rangle$  not-col-permutation-3 by blast
    moreover have  $\neg Col\ B\ C\ T$ 
    by (metis Col-def Out-def  $\langle B\ Out\ A\ X \rangle \langle B\ Out\ C\ Y \rangle \langle Col\ X\ T\ Y \rangle \langle T \neq Y \rangle$ 
       $\langle \neg Col\ A\ B\ C \rangle$  col-permutation-1 colx not-col-distincts)
    moreover have  $T\ InAngle\ C\ B\ A$ 
    using  $\langle T\ InAngle\ A\ B\ C \rangle$  l11-24 by blast
    ultimately show ?thesis
    using in-angle-one-side invert-one-side by blast
  qed
  hence  $B\ C\ OS\ X\ T$ 
  using  $\langle B\ Out\ A\ X \rangle$  col-trivial-3 one-side-symmetry os-out-os by blast
  hence  $Y\ B\ OS\ X\ T$ 
  using  $\langle B\ Out\ C\ Y \rangle \langle Y \neq B \rangle$  col2-os--os col-trivial-3 out-col by blast
  moreover hence  $Y\ Out\ X\ T$ 
  using Col-def  $\langle Col\ X\ T\ Y \rangle$  col-one-side-out by blast
  ultimately have  $Bet\ X\ T\ Y$ 
  using out2--bet  $\langle X\ Out\ T\ Y \rangle$  by blast
}
moreover
{
  assume  $T \neq X$  and  $T = Y$ 
  have  $Bet\ X\ T\ Y$ 
  by (simp add:  $\langle T = Y \rangle$  between-trivial)
}
moreover
{
  assume  $T = X$  and  $T \neq Y$ 
  have  $Bet\ X\ T\ Y$ 
  by (simp add:  $\langle T = X \rangle$  between-trivial2)
}
ultimately show ?thesis
by auto
qed
hence  $\exists\ X\ Y. B\ Out\ A\ X \wedge B\ Out\ C\ Y \wedge Bet\ X\ T\ Y$ 
using  $\langle B\ Out\ A\ X \rangle \langle B\ Out\ C\ Y \rangle$  by auto
}
hence weak-tarski-s-parallel-postulate
using weak-tarski-s-parallel-postulate-def by blast
}
ultimately
have weak-tarski-s-parallel-postulate
by blast
thus ?thesis
by blast
qed

```

lemma *weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate-aux* :

assumes *weak-tarski-s-parallel-postulate*

shows $\forall A B C P T.$

$Per A B C \wedge T InAngle A B C \wedge$

$P \neq T \wedge P B A CongA P B C \wedge Per B P T \wedge Coplanar A B C P$

$\longrightarrow$

$((\exists X. (B Out A X \wedge Col T P X)) \vee (\exists Y. (B Out C Y \wedge Col T P Y)))$

proof –

{

fix $A B C P T$

assume $Per A B C$ **and** $T InAngle A B C$ **and**

$P \neq T$ **and** $P B A CongA P B C$ **and** $Per B P T$ **and** $Coplanar A B C P$

have $P InAngle A B C$

by (*metis conga-inangle-per2--inangle* $\langle Coplanar A B C P \rangle$

$\langle P B A CongA P B C \rangle \langle Per A B C \rangle \langle Per B P T \rangle \langle T InAngle A B C \rangle$)

have $Acute P B A$

by (*meson conga-inangle-per--acute* $\langle P B A CongA P B C \rangle$

$\langle P InAngle A B C \rangle \langle Per A B C \rangle$ *acute-sym*)

have $Acute P B C$

by (*meson acute-conga--acute* $\langle Acute P B A \rangle \langle P B A CongA P B C \rangle$)

have $B \neq P$

using $\langle Acute P B C \rangle$ *acute-distincts* **by** *blast*

have $B \neq A$

using $\langle T InAngle A B C \rangle$ *inangle-distincts* **by** *blast*

have $B \neq C$

using $\langle P B A CongA P B C \rangle$ *conga-distinct* **by** *blast*

have $B P Perp P T$

by (*simp add:* $\langle B \neq P \rangle \langle P \neq T \rangle \langle Per B P T \rangle$ *per-perp*)

have $\neg Col A B C$

using $\langle Per A B C \rangle \langle B \neq A \rangle \langle B \neq C \rangle$ *per-not-col* **by** *auto*

have $\neg Col B P T$

using $\langle Per B P T \rangle \langle B \neq P \rangle \langle P \neq T \rangle$ *per-col-eq* **by** *blast*

 {

assume $Col A B T$

hence $B Out A T$

using $\langle T InAngle A B C \rangle \langle \neg Col A B C \rangle$ *bet-col*

bet-in-angle-bet or-bet-out **by** *blast*

hence $(\exists X. (B Out A X \wedge Col T P X)) \vee (\exists Y. (B Out C Y \wedge Col T P Y))$

using *col-trivial-3* **by** *blast*

 }

moreover

{

assume $\neg Col A B T$

have $\exists U V. (B Out A U \wedge B Out C V \wedge Bet U T V)$

using *assms weak-tarski-s-parallel-postulate-def*

$\langle Per A B C \rangle \langle T InAngle A B C \rangle$ **by** *blast*

then obtain $U V$ **where** $B Out A U$ **and** $B Out C V$ **and** $Bet U T V$

by *blast*

have $Col B A U$

by (*simp add:* $\langle B Out A U \rangle$ *out-col*)

have $Col B C V$

using $\langle B Out C V \rangle$ *out-col* **by** *blast*

have $Coplanar A C U V$

using *Coplanar-def* $\langle Col B A U \rangle \langle Col B C V \rangle$ *not-col-permutation-2* **by** *blast*

have $Coplanar A B C T$

using $\langle T InAngle A B C \rangle$ *coplanar-perm-9 inangle--coplanar* **by** *blast*

have $Coplanar B U V T$

using $\langle Bet U T V \rangle$ *bet--coplanar ncoplanar-perm-11* **by** *blast*

have $(\exists X. (B Out A X \wedge Col T P X)) \vee (\exists Y. (B Out C Y \wedge Col T P Y))$

proof –

 {

assume $Col P T U$

hence $(\exists X. (B Out A X \wedge Col T P X)) \vee (\exists Y. (B Out C Y \wedge Col T P Y))$

 }

```

    using  $\langle B \text{ Out } A \ U \rangle$  not-col-permutation-4 by blast
  }
  moreover
  {
    assume  $\neg \text{Col } P \ T \ U$ 
    have  $(\exists X. (B \text{ Out } A \ X \wedge \text{Col } T \ P \ X)) \vee (\exists Y. (B \text{ Out } C \ Y \wedge \text{Col } T \ P \ Y))$ 
    proof -
      {
        assume  $\text{Col } P \ T \ V$ 
        hence  $(\exists X. (B \text{ Out } A \ X \wedge \text{Col } T \ P \ X)) \vee (\exists Y. (B \text{ Out } C \ Y \wedge \text{Col } T \ P \ Y))$ 
        using  $\langle B \text{ Out } C \ V \rangle$  not-col-permutation-4 by blast
      }
    moreover
    {
      assume  $\neg \text{Col } P \ T \ V$ 
      have  $P \ T \ TS \ B \ U \vee P \ T \ OS \ B \ U$ 
      proof -
        have Coplanar  $P \ T \ B \ U$ 
        proof -
          have Coplanar  $A \ B \ C \ B$ 
          using ncop-distincts by blast
          moreover have Coplanar  $A \ B \ C \ U$ 
          using Col-cases  $\langle \text{Col } B \ A \ U \rangle$  ncop--ncols by blast
          ultimately show ?thesis
          using  $\langle \neg \text{Col } A \ B \ C \rangle \langle \text{Coplanar } A \ B \ C \ P \rangle \langle \text{Coplanar } A \ B \ C \ T \rangle$ 
            coplanar-pseudo-trans by blast
        qed
        moreover have  $\neg \text{Col } U \ P \ T$ 
        using  $\langle \neg \text{Col } P \ T \ U \rangle$  col-permutation-1 by blast
        ultimately show ?thesis
        using  $\langle \neg \text{Col } B \ P \ T \rangle$  cop-nos--ts by blast
      qed
      have  $(\exists X. (B \text{ Out } A \ X \wedge \text{Col } T \ P \ X)) \vee (\exists Y. (B \text{ Out } C \ Y \wedge \text{Col } T \ P \ Y))$ 
      proof -
        {
          assume  $P \ T \ TS \ B \ U$ 
          then obtain  $X$  where Col  $X \ P \ T$  and Bet  $B \ X \ U$ 
          using TS-def by blast
          have  $X \neq B$ 
          using  $\langle \text{Col } X \ P \ T \rangle \langle \neg \text{Col } B \ P \ T \rangle$  by blast
          hence  $B \text{ Out } U \ X$ 
          using Out-def  $\langle B \text{ Out } A \ U \rangle \langle \text{Bet } B \ X \ U \rangle$  by presburger
          hence  $B \text{ Out } A \ X$ 
          using  $\langle B \text{ Out } A \ U \rangle$  l6-7 by blast
          hence  $(\exists X. (B \text{ Out } A \ X \wedge \text{Col } T \ P \ X)) \vee (\exists Y. (B \text{ Out } C \ Y \wedge \text{Col } T \ P \ Y))$ 
          using  $\langle \text{Col } X \ P \ T \rangle$  col-permutation-3 by blast
        }
      }
    moreover
    {
      assume  $P \ T \ OS \ B \ U$ 
      have  $P \ T \ TS \ B \ V$ 
      proof -
        have  $P \ T \ TS \ U \ V$ 
        by (metis  $\langle \text{Bet } U \ T \ V \rangle \langle \neg \text{Col } P \ T \ U \rangle \langle \neg \text{Col } P \ T \ V \rangle$ 
          bet--ts col-trivial-2 invert-two-sides
          not-col-permutation-4)
        moreover have  $P \ T \ OS \ U \ B$ 
        using  $\langle P \ T \ OS \ B \ U \rangle$  one-side-symmetry by blast
        ultimately show ?thesis
        using l9-8-2 by blast
      qed
      then obtain  $Y$  where Col  $Y \ P \ T$  and Bet  $B \ Y \ V$ 
      using TS-def by blast
      have  $B \text{ Out } C \ Y$ 
      by (metis Col-def  $\langle B \text{ Out } C \ V \rangle \langle \text{Bet } B \ Y \ V \rangle \langle \text{Col } Y \ P \ T \rangle$ 
         $\langle P \ T \ OS \ B \ U \rangle$  between-equality)
    }
  }

```

```

      between-symmetry col123--nos l6-7 not-out-bet)
    hence  $(\exists X. (B \text{ Out } A X \wedge \text{Col } T P X)) \vee (\exists Y. (B \text{ Out } C Y \wedge \text{Col } T P Y))$ 
    using  $\langle \text{Col } Y P T \rangle$  not-col-permutation-3 by blast
  }
  ultimately show ?thesis
  using  $\langle P T TS B U \vee P T OS B U \rangle$  by blast
qed
}
ultimately show ?thesis by blast
qed
}
ultimately show ?thesis by blast
qed
}
ultimately have  $(\exists X. (B \text{ Out } A X \wedge \text{Col } T P X)) \vee (\exists Y. (B \text{ Out } C Y \wedge \text{Col } T P Y))$ 
by blast
}
thus ?thesis by blast
qed

```

lemma *weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate:*

assumes *weak-tarski-s-parallel-postulate*

shows *weak-inverse-projection-postulate*

proof –

```

{
  fix A B C P T
  assume Per A B C and T InAngle A B C and
    P  $\neq$  T and P B A CongA P B C and Coplanar A B C P and Per B P T
  have  $\neg B \text{ Out } A C$ 
    by (metis Col-def Out-def Bet-perm per-not-col  $\langle \text{Per } A B C \rangle$ )
  have B P Perp P T
    using CongA-def  $\langle P B A \text{ CongA } P B C \rangle \langle P \neq T \rangle \langle \text{Per } B P T \rangle$  per-perp by auto
  have  $((\exists X. (B \text{ Out } A X \wedge \text{Col } T P X)) \vee (\exists Y. (B \text{ Out } C Y \wedge \text{Col } T P Y)))$ 
    using assms weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate-aux
       $\langle \text{Coplanar } A B C P \rangle \langle P B A \text{ CongA } P B C \rangle \langle P \neq T \rangle$ 
       $\langle \text{Per } A B C \rangle \langle \text{Per } B P T \rangle \langle T \text{ InAngle } A B C \rangle$  by blast

```

moreover

```

{
  assume  $\exists X. (B \text{ Out } A X \wedge \text{Col } T P X)$ 
  then obtain X where B Out A X and Col T P X
    by blast
  obtain Y where P Midpoint X Y
    using symmetric-point-construction by blast
  {
    assume X = Y
    hence B Out A P
      using  $\langle B \text{ Out } A X \rangle \langle P \text{ Midpoint } X Y \rangle$  l8-20-2 by blast
    hence B Out P C
      by (metis Col-def Out-def  $\langle P B A \text{ CongA } P B C \rangle \langle \neg B \text{ Out } A C \rangle$ 
        between-symmetry col-conga--conga eq-conga-out)
    hence False
      using  $\langle B \text{ Out } A P \rangle \langle \neg B \text{ Out } A C \rangle$  l6-7 by blast
  }

```

hence $X \neq Y$

by blast

have $Y X \text{ ReflectL } B P$

proof –

```

  have  $(\exists X0. X0 \text{ Midpoint } X Y \wedge \text{Col } B P X0)$ 
    using  $\langle P \text{ Midpoint } X Y \rangle$  col-trivial-2 by blast
  moreover have  $(B P \text{ Perp } X Y \vee X = Y)$ 
    by (metis Perp-perm perp-col0
       $\langle B P \text{ Perp } P T \rangle \langle \text{Col } T P X \rangle \langle P \text{ Midpoint } X Y \rangle$ 
      col-trivial-2 midpoint-col midpoint-distinct-1)
  ultimately show ?thesis
    by (simp add: ReflectL-def)

```

qed


```

hence  $X Y \text{ ReflectL } B P$ 
  using l10-4-spec by blast
moreover hence  $B \text{ Out } C Y$ 
  using  $\langle B \text{ Out } A X \rangle \langle P B A \text{ CongA } P B C \rangle \langle \neg B \text{ Out } A C \rangle \langle \text{Coplanar } A B C P \rangle$ 
    conga-cop-out-reflectl-out by blast
moreover have  $\text{Col } T P X$ 
  by (simp add:  $\langle \text{Col } T P X \rangle$ )
hence  $\text{Col } P X T$ 
  using not-col-permutation-2 by blast
have  $P \neq X$ 
  using  $\langle P \text{ Midpoint } X Y \rangle \langle X = Y \implies \text{False} \rangle$  is-midpoint-id by blast
hence  $\text{Col } P X Y$ 
  using  $\langle P \text{ Midpoint } X Y \rangle$  midpoint-col by blast
moreover hence  $\text{Col } T P Y$ 
  using  $\langle \text{Col } T P X \rangle \langle P \neq X \rangle$  col-trivial-2 colx by blast
ultimately
have  $\exists X Y. (B \text{ Out } A X \wedge \text{Col } T P X \wedge B \text{ Out } C Y \wedge \text{Col } T P Y)$ 
  using  $\langle B \text{ Out } A X \rangle \langle \text{Col } T P X \rangle$  by blast
}
moreover
{
  assume  $\exists Y. (B \text{ Out } C Y \wedge \text{Col } T P Y)$ 
  then obtain  $Y$  where  $B \text{ Out } C Y$  and  $\text{Col } T P Y$ 
    by blast
  obtain  $X$  where  $P \text{ Midpoint } Y X$ 
    using symmetric-point-construction by blast
  {
    assume  $X = Y$ 
    hence  $X = P$ 
      using  $\langle P \text{ Midpoint } Y X \rangle$  l7-17-bis l7-3-2 by blast
    hence  $Y = P$ 
      by (simp add:  $\langle X = Y \rangle$ )
    have  $B \text{ Out } A P$ 
      proof -
        have  $B \text{ Out } P C$ 
          using  $\langle B \text{ Out } C Y \rangle \langle Y = P \rangle$  l6-6 by blast
        moreover have  $P B C \text{ CongA } P B A$ 
          using  $\langle P B A \text{ CongA } P B C \rangle$  conga-sym-equiv by auto
        ultimately show ?thesis
          using Out-cases out-conga-out by blast
      qed
    have  $B \text{ Out } C P$ 
      using  $\langle B \text{ Out } C Y \rangle \langle X = P \rangle \langle X = Y \rangle$  by blast
    moreover hence  $B \text{ Out } P C$ 
      using l6-6 by blast
    ultimately have  $\text{False}$ 
      using  $\langle \neg B \text{ Out } A C \rangle$  l6-7  $\langle B \text{ Out } A P \rangle$  by blast
  }
}
hence  $X \neq Y$ 
  by blast
have  $B \text{ Out } A X$ 
  proof -
    have  $\neg B \text{ Out } C A$ 
      using  $\langle \neg B \text{ Out } A C \rangle$  l6-6 by blast
    moreover have  $\text{Coplanar } C B A P$ 
      using  $\langle \text{Coplanar } A B C P \rangle$  coplanar-perm-14 by presburger
    moreover have  $P B C \text{ CongA } P B A$ 
      using  $\langle P B A \text{ CongA } P B C \rangle$  conga-sym-equiv by auto
    moreover have  $Y X \text{ ReflectL } B P$ 
      proof -
        have  $\exists X0. X0 \text{ Midpoint } Y X \wedge \text{Col } B P X0$ 
          using  $\langle P \text{ Midpoint } Y X \rangle$  col-trivial-2 by blast
        moreover have  $B P \text{ Perp } Y X \vee Y = X$ 
          by (metis col-permutation-4 col-trivial-3 midpoint-distinct-1
             $\langle B P \text{ Perp } P T \rangle \langle \text{Col } T P Y \rangle \langle P \text{ Midpoint } Y X \rangle$  colx
            midpoint-col perp-col2-bis)
      qed
  qed

```

```

ultimately show ?thesis
  by (metis ReflectL-def l10-4-spec)
qed
ultimately show ?thesis
  using ⟨B Out C Y⟩ conga-cop-out-reflectl-out by blast
qed
have P ≠ X
  using ⟨P Midpoint Y X⟩ ⟨X = Y ⟹ False⟩ is-midpoint-id-2 by blast
have Col P X Y
  using ⟨P Midpoint Y X⟩ Col-def Midpoint-def by blast
hence Col P X T
  by (metis ⟨Col T P Y⟩ ⟨P Midpoint Y X⟩ col-permutation-1
    col-trivial-3 colx l7-2 midpoint-distinct-2)
moreover hence Col T P X
  using not-col-permutation-1 by blast
moreover have Col T P Y
  by (simp add: ⟨Col T P Y⟩)
ultimately
have ∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y)
  using ⟨B Out C Y⟩ ⟨B Out A X⟩ by blast
}
ultimately have ∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y)
  by blast
}
hence LEMME: ∀ A B C P T.
  Per A B C ∧ T InAngle A B C ∧
  P ≠ T ∧ P B A CongA P B C ∧ Coplanar A B C P ∧ Per B P T ⟹
  (∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y))
  by blast
moreover
{
  assume ∀ A B C P T.
    Per A B C ∧ T InAngle A B C ∧
    P ≠ T ∧ P B A CongA P B C ∧ Coplanar A B C P ∧ Per B P T ⟹
    (∃ X Y. (B Out A X ∧ Col T P X ∧ B Out C Y ∧ Col T P Y))
  {
    fix A B C D E F P Q
    assume Acute A B C and Per D E F and A B C A B C SumA D E F and
      B Out A P and P ≠ Q and Per B P Q and Coplanar A B C Q
    have ¬ Col A B C
      using ⟨Per D E F⟩
      by (metis col2-suma--col suma-distincts ⟨A B C A B C SumA D E F⟩ per-not-col)
    have ¬ Col B P Q
      by (metis Col-def per-col-eq ⟨B Out A P⟩ ⟨P ≠ Q⟩
        ⟨Per B P Q⟩ between-trivial l6-4-1)
    have A B C CongA P B C
      by (metis ⟨Acute A B C⟩ ⟨B Out A P⟩ acute-distincts bet-out-1
        conga-sym-equiv not-bet-distincts out2--conga)
    have ¬ Col P B C
      by (metis Out-def ⟨B Out A P⟩ ⟨¬ Col A B C⟩ col-out2-col
        col-permutation-3 col-trivial-3 l6-16-1 out-trivial)
    have B P Perp P Q
      by (metis ⟨B Out A P⟩ ⟨P ≠ Q⟩ ⟨Per B P Q⟩ out-diff2 per-perp)
    have C B A A B C SumA D E F
      by (simp add: ⟨A B C A B C SumA D E F⟩ suma-left-comm)
    then obtain J where A B J CongA A B C and ¬ B A OS C J and
      Coplanar C B A J and C B J CongA D E F
      using SumA-def by blast
    have ∃ Q'. (P ≠ Q') ∧ Col P Q Q' ∧ Q' InAngle C B P
    proof -
      have ∃ Q0. Col Q P Q0 ∧ B P OS C Q0
    proof -
      have Q ≠ P
        using ⟨P ≠ Q⟩ by blast
      moreover have Col B P P
        by (simp add: col-trivial-2)

```

```

moreover have  $Col\ Q\ P\ P$ 
  by (simp add: col-trivial-2)
moreover have  $\neg\ Col\ B\ P\ C$ 
  using  $\langle \neg\ Col\ P\ B\ C \rangle$  not-col-permutation-4 by blast
moreover have  $Coplanar\ B\ P\ Q\ C$ 
  by (metis (no-types, opaque-lifting) Col-def Out-def  $\langle B\ Out\ A\ P \rangle$ 
     $\langle Coplanar\ A\ B\ C\ Q \rangle \langle \neg\ Col\ A\ B\ C \rangle$ 
    coplanar-pseudo-trans-lem1 ncop--ncols ncop-distincts)
ultimately show ?thesis
  using  $\langle \neg\ Col\ B\ P\ Q \rangle$  cop-not-par-same-side by blast
qed
then obtain  $Q0$  where  $Col\ Q\ P\ Q0$  and  $B\ P\ OS\ C\ Q0$ 
  by blast
{
  assume  $B\ C\ OS\ P\ Q0$ 
  hence  $\exists\ Q'. (P \neq Q') \wedge Col\ P\ Q\ Q' \wedge Q'\ InAngle\ C\ B\ P$ 
  using  $\langle B\ P\ OS\ C\ Q0 \rangle \langle Col\ Q\ P\ Q0 \rangle$  not-col-permutation-4
    os2--inangle os-distincts by blast
}
moreover
{
  assume  $\neg\ B\ C\ OS\ P\ Q0$ 
  have  $\exists\ Q'. Col\ P\ Q\ Q' \wedge Col\ B\ C\ Q'$ 
  proof (cases Col B C Q0)
    case True
      thus ?thesis
      using  $\langle Col\ Q\ P\ Q0 \rangle$  not-col-permutation-4 by blast
    next
      case False
        have  $\exists\ Q'. Col\ Q'\ B\ C \wedge Bet\ P\ Q'\ Q0$ 
        proof –
          have  $Coplanar\ B\ C\ P\ Q0$ 
            using False  $\langle B\ P\ OS\ C\ Q0 \rangle$  inangle--coplanar ncoplanar-perm-22
              os-ts--inangle ts--coplanar two-sides-cases by blast
          moreover have  $\neg\ Col\ P\ B\ C$ 
            using  $\langle \neg\ Col\ P\ B\ C \rangle$  by auto
          moreover have  $\neg\ Col\ Q0\ B\ C$ 
            by (simp add: False not-col-permutation-2)
          moreover have  $\neg\ B\ C\ OS\ P\ Q0$ 
            using  $\langle \neg\ B\ C\ OS\ P\ Q0 \rangle$  by auto
          ultimately show ?thesis
            using TS-def cop--one-or-two-sides by blast
        qed
        then obtain  $Q'$  where  $Col\ Q'\ B\ C$  and  $Bet\ P\ Q'\ Q0$ 
          by blast
        have  $Col\ P\ Q\ Q'$ 
          using  $\langle P \neq Q \rangle \langle Bet\ P\ Q'\ Q0 \rangle \langle Col\ Q\ P\ Q0 \rangle$  bet-col bet-neq12--neq col2--eq
            col-permutation-4 by blast
        moreover have  $Col\ B\ C\ Q'$ 
          using  $\langle Col\ Q'\ B\ C \rangle$  not-col-permutation-2 by blast
        ultimately show ?thesis
          by blast
      qed
        then obtain  $Q'$  where  $Col\ P\ Q\ Q'$  and  $Col\ B\ C\ Q'$ 
          by blast
        have  $P \neq Q'$ 
          using  $\langle Col\ B\ C\ Q' \rangle \langle \neg\ Col\ P\ B\ C \rangle$  col-permutation-2 by blast
        moreover have  $Col\ P\ Q\ Q'$ 
          using  $\langle Col\ P\ Q\ Q' \rangle$  by auto
        moreover have  $Q'\ InAngle\ C\ B\ P$ 
        proof –
          have  $P \neq B$ 
            using  $\langle B\ P\ OS\ C\ Q0 \rangle$  os-distincts by blast
          moreover have  $B\ Out\ C\ Q'$ 
          proof –
            have Acute P B C

```

```

    by (meson acute-conga--acute ⟨A B C CongA P B C⟩ ⟨Acute A B C⟩)
  moreover have B P Perp P Q'
    by (meson perp-col1 ⟨B P Perp P Q⟩ ⟨Col P Q Q'⟩ ⟨P ≠ Q'⟩)
  ultimately show ?thesis
    using ⟨Col B C Q'⟩ acute-col-perp--out-1 l6-6 by blast
qed
ultimately show ?thesis
  using out321--inangle by auto
qed
ultimately have  $\exists Q'. (P \neq Q') \wedge Col P Q Q' \wedge Q' InAngle C B P$ 
  by blast
}
ultimately show ?thesis
  by blast
qed
then obtain Q' where P ≠ Q' and Col P Q Q' and Q' InAngle C B P
  by blast
have B Out P A
  by (simp add: ⟨B Out A P⟩ l6-6)
have C ≠ B
  using ⟨¬ Col P B C⟩ col-trivial-2 by auto
have P ≠ B
  using ⟨B Out A P⟩ out-distinct by presburger
have J ≠ B
  using ⟨A B J CongA A B C⟩ conga-diff2 by blast
have D ≠ E
  using ⟨C B J CongA D E F⟩ conga-diff45 by auto
{
  assume E = F
  hence A B C A B C SumA D E E
    using ⟨A B C A B C SumA D E F⟩ by blast
  hence False
    using ⟨C B J CongA D E F⟩ ⟨E = F⟩ conga-diff56 by blast
}
hence F ≠ E
  by blast
have A B C CongA A B J
  using ⟨A B J CongA A B C⟩ conga-sym-equiv by auto
hence ¬ Col A B J
  using ⟨¬ Col A B C⟩ ⟨A B J CongA A B C⟩ col-conga-col by blast
have P InAngle C B J
proof -
  have A InAngle C B J
  proof -
    have B A TS C J
    proof -
      have Coplanar B A C J
        using ⟨Coplanar C B A J⟩ coplanar-perm-8 by presburger
      moreover have ¬ Col C B A
        using NCol-cases ⟨¬ Col A B C⟩ by blast
      moreover have ¬ Col J B A
        using Col-cases ⟨¬ Col A B J⟩ by blast
      ultimately show ?thesis
        using ⟨¬ B A OS C J⟩ cop-nts--os by blast
    qed
    moreover have B C OS J A
  proof -
    have Coplanar C B A J
      by (simp add: ⟨Coplanar C B A J⟩)
    moreover have ¬ Col A C B
      using NCol-cases ⟨¬ Col A B C⟩ by blast
    have ¬ Col C B J
  proof -
    have ¬ Col D E F
      using ⟨D ≠ E⟩ ⟨F ≠ E⟩ ⟨Per D E F⟩ per-not-col by auto
    moreover have D E F CongA C B J

```

```

    by (simp add: ⟨C B J CongA D E F⟩ conga-sym-equiv)
  ultimately show ?thesis
    by (meson col-conga-col ⟨C B J CongA D E F⟩)
qed
moreover hence ¬ Col J C B
  using col-permutation-1 by blast
have SAMS C B A A B C
  using ⟨Acute A B C⟩ acute2--sams acute-sym by blast
moreover hence ¬ C B TS A J
  using ⟨A B J CongA A B C⟩ ⟨¬ B A OS C J⟩ conga-sams-nos--nts by blast
ultimately show ?thesis
  using invert-one-side one-side-symmetry cop-nts--os
    ⟨¬ Col A C B⟩ ⟨¬ Col J C B⟩ by blast
qed
ultimately show ?thesis
  using os-ts--inangle by blast
qed
moreover have B Out C C
  using ⟨C ≠ B⟩ out-trivial by auto
moreover have B Out J J
  using ⟨J ≠ B⟩ out-trivial by auto
ultimately show ?thesis
  using ⟨B Out P A⟩ l11-25 by blast
qed
hence Q' InAngle C B J
  using ⟨Q' InAngle C B P⟩ in-angle-trans by blast
have ∃ Y. B Out C Y ∧ Col Q' P Y
proof -
  have D E F CongA C B J
    by (simp add: ⟨C B J CongA D E F⟩ conga-sym-equiv)
  hence Per C B J
    using ⟨Per D E F⟩ l11-17 by blast
  moreover have Q' InAngle C B J
    using ⟨Q' InAngle C B J⟩ by auto
  moreover have P ≠ Q'
    by (simp add: ⟨P ≠ Q'⟩)
  moreover have P B C CongA P B J
    by (metis col-conga--conga not-conga-sym out-col
      ⟨A B J CongA A B C⟩ ⟨B Out A P⟩ ⟨P ≠ B⟩)
  moreover have Coplanar C B J P
    by (meson ⟨P InAngle C B J⟩ coplanar-perm-9 inangle--coplanar)
  moreover have Per B P Q'
    using ⟨Col P Q Q'⟩ ⟨P ≠ Q⟩ ⟨Per B P Q⟩ per-col by blast
  ultimately show ?thesis
    using LEMME by blast
qed
then obtain Y where B Out C Y and Col Q' P Y
  by blast
hence ∃ Y. B Out C Y ∧ Col P Q Y
  by (metis ⟨Col P Q Q'⟩ ⟨P ≠ Q'⟩ col-trivial-3 colx)
}
hence ∀ A B C D E F P Q.
  Acute A B C ∧ Per D E F ∧ A B C A B C SumA D E F ∧
  B Out A P ∧ P ≠ Q ∧ Per B P Q ∧ Coplanar A B C Q
→
(∃ Y. B Out C Y ∧ Col P Q Y)
  by blast
}
ultimately show ?thesis
  using weak-inverse-projection-postulate-def by fastforce
qed

lemma P31-P32:
shows Postulate31 ↔ Postulate32
using Postulate31-def Postulate32-def
  weak-inverse-projection-postulate--weak-tarski-s-parallel-postulate

```

weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate **by** blast

lemma *P31-P04*:

shows *Postulate31* $\longleftrightarrow$ *Postulate04*

using *Postulate04-def Postulate31-def*

bachmann-s-lotschnittaxiom--weak-inverse-projection-postulate

weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom **by** blast

lemma *P04-P33*:

shows *Postulate04* $\longleftrightarrow$ *Postulate33*

using *Postulate04-def Postulate33-def*

bachmann-s-lotschnittaxiom--weak-triangle-circumscription-principle

weak-triangle-circumscription-principle--bachmann-s-lotschnittaxiom **by** blast

lemma *equivalent-postulates-without-any-continuity-bis*:

shows *Postulate04* $\longleftrightarrow$ *Postulate33* $\wedge$

Postulate31 $\longleftrightarrow$ *Postulate04* $\wedge$

Postulate31 $\longleftrightarrow$ *Postulate32*

proof –

have *Postulate31* $\longleftrightarrow$ *Postulate32*

using *Postulate31-def Postulate32-def*

weak-inverse-projection-postulate--weak-tarski-s-parallel-postulate

weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate **by** blast

moreover have *Postulate31* $\longleftrightarrow$ *Postulate04*

using *Postulate04-def Postulate31-def*

bachmann-s-lotschnittaxiom--weak-inverse-projection-postulate

weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom **by** blast

moreover have *Postulate04* $\longleftrightarrow$ *Postulate33*

using *Postulate04-def Postulate33-def*

bachmann-s-lotschnittaxiom--weak-triangle-circumscription-principle

weak-triangle-circumscription-principle--bachmann-s-lotschnittaxiom **by** blast

ultimately show *?thesis*

by blast

qed

lemma *P4-P34*:

assumes *Postulate04*

shows *Postulate34*

using *P31-P04 P31-P32 Postulate32-def Postulate34-def*

assms bachmann-s-lotschnittaxiom--legendre-s-parallel-postulate

weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom

weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate **by** blast

lemma *P01--P35*:

assumes *Postulate01*

shows *Postulate35*

using *Postulate01-def Postulate35-def playfair--existential-playfair*

tarski-s-euclid-implies-playfair-s-postulate assms **by** blast

lemma *P35--P01*:

assumes *greenberg-s-axiom and*

Postulate01

shows *Postulate35*

proof –

have *Postulate35* $\longrightarrow$ *Postulate27*

by (*simp add: Postulate27-def Postulate35-def existential-playfair--rah*)

moreover

have *Postulate27* $\longrightarrow$ *Postulate03*

using *Cycle-3* **by** blast

moreover

have *Postulate03* $\longrightarrow$ *Postulate12*

by (*simp add: InterCycle1 assms(1)*)

moreover have *Postulate12* $\longrightarrow$ *Postulate09*

by (*simp add: Postulate09-def Postulate12-def playfair--universal-posidonius-postulate*

playfair-bis--playfair

universal-posidonius-postulate--perpendicular-transversal-postulate)

moreover
have *Postulate09* $\longrightarrow$ *Postulate01*
using *Cycle-1 Cycle-2* **by** *blast*
ultimately show *?thesis*
using *P01--P35* *assms(2)* **by** *fastforce*
qed

theorem *stronger-legendre-s-first-theorem*:
assumes *aristotle-s-axiom*
shows $\forall A B C D E F. C A B A B C \text{ Sum} A D E F \longrightarrow \text{SAMS } D E F B C A$
using *aristotle--obtuse-case-elimination* *assms t22-20* **by** *blast*

theorem *legendre-s-first-theorem*:
assumes *archimedes-axiom*
shows $\forall A B C D E F. C A B A B C \text{ Sum} A D E F \longrightarrow \text{SAMS } D E F B C A$
using *assms stronger-legendre-s-first-theorem t22-24* **by** *blast*

theorem *legendre-s-second-theorem*:
assumes *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
shows *triangle-postulate*
using *assms rah--triangle existential-triangle--rah* **by** *blast*

lemma *legendre-s-third-theorem-aux*:
assumes *aristotle-s-axiom* **and**
triangle-postulate
shows *euclid-s-parallel-postulate*
using *Cycle-2 Postulate01-def Postulate13-def*
alternate-interior--proclus *aristotle--greenberg*
assms(1) assms(2) playfair--alternate-interior
playfair-bis--playfair tarski-s-implies-euclid-s-parallel-postulate
triangle--playfair-bis **by** *fastforce*

theorem *legendre-s-third-theorem*:
assumes *archimedes-axiom* **and**
triangle-postulate
shows *euclid-s-parallel-postulate*
by (*simp add: assms(1) assms(2) legendre-s-third-theorem-aux t22-24*)

lemma *legendre-aux*:
assumes $\neg \text{hypothesis-of-obtuse-saccheri-quadrilaterals}$
shows $\forall A B C D B1 C1 P Q R S T U V W X.$
 $\neg \text{Col } A B C \wedge A C B \text{ Cong} A C B D \wedge$
 $\text{Cong } A C B D \wedge B C \text{ TS } A D \wedge A \text{ Out } B B1 \wedge A \text{ Out } C C1 \wedge \text{Bet } B1 D C1 \wedge$
 $\text{Defect } A B C P Q R \wedge \text{Defect } A B1 C1 S T U \wedge P Q R P Q R \text{ Sum} A V W X \longrightarrow$
 $(\text{SAMS } P Q R P Q R \wedge V W X \text{ Le} A S T U)$

proof –
{
fix *A B C D B1 C1 P Q R S T U V W X*
assume $\neg \text{Col } A B C$ **and** *A C B Cong A C B D* **and**
Cong A C B D **and** *B C TS A D* **and** *A Out B B1* **and**
A Out C C1 **and** *Bet B1 D C1* **and**
Defect A B C P Q R **and** *Defect A B1 C1 S T U* **and**
P Q R P Q R Sum A V W X
have $A \neq B$
using $\langle B C \text{ TS } A D \rangle$ *ts-distincts* **by** *blast*
have $\text{Cong } A B D C \wedge (A \neq B) \longrightarrow (C A B \text{ Cong} A B D C \wedge C B A \text{ Cong} A B C D)$
proof –
have *A C B Cong A D B C*
using $\langle A C B \text{ Cong} A C B D \rangle$ *conga-right-comm* **by** *blast*
moreover **have** *Cong C A B D*
using $\langle \text{Cong } A C B D \rangle$ *not-cong-2134* **by** *blast*
moreover **have** *Cong C B B C*
by (*simp add: cong-pseudo-reflexivity*)
ultimately show *?thesis*
using $\langle A \neq B \rangle$ *l11-49* **by** *blast*
qed

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have Cong A B D C
  by (metis cong2-conga-cong ⟨A C B CongA C B D⟩ ⟨Cong A C B D⟩
      cong-pseudo-reflexivity cong-right-commutativity conga-right-comm)
have C A B CongA B D C
  by (simp add: ⟨A ≠ B⟩ ⟨Cong A B D C⟩
      ⟨Cong A B D C ∧ A ≠ B ⟶ C A B CongA B D C ∧ C B A CongA B C D⟩ )
have C B A CongA B C D
  by (simp add: ⟨A ≠ B⟩ ⟨Cong A B D C⟩
      ⟨Cong A B D C ∧ A ≠ B ⟶ C A B CongA B D C ∧ C B A CongA B C D⟩ )
have A C ParStrict B D
proof -
  have A C Par B D
    by (meson conga-right-comm ⟨A C B CongA C B D⟩ ⟨B C TS A D⟩
        invert-two-sides l12-21-b par-left-comm)
  moreover have Col B D B
    by (simp add: col-trivial-3)
  moreover have ¬ Col A C B
    using ⟨¬ Col A B C⟩ col-permutation-5 by blast
  ultimately show ?thesis
    using par-not-col-strict by blast
qed
have A B ParStrict C D
proof -
  have A B Par C D
  proof -
    have B C TS A D
      using ⟨B C TS A D⟩ by auto
    moreover have A B C CongA D C B
      by (simp add: ⟨C B A CongA B C D⟩ conga-comm)
    ultimately show ?thesis
      using l12-21-b par-left-comm by blast
  qed
  moreover have Col C D C
    by (simp add: col-trivial-3)
  moreover have ¬ Col A B C
    using ⟨¬ Col A B C⟩ by blast
  ultimately show ?thesis
    using par-not-col-strict by blast
qed
have ¬ Col B C D
  using ⟨A B ParStrict C D⟩ par-strict-not-col-2 by auto
have ¬ Col C D A
  using ⟨A B ParStrict C D⟩ col-trivial-3 l9-19 pars--os3412 by blast
have ¬ Col A B D
  by (meson Par-strict-perm ⟨A B ParStrict C D⟩ l12-6 one-side-not-col123)
have B D ParStrict A C1
  by (metis Par-strict-cases ⟨A C ParStrict B D⟩ ⟨A Out C C1⟩
      out-col out-diff2 par-strict-col-par-strict)
have ¬ Col B D C1
  by (metis Par-strict-perm ⟨A Out C C1⟩ ⟨B D ParStrict A C1⟩
      diff-bet-ex3 l6-16-1 l6-3-1 l9-19 pars--os3412)
have B ≠ B1
  using ⟨Bet B1 D C1⟩ ⟨¬ Col B D C1⟩ bet-col by blast
have ¬ Col B1 B D
  by (metis ⟨A Out B B1⟩ ⟨B ≠ B1⟩ ⟨¬ Col A B D⟩ col-trivial-2 colx out-col)
have B D TS B1 C1
  using TS-def ⟨Bet B1 D C1⟩ ⟨¬ Col B D C1⟩ ⟨¬ Col B1 B D⟩
      col-trivial-3 not-col-permutation-2 by blast
hence ¬ Col B1 B D ∧ B D TS B1 C1 ∧ Bet A B B1
  by (metis ⟨A Out B B1⟩ ⟨B D ParStrict A C1⟩ ⟨¬ Col B1 B D⟩
      col-trivial-3 l12-6 l9-9 not-out-bet out-col out-two-sides-two-sides)
have Bet A B B1
  by (simp add: ⟨¬ Col B1 B D ∧ B D TS B1 C1 ∧ Bet A B B1⟩)
have C D ParStrict A B1
  by (metis Col-def ⟨A B ParStrict C D⟩ ⟨Bet A B B1⟩
      between-identity col-trivial-3 par-strict-col2-par-strict)

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    par-strict-symmetry)
have  $\neg \text{Col } C \ D \ B1$ 
  by (meson Par-strict-perm  $\langle C \ D \ \text{ParStrict } A \ B1 \rangle$  l12-6 one-side-not-col123)
have  $C \neq C1$ 
  using  $\langle \text{Bet } B1 \ D \ C1 \rangle \langle \neg \text{Col } C \ D \ B1 \rangle$  bet-col between-symmetry by blast
have  $C \ D \ \text{TS } A \ C1$ 
proof -
  have  $C \ D \ \text{TS } B1 \ C1$ 
    by (metis  $\langle \text{Bet } B1 \ D \ C1 \rangle \langle \neg \text{Col } B \ D \ C1 \rangle \langle \neg \text{Col } C \ D \ B1 \rangle$ 
        bet--ts col-trivial-2 invert-two-sides not-col-permutation-4)
  moreover have  $C \ D \ \text{OS } B1 \ A$ 
    using  $\langle C \ D \ \text{ParStrict } A \ B1 \rangle$  l12-6 one-side-symmetry by blast
  ultimately show ?thesis
    using l9-8-2 by blast
qed
hence  $\text{Bet } A \ C \ C1$ 
  using  $\langle A \ \text{Out } C \ C1 \rangle$  col-two-sides-bet not-out-bet out-col by blast
have  $\exists Z. \text{Bet } B \ Z \ D \wedge \text{Bet } C \ Z \ B1$ 
proof -
  have  $B \ D \ \text{TS } C \ B1$ 
  proof -
    have  $B \ D \ \text{TS } C1 \ B1$ 
      using  $\langle B \ D \ \text{TS } B1 \ C1 \rangle$  l9-2 by blast
    moreover have  $B \ D \ \text{OS } C1 \ C$ 
      by (metis bet-col1 l12-6 par-strict-col2-par-strict
           $\langle B \ D \ \text{ParStrict } A \ C1 \rangle \langle \text{Bet } A \ C \ C1 \rangle \langle C \neq C1 \rangle$  not-bet-distincts)
    ultimately show ?thesis
      using l9-8-2 by blast
  qed
  moreover have  $C \ B1 \ \text{TS } B \ D$ 
proof -
  have  $C \ D \ \text{OS } B1 \ B$ 
  proof -
    have  $C \ D \ \text{ParStrict } B \ A$ 
      using Par-strict-perm  $\langle A \ B \ \text{ParStrict } C \ D \rangle$  by blast
    moreover have  $\text{Col } B \ A \ B1$ 
      using  $\langle A \ \text{Out } B \ B1 \rangle$  col-permutation-4 out-col by blast
    ultimately show ?thesis
      using  $\langle B \neq B1 \rangle$  one-side-symmetry par-strict-all-one-side by blast
  qed
  have  $C \ B \ \text{OS } B1 \ C1$ 
proof -
  have  $C \ B \ \text{TS } B1 \ A$ 
    by (meson  $\langle B \neq B1 \rangle \langle \text{Bet } A \ B \ B1 \rangle \langle \neg \text{Col } A \ B \ C \rangle$  bet--ts invert-two-sides
        l9-2 not-col-permutation-1)
  moreover have  $C \ B \ \text{TS } C1 \ A$ 
    using Col-cases  $\langle \text{Bet } A \ C \ C1 \rangle \langle C \neq C1 \rangle \langle \neg \text{Col } A \ B \ C \rangle$  bet--ts l9-2 by blast
  ultimately show ?thesis
    using OS-def by blast
qed
hence  $C \ B \ \text{OS } B1 \ D$ 
  using  $\langle \text{Bet } B1 \ D \ C1 \rangle$  l9-17 by blast
moreover
  have  $C \ D \ \text{OS } B1 \ B$ 
    using  $\langle C \ D \ \text{OS } B1 \ B \rangle$  by blast
  ultimately show ?thesis
    using l9-31 by auto
qed
ultimately show ?thesis
  using ts2--ex-bet2 by blast
qed
then obtain  $Z$  where  $\text{Bet } B \ Z \ D$  and  $\text{Bet } C \ Z \ B1$ 
  by blast
obtain  $G \ H \ I$  where Defect  $A \ B1 \ C \ G \ H \ I$ 
  using ex-defect
  by (metis Col-def  $\langle A \ \text{Out } B \ B1 \rangle \langle \text{Bet } A \ B \ B1 \rangle \langle \neg \text{Col } A \ B \ C \rangle$ 

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    not-col-distincts out-diff2)
obtain  $J\ K\ L$  where  $\text{Defect } B1\ C\ C1\ J\ K\ L$ 
  using  $\text{ex-defect}$ 
  by ( $\text{metis } \text{Col-def } \text{TS-def } \langle B\ C\ \text{TS}\ A\ D \rangle \langle \text{Bet } B\ Z\ D \rangle \langle \text{Bet } C\ Z\ B1 \rangle$ 
     $\langle C \neq C1 \rangle \langle \text{Defect } A\ B1\ C1\ S\ T\ U \rangle \text{bet-neq12--neq defect-distincts}$ )
have  $C \neq B1$ 
  using  $\langle \neg \text{Col } C\ D\ B1 \rangle \text{col-trivial-3}$  by  $\text{blast}$ 
have  $B1 \neq B$ 
  using  $\langle B \neq B1 \rangle$  by  $\text{auto}$ 
have  $C \neq B$ 
  using  $\langle \neg \text{Col } B\ C\ D \rangle \text{col-trivial-1}$  by  $\text{fastforce}$ 
obtain  $M\ N\ O0$  where  $\text{Defect } C\ B\ B1\ M\ N\ O0$ 
  using  $\text{ex-defect } \langle C \neq B \rangle \langle C \neq B1 \rangle \langle B \neq B1 \rangle$  by  $\text{blast}$ 
have  $B1 \neq C$ 
  using  $\langle C \neq B1 \rangle$  by  $\text{auto}$ 
have  $C \neq D$ 
  using  $\langle \neg \text{Col } B\ C\ D \rangle \text{col-trivial-2}$  by  $\text{blast}$ 
have  $B1 \neq D$ 
  using  $\langle B\ D\ \text{TS}\ B1\ C1 \rangle \text{ts-distincts}$  by  $\text{blast}$ 
obtain  $A'\ B'\ C'$  where  $\text{Defect } B1\ C\ D\ A'\ B'\ C'$ 
  using  $\text{ex-defect } \langle B1 \neq D \rangle \langle C \neq B1 \rangle \langle C \neq D \rangle$  by  $\text{presburger}$ 
have  $D \neq C1$ 
  using  $\langle C\ D\ \text{TS}\ A\ C1 \rangle \text{ts-distincts}$  by  $\text{blast}$ 
obtain  $D'\ E'\ F'$  where  $\text{Defect } C\ D\ C1\ D'\ E'\ F'$ 
  using  $\text{ex-defect } \langle C \neq C1 \rangle \langle C \neq D \rangle \langle D \neq C1 \rangle$  by  $\text{presburger}$ 
obtain  $M'\ N'\ O'$  where  $\text{Defect } B1\ B\ D\ M'\ N'\ O'$ 
  using  $\text{ex-defect } \langle B \neq B1 \rangle \langle B1 \neq D \rangle \langle C\ A\ B\ \text{CongA } B\ D\ C \rangle$ 
     $\text{conga-distinct}$  by  $\text{presburger}$ 
have  $G \neq H$ 
  using  $\langle \text{Defect } A\ B1\ C\ G\ H\ I \rangle \text{defect-distincts}$  by  $\text{blast}$ 
have  $I \neq H$ 
  using  $\langle \text{Defect } A\ B1\ C\ G\ H\ I \rangle \text{defect-distincts}$  by  $\text{blast}$ 
have  $M \neq N$ 
  using  $\langle \text{Defect } C\ B\ B1\ M\ N\ O0 \rangle \text{defect-distincts}$  by  $\text{blast}$ 
have  $N \neq O0$ 
  using  $\langle \text{Defect } C\ B\ B1\ M\ N\ O0 \rangle \text{defect-distincts}$  by  $\text{blast}$ 
have  $A' \neq B'$ 
  using  $\langle \text{Defect } B1\ C\ D\ A'\ B'\ C' \rangle \text{defect-distincts}$  by  $\text{blast}$ 
have  $C' \neq B'$ 
  using  $\langle \text{Defect } B1\ C\ D\ A'\ B'\ C' \rangle \text{defect-distincts}$  by  $\text{blast}$ 
obtain  $G'\ H'\ I'$  where  $G\ H\ I\ A'\ B'\ C'\ \text{SumA } G'\ H'\ I'$ 
  using  $\langle A' \neq B' \rangle \langle C' \neq B' \rangle \langle G \neq H \rangle \langle I \neq H \rangle \text{ex-suma}$  by  $\text{presburger}$ 
have  $O0 \neq N$ 
  using  $\langle N \neq O0 \rangle$  by  $\text{blast}$ 
then obtain  $J'\ K'\ L'$  where  $M\ N\ O0\ A'\ B'\ C'\ \text{SumA } J'\ K'\ L'$ 
  using  $\langle A' \neq B' \rangle \langle C' \neq B' \rangle \langle M \neq N \rangle \text{ex-suma}$  by  $\text{presburger}$ 
have  $\text{SAMS } G\ H\ I\ J\ K\ L \wedge G\ H\ I\ J\ K\ L\ \text{SumA } S\ T\ U$ 
  using  $\text{assms } \langle \text{Bet } A\ C\ C1 \rangle \text{t22-16-1bis } \langle \text{Defect } A\ B1\ C\ G\ H\ I \rangle \langle \text{Defect } A\ B1\ C1\ S\ T\ U \rangle$ 
     $\langle \text{Defect } B1\ C\ C1\ J\ K\ L \rangle$  by  $\text{force}$ 
hence  $\text{SAMS } G\ H\ I\ J\ K\ L$ 
  by  $\text{blast}$ 
have  $G\ H\ I\ J\ K\ L\ \text{SumA } S\ T\ U$ 
  using  $\langle \text{SAMS } G\ H\ I\ J\ K\ L \wedge G\ H\ I\ J\ K\ L\ \text{SumA } S\ T\ U \rangle$  by  $\text{auto}$ 
have  $\text{SAMS } A'\ B'\ C'\ D'\ E'\ F' \wedge A'\ B'\ C'\ D'\ E'\ F'\ \text{SumA } J\ K\ L$ 
  using  $\text{assms } \langle \text{Bet } B1\ D\ C1 \rangle \text{t22-16-1bis } \langle \text{Defect } B1\ C\ C1\ J\ K\ L \rangle \langle \text{Defect } B1\ C\ D\ A'\ B'\ C' \rangle$ 
     $\langle \text{Defect } C\ D\ C1\ D'\ E'\ F' \rangle$  by  $\text{blast}$ 
hence  $\text{SAMS } A'\ B'\ C'\ D'\ E'\ F'$ 
  by  $\text{blast}$ 
have  $A'\ B'\ C'\ D'\ E'\ F'\ \text{SumA } J\ K\ L$ 
  by ( $\text{simp add: } \langle \text{SAMS } A'\ B'\ C'\ D'\ E'\ F' \wedge A'\ B'\ C'\ D'\ E'\ F'\ \text{SumA } J\ K\ L \rangle$ )
have  $\text{SAMS } P\ Q\ R\ M\ N\ O0 \wedge P\ Q\ R\ M\ N\ O0\ \text{SumA } G\ H\ I$ 
proof –
  have  $\text{Defect } A\ C\ B\ P\ Q\ R$ 
    using  $\langle \text{Defect } A\ B\ C\ P\ Q\ R \rangle \text{defect-perm-132}$  by  $\text{blast}$ 
  moreover have  $\text{Defect } A\ C\ B1\ G\ H\ I$ 
    using  $\langle \text{Defect } A\ B1\ C\ G\ H\ I \rangle \text{defect-perm-132}$  by  $\text{blast}$ 

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ultimately show ?thesis
 using ⟨Bet A B B1⟩ ⟨Defect C B B1 M N O0⟩ assms t22-16-1bis by blast

qed

hence SAMS P Q R M N O0

by blast

have P Q R M N O0 SumA G H I

by (simp add: ⟨SAMS P Q R M N O0 ∧ P Q R M N O0 SumA G H I⟩)

have SAMS G H I A' B' C'

proof –

have G H I LeA G H I

by (simp add: ⟨G ≠ H⟩ ⟨I ≠ H⟩ lea-refl)

moreover have A' B' C' LeA J K L

using ⟨SAMS A' B' C' D' E' F'⟩ ⟨A' B' C' D' E' F' SumA J K L⟩

sams-suma--lea123789 by blast

ultimately show ?thesis

using ⟨SAMS G H I J K L⟩ sams-lea2--sams by blast

qed

have SAMS M N O0 A' B' C'

proof –

have M N O0 LeA G H I

using ⟨SAMS P Q R M N O0⟩ ⟨P Q R M N O0 SumA G H I⟩

sams-suma--lea456789 by auto

moreover have A' B' C' LeA A' B' C'

using ⟨A' ≠ B'⟩ ⟨C' ≠ B'⟩ lea-refl by auto

ultimately show ?thesis

using ⟨SAMS G H I A' B' C'⟩ sams-lea2--sams by blast

qed

have G' H' I' D' E' F' SumA S T U

using ⟨SAMS G H I A' B' C'⟩ ⟨SAMS A' B' C' D' E' F'⟩

⟨G H I A' B' C' SumA G' H' I'⟩ ⟨A' B' C' D' E' F' SumA J K L⟩

⟨G H I J K L SumA S T U⟩ suma-assoc by blast

have SAMS G' H' I' D' E' F'

using ⟨SAMS G H I A' B' C'⟩ ⟨SAMS A' B' C' D' E' F'⟩

⟨G H I A' B' C' SumA G' H' I'⟩ ⟨A' B' C' D' E' F' SumA J K L⟩

⟨SAMS G H I J K L⟩ sams-assoc by blast

have P Q R J' K' L' SumA G' H' I'

using ⟨SAMS P Q R M N O0⟩ ⟨SAMS M N O0 A' B' C'⟩

⟨P Q R M N O0 SumA G H I⟩ ⟨M N O0 A' B' C' SumA J' K' L'⟩

⟨G H I A' B' C' SumA G' H' I'⟩ suma-assoc-1 by blast

have SAMS P Q R J' K' L'

using ⟨SAMS P Q R M N O0⟩ ⟨SAMS M N O0 A' B' C'⟩

⟨P Q R M N O0 SumA G H I⟩ ⟨M N O0 A' B' C' SumA J' K' L'⟩

⟨SAMS G H I A' B' C'⟩ sams-assoc-1 by blast

have P ≠ Q

using ⟨P Q R P Q R SumA V W X⟩ suma-distincts by auto

have Q ≠ R

using ⟨P Q R P Q R SumA V W X⟩ suma-distincts by blast

have SAMS P Q R M' N' O' ∧ P Q R M' N' O' SumA J' K' L'

proof –

have Defect C B D P Q R

proof –

have A B C CongA D C B

by (simp add: ⟨C B A CongA B C D⟩ conga-comm)

moreover have B C A CongA C B D

using ⟨A C B CongA C B D⟩ conga-left-comm by blast

ultimately show ?thesis

using ⟨Defect A B C P Q R⟩ ⟨C A B CongA B D C⟩ conga3-defect--defect

defect-perm-231 by blast

qed

moreover have Defect C D B1 A' B' C'

by (simp add: ⟨Defect B1 C D A' B' C'⟩ defect-perm-231)

ultimately show ?thesis

using ⟨Defect C B B1 M N O0⟩ ⟨Defect B1 B D M' N' O'⟩ ⟨Bet C Z B1⟩

⟨Bet B Z D⟩ ⟨SAMS M N O0 A' B' C'⟩ ⟨M N O0 A' B' C' SumA J' K' L'⟩

assms t22-16-2 by blast

qed

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have P Q R LeA J' K' L'
  using ⟨SAMS P Q R M' N' O' ∧ P Q R M' N' O' SumA J' K' L'⟩
    sams-suma--lea123789 by blast
have SAMS P Q R P Q R
proof –
  have P Q R LeA P Q R
    using ⟨P ≠ Q⟩ ⟨Q ≠ R⟩ lea-refl by force
  thus ?thesis
    using ⟨SAMS P Q R J' K' L'⟩ ⟨P Q R LeA J' K' L'⟩ sams-lea2--sams by blast
qed
moreover have V W X LeA S T U
proof –
  have V W X LeA G' H' I'
    using ⟨P Q R LeA J' K' L'⟩ ⟨SAMS P Q R J' K' L'⟩
      ⟨P Q R P Q R SumA V W X⟩ ⟨P Q R J' K' L' SumA G' H' I'⟩
        sams-lea456-suma2--lea by blast
  moreover have G' H' I' LeA S T U
    using ⟨G' H' I' D' E' F' SumA S T U⟩ ⟨SAMS G' H' I' D' E' F'⟩
      sams-suma--lea123789 by blast
  ultimately show ?thesis
    using lea-trans by blast
qed
ultimately have SAMS P Q R P Q R ∧ V W X LeA S T U
  by blast
}
thus ?thesis
  by blast
qed

```

```

lemma legendre-aux1:
shows ∀ A B C B' C'.
  ¬ Col A B C ∧ A Out B B' ∧ A Out C C' ⟶
  (∃ D'. D' InAngle B A C ∧ A C' B' CongA C' B' D' ∧
    Cong A C' B' D' ∧ B' C' TS A D')
proof –
{
  fix A B C B' C'
  assume ¬ Col A B C and A Out B B' and A Out C C'
  have A ≠ B'
    using Out-def ⟨A Out B B'⟩ by force
  have A ≠ C'
    using ⟨A Out C C'⟩ out-diff2 by blast
  have B' ≠ C'
    using ⟨A Out B B'⟩ ⟨A Out C C'⟩ ⟨¬ Col A B C⟩ l6-6 l6-7 out-col by blast
  {
    assume Col A B' C'
    hence Col A B C
      by (metis Col-perm ⟨A Out B B'⟩ ⟨A Out C C'⟩ ⟨A ≠ B'⟩
        ⟨A ≠ C'⟩ col-transitivity-1 out-col)
    hence False
      using ⟨¬ Col A B C⟩ by blast
  }
  hence ¬ Col A B' C'
    by blast
  obtain M where M Midpoint B' C'
    using midpoint-existence by blast
  have A ≠ M
    using Bet-cases Col-def Midpoint-def ⟨Col A B' C' ⟶ False⟩
      ⟨M Midpoint B' C'⟩ by blast
  have M ≠ B'
    using ⟨Col A B' C' ⟶ False⟩ ⟨M Midpoint B' C'⟩
      col-trivial-2 is-midpoint-id by blast
  have M ≠ C'
    using ⟨M Midpoint B' C'⟩ ⟨M ≠ B'⟩ is-midpoint-id-2 by blast
  obtain D' where M Midpoint A D'
    using symmetric-point-construction by blast
}

```

```

{
  assume Col A M B'
  hence Col A B' C'
  by (metis Col-def Midpoint-def ⟨M Midpoint B' C'⟩ ⟨M ≠ B'⟩ l6-16-1)
  hence False
  using ⟨Col A B' C' ⟹ False⟩ by blast
}
hence ¬ Col A M B'
  by blast
have A ≠ D'
  using ⟨M Midpoint A D'⟩ ⟨¬ Col A M B'⟩ col-trivial-1 l7-3 by blast
hence M ≠ D'
  using ⟨M Midpoint A D'⟩ is-midpoint-id-2 by blast
have Col A D' M
  using Bet-cases Col-def Midpoint-def ⟨M Midpoint A D'⟩ by blast
have Col B' C' M using ⟨M Midpoint B' C'⟩
  by (meson midpoint-col not-col-permutation-2)
{
  assume Col D' B' C'
  hence Col A B' C'
  by (metis (full-types) ⟨Col A D' M⟩ ⟨Col B' C' M⟩
    ⟨M ≠ D'⟩ col-permutation-2 colx)
  hence False
  using ⟨Col A B' C' ⟹ False⟩ by blast
}
hence ¬ Col D' B' C'
  by blast
have D' InAngle B A C
proof -
  have D' InAngle B' A C'
  proof -
    have Bet B' M C'
      using Midpoint-def ⟨M Midpoint B' C'⟩ by blast
    moreover have Bet A M D'
      using Midpoint-def ⟨M Midpoint A D'⟩ by auto
    moreover hence M = A ∨ A Out M D'
      using bet-out by blast
    ultimately show ?thesis
      using InAngle-def ⟨A ≠ B'⟩ ⟨A ≠ C'⟩ ⟨A ≠ D'⟩ by auto
  qed
  moreover have A Out D' D'
    using ⟨A ≠ D'⟩ out-trivial by auto
  ultimately show ?thesis
    using ⟨A Out B B'⟩ ⟨A Out C C'⟩ l11-25 by blast
qed
moreover have A C' B' CongA C' B' D'
proof -
  have A M C' CongA D' M B'
  proof -
    have Bet A M D'
      using ⟨M Midpoint A D'⟩ by (simp add: Midpoint-def)
    moreover have Bet C' M B'
      using Bet-cases Midpoint-def ⟨M Midpoint B' C'⟩ by blast
    ultimately show ?thesis
      using ⟨M ≠ B'⟩ ⟨M ≠ C'⟩ ⟨M ≠ D'⟩ ⟨A ≠ M⟩ l11-14 by blast
  qed
  moreover have Cong M A M D'
    using Cong-cases ⟨M Midpoint A D'⟩ midpoint-cong by blast
  moreover have Cong M C' M B'
    using ⟨M Midpoint B' C'⟩ midpoint-cong not-cong-4312 by blast
  ultimately show ?thesis
    by (meson Mid-cases ⟨A ≠ C'⟩ ⟨B' ≠ C'⟩ ⟨M Midpoint A D'⟩
      ⟨M Midpoint B' C'⟩ conga-right-comm symmetry-preserves-conga)
qed
moreover have Cong A C' B' D'
  by (metis l7-13 ⟨M Midpoint A D'⟩ ⟨M Midpoint B' C'⟩ cong-3421 l7-2)

```

```

moreover have  $B' C' TS A D'$ 
  using Midpoint-def TS-def  $\langle Col A B' C' \implies False \rangle \langle Col B' C' M \rangle$ 
     $\langle M \text{ Midpoint } A D' \rangle \langle \neg Col D' B' C' \rangle$  not-col-permutation-1 by blast
ultimately have  $\exists D'. D' InAngle B A C \wedge A C' B' CongA C' B' D' \wedge$ 
   $Cong A C' B' D' \wedge B' C' TS A D'$ 
  by blast
}
thus ?thesis
  by blast
qed

lemma legendre-aux2:
assumes  $\neg hypothesis\text{-of-obtuse-saccheri-quadrilaterals}$ 
shows  $\forall A B C.$ 
   $\neg Col A B C \wedge Acute B A C \wedge$ 
   $(\forall T. T InAngle B A C \longrightarrow (\exists X Y. A Out B X \wedge A Out C Y \wedge Bet X T Y))$ 
   $\longrightarrow$ 
   $(\forall P Q R S T U. Defect A B C P Q R \wedge GradAExp P Q R S T U \longrightarrow$ 
     $(\exists B' C' P' Q' R'. (A Out B B' \wedge A Out C C' \wedge$ 
       $Defect A B' C' P' Q' R' \wedge S T U LeA P' Q' R')))$ 
proof –
{
  fix  $A B C$ 
  assume  $\neg Col A B C$  and
     $Acute B A C$  and
     $\forall T. T InAngle B A C \longrightarrow (\exists X Y. A Out B X \wedge A Out C Y \wedge Bet X T Y)$ 
  {
    fix  $P Q R S T U$ 
    assume  $Defect A B C P Q R$  and
       $GradAExp P Q R S T U$ 
    let  $?th = (\exists B' C' P' Q' R'. (A Out B B' \wedge A Out C C' \wedge$ 
       $Defect A B' C' P' Q' R' \wedge S T U LeA P' Q' R'))$ 
    have  $?th$ 
    proof (rule GradAExp.induct [OF  $\langle GradAExp P Q R S T U \rangle$ ])
      show  $\bigwedge D E F.$ 
       $P Q R CongA D E F \implies$ 
       $\exists B' C' P' Q' R'. A Out B B' \wedge A Out C C' \wedge Defect A B' C' P' Q' R' \wedge D E F LeA P' Q' R'$ 
      by (metis  $\langle Defect A B C P Q R \rangle bet\text{-out-1 conga--lea456123$  defect-distincts not-bet-distincts)
    }
  {
    fix  $S T U G H I$ 
    assume  $GradAExp P Q R S T U$  and
       $P1: \exists B' C' P' Q' R'. A Out B B' \wedge A Out C C' \wedge Defect A B' C' P' Q' R' \wedge$ 
       $S T U LeA P' Q' R'$  and
       $SAMS S T U S T U$  and
       $S T U S T U SumA G H I$ 
    obtain  $B' C' P' Q' R'$  where  $A Out B B'$  and  $A Out C C'$  and
       $Defect A B' C' P' Q' R'$  and  $S T U LeA P' Q' R'$  using  $P1$  by blast
    obtain  $D'$  where  $D' InAngle B A C$  and  $A C' B' CongA C' B' D'$  and
       $Cong A C' B' D'$  and  $B' C' TS A D'$ 
    using legendre-aux1  $\langle A Out B B' \rangle \langle A Out C C' \rangle \langle \neg Col A B C \rangle$  by blast
    have  $\neg Col A B' C'$ 
    using TS-def  $\langle B' C' TS A D' \rangle$  by presburger
    obtain  $B'' C''$  where  $A Out B B''$  and  $A Out C C''$  and  $Bet B'' D' C''$ 
    using  $\langle D' InAngle B A C \rangle$ 
     $\langle \forall T. T InAngle B A C \longrightarrow (\exists X Y. A Out B X \wedge A Out C Y \wedge Bet X T Y) \rangle$  by blast
    have  $Q' \neq P'$ 
    using  $\langle Defect A B' C' P' Q' R' \rangle$  defect-distincts by blast
    have  $Q' \neq R'$ 
    using  $\langle Defect A B' C' P' Q' R' \rangle$  defect-distincts by blast
    have  $A \neq C''$ 
    using  $\langle A Out C C'' \rangle$  l6-3-1 by blast
    have  $A \neq B''$ 
    using  $\langle A Out B B'' \rangle$  out-distinct by blast
  }
}

```

have $B'' \neq C''$
by (*metis* *Col-def* *Out-def* $\langle A \text{ Out } B B' \rangle \langle A \text{ Out } C C' \rangle$
 $\langle \neg \text{Col } A B C \rangle$ *between-symmetry* *l6-7*)
then obtain $P'' Q'' R''$ **where** *Defect* $A B'' C'' P'' Q'' R''$
using *ex-defect* $\langle A \neq B'' \rangle \langle A \neq C'' \rangle$ **by** *presburger*
moreover
obtain $V W X$ **where** $P' Q' R' P' Q' R' \text{ SumA } V W X$
using *ex-suma* $\langle Q' \neq P' \rangle \langle Q' \neq R' \rangle$ **by** *presburger*
have *SAMS* $P' Q' R' P' Q' R' \wedge V W X \text{ LeA } P'' Q'' R''$
proof –
have $A \text{ Out } B' B''$
using $\langle A \text{ Out } B B' \rangle \langle A \text{ Out } B B' \rangle$ *l6-6 l6-7* **by** *blast*
moreover have $A \text{ Out } C' C''$
using $\langle A \text{ Out } C C' \rangle \langle A \text{ Out } C C' \rangle$ *l6-6 l6-7* **by** *blast*
ultimately show *?thesis*
using *assms* $\langle \neg \text{Col } A B' C' \rangle \langle A C' B' \text{ CongA } C' B' D' \rangle \langle \text{Cong } A C' B' D' \rangle$
 $\langle B' C' \text{ TS } A D' \rangle \langle P' Q' R' P' Q' R' \text{ SumA } V W X \rangle \langle \text{Bet } B'' D' C'' \rangle$
 $\langle \text{Defect } A B' C' P' Q' R' \rangle \langle \text{Defect } A B'' C'' P'' Q'' R'' \rangle$ *legendre-aux* **by** *blast*
qed
have $G H I \text{ LeA } P'' Q'' R''$
proof –
have $G H I \text{ LeA } V W X$
using *sams-lea2-suma2--lea* [**where** $?A=S$ **and** $?B=T$ **and** $?C=U$ **and**
 $?D=S$ **and** $?E=T$ **and** $?F=U$ **and** $?A'=P'$ **and** $?B'=Q'$ **and** $?C'=R'$ **and**
 $?D'=P'$ **and** $?E'=Q'$ **and** $?F'=R'$] $\langle S T U \text{ LeA } P' Q' R' \rangle \langle S T U \text{ LeA } P' Q' R' \rangle$
 $\langle \text{SAMS } P' Q' R' P' Q' R' \wedge V W X \text{ LeA } P'' Q'' R'' \rangle$
 $\langle S T U S T U \text{ SumA } G H I \rangle \langle P' Q' R' P' Q' R' \text{ SumA } V W X \rangle$
by *blast*
moreover have $V W X \text{ LeA } P'' Q'' R''$
using $\langle \text{SAMS } P' Q' R' P' Q' R' \wedge V W X \text{ LeA } P'' Q'' R'' \rangle$ **by** *blast*
ultimately show *?thesis*
using *lea-trans* **by** *blast*
qed
ultimately have $\exists B' C' P' Q' R'. A \text{ Out } B B' \wedge A \text{ Out } C C' \wedge$
 $\text{Defect } A B' C' P' Q' R' \wedge G H I \text{ LeA } P' Q' R'$
using $\langle A \text{ Out } B B' \rangle \langle A \text{ Out } C C' \rangle$ **by** *blast*
}
thus $\bigwedge D E F G H I.$
 $\text{GradAExp } P Q R D E F \implies$
 $\exists B' C' P' Q' R'.$
 $A \text{ Out } B B' \wedge A \text{ Out } C C' \wedge \text{Defect } A B' C' P' Q' R' \wedge D E F \text{ LeA } P' Q' R' \implies$
 $\text{SAMS } D E F D E F \implies$
 $D E F D E F \text{ SumA } G H I \implies$
 $\exists B' C' P' Q' R'.$
 $A \text{ Out } B B' \wedge$
 $A \text{ Out } C C' \wedge \text{Defect } A B' C' P' Q' R' \wedge G H I \text{ LeA } P' Q' R'$
by *blast*
qed
}
hence $(\forall P Q R S T U.$
 $\text{Defect } A B C P Q R \wedge \text{GradAExp } P Q R S T U \implies$
 $(\exists B' C' P' Q' R'.$
 $(A \text{ Out } B B' \wedge A \text{ Out } C C' \wedge$
 $\text{Defect } A B' C' P' Q' R' \wedge S T U \text{ LeA } P' Q' R'))$ **by** *blast*
}
thus *?thesis*
by *blast*
qed

lemma *legendre-s-fourth-theorem-aux*:
assumes *archimedes-axiom* **and**
legendre-s-parallel-postulate
shows *postulate-of-right-saccheri-quadrilaterals*
proof –
obtain $A B C$ **where** $\neg \text{Col } B A C$ **and** *Acute* $B A C$ **and**
 $\forall T. (T \text{ InAngle } B A C) \longrightarrow (\exists X Y. A \text{ Out } B X \wedge A \text{ Out } C Y \wedge \text{Bet } X T Y)$

```

    using assms(2) legendre-s-parallel-postulate-def by blast
  have  $A \neq B$ 
    using  $\langle \text{Acute } B \ A \ C \rangle$  acute-distincts by auto
  have  $A \neq C$ 
    using  $\langle \neg \text{Col } B \ A \ C \rangle$  not-col-distincts by auto
  have  $B \neq C$ 
    using  $\langle \neg \text{Col } B \ A \ C \rangle$  col-trivial-3 by auto
  obtain  $P \ Q \ R$  where Defect  $A \ B \ C \ P \ Q \ R$ 
    using  $\langle A \neq B \rangle \langle A \neq C \rangle \langle B \neq C \rangle$  ex-defect by presburger
{
  assume  $\text{Col } P \ Q \ R$ 
  hence  $\neg$  hypothesis-of-obtuse-saccheri-quadrilaterals
    using archi--obtuse-case-elimination assms(1) by presburger
  moreover have  $\neg \text{Col } A \ B \ C$ 
    by (simp add:  $\langle \neg \text{Col } B \ A \ C \rangle$  not-col-permutation-4)
  moreover have  $Q \text{ Out } P \ R$ 
  proof -
    {
      assume  $\text{Bet } P \ Q \ R$ 
      have Defect  $B \ A \ C \ P \ Q \ R$ 
        using  $\langle \text{Defect } A \ B \ C \ P \ Q \ R \rangle$  defect-perm-213 by blast
      then obtain  $D \ E \ F \ G \ H \ I$  where  $B \ A \ C \ A \ C \ B \text{ SumA } G \ H \ I$  and
         $G \ H \ I \ C \ B \ A \text{ SumA } D \ E \ F$  and  $D \ E \ F \text{ SuppA } P \ Q \ R$ 
        using Defect-def TriSumA-def by auto
      have  $\text{Col } B \ A \ C$ 
      proof -
        have  $P \ Q \ R \text{ SuppA } D \ E \ F$ 
          by (simp add:  $\langle D \ E \ F \text{ SuppA } P \ Q \ R \rangle$  suppa-sym)
        hence  $E \text{ Out } D \ F$ 
          using  $\langle \text{Bet } P \ Q \ R \rangle$  bet-suppa--out by blast
        moreover
        have  $\text{SAMS } G \ H \ I \ C \ B \ A$ 
          using  $\langle B \ A \ C \ A \ C \ B \text{ SumA } G \ H \ I \rangle$  assms(1) legendre-s-first-theorem by blast
        hence  $C \ B \ A \text{ LeA } D \ E \ F$ 
          using  $\langle G \ H \ I \ C \ B \ A \text{ SumA } D \ E \ F \rangle$  sams-suma--lea456789 by blast
        ultimately show ?thesis
          using l6-6 out-col out-lea--out by blast
      qed
      hence False
        by (simp add:  $\langle \neg \text{Col } B \ A \ C \rangle$ )
    }
  hence  $\neg \text{Bet } P \ Q \ R$ 
    by blast
  thus ?thesis
    using  $\langle \text{Col } P \ Q \ R \rangle$  l6-4-2 by blast
qed
ultimately
have postulate-of-right-saccheri-quadrilaterals
  using  $\langle \text{Defect } A \ B \ C \ P \ Q \ R \rangle$  defect-ncol-out--rah
  existential-playfair--rah-1 by blast
}
moreover
{
  assume  $\neg \text{Col } P \ Q \ R$ 
  obtain  $S \ T \ U$  where GradAExp  $P \ Q \ R \ S \ T \ U$  and Obtuse  $S \ T \ U$ 
    using archi--gradaexp-destruction  $\langle \neg \text{Col } P \ Q \ R \rangle$  assms(1) by blast
  have  $\neg$  hypothesis-of-obtuse-saccheri-quadrilaterals
    by (simp add: archi--obtuse-case-elimination assms(1))
  hence P2:  $\forall A \ B \ C. \neg \text{Col } A \ B \ C \wedge \text{Acute } B \ A \ C \wedge$ 
 $(\forall T. T \text{ InAngle } B \ A \ C \longrightarrow (\exists X \ Y. A \text{ Out } B \ X \wedge A \text{ Out } C \ Y \wedge \text{Bet } X \ T \ Y))$ 
 $\longrightarrow$ 
 $(\forall P \ Q \ R \ S \ T \ U. \text{Defect } A \ B \ C \ P \ Q \ R \wedge \text{GradAExp } P \ Q \ R \ S \ T \ U \longrightarrow$ 
 $(\exists B' \ C' \ P' \ Q' \ R'. (A \text{ Out } B \ B' \wedge A \text{ Out } C \ C' \wedge$ 
 $\text{Defect } A \ B' \ C' \ P' \ Q' \ R' \wedge S \ T \ U \text{ LeA } P' \ Q' \ R'))$ 
    using legendre-aux2 by blast

```


moreover have $\neg \text{Col } A \ B \ C$
by (*simp add*: $\langle \neg \text{Col } B \ A \ C \rangle$ *not-col-permutation-4*)
ultimately have $\exists B' \ C' \ P' \ Q' \ R'. (A \ \text{Out } B \ B' \wedge A \ \text{Out } C \ C' \wedge$
 $\text{Defect } A \ B' \ C' \ P' \ Q' \ R' \wedge S \ T \ U \ \text{LeA } P' \ Q' \ R')$
using $\langle \forall T. T \ \text{InAngle } B \ A \ C \longrightarrow (\exists X \ Y. A \ \text{Out } B \ X \wedge A \ \text{Out } C \ Y \wedge \text{Bet } X \ T \ Y) \rangle \langle \text{Acute } B \ A \ C \rangle$
 $\langle \text{Defect } A \ B \ C \ P \ Q \ R \rangle \langle \text{GradAExp } P \ Q \ R \ S \ T \ U \rangle$ **by blast**
then obtain $B' \ C' \ P' \ Q' \ R'$ **where** $A \ \text{Out } B \ B'$ **and** $A \ \text{Out } C \ C'$ **and**
 $\text{Defect } A \ B' \ C' \ P' \ Q' \ R'$ **and** $S \ T \ U \ \text{LeA } P' \ Q' \ R'$
by blast
have $\neg \text{SAMS } P' \ Q' \ R' \ P' \ Q' \ R'$
using *lea-obtuse-obtuse* $\langle \text{Obtuse } S \ T \ U \rangle \langle S \ T \ U \ \text{LeA } P' \ Q' \ R' \rangle$ *obtuse--nsams* **by blast**
obtain D' **where** $D' \ \text{InAngle } B \ A \ C$ **and** $A \ C' \ B' \ \text{CongA } C' \ B' \ D'$ **and**
 $\text{Cong } A \ C' \ B' \ D'$ **and** $B' \ C' \ \text{TS } A \ D'$
using *legendre-aux1* $\langle A \ \text{Out } B \ B' \rangle \langle A \ \text{Out } C \ C' \rangle \langle \neg \text{Col } A \ B \ C \rangle$ **by blast**
have $\neg \text{Col } A \ B' \ C'$
using *TS-def* $\langle B' \ C' \ \text{TS } A \ D' \rangle$ **by auto**
obtain $B'' \ C''$ **where** $A \ \text{Out } B \ B''$ **and** $A \ \text{Out } C \ C''$ **and** $\text{Bet } B'' \ D' \ C''$
using $\langle D' \ \text{InAngle } B \ A \ C \rangle$
 $\langle \forall T. T \ \text{InAngle } B \ A \ C \longrightarrow (\exists X \ Y. A \ \text{Out } B \ X \wedge A \ \text{Out } C \ Y \wedge \text{Bet } X \ T \ Y) \rangle$
by blast
have $A \neq B''$
using *Out-def* $\langle A \ \text{Out } B \ B'' \rangle$ **by auto**
have $A \neq C''$
using $\langle A \ \text{Out } C \ C'' \rangle$ *l6-3-1* **by blast**
have $B'' \neq C''$
using $\langle A \ \text{Out } B \ B'' \rangle \langle A \ \text{Out } C \ C'' \rangle \langle \neg \text{Col } A \ B \ C \rangle$ *l6-6 l6-7 out-col* **by blast**
then obtain $S' \ T' \ U'$ **where** $\text{Defect } A \ B'' \ C'' \ S' \ T' \ U'$
using *ex-defect* $\langle A \neq B'' \rangle \langle A \neq C'' \rangle$ **by presburger**
have $P' \neq Q'$
using $\langle \text{Defect } A \ B' \ C' \ P' \ Q' \ R' \rangle$ *defect-distincts* **by blast**
have $R' \neq Q'$
using $\langle S \ T \ U \ \text{LeA } P' \ Q' \ R' \rangle$ *lea-distincts* **by blast**
then obtain $V \ W \ X$ **where** $P' \ Q' \ R' \ P' \ Q' \ R' \ \text{SumA } V \ W \ X$
using *ex-suma* $\langle P' \neq Q' \rangle$ **by presburger**
have $\text{SAMS } P' \ Q' \ R' \ P' \ Q' \ R'$
proof –
have $A \ \text{Out } B' \ B''$
using *Out-cases* $\langle A \ \text{Out } B \ B'' \rangle \langle A \ \text{Out } B \ B' \rangle$ *l6-7* **by blast**
moreover have $A \ \text{Out } C' \ C''$
using $\langle A \ \text{Out } C \ C'' \rangle \langle A \ \text{Out } C \ C' \rangle$ *l6-6 l6-7* **by blast**
ultimately show *?thesis*
using *legendre-aux* $\langle P' \ Q' \ R' \ P' \ Q' \ R' \ \text{SumA } V \ W \ X \rangle \langle \text{Defect } A \ B' \ C' \ P' \ Q' \ R' \rangle$
 $\langle \text{Defect } A \ B'' \ C'' \ S' \ T' \ U' \rangle \langle \neg \text{Col } A \ B' \ C' \rangle \langle A \ C' \ B' \ \text{CongA } C' \ B' \ D' \rangle$
 $\langle \text{Cong } A \ C' \ B' \ D' \rangle \langle B' \ C' \ \text{TS } A \ D' \rangle \langle \text{Bet } B'' \ D' \ C'' \rangle$
 $\langle \neg \text{HypothesisObtuseSaccheriQuadrilaterals} \rangle$ **by blast**
qed
hence *False* **using** $\langle \neg \text{SAMS } P' \ Q' \ R' \ P' \ Q' \ R' \rangle$ **by blast**
hence $\forall A \ B \ C \ D. \text{Saccheri } A \ B \ C \ D \longrightarrow \text{Per } A \ B \ C$
by blast
}
ultimately show *?thesis*
using *postulate-of-right-saccheri-quadrilaterals-def* **by blast**
qed

theorem *legendre-s-fourth-theorem*:

assumes *archimedes-axiom* **and**

legendre-s-parallel-postulate

shows *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*

using *assms(1) assms(2) legendre-s-fourth-theorem-aux rah--triangle*

triangle--existential-triangle **by blast**

lemma *P34-P21*:

assumes *archimedes-axiom* **and**

Postulate34

shows *Postulate21*

using *assms(1) assms(2) legendre-s-fourth-theorem Postulate21-def Postulate34-def* **by blast**

```

lemma P34-P27:
  assumes archimedes-axiom and
    Postulate34
  shows Postulate27
proof -
  have Postulate21
    using P34-P21 assms(1) assms(2) by blast
  thus ?thesis
    using Cycle-3 by blast
qed

lemma P25-33:
  assumes Postulate25
  shows Postulate33
  using Postulate25-def Postulate33-def assms
    thales-converse-postulate--weak-triangle-circumscription-principle by auto

lemma P23-33:
  assumes Postulate23
  shows Postulate33
proof -
  have Postulate25
    using assms(1) Cycle-3 by blast
  thus ?thesis
    using P25-33 by blast
qed

lemma P01-35:
  assumes Postulate01
  shows Postulate35
  using Postulate01-def Postulate35-def assms playfair--existential-playfair
    tarski-s-euclid-implies-playfair-s-postulate by blast

lemma P35-27:
  assumes Postulate35
  shows Postulate27
  using Postulate27-def Postulate35-def assms existential-playfair--rah by blast

lemma Thm10-1:
  assumes archimedes-axiom
  shows (Postulate01  $\longleftrightarrow$  Postulate02)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate03)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate04)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate05)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate06)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate07)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate08)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate09)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate10)
  using InterAx5 InterCycle3 Postulate01-def Postulate03-def playfair--alternate-interior
    tarski-s-euclid-implies-playfair-s-postulate alternate-interior--triangle
    Cycle-2 Postulate03-def Postulate20-def assms legendre-s-third-theorem InterAx5
    Cycle-3 P04-P33 P25-33 P34-P27 P4-P34 assms InterCycle1bis Postulate02-def
    Postulate12-def playfair-bis--playfair Postulate07-def Postulate08-def
    alternate-interior--consecutive-interior consecutive-interior--alternate-interior
    equivalent-postulates-without-decidability-of-intersection-of-lines
    equivalent-postulates-without-decidability-of-intersection-of-lines
  by blast

lemma Thm10-2:
  assumes archimedes-axiom
  shows (Postulate01  $\longleftrightarrow$  Postulate11)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate12)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate13)  $\wedge$ 
    (Postulate01  $\longleftrightarrow$  Postulate14)  $\wedge$ 

```

$(Postulate01 \longleftrightarrow Postulate15) \wedge$
 $(Postulate01 \longleftrightarrow Postulate16) \wedge$
 $(Postulate01 \longleftrightarrow Postulate17) \wedge$
 $(Postulate01 \longleftrightarrow Postulate18) \wedge$
 $(Postulate01 \longleftrightarrow Postulate19) \wedge$
 $(Postulate01 \longleftrightarrow Postulate20)$
using *InterAx5 InterCycle1bis InterCycle3 Postulate02-def Postulate03-def*
Postulate12-def playfair--alternate-interior alternate-interior--triangle
playfair-bis--playfair equivalent-postulates-without-decidability-of-intersection-of-lines
Cycle-2 Cycle-1 Postulate11-def Postulate15-def
inter-dec-plus-par-perp-perp-imply-triangle-circumscription
universal-posidonius-postulate--perpendicular-transversal-postulate
by *blast*

lemma *Thm10-3:*

assumes *archimedes-axiom*
shows $(Postulate01 \longleftrightarrow Postulate21) \wedge$
 $(Postulate01 \longleftrightarrow Postulate22) \wedge$
 $(Postulate01 \longleftrightarrow Postulate23) \wedge$
 $(Postulate01 \longleftrightarrow Postulate24) \wedge$
 $(Postulate01 \longleftrightarrow Postulate25) \wedge$
 $(Postulate01 \longleftrightarrow Postulate26) \wedge$
 $(Postulate01 \longleftrightarrow Postulate27) \wedge$
 $(Postulate01 \longleftrightarrow Postulate28) \wedge$
 $(Postulate01 \longleftrightarrow Postulate29) \wedge$
 $(Postulate01 \longleftrightarrow Postulate30)$
using *Postulate03-def Postulate21-def legendre-s-second-theorem*
triangle--existential-triangle assms Thm10-1 Cycle-3 **by** *blast*

lemma *Thm10-4:*

assumes *archimedes-axiom*
shows $(Postulate01 \longleftrightarrow Postulate31) \wedge$
 $(Postulate01 \longleftrightarrow Postulate32) \wedge$
 $(Postulate01 \longleftrightarrow Postulate33) \wedge$
 $(Postulate01 \longleftrightarrow Postulate34) \wedge$
 $(Postulate01 \longleftrightarrow Postulate35)$
using *assms Thm10-1 P31-P04 P31-P04 P31-P32 P04-P33 P4-P34 P34-P27 Thm10-3*
P01--P35 P35-27 **by** *blast*

theorem *Thm10:*

assumes *archimedes-axiom*
shows $(Postulate01 \longleftrightarrow Postulate02) \wedge$
 $(Postulate01 \longleftrightarrow Postulate03) \wedge$
 $(Postulate01 \longleftrightarrow Postulate04) \wedge$
 $(Postulate01 \longleftrightarrow Postulate05) \wedge$
 $(Postulate01 \longleftrightarrow Postulate06) \wedge$
 $(Postulate01 \longleftrightarrow Postulate07) \wedge$
 $(Postulate01 \longleftrightarrow Postulate08) \wedge$
 $(Postulate01 \longleftrightarrow Postulate09) \wedge$
 $(Postulate01 \longleftrightarrow Postulate10) \wedge$
 $(Postulate01 \longleftrightarrow Postulate11) \wedge$
 $(Postulate01 \longleftrightarrow Postulate12) \wedge$
 $(Postulate01 \longleftrightarrow Postulate13) \wedge$
 $(Postulate01 \longleftrightarrow Postulate14) \wedge$
 $(Postulate01 \longleftrightarrow Postulate15) \wedge$
 $(Postulate01 \longleftrightarrow Postulate16) \wedge$
 $(Postulate01 \longleftrightarrow Postulate17) \wedge$
 $(Postulate01 \longleftrightarrow Postulate18) \wedge$
 $(Postulate01 \longleftrightarrow Postulate19) \wedge$
 $(Postulate01 \longleftrightarrow Postulate20) \wedge$
 $(Postulate01 \longleftrightarrow Postulate21) \wedge$
 $(Postulate01 \longleftrightarrow Postulate22) \wedge$
 $(Postulate01 \longleftrightarrow Postulate23) \wedge$
 $(Postulate01 \longleftrightarrow Postulate24) \wedge$
 $(Postulate01 \longleftrightarrow Postulate25) \wedge$

(Postulate01 $\longleftrightarrow$ Postulate26) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate27) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate28) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate29) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate30) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate31) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate32) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate33) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate34) $\wedge$
 (Postulate01 $\longleftrightarrow$ Postulate35)

using InterAx5 InterCycle3 Postulate01-def Postulate03-def playfair--alternate-interior
 tarski-s-euclid-implies-playfair-s-postulate alternate-interior--triangle
 Cycle-2 Postulate03-def Postulate20-def assms legendre-s-third-theorem InterAx5
 Cycle-3 P04-P33 P25-33 P34-P27 P4-P34 assms InterCycle1bis Postulate02-def
 Postulate12-def playfair-bis--playfair Postulate07-def Postulate08-def
 alternate-interior--consecutive-interior consecutive-interior--alternate-interior
 equivalent-postulates-without-decidability-of-intersection-of-lines
 equivalent-postulates-without-decidability-of-intersection-of-lines
 InterAx5 InterCycle1bis InterCycle3 Postulate02-def Postulate03-def
 Postulate12-def playfair--alternate-interior alternate-interior--triangle
 playfair-bis--playfair equivalent-postulates-without-decidability-of-intersection-of-lines
 Cycle-2 Cycle-1 Postulate11-def Postulate15-def
 inter-dec-plus-par-perp-perp-imply-triangle-circumscription
 universal-posidonius-postulate--perpendicular-transversal-postulate
 Postulate03-def Postulate21-def legendre-s-second-theorem
 triangle--existential-triangle assms Thm10-1 Cycle-3 assms Thm10-1 P31-P04
 P31-P04 P31-P32 P04-P33 P4-P34 P34-P27 Thm10-3 P01--P35 P35-27 **by** blast

theorem equivalent-postulates-assuming-greenberg-s-axiom:

assumes greenberg-s-axiom

shows (tarski-s-parallel-postulate $\longleftrightarrow$ alternate-interior-angles-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ alternative-playfair-s-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ alternative-playfair-s-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ alternative-proclus-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ alternative-strong-parallel-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ consecutive-interior-angles-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ euclid-5) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ euclid-s-parallel-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ existential-playfair-s-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ existential-thales-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ inverse-projection-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ midpoint-converse-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ perpendicular-transversal-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ postulate-of-transitivity-of-parallelism) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ playfair-s-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ posidonius-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ universal-posidonius-postulate) $\wedge$
 (tarski-s-parallel-postulate
 $\longleftrightarrow$ postulate-of-existence-of-a-right-lambert-quadrilateral) $\wedge$
 (tarski-s-parallel-postulate
 $\longleftrightarrow$ postulate-of-existence-of-a-right-saccheri-quadrilateral) $\wedge$
 (tarski-s-parallel-postulate
 $\longleftrightarrow$ postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ postulate-of-existence-of-similar-triangles) $\wedge$
 (tarski-s-parallel-postulate
 $\longleftrightarrow$ postulate-of-parallelism-of-perpendicular-transversals) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ postulate-of-right-lambert-quadrilaterals) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ postulate-of-right-saccheri-quadrilaterals) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ postulate-of-transitivity-of-parallelism) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ proclus-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ strong-parallel-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ tarski-s-parallel-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ thales-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ thales-converse-postulate) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ triangle-circumscription-principle) $\wedge$
 (tarski-s-parallel-postulate $\longleftrightarrow$ triangle-postulate)

```

proof –
  have aristotle-s-axiom
    using aristotle-s-axiom-def assms greenberg--aristotle by blast
  have tarski-s-parallel-postulate  $\longleftrightarrow$  alternate-interior-angles-postulate
    using InterAx5 Postulate01-def Postulate02-def alternate-interior--playfair-bis
      playfair--alternate-interior playfair-bis--playfair
      tarski-s-euclid-implies-playfair-s-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  alternative-playfair-s-postulate
    using alternate-interior--playfair-bis calculation playfair--alternate-interior
      playfair-bis--playfair by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  alternative-proclus-postulate
    using inverse-projection-postulate--proclus-bis original-euclid--original-spp
      original-spp--inverse-projection-postulate proclus-bis--proclus
      proclus-s-postulate-implies-strong-parallel-postulate
      strong-parallel-postulate-implies-tarski-s-euclid
      tarski-s-implies-euclid-s-parallel-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  alternative-strong-parallel-postulate
    using calculation(3) inverse-projection-postulate--proclus-bis
      original-euclid--original-spp original-spp--inverse-projection-postulate
      tarski-s-implies-euclid-s-parallel-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  consecutive-interior-angles-postulate
    using alternate-interior--consecutive-interior calculation(1)
      consecutive-interior--alternate-interior by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  euclid-5
    using calculation(4) euclid-5--original-euclid original-euclid--original-spp
      tarski-s-euclid-implies-euclid-5 by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  euclid-s-parallel-postulate
    using calculation(4) original-euclid--original-spp
      tarski-s-implies-euclid-s-parallel-postulate by linarith
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  existential-playfair-s-postulate
    using Postulate35-def Cycle-3 InterCycle1 P01--P35 P35-27 Postulate01-def
      Postulate12-def greenberg-s-axiom calculation(2) by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  existential-thales-postulate
    using Postulate26-def Cycle-3 InterCycle1 P01--P35 P35-27 Postulate01-def
      Postulate12-def greenberg-s-axiom calculation(2) by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  inverse-projection-postulate
    using calculation(3) calculation(4) inverse-projection-postulate--proclus-bis
      original-spp--inverse-projection-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  midpoint-converse-postulate
    using Postulate06-def calculation(2) calculation(8)
      midpoint-converse-postulate-implies-playfair playfair--existential-playfair
      playfair-bis--playfair playfair-s-postulate-implies-midpoint-converse-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  perpendicular-transversal-postulate
    using Postulate09-def inter-dec-plus-par-perp-perp-imp-ly-triangle-circumscription
      playfair--universal-posidonius-postulate
      tarski-s-euclid-implies-playfair-s-postulate
      triangle-circumscription-implies-tarski-s-euclid
      universal-posidonius-postulate--perpendicular-transversal-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  postulate-of-transitivity-of-parallelism
    using InterAx5 Postulate01-def Postulate02-def par-trans-implies-playfair
      playfair-implies-par-trans tarski-s-euclid-implies-playfair-s-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  playfair-s-postulate
    using calculation(13) playfair-implies-par-trans
      tarski-s-euclid-implies-playfair-s-postulate by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  posidonius-postulate
    using Postulate22-def Cycle-3 Postulate26-def calculation(9) by blast
  moreover have tarski-s-parallel-postulate  $\longleftrightarrow$  universal-posidonius-postulate
    using Postulate11-def calculation(12) playfair--universal-posidonius-postulate
      tarski-s-euclid-implies-playfair-s-postulate
      universal-posidonius-postulate--perpendicular-transversal-postulate by blast
  moreover have tarski-s-parallel-postulate
     $\longleftrightarrow$  postulate-of-existence-of-a-right-lambert-quadrilateral
    using Cycle-3 Postulate22-def Postulate30-def calculation(15) by blast
  moreover have tarski-s-parallel-postulate
     $\longleftrightarrow$  postulate-of-existence-of-a-right-saccheri-quadrilateral
    using Cycle-3 Postulate28-def Postulate30-def calculation(17) by blast

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moreover have *tarski-s-parallel-postulate*
 $\longleftrightarrow$ *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
using *Cycle-3 Postulate21-def Postulate22-def calculation(15)* **by** *blast*
moreover have *tarski-s-parallel-postulate*
 $\longleftrightarrow$ *postulate-of-existence-of-similar-triangles*
using *Cycle-3 Postulate22-def Postulate23-def calculation(15)* **by** *blast*
moreover have *tarski-s-parallel-postulate*
 $\longleftrightarrow$ *postulate-of-parallelism-of-perpendicular-transversals*
using *calculation(12) par-perp-2-par-implies-par-perp-perp*
par-perp-perp-implies-par-perp-2-par **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *postulate-of-right-lambert-quadrilaterals*
using *Cycle-3 Postulate26-def Postulate29-def calculation(9)* **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *postulate-of-right-saccheri-quadrilaterals*
using *calculation(18) calculation(9) rah--existential-saccheri thales-existence--rah* **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *proclus-postulate*
using *Cycle-2 Postulate13-def Postulate01-def* **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *strong-parallel-postulate*
using *calculation(24) proclus-s-postulate-implies-strong-parallel-postulate*
strong-parallel-postulate-implies-tarski-s-euclid **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *thales-postulate*
using *Cycle-3 Postulate24-def Postulate26-def calculation(9)* **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *thales-converse-postulate*
using *Cycle-3 Postulate25-def Postulate30-def calculation(17)* **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *triangle-circumscription-principle*
using *inter-dec-plus-par-perp-perp-imp-ly-triangle-circumscription calculation(12)*
triangle-circumscription-implies-tarski-s-euclid **by** *blast*
moreover have *tarski-s-parallel-postulate* $\longleftrightarrow$ *triangle-postulate*
using *Cycle-3 Postulate03-def Postulate30-def calculation(17)* **by** *blast*
ultimately show *?thesis*
by *blast*
qed

theorem *equivalent-postulates-assuming-archimedesdes-axiom:*

assumes *archimedes-axiom*

shows *alternate-interior-angles-postulate* $\longleftrightarrow$ *alternative-playfair-s-postulate* $\wedge$
alternative-playfair-s-postulate $\longleftrightarrow$ *alternative-proclus-postulate* $\wedge$
alternative-proclus-postulate $\longleftrightarrow$ *alternative-strong-parallel-postulate* $\wedge$
alternative-strong-parallel-postulate $\longleftrightarrow$ *bachmann-s-lotschnittaxiom* $\wedge$
bachmann-s-lotschnittaxiom $\longleftrightarrow$ *consecutive-interior-angles-postulate* $\wedge$
consecutive-interior-angles-postulate $\longleftrightarrow$ *euclid-5* $\wedge$
euclid-5 $\longleftrightarrow$ *euclid-s-parallel-postulate* $\wedge$
euclid-s-parallel-postulate $\longleftrightarrow$ *existential-playfair-s-postulate* $\wedge$
existential-playfair-s-postulate $\longleftrightarrow$ *existential-thales-postulate* $\wedge$
existential-thales-postulate $\longleftrightarrow$ *inverse-projection-postulate* $\wedge$
inverse-projection-postulate $\longleftrightarrow$ *legendre-s-parallel-postulate* $\wedge$
legendre-s-parallel-postulate $\longleftrightarrow$ *midpoint-converse-postulate* $\wedge$
midpoint-converse-postulate $\longleftrightarrow$ *perpendicular-transversal-postulate* $\wedge$
perpendicular-transversal-postulate $\longleftrightarrow$ *postulate-of-transitivity-of-parallelism* $\wedge$
postulate-of-transitivity-of-parallelism $\longleftrightarrow$ *playfair-s-postulate* $\wedge$
playfair-s-postulate $\longleftrightarrow$ *posidonius-postulate* $\wedge$
posidonius-postulate $\longleftrightarrow$ *universal-posidonius-postulate* $\wedge$
universal-posidonius-postulate $\longleftrightarrow$
postulate-of-existence-of-a-right-lambert-quadrilateral $\wedge$
postulate-of-existence-of-a-right-lambert-quadrilateral $\longleftrightarrow$
postulate-of-existence-of-a-right-saccheri-quadrilateral $\wedge$
postulate-of-existence-of-a-right-saccheri-quadrilateral $\longleftrightarrow$
postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights $\wedge$
postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights $\longleftrightarrow$
postulate-of-existence-of-similar-triangles $\wedge$
postulate-of-existence-of-similar-triangles $\longleftrightarrow$
postulate-of-parallelism-of-perpendicular-transversals $\wedge$
postulate-of-parallelism-of-perpendicular-transversals $\longleftrightarrow$
postulate-of-right-lambert-quadrilaterals $\wedge$
postulate-of-right-lambert-quadrilaterals $\longleftrightarrow$ *postulate-of-right-saccheri-quadrilaterals* $\wedge$
postulate-of-right-saccheri-quadrilaterals $\longleftrightarrow$ *postulate-of-transitivity-of-parallelism* $\wedge$
postulate-of-transitivity-of-parallelism $\longleftrightarrow$ *proclus-postulate* $\wedge$

```

proclus-postulate  $\longleftrightarrow$  strong-parallel-postulate  $\wedge$ 
strong-parallel-postulate  $\longleftrightarrow$  tarski-s-parallel-postulate  $\wedge$ 
tarski-s-parallel-postulate  $\longleftrightarrow$  thales-postulate  $\wedge$ 
thales-postulate  $\longleftrightarrow$  thales-converse-postulate  $\wedge$ 
thales-converse-postulate  $\longleftrightarrow$  triangle-circumscription-principle  $\wedge$ 
triangle-circumscription-principle  $\longleftrightarrow$  triangle-postulate  $\wedge$ 
triangle-postulate  $\longleftrightarrow$  weak-inverse-projection-postulate  $\wedge$ 
weak-inverse-projection-postulate  $\longleftrightarrow$  weak-tarski-s-parallel-postulate  $\wedge$ 
weak-tarski-s-parallel-postulate  $\longleftrightarrow$  weak-triangle-circumscription-principle
using alternate-interior--consecutive-interior alternate-interior--playfair-bis assms
bachmann-s-lotschnittaxiom--legendre-s-parallel-postulate consecutive-interior--alternate-interior
euclid-5--original-euclid existential-playfair--rah existential-saccheri--rah
inter-dec-plus-par-perp-perp-imply-triangle-circumscription
inverse-projection-postulate--proclus-bis legendre-s-fourth-theorem-aux legendre-s-second-theorem
legendre-s-third-theorem midpoint-converse-postulate-implies-playfair original-euclid--original-spp
original-spp--inverse-projection-postulate par-perp-2-par-implies-par-perp-perp
par-perp-perp-implies-par-perp-2-par playfair--alternate-interior playfair--existential-playfair
playfair--universal-posidonius-postulate playfair-bis--playfair
playfair-s-postulate-implies-midpoint-converse-postulate posidonius-postulate--rah
proclus-bis--proclus proclus-s-postulate-implies-strong-parallel-postulate
rah--existential-saccheri rah--posidonius rah--rectangle-principle rah--similar
rah--thales-postulate rah--triangle rectangle-existence--rah
rectangle-principle--rectangle-existence similar--rah
strong-parallel-postulate-implies-tarski-s-euclid tarski-s-euclid-implies-euclid-5
tarski-s-euclid-implies-playfair-s-postulate thales-converse-postulate--thales-existence
thales-converse-postulate--weak-triangle-circumscription-principle thales-existence--rah
thales-postulate--thales-converse-postulate triangle--existential-triangle
triangle-circumscription-implies-tarski-s-euclid
universal-posidonius-postulate--perpendicular-transversal-postulate
weak-inverse-projection-postulate--weak-tarski-s-parallel-postulate
weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate
weak-triangle-circumscription-principle--bachmann-s-lotschnittaxiom by argo

```

4 Szmielew: hyperbolic plane postulate

4.1 Definition

definition *hyperbolic-plane-postulate* ::

bool

(*HyperbolicPlanePostulate*) **where**

hyperbolic-plane-postulate $\equiv$

$\forall A1 A2 P.$

$\neg Col A1 A2 P$

$\longrightarrow$

$(\exists B1 B2 C1 C2.$

$A1 A2 Par B1 B2 \wedge Col P B1 B2 \wedge A1 A2 Par C1 C2 \wedge Col P C1 C2 \wedge \neg Col C1 B1 B2)$

4.2 Propositions

lemma *hpp--nP35*:

assumes *greenberg-s-axiom*

shows *hyperbolic-plane-postulate* $\longleftrightarrow \neg Postulate35$

proof –

have *hyperbolic-plane-postulate* $\longrightarrow \neg$ *existential-playfair-s-postulate*

using *existential-playfair-s-postulate-def hyperbolic-plane-postulate-def* **by** *blast*

moreover

have (*greenberg-s-axiom* $\wedge \neg$ *existential-playfair-s-postulate*) $\longrightarrow$ *hyperbolic-plane-postulate*

using *existential-playfair-s-postulate-def hyperbolic-plane-postulate-def*

col-permutation-5 par-right-comm **by** *blast*

ultimately show *?thesis*

using *Postulate35-def assms* **by** *blast*

qed

lemma *aah--hpp*:

assumes *hypothesis-of-acute-saccheri-quadrilaterals*

```

shows hyperbolic-plane-postulate
proof –
{
  fix A1 A2 P
  assume  $\neg \text{Col } A1 \ A2 \ P$ 
  then obtain Q where Col A1 A2 Q and A1 A2 Perp P Q
    using l8-18-existence  $\langle \neg \text{Col } A1 \ A2 \ P \rangle$  by blast
  then obtain X where A1  $\neq$  X and A2  $\neq$  X and Q  $\neq$  X and Col A1 A2 X
    using diff-col-ex3 by blast
  obtain Y where Bet X Q Y and Cong X Q Q Y
    using cong-4312 segment-construction by blast
  have A1  $\neq$  A2
    using  $\langle A1 \ A2 \ \text{Perp } P \ Q \rangle$  perp-not-eq-1 by blast
  have Q  $\neq$  Y
    using  $\langle \text{Cong } X \ Q \ Q \ Y \rangle$   $\langle Q \neq X \rangle$  cong-identity-inv by auto
  have P  $\neq$  Q
    using  $\langle \text{Col } A1 \ A2 \ Q \rangle$   $\langle \neg \text{Col } A1 \ A2 \ P \rangle$  by auto
  have Col A1 A2 Y
    by (metis  $\langle \text{Bet } X \ Q \ Y \rangle$   $\langle \text{Col } A1 \ A2 \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ X \rangle$   $\langle Q \neq X \rangle$  bet-col colx)
  have Per P Q X
    by (meson l8-16-1  $\langle A1 \ A2 \ \text{Perp } P \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ X \rangle$ )
  then obtain B1 where Saccheri Q P B1 X
    using  $\langle P \neq Q \rangle$   $\langle Q \neq X \rangle$  per--ex-saccheri by blast
  hence Q X Par P B1
    by (simp add: sac--par1423)
  have Per P Q Y
    by (meson l8-16-1  $\langle A1 \ A2 \ \text{Perp } P \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ Y \rangle$ )
  then obtain C1 where Saccheri Q P C1 Y
    using  $\langle P \neq Q \rangle$   $\langle Q \neq Y \rangle$  per--ex-saccheri by blast
  hence Q Y Par P C1
    using sac--par1423 by blast
  have A1 A2 Par B1 P
    using Par-cases  $\langle A1 \neq A2 \rangle$   $\langle \text{Col } A1 \ A2 \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ X \rangle$   $\langle Q \ X \ \text{Par } P \ B1 \rangle$ 
      par-col2-par-bis by blast
  moreover have Col P B1 P
    using col-trivial-3 by auto
  moreover have A1 A2 Par C1 P
    using Par-cases  $\langle A1 \neq A2 \rangle$   $\langle \text{Col } A1 \ A2 \ Q \rangle$   $\langle \text{Col } A1 \ A2 \ Y \rangle$   $\langle Q \ Y \ \text{Par } P \ C1 \rangle$ 
      par-col2-par-bis by blast
  moreover have Col P C1 P
    by (simp add: col-trivial-3)
  moreover
  {
    assume Col C1 B1 P
    have Q P ParStrict B1 X
      by (simp add:  $\langle \text{Saccheri } Q \ P \ B1 \ X \rangle$  sac--pars1234)
    have Q P ParStrict C1 Y
      using  $\langle \text{Saccheri } Q \ P \ C1 \ Y \rangle$  sac--pars1234 by auto
    have P Q TS Y X
      by (metis NCol-perm invert-two-sides  $\langle \text{Bet } X \ Q \ Y \rangle$   $\langle \text{Col } A1 \ A2 \ Q \rangle$ 
         $\langle \text{Col } A1 \ A2 \ X \rangle$   $\langle Q \neq X \rangle$   $\langle Q \neq Y \rangle$   $\langle \neg \text{Col } A1 \ A2 \ P \rangle$  bet-ts col-trivial-2 l6-21 l9-2)
    hence P Q TS X C1
      by (meson Par-strict-cases l9-8-2  $\langle Q \ P \ \text{ParStrict } C1 \ Y \rangle$  l12-6 l9-2)
    hence P Q TS B1 C1
      using  $\langle Q \ P \ \text{ParStrict } B1 \ X \rangle$  l12-6 l9-8-2 par-strict-comm by blast
    hence Bet B1 P C1
      using Col-cases  $\langle \text{Col } C1 \ B1 \ P \rangle$  col-two-sides-bet by blast
    have B1  $\neq$  P
      using calculation(1) par-distincts by auto
    have C1  $\neq$  P
      using calculation(3) par-distinct by blast
    have  $\neg B1 \ P \ C1 \ \text{LtA } X \ Q \ Y$ 
      using  $\langle \text{Bet } B1 \ P \ C1 \rangle$   $\langle \text{Bet } X \ Q \ Y \rangle$  bet2-lta--lta lta-distincts by blast
    moreover
    have B1 P C1 LtA X Q Y
  }
  proof –

```



```

have  $\neg \text{Col } P \ Q \ X$ 
  using  $\text{TS-def } \langle P \ Q \ \text{TS } X \ C1 \rangle$   $\text{not-col-permutation-1}$  by presburger
have  $Q \ P \ \text{TS } X \ Y$ 
  using  $\langle P \ Q \ \text{TS } Y \ X \rangle$   $\text{invert-two-sides l9-2}$  by blast
have  $\text{Acute } B1 \ P \ Q$ 
  using  $\langle \text{Saccheri } Q \ P \ B1 \ X \rangle$   $\text{acute-sym assms}$ 
   $\text{hypothesis-of-acute-saccheri-quadrilaterals-def}$  by blast
hence  $B1 \ P \ Q \ \text{LtA } P \ Q \ X$ 
  using  $\text{acute-per-lta } \langle P \neq Q \rangle \langle Q \neq X \rangle \langle \text{Per } P \ Q \ X \rangle$  by blast
moreover have  $\text{Acute } Q \ P \ C1$ 
  using  $\langle \text{Saccheri } Q \ P \ C1 \ Y \rangle$   $\text{assms hypothesis-of-acute-saccheri-quadrilaterals-def}$  by blast
hence  $Q \ P \ C1 \ \text{LtA } P \ Q \ Y$ 
  using  $\text{acute-per-lta } \langle P \neq Q \rangle \langle \text{Per } P \ Q \ Y \rangle \langle Q \neq Y \rangle$  by blast
moreover have  $\text{SAMS } P \ Q \ X \ P \ Q \ Y$ 
  using  $\langle P \neq Q \rangle \langle \text{Per } P \ Q \ X \rangle \langle \text{Per } P \ Q \ Y \rangle \langle Q \neq X \rangle \langle Q \neq Y \rangle$   $\text{per2--sams}$  by force
moreover have  $B1 \ P \ Q \ Q \ P \ C1 \ \text{SumA } B1 \ P \ C1$ 
  using  $\langle B1 \neq P \rangle \langle \text{Bet } B1 \ P \ C1 \rangle \langle C1 \neq P \rangle \langle P \neq Q \rangle$   $\text{bet--suma}$  by force
moreover have  $P \ Q \ X \ P \ Q \ Y \ \text{SumA } X \ Q \ Y$ 
  using  $\langle \text{Bet } X \ Q \ Y \rangle \langle P \neq Q \rangle \langle Q \neq X \rangle \langle Q \neq Y \rangle$   $\text{bet--suma suma-left-comm}$  by presburger
ultimately show  $?thesis$ 
  using  $\text{sams-lta2-suma2--lta}$  by blast
qed
ultimately have  $\text{False}$ 
  by blast
}
ultimately have  $\exists \ B1 \ B2 \ C1 \ C2.$ 
 $A1 \ A2 \ \text{Par } B1 \ B2 \wedge \text{Col } P \ B1 \ B2 \wedge A1 \ A2 \ \text{Par } C1 \ C2 \wedge \text{Col } P \ C1 \ C2 \wedge \neg \text{Col } C1 \ B1 \ B2$ 
  by blast
}
thus  $?thesis$ 
  using  $\text{hyperbolic-plane-postulate-def}$  by blast
qed

```

```

theorem szmielew-s-theorem:
assumes  $\text{aristotle-s-axiom}$ 
shows  $\forall \ P:: \text{bool.}$ 
   $(\text{playfair-s-postulate} \longrightarrow P) \wedge (\text{hyperbolic-plane-postulate} \longrightarrow \neg P)$ 
   $\longrightarrow$ 
   $(P \longleftrightarrow \text{playfair-s-postulate})$ 
proof –
{
  fix  $P:: \text{bool}$ 
  assume  $\text{playfair-s-postulate} \longrightarrow P$  and
     $\text{hyperbolic-plane-postulate} \longrightarrow \neg P$ 
  have  $\text{hypothesis-of-acute-saccheri-quadrilaterals} \vee$ 
     $\text{hypothesis-of-right-saccheri-quadrilaterals}$ 
    by  $(\text{simp add: aristotle--acute-or-right assms})$ 
  moreover
  have  $\text{greenberg-s-axiom}$ 
    by  $(\text{simp add: aristotle--greenberg assms})$ 
  hence  $\text{playfair-s-postulate} \longleftrightarrow \text{postulate-of-existence-of-a-right-saccheri-quadrilateral}$ 
    using  $\text{equivalent-postulates-assuming-greenberg-s-axiom assms}$  by blast
  {
    assume  $\text{hypothesis-of-acute-saccheri-quadrilaterals}$ 
    hence  $P \longrightarrow \text{playfair-s-postulate}$ 
    using  $\langle \text{HyperbolicPlanePostulate} \longrightarrow \neg P \rangle$   $\text{aah--hpp}$  by blast
  }
  moreover
  {
    assume  $\text{hypothesis-of-right-saccheri-quadrilaterals}$ 
    assume  $P$ 
    have  $\neg \text{hyperbolic-plane-postulate}$ 
      using  $\langle \text{HyperbolicPlanePostulate} \longrightarrow \neg P \rangle \langle P \rangle$  by fastforce
    hence  $\text{Postulate35}$ 
      using  $\text{hpp--nP35 assms}$  by  $(\text{simp add: } \langle \text{GreenBergsAxiom} \rangle)$ 
  }
}

```

```

    hence playfair-s-postulate
    using equivalent-postulates-assuming-greenberg-s-axiom assms
        Postulate35-def GreenBergsAxiom by fastforce
  }
  ultimately
  have  $P \longrightarrow \text{playfair-s-postulate}$ 
    by blast
  hence  $P \longleftrightarrow \text{playfair-s-postulate}$ 
    using  $\langle \text{PlayfairSPostulate} \longrightarrow P \rangle$  by blast
}
thus ?thesis
  by blast
qed

end
end

```

theory *Tarski-Neutral-Continuity*

imports
Tarski-Neutral

begin

context *Tarski-neutral-dimensionless*

begin

5 Continuity

5.1 Definitions

definition *OnCircle* ::

```

['p, 'p, 'p]  $\Rightarrow$  bool
(- OnCircle - - [99, 99, 99] 50)
where
   $P \text{ OnCircle } A \ B \equiv \text{Cong } A \ P \ A \ B$ 

```

definition *InCircle* ::

```

['p, 'p, 'p]  $\Rightarrow$  bool
(- InCircle - - [99, 99, 99] 50)
where
   $P \text{ InCircle } A \ B \equiv A \ P \ Le \ A \ B$ 

```

definition *OutCircle* ::

```

['p, 'p, 'p]  $\Rightarrow$  bool
(- OutCircle - - [99, 99, 99] 50)
where
   $P \text{ OutCircle } A \ B \equiv A \ B \ Le \ A \ P$ 

```

definition *InCircleS* ::

```

['p, 'p, 'p]  $\Rightarrow$  bool
(- InCircleS - - [99, 99, 99] 50)
where
   $P \text{ InCircleS } A \ B \equiv A \ P \ Lt \ A \ B$ 

```

definition *OutCircleS* ::

```

['p, 'p, 'p]  $\Rightarrow$  bool
(- OutCircleS - - [99, 99, 99] 50)

```

where

$$P \text{ OutCircleS } A \ B \equiv A \ B \text{ Lt } A \ P$$

definition *Diam* ::

$$[p, p, p, p] \Rightarrow \text{bool} \\ (\text{Diam} \text{ - - - - } [99, 99, 99, 99] \ 50)$$

where

$$\text{Diam } A \ B \ PO \ P \equiv \text{Bet } A \ PO \ B \wedge A \text{ OnCircle } PO \ P \wedge B \text{ OnCircle } PO \ P$$

definition *EqC* ::

$$[p, p, p, p] \Rightarrow \text{bool} \\ (\text{EqC} \text{ - - - - } [99, 99, 99, 99] \ 50)$$

where

$$\text{EqC } A \ B \ C \ D \equiv$$

$$\forall X. (X \text{ OnCircle } A \ B \longleftrightarrow X \text{ OnCircle } C \ D)$$

definition *InterCCAt* ::

$$[p, p, p, p, p, p] \Rightarrow \text{bool} \\ (\text{InterCCAt} \text{ - - - - - } [99, 99, 99, 99, 99, 99] \ 50)$$

where

$$\text{InterCCAt } A \ B \ C \ D \ P \ Q \equiv$$

$$\neg \text{EqC } A \ B \ C \ D \wedge$$

$$P \neq Q \wedge P \text{ OnCircle } C \ D \wedge Q \text{ OnCircle } C \ D \wedge P \text{ OnCircle } A \ B \wedge Q \text{ OnCircle } A \ B$$

definition *InterCC* ::

$$[p, p, p, p] \Rightarrow \text{bool} \\ (\text{InterCC} \text{ - - - - } [99, 99, 99, 99] \ 50)$$

where

$$\text{InterCC } A \ B \ C \ D \equiv \exists P \ Q. \text{InterCCAt } A \ B \ C \ D \ P \ Q$$

definition *TangentCC* ::

$$[p, p, p, p] \Rightarrow \text{bool} \\ (\text{TangentCC} \text{ - - - - } [99, 99, 99, 99] \ 50)$$

where

$$\text{TangentCC } A \ B \ C \ D \equiv$$

$$\exists X. (X \text{ OnCircle } A \ B \wedge X \text{ OnCircle } C \ D \wedge$$

$$(\forall Y. (Y \text{ OnCircle } A \ B \wedge Y \text{ OnCircle } C \ D) \longrightarrow X = Y))$$

definition *Tangent* ::

$$[p, p, p, p] \Rightarrow \text{bool} \\ (\text{Tangent} \text{ - - - - } [99, 99, 99, 99] \ 50)$$

where

$$\text{Tangent } A \ B \ PO \ P \equiv$$

$$\exists X. (\text{Col } A \ B \ X \wedge X \text{ OnCircle } PO \ P \wedge$$

$$(\forall Y. (\text{Col } A \ B \ Y \wedge Y \text{ OnCircle } PO \ P) \longrightarrow X = Y))$$

definition *TangentAt* ::

$$[p, p, p, p, p, p] \Rightarrow \text{bool} \\ (\text{TangentAt} \text{ - - - - - } [99, 99, 99, 99, 99] \ 50)$$

where

$$\text{TangentAt } A \ B \ PO \ P \ T \equiv \text{Tangent } A \ B \ PO \ P \wedge \text{Col } A \ B \ T \wedge T \text{ OnCircle } PO \ P$$

definition *Concyclic* ::

$$[p, p, p, p] \Rightarrow \text{bool}$$

(*Concyclic* - - - [99,99,99,99] 50)

where

$Concyclic\ A\ B\ C\ D \equiv Coplanar\ A\ B\ C\ D \wedge$
 $(\exists\ PO\ P.\ A\ OnCircle\ PO\ P \wedge B\ OnCircle\ PO\ P \wedge$
 $C\ OnCircle\ PO\ P \wedge D\ OnCircle\ PO\ P)$

definition *ConcyclicGen* ::

$[p, p, p, p] \Rightarrow bool$

(*ConcyclicGen* - - - [99,99,99,99] 50)

where

$ConcyclicGen\ A\ B\ C\ D \equiv$
 $Concyclic\ A\ B\ C\ D \vee$
 $(Col\ A\ B\ C \wedge Col\ A\ B\ D \wedge Col\ A\ C\ D \wedge Col\ B\ C\ D)$

definition *Concyclic2* ::

$[p, p, p, p] \Rightarrow bool$

(*Concyclic2* - - - [99,99,99,99] 50)

where

$Concyclic2\ A\ B\ C\ D \equiv Coplanar\ A\ B\ C\ D \wedge$
 $(\exists\ P.\ Cong\ P\ A\ P\ B \wedge Cong\ P\ A\ P\ C \wedge Cong\ P\ A\ P\ D)$

definition *segment-circle* :: bool

(*SegmentCircle* 50)

where

$segment-circle \equiv \forall\ A\ B\ P\ Q.\ (P\ InCircle\ A\ B \wedge Q\ OutCircle\ A\ B) \longrightarrow$
 $(\exists\ Z.\ Bet\ P\ Z\ Q \wedge Z\ OnCircle\ A\ B)$

definition *one-point-line-circle* :: bool

(*OnePointLineCircle* 50)

where

$one-point-line-circle \equiv \forall\ A\ B\ U\ V\ P.$
 $Col\ U\ V\ P \wedge U \neq V \wedge Bet\ A\ P\ B \longrightarrow (\exists\ Z.\ Col\ U\ V\ Z \wedge Z\ OnCircle\ A\ B)$

definition *two-points-line-circle* :: bool

(*TwoPointLineCircle* 50)

where

$TwoPointLineCircle \equiv \forall\ A\ B\ U\ V\ P.$
 $(Col\ U\ V\ P \wedge U \neq V \wedge Bet\ A\ P\ B) \longrightarrow (\exists\ Z1\ Z2.\ Col\ U\ V\ Z1 \wedge Z1\ OnCircle\ A\ B \wedge$
 $Col\ U\ V\ Z2 \wedge Z2\ OnCircle\ A\ B \wedge$
 $Bet\ Z1\ P\ Z2 \wedge (P \neq B \longrightarrow Z1 \neq Z2))$

definition *circle-circle* :: bool

(*CircleCircle* 50)

where

$circle-circle \equiv \forall\ A\ B\ C\ D\ P\ Q.$
 $(P\ OnCircle\ C\ D \wedge Q\ OnCircle\ C\ D \wedge P\ InCircle\ A\ B \wedge Q\ OutCircle\ A\ B)$
 $\longrightarrow (\exists\ Z.\ Z\ OnCircle\ A\ B \wedge Z\ OnCircle\ C\ D)$

definition *circle-circle-bis* :: bool

(*CircleCircleBis* 50)

where

$circle-circle-bis \equiv \forall A B C D P Q.$
 $(P \text{ OnCircle } C D \wedge P \text{ InCircle } A B \wedge Q \text{ OnCircle } A B \wedge Q \text{ InCircle } C D)$
 $\longrightarrow (\exists Z. Z \text{ OnCircle } A B \wedge Z \text{ OnCircle } C D)$

definition *circle-circle-axiom* :: bool
 $(CircleCircleAxiom \ 50)$
where
 $circle-circle-axiom \equiv \forall A B C D B' D'.$
 $(Cong \ A \ B' \ A \ B \wedge Cong \ C \ D' \ C \ D \wedge Bet \ A \ D' \ B \wedge Bet \ C \ B' \ D)$
 $\longrightarrow (\exists Z. Cong \ A \ Z \ A \ B \wedge Cong \ C \ Z \ C \ D)$

definition *circle-circle-two* :: bool
 $(CircleCircleTwo \ 50)$
where
 $circle-circle-two \equiv \forall A B C D P Q.$
 $(P \text{ OnCircle } C D \wedge Q \text{ OnCircle } C D \wedge P \text{ InCircle } A B \wedge Q \text{ OutCircle } A B)$
 $\longrightarrow (\exists Z1 \ Z2. Z1 \text{ OnCircle } A B \wedge Z1 \text{ OnCircle } C D \wedge$
 $Z2 \text{ OnCircle } A B \wedge Z2 \text{ OnCircle } C D \wedge$
 $((P \text{ InCircleS } A B \wedge Q \text{ OutCircleS } A B) \longrightarrow Z1 \neq Z2))$

definition *euclid-s-prop-1-22* :: bool
 $(EuclidsProp122 \ 50)$
where
 $euclid-s-prop-1-22 \equiv \forall A B C D E F A' B' C' D' E' F'.$
 $(A \ B \ C \ D \ SumS \ E' \ F' \wedge A \ B \ E \ F \ SumS \ C' \ D' \wedge C \ D \ E \ F \ SumS \ A' \ B' \wedge$
 $E \ F \ Le \ E' \ F' \wedge C \ D \ Le \ C' \ D' \wedge A \ B \ Le \ A' \ B') \longrightarrow$
 $(\exists P \ Q \ R. Cong \ P \ Q \ A \ B \wedge Cong \ P \ R \ C \ D \wedge Cong \ Q \ R \ E \ F)$

definition *Nested* ::
 $[(nat \Rightarrow 'p \Rightarrow bool), (nat \Rightarrow 'p \Rightarrow bool)] \Rightarrow bool$
 $(Nested \ - \ - \ [99,99] \ 50)$
where
 $Nested \ A \ B \equiv$
 $(\forall n. (\exists An \ Bn. A \ n \ An \wedge B \ n \ Bn)) \wedge$
 $(\forall n \ An \ Am \ Bm \ Bn.$
 $(A \ n \ An \wedge A \ (Suc \ n) \ Am \wedge B \ (Suc \ n) \ Bm \wedge B \ n \ Bn)$
 $\longrightarrow$
 $(Bet \ An \ Am \ Bm \wedge Bet \ Am \ Bm \ Bn \wedge Am \neq Bm))$

definition *CantorsAxiom* :: bool
 $(CantorsAxiom \ 50)$
where
 $CantorsAxiom \equiv \forall A \ B. Nested \ A \ B \longrightarrow$
 $(\exists X. \forall n \ An \ Bn. (A \ n \ An \wedge B \ n \ Bn) \longrightarrow Bet \ An \ X \ Bn)$

definition *DedekindsAxiom* :: bool
 $(DedekindsAxiom \ 50)$
where
 $DedekindsAxiom \equiv \forall Alpha \ Beta.$
 $(\exists A. \forall X \ Y. (Alpha \ X \wedge Beta \ Y) \longrightarrow Bet \ A \ X \ Y) \longrightarrow$
 $(\exists B. \forall X \ Y. (Alpha \ X \wedge Beta \ Y) \longrightarrow Bet \ X \ B \ Y)$

definition *DedekindVariant* :: bool
 $(DedekindVariant \ 50)$
where
 $DedekindVariant \equiv \forall Alpha \ Beta :: 'p \Rightarrow bool. \forall A \ C.$

$(Alpha\ A \wedge Beta\ C \wedge$
 $(\forall\ P.\ A\ Out\ P\ C \longrightarrow (Alpha\ P \vee Beta\ P)) \wedge$
 $(\forall\ X\ Y.\ (Alpha\ X \wedge Beta\ Y) \longrightarrow (Bet\ A\ X\ Y \wedge X \neq Y)))$
 $\longrightarrow$
 $(\exists\ B.\ \forall\ X\ Y.\ (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ X\ B\ Y)$

definition $Alpha'Fun :: 'p \Rightarrow ('p \Rightarrow bool) \Rightarrow ('p \Rightarrow bool)$

where

$Alpha'Fun\ A\ Alpha \equiv \lambda\ X'::'p.\ X' = A \vee (\exists\ X::'p.\ Alpha\ X \wedge Bet\ A\ X'\ X)$

definition $Beta'Fun :: 'p \Rightarrow 'p \Rightarrow ('p \Rightarrow bool) \Rightarrow ('p \Rightarrow bool)$

where

$Beta'Fun\ A\ C\ Alpha \equiv \lambda\ Y'::'p.\ A\ Out\ Y'\ C \wedge \neg (\exists\ X::'p.\ Alpha\ X \wedge Bet\ A\ Y'\ X)$

definition $Eq ::$

$['p, 'p] \Rightarrow bool$
 $(-\ Eq - [99,99] 50)$

where

$A\ Eq\ B \equiv A = B$

inductive $FOF :: bool \Rightarrow bool$

where

$eq\text{-}fof : FOF\ (A\ Eq\ B)\ \mathbf{for}\ A\ B :: 'p$
 $bet\text{-}fof : FOF\ (Bet\ A\ B\ C)\ \mathbf{for}\ A\ B\ C :: 'p$
 $cong\text{-}fof : FOF\ (Cong\ A\ B\ C\ D)\ \mathbf{for}\ A\ B\ C\ D :: 'p$
 $not\text{-}fof : FOF\ (\neg\ P)\ \mathbf{if}\ FOF\ P\ \mathbf{for}\ P :: bool$
 $and\text{-}fof : FOF\ (P \wedge Q)\ \mathbf{if}\ FOF\ P\ \mathbf{and}\ FOF\ Q\ \mathbf{for}\ P\ Q :: bool$
 $or\text{-}fof : FOF\ (P \vee Q)\ \mathbf{if}\ FOF\ P\ \mathbf{and}\ FOF\ Q\ \mathbf{for}\ P\ Q :: bool$
 $implies\text{-}fof : FOF\ (P \longrightarrow Q)\ \mathbf{if}\ FOF\ P\ \mathbf{and}\ FOF\ Q\ \mathbf{for}\ P\ Q :: bool$
 $forall\text{-}fof : FOF\ (\forall\ A::'p.\ P\ A)\ \mathbf{if}\ \forall\ A.\ FOF\ (P\ A)\ \mathbf{for}\ P :: 'p \Rightarrow bool$

 $exists\text{-}fof : FOF\ (\exists\ A::'p.\ P\ A)\ \mathbf{if}\ FOF\ (P\ A)\ \mathbf{for}\ P :: 'p \Rightarrow bool$

definition $FirstOrderDedekind :: bool$

$(FirstOrderDedekind\ 50)$

where

$FirstOrderDedekind \equiv$

$\forall\ Alpha\ Beta :: 'p \Rightarrow bool.$

$(\forall\ X :: 'p.\ FOF\ (Alpha\ X)) \wedge$

$(\forall\ Y :: 'p.\ FOF\ (Beta\ Y)) \wedge$

$(\exists\ A.\ \forall\ X\ Y.\ (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ A\ X\ Y)$

$\longrightarrow$

$(\exists\ B.\ \forall\ X\ Y.\ (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ X\ B\ Y)$

definition $AlphaFun :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow$
 $'p \Rightarrow 'p \Rightarrow ('p \Rightarrow bool)$

where

$AlphaFun\ A\ B\ C\ D\ P\ Q \equiv$

$\lambda\ X.\ Bet\ P\ X\ Q \wedge (\exists\ X0.\ X0\ OnCircle\ C\ D \wedge C\ Out\ X\ X0 \wedge X0\ InCircle\ A\ B)$

definition $BetaFun :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow$
 $'p \Rightarrow 'p \Rightarrow ('p \Rightarrow bool)$

where

$BetaFun\ A\ B\ C\ D\ P\ Q \equiv$

$\lambda\ Y.\ Bet\ P\ Y\ Q \wedge (\exists\ Y0.\ Y0\ OnCircle\ C\ D \wedge C\ Out\ Y\ Y0 \wedge Y0\ OutCircle\ A\ B)$

definition $DedekindLemFun :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow$
 $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$

where

$DedekindLemFun \equiv$

$\lambda\ A\ B\ C\ D\ P\ Q.\ \lambda\ R.\ \lambda\ X\ Y.\ (Bet\ P\ X\ Q \wedge$

$(\exists\ X0.\ X0\ OnCircle\ C\ D \wedge C\ Out\ X\ X0 \wedge X0\ InCircle\ A\ B) \wedge$

$Bet\ P\ Y\ Q \wedge$

$(\exists\ Y0.\ Y0\ OnCircle\ C\ D \wedge C\ Out\ Y\ Y0 \wedge Y0\ OutCircle\ A\ B))$

$\longrightarrow$

$Bet\ X\ R\ Y$

5.2 Propositions

lemma *inc112*:
 shows $A \text{ InCircle } A \ B$
 by (*meson InCircle-def l5-5-2 not-bet-distincts segment-construction-0*)

lemma *onc212*:
 shows $B \text{ OnCircle } A \ B$
 by (*simp add: OnCircle-def cong-reflexivity*)

lemma *onc-sym*:
 assumes $P \text{ OnCircle } A \ B$
 shows $B \text{ OnCircle } A \ P$
 using *Cong-cases OnCircle-def assms* **by** *auto*

lemma *ninc--outcs*:
 assumes $\neg (X \text{ InCircle } PO \ P)$
 shows $X \text{ OutCircleS } PO \ P$
 using *nle--lt InCircle-def OutCircleS-def assms* **by** *blast*

lemma *inc--outc-1*:
 assumes $P \text{ InCircle } A \ B$
 shows $B \text{ OutCircle } A \ P$
 using *InCircle-def OutCircle-def assms* **by** *blast*

lemma *inc--outc-2*:
 assumes $B \text{ OutCircle } A \ P$
 shows $P \text{ InCircle } A \ B$
 using *InCircle-def OutCircle-def assms* **by** *blast*

lemma *inc--outc*:
 shows $P \text{ InCircle } A \ B \longleftrightarrow B \text{ OutCircle } A \ P$
 by (*simp add: InCircle-def OutCircle-def*)

lemma *incs--outcs-1*:
 assumes $P \text{ InCircleS } A \ B$
 shows $B \text{ OutCircleS } A \ P$
 using *InCircleS-def OutCircleS-def assms* **by** *blast*

lemma *incs--outcs-2*:
 assumes $B \text{ OutCircleS } A \ P$
 shows $P \text{ InCircleS } A \ B$
 using *InCircleS-def OutCircleS-def assms* **by** *blast*

lemma *incs--outcs*:
 shows $P \text{ InCircleS } A \ B \longleftrightarrow B \text{ OutCircleS } A \ P$
 using *incs--outcs-1 incs--outcs-2* **by** *blast*

lemma *onc--inc*:
 assumes $P \text{ OnCircle } A \ B$
 shows $P \text{ InCircle } A \ B$
 using *InCircle-def OnCircle-def assms cong--le* **by** *blast*

lemma *onc--outc*:
 assumes $P \text{ OnCircle } A \ B$
 shows $P \text{ OutCircle } A \ B$
 by (*simp add: assms inc--outc-1 onc--inc onc-sym*)

lemma *inc-outc--onc*:
 assumes $P \text{ InCircle } A \ B$ **and**
 $P \text{ OutCircle } A \ B$
 shows $P \text{ OnCircle } A \ B$
 using *InCircle-def OnCircle-def OutCircle-def assms(1) assms(2) le-anti-symmetry* **by** *blast*

lemma *incs--inc*:
 assumes $P \text{ InCircleS } A \ B$
 shows $P \text{ InCircle } A \ B$

```

using InCircle-def InCircleS-def Lt-def assms by blast

lemma outcs--outc:
  assumes P OutCircleS A B
  shows P OutCircle A B
  using assms inc--outc incs--inc incs--outcs by blast

lemma incs--noutc-1:
  assumes P InCircleS A B
  shows  $\neg$  P OutCircle A B
  using InCircleS-def OutCircle-def assms le--nlt by blast

lemma incs--noutc-2:
  assumes  $\neg$  P OutCircle A B
  shows P InCircleS A B
  using assms inc--outc-1 incs--outcs-2 ninc--outcs by blast

lemma incs--noutc:
  shows P InCircleS A B  $\longleftrightarrow$   $\neg$  P OutCircle A B
  using incs--noutc-1 incs--noutc-2 by blast

lemma outcs--ninc-1:
  assumes P OutCircleS A B
  shows  $\neg$  P InCircle A B
  by (simp add: assms inc--outc incs--noutc-1 incs--outcs-2)

lemma outcs--ninc-2:
  assumes  $\neg$  P InCircle A B
  shows P OutCircleS A B
  by (simp add: assms ninc--outcs)

lemma outcs--ninc:
  shows P OutCircleS A B  $\longleftrightarrow$   $\neg$  P InCircle A B
  using inc--outc-1 incs--noutc-1 incs--outcs-2 ninc--outcs by blast

lemma inc--noutcs-1:
  assumes P InCircle A B
  shows  $\neg$  P OutCircleS A B
  using assms outcs--ninc-1 by blast

lemma inc--noutcs-2:
  assumes  $\neg$  P OutCircleS A B
  shows P InCircle A B
  using assms outcs--ninc-2 by blast

lemma inc--noutcs:
  P InCircle A B  $\longleftrightarrow$   $\neg$  P OutCircleS A B
  by (simp add: outcs--ninc)

lemma outc--nincs-1:
  assumes P OutCircle A B
  shows  $\neg$  P InCircleS A B
  using assms incs--noutc-1 by blast

lemma outc--nincs-2:
  assumes  $\neg$  P InCircleS A B
  shows P OutCircle A B
  using assms incs--noutc by blast

lemma outc--nincs:
  shows P OutCircle A B  $\longleftrightarrow$   $\neg$  P InCircleS A B
  using incs--noutc by force

lemma inc-eq:
  assumes P InCircle A A
  shows A = P

```


by (metis OnCircle-def assms bet-cong-eq between-trivial2 inc112 inc--outc inc-outc--onc)

lemma outc-eq:

assumes A OutCircle A B

shows $A = B$

by (meson OnCircle-def assms bet-cong-eq between-trivial2 inc112 inc-outc--onc)

lemma onc2--cong:

assumes A OnCircle PO P **and**

B OnCircle PO P

shows Cong PO A PO B

by (meson OnCircle-def assms(1) assms(2) cong-transitivity onc-sym)

lemma bet-inc2--incs:

assumes $X \neq U$ **and**

$X \neq V$ **and**

Bet U X V **and**

U InCircle PO P **and**

V InCircle PO P

shows X InCircleS PO P

proof –

{

assume PO U Le PO V

hence PO X Lt PO V

using Le-cases assms(1) assms(2) assms(3) bet-le--lt lt-comm **by** blast

hence X InCircleS PO P

using InCircleS-def InCircle-def assms(5) le3456-lt--lt **by** blast

}

moreover

{

assume PO V Le PO U

hence PO X Lt PO U

using assms(1) assms(2) assms(3) bet-le--lt between-symmetry le-comm lt-comm **by** blast

hence X InCircleS PO P

by (meson InCircle-def OutCircle-def assms(4) le--nlt le-transitivity outc--nincs-2)

}

ultimately show ?thesis

using local.le-cases **by** blast

qed

lemma bet-incs2--incs:

assumes Bet U X V **and**

U InCircleS PO P **and**

V InCircleS PO P

shows X InCircleS PO P

by (metis assms(1) assms(2) assms(3) bet-inc2--incs incs--inc)

lemma bet-inc2--inc:

assumes U InCircle A B **and**

V InCircle A B **and**

Bet U P V

shows P InCircle A B

by (metis assms(1) assms(2) assms(3) bet-inc2--incs incs--inc)

lemma col-inc-onc2--bet:

assumes $U \neq V$ **and**

U OnCircle A B **and**

V OnCircle A B **and**

Col U V P **and**

P InCircle A B

```

shows Bet U P V
proof (cases  $P = U$ )
case True
thus ?thesis
by (simp add: between-trivial2)
next
case False
hence  $P \neq U$ 
by simp
show ?thesis
proof (cases  $P = V$ )
case True
thus ?thesis
using between-trivial by force
next
case False
hence  $P \neq V$ 
by simp
have Cong A U A V
using assms(2) assms(3) onc2--cong by auto
have U Out P V
by (metis Col-cases  $\langle P \neq U \rangle$  assms(1) assms(2) assms(3) assms(4)
assms(5) bet-inc2--incs incs--noutc-1 onc--inc onc--outc or-bet-out)
moreover have V Out U P
by (metis False assms(2) assms(3) assms(4) assms(5) bet-inc2--incs
calculation l6-3-1 onc--inc onc--outc or-bet-out outc--nincs)
ultimately show ?thesis
using out2--bet by blast
qed
qed

```

```

lemma onc2-out--outcs:
assumes  $U \neq V$  and
U OnCircle A B and
V OnCircle A B and
P Out U V
shows P OutCircleS A B
by (meson Col-cases assms(1) assms(2) assms(3) assms(4)
col-inc-onc2--bet not-bet-and-out out-col outcs--ninc-2)

```

```

lemma col-inc2-outcs--out:
assumes U InCircle A B and
V InCircle A B and
Col U V P and
P OutCircleS A B
shows P Out U V
by (meson Col-cases assms(1) assms(2) assms(3) assms(4)
bet-inc2--inc or-bet-out outcs--ninc-1)

```

```

lemma col-onc2--mid:
assumes  $U \neq V$  and
U OnCircle A B and
V OnCircle A B and
Col U V A
shows A Midpoint U V
using Col-cases assms(1) assms(2) assms(3) assms(4) l7-20 onc2--cong by blast

```

```

lemma chord-completion:

```

```

assumes  $U \text{ OnCircle } A \ B$  and
   $P \text{ InCircle } A \ B$ 
shows  $\exists V. V \text{ OnCircle } A \ B \wedge \text{Bet } U \ P \ V$ 
proof (cases  $U = A$ )
  case True
    then show ?thesis
      using OnCircle-def assms(1) assms(2) between-trivial2
        cong-reverse-identity inc-eq by blast
  next
    case False
    hence  $U \neq A$ 
      by simp
    have  $\exists A'. U \neq A' \wedge \text{Col } U \ P \ A' \wedge \text{Per } A \ A' \ U$ 
    proof (cases  $\text{Col } U \ P \ A$ )
      case True
        thus ?thesis
          using False l8-20-1-R1 by blast
      next
        case False
        hence  $\neg \text{Col } U \ P \ A$ 
          by simp
        obtain  $A'$  where  $\text{Col } U \ P \ A'$  and  $U \ P \ \text{Perp } A \ A'$ 
          using False l8-18-existence by blast
        {
          assume  $U = A'$ 
          have  $U \neq P$ 
            using False not-col-distincts by auto
          have  $\text{Per } P \ U \ A \vee \text{Obtuse } P \ U \ A$ 
            using Perp-perm  $\langle U = A' \rangle \langle U \ P \ \text{Perp } A \ A' \rangle$  perp-per-2 by blast
          hence  $U \ A \ \text{Lt } P \ A$ 
            using  $\langle U \neq A \rangle \langle U \neq P \rangle$  l11-46 by auto
          hence False
            by (meson Cong-cases InCircleS-def OnCircle-def assms(1) assms(2)
              cong2-lt--lt cong-reflexivity inc--outc incs--noutc)
        }
        hence  $U \neq A'$ 
          by auto
        moreover have  $\text{Per } A \ A' \ U$ 
          using  $\langle \text{Col } U \ P \ A' \rangle \langle U \ P \ \text{Perp } A \ A' \rangle$  col-trivial-3 l8-16-1 by blast
        ultimately show ?thesis
          using  $\langle \text{Col } U \ P \ A' \rangle$  by blast
    qed
  then obtain  $A'$  where  $U \neq A'$  and  $\text{Col } U \ P \ A'$  and  $\text{Per } A \ A' \ U$ 
    by auto
  obtain  $V$  where  $A' \text{ Midpoint } U \ V$ 
    using symmetric-point-construction by auto
  hence  $\text{Cong } A \ U \ A \ V$ 
    using  $\langle \text{Per } A \ A' \ U \rangle$  per-double-cong by auto
  hence  $V \text{ OnCircle } A \ B$ 
    using OnCircle-def assms(1) cong-inner-transitivity by blast
  have  $\text{Col } U \ V \ P$ 
    by (meson Midpoint-def  $\langle A' \text{ Midpoint } U \ V \rangle \langle \text{Col } U \ P \ A' \rangle \langle U \neq A' \rangle$ 
      bet-col col-permutation-5 col-transitivity-1)
  thus ?thesis using col-inc-onc2--bet
    by (metis  $\langle A' \text{ Midpoint } U \ V \rangle \langle U \neq A' \rangle \langle V \text{ OnCircle } A \ B \rangle$ 
      assms(1) assms(2) midpoint-distinct-2)
  qed

```

```

lemma outcs-exists:
shows  $\exists Q. Q \text{ OutCircleS } P \ O \ P$ 
by (meson bet-inc2--incs inc--noutcs-2 incs--outcs point-construction-different)

```

```

lemma outcs-exists1:
  assumes  $X \neq PO$ 
  shows  $\exists Q. PO \text{ Out } X \ Q \wedge Q \text{ OutCircleS } PO \ P$ 
proof -
  obtain  $Q$  where  $Bet \ PO \ X \ Q$  and  $Cong \ X \ Q \ PO \ P$ 
  using segment-construction by blast
  have  $PO \text{ Out } X \ Q$ 
  using  $\langle Bet \ PO \ X \ Q \rangle$  assms bet-out by auto
  moreover have  $\neg Cong \ PO \ P \ PO \ Q$ 
  using  $\langle Bet \ PO \ X \ Q \rangle \langle Cong \ X \ Q \ PO \ P \rangle$  assms bet-cong-eq between-trivial
  cong-transitivity by blast
  hence  $Q \text{ OutCircleS } PO \ P$ 
  using OutCircleS-def by (meson Bet-cases  $\langle Bet \ PO \ X \ Q \rangle \langle Cong \ X \ Q \ PO \ P \rangle$ 
    assms bet-lt1213 cong2-lt-lt cong-pseudo-reflexivity not-cong-1243)
  ultimately show ?thesis
  by auto
qed

```

```

lemma incs-exists:
  assumes  $PO \neq P$ 
  shows  $\exists Q. Q \text{ InCircleS } PO \ P$ 
  using assms incs--noutc outc-eq by blast

```

```

lemma incs-exists1:
  assumes  $X \neq PO$  and
     $P \neq PO$ 
  shows  $\exists Q. PO \text{ Out } X \ Q \wedge Q \text{ InCircleS } PO \ P$ 
proof -
  obtain  $M$  where  $M \text{ Midpoint } PO \ P$ 
  using midpoint-existence by blast
  obtain  $Q$  where  $PO \text{ Out } X \ Q$  and  $Cong \ PO \ Q \ PO \ M$ 
  by (metis Midpoint-def  $\langle M \text{ Midpoint } PO \ P \rangle$  assms(1) assms(2) between-cong-2
    between-trivial2 point-construction-different segment-construction-3)
  have  $PO \ M \ Lt \ PO \ P$ 
  using  $\langle M \text{ Midpoint } PO \ P \rangle$  assms(2) mid-lt by presburger
  hence  $Q \text{ InCircleS } PO \ P$ 
  using InCircleS-def  $\langle Cong \ PO \ Q \ PO \ M \rangle$  cong2-lt-lt cong-reflexivity not-cong-3412 by blast
  thus ?thesis
  using  $\langle PO \text{ Out } X \ Q \rangle$  by blast
qed

```

```

lemma onc-exists:
  assumes  $X \neq PO$  and
     $PO \neq P$ 
  shows  $\exists Q. Q \text{ OnCircle } PO \ P \wedge PO \text{ Out } X \ Q$ 
  by (meson OnCircle-def assms(1) assms(2) l6-11-existence l6-6)

```

```

lemma diam-points:
  shows  $\exists Q1 \ Q2. Bet \ Q1 \ PO \ Q2 \wedge Col \ Q1 \ Q2 \ X \wedge Q1 \text{ OnCircle } PO \ P \wedge Q2 \text{ OnCircle } PO \ P$ 
proof (cases  $P = PO$ )
  case True
  thus ?thesis
  using between-trivial col-trivial-1 onc212 by blast
next
  case False
  hence  $P \neq PO$ 
  by simp
  show ?thesis

```

```

proof (cases  $X = PO$ )
  case True
    thus ?thesis
      by (meson Col-def OnCircle-def between-symmetry segment-construction)
next
  case False
    hence  $X \neq PO$ 
      by simp
    obtain  $Q1$  where  $Q1 \text{ OnCircle } PO \ P$  and  $PO \text{ Out } X \ Q1$ 
      using False  $\langle P \neq PO \rangle$  onc-exists by metis
    obtain  $Q2$  where  $Bet \ Q1 \ PO \ Q2$  and  $Cong \ PO \ Q2 \ PO \ Q1$ 
      using segment-construction by blast
    have  $Col \ Q1 \ Q2 \ X$ 
      by (metis Col-def Out-cases  $\langle Bet \ Q1 \ PO \ Q2 \rangle \langle PO \text{ Out } X \ Q1 \rangle$ 
        between-trivial2 col-transitivity-2 not-bet-and-out out-col)
    moreover have  $Q2 \text{ OnCircle } PO \ P$ 
      by (meson OnCircle-def  $\langle Cong \ PO \ Q2 \ PO \ Q1 \rangle \langle Q1 \text{ OnCircle } PO \ P \rangle$  onc2--cong onc-sym)
    ultimately show ?thesis
      using  $\langle Bet \ Q1 \ PO \ Q2 \rangle \langle Q1 \text{ OnCircle } PO \ P \rangle$  by blast
qed
qed

```

```

lemma symmetric-oncircle:
  assumes  $PO \text{ Midpoint } X \ Y$  and
     $X \text{ OnCircle } PO \ P$ 
  shows  $Y \text{ OnCircle } PO \ P$ 
  by (meson OnCircle-def assms(1) assms(2) midpoint-cong not-cong-4312 onc2--cong onc-sym)

```

```

lemma mid-onc2--per:
  assumes  $U \text{ OnCircle } PO \ P$  and
     $V \text{ OnCircle } PO \ P$  and
     $X \text{ Midpoint } U \ V$ 
  shows  $Per \ PO \ X \ U$ 
  using Per-def assms(1) assms(2) assms(3) onc2--cong by blast

```

```

lemma mid-onc2--perp:
  assumes  $PO \neq X$  and
     $A \neq B$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and
     $X \text{ Midpoint } A \ B$ 
  shows  $PO \ X \ Perp \ A \ B$ 
proof –
  have  $Per \ PO \ X \ A$ 
    using assms(3) assms(4) assms(5) mid-onc2--per by blast
  thus ?thesis
    by (metis Perp-cases assms(1) assms(2) assms(5) col-per-perp
      is-midpoint-id midpoint-col)
qed

```

```

lemma col-onc2--perp--mid:
  assumes  $PO \neq X$  and
     $A \neq B$  and
     $Col \ A \ B \ X$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and

```

$PO \ X \ Perp \ A \ B$
shows $X \ Midpoint \ A \ B$
proof –
obtain M **where** $M \ Midpoint \ A \ B$
using *midpoint-existence* **by** *blast*
have $\neg \ Col \ A \ B \ PO$
using *assms(1) assms(3) assms(6) l8-14-1 perp-col2-bis* **by** *blast*
hence $PO \neq M$
using $\langle M \ Midpoint \ A \ B \rangle$ *midpoint-col not-col-permutation-2* **by** *blast*
hence $A \ B \ Perp \ PO \ M$
using *Perp-cases* $\langle M \ Midpoint \ A \ B \rangle$ *assms(2) assms(4) assms(5) mid-onc2--perp* **by** *blast*
hence $X = M$
by (*meson NCol-cases Perp-perm* $\langle M \ Midpoint \ A \ B \rangle$ *assms(3) assms(6)*
l8-18-uniqueness midpoint-col)
thus *?thesis*
by (*simp add:* $\langle M \ Midpoint \ A \ B \rangle$)
qed

lemma *circle-circle-os*:
assumes $I \ OnCircle \ A \ B$ **and**
 $I \ OnCircle \ C \ D$ **and**
 $\neg \ Col \ A \ C \ I$ **and**
 $\neg \ Col \ A \ C \ P$
shows $\exists \ Z. \ Z \ OnCircle \ A \ B \wedge Z \ OnCircle \ C \ D \wedge A \ C \ OS \ P \ Z$
proof –
obtain X **where** $Col \ A \ C \ X$ **and** $A \ C \ Perp \ I \ X$
using *assms(3) l8-18-existence* **by** *fastforce*
obtain $Z0$ **where** $A \ C \ Perp \ Z0 \ X$ **and** $A \ C \ OS \ P \ Z0$
using $\langle Col \ A \ C \ X \rangle$ *assms(4) l10-15* **by** *blast*
obtain Z **where** $X \ Out \ Z \ Z0$ **and** $Cong \ X \ Z \ X \ I$
by (*metis Out-def* $\langle A \ C \ OS \ P \ Z0 \rangle$ $\langle Col \ A \ C \ X \rangle$ *assms(3) col-trivial-2*
l6-11-existence l9-19)
have $A \ C \ Perp \ X \ Z$
by (*metis* $\langle A \ C \ Perp \ Z0 \ X \rangle$ $\langle X \ Out \ Z \ Z0 \rangle$ *between-trivial*
l6-6 not-bet-and-out out-col perp-col1 perp-right-comm)
have $Cong \ A \ Z \ A \ I$
proof (*cases* $A = X$)
case *True*
thus *?thesis*
by (*simp add:* $\langle Cong \ X \ Z \ X \ I \rangle$)
next
case *False*
hence $A \neq X$
by *simp*
show *?thesis*
proof (*rule l10-12* [**where** $?B = X$ **and** $?B' = X$], *insert* $\langle Cong \ X \ Z \ X \ I \rangle$)
show $Per \ A \ X \ Z$
by (*meson False* $\langle A \ C \ Perp \ X \ Z \rangle$ $\langle Col \ A \ C \ X \rangle$ *l8-16-1 not-col-distincts perp-col0*)
show $Per \ A \ X \ I$
by (*metis NCol-cases* $\langle A \ C \ Perp \ I \ X \rangle$ $\langle Col \ A \ C \ X \rangle$ *col-trivial-2 l8-16-1 l8-2*)
show $Cong \ A \ X \ A \ X$
using *cong-reflexivity* **by** *auto*
qed
qed
hence $Z \ OnCircle \ A \ B$
using *OnCircle-def* *assms(1) cong-transitivity* **by** *blast*
moreover
have $Cong \ C \ Z \ C \ I$
proof –
show *?thesis*
proof (*rule l10-12* [**where** $?B = X$ **and** $?B' = X$], *insert* $\langle Cong \ X \ Z \ X \ I \rangle$)
show $Per \ C \ X \ Z$
by (*metis NCol-perm* $\langle A \ C \ Perp \ X \ Z \rangle$ $\langle Col \ A \ C \ X \rangle$ *col-trivial-3 l8-16-1 l8-2 perp-comm*)
show $Per \ C \ X \ I$

```

    using ⟨A C Perp I X⟩ ⟨Col A C X⟩ col-trivial-2 l8-16-1 l8-2 by blast
  show Cong C X C X
    by (simp add: cong-reflexivity)
qed
qed
hence Z OnCircle C D
  by (meson OnCircle-def assms(2) onc2--cong onc-sym)
moreover
have A C OS Z0 Z
  by (metis Out-def ⟨A C OS P Z0⟩ ⟨Col A C X⟩ ⟨X Out Z Z0⟩ col124--nos l9-19-R2)
hence A C OS P Z
  using ⟨A C OS P Z0⟩ one-side-transitivity by blast
ultimately show ?thesis
  by blast
qed

```

```

lemma circle-circle-cop:
  assumes I OnCircle A B and
    I OnCircle C D
  shows  $\exists Z. Z \text{ OnCircle } A B \wedge Z \text{ OnCircle } C D \wedge \text{Coplanar } A C P Z$ 
proof (cases Col A C P)
  case True
  then show ?thesis
    using assms(1) assms(2) ncop--ncol by blast
next
  case False
  hence  $\neg \text{Col } A C P$ 
    by simp
  show ?thesis
proof (cases Col A C I)
  case True
  then show ?thesis
    using assms(1) assms(2) ncop--ncols by blast
next
  case False
  obtain Z where Z OnCircle A B and Z OnCircle C D and A C OS P Z
    using False  $\langle \neg \text{Col } A C P \rangle$  assms(1) assms(2) circle-circle-os by force
  moreover have Coplanar A C P Z
    by (simp add: calculation(3) os--coplanar)
  ultimately show ?thesis
    by blast
qed
qed

```

```

lemma line-circle-two-points:
  assumes  $U \neq V$  and
    Col U V W and
    U OnCircle PO P and
    V OnCircle PO P and
    W OnCircle PO P
  shows  $W = U \vee W = V$ 
  by (metis assms(1) assms(2) assms(3) assms(4) assms(5)
    cong2--ncol not-cong-4321 onc2--cong)

```

```

lemma onc2-mid--incs:
  assumes  $U \neq V$  and
    U OnCircle PO P and
    V OnCircle PO P and
    M Midpoint U V
  shows M InCircleS PO P

```

by (metis Midpoint-def assms(1) assms(2) assms(3) assms(4)
bet-inc2--incs midpoint-not-midpoint midpoint-out onc--inc out-diff1)

lemma circle-cases:

shows $X \text{ OnCircle } PO \ P \vee X \text{ InCircleS } PO \ P \vee X \text{ OutCircleS } PO \ P$
using inc-outc--onc outc--nincs-2 outcs--ninc-2 **by** blast

lemma inc--radius:

assumes $X \text{ InCircle } PO \ P$
shows $\exists Y. Y \text{ OnCircle } PO \ P \wedge \text{Bet } PO \ X \ Y$
by (meson Bet-cases OnCircle-def assms between-exchange3
chord-completion segment-construction)

lemma inc-onc2-out--eq:

assumes $A \text{ InCircle } PO \ P$ **and**
 $B \text{ OnCircle } PO \ P$ **and**
 $C \text{ OnCircle } PO \ P$ **and**
 $A \text{ Out } B \ C$
shows $B = C$
using assms(1) assms(2) assms(3) assms(4) inc--noutcs-1
onc2-out--outcs **by** blast

lemma onc-not-center:

assumes $PO \neq P$ **and**
 $A \text{ OnCircle } PO \ P$
shows $A \neq PO$
using OnCircle-def assms(1) assms(2) cong-reverse-identity **by** blast

lemma onc2-per--mid:

assumes $U \neq V$ **and**
 $M \neq U$ **and**
 $U \text{ OnCircle } PO \ P$ **and**
 $V \text{ OnCircle } PO \ P$ **and**
 $\text{Col } M \ U \ V$ **and**
 $\text{Per } PO \ M \ U$
shows $M \text{ Midpoint } U \ V$

proof –

obtain M' **where** $M' \text{ Midpoint } U \ V$
using midpoint-existence **by** auto
have $\text{Per } PO \ M' \ U$
using $\langle M' \text{ Midpoint } U \ V \rangle$ assms(3) assms(4) mid-onc2--per **by** blast
have $M = M' \vee \neg \text{Col } M' \ U \ V$
by (metis $\langle M' \text{ Midpoint } U \ V \rangle$ $\langle \text{Per } PO \ M' \ U \rangle$ assms(1) assms(2)
assms(5) assms(6) col-per2-cases midpoint-distinct-3)
hence $M = M'$
using $\langle M' \text{ Midpoint } U \ V \rangle$ midpoint-col **by** auto
thus ?thesis
by (simp add: $\langle M' \text{ Midpoint } U \ V \rangle$)

qed

lemma cong-chord-cong-center:

assumes $A \text{ OnCircle } PO \ P$ **and**
 $B \text{ OnCircle } PO \ P$ **and**
 $C \text{ OnCircle } PO \ P$ **and**
 $D \text{ OnCircle } PO \ P$ **and**
 $M \text{ Midpoint } A \ B$ **and**
 $N \text{ Midpoint } C \ D$ **and**
 $\text{Cong } A \ B \ C \ D$
shows $\text{Cong } PO \ N \ PO \ M$

proof –


```

have Cong M B N D
  by (meson assms(5) assms(6) assms(7) cong-commutativity cong-cong-half-1 l7-2)
have A M B PO IFSC C N D PO
  using IFSC-def ⟨Cong M B N D⟩ assms(1) assms(2) assms(3) assms(4)
    assms(5) assms(6) assms(7) midpoint-bet not-cong-4321 onc2--cong by blast
hence Cong M PO N PO
  by (simp add: l4-2)
thus ?thesis
  using not-cong-4321 by blast
qed

```

```

lemma cong-chord-cong-center1:
  assumes A ≠ B and
    C ≠ D and
    M ≠ A and
    N ≠ C
    A OnCircle PO P and
    B OnCircle PO P and
    C OnCircle PO P and
    D OnCircle PO P and
    Col M A B and
    Col N C D and
    Per PO M A and
    Per PO N C and
    Cong A B C D
  shows Cong PO N PO M
  using assms(1) assms(10) assms(11) assms(12) assms(13) assms(2) assms(3)
    assms(4) assms(5) assms(6) assms(7) assms(8) assms(9)
    cong-chord-cong-center onc2-per--mid by presburger

```

```

lemma onc-sym--onc:
  assumes Bet PO A B and
    A OnCircle PO P and
    B OnCircle PO P and
    X OnCircle PO P and
    X Y ReflectL A B
  shows Y OnCircle PO P
  using assms(1) assms(2) assms(3) assms(4) assms(5)
    between-cong image-spec--eq onc2--cong by blast

```

```

lemma mid-onc--diam:
  assumes A OnCircle PO P and
    PO Midpoint A B
  shows Diam A B PO P
  using Diam-def assms(1) assms(2) midpoint-bet symmetric-oncircle by blast

```

```

lemma chord-le-diam:
  assumes Diam A B PO P and
    U OnCircle PO P and
    V OnCircle PO P
  shows U V Le A B
proof (rule triangle-inequality-2 [where ?B=PO and ?B'=PO])
  show Bet A PO B
    using Diam-def assms(1) by auto
  show Cong U PO A PO
    using Diam-def assms(1) assms(2) not-cong-4321 onc2--cong by blast
  show Cong PO V PO B
    using Diam-def assms(1) assms(3) onc2--cong by auto
qed

```

```

lemma chord-lt-diam:
  assumes ¬ Col PO U V and
    Diam A B PO P and

```

```

    U OnCircle PO P and
    V OnCircle PO P
shows U V Lt A B
proof -
  have U V Le A B
    using assms(2) assms(3) assms(4) chord-le-diam by blast
  {
    assume Cong U V A B
    obtain O' where O' Midpoint U V
      using midpoint-existence by blast
    have Cong PO A PO B
      using Diam-def assms(2) onc2--cong by blast
    hence Cong A PO U O'
      by (meson Diam-def ⟨Cong U V A B⟩ ⟨O' Midpoint U V⟩ assms(2)
        cong-cong-half-1 cong-symmetry midpoint-def not-cong-2134)
    hence Cong B PO V O'
      by (meson Midpoint-def ⟨Cong PO A PO B⟩ ⟨O' Midpoint U V⟩
        cong-4321 cong-right-commutativity cong-transitivity)
    have Col PO U V
    proof (rule l4-13 [where ?A=PO and ?B=A and ?C=B])
      show Col PO A B
        using Col-def Diam-def assms(2) between-symmetry by blast
      show PO A B Cong3 PO U V
        using Cong3-def Diam-def ⟨Cong U V A B⟩ assms(2)
          assms(3) assms(4) cong-symmetry onc2--cong by blast
    qed
    hence False
      by (simp add: assms(1))
  }
  thus ?thesis
    using Lt-def ⟨U V Le A B⟩ by blast
qed

lemma inc2-le-diam:
  assumes Diam A B PO P and
    U InCircle PO P and
    V InCircle PO P
  shows U V Le A B
proof -
  have Bet A PO B and A OnCircle PO P and B OnCircle PO P
    using assms(1) Diam-def by auto
  obtain W where Bet U PO W and Cong PO W PO V
    using segment-construction by blast
  have U V Le U W
    using ⟨Bet U PO W⟩ ⟨Cong PO W PO V⟩ not-cong-3412 triangle-inequality by blast
  have U W Le A B
  proof (rule bet2-le2--le [where ?P=PO and ?Q=PO], insert ⟨Bet A PO B⟩)
    show Bet U PO W
      by (simp add: ⟨Bet U PO W⟩)
    show PO U Le PO A
      by (meson InCircle-def OnCircle-def ⟨A OnCircle PO P⟩
        assms(2) between-trivial l5-5-2 le-transitivity onc-sym)
    show PO W Le PO B
      by (meson InCircle-def OnCircle-def ⟨B OnCircle PO P⟩
        ⟨Cong PO W PO V⟩ assms(3) between-trivial l5-5-2 le-transitivity onc-sym)
  qed
  thus ?thesis
    using ⟨U V Le U W⟩ le-transitivity by blast
qed

lemma onc-col-diam--eq:
  assumes Diam A B PO P and
    X OnCircle PO P and
    Col A B X
  shows X = A ∨ X = B
  by (metis Diam-def assms(1) assms(2) assms(3) bet-neq21--neq

```

cong-reverse-identity line-circle-two-points onc2--cong)

```

lemma bet-onc-le-a:
  assumes Diam A B PO P and
    Bet B PO T and
    X OnCircle PO P
  shows T A Le T X
proof –
  have Cong PO X PO A
    using Diam-def assms(1) assms(3) onc2--cong by blast
  show ?thesis
  proof (cases PO = P)
    case True
    then show ?thesis
      by (metis OnCircle-def <Cong PO X PO A> assms(3)
        between-cong between-trivial2 cong-identity-inv le-reflexivity)
    next
    case False
    hence PO ≠ P
      by simp
    show ?thesis
    proof (cases T = PO)
      case True
      then show ?thesis
        by (simp add: <Cong PO X PO A> cong--le3412)
      next
      case False
      hence PO Out T A
        by (metis Diam-def <PO ≠ P> assms(1) assms(2) between-symmetry l6-2 onc-not-center)
      thus ?thesis
        using <Cong PO X PO A> triangle-reverse-inequality by auto
    qed
  qed
qed

```

```

lemma bet-onc-lt-a :
  assumes Diam A B PO P and
    PO ≠ P and
    PO ≠ T and
    X ≠ A and
    Bet B PO T and
    X OnCircle PO P
  shows T A Lt T X
proof –
  have T A Le T X
    using assms(1) assms(5) assms(6) bet-onc-le-a by blast
  have T A Lt T X ∨ Cong T A T X
    using Lt-def <T A Le T X> by auto
  {
    assume Cong T A T X
    have Bet PO A T ∨ Bet PO T A
      by (metis Bet-cases Diam-def OnCircle-def assms(1) assms(2) assms(5)
        between-cong between-trivial2 l5-2)
    moreover have Cong PO A PO X
      using Diam-def assms(1) assms(6) onc2--cong by blast
    have Bet PO A T ⟶ T A Lt T X
      using <Cong PO A PO X> <Cong T A T X> assms(4) l4-19 by blast
    moreover have Bet PO T A ⟶ T A Lt T X
      by (metis Out-def <Cong PO A PO X> <Cong T A T X> assms(3) assms(4)
        bet-out between-cong between-exchange4 l5-2 not-cong-3412
        triangle-strict-reverse-inequality)
    ultimately have T A Lt T X
      by blast
  }
thus ?thesis
  using <T A Lt T X ∨ Cong T A T X> by blast

```

qed

lemma *bet-onc-le-b*:

assumes *Diam A B PO P* and

Bet A PO T and

X OnCircle PO P

shows *T X Le T A*

by (*meson Bet-cases Diam-def assms(1) assms(2) assms(3) onc2--cong triangle-inequality*)

lemma *bet-onc-lt-b*:

assumes *Diam A B PO P* and

T ≠ PO and

X ≠ A and

Bet A PO T and

X OnCircle PO P

shows *T X Lt T A*

proof –

have *T X Le T A*

using *assms(1) assms(4) assms(5) bet-onc-le-b* by *auto*

have *T X Lt T A ∨ Cong T A T X*

using *Lt-def ⟨T X Le T A⟩ not-cong-3412* by *blast*

moreover

{

assume *Cong T A T X*

have *Bet T PO A*

using *Bet-cases assms(4)* by *auto*

have *T PO A Cong3 T PO X*

using *Cong3-def Diam-def ⟨Cong T A T X⟩ assms(1) assms(5)*

cong-reflexivity onc2--cong by *blast*

hence *Bet T PO X*

using *l4-6 [where ?A=T and ?B=PO and ?C=A] ⟨Bet T PO A⟩* by *blast*

hence *False*

using *Cong3-def ⟨Bet T PO A⟩ ⟨T PO A Cong3 T PO X⟩*

assms(2) assms(3) between-cong-3 by *blast*

}

ultimately show *?thesis*

by *blast*

qed

lemma *incs2-lt-diam*:

assumes *Diam A B PO P* and

U InCircleS PO P and

V InCircleS PO P

shows *U V Lt A B*

proof (cases *PO = P*)

case *True*

thus *?thesis*

using *InCircleS-def assms(2) lt-diff* by *blast*

next

case *False*

hence *P ≠ PO*

by *simp*

have *A PO Lt A B ∧ B PO Lt A B*

by (*metis Diam-def False Lt-cases assms(1) bet--lt1213 bet--lt2313 onc-not-center*)

show *?thesis*

proof (cases *PO = U*)

case *True*

hence *PO = U*

by *simp*

have *PO V Lt PO A*

proof (rule *cong2-lt-lt* [where *?A=PO* and *?B=V* and *?C=PO* and *?D=P*])

show *PO V Lt PO P*

using *InCircleS-def assms(3)* by *auto*

show *Cong PO V PO V*

by (*simp add: cong-reflexivity*)

show *Cong PO P PO A*

```

    using Diam-def OnCircle-def assms(1) not-cong-3412 by blast
qed
thus ?thesis
  by (metis True <A PO Lt A B ∧ B PO Lt A B> lt-right-comm lt-transitivity)
next
case False
hence PO ≠ U
  by simp
obtain V' where Bet U PO V' and Cong PO V' PO V
  using segment-construction by blast
hence U V Le U V'
  using not-cong-3412 triangle-inequality by blast
have U V' Lt A B
proof (rule bet2-lt--lt [where ?Po=PO and ?PO=PO])
  show Bet U PO V'
    by (simp add: <Bet U PO V'>)
  show Bet A PO B
    using Diam-def assms(1) by blast
  show PO U Lt PO A
  proof (rule cong2-lt--lt [where ?A=PO and ?B=U and ?C=PO and ?D=P])
    show PO U Lt PO P
      using InCircleS-def assms(2) by auto
    show Cong PO U PO U
      by (simp add: cong-reflexivity)
    show Cong PO P PO A
      using assms(1) Diam-def OnCircle-def onc-sym by blast
  qed
  show PO V' Lt PO B
  proof (rule cong2-lt--lt [where ?A=PO and ?B=V and ?C=PO and ?D=P])
    show PO V Lt PO P
      using InCircleS-def assms(3) by auto
    show Cong PO V PO V'
      using Cong-cases <Cong PO V' PO V> by blast
    show Cong PO P PO B
      using assms(1) Diam-def OnCircle-def onc-sym by blast
  qed
qed
then show ?thesis
  using <U V Le U V'> le1234-lt--lt by blast
qed
qed

lemma incs-onc-diam--lt:
  assumes Diam A B PO P and
    U InCircleS PO P and
    V OnCircle PO P
  shows U V Lt A B
proof -
  obtain U' where Bet V PO U' and Cong PO U' PO U
    using segment-construction by fastforce
  have V U' Lt A B
  proof (rule bet2-lt-le--lt [where ?Po=PO and ?PO=PO])
    show Bet V PO U'
      using <Bet V PO U'> by auto
    show Bet A PO B
      using assms(1) Diam-def by blast
    show Cong PO V PO A
      using Diam-def assms(1) assms(3) onc2--cong by blast
    show PO U' Lt PO B
      by (meson Diam-def InCircleS-def OnCircle-def
        <Cong PO U' PO U> assms(1) assms(2) cong2-lt--lt not-cong-3412)
  qed
  have V U Le V U'
    using <Bet V PO U'> <Cong PO U' PO U> not-cong-3412 triangle-inequality by blast
  thus ?thesis
    using Lt-cases <V U' Lt A B> le1234-lt--lt by blast

```

qed

lemma *diam-cong-incs--outcs*:

assumes *Diam A B PO P* and

Cong A B U V and

U InCircleS PO P

shows *V OutCircleS PO P*

proof (cases *PO = P*)

case *True*

then show ?thesis

using *assms(3) inc112 inc--noutcs-1 incs--outcs* by auto

next

case *False*

hence *PO ≠ P*

by *simp*

have *V OnCircle PO P ∨ V InCircleS PO P ∨ V OutCircleS PO P*

using *circle-cases* by blast

moreover have *V OnCircle PO P → V OutCircleS PO P*

using *assms(1) assms(2) assms(3) cong2-lt--lt cong-reflexivity*

incs-onc-diam--lt nlt by blast

moreover

{

assume *V InCircleS PO P*

hence *U V Lt A B*

using *assms(1) assms(3) incs2-lt-diam* by auto

hence *V OutCircleS PO P*

using *assms(2) cong--nlt not-cong-3412* by blast

}

ultimately show ?thesis

by blast

qed

lemma *diam-uniqueness*:

assumes *Diam A B PO P* and

Cong A X A B and

X OnCircle PO P

shows *X = B*

proof (cases *PO = P*)

case *True*

then show ?thesis

by (metis *Diam-def OnCircle-def assms(1) assms(3) cong-identity-inv*)

next

case *False*

hence *PO ≠ P*

by *simp*

have *Bet A PO X*

by (metis *Cong-cases Diam-def assms(1) assms(2) assms(3) bet-col between-trivial2 l4-18 onc2--cong*)

thus ?thesis

by (metis *Cong-cases Diam-def OnCircle-def assms(1) assms(2) between-cong between-trivial2 l5-1*)

qed

lemma *onc3--ncol*:

assumes *A ≠ B* and

A ≠ C and

B ≠ C and

A OnCircle PO P and

B OnCircle PO P and

C OnCircle PO P

shows $\neg \text{Col } A B C$

using *onc2--cong assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)*
line-circle-two-points by blast

lemma *diam-exists*:

shows $\exists A B. \text{Diam } A B PO P \wedge \text{Col } A B T$

```

using diam-points Diam-def by blast

lemma chord-intersection:
  assumes A OnCircle PO P and
    B OnCircle PO P and
    X OnCircle PO P and
    Y OnCircle PO P and
    A B TS X Y
  shows X Y TS A B
proof –
  obtain T where Col T A B and Bet X T Y
    using TS-def assms(5) by blast
  have  $\neg$  Col A X Y
    using onc3--ncol[where ?PO=PO and ?P = P]
      assms(1) assms(3) assms(4) assms(5) ts-distincts by presburger
  moreover have  $\neg$  Col B X Y
    using onc3--ncol[where ?PO=PO and ?P = P]
      assms(2) assms(3) assms(4) assms(5) ts-distincts by presburger
  moreover have Col T X Y
    using  $\langle$ Bet X T Y $\rangle$  bet-col col-permutation-4 by blast
  moreover have Bet A T B
    using col-inc-onc2--bet  $\langle$ Bet X T Y $\rangle$   $\langle$ Col T A B $\rangle$  assms(1) assms(2) assms(3) assms(4)
      assms(5) bet-inc2--inc not-col-permutation-2 onc--inc ts-distincts by blast
  ultimately show ?thesis
    using TS-def by blast
qed

lemma ray-cut-chord:
  assumes Diam A B PO P and
    X OnCircle PO P and
    Y OnCircle PO P and
    A B TS X Y and
    X Y OS PO A
  shows X Y TS PO B
proof (rule l9-8-2 [where ?A=A])
  show X Y TS A B
    using Diam-def assms(1) assms(2) assms(3) assms(4) chord-intersection by blast
  show X Y OS A PO
    by (simp add: assms(5) one-side-symmetry)
qed

lemma center-col--diam:
  assumes A  $\neq$  B and
    Col PO A B and
    A OnCircle PO P and
    B OnCircle PO P
  shows Diam A B PO P
by (metis Col-def Midpoint-def NCol-cases assms(1) assms(2) assms(3)
  assms(4) between-cong between-symmetry mid-onc--diam not-cong-3421 onc2--cong)

lemma diam--midpoint:
  assumes Diam A B PO P
  shows PO Midpoint A B
  using Diam-def Midpoint-def
  by (metis Col-def assms bet-neq21--neq col-onc2--mid l7-3-2)

lemma diam-sym:
  assumes Diam A B PO P
  shows Diam B A PO P
  using Diam-def assms between-symmetry by blast

lemma diam-end-uniqueness:
  assumes Diam A B PO P and
    Diam A C PO P
  shows B = C
  by (meson assms(1) assms(2) diam--midpoint symmetric-point-uniqueness)

```

```

lemma center-onc2-mid--ncol:
  assumes
     $\neg \text{Col } PO \ A \ B$  and

     $M \text{ Midpoint } A \ B$ 
  shows  $\neg \text{Col } A \ PO \ M$ 
  by (metis assms(2) assms(1) col-transitivity-2 col-trivial-3
      midpoint-col midpoint-distinct-1)

lemma bet-chord--diam-or-ncol:
  assumes  $A \neq B$  and
     $T \neq A$  and
     $T \neq B$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and
     $Bet \ A \ T \ B$ 
  shows  $\text{Diam } A \ B \ PO \ P \vee \neg \text{Col } PO \ A \ T \wedge \neg \text{Col } PO \ B \ T$ 
  by (metis Col-def assms(1) assms(2) assms(3) assms(4) assms(5)
      assms(6) center-col--diam l6-16-1)

lemma mid-chord--diam-or-ncol:
  assumes  $A \neq B$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and
     $T \text{ Midpoint } A \ B$ 
  shows  $\text{Diam } A \ B \ PO \ P \vee \neg \text{Col } PO \ A \ T \wedge \neg \text{Col } PO \ B \ T$ 
  using assms(1) assms(2) assms(3) assms(4) bet-chord--diam-or-ncol
      midpoint-bet midpoint-distinct-1 by blast

lemma cop-mid-onc2-perp--col:
  assumes  $A \neq B$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and
     $X \text{ Midpoint } A \ B$  and
     $X \ Y \text{ Perp } A \ B$  and
     $\text{Coplanar } PO \ A \ B \ Y$ 
  shows  $\text{Col } X \ Y \ PO$ 
  proof (cases  $PO = X$ )
  case True
  thus ?thesis
    using not-col-distincts by blast
  next
  case False
  hence  $PO \neq X$ 
  by simp
  show ?thesis
  proof (rule cop-perp2--col [where ?A = A and ?B = B])
  show  $\text{Coplanar } A \ B \ Y \ PO$ 
    by (simp add: assms(6) coplanar-perm-9)
  show  $X \ Y \text{ Perp } A \ B$ 
    by (simp add: assms(5))
  show  $X \ PO \text{ Perp } A \ B$ 
    using False assms(1) assms(2) assms(3) assms(4) mid-onc2--perp
        perp-left-comm by presburger
  qed
qed

lemma cong2-cop2-onc3--eq:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $B \neq C$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and
     $C \text{ OnCircle } PO \ P$  and
     $\text{Coplanar } A \ B \ C \ PO$  and

```


Cong $X A X B$ **and**
Cong $X A X C$ **and**
Coplanar $A B C X$
shows $X = PO$
by (*meson* *assms*(1) *assms*(10) *assms*(2) *assms*(3) *assms*(4) *assms*(5)
assms(6) *assms*(7) *assms*(8) *assms*(9) *cong4-cop2--eq* *not-cong-2143* *onc2--cong*)

lemma *tree-points-onc-cop*:
assumes $PO \neq P$
shows $\exists A B C. A \neq B \wedge A \neq C \wedge B \neq C \wedge$
 $A \text{ OnCircle } PO P \wedge B \text{ OnCircle } PO P \wedge C \text{ OnCircle } PO P \wedge$
 $\text{Coplanar } A B C PO$

proof –
obtain X **where** $\neg \text{Col } PO P X$
using *assms* *not-col-exists* **by** *presburger*
obtain $B C$ **where** $\text{Bet } B PO C$ **and** $\text{Col } B C X$ **and** $B \text{ OnCircle } PO P$ **and** $C \text{ OnCircle } PO P$
using *diam-points* **by** *force*
have $P \neq B$
by (*metis* *Col-def* $\langle \text{Bet } B PO C \rangle \langle \text{Col } B C X \rangle \langle \neg \text{Col } PO P X \rangle$
bet-neq21--neq col3 not-col-distincts)
moreover **have** $P \neq C$
by (*metis* *Out-def* $\langle \text{Bet } B PO C \rangle \langle \text{Col } B C X \rangle \langle \neg \text{Col } PO P X \rangle$ *calculation col3*
not-col-distincts out-col)
moreover **have** $B \neq C$
using $\langle B \text{ OnCircle } PO P \rangle \langle \text{Bet } B PO C \rangle$ *assms* *between-identity* *onc-not-center* **by** *blast*
moreover **have** $P \text{ OnCircle } PO P$
using *onc212* **by** *auto*
moreover **have** $\text{Coplanar } P B C PO$
by (*metis* *Bet-cases* *Col-def* $\langle \text{Bet } B PO C \rangle$ *ncop--ncols*)
ultimately show *?thesis*
using $\langle B \text{ OnCircle } PO P \rangle \langle C \text{ OnCircle } PO P \rangle$ **by** *blast*
qed

lemma *tree-points-onc-cop2*:
assumes $PO \neq P$
shows $\exists A B C. A \neq B \wedge A \neq C \wedge B \neq C \wedge$
 $A \text{ OnCircle } PO P \wedge B \text{ OnCircle } PO P \wedge C \text{ OnCircle } PO P \wedge$
 $\text{Coplanar } A B C PO \wedge \text{Coplanar } A B C Q$

proof (*cases* $PO = Q$)
case *True*
then show *?thesis*
using *assms* *tree-points-onc-cop* **by** *auto*
next
case *False*
hence $PO \neq Q$
by *simp*
then obtain X **where** $\neg \text{Col } PO Q X$
using *not-col-exists* **by** *blast*
obtain $B C$ **where** $\text{Bet } B PO C$ **and** $\text{Col } B C X$ **and** $B \text{ OnCircle } PO P$ **and** $C \text{ OnCircle } PO P$
using *diam-points* **by** *force*
obtain A **where** $PO \text{ Out } A Q$ **and** $\text{Cong } PO A PO P$
using $\langle PO \neq Q \rangle$ *assms* *l6-11-existence* **by** *metis*
have $A \neq B$
by (*metis* *Col-def* $\langle \text{Bet } B PO C \rangle \langle \text{Col } B C X \rangle \langle PO \text{ Out } A Q \rangle \langle \neg \text{Col } PO Q X \rangle$
bet-out--bet between-equality between-trivial between-trivial2 colx out-col)
moreover **have** $A \neq C$
by (*metis* *Bet-perm* *False* $\langle \text{Bet } B PO C \rangle \langle \text{Col } B C X \rangle \langle PO \text{ Out } A Q \rangle$
 $\langle \neg \text{Col } PO Q X \rangle$ *bet-out-1 bet-out--bet calculation colx*
not-col-permutation-4 not-col-permutation-5 out-col)
moreover **have** $B \neq C$
using $\langle B \text{ OnCircle } PO P \rangle \langle \text{Bet } B PO C \rangle$ *assms* *bet-neq12--neq* *onc-not-center* **by** *blast*
moreover **have** $A \text{ OnCircle } PO P$
by (*simp* *add*: *OnCircle-def* $\langle \text{Cong } PO A PO P \rangle$)
moreover **have** $\text{Coplanar } A B C PO$
using $\langle \text{Bet } B PO C \rangle$ *bet--coplanar* *coplanar-perm-19* **by** *blast*
moreover **hence** $\text{Coplanar } A B C Q$

by (metis Out-def ⟨PO Out A Q⟩ col-cop--cop coplanar-perm-8 not-col-permutation-4 out-col)
ultimately show ?thesis
using ⟨B OnCircle PO P⟩ ⟨C OnCircle PO P⟩ by blast
qed

lemma tree-points-onc:
assumes PO ≠ P
shows ∃ A B C.
A ≠ B ∧ A ≠ C ∧ B ≠ C ∧ A OnCircle PO P ∧ B OnCircle PO P ∧ C OnCircle PO P
proof –
obtain A B C where A ≠ B and A ≠ C and B ≠ C and
A OnCircle PO P and B OnCircle PO P and C OnCircle PO P and
Coplanar A B C PO
using assms tree-points-onc-cop by force
thus ?thesis
by blast
qed

lemma bet-cop-onc2--ex-onc-os-out:
assumes A ≠ I and
B ≠ I and
¬ Col A B C and
¬ Col A B PO and
A OnCircle PO P and
B OnCircle PO P and
Bet A I B and
Coplanar A B C PO
shows ∃ C1. C1 Out PO I ∧ C1 OnCircle PO P ∧ A B OS C C1
proof –
obtain C1 C2 where Bet C1 PO C2 and Col C1 C2 I and
C1 OnCircle PO P and C2 OnCircle PO P
using diam-points by blast
have A B TS C1 C2
proof (rule chord-intersection [where ?PO=PO and ?P=P])
show C1 OnCircle PO P
using ⟨C1 OnCircle PO P⟩ by auto
show C2 OnCircle PO P
by (simp add: ⟨C2 OnCircle PO P⟩)
show A OnCircle PO P
by (simp add: assms(5))
show B OnCircle PO P
by (simp add: assms(6))
have C1 ≠ C2
by (metis ⟨Bet C1 PO C2⟩ ⟨C1 OnCircle PO P⟩ assms(4) assms(5)
bet-neq12--neq col-trivial-3 onc-not-center onc-sym)
show C1 C2 TS A B
proof –
{
assume Col A C1 C2
hence Col A B PO
by (metis ⟨Bet C1 PO C2⟩ ⟨C1 ≠ C2⟩ ⟨Col C1 C2 I⟩ assms(1)
assms(7) bet-col1 between-trivial col-transitivity-1
not-col-permutation-3 not-col-permutation-5)
hence False
by (simp add: assms(4))
}
moreover
{
assume Col B C1 C2
hence Col A B PO
by (metis ⟨Col C1 C2 I⟩ assms(2) assms(7) bet-col bet-col1
calculation col-permutation-1 l6-21)
hence False
by (simp add: assms(4))
}

```

    moreover have Col I C1 C2
      using NCol-cases ⟨Col C1 C2 I⟩ by blast
    moreover have Bet A I B
      by (simp add: assms(7))
    ultimately show ?thesis
      using TS-def by blast
  qed
qed
have Bet C1 I C2
  by (meson NCol-cases ⟨A B TS C1 C2⟩ ⟨Col C1 C2 I⟩ assms(7) bet-col l9-18)
have ¬ Col C1 A B
  using TS-def ⟨A B TS C1 C2⟩ by auto
have Coplanar A B C C1
proof (rule coplanar-trans-1 [where ?P=PO])
  show ¬ Col PO A B
    by (simp add: assms(4) not-col-permutation-2)
  show Coplanar PO A B C
    using assms(8) coplanar-perm-18 by blast
  show Coplanar PO A B C1
    using ⟨A B TS C1 C2⟩ ⟨Bet C1 PO C2⟩ bet-cop--cop coplanar-perm-18 ts--coplanar by blast
qed
hence A B TS C C1 ∨ A B OS C C1
  using cop--one-or-two-sides Col-cases ⟨¬ Col C1 A B⟩ assms(3) by blast
moreover
{
  assume A B TS C C1
  have C2 Out PO I
    by (metis Bet-cases Col-def OnCircle-def Out-cases TS-def
      ⟨A B TS C1 C2⟩ ⟨Bet C1 I C2⟩ ⟨Bet C1 PO C2⟩ ⟨C2 OnCircle PO P⟩
      assms(6) assms(7) bet-out-1 between-trivial2 cong-reverse-identity
      l6-7 not-cong-3421)
  moreover have A B OS C C2
    using OS-def ⟨A B TS C C1⟩ ⟨A B TS C1 C2⟩ l9-2 by blast
  ultimately have C2 Out PO I ∧ C2 OnCircle PO P ∧ A B OS C C2
    using ⟨C2 OnCircle PO P⟩ by blast
}
moreover
{
  assume A B OS C C1
  have C1 Out PO I
    by (metis Col-def ⟨A B TS C1 C2⟩ ⟨Bet C1 I C2⟩
      ⟨Bet C1 PO C2⟩ ⟨C1 OnCircle PO P⟩ ⟨C2 OnCircle PO P⟩
      ⟨¬ Col C1 A B⟩ assms(7) bet2--out between-symmetry l9-9
      onc-not-center onc-sym one-side-reflexivity)
  hence C1 Out PO I ∧ C1 OnCircle PO P ∧ A B OS C C1
    using ⟨C1 OnCircle PO P⟩ ⟨A B OS C C1⟩
    by blast
}
ultimately show ?thesis
  by blast
qed

lemma eqc-chara-1:
  assumes EqC A B C D
  shows A = C ∧ Cong A B C D
proof -
  have A = C
  proof (cases A = B)
    case True
    thus ?thesis
      by (metis EqC-def OnCircle-def assms cong-identity-inv onc212 tree-points-one)
  next
    case False
    hence A ≠ B
    by simp
    then obtain B0 B1 B2 where B0 ≠ B1 and B0 ≠ B2 and B1 ≠ B2 and

```

```

    B0 OnCircle A B and B1 OnCircle A B and B2 OnCircle A B and
    Coplanar B0 B1 B2 A and Coplanar B0 B1 B2 C
    using tree-points-onc-cop2 by blast
    thus ?thesis
    by (meson EqC-def assms cong2-cop2-onc3--eq onc2--cong)
qed
moreover have Cong A B C D
  using EqC-def OnCircle-def assms calculation onc212 by blast
ultimately show ?thesis
  by blast
qed

```

```

lemma eqc-chara-2:
  assumes A = C and
    Cong A B C D
  shows EqC A B C D
  by (metis Cong-cases EqC-def OnCircle-def assms(1) assms(2) onc2--cong)

```

```

lemma eqc-chara:
  shows EqC A B C D  $\longleftrightarrow$  (A = C  $\wedge$  Cong A B C D)
  using eqc-chara-1 eqc-chara-2 by blast

```

```

lemma neqc-chara-1:
  assumes A  $\neq$  B and
     $\neg$  EqC A B C D
  shows  $\exists X. X \text{ OnCircle } A B \wedge \neg X \text{ OnCircle } C D$ 
proof (cases A = C)
  case True
  thus ?thesis
    by (metis OnCircle-def assms(2) eqc-chara-2 onc212)
next
  case False
  hence A  $\neq$  C
  by simp
  obtain B0 B1 B2 where B0  $\neq$  B1 and B0  $\neq$  B2 and B1  $\neq$  B2 and
    B0 OnCircle A B and B1 OnCircle A B and B2 OnCircle A B and
    Coplanar B0 B1 B2 A and Coplanar B0 B1 B2 C
  using tree-points-onc-cop2 assms(1) by blast
  show ?thesis
proof (cases Cong C B0 C D)
  case True
  thus ?thesis
    by (meson False  $\langle$ B0 OnCircle A B $\rangle$   $\langle$ B0  $\neq$  B1 $\rangle$   $\langle$ B0  $\neq$  B2 $\rangle$ 
       $\langle$ B1 OnCircle A B $\rangle$   $\langle$ B1  $\neq$  B2 $\rangle$   $\langle$ B2 OnCircle A B $\rangle$   $\langle$ Coplanar B0 B1 B2 A $\rangle$ 
       $\langle$ Coplanar B0 B1 B2 C $\rangle$  cong2-cop2-onc3--eq onc2--cong)
next
  case False
  hence  $\neg$  Cong C B0 C D
  by simp
  show ?thesis
proof (cases Cong C B1 C D)
  case True
  then show ?thesis
    using False OnCircle-def  $\langle$ B0 OnCircle A B $\rangle$  by blast
next
  case False
  hence  $\neg$  Cong C B1 C D
  by simp
  show ?thesis
proof (cases Cong C B2 C D)
  case True
  then show ?thesis
    using False OnCircle-def  $\langle$ B1 OnCircle A B $\rangle$  by blast
next

```

```

    case False
    hence  $\neg \text{Cong } C \ B2 \ C \ D$ 
    by simp
    then show ?thesis
    using OnCircle-def  $\langle B2 \ \text{OnCircle } A \ B \rangle$  by blast
qed
qed
qed
qed

lemma neqc-chara-2:
  assumes
     $X \ \text{OnCircle } A \ B$  and
     $\neg X \ \text{OnCircle } C \ D$ 
  shows  $\neg \text{EqC } A \ B \ C \ D$ 
  using EqC-def assms(2) assms(1) by blast

lemma neqc-chara:
  assumes  $A \neq B$ 
  shows  $\neg \text{EqC } A \ B \ C \ D \longleftrightarrow (\exists X. X \ \text{OnCircle } A \ B \wedge \neg X \ \text{OnCircle } C \ D)$ 
  using assms neqc-chara-1 neqc-chara-2 by blast

lemma eqc-refl:
  shows  $\text{EqC } A \ B \ A \ B$ 
  by (simp add: EqC-def)

lemma eqc-sym:
  assumes  $\text{EqC } A \ B \ C \ D$ 
  shows  $\text{EqC } C \ D \ A \ B$ 
  using EqC-def assms by auto

lemma eqc-trans:
  assumes  $\text{EqC } A \ B \ C \ D$  and
     $\text{EqC } C \ D \ E \ F$ 
  shows  $\text{EqC } A \ B \ E \ F$ 
  by (metis OnCircle-def assms(1) assms(2) eqc-chara not-cong-3412 onc2--cong)

lemma cop2-onc6--eqc:
  assumes  $A \neq B$  and
     $B \neq C$  and
     $A \neq C$  and
     $A \ \text{OnCircle } PO \ P$  and
     $B \ \text{OnCircle } PO \ P$  and
     $C \ \text{OnCircle } PO \ P$  and
    Coplanar  $A \ B \ C \ PO$  and
     $A \ \text{OnCircle } O' \ P'$  and
     $B \ \text{OnCircle } O' \ P'$  and
     $C \ \text{OnCircle } O' \ P'$  and
    Coplanar  $A \ B \ C \ O'$ 
  shows  $\text{EqC } PO \ P \ O' \ P'$ 
proof -
  have  $PO = O'$ 
  by (meson assms(1) assms(10) assms(11) assms(2) assms(3) assms(4)
    assms(5) assms(6) assms(7) assms(8) assms(9)
    cong2-cop2-onc3--eq onc2--cong)
  thus ?thesis
  by (metis OnCircle-def assms(10) assms(6) eqc-chara-2 onc212 onc2--cong)
qed

```

```

lemma concyclic-aux:
  assumes Concyclic A B C D
  shows  $\exists PO P. A \text{ OnCircle } PO P \wedge B \text{ OnCircle } PO P \wedge C \text{ OnCircle } PO P \wedge$ 
          $D \text{ OnCircle } PO P \wedge \text{Coplanar } A B C PO$ 
proof-
  have 1: Coplanar A B C D  $\wedge$ 
    ( $\exists O1 P1. A \text{ OnCircle } O1 P1 \wedge B \text{ OnCircle } O1 P1 \wedge C \text{ OnCircle } O1 P1 \wedge D \text{ OnCircle } O1 P1$ )
    using Concyclic-def assms by blast
  then obtain O1 P1 where A OnCircle O1 P1 and B OnCircle O1 P1 and
    C OnCircle O1 P1 and D OnCircle O1 P1
    by blast
  have Coplanar A B C D
    using 1 by blast
  show ?thesis
  proof (cases Col A B C)
    case True
    then show ?thesis
      using  $\langle A \text{ OnCircle } O1 P1 \rangle \langle B \text{ OnCircle } O1 P1 \rangle \langle C \text{ OnCircle } O1 P1 \rangle$ 
         $\langle D \text{ OnCircle } O1 P1 \rangle \text{col--coplanar}$  by blast
  next
    case False
    hence  $\neg \text{Col } A B C$ 
      by simp
    obtain PO where Coplanar A B C PO and  $\forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1$ 
      using l11-62-existence by blast
    have  $\forall A B. (A \text{ OnCircle } O1 P1 \wedge B \text{ OnCircle } O1 P1) \longrightarrow \text{Cong } O1 A O1 B$ 
      using onc2--cong by blast
    have A OnCircle PO A
      by (simp add: onc212)
    moreover have Cong PO B PO A
    proof (rule cong2-per2--cong [where ?C=O1 and ?C'=O1], insert  $\langle \text{Coplanar } A B C PO \rangle$ )
      show Per B PO O1
        using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1 \rangle \text{ncop-distincts}$  by blast
      show Per A PO O1
        using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1 \rangle \text{ncop-distincts}$  by blast
      show Cong B O1 A O1
        using  $\langle A \text{ OnCircle } O1 P1 \rangle \langle B \text{ OnCircle } O1 P1 \rangle \text{cong-4321 onc2--cong}$  by blast
      show Cong PO O1 PO O1
        by (simp add: cong-reflexivity)
    qed
    hence B OnCircle PO A
      by (simp add: OnCircle-def)
    moreover have Cong PO C PO A
    proof (rule cong2-per2--cong [where ?C=O1 and ?C'=O1])
      show Per C PO O1
        using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1 \rangle \text{ncop-distincts}$  by blast
      show Per A PO O1
        using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1 \rangle \text{ncop-distincts}$  by blast
      show Cong C O1 A O1
        using  $\langle A \text{ OnCircle } O1 P1 \rangle \langle C \text{ OnCircle } O1 P1 \rangle \text{cong-4321 onc2--cong}$  by blast
      show Cong PO O1 PO O1
        by (simp add: cong-reflexivity)
    qed
    hence C OnCircle PO A
      by (simp add: OnCircle-def)
    moreover have Cong PO D PO A
    proof (rule cong2-per2--cong [where ?C=O1 and ?C'=O1])
      show Per D PO O1
        using Concyclic-def  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1 \rangle \text{assms}$  by blast
      show Per A PO O1
        using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E PO O1 \rangle \text{ncop-distincts}$  by blast
      show Cong D O1 A O1
        using  $\langle A \text{ OnCircle } O1 P1 \rangle \langle D \text{ OnCircle } O1 P1 \rangle$ 
           $\langle \forall A B. A \text{ OnCircle } O1 P1 \wedge B \text{ OnCircle } O1 P1 \longrightarrow \text{Cong } O1 A O1 B \rangle$ 
          not-cong-4321 by blast
    qed
  qed

```

```

    show Cong PO O1 PO O1
    using cong-reflexivity by auto
qed
hence D OnCircle PO A
  by (simp add: OnCircle-def)
ultimately show ?thesis
  using ⟨Coplanar A B C PO⟩ by blast
qed
qed

lemma concyclic-perm-1:
  assumes Concyclic A B C D
  shows Concyclic B C D A
  using Concyclic-def assms coplanar-perm-9 by blast

lemma concyclic-gen-perm-1:
  assumes ConcyclicGen A B C D
  shows ConcyclicGen B C D A
  by (metis Col-cases ConcyclicGen-def assms concyclic-perm-1)

lemma concyclic-perm-2:
  assumes Concyclic A B C D
  shows Concyclic B A C D
  by (meson Concyclic-def assms concyclic-perm-1 coplanar-perm-20)

lemma concyclic-gen-perm-2:
  assumes ConcyclicGen A B C D
  shows ConcyclicGen B A C D
  using ConcyclicGen-def assms concyclic-perm-2 not-col-permutation-4 by blast

lemma concyclic-trans-1:
  assumes ¬ Col P Q R and
    Concyclic P Q R A and
    Concyclic P Q R B
  shows Concyclic Q R A B
proof (simp add: Concyclic-def, rule conjI)
  show Coplanar Q R A B
    using Concyclic-def assms(1) assms(2) assms(3) coplanar-trans-1 by presburger
  show ∃ PO P. Q OnCircle PO P ∧ R OnCircle PO P ∧ A OnCircle PO P ∧ B OnCircle PO P
  proof -
    obtain PO M where P OnCircle PO M and Q OnCircle PO M and R OnCircle PO M and
      A OnCircle PO M and Coplanar P Q R PO
    using concyclic-aux assms(2) by blast
    obtain O' M' where P OnCircle O' M' and Q OnCircle O' M' and R OnCircle O' M' and
      B OnCircle O' M' and Coplanar P Q R O'
    using concyclic-aux assms(3) by blast
    have B OnCircle PO M
      using cop2-onc6--eqc ⟨B OnCircle O' M'⟩ ⟨Coplanar P Q R O'⟩
        ⟨Coplanar P Q R PO⟩ ⟨P OnCircle O' M'⟩ ⟨P OnCircle PO M⟩
        ⟨Q OnCircle O' M'⟩ ⟨Q OnCircle PO M⟩ ⟨R OnCircle O' M'⟩
        ⟨R OnCircle PO M⟩ assms(1) neqc-chara-2 not-col-distincts by blast
    thus ?thesis
      using ⟨A OnCircle PO M⟩ ⟨Q OnCircle PO M⟩ ⟨R OnCircle PO M⟩ by blast
  qed
qed

lemma concyclic-gen-trans-1:
  assumes ¬ Col P Q R and
    ConcyclicGen P Q R A and
    ConcyclicGen P Q R B
  shows ConcyclicGen Q R A B
  using ConcyclicGen-def assms(1) assms(2) assms(3) concyclic-trans-1 by presburger

lemma concyclic-pseudo-trans:
  assumes ¬ Col P Q R and
    Concyclic P Q R A and

```

```

Concyclic P Q R B and
Concyclic P Q R C and
Concyclic P Q R D
shows Concyclic A B C D
proof (simp add: Concyclic-def, rule conjI)
show Coplanar A B C D
proof -
  have Coplanar P Q R A
    using Concyclic-def assms(2) by blast
  moreover have Coplanar P Q R B
    using Concyclic-def assms(3) by blast
  moreover have Coplanar P Q R C
    using Concyclic-def assms(4) by blast
  moreover have Coplanar P Q R D
    using Concyclic-def assms(5) by blast
  ultimately show ?thesis
    using assms(1) coplanar-pseudo-trans by blast
qed
show  $\exists PO P. A \text{ OnCircle } PO P \wedge B \text{ OnCircle } PO P \wedge C \text{ OnCircle } PO P \wedge D \text{ OnCircle } PO P$ 
proof -
  obtain OA MA where P OnCircle OA MA and Q OnCircle OA MA and R OnCircle OA MA and
    A OnCircle OA MA and Coplanar P Q R OA
    using concyclic-aux assms(2) by blast
  obtain OB MB where P OnCircle OB MB and Q OnCircle OB MB and R OnCircle OB MB and
    B OnCircle OB MB and Coplanar P Q R OB
    using concyclic-aux assms(3) by blast
  obtain OC MC where P OnCircle OC MC and Q OnCircle OC MC and R OnCircle OC MC and
    C OnCircle OC MC and Coplanar P Q R OC
    using concyclic-aux assms(4) by blast
  obtain OD MD where P OnCircle OD MD and Q OnCircle OD MD and R OnCircle OD MD and
    D OnCircle OD MD and Coplanar P Q R OD
    using concyclic-aux assms(5) by blast
  have A OnCircle OA MA
    by (simp add:  $\langle A \text{ OnCircle } OA MA \rangle$ )
  moreover have B OnCircle OA P
    using OnCircle-def  $\langle B \text{ OnCircle } OB MB \rangle \langle \text{Coplanar } P Q R OA \rangle \langle \text{Coplanar } P Q R OB \rangle$ 
     $\langle P \text{ OnCircle } OA MA \rangle \langle P \text{ OnCircle } OB MB \rangle \langle Q \text{ OnCircle } OA MA \rangle$ 
     $\langle Q \text{ OnCircle } OB MB \rangle \langle R \text{ OnCircle } OA MA \rangle \langle R \text{ OnCircle } OB MB \rangle$ 
    assms(1) cong2-cop2-onc3--eq not-col-distincts onc2--cong by blast
  moreover have C OnCircle OA P
    using OnCircle-def  $\langle C \text{ OnCircle } OC MC \rangle \langle \text{Coplanar } P Q R OA \rangle \langle \text{Coplanar } P Q R OC \rangle$ 
     $\langle P \text{ OnCircle } OA MA \rangle \langle P \text{ OnCircle } OC MC \rangle \langle Q \text{ OnCircle } OA MA \rangle$ 
     $\langle Q \text{ OnCircle } OC MC \rangle \langle R \text{ OnCircle } OA MA \rangle \langle R \text{ OnCircle } OC MC \rangle$ 
    assms(1) cong2-cop2-onc3--eq not-col-distincts onc2--cong by blast
  moreover have D OnCircle OA P
    using OnCircle-def  $\langle \text{Coplanar } P Q R OA \rangle \langle \text{Coplanar } P Q R OD \rangle \langle D \text{ OnCircle } OD MD \rangle$ 
     $\langle P \text{ OnCircle } OA MA \rangle \langle P \text{ OnCircle } OD MD \rangle \langle Q \text{ OnCircle } OA MA \rangle$ 
     $\langle Q \text{ OnCircle } OD MD \rangle \langle R \text{ OnCircle } OA MA \rangle \langle R \text{ OnCircle } OD MD \rangle$ 
    assms(1) cong2-cop2-onc3--eq not-col-distincts onc2--cong by blast
  ultimately show ?thesis
    using OnCircle-def  $\langle P \text{ OnCircle } OA MA \rangle \text{onc2--cong}$  by blast
qed
qed
lemma concyclic-gen-pseudo-trans:
  assumes  $\neg \text{Col } P Q R$  and
    ConcyclicGen P Q R A
    ConcyclicGen P Q R B
    ConcyclicGen P Q R C
    ConcyclicGen P Q R D
  shows ConcyclicGen A B C D
  using ConcyclicGen-def assms(1) assms(2) assms(3) assms(4) assms(5)
    concyclic-pseudo-trans by presburger
lemma Concyclic--Concyclic2:
  assumes Concyclic A B C D

```


shows *Concyclic2 A B C D*
proof –
 have *Coplanar A B C D*
 using *Concyclic-def assms* **by** *blast*
 moreover
 obtain *PO P* where *A OnCircle PO P* and *B OnCircle PO P* and
C OnCircle PO P and *D OnCircle PO P*
 using *Concyclic-def assms* **by** *blast*
 hence $\exists P. \text{Cong } P A P B \wedge \text{Cong } P A P C \wedge \text{Cong } P A P D$
 using *onc2--cong* **by** *blast*
 ultimately show *?thesis*
 using *Concyclic2-def* **by** *blast*
qed

lemma *Concyclic2--Concyclic*:
 assumes *Concyclic2 A B C D*
 shows *Concyclic A B C D*
proof –
 have *Coplanar A B C D*
 using *assms Concyclic2-def* **by** *blast*
 moreover
 obtain *P* where *Cong P A P B* and *Cong P A P C* and *Cong P A P D*
 using *assms Concyclic2-def* **by** *blast*
 have $\exists PO P. A \text{ OnCircle } PO P \wedge B \text{ OnCircle } PO P \wedge C \text{ OnCircle } PO P \wedge D \text{ OnCircle } PO P$
 by (*meson OnCircle-def onc212 onc-sym*
 $\langle \wedge \text{thesis}. (\wedge P. [\text{Cong } P A P B; \text{Cong } P A P C; \text{Cong } P A P D] \implies \text{thesis}) \implies \text{thesis} \rangle$)
 ultimately show *?thesis*
 using *Concyclic-def* **by** *auto*
qed

lemma *TangentCC-Col*:
 assumes *TangentCC A B C D* and
X OnCircle A B and
X OnCircle C D
 shows *Col X A C*
proof (*cases A = C*)
 case *True*
 then show *?thesis*
 by (*simp add: col-trivial-2*)
 next
 case *False*
 hence $A \neq C$
 by *simp*
 then obtain *Y* where *A C Perp X Y* $\vee X = Y$ and
 $\exists M. \text{Col } A C M \wedge M \text{ Midpoint } X Y \wedge X Y \text{ Reflect } A C$
 using *ex-sym1* **by** *blast*
 then obtain *M* where *Col A C M* and *M Midpoint X Y* and
X Y Reflect A C **by** *blast*
 have *Cong X A Y A*
 using $\langle X Y \text{ Reflect } A C \rangle$ *image-triv l10-10 not-cong-4321* **by** *blast*
 have *Cong X C Y C*
 using *False* $\langle X Y \text{ Reflect } A C \rangle$ *is-image-col-cong not-col-distincts* **by** *blast*
 have $X \text{ OnCircle } A B \wedge X \text{ OnCircle } C D$
 using *assms(2) assms(3)* **by** *blast*
 hence $\exists X. X \text{ OnCircle } A B \wedge X \text{ OnCircle } C D$
 by *blast*
 have *Y OnCircle A B*
 by (*meson OnCircle-def* $\langle \text{Cong } X A Y A \rangle$ *assms(2) cong-commutativity cong-inner-transitivity*)
 have *Y OnCircle C D*
 by (*meson OnCircle-def* $\langle \text{Cong } X C Y C \rangle$ $\langle X \text{ OnCircle } A B \wedge X \text{ OnCircle } C D \rangle$
cong-inner-transitivity not-cong-2134)
 hence $X = Y$

```

  using TangentCC-def ⟨Y OnCircle A B⟩ assms(1) assms(2) assms(3) by blast
{
  assume A ≠ C and X X ReflectL A C
  hence ?thesis
    using Col-cases ⟨Col A C M⟩ ⟨M Midpoint X Y⟩ ⟨X = Y⟩ l8-20-2 by blast
}
moreover
{
  assume A = C and A Midpoint X X
  hence ?thesis
    using False by blast
}
ultimately show ?thesis
  using Col-cases ⟨X = Y⟩ ⟨∃ M. Col A C M ∧ M Midpoint X Y ∧ X Y Reflect A C⟩ l7-3 by blast
qed

```

```

lemma tangent-neq:
  assumes PO ≠ P and
    Tangent A B PO P
  shows A ≠ B
  by (metis OnCircle-def Tangent-def assms(1,2) between-equality-2 between-symmetry
    col-trivial-1 cong-reflexivity segment-construction)

```

```

lemma diam-not-tangent:
  assumes P ≠ PO and
    Col PO A B
  shows ¬ Tangent A B PO P
proof -
{
  assume Tangent A B PO P
  then obtain Q where Col A B Q and Q OnCircle PO P and
    *: ∀ Y. (Col A B Y ∧ Y OnCircle PO P) ⟶ Q = Y
    using Tangent-def by auto
  have False
  proof (cases A = B)
    case True
    then show ?thesis
      by (metis ⟨Tangent A B PO P⟩ assms(1) tangent-neq)
  next
    case False
    hence A ≠ B
      by simp
    obtain C where A ≠ C and B ≠ C and PO ≠ C and Col A B C
      by (metis diff-col-ex diff-col-ex3)
    obtain Q1 Q2 where Bet Q1 PO Q2 and Col Q1 Q2 C and
      Q1 OnCircle PO P and Q2 OnCircle PO P
      using diam-points by blast
    hence Q1 ≠ Q2
      using assms(1) between-identity cong-reverse-identity onc212 onc2--cong by blast
    moreover have Q = Q1
    proof -
      have Col A B Q1
        by (metis Col-cases Col-def ⟨Col A B C⟩ ⟨Col Q1 Q2 C⟩ ⟨PO ≠ C⟩
          ⟨Bet Q1 PO Q2⟩ assms(2) calculation l6-21)
      moreover have Q1 OnCircle PO P
        using ⟨Q1 OnCircle PO P⟩ by auto
      ultimately show ?thesis
        using * by blast
    qed
    moreover have Q = Q2
    proof -
      have Col A B Q2
        by (metis ⟨Col A B C⟩ ⟨Col A B Q⟩ ⟨Col Q1 Q2 C⟩ ⟨PO ≠ C⟩ ⟨Bet Q1 PO Q2⟩
          assms(2) bet-col calculation(2) colx not-col-permutation-2)

```

```

    moreover have  $Q2 \text{ OnCircle } PO \ P$ 
    by (simp add:  $\langle Q2 \text{ OnCircle } PO \ P \rangle$ )
    ultimately show ?thesis
    using * by blast
  qed
  ultimately show ?thesis
  by blast
qed
}
thus ?thesis
by blast
qed

```

```

lemma tangent-out:
  assumes  $X \neq T$  and
     $Col \ A \ B \ X$  and
     $TangentAt \ A \ B \ PO \ P \ T$ 
  shows  $X \text{ OutCircleS } PO \ P$ 
proof (cases  $PO = P$ )
case True
then show ?thesis
  using OnCircle-def TangentAt-def assms(1) assms(3) cong-identity inc-eq
  outcs--ninc-2 by blast
next
case False
have  $Tangent \ A \ B \ PO \ P$ 
  using assms(3) TangentAt-def by blast
have  $Col \ A \ B \ T$ 
  using assms(3) TangentAt-def by blast
have  $T \text{ OnCircle } PO \ P$ 
  using assms(3) TangentAt-def by blast
{
  assume  $X \text{ InCircle } PO \ P$ 
  then obtain  $T'$  where  $T' \text{ OnCircle } PO \ P$  and  $Bet \ T \ X \ T'$ 
    using TangentAt-def assms(3) chord-completion by blast
  have  $A \neq B$ 
    using False TangentAt-def assms(3) tangent-neq by blast
  obtain  $TT$  where  $Col \ A \ B \ TT$  and  $TT \text{ OnCircle } PO \ P$  and
     $\forall Y. (Col \ A \ B \ Y \wedge Y \text{ OnCircle } PO \ P) \longrightarrow TT = Y$ 
    using Tangent-def  $\langle Tangent \ A \ B \ PO \ P \rangle$  by auto
  have  $TT = T$ 
    by (simp add:  $\langle Col \ A \ B \ T \rangle \langle T \text{ OnCircle } PO \ P \rangle \langle \forall Y. Col \ A \ B \ Y \wedge Y \text{ OnCircle } PO \ P \longrightarrow TT = Y \rangle$ )
  have  $T = T'$ 
    by (metis  $\langle Col \ A \ B \ T \rangle \langle Bet \ T \ X \ T' \rangle \langle T' \text{ OnCircle } PO \ P \rangle \langle TT = T \rangle$ 
       $\langle \forall Y. Col \ A \ B \ Y \wedge Y \text{ OnCircle } PO \ P \longrightarrow TT = Y \rangle$  assms(1) assms(2) bet-col colx)
  have  $Col \ A \ X \ T$ 
    using  $\langle Bet \ T \ X \ T' \rangle \langle T = T' \rangle$  assms(1) bet-neq12--neq by blast
  hence  $X = T$ 
    using  $\langle Bet \ T \ X \ T' \rangle \langle T = T' \rangle$  between-identity by blast
}
then show ?thesis
  using assms(1) ninc--outcs by blast
qed

```

```

lemma tangentat-perp:
  assumes  $PO \neq P$  and
     $TangentAt \ A \ B \ PO \ P \ T$ 
  shows  $A \ B \ Perp \ PO \ T$ 
proof -
  have  $Tangent \ A \ B \ PO \ P$ 
    using assms(2) TangentAt-def by blast
  have  $Col \ A \ B \ T$ 

```

```

  using assms(2) TangentAt-def by blast
have T OnCircle PO P
  using assms(2) TangentAt-def by blast
have A ≠ B
  using ⟨Tangent A B PO P⟩ assms(1) tangent-neq by blast
have ¬ Col A B PO
  by (metis NCol-cases ⟨Tangent A B PO P⟩ assms(1) diam-not-tangent)
obtain R where Col A B R and A B Perp PO R
  using ⟨¬ Col A B PO⟩ l8-18-existence by blast
show ?thesis
proof (cases T = R)
  case True
  then show ?thesis
    using ⟨A B Perp PO R⟩ by auto
next
  case False
  hence T ≠ R
    by simp
  obtain T' where R Midpoint T T'
    using symmetric-point-construction by auto
  show ?thesis
  proof (cases A = R)
    case True
    hence T R Perp R PO
      by (metis False ⟨A B Perp PO R⟩ ⟨Col A B T⟩ perp-col perp-comm)
    hence R PerpAt T R R PO
      using col-trivial-1 col-trivial-3 l8-14-2-1b-bis by auto
    hence Per PO R T
      using l8-2 perp-in-per-2 by blast
    then obtain T'' where R Midpoint T T'' and Cong PO T PO T''
      using Per-def by blast
    hence T' = T''
      using SymR-uniq-aux ⟨R Midpoint T T'⟩ by blast
    have T ≠ T'
      using False ⟨R Midpoint T T'⟩ l8-20-2 by blast
    have T' OnCircle PO P
      using OnCircle-def ⟨Cong PO T PO T''⟩ ⟨T OnCircle PO P⟩ ⟨T' = T''⟩
      cong-inner-transitivity by blast
    have T' OutCircleS PO P
      by (metis Col-def Midpoint-def True ⟨Col A B T⟩ ⟨R Midpoint T T''⟩ ⟨T ≠ T'⟩
      ⟨T' = T''⟩ assms(2) col-transitivity-2 l4-19 not-bet-distincts tangent-out)
    hence PO P Lt PO T'
      by (simp add: OutCircleS-def)
    hence PO P Le PO T'
      using lt-le by auto
    have ¬ Cong PO P PO T'
      using ⟨PO P Lt PO T'⟩ cong-nlt by blast
    have Cong PO T' PO P
      using OnCircle-def ⟨T' OnCircle PO P⟩ by blast
    then show ?thesis
      using Cong-cases ⟨¬ Cong PO P PO T'⟩ by blast
  next
    case False
    hence A ≠ R
      by simp
    hence A R Perp PO R
      using ⟨A B Perp PO R⟩ ⟨Col A B R⟩ perp-col by blast
    have Col R A T
      by (metis ⟨A ≠ B⟩ ⟨Col A B R⟩ ⟨Col A B T⟩ col-permutation-4 col-transitivity-2)
    have R T Perp PO R
      using Perp-perm ⟨A B Perp PO R⟩ ⟨Col A B R⟩ ⟨Col A B T⟩ ⟨T ≠ R⟩ perp-col0 by blast
    hence R PerpAt T R R PO
      using Perp-in-perm perp-perp-in by blast
    have Per PO R T
      using ⟨R T Perp PO R⟩ col-trivial-2 col-trivial-3 l8-16-1 by blast
    then obtain T'' where R Midpoint T T'' and Cong PO T PO T''

```

```

    using Per-def by blast
  hence  $T' = T''$ 
    using  $\langle R \text{ Midpoint } T \ T' \rangle$  symmetric-point-uniqueness by blast
  have  $T \neq T'$ 
    using  $\langle R \text{ Midpoint } T \ T' \rangle \langle T \neq R \rangle$  l7-3 by blast
  have  $T' \text{ OnCircle } PO \ P$ 
    using OnCircle-def  $\langle Cong \ PO \ T \ PO \ T'' \rangle \langle T \text{ OnCircle } PO \ P \rangle \langle T' = T'' \rangle$ 
      cong-inner-transitivity by blast
  have  $Col \ A \ B \ T'$ 
    by (metis  $\langle Col \ A \ B \ R \rangle \langle Col \ A \ B \ T \rangle \langle R \text{ Midpoint } T \ T'' \rangle \langle T \neq R \rangle$ 
       $\langle T' = T'' \rangle$  colx midpoint-col)
  hence  $T' \text{ OutCircleS } PO \ P$ 
    by (metis  $\langle T \neq T' \rangle$  assms(2) tangent-out)
  hence  $T \ R \text{ Perp } R \ PO$ 
    using Perp-cases  $\langle R \ T \text{ Perp } PO \ R \rangle$  by blast
  have  $PO \ P \text{ Le } PO \ T'$ 
    using OnCircle-def  $\langle T' \text{ OnCircle } PO \ P \rangle$  cong--le3412 by auto
  have  $\neg Cong \ PO \ P \ PO \ T'$ 
    using  $\langle T' \text{ OnCircle } PO \ P \rangle \langle T' \text{ OutCircleS } PO \ P \rangle$  onc--inc outcs--ninc by blast
  moreover have  $Cong \ PO \ P \ PO \ T'$ 
    using Cong-cases OnCircle-def  $\langle T' \text{ OnCircle } PO \ P \rangle$  by blast
  ultimately show ?thesis
    by blast
qed
qed
qed

```

lemma tangency-chara-R1:

```

  assumes  $P \neq PO$  and
     $\exists X. (X \text{ OnCircle } PO \ P \wedge X \text{ PerpAt } A \ B \ PO \ X)$ 
  shows  $Tangent \ A \ B \ PO \ P$ 
proof -
  obtain  $T$  where  $T \text{ OnCircle } PO \ P$  and  $T \text{ PerpAt } A \ B \ PO \ T$ 
    using assms(2) by blast
  hence  $Col \ A \ B \ T \wedge Col \ PO \ T \ T$ 
    using perp-in-col by blast
  hence  $Col \ A \ B \ T$ 
    by blast
  moreover
  {
    fix  $Y$ 
    assume  $Col \ A \ B \ Y$  and  $Y \text{ OnCircle } PO \ P$ 
    have  $T = Y$ 
    proof (cases  $T = Y$ )
      case True
      then show ?thesis
        by blast
    next
      case False
      hence  $T \ Y \text{ Perp } PO \ T$ 
        using  $\langle Col \ A \ B \ Y \rangle \langle T \text{ PerpAt } A \ B \ PO \ T \rangle$  calculation perp-col2 perp-in-perp by blast
      have  $T \text{ PerpAt } T \ Y \ PO \ T$ 
        using  $\langle T \ Y \text{ Perp } PO \ T \rangle$  col-trivial-3 l8-15-1 by auto
      hence False
        by (metis  $\langle T \text{ OnCircle } PO \ P \rangle \langle Y \text{ OnCircle } PO \ P \rangle$  col-onc2-perp--mid
          midpoint-out not-col-distincts out-diff1 perp-in-perp-bis perp-in-sym perp-not-eq-1)
      then show ?thesis
        by blast
    qed
  }
  ultimately show ?thesis
    using Tangent-def  $\langle T \text{ OnCircle } PO \ P \rangle$  by blast
qed

```

```

lemma tangency-chara-R2:
  assumes  $P \neq PO$  and
    Tangent  $A B PO P$ 
  shows  $\exists X. (X \text{ OnCircle } PO P \wedge X \text{ PerpAt } A B PO X)$ 
proof -
  obtain  $T$  where  $Col A B T$  and  $T \text{ OnCircle } PO P$  and
     $\forall Y. (Col A B Y \wedge Y \text{ OnCircle } PO P) \longrightarrow T = Y$ 
    using assms(2) Tangent-def by auto
  then have TangentAt  $A B PO P T$ 
    using TangentAt-def assms(2) by blast
  have  $A B \text{ Perp } PO T$ 
    by (metis  $\langle \text{TangentAt } A B PO P T \rangle$  assms(1) tangentat-perp)
  hence  $T \text{ PerpAt } A B PO T$ 
    using  $\langle Col A B T \rangle$  l8-15-1 by blast
  thus ?thesis
    using  $\langle T \text{ OnCircle } PO P \rangle$  by blast
qed

lemma tangency-chara:
  assumes  $P \neq PO$ 
  shows  $(\exists X. (X \text{ OnCircle } PO P \wedge X \text{ PerpAt } A B PO X)) \longleftrightarrow \text{Tangent } A B PO P$ 
  using assms tangency-chara-R1 tangency-chara-R2 by blast

lemma tangency-chara2-R1:
  assumes  $Q \text{ OnCircle } PO P$  and
    Col  $Q A B$  and
     $(\forall X. Col A B X \longrightarrow X = Q \vee X \text{ OutCircleS } PO P)$ 
  shows Tangent  $A B PO P$ 
proof -
  have Col  $A B Q$ 
    by (simp add: assms(2) col-permutation-1)
  moreover have  $\forall Y. (Col A B Y \wedge Y \text{ OnCircle } PO P) \longrightarrow Q = Y$ 
    using assms(3) onc-inc outcs--ninc-1 by blast
  ultimately show ?thesis
    using assms(1) Tangent-def by blast
qed

lemma tangency-chara2-R2:
  assumes  $Q \text{ OnCircle } PO P$  and
    Col  $Q A B$  and
    Tangent  $A B PO P$ 
  shows  $\forall X. Col A B X \longrightarrow X = Q \vee X \text{ OutCircleS } PO P$ 
  by (metis Col-cases TangentAt-def assms(1) assms(2) assms(3) tangent-out)

lemma tangency-chara2:
  assumes  $Q \text{ OnCircle } PO P$  and
    Col  $Q A B$ 
  shows  $(\forall X. Col A B X \longrightarrow X = Q \vee X \text{ OutCircleS } PO P) \longleftrightarrow \text{Tangent } A B PO P$ 
  using assms(1) assms(2) tangency-chara2-R1 tangency-chara2-R2 by blast

lemma tangency-chara3-R1:
  assumes  $A \neq B$  and
     $Q \text{ OnCircle } PO P$  and
    Col  $Q A B$  and
     $\forall X. (Col A B X \longrightarrow X \text{ OutCircle } PO P)$ 
  shows Tangent  $A B PO P$ 
proof -
  have  $(\forall X. Col A B X \longrightarrow X = Q \vee X \text{ OutCircleS } PO P) \longleftrightarrow \text{Tangent } A B PO P$ 
    by (simp add: assms(2) assms(3) tangency-chara2)
  moreover
  {
    fix  $X$ 
    assume Col  $A B X$ 
    {
      assume  $X \neq Q$ 
      have  $X \text{ OutCircle } PO P$ 

```

```

    by (simp add: ⟨Col A B X⟩ assms(4))
  hence PO P Le PO X
  by (simp add: OutCircle-def)
{
  assume Cong PO P PO X
  obtain M where M Midpoint X Q
    using MidR-uniq-aux by blast
  hence M InCircleS PO P
    using OnCircle-def ⟨Cong PO P PO X⟩ ⟨X ≠ Q⟩ assms(2) onc2-mid--incs
      onc-sym by presburger
  have Col A B M
    using NCol-cases ⟨Col A B X⟩ ⟨M Midpoint X Q⟩ ⟨X ≠ Q⟩ assms(3) colx midpoint-col by blast
  hence M OutCircle PO P
    using assms(4) by blast
  hence ¬ PO M Lt PO P
    using ⟨M InCircleS PO P⟩ outc--nincs by auto
  hence False
    using ⟨M InCircleS PO P⟩ ⟨M OutCircle PO P⟩ outc--nincs by blast
}
  hence PO P Lt PO X
    using Lt-def ⟨PO P Le PO X⟩ by blast
  hence X OutCircleS PO P
    by (simp add: OutCircleS-def)
}
  hence X = Q ∨ X OutCircleS PO P
    by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma tangency-chara3-R2:
  assumes Q OnCircle PO P and
    Col Q A B and
    Tangent A B PO P
  shows ∀ X. (Col A B X ⟶ X OutCircle PO P)
  using assms(2) assms(3) assms(1) onc--outc outcs--outc tangency-chara2 by blast

```

```

lemma tangency-chara3:
  assumes A ≠ B and
    Q OnCircle PO P and
    Col Q A B
  shows (∀ X. (Col A B X ⟶ X OutCircle PO P)) ⟷ Tangent A B PO P
  using assms(1) assms(2) assms(3) tangency-chara3-R1 tangency-chara3-R2 by blast

```

```

lemma intercc--neg:
  assumes InterCC A B C D
  shows A ≠ C
proof -
  obtain P Q where InterCCAt A B C D P Q
    using InterCC-def assms by blast
{
  assume A = C
  hence Cong A B A D
    by (metis InterCCAt-def ⟨InterCCAt A B C D P Q⟩ onc2--cong onc-sym)
  hence False
    using InterCCAt-def ⟨A = C⟩ ⟨InterCCAt A B C D P Q⟩ eqc-chara-2 by force
}
  thus ?thesis
    by blast
qed

```

```

lemma tangentcc--neq:
  assumes  $A \neq B$  and
    TangentCC  $A B C D$ 
  shows  $A \neq C$ 
  by (metis OnCircle-def TangentCC-def assms(1) assms(2) eqc-chara negc-chara-2
    onc-sym tree-points-onc-cop)

lemma interccat--neq:
  assumes InterCCAt  $A B C D P Q$ 
  shows  $A \neq C$ 
  using InterCC-def assms intercc--neq by blast

lemma interccat--ncol:
  assumes InterCCAt  $A B C D P Q$ 
  shows  $\neg \text{Col } A C P$ 
proof -
  {
    assume Col  $A C P$ 
    have  $P = Q$ 
    proof (rule l4-18 [where ?A=A and ?B=C], simp-all add:  $\langle \text{Col } A C P \rangle$ )
      show  $A \neq C$ 
      using assms interccat--neq by auto
      show Cong  $A P A Q$ 
      by (meson InterCCAt-def assms onc2--cong)
      show Cong  $C P C Q$ 
      by (meson InterCCAt-def assms onc2--cong)
    qed
    hence False
    using InterCCAt-def assms by force
  }
  thus ?thesis
  by blast
qed

lemma cop-onc2--oreq:
  assumes InterCCAt  $A B C D P Q$  and
    Coplanar  $A C P Q$  and
    Z OnCircle  $A B$  and
    Z OnCircle  $C D$  and
    Coplanar  $A C P Z$ 
  shows  $Z = P \vee Z = Q$ 
proof (cases  $Z = Q$ )
  case True
  then show ?thesis
  by simp
next
  case False
  hence  $Z \neq Q$ 
  by simp
  obtain M where M Midpoint  $Q P$ 
  using MidR-uniq-aux by blast
  hence Per  $A M Q$ 
  by (meson InterCCAt-def Per-def assms(1) onc2--cong)
  have Per  $C M Q$ 
  by (meson InterCCAt-def Per-def  $\langle M \text{ Midpoint } Q P \rangle$  assms(1) onc2--cong)
  obtain N where N Midpoint  $Z Q$ 
  using MidR-uniq-aux by blast
  hence Per  $A N Q$ 
  by (meson InterCCAt-def assms(1) assms(3) l7-2 mid-onc2--per)
  have Per  $C N Q$ 
  by (meson InterCCAt-def  $\langle N \text{ Midpoint } Z Q \rangle$  assms(1) assms(4) l7-2 mid-onc2--per)
  have Coplanar  $Q A C M$ 
  by (meson Midpoint-def  $\langle M \text{ Midpoint } Q P \rangle$  assms(2) bet-cop2--cop
    coplanar-perm-18 ncop-distincts)
  have Col  $A C M$ 

```



```

proof -
  have Coplanar Q A C M
    using ‹Coplanar Q A C M› by auto
  moreover have Q ≠ M
    by (metis InterCCAt-def ‹M Midpoint Q P› assms(1) midpoint-distinct-1)
  ultimately show ?thesis
    by (simp add: ‹Per A M Q› ‹Per C M Q› cop-per2--col)
qed
have A ≠ C
  using assms(1) interccat--neq by auto
have Col A C N
proof -
  have Coplanar Q A C N
    by (meson Midpoint-def ‹N Midpoint Z Q› assms(1) assms(2) assms(5)
        bet-cop2--cop coplanar-pseudo-trans interccat--ncol ncop-distincts)
  moreover have Q ≠ N
    using False ‹N Midpoint Z Q› is-midpoint-id l7-2 by blast
  ultimately show ?thesis
    using ‹Per A N Q› ‹Per C N Q› cop-per2--col by blast
qed
have A C Perp Q P
proof (cases A = M)
  case True
  then show ?thesis
    by (metis Cong-cases False Mid-perm OnCircle-def ‹A ≠ C› ‹M Midpoint Q P›
        ‹N Midpoint Z Q› ‹Per A N Q› ‹Per C M Q› between-equality between-symmetry
        cong-identity-inv mid-onc2--perp midpoint-bet onc212 outer-transitivity-between2
        per-double-cong perp-left-comm symmetric-point-construction)
  next
  case False
  hence M PerpAt A M M Q
    by (metis InterCCAt-def ‹M Midpoint Q P› ‹Per A M Q› assms(1) midpoint-distinct-1
        per-perp-in)
  hence M A Perp Q M
    using perp-comm perp-in-perp by blast
  hence A M Perp M Q
    using Perp-cases by auto
  have Col A M C
    using Col-cases ‹Col A C M› by auto
  hence A C Perp M Q
    using perp-col ‹A M Perp M Q› ‹A ≠ C› by blast
  then show ?thesis
    using ‹M Midpoint Q P› col-trivial-2 l7-3 midpoint-col perp-col2-bis by blast
qed
have Col Q N Z
  using ‹N Midpoint Z Q› col-permutation-2 midpoint-col by blast
have N ≠ Q
  using False Midpoint-def ‹N Midpoint Z Q› cong-identity-inv by blast
have Q Z Perp C A
  using perp-col
  by (metis (full-types) False Perp-cases ‹A ≠ C› ‹Col A C N› ‹Col Q N Z›
      ‹N ≠ Q› ‹Per A N Q› ‹Per C N Q› col-per-perp col-permutation-2 not-col-permutation-4)
hence A C Perp Q Z
  using Perp-cases by blast
have Q P Par Q Z
proof (rule l12-9 [where ?C1.0=A and ?C2.0=C], simp-all add: assms(2) assms(5))
  show Coplanar A C Q Q
    using ncop-distincts by blast
  show Coplanar A C Q Z
    using Coplanar-def ‹Col A C N› ‹Col Q N Z› col-permutation-5 by blast
  show Q P Perp A C
    using Perp-perm ‹A C Perp Q P› by blast
  show Q Z Perp A C
    using Perp-perm ‹Q Z Perp C A› by blast
qed
moreover {

```

```

  assume  $Q \ P \ \text{ParStrict} \ Q \ Z$ 
  hence  $Z = P$ 
    using not-par-strict-id by blast
}
moreover {
  assume  $Q \neq P \wedge Q \neq Z \wedge \text{Col } Q \ Q \ Z \wedge \text{Col } P \ Q \ Z$ 
  hence  $Z = P \vee Z = Q$ 
    using line-circle-two-points by (metis InterCCAt-def assms(1) assms(3))
  hence  $Z = P$ 
    by (simp add: False)
}
ultimately show ?thesis
  using Par-def by blast
qed

```

```

lemma tangent-construction:
  assumes segment-circle and
     $X \ \text{OutCircle} \ PO \ P$ 
  shows  $\exists \ Y. \ \text{Tangent } X \ Y \ PO \ P$ 
proof (cases  $PO = P$ )
  case True
  then show ?thesis
    by (metis bet-col inc-eq ninc--outcs onc212 onc--outc outcs--outc
      point-construction-different tangency-chara3)
next
  case False
  hence  $PO \neq X$ 
    using assms(2) outc-eq by blast
  have  $X \ \text{OnCircle} \ PO \ P \vee X \ \text{InCircleS} \ PO \ P \vee X \ \text{OutCircleS} \ PO \ P$ 
    by (simp add: circle-cases)
  moreover {
    assume  $X \ \text{OnCircle} \ PO \ P$ 
    moreover
    obtain  $Y$  where  $X \ Y \ \text{Perp} \ PO \ X$ 
      using  $\langle PO \neq X \rangle$  perp-exists by fastforce
    have  $X \ \text{PerpAt} \ X \ Y \ PO \ X$ 
      using  $\langle X \ Y \ \text{Perp} \ PO \ X \rangle$  perp-perp-in by auto
    ultimately have ?thesis
      by (metis False tangency-chara-R1)
  }
  moreover have  $X \ \text{InCircleS} \ PO \ P \longrightarrow ?thesis$ 
    using assms(2) outc--nincs by auto
  moreover {
    assume  $X \ \text{OutCircleS} \ PO \ P$ 
    obtain  $U$  where  $U \ \text{OnCircle} \ PO \ P$  and  $PO \ \text{Out} \ X \ U$ 
      using False  $\langle PO \neq X \rangle$  onc-exists by metis
    have  $\text{Bet } PO \ U \ X$ 
    proof -
      {
        assume  $\text{Bet } PO \ X \ U$ 
        hence  $PO \ X \ \text{Le} \ PO \ U$ 
          by (simp add: bet--le1213)
        moreover have  $PO \ U \ \text{Lt} \ PO \ X$ 
          by (meson OnCircle-def OutCircleS-def  $\langle U \ \text{OnCircle} \ PO \ P \rangle$   $\langle X \ \text{OutCircleS} \ PO \ P \rangle$ 
            cong--le3412 le1234-lt--lt onc-sym)
        ultimately have ?thesis
          using le--nlt by auto
      }
    thus ?thesis
      using Out-def  $\langle PO \ \text{Out} \ X \ U \rangle$  by blast
  }
qed
  obtain  $R$  where  $U \ R \ \text{Perp} \ PO \ U$ 
    by (metis  $\langle PO \ \text{Out} \ X \ U \rangle$  out-diff2 perp-exists)
  obtain  $W$  where  $PO \ \text{Midpoint} \ X \ W$ 

```

```

    using symmetric-point-construction by blast
  obtain T where Bet U R T  $\vee$  Bet U T R and Cong U T W X
    by (metis segment-construction segment-construction-2)
  have U InCircleS PO X
    by (metis OutCircle-def  $\langle$ Bet PO U X $\rangle$   $\langle$ U OnCircle PO P $\rangle$   $\langle$ X OutCircleS PO P $\rangle$ 
        bet-le-eq between-symmetry incs--noutc-2 le-comm onc--inc outcs--ninc)
  hence U InCircle PO X
    using incs--inc by auto
  have T OutCircleS PO X
    using  $\langle$ Cong U T W X $\rangle$   $\langle$ PO Midpoint X W $\rangle$   $\langle$ U InCircleS PO X $\rangle$  diam-cong-incs--outcs
        mid-onc--diam not-cong-3421 onc212 by blast
  hence T OutCircle PO X
    using outcs--outc by auto
  hence  $\exists$  Z. Bet U Z T  $\wedge$  Z OnCircle PO X
    using assms(1) segment-circle-def  $\langle$ U InCircle PO X $\rangle$  by blast
  then obtain Y where Bet U Y T and Y OnCircle PO X
    by blast
  hence  $\exists$  Q. Q OnCircle PO P  $\wedge$  PO Out Y Q
    using False  $\langle$ PO  $\neq$  X $\rangle$  onc-exists onc-not-center by presburger
  then obtain V where V OnCircle PO P and PO Out Y V
    by blast
  have Bet PO V Y
    by (meson OnCircle-def Out-cases  $\langle$ PO  $-$  U  $-$  X $\rangle$   $\langle$ PO Out Y V $\rangle$   $\langle$ U OnCircle PO P $\rangle$ 
         $\langle$ V OnCircle PO P $\rangle$   $\langle$ Y OnCircle PO X $\rangle$  cong-preserves-bet onc2--cong onc-sym)
  have Cong PO X PO Y
    using  $\langle$ Y OnCircle PO X $\rangle$  onc212 onc2--cong by blast
  have X PO V CongA Y PO U
    by (simp add: Out-cases  $\langle$ PO Out X U $\rangle$   $\langle$ PO Out Y V $\rangle$  conga-right-comm out2--conga)
  hence Cong V X U Y
    by (meson OnCircle-def  $\langle$ Cong PO X PO Y $\rangle$   $\langle$ U OnCircle PO P $\rangle$   $\langle$ V OnCircle PO P $\rangle$ 
        cong-4321 cong-commutativity cong-transitivity l11-49)
  hence PO U Y CongA PO V X
    by (metis Out-def  $\langle$ Cong PO X PO Y $\rangle$   $\langle$ PO Out X U $\rangle$   $\langle$ PO Out Y V $\rangle$   $\langle$ U InCircleS PO X $\rangle$ 
         $\langle$ U OnCircle PO P $\rangle$   $\langle$ V OnCircle PO P $\rangle$   $\langle$ Y OnCircle PO X $\rangle$  l11-51 not-cong-3412
        onc2--cong onc--outc outc--nincs-1)
  have Per PO V X
  proof -
    have U Y Perp PO U
      by (metis  $\langle$ PO U Y CongA PO V X $\rangle$   $\langle$ U R Perp PO U $\rangle$   $\langle$ (U  $-$  R  $-$  T)  $\vee$  (U  $-$  T  $-$  R) $\rangle$ 
           $\langle$ U  $-$  Y  $-$  T $\rangle$  bet-out conga-diff2 l6-6 out-col out-diff2 perp-col perp-not-eq-1)
    hence U PerpAt U Y PO U
      using  $\langle$ U Y Perp PO U $\rangle$  perp-perp-in by force
    hence U PerpAt PO U U Y
      using Perp-in-cases by blast
    hence Per PO U Y
      using perp-in-per by auto
    thus ?thesis
      using  $\langle$ PO U Y CongA PO V X $\rangle$  l11-17 by blast
  qed
  have V PerpAt X V PO V
    by (metis  $\langle$ PO U Y CongA PO V X $\rangle$   $\langle$ Per PO V X $\rangle$  conga-distinct l8-2 per-perp-in
        perp-in-right-comm)
  hence ?thesis
    by (metis  $\langle$ V OnCircle PO P $\rangle$  False tangency-chara-R1)
}
ultimately show ?thesis
  by blast
qed

```

```

lemma dedekind-equiv-R1:
  assumes DedekindsAxiom
  shows DedekindVariant
  using DedekindsAxiom-def DedekindVariant-def assms by metis

```

```

lemma dedekind-equiv-R2:
  assumes DedekindVariant

```

```

shows DedekindsAxiom
proof -
{
  fix Alpha Beta :: 'p  $\Rightarrow$  bool
  assume  $\exists A. (\forall X Y. (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ A\ X\ Y)$ 
  then obtain A where  $\forall X Y. (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ A\ X\ Y$ 
    by blast
  have  $(\exists B. \forall X Y. (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ X\ B\ Y)$ 
  proof (cases Beta A)
    case True
    thus ?thesis
      using  $\langle \forall X Y. Alpha\ X \wedge Beta\ Y \longrightarrow Bet\ A\ X\ Y \rangle$  between-identity by blast
  next
    case False
    hence  $\neg Beta\ A$ 
      by simp
    show ?thesis
  proof (cases  $\exists C. Beta\ C$ )
    case True
    then obtain C where Beta C
      by auto
    show ?thesis
  proof (cases  $\exists P. Alpha\ P \wedge Beta\ P$ )
    case True
    thus ?thesis
      by (meson  $\langle \exists A. \forall X Y. Alpha\ X \wedge Beta\ Y \longrightarrow Bet\ A\ X\ Y \rangle$  between-exchange3)
  next
    case False
    hence  $\neg (\exists P. Alpha\ P \wedge Beta\ P)$ 
      by fast
    show ?thesis
  proof (cases  $\exists X. Alpha\ X \wedge X \neq A$ )
    case True
    hence  $\exists X. Alpha\ X \wedge X \neq A$ 
      by simp
    have  $\exists B. \forall X Y. Alpha'Fun\ A\ Alpha\ X \wedge Beta'Fun\ A\ C\ Alpha\ Y \longrightarrow Bet\ X\ B\ Y$ 
  proof -
    have  $A = A \vee (\exists X :: 'p. Alpha\ X \wedge Bet\ A\ A\ X)$ 
      by blast
    hence Alpha'Fun A Alpha A
      using Alpha'Fun-def by force
    moreover
    have  $A\ Out\ C\ C \wedge \neg (\exists X :: 'p. Alpha\ X \wedge Bet\ A\ C\ X)$ 
      by (metis False  $\langle Beta\ C \rangle \langle \forall X Y. Alpha\ X \wedge Beta\ Y \longrightarrow Bet\ A\ X\ Y \rangle \langle \neg Beta\ A \rangle$ 
        between-equality-2 out-trivial)
    hence Beta'Fun A C Alpha C
      using Beta'Fun-def by simp
    moreover
    {
      fix P
      assume A Out P C
      have  $(P = A \vee (\exists X :: 'p. Alpha\ X \wedge Bet\ A\ P\ X)) \vee$ 
         $(A\ Out\ P\ C \wedge \neg (\exists X :: 'p. Alpha\ X \wedge Bet\ A\ P\ X))$ 
      using  $\langle A\ Out\ P\ C \rangle$  by fastforce
      hence Alpha'Fun A Alpha P  $\vee$  Beta'Fun A C Alpha P
        using Alpha'Fun-def Beta'Fun-def by simp
    }
    hence  $\forall P. A\ Out\ P\ C \longrightarrow (Alpha'Fun\ A\ Alpha\ P \vee Beta'Fun\ A\ C\ Alpha\ P)$  by blast
  moreover {
    fix X' Y'
    assume Alpha'Fun A Alpha X' and Beta'Fun A C Alpha Y'
    hence  $X' = A \vee (\exists X :: 'p. Alpha\ X \wedge Bet\ A\ X'\ X)$ 
      using Alpha'Fun-def by simp
    have  $A\ Out\ Y'\ C \wedge \neg (\exists X :: 'p. Alpha\ X \wedge Bet\ A\ Y'\ X)$ 
      using Beta'Fun-def  $\langle Beta'Fun\ A\ C\ Alpha\ Y' \rangle$  by presburger
    have  $Bet\ A\ X'\ Y' \wedge X' \neq Y'$ 

```

```

proof (cases  $X' = A$ )
  case True
  thus ?thesis
    using  $\langle A \text{ Out } Y' C \wedge (\nexists X. \text{Alpha } X \wedge \text{Bet } A Y' X) \rangle$ 
       $\langle \exists X. \text{Alpha } X \wedge X \neq A \rangle$  between-trivial2 by blast
  next
  case False
  hence  $X' \neq A$ 
  by simp
  then obtain  $X$  where  $\text{Alpha } X$  and  $\text{Bet } A X' X$ 
  using  $\langle X' = A \vee (\exists X. \text{Alpha } X \wedge \text{Bet } A X' X) \rangle$  by blast
  have  $A \text{ Out } X' Y'$ 
  by (metis False  $\langle A \text{ Out } Y' C \wedge (\nexists X. \text{Alpha } X \wedge \text{Bet } A Y' X) \rangle$ 
     $\langle \text{Beta } C \rangle \langle X' = A \vee (\exists X. \text{Alpha } X \wedge \text{Bet } A X' X) \rangle$ 
     $\langle \forall X Y. \text{Alpha } X \wedge \text{Beta } Y \longrightarrow \text{Bet } A X Y \rangle$  bet2-out-out bet-out-1
    l6-6 not-bet-distincts out-diff1)
  hence  $Y' \text{ Out } A X'$ 
  by (metis Bet-cases Out-def  $\langle A \text{ Out } Y' C \wedge (\nexists X. \text{Alpha } X \wedge \text{Bet } A Y' X) \rangle$ 
     $\langle \text{Alpha } X \rangle \langle \text{Bet } A X' X \rangle$  between-exchange4)
  thus ?thesis
  by (simp add:  $\langle A \text{ Out } X' Y' \rangle$  out2--bet out-diff2)
qed
}
ultimately
have  $\exists B. \forall X Y. (\text{Alpha}'\text{Fun } A \text{ Alpha } X \wedge \text{Beta}'\text{Fun } A C \text{ Alpha } Y) \longrightarrow \text{Bet } X B Y$ 
  using DedekindVariant-def by (metis assms)
thus ?thesis by blast
qed
moreover
let ?Alpha' = Alpha'Fun A Alpha
let ?Beta' = Beta'Fun A C Alpha
{
  assume  $\exists B. \forall X Y. ?\text{Alpha}' X \wedge ?\text{Beta}' Y \longrightarrow \text{Bet } X B Y$ 
  then obtain  $B$  where  $\forall X Y. ?\text{Alpha}' X \wedge ?\text{Beta}' Y \longrightarrow \text{Bet } X B Y$ 
  by blast
  {
    fix  $X Y$ 
    assume  $\text{Alpha } X$  and  $\text{Beta } Y$ 
    have ?Alpha'  $X$ 
    using  $\langle \text{Alpha } X \rangle$  between-trivial Alpha'Fun-def by auto
    moreover
    have  $A \text{ Out } Y C \wedge \neg (\exists X::'p. \text{Alpha } X \wedge \text{Bet } A Y X)$ 
    by (metis False Out-def True  $\langle \text{Beta } C \rangle \langle \text{Beta } Y \rangle$ 
       $\langle \forall X Y. \text{Alpha } X \wedge \text{Beta } Y \longrightarrow \text{Bet } A X Y \rangle \langle \neg \text{Beta } A \rangle$  between-equality-2 l6-7)
    hence  $\text{Beta}'\text{Fun } A C \text{ Alpha } Y$ 
    using Beta'Fun-def by simp
    ultimately have  $\text{Bet } X B Y$ 
    by (simp add:  $\langle \forall X Y. \text{Alpha}'\text{Fun } A \text{ Alpha } X \wedge \text{Beta}'\text{Fun } A C \text{ Alpha } Y \longrightarrow \text{Bet } X B Y \rangle$ )
  }
  hence ?thesis
  by blast
}
ultimately show ?thesis
  by fast
next
case False
thus ?thesis
  using not-bet-distincts by blast
qed
qed
next
case False
thus ?thesis
  by blast
qed
qed

```

```

}
thus ?thesis
  using DedekindsAxiom-def by blast
qed

lemma dedekind-equiv:
  shows DedekindsAxiom  $\longleftrightarrow$  DedekindVariant
  using dedekind-equiv-R1 dedekind-equiv-R2 by blast

lemma dedekind--fod:
  assumes DedekindsAxiom
  shows FirstOrderDedekind
  using DedekindsAxiom-def assms FirstOrderDedekind-def by blast

lemma circle-circle-aux:
  assumes  $\forall A B C D P Q. (P \text{ OnCircle } C D \wedge Q \text{ OnCircle } C D \wedge$ 
     $P \text{ InCircleS } A B \wedge Q \text{ OutCircleS } A B \wedge$ 
     $(A C \text{ OS } P Q \vee (\text{Col } P A C \wedge \neg \text{Col } Q A C) \vee (\neg \text{Col } P A C \wedge \text{Col } Q A C)) \longrightarrow$ 
     $(\exists Z. Z \text{ OnCircle } A B \wedge Z \text{ OnCircle } C D))$ 
  shows circle-circle
proof -
{
  fix A B C D P Q
  assume  $P \text{ OnCircle } C D$  and  $Q \text{ OnCircle } C D$  and  $P \text{ InCircleS } A B$  and
     $Q \text{ OutCircleS } A B$  and  $\text{Coplanar } A C P Q$  and  $\neg \text{Col } P A C \vee \neg \text{Col } Q A C$ 
  have  $\exists Z. Z \text{ OnCircle } A B \wedge Z \text{ OnCircle } C D$ 
  proof (cases  $\text{Col } P A C$ )
    case True
    hence  $\neg \text{Col } Q A C$ 
    using  $\neg \text{Col } P A C \vee \neg \text{Col } Q A C$  by auto
    hence  $A C \text{ OS } P Q \vee (\text{Col } P A C \wedge \neg \text{Col } Q A C) \vee (\neg \text{Col } P A C \wedge \text{Col } Q A C)$ 
    using True by auto
    thus ?thesis using assms
       $\langle P \text{ OnCircle } C D \rangle \langle Q \text{ OnCircle } C D \rangle \langle P \text{ InCircleS } A B \rangle$ 
       $\langle Q \text{ OutCircleS } A B \rangle$  by blast
  next
    case False
    hence  $\neg \text{Col } P A C$ 
    by simp
    show ?thesis
  proof (cases  $\text{Col } Q A C$ )
    case True
    then show ?thesis
      using assms False  $\langle P \text{ InCircleS } A B \rangle \langle P \text{ OnCircle } C D \rangle \langle Q \text{ OnCircle } C D \rangle$ 
       $\langle Q \text{ OutCircleS } A B \rangle$  by blast
  next
    case False
    hence  $\neg \text{Col } Q A C$ 
    by simp
    hence  $A C \text{ TS } P Q \vee A C \text{ OS } P Q$ 
    using cop--one-or-two-sides  $\langle \text{Coplanar } A C P Q \rangle \langle \neg \text{Col } P A C \rangle$  by blast
  moreover {
    assume  $A C \text{ TS } P Q$ 
    obtain  $P'$  where  $P P' \text{ Reflect } A C$ 
    using l10-6-existence by blast
    have  $P' \text{ OnCircle } C D$ 
    by (meson OnCircle-def  $\langle P \text{ OnCircle } C D \rangle \langle P P' \text{ Reflect } A C \rangle$ 
      image-triv is-image-rev l10-10 onc2--cong)
    moreover have  $P' \text{ InCircleS } A B$ 
    by (meson InCircleS-def OutCircleS-def  $\langle P \text{ InCircleS } A B \rangle$ 
       $\langle P P' \text{ Reflect } A C \rangle$  circle-cases cong--nlt image-triv l10-10 lt-transitivity
      not-cong-4321 onc2--cong)
    moreover have  $A C \text{ TS } P' P$ 
    by (metis  $\langle P P' \text{ Reflect } A C \rangle \langle \neg \text{Col } P A C \rangle$  bet-col between-trivial l10-14 l10-8 l9-2)
  }
}

```

```

    hence  $A C OS P' Q$ 
    using  $OS-def \langle A C TS P Q \rangle$  l9-2 by blast
    hence  $A C OS P' Q \vee Col P' A C \wedge \neg Col Q A C \vee \neg Col P' A C \wedge Col Q A C$ 
    by blast
    have ?thesis using assms  $\langle A C OS P' Q \rangle \langle Q OnCircle C D \rangle \langle Q OutCircleS A B \rangle$ 
    calculation(1) calculation(2) by blast
  }
  moreover have  $A C OS P Q \longrightarrow ?thesis$ 
  using assms  $\langle P InCircleS A B \rangle \langle P OnCircle C D \rangle \langle Q OnCircle C D \rangle$ 
   $\langle Q OutCircleS A B \rangle$  by blast
  ultimately show ?thesis
  by blast
qed
qed
}
moreover
{
  assume H1:  $\forall A B C D P Q. P OnCircle C D \wedge Q OnCircle C D \wedge$ 
     $P InCircleS A B \wedge Q OutCircleS A B \wedge$ 
     $Coplanar A C P Q \wedge (\neg Col P A C \vee \neg Col Q A C)$ 
     $\longrightarrow (\exists Z. Z OnCircle A B \wedge Z OnCircle C D)$ 
  {
    fix  $A B C D P Q$ 
    assume  $P OnCircle C D$  and  $Q OnCircle C D$  and  $P InCircle A B$  and  $Q OutCircle A B$ 
    have  $\exists Q'. Q' OnCircle C D \wedge Q' OutCircle A B \wedge Col Q' A C$ 
    proof -
      obtain  $Q'$  where  $Bet A C Q'$  and  $Cong C Q' C D$ 
      using segment-construction by blast
      have  $Q' OnCircle C D$ 
      by (simp add: OnCircle-def  $\langle Cong C Q' C D \rangle$ )
      moreover have  $A Q Le A Q'$ 
      using  $\langle Bet A C Q' \rangle \langle Q OnCircle C D \rangle$  calculation onc2--cong triangle-inequality by auto
      hence  $Q' OutCircle A B$ 
      using OutCircle-def  $\langle Q OutCircle A B \rangle$  le-transitivity by blast
      moreover have  $Col Q' A C$ 
      by (simp add: Col-def  $\langle Bet A C Q' \rangle$ )
      ultimately show ?thesis by blast
    qed
    then obtain  $Q''$  where  $Q'' OnCircle C D$  and  $Q'' OutCircle A B$  and  $Col Q'' A C$ 
    by blast
    have  $\exists Z. Z OnCircle A B \wedge Z OnCircle C D$ 
    proof (cases  $Cong A P A B$ )
      case True
      then show ?thesis
      using OnCircle-def  $\langle P OnCircle C D \rangle$  by blast
    next
      case False
      hence  $\neg Cong A P A B$ 
      by simp
      show ?thesis
      proof (cases  $Cong A B A Q''$ )
        case True
        then show ?thesis
        using OnCircle-def  $\langle Q'' OnCircle C D \rangle$  onc-sym by blast
      next
        case False
        hence  $\neg Cong A B A Q''$ 
        by simp
        have  $A P Le A B$ 
        using InCircle-def  $\langle P InCircle A B \rangle$  by auto
        hence  $A P Lt A B$ 
        by (simp add: Lt-def  $\langle \neg Cong A P A B \rangle$ )
        hence  $P InCircleS A B$ 
        using InCircleS-def by blast
        have  $Q'' OutCircleS A B$ 
        using False Lt-def OutCircleS-def OutCircle-def  $\langle Q'' OutCircle A B \rangle$  by blast
      qed
    qed
  }
}

```

```

{
  assume  $A = C$ 
  have  $A \ B \ Lt \ A \ P$ 
    using  $OutCircleS-def \langle A = C \rangle \langle P \ OnCircle \ C \ D \rangle \langle Q'' \ OnCircle \ C \ D \rangle$ 
     $\langle Q'' \ OutCircleS \ A \ B \rangle \text{cong2-lt-lt cong-reflexivity onc2--cong}$  by blast
  hence  $False$ 
    using  $\langle A \ P \ Lt \ A \ B \rangle \text{not-and-lt}$  by blast
}
hence  $A \neq C$ 
  by blast
{
  assume  $C = D$ 
  hence  $A \ C \ Lt \ A \ B \wedge A \ B \ Lt \ A \ C$ 
    using  $OnCircle-def \ OutCircleS-def \langle A \ P \ Lt \ A \ B \rangle \langle P \ OnCircle \ C \ D \rangle$ 
     $\langle Q'' \ OnCircle \ C \ D \rangle \langle Q'' \ OutCircleS \ A \ B \rangle \text{cong-identity}$  by blast
  hence  $False$ 
    using  $lt--le \ lt--nle$  by blast
}
hence  $C \neq D$ 
  by blast
show ?thesis
proof (cases  $Col \ P \ A \ C$ )
  case True
    obtain  $R$  where  $Per \ A \ C \ R$  and  $Cong \ C \ R \ C \ D$ 
      using  $exists-cong-per$  by blast
    hence  $C \neq R$ 
      using  $\langle C \neq D \rangle \text{cong-diff-3}$  by blast
    hence  $A \neq R$ 
      using  $\langle Per \ A \ C \ R \rangle \text{l8-8}$  by blast
    have  $\neg Col \ A \ C \ R$ 
      using  $\langle A \neq C \rangle \langle C \neq R \rangle \langle Per \ A \ C \ R \rangle \text{col-permutation-1 l8-8 per-col}$  by blast
    have  $R \ OnCircle \ A \ B \longrightarrow ?thesis$ 
      using  $OnCircle-def \langle Cong \ C \ R \ C \ D \rangle$  by auto
    have  $Coplanar \ A \ C \ R \ Q''$ 
      using  $Col-cases \langle Col \ Q'' \ A \ C \rangle \text{ncop--ncols}$  by blast
    have  $Coplanar \ A \ C \ P \ R$ 
      using  $True \ ncop--ncol \ not-col-permutation-2$  by blast
    have  $\neg Col \ R \ A \ C \vee \neg Col \ Q \ A \ C$ 
      using  $Col-cases \langle \neg Col \ A \ C \ R \rangle$  by blast
    have  $\neg Col \ P \ A \ C \vee \neg Col \ R \ A \ C$ 
      using  $Col-cases \langle \neg Col \ A \ C \ R \rangle$  by blast
    have  $R \ OnCircle \ C \ D$ 
      by (simp add:  $OnCircle-def \langle Cong \ C \ R \ C \ D \rangle$ )
    hence  $R \ InCircleS \ A \ B \longrightarrow ?thesis$ 
      using  $H1 \ Col-cases \langle Coplanar \ A \ C \ R \ Q'' \rangle \langle Q'' \ OnCircle \ C \ D \rangle$ 
       $\langle Q'' \ OutCircleS \ A \ B \rangle \langle \neg Col \ A \ C \ R \rangle$  by blast
    moreover have  $R \ OutCircleS \ A \ B \longrightarrow ?thesis$ 
      using  $H1 \langle Coplanar \ A \ C \ P \ R \rangle \langle P \ InCircleS \ A \ B \rangle \langle P \ OnCircle \ C \ D \rangle$ 
       $\langle R \ OnCircle \ C \ D \rangle \langle \neg Col \ P \ A \ C \vee \neg Col \ R \ A \ C \rangle$  by blast
    ultimately show ?thesis
      using  $\langle R \ OnCircle \ C \ D \rangle \text{circle-cases}$  by blast
  next
    case False
      moreover have  $Coplanar \ A \ C \ P \ Q''$ 
        using  $NCol-cases \langle Col \ Q'' \ A \ C \rangle \text{ncop--ncols}$  by blast
      ultimately show ?thesis
        using  $H1 \langle P \ OnCircle \ C \ D \rangle \langle Q'' \ OnCircle \ C \ D \rangle \langle P \ InCircleS \ A \ B \rangle$ 
         $\langle Q'' \ OutCircleS \ A \ B \rangle$  by blast
      qed
    qed
  qed
}
hence  $circle-circle$ 
  using  $circle-circle-def$ 
  by blast
}

```


ultimately show ?thesis by blast

qed

lemma fod--circle-circle:

assumes FirstOrderDedekind

shows CircleCircle

proof -

```

{
  fix A B C D P Q
  assume P OnCircle C D and Q OnCircle C D and P InCircleS A B and Q OutCircleS A B
    and A C OS P Q  $\vee$  (Col P A C  $\wedge$   $\neg$  Col Q A C)  $\vee$  ( $\neg$  Col P A C  $\wedge$  Col Q A C)
  have A  $\neq$  C
    using  $\langle A C OS P Q \vee Col P A C \wedge \neg Col Q A C \vee \neg Col P A C \wedge Col Q A C \rangle$ 
      col-trivial-2 os-distincts by blast
  {
    assume P = Q
    have A P Lt A B
      using InCircleS-def  $\langle P InCircleS A B \rangle$  by force
    have A B Lt A P
      using OutCircleS-def  $\langle P = Q \rangle \langle Q OutCircleS A B \rangle$  by fastforce
    hence False
      using not-and-lt
      using  $\langle A P Lt A B \rangle$  by blast
  }
  hence P  $\neq$  Q
    by blast
  hence C  $\neq$  D
    using  $\langle P OnCircle C D \rangle \langle Q OnCircle C D \rangle$  inc-eq onc--inc by blast
  {
    fix X Y
    assume Bet P X Q and Bet P Y Q and
      X  $\neq$  P and X  $\neq$  Q and Y  $\neq$  P and Y  $\neq$  Q
    have A C OS P Q  $\longrightarrow$  A C OS X Y
      by (meson  $\langle Bet P X Q \rangle \langle Bet P Y Q \rangle$  l9-17 one-side-symmetry one-side-transitivity)
    moreover {
      assume Col P A C and  $\neg$  Col Q A C
      have A C OS Q X
        by (metis  $\langle Bet P X Q \rangle \langle Col P A C \rangle \langle X \neq P \rangle \langle \neg Col Q A C \rangle$  bet-col
          bet-out colx not-col-permutation-1 one-side-symmetry out-one-side-1)
      moreover have A C OS Q Y
        by (metis  $\langle Bet P Y Q \rangle \langle Col P A C \rangle \langle Y \neq P \rangle \langle \neg Col Q A C \rangle$  bet-col
          bet-out colx not-col-permutation-2 one-side-symmetry out-one-side-1)
      ultimately have A C OS X Y
        using one-side-symmetry one-side-transitivity by blast
    }
    moreover {
      assume  $\neg$  Col P A C and Col Q A C
      have A C OS P X
        by (metis  $\langle Bet P X Q \rangle \langle Col Q A C \rangle \langle X \neq Q \rangle \langle \neg Col P A C \rangle$  bet-col
          bet-col1 bet-out-1 l6-21 l9-19-R2 not-col-permutation-1 one-side-symmetry)
      moreover have A C OS P Y
        by (metis  $\langle Bet P Y Q \rangle \langle Col Q A C \rangle \langle Y \neq Q \rangle \langle \neg Col P A C \rangle$  bet-col
          bet-col1 bet-out-1 l6-21 not-col-permutation-1
          one-side-symmetry out-one-side-1)
      ultimately have A C OS X Y
        using one-side-symmetry one-side-transitivity by blast
    }
    ultimately have A C OS X Y
      using  $\langle A C OS P Q \vee Col P A C \wedge \neg Col Q A C \vee \neg Col P A C \wedge Col Q A C \rangle$  by fastforce
  }
}
{
  fix X Y
  assume Bet P Y Q and Bet P X Y
  hence  $\neg$  A C TS X Y
    by (metis TS-def  $\langle A C OS P Q \vee Col P A C \wedge \neg Col Q A C \vee \neg Col P A C \wedge Col Q A C \rangle$ 
      bet-ts--ts between-symmetry l9-2 one-side-chara)
}

```

```

}
have A C P LtA A C Q
proof -
  have C ≠ Q
    using ⟨C ≠ D⟩ ⟨Q OnCircle C D⟩ onc-not-center by blast
  moreover have Cong C Q C P
    using ⟨P OnCircle C D⟩ ⟨Q OnCircle C D⟩ onc2--cong by auto
  moreover have P A Lt Q A
    by (meson InCircleS-def OutCircleS-def ⟨P InCircleS A B⟩
        ⟨Q OutCircleS A B⟩ lt-comm lt-transitivity)
  ultimately show ?thesis
    using ⟨A ≠ C⟩ cong-reflexivity t18-19 by auto
qed
hence ¬ A C P CongA A C Q
  using lta-not-conga by auto
{
  assume Col P Q C
  {
    assume A C OS P Q
    hence A C P CongA A C Q
      by (metis NCol-cases TS-def ⟨A C P LtA A C Q⟩ ⟨Col P Q C⟩
          invert-one-side lta-os--ts one-side-symmetry)
    hence False
      by (simp add: ⟨¬ A C P CongA A C Q⟩)
  }
  moreover {
    assume Col P A C and ¬ Col Q A C
    hence False
      by (metis ⟨C ≠ D⟩ ⟨Col P Q C⟩ ⟨P OnCircle C D⟩ col-permutation-1
          col-transitivity-2 onc-not-center)
  }
  moreover {
    assume ¬ Col P A C and Col Q A C
    hence False
      by (metis ⟨C ≠ D⟩ ⟨Col P Q C⟩ ⟨Q OnCircle C D⟩
          col-permutation-1 col-transitivity-2 onc-not-center)
  }
  ultimately have False
    using ⟨A C OS P Q ∨ Col P A C ∧ ¬ Col Q A C ∨ ¬ Col P A C ∧ Col Q A C⟩ by fastforce
}
hence ¬ Col P Q C
  by auto
{
  fix X Y X0 Y0
  assume Bet P Y Q and X ≠ Y and
    X0 OnCircle C D and C Out X X0 and
    Y0 OnCircle C D and C Out Y Y0 and Bet P X Y
  have Cong C Y0 C X0
    using ⟨X0 OnCircle C D⟩ ⟨Y0 OnCircle C D⟩ onc2--cong by auto
  moreover have X0 C A LtA Y0 C A
  proof (rule conga-preserves-lta [where ?A=A and ?B=C and ?C=X and
    ?D=A and ?E=C and ?F=Y])
    show A C X CongA X0 C A
      by (simp add: Out-cases ⟨A ≠ C⟩ ⟨C Out X X0⟩ conga-right-comm
          out2--conga out-trivial)
    show A C Y CongA Y0 C A
      by (simp add: ⟨A ≠ C⟩ ⟨C Out Y Y0⟩ conga-right-comm
          conga-sym-equiv out2--conga out-trivial)
    show A C X LtA A C Y
  proof -
    have X InAngle Y C A
    proof (rule in-angle-trans [where ?D=P])
      show X InAngle Y C P
        by (metis InAngle-def Out-def ⟨Bet P X Y⟩ ⟨C Out X X0⟩
            ⟨C Out Y Y0⟩ ⟨¬ Col P Q C⟩ between-symmetry col-trivial-2
            not-col-permutation-1 out-trivial)
    qed
  qed
}

```

```

show P InAngle Y C A
proof (rule in-angle-trans2 [where ?A=Q])
  show Y InAngle Q C P
  by (metis InAngle-def Out-def ⟨Bet P Y Q⟩ ⟨C Out Y Y0⟩
    ⟨¬ Col P Q C⟩ between-symmetry not-col-distincts out-trivial)
  show P InAngle Q C A
  proof -
    have A C OS P Q ⟶ ?thesis
    by (meson ⟨A C P LtA A C Q⟩ invert-one-side l11-24 lta-os--ts
      one-side-symmetry os-ts--inangle)
    moreover {
      assume Col P A C and
        ¬ Col Q A C
      hence C Out A P
      using ⟨A C P LtA A C Q⟩ col-lta--out col-permutation-1 by blast
      hence ?thesis
      by (metis ⟨¬ Col P Q C⟩ col-trivial-2 out341--inangle)
    }
    moreover {
      assume ¬ Col P A C and Col Q A C
      {
        assume C Out A Q
        have A C Q LeA A C P
        using NCol-perm ⟨A C P LtA A C Q⟩ ⟨C Out A Q⟩ ⟨Col Q A C⟩
          col-lta--bet not-bet-and-out by blast
        hence False
        using ⟨A C P LtA A C Q⟩ lea--nlta by auto
      }
      hence ¬ C Out A Q
      by blast
      hence Bet Q C A
      using ⟨Col Q A C⟩ between-symmetry col-permutation-1 l6-4-2 by blast
      hence ?thesis
      using InAngle-def ⟨A ≠ C⟩ ⟨Y InAngle Q C P⟩ by auto
    }
    ultimately show ?thesis
    using ⟨A C OS P Q ∨ Col P A C ∧ ¬ Col Q A C ∨ ¬ Col P A C ∧ Col Q A C⟩
      by blast
  qed
qed
qed
hence X InAngle A C Y
using l11-24 by blast

hence A C X LeA A C Y
by (simp add: inangle--lea)
moreover {
  assume A C X CongA A C Y
  moreover have Coplanar A C X Y
  using ⟨X InAngle A C Y⟩ coplanar-perm-8 inangle--coplanar by blast
  ultimately have C Out X Y ∨ A C TS X Y
  using conga-cop--or-out-ts [where ?A=A and ?B=C and ?C=X and ?C'=Y]
    by blast
  moreover have C Out X Y ⟶ False
  by (metis Col-def NCol-perm ⟨Bet P X Y⟩ ⟨Bet P Y Q⟩ ⟨X ≠ Y⟩
    ⟨¬ Col P Q C⟩ between-exchange3 l6-16-1 out-col)
  moreover have A C TS X Y ⟶ False
  by (simp add: ⟨Bet P X Y⟩ ⟨Bet P Y Q⟩
    ⟨∧ Y X. [Bet P Y Q; Bet P X Y] ⟹ ¬ A C TS X Y⟩)
  ultimately have False
  by blast
}
hence ¬ A C X CongA A C Y
by blast
ultimately show ?thesis

```

```

    using LtA-def by blast
  qed
qed
ultimately have A X0 Lt A Y0
  using cong-reflexivity t18-18 by blast
}
hence 1:  $\forall X Y X0 Y0. \text{Bet } P Y Q \wedge X \neq Y \wedge X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge$ 
 $Y0 \text{ OnCircle } C D \wedge C \text{ Out } Y Y0 \wedge \text{Bet } P X Y \longrightarrow A X0 \text{ Lt } A Y0$ 
  by blast
have  $\exists R. \forall X Y.$ 
 $(\text{Bet } P X Q \wedge (\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B) \wedge$ 
 $\text{Bet } P Y Q \wedge (\exists Y0. Y0 \text{ OnCircle } C D \wedge C \text{ Out } Y Y0 \wedge Y0 \text{ OutCircle } A B) \longrightarrow$ 
 $\text{Bet } X R Y)$ 
proof -
{
  fix X
  have ((AlphaFun A B C D P Q) X)
    = (Bet P X Q  $\wedge$  ( $\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B$ ))
    using AlphaFun-def by simp
  have FOF (Bet P X Q)
    by (simp add: bet-fof)
  moreover have FOF ( $\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B$ )
  proof (cases  $\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B$ )
    case True
    then obtain X0 where X0 OnCircle C D and C Out X X0 and X0 InCircle A B
      by auto
    have FOF (X0 OnCircle C D  $\wedge$  C Out X X0  $\wedge$  X0 InCircle A B)
    proof -
      have FOF (X0 OnCircle C D)
        using  $\langle X0 \text{ OnCircle } C D \rangle$  calculation implies-fof by fastforce
      moreover have FOF (C Out X X0)
        using  $\langle C \text{ Out } X X0 \rangle \langle X0 \text{ OnCircle } C D \rangle$  calculation by auto
      moreover have FOF (X0 InCircle A B)
        using  $\langle C \text{ Out } X X0 \rangle \langle X0 \text{ InCircle } A B \rangle$  calculation(2) by auto
      ultimately show ?thesis
        using and-fof by auto
    qed
    thus ?thesis
      using exists-fof  $\langle C \text{ Out } X X0 \rangle \langle X0 \text{ InCircle } A B \rangle \langle X0 \text{ OnCircle } C D \rangle$  by fast
  next
    case False
    hence  $\forall X0. \neg(X0 \text{ OnCircle } C D) \vee \neg(C \text{ Out } X X0) \vee \neg(X0 \text{ InCircle } A B)$ 
      by blast
    {
      fix X0
      have  $\neg(X0 \text{ OnCircle } C D)$ 
         $\longrightarrow \text{FOF } (\neg(X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B))$ 
        using calculation implies-fof by fastforce
      moreover have  $\neg(C \text{ Out } X X0)$ 
         $\longrightarrow \text{FOF } (\neg(X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B))$ 
        using  $\langle \text{FOF } (\text{Bet } P X Q) \rangle$  implies-fof by fastforce
      moreover have  $\neg(X0 \text{ InCircle } A B)$ 
         $\longrightarrow \text{FOF } (\neg(X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B))$ 
        using  $\langle \text{FOF } (\text{Bet } P X Q) \rangle$  implies-fof by fastforce
      ultimately have  $\text{FOF } (\neg(X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B))$ 
        using False by blast
    }
    hence  $\text{FOF } (\forall X0. \neg(X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B))$ 
      using False by auto
    hence  $\text{FOF } (\neg(\exists X0. (X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B)))$ 
      by force
    hence  $\text{FOF } (\neg(\neg(\exists X0. (X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B))))$ 
      using not-fof by blast
    thus ?thesis
      by argo
  }
qed

```

```

ultimately
have FOF (Bet P X Q  $\wedge$  ( $\exists$  X0. X0 OnCircle C D  $\wedge$  C Out X X0  $\wedge$  X0 InCircle A B))
  using and-fof by blast
}
hence  $\forall$  X::'p. FOF ((AlphaFun A B C D P Q) X)
  using AlphaFun-def by simp
moreover
{
  fix Y::'p
  have ((BetaFun A B C D P Q) Y)
    = (Bet P Y Q  $\wedge$  ( $\exists$  Y0. Y0 OnCircle C D  $\wedge$  C Out Y Y0  $\wedge$  Y0 OutCircle A B))
    using BetaFun-def by simp
  have FOF (Bet P Y Q)
    by (simp add: bet-fof)
  moreover
  have FOF ( $\exists$  X0. X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B)
  proof (cases  $\exists$  X0. X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B)
    case True
    then obtain X0 where X0 OnCircle C D and C Out Y X0 and X0 OutCircle A B
      by auto
    show ?thesis
    proof -
      have FOF (X0 OnCircle C D)
        using  $\langle$ X0 OnCircle C D $\rangle$  calculation implies-fof by fastforce
      moreover have FOF (C Out Y X0)
        using  $\langle$ C Out Y X0 $\rangle$   $\langle$ X0 OnCircle C D $\rangle$  calculation by auto
      ultimately have FOF (X0 OnCircle C D  $\wedge$  C Out Y X0)
        using and-fof by blast
      moreover have FOF (X0 OutCircle A B)
        using  $\langle$ X0 OutCircle A B $\rangle$  calculation implies-fof by force
      hence FOF (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B)
        using  $\langle$ C Out Y X0 $\rangle$   $\langle$ X0 OnCircle C D $\rangle$  by auto
      thus ?thesis using exists-fof by fast
    qed
  next
  case False
  hence  $\neg$  ( $\exists$  X0. X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B)
    by blast
  hence  $\forall$  X0.  $\neg$ (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B)
    by blast
  hence  $\forall$  X0.  $\neg$ (X0 OnCircle C D)  $\vee$   $\neg$  (C Out Y X0)  $\vee$   $\neg$  (X0 OutCircle A B)
    by blast
  {
    fix X0
    have  $\neg$ (X0 OnCircle C D)
       $\longrightarrow$  FOF ( $\neg$  (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B))
      using calculation implies-fof by fastforce
    moreover have  $\neg$  (C Out Y X0)
       $\longrightarrow$  FOF ( $\neg$  (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B))
      using  $\langle$ FOF (Bet P Y Q) $\rangle$  implies-fof by fastforce
    moreover have  $\neg$  (X0 OutCircle A B)
       $\longrightarrow$  FOF ( $\neg$  (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B))
      using  $\langle$ FOF (Bet P Y Q) $\rangle$  implies-fof by fastforce
    ultimately have FOF ( $\neg$  (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B))
      using False by blast
  }
  hence FOF ( $\forall$  X0.  $\neg$ (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B))
    using False by auto
  hence FOF ( $\neg$  ( $\exists$  X0. (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B)))
    by force
  hence FOF ( $\neg$  ( $\neg$  ( $\exists$  X0. (X0 OnCircle C D  $\wedge$  C Out Y X0  $\wedge$  X0 OutCircle A B))))
    using not-fof by blast
  thus ?thesis
    by argo
}
qed
ultimately have FOF (Bet P Y Q  $\wedge$ 

```

```

      (∃ Y0. Y0 OnCircle C D ∧ C Out Y Y0 ∧ Y0 OutCircle A B))
    using and-fof by blast
  hence FOF ((BetaFun A B C D P Q) Y)
    using ⟨BetaFun A B C D P Q Y
      = (Bet P Y Q ∧ (∃ Y0. Y0 OnCircle C D ∧ C Out Y Y0 ∧ Y0 OutCircle A B))⟩
    by force
}
hence ∀ Y::'p. FOF ((BetaFun A B C D P Q) Y)
  using forall-fof BetaFun-def by simp
moreover
have ∃ R. (∀ X Y. ((AlphaFun A B C D P Q) X) ∧ ((BetaFun A B C D P Q) Y) → Bet R X Y)
proof -
{
  fix X Y
  assume (AlphaFun A B C D P Q) X and
    (BetaFun A B C D P Q) Y
  have Bet P X Q ∧ (∃ X0. X0 OnCircle C D ∧ C Out X X0 ∧ X0 InCircle A B)
    using ⟨(AlphaFun A B C D P Q) X⟩ AlphaFun-def by auto
  then obtain X0 where X0 OnCircle C D and C Out X X0 and X0 InCircle A B
    by blast
  have Bet P Y Q ∧ (∃ Y0. Y0 OnCircle C D ∧ C Out Y Y0 ∧ Y0 OutCircle A B)
    using ⟨(BetaFun A B C D P Q) Y⟩ BetaFun-def by auto
  then obtain Y0 where Y0 OnCircle C D and C Out Y Y0 and Y0 OutCircle A B
    by blast
  have Bet P X Q
    using ⟨(AlphaFun A B C D P Q) X⟩ AlphaFun-def by simp
  moreover have Bet P Y Q
    using ⟨(BetaFun A B C D P Q) Y⟩ BetaFun-def by simp
  moreover {
    assume Bet P Y X
    have Bet P X Y
    proof (cases X = Y)
      case True
      thus ?thesis
        using ⟨Bet P Y X⟩ by blast
    next
      case False
      hence X ≠ Y
        by simp
      moreover have A Y0 Lt A X0
        using 1 by (metis False ⟨Bet P X Q⟩ ⟨C Out X X0⟩ ⟨C Out Y Y0⟩
          ⟨Bet P Y X⟩ ⟨X0 OnCircle C D⟩ ⟨Y0 OnCircle C D⟩)
      moreover have A X0 Le A Y0
        using InCircle-def OutCircle-def ⟨X0 InCircle A B⟩
          ⟨Y0 OutCircle A B⟩ le-transitivity by blast
      ultimately show ?thesis
        using le--nlt by blast
    qed
  }
  ultimately have Bet P X Y using l5-3 by blast
}
thus ?thesis by blast
qed
moreover
hence ∃ R. ∀ X Y. (Bet P X Q ∧
  (∃ X0. X0 OnCircle C D ∧ C Out X X0 ∧ X0 InCircle A B) ∧
  Bet P Y Q ∧
  (∃ Y0. Y0 OnCircle C D ∧ C Out Y Y0 ∧ Y0 OutCircle A B)
  → Bet R X Y)
  using AlphaFun-def BetaFun-def by auto
ultimately have ∃ R. (∀ X Y. ((AlphaFun A B C D P Q) X) ∧ ((BetaFun A B C D P Q) Y)
  → Bet X R Y)
  using assms(1) FirstOrderDedekind-def AlphaFun-def BetaFun-def by blast
thus ?thesis
  using AlphaFun-def BetaFun-def by simp
qed

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hence $\exists R. \forall X Y. ((\text{Bet } P X Q \wedge (\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B) \wedge$
 $\text{Bet } P Y Q \wedge (\exists Y0. Y0 \text{ OnCircle } C D \wedge C \text{ Out } Y Y0 \wedge Y0 \text{ OutCircle } A B))$
 $\longrightarrow \text{Bet } X R Y)$ **by** *blast*
hence $\exists R. \forall X Y. \text{DedekindLemFun } A B C D P Q R X Y$
using *DedekindLemFun-def* **by** *presburger*
then obtain R **where** $\forall X Y. \text{DedekindLemFun } A B C D P Q R X Y$
by *blast*
hence $\forall X Y. \text{Bet } P X Q \wedge (\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } X X0 \wedge X0 \text{ InCircle } A B) \wedge$
 $\text{Bet } P Y Q \wedge (\exists Y0. Y0 \text{ OnCircle } C D \wedge C \text{ Out } Y Y0 \wedge Y0 \text{ OutCircle } A B)$
 $\longrightarrow \text{Bet } X R Y$
using *DedekindLemFun-def* **by** *simp*
moreover
have $\exists X0. X0 \text{ OnCircle } C D \wedge C \text{ Out } P X0 \wedge X0 \text{ InCircle } A B$
using $\langle C \neq D \rangle \langle P \text{ InCircleS } A B \rangle \langle P \text{ OnCircle } C D \rangle$ *incs--inc*
onc-not-center out-trivial **by** *blast*
moreover
then obtain $X0$ **where** $X0 \text{ OnCircle } C D$ **and** $C \text{ Out } P X0$ **and** $X0 \text{ InCircle } A B$
by *blast*
have $\exists Y0. Y0 \text{ OnCircle } C D \wedge C \text{ Out } Q Y0 \wedge Y0 \text{ OutCircle } A B$
using $\langle Q \text{ OutCircleS } A B \rangle \langle Q \text{ OnCircle } C D \rangle \langle A C P \text{ LtA } A C Q \rangle$ *lta-distincts*
out-trivial outcs--outc **by** *blast*
moreover
then obtain $Y0$ **where** $Y0 \text{ OnCircle } C D$ **and** $C \text{ Out } Q Y0$ **and** $Y0 \text{ OutCircle } A B$
by *blast*
have $\text{Bet } P P Q$
using *between-trivial2* **by** *auto*
moreover have $\text{Bet } P Q Q$
by (*simp add: between-trivial*)
ultimately have $\text{Bet } P R Q$
using *DedekindLemFun-def* **by** *blast*
have $R \neq C$
using $\langle \text{Bet } P Q Q \rangle \langle \text{Bet } P R Q \rangle \langle \neg \text{Col } P Q C \rangle$ *bet-col1* **by** *blast*
obtain Z **where** $Z \text{ OnCircle } C D$ **and** $C \text{ Out } R Z$
using $\langle C \neq D \rangle \langle R \neq C \rangle$ *onc-exists* **by** *blast*
have $A \neq B$
using $\langle P \text{ InCircleS } A B \rangle$ *inc--outc inc-eq incs--inc incs--noutc-1* **by** *blast*
{
 assume $Z \text{ InCircleS } A B$
 {
 assume $Q = R$
 hence $Q = Z$
 by (*metis OnCircle-def* $\langle C \text{ Out } R Z \rangle \langle C \neq D \rangle \langle Q \text{ OnCircle } C D \rangle$
 $\langle Z \text{ OnCircle } C D \rangle$ *between-cong between-trivial2*
l6-11-uniqueness out-trivial)
 hence $\neg Z \text{ OnCircle } C D$
 using *outcs--ninc* $\langle Q \text{ OutCircleS } A B \rangle \langle Z \text{ InCircleS } A B \rangle$ *incs--inc* **by** *blast*
 hence *False*
 using $\langle Z \text{ OnCircle } C D \rangle$ **by** *auto*
 }
} **hence** $Q \neq R$ **by** *blast*
have $\neg \text{Col } C Q R$
using *Col-perm* $\langle \text{Bet } P R Q \rangle \langle Q \neq R \rangle \langle \neg \text{Col } P Q C \rangle$ *bet-col col2--eq* **by** *blast*
have $\exists T. T \text{ OnCircle } A B \wedge \text{Bet } A Z T$
using $\langle Z \text{ InCircleS } A B \rangle$ *inc--radius incs--inc* **by** *blast*
then obtain T **where** $T \text{ OnCircle } A B$ **and** $\text{Bet } A Z T$
by *blast*
have $T \neq Z$
using $\langle T \text{ OnCircle } A B \rangle \langle Z \text{ InCircleS } A B \rangle$ *onc--outc outc--nincs-1* **by** *blast*
obtain I **where** $\text{Per } I Z C$ **and** $\text{Cong } I Z T Z$ **and** $C R \text{ OS } I Q$
using *ex-per-cong* **by** (*metis Col-cases Col-def Out-def* $\langle C \text{ Out } R Z \rangle$
 $\langle T \neq Z \rangle \langle \neg \text{Col } C Q R \rangle$)
hence $\neg \text{Col } I Z C$
by (*metis* $\langle C \text{ Out } R Z \rangle \langle T \neq Z \rangle$ *cong-identity cong-symmetry out-distinct per-not-col*)
obtain $X0$ **where** $X0 \text{ OnCircle } C D$ **and** $C \text{ Out } I X0$
by (*metis* $\langle C \neq D \rangle \langle \neg \text{Col } I Z C \rangle$ *col-trivial-3 onc-exists*)
have $C X0 \text{ Lt } C I$

```

proof -
{
  assume  $Z\ C\ Lt\ I\ C$ 
  moreover have  $Cong\ Z\ C\ C\ X0$ 
    using  $Cong\ cases\ \langle X0\ OnCircle\ C\ D\rangle\ \langle Z\ OnCircle\ C\ D\rangle\ onc2\text{--}cong$  by blast
  moreover have  $Cong\ I\ C\ C\ I$ 
    by (simp add:  $cong\text{--}pseudo\text{--}reflexivity$ )
  ultimately have  $?thesis$ 
    using  $cong2\text{--}lt\text{--}lt$  by blast
}
thus  $?thesis$ 
  by (metis  $Col\text{--}def\ \langle Per\ I\ Z\ C\rangle\ \langle \neg\ Col\ I\ Z\ C\rangle\ between\text{--}trivial2\ lt\text{--}left\text{--}comm\ per\text{--}lt$ )
qed
have  $X0 \neq I$ 
  using  $\langle C\ X0\ Lt\ C\ I\rangle\ nlt$  by auto
have  $\neg\ Col\ I\ X0\ Z$ 
  by (metis  $Col\text{--}perm\ \langle C\ Out\ I\ X0\rangle\ \langle X0 \neq I\rangle\ \langle \neg\ Col\ I\ Z\ C\rangle\ col2\text{--}eq\ out\text{--}col$ )
have  $Obtuse\ I\ X0\ Z$ 
proof (rule  $acute\text{--}bet\text{--}obtuse$  [where  $?A = C$ ])
  show  $Bet\ C\ X0\ I$ 
    by (metis  $Out\text{--}def\ \langle C\ X0\ Lt\ C\ I\rangle\ \langle C\ Out\ I\ X0\rangle\ bet\text{--}lt1213\ not\text{--}and\text{--}lt$ )
  show  $I \neq X0$ 
    using  $\langle X0 \neq I\rangle$  by auto
  show  $Acute\ C\ X0\ Z$ 
    by (metis  $Col\text{--}def\ \langle X0\ OnCircle\ C\ D\rangle\ \langle C\ Out\ I\ X0\rangle\ \langle Z\ OnCircle\ C\ D\rangle\ \langle \neg\ Col\ I\ X0\ Z\rangle\ between\text{--}trivial\ cong\text{--}acute\ onc2\text{--}cong\ out\text{--}distinct$ )
qed
hence  $X0\ Z\ Lt\ I\ Z$ 
  by (metis  $\langle X0 \neq I\rangle\ \langle \neg\ Col\ I\ X0\ Z\rangle\ col\text{--}trivial\text{--}2\ l11\text{--}46$ )
have  $X0\ InCircleS\ A\ B$ 
proof -
  have  $Z\ X0\ Le\ Z\ T$ 
    by (meson  $\langle Cong\ I\ Z\ T\ Z\rangle\ \langle X0\ Z\ Lt\ I\ Z\rangle\ cong\text{--}le3412\ le\text{--}nlt\ le\text{--}right\text{--}comm\ le\text{--}transitivity\ local.le\text{--}cases\ not\text{--}cong\text{--}3421$ )
  then obtain  $M$  where  $Bet\ Z\ M\ T$  and  $Cong\ Z\ M\ Z\ X0$ 
    using  $le\text{--}bet$  by blast
  {
    assume  $M = T$ 
    have  $Z\ M\ Lt\ Z\ M$ 
    proof (rule  $cong2\text{--}lt\text{--}lt$  [where  $?A = X0$  and  $?B = Z$  and  $?C = I$  and  $?D = Z$ ])
      show  $X0\ Z\ Lt\ I\ Z$ 
        by (simp add:  $\langle X0\ Z\ Lt\ I\ Z\rangle$ )
      show  $Cong\ X0\ Z\ Z\ M$ 
        using  $Cong\ cases\ \langle Cong\ Z\ M\ Z\ X0\rangle$  by blast
      show  $Cong\ I\ Z\ Z\ M$ 
      proof (rule  $cong\text{--}transitivity$  [where  $?C=T$  and  $?D=Z$ ])
        show  $Cong\ I\ Z\ T\ Z$ 
          by (simp add:  $\langle Cong\ I\ Z\ T\ Z\rangle$ )
        show  $Cong\ T\ Z\ Z\ M$ 
          using  $\langle M = T\rangle\ cong\text{--}pseudo\text{--}reflexivity$  by auto
      qed
    qed
    hence  $False$ 
      using  $nlt$  by auto
  }
  hence  $M \neq T$ 
    by auto
  have  $A\ X0\ Le\ A\ M$ 
    using  $\langle Bet\ A\ Z\ T\rangle\ \langle Bet\ Z\ M\ T\rangle\ \langle Cong\ Z\ M\ Z\ X0\rangle\ between\text{--}inner\text{--}transitivity\ cong\text{--}symmetry\ triangle\text{--}inequality$  by blast
  moreover have  $A\ M\ Lt\ A\ T$ 
    using  $\langle Bet\ A\ Z\ T\rangle\ \langle Bet\ Z\ M\ T\rangle\ \langle M \neq T\rangle\ bet\text{--}lt1213\ between\text{--}exchange2$  by blast
  hence  $A\ M\ Lt\ A\ B$ 
    using  $OnCircle\text{--}def\ \langle T\ OnCircle\ A\ B\rangle\ cong2\text{--}lt\text{--}lt\ cong\text{--}reflexivity$  by blast
  ultimately show  $?thesis$ 
    by (simp add:  $InCircleS\text{--}def\ le1234\text{--}lt\text{--}lt$ )

```



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qed
have X0 InAngle R C Q
proof (rule l11-25 [where ?P=X0 and ?A=Z and ?C=Q])
  show X0 InAngle Z C Q
  proof -
    have Z C X0 LeA Z C Q
    proof -
      have Cong C Q C X0
      using ⟨Q OnCircle C D⟩ ⟨X0 OnCircle C D⟩ onc2--cong by blast
    moreover have Cong C Z C Z
    by (simp add: cong-reflexivity)
  moreover
  have A T Le A Q
  proof (rule l5-6 [where ?A=A and ?B=B and ?C=A and ?D=Q])
    show A B Le A Q
    using InCircle-def ⟨Q OutCircleS A B⟩ incs--inc incs--outcs-2 by blast
  show Cong A B A T
  using ⟨T OnCircle A B⟩ onc212 onc2--cong by blast
  show Cong A Q A Q
  using cong-reflexivity by blast
qed
have T Z Le Q Z
proof (cases A = Z)
  case True
  then show ?thesis
  using Le-cases ⟨A T Le A Q⟩ by blast
next
case False
obtain M where Bet A T M and Cong A M A Q
  using ⟨A T Le A Q⟩ l5-5-1 by blast
have Bet A Z M
  using ⟨Bet A T M⟩ ⟨Bet A Z T⟩ between-exchange4 by blast
have T Z Le M Z
  using Bet-cases ⟨Bet A T M⟩ ⟨Bet A Z T⟩ bet--le2313 between-exchange3 by blast
moreover have A Out Z M
  using False ⟨Bet A Z M⟩ bet-out by force
hence M Z Le Q Z
  using Le-cases ⟨Cong A M A Q⟩ not-cong-3412
  triangle-reverse-inequality by blast
ultimately show ?thesis
  using le-transitivity by blast
qed
hence I Z Le Q Z
  using ⟨Cong I Z T Z⟩ cong--le le-transitivity by blast
hence X0 Z Lt Q Z
  using ⟨X0 Z Lt I Z⟩ le3456-lt--lt by blast
ultimately
show ?thesis
  by (metis Cong-perm cong--le3412 cong-diff-3 le--nlt nlta--lea
    not-and-lt t18-18)
qed
moreover have C R OS Q X0
  using ⟨C Out I X0⟩ ⟨C R OS I Q⟩ not-col-distincts one-side-symmetry
  os-out-os by blast
hence Z C OS Q X0
  by (metis Out-def ⟨C Out R Z⟩ col-one-side invert-one-side out-col)
ultimately show ?thesis
  using lea-in-angle by blast
qed
show C Out R Z
  by (simp add: ⟨C Out R Z⟩)
show C Out Q Q
  using ⟨C Out Q Y0⟩ out-diff1 out-trivial by blast
show C Out X0 X0
  using Out-def ⟨C Out I X0⟩ out-trivial by presburger
qed

```

```

then obtain  $X$  where  $Bet\ R\ X\ Q$  and  $(X = C \vee C\ Out\ X\ X0)$ 
  using InAngle-def by auto
moreover
have  $X = C \longrightarrow False$ 
  using Col-def  $\langle \neg\ Col\ C\ Q\ R \rangle$  calculation(1) by auto
moreover
{
  assume  $C\ Out\ X\ X0$ 
  have  $\exists\ X1. X1\ OnCircle\ C\ D \wedge C\ Out\ X\ X1 \wedge X1\ InCircle\ A\ B$ 
    using  $\langle C\ Out\ X\ X0 \rangle \langle X0\ InCircleS\ A\ B \rangle \langle X0\ OnCircle\ C\ D \rangle$  incs--inc by blast
  hence  $Bet\ X\ R\ Q$ 
    by (metis (full-types) Out-def  $\langle Bet\ P\ Q\ Q \rangle \langle Bet\ P\ R\ Q \rangle \langle C\ Out\ Q\ Y0 \rangle$ 
       $\langle Q\ OnCircle\ C\ D \rangle \langle Q\ OutCircleS\ A\ B \rangle$ 
       $\langle \forall\ X\ Y. Bet\ P\ X\ Q \wedge (\exists\ X0. X0\ OnCircle\ C\ D \wedge C\ Out\ X\ X0 \wedge X0\ InCircle\ A\ B) \rangle \wedge$ 
       $Bet\ P\ Y\ Q \wedge (\exists\ Y0. Y0\ OnCircle\ C\ D \wedge C\ Out\ Y\ Y0 \wedge Y0\ OutCircle\ A\ B) \rangle$ 
       $\longrightarrow Bet\ X\ R\ Y$ )
    between-exchange2 calculation(1) out-trivial outcs--outc)
  hence  $X = R$ 
    using between-equality calculation(1) by blast
  hence False
    using Col-def  $\langle C\ Out\ I\ X0 \rangle \langle C\ Out\ X\ X0 \rangle \langle C\ R\ OS\ I\ Q \rangle$  between-trivial
    col123--nos os-out-os out-col by blast
}
ultimately
have False
  by blast
hence  $\exists\ Z. Z\ OnCircle\ A\ B \wedge Z\ OnCircle\ C\ D$ 
  by blast
}
moreover
{
  assume  $Z\ OutCircleS\ A\ B$ 
  have  $P \neq R$  by (metis OnCircle-def Out-def  $\langle C\ Out\ R\ Z \rangle \langle P\ InCircleS\ A\ B \rangle$ 
     $\langle P\ OnCircle\ C\ D \rangle \langle Z\ OnCircle\ C\ D \rangle \langle Z\ OutCircleS\ A\ B \rangle$  incs--inc
    l6-11-uniqueness out-trivial outcs--ninc-1)
  hence  $\neg\ Col\ C\ P\ R$ 
    using Col-def  $\langle Bet\ P\ R\ Q \rangle \langle \neg\ Col\ P\ Q\ C \rangle$  col2--eq by blast
  obtain  $T$  where  $T\ OnCircle\ A\ B$  and  $A\ Out\ Z\ T$ 
    by (metis  $\langle A \neq B \rangle \langle Z\ OutCircleS\ A\ B \rangle$  inc112 onc-exists outcs--ninc-1)
  have  $A\ T\ Le\ A\ Z$  proof (rule l5-6 [where  $?A=A$  and  $?B=B$  and  $?C=A$  and  $?D=Z$ ])
    show  $A\ B\ Le\ A\ Z$ 
      using OutCircle-def  $\langle Z\ OutCircleS\ A\ B \rangle$  outcs--outc by blast
    show  $Cong\ A\ B\ A\ T$ 
      using OnCircle-def  $\langle T\ OnCircle\ A\ B \rangle$  not-cong-3412 by blast
    show  $Cong\ A\ Z\ A\ Z$ 
      by (simp add: cong-reflexivity)
qed
hence  $Bet\ A\ T\ Z$ 
  by (simp add:  $\langle A\ Out\ Z\ T \rangle$  l6-13-1 l6-6)
{
  assume  $T = Z$ 
  hence  $\neg\ Z\ InCircle\ A\ B$ 
    using  $\langle Z\ OutCircleS\ A\ B \rangle$  outcs--ninc by blast
  hence False
    using  $\langle T = Z \rangle \langle T\ OnCircle\ A\ B \rangle$  onc--inc by blast
}
hence  $T \neq Z$ 
  by auto
obtain  $I$  where  $Per\ I\ Z\ C$  and  $Cong\ I\ Z\ T\ Z$  and  $C\ R\ OS\ I\ P$ 
  by (metis Col-cases Col-def Out-def  $\langle C\ Out\ R\ Z \rangle \langle T \neq Z \rangle \langle \neg\ Col\ C\ P\ R \rangle$  ex-per-cong)
hence  $\neg\ Col\ I\ Z\ C$ 
  using  $\langle C\ Out\ R\ Z \rangle \langle T \neq Z \rangle$  cong-reverse-identity l8-9 out-diff2 by blast
obtain  $Y0$  where  $Y0\ OnCircle\ C\ D$  and  $C\ Out\ I\ Y0$ 
  by (metis  $\langle C\ R\ OS\ I\ P \rangle \langle C \neq D \rangle$  onc-exists os-distincts)
have  $Z\ C\ Lt\ I\ C$ 
  using  $\langle Per\ I\ Z\ C \rangle \langle \neg\ Col\ I\ Z\ C \rangle$  l11-46 not-col-distincts by blast

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have Cong Z C C Y0
  using Cong-cases ⟨Y0 OnCircle C D⟩ ⟨Z OnCircle C D⟩ onc2--cong by blast
have Cong I C C I
  by (simp add: cong-pseudo-reflexivity)
hence C Y0 Lt C I
  using ⟨Cong Z C C Y0⟩ ⟨Z C Lt I C⟩ cong2-lt--lt by blast
have Y0 ≠ I
  using ⟨C Y0 Lt C I⟩ nlt by blast
have ¬ Col I Y0 Z
  by (metis Col-cases Out-def ⟨C Out I Y0⟩ ⟨Y0 ≠ I⟩ ⟨¬ Col I Z C⟩
      bet-col col-transitivity-1)
have Obtuse I Y0 Z
proof (rule acute-bet--obtuse [where ?A=C])
  show Bet C Y0 I
    using l6-13-1 by (meson Lt-def ⟨C Out I Y0⟩ ⟨C Y0 Lt C I⟩ l6-6)
  show I ≠ Y0
    using ⟨Y0 ≠ I⟩ by auto
  show Acute C Y0 Z
    by (metis Out-def ⟨C Out I Y0⟩ ⟨Cong Z C C Y0⟩ ⟨¬ Col I Y0 Z⟩
        col-trivial-2 cong-3421 cong--acute)
qed
hence Per I Y0 Z ∨ Obtuse I Y0 Z
  by blast
have Y0 Z Lt I Z
  using l11-46 by (metis NCol-cases ⟨C Out I Y0⟩ ⟨Per I Y0 Z ∨ Obtuse I Y0 Z⟩
      ⟨Y0 ≠ I⟩ ⟨¬ Col I Z C⟩ out-col)
have Y0 OutCircleS A B
proof -
  have Z Y0 Le Z T
    by (metis Le-cases ⟨Cong I Z T Z⟩ ⟨Y0 Z Lt I Z⟩ cong--le
        le-transitivity local.le-cases lt--nle)
  then obtain M where Bet Z M T and Cong Z M Z Y0
    using le-bet by blast
  hence T ≠ M
    by (metis Lt-def ⟨Cong I Z T Z⟩ ⟨Y0 Z Lt I Z⟩ cong-4312 cong-inner-transitivity)
  have A B Lt A M
proof (rule cong2-lt--lt [where ?A=A and ?B=T and ?C=A and ?D=M])
  show A T Lt A M
    by (meson ⟨Bet A T Z⟩ ⟨Bet Z M T⟩ ⟨T ≠ M⟩ bet--lt1213
        between-inner-transitivity between-symmetry)
  show Cong A T A B
    using OnCircle-def ⟨T OnCircle A B⟩ by auto
  show Cong A M A M
    by (simp add: cong-reflexivity)
qed
moreover have Z Out A M
  by (metis Col-def ⟨Bet A T Z⟩ ⟨Bet Z M T⟩ ⟨Cong Z M Z Y0⟩
      ⟨T = Z ⟹ False⟩ ⟨¬ Col I Y0 Z⟩ bet-out bet-out-1 between-trivial2
      cong-reverse-identity l6-6 l6-7)
hence A M Le A Y0
  using ⟨Cong Z M Z Y0⟩ not-cong-3412 triangle-reverse-inequality by blast
ultimately show ?thesis
  using OutCircleS-def le3456-lt--lt by blast
qed
have Y0 InAngle P C Z
proof -
  have Z C Y0 LeA Z C P
proof -
  have Cong C P C Y0
    using ⟨P OnCircle C D⟩ ⟨Y0 OnCircle C D⟩ onc2--cong by auto
  moreover have Cong C Z C Z
    by (simp add: cong-reflexivity)
  moreover have T Z Le P Z
proof -
  have A P Le A T
    by (meson OutCircle-def ⟨P InCircleS A B⟩ ⟨T OnCircle A B⟩

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```

    cong--le le-transitivity local.le-cases onc212 onc2--cong outc--nincs-1)
  then obtain M where Bet A M T and Cong A M A P
    using le-bet by blast
  have Bet A M Z
    using ⟨Bet A M T⟩ ⟨Bet A T Z⟩ between-exchange4 by blast
  show ?thesis
  proof (cases M = A)
    case True
    thus ?thesis
      by (metis ⟨Bet A T Z⟩ ⟨Cong A M A P⟩ cong-diff-3 l5-12-a)
  next
    case False
    have T Z Le M Z
      using ⟨Bet A M T⟩ ⟨Bet A T Z⟩ bet--le2313 between-exchange3 by blast
    moreover have A Out Z M
      using False ⟨Bet A M Z⟩ bet-out l6-6 by presburger
    hence M Z Le P Z
      using ⟨Cong A M A P⟩ cong-symmetry le-comm triangle-reverse-inequality by blast
    ultimately show ?thesis
      using le-transitivity by blast
  qed
  qed
  hence I Z Le P Z
    using ⟨Cong I Z T Z⟩ cong--le le-transitivity by blast
  hence Y0 Z Lt P Z
    using ⟨Y0 Z Lt I Z⟩ le3456-lt--lt by blast
  ultimately show ?thesis
    by (metis Out-def ⟨C Out I Y0⟩ ⟨C Out P X0⟩ ⟨C Out R Z⟩
        nlea--lta not-and-lt not-cong-3412 t18-18)
  qed
  moreover have C R OS P Y0
    using ⟨C Out I Y0⟩ ⟨C R OS I P⟩ not-col-distincts one-side-symmetry
        os-out-os by blast
  hence Z C OS P Y0
    by (metis Out-def ⟨C Out R Z⟩ col-one-side invert-one-side out-col)
  ultimately show ?thesis
    using l11-24 lea-in-angle by blast
  qed
  hence Y0 InAngle P C R
    using ⟨C Out R Z⟩ in-angle-trans inangle-distincts out341--inangle by blast
  obtain Y where Bet P Y R and Y = C ∨ C Out Y Y0
    using InAngle-def ⟨Y0 InAngle P C R⟩ by blast
  have Bet P R Y
  proof –
    have Bet P P Q
      by (simp add: ⟨Bet P P Q⟩)
    moreover have ∃ X0. (X0 OnCircle C D ∧ C Out P X0 ∧ X0 InCircle A B)
      using ⟨∃ X0. X0 OnCircle C D ∧ C Out P X0 ∧ X0 InCircle A B⟩ by blast
    moreover have Bet P Y Q
      using ⟨Bet P R Q⟩ ⟨Bet P Y R⟩ between-exchange4 by blast
    moreover have ∃ Y1. (Y1 OnCircle C D ∧ C Out Y Y1 ∧ Y1 OutCircle A B)
      using Col-def ⟨Col P Q C ⟹ False⟩ ⟨Y = C ∨ C Out Y Y0⟩
        ⟨Y0 OnCircle C D⟩ ⟨Y0 OutCircleS A B⟩ between-symmetry calculation(3)
        outcs--outc by blast
    ultimately show ?thesis
      using ⟨∀ X Y. Bet P X Q ∧ (∃ X0. X0 OnCircle C D ∧ C Out X X0 ∧ X0 InCircle A B) ∧
          Bet P Y Q ∧ (∃ Y0. Y0 OnCircle C D ∧ C Out Y Y0 ∧ Y0 OutCircle A B)
          ⟹ Bet X R Y⟩ by presburger
  qed
  hence Y = R
    using ⟨Bet P Y R⟩ between-equality-2 by auto
  hence False
    by (metis ⟨C Out I Y0⟩ ⟨C R OS I P⟩ ⟨Y = C ∨ C Out Y Y0⟩ col123--nos
        not-col-distincts os-out-os out-col)
}
ultimately have ∃ Z. Z OnCircle A B ∧ Z OnCircle C D

```

```

    using ⟨Z OnCircle C D⟩ circle-cases by blast
  }
  thus ?thesis
    using circle-circle-aux by blast
qed

lemma nested--diff0:
  assumes Nested A B
  shows  $\forall n \text{ An Bn. } A \text{ n An} \wedge B \text{ n Bn} \longrightarrow \text{An} \neq \text{Bn}$ 
proof -
  have  $(\forall n. (\exists \text{An Bn. } A \text{ n An} \wedge B \text{ n Bn})) \wedge (\forall n \text{ An Am Bm Bn.}$ 
     $(A \text{ n An} \wedge A (\text{Suc } n) \text{Am} \wedge B (\text{Suc } n) \text{Bm} \wedge B \text{ n Bn})$ 
     $\longrightarrow (\text{Bet An Am Bm} \wedge \text{Bet Am Bm Bn} \wedge \text{Am} \neq \text{Bm}))$ 
    using assms Nested-def by simp
  {
    fix n
    have  $\forall \text{An Bn. } A \text{ n An} \wedge B \text{ n Bn} \longrightarrow \text{An} \neq \text{Bn}$ 
    proof (induction n)
      case 0
      thus ?case
        using ⟨ $(\forall n. \exists \text{An Bn. } A \text{ n An} \wedge B \text{ n Bn}) \wedge$ 
           $(\forall n \text{ An Am Bm Bn. } A \text{ n An} \wedge A (\text{Suc } n) \text{Am} \wedge B (\text{Suc } n) \text{Bm} \wedge B \text{ n Bn}$ 
             $\longrightarrow \text{Bet An Am Bm} \wedge \text{Bet Am Bm Bn} \wedge \text{Am} \neq \text{Bm})$ ⟩
          between-equality between-symmetry by blast
    next
      case (Suc n)
      thus ?case
        using ⟨ $(\forall n. \exists \text{An Bn. } A \text{ n An} \wedge B \text{ n Bn}) \wedge$ 
           $(\forall n \text{ An Am Bm Bn. } A \text{ n An} \wedge A (\text{Suc } n) \text{Am} \wedge B (\text{Suc } n) \text{Bm} \wedge B \text{ n Bn}$ 
             $\longrightarrow \text{Bet An Am Bm} \wedge \text{Bet Am Bm Bn} \wedge \text{Am} \neq \text{Bm})$ ⟩ by blast
    qed
  }
  thus ?thesis by blast
qed

lemma nested--sym:
  assumes Nested A B
  shows Nested B A
proof -
  have  $(\forall n. (\exists \text{An Bn. } A \text{ n An} \wedge B \text{ n Bn})) \wedge (\forall n \text{ An Am Bm Bn.}$ 
     $(A \text{ n An} \wedge A (\text{Suc } n) \text{Am} \wedge B (\text{Suc } n) \text{Bm} \wedge B \text{ n Bn})$ 
     $\longrightarrow (\text{Bet An Am Bm} \wedge \text{Bet Am Bm Bn} \wedge \text{Am} \neq \text{Bm}))$ 
    using assms Nested-def by simp
  hence  $(\forall n. (\exists \text{An Bn. } B \text{ n An} \wedge A \text{ n Bn}))$ 
    by simp
  moreover {
    fix n
    have  $(\forall \text{An Am Bm Bn.}$ 
       $(B \text{ n An} \wedge B (\text{Suc } n) \text{Am} \wedge A (\text{Suc } n) \text{Bm} \wedge A \text{ n Bn}) \longrightarrow$ 
       $(\text{Bet An Am Bm} \wedge \text{Bet Am Bm Bn} \wedge \text{Am} \neq \text{Bm}))$ 
      using ⟨ $(\forall n. \exists \text{An Bn. } A \text{ n An} \wedge B \text{ n Bn}) \wedge$ 
         $(\forall n \text{ An Am Bm Bn. } A \text{ n An} \wedge A (\text{Suc } n) \text{Am} \wedge B (\text{Suc } n) \text{Bm} \wedge B \text{ n Bn}$ 
           $\longrightarrow \text{Bet An Am Bm} \wedge \text{Bet Am Bm Bn} \wedge \text{Am} \neq \text{Bm})$ ⟩ between-symmetry by blast
    }
  ultimately show ?thesis
    using Nested-def by presburger
qed

lemma nested--ex-left:
  assumes Nested A B
  shows  $\exists \text{An. } A \text{ n An}$ 
  using Nested-def assms by presburger

lemma nested--ex-right:
  assumes Nested A B
  shows  $\exists \text{Bn. } B \text{ n Bn}$ 

```

using *assms nested--ex-left nested-sym* by *blast*

lemma *nested-aux1-1*:

fixes *n::nat*

assumes *Nested A B*

shows $\forall Am Bm. A\ n\ An \wedge A\ (Suc\ n)\ Am \wedge B\ (Suc\ n)\ Bm \longrightarrow Bet\ An\ Am\ Bm$

proof -

have $(\forall n. (\exists An Bn. A\ n\ An \wedge B\ n\ Bn)) \wedge$

$(\forall n An Am Bm Bn.$

$(A\ n\ An \wedge A\ (Suc\ n)\ Am \wedge B\ (Suc\ n)\ Bm \wedge B\ n\ Bn)$

$\longrightarrow$

$(Bet\ An\ Am\ Bm \wedge Bet\ Am\ Bm\ Bn \wedge Am \neq Bm))$

using *assms Nested-def* by *auto*

thus *?thesis*

by *blast*

qed

lemma *nested-aux1-2*:

fixes *n::nat*

assumes *Nested A B*

shows $(n < m \wedge (Suc\ n) \leq m) \longrightarrow ((\forall Am Bm. A\ n\ An \wedge A\ m\ Am \wedge B\ m\ Bm \longrightarrow Bet\ An\ Am\ Bm))$

proof (induction *m*)

show $(n < 0 \wedge Suc\ n \leq 0) \longrightarrow ((\forall Am Bm. A\ n\ An \wedge A\ 0\ Am \wedge B\ 0\ Bm \longrightarrow Bet\ An\ Am\ Bm))$

by *simp*

{

fix *m*

assume $(n < m \wedge Suc\ n \leq m) \longrightarrow (\forall Am Bm. A\ n\ An \wedge A\ m\ Am \wedge B\ m\ Bm \longrightarrow Bet\ An\ Am\ Bm)$

{

assume $n < Suc\ m \wedge Suc\ n \leq Suc\ m$

hence $Suc\ n = Suc\ m \vee Suc\ n \leq m$

using *le-Suc-eq* by *blast*

hence $n = m \vee Suc\ n \leq m$

by *simp*

moreover

have $n = m \longrightarrow (Suc\ n \leq Suc\ m \longrightarrow$

$(\forall Am Bm. A\ n\ An \wedge A\ (Suc\ m)\ Am \wedge B\ (Suc\ m)\ Bm \longrightarrow Bet\ An\ Am\ Bm))$

using *assms nested-aux1-1* by *blast*

moreover {

fix *Am' Bm'*

assume $Suc\ n \leq m$ and $A\ n\ An$ and $A\ (Suc\ m)\ Am'$ and $B\ (Suc\ m)\ Bm'$

obtain $Am\ Bm$ where $A\ m\ Am$ and $B\ m\ Bm$

using *assms(1) Nested-def* by *fastforce*

hence $Bet\ An\ Am\ Bm$

using *Suc-le-eq* $\langle A\ n\ An \rangle \langle Suc\ n \leq m \rangle$

$\langle n < m \wedge Suc\ n \leq m \longrightarrow (\forall Am Bm. A\ n\ An \wedge A\ m\ Am \wedge B\ m\ Bm \longrightarrow Bet\ An\ Am\ Bm) \rangle$

by *blast*

have $Bet\ Am\ Am'\ Bm' \wedge Bet\ Am'\ Bm'\ Bm \wedge Am' \neq Bm'$

using *Nested-def* $\langle A\ (Suc\ m)\ Am' \rangle \langle A\ m\ Am \rangle \langle B\ (Suc\ m)\ Bm' \rangle \langle B\ m\ Bm \rangle$ *assms* by *presburger*

hence $Bet\ An\ Am\ Bm'$

by (*meson* $\langle Bet\ An\ Am\ Bm \rangle$ *bet-out-1* *bet-out--bet* *between-inner-transitivity*)

hence $Bet\ An\ Am'\ Bm'$

using $\langle Bet\ Am\ Am'\ Bm' \wedge Bet\ Am'\ Bm'\ Bm \wedge Am' \neq Bm' \rangle$ *between-exchange2* by *blast*

}

ultimately have $(n < Suc\ m \wedge (Suc\ n \leq Suc\ m))$

$\longrightarrow (\forall Am Bm. A\ n\ An \wedge A\ (Suc\ m)\ Am \wedge B\ (Suc\ m)\ Bm \longrightarrow Bet\ An\ Am\ Bm)$

by *blast*

}

}

thus $\bigwedge m. ((n < m \wedge Suc\ n \leq m) \longrightarrow (\forall Am Bm. A\ n\ An \wedge A\ m\ Am \wedge B\ m\ Bm \longrightarrow Bet\ An\ Am\ Bm)) \implies$

$((n < Suc\ m) \wedge Suc\ n \leq Suc\ m) \longrightarrow$

$(\forall Am Bm. A\ n\ An \wedge A\ (Suc\ m)\ Am \wedge B\ (Suc\ m)\ Bm \longrightarrow Bet\ An\ Am\ Bm)$

by *blast*

qed

lemma *nested-aux1*:

fixes *n::nat*

```

assumes Nested A B and
   $n < m$ 
shows  $\forall Am Bm. A\ n\ An \wedge A\ m\ Am \wedge B\ m\ Bm \longrightarrow Bet\ An\ Am\ Bm$ 
proof –
  have  $m = (Suc\ n) \vee (Suc\ n) \leq m$ 
    using less-eq-Suc-le assms(2) by blast
  moreover have  $(m = Suc\ n) \longrightarrow ?thesis$ 
    using nested-aux1-1 assms(2) assms(1) by blast
  moreover
  {
    fix  $Am\ Bm$ 
    assume  $(Suc\ n) \leq m$  and  $A\ n\ An$  and  $A\ m\ Am$  and  $B\ m\ Bm$ 
    hence  $Bet\ An\ Am\ Bm$ 
    using assms(1) nested-aux1-2 assms(2) by blast
  }
  ultimately show ?thesis
    by blast
qed

lemma nested-aux2:
assumes Nested A B and
   $n < m$  and
   $A\ n\ An$  and
   $A\ m\ Am$  and
   $B\ n\ Bn$ 
shows  $Bet\ An\ Am\ Bn$ 
proof –
  obtain  $Bm$  where  $B\ m\ Bm$ 
    using assms(1) nested--ex-right by blast
  have  $Bet\ An\ Am\ Bm$ 
    using  $\langle A\ m\ Am \rangle \langle A\ n\ An \rangle \langle B\ m\ Bm \rangle$  assms(1) assms(2) nested-aux1 by blast
  moreover have  $Bet\ Am\ Bm\ Bn$ 
    using  $\langle A\ m\ Am \rangle \langle B\ m\ Bm \rangle \langle B\ n\ Bn \rangle$  assms(1) assms(2)
    between-symmetry nested-aux1 nested-sym by blast
  moreover have  $Am \neq Bm$ 
    using  $\langle A\ m\ Am \rangle \langle B\ m\ Bm \rangle$  assms(1) nested--diff0 by auto
  ultimately show ?thesis
    using outer-transitivity-between by blast
qed

lemma nested--bet:
assumes Nested A B and
   $n < n1$  and
   $A\ n\ An$  and
   $A\ n1\ An1$  and
   $B\ m\ Bm$ 
shows  $Bet\ An\ An1\ Bm$ 
proof –
  have Nested B A
    using assms(1) nested-sym by blast
  have  $n1 = m \longrightarrow Bet\ An\ An1\ Bm$ 
    using nested-aux1 assms(1) assms(2) assms(3) assms(4) assms(5) by blast
  moreover {
    assume  $n1 \neq m$ 
    obtain  $Bn1$  where  $B\ n1\ Bn1$ 
      using  $\langle Nested\ B\ A \rangle$  nested--ex-left by blast
    have  $Bet\ An\ An1\ Bn1$ 
      using  $\langle B\ n1\ Bn1 \rangle$  assms(1) assms(2) assms(3) assms(4) nested-aux1 by blast
    {
      assume  $n1 < m$ 
      have  $Bet\ Bn1\ Bm\ An1$ 
        using nested-aux2 [where  $?A=B$  and  $?B=A$  and  $?n=n1$  and  $?m=m$ ]
         $\langle Nested\ B\ A \rangle \langle n1 < m \rangle \langle B\ n1\ Bn1 \rangle$  assms(5) assms(4) by blast
      hence  $Bet\ An1\ Bm\ Bn1$ 
        using Bet-cases by presburger
      hence  $Bet\ An\ An1\ Bm$ 
    }
  }

```

```

    using ⟨Bet An An1 Bn1⟩ between-inner-transitivity by blast
  }
  moreover {
    assume  $m < n1$ 
    have  $An1 \neq Bn1$ 
    using ⟨B n1 Bn1⟩ assms(1) assms(4) nested--diff0 by auto
    moreover have Bet An1 Bn1 Bm
    using ⟨B n1 Bn1⟩ ⟨Nested B A⟩ ⟨ $m < n1$ ⟩ assms(4) assms(5)
      between-symmetry nested-aux1 by blast
    ultimately have Bet An An1 Bm
    using ⟨Bet An An1 Bn1⟩ outer-transitivity-between by blast
  }
  ultimately have Bet An An1 Bm
  using ⟨ $n1 \neq m$ ⟩ linorder-neqE-nat by blast
}
ultimately show ?thesis
by blast
qed

```

```

lemma nested--diff:
  assumes Nested A B and
    A n An and
    B m Bm
  shows  $An \neq Bm$ 
proof -
  have  $n = m \longrightarrow ?thesis$ 
  using assms(1) assms(2) assms(3) nested--diff0 by blast
  moreover {
    assume  $n < m$ 
    obtain Am where A m Am
    using Nested-def assms(1) by presburger
    hence Bet An Am Bm
    using ⟨ $n < m$ ⟩ assms(1) assms(2) assms(3) nested-aux1 by blast
    hence ?thesis
    using ⟨A m Am⟩ assms(1) assms(3) bet-neq12--neq nested--diff0 by blast
  }
  moreover {
    assume  $m < n$ 
    obtain Bn where B n Bn
    using Nested-def assms(1) by presburger
    have Nested B A
    by (simp add: assms(1) nested-sym)
    hence Bet Bm Bn An
    using ⟨B n Bn⟩ ⟨ $m < n$ ⟩ assms(2) assms(3) nested--bet by blast
    hence ?thesis
    using ⟨B n Bn⟩ assms(1) assms(2) bet-neq12--neq nested--diff0 by blast
  }
  ultimately show ?thesis
  using nat-neq-iff by blast
qed

```

```

lemma dedekind--cantor:
  assumes DedekindsAxiom
  shows CantorsAxiom
proof -
  {
    fix A B
    assume Nested A B
    obtain A0 where A 0 A0
    using Nested-def ⟨Nested A B⟩ by presburger
    hence  $\forall X Y. (\exists n. 0 < n \wedge A n X) \wedge (\exists m. B m Y) \longrightarrow Bet A0 X Y$ 
    using ⟨Nested A B⟩ nested--bet by blast
    hence  $\exists A0. \forall X Y. (\exists n. 0 < n \wedge A n X) \wedge (\exists m. B m Y) \longrightarrow Bet A0 X Y$ 
    by blast
    hence  $\exists X. \forall An Bm. (\exists n. 0 < n \wedge A n An) \wedge (\exists m. B m Bm) \longrightarrow Bet An X Bm$ 
    using assms DedekindsAxiom-def by simp
  }

```



```

then obtain  $X$  where  $\forall An Bm. (\exists n. 0 < n \wedge A n An) \wedge (\exists m. B m Bm) \longrightarrow Bet An X Bm$ 
  by blast
{
  fix  $n An Bn$ 
  assume  $A n An$  and  $B n Bn$ 
  have  $0 < n \longrightarrow Bet An X Bn$ 
    using  $\langle A n An \rangle \langle B n Bn \rangle$ 
       $\langle \forall An Bm. (\exists n > 0. A n An) \wedge (\exists m. B m Bm) \longrightarrow Bet An X Bm \rangle$  by auto
  moreover
  {
    assume  $n = 0$ 
    obtain  $A1$  where  $A 1 A1$ 
      using nested--ex-left  $\langle Nested A B \rangle$  by blast
    hence  $Bet An A1 Bn$ 
      using  $\langle A n An \rangle \langle B n Bn \rangle \langle Nested A B \rangle \langle n = 0 \rangle$  nested-aux2 by blast
    hence  $Bet An X Bn$ 
      using  $\langle A 1 A1 \rangle \langle B n Bn \rangle$  between-exchange2
       $\langle \forall An Bm. (\exists n > 0. A n An) \wedge (\exists m. B m Bm) \longrightarrow Bet An X Bm \rangle$  by blast
  }
  ultimately have  $Bet An X Bn$ 
    by blast
}
hence  $\exists X. \forall n An Bn. (A n An \wedge B n Bn) \longrightarrow Bet An X Bn$ 
  by blast
}
thus ?thesis
  using CantorsAxiom-def by blast
qed

end
end

```

theory Tarski-Neutral-Continuous

imports
Tarski-Neutral-Continuity

begin

6 Tarski Neutral Continuous

6.1 Tarski's axiom system for Neutral Continuous

locale Tarski-Neutral-Continuous = Tarski-neutral-dimensionless +

assumes continuity : $\forall Alpha Beta.$

$(\exists A. \forall X Y. (Alpha X \wedge Beta Y) \longrightarrow Bet A X Y) \longrightarrow$

$(\exists B. \forall X Y. (Alpha X \wedge Beta Y) \longrightarrow Bet X B Y)$

context Tarski-Neutral-Continuous

begin

6.2 Definitions

6.3 Propositions

theorem continuity-DedekindAxiom :

shows DedekindsAxiom

using DedekindsAxiom-def continuity by blast

end

end

theory Tarski-Neutral-Archimedes-Continuity

```

imports
  Tarski-Neutral-Archimedes
  Tarski-Neutral-Continuity
begin

context Tarski-neutral-dimensionless

begin

```

7 Archimedes continuity

7.1 Definitions

```

definition AlphaTmp :: 'p  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool
  where
    AlphaTmp A B  $\equiv$   $\lambda$  X. X = A  $\vee$  (A Out B X  $\wedge$  Reach A B A X)

definition BetaTmp :: 'p  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool
  where
    BetaTmp A B  $\equiv$   $\lambda$  X. A Out B X  $\wedge$   $\neg$  Reach A B A X

definition NestedBis :: (nat  $\Rightarrow$  'p  $\Rightarrow$  bool)  $\Rightarrow$  (nat  $\Rightarrow$  'p  $\Rightarrow$  bool) $\Rightarrow$ bool
  where
    NestedBis A B  $\equiv$ 
    ( $\forall$  n.  $\exists$  An Bn. A n An  $\wedge$  B n Bn)  $\wedge$ 
    ( $\forall$  n An An'. A n An  $\wedge$  A n An'  $\longrightarrow$  An = An')  $\wedge$ 
    ( $\forall$  n Bn Bn'. B n Bn  $\wedge$  B n Bn'  $\longrightarrow$  Bn = Bn')  $\wedge$ 
    ( $\forall$  n An Am Bm Bn. A n An  $\wedge$  A (Suc n) Am  $\wedge$  B (Suc n) Bm  $\wedge$  B n Bn  $\longrightarrow$ 
    Bet An Am Bm  $\wedge$  Bet Am Bm Bn  $\wedge$  Am  $\neq$  Bm)

definition CantorVariant :: bool
  where
    CantorVariant  $\equiv$   $\forall$  A B. NestedBis A B  $\longrightarrow$ 
    ( $\exists$  X.  $\forall$  n An Bn. A n An  $\wedge$  B n Bn  $\longrightarrow$  Bet An X Bn)

definition inj :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  bool
  where
    inj f  $\equiv$   $\forall$  A B :: 'p. f A = f B  $\longrightarrow$  A = B

definition pres-bet :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  bool
  where
    pres-bet f  $\equiv$   $\forall$  A B C. Bet A B C  $\longrightarrow$  Bet (f A) (f B) (f C)

definition pres-cong :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  bool
  where
    pres-cong f  $\equiv$   $\forall$  A B C D. Cong A B C D  $\longrightarrow$  Cong (f A) (f B) (f C) (f D)

definition extension :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  bool
  where
    extension f  $\equiv$  inj f  $\wedge$  pres-bet f  $\wedge$  pres-cong f

definition inj-line :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool
  where
    inj-line f P Q  $\equiv$   $\forall$  A B. Col P Q A  $\wedge$  Col P Q B  $\wedge$  f A = f B  $\longrightarrow$  A = B

definition pres-bet-line :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool
  where
    pres-bet-line f P Q  $\equiv$ 
     $\forall$  A B C. Col P Q A  $\wedge$  Col P Q B  $\wedge$  Col P Q C  $\wedge$  Bet A B C
     $\longrightarrow$ 
    Bet (f A) (f B) (f C)

definition pres-cong-line :: ('p  $\Rightarrow$  'p)  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool
  where

```

$\text{pres-cong-line } f P Q \equiv$
 $\forall A B C D. \text{Col } P Q A \wedge \text{Col } P Q B \wedge \text{Col } P Q C \wedge \text{Col } P Q D \wedge \text{Cong } A B C D$
 $\longrightarrow$
 $\text{Cong } (f A) (f B) (f C) (f D)$

definition *line-extension* :: ('p⇒'p)⇒'p⇒'p⇒bool

where

line-extension $f P Q \equiv P \neq Q \wedge \text{inj-line } f P Q \wedge \text{pres-bet-line } f P Q \wedge \text{pres-cong-line } f P Q$

definition *line-completeness* :: bool

where

line-completeness $\equiv \text{archimedes-axiom} \wedge$

$(\forall f. \forall P Q. \text{line-extension } f P Q$

$\longrightarrow$

$(\forall A. \text{Col } (f P) (f Q) A \longrightarrow (\exists B. \text{Col } P Q B \wedge f B = A)))$

7.2 Propositions

lemma *archimedes-aux* :

assumes $\forall A B C. A \text{ Out } B C \longrightarrow \text{Reach } A B A C$

shows *ArchimedesAxiom*

proof –

```

{
  fix A B C D :: 'p
  assume A ≠ B
  have Reach A B C D
  proof (cases C = D)
    case True
    have Grad A B B
    by (simp add: grad-equiv-coq-1)
    moreover have C D Le A B
    by (simp add: True le-trivial)
    ultimately show ?thesis
    using Reach-def by blast
  next
    case False
    hence C ≠ D
    by simp
    obtain E where A Out B E and Cong A E C D
    using False ⟨A ≠ B⟩ segment-construction-3 by blast
    obtain B' where Grad A B B' and A E Le A B'
    using assms(1) Reach-def ⟨A Out B E⟩ by blast
    moreover have C D Le A B'
    using ⟨Cong A E C D⟩ calculation(2) cong--le3412 le-transitivity by blast
    ultimately show ?thesis
    using Reach-def by blast
  qed
}
thus ?thesis
using archimedes-axiom-def by blast
qed

```

lemma *dedekind-variant--archimedes*:

assumes *DedekindVariant*

shows *archimedes-axiom*

proof –

```

{
  fix A B C
  assume A Out B C
  {
    assume ¬ Reach A B A C
    have ∃ X. ∀ P Q. (P = A ∨ (A Out B P ∧ Reach A B A P)) ∧
      (A Out B Q ∧ ¬ Reach A B A Q) ⟶ Bet P X Q
    proof –
      let ?Alpha = (AlphaTmp A B)

```

```

let ?Beta = (BetaTmp A B)
have  $\forall$  Alpha Beta.  $\forall$  A C. (Alpha A  $\wedge$  Beta C  $\wedge$ 
  ( $\forall$  P. A Out P C  $\longrightarrow$  (Alpha P  $\vee$  Beta P))  $\wedge$  ( $\forall$  X Y. (Alpha X  $\wedge$  Beta Y)
 $\longrightarrow$  (Bet A X Y  $\wedge$  X  $\neq$  Y)))  $\longrightarrow$  ( $\exists$  B'.  $\forall$  X Y. (Alpha X  $\wedge$  Beta Y)
 $\longrightarrow$  Bet X B' Y)
  using assms DedekindVariant-def by auto
hence 2:  $\forall$  A C. (?Alpha A  $\wedge$  ?Beta C  $\wedge$  ( $\forall$  P. A Out P C  $\longrightarrow$  (?Alpha P  $\vee$  ?Beta P))  $\wedge$ 
  ( $\forall$  X Y. (?Alpha X  $\wedge$  ?Beta Y)  $\longrightarrow$  (Bet A X Y  $\wedge$  X  $\neq$  Y)))
 $\longrightarrow$  ( $\exists$  B'.  $\forall$  X Y. (?Alpha X  $\wedge$  ?Beta Y)  $\longrightarrow$  Bet X B' Y)
  by simp
have ?Alpha A
  by (simp add: AlphaTmp-def)
moreover have ?Beta C
  by (simp add: BetaTmp-def  $\langle$  A Out B C  $\rangle$   $\langle$   $\neg$  Reach A B A C  $\rangle$ )
moreover have  $\forall$  P. A Out P C  $\longrightarrow$  (?Alpha P  $\vee$  ?Beta P)
  by (meson AlphaTmp-def BetaTmp-def  $\langle$  A Out B C  $\rangle$  l6-6 l6-7)
moreover {
  fix P Q
  assume ?Alpha P and ?Beta Q
  hence P = A  $\vee$  (A Out B P  $\wedge$  Reach A B A P)
    using AlphaTmp-def by metis
  have A Out B Q  $\wedge$   $\neg$  Reach A B A Q
    using BetaTmp-def  $\langle$  BetaTmp A B Q  $\rangle$  by auto
  hence A Out B Q
    by simp
  have  $\neg$  Reach A B A Q
    using  $\langle$  A Out B Q  $\wedge$   $\neg$  Reach A B A Q  $\rangle$  by auto
  have Bet A P Q  $\wedge$  P  $\neq$  Q
  proof -
    have P = A  $\longrightarrow$  ?thesis
      using Out-def  $\langle$  A Out B Q  $\wedge$   $\neg$  Reach A B A Q  $\rangle$  between-trivial2 by auto
    moreover {
      assume A Out B P and Reach A B A P
      have A Out P Q
        using Out-cases  $\langle$  A Out B P  $\rangle$   $\langle$  A Out B Q  $\rangle$  l6-7 by blast
      hence Bet A P Q  $\vee$  Bet A Q P
        by (simp add: Out-def)
      moreover {
        assume Bet A Q P
        obtain B' where Grad A B B' and A P Le A B'
          using Reach-def  $\langle$  Reach A B A P  $\rangle$  by blast
        hence A Q Le A B'
          using  $\langle$  Bet A Q P  $\rangle$  l5-12-a le-transitivity by blast
        hence Reach A B A Q
          using Reach-def  $\langle$  Grad A B B'  $\rangle$  by blast
        hence False
          by (simp add:  $\langle$   $\neg$  Reach A B A Q  $\rangle$ )
        hence Bet A P Q
          by blast
      }
    }
    ultimately have Bet A P Q
      by blast
    moreover have P  $\neq$  Q
      using  $\langle$  Reach A B A P  $\rangle$   $\langle$   $\neg$  Reach A B A Q  $\rangle$  by auto
    ultimately have Bet A P Q  $\wedge$  P  $\neq$  Q
      by simp
  }
  ultimately show ?thesis
    using  $\langle$  P = A  $\vee$  A Out B P  $\wedge$  Reach A B A P  $\rangle$  by fastforce
  qed
}
hence  $\exists$  B'.  $\forall$  X Y. (?Alpha X  $\wedge$  ?Beta Y)  $\longrightarrow$  Bet X B' Y
  using 2 calculation(1) calculation(2) calculation(3) by blast
thus ?thesis
  using AlphaTmp-def BetaTmp-def by simp
qed

```

then obtain X where $\forall P Q. (P = A \vee (A \text{ Out } B P \wedge \text{Reach } A B A P)) \wedge$
 $(A \text{ Out } B Q \wedge \neg \text{Reach } A B A Q) \longrightarrow \text{Bet } P X Q$
 by *blast*
 have $A \neq B$
 using $\langle A \text{ Out } B C \rangle$ *out-diff1* by *blast*
 have $C \neq A$
 using *Out-def* $\langle A \text{ Out } B C \rangle$ by *presburger*
 have $\text{Grad } A B B$
 by (*simp add: grad-equiv-coq-1*)
 have $B = A \vee A \text{ Out } B B \wedge \text{Reach } A B A B$
 using *Reach-def bet--le2313 between-trivial2 grad-equiv-coq-1 out-trivial* by *blast*
 hence $\text{Bet } B X C$
 by (*simp add: \langle A \text{ Out } B C \rangle \langle \neg \text{Reach } A B A C \rangle*
 $\langle \forall P Q. (P = A \vee A \text{ Out } B P \wedge \text{Reach } A B A P) \wedge A \text{ Out } B Q \wedge \neg \text{Reach } A B A Q$
 $\longrightarrow \text{Bet } P X Q \rangle$)
 have $A \text{ Out } B X$
 using $\langle A \text{ Out } B C \rangle \langle \text{Bet } B X C \rangle$ *out-bet-out-1* by *blast*
 have $\text{Bet } A B C \vee \text{Bet } A C B$
 using *Out-def* $\langle A \text{ Out } B C \rangle$ by *auto*
 {
 assume $\text{Bet } A C B$
 have $A C \text{ Le } A B$
 by (*simp add: \langle \text{Bet } A C B \rangle bet--le1213*)
 hence $\text{Reach } A B A C$
 using *Reach-def* $\langle \text{Grad } A B B \rangle$ by *blast*
 hence *False*
 using $\langle \neg \text{Reach } A B A C \rangle$ by *auto*
 }
 hence $\text{Bet } A B C$
 using $\langle \text{Bet } A B C \vee \text{Bet } A C B \rangle$ by *blast*
 {
 assume $\text{Reach } A B A X$
 then obtain $X1$ where $X \text{ Out } C X1$ and $\text{Cong } X X1 A B$
 by (*metis \langle A \neq B \rangle \langle \neg \text{Reach } A B A C \rangle segment-construction-3*)
 have $\text{Bet } A X X1$
 by (*meson Bet-cases \langle \text{Bet } A B C \rangle \langle \text{Bet } B X C \rangle \langle X \text{ Out } C X1 \rangle*
bet-out--bet between-exchange4)
 have $X \text{ Out } X1 C$
 using *Out-cases* $\langle X \text{ Out } C X1 \rangle$ by *blast*
 moreover have $\text{Bet } X1 X C$
 proof –
 have $A \text{ Out } B X1$
 by (*metis \langle A \neq B \rangle \langle \text{Bet } A B C \rangle \langle \text{Bet } A X X1 \rangle \langle \text{Bet } B X C \rangle bet-out*
between-exchange4 between-inner-transitivity)
 moreover
 obtain B' where $\text{Grad } A B B'$ and $A X \text{ Le } A B'$
 using *Reach-def* $\langle \text{Reach } A B A X \rangle$ by *blast*
 obtain $B1'$ where $\text{Bet } A B' B1'$ and $\text{Cong } B' B1' A B$
 using *segment-construction* by *blast*
 hence $\text{Grad } A B B1'$
 using $\langle \text{Grad } A B B' \rangle \langle \text{Grad } A B B \rangle$ *cong-symmetry grad-sum* by *blast*
 moreover have $X X1 \text{ Le } B' B1'$
 by (*meson \langle \text{Cong } B' B1' A B \rangle \langle \text{Cong } X X1 A B \rangle cong--le3412*
cong-reflexivity cong-symmetry l5-6)
 hence $A X1 \text{ Le } A B1'$
 using $\langle A X \text{ Le } A B' \rangle \langle \text{Bet } A B' B1' \rangle \langle \text{Bet } A X X1 \rangle$ *bet2-le2--le1346* by *force*
 ultimately have $\text{Reach } A B A X1$
 using *Reach-def* by *blast*
 show ?thesis
 using $\langle A \text{ Out } B C \rangle \langle \neg \text{Reach } A B A C \rangle \langle A \text{ Out } B X1 \rangle \langle \text{Reach } A B A X1 \rangle$
 $\langle \forall P Q. (P = A \vee A \text{ Out } B P \wedge \text{Reach } A B A P) \wedge A \text{ Out } B Q \wedge \neg \text{Reach } A B A Q$
 $\longrightarrow \text{Bet } P X Q \rangle$ by *blast*
 qed
 ultimately have $\text{Bet } X1 X C \wedge X \text{ Out } X1 C$
 by *simp*
 hence *False*

```

    using not-bet-and-out by blast
  }
  moreover
  {
    assume  $\neg \text{Reach } A \ B \ A \ X$ 
    have  $X \neq B$ 
    using  $\langle A \neq B \rangle \langle B = A \vee A \ \text{Out } B \ B \wedge \text{Reach } A \ B \ A \ B \rangle$  calculation by blast
    have  $X \ A \ \text{Le } A \ B \longrightarrow \text{False}$ 
    by (meson  $\langle \text{Bet } A \ B \ C \rangle \langle \text{Bet } B \ X \ C \rangle \langle X \neq B \rangle$  bet-le-eq between-exchange3
        between-symmetry le-right-comm)
    moreover
    {
      assume  $A \ B \ \text{Le } X \ A$ 
      have  $\text{Bet } A \ B \ X$ 
      using  $\langle \text{Bet } A \ B \ C \rangle \langle \text{Bet } B \ X \ C \rangle$  between-inner-transitivity by blast
      obtain  $X0$  where  $\text{Bet } X \ X0 \ A$  and  $\text{Cong } A \ B \ X \ X0$ 
      using Le-def  $\langle A \ B \ \text{Le } X \ A \rangle$  by blast
      {
        assume  $\neg \text{Reach } A \ B \ A \ X0$ 
        have  $X \ \text{Out } X0 \ B$ 
        by (metis  $\langle A \neq B \rangle \langle \text{Bet } A \ B \ X \rangle \langle \text{Bet } X \ X0 \ A \rangle \langle \text{Cong } A \ B \ X \ X0 \rangle \langle X \neq B \rangle$ 
            bet-out bet-out-1 cong-reverse-identity l6-6 l6-7 not-cong-3421)
        hence False
        by (metis  $\langle A \neq B \rangle \langle \text{Bet } A \ B \ X \rangle \langle \text{Bet } X \ X0 \ A \rangle \langle \text{Cong } A \ B \ X \ X0 \rangle \langle X \ \text{Out } X0 \ B \rangle$ 
             $\langle \forall P \ Q. (P = A \vee A \ \text{Out } B \ P \wedge \text{Reach } A \ B \ A \ P) \wedge A \ \text{Out } B \ Q \wedge \neg \text{Reach } A \ B \ A \ Q$ 
             $\longrightarrow \text{Bet } P \ X \ Q \rangle$ 
             $\langle \neg \text{Reach } A \ B \ A \ X0 \rangle$  bet2--out bet-cong-eq between-symmetry between-trivial2
            not-bet-and-out not-cong-1243)
      }
      hence  $\text{Reach } A \ B \ A \ X0$ 
      by blast
    }
    moreover
    {
      assume  $\text{Reach } A \ B \ A \ X0$ 
      obtain  $B'$  where  $\text{Grad } A \ B \ B'$  and  $A \ X0 \ \text{Le } A \ B'$ 
      using Reach-def calculation by blast
      obtain  $B1'$  where  $\text{Bet } A \ B' \ B1'$  and  $\text{Cong } B' \ B1' \ A \ B$ 
      using segment-construction by blast
      have  $\text{Grad } A \ B \ B1'$ 
      using Cong-cases  $\langle \text{Bet } A \ B' \ B1' \rangle \langle \text{Cong } B' \ B1' \ A \ B \rangle \langle \text{Grad } A \ B \ B' \rangle$ 
        grad-stab by blast
      moreover have  $X0 \ X \ \text{Le } B' \ B1'$ 
      by (meson  $\langle \text{Cong } A \ B \ X \ X0 \rangle \langle \text{Cong } B' \ B1' \ A \ B \rangle$  cong-pseudo-reflexivity
          cong-transitivity l5-6 le-reflexivity)
      hence  $A \ X \ \text{Le } A \ B1'$ 
      using  $\langle A \ X0 \ \text{Le } A \ B' \rangle \langle \text{Bet } A \ B' \ B1' \rangle \langle \text{Bet } X \ X0 \ A \rangle$  bet2-le2--le1346
        between-symmetry by blast
      ultimately have  $\text{Reach } A \ B \ A \ X$ 
      using Reach-def by blast
      hence False
      by (simp add:  $\langle \neg \text{Reach } A \ B \ A \ X \rangle$ )
    }
    ultimately have False
    by blast
  }
  ultimately have False
  using local.le-cases by blast
}
ultimately have False
by blast
}
hence  $\text{Reach } A \ B \ A \ C$ 
by blast
}
thus ?thesis
using archimedes-aux by blast

```

qed

lemma *dedekind--archimedes*:
 assumes *DedekindsAxiom*
 shows *archimedes-axiom*
 using *assms dedekind-equiv dedekind-variant--archimedes* by *blast*

lemma *nested-bis--nested*:
 assumes *NestedBis A B*
 shows *Nested A B*
 using *NestedBis-def Nested-def assms* by *presburger*

lemma *cantor--cantor-variant-1*:
 assumes *CantorsAxiom*
 shows *CantorVariant*
 using *CantorVariant-def CantorsAxiom-def assms nested-bis--nested* by *blast*

lemma *cantor--cantor-variant-2*:
 assumes *CantorVariant*
 shows *CantorsAxiom*

proof –
 {
 fix *A B*
 assume *Nested A B*
 then obtain *fA::nat⇒'p* **where** $\forall x. A x (fA x)$
 by (*metis nested--ex-right nested-sym*)
 obtain *fB::nat⇒'p* **where** $\forall x. B x (fB x)$
 using $\langle \text{Nested } A B \rangle$ *nested--ex-right nested-sym* by *metis*
 let *?f1* = $\lambda n::nat. \lambda An::'p. An = fA n$
 let *?f2* = $\lambda n::nat. \lambda Bn::'p. Bn = fB n$
 have $(\forall n. (\exists An Bn. A n An \wedge B n Bn)) \wedge (\forall n An Am Bm Bn. A n An \wedge A (Suc n) Am \wedge B (Suc n) Bm \wedge B n Bn \longrightarrow (Bet An Am Bm \wedge Bet Am Bm Bn \wedge Am \neq Bm))$
 using $\langle \text{Nested } A B \rangle$ *Nested-def* by *auto*
 have *NestedBis ?f1 ?f2*
proof –
 have $(\forall n. \exists An Bn. ?f1 n An \wedge ?f2 n Bn)$
 by *simp*
 moreover have $(\forall n An An'. ?f1 n An \wedge ?f1 n An' \longrightarrow An = An')$
 by *simp*
 moreover have $(\forall n Bn Bn'. ?f2 n Bn \wedge ?f2 n Bn' \longrightarrow Bn = Bn')$
 by *simp*
 moreover have $(\forall n An Am Bm Bn. ?f1 n An \wedge ?f1 (Suc n) Am \wedge ?f2 (Suc n) Bm \wedge ?f2 n Bn \longrightarrow Bet An Am Bm \wedge Bet Am Bm Bn \wedge Am \neq Bm)$
 using $\langle (\forall n. \exists An Bn. A n An \wedge B n Bn) \wedge (\forall n An Am Bm Bn. A n An \wedge A (Suc n) Am \wedge B (Suc n) Bm \wedge B n Bn \longrightarrow Bet An Am Bm \wedge Bet Am Bm Bn \wedge Am \neq Bm) \rangle$
 $\langle \forall x. A x (fA x) \rangle \langle \forall x. B x (fB x) \rangle$ by *blast*
 ultimately show *?thesis*
 using *NestedBis-def* by *presburger*
 }
 qed
 hence $\exists X. \forall n An Bn. An = fA n \wedge Bn = fB n \longrightarrow Bet An X Bn$
 using *assms(1) CantorVariant-def* by *blast*
 then obtain *X* **where** $\forall n An Bn. An = fA n \wedge Bn = fB n \longrightarrow Bet An X Bn$
 by *blast*
 {
 fix *n An Bn*
 assume *A n An* and *B n Bn*
 have *Bet An (fA (Suc n)) (fB (Suc n))*
 using $\langle (\forall n. \exists An Bn. A n An \wedge B n Bn) \wedge (\forall n An Am Bm Bn. A n An \wedge A (Suc n) Am \wedge B (Suc n) Bm \wedge B n Bn \longrightarrow Bet An Am Bm \wedge Bet Am Bm Bn \wedge Am \neq Bm) \rangle$
 $\langle A n An \rangle \langle \forall x. A x (fA x) \rangle \langle \forall x. B x (fB x) \rangle$ by *blast*
 moreover have *Bet (fA (Suc n)) (fB (Suc n)) Bn*
 using $\langle (\forall n. \exists An Bn. A n An \wedge B n Bn) \wedge$

```

    (∀ n An Am Bm Bn. A n An ∧ A (Suc n) Am ∧ B (Suc n) Bm ∧ B n Bn
    → Bet An Am Bm ∧ Bet Am Bm Bn ∧ Am ≠ Bm) › ⟨B n Bn⟩
  ⟨∀ x. A x (fA x) › ⟨∀ x. B x (fB x) › by blast
  moreover have fA (Suc n) ≠ fB (Suc n)
  using ⟨(∀ n. ∃ An Bn. A n An ∧ B n Bn) ∧
    (∀ n An Am Bm Bn. A n An ∧ A (Suc n) Am ∧ B (Suc n) Bm ∧ B n Bn
    → Bet An Am Bm ∧ Bet Am Bm Bn ∧ Am ≠ Bm) ›
    ⟨∀ x. A x (fA x) › ⟨∀ x. B x (fB x) › by blast
  moreover have Bet (fA (Suc n)) X (fB (Suc n))
  by (simp add: ⟨∀ n An Bn. An = fA n ∧ Bn = fB n → Bet An X Bn⟩)
  ultimately have Bet An X Bn
  by (meson bet3--bet between-exchange4 outer-transitivity-between2)
}
hence ∃ X. ∀ n An Bn. (A n An ∧ B n Bn) → Bet An X Bn
  by blast
}
thus ?thesis
  using CantorsAxiom-def by blast
qed

```

```

lemma cantor--cantor-variant:
  shows CantorsAxiom ⟷ CantorVariant
  using cantor--cantor-variant-1 cantor--cantor-variant-2 by blast

```

```

lemma inj-line-symmetry:
  assumes inj-line f P Q
  shows inj-line f Q P
  by (metis NCol-cases assms inj-line-def)

```

```

lemma pres-bet-line-symmetry:
  assumes pres-bet-line f P Q
  shows pres-bet-line f Q P
  by (metis assms not-col-permutation-4 pres-bet-line-def)

```

```

lemma pres-cong-line-symmetry:
  assumes pres-cong-line f P Q
  shows pres-cong-line f Q P
proof -
  {
    fix A B C D
    assume Col Q P A and Col Q P B and Col Q P C and
      Col Q P D and Cong A B C D
    have Col P Q A
      using NCol-cases ⟨Col Q P A⟩ by blast
    moreover have Col P Q B
      using ⟨Col Q P B⟩ col-permutation-4 by blast
    moreover have Col P Q C
      using NCol-cases ⟨Col Q P C⟩ by blast
    moreover have Col P Q D
      using Col-cases ⟨Col Q P D⟩ by auto
    ultimately have Cong (f A) (f B) (f C) (f D)
      using ⟨Cong A B C D⟩ assms pres-cong-line-def by blast
  }
  thus ?thesis
    using assms not-col-permutation-4 pres-cong-line-def by blast
qed

```

```

lemma line-extension-symmetry:
  assumes line-extension f P Q
  shows line-extension f Q P
proof -
  have Q ≠ P
    using assms line-extension-def by auto
  moreover have inj-line f Q P
    using assms inj-line-symmetry line-extension-def by blast
  moreover have pres-bet-line f Q P

```



```

    by (meson assms line-extension-def pres-bet-line-symmetry)
  moreover have pres-cong-line f Q P
    by (meson assms line-extension-def pres-cong-line-symmetry)
  ultimately show ?thesis
    using line-extension-def by blast
qed

```

```

lemma inj-line-stability:
  assumes Col P Q R and
    P ≠ R and
    inj-line f P Q
  shows inj-line f P R
proof -
  {
    fix A B
    assume Col P R A and Col P R B and f A = f B
    moreover have Col P Q A
      by (meson assms(1) assms(2) calculation(1) colx not-col-distincts)
    moreover have Col P Q B
      using assms(1) assms(2) calculation(2) col-trivial-3 colx by blast
    ultimately have A = B
      using assms(3) inj-line-def by blast
  }
  thus ?thesis
    using inj-line-def by force
qed

```

```

lemma pres-bet-line-stability:
  assumes Col P Q R and
    P ≠ R and
    pres-bet-line f P Q
  shows pres-bet-line f P R
proof -
  {
    fix A B C
    assume Col P R A and Col P R B and Col P R C and Bet A B C
    have Col P Q A
      using ⟨Col P R A⟩ assms(1) assms(2) col-trivial-3 colx by blast
    moreover have Col P Q B
      using ⟨Col P R B⟩ assms(1) assms(2) col-trivial-3 colx by blast
    moreover have Col P Q C
      using ⟨Col P R C⟩ assms(1) assms(2) col-trivial-3 colx by blast
    ultimately have Bet (f A)(f B)(f C)
      using ⟨Bet A B C⟩ assms(3) pres-bet-line-def by blast
  }
  thus ?thesis
    using pres-bet-line-def by force
qed

```

```

lemma pres-cong-line-stability:
  assumes Col P Q R and
    P ≠ R and
    pres-cong-line f P Q
  shows pres-cong-line f P R
proof -
  {
    fix A B C D
    assume Col P R A and Col P R B and Col P R C and Col P R D and Cong A B C D
    have Col P Q A
      using ⟨Col P R A⟩ assms(1) assms(2) col-trivial-3 colx by blast
    moreover have Col P Q B
      using ⟨Col P R B⟩ assms(1) assms(2) col-trivial-3 colx by blast
    moreover have Col P Q C
      by (meson ⟨Col P R C⟩ assms(1) assms(2) colx not-col-distincts)
    moreover have Col P Q D
      by (meson ⟨Col P R D⟩ assms(1) assms(2) colx not-col-distincts)
  }

```

```

ultimately have Cong (f A) (f B) (f C) (f D)
  using ⟨Cong A B C D⟩ assms(3) pres-cong-line-def by blast
}
thus ?thesis
  using pres-cong-line-def by force
qed

```

```

lemma line-extension-stability:
  assumes Col P Q R and
    P ≠ R and
    line-extension f P Q
  shows line-extension f P R
  using assms(1) assms(2) assms(3) inj-line-stability line-extension-def
    pres-bet-line-stability pres-cong-line-stability by auto

```

```

lemma line-extension-reverse-bet:
  assumes line-extension f P Q and
    Col P Q A and
    Col P Q B and
    Col P Q C and
    Bet (f A) (f B) (f C)
  shows Bet A B C
proof -
  have pres-bet-line f P Q
    using assms(1) line-extension-def by auto
  have inj-line f P Q
    using assms(1) line-extension-def by auto
  have Col A B C
    using assms(1) assms(2) assms(3) assms(4) col3 line-extension-def by blast
  hence Bet A B C ∨ Bet B C A ∨ Bet C A B
    by (simp add: Col-def)
  moreover {
    assume Bet B C A
    hence f B = f C
      by (meson assms(1) assms(2) assms(3) assms(4) assms(5) between-equality-2
        between-symmetry line-extension-def pres-bet-line-def)
    hence B = C
      using assms(3) assms(4) inj-line-def ⟨inj-line f P Q⟩ by blast
    hence Bet A B C
      by (simp add: between-trivial)
  }
  moreover {
    assume Bet C A B
    hence f A = f B
      using ⟨pres-bet-line f P Q⟩ assms(2) assms(3) assms(4) assms(5)
        between-equality between-symmetry pres-bet-line-def by blast
    hence A = B
      using ⟨inj-line f P Q⟩ assms(2) assms(3) inj-line-def by blast
    hence Bet A B C
      by (simp add: between-trivial2)
  }
  ultimately show ?thesis
    by blast
qed

```

```

lemma pres-bet-line--col:
  assumes P ≠ Q and
    pres-bet-line f P Q and
    Col P Q A and
    Col P Q B and
    Col P Q C
  shows Col (f A) (f B) (f C)
proof -
  have Col A B C
    using col3 [where ?X=P and ?Y=Q] assms(1) assms(3) assms(4) assms(5) by blast
  thus ?thesis

```

by (meson Col-def assms(2) assms(3) assms(4) assms(5) pres-bet-line-def)
qed

lemma col2-diff-inj-line--diff:
assumes inj-line f P Q and
Col P Q A and
Col P Q B and
A $\neq$ B
shows f A $\neq$ f B
using assms(1) assms(2) assms(3) assms(4) inj-line-def by blast

lemma extension--line-extension:
assumes P $\neq$ Q and
extension f
shows line-extension f P Q
proof -
have inj f
using assms(2) extension-def by auto
have pres-bet f
using assms(2) extension-def by auto
have pres-cong f
using assms(2) extension-def by auto
moreover have inj-line f P Q
using ⟨local.inj f⟩ inj-line-def local.inj-def by blast
moreover have pres-bet-line f P Q
using ⟨pres-bet f⟩ pres-bet-def pres-bet-line-def by auto
moreover have pres-cong-line f P Q
using ⟨pres-cong f⟩ pres-cong-def pres-cong-line-def by auto
ultimately show ?thesis
using line-extension-def assms(1) by blast
qed

lemma extension-reverse-bet:
assumes extension f and
Bet (f A) (f B) (f C)
shows Bet A B C
proof (cases (f A) = (f B))
case True
have inj f
using assms(1) extension-def by auto
hence A = B
using True local.inj-def by blast
then show ?thesis
by (simp add: between-trivial2)
next
case False
hence (f A) $\neq$ (f B)
by blast
have inj f
using assms(1) extension-def by auto
obtain D where Bet A B D and Cong B D B C
using segment-construction by fastforce
have (f C) = (f D)
proof (rule between-cong-3 [where ?A=(f A) and ?B=(f B)])
show f A $\neq$ f B
by (simp add: False)
show Bet (f A) (f B) (f C)
by (simp add: assms(2))
show Bet (f A) (f B) (f D)
using ⟨Bet A B D⟩ assms(1) extension-def pres-bet-def by blast
show Cong (f B) (f C) (f B) (f D)
using ⟨Cong B D B C⟩ assms(1) extension-def not-cong-3412 pres-cong-def by blast
qed
hence C = D
using ⟨local.inj f⟩ local.inj-def by blast
then show ?thesis

by (simp add: $\langle \text{Bet } A \ B \ D \rangle$)
qed

lemma extension-reverse-col:
assumes extension f and
Col (f A) (f B) (f C)
shows Col A B C
proof –
have Bet (f A) (f B) (f C) $\longrightarrow$ Bet A B C
using assms(1) extension-reverse-bet by blast
moreover have Bet (f B) (f C) (f A) $\longrightarrow$ Bet B C A
using assms(1) extension-reverse-bet by blast
moreover have Bet (f C) (f A) (f B) $\longrightarrow$ Bet C A B
using assms(1) extension-reverse-bet by blast
ultimately show ?thesis
using Col-def assms(1) assms(2) extension-reverse-bet by blast
qed

lemma line-completeness-aux:
assumes line-completeness and
archimedes-axiom and
extension f and
 $\neg$ Col P Q R and
Coplanar (f P) (f Q) (f R) A
shows $\exists B. \text{Coplanar } P \ Q \ R \ B \wedge f B = A$
proof –
have archimedes-axiom
using assms(1) line-completeness-def by blast
have $\forall f. \forall P \ Q. \text{line-extension } f \ P \ Q \longrightarrow$
($\forall A. \text{Col } (f \ P) \ (f \ Q) \ A \longrightarrow (\exists B. \text{Col } P \ Q \ B \wedge f B = A)$)
using assms(1) line-completeness-def by blast
{
fix X Y :: 'p
assume $X \neq Y$
hence line-extension f X Y
by (simp add: assms(3) extension--line-extension)
}
obtain S where S Midpoint P Q
using MidR-uniq-aux by blast
show $\exists B. \text{Coplanar } P \ Q \ R \ B \wedge f B = A$
proof (cases Col (f R) (f S) A)
case True
{
assume $R = S$
hence Col P Q R
using $\langle S \text{ Midpoint } P \ Q \rangle$ col-permutation-1 midpoint-col by blast
hence False
by (simp add: assms(4))
}
hence $R \neq S$
by blast
hence $\exists B. \text{Col } R \ S \ B \wedge (f B) = A$
using assms(1) line-completeness-def True $\langle \bigwedge Y \ X. X \neq Y \implies \text{line-extension } f \ X \ Y \rangle$
by presburger
then obtain B where Col R S B and $(f B) = A$
by blast
moreover
hence Coplanar P Q R B
using Coplanar-def Midpoint-def $\langle S \text{ Midpoint } P \ Q \rangle$ bet-col1 between-exchange2
between-symmetry not-col-permutation-5 point-construction-different by meson
ultimately show ?thesis
by blast
next
case False

```

hence  $\neg \text{Col } (f R) (f S) A$ 
  by simp
show ?thesis
proof (cases  $\text{Col } (f P) (f Q) A$ )
  case True
    have  $\exists B. \text{Col } P Q B \wedge (f B) = A$ 
      using assms(1) line-completeness-def by (metis True
         $\langle \bigwedge Y X. X \neq Y \implies \text{line-extension } f X Y \rangle$  assms(4) col-permutation-2 col-trivial-3)
    then obtain B0 where  $\text{Col } P Q B0$  and  $f B0 = A$ 
      by blast
    hence Coplanar P Q R B0
      using ncop--ncols by blast
    thus ?thesis
      using  $\langle f B0 = A \rangle$  by blast
  next
  case False
    hence  $\neg \text{Col } (f P) (f Q) A$ 
      by simp
    hence  $\exists X. \text{Col } A (f S) X \wedge (\text{BetS } (f P) X (f R) \vee \text{BetS } (f Q) X (f R))$ 
      proof -
        have Coplanar (f P) (f Q) (f R) A
          by (simp add: assms(5))
        moreover have  $\neg \text{Col } (f R) (f S) A$ 
          by (simp add:  $\langle \neg \text{Col } (f R) (f S) A \rangle$ )
        moreover have  $\neg \text{Col } (f P) (f Q) A$ 
          by (simp add: False)
        moreover have  $\text{BetS } (f P) (f S)(f Q)$ 
          by (metis BetS-def False Midpoint-def  $\langle S \text{ Midpoint } P Q \rangle$  assms(3) col-trivial-3
            extension-def local.inj-def midpoint-distinct-1 not-col-permutation-5 pres-bet-def)
        ultimately show ?thesis
          by (simp add: hilbert-s-version-of-pasch)
      qed
    then obtain X where  $\text{Col } A (f S) X$  and  $\text{BetS } (f P) X (f R) \vee \text{BetS } (f Q) X (f R)$ 
      by blast
    have  $\exists Y. \text{Coplanar } P Q R Y \wedge (f Y) = X$ 
      proof -
        {
          assume  $\text{BetS } (f P) X (f R)$ 
          hence  $\exists Y. \text{Col } P R Y \wedge (f Y) = X$ 
            using assms(1) line-completeness-def by (metis BetSeq Col-def
               $\langle \bigwedge Y X. X \neq Y \implies \text{line-extension } f X Y \rangle$  between-symmetry)
          hence  $\exists Y. \text{Coplanar } P Q R Y \wedge (f Y) = X$ 
            using ncop--ncols by blast
        }
        moreover {
          assume  $\text{BetS } (f Q) X (f R)$ 
          hence  $\exists Y. \text{Col } Q R Y \wedge (f Y) = X$ 
            using assms(1) line-completeness-def by (metis BetSeq Col-def
               $\langle \bigwedge Y X. X \neq Y \implies \text{line-extension } f X Y \rangle$  between-symmetry)
          hence  $\exists Y. \text{Coplanar } P Q R Y \wedge (f Y) = X$ 
            using ncop--ncols by blast
        }
      }
    ultimately show ?thesis
      using  $\langle \text{BetS } (f P) X (f R) \vee \text{BetS } (f Q) X (f R) \rangle$  by blast
  qed
then obtain Y where Coplanar P Q R Y and  $(f Y) = X$ 
  by blast
{
  assume  $S = Y$ 
  have Bet P S Q
    using Midpoint-def  $\langle S \text{ Midpoint } P Q \rangle$  by auto
  hence Bet (f P) (f S) (f Q)
    using assms(3) extension-def pres-bet-def by blast
  have BetS (f P) (f S) (f R)  $\vee$  BetS (f Q) (f S) (f R)
    by (simp add:  $\langle \text{BetS } (f P) X (f R) \vee \text{BetS } (f Q) X (f R) \rangle$   $\langle S = Y \rangle$   $\langle f Y = X \rangle$ )
  moreover

```

```

have BetS (f P) (f S) (f R) → Col (f P) (f Q) (f R)
  using BetSEq Col-def ⟨Bet (f P) (f S) (f Q)⟩ between-symmetry l5-1 by blast
moreover {
  assume BetS (f Q) (f S) (f R)
  have f S ≠ f Q
    by (metis BetSEq ⟨BetS (f Q) (f S) (f R)⟩)
  moreover have Col (f S) (f Q) (f P)
    by (simp add: Col-def ⟨Bet (f P) (f S) (f Q)⟩)
  moreover have Col (f S) (f Q) (f R)
    using BetSEq Bet-cases Col-def ⟨BetS (f Q) (f S) (f R)⟩ by blast
  ultimately have Col (f P) (f Q) (f R)
    using col3 col-trivial-2 by blast
}
ultimately have Col (f P) (f Q) (f R)
  by blast
hence Col P Q R
  using assms(3) extension-reverse-col by blast
hence False
  by (simp add: assms(4))
}
hence S ≠ Y
  by blast
obtain B where Col S Y B and (f B) = A
  using assms(1) line-completeness-def Col-cases ⟨Col A (f S) X⟩ ⟨S = Y ⇒ False⟩
  ⟨∧ Y X. X ≠ Y ⇒ line-extension f X Y⟩ ⟨f Y = X⟩ by blast
have Coplanar P Q R S
  using Col-def Midpoint-def ⟨S Midpoint P Q⟩ between-symmetry ncop--ncols by blast
hence Coplanar P Q R B
  using col-cop2--cop [where ?U = S and ?V = Y] ⟨Col S Y B⟩ ⟨Coplanar P Q R Y⟩
  ⟨S ≠ Y⟩ by blast
thus ?thesis
  using ⟨f B = A⟩ by blast
qed
qed
qed

```

```

lemma segment-circle--one-point-line-circle-R1:
  assumes SegmentCircle
  shows OnePointLineCircle
proof -
{
  fix A B U V P
  assume Col U V P and U ≠ V and Bet A P B
  have ∃ Z. Col U V Z ∧ Z OnCircle A B
  proof (cases A = B)
    case True
    thus ?thesis
      using ⟨Bet A P B⟩ ⟨Col U V P⟩ between-identity onc212 by blast
  next
    case False
    hence A ≠ B
    by simp
    have P InCircle A B
    by (simp add: InCircle-def ⟨Bet A P B⟩ bet--le1213)
    obtain W where U ≠ W and V ≠ W and P ≠ W and Col U V W
    using ⟨Col U V P⟩ diff-col-ex3 by blast
    show ?thesis
    proof (cases A = P)
      case True
      obtain Q where Bet W A Q and Cong A Q A B
      using segment-construction by blast
      hence Q OutCircle A B

```

```

    using OnCircle-def onc--outc by blast
  then obtain Z where Bet A Z Q and Z OnCircle A B
    using segment-circle-def True ⟨P InCircle A B⟩ assms by blast
  have Col U V Z
    by (metis Col-def OnCircle-def True ⟨Bet A Z Q⟩ ⟨Bet W A Q⟩
      ⟨Col U V P⟩ ⟨Col U V W⟩ ⟨Cong A Q A B⟩ ⟨P ≠ W⟩ ⟨Z OnCircle A B⟩
      bet-cong-eq between-symmetry between-trivial col-transitivity-1)
  thus ?thesis
    using ⟨Z OnCircle A B⟩ by blast
next
case False
hence A ≠ P
  by simp
obtain Q0 where Bet W P Q0 and Cong P Q0 P A
  using segment-construction by blast
obtain Q where Bet P Q0 Q and Cong Q0 Q A B
  using segment-construction by blast
have P Out Q Q0
  by (metis False ⟨Bet P Q0 Q⟩ ⟨Cong P Q0 P A⟩ bet-out cong-reverse-identity l6-6)
hence Q Q0 Le Q A
  using ⟨Cong P Q0 P A⟩ not-cong-3412 triangle-reverse-inequality by blast
hence Q OutCircle A B
  using l5-6 by (meson OutCircle-def ⟨Cong Q0 Q A B⟩
    cong-pseudo-reflexivity not-cong-4312)
then obtain Z where Bet P Z Q and Z OnCircle A B
  using segment-circle-def ⟨P InCircle A B⟩ assms by blast
hence Col U V Z
  by (metis Col-def Out-def ⟨Bet W P Q0⟩ ⟨Col U V P⟩
    ⟨Col U V W⟩ ⟨P Out Q Q0⟩ ⟨P ≠ W⟩ colx)
then show ?thesis
  using ⟨Z OnCircle A B⟩ by blast
qed
qed
}
thus ?thesis
  using one-point-line-circle-def by blast
qed

```

lemma segment-circle--one-point-line-circle-R2:

assumes OnePointLineCircle

shows SegmentCircle

proof –

```

{
  fix A B P Q
  assume P InCircle A B and
    Q OutCircle A B
  have ∃ Z. Bet P Z Q ∧ Z OnCircle A B
  proof (cases A = B)
    case True
    thus ?thesis
      using ⟨P InCircle A B⟩ between-trivial2 inc-eq onc212 by blast
  next
    case False
    hence A ≠ B
      by simp
    show ?thesis
    proof (cases P = Q)
      case True
      then show ?thesis
        using ⟨P InCircle A B⟩ ⟨Q OutCircle A B⟩ inc-outc--onc not-bet-distincts by blast
    next
      case False
      hence P ≠ Q
        by simp
      show ?thesis
      proof (cases Cong A B A Q)

```

```

case True
then show ?thesis
  by (meson OnCircle-def not-bet-distincts onc-sym)
next
case False
hence  $\neg \text{Cong } A \ B \ A \ Q$ 
  by simp
then obtain B' where Cong A B A B' and Bet A P B'
  using OnCircle-def  $\langle P \text{ InCircle } A \ B \rangle$  inc--radius onc-sym by blast
obtain Z1 where Col P Q Z1 and Z1 OnCircle A B'
  using assms one-point-line-circle-def
  by (meson  $\langle \text{Bet } A \ P \ B' \rangle$   $\langle P \neq Q \rangle$  not-col-distincts)
have Z1 OnCircle A B
  using EqC-def  $\langle \text{Cong } A \ B \ A \ B' \rangle$   $\langle Z1 \text{ OnCircle } A \ B' \rangle$  eqc-chara-2 by presburger
have Bet P Z1 Q  $\longrightarrow$  ?thesis
  using  $\langle Z1 \text{ OnCircle } A \ B \rangle$  by auto
moreover
{
  assume Z1 Out P Q
  obtain Z2 where Z2 OnCircle A B and Bet Z1 P Z2
    using  $\langle P \text{ InCircle } A \ B \rangle$   $\langle Z1 \text{ OnCircle } A \ B \rangle$  chord-completion by blast
  have Bet Z1 Z2 Q
  proof -
    have Z1 Out Z2 Q
      by (metis Out-def  $\langle \text{Bet } Z1 \ P \ Z2 \rangle$   $\langle Z1 \text{ Out } P \ Q \rangle$  bet-neq12--neq l6-7)
    moreover have Q Out Z1 Z2
    proof (rule col-inc2-outcs--out [where ?A=A and ?B=B])
      show Z1 InCircle A B
        by (simp add:  $\langle Z1 \text{ OnCircle } A \ B \rangle$  onc--inc)
      show Z2 InCircle A B
        using  $\langle Z2 \text{ OnCircle } A \ B \rangle$  onc--inc by auto
      show Col Z1 Z2 Q
        by (simp add: calculation out-col)
      show Q OutCircleS A B
        using False OnCircle-def  $\langle Q \text{ OutCircle } A \ B \rangle$  circle-cases
        not-cong-3412 outc--nincs-1 by blast
    qed
    ultimately show ?thesis
      using out2--bet by blast
  qed
  hence Bet P Z2 Q
    using  $\langle \text{Bet } Z1 \ P \ Z2 \rangle$  between-exchange3 by blast
  hence ?thesis
    using  $\langle Z2 \text{ OnCircle } A \ B \rangle$  by blast
}
moreover have  $\neg \text{Col } P \ Z1 \ Q \longrightarrow$  ?thesis
  using Col-perm  $\langle \text{Col } P \ Q \ Z1 \rangle$  by blast
ultimately
show ?thesis
  using l6-4-2 by blast
qed
qed
qed
}
thus ?thesis
  by (simp add: segment-circle-def)
qed

```

```

lemma segment-circle--one-point-line-circle:
shows SegmentCircle  $\longleftrightarrow$  OnePointLineCircle
using segment-circle--one-point-line-circle-R1
  segment-circle--one-point-line-circle-R2 by blast

```

```

lemma one-point-line-circle--two-points-line-circle-R1 :
assumes one-point-line-circle
shows two-points-line-circle

```



```

proof -
{
  fix A B U V P
  assume Col U V P and U ≠ V and Bet A P B
  have ∃ Z1 Z2. Col U V Z1 ∧ Z1 OnCircle A B ∧
    Col U V Z2 ∧ Z2 OnCircle A B ∧
    Bet Z1 P Z2 ∧ (P ≠ B ⟶ Z1 ≠ Z2)
  proof (cases P = B)
    case True
    then show ?thesis
      using ⟨Col U V P⟩ between-trivial2 onc212 by blast
  next
    case False
    hence P ≠ B
    by simp
    obtain Z1 where Col U V Z1 and Z1 OnCircle A B
    using ⟨Bet A P B⟩ ⟨Col U V P⟩ ⟨U ≠ V⟩ assms one-point-line-circle-def by blast
    have P InCircle A B
    using ⟨Bet A P B⟩ bet-inc2--inc inc112 onc212 onc--inc by blast
    then obtain Z2 where Z2 OnCircle A B and Bet Z1 P Z2
    using chord-completion ⟨Z1 OnCircle A B⟩ by blast
    have Z1 ≠ P
    using False ⟨Bet A P B⟩ ⟨Z1 OnCircle A B⟩ between-cong onc212 onc2--cong by blast
    thus ?thesis
    by (metis ⟨Col U V P⟩ ⟨Col U V Z1⟩ ⟨Z1 OnCircle A B⟩
      ⟨∧thesis. (∧Z2. [Z2 OnCircle A B ; Bet Z1 P Z2] ⟹ thesis) ⟹ thesis⟩
      bet-col between-equality colx outer-transitivity-between point-construction-different)
  qed
}
thus ?thesis
  using two-points-line-circle-def by blast
qed

```

```

lemma one-point-line-circle--two-points-line-circle-R2 :
  assumes two-points-line-circle
  shows one-point-line-circle
  using assms two-points-line-circle-def one-point-line-circle-def by blast

```

```

lemma one-point-line-circle--two-points-line-circle :
  shows one-point-line-circle ⟷ two-points-line-circle
  using one-point-line-circle--two-points-line-circle-R1
    one-point-line-circle--two-points-line-circle-R2 by blast

```

```

lemma circle-circle-bis--circle-circle-axiom-R1:
  assumes circle-circle-bis
  shows circle-circle-axiom

```

```

proof -
{
  fix A B C D B' D'
  assume Cong A B' A B and Cong C D' C D and
    Bet A D' B and Bet C B' D
  have ∃ Z. Z OnCircle A B ∧ Z OnCircle C D
  proof -
    have D' OnCircle C D
    by (simp add: OnCircle-def ⟨Cong C D' C D⟩)
    moreover have D' InCircle A B
    using ⟨Bet A D' B⟩ bet-inc2--inc inc112 inc--outc incs--inc outc--nincs by blast
    moreover have B' OnCircle A B
    by (simp add: OnCircle-def ⟨Cong A B' A B⟩)
    moreover have B' InCircle C D
    using ⟨Bet C B' D⟩ bet-inc2--inc inc112 onc212 onc--inc by blast
    ultimately show ?thesis
    using assms circle-circle-bis-def by blast
  qed
  hence ∃ Z. Cong A Z A B ∧ Cong C Z C D
  using OnCircle-def by blast

```

```

}
thus ?thesis
  using circle-circle-axiom-def by blast
qed

lemma circle-circle-bis--circle-circle-axiom-R2:
  assumes circle-circle-axiom
  shows circle-circle-bis
proof -
{
  fix A B C D P Q
  assume P OnCircle C D and P InCircle A B and Q OnCircle A B and Q InCircle C D
  hence A P Le A Q
    using InCircle-def OnCircle-def cong-reflexivity l5-6 onc-sym by blast
  then obtain Q' where Bet A P Q' and Cong A Q' A Q
    using l5-5-1 by blast
  have C Q Le C P
    using InCircle-def OnCircle-def ⟨P OnCircle C D⟩ ⟨Q InCircle C D⟩
      cong--le3412 le-transitivity by blast
  then obtain P' where Bet C Q P' and Cong C P' C P
    using l5-5-1 by blast
  have Cong A Q A Q'
    using ⟨Cong A Q' A Q⟩ not-cong-3412 by blast
  moreover have Cong C P C P'
    by (simp add: ⟨Cong C P' C P⟩ cong-symmetry)
  ultimately obtain Z where Cong A Z A Q' and Cong C Z C P'
    using ⟨Bet A P Q'⟩ ⟨Bet C Q P'⟩ assms circle-circle-axiom-def by blast
  have Z OnCircle A B
    using OnCircle-def ⟨Cong A Q' A Q⟩ ⟨Cong A Z A Q'⟩
      ⟨Q OnCircle A B⟩ cong-transitivity by blast
  moreover have Z OnCircle C D
    using OnCircle-def ⟨Cong C P' C P⟩ ⟨Cong C Z C P'⟩
      ⟨P OnCircle C D⟩ cong-transitivity by blast
  ultimately have ∃ Z. Z OnCircle A B ∧ Z OnCircle C D
    by blast
}
thus ?thesis
  using circle-circle-bis-def by blast
qed

lemma circle-circle-bis--circle-circle-axiom:
  shows circle-circle-bis ⟷ circle-circle-axiom
  using circle-circle-bis--circle-circle-axiom-R1
    circle-circle-bis--circle-circle-axiom-R2 by blast

lemma circle-circle--circle-circle-bis:
  assumes circle-circle
  shows circle-circle-bis
proof -
{
  fix A B C D P Q
  assume P OnCircle C D and P InCircle A B and Q OnCircle A B and Q InCircle C D
  have ∃ Z. Z OnCircle A B ∧ Z OnCircle C D
    proof (cases P = Q)
    case True
    thus ?thesis
      using ⟨P OnCircle C D⟩ ⟨Q OnCircle A B⟩ by blast
    next
    case False
    hence P ≠ Q
      by simp
    obtain P' where P' OnCircle C D and Bet P Q P'
      using ⟨P OnCircle C D⟩ ⟨Q InCircle C D⟩ chord-completion by blast
    obtain Q' where Q' OnCircle A B and Bet Q P Q'
      using ⟨P InCircle A B⟩ ⟨Q OnCircle A B⟩ chord-completion by blast
    have P' OutCircle A B

```

```

    by (metis False ⟨Bet P Q P'⟩ ⟨P InCircle A B⟩ ⟨Q OnCircle A B⟩
        bet-inc2--incs incs--inc onc--outc outc--nincs)
  thus ?thesis
    using ⟨P InCircle A B⟩ ⟨P OnCircle C D⟩ ⟨P' OnCircle C D⟩
        assms circle-circle-def by blast
qed
}
thus ?thesis
  using circle-circle-bis-def by blast
qed

lemma circle-circle-bis--one-point-line-circle-aux:
  assumes circle-circle-bis and
    Col U V P and U ≠ V and Bet A P B and ¬ Per A U V
  shows ∃ Z. Col U V Z ∧ Z OnCircle A B
proof (cases Col U V B)
  case True
  thus ?thesis
    using onc212 by blast
next
  case False
  hence ¬ Col U V B
  by simp
  show ?thesis
  proof (cases A = P)
    case True
    obtain W where U ≠ W and V ≠ W and A ≠ W and Col U V W
    using assms(2) diff-col-ex3 by blast
    obtain Z where Bet W A Z and Cong A Z A B
    using segment-construction by blast
    hence Col U V Z
    by (metis True ⟨A ≠ W⟩ ⟨Col U V W⟩ assms(2) bet-col col3 not-col-distincts)
    moreover have Z OnCircle A B
    by (simp add: OnCircle-def ⟨Cong A Z A B⟩)
    ultimately show ?thesis
    by blast
  next
    case False
    hence A ≠ P
    by simp
    obtain C where A C Reflect U V
    using l10-6-existence by blast
    obtain D where B D Reflect U V
    using l10-6-existence by blast
    hence Bet C P D
    using ⟨A C Reflect U V⟩ assms(4) assms(2) assms(3) image-gen-preserves-bet
        l10-6-existence local.image-id by blast
    have Cong B P D P
    using ⟨B D Reflect U V⟩ assms(3) assms(2) is-image-col-cong by blast
    have Cong P D P B
    using Cong-cases ⟨Cong B P D P⟩ by blast
    hence D InCircle A B
    using triangle-inequality InCircle-def assms(4) by blast
    have Cong P B P D
    using ⟨Cong B P D P⟩ not-cong-2143 by blast
    hence B InCircle A B
    by (simp add: onc212 onc--inc)
    have B InCircle C D
    using InCircle-def ⟨Bet C P D⟩ ⟨Cong P B P D⟩ triangle-inequality by blast
    then obtain Z0 where Z0 OnCircle A B and Z0 OnCircle C D
    using assms(1) circle-circle-bis-def ⟨D InCircle A B⟩ onc212 by blast
    then obtain Z where Z OnCircle A B and Z OnCircle C D and Coplanar A C U Z
    using circle-circle-cop by blast
    have Col U V Z
  proof –
    {

```

```

    assume  $A = C$ 
    hence  $A \text{ Reflect } U \text{ } V$ 
    using  $\langle A \text{ } C \text{ Reflect } U \text{ } V \rangle$  by auto
    hence  $Col \text{ } A \text{ } U \text{ } V$ 
    by (simp add: l10-8)
    hence False
    by (metis False  $\neg \text{ } Col \text{ } U \text{ } V \text{ } B$  assms(2) assms(3) assms(4) bet-col
        col-permutation-1 col-transitivity-1)
  }
  moreover have  $Cong \text{ } A \text{ } B \text{ } C \text{ } D$ 
  by (meson  $\langle B \text{ } D \text{ Reflect } U \text{ } V \rangle \langle A \text{ } C \text{ Reflect } U \text{ } V \rangle$  l10-10 not-cong-3412)
  hence  $Cong \text{ } A \text{ } B \text{ } C \text{ } Z$ 
  using OnCircle-def  $\langle Z \text{ OnCircle } C \text{ } D \rangle$  cong-symmetry cong-transitivity by blast
  hence  $Cong \text{ } A \text{ } Z \text{ } C \text{ } Z$ 
  using OnCircle-def  $\langle Z \text{ OnCircle } A \text{ } B \rangle$  cong-inner-transitivity onc-sym by blast
  moreover
  {
    assume  $Col \text{ } C \text{ } U \text{ } A$ 
    have  $A = C \longrightarrow False$ 
    using calculation(1) by blast
    moreover
    {
      assume  $A \neq C$ 
      have  $A \text{ } C \text{ ReflectL } U \text{ } V$ 
      using assms(3)  $\langle A \text{ } C \text{ Reflect } U \text{ } V \rangle$  is-image-is-image-spec by blast
      hence  $U \text{ ReflectLAt } A \text{ } C \text{ } U \text{ } V$ 
      by (meson  $\langle A \neq C \rangle \langle Col \text{ } C \text{ } U \text{ } A \rangle$  image-image-in not-col-distincts
          not-col-permutation-3)
      hence  $U \text{ } V \text{ Perp } C \text{ } A \vee C = A$ 
      using ReflectLAt-def by blast
      hence  $U \text{ } V \text{ Perp } C \text{ } A$ 
      using  $\langle A \neq C \rangle$  by blast
      hence False
      proof (cases  $A = U$ )
      case True
      then show ?thesis
      using False  $\neg \text{ } Col \text{ } U \text{ } V \text{ } B$  assms(2) assms(4) bet-col colx
          not-col-distincts by blast
      next
      case False
      have  $U \text{ } A \text{ Perp } U \text{ } V$ 
      by (metis False Perp-cases  $\langle A \neq C \rangle \langle Col \text{ } C \text{ } U \text{ } A \rangle \langle U \text{ } V \text{ Perp } C \text{ } A \vee C = A \rangle$ 
          not-col-permutation-1 perp-col1)
      thus ?thesis
      using assms(5) perp-per-2 by auto
    }
    qed
  }
  ultimately have False
  by blast
}
hence  $\neg \text{ } Col \text{ } C \text{ } U \text{ } A$ 
by blast
have Coplanar  $C \text{ } U \text{ } A \text{ } V$ 
using  $\langle A \text{ } C \text{ Reflect } U \text{ } V \rangle$  coplanar-perm-8 reflect--coplanar by blast
have Coplanar  $C \text{ } U \text{ } A \text{ } Z$ 
by (simp add:  $\langle Coplanar \text{ } A \text{ } C \text{ } U \text{ } Z \rangle$  coplanar-perm-8)
hence Coplanar  $U \text{ } A \text{ } V \text{ } Z$ 
using  $\langle Coplanar \text{ } C \text{ } U \text{ } A \text{ } V \rangle \neg \text{ } Col \text{ } C \text{ } U \text{ } A$  coplanar-trans-1 by blast
hence Coplanar  $U \text{ } V \text{ } A \text{ } Z$ 
using ncoplanar-perm-2 by blast
ultimately show ?thesis
using  $\langle A \text{ } C \text{ Reflect } U \text{ } V \rangle$  cong-cop-image--col by blast
qed
thus ?thesis
using  $\langle Z \text{ OnCircle } A \text{ } B \rangle$  by blast
qed

```

qed

lemma *circle-circle-bis--one-point-line-circle*:

assumes *circle-circle-bis*

shows *one-point-line-circle*

proof –

```

{
  fix A B U V P
  assume Col U V P and U ≠ V and Bet A P B
  have ∃ Z. Col U V Z ∧ Z OnCircle A B
  proof (cases Per A U V)
    case True
    hence ¬ Per A V U
    using ⟨U ≠ V⟩ l8-7 by blast
    then obtain Z where Col V U Z ∧ Z OnCircle A B
    using Col-cases ⟨Bet A P B⟩ ⟨Col U V P⟩ ⟨U ≠ V⟩
      circle-circle-bis--one-point-line-circle-aux assms(1) by blast
    thus ?thesis
    using Col-cases by blast
  next
  case False
  thus ?thesis
    using ⟨Bet A P B⟩ ⟨Col U V P⟩ ⟨U ≠ V⟩ assms(1)
      circle-circle-bis--one-point-line-circle-aux by auto
  qed
}
thus ?thesis
  using one-point-line-circle-def by blast
qed

```

lemma *circle-circle--circle-circle-two-R1* :

assumes *circle-circle*

shows *circle-circle-two*

proof –

```

{
  fix A B C D P Q
  assume P OnCircle C D and Q OnCircle C D and
    P InCircle A B and Q OutCircle A B
  have ∃ Z1 Z2. Z1 OnCircle A B ∧ Z1 OnCircle C D ∧
    Z2 OnCircle A B ∧ Z2 OnCircle C D ∧
    ((P InCircleS A B ∧ Q OutCircleS A B) → Z1 ≠ Z2)
  proof (cases A = C)
    case True
    have Cong A B A D
    proof –
      have A B Le A D
    proof –
      have A B Le A Q
      using OutCircle-def ⟨Q OutCircle A B⟩ by blast
      moreover have Cong A B A B
      by (simp add: cong-reflexivity)
      moreover have Cong A Q A D
      using OnCircle-def True ⟨Q OnCircle C D⟩ by blast
      ultimately show ?thesis
      using l5-6 by blast
    qed
    moreover have A D Le A B
    proof –
      have A P Le A B
      using InCircle-def ⟨P InCircle A B⟩ by blast
      moreover have Cong A P A D
      using OnCircle-def True ⟨P OnCircle C D⟩ by auto
      moreover have Cong A B A B
      by (simp add: cong-reflexivity)
      ultimately show ?thesis
      using l5-6 by blast
    qed
  qed
}

```

```

qed
ultimately show ?thesis
  using le-anti-symmetry by blast
qed
have P OnCircle A B
  using EqC-def True ⟨Cong A B A D⟩ ⟨P OnCircle C D⟩ eqc-chara-2 by presburger
moreover have P OnCircle C D
  using ⟨P OnCircle C D⟩ by auto
moreover have (P InCircleS A B ∧ Q OutCircleS A B) → P ≠ P
  using calculation(1) onc--outc outc--nincs by blast
ultimately show ?thesis
  by blast
next
case False
obtain Z1 where Z1 OnCircle A B and Z1 OnCircle C D
  using ⟨P InCircle A B⟩ ⟨P OnCircle C D⟩ ⟨Q OnCircle C D⟩ ⟨Q OutCircle A B⟩
    assms circle-circle-def by blast
obtain Z2 where Z1 Z2 Reflect A C
  using l10-6-existence by fastforce
have Cong Z1 C Z2 C
  by (metis False ⟨Z1 Z2 Reflect A C⟩ image-triv is-image-col-cong is-image-rev l10-8)
have Cong Z1 A Z2 A
proof (rule is-image-col-cong [where ?A=A and ?B=C])
  show A ≠ C
    by (simp add: False)
  show Z1 Z2 Reflect A C
    by (simp add: ⟨Z1 Z2 Reflect A C⟩)
  show Col A C A
    by (simp add: col-trivial-3)
qed
have Z2 OnCircle A B
  by (meson OnCircle-def ⟨Cong Z1 A Z2 A⟩ ⟨Z1 OnCircle A B⟩
    cong-transitivity not-cong-4321)
moreover have Z2 OnCircle C D
  by (meson OnCircle-def ⟨Cong Z1 C Z2 C⟩ ⟨Z1 OnCircle C D⟩
    cong-transitivity not-cong-4321)
moreover
{
  assume P InCircleS A B and Q OutCircleS A B
  assume Z1 = Z2
  {
    assume Bet A C Z1
    have Cong C Q C Z1
      using ⟨Q OnCircle C D⟩ ⟨Z1 OnCircle C D⟩ onc2--cong by blast
    hence A Q Le A Z1
      using ⟨Bet A C Z1⟩ triangle-inequality by auto
    hence A Q Le A B
      using InCircle-def ⟨Z1 OnCircle A B⟩ le-transitivity onc--inc by blast
    hence False
      using InCircle-def ⟨Q OutCircleS A B⟩ outcs--ninc-1 by blast
  }
moreover
{
  assume C Out A Z1
  have Cong C P C Z1
    using ⟨P OnCircle C D⟩ ⟨Z1 OnCircle C D⟩ onc2--cong by force
  hence A Z1 Le A P
    using ⟨C Out A Z1⟩ triangle-reverse-inequality by blast
  hence A B Le A P
    using OutCircle-def ⟨Z1 OnCircle A B⟩ le-transitivity onc--outc by blast
  hence False
    using InCircle-def ⟨P InCircleS A B⟩ inc--outc outc--nincs-1 by blast
  }
moreover
{
  assume ¬ Col A C Z1

```

```

    have Col Z1 A C
      using l10-8 ⟨Z1 = Z2⟩ ⟨Z1 Z2 Reflect A C⟩ by blast
    hence False
      using NCol-perm ⟨¬ Col A C Z1⟩ by blast
  }
  ultimately have False
    using not-bet-out by blast
}
ultimately show ?thesis
  using ⟨Z1 OnCircle A B⟩ ⟨Z1 OnCircle C D⟩ by blast
qed
}
thus ?thesis
  using circle-circle-two-def by blast
qed

lemma circle-circle--circle-circle-two-R2 :
  assumes circle-circle-two
  shows circle-circle
proof -
{
  fix A B C D P Q
  assume P OnCircle C D and Q OnCircle C D and
    P InCircle A B and Q OutCircle A B
  then obtain Z1 Z2 where Z1 OnCircle A B ∧ Z1 OnCircle C D ∧
    Z2 OnCircle A B ∧ Z2 OnCircle C D ∧
    ((P InCircleS A B ∧ Q OutCircleS A B) ⟶ Z1 ≠ Z2)
    using assms circle-circle-two-def by blast
  have Z1 OnCircle A B
    by (simp add: ⟨Z1 OnCircle A B ∧ Z1 OnCircle C D ∧ Z2 OnCircle A B ∧
      Z2 OnCircle C D ∧ (P InCircleS A B ∧ Q OutCircleS A B ⟶ Z1 ≠ Z2)⟩)
  moreover have Z1 OnCircle C D
    by (simp add: ⟨Z1 OnCircle A B ∧ Z1 OnCircle C D ∧ Z2 OnCircle A B ∧
      Z2 OnCircle C D ∧ (P InCircleS A B ∧ Q OutCircleS A B ⟶ Z1 ≠ Z2)⟩)
  ultimately have ∃ Z. Z OnCircle A B ∧ Z OnCircle C D
    by blast
}
thus ?thesis
  using circle-circle-def by blast
qed

lemma circle-circle--circle-circle-two :
  shows circle-circle ⟷ circle-circle-two
  using circle-circle--circle-circle-two-R1 circle-circle--circle-circle-two-R2 by blast

lemma euclid-22-aux:
  assumes A B C D SumS E' F' and
    C D E F SumS A' B' and
    E F Le E' F' and
    A B Le A' B' and
    A Out B C1 and
    Cong A C1 C D and
    Bet B A C2 and
    Cong A C2 C D and
    B Out A E1 and
    Cong B E1 E F
  shows Bet C1 E1 C2
proof -
  have Bet C1 A C2
    using assms(5) assms(7) bet-out--bet by blast
  have Col C1 E1 C2
    by (metis Out-def assms(5) assms(7) assms(9) bet-col col3 not-col-permutation-4 out-col)
  moreover
  {
    assume E1 Out C1 C2
    {

```

```

assume Bet E1 C1 C2
have Bet A C1 E1
  using  $\langle \text{Bet } C1 A C2 \rangle \langle \text{Bet } E1 C1 C2 \rangle$  between-inner-transitivity
    between-symmetry by blast
have Bet A E1 B
  by (meson Out-def  $\langle \text{Bet } A C1 E1 \rangle$  assms(5) assms(9) between-inner-transitivity
    between-symmetry not-bet-and-out)
have  $A' B' Lt A B$ 
proof –
  have  $C D Lt A E1$ 
    by (metis Out-def  $\langle \text{Bet } A C1 E1 \rangle \langle E1 Out C1 C2 \rangle$  assms(6) bet--lt1213
      cong2-lt--lt cong-reflexivity)
  moreover have  $E F Le E1 B$ 
    using Cong-cases assms(10) cong--le3412 by blast
  moreover have  $C D E F SumS A' B'$ 
    by (simp add: assms(2))
  moreover have  $A E1 E1 B SumS A B$ 
    by (simp add:  $\langle \text{Bet } A E1 B \rangle$  bet--sums)
  ultimately show ?thesis
    using le-lt12-sums2--lt by blast
qed
hence False
  using assms(4) lt--nle by force
}
moreover {
  assume Bet E1 C2 C1
  hence Bet A C2 E1
    using Bet-cases  $\langle \text{Bet } C1 A C2 \rangle$  between-exchange3 by blast
  hence Bet B C2 E1
    by (metis  $\langle \text{Bet } E1 C2 C1 \rangle$  assms(6) assms(7) assms(8) calculation cong-diff-4
      cong-identity outer-transitivity-between2)
  have  $B C2 Lt B E1$ 
    using Out-def  $\langle \text{Bet } B C2 E1 \rangle \langle E1 Out C1 C2 \rangle$  bet--lt1213 by auto
  moreover have  $Cong B C2 E' F'$ 
    using SumS-def assms(1) assms(7) assms(8) cong-pseudo-reflexivity
      cong-reflexivity sums2--cong56 by blast
  ultimately have  $E' F' Lt E F$ 
    using assms(10) cong2-lt--lt by blast
  hence False
    by (simp add: assms(3) le--nlt)
}
ultimately have False
  using Out-def  $\langle E1 Out C1 C2 \rangle$  by presburger
}
ultimately show ?thesis
  using not-out-bet by blast
qed

lemma circle-circle-bis--euclid-22:
  assumes circle-circle-bis
  shows euclid-s-prop-1-22
proof –
{
  fix  $A B C D E F A' B' C' D' E' F'$ 
  assume  $A B C D SumS E' F'$  and  $A B E F SumS C' D'$  and  $C D E F SumS A' B'$  and
     $E F Le E' F'$  and  $C D Le C' D'$  and  $A B Le A' B'$ 
  have  $\exists R. Cong A B A B \wedge Cong A R C D \wedge Cong B R E F$ 
  proof (cases A = B)
    case True
      obtain  $P$  where  $Cong A P C D$ 
        using segment-construction-0 by blast
      have  $C D Le E F$ 
        using True  $\langle A B E F SumS C' D' \rangle \langle C D Le C' D' \rangle$  cong-reflexivity l5-6
          not-cong-3412 sums--cong2345 by blast
      moreover have  $E F Le C D$ 
        using True  $\langle A B C D SumS E' F' \rangle \langle E F Le E' F' \rangle$  cong-reflexivity l5-6
    }

```



```

    not-cong-3412 sums--cong2345 by blast
ultimately have Cong C D E F
  using le-anti-symmetry by auto
hence Cong A P E F
  using ⟨Cong A P C D⟩ cong-transitivity by blast
hence Cong B P E F
  using True by auto
thus ?thesis
  using True ⟨Cong A P C D⟩ cong-pseudo-reflexivity by blast
next
case False
hence  $A \neq B$ 
  by simp
show ?thesis
proof (cases  $C = D$ )
case True
have B A Le E F
  using True ⟨A B Le A' B'⟩ ⟨C D E F SumS A' B'⟩ cong-pseudo-reflexivity
  l5-6 not-cong-3412 sums--cong2345 by blast
moreover have E F Le B A
  using True ⟨A B C D SumS E' F'⟩ ⟨E F Le E' F'⟩ cong-reflexivity
  l5-6 not-cong-4312 sums--cong1245 by blast
ultimately have Cong B A E F
  using le-anti-symmetry by auto
thus ?thesis
  using True cong-reflexivity cong-trivial-identity by blast
next
case False
hence  $C \neq D$ 
  by simp
show ?thesis
proof (cases  $E = F$ )
case True
thus ?thesis
  by (metis ⟨A B C D SumS E' F'⟩ ⟨A B E F SumS C' D'⟩ ⟨A B Le A' B'⟩
    ⟨C D E F SumS A' B'⟩ ⟨C D Le C' D'⟩ cong-reflexivity le2-sums2--cong34
    le2-sums2--le12 le-trivial sums123312 sums-sym)
next
case False
hence  $E \neq F$ 
  by simp
obtain C1 where A Out B C1 and Cong A C1 C D
  using segment-construction-3 ⟨A ≠ B⟩ ⟨C ≠ D⟩ by force
obtain E1 where B Out A E1 and Cong B E1 E F
  using segment-construction-3 ⟨A ≠ B⟩ ⟨E ≠ F⟩ by metis
obtain C2 where Bet B A C2 and Cong A C2 C D
  using segment-construction by blast
obtain E2 where Bet A B E2 and Cong B E2 E F
  using segment-construction by blast
have Bet C1 E1 C2 using euclid-22-aux [where ?A=A and ?B=B and
  ?C=C and ?D=D and ?E=E and ?F=F and
  ?A'=A' and ?B'=B' and ?E'=E' and ?F'=F']
  using ⟨A B C D SumS E' F'⟩ ⟨A B Le A' B'⟩ ⟨A Out B C1⟩ ⟨B Out A E1⟩
  ⟨Bet B A C2⟩ ⟨C D E F SumS A' B'⟩ ⟨Cong A C1 C D⟩ ⟨Cong A C2 C D⟩
  ⟨Cong B E1 E F⟩ ⟨E F Le E' F'⟩ by blast
have Bet E1 C1 E2 using euclid-22-aux [where ?A=B and ?B=A and
  ?C=E and ?D=F and ?E=C and ?F=D and
  ?A'=B' and ?B'=A' and ?E'=C' and ?F'=D' and ?C1.0=C1
  and ?C2.0=E2 and ?E1.0=C1]
  by (meson ⟨A B E F SumS C' D'⟩ ⟨A B Le A' B'⟩ ⟨A Out B C1⟩ ⟨B Out A E1⟩
    ⟨Bet A B E2⟩ ⟨C D E F SumS A' B'⟩ ⟨C D Le C' D'⟩ ⟨Cong A C1 C D⟩ ⟨Cong B E1 E F⟩
    ⟨Cong B E2 E F⟩ euclid-22-aux le-left-comm sums-left-comm sums-sym)
have E1 OnCircle B E1
  by (simp add: onc212)
moreover have C1 OnCircle A C1
  by (simp add: onc212)

```

```

moreover have  $E1$  InCircle  $A$   $C1$ 
proof (rule bet-inc2--inc [where  $?U=C1$  and  $?V=C2$ ])
  show  $C1$  InCircle  $A$   $C1$ 
    by (simp add: onc212 onc--inc)
  show  $C2$  InCircle  $A$   $C1$ 
    by (meson InCircle-def  $\langle \text{Cong } A \ C1 \ C \ D \rangle \langle \text{Cong } A \ C2 \ C \ D \rangle$  cong--le
      cong-inner-transitivity not-cong-3421)
  show  $Bet \ C1 \ E1 \ C2$ 
    using  $\langle Bet \ C1 \ E1 \ C2 \rangle$  by auto
qed
moreover have  $C1$  InCircle  $B$   $E1$ 
proof (rule bet-inc2--inc [where  $?U=E1$  and  $?V=E2$ ])
  show  $E1$  InCircle  $B$   $E1$ 
    by (simp add: onc212 onc--inc)
  show  $E2$  InCircle  $B$   $E1$ 
    by (meson InCircle-def  $\langle \text{Cong } B \ E1 \ E \ F \rangle \langle \text{Cong } B \ E2 \ E \ F \rangle$  cong--le3412
      cong-inner-transitivity not-cong-4321)
  show  $Bet \ E1 \ C1 \ E2$ 
    using  $\langle Bet \ E1 \ C1 \ E2 \rangle$  by auto
qed
ultimately obtain  $Z$  where  $Z$  OnCircle  $A$   $C1$  and  $Z$  OnCircle  $B$   $E1$ 
  using assms(1) circle-circle-bis-def by blast
then show  $?thesis$ 
  by (meson OnCircle-def  $\langle \text{Cong } A \ C1 \ C \ D \rangle \langle \text{Cong } B \ E1 \ E \ F \rangle$ 
    cong-reflexivity cong-transitivity)
qed
qed
qed
hence  $\exists \ P \ Q \ R. (\text{Cong } P \ Q \ A \ B \wedge \text{Cong } P \ R \ C \ D \wedge \text{Cong } Q \ R \ E \ F)$ 
  by blast
}
thus  $?thesis$ 
using euclid-s-prop-1-22-def by blast
qed

lemma triangle-inequality1:
  assumes  $A \ B \ B \ C \text{ SumS } D \ E$ 
  shows  $A \ C \text{ Le } D \ E$ 
proof –
  obtain  $D'$  where  $Bet \ A \ B \ D'$  and  $\text{Cong } B \ D' \ B \ C$ 
    using segment-construction by blast
  have  $A \ C \text{ Le } A \ D'$ 
    using  $\langle Bet \ A \ B \ D' \rangle \langle \text{Cong } B \ D' \ B \ C \rangle$  cong-symmetry triangle-inequality by blast
  moreover have  $\text{Cong } A \ D' \ D \ E$ 
    using SumS-def  $\langle Bet \ A \ B \ D' \rangle \langle \text{Cong } B \ D' \ B \ C \rangle$  assms cong-reflexivity sums2--cong56 by blast
  ultimately show  $?thesis$ 
    using cong--le le-transitivity by blast
qed

lemma euclid-22--circle-circle:
  assumes euclid-s-prop-1-22
  shows circle-circle
proof –
  {
    fix  $A \ B \ C \ D \ P \ Q$ 
    assume  $P$  OnCircle  $C \ D$  and  $Q$  OnCircle  $C \ D$  and  $P$  InCircle  $A \ B$  and  $Q$  OutCircle  $A \ B$ 
    have  $\exists \ X \ Y \ Z. \text{Cong } X \ Y \ A \ C \wedge \text{Cong } X \ Z \ A \ B \wedge \text{Cong } Y \ Z \ C \ D$ 
    proof –
      obtain  $L1 \ R1$  where  $A \ B \ C \ D \text{ SumS } L1 \ R1$ 
        using ex-sums by blast
      obtain  $L2 \ R2$  where  $A \ C \ C \ D \text{ SumS } L2 \ R2$ 
        using ex-sums by blast
      obtain  $L3 \ R3$  where  $A \ C \ A \ B \text{ SumS } L3 \ R3$ 
        using ex-sums by blast
      have  $C \ D \text{ Le } L3 \ R3$ 
      proof –

```

obtain $R\ S$ where $C\ A\ A\ P\ SumS\ R\ S$
 using *ex-sums* by blast
 have $C\ D\ Le\ R\ S$
 by (*metis Cong-cases OnCircle-def* $\langle C\ A\ A\ P\ SumS\ R\ S \rangle \langle P\ OnCircle\ C\ D \rangle$
cong--le le-transitivity triangle-inequality1)
 moreover have $R\ S\ Le\ L3\ R3$
 using *le2-sums2--le* [where $?A=C$ and $?B=A$ and $?C=A$ and $?D=P$ and
 $?A'=A$ and $?B'=C$ and $?C'=A$ and $?D'=B$] *InCircle-def* $\langle A\ C\ A\ B\ SumS\ L3\ R3 \rangle$
 $\langle C\ A\ A\ P\ SumS\ R\ S \rangle \langle P\ InCircle\ A\ B \rangle$ *le1221* by blast
 ultimately show *?thesis*
 using *le-transitivity* by blast
 qed
 moreover have $A\ C\ C\ Q\ SumS\ L2\ R2$
 by (*meson OnCircle-def* $\langle A\ C\ C\ D\ SumS\ L2\ R2 \rangle \langle Q\ OnCircle\ C\ D \rangle$
cong3-sums--sums cong-reflexivity onc-sym)
 hence $A\ Q\ Le\ L2\ R2$
 using *triangle-inequality1* by auto
 hence $A\ B\ Le\ L2\ R2$
 using *OutCircle-def* $\langle Q\ OutCircle\ A\ B \rangle$ *le-transitivity* by blast
 moreover have $A\ C\ Le\ L1\ R1$
 proof –
 obtain $R\ S$ where $A\ P\ P\ C\ SumS\ R\ S$
 using *ex-sums* by blast
 have $A\ C\ Le\ R\ S$
 using $\langle A\ P\ P\ C\ SumS\ R\ S \rangle$ *triangle-inequality1* by auto
 moreover have $R\ S\ Le\ L1\ R1$
 using *le2-sums2--le* [where $?A=A$ and $?B=P$ and $?C=P$ and $?D=C$ and
 $?A'=A$ and $?B'=B$ and $?C'=C$ and $?D'=D$] by (*meson InCircle-def OnCircle-def*
 $\langle A\ B\ C\ D\ SumS\ L1\ R1 \rangle \langle A\ P\ P\ C\ SumS\ R\ S \rangle \langle P\ InCircle\ A\ B \rangle \langle P\ OnCircle\ C\ D \rangle$
cong--le not-cong-3421 onc-sym)
 ultimately show *?thesis*
 using *le-transitivity* by blast
 qed
 ultimately show *?thesis*
 using $\langle A\ B\ C\ D\ SumS\ L1\ R1 \rangle \langle A\ C\ A\ B\ SumS\ L3\ R3 \rangle \langle A\ C\ C\ D\ SumS\ L2\ R2 \rangle$ *assms*
euclid-s-prop-1-22-def by blast
 qed
 then obtain $X\ Y\ Z$ where *Cong* $X\ Y\ A\ C$ and *Cong* $X\ Z\ A\ B$ and *Cong* $Y\ Z\ C\ D$
 by blast
 have $\exists\ Z. Z\ OnCircle\ A\ B \wedge Z\ OnCircle\ C\ D$
 proof (cases $A = C$)
 case True
 then show *?thesis*
 using *OnCircle-def* $\langle \exists\ X\ Y\ Z. Cong\ X\ Y\ A\ C \wedge Cong\ X\ Z\ A\ B \wedge Cong\ Y\ Z\ C\ D \rangle$
cong-identity cong-inner-transitivity by blast
 next
 case False
 hence $A \neq C$
 by blast
 show *?thesis*
 proof (cases $A = B$)
 case True
 then show *?thesis*
 using $\langle P\ InCircle\ A\ B \rangle \langle P\ OnCircle\ C\ D \rangle$ *inc--outc-1 inc-eq*
inc-outc--onc by blast
 next
 case False
 hence $A \neq B$
 by simp
 have $X \neq Z$
 using False $\langle Cong\ X\ Z\ A\ B \rangle$ *cong-reverse-identity* by force
 have $X \neq Y$
 using $\langle A \neq C \rangle \langle Cong\ X\ Y\ A\ C \rangle$ *cong-diff-3* by blast
 have $\exists\ Z0. Y\ X\ Z\ CongA\ C\ A\ Z0 \wedge Cong\ X\ Z\ A\ Z0$
 proof –
 obtain Z' where $Y\ X\ Z\ CongA\ C\ A\ Z'$

```

    using  $\langle A \neq C \rangle \langle X \neq Y \rangle \langle X \neq Z \rangle$  angle-construction-3 by presburger
  obtain Z0 where A Out Z' Z0 and Cong A Z0 X Z
  by (metis  $\langle X \neq Z \rangle \langle Y X Z \text{ CongA } C A Z' \rangle$  conga-diff56 segment-construction-3)
  have Y X Z CongA C A Z0
  by (metis  $\langle A \text{ Out } Z' Z0 \rangle \langle A \neq C \rangle \langle Y X Z \text{ CongA } C A Z' \rangle$  not-conga
    not-conga-sym out2--conga out-trivial)
  thus ?thesis
    using  $\langle \text{Cong } A Z0 X Z \rangle$  cong-symmetry by blast
qed
then obtain Z0 where Y X Z CongA C A Z0 and Cong X Z A Z0
  by blast
then show ?thesis
  by (meson OnCircle-def  $\langle \text{Cong } X Y A C \rangle \langle \text{Cong } X Z A B \rangle \langle \text{Cong } Y Z C D \rangle$ 
    cong-inner-transitivity l11-49)
qed
qed
}
thus ?thesis
  using circle-circle-def by blast
qed

```

```

theorem equivalent-variants-of-circle-circle:
  shows (circle-circle  $\longleftrightarrow$  circle-circle-two)  $\wedge$ 
    (circle-circle-two  $\longleftrightarrow$  circle-circle-bis)  $\wedge$ 
    (circle-circle-bis  $\longleftrightarrow$  circle-circle-axiom)  $\wedge$ 
    (circle-circle-axiom  $\longleftrightarrow$  euclid-s-prop-1-22)
proof -
  have circle-circle  $\longleftrightarrow$  circle-circle-two
    using circle-circle--circle-circle-two by blast
  moreover have circle-circle-two  $\longleftrightarrow$  circle-circle-bis
    using circle-circle--circle-circle-bis circle-circle--circle-circle-two
      circle-circle-bis--euclid-22 euclid-22--circle-circle by blast
  moreover have circle-circle-bis  $\longleftrightarrow$  circle-circle-axiom
    by (simp add: circle-circle-bis--circle-circle-axiom)
  moreover have circle-circle-axiom  $\longleftrightarrow$  euclid-s-prop-1-22
    using circle-circle--circle-circle-bis circle-circle-bis--circle-circle-axiom
      circle-circle-bis--euclid-22 euclid-22--circle-circle by blast
  ultimately show ?thesis
    by blast
qed

```

```

theorem equivalent-variants-of-line-circle:
  shows (segment-circle  $\longleftrightarrow$  one-point-line-circle)  $\wedge$ 
    (one-point-line-circle  $\longleftrightarrow$  two-points-line-circle)
proof -
  have segment-circle  $\longrightarrow$  one-point-line-circle
    by (simp add: segment-circle--one-point-line-circle)
  moreover have one-point-line-circle  $\longrightarrow$  two-points-line-circle
    by (simp add: one-point-line-circle--two-points-line-circle)
  moreover have two-points-line-circle  $\longrightarrow$  segment-circle
    by (simp add: one-point-line-circle--two-points-line-circle-R2
      segment-circle--one-point-line-circle-R2)
  ultimately show ?thesis
    by blast
qed

```

```

end
end

```

```

theory Hilbert-Neutral

```

```

imports
  Tarski-Neutral

```

```

begin

```

8 Hilbert - Geometry - neutral dimension less

8.1 Axioms

locale *Hilbert-neutral-dimensionless-pre* =
fixes

IncidL :: 'p ⇒ 'b ⇒ bool **and**
IncidP :: 'p ⇒ 'c ⇒ bool **and**
EqL :: 'b ⇒ 'b ⇒ bool **and**
EqP :: 'c ⇒ 'c ⇒ bool **and**
IsL :: 'b ⇒ bool **and**
IsP :: 'c ⇒ bool **and**
BetH :: 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongaH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool

context *Hilbert-neutral-dimensionless-pre*

begin

8.2 Definitions

definition *ColH* :: 'p ⇒ 'p ⇒ 'p ⇒ bool
where

ColH A B C ≡ (∃ l. *IsL* l ∧ *IncidL* A l ∧ *IncidL* B l ∧ *IncidL* C l)

definition *IncidLP* :: 'b ⇒ 'c ⇒ bool **where**

IncidLP l p ≡ *IsL* l ∧ *IsP* p ∧ (∀ A. *IncidL* A l → *IncidP* A p)

definition *cut* :: 'b ⇒ 'p ⇒ 'p ⇒ bool **where**

cut l A B ≡ *IsL* l ∧ ¬ *IncidL* A l ∧ ¬ *IncidL* B l ∧ (∃ I. *IncidL* I l ∧ *BetH* A I B)

definition *outH* :: 'p ⇒ 'p ⇒ 'p ⇒ bool **where**

outH P A B ≡ *BetH* P A B ∨ *BetH* P B A ∨ (P ≠ A ∧ A = B)

definition *disjoint* ::

'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool **where**

disjoint A B C D ≡ ¬ (∃ P. *BetH* A P B ∧ *BetH* C P D)

definition *same-side* :: 'p ⇒ 'p ⇒ 'b ⇒ bool **where**

same-side A B l ≡ *IsL* l ∧ (∃ P. *cut* l A P ∧ *cut* l B P)

definition *same-side'* ::

'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool **where**

same-side' A B X Y ≡

X ≠ Y ∧

(∀ l::'b. (*IsL* l ∧ *IncidL* X l ∧ *IncidL* Y l) → *same-side* A B l)

definition *Para* :: 'b ⇒ 'b ⇒ bool **where**

Para l m ≡ *IsL* l ∧ *IsL* m ∧

(¬(∃ X. *IncidL* X l ∧ *IncidL* X m)) ∧ (∃ p. *IncidLP* l p ∧ *IncidLP* m p)

definition *Bet* :: 'p ⇒ 'p ⇒ 'p ⇒ bool

where

Bet A B C ≡ *BetH* A B C ∨ A = B ∨ B = C

definition *Cong* :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool

where

Cong A B C D ≡ (*CongH* A B C D ∧ A ≠ B ∧ C ≠ D) ∨ (A = B ∧ C = D)

definition *ParaP* :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool

where

ParaP A B C D ≡ ∀ l m.

IsL l ∧ *IsL* m ∧ *IncidL* A l ∧ *IncidL* B l ∧ *IncidL* C m ∧ *IncidL* D m

→

Para l m

definition *is-line* :: 'p ⇒ 'p ⇒ 'b ⇒ bool
where
is-line A B l ≡ (IsL l ∧ A ≠ B ∧ IncidL A l ∧ IncidL B l)

definition *cut'* ::
'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool
where
cut' A B X Y ≡ X ≠ Y ∧ (∀ l. (IsL l ∧ IncidL X l ∧ IncidL Y l) → *cut* l A B)

definition *Midpoint* :: 'p ⇒ 'p ⇒ 'p ⇒ bool
(- *Midpoint* - - [99,99,99] 50)
where M *Midpoint* A B ≡ Bet A M B ∧ Cong A M M B

end

locale *Hilbert-neutral-dimensionless* = *Hilbert-neutral-dimensionless-pre* IncidL IncidP EqL EqP
IsL IsP BetH CongH CongaH
for
IncidL :: 'p ⇒ 'b ⇒ bool **and**
IncidP :: 'p ⇒ 'c ⇒ bool **and**
EqL :: 'b ⇒ 'b ⇒ bool **and**
EqP :: 'c ⇒ 'c ⇒ bool **and**
IsL :: 'b ⇒ bool **and**
IsP :: 'c ⇒ bool **and**
BetH :: 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongaH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool +
fixes PP PQ PR :: 'p
assumes
EqL-refl: IsL l → *EqL* l l **and**
EqL-sym: IsL l1 ∧ IsL l2 ∧ *EqL* l1 l2 → *EqL* l2 l1 **and**
EqL-trans: (*EqL* l1 l2 ∧ *EqL* l2 l3) → *EqL* l1 l3 **and**
EqP-refl: IsP p → *EqP* p p **and**
EqP-sym: *EqP* p1 p2 → *EqP* p2 p1 **and**
EqP-trans: (*EqP* p1 p2 ∧ *EqP* p2 p3) → *EqP* p1 p3 **and**
IncidL-morphism: (IsL l ∧ IsL m ∧ *IncidL* P l ∧ *EqL* l m) → *IncidL* P m **and**
IncidP-morphism: (IsP p ∧ IsP q ∧ *IncidP* M p ∧ *EqP* p q) → *IncidP* M q **and**
Is-line: *IncidL* P l → IsL l **and**
Is-plane: *IncidP* P p → IsP p
assumes
line-existence: A ≠ B → (∃ l. IsL l ∧ (*IncidL* A l ∧ *IncidL* B l)) **and**
line-uniqueness: A ≠ B ∧ IsL l ∧ IsL m ∧
IncidL A l ∧ *IncidL* B l ∧ *IncidL* A m ∧ *IncidL* B m →
EqL l m **and**
two-points-on-line: ∀ l. IsL l → (∃ A B. *IncidL* A l ∧ *IncidL* B l ∧ A ≠ B)
assumes
lower-dim-2: PP ≠ PQ ∧ PQ ≠ PR ∧ PP ≠ PR ∧ ¬ ColH PP PQ PR **and**
plan-existence: ∀ A B C. (¬ ColH A B C) →
(∃ p. IsP p ∧ *IncidP* A p ∧ *IncidP* B p ∧ *IncidP* C p) **and**
one-point-on-plane: ∀ p. ∃ A. IsP p → *IncidP* A p **and**
plane-uniqueness: ¬ ColH A B C ∧ IsP p ∧ IsP q ∧
IncidP A p ∧ *IncidP* B p ∧ *IncidP* C p ∧ *IncidP* A q ∧ *IncidP* B q ∧ *IncidP* C q →
EqP p q **and**
line-on-plane: ∀ A B l p. A ≠ B ∧ IsL l ∧ IsP p ∧
IncidL A l ∧ *IncidL* B l ∧ *IncidP* A p ∧ *IncidP* B p → *IncidLP* l p
assumes
between-diff: BetH A B C → A ≠ C **and**
between-col: BetH A B C → ColH A B C **and**
between-comm: BetH A B C → BetH C B A **and**
between-out: A ≠ B → (∃ C. BetH A B C) **and**
between-only-one: BetH A B C → ¬ BetH B C A **and**
pasch: ¬ ColH A B C ∧ IsL l ∧ IsP p ∧
IncidP A p ∧ *IncidP* B p ∧ *IncidP* C p ∧ *IncidLP* l p ∧ ¬ *IncidL* C l ∧
(*cut* l A B)

$\longrightarrow$
 $(\text{cut } l \ A \ C) \vee (\text{cut } l \ B \ C)$
assumes
cong-permr: $\text{CongH } A \ B \ C \ D \longrightarrow \text{CongH } A \ B \ D \ C$ **and**
cong-existence: $\bigwedge A \ B \ A' \ P :: 'p. A \neq B \wedge A' \neq P \wedge \text{IsL } l \wedge$
 $\text{IncidL } A' \ l \wedge \text{IncidL } P \ l \longrightarrow$
 $(\exists B'. (\text{IncidL } B' \ l \wedge \text{outH } A' \ P \ B' \wedge \text{CongH } A' \ B' \ A \ B))$ **and**
cong-pseudo-transitivity:
 $\text{CongH } A \ B \ C \ D \wedge \text{CongH } A \ B \ E \ F \longrightarrow \text{CongH } C \ D \ E \ F$ **and**
addition :
 $\text{ColH } A \ B \ C \wedge \text{ColH } A' \ B' \ C' \wedge$
 $\text{disjoint } A \ B \ C \wedge \text{disjoint } A' \ B' \ B' \ C' \wedge$
 $\text{CongH } A \ B \ A' \ B' \wedge \text{CongH } B \ C \ B' \ C' \longrightarrow$
 $\text{CongH } A \ C \ A' \ C'$ **and**
conga-refl: $\neg \text{ColH } A \ B \ C \longrightarrow \text{CongaH } A \ B \ C \ A \ B \ C$ **and**
conga-comm : $\neg \text{ColH } A \ B \ C \longrightarrow \text{CongaH } A \ B \ C \ C \ B \ A$ **and**
conga-permlr: $\text{CongaH } A \ B \ C \ D \ E \ F \longrightarrow \text{CongaH } C \ B \ A \ F \ E \ D$ **and**
conga-out-conga: $(\text{CongaH } A \ B \ C \ D \ E \ F \wedge$
 $\text{outH } B \ A \ A' \wedge \text{outH } B \ C \ C' \wedge \text{outH } E \ D \ D' \wedge \text{outH } E \ F \ F') \longrightarrow$
 $\text{CongaH } A' \ B \ C' \ D' \ E \ F'$ **and**
cong-4-existence:
 $\neg \text{ColH } P \ PO \ X \wedge \neg \text{ColH } A \ B \ C \longrightarrow$
 $(\exists Y. (\text{CongaH } A \ B \ C \ X \ PO \ Y \wedge \text{same-side}' \ P \ Y \ PO \ X))$ **and**
cong-4-uniqueness:
 $\neg \text{ColH } P \ PO \ X \wedge \neg \text{ColH } A \ B \ C \wedge$
 $\text{CongaH } A \ B \ C \ X \ PO \ Y \wedge \text{CongaH } A \ B \ C \ X \ PO \ Y' \wedge$
 $\text{same-side}' \ P \ Y \ PO \ X \wedge \text{same-side}' \ P \ Y' \ PO \ X \longrightarrow$
 $\text{outH } PO \ Y \ Y'$ **and**
cong-5 : $\neg \text{ColH } A \ B \ C \wedge \neg \text{ColH } A' \ B' \ C' \wedge$
 $\text{CongH } A \ B \ A' \ B' \wedge \text{CongH } A \ C \ A' \ C' \wedge$
 $\text{CongaH } B \ A \ C \ B' \ A' \ C' \longrightarrow$
 $\text{CongaH } A \ B \ C \ A' \ B' \ C'$

context *Hilbert-neutral-dimensionless*

begin

8.3 Propositions

lemma *betH-distincts*:

assumes *BetH* $A \ B \ C$

shows $A \neq B \wedge B \neq C \wedge A \neq C$

using *assms between-comm between-diff between-only-one* **by** *blast*

lemma *congH-perm*:

assumes $A \neq B$

shows $\text{CongH } A \ B \ B \ A$

proof –

obtain l **where** *IsL* l **and** *IncidL* $A \ l$ **and** *IncidL* $B \ l$

using *assms line-existence* **by** *fastforce*

moreover

{

fix x

assume *IncidL* $x \ l$ **and**

outH $A \ B \ x$ **and**

CongH $A \ x \ A \ B$

hence *CongH* $A \ x \ B \ A$

by (*simp add: cong-permr*)

hence *CongH* $A \ B \ B \ A$

using $\langle \text{CongH } A \ x \ A \ B \rangle$ *cong-pseudo-transitivity* **by** *blast*

}

hence $\forall x. (\text{IncidL } x \ l \wedge \text{outH } A \ B \ x \wedge \text{CongH } A \ x \ A \ B) \longrightarrow \text{CongH } A \ B \ B \ A$

by *blast*

ultimately show *?thesis*

using *assms cong-existence* by *presburger*
qed

lemma *congH-refl*:
assumes $A \neq B$
shows *CongH A B A B*
by (*simp add: assms congH-perm cong-permr*)

lemma *congH-sym*:
assumes $A \neq B$ and
 CongH A B C D
shows *CongH C D A B*
using *assms(1) assms(2) congH-refl cong-pseudo-transitivity* by *blast*

lemma *colH-permut-231*:
assumes *ColH A B C*
shows *ColH B C A*
using *ColH-def assms* by *blast*

lemma *colH-permut-312*:
assumes *ColH A B C*
shows *ColH C A B*
by (*simp add: assms colH-permut-231*)

lemma *colH-permut-213*:
assumes *ColH A B C*
shows *ColH B A C*
using *ColH-def assms* by *auto*

lemma *colH-permut-132*:
assumes *ColH A B C*
shows *ColH A C B*
using *ColH-def assms* by *auto*

lemma *colH-permut-321*:
assumes *ColH A B C*
shows *ColH C B A*
using *ColH-def assms* by *auto*

lemma *other-point-exists*:
fixes $A::'p$
shows $\exists B. A \neq B$
by (*metis lower-dim-2*)

lemma *colH-trivial111*:
shows *ColH A A A*
using *cong-4-existence same-side'-def* by *blast*

lemma *colH-trivial112*:
shows *ColH A A B*
using *colH-permut-231 cong-4-existence same-side'-def* by *blast*

lemma *colH-trivial122*:
shows *ColH A B B*
by (*simp add: colH-permut-312 colH-trivial112*)

lemma *colH-trivial121*:
shows *ColH A B A*
by (*simp add: colH-permut-312 colH-trivial122*)

lemma *colH-dec*:
shows $ColH A B C \vee \neg ColH A B C$
by *blast*

lemma *colH-trans*:
assumes $X \neq Y$ and

$ColH\ X\ Y\ A$ and
 $ColH\ X\ Y\ B$ and
 $ColH\ X\ Y\ C$
shows $ColH\ A\ B\ C$
proof –
obtain l where $IsL\ l$ and $IncidL\ X\ l$ and $IncidL\ Y\ l$ and $IncidL\ A\ l$
using $ColH-def\ assms(2)$ by blast
obtain m where $IsL\ m$ and $IncidL\ X\ m$ and $IncidL\ Y\ m$ and $IncidL\ B\ m$
using $ColH-def\ assms(3)$ by blast
obtain n where $IsL\ n$ and $IncidL\ X\ n$ and $IncidL\ Y\ n$ and $IncidL\ C\ n$
using $ColH-def\ assms(4)$ by blast
have $EqL\ l\ m$
using $line-uniqueness\ \langle IncidL\ B\ m \rangle\ assms(1)\ \langle IncidL\ X\ l \rangle\ \langle IncidL\ Y\ l \rangle$
 $\langle IncidL\ X\ m \rangle\ \langle IncidL\ Y\ m \rangle\ \langle IsL\ m \rangle\ \langle IsL\ l \rangle$ by blast
hence $IncidL\ B\ l$
using $EqL-sym\ IncidL-morphism\ \langle IncidL\ B\ m \rangle\ \langle IsL\ l \rangle\ \langle IsL\ m \rangle$ by blast
moreover have $IncidL\ C\ l$
by ($meson\ IncidL-morphism\ \langle IncidL\ C\ n \rangle\ \langle IncidL\ X\ l \rangle\ \langle IncidL\ X\ n \rangle\ \langle IncidL\ Y\ l \rangle$
 $\langle IncidL\ Y\ n \rangle\ \langle IsL\ l \rangle\ \langle IsL\ n \rangle\ assms(1)\ line-uniqueness$)
ultimately show $?thesis$
using $ColH-def\ \langle IncidL\ A\ l \rangle\ \langle IsL\ l \rangle$ by blast
qed

lemma *bet-colH*:
assumes $Bet\ A\ B\ C$
shows $ColH\ A\ B\ C$
using $Bet-def\ assms\ between-col\ colH-trivial112\ colH-trivial122$ by blast

lemma *ncolH-exists*:
assumes $A \neq B$
shows $\exists\ C.\ \neg\ ColH\ A\ B\ C$
using $assms\ colH-trans\ lower-dim-2$ by blast

lemma *ncolH-distincts*:
assumes $\neg\ ColH\ A\ B\ C$
shows $A \neq B \wedge B \neq C \wedge A \neq C$
using $assms\ colH-trivial112\ colH-trivial121\ colH-trivial122$ by blast

lemma *betH-expand*:
assumes $BetH\ A\ B\ C$
shows $BetH\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C \wedge ColH\ A\ B\ C$
using $assms\ betH-distincts\ between-col$ by presburger

lemma *inter-uniquenessH*:
assumes $A' \neq B'$ and
 $\neg\ ColH\ A\ B\ A'$ and
 $ColH\ A\ B\ X$ and
 $ColH\ A\ B\ Y$ and
 $ColH\ A'\ B'\ X$ and
 $ColH\ A'\ B'\ Y$
shows $X = Y$
proof –
have $A \neq B$
using $assms(2)\ colH-trivial112$ by blast
thus $?thesis$
by ($metis\ (full-types)\ assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)\ assms(6)$
 $colH-permut-231\ colH-trans\ colH-trivial121$)
qed

lemma *inter-incid-uniquenessH*:
assumes $\neg\ IncidL\ P\ l$ and
 $IncidL\ P\ m$ and
 $IncidL\ X\ l$ and
 $IncidL\ Y\ l$ and
 $IncidL\ X\ m$ and
 $IncidL\ Y\ m$

```

shows  $X = Y$ 
proof -
  have  $IsL\ l$ 
    using  $Is\text{-}line\ assms(4)$  by blast
  then obtain  $A\ B$  where  $IncidL\ B\ l$  and  $IncidL\ A\ l$  and  $A \neq B$ 
    using  $two\text{-}points\text{-}on\text{-}line$  by blast
  have  $IsL\ m$ 
    using  $Is\text{-}line\ assms(5)$  by blast
  then obtain  $A'\ B'$  where  $IncidL\ B'\ m$  and  $IncidL\ A'\ m$  and  $A' \neq B'$ 
    using  $two\text{-}points\text{-}on\text{-}line$  by blast
  have  $ColH\ A\ B\ X$ 
    using  $ColH\text{-}def\ Is\text{-}line\ \langle IncidL\ A\ l\ \rangle\ \langle IncidL\ B\ l\ \rangle\ assms(3)$  by blast
  have  $ColH\ A\ B\ Y$ 
    using  $ColH\text{-}def\ Is\text{-}line\ \langle IncidL\ A\ l\ \rangle\ \langle IncidL\ B\ l\ \rangle\ assms(4)$  by blast
  have  $ColH\ A'\ B'\ X$ 
    using  $ColH\text{-}def\ Is\text{-}line\ \langle IncidL\ A'\ m\ \rangle\ \langle IncidL\ B'\ m\ \rangle\ assms(5)$  by blast
  have  $ColH\ A'\ B'\ Y$ 
    using  $ColH\text{-}def\ Is\text{-}line\ \langle IncidL\ A'\ m\ \rangle\ \langle IncidL\ B'\ m\ \rangle\ assms(6)$  by blast
  {
    assume  $ColH\ A\ B\ P$ 
    then obtain  $l00$  where  $IsL\ l00$  and  $IncidL\ A\ l00$  and  $IncidL\ B\ l00$  and  $IncidL\ P\ l00$ 
      using  $ColH\text{-}def$  by blast
    hence  $EqL\ l\ l00$ 
      using  $\langle A \neq B \rangle\ \langle IncidL\ A\ l\ \rangle\ \langle IncidL\ B\ l\ \rangle\ \langle IsL\ l\ \rangle\ line\text{-}uniqueness$  by blast
    hence  $IncidL\ P\ l$ 
      using  $EqL\text{-}sym\ IncidL\text{-}morphism\ \langle IncidL\ P\ l00\ \rangle\ \langle IsL\ l00\ \rangle\ \langle IsL\ l\ \rangle$  by blast
    hence  $False$ 
      using  $assms(1)$  by blast
  }
  hence  $\neg\ ColH\ A\ B\ P$ 
    by blast
  {
    assume  $A' = P$ 
    hence  $X = Y$ 
      using  $\langle A' \neq B' \rangle\ \langle ColH\ A\ B\ X \rangle\ \langle ColH\ A\ B\ Y \rangle\ \langle ColH\ A'\ B'\ X \rangle\ \langle ColH\ A'\ B'\ Y \rangle\ \langle \neg\ ColH\ A\ B\ P \rangle$ 
       $inter\text{-}uniquenessH$  by blast
  }
  moreover
  {
    assume  $A' \neq P$ 
    hence  $X = Y$ 
      using  $IncidL\text{-}morphism\ \langle IsL\ l\ \rangle\ \langle IsL\ m\ \rangle\ assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)$ 
       $assms(6)\ line\text{-}uniqueness$  by blast
  }
  ultimately show  $?thesis$ 
    by blast
qed

```

```

lemma between-only-one':
  assumes  $BetH\ A\ B\ C$ 
  shows  $\neg\ BetH\ B\ A\ C$ 
  using  $assms(1)\ between\text{-}comm\ between\text{-}only\text{-}one$  by blast

```

```

lemma betH-colH:
  assumes  $BetH\ A\ B\ C$ 
  shows  $ColH\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C$ 
  using  $assms\ betH\text{-}expand$  by presburger

```

```

lemma cut-comm:
  assumes  $cut\ l\ A\ B$ 
  shows  $cut\ l\ B\ A$ 
  using  $assms\ between\text{-}comm\ local.cut\text{-}def$  by blast

```

```

lemma line-on-plane':
  assumes  $A \neq B$  and
     $IncidP\ A\ p$  and

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    IncidP B p and
    ColH A B C
shows IncidP C p
using ColH-def IncidLP-def Is-plane assms(1) assms(2) assms(3) assms(4) line-on-plane by blast

lemma inner-pasch-aux:
  assumes  $\neg$  ColH B C P and
    Bet A P C and
    Bet B Q C
  shows  $\exists X. \text{Bet } P \ X \ B \wedge \text{Bet } Q \ X \ A$ 
proof -
  have BetH A P C  $\vee$  A = P  $\vee$  P = C
    using Bet-def assms(2) by presburger
  moreover
  have BetH B Q C  $\vee$  B = Q  $\vee$  Q = C
    using Bet-def assms(3) by blast
  moreover
  let ?t =  $\exists X. \text{Bet } P \ X \ B \wedge \text{Bet } Q \ X \ A$ 
  have BetH A P C  $\wedge$  BetH B Q C  $\longrightarrow$  ?t
  proof -
    {
      assume Q  $\neq$  A and
        BetH A P C and BetH B Q C
      obtain l where IsL l and IncidL Q l and IncidL A l
        using  $\langle Q \neq A \rangle$  line-existence by blast
      {
        assume P = A
        hence ?t
          using Bet-def by blast
      }
      moreover
      {
        assume P  $\neq$  A
        {
          assume Q = C
          hence ?t
            using  $\langle \text{BetH } B \ Q \ C \rangle$  betH-distincts by auto
        }
        moreover
        {
          assume Q  $\neq$  C
          {
            assume IncidL P l
            have ColH A P C
              using  $\langle \text{BetH } A \ P \ C \rangle$  betH-colH by blast
            have ColH B Q C
              by (simp add:  $\langle \text{BetH } B \ Q \ C \rangle$  between-col)
            have ColH A P Q
              using ColH-def  $\langle \text{IncidL } A \ l \rangle \langle \text{IncidL } P \ l \rangle \langle \text{IncidL } Q \ l \rangle \langle \text{IsL } l \rangle$  by blast
            have ColH P C Q
              by (meson ColH-def  $\langle \text{ColH } A \ P \ C \rangle \langle \text{IncidL } A \ l \rangle \langle \text{IncidL } P \ l \rangle \langle \text{IncidL } Q \ l \rangle$ 
                 $\langle P \neq A \rangle$  inter-incid-uniquenessH)
            have ColH B C P
              by (metis inter-uniquenessH  $\langle \text{ColH } B \ Q \ C \rangle \langle \text{ColH } P \ C \ Q \rangle \langle Q \neq C \rangle$ 
                colH-permut-321 colH-trivial122)
            hence False
              using assms(1) by blast
          }
        }
      }
      hence  $\neg$  IncidL P l
        by blast
    }
    {
      assume B = Q
      hence ?t
        using Bet-def by blast
    }
  }
  moreover

```

```

{
  assume  $B \neq Q$ 
  {
    assume  $A = C$ 
    hence ?t
    using  $\langle \text{BetH } A \ P \ C \rangle$  between-diff by blast
  }
  moreover
  {
    assume  $A \neq C$ 
    {
      assume  $\text{IncidL } B \ l$ 
      have  $\text{ColH } A \ B \ Q$ 
      using  $\text{ColH-def } \langle \text{IncidL } A \ l \rangle \langle \text{IncidL } B \ l \rangle \langle \text{IncidL } Q \ l \rangle \langle \text{IsL } l \rangle$  by blast
      have  $\text{ColH } A \ B \ C$ 
      using  $\langle \text{BetH } B \ Q \ C \rangle \langle \text{ColH } A \ B \ Q \rangle$  betH-colH colH-permut-231 colH-trans
        colH-trivial121 by blast
      have  $\text{ColH } A \ P \ C$ 
      by (simp add:  $\langle \text{BetH } A \ P \ C \rangle$  betH-colH)
      have  $\text{ColH } B \ Q \ C$ 
      by (simp add:  $\langle \text{BetH } B \ Q \ C \rangle$  between-col)
      have  $\text{ColH } B \ C \ P$ 
      by (meson  $\text{ColH-def } \langle A \neq C \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{ColH } A \ P \ C \rangle$  inter-incid-uniquenessH)
      hence False
      using assms(1) by blast
    }
    hence  $\neg \text{IncidL } B \ l$ 
    by blast
    {
      assume  $\text{IncidL } C \ l$ 
      have  $\text{ColH } A \ C \ Q$ 
      using  $\text{ColH-def } \langle \text{IncidL } A \ l \rangle \langle \text{IncidL } C \ l \rangle \langle \text{IncidL } Q \ l \rangle \langle \text{IsL } l \rangle$  by blast
      have  $\text{ColH } B \ Q \ C$ 
      using assms(3) bet-colH by auto
      have  $\text{ColH } A \ B \ C$ 
      by (metis  $\langle \text{BetH } B \ Q \ C \rangle \langle \text{ColH } A \ C \ Q \rangle \langle Q \neq C \rangle$  betH-colH colH-permut-231
        colH-trans colH-trivial121)
      have  $\text{ColH } B \ C \ P$ 
      using inter-incid-uniquenessH  $\text{ColH-def } \langle A \neq C \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{IncidL } A \ l \rangle$ 
         $\langle \text{IncidL } C \ l \rangle \langle \neg \text{IncidL } B \ l \rangle$  by fastforce
      hence False
      by (simp add: assms(1))
    }
  }
  have  $\text{cut } l \ B \ C$ 
  using  $\langle \text{BetH } B \ Q \ C \rangle \langle \text{IncidL } C \ l \implies \text{False} \rangle \langle \text{IncidL } Q \ l \rangle \langle \text{IsL } l \rangle \langle \neg \text{IncidL } B \ l \rangle$ 
    local.cut-def by blast
  obtain  $p$  where  $\text{IncidP } B \ p$  and  $\text{IncidP } C \ p$  and  $\text{IncidP } P \ p$ 
  using assms(1) plan-existence by blast
  have  $(\neg \text{ColH } B \ C \ P \wedge \text{IncidP } B \ p \wedge \text{IncidP } C \ p \wedge \text{IncidP } P \ p \wedge \text{IncidLP } l \ p \wedge$ 
     $\neg \text{IncidL } P \ l \wedge \text{cut } l \ B \ C) \longrightarrow \text{cut } l \ B \ P \vee \text{cut } l \ C \ P$ 
  using pasch Is-plane  $\langle \text{IsL } l \rangle$  by blast
  moreover
  have  $\neg \text{ColH } B \ C \ P$ 
  by (simp add: assms(1))
  moreover have  $\text{IncidP } B \ p$ 
  by (simp add:  $\langle \text{IncidP } B \ p \rangle$ )
  moreover have  $\text{IncidP } C \ p$ 
  by (simp add:  $\langle \text{IncidP } C \ p \rangle$ )
  moreover have  $\text{IncidP } P \ p$ 
  by (simp add:  $\langle \text{IncidP } P \ p \rangle$ )
  moreover have  $\text{IncidLP } l \ p$ 
  by (metis colH-permut-231 line-on-plane'
    Hilbert-neutral-dimensionless.ncolH-distincts
    Hilbert-neutral-dimensionless-axioms Is-plane  $\langle \text{IncidL } A \ l \rangle \langle \text{IncidL } Q \ l \rangle$ 
     $\langle \text{IsL } l \rangle \langle Q \neq A \rangle$  assms(2) assms(3) bet-colH calculation(2) calculation(3)
    calculation(4) calculation(5) line-on-plane)

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moreover have  $\neg \text{IncidL } P \text{ } l$ 
  by (simp add:  $\langle \neg \text{IncidL } P \text{ } l \rangle$ )
moreover have  $\text{cut } l \text{ } B \text{ } C$ 
  by (simp add:  $\langle \text{local.cut } l \text{ } B \text{ } C \rangle$ )
moreover
{
  assume  $\text{cut } l \text{ } B \text{ } P$ 
  hence  $\neg \text{IncidL } B \text{ } l$ 
    using  $\langle \neg \text{IncidL } B \text{ } l \rangle$  by auto
  have  $\neg \text{IncidL } P \text{ } l$ 
    by (simp add: calculation(7))
  obtain  $X$  where  $\text{IncidL } X \text{ } l$  and  $\text{BetH } B \text{ } X \text{ } P$ 
    using  $\langle \text{local.cut } l \text{ } B \text{ } P \rangle$   $\text{local.cut-def}$  by blast
  have  $\text{BetH } P \text{ } X \text{ } B \vee P = X \vee X = B$ 
    using  $\langle \text{BetH } B \text{ } X \text{ } P \rangle$   $\text{between-comm}$  by presburger
  moreover have  $\text{BetH } Q \text{ } X \text{ } A \vee Q = X \vee X = A$ 
  proof (cases  $A = X$ )
    case True
      thus ?thesis
        by simp
    next
      case False
        have  $A \neq Q$ 
          using  $\langle Q \neq A \rangle$  by fastforce
        have  $P \neq C$ 
          using  $\text{assms}(1)$   $\text{colH-trivial122}$  by blast
        have  $A \neq B$ 
          using  $\langle \text{IncidL } A \text{ } l \rangle \langle \neg \text{IncidL } B \text{ } l \rangle$  by auto
        have  $X \neq Q$ 
          using  $\langle B \neq Q \rangle \langle \text{BetH } B \text{ } X \text{ } P \rangle \text{assms}(1) \text{assms}(3) \text{bet-colH between-col}$ 
             $\text{colH-trans colH-trivial121}$  by blast
        have  $\text{ColH } A \text{ } X \text{ } Q$ 
          using  $\text{ColH-def Is-line}$   $\langle \text{IncidL } A \text{ } l \rangle \langle \text{IncidL } Q \text{ } l \rangle \langle \text{IncidL } X \text{ } l \rangle$  by blast
        {
          assume  $\text{ColH } A \text{ } C \text{ } Q$ 
          obtain  $l00$  where  $\text{IncidL } A \text{ } l00$  and  $\text{IncidL } C \text{ } l00$  and  $\text{IncidL } Q \text{ } l00$ 
            using  $\text{ColH-def}$   $\langle \text{ColH } A \text{ } C \text{ } Q \rangle$  by blast
          hence  $\text{IncidL } C \text{ } l$ 
            using  $\langle A \neq Q \rangle \langle \text{IncidL } A \text{ } l \rangle \langle \text{IncidL } Q \text{ } l \rangle \text{inter-incid-uniquenessH}$  by blast
          hence False
            using  $\langle \text{IncidL } C \text{ } l \implies \text{False} \rangle$  by auto
        }
      }
    hence  $\neg \text{ColH } A \text{ } C \text{ } Q$ 
      by blast
    have  $P \neq B$ 
      using  $\text{assms}(1)$   $\text{colH-trivial121}$  by fastforce
    obtain  $m$  where  $\text{IsL } m$  and  $\text{IncidL } P \text{ } m$  and  $\text{IncidL } B \text{ } m$ 
      using  $\langle P \neq B \rangle \text{line-existence}$  by blast
    have  $\neg \text{IncidL } Q \text{ } m$ 
      by (meson  $\text{ColH-def}$   $\langle B \neq Q \rangle \langle \text{BetH } B \text{ } Q \text{ } C \rangle \langle \text{IncidL } B \text{ } m \rangle \langle \text{IncidL } P \text{ } m \rangle \text{assms}(1)$ 
         $\text{between-col inter-incid-uniquenessH}$ )
    have  $\neg \text{ColH } A \text{ } B \text{ } P$ 
      by (metis  $\langle \text{BetH } A \text{ } P \text{ } C \rangle \langle P \neq A \rangle \text{assms}(1) \text{betH-colH colH-permut-312}$ 
         $\text{colH-trans colH-trivial122}$ )
    have  $\text{cut } m \text{ } A \text{ } C$ 
      using  $\text{Hilbert-neutral-dimensionless-pre.ColH-def}$   $\langle \text{BetH } A \text{ } P \text{ } C \rangle$ 
         $\langle \text{IncidL } B \text{ } m \rangle \langle \text{IncidL } P \text{ } m \rangle \langle \text{IsL } m \rangle \langle \neg \text{ColH } A \text{ } B \text{ } P \rangle \text{assms}(1)$ 
         $\text{local.cut-def}$  by fastforce
    obtain  $p$  where  $\text{IncidP } A \text{ } p$  and  $\text{IncidP } C \text{ } p$  and  $\text{IncidP } Q \text{ } p$ 
      using  $\langle \neg \text{ColH } A \text{ } C \text{ } Q \rangle \text{plan-existence}$  by blast
    have  $(\neg \text{ColH } A \text{ } C \text{ } Q \wedge \text{IncidP } A \text{ } p \wedge \text{IncidP } C \text{ } p \wedge \text{IncidP } Q \text{ } p \wedge$ 
       $\text{IncidLP } m \text{ } p \wedge \neg \text{IncidL } Q \text{ } m \wedge \text{cut } m \text{ } A \text{ } C) \longrightarrow (\text{cut } m \text{ } A \text{ } Q \vee \text{cut } m \text{ } C \text{ } Q)$ 
      using  $\text{pasch Is-plane}$   $\langle \text{IsL } m \rangle$  by blast
    moreover have  $\neg \text{ColH } A \text{ } C \text{ } Q \wedge \text{IncidP } A \text{ } p \wedge \text{IncidP } C \text{ } p \wedge \text{IncidP } Q \text{ } p \wedge$ 
       $\text{IncidLP } m \text{ } p \wedge \neg \text{IncidL } Q \text{ } m \wedge \text{cut } m \text{ } A \text{ } C$ 
      by (metis  $\text{Is-plane}$   $\langle A \neq C \rangle \langle \text{BetH } A \text{ } P \text{ } C \rangle \langle \text{BetH } B \text{ } Q \text{ } C \rangle \langle \text{IncidL } B \text{ } m \rangle$ )

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    ⟨IncidL P m⟩ ⟨IncidP A p⟩ ⟨IncidP C p⟩ ⟨IncidP Q p⟩
    ⟨IsL m⟩ ⟨P ≠ B⟩ ⟨Q ≠ C⟩
    ⟨¬ ColH A C Q⟩ ⟨¬ IncidL Q m⟩ ⟨local.cut m A C⟩ between-col between-comm
    colH-permut-132 line-on-plane line-on-plane')
  moreover {
    assume cut m A Q
    obtain Y where IncidL Y m ∧ BetH A Y Q
    using ⟨local.cut m A Q⟩ local.cut-def by blast
    have ColH A Y Q
    by (simp add: ⟨IncidL Y m ∧ BetH A Y Q⟩ betH-colH)
    then obtain l0 where IncidL A l0 and IncidL Y l0 and IncidL Q l0
    using ColH-def by blast
    have EqL l0 l
    using Is-line ⟨A ≠ Q⟩ ⟨IncidL A l0⟩ ⟨IncidL A l⟩ ⟨IncidL Q l0⟩
    ⟨IncidL Q l⟩ line-uniqueness by presburger
    hence IncidL Y l
    using ⟨A ≠ Q⟩ ⟨IncidL A l0⟩ ⟨IncidL A l⟩ ⟨IncidL Q l0⟩ ⟨IncidL Q l⟩
    ⟨IncidL Y l0⟩ inter-incid-uniquenessH by blast
  moreover
  have ColH B X P
  by (simp add: ⟨BetH B X P⟩ between-col)
  then obtain m0 where IncidL B m0 and IncidL X m0 and IncidL P m0
  using ColH-def by blast
  have EqL m0 m
  using Is-line ⟨IncidL B m0⟩ ⟨IncidL B m⟩ ⟨IncidL P m0⟩ ⟨IncidL P m⟩
  ⟨P ≠ B⟩ line-uniqueness by presburger
  hence IncidL X m
  using IncidL-morphism Is-line ⟨IncidL X m0⟩ ⟨IsL m⟩ by blast
  ultimately
  have X = Y
  using ⟨IncidL B m⟩ ⟨IncidL X l⟩ ⟨IncidL Y m ∧ BetH A Y Q⟩
  ⟨¬ IncidL B l⟩ inter-incid-uniquenessH by auto
  hence ?thesis
  using ⟨IncidL Y m ∧ BetH A Y Q⟩ between-comm by blast
}
moreover {
  assume cut m C Q
  then obtain I where IncidL I m and BetH C I Q
  using local.cut-def by blast
  have ColH C I Q
  using ⟨BetH C I Q⟩ betH-colH by blast
  obtain lo where IncidL C lo and IncidL I lo and IncidL Q lo
  using ColH-def ⟨ColH C I Q⟩ by auto
  {
    assume IncidL C m
    hence ColH B C P
    using ⟨local.cut m A C⟩ local.cut-def by blast
    hence False
    by (simp add: assms(1))
  }
  have IncidL B lo
  using ColH-def ⟨BetH B Q C⟩ ⟨IncidL C lo⟩ ⟨IncidL Q lo⟩ betH-colH
  inter-incid-uniquenessH by blast
  have I = B
  using ⟨IncidL B lo⟩ ⟨IncidL B m⟩ ⟨IncidL I lo⟩ ⟨IncidL I m⟩
  ⟨IncidL Q lo⟩ ⟨¬ IncidL Q m⟩ inter-incid-uniquenessH by blast
  hence ?thesis
  using ⟨BetH B Q C⟩ ⟨BetH C I Q⟩ between-only-one by blast
}
ultimately show ?thesis
by blast
qed
ultimately have ?t
using Bet-def by auto
}
moreover

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```

}
ultimately show ?thesis
  by blast
qed
moreover have  $BetH\ A\ P\ C \wedge B = Q \longrightarrow ?t$ 
  using Bet-def by blast
moreover have  $BetH\ A\ P\ C \wedge Q = C \longrightarrow ?t$ 
  by (meson Bet-def between-comm)
moreover have  $A = P \wedge BetH\ B\ Q\ C \longrightarrow ?t$ 
  using Bet-def by auto
moreover have  $A = P \wedge B = Q \longrightarrow ?t$ 
  using Bet-def by auto
moreover have  $A = P \wedge Q = C \longrightarrow ?t$ 
  using Bet-def by blast
moreover have  $P = C \wedge BetH\ B\ Q\ C \longrightarrow ?t$ 
  using assms(1) colH-trivial122 by blast
moreover have  $P = C \wedge B = Q \longrightarrow ?t$ 
  using assms(1) colH-trivial122 by blast
moreover have  $P = C \wedge Q = C \longrightarrow ?t$ 
  using assms(1) colH-trivial122 by blast
ultimately show ?thesis
  by blast
qed

lemma betH-trans1:
  assumes  $BetH\ A\ B\ C$  and
     $BetH\ B\ C\ D$ 
  shows  $BetH\ A\ C\ D$ 
proof -
  have  $A \neq B$ 
    using assms(1) betH-distincts by blast
  have  $C \neq B$ 
    using assms(1) betH-distincts by blast
  have  $A \neq C$ 
    using assms(1) betH-expand by blast
  have  $D \neq C$ 
    using assms(2) betH-distincts by blast
  have  $D \neq B$ 
    using assms(2) between-diff by blast
  obtain  $E$  where  $\neg ColH\ A\ C\ E$ 
    using  $\langle A \neq C \rangle$  ncolH-exists by presburger
  hence  $C \neq E$ 
    by (simp add: ncolH-distincts)
  obtain  $F$  where  $BetH\ C\ E\ F$ 
    using  $\langle C \neq E \rangle$  between-out by fastforce
  have  $E \neq F$ 
    using  $\langle BetH\ C\ E\ F \rangle$  betH-distincts by blast
  have  $C \neq F$ 
    using  $\langle BetH\ C\ E\ F \rangle$  between-diff by blast
  have  $ColH\ A\ B\ C$ 
    using assms(1) betH-colH by auto
  have  $ColH\ B\ C\ D$ 
    by (simp add: assms(2) betH-colH)
  have  $ColH\ C\ E\ F$ 
    using  $\langle BetH\ C\ E\ F \rangle$  between-col by auto
  have  $\neg ColH\ F\ C\ B$ 
    using ColH-def  $\langle C \neq B \rangle$   $\langle C \neq F \rangle$   $\langle ColH\ A\ B\ C \rangle$   $\langle ColH\ C\ E\ F \rangle$   $\langle \neg ColH\ A\ C\ E \rangle$ 
    inter-incid-uniquenessH by auto
  have  $\exists X. Bet\ B\ X\ F \wedge Bet\ E\ X\ A$ 
proof -
  have  $Bet\ A\ B\ C$ 
    by (simp add: Bet-def assms(1))
  moreover have  $Bet\ F\ E\ C$ 
    by (simp add: Bet-def  $\langle BetH\ C\ E\ F \rangle$  between-comm)
  ultimately show ?thesis
    using  $\langle \neg ColH\ F\ C\ B \rangle$  inner-pasch-aux by auto

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qed
then obtain  $G$  where  $Bet\ B\ G\ F$  and  $Bet\ E\ G\ A$ 
by blast
{
  assume  $BetH\ B\ G\ F$ 
  hence  $ColH\ B\ G\ F$ 
  by (simp add:  $\langle Bet\ B\ G\ F \rangle bet-colH$ )
  {
    assume  $BetH\ E\ G\ A$ 
    hence  $ColH\ E\ G\ A$ 
    by (simp add:  $betH-colH$ )
    {
      assume  $ColH\ D\ B\ G$ 
      hence  $ColH\ B\ D\ F$ 
      by (metis  $betH-colH\ \langle BetH\ B\ G\ F \rangle colH-permut-231\ colH-trans\ colH-trivial121$ )
      hence False
      by (metis  $\langle ColH\ B\ C\ D \rangle \langle D \neq B \rangle \langle \neg\ ColH\ F\ C\ B \rangle colH-permut-132\ colH-trans\ colH-trivial121$ )
    }
  }
  obtain  $m$  where  $IncidL\ C\ m$  and  $IncidL\ E\ m$ 
  using  $\langle C \neq E \rangle line-existence$  by blast
  {
    assume  $cut\ m\ A\ G \vee cut\ m\ B\ G$ 
    moreover
    {
      assume  $cut\ m\ A\ G$ 
      then obtain  $E'$  where  $IncidL\ E'\ m$  and  $BetH\ A\ E'\ G$ 
      using  $local.cut-def$  by blast
      have  $E = E'$ 
      proof -
        have  $C \neq F$ 
        by (simp add:  $\langle C \neq F \rangle$ )
        moreover have  $ColH\ A\ G\ E$ 
        by (simp add:  $\langle ColH\ E\ G\ A \rangle colH-permut-321$ )
        moreover have  $ColH\ A\ G\ E'$ 
        using  $\langle BetH\ A\ E'\ G \rangle between-col\ colH-permut-132$  by blast
        moreover have  $ColH\ C\ F\ E$ 
        by (simp add:  $\langle ColH\ C\ E\ F \rangle colH-permut-132$ )
        moreover have  $\neg\ ColH\ A\ G\ C$ 
        by (metis  $betH-colH\ \langle BetH\ A\ E'\ G \rangle \langle \neg\ ColH\ A\ C\ E \rangle calculation(2)\ colH-trans\ colH-trivial121$ )
        moreover have  $ColH\ C\ F\ E'$ 
        by (meson  $ColH-def\ \langle C \neq E \rangle \langle IncidL\ C\ m \rangle \langle IncidL\ E\ m \rangle \langle IncidL\ E'\ m \rangle calculation(4)\ inter-incid-uniquenessH$ )
        ultimately show ?thesis
        using  $inter-uniquenessH$  by blast
      qed
      have  $\neg\ Bet\ E\ G\ A$ 
      using  $\langle BetH\ A\ E'\ G \rangle \langle BetH\ E\ G\ A \rangle \langle E = E' \rangle between-only-one$  by blast
      hence False
      using  $\langle Bet\ E\ G\ A \rangle \langle \neg\ Bet\ E\ G\ A \rangle$  by blast
    }
  }
  moreover
  {
    assume  $cut\ m\ B\ G$ 
    obtain  $F'$  where  $IncidL\ F'\ m$  and  $BetH\ B\ F'\ G$ 
    using  $\langle local.cut\ m\ B\ G \rangle local.cut-def$  by blast
    have  $ColH\ C\ E\ F'$ 
    using  $ColH-def\ Is-line\ \langle IncidL\ C\ m \rangle \langle IncidL\ E\ m \rangle \langle IncidL\ F'\ m \rangle$  by blast
    have  $B \neq G$ 
    using  $\langle BetH\ B\ F'\ G \rangle between-diff$  by auto
    have  $\neg\ ColH\ C\ E\ B$ 
    using  $\langle C \neq E \rangle \langle ColH\ C\ E\ F \rangle \langle \neg\ ColH\ F\ C\ B \rangle colH-trans\ colH-trivial121$  by blast
    hence  $F = F'$ 
    using  $inter-uniquenessH\ \langle ColH\ C\ E\ F \rangle \langle B \neq G \rangle \langle BetH\ B\ F'\ G \rangle \langle ColH\ B\ G\ F \rangle \langle ColH\ C\ E\ F' \rangle between-col\ colH-permut-132$  by blast
  }
}

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    hence  $\neg \text{BetH } F \ G \ B$ 
    using  $\langle \text{BetH } B \ F' \ G \rangle$  between-only-one by blast
    hence False
    using  $\langle \text{BetH } B \ G \ F \rangle$  between-comm by blast
  }
  ultimately have False
  by blast
}
hence  $\neg (\text{cut } m \ A \ G \vee \text{cut } m \ B \ G)$ 
by blast
have  $\neg \text{ColH } B \ D \ G$ 
using  $\langle \text{ColH } D \ B \ G \implies \text{False} \rangle$  colH-permut-213 by blast
{
  assume IncidL G m
  hence ColH C E G
  using ColH-def Is-line  $\langle \text{IncidL } C \ m \rangle \langle \text{IncidL } E \ m \rangle$  by blast
  hence ColH A C E
  by (metis betH-colH  $\langle \text{BetH } E \ G \ A \rangle$  colH-permut-231 colH-trans colH-trivial121)
  hence False
  by (simp add:  $\langle \neg \text{ColH } A \ C \ E \rangle$ )
}
hence  $\neg \text{IncidL } G \ m$ 
by blast
{
  assume IncidL D m
  have ColH A C D
  by (meson ColH-def  $\langle C \neq B \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{ColH } B \ C \ D \rangle$  inter-incid-uniquenessH)
  hence ColH A C E
  by (meson ColH-def  $\langle D \neq C \rangle \langle \text{IncidL } C \ m \rangle \langle \text{IncidL } D \ m \rangle \langle \text{IncidL } E \ m \rangle$ 
inter-incid-uniquenessH)
  hence False
  by (simp add:  $\langle \neg \text{ColH } A \ C \ E \rangle$ )
}
hence  $\neg \text{IncidL } D \ m$ 
by blast
{
  assume IncidL B m
  hence ColH A C E
  using ColH-def  $\langle C \neq B \rangle \langle \text{ColH } B \ C \ D \rangle \langle \text{IncidL } C \ m \rangle \langle \neg \text{IncidL } D \ m \rangle$ 
inter-incid-uniquenessH by blast
  hence False
  by (simp add:  $\langle \neg \text{ColH } A \ C \ E \rangle$ )
}
hence  $\neg \text{IncidL } B \ m$ 
by blast
hence cut m D B
using Is-line  $\langle \text{IncidL } C \ m \rangle \langle \neg \text{IncidL } D \ m \rangle$  assms(2) between-comm
local.cut-def by blast
obtain p where IncidP B p and IncidP D p and IncidP G p
using  $\langle \neg \text{ColH } B \ D \ G \rangle$  plan-existence by blast
have  $(\neg \text{ColH } B \ D \ G \wedge \text{IncidP } B \ p \wedge \text{IncidP } D \ p \wedge \text{IncidP } G \ p \wedge$ 
IncidLP m p  $\wedge \neg \text{IncidL } G \ m \wedge \text{cut } m \ B \ D) \longrightarrow (\text{cut } m \ B \ G \vee \text{cut } m \ D \ G)$ 
using IncidLP-def pasch by blast
moreover have  $\neg \text{ColH } B \ D \ G \wedge \text{IncidP } B \ p \wedge \text{IncidP } D \ p \wedge \text{IncidP } G \ p \wedge$ 
IncidLP m p  $\wedge \neg \text{IncidL } G \ m \wedge \text{cut } m \ B \ D$ 
by (metis betH-colH Is-line Is-plane  $\langle \text{BetH } B \ G \ F \rangle \langle C \neq E \rangle \langle C \neq F \rangle \langle \text{ColH } B \ C \ D \rangle$ 
 $\langle \text{ColH } C \ E \ F \rangle \langle D \neq B \rangle \langle \text{IncidL } C \ m \rangle \langle \text{IncidL } E \ m \rangle \langle \text{IncidP } B \ p \rangle \langle \text{IncidP } D \ p \rangle$ 
 $\langle \text{IncidP } G \ p \rangle \langle \neg \text{ColH } B \ D \ G \rangle \langle \neg \text{IncidL } G \ m \rangle \langle \text{local.cut } m \ D \ B \rangle$  colH-permut-312
cut-comm line-on-plane line-on-plane')
ultimately have cut m B G  $\vee$  cut m D G
by blast
moreover
{
  assume cut m B G
  hence ?thesis
  using  $\langle \neg (\text{local.cut } m \ A \ G \vee \text{local.cut } m \ B \ G) \rangle$  by blast

```

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}
moreover
{
  assume  $\text{cut } m \ D \ G$ 
  obtain  $HX$  where  $\text{IncidL } HX \ m$  and  $\text{BetH } D \ HX \ G$ 
  using  $\langle \text{local.cut } m \ D \ G \rangle$   $\text{local.cut-def}$  by  $\text{auto}$ 
  {
    assume  $\text{ColH } G \ D \ A$ 
    have  $\text{ColH } A \ D \ B$ 
    by  $(\text{metis } \langle C \neq B \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{ColH } B \ C \ D \rangle \text{colH-permut-231 colH-trans colH-trivial121})$ 
    hence  $\text{False}$ 
    by  $(\text{metis } \langle \text{ColH } G \ D \ A \rangle \langle \neg \text{ColH } B \ D \ G \rangle \text{assms}(1) \text{assms}(2) \text{between-comm between-only-one'} \text{colH-permut-321 colH-trans colH-trivial122})$ 
  }
  hence  $\neg \text{ColH } G \ D \ A$ 
  by  $\text{blast}$ 
  {
    assume  $\text{IncidL } A \ m$ 
    hence  $\text{ColH } A \ C \ E$ 
    using  $\text{ColH-def } \langle A \neq C \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{IncidL } C \ m \rangle \langle \neg \text{IncidL } B \ m \rangle$ 
     $\text{inter-incid-uniquenessH}$  by  $\text{blast}$ 
    hence  $\text{False}$ 
    by  $(\text{simp add: } \langle \neg \text{ColH } A \ C \ E \rangle)$ 
  }
  hence  $\neg \text{IncidL } A \ m$ 
  by  $\text{blast}$ 
  have  $\text{cut } m \ G \ D$ 
  by  $(\text{simp add: } \langle \text{local.cut } m \ D \ G \rangle \text{cut-comm})$ 
  obtain  $p$  where  $\text{IncidP } G \ p$  and  $\text{IncidP } D \ p$  and  $\text{IncidP } A \ p$ 
  using  $\langle \neg \text{ColH } G \ D \ A \rangle$   $\text{plan-existence}$  by  $\text{blast}$ 
  have  $(\neg \text{ColH } G \ D \ A \wedge \text{IncidP } G \ p \wedge \text{IncidP } D \ p \wedge \text{IncidP } A \ p \wedge$ 
     $\text{IncidLP } m \ p \wedge \neg \text{IncidL } A \ m \wedge \text{cut } m \ G \ D) \longrightarrow (\text{cut } m \ G \ A \vee \text{cut } m \ D \ A)$ 
  using  $\text{pasch Is-line Is-plane } \langle \text{IncidL } HX \ m \rangle$  by  $\text{blast}$ 
  moreover
  have  $\neg \text{ColH } G \ D \ A \wedge \text{IncidP } G \ p \wedge \text{IncidP } D \ p \wedge \text{IncidP } A \ p \wedge$ 
     $\text{IncidLP } m \ p \wedge \neg \text{IncidL } A \ m \wedge \text{cut } m \ G \ D$ 
  proof –
  have  $\neg \text{ColH } G \ D \ A \wedge \text{IncidP } G \ p \wedge \text{IncidP } D \ p \wedge \text{IncidP } A \ p$ 
  using  $\langle \text{IncidP } A \ p \rangle \langle \text{IncidP } D \ p \rangle \langle \text{IncidP } G \ p \rangle \langle \neg \text{ColH } G \ D \ A \rangle$  by  $\text{blast}$ 
  moreover
  have  $\text{ColH } B \ D \ C$ 
  by  $(\text{simp add: } \langle \text{ColH } B \ C \ D \rangle \text{colH-permut-132})$ 
  have  $\text{ColH } B \ A \ D$ 
  by  $(\text{metis } \langle C \neq B \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{ColH } B \ C \ D \rangle \text{colH-permut-231 colH-trans colH-trivial121})$ 
  hence  $\text{IncidP } B \ p$ 
  by  $(\text{metis calculation colH-permut-231 colH-trivial122 line-on-plane'})$ 
  hence  $\text{IncidP } C \ p$ 
  using  $\langle A \neq B \rangle \langle \text{ColH } A \ B \ C \rangle \langle \text{IncidP } A \ p \rangle \text{line-on-plane'}$  by  $\text{blast}$ 
  hence  $\text{IncidLP } m \ p$ 
  by  $(\text{metis betH-colH Is-line Is-plane } \langle \text{BetH } E \ G \ A \rangle \langle C \neq E \rangle \langle \text{IncidL } C \ m \rangle$ 
     $\langle \text{IncidL } E \ m \rangle \langle \text{IncidP } A \ p \rangle \langle \text{IncidP } G \ p \rangle \text{colH-permut-321 line-on-plane line-on-plane'})$ 
  moreover have  $\neg \text{IncidL } A \ m$ 
  by  $(\text{simp add: } \langle \neg \text{IncidL } A \ m \rangle)$ 
  moreover have  $\text{cut } m \ G \ D$ 
  by  $(\text{simp add: } \langle \text{local.cut } m \ G \ D \rangle)$ 
  ultimately show  $?thesis$ 
  by  $\text{blast}$ 
qed
ultimately have  $\text{cut } m \ G \ A \vee \text{cut } m \ D \ A$ 
  by  $\text{blast}$ 
moreover
  {
    assume  $\text{cut } m \ G \ A$ 

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    hence ?thesis
      using  $\langle \neg (local.cut\ m\ A\ G \vee local.cut\ m\ B\ G) \rangle$  cut-comm by blast
  }
  moreover
  {
    assume cut m D A
    then obtain I where IncidL I m and BetH D I A
      using local.cut-def by blast
    have ColH D I A
      using  $\langle BetH\ D\ I\ A \rangle$  between-col by fastforce
    obtain l1 where IsL l1 and IncidL A l1 and IncidL B l1 and IncidL C l1
      using ColH-def  $\langle ColH\ A\ B\ C \rangle$  by blast
    obtain l2 where IsL l2 and IncidL B l2 and IncidL C l2 and IncidL D l2
      using ColH-def  $\langle ColH\ B\ C\ D \rangle$  by blast
    obtain l3 where IsL l3 and IncidL D l3 and IncidL I l3 and IncidL A l3
      using ColH-def  $\langle ColH\ D\ I\ A \rangle$  by blast
    have A  $\neq$  D
      using  $\langle \neg ColH\ G\ D\ A \rangle$  colH-trivial122 by blast
    hence BetH D C A
      by (metis  $\langle BetH\ D\ I\ A \rangle$   $\langle C \neq B \rangle$   $\langle IncidL\ A\ l1 \rangle$   $\langle IncidL\ A\ l3 \rangle$ 
         $\langle IncidL\ B\ l1 \rangle$   $\langle IncidL\ B\ l2 \rangle$   $\langle IncidL\ C\ l1 \rangle$   $\langle IncidL\ C\ l2 \rangle$   $\langle IncidL\ C\ m \rangle$ 
         $\langle IncidL\ D\ l2 \rangle$   $\langle IncidL\ D\ l3 \rangle$   $\langle IncidL\ I\ l3 \rangle$   $\langle IncidL\ I\ m \rangle$   $\langle \neg IncidL\ A\ m \rangle$ 
        inter-incid-uniquenessH)
    hence BetH A C D
      using between-comm by blast
  }
  ultimately have ?thesis
    by blast
}
ultimately have ?thesis
  by blast
}
moreover
{
  assume E = G
  hence ?thesis
    using ColH-def  $\langle BetH\ B\ G\ F \rangle$   $\langle ColH\ C\ E\ F \rangle$   $\langle E \neq F \rangle$   $\langle \neg ColH\ F\ C\ B \rangle$ 
      between-col inter-incid-uniquenessH by fastforce
}
moreover
{
  assume G = A
  hence ?thesis
    by (metis  $\langle A \neq B \rangle$   $\langle BetH\ B\ G\ F \rangle$   $\langle ColH\ A\ B\ C \rangle$   $\langle \neg ColH\ F\ C\ B \rangle$ 
      between-col inter-uniquenessH ncolH-distincts)
}
ultimately have ?thesis
  using Bet-def  $\langle Bet\ E\ G\ A \rangle$  by blast
}
moreover
{
  assume B = G
  hence ?thesis
    using ColH-def  $\langle A \neq B \rangle$   $\langle Bet\ E\ G\ A \rangle$   $\langle ColH\ A\ B\ C \rangle$   $\langle \neg ColH\ A\ C\ E \rangle$ 
      bet-colH inter-incid-uniquenessH by fastforce
}
moreover
{
  assume G = F
  hence False
    by (metis  $\langle Bet\ E\ G\ A \rangle$   $\langle ColH\ C\ E\ F \rangle$   $\langle E \neq F \rangle$   $\langle G = F \rangle$   $\langle \neg ColH\ A\ C\ E \rangle$ 
      bet-colH colH-permut-321 colH-trans colH-trivial122)
  hence ?thesis
    by blast
}
ultimately show ?thesis

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    using Bet-def  $\langle \text{Bet } B \ G \ F \rangle$  by blast
qed

lemma betH-trans2:
  assumes BetH A B C and
    BetH B C D
  shows BetH A B D
  using assms(1) assms(2) betH-trans1 between-comm by blast

lemma betH-trans:
  assumes BetH A B C and
    BetH B C D
  shows BetH A B D  $\wedge$  BetH A C D
  using assms(1) assms(2) betH-trans1 betH-trans2 by blast

lemma not-cut3:
  assumes IncidP A p and
    IncidP B p and
    IncidP C p and
    IncidLP l p and
     $\neg$  IncidL A l and
     $\neg$  ColH A B C and
     $\neg$  cut l A B and
     $\neg$  cut l A C
  shows  $\neg$  cut l B C
  by (meson colH-permut-312 cut-comm IncidLP-def assms(1) assms(2) assms(3)
    assms(4) assms(5) assms(6) assms(7) assms(8) pasch)

lemma betH-trans0:
  assumes BetH A B C and
    BetH A C D
  shows BetH B C D  $\wedge$  BetH A B D
proof –
  have A  $\neq$  B
    using assms(1) betH-distincts by auto
  have A  $\neq$  C
    using assms(1) between-diff by auto
  have A  $\neq$  D
    using assms(2) between-diff by presburger
  obtain G where  $\neg$  ColH A B G
    using  $\langle A \neq B \rangle$  ncolH-exists by presburger
  have B  $\neq$  G
    using  $\langle \neg \text{ColH } A \ B \ G \rangle$  colH-trivial122 by blast
  obtain F where BetH B G F
    using  $\langle B \neq G \rangle$  between-out by blast
  have B  $\neq$  F
    using  $\langle \text{BetH } B \ G \ F \rangle$  between-diff by auto
  {
    assume C = F
    have ColH B F A
      using  $\langle C = F \rangle$  assms(1) between-col colH-permut-231 by presburger
    moreover have ColH B F B
      by (simp add: colH-trivial121)
    moreover have ColH B F G
      using  $\langle \text{BetH } B \ G \ F \rangle$  betH-expand colH-permut-132 by blast
    ultimately have ColH A B G
      using colH-trans  $\langle B \neq F \rangle$  by blast
    hence ?thesis
      by (simp add:  $\langle \neg \text{ColH } A \ B \ G \rangle$ )
  }
  moreover
  {
    assume C  $\neq$  F
    obtain l where IncidL C l and IncidL F l
      using  $\langle C \neq F \rangle$  line-existence by blast
  }

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assume  $\text{cut } l \ A \ B$ 
then obtain  $X$  where  $\text{IncidL } X \ l$  and  $\text{BetH } A \ X \ B$ 
  using  $\text{local.cut-def}$  by  $\text{blast}$ 
hence  $X = C$ 
proof –
  have  $\neg \text{ColH } A \ B \ F$ 
    by ( $\text{meson } \text{ColH-def } \langle B \neq F \rangle \langle \text{BetH } B \ G \ F \rangle \langle \neg \text{ColH } A \ B \ G \rangle$ 
       $\text{between-col inter-incid-uniquenessH}$ )
  moreover have  $\text{ColH } A \ B \ X$ 
    by ( $\text{simp add: } \langle \text{BetH } A \ X \ B \rangle \text{ between-col colH-permut-132}$ )
  moreover have  $\text{ColH } A \ B \ C$ 
    using  $\text{assms}(1) \text{ betH-colH}$  by  $\text{presburger}$ 
  moreover have  $\text{ColH } F \ C \ X$ 
    using  $\text{ColH-def Is-line } \langle \text{IncidL } C \ l \rangle \langle \text{IncidL } F \ l \rangle \langle \text{IncidL } X \ l \rangle$  by  $\text{blast}$ 
  moreover have  $\text{ColH } F \ C \ C$ 
    by ( $\text{simp add: colH-trivial122}$ )
  ultimately show  $?thesis$ 
    by ( $\text{metis inter-uniquenessH}$ )
qed
have  $\text{False}$ 
  using  $\langle \text{BetH } A \ X \ B \rangle \langle X = C \rangle \text{ assms}(1) \text{ between-comm between-only-one}$  by  $\text{blast}$ 
}
hence  $\neg \text{cut } l \ A \ B$ 
  by  $\text{blast}$ 
{
  assume  $\text{cut } l \ B \ G$ 
  obtain  $X$  where  $\text{IncidL } X \ l$  and  $\text{BetH } B \ X \ G$ 
    using  $\langle \text{local.cut } l \ B \ G \rangle \text{ local.cut-def}$  by  $\text{blast}$ 
  hence  $\text{ColH } B \ X \ F$ 
    by ( $\text{meson } \langle \text{BetH } B \ G \ F \rangle \langle B \neq G \rangle \text{ between-col colH-permut-132}$ 
       $\text{colH-trans colH-trivial121}$ )
  hence  $\text{False}$ 
    using  $\text{ColH-def } \langle \text{BetH } B \ G \ F \rangle \langle \text{BetH } B \ X \ G \rangle \langle \text{IncidL } F \ l \rangle \langle \text{IncidL } X \ l \rangle$ 
       $\langle \text{local.cut } l \ B \ G \rangle \text{ between-comm between-only-one inter-incid-uniquenessH}$ 
       $\text{local.cut-def}$  by  $\text{blast}$ 
}
hence  $\neg \text{cut } l \ B \ G$ 
  by  $\text{blast}$ 
obtain  $p$  where  $\text{IncidP } A \ p$  and  $\text{IncidP } B \ p$  and  $\text{IncidP } G \ p$ 
  using  $\langle \neg \text{ColH } A \ B \ G \rangle \text{ plan-existence}$  by  $\text{blast}$ 
have  $\neg \text{cut } l \ A \ G$ 
proof –
  have  $\text{IncidLP } l \ p$ 
    by ( $\text{metis Hilbert-neutral-dimensionless.line-on-plane'}$ 
       $\text{Hilbert-neutral-dimensionless-axioms Is-line Is-plane } \langle \text{BetH } B \ G \ F \rangle \langle C \neq F \rangle$ 
       $\langle \text{IncidL } C \ l \rangle \langle \text{IncidL } F \ l \rangle \langle \text{IncidP } A \ p \rangle \langle \text{IncidP } B \ p \rangle \langle \text{IncidP } G \ p \rangle$ 
       $\text{assms}(1) \text{ betH-expand line-on-plane}$ )
  moreover
  {
    assume  $\text{IncidL } B \ l$ 
    have  $\text{ColH } B \ C \ G$ 
      by ( $\text{meson } \text{ColH-def } \langle B \neq F \rangle \langle \text{BetH } B \ G \ F \rangle \langle \text{IncidL } B \ l \rangle \langle \text{IncidL } C \ l \rangle$ 
         $\langle \text{IncidL } F \ l \rangle \text{ between-col inter-incid-uniquenessH}$ )
    have  $\text{ColH } B \ F \ C$ 
      using  $\text{ColH-def Is-line } \langle \text{IncidL } B \ l \rangle \langle \text{IncidL } C \ l \rangle \langle \text{IncidL } F \ l \rangle$  by  $\text{blast}$ 
    hence  $\text{False}$ 
      using  $\text{betH-expand colH-permut-231 colH-trans colH-trivial121}$ 
       $\langle \text{ColH } B \ C \ G \rangle \langle \neg \text{ColH } A \ B \ G \rangle \text{ assms}(1)$  by  $\text{fast}$ 
  }
hence  $\neg \text{IncidL } B \ l$ 
  by  $\text{blast}$ 
moreover have  $\neg \text{ColH } B \ A \ G$ 
  using  $\langle \neg \text{ColH } A \ B \ G \rangle \text{ colH-permut-213}$  by  $\text{blast}$ 
moreover have  $\neg \text{cut } l \ B \ A$ 
  using  $\langle \neg \text{local.cut } l \ A \ B \rangle \text{ cut-comm}$  by  $\text{blast}$ 
ultimately show  $?thesis$ 

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    using ⟨IncidP A p⟩ ⟨IncidP B p⟩ ⟨IncidP G p⟩ ⟨¬ local.cut l B G⟩ not-cut3 by blast
qed
have cut l A G ∨ cut l D G
proof -
{
  assume ColH A D G
  have ColH A C A
    by (simp add: colH-trivial121)
  moreover have ColH A C D
    by (simp add: assms(2) between-col)
  moreover have ColH A C B
    using assms(1) between-col colH-permut-132 by blast
  ultimately have ColH A D B
    using ⟨A ≠ C⟩ colH-trans by blast
  hence False
    using ⟨A ≠ D⟩ ⟨ColH A D G⟩ ⟨¬ ColH A B G⟩ colH-trans colH-trivial121 by blast
}
hence ¬ ColH A D G
  by blast
then obtain p where IncidP A p and IncidP D p and IncidP G p
  using plan-existence by blast
show ?thesis
proof -
  have ¬ ColH A D G
    by (simp add: ⟨¬ ColH A D G⟩)
  moreover have IsP p
    using Is-plane ⟨IncidP D p⟩ by fastforce
  moreover have IsL l
    using Is-line ⟨IncidL C l⟩ by auto
  moreover have IncidP A p
    by (simp add: ⟨IncidP A p⟩)
  moreover have IncidP D p
    by (simp add: ⟨IncidP D p⟩)
  moreover have IncidP G p
    using ⟨IncidP G p⟩ by auto
  moreover
  have IncidP C p
    by (metis ⟨A ≠ D⟩ assms(2) between-col calculation(4) calculation(5)
      colH-permut-312 line-on-plane')
  have IncidP F p
    by (meson ⟨A ≠ C⟩ ⟨B ≠ G⟩ ⟨BetH B G F⟩ ⟨IncidP C p⟩ assms(1)
      between-col calculation(4) calculation(6) colH-permut-132 line-on-plane')
  hence IncidLP l p
    using ⟨C ≠ F⟩ ⟨IncidL C l⟩ ⟨IncidL F l⟩ ⟨IncidP C p⟩ calculation(2)
      calculation(3) line-on-plane by blast
  moreover
  {
    assume IncidL G l
    have ColH G F C
      using ColH-def Is-line ⟨IncidL C l⟩ ⟨IncidL F l⟩ ⟨IncidL G l⟩ by blast
    hence ColH B C G
      using ⟨BetH B G F⟩ betH-colH colH-permut-231 colH-trans colH-trivial121 by blast
    hence False
      by (metis betH-expand colH-permut-231 colH-trans colH-trivial121
        ⟨¬ ColH A B G⟩ assms(1))
  }
  hence ¬ IncidL G l
    by blast
  moreover have cut l A D
  proof -
  {
    assume IncidL A l
    hence ColH A C F
      using ColH-def Is-line ⟨IncidL C l⟩ ⟨IncidL F l⟩ by blast
    have ColH A B F
      using ⟨ColH A C F⟩ assms(1) betH-colH colH-permut-132 colH-trans

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    colH-trivial121 by blast
  hence False
  by (metis  $\langle B \neq F \rangle \langle \text{BetH } B \ G \ F \rangle \langle \neg \text{ColH } A \ B \ G \rangle$  between-col
    colH-permut-231 colH-trans colH-trivial121)
}
hence  $\neg \text{IncidL } A \ l$ 
  by blast
moreover
{
  assume IncidL D l
  hence ColH C D F
  using ColH-def  $\langle \text{IncidL } C \ l \rangle \langle \text{IncidL } F \ l \rangle \langle \text{IsL } l \rangle$  by auto
  hence ColH A C F
  using assms(2) betH-colH colH-permut-231 colH-trans colH-trivial121 by blast
  hence ColH A B F
  by (metis assms(1) betH-colH colH-permut-312 colH-trans colH-trivial121)
  hence False
  by (metis  $\langle B \neq F \rangle \langle \text{BetH } B \ G \ F \rangle \langle \neg \text{ColH } A \ B \ G \rangle$  between-col
    colH-permut-231 colH-trans colH-trivial121)
}
hence  $\neg \text{IncidL } D \ l$ 
  by blast
moreover have IncidL C l
  by (simp add:  $\langle \text{IncidL } C \ l \rangle$ )
moreover have BetH A C D
  by (simp add: assms(2))
ultimately show ?thesis
  using  $\langle \text{IsL } l \rangle$  local.cut-def by blast
qed
ultimately show ?thesis
  using pasch by blast
qed
moreover
{
  assume cut l D G
  {
    assume ColH D G B
    have ColH A B D
    by (meson  $\langle A \neq C \rangle$  assms(1) assms(2) between-col colH-permut-132
      colH-trans colH-trivial121)
    have  $B \neq D$ 
    using  $\langle \neg \text{local.cut } l \ A \ G \rangle \langle \neg \text{local.cut } l \ B \ G \rangle$  calculation by auto
    hence ColH A B G
    by (meson ColH-def  $\langle \text{ColH } A \ B \ D \rangle \langle \text{ColH } D \ G \ B \rangle$  inter-incid-uniquenessH)
    hence False
    using  $\langle \neg \text{ColH } A \ B \ G \rangle$  by blast
  }
  hence  $\neg \text{ColH } D \ G \ B$ 
  by blast
  {
    assume IncidL B l
    hence ColH B F C
    using ColH-def Is-line  $\langle \text{IncidL } C \ l \rangle \langle \text{IncidL } F \ l \rangle$  by blast
    have  $B \neq C$ 
    using assms(1) betH-distincts by presburger
    hence ColH A B G
    by (metis  $\langle B \neq F \rangle \langle \text{BetH } B \ G \ F \rangle \langle \text{ColH } B \ F \ C \rangle$  assms(1) between-col
      colH-permut-231 colH-trans colH-trivial121)
    hence False
    by (simp add:  $\langle \neg \text{ColH } A \ B \ G \rangle$ )
  }
}
hence  $\neg \text{IncidL } B \ l$ 
  by blast
obtain p where IncidP D p and IncidP G p and IncidP B p
  using  $\langle \neg \text{ColH } D \ G \ B \rangle$  plan-existence by blast

```



```

have (¬ ColH D G B ∧ IncidP D p ∧ IncidP G p ∧ IncidP B p ∧
IncidLP l p ∧ ¬ IncidL B l ∧ cut l D G) → (cut l D B ∨ cut l G B)
  using Is-line Is-plane ⟨IncidL C l⟩ pasch by blast
moreover
have (¬ ColH D G B ∧ IncidP D p ∧ IncidP G p ∧ IncidP B p ∧
IncidLP l p ∧ ¬ IncidL B l ∧ cut l D G)
proof -
  have ¬ ColH D G B ∧ IncidP D p ∧ IncidP G p ∧ IncidP B p
  by (simp add: ⟨IncidP B p⟩ ⟨IncidP D p⟩ ⟨IncidP G p⟩ ⟨¬ ColH D G B⟩)
  moreover have IncidLP l p
  proof -
    have ColH B D C
    by (meson ⟨A ≠ C⟩ assms(1) assms(2) between-col colH-permut-132
colH-trans colH-trivial122)
    hence IncidP C p
    by (metis calculation colH-trivial121 line-on-plane')
    moreover have IncidP F p
    using ⟨BetH B G F⟩ ⟨IncidP B p⟩ ⟨IncidP G p⟩ betH-colH line-on-plane' by blast
    ultimately show ?thesis
    using Is-line Is-plane ⟨C ≠ F⟩ ⟨IncidL C l⟩ ⟨IncidL F l⟩ line-on-plane by blast
  qed
  moreover have ¬ IncidL B l
  by (simp add: ⟨¬ IncidL B l⟩)
  moreover have cut l D G
  by (simp add: ⟨local.cut l D G⟩)
  ultimately show ?thesis
  by blast
qed
ultimately have cut l D B ∨ cut l G B
  using ⟨¬ ColH D G B ∧ IncidP D p ∧ IncidP G p ∧ IncidP B p ∧
IncidLP l p ∧ ¬ IncidL B l ∧ local.cut l D G → local.cut l D B ∨ local.cut l G B⟩
  by blast
moreover
{
  assume cut l D B
  then obtain C' where IncidL C' l and BetH D C' B
  using local.cut-def by blast
  have ColH C C' F
  using ColH-def Is-line ⟨IncidL C l⟩ ⟨IncidL C' l⟩ ⟨IncidL F l⟩ by blast
  have ¬ ColH A B F
  by (metis ⟨BetH B G F⟩ ⟨¬ ColH A B G⟩ betH-colH colH-permut-312
colH-trans colH-trivial122)
  have ColH B D A
  by (metis (full-types) ⟨A ≠ C⟩ assms(1) assms(2) between-col colH-trans
ncolH-distincts)
  hence ColH A B C'
  using ColH-def ⟨BetH D C' B⟩ betH-colH inter-incid-uniquenessH by fastforce
  then have C = C' using inter-uniquenessH
  by (metis (full-types) ⟨ColH C C' F⟩ ⟨¬ ColH A B F⟩ assms(1)
betH-colH ncolH-distincts)
  hence BetH B C D ∧ BetH A B D
  using ⟨BetH D C' B⟩ assms(1) betH-trans between-comm by blast
}
moreover
{
  assume cut l G B
  hence BetH B C D ∧ BetH A B D
  using ⟨¬ local.cut l B G⟩ cut-comm by blast
}
ultimately have BetH B C D ∧ BetH A B D
  by blast
}
ultimately have BetH B C D ∧ BetH A B D
  using ⟨¬ local.cut l A G⟩ by fastforce
}
ultimately show ?thesis

```

by blast
qed

lemma *betH-outh2--betH*:

assumes *BetH A B C* and

outh B C C' and

outh B A A'

shows *BetH A' B C'*

proof –

have *BetH B C C' ∨ BetH B C' C ∨ (B ≠ C ∧ C = C')*

using *assms(2) outh-def* by blast

moreover have *BetH B A A' ∨ BetH B A' A ∨ (B ≠ A ∧ A = A')*

using *assms(3) outh-def* by blast

moreover

{

assume *BetH B C C'*

hence *BetH A' B C*

using *assms(1) betH-trans betH-trans0 between-comm calculation(2)* by blast

}

moreover

{

assume *BetH B C' C*

hence *BetH A' B C*

using *assms(1) betH-trans betH-trans0 between-comm calculation(2)* by blast

}

moreover

{

assume *B ≠ C ∧ C = C'*

hence *BetH A' B C*

using *assms(1) betH-trans betH-trans0 between-comm calculation(2)* by blast

}

ultimately show *?thesis*

by (*meson betH-trans betH-trans0 between-comm*)

qed

lemma *cong-existence'*:

fixes *A B::'p*

assumes *A ≠ B* and

IncidL M l

shows $\exists A' B'. \text{IncidL } A' l \wedge \text{IncidL } B' l \wedge \text{BetH } A' M B' \wedge \text{CongH } M A' A B \wedge \text{CongH } M B' A B$

proof –

have $\exists P. \text{IncidL } P l \wedge M \neq P$

by (*metis Is-line assms(2) two-points-on-line*)

then obtain *P* where *IncidL P l* and *M ≠ P*

by blast

obtain *P'* where *BetH P M P'*

using $\langle M \neq P \rangle$ *between-out* by *presburger*

obtain *A'* where *IncidL A' l* and *outh M P A'* and *CongH M A' A B*

using *Is-line* $\langle \text{IncidL } P l \rangle \langle M \neq P \rangle$ *assms(1) assms(2) cong-existence* by blast

moreover

have *IncidL P' l*

using *ColH-def* $\langle \text{BetH } P M P' \rangle \langle \text{IncidL } P l \rangle$ *assms(2) betH-colH inter-incid-uniquenessH* by blast

then obtain *B'* where *IncidL B' l* and *outh M P' B'* and *CongH M B' A B*

by (*metis Is-line* $\langle \text{BetH } P M P' \rangle$ *assms(1) assms(2) betH-distincts cong-existence*)

ultimately show *?thesis*

using $\langle \text{BetH } P M P' \rangle$ *betH-outh2--betH* by blast

qed

lemma *betH-to-bet*:

assumes *BetH A B C*

shows *Bet A B C ∧ A ≠ B ∧ B ≠ C ∧ A ≠ C*

using *Bet-def assms betH-distincts* by *presburger*

lemma *betH-line*:

assumes *BetH A B C*

shows $\exists l. \text{IncidL } A \ l \wedge \text{IncidL } B \ l \wedge \text{IncidL } C \ l$
by (*meson ColH-def assms betH-colH*)

lemma *bet-identity*:
assumes *Bet A B A*
shows $A = B$
using *Bet-def assms between-diff* **by** *blast*

lemma *morph*:
assumes *IsL l* **and**
 IsL m **and**
 EqL l m
shows $\forall A. \text{IncidL } A \ l \longleftrightarrow \text{IncidL } A \ m$
using *EqL-sym IncidL-morphism assms(1) assms(2) assms(3)* **by** *blast*

lemma *point3-online-exists*:
assumes *IncidL A l* **and**
 IncidL B l
shows $\exists C. \text{IncidL } C \ l \wedge C \neq A \wedge C \neq B$
by (*metis (full-types) assms(2) betH-distincts cong-existence' lower-dim-2*)

lemma *not-betH121*:
shows $\neg \text{BetH } A \ B \ A$
using *between-diff* **by** *blast*

lemma *cong-identity*:
assumes *Cong A B C C*
shows $A = B$
using *Cong-def assms* **by** *presburger*

lemma *cong-inner-transitivity*:
assumes *Cong A B C D* **and**
 Cong A B E F
shows *Cong C D E F*
by (*metis Cong-def assms(1) assms(2) cong-pseudo-transitivity*)

lemma *other-point-on-line*:
assumes *IncidL A l*
shows $\exists B. A \neq B \wedge \text{IncidL } B \ l$
by (*metis Is-line assms two-points-on-line*)

lemma *bet-disjoint*:
assumes *BetH A B C*
shows *disjoint A B B C*
proof –
 {
 assume $\neg \text{disjoint } A \ B \ B \ C$
 then obtain *P* **where** *BetH A P B* **and** *BetH B P C*
 using *disjoint-def* **by** *blast*
 have *BetH P B C* $\wedge$ *BetH A P C*
 using *betH-trans0* $\langle \text{BetH } A \ P \ B \rangle$ *assms* **by** *blast*
 hence *False*
 using $\langle \text{BetH } B \ P \ C \rangle$ *between-only-one'* **by** *blast*
 }
thus *?thesis*
 by *blast*
qed

lemma *addition-betH*:
assumes *BetH A B C* **and**
BetH A' B' C' **and**
CongH A B A' B' **and**
CongH B C B' C'
shows *CongH A C A' C'*
using *addition* *assms(1)* *assms(2)* *assms(3)* *assms(4)* *bet-disjoint between-col* **by** *blast*

lemma *outH-trivial*:
assumes $A \neq B$
shows *outH A B B*
by (*simp add: assms outH-def*)

lemma *same-side-refl*:
assumes *IsL l* **and**
 $\neg \text{IncidL } A \ l$
shows *same-side A A l*
proof –
obtain $x \ x0$ **where** *IncidL x0 l* **and** *IncidL x l* **and** $x \neq x0$
using *two-points-on-line* *assms(1)* **by** *blast*
{
assume $A = x$
hence *same-side A A l*
using $\langle \text{IncidL } x \ l \rangle$ *assms(2)* **by** *fastforce*
}
moreover
{
assume $A \neq x$
obtain $x1$ **where** *BetH A x x1*
by (*metis* $\langle \text{IncidL } x \ l \rangle$ *assms(2)* *between-out*)
have $\neg \text{IncidL } x1 \ l$
using $\langle \text{BetH } A \ x \ x1 \rangle$ $\langle \text{IncidL } x \ l \rangle$ *assms(2)* *betH-expand betH-line*
inter-incid-uniquenessH **by** *blast*
moreover **have** $\exists \ I. \text{IncidL } I \ l \wedge \text{BetH } A \ I \ x1$
using $\langle \text{BetH } A \ x \ x1 \rangle$ $\langle \text{IncidL } x \ l \rangle$ **by** *blast*
ultimately **have** *same-side A A l*
using *assms(1)* *assms(2)* *local.cut-def same-side-def* **by** *blast*
}
ultimately **show** *?thesis*
by *blast*
qed

lemma *same-side-prime-refl*:
assumes $\neg \text{ColH } A \ B \ C$
shows *same-side' C C A B*
using *ColH-def* *assms* *ncolH-distincts same-side'-def same-side-refl* **by** *presburger*

lemma *outH-expand*:
assumes *outH A B C*
shows *outH A B C* $\wedge$ *ColH A B C* $\wedge$ $A \neq C$ $\wedge$ $A \neq B$
by (*metis* *assms* *betH-colH* *colH-permut-132* *colH-trivial122* *outH-def*)

lemma *construction-uniqueness*:
assumes *BetH A B D* **and**
BetH A B E **and**
CongH B D B E
shows $D = E$
proof –
have $A \neq D$
using *assms(1)* *between-diff* **by** *blast*
have $A \neq E$
using *assms(2)* *not-betH121* **by** *blast*
have *ColH A B D*
by (*simp add: assms(1) betH-colH*)
have *ColH A B E*
using *assms(2)* *betH-colH* **by** *fastforce*

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have  $A \neq B$ 
  using assms(1) betH-distincts by auto
then obtain  $C$  where  $\neg \text{ColH } A \ B \ C$ 
  using ncolH-exists by presburger
have CongaH  $A \ C \ D \ A \ C \ E$ 
proof -
  have  $\neg \text{ColH } A \ C \ D$ 
    by (meson  $\langle A \neq D \rangle \langle \text{ColH } A \ B \ D \rangle \langle \neg \text{ColH } A \ B \ C \rangle$  colH-permut-132
      colH-trans colH-trivial121)
  moreover have  $\neg \text{ColH } A \ C \ E$ 
    by (metis  $\langle A \neq E \rangle \langle \text{ColH } A \ B \ E \rangle \langle \neg \text{ColH } A \ B \ C \rangle$  colH-trivial121
      colH-trivial122 inter-uniquenessH)
  moreover have CongH  $A \ C \ A \ C$ 
    by (metis calculation(1) colH-trivial112 congH-refl)
  moreover have CongH  $A \ D \ A \ E$ 
    by (metis addition-betH assms(1) assms(2) assms(3) congH-refl)
  moreover have CongaH  $C \ A \ D \ C \ A \ E$ 
  proof-
    have CongaH  $C \ A \ B \ C \ A \ B$ 
      by (meson  $\langle \neg \text{ColH } A \ B \ C \rangle$  colH-permut-231 conga-refl)
    moreover have outH  $A \ C \ C$ 
      using  $\langle \neg \text{ColH } A \ C \ D \rangle$  ncolH-distincts outH-trivial by presburger
    moreover have outH  $A \ B \ D$ 
      by (simp add: assms(1) outH-def)
    moreover have outH  $A \ B \ E$ 
      by (simp add: assms(2) outH-def)
    ultimately show ?thesis
      using conga-out-conga by blast
  qed
  ultimately show ?thesis
    using cong-5 by blast
qed
moreover have  $\neg \text{ColH } D \ C \ A$ 
  by (metis ncolH-distincts  $\langle A \neq D \rangle \langle \text{ColH } A \ B \ D \rangle \langle \neg \text{ColH } A \ B \ C \rangle$  inter-uniquenessH)
moreover hence  $\neg \text{ColH } A \ C \ D$ 
  using colH-permut-321 by blast
moreover hence  $\neg \text{ColH } C \ A \ D$ 
  using colH-permut-213 by blast
moreover have same-side'  $D \ E \ C \ A$ 
proof -
  obtain  $F$  where BetH  $B \ A \ F$ 
    using  $\langle A \neq B \rangle$  between-out by fastforce
  obtain  $l$  where IncidL  $C \ l$  and IncidL  $A \ l$ 
    using ColH-def colH-trivial122 by blast
  have  $C \neq A$ 
    using calculation(2) colH-trivial122 by auto
  moreover
  have  $\forall l. \text{IsL } l \wedge \text{IncidL } C \ l \wedge \text{IncidL } A \ l \longrightarrow \text{same-side } D \ E \ l$ 
  proof -
    {
      fix  $l$ 
      assume IsL  $l$  and IncidL  $C \ l$  and IncidL  $A \ l$ 
      have cut  $l \ D \ F$ 
      proof -
        {
          assume IncidL  $D \ l$ 
          have ColH  $A \ C \ D$ 
            using ColH-def  $\langle \text{IncidL } A \ l \rangle \langle \text{IncidL } C \ l \rangle \langle \text{IncidL } D \ l \rangle \langle \text{IsL } l \rangle$  by blast
          have False
            using  $\langle \text{ColH } A \ C \ D \rangle \langle \neg \text{ColH } A \ C \ D \rangle$  by auto
        }
      }
      hence  $\neg \text{IncidL } D \ l$ 
        by blast
    }
    moreover
    {
      assume IncidL  $F \ l$ 

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    have ColH A C F
      using ColH-def Is-line ⟨IncidL A l⟩ ⟨IncidL C l⟩ ⟨IncidL F l⟩ by blast
    have False
      using ColH-def ⟨Beth B A F⟩ ⟨IncidL A l⟩ ⟨IncidL C l⟩ ⟨IncidL F l⟩
        ⟨IsL l⟩ ⟨¬ ColH A B C⟩ betH-distincts betH-line inter-incid-uniquenessH by blast
  }
  hence ¬ IncidL F l
    by blast
  moreover have ∃ I. IncidL I l ∧ Beth D I F
    using ⟨Beth B A F⟩ ⟨IncidL A l⟩ assms(1) betH-trans between-comm by blast
  ultimately show ?thesis
    by (simp add: ⟨IsL l⟩ local.cut-def)
qed
moreover have cut l E F
proof -
{
  assume IncidL E l
  have ColH A C E
    using ColH-def ⟨IncidL A l⟩ ⟨IncidL C l⟩ ⟨IncidL E l⟩ ⟨IsL l⟩ by blast
  hence False
    using ColH-def ⟨A ≠ E⟩ ⟨¬ ColH A B C⟩ assms(2) betH-line
      inter-incid-uniquenessH by blast
}
  hence ¬ IncidL E l
    by blast
  moreover
  {
    assume IncidL F l
    have ColH A C F
      using ⟨IncidL F l⟩ ⟨local.cut l D F⟩ local.cut-def by auto
    hence False
      using ⟨IncidL F l⟩ ⟨local.cut l D F⟩ local.cut-def by blast
  }
  hence ¬ IncidL F l
    by blast
  moreover have ∃ I. IncidL I l ∧ Beth E I F
    using ⟨Beth B A F⟩ ⟨IncidL A l⟩ assms(2) betH-trans between-comm by blast
  ultimately show ?thesis
    by (simp add: ⟨IsL l⟩ local.cut-def)
qed
ultimately have same-side D E l
  using ⟨IsL l⟩ same-side-def by blast
}
thus ?thesis
  by blast
qed
ultimately show ?thesis
  using same-side'-def by auto
qed
ultimately have outH C D E
  using same-side-prime-refl by (meson cong-4-uniqueness conga-refl)
thus ?thesis
  using ⟨ColH A B D⟩ ⟨ColH A B E⟩ ⟨¬ ColH A B C⟩ colH-trivial122
    inter-uniquenessH outH-expand by blast
qed

lemma out-distinct:
  assumes outH A B C
  shows A ≠ B ∧ A ≠ C
  using assms outH-expand by auto

lemma out-same-side:
  assumes IncidL A l and
    ¬ IncidL B l and
    outH A B C
  shows same-side B C l

```

```

proof -
  have  $A \neq B$ 
    using assms(1) assms(2) by blast
  obtain  $P$  where  $BetH\ B\ A\ P$ 
    using  $\langle A \neq B \rangle$  between-out by presburger
  {
    assume  $IncidL\ P\ l$ 
    have  $ColH\ B\ A\ P$ 
      by (simp add:  $\langle BetH\ B\ A\ P \rangle$  between-col)
    obtain  $ll$  where  $IncidL\ B\ ll$  and  $IncidL\ A\ ll$  and  $IncidL\ P\ ll$ 
      using  $\langle BetH\ B\ A\ P \rangle$  betH-line by blast
    hence  $EqL\ l\ ll$ 
      by (metis Is-line  $\langle BetH\ B\ A\ P \rangle$   $\langle IncidL\ P\ l \rangle$  assms(1) betH-distincts line-uniqueness)
    hence False
      using Is-line  $\langle IncidL\ B\ ll \rangle$   $\langle IncidL\ P\ l \rangle$  assms(2) morph by blast
  }
  hence  $\neg IncidL\ P\ l$ 
    by blast
  have  $cut\ l\ B\ P$ 
    using assms(1)  $\langle BetH\ B\ A\ P \rangle$  Is-line  $\langle \neg IncidL\ P\ l \rangle$  assms(2) local.cut-def by blast
  moreover have  $cut\ l\ C\ P$ 
  proof -
    have  $\neg IncidL\ C\ l$ 
      using ColH-def assms(1) assms(2) assms(3) inter-incid-uniquenessH
        outH-expand by blast
    moreover have  $BetH\ C\ A\ P$ 
      by (metis betH-outH2--betH betH-to-bet outH-trivial  $\langle BetH\ B\ A\ P \rangle$  assms(3))
    ultimately show ?thesis
      using assms(1) Is-line  $\langle \neg IncidL\ P\ l \rangle$  local.cut-def by blast
  qed
  ultimately show ?thesis
    using Is-line assms(1) same-side-def by blast
qed

```

lemma *between-one*:

```

assumes  $A \neq B$  and
   $A \neq C$  and
   $B \neq C$  and
   $ColH\ A\ B\ C$ 
shows  $BetH\ A\ B\ C \vee BetH\ B\ C\ A \vee BetH\ B\ A\ C$ 
proof -
  obtain  $D$  where  $\neg ColH\ A\ C\ D$ 
    using assms(2) ncolH-exists by presburger
  have  $B \neq D$ 
    using  $\langle \neg ColH\ A\ C\ D \rangle$  assms(4) colH-permut-132 by blast
  obtain  $G$  where  $BetH\ B\ D\ G$ 
    using  $\langle B \neq D \rangle$  between-out by blast
  have  $A \neq D$ 
    using  $\langle \neg ColH\ A\ C\ D \rangle$  colH-trivial121 by force
  then obtain  $lAD$  where  $IsL\ lAD$  and  $IncidL\ A\ lAD$  and  $IncidL\ D\ lAD$ 
    using line-existence by blast
  have  $C \neq D$ 
    using  $\langle \neg ColH\ A\ C\ D \rangle$  colH-trivial122 by blast
  then obtain  $lCD$  where  $IsL\ lCD$  and  $IncidL\ C\ lCD$  and  $IncidL\ D\ lCD$ 
    using line-existence by blast
  {
    assume  $ColH\ B\ G\ C$ 
    hence False
      by (metis  $\langle BetH\ B\ D\ G \rangle$   $\langle \neg ColH\ A\ C\ D \rangle$  assms(3) assms(4) between-col
        colH-permut-312 colH-trans colH-trivial122 not-betH121)
  }
  hence  $\neg ColH\ B\ G\ C$ 
    by blast
  {
    assume  $ColH\ B\ G\ A$ 
    hence False

```

```

    by (meson colH-permut-312 colH-trans colH-trivial122  $\langle \neg \text{ColH } B \ G \ C \rangle$ 
        assms(1) assms(4))
  }
  hence  $\neg \text{ColH } B \ G \ A$ 
    by blast
  have  $\neg \text{IncidL } C \ \text{lAD}$ 
    using ColH-def  $\langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IsL } \text{lAD} \rangle \langle \neg \text{ColH } A \ C \ D \rangle$  by blast
  have  $\neg \text{IncidL } A \ \text{lCD}$ 
    using  $\langle A \neq D \rangle \langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } C \ \text{lCD} \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IncidL } D \ \text{lCD} \rangle$ 
         $\langle \neg \text{IncidL } C \ \text{lAD} \rangle$  inter-incid-uniquenessH by blast
  {
    assume IncidL B lAD
    have ColH A B D
      using ColH-def  $\langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } B \ \text{lAD} \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IsL } \text{lAD} \rangle$  by blast
    hence False
      using  $\langle \neg \text{ColH } A \ C \ D \rangle$  assms(1) assms(4) colH-trans colH-trivial121 by blast
  }
  hence  $\neg \text{IncidL } B \ \text{lAD}$ 
    by blast
  {
    assume IncidL G lAD
    have ColH A D G
      using ColH-def Is-line  $\langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IncidL } G \ \text{lAD} \rangle$  by blast
    hence False
      using betH-colH  $\langle \text{BetH } B \ D \ G \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IncidL } G \ \text{lAD} \rangle \langle \neg \text{IncidL } B \ \text{lAD} \rangle$ 
          betH-line inter-incid-uniquenessH by blast
  }
  hence  $\neg \text{IncidL } G \ \text{lAD}$ 
    by blast
  have cut lAD B G
    using  $\langle \text{BetH } B \ D \ G \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IsL } \text{lAD} \rangle \langle \neg \text{IncidL } B \ \text{lAD} \rangle \langle \neg \text{IncidL } G \ \text{lAD} \rangle$ 
        local.cut-def by blast
  {
    assume IncidL B lCD
    have ColH B C D
      using ColH-def  $\langle \text{IncidL } B \ \text{lCD} \rangle \langle \text{IncidL } C \ \text{lCD} \rangle \langle \text{IncidL } D \ \text{lCD} \rangle \langle \text{IsL } \text{lCD} \rangle$  by blast
    hence False
      by (meson colH-permut-231 colH-trans colH-trivial122  $\langle \neg \text{ColH } A \ C \ D \rangle$  assms(3) assms(4))
  }
  hence  $\neg \text{IncidL } B \ \text{lCD}$ 
    by blast
  {
    assume IncidL G lCD
    have ColH C D G
      using ColH-def  $\langle \text{IncidL } C \ \text{lCD} \rangle \langle \text{IncidL } D \ \text{lCD} \rangle \langle \text{IncidL } G \ \text{lCD} \rangle \langle \text{IsL } \text{lCD} \rangle$  by blast
    hence False
      using betH-colH  $\langle \text{BetH } B \ D \ G \rangle \langle \text{IncidL } D \ \text{lCD} \rangle \langle \text{IncidL } G \ \text{lCD} \rangle \langle \neg \text{IncidL } B \ \text{lCD} \rangle$ 
          betH-line inter-incid-uniquenessH by blast
  }
  hence  $\neg \text{IncidL } G \ \text{lCD}$ 
    by blast
  have cut lCD B G
    using Is-line  $\langle \text{BetH } B \ D \ G \rangle \langle \text{IncidL } D \ \text{lCD} \rangle \langle \neg \text{IncidL } B \ \text{lCD} \rangle$ 
         $\langle \neg \text{IncidL } G \ \text{lCD} \rangle$  local.cut-def by blast
  obtain bgc where IsP bgc and IncidP B bgc and IncidP G bgc and IncidP C bgc
    using  $\langle \neg \text{ColH } B \ G \ C \rangle$  plan-existence by blast
  obtain bga where IsP bga and IncidP B bga and IncidP G bga and IncidP A bga
    using  $\langle \neg \text{ColH } B \ G \ A \rangle$  plan-existence by blast
  have IncidLP lAD bgc
    using line-on-plane
    by (metis  $\langle A \neq D \rangle \langle \text{BetH } B \ D \ G \rangle \langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } D \ \text{lAD} \rangle \langle \text{IncidP } B \ \text{bgc} \rangle \langle \text{IncidP } C \ \text{bgc} \rangle$ 
         $\langle \text{IncidP } G \ \text{bgc} \rangle \langle \text{IsL } \text{lAD} \rangle \langle \text{IsP } \text{bgc} \rangle \langle \neg \text{ColH } B \ G \ C \rangle$  assms(4) between-col colH-permut-132
        colH-permut-321 line-on-plane' same-side'-def same-side-prime-refl)
  have IncidLP lCD bga
    using line-on-plane
    by (smt (verit, best) IncidLP-def IncidP-morphism  $\langle C \neq D \rangle \langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } C \ \text{lCD} \rangle$ 

```



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    ⟨IncidL D lAD⟩ ⟨IncidL D lCD⟩ ⟨IncidLP lAD bgc⟩ ⟨IncidP A bga⟩ ⟨IncidP B bga⟩
    ⟨IncidP B bgc⟩ ⟨IncidP G bga⟩ ⟨IncidP G bgc⟩ ⟨IsL lCD⟩ ⟨IsP bga⟩ ⟨¬ ColH B G A⟩
    assms(1,4) line-on-plane' plane-uniqueness)
show ?thesis
proof -
{
  assume cut lAD B C
  then obtain I where IncidL I lAD and BetH B I C
    using local.cut-def by blast
  have BetH A B C ∨ BetH B C A ∨ BetH B A C
  proof -
    {
      assume cut lCD B A
      {
        assume A = I
        have BetH A B C ∨ BetH B C A ∨ BetH B A C
          by (simp add: ⟨A = I⟩ ⟨BetH B I C⟩)
      }
      moreover
      {
        assume A ≠ I
        have ColH A D I
          using ColH-def Is-line ⟨IncidL A lAD⟩ ⟨IncidL D lAD⟩ ⟨IncidL I lAD⟩ by blast
        have ¬ ColH B C D
          by (meson ⟨¬ ColH A C D⟩ assms(3) assms(4) colH-permut-231
              colH-trans colH-trivial122)
        have ColH B C I
          by (simp add: ⟨BetH B I C⟩ betH-colH colH-permut-132)
        have ColH B C A
          by (simp add: assms(4) colH-permut-231)
        have ColH D A I
          using ⟨ColH A D I⟩ colH-permut-213 by blast
        have ColH D A A
          by (simp add: colH-trivial122)
        hence False using inter-uniquenessH
          by (metis ⟨A ≠ I⟩ ⟨ColH B C A⟩ ⟨ColH B C I⟩ ⟨ColH D A I⟩ ⟨¬ ColH B C D⟩)
        hence BetH A B C ∨ BetH B C A ∨ BetH B A C
          by blast
      }
      ultimately
      have BetH A B C ∨ BetH B C A ∨ BetH B A C
        by fastforce
    }
  moreover
  {
    assume cut lCD G A
    {
      assume A = I
      have BetH A B C ∨ BetH B C A ∨ BetH B A C
        by (simp add: ⟨A = I⟩ ⟨BetH B I C⟩)
    }
    moreover
    {
      assume A ≠ I
      have ColH A D I
        using ColH-def Is-line ⟨IncidL A lAD⟩ ⟨IncidL D lAD⟩ ⟨IncidL I lAD⟩ by blast
      have D ≠ A
        using ⟨A ≠ D⟩ by blast
      moreover have ¬ ColH B C D
        by (meson ⟨¬ ColH A C D⟩ assms(3) assms(4) colH-permut-231
            colH-trans colH-trivial122)
      moreover have ColH B C I
        by (simp add: ⟨BetH B I C⟩ betH-colH colH-permut-132)
      moreover have ColH B C A
        by (simp add: assms(4) colH-permut-231)
      moreover have ColH D A I
    }
  }
}

```

```

    using ⟨ColH A D I⟩ colH-permut-213 by blast
  moreover have ColH D A A
  by (simp add: colH-trivial122)
  hence False
  by (metis ⟨A ≠ I⟩ calculation(2) calculation(3) calculation(4)
    calculation(5) inter-uniquenessH)
  hence BetH A B C ∨ BetH B C A ∨ BetH B A C
  by blast
}
ultimately have BetH A B C ∨ BetH B C A ∨ BetH B A C
by blast
}
ultimately show ?thesis
using ⟨IncidLP LCD bga⟩ ⟨IncidP A bga⟩ ⟨IncidP B bga⟩ ⟨IncidP G bga⟩
  ⟨IsL LCD⟩ ⟨IsP bga⟩ ⟨¬ ColH B G A⟩ ⟨¬ IncidL A LCD⟩
  ⟨local.cut LCD B G⟩ pasch by blast
qed
}
moreover
{
  assume cut lAD G C
  have BetH A B C ∨ BetH B C A ∨ BetH B A C
  proof -
  {
    assume cut LCD B A
    then obtain I where IncidL I LCD and BetH B I A
    using local.cut-def by blast
    {
      assume C = I
      hence BetH A B C ∨ BetH B C A ∨ BetH B A C
      by (simp add: ⟨BetH B I A⟩)
    }
  }
  moreover
  {
    assume C ≠ I
    have ColH C D I
    using ColH-def Is-line ⟨IncidL C LCD⟩ ⟨IncidL D LCD⟩ ⟨IncidL I LCD⟩ by blast
    have D ≠ C
    using ⟨C ≠ D⟩ by blast
    moreover have ¬ ColH A B D
    using ⟨¬ ColH A C D⟩ assms(1) assms(4) colH-trans colH-trivial121 by blast
    moreover have ColH A B I
    by (simp add: ⟨BetH B I A⟩ betH-colH colH-permut-312)
    moreover have ColH A B C
    by (simp add: assms(4))
    moreover have ColH D C I
    by (simp add: ⟨ColH C D I⟩ colH-permut-213)
    moreover have ColH D C C
    by (simp add: colH-trivial122)
    ultimately have False
    using ⟨C ≠ I⟩ inter-uniquenessH by blast
    hence BetH A B C ∨ BetH B C A ∨ BetH B A C
    by blast
  }
  ultimately have BetH A B C ∨ BetH B C A ∨ BetH B A C
  by blast
}
}
moreover {
  assume cut LCD G A
  obtain E where IncidL E lAD and BetH G E C
  using local.cut-def ⟨local.cut lAD G C⟩ by blast
  obtain F where IncidL F LCD and BetH G F A
  using ⟨local.cut LCD G A⟩ local.cut-def by blast
  {
    assume ColH A G E
    hence ColH A C D

```

```

    using ⟨BetH G E C⟩ ⟨¬ ColH B G A⟩ assms(4) betH-colH
      colH-permut-231 colH-trivial121 inter-uniquenessH by blast
  hence False
    using ⟨¬ ColH A C D⟩ by auto
}
hence ¬ ColH A G E
  by blast
have C ≠ F
  using between-col ⟨BetH G E C⟩ ⟨BetH G F A⟩ ⟨¬ ColH A G E⟩
    betH-trans0 colH-permut-312 by blast
obtain lCF where IsL lCF and IncidL C lCF and IncidL F lCF
  using ⟨IncidL C lCD⟩ ⟨IncidL F lCD⟩ ⟨IsL lCD⟩ by blast
{
  assume IncidL E lCF
  hence False
    by (metis (full-types) Hilbert-neutral-dimensionless.betH-expand
      Hilbert-neutral-dimensionless-axioms ⟨BetH G E C⟩ ⟨C ≠ F⟩ ⟨IncidL C lCD⟩
      ⟨IncidL C lCF⟩ ⟨IncidL E lCF⟩ ⟨IncidL F lCD⟩ ⟨IncidL F lCF⟩
      ⟨¬ IncidL G lCD⟩ betH-line inter-incid-uniquenessH)
}
hence ¬ IncidL E lCF
  by blast
have cut lCF A G
  by (metis IncidL-morphism Is-line ⟨BetH G F A⟩ ⟨C ≠ F⟩ ⟨IncidL C lCD⟩
    ⟨IncidL C lCF⟩ ⟨IncidL F lCD⟩ ⟨IncidL F lCF⟩ ⟨¬ IncidL A lCD⟩
    ⟨¬ IncidL G lCD⟩ between-comm line-uniqueness local.cut-def)
obtain p where IncidP A p and IncidP G p and IncidP E p
  using IncidLP-def ⟨IncidL A lAD⟩ ⟨IncidL E lAD⟩ ⟨IncidLP lAD bgc⟩
    ⟨IncidP G bgc⟩ by blast
have IncidP C p
  using ⟨BetH G E C⟩ ⟨IncidP E p⟩ ⟨IncidP G p⟩ betH-colH line-on-plane' by blast
moreover have IncidP F p
  by (metis ⟨BetH G F A⟩ ⟨IncidP A p⟩ ⟨IncidP G p⟩ betH-colH
    colH-permut-312 line-on-plane')
ultimately have IncidLP lCF p
  using Is-plane ⟨C ≠ F⟩ ⟨IncidL C lCF⟩ ⟨IncidL F lCF⟩ ⟨IsL lCF⟩
    line-on-plane by blast
{
  assume cut lCF A E
  then obtain I where IncidL I lCF and BetH A I E
    using local.cut-def by blast
  have I = D
    by (metis (full-types) Hilbert-neutral-dimensionless.betH-colH
      Hilbert-neutral-dimensionless-axioms ⟨BetH A I E⟩ ⟨C ≠ F⟩ ⟨IncidL A lAD⟩
      ⟨IncidL C lCD⟩ ⟨IncidL C lCF⟩ ⟨IncidL D lAD⟩ ⟨IncidL D lCD⟩ ⟨IncidL E lAD⟩
      ⟨IncidL F lCD⟩ ⟨IncidL F lCF⟩ ⟨IncidL I lCF⟩ ⟨¬ IncidL A lCD⟩
      betH-line inter-incid-uniquenessH)
  {
    assume ColH A E C
    have A ≠ E
      using ⟨BetH A I E⟩ not-betH121 by fastforce
    have ColH A D E
      using ⟨BetH A I E⟩ ⟨I = D⟩ between-col by auto
    hence ColH A C D
      by (meson Hilbert-neutral-dimensionless.colH-permut-132
        colH-trans colH-trivial121 Hilbert-neutral-dimensionless-axioms
        ⟨A ≠ E⟩ ⟨ColH A E C⟩)
    hence False
      using ⟨¬ ColH A C D⟩ by blast
  }
}
hence ¬ ColH A E C
  by blast
obtain lBD where IncidL B lBD and IncidL D lBD
  using ⟨B ≠ D⟩ line-existence by blast
have cut lBD A E
  by (metis betH-colH cut-def

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    Is-line  $\langle \text{BetH } A \ I \ E \rangle \langle I = D \rangle \langle \text{IncidL } A \ \text{lAD} \rangle \langle \text{IncidL } B \ \text{lBD} \rangle \langle \text{IncidL } D \ \text{lAD} \rangle$ 
     $\langle \text{IncidL } D \ \text{lBD} \rangle \langle \text{IncidL } E \ \text{lAD} \rangle \langle \neg \text{IncidL } B \ \text{lAD} \rangle$  inter-incid-uniquenessH)
  have IncidLP lBD p
  by (metis IncidLP-def Is-line  $\langle B \neq D \rangle \langle I = D \rangle \langle \text{IncidL } B \ \text{lBD} \rangle \langle \text{IncidL } D \ \text{lBD} \rangle$ 
     $\langle \text{IncidL } I \ \text{lCF} \rangle \langle \text{IncidLP } \text{lCF } p \rangle \langle \text{IncidP } A \ p \rangle \langle \text{IncidP } C \ p \rangle$  assms(2) assms(4)
    colH-permut-312 line-on-plane line-on-plane')
{
  assume cut lBD A C
  then obtain I where IncidL I lBD and BetH A I C
  using local.cut-def by blast
  hence ColH A I C
  using betH-colH by blast
  hence  $I = B$ 
  proof -
    have  $D \neq B$ 
    using  $\langle B \neq D \rangle$  by auto
    moreover have  $\neg \text{ColH } A \ C \ D$ 
    by (simp add:  $\langle \neg \text{ColH } A \ C \ D \rangle$ )
    moreover have ColH A C I
    using  $\langle \text{ColH } A \ I \ C \rangle$  colH-permut-132 by blast
    moreover have ColH A C B
    by (simp add: assms(4) colH-permut-132)
    moreover have ColH D B I
    using ColH-def Is-line  $\langle \text{IncidL } B \ \text{lBD} \rangle \langle \text{IncidL } D \ \text{lBD} \rangle \langle \text{IncidL } I \ \text{lBD} \rangle$  by blast
    moreover have ColH D B B
    by (simp add: colH-trivial122)
    ultimately show ?thesis
    using inter-uniquenessH by blast
  qed
  hence  $\text{BetH } A \ B \ C \vee \text{BetH } B \ C \ A \vee \text{BetH } B \ A \ C$ 
  using  $\langle \text{BetH } A \ I \ C \rangle$  by auto
}
moreover {
  assume cut lBD E C
  then obtain I where IncidL I lBD and BetH E I C
  using local.cut-def by blast
  hence ColH E I C
  by (simp add: betH-colH)
  hence  $I = G$ 
  proof -
    have  $C \neq E$ 
    using  $\langle \text{IncidL } E \ \text{lAD} \rangle \langle \neg \text{IncidL } C \ \text{lAD} \rangle$  by auto
    moreover have  $\neg \text{ColH } D \ B \ C$ 
    by (metis  $\langle \text{BetH } B \ D \ G \rangle \langle \neg \text{ColH } B \ G \ C \rangle$  betH-colH colH-permut-213
      colH-trans colH-trivial122)
    moreover have ColH D B I
    using ColH-def Is-line  $\langle \text{IncidL } B \ \text{lBD} \rangle \langle \text{IncidL } D \ \text{lBD} \rangle \langle \text{IncidL } I \ \text{lBD} \rangle$  by blast
    moreover have ColH D B G
    using  $\langle \text{BetH } B \ D \ G \rangle$  between-col colH-permut-213 by blast
    moreover have ColH C E I
    using  $\langle \text{ColH } E \ I \ C \rangle$  colH-permut-312 by blast
    moreover have ColH C E G
    by (simp add:  $\langle \text{BetH } G \ E \ C \rangle$  between-col between-comm)
    ultimately show ?thesis
    using inter-uniquenessH by blast
  qed
  hence  $\text{BetH } A \ B \ C \vee \text{BetH } B \ C \ A \vee \text{BetH } B \ A \ C$ 
  using  $\langle \text{BetH } E \ I \ C \rangle \langle \text{BetH } G \ E \ C \rangle$  between-only-one' by blast
}
ultimately have  $\text{BetH } A \ B \ C \vee \text{BetH } B \ C \ A \vee \text{BetH } B \ A \ C$ 
using pasch inter-incid-uniquenessH Is-line Is-plane  $\langle C \neq D \rangle \langle \text{IncidL } B \ \text{lBD} \rangle$ 
 $\langle \text{IncidL } C \ \text{lCD} \rangle \langle \text{IncidL } D \ \text{lBD} \rangle \langle \text{IncidL } D \ \text{lCD} \rangle \langle \text{IncidLP } \text{lBD } p \rangle \langle \text{IncidP } A \ p \rangle$ 
 $\langle \text{IncidP } C \ p \rangle \langle \text{IncidP } E \ p \rangle \langle \neg \text{ColH } A \ E \ C \rangle \langle \neg \text{IncidL } B \ \text{lCD} \rangle$ 
 $\langle \text{local.cut } \text{lBD } A \ E \rangle$  by fastforce
}
moreover

```

```

{
  assume cut lCF G E
  then obtain I where IncidL I lCF and BetH G I E
    using local.cut-def by blast
  hence ColH G I E
    using betH-expand by blast
  have I = C
  proof -
    have F ≠ C
      using ⟨C ≠ F⟩ by auto
    moreover have ¬ ColH G E F
      using ⟨BetH G F A⟩ ⟨¬ ColH A G E⟩ betH-colH colH-permut-132
        colH-trans colH-trivial121 by blast
    moreover have ColH G E I
      using ⟨ColH G I E⟩ colH-permut-132 by presburger
    moreover have ColH G E C
      by (simp add: ⟨BetH G E C⟩ betH-colH)
    moreover have ColH F C I
      using ColH-def Is-line ⟨IncidL C lCF⟩ ⟨IncidL F lCF⟩
        ⟨IncidL I lCF⟩ by blast
    moreover have ColH F C C
      by (simp add: colH-trivial122)
    ultimately show ?thesis
      using inter-uniquenessH by blast
  qed
  hence BetH A B C ∨ BetH B C A ∨ BetH B A C
    using ⟨BetH G E C⟩ ⟨BetH G I E⟩ between-comm between-only-one by blast
}
ultimately have BetH A B C ∨ BetH B C A ∨ BetH B A C
  using Is-plane ⟨IncidLP lCF p⟩ ⟨IncidP A p⟩ ⟨IncidP E p⟩ ⟨IncidP G p⟩
    ⟨IsL lCF⟩ ⟨¬ ColH A G E⟩ ⟨¬ IncidL E lCF⟩ ⟨local.cut lCF A G⟩
    pasch by blast
}
ultimately show ?thesis
  using ⟨IncidLP lCD bga⟩ ⟨IncidP A bga⟩ ⟨IncidP B bga⟩ ⟨IncidP G bga⟩
    ⟨IsL lCD⟩ ⟨IsP bga⟩ ⟨¬ ColH B G A⟩ ⟨¬ IncidL A lCD⟩
    ⟨local.cut lCD B G⟩ pasch by blast
qed
}
ultimately show ?thesis
  using pasch ⟨IsL lAD⟩ ⟨IsP bgc⟩ ⟨IncidLP lAD bgc⟩ ⟨IncidP B bgc⟩ ⟨IncidP C bgc⟩
    ⟨IncidP G bgc⟩ ⟨¬ ColH B G C⟩ ⟨¬ IncidL C lAD⟩ ⟨local.cut lAD B G⟩ by blast
qed
qed

```

lemma betH-dec:

shows $BetH A B C \vee \neg BetH A B C$
by blast

lemma cut2-not-cut:

assumes $\neg ColH A B C$ and
cut l A B and
cut l A C

shows $\neg cut l B C$

proof -

have $\neg IncidL A l$
using assms(2) local.cut-def by blast
have $\neg IncidL B l$
using assms(2) local.cut-def by blast
have $\neg IncidL C l$
using assms(3) local.cut-def by blast
obtain AB where IncidL AB l and BetH A AB B
using assms(2) local.cut-def by blast
hence ColH A AB A
using colH-trivial121 by blast

```

obtain  $AC$  where  $\text{IncidL } AC \text{ } l$  and  $\text{BetH } A \text{ } AC \text{ } C$ 
  using  $\text{assms}(3)$   $\text{local.cut-def}$  by  $\text{blast}$ 
hence  $\text{ColH } A \text{ } AC \text{ } C$ 
  by  $(\text{simp add: between-col})$ 
{
  assume  $\text{cut } l \text{ } B \text{ } C$ 
  then obtain  $BC$  where  $\text{IncidL } BC \text{ } l$  and  $\text{BetH } B \text{ } BC \text{ } C$ 
    using  $\text{local.cut-def}$  by  $\text{blast}$ 
  hence  $\text{ColH } B \text{ } BC \text{ } C$ 
    by  $(\text{simp add: between-col})$ 
  have  $\text{ColH } AB \text{ } AC \text{ } BC$ 
    using  $\text{ColH-def Is-line } \langle \text{IncidL } AB \text{ } l \rangle \langle \text{IncidL } AC \text{ } l \rangle \langle \text{IncidL } BC \text{ } l \rangle$  by  $\text{blast}$ 
  have  $AB \neq AC$ 
    using  $\text{colH-trans } \langle \text{BetH } A \text{ } AB \text{ } B \rangle \langle \text{ColH } A \text{ } AB \text{ } A \rangle \langle \text{ColH } A \text{ } AC \text{ } C \rangle$   $\text{assms}(1)$ 
     $\text{betH-expand}$  by  $\text{blast}$ 
  have  $AB \neq BC$ 
    by  $(\text{metis } \langle \text{BetH } A \text{ } AB \text{ } B \rangle \langle \text{ColH } B \text{ } BC \text{ } C \rangle \text{assms}(1) \text{betH-colH colH-trans ncolH-distincts})$ 
  have  $BC \neq AC$ 
    by  $(\text{metis } \langle \text{BetH } A \text{ } AC \text{ } C \rangle \langle \text{BetH } B \text{ } BC \text{ } C \rangle \text{assms}(1) \text{betH-colH between-comm}$ 
     $\text{colH-trans ncolH-distincts})$ 
  have  $\text{BetH } AB \text{ } AC \text{ } BC \vee \text{BetH } AC \text{ } BC \text{ } AB \vee \text{BetH } BC \text{ } AB \text{ } AC$ 
    using  $\langle AB \neq AC \rangle \langle AB \neq BC \rangle \langle BC \neq AC \rangle \langle \text{ColH } AB \text{ } AC \text{ } BC \rangle$   $\text{between-comm between-one}$  by  $\text{blast}$ 
moreover
{
  assume  $\text{BetH } AB \text{ } AC \text{ } BC$ 
  then obtain  $m$  where  $\text{IncidL } A \text{ } m$  and  $\text{IncidL } C \text{ } m$ 
    using  $\langle \text{BetH } A \text{ } AC \text{ } C \rangle$   $\text{betH-line}$  by  $\text{blast}$ 
  have  $\neg \text{ColH } AB \text{ } BC \text{ } B$ 
    using  $\text{ColH-def } \langle AB \neq BC \rangle \langle \text{IncidL } AB \text{ } l \rangle \langle \text{IncidL } BC \text{ } l \rangle \langle \neg \text{IncidL } B \text{ } l \rangle$ 
     $\text{inter-incid-uniquenessH}$  by  $\text{blast}$ 
  have  $\neg \text{IncidL } B \text{ } m$ 
    using  $\text{Hilbert-neutral-dimensionless-pre.ColH-def Is-line } \langle \text{IncidL } A \text{ } m \rangle$ 
     $\langle \text{IncidL } C \text{ } m \rangle$   $\text{assms}(1)$  by  $\text{fastforce}$ 
  have  $\text{cut } m \text{ } AB \text{ } BC$ 
  proof –
  {
    assume  $\text{IncidL } AB \text{ } m$ 
    hence  $\text{ColH } A \text{ } B \text{ } C$ 
    using  $\text{betH-colH } \langle \text{BetH } A \text{ } AB \text{ } B \rangle \langle \text{IncidL } A \text{ } m \rangle \langle \neg \text{IncidL } B \text{ } m \rangle$   $\text{betH-line}$ 
     $\text{inter-incid-uniquenessH}$  by  $\text{blast}$ 
    hence  $?thesis$ 
    by  $(\text{simp add: assms}(1))$ 
  }
  moreover
  {
    assume  $\text{IncidL } BC \text{ } m$ 
    hence  $\text{ColH } A \text{ } B \text{ } C$ 
    using  $\text{betH-colH } \langle \text{BetH } B \text{ } BC \text{ } C \rangle \langle \text{IncidL } C \text{ } m \rangle \langle \neg \text{IncidL } B \text{ } m \rangle$   $\text{betH-line}$ 
     $\text{inter-incid-uniquenessH}$  by  $\text{blast}$ 
    hence  $?thesis$ 
    by  $(\text{simp add: assms}(1))$ 
  }
  moreover have  $\text{IncidL } AC \text{ } m$ 
    using  $\text{betH-colH } \langle \text{BetH } A \text{ } AC \text{ } C \rangle \langle \text{IncidL } A \text{ } m \rangle \langle \text{IncidL } C \text{ } m \rangle$   $\text{betH-line}$ 
     $\text{inter-incid-uniquenessH}$  by  $\text{blast}$ 
  ultimately show  $?thesis$ 
    using  $\text{Is-line } \langle \text{BetH } AB \text{ } AC \text{ } BC \rangle$   $\text{local.cut-def}$  by  $\text{blast}$ 
qed
obtain  $p$  where  $\text{IncidP } AB \text{ } p$  and  $\text{IncidP } BC \text{ } p$  and  $\text{IncidP } B \text{ } p$ 
  using  $\langle \neg \text{ColH } AB \text{ } BC \text{ } B \rangle$   $\text{plan-existence}$  by  $\text{blast}$ 
have  $(\neg \text{ColH } AB \text{ } BC \text{ } B \wedge \text{IncidP } AB \text{ } p \wedge \text{IncidP } BC \text{ } p \wedge \text{IncidP } B \text{ } p \wedge$ 
   $\text{IncidLP } m \text{ } p \wedge \neg \text{IncidL } B \text{ } m \wedge \text{cut } m \text{ } AB \text{ } BC) \longrightarrow (\text{cut } m \text{ } AB \text{ } B \vee \text{cut } m \text{ } BC \text{ } B)$ 
  using  $\text{pasch Is-line Is-plane } \langle \text{IncidL } C \text{ } m \rangle$  by  $\text{blast}$ 
moreover
have  $(\neg \text{ColH } AB \text{ } BC \text{ } B \wedge \text{IncidP } AB \text{ } p \wedge \text{IncidP } BC \text{ } p \wedge \text{IncidP } B \text{ } p \wedge$ 
   $\text{IncidLP } m \text{ } p \wedge \neg \text{IncidL } B \text{ } m \wedge \text{cut } m \text{ } AB \text{ } BC)$ 

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proof –
  have IncidLP m p  $\wedge$   $\neg$  IncidL B m  $\wedge$  cut m AB BC
  by (metis (full-types) Hilbert-neutral-dimensionless.betH-colH
    Hilbert-neutral-dimensionless-axioms Is-line Is-plane  $\langle AB \neq BC \rangle$   $\langle BetH A AC C \rangle$ 
     $\langle BetH B BC C \rangle$   $\langle ColH AB AC BC \rangle$   $\langle IncidL A m \rangle$   $\langle IncidL C m \rangle$   $\langle IncidP AB p \rangle$ 
     $\langle IncidP B p \rangle$   $\langle IncidP BC p \rangle$   $\langle \neg IncidL B m \rangle$   $\langle local.cut m AB BC \rangle$  betH-line
    colH-permut-132 inter-incid-uniquenessH line-on-plane line-on-plane')
  thus ?thesis
  using  $\langle IncidP AB p \rangle$   $\langle IncidP B p \rangle$   $\langle IncidP BC p \rangle$   $\langle \neg ColH AB BC B \rangle$  by auto
qed
ultimately have cut m AB B  $\vee$  cut m BC B by blast
moreover
{
  assume cut m AB B
  then obtain I where IncidL I m and BetH AB I B
  using local.cut-def by blast
  hence ColH AB I B
  using between-col by blast
  have ColH A I C
  using ColH-def Is-line  $\langle IncidL A m \rangle$   $\langle IncidL C m \rangle$   $\langle IncidL I m \rangle$  by blast
  {
    assume A = I
    hence  $\neg BetH A AB B$ 
    using  $\langle A = I \rangle$   $\langle BetH AB I B \rangle$  between-only-one' by blast
    hence False
    using  $\langle BetH A AB B \rangle$  by blast
  }
  hence A  $\neq$  I
  by blast
  have ColH A I A
  using colH-trivial121 by fastforce
  have ColH A I B
  by (meson  $\langle BetH A AB B \rangle$   $\langle BetH AB I B \rangle$  betH-colH betH-trans0 between-comm)
  hence ColH A B C
  using colH-trans  $\langle A \neq I \rangle$   $\langle ColH A I A \rangle$   $\langle ColH A I C \rangle$  by blast
  hence False
  by (simp add: assms(1))
}
moreover
{
  assume cut m BC B
  then obtain C' where IncidL C' m and BetH BC C' B
  using local.cut-def by blast
  have ColH BC C' B
  by (simp add:  $\langle BetH BC C' B \rangle$  betH-colH)
  have B  $\neq$  BC
  using  $\langle IncidL BC b \rangle$   $\langle \neg IncidL B b \rangle$  by auto
  have ColH B BC B
  using colH-trivial121 by auto
  have ColH B BC C'
  using  $\langle ColH BC C' B \rangle$  colH-permut-312 by blast
  have ColH B C C'
  using  $\langle ColH B BC C \rangle$   $\langle B \neq BC \rangle$   $\langle ColH B BC B \rangle$   $\langle ColH B BC C' \rangle$ 
     $\langle ColH B BC C \rangle$  colH-trans by blast
  have C  $\neq$  C'
  using  $\langle BetH B BC C \rangle$   $\langle BetH BC C' B \rangle$  between-only-one by blast
  have ColH A B C
  using ColH-def  $\langle C \neq C' \rangle$   $\langle ColH B C C' \rangle$   $\langle IncidL C m \rangle$   $\langle IncidL C' m \rangle$ 
     $\langle \neg IncidL B m \rangle$  inter-incid-uniquenessH by auto
  hence False
  by (simp add: assms(1))
}
ultimately have False
by blast
}
moreover

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{
  assume BetH AC BC AB
  obtain m where IncidL C m and IncidL B m
  using  $\langle \text{BetH } B \ BC \ C \rangle$  betH-line by blast
  have  $\neg \text{ColH } AB \ AC \ A$ 
  using ColH-def  $\langle AB \neq AC \rangle$   $\langle \text{IncidL } AB \ l \rangle$   $\langle \text{IncidL } AC \ l \rangle$ 
     $\langle \neg \text{IncidL } A \ l \rangle$  inter-incid-uniquenessH by blast
  have  $\neg \text{IncidL } A \ m$ 
  using ColH-def Is-line  $\langle \text{IncidL } B \ m \rangle$   $\langle \text{IncidL } C \ m \rangle$  assms(1) by blast
  have  $\neg \text{IncidL } AC \ m$ 
  using betH-colH  $\langle \text{BetH } A \ AC \ C \rangle$   $\langle \text{IncidL } C \ m \rangle$   $\langle \neg \text{IncidL } A \ m \rangle$ 
    betH-line inter-incid-uniquenessH by blast
  have  $\neg \text{IncidL } AB \ m$ 
  using betH-colH  $\langle \text{BetH } A \ AB \ B \rangle$   $\langle \text{IncidL } B \ m \rangle$   $\langle \neg \text{IncidL } A \ m \rangle$ 
    betH-line inter-incid-uniquenessH by blast
  have IncidL BC m
  using betH-colH  $\langle \text{BetH } B \ BC \ C \rangle$   $\langle \text{IncidL } B \ m \rangle$   $\langle \text{IncidL } C \ m \rangle$ 
    betH-line inter-incid-uniquenessH by blast
  hence cut m AC AB
  using Is-line  $\langle \text{BetH } AC \ BC \ AB \rangle$   $\langle \text{IncidL } BC \ m \rangle$   $\langle \neg \text{IncidL } AB \ m \rangle$ 
     $\langle \neg \text{IncidL } AC \ m \rangle$  local.cut-def by blast
  hence cut m AB AC
  using cut-comm by blast
  obtain p where IncidP AB p and IncidP AC p and IncidP A p
  using  $\langle \neg \text{ColH } AB \ AC \ A \rangle$  plan-existence by blast
  have  $(\neg \text{ColH } AB \ AC \ A \wedge \text{IncidP } AB \ p \wedge \text{IncidP } AC \ p \wedge \text{IncidP } A \ p \wedge$ 
     $\text{IncidLP } m \ p \wedge \neg \text{IncidL } A \ m \wedge \text{cut } m \ AB \ AC) \longrightarrow (\text{cut } m \ AB \ A \vee \text{cut } m \ AC \ A)$ 
  using Is-line Is-plane  $\langle \text{IncidL } C \ m \rangle$  pasch by blast
  moreover
  have  $\neg \text{ColH } AB \ AC \ A \wedge \text{IncidP } AB \ p \wedge \text{IncidP } AC \ p \wedge \text{IncidP } A \ p \wedge$ 
     $\text{IncidLP } m \ p \wedge \neg \text{IncidL } A \ m \wedge \text{cut } m \ AB \ AC$ 
  proof –
  have  $\neg \text{ColH } AB \ AC \ A \wedge \text{IncidP } AB \ p \wedge \text{IncidP } AC \ p \wedge \text{IncidP } A \ p$ 
  by (simp add:  $\langle \text{IncidP } A \ p \rangle$   $\langle \text{IncidP } AB \ p \rangle$   $\langle \text{IncidP } AC \ p \rangle$   $\langle \neg \text{ColH } AB \ AC \ A \rangle$ )
  moreover have IncidLP m p  $\wedge \neg \text{IncidL } A \ m \wedge \text{cut } m \ AB \ AC$ 
  by (metis (mono-tags, lifting) betH-colH line-on-plane Is-line Is-plane
     $\langle AB \neq AC \rangle$   $\langle \text{BetH } A \ AC \ C \rangle$   $\langle \text{BetH } B \ BC \ C \rangle$   $\langle \text{ColH } AB \ AC \ BC \rangle$   $\langle \text{IncidL } BC \ m \rangle$   $\langle \text{IncidL } C \ m \rangle$ 
     $\langle \text{IncidP } A \ p \rangle$   $\langle \text{IncidP } AB \ p \rangle$   $\langle \text{IncidP } AC \ p \rangle$   $\langle \neg \text{IncidL } A \ m \rangle$   $\langle \text{local.cut } m \ AB \ AC \rangle$ 
    line-on-plane)
  ultimately show ?thesis
  by blast
qed
ultimately have cut m AB A  $\vee$  cut m AC A
  by blast
moreover {
  assume cut m AB A
  obtain B' where IncidL B' m and BetH AB B' A
  using  $\langle \text{local.cut } m \ AB \ A \rangle$  local.cut-def by blast
  then have ColH AB B' A
  using betH-colH by force
  have ColH A B B'
  using betH-expand  $\langle \text{BetH } A \ AB \ B \rangle$   $\langle \text{ColH } A \ AB \ A \rangle$   $\langle \text{ColH } AB \ B' \ A \rangle$ 
    colH-permut-312 colH-trans by blast
  hence False
  using ColH-def  $\langle \text{BetH } A \ AB \ B \rangle$   $\langle \text{BetH } AB \ B' \ A \rangle$   $\langle \text{IncidL } B \ m \rangle$   $\langle \text{IncidL } B' \ m \rangle$ 
     $\langle \neg \text{IncidL } A \ m \rangle$  between-only-one inter-incid-uniquenessH by blast
}
moreover {
  assume cut m AC A
  then obtain I where IncidL I m and BetH AC I A
  using local.cut-def by blast
  hence ColH AC I A
  by (simp add: betH-colH)
  have ColH C I A
  by (metis betH-colH  $\langle \text{BetH } AC \ I \ A \rangle$   $\langle \text{ColH } A \ AC \ C \rangle$  colH-permut-312
    colH-trans colH-trivial121)
}

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    hence ColH A B C
    using ColH-def ⟨BetH A AC C⟩ ⟨BetH AC I A⟩ ⟨IncidL C m⟩ ⟨IncidL I m⟩
      ⟨¬ IncidL A m⟩ between-only-one inter-incid-uniquenessH by blast
    hence False
    using assms(1) by fastforce
  }
  ultimately have False
    by blast
}
moreover
{
  assume BetH BC AB AC
  obtain m where IncidL A m and IncidL B m
    using ⟨BetH A AB B⟩ betH-line by blast
  have ¬ ColH BC A C
    by (metis ⟨ColH B BC C⟩ ⟨IncidL BC l⟩ ⟨¬ IncidL C l⟩ assms(1)
      colH-permut-213 colH-trans colH-trivial122)
  have ¬ IncidL C m
    using ColH-def Is-line ⟨IncidL A m⟩ ⟨IncidL B m⟩ assms(1) by blast
  have cut m BC AC
    by (metis (full-types) ncolH-distincts Is-line ⟨AB ≠ AC⟩ ⟨AB ≠ BC⟩
      ⟨BetH A AB B⟩ ⟨BetH BC AB AC⟩ ⟨IncidL A m⟩ ⟨IncidL AB l⟩ ⟨IncidL AC l⟩
      ⟨IncidL B m⟩ ⟨¬ IncidL B l⟩ assms(1) betH-line inter-incid-uniquenessH
      local.cut-def)
  have ¬ IncidL AC m
    using ⟨local.cut m BC AC⟩ local.cut-def by blast
  obtain p where IncidP BC p and IncidP AC p and IncidP C p
    using plan-existence
    by (meson ⟨BC ≠ AC⟩ line-on-plane' ncolH-exists)
  have (¬ ColH BC AC C ∧ IncidP BC p ∧ IncidP AC p ∧ IncidP C p ∧
    IncidLP m p ∧ ¬ IncidL C m ∧ cut m BC AC) ⟶ (cut m BC C ∨ cut m AC C)
    using Is-line Is-plane ⟨IncidL B m⟩ pasch by blast
  moreover have ¬ ColH BC AC C ∧ IncidP BC p ∧ IncidP AC p
    using ColH-def ⟨BC ≠ AC⟩ ⟨IncidL AC l⟩ ⟨IncidL BC l⟩ ⟨IncidP AC p⟩
      ⟨IncidP BC p⟩ ⟨¬ IncidL C l⟩ inter-incid-uniquenessH by auto
  moreover have IncidP C p ∧ IncidLP m p ∧ ¬ IncidL C m ∧ cut m BC AC
    by (metis ncolH-distincts Is-line Is-plane ⟨ColH A AC C⟩ ⟨ColH B BC C⟩
      ⟨IncidL A m⟩ ⟨IncidL B m⟩ ⟨IncidP C p⟩ ⟨¬ IncidL C m⟩ ⟨local.cut m BC AC⟩
      assms(1) calculation(2) colH-permut-321 line-on-plane line-on-plane')
  moreover have ¬ cut m BC C
    by (meson ⟨BetH B BC C⟩ ⟨IncidL B m⟩ betH-trans0 between-comm local.cut-def
      outH-def out-same-side same-side-def)
  moreover have ¬ cut m AC C
    by (meson ⟨BetH A AC C⟩ ⟨IncidL A m⟩ betH-trans0 between-comm local.cut-def
      outH-def out-same-side same-side-def)
  ultimately have False
    by blast
}
ultimately
have False
  by blast
}
thus ?thesis
  by blast
qed

```

lemma strong-pasch:

```

assumes ¬ ColH A B C and
  IncidP A p and
  IncidP B p and
  IncidP C p and
  IncidLP l p and
  ¬ IncidL C l and
  cut l A B

```

```

shows (cut l A C ∧ ¬ cut l B C) ∨ (cut l B C ∧ ¬ cut l A C)
by (meson cut2-not-cut IncidLP-def assms(1) assms(2) assms(3) assms(4))

```

assms(5) assms(6) assms(7) pasch)

lemma *out2-out*:

assumes $C \neq D$ **and**

BetH A B C **and**

BetH A B D

shows $BetH B C D \vee BetH B D C$

proof –

have $A \neq B$

using *assms(2) betH-distincts* **by** *blast*

have *ColH A B B*

by (*simp add: colH-trivial122*)

have *ColH A B C*

by (*simp add: assms(2) between-col*)

have *ColH A B D*

by (*simp add: assms(3) betH-colH*)

hence *ColH B C D*

using *colH-trans* $\langle A \neq B \rangle \langle ColH A B B \rangle \langle ColH A B C \rangle$ **by** *blast*

{

assume *BetH C D B*

hence *?thesis*

using *between-comm* **by** *blast*

}

moreover

{

assume *BetH C B D*

have *ColH A C D*

using $\langle A \neq B \rangle \langle ColH A B C \rangle \langle ColH A B D \rangle$ *colH-trans colH-trivial121* **by** *blast*

moreover have *BetH A C D* $\longrightarrow$ *?thesis*

using *assms(2) betH-trans0* **by** *blast*

moreover have *BetH C D A* $\longrightarrow$ *?thesis*

using *assms(3) betH-trans0 between-comm* **by** *blast*

moreover have *BetH C A D* $\longrightarrow$ *?thesis*

using *assms(2) assms(3) betH-trans0 between-comm between-only-one'* **by** *blast*

ultimately have *?thesis*

by (*metis assms(1) assms(2) assms(3) betH-colH between-one*)

}

ultimately show *?thesis*

by (*metis* $\langle ColH B C D \rangle$ *assms(1) assms(2) assms(3) betH-distincts between-one*)

qed

lemma *out2-out1*:

assumes $C \neq D$ **and**

BetH A B C **and**

BetH A B D

shows $BetH A C D \vee BetH A D C$

by (*meson assms(1) assms(2) assms(3) betH-trans out2-out*)

lemma *betH2-out*:

assumes $B \neq C$ **and**

BetH A B D **and**

BetH A C D

shows $BetH A B C \vee BetH A C B$

proof –

have $A \neq D$

using *assms(2) between-diff* **by** *auto*

moreover have *ColH A D A*

by (*simp add: colH-trivial121*)

moreover have *ColH A D B*

by (*simp add: assms(2) betH-colH colH-permut-132*)

moreover have *ColH A D C*

using *assms(3) betH-colH colH-permut-132* **by** *blast*

ultimately have *ColH A B C*

using *colH-trans* **by** *blast*

moreover have *BetH B C A* $\longrightarrow$ *?thesis*

using *between-comm* **by** *blast*

```

moreover have  $BetH\ C\ A\ B \longrightarrow ?thesis$ 
  using  $assms(2)\ assms(3)\ betH-trans2\ between-only-one'$  by  $blast$ 
ultimately show  $?thesis$ 
  by  $(metis\ assms(1)\ assms(2)\ assms(3)\ betH-colH\ between-comm\ between-one)$ 
qed

lemma segment-construction:
  shows  $\exists\ E.\ Bet\ A\ B\ E \wedge Cong\ B\ E\ C\ D$ 
proof  $(cases\ C = D)$ 
  case  $True$ 
  thus  $?thesis$ 
  using  $Bet-def\ Cong-def$  by  $force$ 
next
  case  $False$ 
  hence  $C \neq D$ 
  by  $blast$ 
  {
    assume  $A = B$ 
    hence  $?thesis$ 
    by  $(metis\ Bet-def\ Cong-def\ betH-distincts\ cong-existence'\ line-existence)$ 
  }
moreover
  {
    assume  $A \neq B$ 
    obtain  $l$  where  $IncidL\ A\ l$  and  $IncidL\ B\ l$ 
    using  $\langle A \neq B \rangle\ line-existence$  by  $auto$ 
    obtain  $F$  where  $\exists\ B'.\ IncidL\ F\ l \wedge IncidL\ B'\ l \wedge BetH\ F\ B\ B' \wedge CongH\ B\ F\ C\ D \wedge CongH\ B\ B'\ C\ D$ 
    using  $False\ \langle IncidL\ B\ l \rangle\ cong-existence'$  by  $presburger$ 
    then obtain  $F'$  where  $IncidL\ F\ l$  and  $IncidL\ F'\ l$  and  $BetH\ F\ B\ F'$  and
       $CongH\ B\ F\ C\ D$  and  $CongH\ B\ F'\ C\ D$ 
    by  $blast$ 
    hence  $ColH\ A\ F\ F'$ 
    using  $ColH-def\ Is-line\ \langle IncidL\ A\ l \rangle$  by  $blast$ 
    have  $A = F \longrightarrow ?thesis$ 
    by  $(metis\ Bet-def\ Cong-def\ \langle BetH\ F\ B\ F' \rangle\ \langle CongH\ B\ F'\ C\ D \rangle\ betH-distincts)$ 
    moreover {
      assume  $A \neq F$ 
      have  $A = F' \longrightarrow ?thesis$ 
      by  $(metis\ Bet-def\ Cong-def\ \langle BetH\ F\ B\ F' \rangle\ \langle CongH\ B\ F\ C\ D \rangle\ betH-distincts\ between-comm)$ 
    }
    moreover
    {
      assume  $A \neq F'$ 
      have  $ColH\ A\ F\ F'$ 
      using  $\langle ColH\ A\ F\ F' \rangle$  by  $blast$ 
      have  $BetH\ A\ F\ F' \longrightarrow ?thesis$ 
      by  $(metis\ Cong-def\ False\ \langle BetH\ F\ B\ F' \rangle\ \langle CongH\ B\ F'\ C\ D \rangle\ betH-to-bet\ betH-trans0\ between-comm)$ 
      moreover have  $BetH\ F\ F'\ A \longrightarrow ?thesis$ 
      by  $(metis\ Bet-def\ Cong-def\ False\ \langle BetH\ F\ B\ F' \rangle\ \langle CongH\ B\ F\ C\ D \rangle\ betH-distincts\ betH-trans0\ between-comm)$ 
      moreover have  $BetH\ F'\ A\ F \longrightarrow ?thesis$ 
      by  $(metis\ (full-types)\ Cong-def\ False\ \langle A \neq B \rangle\ \langle BetH\ F\ B\ F' \rangle\ \langle CongH\ B\ F\ C\ D \rangle\ \langle CongH\ B\ F'\ C\ D \rangle\ betH2-out\ betH-to-bet\ between-comm\ between-only-one\ out2-out)$ 
      ultimately have  $?thesis$ 
      by  $(metis\ between-one\ \langle A = F \longrightarrow (\exists\ E.\ Bet\ A\ B\ E \wedge Cong\ B\ E\ C\ D) \rangle\ \langle A = F' \longrightarrow (\exists\ E.\ Bet\ A\ B\ E \wedge Cong\ B\ E\ C\ D) \rangle\ \langle BetH\ F\ B\ F' \rangle\ \langle ColH\ A\ F\ F' \rangle\ between-comm\ between-diff)$ 
    }
    ultimately have  $?thesis$ 
    by  $blast$ 
  }
  ultimately have  $?thesis$ 
  by  $blast$ 
}
ultimately show  $?thesis$ 
by  $blast$ 

```

qed

lemma *lower-dim-e*:

shows $\exists A B C. \neg (Bet A B C \vee Bet B C A \vee Bet C A B)$
by (*meson bet-colH colH-permut-312 lower-dim-2*)

lemma *outH-dec*:

shows $outH A B C \vee \neg outH A B C$
by *simp*

lemma *out-construction*:

assumes $X \neq Y$ and
 $A \neq B$
shows $\exists C. CongH A C X Y \wedge outH A B C$
by (*metis assms(1) assms(2) cong-existence line-existence*)

lemma *segment-constructionH*:

assumes $A \neq B$ and
 $C \neq D$
shows $\exists E. BetH A B E \wedge CongH B E C D$
by (*metis Bet-def Cong-def assms(1) assms(2) segment-construction*)

lemma *EqL-dec*:

shows $EqL l m \vee \neg EqL l m$
by *simp*

lemma *cut-exists*:

assumes $IsL l$ and
 $\neg IncidL A l$
shows $\exists B. cut l A B$
using *assms(1) assms(2) same-side-def same-side-refl* by *blast*

lemma *outH-col*:

assumes $outH A B C$
shows $ColH A B C$
by (*simp add: assms outH-expand*)

lemma *cut-distinct*:

assumes $cut l A B$
shows $A \neq B$
using *assms local.cut-def not-betH121* by *fastforce*

lemma *same-side-not-cut*:

assumes *same-side* $A B l$
shows $\neg cut l A B$

proof –

obtain P where $cut l A P$ and $cut l B P$
using *assms same-side-def* by *blast*
{
 assume $ColH P A B$
 {
 assume $cut l A B$
 obtain M where $IncidL M l$ and $BetH A M P$
 using $\langle local.cut l A P \rangle local.cut-def$ by *blast*
 obtain N where $IncidL N l$ and $BetH B N P$
 using $\langle local.cut l B P \rangle local.cut-def$ by *blast*
 {
 assume $M = N$
 hence $BetH M A B \vee BetH M B A$
 using $\langle BetH A M P \rangle \langle BetH B N P \rangle \langle local.cut l A B \rangle$ *between-comm*
 cut-distinct out2-out by *blast*
 obtain Q where $IncidL Q l$ and $BetH A Q B$
 using $\langle local.cut l A B \rangle local.cut-def$ by *blast*
 obtain R where $IncidL R l$ and $BetH A R B$
 using $\langle BetH A Q B \rangle \langle IncidL Q l \rangle$ by *blast*
 have $ColH A B M$

```

    using ⟨BetH M A B ∨ BetH M B A⟩ betH-colH between-comm colH-permut-231 by blast
  obtain m where IncidL A m and IncidL B m
    using ⟨BetH A Q B⟩ betH-line by blast
  obtain mm where IncidL P mm and IncidL A mm and IncidL B mm
    using ColH-def ⟨ColH P A B⟩ by blast
  have EqL m mm
    by (metis Is-line ⟨IncidL A m⟩ ⟨IncidL A mm⟩ ⟨IncidL B m⟩ ⟨IncidL B mm⟩
      ⟨ $\bigwedge thesis. (\bigwedge R. \llbracket IncidL R l; BetH A R B \rrbracket \implies thesis) \implies thesis$ ⟩
      betH-colH line-uniqueness)
  obtain nn where IncidL A nn and IncidL R nn and IncidL B nn
    using ⟨BetH A R B⟩ betH-line by blast
  have EqL m nn
    by (metis Is-line ⟨BetH A R B⟩ ⟨IncidL A m⟩ ⟨IncidL A nn⟩ ⟨IncidL B m⟩
      ⟨IncidL B nn⟩ line-uniqueness not-betH121)
  obtain pp where IncidL A pp and IncidL B pp and IncidL M pp
    using ColH-def ⟨ColH A B M⟩ by blast
  have EqL m pp
    by (metis Is-line ⟨BetH M A B ∨ BetH M B A⟩ ⟨IncidL A m⟩ ⟨IncidL A pp⟩
      ⟨IncidL B m⟩ ⟨IncidL B pp⟩ betH-distincts line-uniqueness)
  have R = M
    by (metis betH-colH ⟨IncidL A nn⟩ ⟨IncidL A pp⟩ ⟨IncidL B nn⟩
      ⟨IncidL B pp⟩ ⟨IncidL M l⟩ ⟨IncidL M pp⟩ ⟨IncidL R l⟩ ⟨IncidL R nn⟩
      ⟨local.cut l A B⟩ inter-incid-uniquenessH local.cut-def)
  have BetH M A B  $\longrightarrow$  False
    using ⟨BetH A R B⟩ ⟨R = M⟩ between-only-one' by blast
  moreover have BetH M B A  $\longrightarrow$  False
    using ⟨BetH A R B⟩ ⟨R = M⟩ between-only-one by blast
  ultimately have False
    using ⟨BetH M A B ∨ BetH M B A⟩ by blast
}
hence M  $\neq$  N
  by blast
have A  $\neq$  B
  using ⟨local.cut l A B⟩ cut-distinct by auto
obtain T where IncidL T l and BetH A T B
  using ⟨local.cut l A B⟩ local.cut-def by blast
obtain m where IncidL P m and IncidL A m and IncidL B m
  using ColH-def ⟨ColH P A B⟩ by blast
obtain mm where IncidL A mm and IncidL M mm and IncidL P mm
  using ⟨BetH A M P⟩ betH-line by blast
obtain nn where IncidL B nn and IncidL N nn and IncidL P nn
  using ⟨BetH B N P⟩ betH-line by blast
have M = N using inter-incid-uniquenessH
  by (metis Hilbert-neutral-dimensionless-pre.cut-def ⟨BetH B N P⟩
    ⟨IncidL A m⟩ ⟨IncidL A mm⟩ ⟨IncidL B m⟩ ⟨IncidL M l⟩ ⟨IncidL M mm⟩
    ⟨IncidL N l⟩ ⟨IncidL P m⟩ ⟨IncidL P mm⟩
    ⟨ $\bigwedge thesis. (\bigwedge nn. \llbracket IncidL B nn; IncidL N nn; IncidL P nn \rrbracket \implies thesis) \implies thesis$ ⟩
    ⟨local.cut l A P⟩ betH-expand)
hence False
  by (simp add: ⟨M  $\neq$  N⟩)
}
hence  $\neg$  cut l A B
  by blast
}
moreover
{
  assume  $\neg$  ColH P A B
  hence ?thesis
    using ⟨local.cut l A P⟩ ⟨local.cut l B P⟩ cut2-not-cut cut-comm by blast
}
ultimately show ?thesis
  by blast
qed

lemma IncidLP-morphism:
  assumes

```

```

  IsL m and
  IsP q and
  IncidLP l p and
  EqL l m and
  EqP p q
shows IncidLP m q
proof -
{
  fix A
  assume IncidL A m
  have IncidP A q
    by (metis Hilbert-neutral-dimensionless-pre.IncidLP-def IncidP-morphism
        ⟨IncidL A m⟩ assms(1) assms(2) assms(3) assms(4) assms(5) morph)
}
thus ?thesis
  using IncidLP-def assms(1) assms(2) by blast
qed

lemma same-side--plane:
  assumes same-side A B l
  shows  $\exists p. \text{IncidP } A p \wedge \text{IncidP } B p \wedge \text{IncidLP } l p$ 
proof -
  obtain P where cut l A P and cut l B P
    using assms same-side-def by blast
  then obtain X where IncidL X l and BetH A X P
    using local.cut-def by blast
  {
    assume ColH A B P
    obtain Y where  $X \neq Y$  and IncidL Y l
      using ⟨IncidL X l⟩ other-point-on-line by blast
    {
      assume ColH X Y A
      obtain m where IncidL X m and IncidL Y m and IncidL A m
        using ColH-def ⟨ColH X Y A⟩ by auto
      hence  $X = Y$ 
        using inter-incid-uniquenessH ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨local.cut l A P⟩
        local.cut-def by blast
      hence False
        by (simp add: ⟨ $X \neq Y$ ⟩)
    }
    hence  $\neg \text{ColH } X Y A$ 
      by blast
    obtain p where IncidP X p and IncidP Y p and IncidP A p
      using ⟨ColH X Y A  $\implies$  False⟩ plan-existence by blast
    hence ?thesis
      by (metis betH-colH Is-line Is-plane ⟨BetH A X P⟩ ⟨ColH A B P⟩
          ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨ $X \neq Y$ ⟩ colH-permut-132 line-on-plane line-on-plane')
  }
  moreover
  {
    assume  $\neg \text{ColH } A B P$ 
    obtain p where IncidP A p and IncidP B p and IncidP P p
      using ⟨ $\neg \text{ColH } A B P$ ⟩ plan-existence by blast
    obtain Y where IncidL Y l and BetH B Y P
      using ⟨local.cut l B P⟩ local.cut-def by blast
    have IncidP X p
      by (metis ncolH-distincts ⟨BetH A X P⟩ ⟨IncidP A p⟩ ⟨IncidP P p⟩
          ⟨ $\neg \text{ColH } A B P$ ⟩ between-col colH-permut-132 line-on-plane')
    moreover have IncidP Y p
      using ⟨BetH B Y P⟩ ⟨IncidP B p⟩ ⟨IncidP P p⟩ betH-colH colH-permut-132
      line-on-plane' by blast
    ultimately have IncidLP l p using line-on-plane
      by (metis Is-line Is-plane ⟨BetH A X P⟩ ⟨BetH B Y P⟩ ⟨IncidL X l⟩
          ⟨IncidL Y l⟩ ⟨ $\neg \text{ColH } A B P$ ⟩ between-col between-comm colH-permut-231
          colH-trivial112 out2-out1)
    hence ?thesis
  }

```

```

    using ⟨IncidP A p⟩ ⟨IncidP B p⟩ by blast
  }
  ultimately show ?thesis
    by blast
qed

lemma same-side-prime--plane:
  assumes same-side' A B C D
  shows  $\exists p. \text{IncidP } A \ p \wedge \text{IncidP } B \ p \wedge \text{IncidP } C \ p \wedge \text{IncidP } D \ p$ 
proof -
  obtain l where IsL l and IncidL C l and IncidL D l
    by (metis line-existence lower-dim-2)
  obtain p where IsP p and IncidP A p and IncidP B p and IncidLP l p
    by (meson Is-plane ⟨IncidL C l⟩ ⟨IncidL D l⟩ ⟨IsL l⟩ assms
      same-side'-def same-side--plane)
  thus ?thesis
    using IncidLP-def ⟨IncidL C l⟩ ⟨IncidL D l⟩ by blast
qed

lemma cut-same-side-cut:
  assumes cut l P X and
    same-side X Y l
  shows cut l P Y
proof -
  have  $\neg \text{cut } l \ X \ Y$ 
    by (simp add: assms(2) same-side-not-cut)
  have  $X = Y \longrightarrow ?thesis$ 
    using assms(1) by fastforce
  moreover {
    assume  $X \neq Y$ 
    obtain A where IncidL A l and BetH P A X
      using assms(1) local.cut-def by blast
    {
      assume ColH P X Y
      {
        assume IncidL Y l
        hence False
          using ⟨IncidL Y l⟩ assms(2) local.cut-def same-side-def by force
      }
      hence  $\neg \text{IncidL } Y \ l$ 
        by blast
      have IncidL A l
        by (simp add: ⟨IncidL A l⟩)
      moreover have BetH P A Y
      proof -
        have BetH P X Y  $\longrightarrow$  BetH P A Y
          using ⟨BetH P A X⟩ betH-trans0 by blast
        moreover have BetH X Y P  $\longrightarrow$  BetH P A Y
          by (metis ⟨BetH P A X⟩ ⟨IncidL A l⟩  $\langle \neg \text{IncidL } Y \ l \rangle$   $\langle \neg \text{local.cut } l \ X \ Y \rangle$ 
            assms(1) betH2-out betH-trans0 between-comm local.cut-def)
        moreover have BetH X P Y  $\longrightarrow$  BetH P A Y
          by (meson  $\langle \neg \text{IncidL } Y \ l \rangle$   $\langle \neg \text{local.cut } l \ X \ Y \rangle$  assms(1) betH-trans0
            between-comm local.cut-def)
        ultimately show ?thesis
          by (metis betH-to-bet between-one cut-comm ⟨BetH P A X⟩ ⟨ColH P X Y⟩
             $\langle \neg \text{local.cut } l \ X \ Y \rangle$  assms(1))
      qed
    }
    qed
    ultimately have ?thesis
      using  $\langle \neg \text{IncidL } Y \ l \rangle$  assms(1) local.cut-def by blast
  }
  moreover
  {
    assume  $\neg \text{ColH } P \ X \ Y$ 
    {
      assume IncidL Y l
      obtain T where cut l X T and cut l Y T
    }
  }

```

```

    using assms(2) same-side-def by blast
  hence False
    using ⟨IncidL Y l⟩ local.cut-def by blast
}
hence ¬ IncidL Y l
  by blast
obtain p where IncidP P p and IncidP X p and IncidP Y p
  using ⟨¬ ColH P X Y⟩ plan-existence by blast
have cut l X Y ∨ cut l P Y
proof -
  let ?A=X
  let ?B=P
  let ?C=Y
  have ¬ ColH ?A ?B ?C
    using ⟨¬ ColH P X Y⟩ colH-permut-213 by blast
  moreover have IsL l
    using assms(2) same-side-def by blast
  moreover have IsP p
    using Is-plane ⟨IncidP P p⟩ by blast
  moreover have IncidP ?A p
    by (simp add: ⟨IncidP X p⟩)
  moreover have IncidP ?B p
    using ⟨IncidP P p⟩ by auto
  moreover have IncidP ?C p
    by (simp add: ⟨IncidP Y p⟩)
  moreover have IncidLP l p
proof -
  {
    fix A'
    assume IncidL A' l
    have ColH P A X
      by (simp add: ⟨BetH P A X⟩ between-col)

    obtain I where IncidL I l and BetH P I X
      by (meson ⟨∧thesis. (∧A. [IncidL A l; BetH P A X] ⟹ thesis) ⟹ thesis⟩)
    hence ColH P I X
      using between-col by blast
    have I = A' ⟶ IncidP A' p
      by (metis betH-colH ⟨BetH P I X⟩ calculation(4) calculation(5)
        colH-permut-132 line-on-plane')
    moreover
    obtain p' where IsP p' and IncidP X p' and IncidP Y p' and IncidLP l p'
      using Is-plane assms(2) same-side--plane by blast
    have IncidP P p'
      using IncidLP-def ⟨BetH P A X⟩ ⟨ColH P A X⟩ ⟨IncidL A l⟩
        ⟨IncidLP l p'⟩ ⟨IncidP X p'⟩ betH-distincts colH-permut-231
        line-on-plane' by blast
    have IncidP A' p'
      using IncidLP-def ⟨IncidL A' l⟩ ⟨IncidLP l p'⟩ by blast
    have IncidP A p'
      using IncidLP-def ⟨IncidL A l⟩ ⟨IncidLP l p'⟩ by blast
    have EqP p p'
      by (meson plane-uniqueness ⟨IncidP P p'⟩ ⟨IncidP P p⟩
        ⟨IncidP X p'⟩ ⟨IncidP X p⟩ ⟨IncidP Y p'⟩ ⟨IncidP Y p⟩ ⟨IsP p'⟩
        ⟨IsP p⟩ ⟨¬ ColH P X Y⟩)
    hence IncidP A' p
      by (meson EqP-sym IncidP-morphism ⟨IncidP A' p'⟩ ⟨IsP p'⟩ ⟨IsP p⟩)
  }
  thus ?thesis
    using IncidLP-def calculation(2) calculation(3) by blast
qed
moreover have ¬ IncidL ?C l
  by (simp add: ⟨¬ IncidL Y l⟩)
moreover have cut l ?A ?B
  by (simp add: assms(1) cut-comm)
ultimately

```



```

    show ?thesis using pasch by blast
qed
have IncidP A p
  using betH-colH ⟨BetH P A X⟩ ⟨IncidP P p⟩ ⟨IncidP X p⟩ colH-permut-132
    line-on-plane' by blast
obtain q where IncidP X q and IncidP Y q and IncidLP l q
  using assms(2) same-side--plane by blast
have IncidP P q
  using line-on-plane' [of - - p P]
  by (metis (mono-tags, lifting) Hilbert-neutral-dimensionless-pre.IncidLP-def ⟨BetH P A X⟩
    ⟨IncidL A l⟩ ⟨IncidLP l q⟩ ⟨IncidP X q⟩ betH-colH between-comm line-on-plane')
have EqP p q
  using Is-plane ⟨IncidP P p⟩ ⟨IncidP P q⟩ ⟨IncidP X p⟩ ⟨IncidP X q⟩
    ⟨IncidP Y p⟩ ⟨IncidP Y q⟩ ⟨¬ ColH P X Y⟩ plane-uniqueness by presburger
hence IncidLP l p
  by (meson EqL-refl EqP-sym IncidLP-morphism Is-line Is-plane
    ⟨IncidL A l⟩ ⟨IncidLP l q⟩ ⟨IncidP P p⟩)
have ?thesis
  using ⟨¬ local.cut l X Y⟩ ⟨local.cut l X Y ∨ local.cut l P Y⟩ by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma isosceles-congaH:
  assumes ¬ ColH A B C and
    CongH A B A C
  shows CongaH A B C A C B
  by (metis assms(1) assms(2) colH-permut-213 colH-permut-312 colH-trivial112
    congH-sym cong-5 conga-comm)

```

```

lemma cong-distincts:
  assumes A ≠ B and
    Cong A B C D
  shows C ≠ D
  using assms(1) assms(2) cong-identity by blast

```

```

lemma cong-sym:
  assumes Cong A B C D
  shows Cong C D A B
  using Cong-def assms congH-sym by presburger

```

```

lemma cong-trans:
  assumes Cong A B C D and
    Cong C D E F
  shows Cong A B E F
  by (meson assms(1) assms(2) cong-inner-transitivity cong-sym)

```

```

lemma betH-not-congH:
  assumes BetH A B C
  shows ¬ CongH A B A C
  by (metis assms betH-expand betH-trans between-comm between-out construction-uniqueness)

```

```

lemma congH-permlr:
  assumes
    C ≠ D and
    CongH A B C D
  shows CongH B A D C
  by (metis assms(2) assms(1) congH-sym cong-permr)

```

```

lemma congH-perms:

```

```

assumes  $A \neq B$  and
   $C \neq D$  and
   $\text{CongH } A \ B \ C \ D$ 
shows  $\text{CongH } B \ A \ C \ D \wedge \text{CongH } A \ B \ D \ C \wedge \text{CongH } C \ D \ A \ B \wedge$ 
 $\text{CongH } D \ C \ A \ B \wedge \text{CongH } C \ D \ B \ A \wedge \text{CongH } D \ C \ B \ A \wedge \text{CongH } B \ A \ D \ C$ 
using  $\text{assms}(1) \ \text{assms}(2) \ \text{assms}(3) \ \text{congH-sym} \ \text{cong-permr}$  by presburger

lemma congH-perml:
assumes
   $C \neq D$  and
   $\text{CongH } A \ B \ C \ D$ 
shows  $\text{CongH } B \ A \ C \ D$ 
by (metis  $\text{assms}(2) \ \text{assms}(1) \ \text{congH-perms}$ )

lemma bet-cong3-bet:
assumes  $A' \neq B'$  and
   $B' \neq C'$  and
   $A' \neq C'$  and
   $\text{BetH } A \ B \ C$  and
   $\text{ColH } A' \ B' \ C'$  and
   $\text{CongH } A \ B \ A' \ B'$  and
   $\text{CongH } B \ C \ B' \ C'$  and
   $\text{CongH } A \ C \ A' \ C'$ 
shows  $\text{BetH } A' \ B' \ C'$ 
proof –
{
  assume  $\text{BetH } B' \ C' \ A'$ 
  obtain  $B''$  where  $\text{BetH } B \ C \ B''$  and  $\text{CongH } C \ B'' \ B \ C$ 
    using  $\text{assms}(4) \ \text{betH-to-bet} \ \text{segment-constructionH}$  by blast
  have  $\text{BetH } A \ B \ B'' \wedge \text{BetH } A \ C \ B''$ 
    using  $\text{betH-trans} \ \langle \text{BetH } B \ C \ B'' \rangle \ \text{assms}(4)$  by blast
  hence  $\text{BetH } A \ B \ B''$ 
    by blast
  have  $\text{BetH } A \ C \ B''$ 
    using  $\langle \text{BetH } A \ B \ B'' \wedge \text{BetH } A \ C \ B'' \rangle$  by blast
  have  $\text{CongH } A \ B'' \ A' \ B'$ 
proof –
    have  $\text{ColH } A \ C \ B''$ 
      by (simp  $\text{add:} \ \langle \text{BetH } A \ C \ B'' \rangle \ \text{between-col}$ )
    moreover have  $\text{disjoint } A \ C \ C \ B''$ 
      by (simp  $\text{add:} \ \langle \text{BetH } A \ C \ B'' \rangle \ \text{bet-disjoint}$ )
    moreover have  $\text{disjoint } A' \ C' \ C' \ B'$ 
      by (simp  $\text{add:} \ \langle \text{BetH } B' \ C' \ A' \rangle \ \text{bet-disjoint} \ \text{between-comm}$ )
    moreover have  $\text{CongH } C \ B'' \ C' \ B'$ 
      by (metis  $\text{betH-to-bet} \ \text{congH-sym} \ \langle \text{BetH } B \ C \ B'' \rangle \ \langle \text{CongH } C \ B'' \ B \ C \rangle$ 
 $\text{assms}(7) \ \text{cong-permr} \ \text{cong-pseudo-transitivity}$ )
    ultimately show ?thesis
      using  $\text{addition} \ \text{assms}(5) \ \text{assms}(8) \ \text{colH-permut-132}$  by blast
  qed
  have  $\text{CongH } A \ B'' \ A \ B$ 
    by (metis  $\langle \text{BetH } A \ B \ B'' \wedge \text{BetH } A \ C \ B'' \rangle \ \langle \text{CongH } A \ B'' \ A' \ B' \rangle \ \text{assms}(6)$ 
 $\text{betH-colH} \ \text{congH-sym} \ \text{cong-pseudo-transitivity}$ )
  have  $\neg \text{CongH } A \ B \ A \ B''$ 
    by (simp  $\text{add:} \ \langle \text{BetH } A \ B \ B'' \rangle \ \text{betH-not-congH}$ )
  have  $\text{CongH } A \ B \ A \ B''$ 
    using  $\langle \text{BetH } A \ B \ B'' \rangle \ \langle \text{CongH } A \ B'' \ A \ B \rangle \ \text{betH-to-bet} \ \text{congH-sym}$  by blast
  hence False
    by (simp  $\text{add:} \ \langle \neg \text{CongH } A \ B \ A \ B'' \rangle$ )
  hence ?thesis
    by blast
}
moreover
{
  assume  $\text{BetH } C' \ A' \ B'$ 
  obtain  $B''$  where  $\text{BetH } C \ A \ B''$  and  $\text{CongH } A \ B'' \ A \ B$ 
    by (metis  $\text{assms}(4) \ \text{betH-expand} \ \text{segment-constructionH}$ )

```

```

have CongH C B'' C' B'
proof -
  have disjoint C A A B''
    using ⟨BetH C A B'⟩ bet-disjoint by force
  moreover have disjoint C' A' A' B'
    by (simp add: ⟨BetH C' A' B'⟩ bet-disjoint)
  moreover have CongH C A C' A'
    by (simp add: assms(3) assms(8) congH-permlr)
  moreover have CongH A B'' A' B'
    by (metis ⟨BetH C A B'⟩ ⟨CongH A B'' A B⟩ assms(6) betH-distincts
      congH-sym cong-pseudo-transitivity)
  ultimately show ?thesis
    using ⟨BetH C A B'⟩ ⟨BetH C' A' B'⟩ addition-betH by blast
qed
have BetH B A B'' ∧ BetH C B B''
  using betH-trans0 ⟨BetH C A B'⟩ assms(4) between-comm by blast
hence BetH B A B''
  by blast
have BetH C B B''
  using ⟨BetH B A B'' ∧ BetH C B B''⟩ by auto
have ¬ CongH C B C B''
  by (simp add: ⟨BetH C B B'⟩ betH-not-congH)
hence ?thesis
  by (metis betH-colH ⟨BetH C B B'⟩ ⟨CongH C B'' C' B'⟩ assms(2)
    assms(7) congH-perms cong-pseudo-transitivity)
}
ultimately show ?thesis
  using assms(1) assms(2) assms(3) assms(5) between-comm between-one by blast
qed

```

lemma betH-congH3-outH-betH:

```

assumes BetH A B C and
  outH A' B' C' and
  CongH A C A' C' and
  CongH A B A' B'
shows BetH A' B' C'
proof -
  {
    assume BetH A' C' B'
    obtain C'' where BetH A' B' C'' and CongH B' C'' B C
      using ⟨BetH A' C' B'⟩ assms(1) betH-distincts segment-constructionH by blast
    hence CongH A' C'' A C
      using addition-betH assms(1) assms(4) betH-distincts congH-sym by presburger
    have BetH C' B' C'' ∧ BetH A' C' C''
      using ⟨BetH A' B' C''⟩ ⟨BetH A' C' B'⟩ betH-trans0 by blast
    hence ¬ CongH A' C' A' C''
      by (simp add: betH-not-congH)
    moreover
    have CongH A' C' A' C''
      by (metis betH-colH congH-sym ⟨BetH A' B' C''⟩ ⟨CongH A' C'' A C⟩
        assms(3) cong-pseudo-transitivity)
    ultimately have ?thesis
      by simp
  }
  moreover have (A' ≠ B' ∧ B' ≠ C') → ?thesis
    using assms(2) calculation outH-def by auto
  ultimately show ?thesis
    by (metis betH-distincts assms(1) assms(2) assms(3) assms(4) betH-not-congH
      congH-sym cong-pseudo-transitivity outH-def)
qed

```

lemma outH-sym:

```

assumes A ≠ C and
  outH A B C
shows outH A C B
using assms(2) outH-def by fastforce

```

lemma *soustraction-betH*:
assumes $BetH\ A\ B\ C$ **and**
 $BetH\ A'\ B'\ C'$ **and**
 $CongH\ A\ B\ A'\ B'$ **and**
 $CongH\ A\ C\ A'\ C'$
shows $CongH\ B\ C\ B'\ C'$
proof –
obtain $C1$ **where** $BetH\ A'\ B'\ C1$ **and** $CongH\ B'\ C1\ B\ C$
using $assms(1)\ assms(2)\ betH\text{-to-bet}\ segment\text{-constructionH}$ **by** *blast*
hence $CongH\ A\ C\ A'\ C1$
using $addition\text{-betH}\ assms(1)\ assms(3)\ betH\text{-distincts}\ congH\text{-sym}$ **by** *presburger*
obtain X **where** $BetH\ B\ A\ X$
by $(metis\ assms(1)\ between\text{-out})$
obtain X' **where** $BetH\ B'\ A'\ X'$
by $(metis\ assms(2)\ between\text{-out})$
have $BetH\ X\ A\ C$
using $\langle BetH\ B\ A\ X \rangle\ assms(1)\ betH\text{-trans}\ between\text{-comm}$ **by** *blast*
have $BetH\ X'\ A'\ C1$
using $\langle BetH\ A'\ B'\ C1 \rangle\ \langle BetH\ B'\ A'\ X' \rangle\ betH\text{-trans}\ between\text{-comm}$ **by** *blast*
have $C' = C1$
by $(meson\ \langle BetH\ A'\ B'\ C1 \rangle\ \langle CongH\ A\ C\ A'\ C1 \rangle\ assms(2)\ assms(4)\ betH\text{-not-congH}\ cong\text{-pseudo-transitivity}\ out2\text{-out1})$
thus *?thesis*
using $\langle BetH\ A'\ B'\ C1 \rangle\ \langle CongH\ B'\ C1\ B\ C \rangle\ betH\text{-colH}\ congH\text{-sym}$ **by** *blast*
qed

lemma *ncolH-expand*:
assumes $\neg\ ColH\ A\ B\ C$
shows $\neg\ ColH\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C$
using $assms\ colH\text{-trivial112}\ colH\text{-trivial121}\ colH\text{-trivial122}$ **by** *blast*

lemma *betH-outH--outH*:
assumes $BetH\ A\ B\ C$ **and**
 $outH\ B\ C\ D$
shows $outH\ A\ C\ D$
by $(metis\ betH\text{-expand}\ betH\text{-outH2--betH}\ betH\text{-trans1}\ Hilbert\text{-neutral-dimensionless-pre.outH-def}\ assms(1)\ assms(2))$

lemma *th12*:
assumes $\neg\ ColH\ A\ B\ C$ **and**
 $\neg\ ColH\ A'\ B'\ C'$ **and**
 $CongH\ A\ B\ A'\ B'$ **and**
 $CongH\ A\ C\ A'\ C'$ **and**
 $CongaH\ B\ A\ C\ B'\ A'\ C'$
shows $CongaH\ A\ B\ C\ A'\ B'\ C' \wedge CongaH\ A\ C\ B\ A'\ C'\ B' \wedge CongH\ B\ C\ B'\ C'$
proof $(intro\ conjI)$
show $CongaH\ A\ B\ C\ A'\ B'\ C'$
by $(simp\ add:\ assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)\ cong\text{-}5)$
show $CongaH\ A\ C\ B\ A'\ C'\ B'$
by $(meson\ assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)\ colH\text{-permut-132}\ cong\text{-}5\ conga\text{-}permlr)$
obtain D' **where** $CongH\ B'\ D'\ B\ C$ **and** $outH\ B'\ C'\ D'$
by $(metis\ assms(1)\ assms(2)\ colH\text{-trivial122}\ out\text{-construction})$
show $CongH\ B\ C\ B'\ C'$
proof $(cases\ B' = D')$
case *True*
thus *?thesis*
using $\langle outH\ B'\ C'\ D' \rangle\ outH\text{-expand}$ **by** *blast*
next
case *False*
hence $B' \neq D'$
by *simp*
have $\neg\ ColH\ B\ A\ C$

```

    using assms(1) colH-permut-213 by blast
  moreover have  $\neg \text{ColH } B' A' D'$ 
    using ColH-def  $\langle \text{outH } B' C' D' \rangle$  assms(2) inter-incid-uniquenessH
    outH-expand by fastforce
  moreover have CongH B A B' A'
    by (metis assms(3) calculation(2) congH-permlr ncolH-distincts)
  moreover have CongH B C B' D'
    by (simp add: False  $\langle \text{CongH } B' D' B C \rangle$  congH-sym)
  moreover have CongaH A B C A' B' D'
    using  $\langle \text{CongaH } A B C A' B' C' \rangle \langle \text{outH } B' C' D' \rangle$  calculation(1) calculation(2)
    conga-out-conga ncolH-expand outH-trivial by blast
  ultimately have CongaH B A C B' A' D'
    using cong-5 by blast
  have  $\neg \text{ColH } C' A' B'$ 
    using assms(2) colH-permut-231 by blast
  moreover have  $\neg \text{ColH } B A C$ 
    using  $\langle \neg \text{ColH } B A C \rangle$  by blast
  moreover have same-side' C' C' A' B'
    using assms(2) same-side-prime-refl by blast
  moreover have same-side' C' D' A' B'
    by (metis (no-types, lifting) ColH-def ncolH-distincts  $\langle \text{outH } B' C' D' \rangle$ 
    assms(2) out-same-side same-side'-def)
  ultimately have outH A' C' D'
    using  $\langle \text{CongaH } B A C B' A' D' \rangle$  assms(5) cong-4-uniqueness by presburger
  have  $C' = D'$ 
    by (metis (full-types) colH-trivial121 outH-expand  $\langle \neg \text{ColH } C' A' B' \rangle$ 
     $\langle \text{outH } A' C' D' \rangle \langle \text{outH } B' C' D' \rangle$  colH-permut-213 inter-uniquenessH)
  thus ?thesis
    by (simp add:  $\langle \text{CongH } B C B' D' \rangle$ )
qed
qed

```

lemma th14:

```

  assumes  $\neg \text{ColH } A B C$  and
     $\neg \text{ColH } A' B' C'$  and
    CongaH A B C A' B' C' and
    BetH A B D and
    BetH A' B' D'
  shows CongaH C B D C' B' D'
proof -
  obtain A'' where CongH B' A'' A B and outH B' A' A''
    by (metis assms(4) assms(5) betH-distincts out-construction)
  obtain C'' where CongH B' C'' B C and outH B' C' C''
    by (metis assms(1) assms(2) colH-trivial122 out-construction)
  obtain D'' where CongH B' D'' B D and outH B' D' D''
    using assms(4) assms(5) betH-expand out-construction by blast
  have CongaH B A C B' A'' C''  $\wedge$  CongaH B C A B' C'' A''  $\wedge$  CongaH A C A'' C''
  proof (rule th12)
    show  $\neg \text{ColH } B A C$ 
      using assms(1) colH-permut-213 by blast
    show  $\neg \text{ColH } B' A'' C''$ 
      by (metis (mono-tags, opaque-lifting)  $\langle \text{outH } B' A' A'' \rangle \langle \text{outH } B' C' C'' \rangle$ 
      assms(2) colH-trans colH-trivial121 colH-trivial122 outH-expand)
    show CongH B A B' A''
      by (metis  $\langle \text{CongH } B' A'' A B \rangle \langle \text{outH } B' A' A'' \rangle$  congH-perms congH-sym out-distinct)
    show CongH B C B' C''
      using  $\langle \text{CongH } B' C'' B C \rangle \langle \text{outH } B' C' C'' \rangle$  congH-sym out-distinct by presburger
    show CongaH A B C A'' B' C''
      using  $\langle \neg \text{ColH } B A C \rangle \langle \text{outH } B' A' A'' \rangle \langle \text{outH } B' C' C'' \rangle$  assms(3)
      conga-out-conga ncolH-distincts outH-trivial by blast
  qed
  have BetH A'' B' D''
    using  $\langle \text{outH } B' A' A'' \rangle \langle \text{outH } B' D' D'' \rangle$  assms(5) betH-outH2--betH by blast
  have CongH A D A'' D''
    by (metis (full-types)  $\langle \text{BetH } A'' B' D'' \rangle \langle \text{CongH } B' A'' A B \rangle$ 
     $\langle \text{CongH } B' D'' B D \rangle$  addition-betH assms(1) assms(4) colH-trivial112)

```

```

    congH-perml congH-sym not-betH121)
{
  assume ColH A C D
  have A ≠ D
    using between-diff assms(4) by blast
  moreover have ColH A D A
    by (simp add: colH-trivial121)
  moreover have ColH A D B
    by (simp add: assms(4) between-col colH-permut-132)
  moreover have ColH A D C
    by (simp add: ⟨ColH A C D⟩ colH-permut-132)
  hence False
    using assms(1) calculation(1) calculation(2) calculation(3) colH-trans by blast
}
hence ¬ ColH A C D
  by blast
{
  assume ColH A'' C'' D''
  have ColH B' A' A'' ∧ B' ≠ A'' ∧ B' ≠ A'
    using ⟨outH B' A' A'⟩ outH-expand by blast
  hence ColH B' A' A''
    by simp
  moreover have B' ≠ A''
    using ⟨ColH B' A' A'' ∧ B' ≠ A'' ∧ B' ≠ A'⟩ by blast
  moreover have B' ≠ A'
    by (simp add: ⟨ColH B' A' A'' ∧ B' ≠ A'' ∧ B' ≠ A'⟩)
  moreover have ColH B' C' C'' ∧ B' ≠ C'' ∧ B' ≠ C'
    using ⟨outH B' C' C'⟩ outH-expand by presburger
  hence ColH B' C' C''
    by simp
  moreover have B' ≠ C''
    using ⟨ColH B' C' C'' ∧ B' ≠ C'' ∧ B' ≠ C'⟩ by blast
  have B' ≠ C'
    using ⟨ColH B' C' C'' ∧ B' ≠ C'' ∧ B' ≠ C'⟩ by blast
  moreover have ColH B' D' D'' ∧ B' ≠ D'' ∧ B' ≠ D'
    using ⟨outH B' D' D'⟩ outH-expand by blast
  hence ColH B' D' D''
    by simp
  moreover have B' ≠ D''
    by (simp add: ⟨ColH B' D' D'' ∧ B' ≠ D'' ∧ B' ≠ D'⟩)
  moreover have B' ≠ D'
    using ⟨ColH B' D' D'' ∧ B' ≠ D'' ∧ B' ≠ D'⟩ by blast
  ultimately have ColH A'' B' D''
    using ⟨BetH A'' B' D'⟩ between-col by blast
  hence ColH A'' D'' B'
    using colH-permut-132 by blast
  have A'' ≠ B'
    using ⟨B' ≠ A'⟩ by blast
  have ColH A'' B' A'
    by (simp add: ⟨ColH B' A' A'⟩ colH-permut-312)
  have ColH A'' B' B'
    by (simp add: colH-trivial122)
  have A'' ≠ D''
    using ⟨BetH A'' B' D'⟩ not-betH121 by auto
  have ColH A'' D'' A'
    using ⟨A'' ≠ B'⟩ ⟨ColH A'' B' A'⟩ ⟨ColH A'' B' D'⟩
      colH-trans colH-trivial121 by presburger
  moreover have ColH A'' D'' B'
    by (simp add: ⟨ColH A'' D'' B'⟩)
  moreover have ColH A'' D'' C'
  proof -
    have ColH A'' C' D''
      by (metis ⟨ColH A'' C'' D''⟩ ⟨ColH B' C' C'' ∧ B' ≠ C'' ∧ B' ≠ C'⟩ calculation(2)
        colH-permut-213 colH-trans colH-trivial121)
    thus ?thesis
      using colH-permut-132 by blast
  }
}

```

```

qed
ultimately have ColH A' B' C'
  using ⟨A'' ≠ D''⟩ colH-trans by blast
hence False
  by (simp add: assms(2))
}
have CongaH A C D A'' C'' D'' ∧ CongaH A D C A'' D'' C'' ∧ CongH C D C'' D''
proof (rule th12)
  show ¬ ColH A C D
    using ⟨ColH A C D ⟹ False⟩ by auto
  show ¬ ColH A'' C'' D''
    using ⟨ColH A'' C'' D'' ⟹ False⟩ by auto
  show CongH A C A'' C''
    using ⟨CongaH B A C B' A'' C'' ∧ CongaH B C A B' C'' A'' ∧ CongH A C A'' C''⟩ by blast
  show CongH A D A'' D''
    by (simp add: ⟨CongH A D A'' D''⟩)
  have CongaH C A B C'' A'' B'
    using ⟨CongaH B A C B' A'' C'' ∧ CongaH B C A B' C'' A'' ∧ CongH A C A'' C''⟩
      conga-permlr by presburger
  moreover have outH A C C
    by (metis assms(1) colH-trivial121 outH-trivial)
  moreover have outH A'' C'' C''
    by (metis ⟨¬ ColH A'' C'' D''⟩ colH-trivial112 outH-trivial)
  moreover have outH A B D
    by (simp add: assms(4) outH-def)
  moreover have outH A'' B' D''
    using ⟨BetH A'' B' D''⟩ outH-def by auto
  ultimately show CongaH C A D C'' A'' D''
    using conga-out-conga by blast
qed
have ¬ ColH D B C
  using assms(1) assms(4) betH-colH between-comm colH-trans colH-trivial122 by blast
moreover have ¬ ColH D'' B' C''
  using ⟨BetH A'' B' D''⟩ ⟨ColH A'' C'' D'' ⟹ False⟩ betH-colH between-comm
    colH-trans colH-trivial121 by blast
moreover have CongH D B D'' B'
  by (metis betH-distincts outH-expand ⟨CongH B' D'' B D⟩
    ⟨outH B' D' D''⟩ assms(4) congH-perms)
moreover have CongH D C D'' C''
  by (metis ⟨CongaH A C D A'' C'' D'' ∧ CongaH A D C A'' D'' C'' ∧ CongH C D C'' D''⟩
    calculation(2) colH-trivial121 congH-permlr)
moreover have CongaH B D C B' D'' C''
proof -
  have CongaH A D C A'' D'' C''
    using ⟨CongaH A C D A'' C'' D'' ∧ CongaH A D C A'' D'' C'' ∧ CongH C D C'' D''⟩ by auto
  moreover have outH D A B
    using assms(4) between-comm outH-def by blast
  moreover have outH D C C
    by (metis ⟨¬ ColH D B C⟩ colH-trivial121 outH-trivial)
  moreover have outH D'' A'' B'
    using between-comm ⟨BetH A'' B' D''⟩ outH-def by blast
  moreover have outH D'' C'' C''
    by (metis ⟨¬ ColH D'' B' C''⟩ colH-trivial121 outH-trivial)
  ultimately show ?thesis
    using conga-out-conga by blast
qed
ultimately have CongaH D B C D'' B' C''
  using th12 by blast
have CongaH D B C D' B' C'
proof -
  have outH B D D
    using assms(4) betH-distincts outH-trivial by blast
  moreover have outH B C C
    by (metis ⟨¬ ColH A C D⟩ assms(4) between-col outH-trivial)
  moreover have outH B' D'' D'
    using ⟨BetH A'' B' D''⟩ ⟨outH B' D' D''⟩ betH-distincts outH-sym by presburger

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moreover have  $outH\ B'\ C''\ C'$ 
  using  $\langle outH\ B'\ C'\ C'' \rangle\ outH\text{-sym}\ out\text{-distinct}$  by blast
ultimately show ?thesis
  using  $\langle CongaH\ D\ B\ C\ D''\ B'\ C'' \rangle\ conga\text{-out-conga}$  by blast
qed
thus ?thesis
  using conga-permlr by blast
qed

lemma congH-colH-betH:
assumes  $A \neq B$  and
   $A \neq I$  and
   $B \neq I$  and
   $CongH\ I\ A\ I\ B$  and
   $ColH\ I\ A\ B$ 
shows  $BetH\ A\ I\ B$ 
proof –
  have  $BetH\ I\ A\ B \longrightarrow ?thesis$ 
    using assms(4) betH-not-congH by blast
moreover have  $BetH\ A\ B\ I \longrightarrow ?thesis$ 
  by (metis assms(2) assms(4) betH-not-congH between-comm congH-sym)
ultimately show ?thesis
  by (metis assms(1) assms(2) assms(3) assms(5) between-one)
qed

lemma plane-separation:
assumes  $\neg ColH\ A\ X\ Y$  and
   $\neg ColH\ B\ X\ Y$  and
   $IncidP\ A\ p$  and
   $IncidP\ B\ p$  and
   $IncidP\ X\ p$  and
   $IncidP\ Y\ p$ 
shows  $cut'\ A\ B\ X\ Y \vee same\text{-side}'\ A\ B\ X\ Y$ 
proof –
obtain  $l$  where  $IsL\ l$  and  $IncidL\ X\ l$  and  $IncidL\ Y\ l$ 
  using ColH-def colH-trivial122 by blast
obtain  $C$  where  $cut\ l\ A\ C$ 
  using ColH-def  $\langle IncidL\ X\ l \rangle\ \langle IncidL\ Y\ l \rangle\ \langle IsL\ l \rangle\ assms(1)$  cut-exists by blast
have  $A = B \longrightarrow ?thesis$ 
  using assms(2) colH-permut-312 same-side-prime-refl by blast
moreover {
  assume  $A \neq B$ 
  have  $A \neq C$ 
    using  $\langle local.cut\ l\ A\ C \rangle\ cut\text{-distinct}$  by auto
  {
    assume  $B = C$ 
    obtain  $m$  where  $IncidL\ X\ m \wedge IncidL\ Y\ m$ 
      using  $\langle IncidL\ X\ l \rangle\ \langle IncidL\ Y\ l \rangle$  by auto
    obtain  $I$  where  $IncidL\ I\ l$  and  $BetH\ A\ I\ C$ 
      using  $\langle local.cut\ l\ A\ C \rangle\ local.cut\text{-def}$  by blast
    have  $\neg IncidL\ A\ l$ 
      using  $\langle local.cut\ l\ A\ C \rangle\ local.cut\text{-def}$  by auto
    have  $\neg IncidL\ C\ l$ 
      using  $\langle local.cut\ l\ A\ C \rangle\ local.cut\text{-def}$  by blast
    have  $cut\ l\ A\ C$ 
      by (simp add:  $\langle local.cut\ l\ A\ C \rangle$ )
    hence  $cut\ l\ A\ B$ 
      by (simp add:  $\langle B = C \rangle$ )
    have  $X \neq Y$ 
      using assms(1) colH-trivial122 by blast
    moreover {
      fix  $k$ 
      assume  $IsL\ k$  and  $IncidL\ X\ k$  and  $IncidL\ Y\ k$ 
      hence  $cut\ k\ A\ B$ 
        by (metis inter-incid-uniquenessH  $\langle B = C \rangle\ \langle BetH\ A\ I\ C \rangle\ \langle IncidL\ I\ l \rangle$ )
    }
  }

```



```

      ⟨IncidL X k⟩ ⟨IncidL X l⟩ ⟨IncidL Y k⟩ ⟨IncidL Y l⟩ ⟨IsL k⟩ ⟨¬ IncidL A l⟩
      ⟨¬ IncidL C l⟩ calculation local.cut-def)
    }
  ultimately have cut' A B X Y
    using cut'-def by presburger
  hence ?thesis
    by blast
}
moreover
{
  assume B ≠ C
  hence ?thesis
  proof (cases ColH A C B)
    case True
    hence ColH A C B
    by simp
    obtain I where IncidL I l and BetH A I C
    using ⟨local.cut l A C⟩ local.cut-def by blast
    hence ColH A I C
    using between-col by blast
    have ColH A I B
    by (meson True ⟨A ≠ C⟩ ⟨ColH A I C⟩ colH-permut-132 colH-trans colH-trivial121)
    have ¬ IncidL A l
    using ⟨local.cut l A C⟩ local.cut-def by auto
    {
      assume BetH A I B
      {
        fix m
        assume IsL m and IncidL X m and IncidL Y m
        hence EqL m l
        by (metis ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ assms(2)
            colH-trivial122 line-uniqueness)
        hence cut m A B
        using ColH-def ⟨BetH A I B⟩ ⟨IncidL I l⟩ ⟨IncidL X m⟩ ⟨IncidL Y m⟩
            ⟨IsL l⟩ ⟨IsL m⟩ ⟨¬ IncidL A l⟩ assms(2) local.cut-def morph by auto
      }
      hence cut' A B X Y
      using assms(1) colH-trivial122 cut'-def by auto
    }
    hence ?thesis
    by blast
  }
  moreover {
    assume BetH I B A
    {
      fix m
      assume IsL m and IncidL X m and IncidL Y m
      hence EqL m l
      by (metis ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ assms(2)
          colH-trivial122 line-uniqueness)
      hence same-side A B m
      by (metis inter-incid-uniquenessH ncolH-distincts out-same-side
          outH-def ⟨BetH I B A⟩ ⟨IncidL I l⟩ ⟨IncidL X l⟩ ⟨IncidL X m⟩ ⟨IncidL Y l⟩
          ⟨IncidL Y m⟩ ⟨¬ IncidL A l⟩ assms(2))
    }
    hence same-side' A B X Y
    using assms(1) colH-trivial122 same-side'-def by auto
    hence ?thesis
    by blast
  }
}
moreover {
  assume BetH I A B
  {
    fix m
    assume IsL m and IncidL X m and IncidL Y m
    hence EqL m l
    by (metis ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ assms(2)
        colH-trivial122 line-uniqueness)
  }
}

```

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      colH-trivial122 line-uniqueness)
    hence same-side A B m
    by (metis inter-incid-uniquenessH ncolH-expand out-same-side
      outH-def ⟨BetH I A B⟩ ⟨IncidL I l⟩ ⟨IncidL X l⟩ ⟨IncidL X m⟩ ⟨IncidL Y l⟩
      ⟨IncidL Y m⟩ ⟨¬ IncidL A l⟩ assms(2))
  }
  hence same-side' A B X Y
  using assms(1) colH-trivial122 same-side'-def by auto
  hence ?thesis
  by blast
}
ultimately show ?thesis
by (metis ColH-def Is-line ⟨A ≠ B⟩ ⟨BetH A I C⟩ ⟨ColH A I B⟩
  ⟨IncidL I l⟩ ⟨IncidL X l⟩ ⟨IncidL Y l⟩ assms(2) betH-colH between-one)
next
case False
have IncidLP l p
  using ncolH-expand Is-plane ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩
  assms(2) assms(5) assms(6) line-on-plane by blast
obtain I where IncidL I l and BetH A I C
  using ⟨local.cut l A C⟩ local.cut-def by blast
hence ColH A I C
  by (simp add: between-col)
hence IncidP I p
  using IncidLP-def ⟨IncidL I l⟩ ⟨IncidLP l p⟩ by blast
hence IncidP C p
  using ⟨BetH A I C⟩ ⟨ColH A I C⟩ assms(3) betH-distincts line-on-plane' by blast
{
  assume cut l A B
  {
    fix m
    assume IsL m and IncidL X m and IncidL Y m
    hence EqL m l
      by (metis ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ assms(2)
        colH-trivial122 line-uniqueness)
    hence cut m A B
      by (meson ⟨IsL m⟩ ⟨local.cut l A B⟩ local.cut-def morph)
  }
  hence ?thesis
  using assms(1) colH-trivial122 cut'-def by force
}
moreover
{
  assume cut l C B
  have X ≠ Y
    using assms(1) colH-trivial122 by blast
  {
    fix m
    assume IsL m and IncidL X m and IncidL Y m
    hence EqL m l
      by (metis ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ assms(2)
        colH-trivial122 line-uniqueness)
    have cut m A C
      by (meson ⟨EqL m l⟩ ⟨IsL m⟩ ⟨local.cut l A C⟩ local.cut-def morph)
    moreover have cut m B C
      by (metis ⟨EqL m l⟩ ⟨local.cut l C B⟩ calculation cut-comm local.cut-def morph)
    ultimately have same-side A B m
      using ⟨IsL m⟩ same-side-def by blast
  }
  hence same-side' A B X Y
  using assms(1) colH-trivial122 same-side'-def by auto
  hence ?thesis
  by blast
}
moreover
have ¬ IncidL B l

```

```

    using ColH-def ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ assms(2) by blast
  ultimately show ?thesis
    using pasch False ⟨IncidLP l p⟩ ⟨IncidP C p⟩ ⟨local.cut l A C⟩
      assms(3) assms(4) strong-pasch by blast
  qed
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

lemma same-side-comm:
  assumes same-side A B l
  shows same-side B A l
  using assms same-side-def by blast

lemma same-side-not-incid:
  assumes same-side A B l
  shows  $\neg$  IncidL A l  $\wedge$   $\neg$  IncidL B l
  using assms local.cut-def same-side-def by auto

lemma out-same-side':
  assumes  $X \neq Y$  and
    IncidL X l and
    IncidL Y l and
    IncidL A l and
     $\neg$  IncidL B l and
    outH A B C
  shows same-side' B C X Y
proof -
  have same-side B C l
    using assms(4) assms(5) assms(6) out-same-side by blast
  {
    fix m
    assume IsL m and
      IncidL X m and
      IncidL Y m
    hence EqL m l
      using Is-line assms(1) assms(2) assms(3) line-uniqueness by presburger
    hence same-side B C m
      by (meson Is-line ⟨IsL m⟩ assms(4) assms(5) assms(6) morph out-same-side)
  }
  thus ?thesis
    by (simp add: assms(1) same-side'-def)
qed

lemma same-side-trans:
  assumes same-side A B l and
    same-side B C l
  shows same-side A C l
  by (meson assms(1) assms(2) cut-comm cut-same-side-cut same-side-def)

lemma colH-IncidL--IncidL:
  assumes  $A \neq B$  and
    IncidL A l and
    IncidL B l and
    ColH A B C
  shows IncidL C l
  using ColH-def assms(1) assms(2) assms(3) assms(4) inter-incid-uniquenessH by blast

lemma IncidL-not-IncidL--not-colH:
  assumes  $A \neq B$  and
    IncidL A l and
    IncidL B l and

```

$\neg \text{IncidL } C \ l$
shows $\neg \text{ColH } A \ B \ C$
using *assms(1) assms(2) assms(3) assms(4) colH-IncidL--IncidL* **by** *blast*

lemma *same-side-prime-not-colH*:

assumes *same-side' A B C D*

shows $\neg \text{ColH } A \ C \ D \wedge \neg \text{ColH } B \ C \ D$

proof –

{
 fix *l*
 assume *IncidL C l and IncidL D l*
 have *same-side A B l*
 using *Is-line ⟨IncidL C l⟩ ⟨IncidL D l⟩ assms same-side'-def* **by** *auto*
 hence *?thesis*
 by (*meson ColH-def assms same-side'-def same-side-not-incid*)
 }
thus *?thesis*
 by (*metis assms line-existence same-side'-def*)

qed

lemma *OS2--TS*:

assumes *same-side' Y Z PO X and*

same-side' X Y PO Z

shows *cut' X Z PO Y*

proof –

obtain *Z' where BetH Z PO Z'*
 using *assms(2) between-out same-side'-def* **by** *force*
have *PO ≠ X*
 using *assms(1) same-side'-def* **by** *blast*
have *PO ≠ Z*
 using *assms(2) same-side'-def* **by** *force*
 {
 fix *l*
 assume *IsL l and IncidL PO l and IncidL X l*
 have $\neg \text{IncidL } Y \ l$
 using *ColH-def ⟨IncidL PO l⟩ ⟨IncidL X l⟩ ⟨IsL l⟩ assms(1)*
 same-side-prime-not-colH **by** *blast*
 moreover have $\neg \text{IncidL } Z \ l$
 using *ColH-def ⟨IncidL PO l⟩ ⟨IncidL X l⟩ ⟨IsL l⟩ assms(1)*
 same-side-prime-not-colH **by** *blast*
 ultimately have *cut l Z' Z*
 by (*metis IncidL-not-IncidL--not-colH Is-line ⟨BetH Z PO Z'⟩ ⟨IncidL PO l⟩*
 betH-distincts between-col between-comm local.cut-def)
 hence *cut l Y Z'*
 by (*meson ⟨IncidL PO l⟩ ⟨IncidL X l⟩ ⟨IsL l⟩ assms(1) cut-comm*
 cut-same-side-cut same-side'-def same-side-comm)
 }
hence *cut' Y Z' PO X*
 using *⟨PO ≠ X⟩ cut'-def* **by** *auto*
 {
 fix *l*
 assume *IsL l and IncidL PO l and IncidL Y l*
 have $\neg \text{ColH } Y \ PO \ Z$
 using *assms(2) same-side-prime-not-colH* **by** *blast*
 have *PO ≠ Y*
 using *⟨¬ ColH Y PO Z⟩ colH-trivial112* **by** *auto*
 have *PO ≠ Z'*
 using *⟨BetH Z PO Z'⟩ betH-distincts* **by** *auto*
 have *Z ≠ Z'*
 using *⟨BetH Z PO Z'⟩ not-betH121* **by** *auto*
 }

```

assume  $\text{IncidL } Z' \text{ } l$ 
hence  $\text{ColH } PO \text{ } Y \text{ } Z'$ 
  using  $\text{ColH-def } \langle \text{IncidL } PO \text{ } l \rangle \langle \text{IncidL } Y \text{ } l \rangle \langle \text{IsL } l \rangle$  by blast
moreover have  $\text{ColH } Z \text{ } PO \text{ } Z'$ 
  by (simp add:  $\langle \text{BetH } Z \text{ } PO \text{ } Z' \rangle$  betH-colH)
ultimately have  $\text{ColH } Y \text{ } PO \text{ } Z$ 
  by (meson ColH-def  $\langle PO \neq Z' \rangle$  inter-incid-uniquenessH)
have False
  using  $\text{ColH-def } \langle \text{BetH } Z \text{ } PO \text{ } Z' \rangle \langle \text{IncidL } PO \text{ } l \rangle \langle \text{IncidL } Y \text{ } l \rangle$ 
     $\langle \text{IncidL } Z' \text{ } l \rangle \langle \text{IsL } l \rangle \langle PO \neq Z' \rangle \langle \neg \text{ColH } Y \text{ } PO \text{ } Z \rangle$  betH-line
    inter-incid-uniquenessH by blast
}
hence  $\text{cut } l \text{ } Z \text{ } Z'$ 
  using  $\text{ColH-def } \langle \text{BetH } Z \text{ } PO \text{ } Z' \rangle \langle \text{IncidL } PO \text{ } l \rangle \langle \text{IncidL } Y \text{ } l \rangle$ 
     $\langle \text{IsL } l \rangle \langle \neg \text{ColH } Y \text{ } PO \text{ } Z \rangle$  local.cut-def by blast
moreover
obtain  $m$  where  $\text{IsL } m$  and  $\text{IncidL } PO \text{ } m$  and  $\text{IncidL } X \text{ } m$ 
  using  $\langle PO \neq X \rangle$  line-existence by blast
have  $\text{cut } m \text{ } Y \text{ } Z'$ 
  using  $\text{Is-line } \langle \text{IncidL } PO \text{ } m \rangle$ 
     $\langle \text{IncidL } X \text{ } m \rangle \langle \bigwedge l. [\text{IsL } l; \text{IncidL } PO \text{ } l; \text{IncidL } X \text{ } l] \implies \text{local.cut } l \text{ } Y \text{ } Z' \rangle$  by blast
have  $\neg \text{IncidL } Y \text{ } m$ 
  using  $\langle \text{local.cut } m \text{ } Y \text{ } Z' \rangle$  local.cut-def by blast
have  $\neg \text{IncidL } Z' \text{ } m$ 
  using  $\langle \text{local.cut } m \text{ } Y \text{ } Z' \rangle$  local.cut-def by auto
obtain  $X'$  where  $\text{IncidL } X' \text{ } m$  and  $\text{BetH } Y \text{ } X' \text{ } Z'$ 
  using  $\langle \text{local.cut } m \text{ } Y \text{ } Z' \rangle$  local.cut-def by blast
have same-side  $Z' \text{ } X' \text{ } l$ 
  using  $\langle \text{BetH } Y \text{ } X' \text{ } Z' \rangle \langle \text{IncidL } Y \text{ } l \rangle \langle \text{IncidL } Z' \text{ } l \implies \text{False} \rangle$  outH-def out-same-side by blast
moreover have same-side  $X' \text{ } X \text{ } l$ 
proof –
  have  $\text{ColH } PO \text{ } X \text{ } X'$ 
    using  $\text{ColH-def Is-line } \langle \text{IncidL } PO \text{ } m \rangle \langle \text{IncidL } X \text{ } m \rangle \langle \text{IncidL } X' \text{ } m \rangle$  by blast
  have  $\neg \text{ColH } Y \text{ } PO \text{ } X$ 
    using assms(1) same-side-prime-not-colH by presburger
  have  $\text{outH } PO \text{ } X \text{ } X'$ 
  proof (cases  $X = X'$ )
    case True
      thus ?thesis
      using  $\langle PO \neq X \rangle$  outH-trivial by auto
    next
      case False
        hence  $X \neq X'$ 
        by simp
        have  $\text{ColH } Z \text{ } PO \text{ } Z'$ 
          by (simp add:  $\langle \text{BetH } Z \text{ } PO \text{ } Z' \rangle$  betH-colH)
        have  $\text{ColH } Y \text{ } X' \text{ } Z'$ 
          by (simp add:  $\langle \text{BetH } Y \text{ } X' \text{ } Z' \rangle$  betH-colH)
        have  $PO = X' \longrightarrow ?thesis$ 
          using  $\langle \text{IncidL } PO \text{ } l \rangle$  calculation(2) same-side-not-incid by blast
        moreover
          {
            assume  $PO \neq X'$ 
            have  $\text{ColH } PO \text{ } X \text{ } X'$ 
              by (simp add:  $\langle \text{ColH } PO \text{ } X \text{ } X' \rangle$ )
            hence  $\text{BetH } PO \text{ } X \text{ } X' \vee \text{BetH } X \text{ } X' \text{ } PO \vee \text{BetH } X \text{ } PO \text{ } X'$ 
              using  $\text{False } \langle PO \neq X' \rangle \langle PO \neq X \rangle$  between-one by auto
            moreover have  $\text{BetH } PO \text{ } X \text{ } X' \longrightarrow ?thesis$ 
              using outH-def by auto
            moreover have  $\text{BetH } X \text{ } X' \text{ } PO \longrightarrow ?thesis$ 
              using between-comm outH-def by blast
            moreover {
              assume  $\text{BetH } X \text{ } PO \text{ } X'$ 
              obtain  $lo$  where  $\text{IsL } lo$  and  $\text{IncidL } PO \text{ } lo$  and  $\text{IncidL } Z \text{ } lo$ 
                using  $\langle PO \neq Z \rangle$  line-existence by blast
              have same-side  $X \text{ } Y \text{ } lo$ 

```

```

    using ⟨IncidL PO lo⟩ ⟨IncidL Z lo⟩ ⟨IsL lo⟩ assms(2) same-side'-def by auto
  have cut lo X X'
  by (metis Is-line ⟨BetH X PO X'⟩ ⟨ColH PO X X'⟩ ⟨IncidL PO lo⟩
    ⟨PO ≠ X'⟩ ⟨same-side X Y lo⟩ colH-IncidL--IncidL colH-permut-312
    local.cut-def same-side-not-incid)
  moreover
  have ¬ IncidL X' lo
    using calculation local.cut-def by blast
  moreover have IncidL Z' lo
  by (metis ⟨ColH Z PO Z'⟩ ⟨IncidL PO lo⟩ ⟨IncidL Z lo⟩
    ⟨PO ≠ Z⟩ colH-IncidL--IncidL)
  ultimately have same-side X' Y lo
  by (meson ⟨BetH Y X' Z'⟩ between-comm outH-def out-same-side)
  hence ?thesis
  using ⟨local.cut lo X X'⟩ ⟨same-side X Y lo⟩ cut-same-side-cut
    same-side-not-cut by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed
thus ?thesis
  by (meson ⟨IncidL PO l⟩ ⟨IncidL PO m⟩ ⟨IncidL X m⟩ ⟨IncidL Y l⟩
    ⟨PO ≠ X⟩ ⟨¬ IncidL Y m⟩ inter-incid-uniquenessH out-same-side same-side-comm)
qed
ultimately have same-side Z' X l
  using same-side-trans by blast
hence cut l X Z
  using ⟨local.cut l Z Z'⟩ cut-comm cut-same-side-cut by blast
}
thus ?thesis
  by (metis assms(2) colH-trivial112 cut'-def same-side-prime-not-colH)
qed

```

lemma *th15-aux-1*:

```

assumes ¬ ColH H PO L and
  ¬ ColH H' O' L' and
  ¬ ColH K PO L and
  ¬ ColH K' O' L' and
  ¬ ColH H PO K and
  ¬ ColH H' O' K' and
  same-side' H K PO L and
  same-side' H' K' O' L' and
  cut' K L PO H and
  CongaH H PO L H' O' L' and
  CongaH K PO L K' O' L'
shows CongaH H PO K H' O' K'
proof -
  obtain K'' where CongH O' K'' PO K and outH O' K' K''
    by (metis assms(3) assms(4) colH-trivial112 out-construction)
  obtain L'' where CongH O' L'' PO L and outH O' L' L''
    by (metis assms(3) assms(4) colH-trivial122 out-construction)
  have CongaH K PO L K'' O' L''
    using ⟨outH O' K' K''⟩ ⟨outH O' L' L''⟩ assms(1) assms(11) assms(5)
      conga-out-conga ncolH-distincts outH-trivial by blast
  have PO ≠ H
    using assms(1) colH-trivial112 by blast
  obtain l where IsL l and IncidL PO l and IncidL H l
    using ⟨PO ≠ H⟩ line-existence by fastforce
  hence cut l K L
    using assms(9) cut'-def by auto
  have ¬ IncidL K l
    using ⟨local.cut l K L⟩ local.cut-def by blast
  have ¬ IncidL L l

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    using ⟨local.cut l K L⟩ local.cut-def by auto
  obtain I where IncidL I l and BetH K I L
    using ⟨local.cut l K L⟩ local.cut-def by blast
  have PO ≠ I
    using ⟨BetH K I L⟩ assms(3) betH-colH by blast
  have H = I ⟶ outH PO I H
    using ⟨PO ≠ I⟩ outH-trivial by force
  moreover {
    assume H ≠ I
    have ColH PO I H
      using ColH-def Is-line ⟨IncidL H l⟩ ⟨IncidL I l⟩ ⟨IncidL PO l⟩ by blast
    hence BetH PO I H ∨ BetH I H PO ∨ BetH I PO H
      using between-one ⟨H ≠ I⟩ ⟨PO ≠ H⟩ ⟨PO ≠ I⟩ by blast
    moreover have BetH PO I H ⟶ outH PO I H
      by (simp add: outH-def)
    moreover have BetH I H PO ⟶ outH PO I H
      using between-comm outH-def by blast
    moreover {
      assume BetH I PO H
      have PO ≠ L
        using ⟨IncidL PO l⟩ ⟨¬ IncidL L l⟩ by auto
      obtain m where IsL m and IncidL PO m and IncidL L m
        using ColH-def colH-trivial121 by blast
      have same-side H K m
        using ⟨IncidL L m⟩ ⟨IncidL PO m⟩ ⟨IsL m⟩ assms(7) same-side'-def by auto
      have cut m H I
        by (metis IncidL-not-IncidL--not-colH Is-line ⟨BetH I PO H⟩ ⟨ColH PO I H⟩
          ⟨IncidL PO m⟩ ⟨PO ≠ I⟩ ⟨same-side H K m⟩
          cut-comm local.cut-def same-side-not-incid)
      moreover have same-side I K m
        by (meson ⟨BetH K I L⟩ ⟨IncidL L m⟩ between-comm calculation local.cut-def
          outH-def out-same-side)
      ultimately have cut m H K
        using cut-same-side-cut by blast
      hence outH PO I H
        using ⟨same-side H K m⟩ same-side-not-cut by auto
    }
  }
  ultimately have outH PO I H
    by blast
}
ultimately have outH PO I H
  by blast
have outH PO L L
  using assms(7) outH-trivial same-side'-def by force
obtain I' where CongH O' I' PO I and outH O' H' I'
  by (metis ⟨PO ≠ I⟩ assms(2) colH-trivial112 out-construction)
hence CongaH I PO L I' O' L''
  using ⟨PO ≠ H⟩ ⟨outH O' L' L''⟩ ⟨outH PO I H⟩ ⟨outH PO L L⟩ assms(10)
  conga-out-conga outH-sym by blast
have O' ≠ I'
  using ⟨outH O' H' I'⟩ out-distinct by auto
have I' ≠ L''
  by (metis (mono-tags, opaque-lifting) ⟨O' ≠ I'⟩ ⟨outH O' H' I'⟩ ⟨outH O' L' L''⟩
    assms(2) colH-trans ncolH-expand outH-col)
hence ∃ I'. (O' ≠ I' ∧ I' ≠ L'' ∧ outH PO H I ∧ outH O' H' I' ∧
  ColH O' I' H' ∧ CongH O' I' PO I ∧ CongaH I PO L I' O' L'')
  using ⟨CongH O' I' PO I⟩ ⟨CongaH I PO L I' O' L''⟩ ⟨O' ≠ I'⟩ ⟨PO ≠ H⟩
  ⟨outH O' H' I'⟩ ⟨outH PO I H⟩ outH-col outH-sym by blast
then obtain I' where O' ≠ I' and I' ≠ L'' and
  outH PO H I and outH O' H' I' and
  ColH O' I' H' and CongH O' I' PO I and CongaH I PO L I' O' L''
  by blast
have PO ≠ L
  using ⟨IncidL PO l⟩ ⟨¬ IncidL L l⟩ by blast
have O' ≠ L''
  using ⟨outH O' L' L''⟩ out-distinct by blast

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have ColH O' L' L''
  using ⟨outH O' L' L'⟩ outH-expand by blast
have CongaH PO I L O' I' L'' ∧ CongaH PO L I O' L'' I' ∧ CongH I L I' L''
proof -
  have ¬ ColH PO I L
    using ⟨IncidL I l⟩ ⟨IncidL PO l⟩ ⟨PO ≠ I⟩ ⟨¬ IncidL L l⟩
      colH-IncidL--IncidL by blast
  moreover
  {
    assume ColH O' I' L''
    have False
      by (metis ⟨ColH O' I' H'⟩ ⟨ColH O' I' L''⟩ ⟨ColH O' L' L''⟩ ⟨O' ≠ I'⟩ ⟨O' ≠ L''⟩
        assms(2) colH-permut-312 inter-uniquenessH ncolH-expand)
  }
  hence ¬ ColH O' I' L''
    by blast
  moreover have CongH PO I O' I'
    using ⟨CongH O' I' PO I⟩ ⟨O' ≠ I'⟩ congH-sym by blast
  moreover have CongH PO L O' L''
    using ⟨CongH O' L'' PO L⟩ ⟨O' ≠ L''⟩ congH-sym by blast
  moreover have CongaH I PO L I' O' L''
    by (simp add: ⟨CongaH I PO L I' O' L''⟩)
  ultimately show ?thesis
    using th12 by blast
qed
have CongaH PO I L O' I' L''
  using ⟨CongaH PO I L O' I' L'' ∧ CongaH PO L I O' L'' I' ∧ CongH I L I' L''⟩ by auto
have CongaH PO L I O' L'' I'
  using ⟨CongaH PO I L O' I' L'' ∧ CongaH PO L I O' L'' I' ∧ CongH I L I' L''⟩ by auto
have CongH I L I' L''
  using ⟨CongaH PO I L O' I' L'' ∧ CongaH PO L I O' L'' I' ∧ CongH I L I' L''⟩ by blast
have PO ≠ K
  using ⟨IncidL PO l⟩ ⟨¬ IncidL K l⟩ by auto
have PO ≠ L
  by (simp add: ⟨PO ≠ L⟩)
have ColH O' L' L''
  using ⟨ColH O' L' L''⟩ by blast
have O' ≠ L'
  using ⟨outH O' L' L'⟩ outH-expand by presburger
have CongaH PO K L O' K'' L'' ∧ CongaH PO L K O' L'' K'' ∧ CongH K L K'' L''
proof -
  have ¬ ColH PO K L
    using assms(3) colH-permut-213 by blast
  moreover
  {
    assume ColH O' K'' L''
    have O' ≠ K''
      using ⟨outH O' K' K''⟩ out-distinct by presburger
    have K'' ≠ L''
      by (metis ⟨ColH O' L' L''⟩ ⟨outH O' K' K''⟩ assms(4) colH-trans
        ncolH-expand outH-expand)
    have O' ≠ K'
      using assms(6) colH-trivial122 by blast
    have K' ≠ L'
      using assms(4) ncolH-expand by presburger
    have ColH O' K' L'
      by (metis ⟨ColH O' K'' L''⟩ ⟨ColH O' L' L''⟩ ⟨O' ≠ K''⟩ ⟨O' ≠ L''⟩
        ⟨outH O' K' K''⟩ colH-trans ncolH-expand outH-col)
    have False
      using ⟨ColH O' K' L'⟩ assms(4) colH-permut-213 by blast
  }
  hence ¬ ColH O' K'' L''
    by blast
  moreover have CongH PO K O' K''
    using ⟨CongH O' K'' PO K⟩ ⟨outH O' K' K''⟩ congH-sym out-distinct by presburger
  moreover have CongH PO L O' L''

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    using ⟨CongH O' L'' PO L⟩ ⟨O' ≠ L''⟩ congH-sym by presburger
  moreover have CongaH K PO L K'' O' L''
    by (simp add: ⟨CongaH K PO L K'' O' L''⟩)
  ultimately show ?thesis
    using th12 by blast
qed
have CongaH PO K L O' K'' L''
  using ⟨CongaH PO K L O' K'' L'' ∧ CongaH PO L K O' L'' K'' ∧ CongH K L K'' L''⟩ by auto
have CongaH PO L K O' L'' K''
  using ⟨CongaH PO K L O' K'' L'' ∧ CongaH PO L K O' L'' K'' ∧ CongH K L K'' L''⟩ by auto
have CongH K L K'' L''
  using ⟨CongaH PO K L O' K'' L'' ∧ CongaH PO L K O' L'' K'' ∧ CongH K L K'' L''⟩ by auto
have BetH K'' I' L''
proof -
  have outH L PO PO
    using ⟨PO ≠ L⟩ outH-def by auto
  have outH L'' O' O'
    using ⟨O' ≠ L''⟩ outH-def by auto
  have outH L'' I' I'
    using ⟨I' ≠ L''⟩ outH-def by blast
  have outH L I K
    using ⟨BetH K I L⟩ between-comm outH-def by blast
  hence CongaH PO L K O' L'' I'
    using conga-out-conga ⟨CongaH PO L I O' L'' I'⟩ ⟨outH L PO PO⟩ ⟨outH L'' I' I'⟩
    ⟨outH L'' O' O'⟩ by blast
  moreover
  have same-side' H' I' L'' O'
proof -
  obtain m where IsL m and IncidL O' m and IncidL L'' m
    using ⟨O' ≠ L''⟩ line-existence by blast
  have L'' ≠ O'
    using ⟨O' ≠ L''⟩ by auto
  moreover have IncidL L'' m
    by (simp add: ⟨IncidL L'' m⟩)
  moreover have IncidL O' m
    using ⟨IncidL O' m⟩ by auto
  moreover
  {
    assume IncidL H' m
    hence ColH O' L'' H'
      using ColH-def ⟨IsL m⟩ calculation(2) calculation(3) by blast
    hence False
      by (metis ⟨ColH O' L' L''⟩ assms(2) calculation(1) colH-trans ncolH-distincts)
  }
  hence ¬ IncidL H' m
    by blast
  moreover have outH O' H' I'
    using ⟨outH O' H' I'⟩ by blast
  ultimately show ?thesis using out-same-side'
    by blast
qed
moreover
have L'' ≠ O'
  using ⟨O' ≠ L''⟩ by blast
moreover
have ∀ l. (IsL l ∧ IncidL L'' l ∧ IncidL O' l) ⟶ same-side H' K'' l
proof -
  {
    fix l
    assume IsL l and
      IncidL L'' l and IncidL O' l
    have ColH O' L' L''
      by (simp add: ⟨ColH O' L' L''⟩)
    obtain lo where IncidL O' lo and IncidL L' lo and
      IncidL L'' lo and IncidL O' lo
      by (metis ⟨outH O' L' L''⟩ betH-line line-existence outH-def)
  }

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have EqL l lo
  using Is-line ⟨IncidL L'' l⟩ ⟨IncidL L'' lo⟩ ⟨IncidL O' l⟩ ⟨IncidL O' lo⟩
  calculation(3) line-uniqueness by presburger
hence same-side K' K'' lo
  using ColH-def Is-line ⟨IncidL L' lo⟩ ⟨IncidL O' lo⟩ ⟨outH O' K' K''⟩ assms(4)
  out-same-side by blast
hence same-side H' K' lo
  using Is-line ⟨IncidL L' lo⟩ ⟨IncidL O' lo⟩ assms(8) same-side'-def by force
hence same-side H' K'' l using same-side-trans
  by (metis (no-types, lifting) ColH-def ⟨IncidL L' lo⟩ ⟨IncidL L'' l⟩
    ⟨IncidL L'' lo⟩ ⟨IncidL O' l⟩ ⟨IncidL O' lo⟩ ⟨IsL l⟩ ⟨outH O' K' K''⟩ assms(4)
    assms(8) calculation(3) inter-incid-uniquenessH out-same-side same-side'-def)
}
thus ?thesis
  by blast
qed
ultimately have same-side' H' K'' L'' O'
  using same-side'-def by blast
thus ?thesis
  by (metis ⟨BetH K I L⟩ ⟨CongaH PO L K O' L'' I'⟩ ⟨same-side' H' I' L'' O'⟩ assms(3)
    ⟨CongaH PO I L O' I' L'' ∧ CongaH PO L I O' L'' I' ∧ CongH I L I' L''⟩
    ⟨CongaH PO K L O' K'' L'' ∧ CongaH PO L K O' L'' K'' ∧ CongH K L K'' L''⟩
    betH-congH3-outH-betH between-comm colH-permut-312 congH-permlr
    cong-4-uniqueness out-distinct same-side-prime-not-colH)
qed
have CongaH K PO I K'' O' I' ∧ CongaH K I PO K'' I' O' ∧ CongH PO I O' I'
proof -
  have CongH I' K'' I K
  proof -
    have BetH L'' I' K''
      using ⟨BetH K'' I' L''⟩ between-comm by blast
    moreover have BetH L I K
      by (simp add: ⟨BetH K I L⟩ between-comm)
    moreover have CongH L'' I' L I
      using ⟨BetH K I L⟩ ⟨CongH I L I' L''⟩ ⟨I' ≠ L''⟩ betH-distincts
      congH-perms by blast
    moreover have CongH L'' K'' L K
      by (metis ⟨CongH K L K'' L''⟩ assms(3) calculation(1) colH-trivial121
        congH-perms not-betH121)

    ultimately show ?thesis
      using soustraction-betH by blast
  qed
have ¬ ColH K PO I
  by (metis IncidL-not-IncidL--not-colH ⟨IncidL I l⟩ ⟨IncidL PO l⟩ ⟨PO ≠ I⟩
    ⟨¬ IncidL K l⟩ colH-permut-321)
moreover
{
  assume ColH K'' O' I'
  have O' ≠ K''
    using ⟨outH O' K' K''⟩ out-distinct by blast
  have ColH O' K' K''
    using ⟨outH O' K' K''⟩ outH-expand by auto
  hence ColH K' O' L'
    by (metis colH-permut-231 colH-trans colH-trivial121 ⟨ColH K'' O' I'⟩
      ⟨ColH O' I' H'⟩ ⟨O' ≠ I'⟩ ⟨O' ≠ K''⟩ assms(6))
  hence False
    by (simp add: assms(4))
}
hence ¬ ColH K'' O' I'
  by blast
moreover have CongH K PO K'' O'
  using outH-expand ⟨CongH O' K'' PO K⟩ ⟨PO ≠ K⟩ ⟨outH O' K' K''⟩
  congH-perms by blast
moreover have CongH K I K'' I'
  by (metis ⟨CongH I' K'' I K⟩ ⟨IncidL I l⟩ ⟨¬ IncidL K l⟩ calculation(2))

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      colH-trivial121 congH-perms)
moreover have CongaH PO K I O' K'' I'
  using ⟨BetH K I L⟩ ⟨BetH K'' I' L'⟩ ⟨CongaH PO K L O' K'' L'⟩ ⟨PO ≠ K⟩
    calculation(2) conga-out-conga ncolH-expand outH-def by blast
ultimately
show ?thesis using th12 by blast
qed
have CongaH K PO I K'' O' I'
  using ⟨CongaH K PO I K'' O' I' ∧ CongaH K I PO K'' I' O' ∧ CongH PO I O' I'⟩ by blast
have CongaH K I PO K'' I' O'
  using ⟨CongaH K PO I K'' O' I' ∧ CongaH K I PO K'' I' O' ∧ CongH PO I O' I'⟩ by auto
have CongH PO I O' I'
  using ⟨CongH O' I' PO I⟩ ⟨O' ≠ I'⟩ congH-sym by auto
have outH PO K K
  using ⟨PO ≠ K⟩ outH-def by force
moreover have outH PO I H
  by (simp add: ⟨outH PO I H⟩)
moreover have outH O' K'' K'
  using ⟨outH O' K' K''⟩ outH-sym out-distinct by blast
moreover have outH O' I' H'
  by (simp add: ⟨O' ≠ I'⟩ ⟨outH O' H' I'⟩ outH-sym)
ultimately show ?thesis
  using ⟨CongaH K PO I K'' O' I'⟩ conga-out-conga conga-permlr by blast
qed

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lemma th15-aux:

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assumes ¬ ColH H PO L and
  ¬ ColH H' O' L' and
  ¬ ColH K PO L and
  ¬ ColH K' O' L' and
  ¬ ColH H PO K and
  ¬ ColH H' O' K' and
  same-side' H K PO L and
  same-side' H' K' O' L' and
  CongaH H PO L H' O' L' and
  CongaH K PO L K' O' L'
shows CongaH H PO K H' O' K'
proof –
obtain p where IsP p and IncidP H p and IncidP K p and
  IncidP PO p and IncidP L p
  using Is-plane assms(7) same-side-prime--plane by blast
moreover have cut' K L PO H → CongaH H PO K H' O' K'
  using assms(1) assms(10) assms(2) assms(3) assms(4) assms(5) assms(6)
    assms(7) assms(8) assms(9) th15-aux-1 by auto
moreover {
  assume same-side' K L PO H
  moreover have ¬ ColH K PO H
    using assms(5) colH-permut-321 by blast
  moreover have ¬ ColH K' O' H'
    using assms(6) colH-permut-321 by blast
  moreover
  have PO ≠ L
    using assms(3) colH-trivial122 by blast
  {
    fix l
    assume IsL l and IncidL PO l and IncidL L l
    hence same-side K H l
      by (metis assms(7) same-side'-def same-side-comm)
  }
  hence same-side' K H PO L
    using ⟨PO ≠ L⟩ same-side'-def by auto
  moreover have O' ≠ L'
    using assms(4) colH-trivial122 by force
  {
    fix l
    assume IsL l and IncidL O' l and IncidL L' l

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    hence same-side  $K' H' l$ 
    by (meson  $assms(8)$  same-side'-def same-side-comm)
  }
  hence same-side'  $K' H' O' L'$ 
  by (simp add:  $\langle O' \neq L' \rangle$  same-side'-def)
  moreover have  $cut' H L PO K$ 
  by (simp add:  $OS2--TS$   $assms(7)$  calculation(1))
  ultimately have  $CongaH H PO K H' O' K'$ 
  using  $assms(1)$   $assms(2)$   $assms(3)$   $assms(4)$   $assms(9)$   $assms(10)$ 
  conga-permlr th15-aux-1 by presburger
}
moreover have  $\neg ColH K PO H$ 
using  $assms(5)$  colH-permut-321 by blast
moreover have  $\neg ColH L PO H$ 
using  $assms(1)$  colH-permut-321 by blast
ultimately show ?thesis
using plane-separation by blast
qed

lemma th15:
  assumes  $\neg ColH H PO L$  and
     $\neg ColH H' O' L'$  and
     $\neg ColH K PO L$  and
     $\neg ColH K' O' L'$  and
     $\neg ColH H PO K$  and
     $\neg ColH H' O' K'$  and
     $(cut' H K PO L \wedge cut' H' K' O' L') \vee (same-side' H K PO L \wedge same-side' H' K' O' L')$  and
     $CongaH H PO L H' O' L'$  and
     $CongaH K PO L K' O' L'$ 
  shows  $CongaH H PO K H' O' K'$ 
proof -
{
  assume same-side'  $H K PO L$  and same-side'  $H' K' O' L'$ 
  hence ?thesis using th15-aux
  using  $assms(1)$   $assms(2)$   $assms(3)$   $assms(4)$   $assms(5)$   $assms(6)$ 
   $assms(8)$   $assms(9)$  by presburger
}
moreover
{
  assume  $cut' H K PO L$  and
     $cut' H' K' O' L'$ 
  obtain  $SH$  where  $BetH H PO SH$ 
  by (metis  $assms(1)$  between-out colH-trivial112)
  obtain  $SH'$  where  $BetH H' O' SH'$ 
  by (metis  $assms(2)$  between-out colH-trivial112)
  have  $CongaH SH PO L SH' O' L'$ 
  using  $\langle BetH H PO SH \rangle \langle BetH H' O' SH' \rangle$   $assms(1)$   $assms(2)$   $assms(8)$ 
  conga-permlr th14 by blast
  have  $H \neq PO$ 
  using  $assms(1)$  colH-trivial112 by force
  have  $PO \neq SH$ 
  using  $\langle BetH H PO SH \rangle$  betH-distincts by blast
  have  $H \neq SH$ 
  using  $\langle BetH H PO SH \rangle$  not-betH121 by auto
  have  $H' \neq O'$ 
  using  $assms(6)$  colH-trivial112 by force
  have  $H' \neq SH'$ 
  using  $\langle BetH H' O' SH' \rangle$  not-betH121 by force
  have  $O' \neq SH'$ 
  using  $\langle BetH H' O' SH' \rangle$  betH-distincts by blast
  have  $CongaH SH PO K SH' O' K'$ 
  proof -
    have  $ColH H PO SH$ 
    by (simp add:  $\langle BetH H PO SH \rangle$  betH-colH)
    have  $ColH H' O' SH'$ 
    by (simp add:  $\langle BetH H' O' SH' \rangle$  betH-colH)

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moreover have  $\neg \text{ColH SH PO L}$ 
  by (metis  $\langle \text{ColH H PO SH} \rangle \langle \text{PO} \neq \text{SH} \rangle$  assms(1) colH-trans ncolH-expand)
moreover have  $\neg \text{ColH SH}' O' L'$ 
  by (metis  $\langle \text{ColH H}' O' \text{SH}' \rangle \langle O' \neq \text{SH}' \rangle$  assms(2) colH-permut-321
    colH-trans colH-trivial122)
moreover have  $\neg \text{ColH SH PO K}$ 
  by (metis  $\langle \text{ColH H PO SH} \rangle \langle \text{PO} \neq \text{SH} \rangle$  assms(5) colH-permut-321
    colH-trans colH-trivial122)
moreover have  $\neg \text{ColH SH}' O' K'$ 
  by (metis  $\langle \text{ColH H}' O' \text{SH}' \rangle \langle O' \neq \text{SH}' \rangle$  assms(6) colH-permut-321
    colH-trans colH-trivial122)
moreover
{
  fix l
  assume IsL l and IncidL PO l and IncidL L l
  hence cut l SH H
    by (meson ColH-def  $\langle \text{BetH H PO SH} \rangle$  assms(1) between-comm
      calculation(2) local.cut-def)
  moreover
  have cut l H K
    using  $\langle \text{cut}' H K PO L \rangle \langle \text{IncidL L l} \rangle \langle \text{IncidL PO l} \rangle \langle \text{IsL l} \rangle$  cut'-def by auto
  hence cut l K H
    using cut-comm by blast
  ultimately have same-side SH K l
    using same-side-def  $\langle \text{IsL l} \rangle$  by blast
}
moreover have  $\text{PO} \neq L$ 
  using calculation(2) colH-trivial122 by blast
ultimately have same-side' SH K PO L
  using same-side'-def by presburger
moreover
{
  fix l
  assume IsL l and IncidL O' l and IncidL L' l
  hence cut l SH' H'
    by (meson ColH-def  $\langle \text{BetH H}' O' \text{SH}' \rangle \langle \neg \text{ColH SH}' O' L' \rangle$ 
      assms(2) between-comm local.cut-def)
  moreover
  have cut l H' K'
    using  $\langle \text{cut}' H' K' O' L' \rangle \langle \text{IncidL L' l} \rangle \langle \text{IncidL O' l} \rangle \langle \text{IsL l} \rangle$  cut'-def by auto
  hence cut l K' H'
    using cut-comm by blast
  ultimately have same-side SH' K' l
    using same-side-def  $\langle \text{IsL l} \rangle$  by blast
}
moreover have  $O' \neq L'$ 
  using assms(2) colH-trivial122 by force
ultimately have same-side' SH' K' O' L'
  using same-side'-def by presburger
thus ?thesis
  using th15-aux  $\langle \text{CongaH SH PO L SH}' O' L' \rangle \langle \neg \text{ColH SH PO K} \rangle$ 
     $\langle \neg \text{ColH SH}' O' K' \rangle$  assms(9) same-side-prime-not-colH
     $\langle \text{same-side}' \text{SH K PO L} \rangle$  by blast
qed
moreover have  $\neg \text{ColH SH PO K}$ 
  using  $\langle \text{BetH H PO SH} \rangle$  assms(5) betH-colH between-comm colH-trans colH-trivial122 by blast
moreover have  $\neg \text{ColH SH}' O' K'$ 
  using  $\langle \text{BetH H}' O' \text{SH}' \rangle$  assms(6) betH-colH between-comm colH-trans colH-trivial122 by blast
moreover have BetH SH PO H
  by (simp add:  $\langle \text{BetH H PO SH} \rangle$  between-comm)
moreover have BetH SH' O' H'
  by (simp add:  $\langle \text{BetH H}' O' \text{SH}' \rangle$  between-comm)
ultimately have ?thesis
  using conga-permlr th14 by blast
}
ultimately show ?thesis

```

```

    using assms(7) by fastforce
qed

lemma th17:
  assumes  $\neg \text{ColH } X \ Y \ Z1$  and
     $\neg \text{ColH } X \ Y \ Z2$  and
     $\text{ColH } X \ I \ Y$  and
     $\text{BetH } Z1 \ I \ Z2$  and
     $\text{CongH } X \ Z1 \ X \ Z2$  and
     $\text{CongH } Y \ Z1 \ Y \ Z2$ 
  shows  $\text{CongaH } X \ Y \ Z1 \ X \ Y \ Z2$ 
proof (cases  $\text{ColH } Y \ Z1 \ Z2$ )
  case True
  hence  $\text{ColH } Y \ Z1 \ Z2$ 
  by simp
  show ?thesis
  proof (cases  $\text{ColH } X \ Z1 \ Z2$ )
    case True
    hence  $\text{ColH } X \ Z1 \ Z2$ 
    by simp
    have  $\text{ColH } Z1 \ I \ Z2$ 
    by (simp add: assms(4) betH-colH)
    have  $Z1 \neq Z2$ 
    using assms(4) not-betH121 by blast
    hence  $\text{ColH } X \ Y \ Z2$ 
    by (metis True  $\langle \text{ColH } Y \ Z1 \ Z2 \rangle$  colH-permut-312 colH-trivial122 inter-uniquenessH)
    thus ?thesis
    using assms(2) by auto
  next
  case False
  hence  $\neg \text{ColH } X \ Z1 \ Z2$ 
  by simp
  have  $\text{CongaH } X \ Z1 \ Z2 \ X \ Z2 \ Z1$ 
  by (simp add: False assms(5) isosceles-congaH)
  have  $\neg \text{ColH } Z1 \ Y \ X$ 
  using assms(1) colH-permut-321 by blast
  moreover have  $\neg \text{ColH } Z2 \ Y \ X$ 
  using assms(2) colH-permut-321 by blast
  moreover have  $\text{CongH } Z1 \ Y \ Z2 \ Y$ 
  using assms(2) assms(6) congH-permlr ncolH-expand by blast
  moreover have  $\text{CongH } Z1 \ X \ Z2 \ X$ 
  using False assms(5) congH-permlr ncolH-distincts by presburger
  moreover
  have  $\text{BetH } Z1 \ Y \ Z2$ 
  by (metis False True assms(1) assms(6) calculation(2) colH-trivial112
    colH-trivial122 congH-colH-betH)
  have  $\text{CongaH } X \ Z1 \ Y \ X \ Z2 \ Y$ 
  proof -
    have  $\text{outH } Z1 \ X \ X$ 
    by (metis False colH-trivial112 outH-trivial)
    moreover have  $\text{outH } Z1 \ Z2 \ Y$ 
    by (simp add:  $\langle \text{BetH } Z1 \ Y \ Z2 \rangle$  outH-def)
    moreover have  $\text{outH } Z2 \ X \ X$ 
    by (metis False colH-trivial121 outH-trivial)
    moreover have  $\text{outH } Z2 \ Z1 \ Y$ 
    using  $\langle \text{BetH } Z1 \ Y \ Z2 \rangle$  between-comm outH-def by blast
    ultimately show ?thesis
    using  $\langle \text{CongaH } X \ Z1 \ Z2 \ X \ Z2 \ Z1 \rangle$  conga-out-conga by blast
  qed
  hence  $\text{CongaH } Y \ Z1 \ X \ Y \ Z2 \ X$ 
  using conga-permlr by presburger
  ultimately have  $\text{CongaH } Z1 \ Y \ X \ Z2 \ Y \ X$ 
  using cong-5 by blast
  thus ?thesis
  using conga-permlr by blast
qed

```

```

next
case False
hence  $\neg \text{ColH } Y \ Z1 \ Z2$ 
by simp
show ?thesis
proof (cases  $\text{ColH } X \ Z1 \ Z2$ )
case True
hence  $\text{ColH } X \ Z1 \ Z2$ 
by simp
have  $\text{CongaH } Y \ Z1 \ Z2 \ Y \ Z2 \ Z1$ 
by (simp add: False assms(6) isosceles-congaH)
have  $\text{CongaH } Z1 \ Y \ X \ Z2 \ Y \ X$ 
proof -
have  $\neg \text{ColH } Z1 \ Y \ X$ 
using assms(1) colH-permut-321 by blast
moreover have  $\neg \text{ColH } Z2 \ Y \ X$ 
using assms(2) colH-permut-321 by blast
moreover have  $\text{CongH } Z1 \ Y \ Z2 \ Y$ 
by (metis False assms(6) colH-trivial121 congH-permlr)
moreover have  $\text{CongH } Z1 \ X \ Z2 \ X$ 
using assms(2) assms(5) congH-permlr ncolH-expand by presburger
moreover
have  $\text{BetH } Z1 \ X \ Z2$ 
by (metis False True assms(2) assms(5) calculation(1)
colH-trivial121 colH-trivial122 congH-colH-betH)
have  $\text{CongaH } Y \ Z1 \ X \ Y \ Z2 \ X$ 
proof -
have  $\text{outH } Z1 \ Y \ Y$ 
by (metis False colH-trivial112 outH-trivial)
moreover have  $\text{outH } Z1 \ Z2 \ X$ 
by (simp add:  $\langle \text{BetH } Z1 \ X \ Z2 \rangle$  outH-def)
moreover have  $\text{outH } Z2 \ Y \ Y$ 
using  $\langle \neg \text{ColH } Z2 \ Y \ X \rangle$  ncolH-expand outH-trivial by blast
moreover have  $\text{outH } Z2 \ Z1 \ X$ 
using  $\langle \text{BetH } Z1 \ X \ Z2 \rangle$  between-comm outH-def by blast
ultimately show ?thesis
using  $\langle \text{CongaH } Y \ Z1 \ Z2 \ Y \ Z2 \ Z1 \rangle$  conga-out-conga by blast
qed
ultimately show ?thesis
using cong-5 by blast
qed
thus ?thesis
using conga-permlr by blast
next
case False
hence  $\neg \text{ColH } X \ Z1 \ Z2$ 
by simp
have  $\text{CongaH } X \ Z1 \ Z2 \ X \ Z2 \ Z1$ 
by (simp add: False assms(5) isosceles-congaH)
have  $\text{CongaH } Y \ Z1 \ Z2 \ Y \ Z2 \ Z1$ 
by (simp add:  $\langle \neg \text{ColH } Y \ Z1 \ Z2 \rangle$  assms(6) isosceles-congaH)
have  $\text{CongaH } X \ Z1 \ Y \ X \ Z2 \ Y$ 
proof -
have  $\neg \text{ColH } X \ Z1 \ Z2$ 
by (simp add: False)
moreover have  $\neg \text{ColH } X \ Z2 \ Z1$ 
using calculation colH-permut-132 by blast
moreover have  $\neg \text{ColH } Y \ Z1 \ Z2$ 
by (simp add:  $\langle \neg \text{ColH } Y \ Z1 \ Z2 \rangle$ )
moreover have  $\neg \text{ColH } Y \ Z2 \ Z1$ 
using calculation(3) colH-permut-132 by blast
moreover have  $\neg \text{ColH } X \ Z1 \ Y$ 
using assms(1) colH-permut-132 by blast
moreover have  $\neg \text{ColH } X \ Z2 \ Y$ 
using assms(2) colH-permut-132 by blast
moreover

```

```

have cut' X Y Z1 Z2  $\wedge$  cut' X Y Z2 Z1  $\vee$  same-side' X Y Z1 Z2  $\wedge$  same-side' X Y Z2 Z1
proof -
  obtain p where IsP p and IncidP X p and IncidP Y p and IncidP Z1 p
  using assms(1) plan-existence by blast
  hence IncidP Z2 p
  by (metis betH-colH assms(2) assms(3) assms(4) colH-permut-312
      line-on-plane' ncolH-distincts)
  thus ?thesis
  by (metis False  $\langle$ IncidP X p $\rangle$   $\langle$ IncidP Y p $\rangle$   $\langle$ IncidP Z1 p $\rangle$  calculation(2)
      calculation(3) calculation(4) plane-separation same-side'-def)
qed
ultimately show ?thesis
  by (simp add:  $\langle$ CongaH Y Z1 Z2 Y Z2 Z1 $\rangle$   $\langle$ CongaH X Z1 Z2 X Z2 Z1 $\rangle$  th15)
qed
have CongaH Z1 Y X Z2 Y X
proof -
  have  $\neg$  ColH Z1 Y X
  using assms(1) colH-permut-321 by blast
  moreover have  $\neg$  ColH Z2 Y X
  using assms(2) colH-permut-321 by blast
  moreover have CongH Z1 Y Z2 Y
  by (metis  $\langle$  $\neg$  ColH Y Z1 Z2 $\rangle$  assms(6) colH-trivial121 congH-permlr)
  moreover have CongH Z1 X Z2 X
  by (metis assms(5) calculation(2) colH-trivial121 congH-permlr)
  moreover have CongaH Y Z1 X Y Z2 X
  using  $\langle$ CongaH X Z1 Y X Z2 Y $\rangle$  conga-permlr by blast
  ultimately show ?thesis
  using cong-5 by blast
qed
thus ?thesis
  using conga-permlr by blast
qed
qed

```

lemma congaH-existence-congH:

```

assumes U  $\neq$  V and
   $\neg$  ColH P PO X and
   $\neg$  ColH A B C
shows  $\exists$  Y. (CongaH A B C X PO Y  $\wedge$  same-side' P Y PO X  $\wedge$  CongH PO Y U V)
proof -
  have A  $\neq$  B
  using assms(3) colH-trivial112 by blast
  have C  $\neq$  B
  using assms(3) colH-trivial122 by blast
  have PO  $\neq$  X
  using assms(2) colH-trivial122 by blast
  {
    fix x
    assume CongaH A B C X PO x and
      same-side' P x PO X
    obtain Yaux where CongaH A B C X PO Yaux and same-side' P Yaux PO X
    using  $\langle$ CongaH A B C X PO x $\rangle$   $\langle$ same-side' P x PO X $\rangle$  by blast
    hence  $\neg$  ColH Yaux PO X
    using same-side-prime-not-colH by blast
    {
      assume PO = Yaux
      hence ColH Yaux PO X
      by (simp add: colH-trivial112)
      hence False
      by (simp add:  $\langle$  $\neg$  ColH Yaux PO X $\rangle$ )
    }
  }
  {
    fix Y
    assume CongH PO Y U V and outH PO Yaux Y
    have CongaH A B C X PO Y
    proof -

```



```

have CongaH A B C X PO Yaux
  using ⟨CongaH A B C X PO Yaux⟩ by blast
moreover have outH B A A
  using ⟨A ≠ B⟩ outH-trivial by auto
moreover have outH B C C
  using ⟨C ≠ B⟩ outH-trivial by presburger
moreover have outH PO X X
  using ⟨PO ≠ X⟩ outH-trivial by blast
moreover have outH PO Yaux Y
  using ⟨outH PO Yaux Y⟩ by auto
ultimately show ?thesis
  using conga-out-conga by blast
qed
moreover have same-side' P Y PO X
  using ColH-def ⟨¬ ColH Yaux PO X⟩ ⟨outH PO Yaux Y⟩
    ⟨same-side' P Yaux PO X⟩ out-same-side same-side'-def same-side-trans by fastforce
ultimately have ∃ Y. CongaH A B C X PO Y ∧ same-side' P Y PO X ∧ CongH PO Y U V
  using ⟨CongH PO Y U V⟩ by auto
}
hence ?thesis
  using ⟨PO = Yaux ⟹ False⟩ assms(1) out-construction by blast
}
thus ?thesis
  using assms(2) assms(3) cong-4-existence by blast
qed

lemma th18-aux:
  assumes ¬ ColH A B C and
    ¬ ColH A' B' C' and
    CongH A B A' B' and
    CongH A C A' C' and
    CongH B C B' C'
  shows CongaH B A C B' A' C'
proof -
  have A ≠ B
    using assms(1) colH-trivial112 by blast
  moreover
  have C' ≠ B'
    using assms(2) colH-trivial122 by blast
  have A' ≠ C'
    using assms(2) colH-trivial121 by force
  {
    fix B0
    assume CongaH C A B C' A' B0 and same-side' B' B0 A' C' and CongH A' B0 A B
    {
      fix P
      assume BetH B' C' P
      {
        fix B''
        assume CongaH C A B C' A' B'' and same-side' P B'' A' C' and CongH A' B'' A B
        have ¬ ColH A' C' B0
          using ⟨same-side' B' B0 A' C'⟩ colH-permut-312 same-side-prime-not-colH by blast
        have ¬ ColH A' C' B''
          using ⟨same-side' P B'' A' C'⟩ colH-permut-312 same-side-prime-not-colH by blast
        have CongH B C B0 C'
          by (metis ⟨CongH A' B0 A B⟩ ⟨CongaH C A B C' A' B0⟩
            ⟨same-side' B' B0 A' C'⟩ assms(1) assms(4) colH-permut-213 colH-trivial112
            congH-sym conga-permlr same-side-prime-not-colH th12)
        have CongH B C B'' C'
          by (metis ⟨CongH A' B'' A B⟩ ⟨CongaH C A B C' A' B''⟩ ⟨¬ ColH A' C' B''⟩
            assms(1) assms(4) colH-permut-132 congH-sym cong-permlr ncolH-distincts th12)
        have A' ≠ B''
          using ⟨¬ ColH A' C' B''⟩ colH-trivial121 by blast
        have A' ≠ B0
          using ⟨¬ ColH A' C' B0⟩ colH-trivial121 by fastforce
        have CongH A' B'' A' B0

```

by (meson $\langle A' \neq B'' \rangle \langle A' \neq B0 \rangle \langle CongH A' B'' A B \rangle \langle CongH A' B0 A B \rangle$
 congH-sym cong-pseudo-transitivity)
 have CongH B'' C' B0 C'
 using $\langle CongH B C B'' C' \rangle \langle CongH B C B0 C' \rangle$ cong-pseudo-transitivity by blast
 have CongH B'' C' B' C'
 using $\langle CongH B C B'' C' \rangle$ assms(5) cong-pseudo-transitivity by blast
 obtain l where IsL l and IncidL A' l and IncidL C' l
 using $\langle A' \neq C' \rangle$ line-existence by blast
 have cut l B' P
 by (meson ColH-def Is-line $\langle BetH B' C' P \rangle \langle IncidL A' l \rangle \langle IncidL C' l \rangle$
 $\langle same-side' P B'' A' C' \rangle$ assms(2) local.cut-def same-side-prime-not-colH)
 have cut l B' B''
 using $\langle IncidL A' l \rangle \langle IncidL C' l \rangle \langle IsL l \rangle \langle local.cut l B' P \rangle$
 $\langle same-side' P B'' A' C' \rangle$ cut-same-side-cut same-side'-def by blast
 have $\neg IncidL B' l$
 using $\langle local.cut l B' P \rangle$ local.cut-def by blast
 moreover have $\neg IncidL B'' l$
 using $\langle local.cut l B' B'' \rangle$ local.cut-def by blast
 moreover have $\exists I. IncidL I l \wedge BetH B' I B''$
 using $\langle local.cut l B' B'' \rangle$ local.cut-def by auto
 ultimately obtain I' where ColH A' I' C' and BetH B' I' B''
 using ColH-def Is-line $\langle IncidL A' l \rangle \langle IncidL C' l \rangle$ by blast
 have cut l B'' B0
 by (meson $\langle IncidL A' l \rangle \langle IncidL C' l \rangle \langle IsL l \rangle \langle local.cut l B' B'' \rangle$
 $\langle same-side' B' B0 A' C' \rangle$ cut-comm cut-same-side-cut same-side'-def)
 then obtain I where ColH A' I C' $\wedge$ BetH B0 I B''
 by (meson ColH-def $\langle IncidL A' l \rangle \langle IncidL C' l \rangle$ cut-comm local.cut-def)
 have CongaH C' A' B'' C' A' B0
 proof –
 have $\neg ColH C' A' B''$
 using $\langle \neg ColH A' C' B'' \rangle$ colH-permut-213 by blast
 moreover have $\neg ColH C' A' B0$
 using $\langle \neg ColH A' C' B0 \rangle$ colH-permut-213 by blast
 moreover have ColH C' I A'
 using $\langle ColH A' I C' \wedge BetH B0 I B'' \rangle$ colH-permut-321 by presburger
 moreover have BetH B'' I B0
 by (simp add: $\langle ColH A' I C' \wedge BetH B0 I B'' \rangle$ between-comm)
 moreover have CongH C' B'' C' B0
 by (metis $\langle CongH B'' C' B0 C' \rangle$ calculation(2) colH-trivial121 congH-permlr)
 ultimately show ?thesis
 using th17 $\langle CongH A' B'' A' B0 \rangle$ by blast
 qed
 moreover have outH A' B0 B'
 proof –
 have $\neg ColH B' A' C'$
 using assms(2) colH-permut-213 by blast
 moreover have $\neg ColH C' A' B''$
 using $\langle \neg ColH A' C' B'' \rangle$ colH-permut-213 by blast
 moreover have CongaH C' A' B'' C' A' B'
 proof –
 have $\neg ColH C' A' B'$
 using $\langle \neg ColH B' A' C' \rangle$ colH-permut-321 by blast
 moreover have CongH C' B'' C' B'
 using $\langle C' \neq B' \rangle \langle CongH B'' C' B' C' \rangle$ congH-permlr by presburger
 moreover have CongH A' B'' A' B'
 by (metis cong-pseudo-transitivity $\langle A' \neq B'' \rangle \langle CongH A' B'' A B \rangle$ assms(3) congH-sym)
 ultimately show ?thesis
 using th17 [of C' A' B'' B'] between-comm $\langle BetH B' I' B'' \rangle \langle ColH A' I' C' \rangle$
 colH-permut-321 $\langle \neg ColH C' A' B'' \rangle$ by blast
 qed
 moreover have same-side' B' B' A' C'
 using calculation(1) colH-permut-312 same-side-prime-refl by blast
 ultimately show ?thesis
 using $\langle CongaH C' A' B'' C' A' B0 \rangle \langle same-side' B' B0 A' C' \rangle$
 cong-4-uniqueness by blast
 qed

```

ultimately have CongaH B A C B' A' C'
  by (metis ⟨A ≠ B⟩ ⟨A' ≠ B0⟩ ⟨CongH A' B0 A B⟩ ⟨CongaH C A B C' A' B0⟩
    ⟨outH A' B0 B'⟩ assms(3) betH-not-congH congH-perms cong-pseudo-transitivity conga-permlr outH-def)
}
moreover have ¬ ColH P A' C'
  by (metis betH-expand ⟨BetH B' C' P⟩ assms(2) colH-permut-312
    colH-trans colH-trivial121)
moreover have ¬ ColH C A B
  using assms(1) colH-permut-231 by blast
ultimately have CongaH B A C B' A' C'
  using ⟨A ≠ B⟩ congaH-existence-congH by force
}
hence CongaH B A C B' A' C'
  using ⟨C' ≠ B'⟩ between-out by presburger
}
moreover have ¬ ColH B' A' C'
  using assms(2) colH-permut-213 by blast
moreover have ¬ ColH C A B
  using assms(1) colH-permut-231 by blast
ultimately show ?thesis
  using congaH-existence-congH by blast
qed

```

lemma th19:

```

assumes ¬ ColH PO A B and
  ¬ ColH O1 A1 B1 and
  ¬ ColH O2 A2 B2 and
  CongaH A PO B A1 O1 B1 and
  CongaH A PO B A2 O2 B2
shows CongaH A1 O1 B1 A2 O2 B2
proof -
  have PO ≠ A
    using assms(1) colH-trivial112 by blast
  have PO ≠ B
    using assms(1) colH-trivial121 by blast
  have O1 ≠ A1
    using assms(2) colH-trivial112 by blast
  have O1 ≠ B1
    using assms(2) colH-trivial121 by blast
  have O2 ≠ A2
    using assms(3) colH-trivial112 by force
  have O2 ≠ B2
    using assms(3) colH-trivial121 by blast
  {
    fix A1'
    assume CongH O1 A1' PO A and outH O1 A1 A1'
    {
      fix A2'
      assume CongH O2 A2' PO A and outH O2 A2 A2'
      {
        fix B1'
        assume CongH O1 B1' PO B and outH O1 B1 B1'
        {
          fix B2'
          assume CongH O2 B2' PO B and outH O2 B2 B2'
          have O1 ≠ A1'
            using ⟨outH O1 A1 A1'⟩ outH-expand by blast
          have O2 ≠ A2'
            using ⟨outH O2 A2 A2'⟩ outH-expand by auto
          have O1 ≠ B1'
            using ⟨outH O1 B1 B1'⟩ out-distinct by auto
          have O2 ≠ B2'
            using ⟨outH O2 B2 B2'⟩ outH-expand by blast
          {
            assume ColH O1 A1' B1'
            have ColH O1 A1 A1'

```

```

    by (simp add: ⟨outH O1 A1 A1'⟩ outH-expand)
  have ColH O1 B1 B1'
    by (simp add: ⟨outH O1 B1 B1'⟩ outH-expand)
  hence ColH O1 A1 B1
    by (metis colH-trivial121 ⟨ColH O1 A1 A1'⟩ ⟨ColH O1 A1' B1'⟩
      ⟨ColH O1 B1 B1'⟩ ⟨O1 ≠ A1'⟩ ⟨O1 ≠ B1'⟩ colH-permut-231 inter-uniquenessH)
  hence False
    using assms(2) by blast
}
have ColH O2 A2 A2'
  using ⟨outH O2 A2 A2'⟩ outH-expand by blast
have ColH O2 B2 B2'
  by (simp add: ⟨outH O2 B2 B2'⟩ outH-expand)
hence ¬ ColH O2 A2' B2'
  by (metis ⟨ColH O2 A2 A2'⟩ ⟨O2 ≠ A2'⟩ ⟨O2 ≠ B2'⟩ assms(3)
    colH-permut-312 colH-trans colH-trivial122)
have CongH A B A1' B1'
proof -
  have CongH PO A O1 A1'
    using ⟨CongH O1 A1' PO A⟩ ⟨O1 ≠ A1'⟩ ⟨PO ≠ A⟩ congH-perms by blast
  moreover have CongH PO B O1 B1'
    by (simp add: ⟨CongH O1 B1' PO B⟩ ⟨O1 ≠ B1'⟩ congH-sym)
  moreover have CongaH A PO B A1' O1 B1'
    using ⟨PO ≠ A⟩ ⟨PO ≠ B⟩ ⟨outH O1 A1 A1'⟩ ⟨outH O1 B1 B1'⟩
      assms(4) conga-out-conga outH-trivial by blast
  ultimately show ?thesis
    using assms(1) th12 ⟨ColH O1 A1' B1' ⟹ False⟩ by blast
qed
have CongH A B A2' B2'
proof -
  have CongH PO A O2 A2'
    using ⟨CongH O2 A2' PO A⟩ ⟨O2 ≠ A2'⟩ congH-sym by auto
  moreover have CongH PO B O2 B2'
    by (simp add: ⟨CongH O2 B2' PO B⟩ ⟨O2 ≠ B2'⟩ congH-sym)
  moreover have CongaH A PO B A2' O2 B2'
    using ⟨PO ≠ A⟩ ⟨PO ≠ B⟩ ⟨outH O2 A2 A2'⟩ ⟨outH O2 B2 B2'⟩
      assms(5) conga-out-conga outH-trivial by blast
  ultimately show ?thesis
    using assms(1) th12 ⟨¬ ColH O2 A2' B2'⟩ by blast
qed
have CongH A1' B1' A2' B2'
  using ⟨CongH A B A1' B1'⟩ ⟨CongH A B A2' B2'⟩ cong-pseudo-transitivity by blast
have CongaH A1' O1 B1' A2' O2 B2'
proof -
  have CongH O1 A1' O2 A2'
    by (meson ⟨CongH O1 A1' PO A⟩ ⟨CongH O2 A2' PO A⟩ ⟨O1 ≠ A1'⟩
      ⟨O2 ≠ A2'⟩ congH-sym cong-pseudo-transitivity)
  moreover have CongH O1 B1' O2 B2'
    by (meson ⟨CongH O1 B1' PO B⟩ ⟨CongH O2 B2' PO B⟩ ⟨O1 ≠ B1'⟩
      ⟨O2 ≠ B2'⟩ congH-sym cong-pseudo-transitivity)
  ultimately show ?thesis
    using th18-aux ⟨CongH A1' B1' A2' B2'⟩ ⟨ColH O1 A1' B1' ⟹ False⟩
      ⟨¬ ColH O2 A2' B2'⟩ by blast
qed
hence CongaH A1 O1 B1 A2 O2 B2
  using ⟨O1 ≠ A1'⟩ ⟨O1 ≠ B1'⟩ ⟨O2 ≠ A2'⟩ ⟨O2 ≠ B2'⟩ ⟨outH O1 A1 A1'⟩
    ⟨outH O1 B1 B1'⟩ ⟨outH O2 A2 A2'⟩ ⟨outH O2 B2 B2'⟩ conga-out-conga outH-sym by blast
}
hence CongaH A1 O1 B1 A2 O2 B2
  using ⟨O2 ≠ B2⟩ ⟨PO ≠ B⟩ out-construction by blast
}
hence CongaH A1 O1 B1 A2 O2 B2
  using ⟨O1 ≠ B1⟩ ⟨PO ≠ B⟩ out-construction by blast
}
hence CongaH A1 O1 B1 A2 O2 B2
  using ⟨O2 ≠ A2⟩ ⟨PO ≠ A⟩ out-construction by blast

```

```

}
thus ?thesis
  using ⟨O1 ≠ A1⟩ ⟨PO ≠ A⟩ out-construction by force
qed

```

```

lemma congaH-sym:
  assumes ¬ ColH A B C and
    ¬ ColH D E F and
    CongaH A B C D E F
  shows CongaH D E F A B C
  by (meson assms(1) assms(2) assms(3) colH-permut-213 conga-refl th19)

```

```

lemma congaH-commr:
  assumes ¬ ColH A B C and
    ¬ ColH D E F and
    CongaH A B C D E F
  shows CongaH A B C F E D
  using th19 [of - - - B A C E F D]
  by (meson assms(1,2,3) colH-permut-132 colH-permut-213 conga-comm conga-permlr)

```

```

lemma cong-preserves-col:
  assumes BetH A B C and
    CongH A B A' B' and
    CongH B C B' C' and
    CongH A C A' C'
  shows ColH A' B' C'
proof -
  have ColH A B C
    by (simp add: assms(1) betH-colH)
  have A ≠ B
    using assms(1) betH-distincts by blast
  have C ≠ B
    using assms(1) betH-distincts by blast
  {
    assume ¬ ColH A' B' C'
    {
      assume A' = C'
      hence False
        using ⟨¬ ColH A' B' C'⟩ colH-trivial121 by blast
    }
  }
  then obtain B'' where CongH A' B'' A B and outH A' C' B''
    using ⟨A ≠ B⟩ out-construction by blast
  hence ColH A' C' B''
    using outH-expand by blast
  have A' ≠ B''
    using ⟨outH A' C' B''⟩ out-distinct by blast
  have outH A' B'' C'
    by (simp add: ⟨A' ≠ B''⟩ ⟨outH A' C' B''⟩ outH-sym)
  hence BetH A' B'' C'
    using ⟨A' ≠ B''⟩ ⟨CongH A' B'' A B⟩ assms(1) assms(4)
    betH-congH3-outH-betH congH-sym by blast
  have A' ≠ B'
    using ⟨¬ ColH A' B' C'⟩ colH-trivial112 by auto
  obtain C'' where BetH A' B' C'' and CongH B' C'' B C
    using ⟨A' ≠ B'⟩ ⟨C ≠ B⟩ segment-constructionH by metis
  hence ColH A' B' C''
    using betH-colH by force
  have B' ≠ C''
    using ⟨BetH A' B' C''⟩ betH-distincts by blast
  have A' ≠ C''
    using ⟨BetH A' B' C''⟩ not-betH121 by blast
  have CongH B C B'' C'
    using ⟨A' ≠ B''⟩ ⟨BetH A' B'' C'⟩ ⟨CongH A' B'' A B⟩ assms(1) assms(4)
    congH-sym soustraction-betH by presburger
  have CongH A' C'' A' C'

```

```

    by (meson ⟨B' ≠ C'⟩ ⟨BetH A' B' C'⟩ ⟨ColH A B C⟩ ⟨ColH A' B' C'⟩
        ⟨CongH B' C'' B C⟩ addition assms(1) assms(2) assms(4) bet-disjoint
        congH-sym cong-pseudo-transitivity)
  have ¬ ColH A' C'' C'
    using ⟨A' ≠ C'⟩ ⟨ColH A' B' C'⟩ ⟨¬ ColH A' B' C'⟩ colH-permut-132
        colH-trans colH-trivial121 by blast
  have CongaH A' C'' C' A' C' C''
    by (simp add: ⟨CongH A' C'' A' C'⟩ ⟨¬ ColH A' C'' C'⟩ isosceles-congaH)
  have ¬ ColH B' C'' C'
    by (meson ⟨B' ≠ C'⟩ ⟨ColH A' B' C'⟩ ⟨¬ ColH A' B' C'⟩ colH-permut-231
        colH-trans colH-trivial121)
  have CongaH B' C'' C' B' C' C''
    by (meson ⟨B' ≠ C'⟩ ⟨CongH B' C'' B C'⟩ ⟨¬ ColH B' C'' C'⟩ assms(3)
        congH-sym cong-pseudo-transitivity isosceles-congaH)
  have CongaH A' C'' C' B' C'' C'
    using ⟨A' ≠ C'⟩ ⟨BetH A' B' C'⟩ ⟨ColH A' B' C'⟩ ⟨¬ ColH A' B' C'⟩
        ⟨¬ ColH A' C'' C'⟩ between-comm conga-out-conga conga-refl outH-def by blast
  have CongaH A' C' C'' B' C' C''
  proof (rule th19 [of C'' A' C' C' A' C'' C' B' C''], insert ⟨CongaH A' C'' C' A' C' C''⟩)
    show ¬ ColH C'' A' C'
      using ⟨¬ ColH A' C'' C'⟩ colH-permut-213 by blast
    show ¬ ColH C' A' C''
      using ⟨¬ ColH C'' A' C'⟩ colH-permut-321 by blast
    show ¬ ColH C' B' C''
      using ⟨¬ ColH B' C'' C'⟩ colH-permut-231 by blast
    show CongaH A' C'' C' B' C' C''
      by (metis (no-types) ⟨CongaH A' C'' C' B' C'' C'⟩ ⟨CongaH B' C'' C' B' C' C''⟩
          ⟨¬ ColH A' C'' C'⟩ ⟨¬ ColH B' C'' C'⟩ ⟨¬ ColH C' B' C''⟩ ⟨¬ ColH C'' A' C'⟩
          colH-permut-321 congaH-sym th19)
  qed
  have outH C' A' B'
  proof -
    have ¬ ColH B' C' C''
      using ⟨¬ ColH B' C'' C'⟩ colH-permut-132 by blast
    moreover have ¬ ColH A' C' C''
      using ⟨¬ ColH A' C'' C'⟩ colH-permut-132 by blast
    moreover have CongaH A' C' C'' C'' C' A'
      by (simp add: calculation(2) conga-comm)
    moreover have CongaH A' C' C'' C'' C' B'
      using ⟨CongaH A' C' C'' B' C' C''⟩ calculation(1) calculation(2)
          congaH-commr by blast
    moreover have same-side' B' A' C' C''
      by (metis ColH-def ⟨BetH A' B' C''⟩ between-comm calculation(1)
          colH-trivial122 outH-def out-same-side')
    moreover have same-side' B' B' C' C''
      using calculation(1) colH-permut-312 same-side-prime-refl by blast
    ultimately show ?thesis
      using cong-4-uniqueness by blast
  qed
  hence False
    using ⟨¬ ColH A' B' C'⟩ colH-permut-231 outH-col by blast
}
thus ?thesis by blast
qed

```

lemma *cong-preserves-col-stronger*:

assumes $A \neq B$ **and**

$A \neq C$ **and**

$B \neq C$ **and**

$\text{ColH } A \ B \ C$ **and**

$\text{CongH } A \ B \ A' \ B'$ **and**

$\text{CongH } B \ C \ B' \ C'$ **and**

$\text{CongH } A \ C \ A' \ C'$

shows $\text{ColH } A' \ B' \ C'$

by (metis ncolH-expand assms(1) assms(2) assms(3) assms(4) assms(5)

assms(6) assms(7) between-one colH-permut-213 colH-permut-312

congH-permlr cong-preserves-col)

lemma *betH-congH2--False*:
assumes *BetH A B C* **and**
BetH A' C' B' **and**
CongH A B A' B' **and**
CongH A C A' C'
shows *False*
proof –
have *ColH A B C*
using *assms(1) betH-colH* **by** *auto*
have *ColH A' C' B'*
by (*simp add: assms(2) betH-colH*)
have $A \neq B$
using *assms(1) betH-distincts* **by** *blast*
have $C \neq B$
using *assms(1) betH-distincts* **by** *blast*
have $A' \neq B'$
using *assms(2) not-betH121* **by** *force*
obtain *C0* **where** *BetH A' B' C0* **and** *CongH B' C0 B C*
using $\langle A' \neq B' \rangle \langle C \neq B \rangle$ *segment-constructionH* **by** *metis*
hence *CongH A' C0 A C*
using $\langle A \neq B \rangle$ *addition-betH assms(1) assms(3) congH-sym* **by** *blast*
have *BetH A' C' C0*
using $\langle BetH A' B' C0 \rangle$ *assms(2) betH-trans0* **by** *blast*
moreover have *CongH A' C' A' C0*
by (*metis* $\langle BetH A' B' C0 \rangle \langle CongH A' C0 A C \rangle$ *assms(4) betH-distincts*
congH-sym cong-pseudo-transitivity)
ultimately show *?thesis*
by (*simp add: betH-not-congH*)
qed

lemma *cong-preserves-bet*:
assumes $A' \neq B'$ **and**
 $B' \neq C'$ **and**
 $A' \neq C'$ **and**
BetH A B C **and**
CongH A B A' B' **and**
CongH B C B' C' **and**
CongH A C A' C'
shows *BetH A' B' C'*
by (*meson assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)*
assms(7) bet-cong3-bet cong-preserves-col)

lemma *axiom-five-segmentsH*:
assumes $A \neq D$ **and**
 $A' \neq D'$ **and**
 $B \neq D$ **and**
 $B' \neq D'$ **and**
 $C \neq D$ **and**
 $C' \neq D'$ **and**
CongH A B A' B' **and**
CongH B C B' C' **and**
CongH A D A' D' **and**
CongH B D B' D' **and**
BetH A B C **and**
BetH A' B' C' **and**
 $A' \neq D'$
shows *CongH C D C' D'*
proof –
have *ColH A B C*
by (*simp add: assms(11) betH-colH*)
have *ColH A' B' C'*
by (*simp add: assms(12) betH-colH*)
have $A \neq B$
using *assms(11) betH-distincts* **by** *blast*

```

have A ≠ C
  using assms(11) not-betH121 by blast
have A' ≠ B'
  using assms(12) betH-distincts by auto
have A' ≠ C'
  using assms(12) not-betH121 by blast
have C' ≠ B'
  using assms(12) betH-distincts by blast
have CongH A C A' C'
  using addition-betH assms(11) assms(12) assms(7) assms(8) by blast
{
  assume ColH A B D
  hence BetH A B D ∨ BetH B D A ∨ BetH B A D
    by (simp add: ⟨A ≠ B⟩ assms(1) assms(3) between-one)
  moreover
  {
    assume BetH A B D
    hence BetH A' B' D'
      using ⟨A' ≠ B'⟩ assms(10) assms(2) assms(4) assms(7) assms(9)
        cong-preserves-bet by blast
    hence BetH A' D' C' ∨ BetH A' C' D'
      using assms(12) assms(6) out2-out1 by auto
    moreover
    {
      assume BetH A' D' C'
      hence BetH A D C
        by (meson ⟨BetH A B D⟩ ⟨CongH A C A' C'⟩ assms(11) assms(5)
          assms(9) betH-congH2--False out2-out1)
      hence ?thesis
        using ⟨BetH A' D' C'⟩ ⟨CongH A C A' C'⟩ assms(9) betH-colH
          congH-permlr soustraction-betH by presburger
    }
  }
  moreover
  {
    assume BetH A' C' D'
    hence BetH A C D
      by (meson ⟨BetH A B D⟩ ⟨CongH A C A' C'⟩ assms(11) assms(5)
        assms(9) betH-congH2--False out2-out1)
    hence ?thesis
      using ⟨BetH A' C' D'⟩ ⟨CongH A C A' C'⟩ assms(9)
        soustraction-betH by blast
  }
  ultimately have ?thesis
    by blast
}
moreover
{
  assume BetH B D A
  hence BetH B' D' A'
    by (metis ⟨A' ≠ B'⟩ assms(10) assms(2) assms(4) assms(7)
      assms(9) congH-permlr cong-preserves-bet)
  hence BetH C D A
    using betH-trans0 ⟨BetH B D A⟩ assms(11) between-comm by blast
  hence BetH C' D' A'
    using betH-trans0 ⟨BetH B' D' A'⟩ assms(12) between-comm by blast
  have BetH A D C
    by (simp add: ⟨BetH C D A⟩ between-comm)
  moreover have BetH A' D' C'
    by (simp add: ⟨BetH C' D' A'⟩ between-comm)
  ultimately have ?thesis
    using ⟨CongH A C A' C'⟩ assms(9) betH-expand congH-permlr
      soustraction-betH by presburger
}
moreover
{
  assume BetH B A D

```



```

hence  $BetH\ B'\ A'\ D'$ 
  by (metis  $\langle A' \neq B' \rangle$  assms(10) assms(2) assms(4) assms(7) assms(9)
      congH-permlr cong-preserves-bet)
have  $BetH\ C'\ B'\ D'$ 
  using  $\langle BetH\ B'\ A'\ D' \rangle$  assms(12) betH-trans between-comm by blast
have  $BetH\ C\ B\ D$ 
  using  $\langle BetH\ B\ A\ D \rangle$  assms(11) betH-trans between-comm by blast
moreover have  $CongH\ C\ B\ C'\ B'$ 
  using  $\langle C' \neq B' \rangle$  assms(8) congH-permlr by presburger
ultimately have ?thesis
  using  $\langle BetH\ C'\ B'\ D' \rangle$  assms(10) addition-betH by blast
}
ultimately have ?thesis
  by blast
}
moreover
{
  assume  $\neg ColH\ A\ B\ D$ 
  hence  $\neg ColH\ A'\ B'\ D'$ 
    using cong-preserves-col-stronger  $\langle A \neq B \rangle$   $\langle A' \neq B' \rangle$  assms(1) assms(10)
        assms(2) assms(3) assms(4) assms(7) assms(9) congH-sym by blast
  hence  $CongaH\ B\ A\ D\ B'\ A'\ D'$ 
    using  $\langle \neg ColH\ A\ B\ D \rangle$  assms(10) assms(7) assms(9) th18-aux by blast
  have  $\neg ColH\ A\ C\ D$ 
    using  $\langle A \neq C \rangle$   $\langle ColH\ A\ B\ C \rangle$   $\langle \neg ColH\ A\ B\ D \rangle$  colH-permut-132 colH-trans
        colH-trivial121 by blast
  moreover have  $\neg ColH\ A'\ C'\ D'$ 
    using  $\langle A' \neq C' \rangle$   $\langle ColH\ A'\ B'\ C' \rangle$   $\langle \neg ColH\ A'\ B'\ D' \rangle$  colH-permut-132
        colH-trans colH-trivial121 by blast
  moreover have  $CongaH\ C\ A\ D\ C'\ A'\ D'$ 
    using  $\langle CongaH\ B\ A\ D\ B'\ A'\ D' \rangle$  assms(1) assms(11) assms(12) assms(2)
        conga-out-conga outH-def by blast
  ultimately have ?thesis
    using  $\langle CongH\ A\ C\ A'\ C' \rangle$  assms(9) th12 by blast
}
ultimately show ?thesis
  by blast
qed

```

lemma five-segment:

```

assumes  $Cong\ A\ B\ A'\ B'$  and
   $Cong\ B\ C\ B'\ C'$  and
   $Cong\ A\ D\ A'\ D'$  and
   $Cong\ B\ D\ B'\ D'$  and
   $Bet\ A\ B\ C$  and
   $Bet\ A'\ B'\ C'$  and
   $A \neq B$ 

```

shows $Cong\ C\ D\ C'\ D'$

proof –

```

{
  assume  $BetH\ A\ B\ C$ 
  {
    assume  $BetH\ A'\ B'\ C'$ 
    have  $ColH\ A\ B\ C$ 
      using  $\langle BetH\ A\ B\ C \rangle$  betH-colH by blast
    have  $ColH\ A'\ B'\ C'$ 
      by (simp add: assms(6) bet-colH)
    have  $C \neq B$ 
      using  $\langle BetH\ A\ B\ C \rangle$  betH-distincts by blast
    have  $A \neq C$ 
      using  $\langle BetH\ A\ B\ C \rangle$  betH-distincts by blast
    have  $A' \neq B'$ 
      using  $\langle BetH\ A'\ B'\ C' \rangle$  betH-distincts by blast
    have  $C' \neq B'$ 
      using  $\langle BetH\ A'\ B'\ C' \rangle$  betH-distincts by blast
    have  $A' \neq C'$ 

```

```

    using ⟨BetH A' B' C'⟩ betH-distincts by blast
have CongH A B A' B'
    using Cong-def ⟨A' ≠ B'⟩ assms(1) by presburger
have CongH B C B' C'
    using Cong-def ⟨C ≠ B⟩ assms(2) by blast
have CongH A D A' D' ∧ A ≠ D ∧ A' ≠ D' ∨ (A = D ∧ A' = D')
    using Cong-def assms(3) by force
moreover
have CongH B D B' D' ∧ B ≠ D ∧ B' ≠ D' ∨ (B = D ∧ B' = D')
    using Cong-def assms(4) by presburger
{
  assume CongH A D A' D' ∧ A ≠ D ∧ A' ≠ D'
  hence CongH A D A' D'
    by blast
  have A ≠ D
    by (simp add: ⟨CongH A D A' D' ∧ A ≠ D ∧ A' ≠ D'⟩)
  have A' ≠ D'
    by (simp add: ⟨CongH A D A' D' ∧ A ≠ D ∧ A' ≠ D'⟩)
  {
    assume CongH B D B' D' ∧ B ≠ D ∧ B' ≠ D'
    hence CongH B D B' D'
      by blast
    have B ≠ D
      by (simp add: ⟨CongH B D B' D' ∧ B ≠ D ∧ B' ≠ D'⟩)
    have B' ≠ D'
      by (simp add: ⟨CongH B D B' D' ∧ B ≠ D ∧ B' ≠ D'⟩)
    {
      assume C = D
      have CongH B' C' B' D'
        using ⟨C = D⟩ ⟨CongH B C B' C'⟩ ⟨CongH B D B' D' ∧ B ≠ D ∧ B' ≠ D'⟩
        cong-pseudo-transitivity by blast
      hence C' = D'
        using ⟨A' ≠ B'⟩ ⟨A' ≠ D'⟩ ⟨B' ≠ D'⟩ ⟨BetH A B C⟩ ⟨BetH A' B' C'⟩
        ⟨C = D⟩ ⟨CongH A B A' B'⟩ ⟨CongH A D A' D'⟩ ⟨CongH B D B' D'⟩
        cong-preserves-bet construction-uniqueness by blast
      hence ?thesis
        by (simp add: Cong-def ⟨C = D⟩)
    }
  }
moreover
{
  assume C ≠ D
  {
    assume C' = D'
    hence CongH B D B C
      by (metis ⟨B ≠ D⟩ ⟨C ≠ B⟩ ⟨CongH B C B' C'⟩ ⟨CongH B D B' D'⟩
      congH-sym cong-pseudo-transitivity)
    have BetH A B D
      using ⟨A ≠ D⟩ ⟨B ≠ D⟩ ⟨BetH A' B' C'⟩ ⟨C' = D'⟩ ⟨CongH A B A' B'⟩
      ⟨CongH A D A' D'⟩ ⟨CongH B D B' D'⟩ assms(7) congH-sym cong-preserves-bet by presburger
    moreover have CongH B C B D
      using ⟨B ≠ D⟩ ⟨CongH B D B C⟩ congH-sym by auto
    ultimately have C = D
      using construction-uniqueness ⟨BetH A B C⟩ by presburger
    hence False
      using ⟨C ≠ D⟩ by blast
  }
  hence C' ≠ D'
    by blast
  hence ?thesis
    by (meson Cong-def ⟨BetH A B C⟩ ⟨BetH A' B' C'⟩ ⟨CongH A B A' B'⟩
    ⟨CongH A D A' D' ∧ A ≠ D ∧ A' ≠ D'⟩ ⟨CongH B C B' C'⟩
    ⟨CongH B D B' D' ∧ B ≠ D ∧ B' ≠ D'⟩ axiom-five-segmentsH calculation)
}
ultimately have ?thesis
  by blast
}

```

```

    moreover
    {
      assume  $B = D \wedge B' = D'$ 
      hence ?thesis
      using Cong-def  $\langle C \neq B \rangle \langle C' \neq B' \rangle \langle CongH\ B\ C\ B'\ C' \rangle\ congH-permlr$  by presburger
    }
    ultimately have ?thesis
      using  $\langle CongH\ B\ D\ B'\ D' \wedge B \neq D \wedge B' \neq D' \vee B = D \wedge B' = D' \rangle$  by fastforce
  }
  moreover
  {
    assume  $A = D \wedge A' = D'$ 
    hence ?thesis
    using Cong-def  $\langle A \neq C \rangle \langle A' \neq C' \rangle \langle BetH\ A\ B\ C \rangle \langle BetH\ A'\ B'\ C' \rangle$ 
       $\langle CongH\ A\ B\ A'\ B' \rangle \langle CongH\ B\ C\ B'\ C' \rangle$  addition-betH congH-permlr by presburger
  }
  ultimately have ?thesis
    by blast
}
moreover
{
  assume  $A' = B'$ 
  hence ?thesis
  using assms(1) assms(7) cong-identity by blast
}
moreover
{
  assume  $B' = C'$ 
  hence ?thesis
  using Cong-def  $\langle BetH\ A\ B\ C \rangle$  assms(2) betH-to-bet by auto
}
ultimately have ?thesis
  using Bet-def assms(6) by blast
}
moreover
{
  assume  $A = B$ 
  hence ?thesis
  using assms(7) by blast
}
moreover
{
  assume  $B = C$ 
  hence ?thesis
  using Cong-def assms(2) assms(4) by presburger
}
ultimately show ?thesis
  using Bet-def assms(5) by blast
qed

```

lemma bet-comm:

```

  assumes Bet A B C
  shows Bet C B A
  using Bet-def assms between-comm by presburger

```

lemma bet-trans:

```

  assumes Bet A B D and
    Bet B C D
  shows Bet A B C
proof -
  {
    assume BetH A B D
    hence ?thesis
    using Bet-def assms(2) betH-trans0 between-comm by blast
  }
  moreover

```

```

{
  assume  $A = B \vee B = D$ 
  hence ?thesis
    using Bet-def assms(2) bet-identity by blast
}
ultimately show ?thesis
  using Bet-def assms(1) by blast
qed

```

```

lemma cong-transitivity:
  assumes Cong  $A B E F$  and
    Cong  $C D E F$ 
  shows Cong  $A B C D$ 
  using assms(1) assms(2) cong-sym cong-trans by blast

```

```

lemma cong-permT:
  shows Cong  $A B B A$ 
  by (metis Cong-def congH-perm)

```

```

lemma pasch-general-case:
  assumes Bet  $A P C$  and
    Bet  $B Q C$  and
     $A \neq P$  and
     $P \neq C$  and
     $B \neq Q$  and
     $Q \neq C$  and
     $\neg (Bet A B C \vee Bet B C A \vee Bet C A B)$ 
  shows  $\exists x. Bet P x B \wedge Bet Q x A$ 
proof (cases  $A = B$ )
  case True
  thus ?thesis
    using Bet-def by auto

```

```

next
  case False
  {
    assume  $B = C$ 
    hence ?thesis
      using Bet-def assms(7) by presburger
  }
  moreover
  {
    assume  $B \neq C$ 
    {
      assume  $A = C$ 
      hence ?thesis
        using Bet-def assms(7) by blast
    }
    moreover
    {
      assume  $A \neq C$ 
      have ColH  $A P C$ 
        by (simp add: assms(1) bet-colH)
      {
        assume ColH  $B C P$ 
        hence ColH  $A B C$ 
          by (metis colH-permut-231 colH-trivial121 <ColH A P C> assms(4) inter-uniquenessH)
        hence False
          by (metis Bet-def assms(7) between-comm between-one)
      }
      hence  $\neg ColH B C P$ 
        by blast
      hence ?thesis
        using assms(1) assms(2) inner-pasch-aux by blast
    }
  }
  ultimately have ?thesis
    by blast

```

```

}
ultimately show ?thesis
  by blast
qed

```

```

lemma lower-dim-l:
  shows  $\neg (Bet\ PP\ PQ\ PR \vee Bet\ PQ\ PR\ PP \vee Bet\ PR\ PP\ PQ)$ 
  using bet-colH colH-permut-231 lower-dim-2 by blast

```

```

lemma ColH-bets:
  assumes ColH A B C
  shows  $Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$ 
  by (metis Bet-def assms bet-comm between-one)

```

```

end
end

```

```

theory Tarski-Neutral-Hilbert

```

```

imports
  Tarski-Neutral
  Hilbert-Neutral

```

```

begin

```

9 Tarski neutral dimensionless - Hilbert

```

context Tarski-neutral-dimensionless

```

```

begin

```

9.1 Definition

```

definition isLine:: 'p×'p⇒bool where
  isLine l ≡ (fst l ≠ snd l)

```

```

definition Line :: 'p ⇒'p ⇒'p×'p where
  Line A B = (if A ≠ B then (Pair A B) else undefined)

```

```

definition IncidentL :: 'p⇒'p×'p⇒bool where
  IncidentL A l ≡ isLine l ∧ Col A (fst l) (snd l)

```

```

definition EqTL:: 'p×'p⇒'p×'p⇒bool
  (- =l= - [99,99] 50) where
  l1 =l= l2 ≡ isLine l1 ∧ isLine l2 ∧ (∀ X. (IncidentL X l1 ⟷ IncidentL X l2))

```

```

definition Col-H:: 'p⇒'p⇒'p⇒bool where
  Col-H A B C ≡ ∃ l. (isLine l ∧ IncidentL A l ∧ IncidentL B l ∧ IncidentL C l)

```

```

definition isPlane :: 'p×'p×'p⇒bool where
  isPlane pl ≡ (∀ P Q R::'p. pl = (P,Q,R) ⟶ ¬ Col P Q R)

```

```

definition Plane :: 'p⇒'p⇒'p⇒'p×'p×'p where
  Plane A B C = (A,B,C)

```

```

definition IncidentP :: 'p⇒'p×'p×'p⇒bool
  where
  IncidentP A pl ≡ (isPlane pl) ∧ (∃ P Q R. pl = Plane P Q R ∧ Coplanar A P Q R)

```

```

definition EqTP:: 'p×'p×'p⇒'p×'p×'p⇒bool (- =p= - [99,99] 50) where
  p1 =p= p2 ≡ isPlane p1 ∧ isPlane p2 ∧ (∀ X. (IncidentP X p1 ⟷ IncidentP X p2))

```

```

definition IncidentLP :: 'p×'p⇒'p×'p×'p⇒bool where
  IncidentLP l p ≡ isLine l ∧ isPlane p ∧ (∀ A. IncidentL A l ⟶ IncidentP A p)

```

definition *Between-H* :: ' $p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ ' **where**

Between-H $A B C \equiv \text{Bet } A B C \wedge A \neq B \wedge B \neq C \wedge A \neq C$

definition *cut-H* :: ' $p \times p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ ' **where**

cut-H $l A B \equiv \text{isLine } l \wedge \neg \text{IncidentL } A l \wedge \neg \text{IncidentL } B l \wedge$
 $(\exists I. \text{IncidentL } I l \wedge \text{Between-H } A I B)$

definition *outH* :: ' $p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ ' **where**

outH $P A B \equiv \text{Between-H } P A B \vee \text{Between-H } P B A \vee (P \neq A \wedge A = B)$

definition *same-side-scott* :: ' $p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ '

where *same-side-scott* $E A B \equiv E \neq A \wedge E \neq B \wedge \text{Col-H } E A B \wedge \neg \text{Between-H } A E B$

definition *disjoint-H* :: ' $p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ '

where *disjoint-H* $A B C D \equiv \neg (\exists P. \text{Between-H } A P B \wedge \text{Between-H } C P D)$

definition *same-side-H* :: ' $p \Rightarrow p \Rightarrow p \times p \Rightarrow \text{bool}$ '

where *same-side-H* $A B l \equiv \text{isLine } l \wedge (\exists P. \text{cut-H } l A P \wedge \text{cut-H } l B P)$

definition *same-side'-H* :: ' $p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ '

where

same-side'-H $A B X Y \equiv X \neq Y \wedge (\forall l. \text{isLine } l \wedge (\text{IncidentL } X l \wedge \text{IncidentL } Y l) \rightarrow \text{same-side-H } A B l)$

definition *CongA-H* :: ' $p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow p \Rightarrow \text{bool}$ ' **where**

CongA-H $A B C D E F \equiv A B C \text{ CongA } D E F$

definition *Para-H* :: ' $p \times p \Rightarrow p \times p \Rightarrow \text{bool}$ ' **where**

Para-H $l m \equiv \text{isLine } l \wedge \text{isLine } m \wedge (\neg (\exists X. \text{IncidentL } X l \wedge \text{IncidentL } X m)) \wedge$
 $(\exists p. \text{IncidentLP } l p \wedge \text{IncidentLP } m p)$

9.2 Propositions

lemma *EqL--diff-left*:

assumes $l1 = l = l1$

shows $\text{fst } l1 \neq \text{snd } l1$

using *EqTL-def* *assms* *isLine-def* **by** *blast*

lemma *EqL--diff-right*:

assumes $l1 = l = l2$

shows $(\text{fst } l2) \neq (\text{snd } l2)$

using *EqTL-def* *assms* *isLine-def* **by** *blast*

lemma *axiom-line-existence*:

assumes $A \neq B$

shows $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l$

proof –

let $?l = \text{Pair } A B$

have *isLine* $?l$

by (*simp* *add*: *assms* *isLine-def*)

moreover **have** *IncidentL* $A ?l$

by (*simp* *add*: *IncidentL-def* $\langle \text{isLine } (A, B) \rangle$ *col-trivial-1*)

moreover **have** *IncidentL* $B ?l$

by (*simp* *add*: *IncidentL-def* $\langle \text{isLine } (A, B) \rangle$ *col-trivial-3*)

ultimately **show** *?thesis*

by *blast*

qed

lemma *incident-eq*:

assumes $A \neq B$ **and**

IncidentL $A l$ **and**

IncidentL $B l$

shows $\text{Line } A B = l = l$

proof –

have *Col* $A (\text{fst } l) (\text{snd } l)$

using *IncidentL-def* *assms* (2) **by** *presburger*

```

have Col B (fst l) (snd l)
  using IncidentL-def assms(3) by blast

have isLine (Line A B)
  by (simp add: Line-def assms(1) isLine-def)
moreover have isLine l
  using IncidentL-def assms(2) by blast
{
  fix X
  assume IncidentL X (Line A B)
  hence Col X A B
    by (simp add: IncidentL-def Line-def assms(1))
  have Col X (fst l) (snd l)
    by (meson ⟨Col A (fst l) (snd l)⟩ ⟨Col B (fst l) (snd l)⟩ ⟨Col X A B⟩
      assms(1) col-permutation-1 colx)
  hence IncidentL X l
    by (simp add: IncidentL-def ⟨isLine l⟩)
}
moreover
{
  fix X
  assume IncidentL X l
  have Col X A B
    by (metis (no-types, lifting) IncidentL-def ⟨Col A (fst l) (snd l)⟩
      ⟨Col B (fst l) (snd l)⟩ ⟨IncidentL X l⟩ col-permutation-5 isLine-def l6-16-1)
  have IncidentL X (Line A B)
    using IncidentL-def Line-def ⟨Col X A B⟩ assms(1) calculation(1) by auto
}
ultimately show ?thesis
  using EqTL-def ⟨isLine l⟩ by blast
qed

```

```

lemma eq-transitivity:
  assumes l =l= m and
    m =l= n
  shows l =l= n
  using EqTL-def assms(1) assms(2) by presburger

```

```

lemma eq-reflexivity:
  assumes isLine l
  shows l =l= l
  using EqTL-def assms by auto

```

```

lemma eq-symmetry:
  assumes l =l= m
  shows m =l= l
  using EqTL-def assms by blast

```

```

lemma eq-incident:
  assumes l =l= m
  shows IncidentL A l ⟷ IncidentL A m
  using EqTL-def assms by blast

```

```

lemma axiom-Incidl-morphism:
  assumes IncidentL P l and
    l =l= m
  shows IncidentL P m
  using assms(1) assms(2) eq-incident by blast

```

```

lemma axiom-line-uniqueness:
  assumes A ≠ B and
    IncidentL A l and IncidentL B l and

```

```

    IncidentL A m and IncidentL B m
shows l =l= m
proof -
  have Line A B =l= l
    by (simp add: assms(1) assms(2) assms(3) incident-eq)
  moreover have Line A B =l= m
    by (simp add: assms(1) assms(4) assms(5) incident-eq)
  ultimately show ?thesis
    using eq-symmetry eq-transitivity by blast
qed

```

```

lemma axiom-two-points-on-line:
  assumes isLine l
  shows  $\exists A B. \text{IncidentL } B \ l \wedge \text{IncidentL } A \ l \wedge A \neq B$ 
proof -
  let ?A = fst l
  let ?B = snd l
  have IncidentL ?B l
    by (simp add: IncidentL-def assms col-trivial-3)
  moreover have IncidentL ?A l
    by (simp add: IncidentL-def assms col-trivial-1)
  moreover have ?A  $\neq$  ?B
    using assms isLine-def by blast
  ultimately show ?thesis
    by blast
qed

```

```

lemma cols-coincide-1:
  assumes Col-H A B C
  shows Col A B C
proof -
  obtain l where IncidentL A l and IncidentL B l and IncidentL C l
    using Col-H-def assms by blast
  have (fst l)  $\neq$  (snd l)
    using IncidentL-def  $\langle \text{IncidentL } B \ l \rangle$  isLine-def by blast
  have Col A (fst l) (snd l)
    using IncidentL-def  $\langle \text{IncidentL } A \ l \rangle$  by auto
  moreover have Col B (fst l) (snd l)
    using IncidentL-def  $\langle \text{IncidentL } B \ l \rangle$  by auto
  moreover have Col C (fst l) (snd l)
    using IncidentL-def  $\langle \text{IncidentL } C \ l \rangle$  by auto
  ultimately show ?thesis
    by (metis  $\langle \text{fst } l \neq \text{snd } l \rangle$  col2--eq col-permutation-4 col-transitivity-1)
qed

```

```

lemma cols-coincide-2:
  assumes Col A B C
  shows Col-H A B C
proof (cases A = B)
  case True
  thus ?thesis
    by (metis Col-H-def axiom-line-existence diff-col-ex)
next
  case False
  let ?l = Line A B
  have isLine ?l
    by (simp add: False Line-def isLine-def)
  moreover have IncidentL A ?l
    using False IncidentL-def Line-def calculation col-trivial-1 by auto
  moreover have IncidentL B ?l
    using False IncidentL-def Line-def calculation(1) col-trivial-3 by auto
  moreover have IncidentL C ?l
    by (metis Col-perm False IncidentL-def Line-def assms calculation(1))

```



```

      fst-def prod.simps(2) snd-def)
ultimately show ?thesis
  using Col-H-def by blast
qed

lemma cols-coincide:
  shows  $Col\ A\ B\ C \longleftrightarrow Col-H\ A\ B\ C$ 
  using cols-coincide-1 cols-coincide-2 by blast

lemma ncols-coincide:
  shows  $\neg Col\ A\ B\ C \longleftrightarrow \neg Col-H\ A\ B\ C$ 
  by (simp add: cols-coincide)

lemma lower-dim':
  shows  $\exists\ PA\ PB\ PC. PA \neq PB \wedge PB \neq PC \wedge$ 
 $PA \neq PC \wedge \neg Col-H\ PA\ PB\ PC$ 
  by (metis col-trivial-2 col-trivial-3 cols-coincide-1 not-col-exists
    point-construction-different)

lemma axiom-plane-existence:
  assumes  $\neg Col-H\ A\ B\ C$ 
  shows  $\exists\ p. IncidentP\ A\ p \wedge IncidentP\ B\ p \wedge IncidentP\ C\ p$ 
proof -
  let ?p = Plane A B C
  have isPlane ?p
    by (simp add: Plane-def assms isPlane-def ncols-coincide)
  have IncidentP A ?p
    using IncidentP-def <isPlane (Plane A B C)> ncop-distincts by blast
  moreover have IncidentP B ?p
    using IncidentP-def <isPlane (Plane A B C)> ncop-distincts by blast
  moreover have IncidentP C ?p
    using IncidentP-def <isPlane (Plane A B C)> ncop-distincts by blast
  ultimately show ?thesis
    by blast
qed

lemma incidentp-eqp:
  assumes  $\neg Col-H\ A\ B\ C$  and
    IncidentP A p and
    IncidentP B p and
    IncidentP C p
  shows (Plane A B C) = p
proof -
  have isPlane (Plane A B C)
    using Plane-def Tarski-neutral-dimensionless.cols-coincide-2
    Tarski-neutral-dimensionless-axioms assms(1) isPlane-def by fastforce
  have isPlane p
    using IncidentP-def assms(3) by blast
  obtain PA QA RA where  $p = Plane\ PA\ QA\ RA$  and
    Coplanar A PA QA RA
    using assms(2) IncidentP-def by blast
  hence  $p = (PA, QA, RA)$ 
    by (simp add: Plane-def)
  hence Coplanar B PA QA RA
    using IncidentP-def Plane-def assms(3) by auto
  have Coplanar C PA QA RA
    using IncidentP-def Plane-def assms(4) by (simp add: <p = (PA, QA, RA)>)
  have  $\neg Col\ PA\ QA\ RA$ 
    using Plane-def <isPlane p> <p = (PA, QA, RA)> isPlane-def by blast
  have  $\neg Col\ A\ B\ C$ 
    using Plane-def by (simp add: assms(1) ncols-coincide)
  {
    fix X
    assume IncidentP X (Plane A B C)
    hence isPlane (Plane A B C)
      by (simp add: <isPlane (Plane A B C)>)
  }

```

```

obtain  $P \ Q \ R$  where  $(\text{Plane } A \ B \ C) = \text{Plane } P \ Q \ R$  and  $\text{Coplanar } X \ P \ Q \ R$ 
using  $\text{IncidentP-def } \langle \text{IncidentP } X \ (\text{Plane } A \ B \ C) \rangle$  by blast
hence  $A = P \wedge B = Q \wedge C = R$ 
using  $\text{Plane-def}$  by simp
hence  $\text{Coplanar } X \ A \ B \ C$ 
using  $\langle \text{Coplanar } X \ P \ Q \ R \rangle$  by auto
moreover
have  $\text{Coplanar } PA \ A \ B \ C$ 
by  $(\text{meson } \langle \text{Coplanar } A \ PA \ QA \ RA \rangle \langle \text{Coplanar } B \ PA \ QA \ RA \rangle \langle \text{Coplanar } C \ PA \ QA \ RA \rangle$ 
 $\langle \neg \text{Col } PA \ QA \ RA \rangle \text{coplanar-perm-9 coplanar-pseudo-trans ncop-distincts})$ 
moreover have  $\text{Coplanar } QA \ A \ B \ C$ 
by  $(\text{meson } \langle \text{Coplanar } A \ PA \ QA \ RA \rangle \langle \text{Coplanar } B \ PA \ QA \ RA \rangle \langle \text{Coplanar } C \ PA \ QA \ RA \rangle$ 
 $\langle \neg \text{Col } PA \ QA \ RA \rangle \text{coplanar-perm-9 coplanar-pseudo-trans ncop-distincts})$ 
moreover have  $\text{Coplanar } RA \ A \ B \ C$ 
by  $(\text{meson } \langle \text{Coplanar } A \ PA \ QA \ RA \rangle \langle \text{Coplanar } B \ PA \ QA \ RA \rangle \langle \text{Coplanar } C \ PA \ QA \ RA \rangle$ 
 $\langle \neg \text{Col } PA \ QA \ RA \rangle \text{coplanar-perm-9 coplanar-pseudo-trans ncop-distincts})$ 
ultimately
have  $\text{Coplanar } X \ PA \ QA \ RA$ 
by  $(\text{meson } \langle \neg \text{Col } A \ B \ C \rangle \text{coplanar-perm-9 coplanar-pseudo-trans})$ 
hence  $\exists \ P \ Q \ R. \ p = \text{Plane } P \ Q \ R \wedge \text{Coplanar } X \ P \ Q \ R$ 
using  $\langle p = \text{Plane } PA \ QA \ RA \rangle$  by blast
}
moreover
{
fix  $X$ 
assume  $\text{IncidentP } X \ p$ 
hence  $\text{isPlane } p$ 
by  $(\text{simp add: } \langle \text{isPlane } p \rangle)$ 
obtain  $P \ Q \ R$  where  $p = \text{Plane } P \ Q \ R$  and
 $\text{Coplanar } X \ P \ Q \ R$ 
using  $\text{IncidentP-def } \langle \text{IncidentP } X \ p \rangle$  by blast
have  $P = PA \wedge Q = QA \wedge R = RA$ 
using  $\text{Plane-def } \langle p = (PA, QA, RA) \rangle \langle p = \text{Plane } P \ Q \ R \rangle$  by auto
hence  $\text{Coplanar } X \ PA \ QA \ RA$ 
using  $\langle \text{Coplanar } X \ P \ Q \ R \rangle$  by auto
hence  $\text{Coplanar } X \ A \ B \ C$ 
by  $(\text{meson } \langle \text{Coplanar } A \ PA \ QA \ RA \rangle \langle \text{Coplanar } B \ PA \ QA \ RA \rangle \langle \text{Coplanar } C \ PA \ QA \ RA \rangle$ 
 $\langle \neg \text{Col } PA \ QA \ RA \rangle \text{coplanar-perm-9 coplanar-pseudo-trans})$ 
hence  $\text{IncidentP } X \ (\text{Plane } A \ B \ C)$ 
using  $\text{IncidentP-def } \langle \text{isPlane } (\text{Plane } A \ B \ C) \rangle$  by blast
}
ultimately show  $?thesis$ 
by  $(\text{metis EqTP-def IncidentP-def } \langle \text{isPlane } (\text{Plane } A \ B \ C) \rangle \langle \text{isPlane } p \rangle)$ 
qed

```

```

lemma eqp-transitivity:
assumes  $p = p = q$  and
 $q = p = r$ 
shows  $p = p = r$ 
using  $\text{EqTP-def assms}(1) \text{ assms}(2)$  by presburger

```

```

lemma eqp-reflexivity:
assumes  $\text{isPlane } p$ 
shows  $p = p = p$ 
by  $(\text{simp add: EqTP-def assms})$ 

```

```

lemma eqp-symmetry:
assumes  $p = p = q$ 
shows  $q = p = p$ 
using  $\text{EqTP-def assms}$  by presburger

```

```

lemma eqp-incidentp:

```

assumes $p = p = q$
shows $\text{IncidentP } A \ p \longleftrightarrow \text{IncidentP } A \ q$
using *EqTP-def* *assms* **by** *blast*

lemma *axiom-Incidp-morphism* :
assumes $\text{IncidentP } M \ p$ **and**
 $\text{EqTP } p \ q$
shows $\text{IncidentP } M \ q$
using *assms*(1) *assms*(2) *eqp-incidentp* **by** *blast*

lemma *axiom-plane-uniqueness*:
assumes $\neg \text{Col-H } A \ B \ C$ **and**
 $\text{IncidentP } A \ p$ **and**
 $\text{IncidentP } B \ p$ **and**
 $\text{IncidentP } C \ p$ **and**
 $\text{IncidentP } A \ q$ **and**
 $\text{IncidentP } B \ q$ **and**
 $\text{IncidentP } C \ q$
shows $p = p = q$
proof –
have $\text{Plane } A \ B \ C = p = p$
using *assms*(1) *assms*(2) *assms*(3) *assms*(4) *incidentp-eqp* **by** *auto*
moreover have $\text{Plane } A \ B \ C = p = q$
using *assms*(1) *assms*(5) *assms*(6) *assms*(7) *incidentp-eqp* **by** *auto*
ultimately show *?thesis*
using *eqp-symmetry* *eqp-transitivity* **by** *blast*
qed

lemma *axiom-one-point-on-plane*:
assumes $\text{isPlane } p$
shows $\exists A. \text{IncidentP } A \ p$
proof –
obtain $A \ B \ C$ **where** $p = \text{Plane } A \ B \ C$
by (*metis* *Plane-def* *surj-pair*)
thus *?thesis*
using *IncidentP-def* *assms* *coplanar-trivial* **by** *blast*
qed

lemma *axiom-line-on-plane*:
assumes $A \neq B$ **and**
 $\text{IncidentL } A \ l$ **and**
 $\text{IncidentL } B \ l$ **and**
 $\text{IncidentP } A \ p$ **and**
 $\text{IncidentP } B \ p$
shows $\text{IncidentLP } l \ p$
proof –
have $\text{isLine } l \wedge \text{Col } A \ (\text{fst } l) \ (\text{snd } l)$
using *IncidentL-def* *assms*(2) **by** *blast*
let $?U = (\text{fst } l)$
let $?V = (\text{snd } l)$
have $l = \text{Line } ?U \ ?V$
by (*metis* *IncidentL-def* *Line-def* *assms*(2) *isLine-def* *prod.exhaust-sel*)
hence $\text{Col } A \ ?U \ ?V$
by (*metis* *IncidentL-def* *assms*(2))
have $\text{Col } B \ ?U \ ?V$
by (*metis* *IncidentL-def* *assms*(3))
have $\text{isLine } l$
using *IncidentL-def* *assms*(3) **by** *blast*
hence $\text{fst } l \neq \text{snd } l$
by (*simp* *add*: *isLine-def*)
hence $?U \neq ?V$
by *blast*

```

have Col A B ?U
  using ⟨Col A (fst l) (snd l)⟩ ⟨Col B (fst l) (snd l)⟩ ⟨fst l ≠ snd l⟩
  col2--eq col-permutation-5 by blast
have Col A B ?V
  by (metis Col-cases ⟨Col A B (fst l)⟩ ⟨Col A (fst l) (snd l)⟩
    ⟨Col B (fst l) (snd l)⟩ col-transitivity-1)
obtain PA QA RA where p = Plane PA QA RA and Coplanar A PA QA RA
  using IncidentP-def assms(4) by blast
hence ¬ Col PA QA RA
  using IncidentP-def Plane-def assms(4) isPlane-def by blast
obtain PB QB RB where p = Plane PB QB RB and Coplanar B PB QB RB
  using IncidentP-def assms(5) by blast
hence ¬ Col PB QB RB
  using IncidentP-def Plane-def assms(5) isPlane-def by auto
have Coplanar B PA QA RA
  using Plane-def ⟨Coplanar B PB QB RB⟩ ⟨p = Plane PA QA RA⟩ ⟨p = Plane PB QB RB⟩ by force
have Coplanar ?U PA QA RA
  using ⟨Coplanar A PA QA RA⟩ ⟨Coplanar B PA QA RA⟩
  ncoplanar-perm-23 ⟨Col A B (fst l)⟩ assms(1) col-cop2--cop by blast
have Coplanar ?V PA QA RA
  by (meson ncoplanar-perm-23 ⟨Col A B (snd l)⟩ ⟨Coplanar A PA QA RA⟩
    ⟨Coplanar B PA QA RA⟩ assms(1) col-cop2--cop)
have isLine l
  using IncidentL-def assms(3) by blast
moreover have isPlane p
  using IncidentP-def assms(5) by blast
moreover
{
  fix A'
  assume IncidentL A' l
  hence Col A' ?U ?V
    using IncidentL-def by blast
  have Coplanar A' PA QA RA
    by (meson ncoplanar-perm-23 IncidentL-def ⟨Coplanar (fst l) PA QA RA⟩
      ⟨Coplanar (snd l) PA QA RA⟩ ⟨IncidentL A' l⟩
      col-cop2--cop isLine-def not-col-permutation-2)
  hence ∃ P Q R. p = Plane P Q R ∧ Coplanar A' P Q R
    using ⟨p = Plane PA QA RA⟩ by blast
  hence IncidentP A' p
    by (simp add: IncidentP-def calculation(2))
}
ultimately show ?thesis
  using IncidentLP-def by blast
qed

```

```

lemma axiom-between-col:
  assumes Between-H A B C
  shows Col-H A B C
  using Between-H-def Col-def assms cols-coincide-2 by presburger

```

```

lemma axiom-between-diff:
  assumes Between-H A B C
  shows A ≠ C
  using Between-H-def assms by auto

```

```

lemma axiom-between-comm:
  assumes Between-H A B C
  shows Between-H C B A
  by (metis Between-H-def assms between-symmetry)

```

```

lemma axiom-between-out:
  assumes  $A \neq B$ 
  shows  $\exists C. \text{Between-H } A B C$ 
  by (metis Between-H-def assms bet-neq12--neq point-construction-different)

lemma axiom-between-only-one:
  assumes Between-H  $A B C$ 
  shows  $\neg \text{Between-H } B C A$ 
  by (metis Bet-cases Between-H-def assms between-equality)

lemma between-one:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $B \neq C$  and
    Col  $A B C$ 
  shows Between-H  $A B C \vee \text{Between-H } B C A \vee \text{Between-H } B A C$ 
  using Bet-cases Between-H-def Col-def assms(1) assms(2) assms(3) assms(4) by auto

lemma axiom-between-one:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $B \neq C$  and
    Col-H  $A B C$ 
  shows Between-H  $A B C \vee \text{Between-H } B C A \vee \text{Between-H } B A C$ 
  by (simp add: assms(1) assms(2) assms(3) assms(4) between-one cols-coincide-1)

lemma cut-two-sides:
  shows cut-H  $l A B \longleftrightarrow (fst\ l) (snd\ l) TS\ A\ B$ 
proof -
  {
    assume cut-H  $l A B$ 
    have (fst  $l$ ) (snd  $l$ )  $TS\ A\ B$ 
      by (meson Between-H-def TS-def IncidentL-def cut-H  $l A B$  cut-H-def)
  }
  moreover
  {
    assume (fst  $l$ ) (snd  $l$ )  $TS\ A\ B$ 
    then obtain  $T$  where Col  $T (fst\ l) (snd\ l)$  and Bet  $A\ T\ B$ 
      using TS-def by blast
    have  $\neg IncidentL\ A\ l$ 
      using IncidentL-def TS-def (fst  $l$ ) (snd  $l$ )  $TS\ A\ B$  by auto
    moreover have  $\neg IncidentL\ B\ l$ 
      using IncidentL-def TS-def (fst  $l$ ) (snd  $l$ )  $TS\ A\ B$  by blast
    moreover have  $\exists I. isLine\ l \wedge IncidentL\ I\ l \wedge \text{Between-H } A\ I\ B$ 
      by (metis Between-H-def IncidentL-def TS-def (fst  $l$ ) (snd  $l$ )  $TS\ A\ B$ 
        isLine-def ts-distincts)
    ultimately have cut-H  $l A B$ 
      using cut-H-def by presburger
  }
  ultimately show ?thesis
    by blast
qed

lemma cop-plane-aux:
  assumes Coplanar  $A B C D$  and
     $A \neq B$ 
  shows  $\exists p. IncidentP\ A\ p \wedge IncidentP\ B\ p \wedge IncidentP\ C\ p \wedge IncidentP\ D\ p$ 
proof -

```

```

obtain  $X$  where  $(Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee$ 
 $(Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee$ 
 $(Col\ A\ D\ X \wedge Col\ B\ C\ X)$ 
using Coplanar-def assms(1) by auto
{
  assume  $Col\ A\ B\ C$ 
  have  $\exists\ p. IncidentP\ A\ p \wedge IncidentP\ B\ p \wedge IncidentP\ C\ p \wedge IncidentP\ D\ p$ 
  proof (cases Col A B D)
    case True
    hence  $Col\ A\ B\ D$ 
    by blast
    obtain  $E$  where  $\neg\ Col\ A\ B\ E$ 
    using assms(2) not-col-exists by auto
    hence  $\neg\ Col-H\ A\ B\ E$ 
    using cols-coincide-1 by blast
    let  $?p = Plane\ A\ B\ E$ 
    have  $IncidentP\ A\ ?p$ 
    using EqTP-def  $\langle \neg\ Col-H\ A\ B\ E \rangle$  axiom-plane-existence incidentp-eqp by blast
    moreover have  $IncidentP\ B\ ?p$ 
    using IncidentP-def calculation ncop-distincts by blast
    moreover have  $IncidentP\ C\ ?p$ 
    by (meson Col-cases IncidentP-def  $\langle Col\ A\ B\ C \rangle$  calculation(1) ncop--ncols)
    moreover have  $IncidentP\ D\ ?p$ 
    by (meson Col-cases IncidentP-def True calculation(3) ncop--ncols)
    ultimately show ?thesis
    by blast
  next
  case False
  hence  $\neg\ Col-H\ A\ B\ D$ 
  using cols-coincide-1 by blast
  let  $?p = Plane\ A\ B\ D$ 
  have  $IncidentP\ A\ ?p$ 
  by (meson EqTP-def  $\langle \neg\ Col-H\ A\ B\ D \rangle$  axiom-plane-existence incidentp-eqp)
  moreover have  $IncidentP\ B\ ?p$ 
  using IncidentP-def calculation ncop-distincts by blast
  moreover have  $IncidentP\ C\ ?p$ 
  using IncidentP-def assms(1) calculation(1) coplanar-perm-12 by blast
  moreover have  $IncidentP\ D\ ?p$ 
  using IncidentP-def calculation(3) ncop-distincts by blast
  ultimately show ?thesis
  by blast
qed
}
moreover
{
  assume  $\neg\ Col\ A\ B\ C$ 
  let  $?p = Plane\ A\ B\ C$ 
  have  $IncidentP\ A\ ?p$ 
  by (simp add: IncidentP-def Plane-def  $\langle \neg\ Col\ A\ B\ C \rangle$  coplanar-trivial isPlane-def)
  moreover have  $IncidentP\ B\ ?p$ 
  using IncidentP-def calculation ncop-distincts by blast
  moreover have  $IncidentP\ C\ ?p$ 
  using IncidentP-def calculation(2) ncop-distincts by blast
  moreover have  $IncidentP\ D\ ?p$ 
  using IncidentP-def assms(1) calculation(3) coplanar-perm-18 by blast
  ultimately have  $\exists\ p. IncidentP\ A\ p \wedge IncidentP\ B\ p \wedge IncidentP\ C\ p \wedge IncidentP\ D\ p$ 
  by blast
}
ultimately show ?thesis
by blast
qed

lemma cop-plane:
assumes Coplanar A B C D
shows  $\exists\ p. IncidentP\ A\ p \wedge IncidentP\ B\ p \wedge IncidentP\ C\ p \wedge IncidentP\ D\ p$ 
by (metis assms cop-plane-aux coplanar-trivial diff-col-ex ncoplanar-perm-22)

```

lemma *plane-cop*:
assumes *IncidentP A p and*
IncidentP B p and
IncidentP C p and
IncidentP D p
shows *Coplanar A B C D*
proof –
obtain *PA QA RA* **where** *p = Plane PA QA RA and Coplanar A PA QA RA*
using *IncidentP-def assms(1)* **by** *auto*
hence *p = (PA,QA,RA)*
using *Plane-def* **by** *blast*
moreover
obtain *PB QB RB* **where** *p = Plane PB QB RB and Coplanar B PB QB RB*
using *IncidentP-def assms(2)* **by** *auto*
hence *p = (PB,QB,RB)*
using *Plane-def* **by** *blast*
moreover
obtain *PC QC RC* **where** *p = Plane PC QC RC and Coplanar C PC QC RC*
using *IncidentP-def assms(3)* **by** *auto*
hence *p = (PC,QC,RC)*
using *Plane-def* **by** *blast*
moreover
obtain *PD QD RD* **where** *p = Plane PD QD RD and Coplanar D PD QD RD*
using *IncidentP-def assms(4)* **by** *auto*
hence *p = (PD,QD,RD)*
using *Plane-def* **by** *blast*
ultimately have *Coplanar B PA QA RA ∧ Coplanar C PA QA RA ∧ Coplanar D PA QA RA*
using *⟨Coplanar B PB QB RB⟩ ⟨Coplanar C PC QC RC⟩ ⟨Coplanar D PD QD RD⟩* **by** *fastforce*
have *isPlane p*
using *IncidentP-def assms(1)* **by** *blast*
hence $\neg \text{Col } PA \ QA \ RA$
using *isPlane-def ⟨p = (PA, QA, RA)⟩* **by** *blast*
thus *?thesis*
by (*meson ⟨Coplanar A PA QA RA⟩ coplanar-perm-18 coplanar-pseudo-trans*
⟨Coplanar B PA QA RA ∧ Coplanar C PA QA RA ∧ Coplanar D PA QA RA⟩)
qed

lemma *axiom-pasch*:
assumes $\neg \text{Col-H } A \ B \ C$ **and**
IncidentP A p and
IncidentP B p and
IncidentP C p and
IncidentLP l p and
 $\neg \text{IncidentL } C \ l$ **and**
cut-H l A B
shows *cut-H l A C ∨ cut-H l B C*
proof –
let *?A = (fst l)*
let *?B = (snd l)*
have *?A ?B TS A B*
using *assms(7) cut-two-sides* **by** *auto*
hence *1: ¬ Col A ?A ?B ∧ ¬ Col B ?A ?B ∧ (∃ T. Col T ?A ?B ∧ Bet A T B)*
using *TS-def* **by** *blast*
hence $\neg \text{Col } A \ ?A \ ?B$
by *blast*
have $\neg \text{Col } B \ ?A \ ?B$
by (*simp add: 1*)
obtain *T* **where** *Col T ?A ?B and Bet A T B*
using *1* **by** *auto*
have $\neg \text{Col } A \ B \ C$
using *assms(1) cols-coincide-2* **by** *blast*
have $\neg \text{Col } C \ ?A \ ?B$
using *IncidentLP-def IncidentL-def assms(5) assms(6)* **by** *blast*
have *Coplanar ?A ?B A C*
proof (*rule plane-cop, insert assms(2) assms(4)*)

```

show IncidentP (fst l) p
  using IncidentLP-def IncidentL-def assms(5) col-trivial-1 by blast
show IncidentP (snd l) p
  using IncidentLP-def IncidentL-def assms(5) col-trivial-3 by blast
qed
have  $?A \ ?B \ TS \ A \ C \vee ?A \ ?B \ OS \ A \ C$ 
  using  $\langle \text{Coplanar } ?A \ ?B \ A \ C \rangle \ 1 \ \neg \text{Col } C \ ?A \ ?B \rangle \text{ cop-nos--ts}$  by blast
moreover
{
  assume  $?A \ ?B \ TS \ A \ C$ 
  have  $\text{cut-H } l \ A \ C \vee \text{cut-H } l \ B \ C$ 
  by (simp add:  $\langle ?A \ ?B \ TS \ A \ C \rangle$  cut-two-sides)
}
moreover
{
  assume  $?A \ ?B \ OS \ A \ C$ 
  have  $\text{cut-H } l \ A \ C \vee \text{cut-H } l \ B \ C$ 
  using  $\langle ?A \ ?B \ OS \ A \ C \rangle \ \langle ?A \ ?B \ TS \ A \ B \rangle \text{ cut-two-sides l9-2 l9-8-2}$  by blast
}
ultimately show ?thesis
by blast
qed

```

```

lemma Incid-line:
assumes  $A \neq B$  and
  IncidentL  $A \ l$  and
  IncidentL  $B \ l$  and
  Col  $P \ A \ B$ 
shows IncidentL  $P \ l$ 
by (meson Col-H-def assms(1) assms(2) assms(3) assms(4) cols-coincide-2
  eq-incident incident-eq)

```

```

lemma out-outH:
assumes  $P \text{ Out } A \ B$ 
shows outH  $P \ A \ B$ 
using Between-H-def Out-def assms outH-def by auto

```

```

lemma axiom-hcong-1-existence:
assumes  $A \neq B$  and
   $A' \neq P$  and
  IncidentL  $A' \ l$  and
  IncidentL  $P \ l$ 
shows  $\exists B'. \text{IncidentL } B' \ l \wedge \text{outH } A' \ P \ B' \wedge \text{Cong } A' \ B' \ A \ B$ 
proof –
obtain  $B'$  where  $A' \text{ Out } B' \ P \wedge \text{Cong } A' \ B' \ A \ B$ 
  using assms(1) assms(2) l6-11-existence by presburger
have IncidentL  $B' \ l$ 
  using Col-perm Incid-line  $\langle A' \text{ Out } B' \ P \wedge \text{Cong } A' \ B' \ A \ B \rangle$  assms(2) assms(3)
  assms(4) out-col by blast
moreover have outH  $A' \ P \ B'$ 
  by (simp add:  $\langle A' \text{ Out } B' \ P \wedge \text{Cong } A' \ B' \ A \ B \rangle$  l6-6 out-outH)
ultimately show ?thesis
  using  $\langle A' \text{ Out } B' \ P \wedge \text{Cong } A' \ B' \ A \ B \rangle$  by blast
qed

```

```

lemma axiom-hcong-1-uniqueness:
assumes
  IncidentL  $M \ l$  and
  IncidentL  $A' \ l$  and

  IncidentL  $A'' \ l$  and

```



```

    Between-H  $A' M B'$  and
    Cong  $M A' A B$  and
    Cong  $M B' A B$  and
    Between-H  $A'' M B''$  and
    Cong  $M A'' A B$  and
    Cong  $M B'' A B$ 
  shows  $(A' = A'' \wedge B' = B'') \vee (A' = B'' \wedge B' = A'')$ 
proof -
  let  $?A = \text{fst } l$ 
  let  $?B = \text{snd } l$ 
  have  $A'' \neq M$ 
    using Between-H-def assms(7) by blast
  {
    assume  $M \text{ Out } A' A''$ 
    have  $M \text{ Out } A'' A''$ 
      by (simp add:  $\langle A'' \neq M \rangle \text{ out-trivial}$ )
    hence  $A' = A''$ 
      using l6-11-uniqueness  $\langle M \text{ Out } A' A'' \rangle$  assms(5) assms(8) by blast
    hence  $A' = A'' \wedge B' = B''$ 
      by (metis Between-H-def assms(7) assms(9) assms(4) assms(6) construction-uniqueness)
  }
  moreover
  {
    assume  $\neg M \text{ Out } A' A''$ 
    have  $\text{Col } A' M A''$ 
      using col3 [where  $?X = ?A$  and  $?Y = ?B$ ]
      by (meson Col-H-def IncidentL-def assms(1) assms(2) assms(3) ncols-coincide)
    hence  $\text{Bet } A' M A''$ 
      using not-out-bet  $\langle \neg M \text{ Out } A' A'' \rangle$  by blast
    have  $A' = B''$ 
      by (metis Between-H-def  $\langle \text{Bet } A' M A'' \rangle$  assms(7) assms(9) assms(5)
        between-symmetry construction-uniqueness)
    hence  $A' = B'' \wedge B' = A''$ 
      by (metis Between-H-def  $\langle \text{Bet } A' M A'' \rangle$  assms(8) assms(4) assms(6) construction-uniqueness)
  }
  ultimately show ?thesis
    by blast
qed

```

```

lemma axiom-hcong-scott:
  assumes  $A \neq C$  and
     $P \neq Q$ 
  shows  $\exists B. \text{same-side-scott } A B C \wedge \text{Cong } P Q A B$ 
proof -
  obtain  $X$  where  $A \text{ Out } X C$  and  $\text{Cong } A X P Q$ 
    using assms(1) assms(2) l6-11-existence by metis
  have  $A \neq X$ 
    using  $\langle A \text{ Out } X C \rangle$  l6-3-1 by blast
  moreover have  $\text{Col-H } A X C$ 
    by (simp add:  $\langle A \text{ Out } X C \rangle$  cols-coincide-2 out-col)
  moreover have  $\neg \text{Between-H } X A C$ 
    using Between-H-def  $\langle A \text{ Out } X C \rangle$  not-bet-and-out by force
  moreover have  $\text{Cong } P Q A X$ 
    using Cong-cases  $\langle \text{Cong } A X P Q \rangle$  by blast
  ultimately show ?thesis
    using assms(1) same-side-scott-def by blast
qed

```

```

lemma col-disjoint-bet:

```

```

assumes Col-H A B C and
  disjoint-H A B B C
shows Bet A B C
proof –
  have Col A B C
    by (simp add: assms(1) cols-coincide-1)
  show ?thesis
  proof (cases A = B)
    case True
    thus ?thesis
      by (simp add: between-trivial2)
  next
    case False
    hence  $A \neq B$ 
      by blast
    show ?thesis
    proof (cases B = C)
      case True
      thus ?thesis
        by (simp add: between-trivial)
    next
      case False
      hence  $B \neq C$ 
        by blast
      {
        assume Bet B C A
        obtain M where M Midpoint B C
          using midpoint-existence by blast
        have Between-H A M B
          by (metis Between-H-def False Midpoint-def  $\langle A \neq B \rangle \langle Bet B C A \rangle$ 
             $\langle M \text{ Midpoint } B C \rangle$  axiom-between-comm between-equality-2 between-exchange4
            midpoint-not-midpoint midpoint-out out-diff1)
        moreover have Between-H B M C
          by (metis Between-H-def False Midpoint-def  $\langle M \text{ Midpoint } B C \rangle$ 
            calculation is-midpoint-id-2)
        ultimately have  $\exists P. \text{Between-H } A P B \wedge \text{Between-H } B P C$ 
          by blast
        hence False
          using assms(2) disjoint-H-def by blast
      }
    moreover
    {
      assume Bet C A B
      obtain M where M Midpoint A B
        using midpoint-existence by blast
      have Between-H A M B
        by (metis Between-H-def  $\langle A \neq B \rangle \langle M \text{ Midpoint } A B \rangle$  midpoint-bet midpoint-distinct-1)
      moreover have Between-H B M C
        by (metis Bet-cases Between-H-def False  $\langle Bet B C A \implies False \rangle \langle Bet C A B \rangle$ 
          between-exchange2 calculation)
      ultimately
      have  $\exists P. \text{Between-H } A P B \wedge \text{Between-H } B P C$ 
        by blast
      hence False
        using assms(2) disjoint-H-def by blast
    }
    ultimately show ?thesis
      using  $\langle Col A B C \rangle$  Col-def by blast
  qed
qed
qed

lemma axiom-hcong-3:
assumes Col-H A B C and
  Col-H A' B' C' and
  disjoint-H A B B C and

```

disjoint-H $A' B' B' C'$ **and**
Cong $A B A' B'$ **and**
Cong $B C B' C'$
shows *Cong* $A C A' C'$
by (*meson* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) *col-disjoint-bet* l2-11-b)

lemma *exists-not-incident*:
assumes $A \neq B$
shows $\exists C. \neg \text{IncidentL } C \text{ (Line } A \ B)$
using *Col-H-def IncidentL-def lower-dim'* **by** *blast*

lemma *same-side-one-side*:
assumes *same-side-H* $A \ B \ l$
shows (*fst* l) (*snd* l) *OS* $A \ B$
by (*meson OS-def cut-two-sides assms same-side-H-def*)

lemma *one-side-same-side*:
assumes (*fst* l)(*snd* l) *OS* $A \ B$
shows *same-side-H* $A \ B \ l$
proof –
have *isLine* l
using *assms isLine-def os-distincts* **by** *presburger*
thus *?thesis*
using *OS-def assms cut-two-sides same-side-H-def* **by** *presburger*
qed

lemma *OS-distinct*:
assumes $P \ Q \ OS \ A \ B$
shows $P \neq Q$
using *assms os-distincts* **by** *auto*

lemma *OS-same-side'*:
assumes $P \ Q \ OS \ A \ B$
shows *same-side'-H* $A \ B \ P \ Q$
proof –
have $P \neq Q$
using *assms os-distincts* **by** *blast*
moreover
{
fix l
assume *IncidentL* $P \ l$ **and**
IncidentL $Q \ l$
hence *isLine* l
using *IncidentL-def* **by** *auto*
have (*fst* l) $\neq$ (*snd* l)
using $\langle \text{isLine } l \rangle \text{ isLine-def}$ **by** *auto*
have *Col* P (*fst* l) (*snd* l)
using *IncidentL-def* $\langle \text{IncidentL } P \ l \rangle$ **by** *auto*
have *Col* Q (*fst* l) (*snd* l)
using *IncidentL-def* $\langle \text{IncidentL } Q \ l \rangle$ **by** *auto*
have (*fst* l) (*snd* l) *OS* $A \ B$
by (*metis* (*no-types*, *lifting*) $\langle \text{Col } P \text{ (fst } l) \text{ (snd } l) \rangle \langle \text{Col } Q \text{ (fst } l) \text{ (snd } l) \rangle$
 $\langle \text{fst } l \rangle \neq \langle \text{snd } l \rangle \rangle \text{ assms col2-os--os l6-16-1 not-col-permutation-5})$
hence *same-side-H* $A \ B \ l$
using *one-side-same-side* **by** *blast*
}
ultimately show *?thesis*
using *same-side'-H-def* **by** *auto*
qed

lemma *same-side-OS*:
assumes *same-side'-H* $P \ Q \ A \ B$
shows $A \ B \ OS \ P \ Q$
proof –

```

obtain  $X$  where  $IncidentL\ A\ X$  and  $IncidentL\ B\ X$ 
using  $assms\ axiom-line-existence\ same-side'-H-def$  by  $blast$ 
hence  $same-side-H\ P\ Q\ X$ 
using  $EqTL-def\ assms\ incident-eq\ same-side'-H-def$  by  $blast$ 
thus  $?thesis$ 
by ( $metis\ IncidentL-def\ assms\ axiom-line-existence\ col2-os--os\ col-permutation-1$ 
 $same-side'-H-def\ same-side-one-side$ )
qed

```

```

lemma  $outH-out$ :
assumes  $outH\ P\ A\ B$ 
shows  $P\ Out\ A\ B$ 
using  $Between-H-def\ Out-def\ assms\ outH-def\ out-trivial$  by  $force$ 

```

```

lemma  $incident-col$ :
assumes  $IncidentL\ M\ l$ 
shows  $Col\ M\ (fst\ l)(snd\ l)$ 
using  $IncidentL-def\ assms$  by  $auto$ 

```

```

lemma  $col-incident$ :
assumes  $(fst\ l) \neq (snd\ l)$  and
 $Col\ M\ (fst\ l)(snd\ l)$ 
shows  $IncidentL\ M\ l$ 
by ( $simp\ add: IncidentL-def\ assms(1)\ assms(2)\ isLine-def$ )

```

```

lemma  $Bet-Between-H$ :
assumes  $Bet\ A\ B\ C$  and
 $A \neq B$  and
 $B \neq C$ 
shows  $Between-H\ A\ B\ C$ 
using  $Between-H-def\ assms(1)\ assms(2)\ assms(3)\ bet-neq12--neq$  by  $presburger$ 

```

```

lemma  $axiom-cong-5'$ :
assumes
 $\neg Col-H\ A'\ B'\ C'$  and
 $Cong\ A\ B\ A'\ B'$  and
 $Cong\ A\ C\ A'\ C'$  and
 $B\ A\ C\ CongA\ B'\ A'\ C'$ 
shows  $A\ B\ C\ CongA\ A'\ B'\ C'$ 
by ( $metis\ assms(2)\ assms(3)\ assms(4)\ assms(1)\ cols-coincide-2\ cong-diff-4$ 
 $l11-49\ not-col-distincts$ )

```

```

lemma  $axiom-cong-5'-bis$ :
assumes  $\neg Col-H\ A\ B\ C$  and
 $Cong\ A\ B\ A'\ B'$  and
 $Cong\ A\ C\ A'\ C'$  and
 $B\ A\ C\ CongA\ B'\ A'\ C'$ 
shows  $A\ B\ C\ CongA\ A'\ B'\ C'$ 
using  $assms(1)\ assms(3)\ assms(4)\ assms(2)\ col-trivial-2\ cols-coincide-2\ l11-49$  by  $blast$ 

```

```

lemma  $axiom-hcong-4-existence$ :
assumes  $\neg Col-H\ P\ PO\ X$  and
 $\neg Col-H\ A\ B\ C$ 
shows  $\exists\ Y.\ (A\ B\ C\ CongA\ X\ PO\ Y \wedge same-side'-H\ P\ Y\ PO\ X)$ 
proof –
have  $\neg Col\ P\ PO\ X$ 
by ( $simp\ add: assms(1)\ ncols-coincide$ )
have  $\neg Col\ A\ B\ C$ 
using  $assms(2)\ cols-coincide-2$  by  $blast$ 
have  $\neg Col\ X\ PO\ P$ 
using  $\langle \neg Col\ P\ PO\ X \rangle\ not-col-permutation-3$  by  $blast$ 

```

obtain Y **where** $A B C \text{ CongA } X PO Y$ **and** $X PO OS Y P$
using $\langle \neg \text{ Col } A B C \rangle \langle \neg \text{ Col } X PO P \rangle$ *angle-construction-1* **by** *blast*
thus *?thesis*
using *OS-same-side'* *invert-one-side one-side-symmetry* **by** *blast*
qed

lemma *same-side-trans*:
assumes *same-side-H* $A B l$ **and**
 $\text{same-side-H } B C l$
shows *same-side-H* $A C l$
using *assms(1)* *assms(2)* *one-side-same-side one-side-transitivity same-side-one-side* **by** *blast*

lemma *same-side-sym*:
assumes *same-side-H* $A B l$
shows *same-side-H* $B A l$
by (*meson assms same-side-H-def*)

lemma *axiom-hcong-4-uniqueness*:
assumes $\neg \text{ Col-H } P PO X$ **and**
 $\neg \text{ Col-H } A B C$ **and**
 $A B C \text{ CongA } X PO Y$ **and**
 $A B C \text{ CongA } X PO Y'$ **and**
same-side'-H $P Y PO X$ **and**
same-side'-H $P Y' PO X$
shows *outH* $PO Y Y'$

proof –
have $\neg \text{ Col } P PO X$
using *assms(1)* *cols-coincide-2* **by** *blast*
have $\neg \text{ Col } A B C$
using *assms(2)* *cols-coincide-2* **by** *blast*
have $X PO Y \text{ CongA } X PO Y'$
using *assms(3)* *assms(4)* *conga-sym not-conga* **by** *blast*
moreover
have $X PO OS Y Y'$
by (*meson assms(5) assms(6) invert-one-side one-side-symmetry*
one-side-transitivity same-side-OS)
ultimately have $PO \text{ Out } Y Y'$
by (*simp add: conga-os--out*)
thus *?thesis*
by (*simp add: out-outH*)
qed

lemma *axiom-conga-comm*:
assumes $\neg \text{ Col-H } A B C$
shows $A B C \text{ CongA } C B A$
by (*metis assms col-trivial-1 col-trivial-2 cols-coincide-2 conga-pseudo-refl*)

lemma *axiom-congaH-outH-congaH*:
assumes $A B C \text{ CongA } D E F$ **and**
 $\text{Between-H } B A A' \vee \text{Between-H } B A' A \vee B \neq A \wedge A = A'$ **and**
 $\text{Between-H } B C C' \vee \text{Between-H } B C' C \vee B \neq C \wedge C = C'$ **and**
 $\text{Between-H } E D D' \vee \text{Between-H } E D' D \vee E \neq D \wedge D = D'$ **and**
 $\text{Between-H } E F F' \vee \text{Between-H } E F' F \vee E \neq F \wedge F = F'$
shows $A' B C' \text{ CongA } D' E F'$
proof (*rule l11-10 [where ?A = A and ?C = C and ?D = D and ?F = F], insert assms(1)*)
show $B \text{ Out } A' A$
using *Between-H-def Out-def assms(2) out-trivial* **by** *force*
show $B \text{ Out } C' C$
using *assms(3) outH-def outH-out* **by** *blast*
show $E \text{ Out } D' D$
using *assms(4) outH-def outH-out* **by** *blast*
show $E \text{ Out } F' F$
using *assms(5) outH-def outH-out* **by** *blast*
qed

lemma *axiom-conga-permlr*:

```

assumes  $A B C \text{ Cong} A D E F$ 
shows  $C B A \text{ Cong} A F E D$ 
by (simp add: assms conga-comm)

```

```

lemma axiom-conga-refl:
  assumes  $\neg \text{Col-H } A B C$ 
  shows  $A B C \text{ Cong} A A B C$ 
  using assms axiom-conga-comm conga-right-comm by blast

```

```

end
end

```

```

theory Tarski-Neutral-2D

```

```

imports
  Tarski-Neutral

```

```

begin

```

10 Tarski's axiom system for neutral geometry: 2D

10.1 Definitions

```

locale Tarski-neutral-2D = Tarski-neutral-dimensionless +
  assumes upper-dim:  $\forall a b c p q.$ 
     $p \neq q \wedge$ 
     $\text{Cong } a p a q \wedge$ 
     $\text{Cong } b p b q \wedge$ 
     $\text{Cong } c p c q$ 
     $\longrightarrow$ 
     $(\text{Bet } a b c \vee \text{Bet } b c a \vee \text{Bet } c a b)$ 

```

10.2 Propositions

```

context Tarski-neutral-2D

```

```

begin

```

```

lemma all-coplanar:
  Coplanar A B C D
proof –
  have  $\forall A B C P Q. P \neq Q \longrightarrow \text{Cong } A P A Q \longrightarrow \text{Cong } B P B Q \longrightarrow \text{Cong } C P C Q \longrightarrow$ 
     $(\text{Bet } A B C \vee \text{Bet } B C A \vee \text{Bet } C A B)$ 
    using upper-dim by blast
  then show ?thesis using upper-dim-implies-all-coplanar
  by (smt Tarski-neutral-dimensionless.not-col-permutation-2 Tarski-neutral-dimensionless-axioms all-coplanar-axiom-def
all-coplanar-implies-upper-dim coplanar-perm-9 ncop--ncol os--coplanar ts--coplanar upper-dim-implies-not-one-side-two-sides)
qed

```

```

lemma per2--col:
  assumes Per A X C and
     $X \neq C$  and
    Per B X C
  shows Col A B X
  using all-coplanar-axiom-def all-coplanar-upper-dim assms(1) assms(2) assms(3) upper-dim upper-dim-implies-per2--col
by blast

```

```

lemma perp2--col:
  assumes  $X Y \text{ Perp } A B$  and
     $X Z \text{ Perp } A B$ 
  shows Col X Y Z
  by (meson Tarski-neutral-dimensionless.cop-perp2--col Tarski-neutral-dimensionless-axioms all-coplanar assms(1) assms(2))

```

```

lemma l12-9-2D:
  assumes  $A1 A2 \text{ Perp } C1 C2$  and

```

```

    B1 B2 Perp C1 C2
shows A1 A2 Par B1 B2
using l12-9 all-coplanar assms(1) assms(2) by auto

lemma perp-in2--col:
  assumes P PerpAt A B X Y and
    P PerpAt A' B' X Y
  shows Col A B A'
  using cop4-perp-in2--col all-coplanar assms by blast

lemma perp2-trans:
  assumes P Perp2 A B C D and
    P Perp2 C D E F
  shows P Perp2 A B E F
proof -
  obtain X Y where P1: Col P X Y  $\wedge$  X Y Perp A B  $\wedge$  X Y Perp C D
  using Perp2-def assms(1) by blast
  obtain X' Y' where P2: Col P X' Y'  $\wedge$  X' Y' Perp C D  $\wedge$  X' Y' Perp E F
  using Perp2-def assms(2) by blast
  {
    assume X Y Par X' Y'
    then have P3: X Y ParStrict X' Y'  $\vee$  (X  $\neq$  Y  $\wedge$  X'  $\neq$  Y'  $\wedge$  Col X X' Y'  $\wedge$  Col Y X' Y')
    using Par-def by blast
    {
      assume X Y ParStrict X' Y'
      then have P Perp2 A B E F
      using P1 P2 par-not-col by auto
    }
    {
      assume X  $\neq$  Y  $\wedge$  X'  $\neq$  Y'  $\wedge$  Col X X' Y'  $\wedge$  Col Y X' Y'
      then have P Perp2 A B E F
      by (meson P1 P2 Perp2-def col-permutation-1 perp-col2)
    }
    then have P Perp2 A B E F
    using P3  $\langle$  X Y ParStrict X' Y'  $\implies$  P Perp2 A B E F  $\rangle$  by blast
  }
  {
    assume  $\neg$  X Y Par X' Y'
    then have P Perp2 A B E F
    using P1 P2 l12-9-2D by blast
  }
  thus ?thesis
  using  $\langle$  X Y Par X' Y'  $\implies$  P Perp2 A B E F  $\rangle$  by blast
qed

lemma perp2-par:
  assumes PO Perp2 A B C D
  shows A B Par C D
  using Perp2-def l12-9-2D Perp-perm assms by blast

lemma not-par-strict-inter-exists:
  assumes  $\neg$  A1 B1 ParStrict A2 B2
  shows  $\exists$  X. Col X A1 B1  $\wedge$  Col X A2 B2
  using ParStrict-def all-coplanar assms by presburger

lemma not-par-inter-exists:
  assumes  $\neg$  A1 B1 Par A2 B2
  shows  $\exists$  X. Col X A1 B1  $\wedge$  Col X A2 B2
  using all-coplanar assms cop-npar--inter-exists by blast

end
end

theory Tarski-Neutral-Continuity-2D

```

```

imports
  Tarski-Neutral-2D
  Tarski-Neutral-Continuity
begin

```

```

context Tarski-neutral-2D

```

```

begin

```

11 Tarski Neutral Continuity 2D

11.1 Definitions

11.2 Propositions

```

lemma mid-onc2-perp--col:
  assumes  $A \neq B$  and
     $A$  OnCircle  $PO$   $P$  and
     $B$  OnCircle  $PO$   $P$  and
     $X$  Midpoint  $A$   $B$  and
     $X$   $Y$  Perp  $A$   $B$ 
  shows Col  $X$   $Y$   $PO$ 
  using all-coplanar assms(1) assms(2) assms(3) assms(4) assms(5)
    cop-mid-onc2-perp--col by blast

```

```

lemma mid2-onc4--eq:
  assumes  $B \neq C$  and
     $A \neq B$  and
     $A$  OnCircle  $PO$   $P$  and
     $B$  OnCircle  $PO$   $P$  and
     $C$  OnCircle  $PO$   $P$  and
     $D$  OnCircle  $PO$   $P$  and
     $X$  Midpoint  $A$   $C$  and
     $X$  Midpoint  $B$   $D$ 
  shows  $X = PO$ 
proof –
  have Per  $PO$   $X$   $A$ 
    using assms(3) assms(5) assms(7) mid-onc2--per by blast
  have Per  $PO$   $X$   $B$ 
    using assms(4) assms(6) assms(8) mid-onc2--per by auto
  {
    assume  $X \neq PO$ 
    have Col  $A$   $B$   $X$ 
      by (meson Per-cases  $\langle$ Per  $PO$   $X$   $A$  $\rangle$   $\langle$ Per  $PO$   $X$   $B$  $\rangle$   $\langle$  $X \neq PO$  $\rangle$  per2--col)
    have Col  $B$   $X$   $D$ 
      using Col-def Midpoint-def assms(8) by blast
    have Col  $A$   $X$   $C$ 
      by (simp add: assms(7) bet-col midpoint-bet)
    have  $A = X \longrightarrow$  False
      by (metis  $\langle$ Per  $PO$   $X$   $B$  $\rangle$  assms(2) assms(3) assms(4)
        col-trivial-3 is-midpoint-id-2 onc2-per--mid)
    moreover have  $A \neq X \longrightarrow$  False
      by (metis  $\langle$ Col  $A$   $B$   $X$  $\rangle$   $\langle$ Col  $A$   $X$   $C$  $\rangle$  assms(1) assms(2) assms(3)
        assms(4) assms(5) assms(7) col-trivial-3 colx
        line-circle-two-points midpoint-distinct-2)
    ultimately have False
      by blast
  }
thus ?thesis
  by auto
qed

```



```

lemma cong2-onc3--eq:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $B \neq C$  and
     $A \text{ OnCircle } PO \ P$  and
     $B \text{ OnCircle } PO \ P$  and
     $C \text{ OnCircle } PO \ P$  and
     $Cong \ X \ A \ X \ B$  and
     $Cong \ X \ A \ X \ C$ 
  shows  $X = PO$ 
proof -
  have Coplanar  $A \ B \ C \ PO$ 
    using all-coplanar by simp
  moreover have Coplanar  $A \ B \ C \ X$ 
    using all-coplanar by simp
  ultimately show ?thesis
    using assms(1) assms(2) assms(3) assms(4) assms(5)
      assms(6) assms(7) assms(8) cong2-cop2-onc3--eq by blast
qed

lemma onc2-mid-cong-col:
  assumes  $U \neq V$  and
     $U \text{ OnCircle } PO \ P$  and
     $V \text{ OnCircle } PO \ P$  and
     $M \text{ Midpoint } U \ V$  and
     $Cong \ U \ X \ V \ X$ 
  shows  $Col \ PO \ X \ M$ 
  by (meson Col-def Cong-cases assms(1) assms(2) assms(3) assms(4)
      assms(5) midpoint-cong onc2--cong upper-dim)

lemma cong-onc3-cases:
  assumes  $Cong \ A \ X \ A \ Y$  and
     $A \text{ OnCircle } PO \ P$  and
     $X \text{ OnCircle } PO \ P$  and
     $Y \text{ OnCircle } PO \ P$ 
  shows  $X = Y \vee X \ Y \text{ ReflectL } PO \ A$ 
proof (cases  $X = Y$ )
  case True
  thus ?thesis
    by blast
next
  case False
  hence  $X \neq Y$ 
    by simp
  obtain  $M$  where  $M \text{ Midpoint } X \ Y$ 
    using midpoint-existence by blast
  hence  $Per \ PO \ M \ X$ 
    using assms(3) assms(4) mid-onc2--per by blast
  have  $Per \ A \ M \ X$ 
    using Per-def  $\langle M \text{ Midpoint } X \ Y \rangle$  assms(1) by auto
  have  $M \neq X$ 
    using False  $\langle M \text{ Midpoint } X \ Y \rangle$  is-midpoint-id by force
  hence  $Col \ A \ PO \ M$ 
    using  $\langle Per \ A \ M \ X \rangle \ \langle Per \ PO \ M \ X \rangle$  per2--col by auto
  have  $M \text{ Midpoint } Y \ X$ 
    using Mid-cases  $\langle M \text{ Midpoint } X \ Y \rangle$  by blast
  moreover have  $Col \ PO \ A \ M$ 
    using  $\langle Col \ A \ PO \ M \rangle$  col-permutation-4 by blast
  moreover have  $PO \ A \ Perp \ Y \ X$ 
  proof (cases  $M = PO$ )
    case True
    thus ?thesis
      by (metis False OnCircle-def assms(1) assms(2) assms(3)
          calculation(1) cong-diff-3 cong-reflexivity
          mid-onc2--perp onc2--cong perp-left-comm)
  next

```

```

case False
then show ?thesis
  by (metis  $\langle M \text{ Midpoint } X \ Y \rangle \langle X \neq Y \rangle$  assms(2) assms(3)
    assms(4) calculation(2) col-permutation-5 cong-diff
    mid-onc2--perp onc2--cong perp-col perp-right-comm)
qed
ultimately show ?thesis
  using ReflectL-def by blast
qed

lemma bet-cong-onc3-cases:
assumes  $T \neq PO$  and
  Bet  $A \ PO \ T$  and
  Cong  $T \ X \ T \ Y$  and
   $A \ OnCircle \ PO \ P$  and
   $X \ OnCircle \ PO \ P$  and
   $Y \ OnCircle \ PO \ P$ 
shows  $X = Y \vee X \ Y \ ReflectL \ PO \ A$ 
using Col-cases assms(1) assms(2) assms(3) assms(4) assms(5)
  assms(6) bet-col cong-onc3-cases l4-17 onc2--cong by blast

lemma prop-7-8:
assumes Diam  $A \ B \ PO \ P$  and
  Bet  $A \ PO \ T$  and
   $X \ OnCircle \ PO \ P$  and
   $Y \ OnCircle \ PO \ P$  and
   $A \ PO \ X \ LeA \ A \ PO \ Y$ 
shows  $T \ Y \ Le \ T \ X$ 
proof (cases  $PO = P$ )
  case True
  thus ?thesis
    by (metis assms(4) assms(5) lea-distincts onc-not-center onc-sym)
next
  case False
  hence  $PO \neq P$ 
  by simp
  show ?thesis
  proof (cases  $PO = T$ )
    case True
    thus ?thesis
      using assms(3) assms(4) cong--le onc2--cong by auto
  next
    case False
    hence  $PO \neq T$ 
    by simp

  show ?thesis
  proof (cases Cong  $A \ X \ A \ Y$ )
    case True
    thus ?thesis
      by (metis assms(2) assms(3) assms(4) assms(5) bet-col
        cong--le l4-17 lea-distincts not-cong-3412 onc2--cong)
  next
    case False
    have  $T \ X \ Le \ T \ A$ 
      using assms(1) assms(2) assms(3) bet-onc-le-b by auto
    have  $Y \ PO \ T \ LeA \ X \ PO \ T$ 
      by (metis  $\langle PO \neq T \rangle$  assms(2) assms(5) l11-36-aux1 lea-comm lea-distincts)
    have Cong  $PO \ X \ PO \ Y$ 
      using assms(3) assms(4) onc2--cong by blast
    thus ?thesis
      using t18-18 by (metis  $\langle PO \neq T \rangle \langle Y \ PO \ T \ LeA \ X \ PO \ T \rangle$  cong-identity
        cong-reflexivity le-reflexivity lta--nlea nlt--le not-cong-3412 t18-19)
  qed
qed
qed

```

lemma *Prop-7-8-uniqueness:*
assumes $T \neq PO$ **and**
 $X \neq Y$ **and**

Cong $T X T Y$ **and**
Cong $T X T Z$ **and**

 $X \text{ OnCircle } PO P$ **and**
 $Y \text{ OnCircle } PO P$ **and**
 $Z \text{ OnCircle } PO P$
shows $Z = X \vee Z = Y$
by (*metis* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(6) *assms*(7) *cong2-nc3--eq*)

lemma *chords-midpoints-col-par:*
assumes
 $A \text{ OnCircle } PO P$ **and**
 $B \text{ OnCircle } PO P$ **and**
 $C \text{ OnCircle } PO P$ **and**
 $D \text{ OnCircle } PO P$ **and**
 $M \text{ Midpoint } A B$ **and**
 $N \text{ Midpoint } C D$ **and**
 $Col PO M N$ **and**
 $\neg Col PO A B$ **and**
 $\neg Col PO C D$
shows $A B \text{ Par } C D$
proof –
have $PO M \text{ Perp } A B$
by (*metis* *assms*(1) *assms*(2) *assms*(5) *assms*(8) *mid-nc2--perp* *midpoint-col* *not-col-distincts*)
moreover have $PO N \text{ Perp } C D$
by (*metis* *assms*(9) *assms*(3) *assms*(4) *assms*(6) *mid-nc2--perp* *midpoint-col* *not-col-distincts*)
hence $PO M \text{ Perp } C D$
by (*metis* *assms*(5) *assms*(8) *assms*(7) *col-permutation-5* *midpoint-col* *perp-col*)
ultimately show *?thesis*
by (*meson* *Perp-cases* *l12-9-2D*)
qed

lemma *nc3-mid2--ncol:*
assumes
 $A \text{ OnCircle } PO P$ **and**
 $B \text{ OnCircle } PO P$ **and**
 $C \text{ OnCircle } PO P$ **and**
 $A' \text{ Midpoint } A C$ **and**
 $B' \text{ Midpoint } B C$ **and**
 $\neg Col A B C$
shows $\neg Col PO A' B' \vee A' = PO \vee B' = PO$
proof (*cases* $Col PO A C$)
case *True*
hence $A = C \vee PO \text{ Midpoint } A C$
using *assms*(1) *assms*(3) *l7-20 not-col-permutation-4* *nc2--cong* **by** *blast*
thus *?thesis*
using *MidR-uniq-aux* *assms*(4) *assms*(6) *col-trivial-3* **by** *blast*
next
case *False*
hence $\neg Col PO A C$
by *simp*
show *?thesis*
proof (*cases* $Col PO B C$)
case *True*
hence $B = C \vee PO \text{ Midpoint } B C$
by (*metis* *assms*(2) *assms*(3) *col-permutation-4* *l7-20-bis* *nc2--cong*)
thus *?thesis*
using *assms*(5) *assms*(6) *col-trivial-2* *l7-17* **by** *blast*
next
case *False*
hence $\neg Col PO B C$

```

    by simp
  {
    assume Col PO A' B'
    hence A C Par B C
      using False  $\langle \neg \text{Col PO A C} \rangle$  assms(2) assms(3) assms(4)
      assms(5) assms(1) chords-midpoints-col-par by force
    hence False
      using assms(6) par-comm par-id-2 by blast
  }
  thus ?thesis
    by blast
qed
qed

```

```

lemma onc4-cong2--eq:
  assumes A  $\neq$  B and
    C  $\neq$  D and
     $\neg$  A B Par C D and
    A OnCircle PO P and
    B OnCircle PO P and
    C OnCircle PO P and
    D OnCircle PO P and
    Cong A X B X and
    Cong C X D X
  shows PO = X
proof (cases PO = P)
  case True
  then show ?thesis
    using assms(2) assms(6) assms(7) inc-eq onc--inc by blast
next
  case False
  obtain M where M Midpoint A B
    using midpoint-existence by blast
  hence Col PO X M
    using assms(1) assms(4) assms(5) assms(8) onc2-mid-cong-col by blast
  obtain N where N Midpoint C D
    using midpoint-existence by blast
  hence Col PO X N
    using assms(2) assms(6) assms(7) assms(9) onc2-mid-cong-col by blast
  show ?thesis
proof (cases PO = X)
  case True
  thus ?thesis
    by simp
next
  case False
  hence PO  $\neq$  X
    by simp
  have Col PO M N
    using False  $\langle \text{Col PO X M} \rangle$   $\langle \text{Col PO X N} \rangle$  col-transitivity-1 by blast
  have X = M  $\vee \neg$  Col A B X  $\wedge$  M PerpAt X M A B
    by (simp add:  $\langle \text{M Midpoint A B} \rangle$  assms(1) assms(8) cong-perp-or-mid)
  moreover have X = N  $\vee \neg$  Col C D X  $\wedge$  N PerpAt X N C D
    using  $\langle \text{N Midpoint C D} \rangle$  assms(2) assms(9) cong-perp-or-mid by auto
  moreover
  {
    assume X = M and X = N
    hence PO = X
      by (metis  $\langle \text{M Midpoint A B} \rangle$   $\langle \text{N Midpoint C D} \rangle$  assms(1) assms(3)
        assms(4) assms(5) assms(6) assms(7) l12-17 mid2-onc4--eq
        par-reflexivity symmetric-point-uniqueness)
  }
  moreover
  {

```

```

assume  $X = M$  and  $\neg \text{Col } C \ D \ X$  and  $N \text{ PerpAt } X \ N \ C \ D$ 
hence  $M \ N \text{ Perp } C \ D$ 
using perp-in-perp by blast

have  $PO = X$ 
proof (cases Col PO A B)
  case True
    thus ?thesis
    by (metis  $\langle M \text{ Midpoint } A \ B \rangle \langle X = M \rangle$  assms(1) assms(4) assms(5)
      assms(8) col-onc2--mid cong-perp-or-mid midpoint-col not-col-permutation-2)
  next
    case False
    have  $PO \ M \text{ Perp } A \ B$ 
    using  $\langle M \text{ Midpoint } A \ B \rangle \langle PO \neq X \rangle \langle X = M \rangle$  assms(1) assms(4)
      assms(5) mid-onc2--perp by blast
    hence  $M \ PO \text{ Perp } C \ D$ 
    by (metis Perp-cases  $\langle \text{Col } PO \ M \ N \rangle \langle \text{Col } PO \ X \ M \rangle \langle M \ N \text{ Perp } C \ D \rangle$ 
       $\langle X = M \rangle$  assms(3) calculation(3) l12-9-2D perp-col0)
    then show ?thesis
    by (meson Perp-cases  $\langle PO \ M \text{ Perp } A \ B \rangle$  assms(3) l12-9-2D)
  qed
}
moreover
{
  assume  $\neg \text{Col } A \ B \ X$  and  $M \text{ PerpAt } X \ M \ A \ B$  and  $X = N$ 

  have  $PO = X$ 
  proof (cases Col PO C D)
    case True
    have  $C = D \vee PO \text{ Midpoint } C \ D$ 
    using Col-cases True assms(6) assms(7) col-onc2--mid by blast
    thus ?thesis
    by (simp add:  $\langle N \text{ Midpoint } C \ D \rangle \langle X = N \rangle$  assms(2) l7-17)
  next
    case False
    have  $PO \ N \text{ Perp } C \ D$ 
    using  $\langle N \text{ Midpoint } C \ D \rangle \langle PO \neq X \rangle \langle X = N \rangle$  assms(2) assms(6) assms(7)
      mid-onc2--perp by auto
    have  $N \ M \text{ Perp } A \ B$ 
    using  $\langle M \text{ PerpAt } X \ M \ A \ B \rangle \langle X = N \rangle$  perp-in-perp by auto
    hence  $A \ B \text{ Par } C \ D$ 
    by (metis Col-perm Perp-cases  $\langle \text{Col } PO \ M \ N \rangle \langle \text{Col } PO \ X \ N \rangle$ 
       $\langle PO \ N \text{ Perp } C \ D \rangle \langle X = N \rangle$  calculation(3) l12-9-2D perp-col0)
    thus ?thesis
    by (simp add: assms(3))
  qed
}
moreover
{
  assume  $\neg \text{Col } A \ B \ X$  and  $M \text{ PerpAt } X \ M \ A \ B$  and
     $\neg \text{Col } C \ D \ X$  and  $N \text{ PerpAt } X \ N \ C \ D$ 
  have  $X \ M \text{ Perp } A \ B$ 
  using  $\langle M \text{ PerpAt } X \ M \ A \ B \rangle$  perp-in-perp by auto
  hence  $X \ PO \text{ Perp } A \ B$ 
  by (metis Col-cases False  $\langle \text{Col } PO \ X \ M \rangle$  perp-col)
  moreover have  $X \ N \text{ Perp } C \ D$ 
  using  $\langle N \text{ PerpAt } X \ N \ C \ D \rangle$  perp-in-perp by auto
  hence  $X \ PO \text{ Perp } C \ D$ 
  by (metis False  $\langle \text{Col } PO \ X \ N \rangle$  not-col-permutation-2 perp-col)
  ultimately have  $PO = X$ 
  by (meson Perp-cases assms(3) l12-9-2D)
}
ultimately show ?thesis
by blast
qed
qed

```

```

lemma onc2--oreq:
  assumes InterCCAt  $A\ B\ C\ D\ P\ Q$  and
     $Z\ \text{OnCircle}\ A\ B$  and
     $Z\ \text{OnCircle}\ C\ D$ 
  shows  $Z = P \vee Z = Q$ 
  by (metis InterCCAt-def assms(1) assms(2) assms(3) cong2-onc3--eq interccat--neq onc2--cong)

end
end

```

```

theory Tarski-Neutral-3D

```

```

imports
  Tarski-Neutral

```

```

begin

```

12 Tarski's axiom system for neutral geometry: 3D

12.1 Definitions

```

locale Tarski-neutral-3D = Tarski-neutral-dimensionless +
  fixes TS1 and TS2 and TS3 and TS4
  assumes lower-dim-3:  $\neg (\exists\ X.$ 
     $((\text{Bet}\ TS1\ TS2\ X \vee \text{Bet}\ TS2\ X\ TS1 \vee \text{Bet}\ X\ TS1\ TS2) \wedge$ 
       $(\text{Bet}\ TS3\ TS4\ X \vee \text{Bet}\ TS4\ X\ TS3 \vee \text{Bet}\ X\ TS3\ TS4) \vee$ 
       $(\text{Bet}\ TS1\ TS3\ X \vee \text{Bet}\ TS3\ X\ TS1 \vee \text{Bet}\ X\ TS1\ TS3) \wedge$ 
       $(\text{Bet}\ TS2\ TS4\ X \vee \text{Bet}\ TS4\ X\ TS2 \vee \text{Bet}\ X\ TS2\ TS4) \vee$ 
       $(\text{Bet}\ TS1\ TS4\ X \vee \text{Bet}\ TS4\ X\ TS1 \vee \text{Bet}\ X\ TS1\ TS4) \wedge$ 
       $(\text{Bet}\ TS2\ TS3\ X \vee \text{Bet}\ TS3\ X\ TS2 \vee \text{Bet}\ X\ TS2\ TS3)))$ 
  assumes upper-dim-3:  $\forall\ A\ B\ C\ P\ Q\ R.$ 
     $P \neq Q \wedge Q \neq R \wedge P \neq R \wedge$ 
     $\text{Cong}\ A\ P\ A\ Q \wedge \text{Cong}\ B\ P\ B\ Q \wedge \text{Cong}\ C\ P\ C\ Q \wedge$ 
     $\text{Cong}\ A\ P\ A\ R \wedge \text{Cong}\ B\ P\ B\ R \wedge \text{Cong}\ C\ P\ C\ R \longrightarrow$ 
     $(\text{Bet}\ A\ B\ C \vee \text{Bet}\ B\ C\ A \vee \text{Bet}\ C\ A\ B)$ 

```

```

context Tarski-neutral-3D

```

```

begin

```

12.2 Propositions

```

lemma not-coplanar-S1-S2-S3-S4:
  shows  $\neg \text{Coplanar}\ TS1\ TS2\ TS3\ TS4$ 
proof -
  {
    assume Coplanar  $TS1\ TS2\ TS3\ TS4$ 
    then obtain  $X$  where  $(\text{Col}\ TS1\ TS2\ X \wedge \text{Col}\ TS3\ TS4\ X) \vee$ 
       $(\text{Col}\ TS1\ TS3\ X \wedge \text{Col}\ TS2\ TS4\ X) \vee$ 
       $(\text{Col}\ TS1\ TS4\ X \wedge \text{Col}\ TS2\ TS3\ X)$ 
    using Coplanar-def by auto
    hence False using lower-dim-3 Col-def by blast
  }
  thus ?thesis
    by blast
qed

end
end

```

```

theory Tarski-Neutral-3D-Hilbert

```

```

imports

```

begin

13 Tarski neutral dimensionless - Hilbert

context *Tarski-neutral-3D*

begin

13.1 Definition

13.2 Propositions

lemma *lower-dim-3'*:

shows $\neg (\exists p. \text{isPlane } p \wedge \text{IncidentP } TS1 \ p \wedge \text{IncidentP } TS2 \ p \wedge$
 $\text{IncidentP } TS3 \ p \wedge \text{IncidentP } TS4 \ p)$

proof –

{
 assume $\exists p. \text{isPlane } p \wedge \text{IncidentP } TS1 \ p \wedge \text{IncidentP } TS2 \ p \wedge$
 $\text{IncidentP } TS3 \ p \wedge \text{IncidentP } TS4 \ p$
 then obtain *p* where *isPlane p* and *IncidentP TS1 p* and *IncidentP TS2 p* and
 $\text{IncidentP } TS3 \ p \wedge \text{IncidentP } TS4 \ p$
 by blast
 hence *Coplanar TS1 TS2 TS3 TS4*
 using *plane-cop* by blast
 hence *False* using *not-coplanar-S1-S2-S3-S4* by simp
}

thus ?thesis

by blast

qed

end

end

theory *Tarski-Euclidean*

imports

Tarski-Neutral

Tarski-Postulate-Parallels

begin

14 Tarski Euclidean

14.1 Tarski's axiom system for Euclidean

locale *Tarski-Euclidean* = *Tarski-neutral-dimensionless* +

assumes *tarski-s-parallel-postulate*:

$\forall A \ B \ C \ D \ T. (\text{Bet } A \ D \ T \wedge \text{Bet } B \ D \ C \wedge A \neq D)$
 $\longrightarrow$
 $(\exists X \ Y. \text{Bet } A \ B \ X \wedge \text{Bet } A \ C \ Y \wedge \text{Bet } X \ T \ Y)$

context *Tarski-Euclidean*

begin

14.2 Definitions

14.3 Propositions

theorem *tarski-s-parallel-postulate-thm*:

shows *tarski-s-parallel-postulate*

using *tarski-s-parallel-postulate tarski-s-parallel-postulate-def* by blast

lemma *Post02*:
shows *Postulate02*
using *Postulate02-def tarski-s-euclid-implies-playfair-s-postulate*
tarski-s-parallel-postulate tarski-s-parallel-postulate-def **by** *blast*

theorem *playfair-s-postulate-thm*:
shows *playfair-s-postulate*
using *Post02 Postulate02-def* **by** *auto*

lemma *Post03*:
shows *Postulate03*
using *Postulate03-def alternate-interior--triangle*
playfair--alternate-interior tarski-s-euclid-implies-playfair-s-postulate
tarski-s-parallel-postulate tarski-s-parallel-postulate-def **by** *blast*

theorem *triangle-postulate-thm*:
shows *triangle-postulate*
using *Post03 Postulate03-def* **by** *blast*

lemma *Post04*:
shows *Postulate04*
using *P31-P04 Postulate31-def*
inverse-projection-postulate-def weak-inverse-projection-postulate-def
original-euclid--original-spp
original-spp--inverse-projection-postulate tarski-s-implies-euclid-s-parallel-postulate
tarski-s-parallel-postulate tarski-s-parallel-postulate-def
by *blast*

theorem *bachmann-s-lotschnittaxiom-thm*:
shows *bachmann-s-lotschnittaxiom*
using *Post04 Postulate04-def* **by** *blast*

lemma *Post05*:
shows *Postulate05*
using *Post02 Postulate02-def Postulate05-def playfair-implies-par-trans* **by** *blast*

theorem *postulate-of-transitivity-of-parallelism-thm*:
shows *postulate-of-transitivity-of-parallelism*
using *Post05 Postulate05-def* **by** *blast*

lemma *Post06*:
shows *Postulate06*
using *Post02 Postulate02-def Postulate06-def*
playfair-s-postulate-implies-midpoint-converse-postulate **by** *fastforce*

theorem *midpoint-converse-postulate-thm*:
shows *midpoint-converse-postulate*
using *Post06 Postulate06-def* **by** *blast*

lemma *Post07*:
shows *Postulate07*
using *Post02 Postulate02-def Postulate07-def playfair--alternate-interior* **by** *blast*

theorem *alternate-interior-angles-postulate-thm*:
shows *alternate-interior-angles-postulate*
using *Post07 Postulate07-def* **by** *auto*

lemma *Post08*:
shows *Postulate08*
using *Post07 Postulate07-def Postulate08-def*
alternate-interior--consecutive-interior **by** *blast*

theorem *consecutive-interior-angles-postulate-thm*:
shows *consecutive-interior-angles-postulate*
using *Post08 Postulate08-def* **by** *blast*

lemma *Post09*:
 shows *Postulate09*
 using *Post02 equivalent-postulates-without-decidability-of-intersection-of-lines* **by** *blast*

theorem *perpendicular-transversal-postulate-thm*:
 shows *perpendicular-transversal-postulate*
 using *Post09 Postulate09-def* **by** *blast*

lemma *Post10*:
 shows *Postulate10*
 using *Post05 equivalent-postulates-without-decidability-of-intersection-of-lines* **by** *blast*

theorem *postulate-of-parallelism-of-perpendicular-transversals-thm*:
 shows *postulate-of-parallelism-of-perpendicular-transversals*
 using *Post10 Postulate10-def* **by** *blast*

lemma *Post11*:
 shows *Postulate11*
 using *Post02 equivalent-postulates-without-decidability-of-intersection-of-lines* **by** *blast*

theorem *universal-posidonius-postulate-thm*:
 shows *universal-posidonius-postulate*
 using *Post11 Postulate11-def* **by** *blast*

lemma *Post12*:
 shows *Postulate12*
by (*simp add: Post05 equivalent-postulates-without-decidability-of-intersection-of-lines*)

theorem *alternative-playfair-s-postulate-thm*:
 shows *alternative-playfair-s-postulate*
 using *Post12 Postulate12-def* **by** *blast*

lemma *Post13*:
 shows *Postulate13*
 using *Cycle-2 InterAx5 Post02* **by** *fastforce*

theorem *proclus-postulate-thm*:
 shows *proclus-postulate*
 using *Post13 Postulate13-def* **by** *blast*

lemma *Post14*:
 shows *Postulate14*
 using *Cycle-2 InterAx5 Post02* **by** *blast*

theorem *alternative-proclus-postulate-thm*:
 shows *alternative-proclus-postulate*
 using *Post14 Postulate14-def* **by** *auto*

lemma *Post15*:
 shows *Postulate15*
by (*simp add: InterCycle4 Post09*)

theorem *triangle-circumscription-principle-thm*:
 shows *triangle-circumscription-principle*
 using *Post15 Postulate15-def* **by** *blast*

lemma *Post16*:
 shows *Postulate16*
 using *Postulate16-def original-euclid--original-spp*
 original-spp--inverse-projection-postulate tarski-s-implies-euclid-s-parallel-postulate
 tarski-s-parallel-postulate tarski-s-parallel-postulate-def **by** *fastforce*

theorem *inverse-projection-postulate-thm*:
 shows *inverse-projection-postulate*
 using *Post16 Postulate16-def* **by** *blast*

```

lemma Post17:
  shows Postulate17
  using Postulate17-def tarski-s-euclid-implies-euclid-5
    tarski-s-parallel-postulate tarski-s-parallel-postulate-def by blast

theorem euclid-5-thm:
  shows euclid-5
  using Post17 Postulate17-def by blast

lemma Post18:
  shows Postulate18
  using Cycle-2 Post14 by fastforce

theorem strong-parallel-postulate-thm:
  shows strong-parallel-postulate
  by (simp add: proclus-postulate-thm proclus-s-postulate-implies-strong-parallel-postulate)

lemma Post19:
  shows Postulate19
  using Cycle-2 Post18 by force

theorem alternative-strong-parallel-postulate-thm:
  shows alternative-strong-parallel-postulate
  using Post19 Postulate19-def by blast

lemma Post20:
  shows Postulate20
  using Cycle-2 Post17 by blast

theorem euclid-s-parallel-postulate-thm:
  shows euclid-s-parallel-postulate
  using Post20 Postulate20-def by blast

lemma Post21:
  shows Postulate21
  using Post03 Tarski-Euclidean-def Cycle-3 by blast

theorem postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-thm:
  shows postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights
  using Post21 Postulate21-def by blast

lemma Post22:
  shows Postulate22
  using Post21 Tarski-Euclidean-def Cycle-3 by blast

theorem posidonius-postulate-thm:
  shows posidonius-postulate
  using Post22 Postulate22-def by fastforce

lemma Post23:
  assumes Postulate01
  shows Postulate23
  using Post22 Tarski-Euclidean-def Cycle-3 by blast

theorem postulate-of-existence-of-similar-triangles-thm:
  shows postulate-of-existence-of-similar-triangles
  using Post23 Postulate01-def Postulate23-def tarski-s-parallel-postulate-thm by auto

lemma Post24:
  shows Postulate24
  using Post23 Postulate01-def Tarski-Euclidean-def Cycle-3
    tarski-s-parallel-postulate tarski-s-parallel-postulate-def by blast

theorem thales-postulate-thm:
  shows thales-postulate

```

using *Post24 Postulate24-def* **by** *blast*

lemma *Post25*:
 shows *Postulate25*
 using *Post24 Postulate24-def Postulate25-def*
thales-postulate--thales-converse-postulate **by** *blast*

theorem *thales-converse-postulate-thm*:
 shows *thales-converse-postulate*
 using *Post25 Postulate25-def* **by** *blast*

lemma *Post26*:
 shows *Postulate26*
 using *Post24 Postulate24-def Postulate26-def thales-converse-postulate--thales-existence*
thales-postulate--thales-converse-postulate **by** *blast*

theorem *existential-thales-postulate-thm*:
 shows *existential-thales-postulate*
 using *Post26 Postulate26-def* **by** *blast*

lemma *Post27*:
 shows *Postulate27*
 using *Post26 Tarski-Euclidean-def Cycle-3* **by** *blast*

theorem *postulate-of-right-saccheri-quadrilaterals-thm*:
 shows *postulate-of-right-saccheri-quadrilaterals*
 using *Post27 Postulate27-def* **by** *auto*

lemma *Post28*:
 shows *Postulate28*
 using *Post27 Tarski-Euclidean-def Cycle-3* **by** *blast*

theorem *postulate-of-existence-of-a-right-saccheri-quadrilateral-thm*:
 shows *postulate-of-existence-of-a-right-saccheri-quadrilateral*
 using *Post28 Postulate28-def* **by** *blast*

lemma *Post29*:
 shows *Postulate29*
 using *Cycle-3 Post28* **by** *blast*

theorem *postulate-of-right-lambert-quadrilaterals-thm*:
 shows *postulate-of-right-lambert-quadrilaterals*
 using *Post29 Postulate29-def* **by** *blast*

lemma *Post30*:
 shows *Postulate30*
 using *Cycle-3 Post03* **by** *blast*

theorem *postulate-of-existence-of-a-right-lambert-quadrilateral-thm*:
 shows *postulate-of-existence-of-a-right-lambert-quadrilateral*
 using *Post30 Postulate30-def* **by** *blast*

lemma *Post31*:
 shows *Postulate31*
 using *Post16 Postulate16-def Postulate31-def*
inverse-projection-postulate-def weak-inverse-projection-postulate-def **by** *blast*

theorem *weak-inverse-projection-postulate-thm*:
 shows *weak-inverse-projection-postulate*
 using *Post31 Postulate31-def* **by** *blast*

lemma *Post32*:
 shows *Postulate32*
 using *P31-P32 Post31* **by** *blast*

theorem *weak-tarski-s-parallel-postulate-thm*:

shows *weak-tarski-s-parallel-postulate*
using *Post32 Postulate32-def* **by** *blast*

lemma *Post33*:
shows *Postulate33*
using *P04-P33 P31-P04 Post31* **by** *auto*

theorem *weak-triangle-circumscription-principle-thm*:
shows *weak-triangle-circumscription-principle*
using *Post33 Postulate33-def* **by** *blast*

lemma *Post34*:
shows *Postulate34*
using *P31-P04 P4-P34 Post31* **by** *blast*

theorem *legendre-s-parallel-postulate-thm*:
shows *legendre-s-parallel-postulate*
using *Post34 Postulate34-def* **by** *blast*

theorem *TarskiPP*:
shows *tarski-s-parallel-postulate*
by (*simp add: tarski-s-parallel-postulate-thm*)

lemma *parallel-uniqueness*:
assumes *A1 A2 Par B1 B2 and*
Col P B1 B2 and
A1 A2 Par C1 C2 and
Col P C1 C2
shows *Col C1 B1 B2 $\wedge$ Col C2 B1 B2*
using *TarskiPP assms(1) assms(2) assms(3) assms(4) playfair-s-postulate-def*
tarski-s-euclid-implies-playfair-s-postulate **by** *blast*

lemma *par-trans*:
assumes *A1 A2 Par B1 B2 and*
B1 B2 Par C1 C2
shows *A1 A2 Par C1 C2*
using *TarskiPP assms(1) assms(2) playfair-implies-par-trans*
postulate-of-transitivity-of-parallelism-def
tarski-s-euclid-implies-playfair-s-postulate **by** *blast*

lemma *l12-16*:
assumes *A1 A2 Par B1 B2 and*
Coplanar B1 B2 C1 C2 and
X Inter A1 A2 C1 C2
shows $\exists Y. Y \text{ Inter } B1 B2 C1 C2$
by (*metis Inter-def TarskiPP assms(1) assms(2) assms(3)*
cop-npar--inter inter--npar par-neq2 playfair-implies-par-trans
postulate-of-transitivity-of-parallelism-def
tarski-s-euclid-implies-playfair-s-postulate)

lemma *par-dec*:
shows $A B \text{ Par } C D \vee \neg A B \text{ Par } C D$
by *auto*

lemma *par-not-par*:
assumes *A B Par C D and*
 $\neg A B \text{ Par } P Q$
shows $\neg C D \text{ Par } P Q$
using *assms(1) assms(2) par-trans* **by** *blast*

lemma *cop-par--inter*:
assumes *A B Par C D and*
 $\neg A B \text{ Par } P Q$ **and**
Coplanar C D P Q
shows $\exists Y. Col P Q Y \wedge Col C D Y$
proof –

```

have  $\neg C D \text{ Par } P Q$ 
  using assms(1) assms(2) par-trans by blast
hence  $\exists Y. \text{Col } Y C D \wedge \text{Col } Y P Q$ 
  using assms(3) cop-npar--inter-exists by blast
thus ?thesis
  using col-permutation-1 by blast
qed

lemma l12-19:
  assumes  $\neg \text{Col } A B C$  and
     $A B \text{ Par } C D$  and
     $B C \text{ Par } D A$ 
  shows  $\text{Cong } A B C D \wedge \text{Cong } B C D A \wedge B D \text{ TS } A C \wedge A C \text{ TS } B D$ 
proof -
  obtain  $P$  where  $P \text{ Midpoint } A C$ 
    using midpoint-existence by blast
  obtain  $D'$  where  $P \text{ Midpoint } B D'$ 
    using symmetric-point-construction by blast
  have  $\text{Cong } C D' A B$ 
    using  $\langle P \text{ Midpoint } A C \rangle \langle P \text{ Midpoint } B D' \rangle$  l7-13 by blast
  have  $\text{Cong } B C D' A$ 
    using l7-13  $\langle P \text{ Midpoint } A C \rangle \langle P \text{ Midpoint } B D' \rangle$  assms(1)
      mid-par-cong2 not-col-distincts by presburger
  have  $A B \text{ Par } C D'$ 
    using  $\langle P \text{ Midpoint } A C \rangle \langle P \text{ Midpoint } B D' \rangle$  assms(2)
      l12-17 par-neq1 by blast
  have  $C D \text{ Par } C D'$ 
    by (meson  $\langle A B \text{ Par } C D' \rangle$  assms(2) par-not-par par-symmetry)
  have  $\text{Col } C D D'$ 
    by (simp add:  $\langle C D \text{ Par } C D' \rangle$  par-id)
  have  $B C \text{ Par } D' A$ 
    using  $\langle P \text{ Midpoint } A C \rangle \langle P \text{ Midpoint } B D' \rangle$  assms(3)
      mid-par-cong2 par-neq1 by blast
  have  $D A \text{ Par } D' A$ 
    by (meson  $\langle B C \text{ Par } D' A \rangle$  assms(3) par-not-par par-symmetry)
  have  $\text{Col } A D D'$ 
    using Par-perm  $\langle D A \text{ Par } D' A \rangle$  par-id by blast
  have  $D = D'$ 
    by (metis  $\langle C D \text{ Par } C D' \rangle \langle \text{Col } A D D' \rangle$  assms(1) assms(2)
      col-trivial-2 col-trivial-3 l6-21 not-strict-par2 par-symmetry)
  have  $\text{Cong } A B C D$ 
    using  $\langle \text{Cong } C D' A B \rangle \langle D = D' \rangle$  not-cong-3412 by blast
  moreover
  have  $\text{Cong } B C D A$ 
    by (simp add:  $\langle \text{Cong } B C D' A \rangle \langle D = D' \rangle$ )
  moreover
  have  $B D \text{ TS } A C$ 
    by (metis  $\langle D = D' \rangle \langle P \text{ Midpoint } A C \rangle \langle P \text{ Midpoint } B D' \rangle$ 
      assms(1) assms(2) col-permutation-5 mid-two-sides par-comm
      par-two-sides-two-sides)
  moreover
  have  $A C \text{ TS } B D$ 
    using assms(2) calculation(3) par-two-sides-two-sides by blast
  ultimately
  show ?thesis
    by blast
qed

lemma l12-20-bis:
  assumes  $A B \text{ Par } C D$  and
     $\text{Cong } A B C D$  and
     $B D \text{ TS } A C$ 
  shows  $B C \text{ Par } D A \wedge \text{Cong } B C D A \wedge A C \text{ TS } B D$ 
proof -
  have  $B \neq C$ 
    using assms(3) ts-distincts by blast

```

```

have A ≠ D
  using assms(3) ts-distincts by blast
have ¬ Col A B C
  by (metis TS-def assms(1) assms(3) not-col-permutation-4
      par-two-sides-two-sides)
obtain P where P Midpoint A C
  using midpoint-existence by blast
obtain D' where P Midpoint B D'
  using symmetric-point-construction by blast
have B C Par D' A
  using ⟨B ≠ C⟩ ⟨P Midpoint A C⟩ ⟨P Midpoint B D'⟩ mid-par-cong2 by blast
have Cong C D' A B
  using ⟨P Midpoint A C⟩ ⟨P Midpoint B D'⟩ l7-13 by blast
have Cong B C D' A
  using ⟨B ≠ C⟩ ⟨P Midpoint A C⟩ ⟨P Midpoint B D'⟩ mid-par-cong2 by blast
have C D Par C D'
proof —
  have A B Par C D
    using assms(1) by blast
  moreover have A B Par C D'
    using ⟨P Midpoint A C⟩ ⟨P Midpoint B D'⟩ calculation l12-17
      par-distinct by blast
  ultimately show ?thesis
    using par-not-par par-symmetry by blast
qed
have Col C D D'
  by (simp add: ⟨C D Par C D'⟩ par-id)
have Cong C D C D'
  by (meson ⟨Cong C D' A B⟩ assms(2) cong-transitivity not-cong-3412)
have D = D' ∨ C Midpoint D D'
  using ⟨Col C D D'⟩ ⟨Cong C D C D'⟩ l7-20 not-col-permutation-4 by blast
{
  assume D = D'
  hence B C Par D A ∧ Cong B C D A ∧ A C TS B D
    using ⟨B C Par D' A⟩ ⟨Cong B C D' A⟩ assms(1) assms(3)
      par-two-sides-two-sides by blast
}
moreover
{
  assume C Midpoint D D'
  have A C TS B D
    by (simp add: assms(1) assms(3) par-two-sides-two-sides)
  hence ¬ Col A C D
    using TS-def not-col-permutation-1 by blast
  hence ¬ Col A C D'
    by (metis ⟨C D Par C D'⟩ not-col-distincts par-col2-par-bis par-id-5)
  have A C TS B D'
    by (meson Tarski-Euclidean-axioms Tarski-Euclidean-def mid-two-sides
        ⟨P Midpoint A C⟩ ⟨P Midpoint B D'⟩ ⟨¬ Col A B C⟩ col-permutation-5)
  hence A C TS D D'
    using TS-def ⟨A C TS B D⟩ ⟨C Midpoint D D'⟩ col-trivial-3 midpoint-bet by blast
  have A C TS B D
    by (simp add: ⟨A C TS B D⟩)
  moreover
  have A C OS B D
    using ⟨A C TS B D'⟩ ⟨A C TS D D'⟩ l9-8-1 by auto
  ultimately
  have False
    using l9-9 by blast
}
ultimately
show ?thesis
  using ⟨D = D' ∨ C Midpoint D D'⟩ by fastforce
qed

```

lemma *l12-20*:

```

assumes  $A \ B \ Par \ C \ D$  and
   $Cong \ A \ B \ C \ D$  and
   $A \ C \ TS \ B \ D$ 
shows  $B \ C \ Par \ D \ A \ \wedge \ Cong \ B \ C \ D \ A \ \wedge \ A \ C \ TS \ B \ D$ 
proof –
  have  $B \ D \ TS \ A \ C$ 
    by (simp add: assms(1) assms(3) par-comm par-two-sides-two-sides)
  thus ?thesis
    using assms(1) assms(2) l12-20-bis by blast
qed

```

```

lemma l12-21-a:
assumes  $A \ C \ TS \ B \ D$  and
   $A \ B \ Par \ C \ D$ 
shows  $B \ A \ C \ CongA \ D \ C \ A$ 
using TarskiPP alternate-interior-angles-postulate-def assms(1) assms(2)
  playfair--alternate-interior tarski-s-euclid-implies-playfair-s-postulate by blast

```

```

lemma l12-21:
assumes  $A \ C \ TS \ B \ D$ 
shows  $B \ A \ C \ CongA \ D \ C \ A \longleftrightarrow A \ B \ Par \ C \ D$ 
using assms l12-21-a l12-21-b by blast

```

```

lemma l12-22-a:
assumes  $P \ Out \ A \ C$  and
   $P \ A \ OS \ B \ D$  and
   $A \ B \ Par \ C \ D$ 
shows  $B \ A \ P \ CongA \ D \ C \ P$ 
proof –
{
  assume  $H1: \forall \ A \ B \ C \ D \ P. (A \neq P \wedge Bet \ P \ A \ C \wedge P \ A \ OS \ B \ D \wedge A \ B \ Par \ C \ D) \longrightarrow B \ A \ P \ CongA \ D \ C \ P$ 
  {
    fix  $A9 \ B9 \ C9 \ D9 \ P9$ 
    assume  $P9 \ Out \ A9 \ C9$  and  $P9 \ A9 \ OS \ B9 \ D9$  and  $A9 \ B9 \ Par \ C9 \ D9$ 
    have  $A9 \neq P9 \wedge C9 \neq P9 \wedge (Bet \ P9 \ A9 \ C9 \vee Bet \ P9 \ C9 \ A9)$ 
      using Out-def  $\langle P9 \ Out \ A9 \ C9 \rangle$  by force
    have  $Bet \ P9 \ A9 \ C9 \longrightarrow B9 \ A9 \ P9 \ CongA \ D9 \ C9 \ P9$ 
      using  $\langle A9 \ B9 \ Par \ C9 \ D9 \rangle \langle A9 \neq P9 \wedge C9 \neq P9 \wedge (Bet \ P9 \ A9 \ C9 \vee Bet \ P9 \ C9 \ A9) \rangle$ 
       $\langle P9 \ A9 \ OS \ B9 \ D9 \rangle$ 
       $\langle \forall \ A \ B \ C \ D \ P. A \neq P \wedge Bet \ P \ A \ C \wedge P \ A \ OS \ B \ D \wedge A \ B \ Par \ C \ D \longrightarrow B \ A \ P \ CongA \ D \ C \ P \rangle$ 
      by presburger
    moreover
    {
      assume  $Bet \ P9 \ C9 \ A9$ 
      have  $C9 \neq P9$ 
        by (simp add:  $\langle A9 \neq P9 \wedge C9 \neq P9 \wedge (Bet \ P9 \ A9 \ C9 \vee Bet \ P9 \ C9 \ A9) \rangle$ )
      moreover
      have  $Bet \ P9 \ C9 \ A9$ 
        by (simp add:  $\langle Bet \ P9 \ C9 \ A9 \rangle$ )
      moreover
      have  $P9 \ C9 \ OS \ D9 \ B9$ 
        using  $\langle A9 \neq P9 \wedge C9 \neq P9 \wedge (Bet \ P9 \ A9 \ C9 \vee Bet \ P9 \ C9 \ A9) \rangle$ 
         $\langle P9 \ A9 \ OS \ B9 \ D9 \rangle \langle P9 \ Out \ A9 \ C9 \rangle$  col-one-side one-side-symmetry
        out-col by blast
      moreover
      have  $C9 \ D9 \ Par \ A9 \ B9$ 
        using  $\langle A9 \ B9 \ Par \ C9 \ D9 \rangle$  par-symmetry by blast
      ultimately
      have  $B9 \ A9 \ P9 \ CongA \ D9 \ C9 \ P9$ 
        by (simp add: H1 conga-sym)
    }
  }
  ultimately
  have  $B9 \ A9 \ P9 \ CongA \ D9 \ C9 \ P9$ 
    using  $\langle A9 \neq P9 \wedge C9 \neq P9 \wedge (Bet \ P9 \ A9 \ C9 \vee Bet \ P9 \ C9 \ A9) \rangle$  by blast
}

```

```

hence  $\forall A B C D P. (P \text{ Out } A C \wedge P A \text{ OS } B D \wedge A B \text{ Par } C D) \longrightarrow B A P \text{ CongA } D C P$ 
  by blast
}
moreover
{
  fix  $A B C D P$ 
  assume  $A \neq P$  and  $Bet P A C$  and  $P A \text{ OS } B D$  and  $A B \text{ Par } C D$ 
  have  $B A P \text{ CongA } D C P$ 
  proof (cases  $A = C$ )
    case True
    have  $A \text{ Out } D B$ 
    by (metis  $True \langle A B \text{ Par } C D \rangle \langle P A \text{ OS } B D \rangle$  bet-col1 l6-6 l9-19
      par-id-2 point-construction-different)
    moreover
    have  $A \text{ Out } P P$ 
    using  $\langle A \neq P \rangle$  out-trivial by presburger
    ultimately
    show ?thesis
    by (simp add: True out2--conga)
  next
  case False
  obtain  $B'$  where  $Bet B A B'$  and  $Cong A B' B A$ 
  using segment-construction by blast
  have  $B A P \text{ CongA } D C P$ 
  proof –
  have  $B A P \text{ CongA } B' A C$ 
  proof –
  have  $B \neq A$ 
  using  $\langle P A \text{ OS } B D \rangle$  os-distincts by blast
  moreover
  have  $A \neq P$ 
  by (simp add: \langle A \neq P \rangle)
  moreover
  have  $B' \neq A$ 
  using  $\langle Cong A B' B A \rangle$  calculation(1) cong-diff-3 by blast
  moreover
  have  $A \neq C$ 
  by (simp add: False)
  ultimately
  show ?thesis
  using  $\langle Bet B A B' \rangle \langle Bet P A C \rangle$  l11-14 by simp
qed
moreover
have  $B' A C \text{ CongA } D C P$ 
proof –
  have  $B' A C \text{ CongA } D C A$ 
proof –
  have  $A C \text{ TS } B' D$ 
proof –
  have  $A C \text{ TS } B B'$ 
  by (metis  $False \langle Bet B A B' \rangle \langle Bet P A C \rangle \langle Cong A B' B A \rangle$ 
     $\langle P A \text{ OS } B D \rangle$  bet--ts bet-col bet-col1 col123--nos colx cong-diff-4)
  moreover
  have  $A C \text{ OS } B D$ 
  by (meson  $False \langle Bet P A C \rangle \langle P A \text{ OS } B D \rangle$  bet-col bet-col1 col2-os--os)
  ultimately
  show ?thesis
  using l9-2 l9-8-2 by blast
qed
moreover
have  $A B' \text{ Par } C D$ 
proof –
  have  $C D \text{ Par } A B$ 
  by (simp add: \langle A B \text{ Par } C D \rangle par-symmetry)
  moreover have  $C D \text{ Par } A B'$ 
  by (metis CongA-def NCol-cases  $\langle B A P \text{ CongA } B' A C \rangle \langle Bet B A B' \rangle$ )

```



```

      bet-col calculation par-col-par)
    ultimately show ?thesis
      using Par-cases by blast
  qed
  ultimately
  show ?thesis
    by (simp add: l12-21)
  qed
  moreover
  have A Out B' B'
    by (metis ‹Cong A B' B A› ‹P A OS B D› cong-reverse-identity
      os-distincts out-trivial)
  moreover
  have A Out C C
    using False out-trivial by auto
  moreover
  have C Out D D
    by (metis ‹A B Par C D› out-trivial par-distinct)
  moreover
  have C Out P A
    by (simp add: False ‹Bet P A C› bet-out-1 l6-6)
  ultimately
  show ?thesis
    using l11-10 by blast
  qed
  ultimately
  show ?thesis
    by (meson not-conga)
  qed
  thus ?thesis
    by blast
  qed
}
ultimately show ?thesis
  using assms(1) assms(2) assms(3) by blast
qed

```

lemma l12-22:

```

  assumes P Out A C and
    P A OS B D
  shows B A P CongA D C P  $\longleftrightarrow$  A B Par C D
  using assms(1) assms(2) l12-22-a l12-22-b by blast

```

lemma l12-23:

```

  assumes  $\neg$  Col A B C
  shows  $\exists B' C'. A C TS B B' \wedge A B TS C C' \wedge Bet B' A C' \wedge$ 
    A B C CongA B A C'  $\wedge$  A C B CongA C A B'

```

proof –

```

  obtain B0 where B0 Midpoint A B
    using midpoint-existence by blast
  obtain C0 where C0 Midpoint A C
    using midpoint-existence by blast
  obtain B' where Bet B C0 B' and Cong C0 B' B C0
    using segment-construction by blast
  obtain C' where Bet C B0 C' and Cong B0 C' C B0
    using segment-construction by blast
  have A C TS B B'
    by (meson ‹Bet B C0 B'› ‹C0 Midpoint A C› ‹Cong C0 B' B C0› assms
      cong-symmetry mid-two-sides midpoint-def not-col-permutation-5)
  moreover
  have A B TS C C'
    by (meson Cong-cases ‹B0 Midpoint A B› ‹Bet C B0 C'› ‹Cong B0 C' C B0›
      assms mid-two-sides midpoint-def)
  moreover
  have A B' Par C B
    by (metis ‹Bet B C0 B'› ‹C0 Midpoint A C› ‹Cong C0 B' B C0› calculation(1))

```

```

    cong-symmetry mid-par-cong2 midpoint-def ts-distincts)
have A C' Par B C
  by (metis ‹B0 Midpoint A B› ‹Bet C B0 C'› ‹Cong B0 C' C B0› calculation(2)
      mid-par-cong2 midpoint-def not-cong-3412 ts-distincts)
have A B' Par A C'
  using Par-perm ‹A B' Par C B› ‹A C' Par B C› par-not-par by blast
hence Col A B' C'
  by (simp add: par-id)
have A C OS B0 C'
proof -
{
  assume Col A C B0
  have Col A B C
  proof -
    have A ≠ B0
      using ‹B0 Midpoint A B› assms col-trivial-1 is-midpoint-id by blast
    moreover have Col A B0 B
      using Col-cases ‹B0 Midpoint A B› midpoint-col by blast
    moreover have Col A B0 C
      using ‹Col A C B0› not-col-permutation-5 by blast
    ultimately show ?thesis
      using col-transitivity-1 by blast
  qed
  hence False
    using assms by blast
}
hence ¬ Col A C B0
  by blast
moreover
have Col A C C
  using not-col-distincts by blast
moreover
have C Out B0 C'
  by (metis ‹Bet C B0 C'› bet-out calculation(1) col-trivial-2)
ultimately
show ?thesis
  using l9-19-R2 by blast
qed
have A C OS B0 B
  by (metis ‹A C OS B0 C'› ‹B0 Midpoint A B› calculation(1) col123--nos
      midpoint-out out-one-side ts-distincts)
have A C OS B C'
  by (metis Mid-cases ‹B0 Midpoint A B› ‹Bet C B0 C'› ‹Cong B0 C' C B0›
      assms calculation(2) mid-par-cong2 midpoint-def not-col-distincts not-cong-3412
      par-two-sides-two-sides ts-ts-os)
have A C TS B' C'
  using ‹A C OS B C'› calculation(1) l9-2 l9-8-2 by blast
hence Bet B' A C'
  using ‹Col A B' C'› col-two-sides-bet by blast
moreover
have A B C CongA B A C'
  by (meson ‹A C' Par B C› calculation(2) conga-comm invert-two-sides l12-21-a par-symmetry)
moreover
have A C B CongA C A B'
  by (meson ‹A B' Par C B› calculation(1) conga-comm invert-two-sides l12-21-a par-symmetry)
ultimately
show ?thesis
  by blast
qed

lemma cop2-npar--inter:
  assumes Coplanar A B X Y and
    Coplanar A' B' X Y and
    ¬ A B Par A' B'
  shows ∃ P. Col P X Y ∧ (Col P A B ∨ Col P A' B')
proof (cases A B Par X Y)

```

```

case True
thus ?thesis
  by (metis Par-cases par-not-par assms(2) assms(3) cop-npar--inter-exists)
next
case False
thus ?thesis
  using assms(1) cop-npar--inter-exists by blast
qed

lemma not-par-one-not-par:
  assumes  $\neg A \ B \ Par \ A' \ B'$ 
  shows  $\neg A \ B \ Par \ X \ Y \vee \neg A' \ B' \ Par \ X \ Y$ 
  using Par-cases assms par-trans by blast

lemma col-par-par-col:
  assumes Col A B C and
    A B Par A' B' and
    B C Par B' C'
  shows Col A' B' C'
  by (metis Par-cases par-not-par assms(1) assms(2) assms(3) col-par par-id-5 par-neq2)

lemma cop-par-perp--perp:
  assumes A B Par C D and
    A B Perp P Q and
    Coplanar C D P Q
  shows C D Perp P Q
proof -
  have universal-posidonius-postulate
    by (simp add: TarskiPP playfair--universal-posidonius-postulate
      tarski-s-euclid-implies-playfair-s-postulate)
  thus ?thesis
    using assms(1) assms(2) assms(3) perpendicular-transversal-postulate-def
      universal-posidonius-postulate--perpendicular-transversal-postulate by blast
qed

lemma cop4-par-perp2--par:
  assumes A B Par C D and
    A B Perp E F and
    C D Perp G H and
    Coplanar A B E G and
    Coplanar A B E H and
    Coplanar A B F G and
    Coplanar A B F H
  shows E F Par G H
proof -
  have perpendicular-transversal-postulate
    using TarskiPP playfair--universal-posidonius-postulate
      tarski-s-euclid-implies-playfair-s-postulate
      universal-posidonius-postulate--perpendicular-transversal-postulate by blast
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
      assms(7) par-perp-perp-implies-par-perp-2-par
      postulate-of-parallelism-of-perpendicular-transversals-def by blast
qed

lemma par-cong-mid-ts:
  assumes A B ParStrict A' B' and
    Cong A B A' B' and
    A A' TS B B'
  shows  $\exists M. \ M \ Midpoint \ A \ A' \wedge M \ Midpoint \ B \ B'$ 
proof -
  have  $\neg Col \ B \ A \ A'$ 
    by (meson assms(1) col-permutation-4 par-strict-not-col-1)
  obtain M where Col M A A' and Bet B M B'
    using TS-def assms(3) by blast
  have Bet A M A'

```

```

  by (metis NCol-perm ⟨Bet B M B'⟩ ⟨Col M A A'⟩ assms(1) assms(3)
      bet-col l9-18 l9-9-bis par-one-or-two-sides)
moreover
have M Midpoint A A'
  by (metis l7-21 ⟨Bet B M B'⟩ ⟨¬ Col B A A'⟩ assms(1) assms(2) assms(3)
      bet-col calculation l12-20 not-col-permutation-4 par-strict-par ts-distincts)
moreover
have M Midpoint B B'
  by (metis l7-21 ⟨Bet B M B'⟩ ⟨¬ Col B A A'⟩ assms(1) assms(2) assms(3)
      bet-col calculation(1) l12-20 not-col-permutation-4 par-strict-par ts-distincts)
ultimately
show ?thesis by blast
qed

lemma par-cong-mid-os:
  assumes A B ParStrict A' B' and
    Cong A B A' B' and
    A A' OS B B'
  shows ∃ M. M Midpoint A B' ∧ M Midpoint B A'
proof -
  obtain X where X Midpoint A' B
    using MidR-uniq-aux by blast
  obtain B'' where Bet A X B'' and Cong X B'' A X
    using segment-construction by blast
  have X Midpoint A B''
    using Cong-cases Midpoint-def ⟨Bet A X B''⟩ ⟨Cong X B'' A X⟩ by blast
  have ¬ Col A B A'
    using assms(1) par-strict-not-col-1 by auto
  hence A B ParStrict B'' A'
    by (meson Mid-perm ⟨X Midpoint A B''⟩ ⟨X Midpoint A' B⟩ midpoint-par-strict)
  have Col B'' B' A' ∧ Col A' B' A'
  proof -
    have B A Par B' A'
      using assms(1) par-comm par-strict-par by blast
    moreover
    have Col A' B' A'
      by (simp add: col-trivial-3)
    moreover
    have B A Par B'' A'
      by (simp add: ⟨A B ParStrict B'' A'⟩ par-strict-left-comm par-strict-par)
    moreover
    have Col A' B'' A'
      by (simp add: col-trivial-3)
    ultimately
    show ?thesis
      using parallel-uniqueness by blast
  qed
  have Cong A B B'' A'
    using ⟨X Midpoint A' B⟩ ⟨X Midpoint A B''⟩ Mid-cases l7-13 by blast
  have Cong A' B' A' B''
    by (meson ⟨Cong A B B'' A'⟩ assms(2) cong-inner-transitivity not-cong-1243)
  have B' = B'' ∨ A' Midpoint B' B''
    by (meson NCol-perm ⟨Col B'' B' A' ∧ Col A' B' A'⟩ ⟨Cong A' B' A' B''⟩ l7-20-bis)
  have B' = B'' ⟶ (X Midpoint A B' ∧ X Midpoint B A)
    using Mid-cases ⟨X Midpoint A B''⟩ ⟨X Midpoint A' B⟩ by blast
moreover
{
  assume A' Midpoint B' B''
  have A A' OS X B''
  proof -
    have ¬ Col A A' X
      by (metis (full-types) ⟨X Midpoint A B''⟩ ⟨X Midpoint A' B⟩
          ⟨¬ Col A B A'⟩ col2--eq col-permutation-1 midpoint-col
          midpoint-midpoint-col)
    moreover
    have Col A A' A

```

```

    by (simp add: col-trivial-3)
  moreover
  have A Out X B''
    using ⟨Bet A X B''⟩ ⟨X Midpoint A B''⟩ bet-neq12--neq calculation(1)
    midpoint-out not-col-distincts by presburger
  ultimately
  show ?thesis
    by (meson l9-19-R2)
qed
have ¬ Col B'' A A'
  using ⟨A A' OS X B''⟩ col-permutation-1 one-side-not-col124 by blast
hence A A' TS B' B''
  by (metis Midpoint-def not-col-distincts ⟨A' Midpoint B' B''⟩
    ⟨Col B'' B' A' ∧ Col A' B' A'⟩ assms(3) col-permutation-1
    col-permutation-3 cop-nts--os ncop--ncols one-side-chara
    one-side-not-col124)
hence A A' TS X B'
  using l9-2 l9-8-2 one-side-symmetry by (meson ⟨A A' OS X B''⟩)
moreover
have A A' OS X B
  by (metis midpoint-out not-col-distincts ⟨X Midpoint A' B⟩
    ⟨¬ Col A B A'⟩ col-permutation-1 invert-one-side out-one-side)
hence A A' OS X B'
  by (meson assms(3) one-side-transitivity)
ultimately
have False
  using l9-9 by auto
}
ultimately
show ?thesis
  using ⟨B' = B'' ∨ A' Midpoint B' B''⟩ by blast
qed

lemma par-strict-cong-mid:
  assumes A B ParStrict A' B' and
    Cong A B A' B'
  shows ∃ M. (M Midpoint A A' ∧ M Midpoint B B') ∨ (M Midpoint A B' ∧ M Midpoint B A')
proof -
  obtain M1 where M1 Midpoint A A'
    using MidR-uniq-aux by blast
  obtain B'' where M1 Midpoint B B''
    using symmetric-point-construction by blast
  have Col B'' A' B'
    by (metis not-col-distincts ⟨M1 Midpoint A A'⟩ ⟨M1 Midpoint B B''⟩
      assms(1) col-par-par-col mid-par-cong1 par-strict-not-col-1 par-strict-par)
  have Cong A' B' A' B''
    by (meson ⟨M1 Midpoint A A'⟩ ⟨M1 Midpoint B B''⟩ assms(2)
      cong-transitivity l7-13 not-cong-3412)
  obtain M2 where M2 Midpoint A B'
    using MidR-uniq-aux by blast
  obtain A'' where M2 Midpoint B A''
    using symmetric-point-construction by blast
  have A B Par B' A''
    using ⟨M2 Midpoint A B'⟩ ⟨M2 Midpoint B A''⟩ assms(1) par-strict-neq1 sym-par by blast
  hence Col A'' A' B'
    by (meson assms(1) par-id-2 par-right-comm par-strict-par par-symmetry par-trans)
  have Cong A' B' B' A''
    by (metis l7-13 ⟨M2 Midpoint A B'⟩ ⟨M2 Midpoint B A''⟩ assms(2)
      cong-transitivity not-cong-4312)
  {
    assume B' = B''
    hence ∃ M. (M Midpoint A A' ∧ M Midpoint B B') ∨ (M Midpoint A B' ∧ M Midpoint B A')
      using ⟨M1 Midpoint A A'⟩ ⟨M1 Midpoint B B''⟩ by blast
  }
  moreover
  {

```

```

assume  $A' \text{ Midpoint } B' B''$ 
have  $\exists M. (M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee (M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ 
  by (meson par-one-or-two-sides assms(1) assms(2) par-cong-mid-os par-cong-mid-ts)
}
ultimately
show ?thesis
  using l7-20  $\langle \text{Col } B'' A' B' \rangle \langle \text{Cong } A' B' A' B'' \rangle$  not-col-permutation-3 by blast
qed

```

```

lemma par-strict-cong-mid1:
assumes  $A B \text{ ParStrict } A' B'$  and
   $\text{Cong } A B A' B'$ 
shows  $(A A' \text{ TS } B B' \wedge (\exists M. M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B')) \vee$ 
 $(A A' \text{ OS } B B' \wedge (\exists M. M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A'))$ 
by (metis par-one-or-two-sides assms(1) assms(2) par-cong-mid-os par-cong-mid-ts)

```

```

lemma par-cong-mid:
assumes  $A B \text{ Par } A' B'$  and
   $\text{Cong } A B A' B'$ 
shows  $\exists M. (M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee$ 
 $(M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ 
proof –
{
  assume  $A B \text{ ParStrict } A' B'$ 
  hence  $\exists M. (M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee (M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ 
    by (simp add: assms(2) par-strict-cong-mid)
}
moreover
{
  assume  $A \neq B \wedge A' \neq B' \wedge \text{Col } A A' B' \wedge \text{Col } B A' B'$ 
  hence  $\exists M. (M \text{ Midpoint } A A' \wedge M \text{ Midpoint } B B') \vee (M \text{ Midpoint } A B' \wedge M \text{ Midpoint } B A')$ 
    using assms(1) assms(2) calculation col-cong-mid by blast
}
ultimately
show ?thesis
  using Par-def assms(1) by presburger
qed

```

```

lemma ts-cong-par-cong-par:
assumes  $A A' \text{ TS } B B'$  and
   $\text{Cong } A B A' B'$  and
   $A B \text{ Par } A' B'$ 
shows  $\text{Cong } A B' A' B \wedge A B' \text{ Par } A' B$ 
by (metis Cong-cases Par-cases assms(1) assms(2) assms(3) l12-20)

```

```

lemma plgs-cong:
assumes ParallelogramStrict  $A B C D$ 
shows  $\text{Cong } A B C D \wedge \text{Cong } A D B C$ 
proof –
have  $A C \text{ TS } B D \wedge A B \text{ Par } C D \wedge \text{Cong } A B C D$ 
  using ParallelogramStrict-def assms by presburger
thus ?thesis
  by (meson l12-20 not-cong-3421)
qed

```

```

lemma plg-cong:
assumes Parallelogram  $A B C D$ 
shows  $\text{Cong } A B C D \wedge \text{Cong } A D B C$ 
proof –
{
  assume ParallelogramStrict  $A B C D$ 
  hence  $\text{Cong } A B C D \wedge \text{Cong } A D B C$ 
    using plgs-cong by blast
}
moreover
{

```

```

    assume ParallelogramFlat A B C D
    hence Cong A B C D  $\wedge$  Cong A D B C
      using plgf-cong by blast
  }
  ultimately
  show ?thesis
    using Parallelogram-def assms by blast
qed

```

```

lemma rmb-cong:
  assumes Rhombus A B C D
  shows Cong A B B C  $\wedge$  Cong A B C D  $\wedge$  Cong A B D A
proof -
  have Parallelogram A B C D
    by (simp add: Rhombus-Plg assms plg-to-parallelogram)
  hence Cong A B C D  $\wedge$  Cong A D B C
    using plg-cong by blast
  thus ?thesis
    by (meson Rhombus-def assms cong-inner-transitivity
      cong-right-commutativity cong-symmetry)
qed

```

```

lemma rmb-per:
  assumes M Midpoint A C and
    Rhombus A B C D
  shows Per A M D
proof -
  obtain M' where M' Midpoint A C  $\wedge$  M' Midpoint B D
    using Plg-def Rhombus-Plg assms(2) by presburger
  hence M = M'
    using assms(1) l7-17 by blast
  show ?thesis
    using Per-def  $\langle M = M' \rangle \langle M' \text{ Midpoint } A \ C \wedge M' \text{ Midpoint } B \ D \rangle$  assms(2)
      l7-2 not-cong-3421 rmb-cong by blast
qed

```

```

lemma per-rmb:
  assumes Plg A B C D and
    M Midpoint A C and
    Per A M B
  shows Rhombus A B C D
  by (meson Per-cases Rhombus-def assms(1) assms(2) assms(3) not-cong-2134 per-double-cong)

```

```

lemma perp-rmb:
  assumes Plg A B C D and
    A C Perp B D
  shows Rhombus A B C D
proof -
  obtain M where M Midpoint A C
    using MidR-uniq-aux by blast
  obtain M' where M' Midpoint A C  $\wedge$  M' Midpoint B D
    using Plg-def assms(1) by presburger
  hence M = M'
    using  $\langle M \text{ Midpoint } A \ C \rangle$  l7-17 by blast
  have A M Perp B M
  proof -
    have A  $\neq$  M
      using  $\langle M = M' \rangle \langle M' \text{ Midpoint } A \ C \wedge M' \text{ Midpoint } B \ D \rangle$  assms(2)
        is-midpoint-id perp-not-eq-1 by blast
    moreover have A C Perp B M
      by (metis Bet-cases Col-def  $\langle M = M' \rangle \langle M' \text{ Midpoint } A \ C \wedge M' \text{ Midpoint } B \ D \rangle$ 
        assms(2) midpoint-bet midpoint-distinct-3 perp-col1)
    moreover have Col A C M
      using Col-cases  $\langle M \text{ Midpoint } A \ C \rangle$  midpoint-col by blast
  ultimately show ?thesis
    using perp-col by blast

```

qed
 hence $M \text{ PerpAt } A \ M \ M \ B$
 using *Perp-cases Perp-in-cases perp-perp-in* by blast
 hence $\text{Per } A \ M \ B$
 by (simp add: *perp-in-per*)
 thus ?thesis
 using $\langle M \text{ Midpoint } A \ C \rangle \text{ assms}(1) \text{ per-rmb}$ by blast
 qed

lemma *plg-conga1*:
 assumes $A \neq B$ and
 $A \neq C$ and
 $\text{Plg } A \ B \ C \ D$
 shows $B \ A \ C \text{ CongA } D \ C \ A$
 proof –
 have $\text{Parallelogram } A \ B \ C \ D$
 by (simp add: *assms(3) plg-to-parallelogram*)
 moreover have $\text{ParallelogramStrict } A \ B \ C \ D \longrightarrow B \ A \ C \text{ CongA } D \ C \ A$
 by (metis *Cong-cases Tarski-neutral-dimensionless.plgs-diff*
Tarski-neutral-dimensionless-axioms $\langle \text{Parallelogram } A \ B \ C \ D \rangle$
cong-pseudo-reflexivity l11-51 plg-cong)
 moreover have $\text{ParallelogramFlat } A \ B \ C \ D \longrightarrow B \ A \ C \text{ CongA } D \ C \ A$
 by (metis *Tarski-neutral-dimensionless.cong-3421 Tarski-neutral-dimensionless-axioms*
 $\langle \text{Parallelogram } A \ B \ C \ D \rangle \text{ assms}(1) \text{ assms}(2) \text{ cong-diff-2 cong-pseudo-reflexivity}$
conga-trivial-1 l11-51 plg-cong)
 ultimately show ?thesis
 using *Parallelogram-def* by blast
 qed

lemma *os-cong-par-cong-par*:
 assumes $A \ A' \text{ OS } B \ B'$ and
 $\text{Cong } A \ B \ A' \ B'$ and
 $A \ B \text{ Par } A' \ B'$
 shows $\text{Cong } A \ A' \ B \ B' \wedge A \ A' \text{ Par } B \ B'$
 by (metis (full-types) *Par-cases Par-def not-col-distincts assms(1)*
assms(2) assms(3) cong-right-commutativity mid-par-cong2
one-side-not-col124 par-cong-mid-os)

lemma *plgs-permut*:
 assumes $\text{ParallelogramStrict } A \ B \ C \ D$
 shows $\text{ParallelogramStrict } B \ C \ D \ A$
 by (meson *Mid-perm ParallelogramStrict-def assms mid-plgs par-cong-mid-ts*
parallelogram-strict-not-col-2 plgs--pars)

lemma *plg-permut*:
 assumes $\text{Parallelogram } A \ B \ C \ D$
 shows $\text{Parallelogram } B \ C \ D \ A$
 proof –
 have $\text{ParallelogramStrict } A \ B \ C \ D \longrightarrow \text{Parallelogram } B \ C \ D \ A$
 using *Parallelogram-def plgs-permut* by blast
 thus ?thesis
 using *Parallelogram-def assms plgf-permut* by blast
 qed

lemma *plgs-mid*:
 assumes $\text{ParallelogramStrict } A \ B \ C \ D$
 shows $\exists M. M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$
 using *ParallelogramStrict-def assms par-cong-mid-ts plgs--pars* by blast

lemma *plg-mid*:
 assumes $\text{Parallelogram } A \ B \ C \ D$
 shows $\exists M. M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$
 proof –
 have $\text{ParallelogramStrict } A \ B \ C \ D \longrightarrow (\exists M. M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D)$
 using *plgs-mid* by blast
 thus ?thesis


```

    using plgf-mid Parallelogram-def assms by blast
qed

lemma plg-mid-2:
  assumes Parallelogram A B C D and
    I Midpoint A C
  shows I Midpoint B D
  using MidR-uniq-aux assms(1) assms(2) plg-mid by fastforce

lemma plgs-not-comm-1:
  assumes ParallelogramStrict A B C D
  shows  $\neg$  ParallelogramStrict A B D C
  by (meson ParallelogramStrict-def l9-9-bis assms plgs-one-side plgs-permut)

lemma plgs-not-comm-2:
  assumes ParallelogramStrict A B C D
  shows  $\neg$  ParallelogramStrict B A C D
  by (meson assms plgs-not-comm-1 plgs-sym)

lemma plgs-not-comm:
  assumes ParallelogramStrict A B C D
  shows  $\neg$  ParallelogramStrict A B D C  $\wedge$   $\neg$  ParallelogramStrict B A C D
  using assms plgs-not-comm-1 plgs-not-comm-2 by blast

lemma plg-not-comm-R1:
  assumes A  $\neq$  B and
    Parallelogram A B C D
  shows  $\neg$  Parallelogram A B D C
proof -
  {
    assume ParallelogramStrict A B C D
    {
      assume ParallelogramStrict A B D C
      have False
        using plgs-not-comm-1
        by (meson  $\langle$ ParallelogramStrict A B C D $\rangle$   $\langle$ ParallelogramStrict A B D C $\rangle$ )
    }
    moreover
    {
      assume ParallelogramFlat A B D C
      have C D Par A B
        using Par-cases ParallelogramStrict-def  $\langle$ ParallelogramStrict A B C D $\rangle$  by blast
      moreover
      have C D ParStrict A B  $\implies$  False
        by (meson  $\langle$ ParallelogramFlat A B D C $\rangle$   $\langle$ ParallelogramStrict A B C D $\rangle$ 
           $\langle$ ParallelogramStrict A B D C  $\implies$  False $\rangle$  mid-plgs parallelogram-strict-not-col-4
          plgf-mid)
      moreover
      have C  $\neq$  D  $\wedge$  A  $\neq$  B  $\wedge$  Col C A B  $\wedge$  Col D A B  $\implies$  False
        using  $\langle$ ParallelogramStrict A B C D $\rangle$  calculation(2) plgs--pars plgs-sym by blast
      ultimately
      have False
        using Par-def by blast
    }
    ultimately
    have  $\neg$  Parallelogram A B D C
      using Parallelogram-def by blast
  }
  moreover
  {
    assume ParallelogramFlat A B C D
    hence  $\neg$  ParallelogramFlat A B D C  $\wedge$   $\neg$  ParallelogramFlat B A C D
      using plgf-not-comm assms(1) by blast
    hence  $\neg$  ParallelogramFlat A B D C
      by blast
    moreover
  }

```

```

have  $\neg$  ParallelogramFlat  $B A C D$ 
  by (simp add:  $\langle \neg$  ParallelogramFlat  $A B D C \wedge \neg$  ParallelogramFlat  $B A C D \rangle$ )
{
  assume ParallelogramStrict  $A B D C$ 
  have  $A D$  TS  $B C$ 
    using ParallelogramStrict-def  $\langle$ ParallelogramStrict  $A B D C \rangle$  by auto
  hence False
    by (meson TS-def  $\langle$ ParallelogramFlat  $A B C D \rangle$   $\langle$ ParallelogramStrict  $A B D C \rangle$ 
        mid-plgs plgf-mid plgs-not-comm)
}
ultimately
have  $\neg$  Parallelogram  $A B D C$ 
  using Parallelogram-def by blast
}
ultimately
show ?thesis
  using Parallelogram-def assms(2) by blast
qed

```

```

lemma plg-not-comm-R2:
  assumes  $A \neq B$  and
    Parallelogram  $A B C D$ 
  shows  $\neg$  Parallelogram  $B A C D$ 
  by (metis assms(1) assms(2) plg-not-comm-R1 plg-sym)

```

```

lemma plg-not-comm:
  assumes  $A \neq B$  and
    Parallelogram  $A B C D$ 
  shows  $\neg$  Parallelogram  $A B D C \wedge \neg$  Parallelogram  $B A C D$ 
  by (simp add: assms(1) assms(2) plg-not-comm-R1 plg-not-comm-R2)

```

```

lemma parallelogram-to-plg:
  assumes Parallelogram  $A B C D$ 
  shows Plg  $A B C D$ 
  using Plg-def assms plg-irreflexive plg-mid by force

```

```

lemma parallelogram-equiv-plg:
  shows Parallelogram  $A B C D \longleftrightarrow$  Plg  $A B C D$ 
  using parallelogram-to-plg plg-to-parallelogram by blast

```

```

lemma plg-conga:
  assumes  $A \neq B$  and
     $B \neq C$  and
    Parallelogram  $A B C D$ 
  shows  $A B C$  CongA  $C D A \wedge B C D$  CongA  $D A B$ 
proof
  show  $A B C$  CongA  $C D A$ 
    by (metis assms(1) assms(2) assms(3) cong-pseudo-reflexivity
        conga-trivial-1 l11-51 plg-cong plg-not-comm-R2 plg-permut)
  show  $B C D$  CongA  $D A B$ 
    by (metis Cong-cases Tarski-neutral-dimensionless.conga-distinct
        Tarski-neutral-dimensionless-axioms  $\langle A B C$  CongA  $C D A \rangle$  assms(3)
        cong-pseudo-reflexivity conga-trivial-1 l11-51 plg-cong)
qed

```

```

lemma half-plgs-R1:
  assumes ParallelogramStrict  $A B C D$  and
     $P$  Midpoint  $A B$  and
     $Q$  Midpoint  $C D$  and
     $M$  Midpoint  $A C$ 
  shows  $P Q$  Par  $A D \wedge$  Cong  $A D P Q$ 
proof -
  obtain  $M'$  where  $M'$  Midpoint  $A C$  and  $M'$  Midpoint  $B D$ 
    using assms(1) plgs-mid by blast
  have  $M = M'$ 

```

```

    using MidR-uniq-aux ⟨M' Midpoint A C⟩ assms(4) by blast
  obtain Q' where Bet P M Q' and Cong M Q' P M
    using segment-construction by blast
  have M Midpoint P Q'
    using Cong-perm ⟨Bet P M Q'⟩ ⟨Cong M Q' P M⟩ midpoint-def by blast
  hence Q' Midpoint C D
    using ⟨M = M'⟩ ⟨M' Midpoint A C⟩ ⟨M' Midpoint B D⟩ assms(2)
      symmetry-preserves-midpoint by blast
  have Q = Q'
    using ⟨Q' Midpoint C D⟩ assms(3) l7-17 by auto
  have A B Par C D
    using ParallelogramStrict-def assms(1) by force
  have Cong A B C D
    using assms(1) plgs-cong by force
  hence Cong A P D Q
    by (metis ⟨Q = Q'⟩ ⟨Q' Midpoint C D⟩ assms(2) cong-cong-half-1 l7-2 not-cong-1243)
  have A P Par Q D
  proof -
    have A ≠ P
      using ⟨A B Par C D⟩ assms(2) is-midpoint-id par-neq1 by blast
    moreover have Col A B P
      using assms(2) col-permutation-1 midpoint-col by blast
    moreover have A B Par Q D
      by (metis ⟨A B Par C D⟩ ⟨Q = Q'⟩ ⟨Q' Midpoint C D⟩ col-permutation-5
        col-transitivity-1 midpoint-col midpoint-distinct-1 par-col2-par-bis)
    ultimately show ?thesis
      using par-col-par-2 by blast
  qed
  have Cong A D P Q ∧ A D Par P Q
  proof -
    have A D OS P B
      by (metis ⟨A B Par C D⟩ assms(1) assms(2) bet-out-1 between-symmetry
        midpoint-bet midpoint-distinct-1 not-col-permutation-5 one-side-not-col124
        out-one-side par-neq1 plgs-one-side)
    have A D OS B C
      using assms(1) invert-one-side plgs-one-side plgs-permut by blast
    have A D OS P Q
    proof -
      have A D OS P B
        by (simp add: ⟨A D OS P B⟩)
      moreover have A D OS B Q
        by (metis Out-cases ⟨A D OS B C⟩ ⟨Q = Q'⟩ ⟨Q' Midpoint C D⟩
          bet-out-1 col-trivial-2 midpoint-bet midpoint-distinct-1
          one-side-symmetry os-out-os)
      ultimately show ?thesis
        using one-side-transitivity by blast
    qed
    moreover
    have A D Par Q P
      by (simp add: ⟨A P Par Q D⟩ ⟨Cong A P D Q⟩ calculation
        os-cong-par-cong-par par-right-comm)
    ultimately
    show ?thesis
      by (simp add: ⟨A P Par Q D⟩ ⟨Cong A P D Q⟩ os-cong-par-cong-par par-right-comm)
  qed
  thus ?thesis
    by (simp add: par-symmetry)
  qed

lemma half-plgs-R2:
  assumes ParallelogramStrict A B C D and
    P Midpoint A B and
    Q Midpoint C D and
    M Midpoint A C
  shows M Midpoint P Q
  proof -

```

```

obtain  $M'$  where  $M'$  Midpoint  $A$   $C$  and  $M'$  Midpoint  $B$   $D$ 
  using  $assms(1)$   $plgs$ -mid by blast
have  $M = M'$ 
  using MidR-uniq-aux  $\langle M'$  Midpoint  $A$   $C \rangle$   $assms(4)$  by blast
obtain  $Q'$  where  $Bet$   $P$   $M$   $Q'$  and  $Cong$   $M$   $Q'$   $P$   $M$ 
  using segment-construction by blast
have  $M$  Midpoint  $P$   $Q'$ 
  using Cong-perm  $\langle Bet$   $P$   $M$   $Q' \rangle$   $\langle Cong$   $M$   $Q'$   $P$   $M \rangle$  midpoint-def by blast
hence  $Q'$  Midpoint  $C$   $D$ 
  using  $\langle M = M' \rangle$   $\langle M'$  Midpoint  $A$   $C \rangle$   $\langle M'$  Midpoint  $B$   $D \rangle$   $assms(2)$ 
    symmetry-preserves-midpoint by blast
have  $Q = Q'$ 
  using  $\langle Q'$  Midpoint  $C$   $D \rangle$   $assms(3)$  l7-17 by auto
thus ?thesis
  by (simp add:  $\langle M$  Midpoint  $P$   $Q' \rangle$   $\langle Q = Q' \rangle$ )
qed

lemma half-plgs:
  assumes ParallelogramStrict  $A$   $B$   $C$   $D$  and
     $P$  Midpoint  $A$   $B$  and
     $Q$  Midpoint  $C$   $D$  and
     $M$  Midpoint  $A$   $C$ 
  shows  $P$   $Q$  Par  $A$   $D$   $\wedge$   $M$  Midpoint  $P$   $Q$   $\wedge$   $Cong$   $A$   $D$   $P$   $Q$ 
  by (meson half-plgs-R1 Tarski-Euclidean-axioms  $assms(1)$   $assms(2)$   $assms(3)$ 
     $assms(4)$  half-plgs-R2)

lemma plgs-two-sides:
  assumes ParallelogramStrict  $A$   $B$   $C$   $D$ 
  shows  $A$   $C$  TS  $B$   $D$   $\wedge$   $B$   $D$  TS  $A$   $C$ 
  using ParallelogramStrict-def  $assms$  col-not-plgs l12-19 l12-20 by blast

lemma plgs-par-strict:
  assumes ParallelogramStrict  $A$   $B$   $C$   $D$ 
  shows  $A$   $B$  ParStrict  $C$   $D$   $\wedge$   $A$   $D$  ParStrict  $B$   $C$ 
  using Par-strict-perm  $assms$  plgs--pars plgs-permut by blast

lemma plgs-half-plgs-aux:
  assumes ParallelogramStrict  $A$   $B$   $C$   $D$  and
     $P$  Midpoint  $A$   $B$  and
     $Q$  Midpoint  $C$   $D$ 
  shows ParallelogramStrict  $A$   $P$   $Q$   $D$ 
proof –
  obtain  $M$  where  $M$  Midpoint  $A$   $C$  and  $M$  Midpoint  $B$   $D$ 
    using  $assms(1)$   $plgs$ -mid by blast
  have  $P$   $Q$  Par  $A$   $D$   $\wedge$   $M$  Midpoint  $P$   $Q$   $\wedge$   $Cong$   $A$   $D$   $P$   $Q$ 
    by (meson  $\langle M$  Midpoint  $A$   $C \rangle$   $assms(1)$   $assms(2)$   $assms(3)$  half-plgs-R1 half-plgs-R2)
  obtain  $N$  where  $(N$  Midpoint  $P$   $A$   $\wedge$   $N$  Midpoint  $Q$   $D) \vee (N$  Midpoint  $P$   $D$   $\wedge$   $N$  Midpoint  $Q$   $A)$ 
    using  $\langle P$   $Q$  Par  $A$   $D$   $\wedge$   $M$  Midpoint  $P$   $Q$   $\wedge$   $Cong$   $A$   $D$   $P$   $Q \rangle$  not-cong-3412
    par-cong-mid by blast
  {
    assume  $N$  Midpoint  $P$   $A$  and  $N$  Midpoint  $Q$   $D$ 
    have Col  $A$   $P$   $B$ 
      by (simp add:  $assms(2)$  bet-col midpoint-bet)
    have Col  $A$   $P$   $N$ 
      by (simp add: Col-def  $\langle N$  Midpoint  $P$   $A \rangle$  midpoint-bet)
    have Col  $N$   $A$   $B$ 
      by (metis NCol-cases  $\langle Col$   $A$   $P$   $B \rangle$   $\langle Col$   $A$   $P$   $N \rangle$   $assms(2)$  col-transitivity-1
        midpoint-distinct-1)
    moreover
    have Col  $N$   $C$   $D$ 
      by (meson  $\langle N$  Midpoint  $Q$   $D \rangle$   $assms(3)$  bet-col between-exchange2 midpoint-bet
        not-col-permutation-4)
    ultimately
    have False
      by (meson Midpoint-def  $\langle N$  Midpoint  $Q$   $D \rangle$   $assms(1)$   $assms(3)$  between-exchange2
        one-side-chara plgs-one-side)
  }

```

```

  hence ParallelogramStrict A P Q D
    by blast
}
moreover
{
  assume N Midpoint P D and N Midpoint Q A
  have  $A \neq P$ 
    by (metis assms(1) assms(2) midpoint-distinct-1 plgs-not-comm-2)
  have  $D \neq Q$ 
    using  $\langle A \neq P \rangle \langle N \text{ Midpoint } P D \rangle \langle N \text{ Midpoint } Q A \rangle$  l7-9-bis by blast
  have  $\neg \text{Col } A P Q$ 
    by (metis one-side-chara assms(1) assms(2) assms(3) col-trivial-2
      colx midpoint-bet midpoint-col midpoint-distinct-1
      not-col-permutation-1 plgs-one-side)
  moreover
  have N Midpoint A Q
    using Mid-cases  $\langle N \text{ Midpoint } Q A \rangle$  by blast
  ultimately
  have ParallelogramStrict A P Q D
    using  $\langle N \text{ Midpoint } P D \rangle$  mid-plgs by blast
}
ultimately
show ?thesis
  using  $\langle N \text{ Midpoint } P A \wedge N \text{ Midpoint } Q D \vee N \text{ Midpoint } P D \wedge N \text{ Midpoint } Q A \rangle$  by fastforce
qed

```

```

lemma plgs-comm2:
  assumes ParallelogramStrict A B C D
  shows ParallelogramStrict B A D C
  by (meson ParallelogramStrict-def par-comm assms cong-commutativity l12-20-bis)

```

```

lemma plgf-comm2:
  assumes ParallelogramFlat A B C D
  shows ParallelogramFlat B A D C
  by (metis Col-cases ParallelogramFlat-def plgf-mid assms mid-plgf-2 mid-plgf-aux)

```

```

lemma plg-comm2:
  assumes Parallelogram A B C D
  shows Parallelogram B A D C
  by (metis assms mid-plg plg-mid plg-permut)

```

```

lemma par-preserves-conga-os:
  assumes A B Par C D and
    Bet A D P and
     $D \neq P$  and
    A D OS B C
  shows B A P CongA C D P
proof -
  obtain T where A D TS B T and A D TS C T
    using OS-def assms(4) by blast
  obtain C' where Bet C D C' and Cong D C' C D
    using segment-construction by blast
  have  $C' \neq D$ 
    using  $\langle \text{Cong } D C' C D \rangle$  assms(1) cong-reverse-identity par-neq2 by blast
  have B A D CongA C' D A
proof -
  have A D TS B C'
    by (metis OS-def  $\langle \text{Bet } C D C' \rangle \langle C' \neq D \rangle$ 
       $\langle \bigwedge \text{thesis. } (\bigwedge T. \llbracket A D TS B T; A D TS C T \rrbracket \implies \text{thesis}) \implies \text{thesis} \rangle$ 
      bet-ts col-trivial-3 invert-two-sides l9-18 l9-8-2)
  moreover
  have A B Par D C'
    using  $\langle \text{Bet } C D C' \rangle$  assms(1) bet-col bet-col1 calculation par-col2-par
      ts-distincts by presburger
  ultimately
  show ?thesis

```

```

    using l12-21-a by blast
qed
have A ≠ B
  using assms(1) par-neq1 by auto
have A ≠ D
  using ⟨A D TS B T⟩ ts-distincts by blast
have A Out P D
  using ⟨A ≠ D⟩ assms(2) bet-out l6-6 by presburger
hence B A D CongA B A P
  by (metis CongA-def ⟨B A D CongA C' D A⟩ bet2--out out2--conga)
have C D P CongA C' D A
  by (metis ⟨A ≠ D⟩ ⟨Bet C D C'⟩ ⟨C' ≠ D⟩ ⟨Cong D C' C D⟩ assms(2)
    assms(3) between-symmetry cong-diff-2 l11-14)
hence B A P CongA P D C
  by (metis ⟨B A D CongA B A P⟩ ⟨B A D CongA C' D A⟩ conga-right-comm
    not-conga not-conga-sym)
thus ?thesis
  using conga-right-comm by blast
qed

lemma cong3-par2-par:
  assumes A ≠ C and
    B A C Cong3 B' A' C' and
    B A Par B' A' and
    B C Par B' C'
  shows A C Par A' C' ∨ ¬ B B' ParStrict A A' ∨ ¬ B B' ParStrict C C'
proof -
  have A ≠ B
    using assms(3) par-neq1 by blast
  have C ≠ B
    using assms(4) par-neq1 by blast
  obtain M where M Midpoint B B'
    using MidR-uniq-aux by blast
  obtain A'' where Bet A M A'' and Cong M A'' A M
    using segment-construction by blast
  obtain C'' where Bet C M C'' and Cong M C'' C M
    using segment-construction by blast
  have M Midpoint A A''
    by (meson Cong-cases Midpoint-def ⟨Bet A M A''⟩ ⟨Cong M A'' A M⟩)
  have M Midpoint C C''
    using Cong-cases ⟨Bet C M C''⟩ ⟨Cong M C'' C M⟩ midpoint-def by blast
  have B A Par B' A''
    using ⟨A ≠ B⟩ ⟨M Midpoint A A''⟩ ⟨M Midpoint B B'⟩ sym-par by auto
  have B C Par B' C''
    using ⟨C ≠ B⟩ ⟨M Midpoint B B'⟩ ⟨M Midpoint C C''⟩ l12-17 by force
  have A C Par A'' C''
    using ⟨M Midpoint A A''⟩ ⟨M Midpoint C C''⟩ assms(1) sym-par by blast
  have B' A' Par B' A''
    using Par-cases ⟨B A Par B' A''⟩ assms(3) par-trans by blast
  have B' C' Par B' C''
    by (meson ⟨B C Par B' C''⟩ assms(4) par-not-par par-symmetry)
  have Col B' A' A''
    using ⟨B' A' Par B' A''⟩ par-id by auto
  have Col B' C' C''
    using ⟨B' C' Par B' C''⟩ par-id by auto
  have Cong B A B' A''
    using ⟨A ≠ B⟩ ⟨M Midpoint A A''⟩ ⟨M Midpoint B B'⟩ mid-par-cong1 by auto
  have Cong B C B' C''
    using ⟨C ≠ B⟩ ⟨M Midpoint B B'⟩ ⟨M Midpoint C C''⟩ mid-par-cong1 by auto
  have A' = A'' ∨ B' Midpoint A' A''
proof -
  have Col A' B' A''
    using Col-cases ⟨Col B' A' A''⟩ by auto
  moreover
  have Cong B' A' B' A''
    by (meson Cong3-def ⟨Cong B A B' A''⟩ assms(2) cong-inner-transitivity)

```

```

ultimately
show ?thesis
  by (simp add: l7-20)
qed
have  $C' = C'' \vee B' \text{ Midpoint } C' C''$ 
proof -
  have  $\text{Col } C' B' C''$ 
  by (simp add:  $\langle B' C' \text{ Par } B' C'' \rangle$  par-id-1)
  moreover
  have  $\text{Cong } B' C' B' C''$ 
  using Cong3-def  $\langle \text{Cong } B C B' C'' \rangle$  assms(2) cong-inner-transitivity by blast
  ultimately
  show ?thesis
  by (simp add: l7-20)
qed
{
  assume  $A' = A''$ 
  {
    assume  $C' = C''$ 
    hence  $A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C'$ 
    by (simp add:  $\langle A C \text{ Par } A'' C'' \rangle \langle A' = A'' \rangle$ )
  }
  moreover
  {
    assume  $B' \text{ Midpoint } C' C''$ 
    hence  $A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C'$ 
    by (metis Midpoint-def  $\langle A' = A'' \rangle \langle \text{Bet } A M A'' \rangle \langle M \text{ Midpoint } B B' \rangle$ 
        between-trivial impossible-case-3 par-strict-not-col-4)
  }
  ultimately
  have  $A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C'$ 
  using  $\langle C' = C'' \vee B' \text{ Midpoint } C' C'' \rangle$  by blast
}
moreover
{
  assume  $B' \text{ Midpoint } A' A''$ 
  {
    assume  $C' = C''$ 
    hence  $A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C'$ 
    by (meson  $\langle M \text{ Midpoint } B B' \rangle \langle M \text{ Midpoint } C C'' \rangle$  midpoint-col par-not-col)
  }
  moreover
  {
    assume  $B' \text{ Midpoint } C' C''$ 
    hence  $A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C'$ 
    by (metis Cong3-def  $\langle A C \text{ Par } A'' C'' \rangle$ 
         $\langle A' = A'' \implies A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C' \rangle$ 
         $\langle A' = A'' \vee B' \text{ Midpoint } A' A'' \rangle$  assms(1) assms(2) cong-identity
        not-par-one-not-par sym-par)
  }
  ultimately
  have  $A C \text{ Par } A' C' \vee \neg B B' \text{ ParStrict } A A' \vee \neg B B' \text{ ParStrict } C C'$ 
  using  $\langle C' = C'' \vee B' \text{ Midpoint } C' C'' \rangle$  by blast
}
ultimately
show ?thesis
  using  $\langle A' = A'' \vee B' \text{ Midpoint } A' A'' \rangle$  by blast
qed

lemma square-perp-rectangle:
  assumes Rectangle A B C D and
    A C Perp B D
  shows Square A B C D
  by (simp add: Rectangle-Plg Rhombus-Rectangle-Square assms(1) assms(2) perp-rmb)

lemma plgs-half-plgs:

```

assumes *ParallelogramStrict* $A\ B\ C\ D$ **and**
 P *Midpoint* $A\ B$ **and**
 Q *Midpoint* $C\ D$
shows *ParallelogramStrict* $A\ P\ Q\ D \wedge \text{ParallelogramStrict } B\ P\ Q\ C$
by (*meson* *assms*(1) *assms*(2) *assms*(3) *l7-2* *plgs-comm2* *plgs-half-plgs-aux*)

lemma *parallel-2-plg*:
assumes $\neg \text{Col } A\ B\ C$ **and**
 $A\ B\ \text{Par } C\ D$ **and**
 $A\ D\ \text{Par } B\ C$
shows *ParallelogramStrict* $A\ B\ C\ D$
by (*metis* *Par-cases* *ParallelogramStrict-def* *assms*(1) *assms*(2) *assms*(3) *l12-19*)

lemma *par-2-plg*:
assumes $\neg \text{Col } A\ B\ C$ **and**
 $A\ B\ \text{Par } C\ D$ **and**
 $A\ D\ \text{Par } B\ C$
shows *Parallelogram* $A\ B\ C\ D$
by (*metis* *Par-cases* *ParallelogramStrict-def* *Parallelogram-def*
assms(1) *assms*(2) *assms*(3) *l12-19*)

lemma *plg-cong-1*:
assumes *Parallelogram* $A\ B\ C\ D$
shows *Cong* $A\ B\ C\ D$
by (*simp* *add*: *assms* *plg-cong*)

lemma *plg-cong-2*:
assumes *Parallelogram* $A\ B\ C\ D$
shows *Cong* $A\ D\ B\ C$
using *assms* *plg-cong* **by** *auto*

lemma *plgs-cong-1*:
assumes *ParallelogramStrict* $A\ B\ C\ D$
shows *Cong* $A\ B\ C\ D$
by (*simp* *add*: *assms* *plgs-cong*)

lemma *plgs-cong-2*:
assumes *ParallelogramStrict* $A\ B\ C\ D$
shows *Cong* $A\ D\ B\ C$
by (*simp* *add*: *assms* *plgs-cong*)

lemma *Plg-perm*:
assumes *Parallelogram* $A\ B\ C\ D$
shows *Parallelogram* $A\ B\ C\ D \wedge \text{Parallelogram } B\ C\ D\ A \wedge$
 $\text{Parallelogram } C\ D\ A\ B \wedge \text{Parallelogram } D\ A\ B\ C \wedge$
 $\text{Parallelogram } A\ D\ C\ B \wedge \text{Parallelogram } D\ C\ B\ A \wedge$
 $\text{Parallelogram } C\ B\ A\ D \wedge \text{Parallelogram } B\ A\ D\ C$
by (*meson* *assms* *plg-comm2* *plg-permut*)

lemma *plg-not-comm-1*:
assumes $A \neq B$ **and**
 $\text{Parallelogram } A\ B\ C\ D$
shows $\neg \text{Parallelogram } A\ B\ D\ C$
by (*simp* *add*: *assms*(1) *assms*(2) *plg-not-comm-R1*)

lemma *plg-not-comm-2*:
assumes $A \neq B$ **and**
 $\text{Parallelogram } A\ B\ C\ D$
shows $\neg \text{Parallelogram } B\ A\ C\ D$
using *assms*(1) *assms*(2) *plg-not-comm-R2* **by** *blast*

lemma *parallelogram-strict-midpoint*:
assumes *ParallelogramStrict* $A\ B\ C\ D$ **and**
 $\text{Col } I\ A\ C$ **and**
 $\text{Col } I\ B\ D$

shows $I \text{ Midpoint } A \ C \wedge I \text{ Midpoint } B \ D$
by (*metis not-col-distincts* *assms(1)* *assms(2)* *assms(3)* *col-not-plgs*
col-permutation-4 *l7-21* *not-cong-4312* *parallelogram-strict-not-col-2* *plgs-cong*)

lemma *rmb-perp*:
assumes $A \neq C$ **and**
 $B \neq D$ **and**
Rhombus $A \ B \ C \ D$
shows $A \ C \text{ Perp } B \ D$

proof –
obtain M **where** $M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$
using *Plg-def Rhombus-Plg* *assms(3)* **by** *presburger*
hence $M \text{ Midpoint } A \ C$
by *simp*
hence $Per \ A \ M \ D$
using *assms(3)* *rmb-per* **by** *blast*
hence $A \ M \text{ Perp } M \ D$
by (*metis* $\langle M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D \rangle$ *assms(1)* *assms(2)*
cong-identity *l7-2* *l8-5* *per-double-cong* *per-perp*)
have $M \text{ Midpoint } B \ D$
using $\langle M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D \rangle$ **by** *blast*
have $A \neq M$
using $\langle M \text{ Midpoint } A \ C \rangle$ *assms(1)* *midpoint-distinct-1* **by** *blast*
moreover
have $M \neq D$
using $\langle M \text{ Midpoint } B \ D \rangle$ *assms(2)* *midpoint-not-midpoint* **by** *auto*
moreover
have $Col \ D \ M \ B$
using $\langle M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D \rangle$ *col-permutation-2* *midpoint-col* **by** *blast*
moreover
have $Col \ A \ M \ C$
by (*simp* *add*: $\langle M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D \rangle$ *bet-col* *midpoint-bet*)
ultimately
show *?thesis*
by (*meson* *perp-col4* $\langle A \ M \text{ Perp } M \ D \rangle$ *assms(1)* *assms(2)* *col-trivial-3* *perp-right-comm*)
qed

lemma *rect-permut*:
assumes *Rectangle* $A \ B \ C \ D$
shows *Rectangle* $B \ C \ D \ A$

proof –
have *Plg* $B \ C \ D \ A$
by (*meson* *Rectangle-Plg* *assms* *parallelogram-to-plg* *plg-permut* *plg-to-parallelogram*)
moreover
have *Cong* $B \ D \ C \ A$
using *Rectangle-def* *assms* *cong-3421* **by** *blast*
ultimately
show *?thesis*
by (*simp* *add*: *Rectangle-def*)
qed

lemma *rect-comm2*:
assumes *Rectangle* $A \ B \ C \ D$
shows *Rectangle* $B \ A \ D \ C$
by (*metis* *Plg-def* *Rectangle-def* *assms* *cong-symmetry*)

lemma *rect-per1*:
assumes *Rectangle* $A \ B \ C \ D$
shows $Per \ B \ A \ D$

proof –
obtain P **where** $P \text{ Midpoint } A \ B$
using *MidR-uniq-aux* **by** *blast*
obtain Q **where** $Q \text{ Midpoint } C \ D$
using *MidR-uniq-aux* **by** *blast*
obtain M **where** $M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$
using *Plg-def* *Rectangle-Plg* *assms* **by** *presburger*

```

have M Midpoint A C
  by (simp add: ⟨M Midpoint A C ∧ M Midpoint B D⟩)
have M Midpoint B D
  by (simp add: ⟨M Midpoint A C ∧ M Midpoint B D⟩)
have Parallelogram A B C D
  by (simp add: Rectangle-Parallelogram assms)
moreover
{
  assume ParallelogramStrict A B C D
  have P Q Par A D ∧ M Midpoint P Q ∧ Cong A D P Q
    by (meson half-plgs-R1 Tarski-Euclidean-axioms ⟨M Midpoint A C⟩
      ⟨P Midpoint A B⟩ ⟨ParallelogramStrict A B C D⟩ ⟨Q Midpoint C D⟩ half-plgs-R2)
  have P Q Par A D
    by (simp add: ⟨P Q Par A D ∧ M Midpoint P Q ∧ Cong A D P Q⟩)
  have M Midpoint P Q
    by (simp add: ⟨P Q Par A D ∧ M Midpoint P Q ∧ Cong A D P Q⟩)
  have Cong A D P Q
    by (simp add: ⟨P Q Par A D ∧ M Midpoint P Q ∧ Cong A D P Q⟩)
  have Per A P Q
  proof —
    have Col P M Q
      using ⟨M Midpoint P Q⟩ bet-col midpoint-bet by blast
    moreover
    have Cong M A M B
      by (meson Rectangle-def ⟨M Midpoint A C ∧ M Midpoint B D⟩
        assms cong-commutativity cong-cong-half-1)
    hence Per M P A
      using ⟨P Midpoint A B⟩ Per-def by blast
    hence Per A P M
      using l8-2 by blast
    ultimately
    show ?thesis
      by (metis ⟨M Midpoint P Q⟩ l8-4 per-col)
  qed
  have A ≠ B
    using ⟨ParallelogramStrict A B C D⟩ plgs-diff by blast
  hence A ≠ P
    using ⟨P Midpoint A B⟩ midpoint-distinct-1 by blast

  have A D Perp A B
  proof —
    have P Q Perp A B
      by (metis Perp-cases ⟨A ≠ B⟩ ⟨A ≠ P⟩ ⟨P Midpoint A B⟩
        ⟨P Q Par A D ∧ M Midpoint P Q ∧ Cong A D P Q⟩ ⟨Per A P Q⟩
        col-per-perp cong-identity l8-2 midpoint-col par-neq2)
    moreover
    have Coplanar A D A B
      using ncop-distincts by blast
    ultimately
    show ?thesis
      using ⟨P Q Par A D⟩ cop-par-perp--perp by blast
  qed
  hence A B Perp D A
    using Perp-perm by blast
  hence Per B A D
    using perp-per-1 by force
}
moreover
{
  assume ParallelogramFlat A B C D
  hence Col A B C ∧ Col A B D ∧ Cong A D C B ∧ ((A ≠ C) ∨ (B ≠ D))
    using ParallelogramFlat-def by blast
  hence (A = B ∧ C = D) ∨ (A = D ∧ C = B)
  proof —
    have Col A C B
      by (simp add: ⟨Col A B C ∧ Col A B D ∧ Cong A D C B ∧ (A ≠ C ∨ B ≠ D)⟩)

```

```

      col-permutation-5)
    moreover
    have  $M$  Midpoint  $A$   $C$ 
      using  $\langle M$  Midpoint  $A$   $C \wedge M$  Midpoint  $B$   $D \rangle$  by auto
    moreover
    have  $M$  Midpoint  $B$   $D$ 
      by (simp add:  $\langle M$  Midpoint  $A$   $C \wedge M$  Midpoint  $B$   $D \rangle$ )
    moreover
    have  $Cong$   $A$   $C$   $B$   $D$ 
      using Rectangle-def assms by blast
    ultimately
    show ?thesis
      using midpoint-cong-uniqueness by blast
  qed
  hence  $Per$   $B$   $A$   $D$ 
    using l8-20-1-R1 l8-5 by presburger
}
ultimately
show ?thesis
  using Parallelogram-def by blast
qed

lemma rect-per2:
  assumes Rectangle  $A$   $B$   $C$   $D$ 
  shows  $Per$   $A$   $B$   $C$ 
  using assms rect-comm2 rect-per1 by blast

lemma rect-per3:
  assumes Rectangle  $A$   $B$   $C$   $D$ 
  shows  $Per$   $B$   $C$   $D$ 
  using assms rect-per2 rect-permut by blast

lemma rect-per4:
  assumes Rectangle  $A$   $B$   $C$   $D$ 
  shows  $Per$   $A$   $D$   $C$ 
  by (meson assms rect-comm2 rect-per3)

lemma plg-per-rect1:
  assumes Plg  $A$   $B$   $C$   $D$  and
     $Per$   $D$   $A$   $B$ 
  shows Rectangle  $A$   $B$   $C$   $D$ 
proof -
  obtain  $P$  where  $P$  Midpoint  $A$   $B$ 
    using MidR-uniq-aux by blast
  obtain  $Q$  where  $Q$  Midpoint  $C$   $D$ 
    using MidR-uniq-aux by blast
  obtain  $M$  where  $M$  Midpoint  $A$   $C \wedge M$  Midpoint  $B$   $D$ 
    using Plg-def assms(1) by presburger
  have Parallelogram  $A$   $B$   $C$   $D$ 
    by (simp add: assms(1) plg-to-parallelogram)
  moreover
  {
    assume ParallelogramStrict  $A$   $B$   $C$   $D$ 
    hence  $P$   $Q$   $Par$   $A$   $D \wedge M$  Midpoint  $P$   $Q \wedge Cong$   $A$   $D$   $P$   $Q$ 
      using  $\langle M$  Midpoint  $A$   $C \wedge M$  Midpoint  $B$   $D \rangle$   $\langle P$  Midpoint  $A$   $B \rangle$ 
       $\langle Q$  Midpoint  $C$   $D \rangle$  half-plgs by blast
    have  $P$   $Q$   $Par$   $A$   $D$ 
      by (simp add:  $\langle P$   $Q$   $Par$   $A$   $D \wedge M$  Midpoint  $P$   $Q \wedge Cong$   $A$   $D$   $P$   $Q \rangle$ )
    have  $M$  Midpoint  $P$   $Q$ 
      by (simp add:  $\langle P$   $Q$   $Par$   $A$   $D \wedge M$  Midpoint  $P$   $Q \wedge Cong$   $A$   $D$   $P$   $Q \rangle$ )
    have  $Cong$   $A$   $D$   $P$   $Q$ 
      by (simp add:  $\langle P$   $Q$   $Par$   $A$   $D \wedge M$  Midpoint  $P$   $Q \wedge Cong$   $A$   $D$   $P$   $Q \rangle$ )
    have  $A \neq D \wedge P \neq Q$ 
      using  $\langle P$   $Q$   $Par$   $A$   $D \rangle$  par-neq1 par-neq2 by blast
    have  $A \neq B$ 
      using  $\langle ParallelogramStrict$   $A$   $B$   $C$   $D \rangle$  plgs-diff by blast
  }

```

```

have A ≠ P
  using ⟨A ≠ B⟩ ⟨P Midpoint A B⟩ midpoint-distinct-1 by blast
have P Q Perp A B
proof -
  have A D Par P Q
    using Par-cases ⟨P Q Par A D⟩ by blast
  moreover
  have A D Perp A B
    by (metis ⟨A ≠ B⟩ ⟨A ≠ D ∧ P ≠ Q⟩ assms(2) per-perp perp-left-comm)
  moreover
  have Coplanar P Q A B
    using ⟨P Midpoint A B⟩ midpoint-col ncop--ncols by blast
  ultimately
  show ?thesis
    using cop-par-perp--perp by blast
qed
have P M Perp A B
  by (metis Bet-cases Col-def Midpoint-def
    ⟨P Q Par A D ∧ M Midpoint P Q ∧ Cong A D P Q⟩
    ⟨P Q Perp A B⟩ is-midpoint-id perp-col)
have A P Perp P M
  by (metis Col-cases Perp-perm ⟨A ≠ P⟩ ⟨P M Perp A B⟩ ⟨P Midpoint A B⟩
    col-trivial-3 midpoint-col perp-col2)
have Per A P M
  using ⟨A P Perp P M⟩ perp-comm perp-per-1 by blast
obtain B' where P Midpoint A B' and Cong M A M B'
  using ⟨P Midpoint A B⟩ ⟨Per A P M⟩ l8-2 per-double-cong by blast
have B = B'
  using SymR-uniq-aux ⟨P Midpoint A B'⟩ ⟨P Midpoint A B⟩ by blast
hence Cong A C B D
  using ⟨Cong M A M B'⟩ ⟨M Midpoint A C ∧ M Midpoint B D⟩
    cong-mid2--cong not-cong-2143 by blast
hence Rectangle A B C D
  by (simp add: assms(1) Rectangle-def)
}
moreover
{
  assume ParallelogramFlat A B C D
  have Rectangle A B C D
    by (metis Col-cases ParallelogramFlat-def plg-permut
      ⟨ParallelogramFlat A B C D⟩ assms(1) assms(2) calculation(1)
      cong-reflexivity l8-9 plg-cong-rectangle plg-not-comm-R1)
}
ultimately
show ?thesis
  using Parallelogram-def by blast
qed

lemma plg-per-rect2:
  assumes Plg A B C D and
    Per C B A
  shows Rectangle A B C D
proof -
  have Plg B A D C
    by (metis Plg-def assms(1))
  thus ?thesis
    by (simp add: assms(2) plg-per-rect1 rect-comm2)
qed

lemma plg-per-rect3:
  assumes Plg A B C D and
    Per A D C
  shows Rectangle A B C D
proof -
  have Plg D A B C
    by (meson assms(1) parallelogram-to-plg plg-permut plg-to-parallelogram)

```

moreover
have $Per\ C\ D\ A$
using $Per\text{-}cases\ assms(2)$ **by** $blast$
ultimately
show $?thesis$
by $(simp\ add:\ plg\text{-}per\text{-}rect1\ rect\text{-}permut)$
qed

lemma $plg\text{-}per\text{-}rect4$:
assumes $Plg\ A\ B\ C\ D$ **and**
 $Per\ B\ C\ D$
shows $Rectangle\ A\ B\ C\ D$
proof –
have $Plg\ B\ A\ D\ C$
using $Plg\text{-}def\ assms(1)$ **by** $auto$
thus $?thesis$
by $(meson\ assms(2)\ plg\text{-}per\text{-}rect3\ rect\text{-}comm2)$
qed

lemma $plg\text{-}per\text{-}rect$:
assumes $Plg\ A\ B\ C\ D$ **and**
 $Per\ D\ A\ B \vee Per\ C\ B\ A \vee Per\ A\ D\ C \vee Per\ B\ C\ D$
shows $Rectangle\ A\ B\ C\ D$
by $(meson\ assms(1)\ assms(2)\ plg\text{-}per\text{-}rect1\ plg\text{-}per\text{-}rect2\ plg\text{-}per\text{-}rect3\ plg\text{-}per\text{-}rect4)$

lemma $rect\text{-}per$:
assumes $Rectangle\ A\ B\ C\ D$
shows $Per\ B\ A\ D \wedge Per\ A\ B\ C \wedge Per\ B\ C\ D \wedge Per\ A\ D\ C$
using $assms\ rect\text{-}comm2\ rect\text{-}per1\ rect\text{-}per3$ **by** $blast$

lemma $plgf\text{-}rect\text{-}id$:
assumes $ParallelogramFlat\ A\ B\ C\ D$ **and**
 $Rectangle\ A\ B\ C\ D$
shows $(A = D \wedge B = C) \vee (A = B \wedge D = C)$
by $(metis\ ParallelogramFlat\text{-}def\ assms(1)\ assms(2)\ cong3\text{-}id\ l8\text{-}9\ plgf\text{-}permut\ rect\text{-}per3)$

lemma $cop\text{-}perp3\text{-}perp$:
assumes $Coplanar\ A\ B\ C\ D$ **and**
 $A\ B\ Perp\ B\ C$ **and**
 $B\ C\ Perp\ C\ D$ **and**
 $C\ D\ Perp\ D\ A$
shows $D\ A\ Perp\ A\ B$
proof –
have $A\ B\ Perp\ D\ A$
proof –
have $A\ B\ Par\ C\ D$
by $(meson\ Perp\text{-}perm\ assms(1)\ assms(2)\ assms(3)\ l12\text{-}9\ ncop\text{-}distincts\ ncoplanar\text{-}perm\text{-}12)$
hence $C\ D\ Par\ A\ B$
using $Par\text{-}cases$ **by** $blast$
moreover
have $Coplanar\ A\ B\ D\ A$
using $ncop\text{-}distincts$ **by** $blast$
ultimately
show $?thesis$
using $assms(4)\ cop\text{-}par\text{-}perp\text{-}perp$ **by** $blast$
qed
thus $?thesis$
using $Perp\text{-}cases$ **by** $blast$
qed

lemma $cop\text{-}perp3\text{-}rect$:
assumes $Coplanar\ A\ B\ C\ D$ **and**
 $A\ B\ Perp\ B\ C$ **and**
 $B\ C\ Perp\ C\ D$ **and**
 $C\ D\ Perp\ D\ A$

shows *Rectangle A B C D*
proof –
have $\neg \text{Col } A \ B \ C$
using *assms(2) col-trivial-2 perp-not-col2* **by** *blast*
have *A B Par C D*
by (*meson Perp-perm assms(1) assms(2) assms(3) l12-9 ncop-distincts*
ncoplanar-perm-12)
hence *A D Par B C*
by (*meson assms(1) assms(2) assms(4) cop4-par-perp2--par ncop-distincts*
par-left-comm par-symmetry)
have *D A Perp A B*
using *assms(1) assms(2) assms(3) assms(4) cop-perp3--perp* **by** *blast*
have *ParallelogramStrict A B C D*
using $\langle A \ B \ \text{Par} \ C \ D \rangle \ \langle A \ D \ \text{Par} \ B \ C \rangle \ \langle \neg \text{Col } A \ B \ C \rangle$ *parallel-2-plg* **by** *presburger*
thus *?thesis*
by (*simp add: Parallelogram-strict-Parallelogram parallelogram-to-plg*
assms(3) perp-comm perp-per-1 plg-per-rect4)
qed

lemma *conga-to-par-os*:
assumes *Bet A D P and*
A D OS B C and
B A P CongA C D P
shows *A B Par C D*
by (*metis CongA-def Out-def assms(1) assms(2) assms(3) bet-col*
between-symmetry col-one-side invert-one-side l12-22 par-right-comm)

lemma *plg-par*:
assumes *A $\neq$ B and*
B $\neq$ C and
Parallelogram A B C D
shows *A B Par C D $\wedge$ A D Par B C*
by (*metis Plg-perm Par-cases l12-17 assms(1) assms(2) assms(3) plg-mid*)

lemma *plg-par-1*:
assumes *A $\neq$ B and*
B $\neq$ C and
Parallelogram A B C D
shows *A B Par C D*
by (*simp add: assms(1) assms(2) assms(3) plg-par*)

lemma *plg-par-2*:
assumes *A $\neq$ B and*
B $\neq$ C and
Parallelogram A B C D
shows *A D Par B C*
using *assms(1) assms(2) assms(3) plg-par* **by** *blast*

lemma *plgs-pars-1*:
assumes *ParallelogramStrict A B C D*
shows *A B ParStrict C D*
by (*simp add: assms plgs--pars*)

lemma *plgs-pars-2*:
assumes *ParallelogramStrict A B C D*
shows *A D ParStrict B C*
using *assms plgs-par-strict* **by** *blast*

lemma *par-cong-cong*:
assumes *A B Par C D and*
Cong A B C D
shows *Cong A C B D $\vee$ Cong A D B C*
proof *cases*
assume *A = B*
thus *?thesis*
using *assms(1) par-neq1* **by** *auto*

```

next
  assume  $A \neq B$ 
  thus ?thesis
proof cases
  assume  $A = D$ 
  thus ?thesis
    using assms(2) not-cong-4321 by blast
next
  assume  $A \neq D$ 
  obtain  $M$  where  $(M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D) \vee (M \text{ Midpoint } A \ D \wedge M \text{ Midpoint } B \ C)$ 
    using assms(1) assms(2) par-cong-mid by blast
  moreover
  {
    assume  $M \text{ Midpoint } A \ C$  and  $M \text{ Midpoint } B \ D$ 
    hence  $\text{Cong } A \ C \ B \ D \vee \text{Cong } A \ D \ B \ C$ 
      using  $\langle A \neq B \rangle \langle A \neq D \rangle$  cong-right-commutativity mid-par-cong by blast
  }
  moreover
  {
    assume  $M \text{ Midpoint } A \ D$  and  $M \text{ Midpoint } B \ C$ 
    hence  $\text{Cong } A \ C \ B \ D \vee \text{Cong } A \ D \ B \ C$ 
      by (metis Cong-cases Mid-cases l7-13)
  }
  ultimately show ?thesis
    by blast
qed
qed

lemma col-cong-cong:
  assumes  $\text{Col } A \ B \ C$  and
     $\text{Col } A \ B \ D$  and
     $\text{Cong } A \ B \ C \ D$ 
  shows  $\text{Cong } A \ C \ B \ D \vee \text{Cong } A \ D \ B \ C$ 
proof cases
  assume  $A = B$ 
  thus ?thesis
    using assms(3) cong-reflexivity cong-reverse-identity by blast
next
  assume  $A \neq B$ 
  show ?thesis
proof cases
  assume  $C = D$ 
  thus ?thesis
    using  $\langle A \neq B \rangle$  assms(3) cong-identity-inv by auto
next
  assume  $C \neq D$ 
  moreover have  $A \ B \ \text{Par } C \ D$ 
    using  $\langle A \neq B \rangle$  assms(1) assms(2) calculation par-col2-par par-reflexivity by blast
  ultimately show ?thesis
    using assms(3) par-cong-cong by blast
qed
qed

lemma par-cong-cop:
  assumes  $A \ B \ \text{Par } C \ D$ 
  shows  $\text{Coplanar } A \ B \ C \ D$ 
  using assms(1) par--coplanar by auto

lemma par-cong-plg :
  assumes  $A \ B \ \text{Par } C \ D$  and
     $\text{Cong } A \ B \ C \ D$ 
  shows  $\text{Plg } A \ B \ C \ D \vee \text{Plg } A \ B \ D \ C$ 
proof -
  obtain  $M$  where  $(M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D) \vee (M \text{ Midpoint } A \ D \wedge M \text{ Midpoint } B \ C)$ 
    using assms(1) assms(2) par-cong-mid by blast
  moreover

```

```

{
  assume  $M \text{ Midpoint } A \ C \wedge M \text{ Midpoint } B \ D$ 
  hence  $Plg \ A \ B \ C \ D \vee Plg \ A \ B \ D \ C$ 
  by (metis Plg-def assms(1) l7-3 par-neq1)
}
moreover
{
  assume  $M \text{ Midpoint } A \ D \wedge M \text{ Midpoint } B \ C$ 
  hence  $Plg \ A \ B \ C \ D \vee Plg \ A \ B \ D \ C$ 
  by (metis Plg-def assms(1) l7-3 par-neq1)
}
ultimately show ?thesis
  by blast
qed

lemma par-cong-plg-2 :
  assumes  $A \ B \text{ Par } C \ D$  and
     $Cong \ A \ B \ C \ D$ 
  shows  $Parallelogram \ A \ B \ C \ D \vee Parallelogram \ A \ B \ D \ C$ 
  by (metis assms(1) assms(2) mid-plg par-cong-mid par-neq2 plg-trivial)

lemma par-cong3-rect:
  assumes  $A \neq C \vee B \neq D$  and
     $A \ B \text{ Par } C \ D$  and
     $Cong \ A \ B \ C \ D$  and
     $Cong \ A \ D \ B \ C$  and
     $Cong \ A \ C \ B \ D$ 
  shows  $Rectangle \ A \ B \ C \ D \vee Rectangle \ A \ B \ D \ C$ 
  by (simp add: Rectangle-def assms(2) assms(3) assms(4) assms(5) par-cong-plg)

lemma pars-par-pars:
  assumes  $A \ B \text{ ParStrict } C \ D$  and
     $A \ D \text{ Par } B \ C$ 
  shows  $A \ D \text{ ParStrict } B \ C$ 
  using Par-def assms(1) assms(2) par-strict-not-col-1 by blast

lemma pars-par-plg:
  assumes  $A \ B \text{ ParStrict } C \ D$  and
     $A \ D \text{ Par } B \ C$ 
  shows  $Plg \ A \ B \ C \ D$ 
proof -
  have  $A \ D \text{ ParStrict } B \ C$ 
    by (simp add: assms(1) assms(2) pars-par-pars)
  have  $A \ B \text{ Par } C \ D$ 
    by (simp add: Par-def assms(1))
  have  $A \neq C$ 
    using assms(1) not-par-strict-id by blast
  moreover
  have  $B \neq D$ 
    using  $\langle A \ D \text{ ParStrict } B \ C \rangle$  col-trivial-2 par-strict-not-col-1 by blast
  moreover
  obtain  $M$  where  $M \text{ Midpoint } A \ C$ 
    using MidR-uniq-aux by blast
  moreover
  have  $M \text{ Midpoint } B \ D$ 
    by (metis Par-cases par-strict-not-col-1  $\langle A \ B \text{ Par } C \ D \rangle$   $\langle M \text{ Midpoint } A \ C \rangle$ 
      assms(1) assms(2) l12-19 l7-17 par-cong-mid-ts)
  ultimately
  show ?thesis
    using Plg-def by blast
qed

lemma not-par-pars-not-cong:
  assumes  $P \text{ Out } A \ B$  and
     $P \text{ Out } A' \ B'$  and
     $A \ A' \text{ ParStrict } B \ B'$ 

```



```

shows  $\neg \text{Cong } A \ A' \ B \ B'$ 
proof -
  have  $\text{Col } P \ A \ B$ 
  by (simp add: assms(1) out-col)
  have  $\text{Col } P \ A' \ B'$ 
  using assms(2) out-col by blast
  {
    assume  $\text{Cong } A \ A' \ B \ B'$ 
    have  $A \neq B$ 
    using assms(3) not-par-strict-id by auto
    have  $A' \neq B'$ 
    using assms(3) between-trivial2 impossible-case-1 by blast
    have  $P \neq A$ 
    using Out-def assms(1) by auto
    have  $P \neq A'$ 
    using Out-def assms(2) by blast
    have  $\neg \text{Col } P \ A \ A'$ 
    by (metis  $\langle P \neq A' \rangle$  assms(2) assms(3) col3 col-transitivity-1
      not-col-permutation-5 out-col par-strict-not-col-4)
    have  $A \ B \ \text{OS} \ A' \ B'$ 
    by (metis Col-cases assms(1) assms(2) assms(3) out-col
      out-one-side-1 par-strict-not-col-1)
    have  $A \ A' \ \text{Par} \ B \ B'$ 
    by (simp add: Par-def assms(3))
    hence False
    using NCol-perm  $\langle A \ B \ \text{OS} \ A' \ B' \rangle$   $\langle \text{Col } P \ A \ B \rangle$   $\langle \text{Col } P \ A' \ B' \rangle$ 
       $\langle \text{Cong } A \ A' \ B \ B' \rangle$  not-strict-par one-side-not-col124
      os-cong-par-cong-par by blast
  }
  thus ?thesis
  by auto
qed

```

```

lemma plg-uniqueness:
  assumes Parallelogram  $A \ B \ C \ D$  and
    Parallelogram  $A \ B \ C \ D'$ 
  shows  $D = D'$ 
  by (metis SymR-uniq-aux assms(1) assms(2) l7-17 plg-mid)

```

```

lemma plgs-trans-trivial:
  assumes ParallelogramStrict  $A \ B \ C \ D$  and
    ParallelogramStrict  $C \ D \ A \ B'$ 
  shows Parallelogram  $A \ B \ B' \ A$ 
  by (metis plg-uniqueness Parallelogram-strict-Parallelogram plg-sym plg-trivial
    assms(1) assms(2) plgs-diff)

```

```

lemma par-strict-trans:
  assumes  $A \ B \ \text{ParStrict} \ C \ D$  and
     $C \ D \ \text{ParStrict} \ E \ F$ 
  shows  $A \ B \ \text{Par} \ E \ F$ 
  using assms(1) assms(2) par-not-par par-strict-par by blast

```

```

lemma plgs-pseudo-trans:
  assumes ParallelogramStrict  $A \ B \ C \ D$  and
    ParallelogramStrict  $C \ D \ E \ F$ 
  shows Parallelogram  $A \ B \ F \ E$ 
proof cases
  assume  $A = E$ 
  thus ?thesis
  using assms(1) assms(2) plgs-trans-trivial by blast
next
  assume  $A \neq E$ 
  have  $C \neq D$ 
  using assms(1) plgs-diff by blast
  obtain  $D'$  where Bet  $D \ C \ D'$  and Cong  $C \ D' \ D \ C$ 

```

```

  using segment-construction by blast
have C ≠ D'
  using ⟨C ≠ D⟩ ⟨Cong C D' D C⟩ cong-diff-4 by presburger
have A B ParStrict C D
  using assms(1) plgs-par-strict by auto
hence C D ParStrict A B
  using par-strict-symmetry by blast
have A D ParStrict B C
  using assms(1) plgs-par-strict by auto
have C D ParStrict E F
  using assms(2) plgs-par-strict by auto
have C F ParStrict D E
  using assms(2) plgs-par-strict by auto
have C D OS A B
  using assms(1) plgs-one-side by blast
have C D OS E F
  using ⟨C D ParStrict E F⟩ l12-6 by auto
have A D D' CongA B C D'
  by (meson Par-def Par-strict-cases ⟨A D ParStrict B C⟩ ⟨Bet D C D'⟩
    ⟨C D OS A B⟩ ⟨C ≠ D'⟩ invert-one-side par-preserves-conga-os)
have E D D' CongA F C D'
  by (meson ParallelogramStrict-def ⟨Bet D C D'⟩ ⟨C D OS E F⟩ ⟨C ≠ D'⟩
    assms(2) invert-one-side l12-20 par-preserves-conga-os)
have ¬ Col A C D
  using ⟨A B ParStrict C D⟩ col-permutation-1 par-strict-not-col-3 by blast
have ¬ Col E C D
  using ⟨C D OS E F⟩ col123--nos not-col-permutation-2 by blast
{
  assume Coplanar C D A E
  have A D E CongA B C F
  proof —
    have C D TS A E ∨ C D OS A E
      using ⟨Coplanar C D A E⟩ ⟨¬ Col A C D⟩ ⟨¬ Col E C D⟩ cop-nts--os by blast
    moreover
    {
      assume C D TS A E
      hence C D TS A F
        using ⟨C D OS E F⟩ l9-2 l9-8-2 by blast
      hence C D TS B F
        using ⟨C D OS A B⟩ l9-8-2 by blast
      have A D E CongA B C F
      proof —
        have D D' TS A E
          using ⟨Bet D C D'⟩ ⟨C D TS A E⟩ bet-col bet-neq23--neq
            col-two-sides invert-two-sides by blast
        moreover have C D' TS B F
          by (meson ⟨Bet D C D'⟩ ⟨C D TS B F⟩ ⟨C ≠ D'⟩ bet-col bet-col1
            col-preserves-two-sides invert-two-sides)
        moreover have A D D' CongA B C D'
          using ⟨A D D' CongA B C D'⟩ by auto
        moreover have D' D E CongA D' C F
          by (simp add: ⟨E D D' CongA F C D'⟩ conga-comm)
        ultimately show ?thesis
          using l11-22a by blast
      qed
    }
  moreover
  {
    assume C D OS A E
    have A D E CongA B C F
    proof —
      have D D' OS A E
        using ⟨Bet D C D'⟩ ⟨C D OS A E⟩ bet-col between-identity
          col-one-side invert-one-side by blast
      moreover have C D' OS B F
        by (metis ⟨Bet D C D'⟩ ⟨C D OS A B⟩ ⟨C D OS A E⟩ ⟨C D OS E F⟩

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      ‹C ≠ D'› bet-col col-one-side col-permutation-4 one-side-symmetry
      one-side-transitivity)
    moreover have A D D' CongA B C D'
      by (simp add: ‹A D D' CongA B C D'›)
    moreover have D' D E CongA D' C F
      by (simp add: ‹E D D' CongA F C D'› conga-comm)
    ultimately show ?thesis
      using l11-22 by blast
  qed
}
ultimately
show ?thesis
  by blast
qed
have Cong A B C D
  using assms(1) plgs-cong by auto
have Cong A D B C
  using assms(1) plgs-cong by auto
have Cong C D E F
  using assms(2) plgs-cong by auto
have Cong C F D E
  using assms(2) plgs-cong by auto
hence Cong D E C F
  using not-cong-3412 by blast
hence Cong A E B F
  using ‹A D E CongA B C F› ‹Cong A D B C› cong2-conga-cong by auto
have A B Par E F
  using ‹A B ParStrict C D› ‹C D ParStrict E F› not-par-one-not-par
    par-strict-par par-strict-symmetry by blast
have Parallelogram A B F E
proof cases
  assume Col A E F
  {
    assume A B ParStrict E F
    hence Parallelogram A B F E
      using Col-cases ‹Col A E F› par-strict-not-col-3 by blast
  }
  moreover
  {
    assume A ≠ B and E ≠ F and
      Col A E F and Col B E F
    have ParallelogramFlat A B F E
    proof -
      have Col A B F
        using ‹Col A E F› ‹Col B E F› ‹E ≠ F› col2--eq col-permutation-5 by blast
      moreover have Col A B E
        using ‹Col A E F› ‹Col B E F› ‹E ≠ F› col-permutation-5 l6-16-1 by blast
      moreover have Cong A B F E
        by (meson ‹Cong A B C D› ‹Cong C D E F› cong-right-commutativity cong-transitivity)
      moreover have Cong A E F B
        using Cong-cases ‹Cong A E B F› by blast
      moreover have A ≠ F ∨ B ≠ E
        using assms(1) assms(2) plgs-not-comm-1 plgs-sym by blast
      ultimately show ?thesis
        using ParallelogramFlat-def by blast
    qed
    hence Parallelogram A B F E
      by (simp add: Parallelogram-def)
  }
ultimately show ?thesis
  using Par-def ‹A B Par E F› by presburger
next
assume ¬ Col A E F
have A E Par B F ∨ ¬ D C ParStrict A B ∨ ¬ D C ParStrict E F
proof -
  have D A E Cong3 C B F

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    using Cong3-def Cong-cases ⟨Cong A D B C⟩ ⟨Cong A E B F⟩ ⟨Cong D E C F⟩ by blast
  moreover have D A Par C B
    using Par-cases Par-def ⟨A D ParStrict B C⟩ by blast
  moreover have D E Par C F
    by (simp add: ⟨C F ParStrict D E⟩ par-strict-par par-symmetry)
  ultimately show ?thesis
    using ⟨A ≠ E⟩ cong3-par2-par by blast
qed
moreover
{
  assume A E Par B F
  hence Parallelogram A B F E
    using Plg-perm ⟨A B Par E F⟩ ⟨¬ Col A E F⟩ par-2-plg par-right-comm by blast
}
moreover
{
  assume ¬ D C ParStrict A B
  hence Parallelogram A B F E
    using Par-strict-perm ⟨A B ParStrict C D⟩ by blast
}
moreover
{
  assume ¬ D C ParStrict E F
  hence Parallelogram A B F E
    using Par-strict-perm ⟨C D ParStrict E F⟩ by blast
}
ultimately show ?thesis
  by blast
qed
}
moreover
{
  assume ¬ Coplanar C D A E
  have Parallelogram A B E F ∨ Parallelogram A B F E
  proof -
    have A B Par E F
      using ⟨A B ParStrict C D⟩ ⟨C D ParStrict E F⟩ par-strict-trans by force
    moreover
    have Cong A B E F
      using assms(1) assms(2) cong-transitivity plgs-cong-1 by blast
    ultimately
    show ?thesis
      using par-cong-plg-2 by blast
  qed
  moreover
  {
    assume Parallelogram A B E F
    hence A E OS B F
    proof -
      have Coplanar A D E A
        using ncop-distincts by auto
      moreover have Coplanar A D E E
        using ncop-distincts by blast
      moreover have Coplanar A E B F
        using ⟨Parallelogram A B E F⟩ coplanar-perm-2 parallelogram--coplanar by blast
      moreover have A D E OSP B F
      proof -
        have A D E OSP B C
        proof -
          have ¬ Coplanar A D E B
            by (meson ⟨A B ParStrict C D⟩ ⟨C D ParStrict A B⟩
              ⟨¬ Coplanar C D A E⟩ l9-30 ncop-distincts ncoplanar-perm-15
              par-strict-not-col-3 par-strict-not-col-4 pars--coplanar)
          moreover have Coplanar A D E A
            using ⟨Coplanar A D E A⟩ by auto
          moreover have Coplanar A D E D

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    using ncop-distincts by blast
  moreover have  $A \ D \ OS \ B \ C$ 
    using  $\langle A \ D \ ParStrict \ B \ C \rangle$  l12-6 by auto
  ultimately show ?thesis
    using cop2-os--osp by blast
qed
moreover
have  $A \ D \ E \ OSP \ C \ F$ 
proof -
  have  $\neg \ Coplanar \ A \ D \ E \ C$ 
    using  $\langle \neg \ Coplanar \ C \ D \ A \ E \rangle$  coplanar-perm-20 by blast
  moreover have  $Coplanar \ A \ D \ E \ D$ 
    using ncop-distincts by blast
  moreover have  $Coplanar \ A \ D \ E \ E$ 
    by (simp add:  $\langle Coplanar \ A \ D \ E \ E \rangle$ )
  moreover have  $D \ E \ OS \ C \ F$ 
    by (simp add:  $\langle C \ F \ ParStrict \ D \ E \rangle$  pars--os3412)
  ultimately show ?thesis
    using cop2-os--osp by blast
qed
ultimately show ?thesis
  using osp-transitivity by blast
qed
ultimately show ?thesis
  using  $\langle A \neq E \rangle$  cop3-osp--os by blast
qed
have  $A \ E \ TS \ B \ F$ 
  by (metis  $\langle A \ E \ OS \ B \ F \rangle \langle Parallelogram \ A \ B \ E \ F \rangle$  col-permutation-5
    l12-19 one-side-not-col123 os-distincts plg-par plg-permut)
hence False
  by (simp add:  $\langle A \ E \ OS \ B \ F \rangle$  l9-9-bis)
}
ultimately
have  $Parallelogram \ A \ B \ F \ E$ 
  by blast
}
ultimately show ?thesis
  by blast
qed

lemma plgf-plgs-trans:
  assumes  $A \neq B$  and
     $ParallelogramFlat \ A \ B \ C \ D$  and
     $ParallelogramStrict \ C \ D \ E \ F$ 
  shows  $ParallelogramStrict \ A \ B \ F \ E$ 
proof (cases  $A = D$ )
case True
  thus ?thesis
    by (metis ParallelogramFlat-def assms(2) assms(3)
      cong-reverse-identity plgs-comm2)
next
case False
  hence  $A \neq D$ 
    by blast
  show ?thesis
  proof (cases  $B = C$ )
  case True
    thus ?thesis
      using False assms(2) cong-diff plgf-cong by blast
  next
  case False
    hence  $B \neq C$ 
      by auto
    have  $C \ D \ ParStrict \ E \ F \wedge C \ F \ ParStrict \ D \ E$ 
      using assms(3) plgs-par-strict by blast
    have  $C \ D \ ParStrict \ E \ F$ 

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  by (simp add:  $\langle C D \text{ ParStrict } E F \wedge C F \text{ ParStrict } D E \rangle$ )
have  $C F \text{ ParStrict } D E$ 
  by (simp add:  $\langle C D \text{ ParStrict } E F \wedge C F \text{ ParStrict } D E \rangle$ )
have  $\text{Cong } C D E F \wedge \text{Cong } C F D E$ 
  using assms(3) plgs-cong by blast
have  $\text{Cong } C D E F$ 
  by (simp add:  $\langle \text{Cong } C D E F \wedge \text{Cong } C F D E \rangle$ )
have  $\text{Cong } C F D E$ 
  by (simp add:  $\langle \text{Cong } C D E F \wedge \text{Cong } C F D E \rangle$ )
have  $C D \text{ OS } E F$ 
  by (simp add: assms(3) plgs-one-side)
then obtain  $x$  where  $C D \text{ TS } E x$  and  $C D \text{ TS } F x$ 
  using OS-def by blast
have  $\neg \text{Col } F C D$ 
  using TS-def  $\langle C D \text{ TS } F x \rangle$  by blast
have  $\neg \text{Col } E C D$ 
  using TS-def  $\langle C D \text{ TS } E x \rangle$  by blast
hence  $D \neq E$ 
  using col-trivial-3 by blast
have  $\text{Col } A B C$ 
  using ParallelogramFlat-def assms(2) by blast
have  $\text{Col } A B D$ 
  using ParallelogramFlat-def assms(2) by blast
have  $\text{Cong } A B C D$ 
  using ParallelogramFlat-def assms(2) by blast
have  $\text{Cong } A D C B$ 
  using ParallelogramFlat-def assms(2) by blast
have  $A \neq C \vee B \neq D$ 
  using ParallelogramFlat-def assms(2) by blast
obtain  $D'$  where  $\text{Bet } D C D'$  and  $\text{Cong } C D' D C$ 
  using segment-construction by blast
have  $D \neq D'$ 
  using  $\langle \text{Bet } D C D' \rangle \langle \text{Cong } A B C D \rangle$  assms(1) between-identity cong-identity by blast
hence  $C \neq D'$ 
  using  $\langle \text{Cong } C D' D C \rangle$  cong-reverse-identity by blast
have  $E D D' \text{ Cong } A F C D'$ 
  by (meson ParallelogramStrict-def  $\langle \text{Bet } D C D' \rangle \langle C D \text{ OS } E F \rangle \langle C \neq D' \rangle$ 
    assms(3) invert-one-side l12-20 par-preserves-conga-os)
show ?thesis
proof (cases  $A = C$ )
case True
hence  $A = C$ 
  by auto
hence  $B = D \vee A \text{ Midpoint } B D$ 
  using assms(2) plgf3-mid by blast
moreover
{
  assume  $A \text{ Midpoint } B D$ 
  have  $A F \text{ TS } E B$ 
  proof -
    have  $A F \text{ TS } D B$ 
    by (metis False bet-ts True  $\langle A \neq C \vee B \neq D \rangle \langle \text{Col } A B D \rangle$ 
       $\langle \text{Cong } A B C D \rangle \langle \text{Cong } A D C B \rangle \neg \text{Col } F C D$  between-cong
      col-permutation-4 third-point)
    moreover
    have  $A F \text{ OS } D E$ 
    by (simp add: True  $\langle C D \text{ ParStrict } E F \wedge C F \text{ ParStrict } D E \rangle$  l12-6)
    ultimately
    show ?thesis
    using l9-8-2 by blast
  qed
  hence  $A F \text{ TS } B E$ 
  using l9-2 by blast
moreover
have  $A B \text{ Par } F E$ 
  by (metis True  $\langle A \neq D \rangle \langle C D \text{ OS } E F \rangle \langle C F \text{ ParStrict } D E \rangle$ )

```

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      ⟨Col A B D⟩ ⟨Cong C F D E⟩ assms(1) not-par-not-col one-side-symmetry
      os-cong-par-cong-par par-not-par par-strict-par)
  moreover
  have Cong A B F E
    using ⟨Cong A B C D⟩ ⟨Cong C D E F ∧ Cong C F D E⟩
      cong-right-commutativity cong-transitivity by blast
  ultimately
  have ParallelogramStrict A B F E
    by (simp add: ParallelogramStrict-def)
}
ultimately
show ?thesis
  using True ⟨A ≠ C ∨ B ≠ D⟩ by blast
next
case False
hence A ≠ C
  by auto
thus ?thesis
proof (cases B = D)
case True
have A = C ∨ B Midpoint A C
  by (metis ParallelogramFlat-def True assms(2) cong-col-mid not-cong-1243)
{
  assume B Midpoint A C
  have B E TS F A
  proof -
    have B E TS C A
      by (metis Bet-cases False True ⟨A = C ∨ B Midpoint A C⟩
        ¬ Col E C D⟩ assms(1) bet--ts col-permutation-1 midpoint-bet)
    moreover
    have B E OS C F
      by (simp add: True ⟨C D ParStrict E F ∧ C F ParStrict D E⟩ pars--os3412)
    ultimately
    show ?thesis
      using l9-8-2 by blast
  qed
  hence B E TS A F
    by (simp add: l9-2)
  moreover
  have B A Par E F
    by (metis True ⟨A = C ∨ B Midpoint A C⟩ ⟨B ≠ C⟩
      ⟨C D ParStrict E F⟩ assms(1) midpoint-col not-par-not-col
      par-left-comm par-not-par par-strict-par)
  moreover
  have Cong B A E F
    using ⟨Cong A B C D⟩ ⟨Cong C D E F⟩ cong-inner-transitivity
      not-cong-4312 by blast
  ultimately
  have ParallelogramStrict B A E F
    using ParallelogramStrict-def by blast
  hence ParallelogramStrict A B F E
    using plgs-comm2 by blast
}
thus ?thesis
  using True ⟨A = C ∨ B Midpoint A C⟩ ⟨A ≠ C ∨ B ≠ D⟩ by fastforce
next
case False
hence B ≠ D
  by auto
have H1: Bet D C A ∧ Bet C A B ∨
  Bet D A C ∧ Bet A C B ∨
  Bet A D B ∧ Bet D B C ∨
  Bet A B D ∧ Bet B D C
  using assms(2) plgf-bet by presburger
have A D E CongA B C F
proof -

```

```

{
  assume Bet D C A  $\wedge$  Bet C A B
  hence D Out A D'
  by (metis Out-def not-col-distincts  $\langle A \neq D \rangle \langle \textit{Bet D C D}' \rangle$ 
     $\langle D \neq D' \rangle \langle \neg \textit{Col E C D} \rangle$  l5-1)
  moreover
  have D Out E E
  using  $\langle D \neq E \rangle$  out-trivial by presburger
  moreover
  have C Out B D'
  by (metis Out-def not-col-distincts  $\langle A \neq C \rangle \langle B \neq C \rangle$ 
     $\langle \textit{Bet D C A} \wedge \textit{Bet C A B} \rangle \langle \textit{Bet D C D}' \rangle \langle C \neq D' \rangle \langle \neg \textit{Col E C D} \rangle$ 
    l5-2 outer-transitivity-between)
  moreover
  have C Out F F
  using  $\langle \neg \textit{Col F C D} \rangle$  not-col-distincts out-trivial by blast
  ultimately
  have A D E CongA B C F
  by (meson  $\langle E D D' \textit{ CongA F C D}' \rangle$  conga-left-comm conga-right-comm l11-10)
}
moreover
{
  assume Bet D A C  $\wedge$  Bet A C B
  hence D Out A D'
  by (meson  $\langle A \neq D \rangle \langle \textit{Bet D C D}' \rangle$  bet-out between-exchange4)
  moreover
  have D Out E E
  using  $\langle D \neq E \rangle$  out-trivial by presburger
  moreover
  have C Out B D'
  by (meson  $\langle A \neq C \rangle \langle A \neq D \rangle \langle B \neq C \rangle \langle \textit{Bet D A C} \wedge \textit{Bet A C B} \rangle$ 
     $\langle \textit{Bet D C D}' \rangle \langle \textit{Cong A B C D} \rangle \langle \textit{Cong A D C B} \rangle \langle \textit{Cong C D' D C} \rangle$ 
    assms(1) bet3-cong3-bet bet-out cong-3421 not-cong-1243)
  moreover
  have C Out F F
  using  $\langle \neg \textit{Col F C D} \rangle$  not-col-distincts out-trivial by blast
  ultimately
  have A D E CongA B C F
  by (meson  $\langle E D D' \textit{ CongA F C D}' \rangle$  conga-comm l11-10)
}
moreover
{
  assume Bet A D B  $\wedge$  Bet D B C
  hence Bet D' D A
  using False  $\langle \textit{Bet D C D}' \rangle$  between-exchange4 between-symmetry
    outer-transitivity-between by blast
  moreover
  have A  $\neq$  D
  using  $\langle A \neq D \rangle$  by auto
  moreover
  have Bet D' C B
  using  $\langle \textit{Bet A D B} \wedge \textit{Bet D B C} \rangle \langle \textit{Bet D C D}' \rangle$  between-exchange3
    between-symmetry by blast
  moreover
  have B  $\neq$  C
  using  $\langle B \neq C \rangle$  by auto
  ultimately
  have A D E CongA B C F
  by (meson  $\langle E D D' \textit{ CongA F C D}' \rangle$  conga-comm l11-13)
}
moreover
{
  assume Bet A B D  $\wedge$  Bet B D C
  hence Bet D' D A
  by (metis Bet-cases False  $\langle \textit{Bet D C D}' \rangle \langle \neg \textit{Col E C D} \rangle$ 
    col-trivial-2 outer-transitivity-between2)
}

```



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moreover
have  $A \neq D$ 
  using  $\langle A \neq D \rangle$  by auto
moreover
have  $Bet\ D'\ C\ B$ 
  using Bet-cases  $\langle Bet\ A\ B\ D \wedge Bet\ B\ D\ C \rangle \langle Bet\ D\ C\ D' \rangle$ 
     $\langle \neg\ Col\ E\ C\ D \rangle$  col-trivial-2 outer-transitivity-between2 by blast
moreover
have  $B \neq C$ 
  using  $\langle B \neq C \rangle$  by auto
ultimately
have  $A\ D\ E\ CongA\ B\ C\ F$ 
  by (meson  $\langle E\ D\ D'\ CongA\ F\ C\ D' \rangle$  conga-comm l11-13)
}
ultimately
show ?thesis
  using H1 by fastforce
qed
have  $Cong\ A\ E\ B\ F$ 
  using  $\langle A\ D\ E\ CongA\ B\ C\ F \rangle \langle Cong\ A\ D\ C\ B \rangle \langle Cong\ C\ F\ D\ E \rangle$ 
    cong2-conga-cong not-cong-1243 not-cong-3412 by blast
hence  $E\ A\ D\ Cong3\ F\ B\ C$ 
  by (meson Cong3-def  $\langle Cong\ A\ D\ C\ B \rangle \langle Cong\ C\ F\ D\ E \rangle$ 
    cong-4321 cong-right-commutativity)
have  $A\ B\ Par\ E\ F$ 
  by (metis ParallelogramFlat-def par-not-par  $\langle C\ D\ ParStrict\ E\ F \rangle$ 
     $\langle \neg\ Col\ E\ C\ D \rangle$  assms(1) assms(2) col-trivial-2 par-col2-par
    par-reflexivity par-strict-par)
have  $E\ A\ D\ CongA\ F\ B\ C$ 
  by (metis  $\langle A \neq D \rangle \langle Col\ A\ B\ D \rangle \langle Cong\ A\ E\ B\ F \rangle \langle E\ A\ D\ Cong3\ F\ B\ C \rangle$ 
    assms(3) cong3-conga cong-reverse-identity not-col-permutation-1
    parallelogram-strict-not-col-2)
have  $A\ B\ OS\ E\ F$ 
  by (metis Par-def  $\langle A\ B\ Par\ E\ F \rangle$ 
     $\langle C\ D\ ParStrict\ E\ F \wedge C\ F\ ParStrict\ D\ E \rangle \langle Col\ A\ B\ D \rangle$  col-par
    l12-6 par-strict-not-col-2 parallel-uniqueness)
have  $A\ E\ Par\ F\ B$ 
proof –
{
  assume  $Bet\ D\ C\ A \wedge Bet\ C\ A\ B$ 
  obtain  $A'$  where  $Bet\ A\ B\ A'$  and  $Cong\ B\ A'\ A\ B$ 
    using segment-construction by blast
  have  $E\ A\ A'\ CongA\ F\ B\ A'$ 
    by (metis  $\langle A \neq C \rangle \langle Bet\ A\ B\ A' \rangle \langle Bet\ D\ C\ A \wedge Bet\ C\ A\ B \rangle$ 
       $\langle Cong\ B\ A'\ A\ B \rangle \langle E\ A\ D\ CongA\ F\ B\ C \rangle$  assms(1) bet-neq12--neq
      cong-diff-3 conga-comm l11-13 outer-transitivity-between
      outer-transitivity-between2)
  hence  $A\ E\ Par\ F\ B$ 
    using  $\langle A\ B\ OS\ E\ F \rangle \langle Bet\ A\ B\ A' \rangle$  conga-to-par-os by blast
}
moreover
{
  assume  $Bet\ D\ A\ C \wedge Bet\ A\ C\ B$ 
  obtain  $A'$  where  $Bet\ A\ B\ A'$  and  $Cong\ B\ A'\ A\ B$ 
    using segment-construction by blast
  have  $A'\ A\ E\ CongA\ A'\ B\ F$ 
proof –
  have  $Bet\ D\ A\ A'$ 
    using  $\langle A \neq C \rangle \langle Bet\ A\ B\ A' \rangle \langle Bet\ D\ A\ C \wedge Bet\ A\ C\ B \rangle$ 
      assms(1) outer-transitivity-between by blast
  moreover
  have  $A' \neq A$ 
    using  $\langle Bet\ A\ B\ A' \rangle$  assms(1) bet-neq12--neq by blast
  moreover
  have  $Bet\ C\ B\ A'$ 
    using  $\langle Bet\ A\ B\ A' \rangle \langle Bet\ D\ A\ C \wedge Bet\ A\ C\ B \rangle$  between-exchange3 by blast

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moreover
have  $A' \neq B$ 
  using  $\langle \text{Cong } B \ A' \ A \ B \rangle$  assms(1) cong-reverse-identity by blast
ultimately
show ?thesis
  by (meson  $\langle E \ A \ D \ \text{CongA } F \ B \ C \rangle$  conga-comm l11-13)
qed
hence  $E \ A \ A' \ \text{CongA } F \ B \ A'$ 
  by (meson conga-comm)
hence  $A \ E \ \text{Par } F \ B$ 
  using  $\langle A \ B \ \text{OS } E \ F \rangle$   $\langle \text{Bet } A \ B \ A' \rangle$  conga-to-par-os by blast
}
moreover
{
  assume  $\text{Bet } A \ D \ B \wedge \text{Bet } D \ B \ C$ 
  obtain  $A'$  where  $\text{Bet } A \ B \ A'$  and  $\text{Cong } B \ A' \ A \ B$ 
  using segment-construction by blast
  have  $A' \ A \ E \ \text{CongA } A' \ B \ F$ 
  proof –
    have  $A \ \text{Out } A' \ D$ 
      by (metis Out-def  $\langle A \neq D \rangle$   $\langle \text{Bet } A \ B \ A' \rangle$ 
         $\langle \text{Bet } A \ D \ B \wedge \text{Bet } D \ B \ C \rangle$  bet-neq12--neq between-exchange4)
    moreover
    have  $A \ \text{Out } E \ E$ 
      using CongA-def  $\langle E \ A \ D \ \text{CongA } F \ B \ C \rangle$  out-trivial by presburger
    moreover
    have  $B \ \text{Out } A' \ C$ 
      by (metis False  $\langle B \neq C \rangle$   $\langle \text{Bet } A \ B \ A' \rangle$   $\langle \text{Bet } A \ D \ B \wedge \text{Bet } D \ B \ C \rangle$ 
         $\langle \text{Cong } A \ B \ C \ D \rangle$   $\langle \text{Cong } A \ D \ C \ B \rangle$   $\langle \text{Cong } B \ A' \ A \ B \rangle$  bet3-cong3-bet
        bet-out calculation(1) cong-left-commutativity cong-symmetry
        l6-6 out-distinct)
    moreover
    have  $B \ \text{Out } F \ F$ 
      using CongA-def  $\langle E \ A \ D \ \text{CongA } F \ B \ C \rangle$  out-trivial by presburger
    ultimately
    show ?thesis
      by (meson  $\langle E \ A \ D \ \text{CongA } F \ B \ C \rangle$  conga-comm l11-10)
  qed
  hence  $E \ A \ A' \ \text{CongA } F \ B \ A'$ 
    by (simp add: conga-comm)
  hence  $A \ E \ \text{Par } F \ B$ 
    using  $\langle A \ B \ \text{OS } E \ F \rangle$   $\langle \text{Bet } A \ B \ A' \rangle$  conga-to-par-os by blast
}
moreover
{
  assume  $\text{Bet } A \ B \ D \wedge \text{Bet } B \ D \ C$ 
  obtain  $A'$  where  $\text{Bet } A \ B \ A'$  and  $\text{Cong } B \ A' \ A \ C$ 
  using segment-construction by blast
  have  $A' \ A \ E \ \text{CongA } A' \ B \ F$ 
  proof –
    have  $A \ \text{Out } A' \ D$ 
      by (metis Out-def  $\langle \text{Bet } A \ B \ A' \rangle$   $\langle \text{Bet } A \ B \ D \wedge \text{Bet } B \ D \ C \rangle$ 
        assms(1) bet-out l5-1)
    moreover
    have  $A \ \text{Out } E \ E$ 
      using CongA-def  $\langle E \ A \ D \ \text{CongA } F \ B \ C \rangle$  out-trivial by presburger
    moreover
    have  $B \ \text{Out } A' \ C$ 
      by (metis Bet-cases False  $\langle B \neq C \rangle$   $\langle \text{Bet } A \ B \ A' \rangle$ 
         $\langle \text{Bet } A \ B \ D \wedge \text{Bet } B \ D \ C \rangle$   $\langle \text{Cong } B \ A' \ A \ C \rangle$  assms(1)
        bet-cong-eq l6-2 outer-transitivity-between)
    moreover
    have  $B \ \text{Out } F \ F$ 
      using CongA-def  $\langle E \ A \ D \ \text{CongA } F \ B \ C \rangle$  out-trivial by presburger
    ultimately
    show ?thesis

```

```

      by (meson ⟨E A D CongA F B C⟩ conga-comm l11-10)
    qed
  hence E A A' CongA F B A'
    by (simp add: conga-comm)
  hence A E Par F B
    using ⟨A B OS E F⟩ ⟨Bet A B A'⟩ conga-to-par-os by blast
}
ultimately
show ?thesis
  using H1 by fastforce
qed
thus ?thesis
  by (meson Par-cases ⟨A B OS E F⟩ ⟨A B Par E F⟩
      one-side-not-col124 parallel-2-plg)
qed
qed
qed
qed
lemma plgf-plgf-plgf:
  assumes A ≠ B and
    ParallelogramFlat A B C D and
    ParallelogramFlat C D E F
  shows ParallelogramFlat A B F E
proof -
  have C ≠ D
    using assms(1) assms(2) plgf-not-comm1 by blast
  hence E ≠ F
    using assms(3) plgf-not-comm1 by blast
  obtain D' C' where ParallelogramStrict C D D' C'
    using ⟨C ≠ D⟩ plgs-existence by fastforce
  hence ParallelogramStrict A B C' D'
    using assms(1) assms(2) plgf-plgs-trans by blast
  have ParallelogramStrict E F C' D'
    by (meson plgf-plgs-trans Tarski-Euclidean-axioms ⟨E ≠ F⟩
        ⟨ParallelogramStrict C D D' C'⟩ assms(3) plgf-sym)
  hence ParallelogramStrict C' D' E F
    using plgs-sym by blast
  hence Parallelogram A B F E
    using ⟨ParallelogramStrict A B C' D'⟩ plgs-pseudo-trans by blast
  thus ?thesis
    by (meson ParallelogramFlat-def ⟨C ≠ D⟩ assms(2) assms(3) colx plg-col-plgf)
qed

lemma plg-pseudo-trans:
  assumes Parallelogram A B C D and
    Parallelogram C D E F
  shows Parallelogram A B F E ∨ (A = B ∧ C = D ∧ E = F ∧ A = E)
proof (cases A = B)
  case True
  thus ?thesis
    by (metis assms(1) assms(2) plg-not-comm-R1 plg-sym plg-trivial1)
next
  case False
  hence A ≠ B
    by auto
  hence C ≠ D
    using False assms(1) plg-not-comm-R1 by blast
  hence E ≠ F
    by (metis ⟨C ≠ D⟩ assms(2) plg-not-comm-R1)
  {
    assume ParallelogramStrict A B C D
    assume ParallelogramStrict C D E F
    have Parallelogram A B F E
      using ⟨ParallelogramStrict A B C D⟩ ⟨ParallelogramStrict C D E F⟩
      plgs-pseudo-trans by blast
  }

```

```

}
moreover
{
  assume ParallelogramStrict A B C D
  assume ParallelogramFlat C D E F
  have Parallelogram A B F E
  by (metis Parallelogram-strict-Parallelogram Plg-perm
    plgf-plgs-trans plgs-sym  $\langle E \neq F \rangle$   $\langle \text{ParallelogramFlat } C \ D \ E \ F \rangle$ 
     $\langle \text{ParallelogramStrict } A \ B \ C \ D \rangle$  plgf-sym)
}
moreover
{
  assume ParallelogramFlat A B C D
  assume ParallelogramStrict C D E F
  have Parallelogram A B F E
  by (meson False plgf-plgs-trans Parallelogram-strict-Parallelogram
     $\langle \text{ParallelogramFlat } A \ B \ C \ D \rangle$   $\langle \text{ParallelogramStrict } C \ D \ E \ F \rangle$ )
}
moreover
{
  assume ParallelogramFlat A B C D
  assume ParallelogramFlat C D E F
  have ParallelogramFlat A B F E
  using False  $\langle \text{ParallelogramFlat } A \ B \ C \ D \rangle$   $\langle \text{ParallelogramFlat } C \ D \ E \ F \rangle$ 
    plgf-plgf-plgf by blast
  hence Parallelogram A B F E
  by (simp add: Parallelogram-def)
}
ultimately show ?thesis
by (metis Parallelogram-def assms(1) assms(2))
qed

```

lemma *Square-Rhombus*:
assumes *Square* A B C D
shows *Rhombus* A B C D
using *Rectangle-def* *Rhombus-def* *Square-def* *assms* **by** *presburger*

lemma *plgs-in-angle*:
assumes *ParallelogramStrict* A B C D
shows D *InAngle* A B C
by (*meson* *ParallelogramStrict-def* *invert-one-side* *os-ts--inangle*
assms *plgs-comm2* *plgs-one-side*)

lemma *par-par-cong-cong-parallelogram*:
assumes $B \neq D$ **and**
Cong A B C D **and**
Cong B C D A **and**
B C Par A D **and**
A B Par C D
shows *Parallelogram* A B C D
by (*metis* *Plg-perm* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5)
 cong-right-commutativity *par-cong-plg-2* *plg-not-comm-R1*)

lemma *degenerated-rect-eq*:
assumes *Rectangle* A B B C
shows A = C
by (*meson* *assms* *l8-2* *l8-7* *rect-per1* *rect-per4*)

lemma *rect-2-rect*:
assumes $A \neq B$ **and**
Rectangle A B C1 D1 **and**
Rectangle A B C2 D2
shows *Rectangle* C1 D1 D2 C2
proof (*cases* C1 = C2)
case *True*
obtain M **where** M *Midpoint* A C1 $\wedge$ M *Midpoint* B D1

```

    using Plg-def Rectangle-Plg assms(2) by presburger
  obtain M' where M' Midpoint A C1  $\wedge$  M' Midpoint B D2
    using Plg-def Rectangle-Plg True assms(3) by presburger
  have Parallelogram C1 D2 D2 C1
    by (metis Rectangle-Parallelogram plg-trivial True assms(1)
      assms(3) plg-not-comm-R2 rect-comm2)
  thus ?thesis
    by (metis MidR-uniq-aux SymR-uniq-aux True
       $\langle M' \text{ Midpoint } A \ C1 \wedge M' \text{ Midpoint } B \ D2 \rangle$ 
       $\langle \bigwedge thesis. (\bigwedge M. M \text{ Midpoint } A \ C1 \wedge M \text{ Midpoint } B \ D1 \implies thesis) \implies thesis \rangle$ 
      parallelogram-equiv-plg plg-cong-1 plg-cong-rectangle)
next
  case False
  hence C1  $\neq$  C2
    by blast
  thus ?thesis
  proof (cases B = C1)
    case True
    thus ?thesis
      using assms(2) assms(3) degenerated-rect-eq rect-comm2 by blast
  next
    case False
    hence B  $\neq$  C1
      by auto
    show ?thesis
    proof (cases B = C2)
      case True
      thus ?thesis
        by (metis assms(2) assms(3) degenerated-rect-eq rect-permut)
    next
      case False
      hence B  $\neq$  C2
        by auto
      have Parallelogram A B C1 D1
        using Rectangle-Parallelogram assms(2) by blast
      have Parallelogram A B C2 D2
        using Rectangle-Parallelogram assms(3) by blast
      have Per A B C1
        using assms(2) rect-per2 by blast
      hence  $\neg \text{Col } A \ B \ C1$ 
        using  $\langle B \neq C1 \rangle$  assms(1) l8-9 by blast
      have Per A B C2
        using assms(3) rect-per2 by blast
      hence  $\neg \text{Col } A \ B \ C2$ 
        using False assms(1) l8-9 by blast
      have Plg D2 C2 C1 D1
        by (meson  $\langle \text{Parallelogram } A \ B \ C1 \ D1 \rangle \langle \text{Parallelogram } A \ B \ C2 \ D2 \rangle$ 
          assms(1) parallelogram-to-plg plg-permut plg-pseudo-trans)
      moreover
      have Per C1 C2 D2
      proof (cases Coplanar A B C1 C2)
        case True
        show ?thesis
        proof -
          have A B Par D2 C2
            using False  $\langle \text{Parallelogram } A \ B \ C2 \ D2 \rangle$  assms(1)
              par-right-comm plg-par by blast
          moreover have A B Perp C2 C1
          proof -
            have B C1 Perp A B
              using Perp-perm  $\langle B \neq C1 \rangle \langle \text{Per } A \ B \ C1 \rangle$ 
                assms(1) per-perp by blast
            moreover have A  $\neq$  B
              by (simp add: assms(1))
            moreover have B  $\neq$  C1
              by (simp add:  $\langle B \neq C1 \rangle$ )
          qed
        qed
      qed
    qed
  qed

```

```

moreover have Col B C1 C2
  using True ⟨Per A B C1⟩ ⟨Per A B C2⟩ calculation(2)
    col-permutation-2 cop-per2--col coplanar-perm-3 l8-2 by blast
moreover have Col B C1 C1
  by (simp add: col-trivial-2)
moreover have C1 ≠ C2
  using ⟨C1 ≠ C2⟩ by auto
ultimately show ?thesis using perp-col0
  by presburger
qed
moreover have Coplanar D2 C2 C2 C1
  using ncop-distincts by blast
ultimately show ?thesis
  using Perp-perm cop-par-perp--perp perp-per-2 by blast
qed
next
case False
have Per B C2 D2
  using assms(3) rect-per3 by auto
have C2 ≠ D2
  using ⟨Parallelogram A B C2 D2⟩ assms(1) cong-identity plg-cong-1 by blast
have  $\neg$  Col B C1 C2
  using False ncop--ncols by blast
hence B OrthAt B C1 C2 B A
  by (metis ⟨Per A B C1⟩ ⟨Per A B C2⟩ assms(1) col-trivial-3
    l11-60-bis l8-2 l8-5 ncop--ncols)
hence C2 OrthAt B C1 C2 C2 D2
  by (meson ⟨B ≠ C2⟩ ⟨C2 ≠ D2⟩ ⟨Parallelogram A B C2 D2⟩
    ⟨Per B C2 D2⟩ l11-61-bis ncop-distincts ncoplanar-perm-12
    parallelogram--coplanar perp-perp)
thus ?thesis
  by (metis OrthAt-def col-trivial-2 ncop-distincts)
qed
ultimately
show ?thesis
  by (simp add: plg-per-rect rect-permut)
qed
qed
qed

lemma ncol123-plg--plgs:
  assumes  $\neg$  Col A B C and
    Parallelogram A B C D
shows ParallelogramStrict A B C D
using ParallelogramFlat-def Parallelogram-def assms(1) assms(2) by presburger

lemma ncol124-plg--plgs:
  assumes  $\neg$  Col A B D and
    Parallelogram A B C D
shows ParallelogramStrict A B C D
by (meson assms(1) assms(2) mid-plgs not-col-permutation-4 plg-mid plgs-comm2)

lemma ncol134-plg--plgs:
  assumes  $\neg$  Col A C D and
    Parallelogram A B C D
shows ParallelogramStrict A B C D
by (meson assms(1) assms(2) col-permutation-2 ncol123-plg--plgs plg-sym plgs-sym)

lemma ncol234-plg--plgs:
  assumes  $\neg$  Col B C D and
    Parallelogram A B C D
shows ParallelogramStrict A B C D
by (meson assms(1) assms(2) col-permutation-2 ncol124-plg--plgs plg-sym plgs-sym)

lemma ncol123-plg--pars1234:
  assumes  $\neg$  Col A B C and

```

Parallelogram A B C D
shows *A B ParStrict C D*
by (*meson* *assms(1) assms(2) mid-plgs plg-mid plgs-pars-1*)

lemma *ncol124-plg--pars1234*:
assumes $\neg$ *Col A B D* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
by (*meson* *assms(1) assms(2) midpoint-par-strict plg-mid*)

lemma *ncol134-plg--pars1234*:
assumes $\neg$ *Col A C D* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
using *Par-strict-perm Plg-perm assms(1) assms(2) col-permutation-3*
ncol124-plg--pars1234 **by** *blast*

lemma *ncol234-plg--pars1234*:
assumes $\neg$ *Col B C D* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
using *Par-strict-perm Plg-perm assms(1) assms(2) ncol123-plg--pars1234*
not-col-permutation-3 **by** *blast*

lemma *ncol123-plg--pars1423*:
assumes $\neg$ *Col A B C* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
using *Par-def assms(1) assms(2) not-col-distincts plg-par* **by** *blast*

lemma *ncol124-plg--pars1423*:
assumes $\neg$ *Col A B D* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
by (*metis* *assms(1) assms(2) ncol124-plg--pars1234 not-col-distincts*
par-strict-not-col-2 pars-par-pars plg-par)

lemma *ncol134-plg--pars1423*:
assumes $\neg$ *Col A C D* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
by (*meson* *assms(1) assms(2) ncol123-plg--pars1423 ncol134-plg--pars1234*
par-strict-not-col-1)

lemma *ncol234-plg--pars1423*:
assumes $\neg$ *Col B C D* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
by (*metis* *assms(1) assms(2) ncol134-plg--pars1423 ncol234-plg--pars1234*
not-col-distincts pars-par-pars plg-par)

lemma *sac-plg*:
assumes *Saccheri A B C D*
shows *Parallelogram A B C D*
proof –
have *A B ParStrict C D*
by (*simp* *add: assms sac--pars1234*)
thus *?thesis*
by (*meson* *par-2-plg Tarski-Euclidean-axioms assms par-strict-not-col-1*
par-strict-par sac--pars1423)

qed

lemma *sac-rectangle*:
assumes *Saccheri A B C D*
shows *Rectangle A B C D*
proof –

```

have Parallelogram A B C D
  by (meson assms sac-plg)
thus ?thesis
  by (simp add: Rectangle-def assms parallelogram-to-plg sac--cong)
qed

lemma exists-square:
  assumes A ≠ B
  shows ∃ C D. Square A B C D
proof -
  obtain C where Per A B C and Cong B C A B
    using exists-cong-per by blast
  obtain D where Saccheri B C D A
    by (metis Per-cases ⟨Cong B C A B⟩ ⟨Per A B C⟩ assms cong-diff-3 per--ex-saccheri)
  hence Rectangle B C D A
    by (simp add: sac-rectangle)
  thus ?thesis
    by (meson Cong-cases Square-def ⟨Cong B C A B⟩ rect-permut)
qed

```

```

lemma euclidT:
  assumes Bet A D T and
    Bet B D C and
    A ≠ D
  shows ∃ X Y. Bet A B X ∧ Bet A C Y ∧ Bet X T Y
proof -
  {
    assume B ≠ D
    {
      assume Col A B C
      {
        assume Bet A B C
        {
          assume A ≠ B
          moreover have Bet A B C
            using ⟨Bet A B C⟩ by blast
          moreover have Bet A B T
            using assms(1) assms(2) between-exchange4
              between-inner-transitivity calculation(2) by blast
          moreover have Bet B C T ⟶ ?thesis
            using calculation(3) not-bet-distincts by blast
          moreover have Bet B T C ⟶ ?thesis
            using not-bet-distincts by blast
          ultimately have ?thesis using l5-2 by blast
        }
        hence ?thesis
          using between-symmetry between-trivial by blast
      }
      moreover have Bet B C A ⟶ ?thesis
        using assms(1) assms(2) assms(3) tarski-s-parallel-postulate by presburger
      moreover have Bet C A B ⟶ ?thesis
        using assms(1) assms(2) assms(3) tarski-s-parallel-postulate by presburger
      ultimately have ?thesis
        using Col-def ⟨Col A B C⟩ by blast
    }
  }
  moreover
  {
    assume ¬ Col A B C
    {
      {
        assume D ≠ C
        hence ?thesis
          using assms(1) assms(2) assms(3) tarski-s-parallel-postulate by blast
      }
    }
  }

```



```

    hence ?thesis
    using assms(1) not-bet-distincts by blast
  }
  hence ?thesis
  by blast
}
ultimately have ?thesis
by blast
}
thus ?thesis
using assms(1) between-symmetry between-trivial by blast
qed

```

```

lemma eqv-refl:
  shows  $A \ B \text{ EqV } A \ B$ 
  using EqV-def plg-trivial by blast

```

```

lemma eqv-sym:
  assumes  $A \ B \text{ EqV } C \ D$ 
  shows  $C \ D \text{ EqV } A \ B$ 
  using EqV-def Plg-perm assms by blast

```

```

lemma eqv-trans:
  assumes  $A \ B \text{ EqV } C \ D$  and
     $C \ D \text{ EqV } E \ F$ 
  shows  $A \ B \text{ EqV } E \ F$ 
proof -
  {
    assume Parallelogram  $A \ B \ D \ C$ 
    hence Parallelogram  $C \ D \ F \ E \longrightarrow ?thesis$ 
      using EqV-def plg-comm2 plg-pseudo-trans by blast
    moreover have  $C = D \wedge F = E \longrightarrow ?thesis$ 
      by (metis assms(1) calculation plg-trivial1)
    ultimately have ?thesis
      using EqV-def assms(2) by blast
  }
  moreover have  $A = B \wedge C = D \longrightarrow ?thesis$ 
    using assms(2) calculation plg-trivial1 by force
  ultimately show ?thesis
    using EqV-def assms(1) by force
qed

```

```

lemma eqv-comm:
  assumes  $A \ B \text{ EqV } C \ D$ 
  shows  $B \ A \text{ EqV } D \ C$ 
  by (metis EqV-def assms eqv-sym plg-sym)

```

```

lemma vector-construction:
  shows  $\exists \ D. \ A \ B \text{ EqV } C \ D$ 
  by (metis EqV-def eqv-comm plg-existence)

```

```

lemma vector-construction-uniqueness:
  assumes  $A \ B \text{ EqV } C \ D$  and
     $A \ B \text{ EqV } C \ D'$ 
  shows  $D = D'$ 
proof -
  {
    assume Parallelogram  $A \ B \ D \ C$ 
    hence Parallelogram  $A \ B \ D' \ C \longrightarrow ?thesis$ 
      by (meson plg-comm2 plg-uniqueness)
    moreover have  $A = B \wedge C = D' \longrightarrow ?thesis$ 
      using  $\langle \text{Parallelogram } A \ B \ D \ C \rangle$  cong-reverse-identity plg-cong by blast
    ultimately have ?thesis
      using EqV-def assms(2) by force
  }

```

```

}
moreover have  $A = B \wedge C = D \longrightarrow ?thesis$ 
  by (metis EqV-def assms(2) cong-reverse-identity plg-cong)
ultimately show  $?thesis$ 
  using EqV-def assms(1) by force
qed

lemma null-vector:
  assumes  $A \ A \ EqV \ B \ C$ 
  shows  $B = C$ 
  using EqV-def assms vector-construction-uniqueness by blast

lemma vector-uniqueness:
  assumes  $A \ B \ EqV \ A \ C$ 
  shows  $B = C$ 
  using assms eqv-refl vector-construction-uniqueness by blast

lemma eqv-trivial:
  shows  $A \ A \ EqV \ B \ B$ 
  by (simp add: EqV-def)

lemma eqv-permut:
  assumes  $A \ B \ EqV \ C \ D$ 
  shows  $A \ C \ EqV \ B \ D$ 
  using EqV-def assms eqv-refl eqv-sym plg-permut by presburger

lemma eqv-par:
  assumes  $A \neq B$  and
     $A \ B \ EqV \ C \ D$ 
  shows  $A \ B \ Par \ C \ D$ 
  by (metis EqV-def Par-cases assms(1) assms(2) eqv-comm par-reflexivity plg-par)

lemma eqv-opp-null:
  assumes  $A \ B \ EqV \ B \ A$ 
  shows  $A = B$ 
  by (meson EqV-def assms plg-not-comm plg-trivial)

lemma eqv-sum:
  assumes  $A \ B \ EqV \ A' \ B'$  and
     $B \ C \ EqV \ B' \ C'$ 
  shows  $A \ C \ EqV \ A' \ C'$ 
  by (meson assms(1) assms(2) eqv-permut eqv-trans)

lemma null-sum:
  shows  $A \ B \ B \ A \ SumV \ C \ C$ 
  using EqV-def SumV-def eqv-permut null-vector by blast

lemma chasles:
  shows  $A \ B \ B \ C \ SumV \ A \ C$ 
  using SumV-def eqv-permut eqv-refl null-vector by blast

lemma eqv-mid :
  assumes  $A \ B \ EqV \ B \ C$ 
  shows  $B \ Midpoint \ A \ C$ 
  using EqV-def assms l7-3-2 plg-comm2 plg-mid-2 by blast

lemma mid-equiv:
  assumes  $A \ Midpoint \ B \ C$ 
  shows  $B \ A \ EqV \ A \ C$ 
  by (metis EqV-def assms midpoint-distinct-1 plg-existence plg-mid-2)

lemma sum-sym:
  assumes  $A \ B \ C \ D \ SumV \ E \ F$ 
  shows  $C \ D \ A \ B \ SumV \ E \ F$ 
proof –
  {

```

```

fix D'0
assume A B EqV D D'0
obtain D' where C D EqV B D'
  using vector-construction by blast
have A D' EqV E F
  using SumV-def ⟨C D EqV B D'⟩ assms by blast
have C B EqV D D'
  by (simp add: ⟨C D EqV B D'⟩ eqv-permut)
have A D EqV B D'0
  by (simp add: ⟨A B EqV D D'0⟩ eqv-permut)
have C D'0 EqV E F
proof (cases A = D'0)
  case True
  have A Midpoint B D
  using True ⟨A B EqV D D'0⟩ eqv-comm eqv-mid by blast
  have Parallelogram C D D' B ⟶ ?thesis
  using Plg-perm True ⟨A D' EqV E F⟩ ⟨A Midpoint B D⟩ eqv-trans mid-equiv plg-mid-2 by blast
  moreover have C = D ∧ B = D' ⟶ ?thesis
  using ⟨A B EqV D D'0⟩ ⟨A D' EqV E F⟩ eqv-sym eqv-trans by blast
  ultimately show ?thesis
  using EqV-def ⟨C D EqV B D'⟩ by blast
next
  case False
  have Parallelogram C D D' B ⟶ ?thesis
  proof -
  {
    assume Parallelogram A B D'0 D
    obtain M0 where M0 Midpoint C D' and M0 Midpoint D B
      by (metis EqV-def MidR-uniqu-aux ⟨C D EqV B D'⟩ plg-mid-2)
    obtain M1 where M1 Midpoint A D'0 and M1 Midpoint B D
      using ⟨Parallelogram A B D'0 D⟩ plg-mid by blast
    have M1 = M0
      using ⟨M0 Midpoint D B⟩ ⟨M1 Midpoint B D⟩ l7-17-bis by auto
    have A ≠ D'0 ∨ D' ≠ C
      by (simp add: False)
    hence Parallelogram A D' D'0 C
      using mid-plg [where ?M = M0] ⟨M0 Midpoint C D'⟩ ⟨M1 = M0⟩ ⟨M1 Midpoint A D'0⟩ l7-2 by presburger
    hence ?thesis
      by (meson EqV-def ⟨A D' EqV E F⟩ eqv-sym eqv-trans)
  }
  moreover have A ≠ B ∧ D'0 ≠ B ⟶ ?thesis
    using EqV-def ⟨A B EqV D D'0⟩ calculation by blast
  ultimately show ?thesis
    using EqV-def False ⟨A B EqV D D'0⟩ plg-trivial1 by blast
qed
moreover have C = D ∧ B = D' ⟶ ?thesis
  using ⟨A B EqV D D'0⟩ ⟨A D' EqV E F⟩ eqv-sym eqv-trans by blast
ultimately show ?thesis
  using EqV-def ⟨C D EqV B D'⟩ by blast
qed
}
thus ?thesis using SumV-def
  by auto
qed

lemma opposite-sum:
  assumes A B C D SumV E F
  shows B A D C SumV F E
proof -
{
  fix D0
  assume D C EqV A D0
  obtain D' where C D EqV B D'
    using vector-construction by blast
  hence A D' EqV E F
    using SumV-def assms by blast

```

```

  have  $D' B \text{ EqV } A D0$ 
    using  $\langle C D \text{ EqV } B D' \rangle \langle D C \text{ EqV } A D0 \rangle \text{ eqv-comm eqv-sym eqv-trans}$  by blast
  hence  $B D0 \text{ EqV } F E$ 
    by (meson  $\langle A D' \text{ EqV } E F \rangle \text{ eqv-comm eqv-permut eqv-trans}$ )
}
thus ?thesis
  using SumV-def by blast
qed

```

```

lemma null-sum-eq:
  assumes  $A B B C \text{ SumV } D D$ 
  shows  $A = C$ 
proof -
  obtain  $D'$  where  $B C \text{ EqV } B D'$ 
    using vector-construction by blast
  hence  $A D' \text{ EqV } D D$ 
    using SumV-def assms by blast
  have  $A = D'$ 
    using  $\langle A D' \text{ EqV } D D \rangle \text{ eqv-sym null-vector}$  by blast
  thus ?thesis
    using  $\langle B C \text{ EqV } B D' \rangle \text{ vector-uniqueness}$  by blast
qed

```

```

lemma is-to-ise:
  assumes  $A B C D \text{ SumV } E F$ 
  shows  $A B C D \text{ SumVExists } E F$ 
  by (meson SumVExists-def SumV-def assms eqv-sym vector-construction)

```

```

lemma ise-to-is:
  assumes  $A B C D \text{ SumVExists } E F$ 
  shows  $A B C D \text{ SumV } E F$ 
proof -
  obtain  $D'$  where  $B D' \text{ EqV } C D$ 
    using SumVExists-def assms by blast
  thus ?thesis
    by (metis SumVExists-def SumV-def assms eqv-sym vector-construction-uniqueness)
qed

```

```

lemma sum-exists:
  shows  $\exists E F. A B C D \text{ SumV } E F$ 
  by (metis SumV-def vector-construction vector-construction-uniqueness)

```

```

lemma sum-uniqueness-pappus:

```

```

  assumes  $A B C D \text{ SumV } E F$  and
     $A B C D \text{ SumV } E' F'$ 
  shows  $E F \text{ EqV } E' F'$ 
  by (metis SumV-def assms(1) assms(2) eqv-sym eqv-trans vector-construction)

```

```

lemma same-dir-refl:
  shows  $A B \text{ SameDir } A B$ 
proof (cases  $A = B$ )
  case True
  thus ?thesis
    using SameDir-def by force
next
  case False
  thus ?thesis
    using SameDir-def by (metis eqv-refl out-trivial)
qed

```

```

lemma same-dir-ts:
  assumes  $A B \text{ SameDir } C D$ 
  shows  $\exists M. \text{Bet } A M D \wedge \text{Bet } B M C$ 
proof -
  have  $(A = B \wedge C = D) \longrightarrow ?thesis$ 

```

```

using between-trivial by blast
moreover {
  assume  $\exists D'. C \text{ Out } D D' \wedge A B \text{ EqV } C D'$ 
  then obtain  $D'$  where  $C \text{ Out } D D'$  and  $A B \text{ EqV } C D'$ 
  by blast
  moreover {
    assume Parallelogram  $A B D' C$ 
    obtain  $M$  where  $M \text{ Midpoint } A D'$  and  $M \text{ Midpoint } B C$ 
    using  $\langle \text{Parallelogram } A B D' C \rangle$  plg-mid by blast
  }
  assume ParallelogramStrict  $A B D' C$ 
  have  $B C \text{ TS } A D'$ 
  by (simp add:  $\langle \text{ParallelogramStrict } A B D' C \rangle$  plgs-two-sides)
  have  $A D' \text{ TS } B C$ 
  by (simp add:  $\langle \text{ParallelogramStrict } A B D' C \rangle$  plgs-two-sides)
  have  $B \neq C$ 
  using  $\langle A D' \text{ TS } B C \rangle$  ts-distincts by presburger
  have  $\neg \text{Col } B C D'$ 
  using  $\langle \text{ParallelogramStrict } A B D' C \rangle$  col-permutation-5 plgs-not-col by blast
  have  $B C \text{ OS } D' D$ 
  using  $\langle \neg \text{Col } B C D' \rangle$  calculation(1) l6-6 not-col-distincts out-one-side-1 by blast
  have  $B C \text{ TS } A D$ 
  using  $\langle B C \text{ OS } D' D \rangle \langle B C \text{ TS } A D' \rangle$  l9-2 l9-8-2 by blast
  have  $\neg \text{Col } A B C$ 
  using TS-def  $\langle B C \text{ TS } A D' \rangle$  by blast
  have  $A B \text{ ParStrict } D' C$ 
  by (simp add:  $\langle \text{ParallelogramStrict } A B D' C \rangle$  plgs-par-strict)
  have  $A C \text{ ParStrict } B D'$ 
  using  $\langle \text{ParallelogramStrict } A B D' C \rangle$  plgs-par-strict by blast
  have  $A \neq B$ 
  using  $\langle \neg \text{Col } A B C \rangle$  not-col-distincts by blast
  have  $C D \text{ Par } A B$ 
  by (metis Out-cases  $\langle B C \text{ TS } A D \rangle \langle \exists D'. C \text{ Out } D D' \wedge A B \text{ EqV } C D' \rangle$ 
    equiv-par out-col par-col-par par-symmetry ts-distincts)
  have  $A B \text{ ParStrict } C D$ 
  using Col-cases Par-def Par-strict-cases  $\langle C D \text{ Par } A B \rangle \langle \neg \text{Col } A B C \rangle$  by blast
  obtain  $T$  where  $\text{Col } T B C$  and  $\text{Bet } A T D$ 
  using TS-def  $\langle B C \text{ TS } A D \rangle$  by blast
  hence  $\text{Col } A B T \longrightarrow ?thesis$ 
  by (metis  $\langle \neg \text{Col } A B C \rangle$  between-trivial2 col-trivial-2 colx)
  moreover
  {
    assume  $\neg \text{Col } A B T$ 
    have  $\text{Col } C D T \longrightarrow ?thesis$ 
    by (metis TS-def  $\langle A B \text{ ParStrict } C D \rangle \langle B C \text{ TS } A D \rangle \langle \text{Bet } A T D \rangle$ 
       $\langle \text{Col } T B C \rangle$  bet-col1 between-trivial l6-16-1 par-strict-not-col-3)
    moreover
    {
      assume  $\neg \text{Col } C D T$ 
      have  $\text{Bet } T B C \longrightarrow ?thesis$ 
      using Par-strict-cases  $\langle A B \text{ ParStrict } C D \rangle \langle \text{Bet } A T D \rangle$ 
         $\langle \text{Col } T B C \rangle \langle \neg \text{Col } A B C \rangle \langle \neg \text{Col } A B T \rangle$  between-trivial
        between-trivial2 impossible-case-4-2 not-col-permutation-2 by blast
    }
    moreover
    {
      assume  $\text{Bet } B C T$ 
      have  $C D \text{ OS } T A$ 
      by (metis  $\langle \text{Bet } A T D \rangle \langle \neg \text{Col } C D T \rangle$  bet-out-1 not-col-distincts out-one-side-1)
      have  $C D \text{ OS } T B$ 
      using  $\langle A B \text{ ParStrict } C D \rangle \langle C D \text{ OS } T A \rangle$  one-side-transitivity
        pars--os3412 by blast
      moreover have  $C D \text{ TS } T B$ 
      using  $\langle B \neq C \rangle \langle \text{Bet } B C T \rangle \langle \neg \text{Col } C D T \rangle$  bet--ts
        between-symmetry by presburger
      ultimately have False
      using l9-9-bis by auto
    }
  }
}

```

```

    hence ?thesis
    by blast
  }
  moreover have  $Bet\ C\ T\ B \longrightarrow ?thesis$ 
    using  $\langle Bet\ A\ T\ D \rangle$  between-symmetry by blast
  ultimately have ?thesis
    using Col-def  $\langle Col\ T\ B\ C \rangle$  by blast
  }
  ultimately have ?thesis
    by blast
}
ultimately have ?thesis
  by blast
}
moreover
{
  assume ParallelogramFlat  $A\ B\ D'\ C$ 
  hence  $Bet\ C\ D'\ A \wedge Bet\ D'\ A\ B \vee Bet\ C\ A\ D' \wedge Bet\ A\ D'\ B \vee$ 
     $Bet\ A\ C\ B \wedge Bet\ C\ B\ D' \vee Bet\ A\ B\ C \wedge Bet\ B\ C\ D'$ 
    by (simp add: plgf-bet)
  have  $A = D' \longrightarrow ?thesis$ 
    by (metis Midpoint-def
       $\langle \bigwedge thesis. (\bigwedge M. \llbracket M\ Midpoint\ A\ D'; M\ Midpoint\ B\ C \rrbracket \Longrightarrow thesis) \Longrightarrow thesis \rangle$ 
      between-identity not-bet-distincts)
  moreover
  {
    assume  $A \neq D'$ 
    {
      assume  $B = C$ 
      have  $A = D' \vee B\ Midpoint\ A\ D'$ 
        using  $\langle A\ B\ EqV\ C\ D' \rangle \langle B = C \rangle$  eqv-mid by auto
      moreover have  $A = D' \longrightarrow ?thesis$ 
        using  $\langle A \neq D' \rangle$  by blast
      moreover
      {
        assume  $B\ Midpoint\ A\ D'$ 
        have  $Bet\ B\ D\ D' \longrightarrow Bet\ A\ B\ D$ 
          using  $\langle B\ Midpoint\ A\ D' \rangle$  between-inner-transitivity midpoint-bet by blast
        moreover have  $Bet\ B\ D'\ D \longrightarrow Bet\ A\ B\ D$ 
          by (metis  $\langle B\ Midpoint\ A\ D' \rangle$  midpoint-bet midpoint-not-midpoint
            outer-transitivity-between)
        ultimately have  $Bet\ A\ B\ D$ 
          using Out-def  $\langle B = C \rangle \langle C\ Out\ D\ D' \rangle$  by blast
        hence ?thesis
          using not-bet-distincts by blast
      }
      ultimately have ?thesis
        by blast
    }
  }
  moreover
  {
    assume  $B \neq C$ 
    have  $Bet\ C\ D'\ A \wedge Bet\ D'\ A\ B \longrightarrow ?thesis$ 
      by (metis Bet-cases  $\langle A \neq D' \rangle$  between-trivial2 outer-transitivity-between2)
    moreover have  $Bet\ C\ A\ D' \wedge Bet\ A\ D'\ B \longrightarrow ?thesis$ 
      by (meson Bet-cases  $\langle A \neq D' \rangle$  between-trivial2 outer-transitivity-between)
    moreover
    {
      assume  $Bet\ A\ C\ B$  and  $Bet\ C\ B\ D'$ 
      have  $Bet\ A\ C\ D$ 
        by (metis  $\langle A\ B\ EqV\ C\ D' \rangle \langle B \neq C \rangle \langle Bet\ A\ C\ B \rangle \langle Bet\ C\ B\ D' \rangle$ 
           $\langle \exists D'. C\ Out\ D\ D' \wedge A\ B\ EqV\ C\ D' \rangle$  bet-out bet-out-out-bet l6-6
          not-bet-distincts vector-construction-uniqueness)
      moreover have  $Bet\ B\ C\ C$ 
        by (simp add: between-trivial)
      ultimately have ?thesis

```

```

    by blast
  }
  moreover have  $Bet\ A\ B\ C \wedge Bet\ B\ C\ D' \longrightarrow ?thesis$ 
    by (meson Out-cases  $\langle B \neq C \rangle \langle C\ Out\ D\ D' \rangle$  bet-out-1 bet-out-out-bet l5-3)
  ultimately have ?thesis
    using  $\langle Bet\ C\ D'\ A \wedge Bet\ D'\ A\ B \vee Bet\ C\ A\ D' \wedge$ 
       $Bet\ A\ D'\ B \vee Bet\ A\ C\ B \wedge Bet\ C\ B\ D' \vee Bet\ A\ B\ C \wedge Bet\ B\ C\ D' \rangle$ 
    by fastforce
  }
  ultimately have ?thesis
    by blast
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  using Parallelogram-def  $\langle Parallelogram\ A\ B\ D'\ C \rangle$  by blast
}
moreover have  $(A = B \wedge C = D') \longrightarrow ?thesis$ 
  using calculation(1) out-diff2 by blast
ultimately have ?thesis
  using EqV-def by blast
}
ultimately show ?thesis
  using SameDir-def assms by presburger
qed

```

lemma one-side-col-out:
 assumes $Col\ A\ X\ Y$ and
 $A\ B\ OS\ X\ Y$
 shows $A\ Out\ X\ Y$
 using assms(1) assms(2) col-one-side-out by force

lemma par-ts-same-dir:
 assumes $A\ B\ ParStrict\ C\ D$ and
 $\exists\ M. Bet\ A\ M\ D \wedge Bet\ B\ M\ C$
 shows $A\ B\ SameDir\ C\ D$
 proof –
 obtain M where $Bet\ A\ M\ D$ and $Bet\ B\ M\ C$
 using assms(2) by force
 obtain D' where $A\ B\ EqV\ C\ D'$
 using vector-construction by blast
 have $A \neq B$
 using assms(1) par-strict-neq1 by blast
 have $C \neq D$
 using assms(1) par-strict-distinct by blast
 have $A \neq M$
 using NCol-perm $\langle Bet\ B\ M\ C \rangle$ assms(1) bet-col par-strict-not-col-1 by blast
 have $B = D' \longrightarrow C\ Out\ D\ D'$
 by (metis Col-def EqV-def Plg-perm $\langle A\ B\ EqV\ C\ D' \rangle$ assms(1)
 not-bet-distincts par-strict-not-col-3 plg-not-comm)
 moreover
 {
 assume $B \neq D'$
 {
 assume Parallelogram $A\ B\ D'\ C$
 have $A\ B\ Par\ D'\ C$
 using $\langle A \neq B \rangle \langle B \neq D' \rangle \langle Parallelogram\ A\ B\ D'\ C \rangle$ plg-par by auto
 have $Col\ C\ D\ D'$
 by (metis Par-strict-cases $\langle Parallelogram\ A\ B\ D'\ C \rangle$ assms(1)
 col-permutation-4 ncol124-plg-pars1234 par-id-1
 par-strict-not-col-1 par-strict-trans)
 }
 assume ParallelogramStrict $A\ B\ D'\ C$
 have $A\ D'\ TS\ B\ C$
 by (simp add: $\langle ParallelogramStrict\ A\ B\ D'\ C \rangle$ plgs-two-sides)
 }
 }

```

have B C TS A D'
  by (simp add: ⟨ParallelogramStrict A B D' C⟩ plgs-two-sides)
have B C TS A D
  by (metis NCol-perm Par-strict-cases assms(1) assms(2) bet-col
    l9-18-R2 par-strict-not-col-2)
have B C OS D D'
proof (rule l9-8-1 [where ?C = A])
  show B C TS D A
    using ⟨B C TS A D⟩ l9-2 by blast
  show B C TS D' A
    by (simp add: ⟨B C TS A D'⟩ l9-2)
qed
hence C Out D D'
  using one-side-col-out [where ?B = B] ⟨B C OS D D'⟩ calculation
    invert-one-side ⟨Col C D D'⟩ by blast
}
moreover have ParallelogramFlat A B D' C ⟶ C Out D D'
  using ParallelogramFlat-def assms(1) par-strict-not-col-1 by force
ultimately have C Out D D'
  using Parallelogram-def ⟨Parallelogram A B D' C⟩ by blast
}
moreover have (A = B ∧ C = D') ⟶ C Out D D'
  by (simp add: ⟨A ≠ B⟩)
ultimately have C Out D D'
  using EqV-def ⟨A B EqV C D'⟩ by presburger
}
ultimately have C Out D D'
  by blast
hence ∃ D'. C Out D D' ∧ A B EqV C D'
  using ⟨A B EqV C D'⟩ by blast
thus ?thesis
  by (simp add: SameDir-def)
qed

```

lemma same-dir-out:

```

assumes A B SameDir A C
shows A Out B C ∨ (A = B ∧ A = C)
proof -
  have A = B ∧ A = C ∨ (∃ D'. A Out C D' ∧ A B EqV A D')
    using SameDir-def assms by presburger
  moreover
  have (∃ D'. A Out C D' ∧ A B EqV A D') ⟶ ?thesis
    using l6-6 vector-uniqueness by blast
  ultimately show ?thesis
    by blast
qed

```

lemma same-dir-out1:

```

assumes A B SameDir B C
shows A Out B C ∨ (A = B ∧ A = C)
proof -
  have A = B ∧ B = C ∨ (∃ D'. B Out C D' ∧ A B EqV B D')
    using SameDir-def assms by presburger
  moreover have (∃ D'. B Out C D' ∧ A B EqV B D') ⟶ ?thesis
    by (metis assms bet-neq23--neq bet-out-1 between-symmetry same-dir-out same-dir-ts)
  ultimately show ?thesis
    by blast
qed

```

lemma same-dir-null:

```

assumes A A SameDir B C
shows B = C
proof -
  have A = A ∧ B = C ∨ (∃ D'. B Out C D' ∧ A A EqV B D')
    using SameDir-def assms by presburger
  moreover have (∃ D'. B Out C D' ∧ A A EqV B D') ⟶ ?thesis

```



```

    using null-vector out-distinct by blast
ultimately show ?thesis
  by blast
qed

lemma plgs-out-plgs:
  assumes ParallelogramStrict A B C D and
    A Out B B' and
    D Out C C' and
    Cong A B' D C'
  shows ParallelogramStrict A B' C' D
proof -
  have A D OS C B
    using assms(1) plgs-comm2 plgs-one-side plgs-permut by blast
  have C B OS A D
    using assms(1) plgs-comm2 plgs-one-side plgs-permut by blast
  have A ≠ B'
    using assms(2) out-distinct by blast
  have D ≠ C'
    using assms(3) out-distinct by blast
  have A B ParStrict C D
    by (simp add: assms(1) plgs-pars-1)
  have A B' Par D C'
  proof (rule par-col-par-2 [where ?B = B])
    show A ≠ B'
      by (simp add: ⟨A ≠ B'⟩)
    show Col A B B'
      by (simp add: assms(2) out-col)
    show A B Par D C'
      by (metis Col-def Out-def ⟨A B ParStrict C D⟩ assms(3)
        not-col-distincts par-col2-par par-strict-par)
  qed
  moreover have A B ParStrict D C' ⟶ A B' ParStrict D C'
    by (metis Col-cases Par-def calculation par-strict-not-col-3)
  ultimately have A B' ParStrict D C'
    using Par-strict-perm ⟨A B ParStrict C D⟩ ⟨D ≠ C'⟩ assms(3)
    out-col par-strict-col-par-strict by blast
  have A D OS B B'
    using ⟨A D OS C B⟩ assms(2) one-side-symmetry one-side-transitivity
    out-out-one-side by blast
  have A D OS C C'
    using ⟨A D OS C B⟩ assms(3) col123--nos invert-one-side out-one-side by blast
  have A D OS B' C'
    by (meson ⟨A D OS B B'⟩ ⟨A D OS C B⟩ ⟨A D OS C C'⟩
      one-side-symmetry one-side-transitivity)
  then obtain M where M Midpoint A C' and M Midpoint B' D
    using par-cong-mid-os ⟨A B' ParStrict D C'⟩ assms(4) by blast
  thus ?thesis using mid-plgs
    using Par-strict-cases ⟨A B' ParStrict D C'⟩ par-strict-not-col-1 by blast
qed

lemma plgs-plgs-bet:
  assumes ParallelogramStrict A B C D and
    Bet A B B' and
    ParallelogramStrict A B' C' D
  shows Bet D C C'
proof -
  have A B ParStrict C D ∧ A D ParStrict B C
    by (simp add: assms(1) plgs-par-strict)
  hence A B Par C D
    by (simp add: par-strict-par)
  moreover have A B Par C' D
    using par-col-par-2 [where ?B = B']
    by (metis Col-def assms(1) assms(2) assms(3) between-symmetry
      par-strict-par plgs-comm2 plgs-not-comm-1 plgs-pars-1)
  ultimately have Col C' C D ∧ Col D C D

```

```

using parallel-uniqueness [where ?A1.0 = A and ?A2.0 = B and ?P = D]
by (metis col-trivial-3)
hence Col C' C D
  by simp
moreover have Bet C' C D  $\longrightarrow$  ?thesis
  using between-symmetry by blast
moreover
{
  assume Bet C D C'
  have B C OS D A
    by (simp add: assms(1) plgs-one-side plgs-permut)
  have D A OS B C
    by (simp add: assms(1) plgs-one-side plgs-permut)
  have B' C' OS D A
    by (simp add: assms(3) plgs-one-side plgs-permut)
  have D A OS B' C'
    by (simp add: assms(3) plgs-one-side plgs-permut)
  have D A OS B' B
  proof (rule out-one-side-1 [where ?X = A])
    show  $\neg$  Col D A B'
      using  $\langle D A OS B' C' \rangle$  col123--nos by blast
    show Col D A A
      by (simp add: col-trivial-2)
    show A Out B' B
      by (metis  $\langle A B Par C' D \rangle$  assms(2) bet-out l6-6 par-neq1)
  qed
  hence D A OS B C'
    using  $\langle D A OS B' C' \rangle$  one-side-symmetry one-side-transitivity by blast
  hence D A OS C C'
    using  $\langle D A OS B C \rangle$  one-side-symmetry one-side-transitivity by blast
  moreover have D A TS C C'
    using  $\langle Bet C D C' \rangle \langle D A OS C C' \rangle$  l9-17 os-distincts by blast
  ultimately have False
    by (simp add: l9-9-bis)
}
moreover
{
  assume Bet D C' C
  have ParallelogramStrict B C D A
    by (simp add: assms(1) plgs-permut)
  have ParallelogramStrict D A B' C'
    by (simp add: assms(3) plgs-permut)
  have C = C'  $\longrightarrow$  ?thesis
    using  $\langle Bet D C' C \rangle$  by blast
  moreover
  {
    assume C  $\neq$  C'
    have Parallelogram B C C' B'
      using  $\langle ParallelogramStrict B C D A \rangle \langle ParallelogramStrict D A B' C' \rangle$ 
      plgs-pseudo-trans by auto
    hence ParallelogramStrict B C C' B'
      by (metis NCol-perm  $\langle C \neq C' \rangle \langle Col C' C D \wedge Col D C D \rangle$  assms(3)
      colx ncol234-plg--plgs not-col-distincts plgs-not-col)
    have B C OS C' B'
      by (simp add:  $\langle ParallelogramStrict B C C' B' \rangle$  plgs-one-side)
    have C' B' OS B C
      by (simp add:  $\langle ParallelogramStrict B C C' B' \rangle$  plgs-one-side)
    have B C OS D A
      using  $\langle ParallelogramStrict B C D A \rangle$  plgs-one-side by blast
    have D A OS B C
      using  $\langle ParallelogramStrict B C D A \rangle$  plgs-one-side by blast
    have B C TS A B'
      by (metis  $\langle A B ParStrict C D \wedge A D ParStrict B C \rangle \langle B C OS C' B' \rangle$  assms(2)
      bet--ts os-distincts par-strict-not-col-3)
    moreover have B C OS C' D
      by (metis  $\langle B C OS C' B' \rangle \langle Bet D C' C \rangle$  bet-out-1 col123--nos invert-one-side

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      os-distincts out-one-side)
    have B C OS A B'
      by (meson ⟨B C OS C' B'⟩ ⟨B C OS C' D⟩ ⟨B C OS D A⟩ one-side-symmetry
        one-side-transitivity)
    ultimately have ?thesis
      by (simp add: l9-9)
  }
  ultimately have ?thesis
    by blast
}
ultimately show ?thesis
  using Col-def by blast
qed

lemma plgf-plgf-bet:
  assumes ParallelogramFlat A B C D and
    Bet A B B' and
    ParallelogramFlat A B' C' D
  shows Bet D C C'
proof (cases A = B)
  case True
  thus ?thesis
    using assms(1) cong-reverse-identity not-bet-distincts plgf-cong by blast
next
  case False
  hence A ≠ B
    by blast
  obtain P where ¬ Col A B P
    using False not-col-exists by blast
  obtain Q where Parallelogram A B P Q
    using False plg-existence by blast
  have ParallelogramStrict A B P Q
    using ⟨Parallelogram A B P Q⟩ ⟨¬ Col A B P⟩ not-col-distincts
      parallel-2-plg plg-par by blast
  have ParallelogramStrict C D Q P
  proof (rule plgf-plgs-trans [where ?C = A and ?D = B])
    show C ≠ D
      using False assms(1) plgf-not-comm1 by blast
    show ParallelogramFlat C D A B
      by (simp add: assms(1) plgf-sym)
    show ParallelogramStrict A B P Q
      by (simp add: ⟨ParallelogramStrict A B P Q⟩)
  qed
  have A ≠ B'
    using False assms(2) between-identity by blast
  obtain P' where A B' EqV Q P'
    using vector-construction by blast
  {
    assume Parallelogram A B' P' Q
    have B ≠ P
      using ⟨¬ Col A B P⟩ col-trivial-2 by blast
    have B' ≠ P'
      by (metis ⟨Parallelogram A B' P' Q⟩ ⟨ParallelogramStrict A B P Q⟩
        cong-identity-inv plg-cong plgs-diff)
    have A B' Par P Q
      by (meson False ⟨A ≠ B'⟩ ⟨B ≠ P⟩ ⟨Parallelogram A B P Q⟩ assms(2)
        bet-col par-col-par-2 plg-par)
    have Col P' P Q ∧ Col Q P Q
      using parallel-uniqueness by (metis Par-cases ⟨A B' EqV Q P'⟩
        ⟨A B' Par P Q⟩ ⟨A ≠ B'⟩ eqv-par par-id-4 par-trans)
    hence Col P' P Q
      by simp
    have ParallelogramStrict A B' P' Q
      by (meson ⟨A ≠ B'⟩ ⟨Parallelogram A B' P' Q⟩ ⟨ParallelogramStrict A B P Q⟩
        assms(2) bet-col col-trivial-3 colx ncol124-plg--plgs
        parallelogram-strict-not-col-4)
  }

```

```

have ParallelogramStrict D C' P' Q
  using plgf-plgs-trans [where ?C = B' and ?D = A]
  by (metis ‹A ≠ B'› ‹ParallelogramStrict A B' P' Q› assms(3)
      plgf-not-comm1 plgf-plgs-trans plgf-sym plgs-comm2)
have Bet Q P P'
  using plgs-plgs-bet ‹ParallelogramStrict A B P Q›
  ‹ParallelogramStrict A B' P' Q› assms(2) by blast
have ?thesis
  using plgs-plgs-bet [where ?A = Q and ?B = P and ?B' = P'] ‹Bet Q P P'›
  ‹ParallelogramStrict C D Q P› ‹ParallelogramStrict D C' P' Q›
  plgs-comm2 plgs-sym by blast
}
thus ?thesis
  using EqV-def ‹A B' EqV Q P'› ‹A ≠ B'› by blast
qed

lemma plg-plg-bet:
  assumes Parallelogram A B C D and
    Bet A B B' and
    Parallelogram A B' C' D
  shows Bet D C C'
proof (cases A = B)
case True
  thus ?thesis
    using assms(1) cong-reverse-identity not-bet-distincts plg-cong by blast
next
case False
  hence A ≠ B
    by blast
  show ?thesis
proof (cases B = C)
case True
  thus ?thesis
    by (metis assms(1) assms(2) assms(3) cong-diff cong-reverse-identity plg-cong)
next
case False
  hence B ≠ C
    by blast
  have A ≠ B'
    using ‹A ≠ B› assms(2) between-identity by blast
  have B' ≠ C'
    by (metis False Plg-perm assms(1) assms(3) plg-not-comm)
  have A B Par C D ∧ A D Par B C
    by (simp add: False ‹A ≠ B› assms(1) plg-par)
  have A B' Par C' D ∧ A D Par B' C'
    by (simp add: ‹A ≠ B'› ‹B' ≠ C'› assms(3) plg-par-1 plg-par-2)
  have A B Par C' D
    using Bet-cases Col-def ‹A B' Par C' D ∧ A D Par B' C'› ‹A ≠ B›
    assms(2) par-col-par-2 by blast
  have Col C' C D ∧ Col D C D
    using ‹A B Par C D ∧ A D Par B C› ‹A B Par C' D› col-trivial-3
    parallel-uniqueness by blast
  {
    assume ParallelogramStrict A B C D and ParallelogramFlat A B' C' D
    hence False
      by (metis Col-cases ParallelogramFlat-def ‹A B' Par C' D ∧ A D Par B' C'›
          ‹Col C' C D ∧ Col D C D› colx par-distincts
          parallelogram-strict-not-col-3 plgf-sym)
  }
  moreover
  {
    assume ParallelogramStrict A B C D and ParallelogramStrict A B' C' D
    hence ?thesis
      using plgs-plgs-bet [where ?A = A and ?B = B and ?B' = B'] assms(2) by blast
  }
  moreover

```

```

have ParallelogramFlat A B C D  $\wedge$  ParallelogramFlat A B' C' D  $\longrightarrow$  ?thesis
  using plgf-plgf-bet assms(2) by blast
moreover
{
  assume ParallelogramFlat A B C D and ParallelogramStrict A B' C' D
  hence False
  by (metis Col-cases ParallelogramFlat-def  $\langle A B \text{ Par } C D \wedge A D \text{ Par } B C \rangle$ 
     $\langle \text{Col } C' C D \wedge \text{Col } D C' D \rangle$  colx par-distincts
    parallelogram-strict-not-col-3 plgf-sym)
}
ultimately show ?thesis
  using Parallelogram-def assms(1) assms(3) by presburger
qed
qed

lemma plgf-out-plgf:
  assumes ParallelogramFlat A B C D and
    A Out B B' and
    D Out C C' and
    Cong A B' D C'
  shows ParallelogramFlat A B' C' D
proof -
  have A  $\neq$  B
    using assms(2) out-distinct by blast
  have A  $\neq$  B'
    using assms(2) out-diff2 by blast
  have D  $\neq$  C
    using assms(3) out-distinct by blast
  have D  $\neq$  C'
    using assms(3) out-distinct by blast
  obtain P where  $\neg \text{Col } A B P$ 
    using  $\langle A \neq B \rangle$  not-col-exists by fastforce
  obtain Q where Parallelogram A B P Q
    using  $\langle A \neq B \rangle$  plg-existence by blast
  have ParallelogramStrict A B P Q
    using ParallelogramFlat-def Parallelogram-def  $\langle \text{Parallelogram } A B P Q \rangle$ 
     $\langle \neg \text{Col } A B P \rangle$  by presburger
  have ParallelogramStrict C D Q P
    by (metis  $\langle D \neq C \rangle \langle \text{ParallelogramStrict } A B P Q \rangle$  assms(1) plgf-plgs-trans plgf-sym)
  obtain P' where A B' EqV Q P'
    using vector-construction by blast
  have B  $\neq$  P
    using  $\langle \neg \text{Col } A B P \rangle$  not-col-distincts by auto
  have B'  $\neq$  P'
    by (metis EqV-def  $\langle A B' \text{ EqV } Q P' \rangle \langle \text{ParallelogramStrict } A B P Q \rangle$ 
      equ-permut plg-not-comm-R1 plgs-diff)
  have A B Par P Q  $\wedge$  A Q Par B P
    using plg-par  $\langle A \neq B \rangle \langle B \neq P \rangle \langle \text{Parallelogram } A B P Q \rangle$  by blast
  moreover have A B' Par P' Q  $\wedge$  A Q Par B' P'
    using plg-par EqV-def  $\langle A B' \text{ EqV } Q P' \rangle \langle A \neq B' \rangle \langle B' \neq P' \rangle$  by auto
  ultimately have Col Q P P'
    by (meson assms(2) col-par-par-col col-permutation-4 out-col par-left-comm par-right-comm)
  {
    assume ParallelogramFlat A B' P' Q
    hence Col A B' P'  $\wedge$  Col A B' Q  $\wedge$  Cong A B' P' Q  $\wedge$  Cong A Q P' B'  $\wedge$  (A  $\neq$  P'  $\vee$  B'  $\neq$  Q)
      using ParallelogramFlat-def by force
    hence False
      by (metis  $\langle \text{ParallelogramStrict } A B P Q \rangle$  assms(2) col-trivial-3
        colx l6-3-1 out-col plgs-not-col)
  }
  hence ParallelogramStrict A B' P' Q
    using EqV-def Parallelogram-def  $\langle A B' \text{ EqV } Q P' \rangle \langle A \neq B' \rangle$  by blast
  hence A B' Par P Q
    using  $\langle A B \text{ Par } P Q \wedge A Q \text{ Par } B P \rangle \langle A \neq B' \rangle$  assms(2) out-col par-col-par-2 by blast
  have P  $\neq$  Q
    using  $\langle A B' \text{ Par } P Q \rangle$  par-distincts by auto

```

```

have  $P' \neq Q$ 
  using  $\langle A \ B' \ \text{Par} \ P' \ Q \wedge A \ Q \ \text{Par} \ B' \ P' \rangle$  par-distincts by auto
have ParallelogramStrict  $D \ C' \ P' \ Q$ 
proof(rule plgs-out-plgs [where  $?B=C$  and  $?C=P$ ])
  show ParallelogramStrict  $D \ C \ P \ Q$ 
    using  $\langle \text{ParallelogramStrict} \ C \ D \ Q \ P \rangle$  plgs-comm2 by blast
  show  $D \ \text{Out} \ C \ C'$ 
    using assms(3) by auto
  show  $Q \ \text{Out} \ P \ P'$ 
    using EqV-def Out-def  $\langle A \ B' \ \text{EqV} \ Q \ P' \rangle$   $\langle P \neq Q \rangle$   $\langle P' \neq Q \rangle$ 
     $\langle \text{Parallelogram} \ A \ B \ P \ Q \rangle$  assms(2) plg-plg-bet by auto
  show Cong  $D \ C' \ Q \ P'$ 
    using  $\langle \text{ParallelogramStrict} \ A \ B' \ P' \ Q \rangle$  assms(4) cong-inner-transitivity
    cong-right-commutativity plgs-cong-1 by blast
qed
have Parallelogram  $A \ B' \ C' \ D$ 
  by (meson  $\langle \text{ParallelogramStrict} \ A \ B' \ P' \ Q \rangle$   $\langle \text{ParallelogramStrict} \ D \ C' \ P' \ Q \rangle$ 
    plgs-permut plgs-pseudo-trans)
thus ?thesis
  by (metis Col-cases ParallelogramFlat-def  $\langle A \neq B \rangle$   $\langle D \neq C \rangle$  assms(1) assms(2)
    assms(3) col-trivial-3 colx out-col plg-col-plgf)
qed

lemma plg-out-plg:
  assumes Parallelogram  $A \ B \ C \ D$  and
     $A \ \text{Out} \ B \ B'$  and
     $D \ \text{Out} \ C \ C'$  and
    Cong  $A \ B' \ D \ C'$ 
  shows Parallelogram  $A \ B' \ C' \ D$ 
proof –
  have ParallelogramStrict  $A \ B \ C \ D \longrightarrow ?thesis$ 
    using Parallelogram-strict-Parallelogram assms(2) assms(3) assms(4)
    plgs-out-plgs by blast
  moreover have ParallelogramFlat  $A \ B \ C \ D \longrightarrow ?thesis$ 
    using Parallelogram-def assms(2) assms(3) assms(4) plgf-out-plgf by blast
  ultimately show ?thesis
    using Parallelogram-def assms(1) by blast
qed

lemma same-dir-sym:
  assumes  $A \ B \ \text{SameDir} \ C \ D$ 
  shows  $C \ D \ \text{SameDir} \ A \ B$ 
proof (cases  $A = B$ )
  case True
  thus ?thesis
    using SameDir-def assms same-dir-null by presburger
next
  case False
  have  $(A = B \wedge C = D) \longrightarrow ?thesis$ 
    using False by auto
  moreover
  {
    assume  $\exists \ D'. \ C \ \text{Out} \ D \ D' \wedge A \ B \ \text{EqV} \ C \ D'$ 
    then obtain  $D'$  where  $C \ \text{Out} \ D \ D'$  and  $A \ B \ \text{EqV} \ C \ D'$ 
      by blast
    obtain  $B'$  where  $C \ D \ \text{EqV} \ A \ B'$ 
      using vector-construction by blast
    have Parallelogram  $A \ B \ D' \ C \vee (A = B \wedge C = D')$ 
      using EqV-def  $\langle A \ B \ \text{EqV} \ C \ D' \rangle$  by force
    hence Parallelogram  $A \ B \ D' \ C$ 
      by (simp add: False)
    have Parallelogram  $C \ D \ B' \ A \vee (C = D \wedge A = B')$ 
      using EqV-def  $\langle C \ D \ \text{EqV} \ A \ B' \rangle$  by auto
    hence Parallelogram  $C \ D \ B' \ A$ 
      using Out-def  $\langle C \ \text{Out} \ D \ D' \rangle$  by blast
  }

```

```

assume Parallelogram  $A B D' C$  and Parallelogram  $C D B' A$ 
have  $B \neq A$ 
  using False by auto
moreover have  $B' \neq A$ 
  by (metis  $\langle C \text{ Out } D D' \rangle \langle \text{Parallelogram } C D B' A \rangle$  out-diff1 plg-not-comm-R1)
moreover have  $Bet A B B' \vee Bet A B' B$ 
proof –
  have  $Bet C D D' \longrightarrow ?thesis$ 
    using EqV-def False  $\langle A B \text{ EqV } C D' \rangle \langle \text{Parallelogram } C D B' A \rangle$ 
    eqv-sym plg-plg-bet by blast
  moreover have  $Bet C D' D \longrightarrow ?thesis$ 
    using Plg-perm  $\langle \text{Parallelogram } A B D' C \rangle \langle \text{Parallelogram } C D B' A \rangle$ 
    plg-plg-bet by blast
  ultimately show  $?thesis$ 
    using Out-def  $\langle C \text{ Out } D D' \rangle$  by blast
qed
ultimately have  $A \text{ Out } B B'$ 
  by (simp add: Out-def)
}
hence  $A \text{ Out } B B'$ 
  using  $\langle \text{Parallelogram } A B D' C \rangle \langle \text{Parallelogram } C D B' A \rangle$  by blast
moreover have  $C D \text{ EqV } A B'$ 
  by (simp add:  $\langle C D \text{ EqV } A B' \rangle$ )
ultimately have  $\exists D'0. A \text{ Out } B D'0 \wedge C D \text{ EqV } A D'0$ 
  by blast
hence  $?thesis$ 
  by (simp add: SameDir-def)
}
ultimately show  $?thesis$ 
  using SameDir-def assms by force
qed

```

lemma *same-dir-trans*:

```

assumes  $A B \text{ SameDir } C D$  and
   $C D \text{ SameDir } E F$ 
shows  $A B \text{ SameDir } E F$ 
proof –
have  $(A = B \wedge C = D) \vee (\exists D'. C \text{ Out } D D' \wedge A B \text{ EqV } C D')$ 
  using SameDir-def assms(1) by presburger
moreover have  $(A = B \wedge C = D) \longrightarrow ?thesis$ 
  using SameDir-def assms(2) same-dir-null by blast
moreover
{
  assume  $\exists D'. C \text{ Out } D D' \wedge A B \text{ EqV } C D'$ 
  then obtain  $D'$  where  $C \text{ Out } D D'$  and  $A B \text{ EqV } C D'$ 
    by blast
  have  $(C = D \wedge E = F) \vee (\exists D'. E \text{ Out } F D' \wedge C D \text{ EqV } E D')$ 
    using SameDir-def assms(2) by auto
  moreover have  $(C = D \wedge E = F) \longrightarrow ?thesis$ 
    using Out-def  $\langle C \text{ Out } D D' \rangle$  by force
  moreover
  {
    assume  $\exists D'. E \text{ Out } F D' \wedge C D \text{ EqV } E D'$ 
    then obtain  $F'$  where  $E \text{ Out } F F'$  and  $C D \text{ EqV } E F'$ 
      by blast
    have  $A = B \longrightarrow (\exists D'0. E \text{ Out } F D'0 \wedge A B \text{ EqV } E D'0)$ 
      by (metis Out-def  $\langle A B \text{ EqV } C D' \rangle \langle C \text{ Out } D D' \rangle$  null-vector)
    moreover
    {
      assume  $A \neq B$ 
      obtain  $F''$  where  $A B \text{ EqV } E F''$ 
        using vector-construction by blast
      have  $C \neq D$ 
        using Out-def  $\langle C \text{ Out } D D' \rangle$  by blast
      have  $C \neq D'$ 
        using  $\langle C \text{ Out } D D' \rangle$  out-distinct by blast
    }
  }
}

```

```

have  $E \neq F$ 
  using  $\langle E \text{ Out } F F' \rangle$  l6-3-1 by blast
have  $E \neq F'$ 
  using  $\langle E \text{ Out } F F' \rangle$  out-diff2 by blast
have  $\text{Parallelogram } A B D' C \vee (A = B \wedge C = D')$ 
  using  $\text{EqV-def } \langle A B \text{ EqV } C D' \rangle$  by blast
hence  $\text{Parallelogram } A B D' C$ 
  using  $\langle A \neq B \rangle$  by blast
have  $\text{Parallelogram } C D F' E \vee (C = D \wedge E = F')$ 
  using  $\text{EqV-def } \langle C D \text{ EqV } E F' \rangle$  by auto
hence  $\text{Parallelogram } C D F' E$ 
  using  $\langle C \neq D \rangle$  by blast
have  $\text{Parallelogram } A B F'' E \vee (A = B \wedge E = F'')$ 
  using  $\text{EqV-def } \langle A B \text{ EqV } E F'' \rangle$  by presburger
hence  $\text{Parallelogram } A B F'' E$ 
  using  $\langle A \neq B \rangle$  by blast
have  $\text{Bet } C D D' \vee \text{Bet } C D' D$ 
  using  $\text{Out-def } \langle C \text{ Out } D D' \rangle$  by presburger
have  $\text{Bet } E F F' \vee \text{Bet } E F' F$ 
  using  $\text{Out-def } \langle E \text{ Out } F F' \rangle$  by force
have  $\text{Parallelogram } C D' F'' E$ 
  using  $\text{Plg-perm } \langle C \neq D' \rangle \langle \text{Parallelogram } A B D' C \rangle \langle \text{Parallelogram } A B F'' E \rangle$ 
  plg-pseudo-trans by blast
have  $E \text{ Out } F F''$ 
proof -
  have  $\text{Parallelogram } A B D' C \vee (A = B \wedge C = D')$ 
    by (simp add:  $\langle \text{Parallelogram } A B D' C \vee A = B \wedge C = D' \rangle$ )
  moreover have  $\text{Parallelogram } C D F' E \vee (C = D \wedge E = F')$ 
    using  $\langle \text{Parallelogram } C D F' E \rangle$  by blast
  moreover have  $\text{Parallelogram } A B F'' E \vee (A = B \wedge E = F'')$ 
    using  $\langle \text{Parallelogram } A B F'' E \vee A = B \wedge E = F'' \rangle$  by blast
  moreover have  $(A = B \wedge C = D') \longrightarrow ?thesis$ 
    by (simp add:  $\langle A \neq B \rangle$ )
  moreover have  $\text{Parallelogram } A B D' C \wedge \text{Parallelogram } C D F' E \wedge$ 
     $\text{Parallelogram } A B F'' E$ 
    using  $\langle \text{Parallelogram } A B D' C \rangle \langle \text{Parallelogram } A B F'' E \rangle$ 
     $\langle \text{Parallelogram } C D F' E \rangle$  by auto
  moreover
  {
    assume  $\text{Parallelogram } A B D' C$  and
       $\text{Parallelogram } C D F' E$  and
       $\text{Parallelogram } A B F'' E$ 
    have  $\text{Bet } C D D' \vee \text{Bet } C D' D$ 
      by (simp add:  $\langle \text{Bet } C D D' \vee \text{Bet } C D' D \rangle$ )
    moreover have  $\text{Bet } E F F' \vee \text{Bet } E F' F$ 
      using  $\langle \text{Bet } E F F' \vee \text{Bet } E F' F \rangle$  by auto
    moreover
    {
      assume  $\text{Bet } C D D'$  and  $\text{Bet } E F F'$ 
      have  $F \neq E$ 
        using  $\langle E \neq F \rangle$  by blast
      moreover have  $F'' \neq E$ 
        using  $\langle A \neq B \rangle \langle \text{Parallelogram } A B F'' E \rangle$  plg-not-comm-R1 by blast
      moreover have  $\text{Bet } E F' F''$ 
      proof (rule plg-plg-bet [where ?A=C and ?B=D and ?B'=D'])
        show  $\text{Parallelogram } C D F' E$ 
          by (simp add:  $\langle \text{Parallelogram } C D F' E \rangle$ )
        show  $\text{Bet } C D D'$ 
          by (simp add:  $\langle \text{Bet } C D D' \rangle$ )
        show  $\text{Parallelogram } C D' F'' E$ 
          by (simp add:  $\langle \text{Parallelogram } C D' F'' E \rangle$ )
      qed
      have  $\text{Bet } E F F''$ 
        using between-exchange4 [where ?C = F']  $\langle \text{Bet } E F F' \rangle \langle \text{Bet } E F' F'' \rangle$  by blast
      hence  $\text{Bet } E F F'' \vee \text{Bet } E F' F$ 
        by blast
    }
  }

```



```

ultimately have ?thesis
  by (simp add: Out-def)
}
moreover {
  assume Bet C D D' and Bet E F' F
  have F ≠ E
    using ⟨E ≠ F⟩ by blast
  moreover have F'' ≠ E
    using ⟨A ≠ B⟩ ⟨Parallelogram A B F'' E⟩ plg-not-comm-R1 by blast
  moreover
  have Bet E F' F''
    using plg-plg-bet [where ?A=C and ?B=D and ?B'=D']
      ⟨Parallelogram C D F' E⟩ ⟨Bet C D D'⟩ ⟨Parallelogram C D' F'' E⟩ by blast
  have Bet E F F'' ∨ Bet E F'' F
    using l5-1 [where ?B = F'] ⟨E ≠ F'⟩ ⟨Bet E F' F⟩ ⟨Bet E F' F''⟩ by blast
  ultimately have ?thesis
    using Out-def by blast
}
moreover
{
  assume Bet C D' D and Bet E F F'
  have F ≠ E
    using ⟨E ≠ F⟩ by blast
  moreover have F'' ≠ E
    using ⟨A ≠ B⟩ ⟨Parallelogram A B F'' E⟩ plg-not-comm-R1 by blast
  moreover have Bet E F F'' ∨ Bet E F'' F
  proof (rule l5-3 [where ?D = F'])
    show Bet E F F'
      by (simp add: ⟨Bet E F F'⟩)
    show Bet E F'' F'
      using plg-plg-bet [where ?A=C and ?B=D' and ?B'=D]
        ⟨Parallelogram C D' F'' E⟩ ⟨Bet C D' D⟩ ⟨Parallelogram C D F' E⟩ by blast
  qed
  ultimately have ?thesis
    by (simp add: Out-def)
}
moreover {
  assume Bet C D' D and Bet E F' F
  have F ≠ E
    using ⟨E ≠ F⟩ by blast
  moreover have F'' ≠ E
    using ⟨A ≠ B⟩ ⟨Parallelogram A B F'' E⟩ plg-not-comm-R1 by blast
  moreover have Bet E F F'' ∨ Bet E F'' F
  proof -
    have Bet E F'' F'
      using plg-plg-bet [where ?A=C and ?B=D' and ?B'=D]
        ⟨Bet C D' D⟩ ⟨Parallelogram C D F' E⟩ ⟨Parallelogram C D' F'' E⟩ by blast
    thus ?thesis
      using ⟨Bet E F' F⟩ between-exchange4 by blast
  qed
  ultimately have ?thesis
    by (simp add: Out-def)
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed
hence ∃ D'0. E Out F D'0 ∧ A B EqV E D'0
  using ⟨A B EqV E F''⟩ by blast
}
ultimately have ∃ D'0. E Out F D'0 ∧ A B EqV E D'0
  by blast
}
hence ?thesis

```

```

    using SameDir-def calculation(1) calculation(2) by presburger
  }
  thus ?thesis
    using calculation(1) calculation(2) by blast
qed

lemma same-dir-comm:
  assumes A B SameDir C D
  shows B A SameDir D C
proof -
  have A = B  $\wedge$  C = D  $\longrightarrow$  ?thesis
    using assms by force
  moreover {
    assume  $\exists D'. C \text{ Out } D D' \wedge A B \text{ EqV } C D'$ 
    then obtain D' where C Out D D' and A B EqV C D'
      by blast
    hence H0: Bet C D D'  $\vee$  Bet C D' D
      using Out-def by simp
    have A  $\neq$  B
      using  $\langle \exists D'. C \text{ Out } D D' \wedge A B \text{ EqV } C D' \rangle$  l6-3-1 null-vector by blast
    obtain C' where B A EqV D C'
      using vector-construction by blast
    have D Out C C'
    proof -
      have C  $\neq$  D
        using  $\langle C \text{ Out } D D' \rangle$  l6-3-1 by blast
      moreover have C'  $\neq$  D
        using  $\langle A \neq B \rangle \langle B A \text{ EqV } D C' \rangle$  eqv-sym null-vector by blast
      moreover
      have Bet D C C'  $\vee$  Bet D C' C
    proof -
      {
        assume Parallelogram A B D' C
        hence Parallelogram C D' D C'  $\vee$  (C = D'  $\wedge$  B = A  $\wedge$  C' = D  $\wedge$  C = C')
          by (metis EqV-def Plg-perm  $\langle A \neq B \rangle \langle B A \text{ EqV } D C' \rangle$  plg-pseudo-trans)
        moreover
        {
          assume Parallelogram C D' D C'
          have ParallelogramFlat C D' D C'
            by (meson Col-def Out-def  $\langle C \text{ Out } D D' \rangle \langle \text{Parallelogram } C D' D C' \rangle$ 
              between-symmetry plg-col-plgf)
          hence H1: Col C D' D  $\wedge$  Col C D' C'  $\wedge$  Cong C D' D C'  $\wedge$ 
            Cong C C' D D'  $\wedge$  (C  $\neq$  D  $\vee$  D'  $\neq$  C')
            using ParallelogramFlat-def by simp
          have Col D' C C'
            using H1 col-permutation-4 by blast
          have Cong D' C D C'
            using H1 not-cong-2134 by blast
          have Cong D' D C C'
            using H1 not-cong-3421 by blast
          hence Bet D' D C  $\longrightarrow$  ?thesis
            using col-cong2-bet1 [where ?A = D]  $\langle \text{Col } D' C C' \rangle \langle \text{Cong } D' C D C' \rangle$  by blast
        }
        moreover {
          assume Bet C D' D
          {
            assume Parallelogram A B D' C
            {
              assume Parallelogram B A C' D
              have C = C'  $\longrightarrow$  ?thesis
                using between-trivial by presburger
            }
            moreover {
              assume C  $\neq$  C'
              have Parallelogram C D' D C'  $\vee$  (C = D'  $\wedge$  B = A  $\wedge$  C' = D  $\wedge$  C = C')
                using  $\langle \text{Parallelogram } C D' D C' \rangle$  by blast
            }
            moreover {
              assume Parallelogram C D' D C'
            }
          }
        }
      }
    }
  }

```

```

have ParallelogramFlat C D' D C'
  by (simp add: ⟨ParallelogramFlat C D' D C'⟩)
moreover have ParallelogramStrict C D' D C' → ?thesis
  using H1 col-not-plgs by blast
moreover have ParallelogramFlat C D' D C' → Bet D C' C
  by (metis ⟨Bet C D' D⟩ ⟨Cong D' C D C'⟩ ⟨Cong D' D C C'⟩
    ⟨Parallelogram C D' D C'⟩ bet-cong-eq between-symmetry
    calculation(2) ncol134-plg--plgs third-point)
ultimately have ?thesis
  by blast
}
moreover have (C = D' ∧ B = A ∧ C' = D ∧ C = C') → ?thesis
  using ⟨A ≠ B⟩ by auto
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  by blast
}
moreover have B = A ∧ D = C' → ?thesis
  using ⟨A ≠ B⟩ by blast
ultimately have ?thesis
  using EqV-def ⟨B A EqV D C'⟩ by presburger
}
moreover have A = B ∧ C = D' → ?thesis
  using ⟨A ≠ B⟩ by blast
ultimately have ?thesis
  using ⟨Parallelogram A B D' C⟩ by blast
}
ultimately have ?thesis
  using Bet-cases H0 by blast
}
moreover have (C = D' ∧ B = A ∧ C' = D ∧ C = C') → ?thesis
  using ⟨A ≠ B⟩ by auto
ultimately have ?thesis
  by blast
}
moreover have A = B ∧ C = D' → ?thesis
  using ⟨A ≠ B⟩ by auto
ultimately show ?thesis
  using EqV-def ⟨A B EqV C D'⟩ by force
qed
hence Bet D C C' ∨ Bet D C' C
  by blast
ultimately show ?thesis
  by (simp add: Out-def)
qed
hence ∃ D'0. D Out C D'0 ∧ B A EqV D D'0
  using ⟨B A EqV D C'⟩ by blast
}
hence (∃ D'. C Out D D' ∧ A B EqV C D') → ?thesis
  by (simp add: SameDir-def)
ultimately show ?thesis
  using SameDir-def assms by auto
qed

lemma bet-same-dir1:
  assumes A ≠ B and

    Bet A B C
  shows A B SameDir A C
  by (metis Out-def SameDir-def assms(1) assms(2) bet-neq21--neq equ-refl)

lemma bet-same-dir2:
  assumes A ≠ B and
    B ≠ C and

```

```

  Bet A B C
shows A B SameDir B C
by (metis Bet-cases assms(1) assms(2) assms(3) bet-same-dir1 same-dir-comm
    same-dir-sym same-dir-trans)

lemma plg-opp-dir:
  assumes Parallelogram A B C D
  shows A B SameDir D C
  by (metis EqV-def SameDir-def assms between-equality not-bet-distincts
      not-col-distincts not-out-bet plg-not-comm)

lemma same-dir-dec:
  shows A B SameDir C D  $\vee$   $\neg$  A B SameDir C D
  by blast

lemma same-or-opp-dir:
  assumes A B Par C D
  shows A B SameDir C D  $\vee$  A B OppDir C D
proof (cases A B SameDir C D)
  case True
  thus ?thesis
  by blast
next
  case False
  hence  $\neg$  A B SameDir C D
  by blast
  obtain C' where A B EqV D C'
  using vector-construction by fastforce
  have D Out C C'
  proof (cases B = C')
    case True
    hence B = C'
    by blast
    have Parallelogram A B B D  $\longrightarrow$  ?thesis
    proof -
      {
        assume ParallelogramStrict A B B D
        hence Col B D B
        by (simp add: col-trivial-3)
        hence False
        using  $\langle$ ParallelogramStrict A B B D $\rangle$  plgs-diff by blast
      }
      hence ParallelogramStrict A B B D  $\longrightarrow$  ?thesis
      by blast
    moreover
    {
      assume ParallelogramFlat A B B D
      hence D = A  $\wedge$  B  $\neq$  D
      using plgf-permut plgf-trivial-neq by blast
      hence ?thesis
      by (metis False True  $\langle$  $\wedge$ thesis. ( $\wedge$ C'. A B EqV D C'  $\implies$  thesis)  $\implies$  thesis $\rangle$ 
          assms bet-same-dir2 eqv-par not-col-distincts not-out-bet par-distincts
          parallel-uniqueness same-dir-sym vector-uniqueness)
    }
    ultimately show ?thesis
    using Parallelogram-def by blast
  qed
  moreover have A = B  $\wedge$  D = B  $\longrightarrow$  ?thesis
  using assms par-neq1 by blast
  ultimately show ?thesis
  using EqV-def True  $\langle$ A B EqV D C' $\rangle$  by blast
next
  case False
  {
    assume Parallelogram A B C' D
    have Col C' C D  $\wedge$  Col D C D
  }

```

```

    by (metis Par-cases ⟨A B EqV D C'⟩ assms equiv-par par-id-4 par-neq1
        par-not-par parallel-uniqueness)
  hence Col C' C D
  by blast
  have C ≠ D
  using assms par-distinct by auto
  moreover have C' ≠ D
  using ⟨Parallelogram A B C' D⟩ assms par-neq1 plg-not-comm-R1 by blast
  moreover have Bet D C C' ∨ Bet D C' C
  proof -
    have Bet C' C D ⟶ ?thesis
    using between-symmetry by blast
  moreover
  {
    assume Bet C D C'
    have A B SameDir D C'
    by (simp add: ⟨Parallelogram A B C' D⟩ plg-opp-dir)
    moreover have C D SameDir D C'
    using ⟨Bet C D C'⟩ ⟨C ≠ D⟩ ⟨C' ≠ D⟩ bet-same-dir2 by force
    ultimately have A B SameDir C D
    by (meson same-dir-sym same-dir-trans)
    hence False
    using ⟨¬ A B SameDir C D⟩ by blast
  }
  ultimately show ?thesis
  using Col-def ⟨Col C' C D⟩ by blast
qed
ultimately have ?thesis
by (simp add: Out-def)
}
moreover have A = B ∧ D = C' ⟶ ?thesis
using assms par-distinct by blast
ultimately show ?thesis
using EqV-def ⟨A B EqV D C'⟩ by blast
qed
hence ∃ D'. D Out C D' ∧ A B EqV D D'
using ⟨A B EqV D C'⟩ by blast
hence A B SameDir D C
by (simp add: SameDir-def)
hence A B OppDir C D
by (simp add: OppDir-def)
thus ?thesis
by blast
qed

```

```

lemma same-dir-id:
  assumes A B SameDir B A
  shows A = B
  using assms bet-neq32--neq same-dir-ts by blast

```

```

lemma opp-dir-id:
  assumes A B OppDir A B
  shows A = B
  using OppDir-def assms same-dir-id by blast

```

```

lemma same-dir-to-null:
  assumes A B SameDir C D and
    A B SameDir D C
  shows A = B ∧ C = D
  by (meson assms(1) assms(2) same-dir-comm same-dir-id same-dir-sym same-dir-trans)

```

```

lemma opp-dir-to-null:
  assumes A B OppDir C D and
    A B OppDir D C
  shows A = B ∧ C = D
  using OppDir-def assms(1) assms(2) same-dir-to-null by auto

```

```

lemma same-not-opp-dir:
  assumes  $A \neq B$  and
     $A \ B \ \text{SameDir} \ C \ D$ 
  shows  $\neg A \ B \ \text{OppDir} \ C \ D$ 
  using OppDir-def assms(1) assms(2) same-dir-to-null by blast

lemma opp-not-same-dir:
  assumes  $A \neq B$  and
     $A \ B \ \text{OppDir} \ C \ D$ 
  shows  $\neg A \ B \ \text{SameDir} \ C \ D$ 
  using assms(1) assms(2) same-not-opp-dir by force

lemma vector-same-dir-cong:
  assumes  $A \neq B$  and
     $C \neq D$ 
  shows  $\exists X \ Y. \ A \ B \ \text{SameDir} \ X \ Y \wedge \text{Cong} \ X \ Y \ C \ D$ 
  by (metis assms(1) assms(2) bet-same-dir2 cong-diff-3 segment-construction)

lemma project-par:
  assumes  $P \ P' \ \text{Proj} \ A \ B \ X \ Y$  and
     $Q \ Q' \ \text{Proj} \ A \ B \ X \ Y$  and
     $P \ Q \ \text{Par} \ X \ Y$ 
  shows  $P' = Q'$ 
proof -
  {
    assume  $P \ P' \ \text{Par} \ X \ Y$ 
    hence  $\text{Col} \ P \ P' \wedge \text{Col} \ Q \ P \ P'$ 
      using parallel-uniqueness assms(3) not-par-one-not-par par-id-5 by blast
    hence  $\text{Col} \ Q \ P \ P'$ 
      by simp
    {
      assume  $Q \ Q' \ \text{Par} \ X \ Y$ 
      have  $\text{Col} \ P \ Q \ Q' \wedge \text{Col} \ Q \ Q \ Q'$ 
        using parallel-uniqueness
        by (meson  $\langle \text{Col} \ Q \ P \ P' \rangle \langle P \ P' \ \text{Par} \ X \ Y \rangle \langle Q \ Q' \ \text{Par} \ X \ Y \rangle$  col-trivial-1 par-symmetry)
      hence  $\text{Col} \ P \ Q \ Q'$ 
        by simp
      have ?thesis
        by (metis Proj-def  $\langle \text{Col} \ P \ Q \ Q' \rangle \langle \text{Col} \ Q \ P \ P' \rangle \langle P \ P' \ \text{Par} \ X \ Y \rangle$  assms(1)
          assms(2) assms(3) col-permutation-4 l6-21 par-distincts project-id)
    }
    moreover have  $Q = Q' \longrightarrow ?thesis$ 
      using l6-21 by (metis NCol-perm Proj-def  $\langle \text{Col} \ Q \ P \ P' \rangle$ 
         $\langle P \ P' \ \text{Par} \ X \ Y \rangle$  assms(1) assms(2) par-distincts par-id-5
        par-reflexivity project-not-col)
    ultimately have ?thesis
      using Proj-def assms(2) by force
  }
  moreover
  {
    assume  $P = P'$ 
    have  $A \neq B$ 
      using Proj-def assms(2) by force
    have  $X \neq Y$ 
      using assms(3) par-distincts by blast
    have  $\neg A \ B \ \text{Par} \ X \ Y$ 
      using Proj-def assms(2) by presburger
    have  $\text{Col} \ A \ B \ Q'$ 
      using Proj-def assms(2) by force
    {
      assume  $Q \ Q' \ \text{Par} \ X \ Y$ 
      hence  $Q \neq Q'$ 
        using par-distinct by presburger
      have  $\text{Col} \ P \ Q \ Q' \wedge \text{Col} \ Q \ Q \ Q'$ 

```

```

    using parallel-uniqueness
  by (metis ‹Q Q' Par X Y› assms(3) not-par-one-not-par par-id-5)
hence Col P Q Q'
  by simp
have P' = Q'
proof (rule l6-21 [where ?A = A and ?B = B and ?C = Q and ?D = P])
  show ¬ Col A B Q
    using ‹Q ≠ Q'› assms(2) project-id by blast
  show Q ≠ P
    using assms(3) par-neq1 by blast
  show Col A B P'
    using Proj-def assms(1) by force
  show Col A B Q'
    by (simp add: ‹Col A B Q'›)
  show Col Q P P'
    using ‹P = P'› not-col-distincts by blast
  show Col Q P Q'
    using ‹Col P Q Q'› not-col-permutation-4 by blast
qed
}
moreover
{
  assume Q = Q'
  have A = P ⟶ A B Par X Y
    by (metis ‹A ≠ B› ‹Col A B Q'› ‹Q = Q'› assms(3) col3
      col-trivial-2 col-trivial-3 par-col-par-2)
  moreover {
    assume A ≠ P
    have A B Par X Y
    proof (rule par-col-par-2 [where ?B = Q])
      show A ≠ B
        by (simp add: ‹A ≠ B›)
      show Col A Q B
        using ‹Col A B Q'› ‹Q = Q'› not-col-permutation-5 by blast
      have A P Par X Y
        using ‹A ≠ B› ‹Col A B Q'› ‹P = P'› ‹Q = Q'› ‹¬ A B Par X Y› assms(1)
          assms(3) par-col2-par-bis par-symmetry project-not-id by blast
      thus A Q Par X Y
        by (metis Proj-def ‹P = P'› assms(1) par-col-par par-distinct
          par-reflexivity postulate-of-transitivity-of-parallelism-def
          postulate-of-transitivity-of-parallelism-thm)
    qed
  }
  ultimately have A B Par X Y
    by blast
  hence False
    by (simp add: ‹¬ A B Par X Y›)
}
ultimately have P' = Q'
  using Proj-def assms(2) by auto
}
ultimately show ?thesis
  using Proj-def assms(1) by force
qed

```

lemma ker-col:

```

assumes P P' Proj A B X Y and
  Q P' Proj A B X Y
shows Col P Q P'
by (metis Par-perm Proj-def assms(1) assms(2) not-col-distincts not-par-one-not-par par-id-2)

```

lemma ker-par:

```

assumes P ≠ Q and
  P P' Proj A B X Y and
  Q P' Proj A B X Y
shows P Q Par X Y

```

```

proof –
  have Col P Q P'
    using assms(2) assms(3) ker-col by blast
  {
    assume P P' Par X Y
    hence ?thesis
      by (metis NCol-cases assms(1) assms(2) assms(3) ker-col par-col-par-2)
  }
  moreover have P = P'  $\longrightarrow$  ?thesis
    by (metis Proj-def assms(1) assms(3) par-left-comm)
  ultimately show ?thesis
    using Proj-def assms(2) by force
qed

lemma project-uniqueness:
  assumes P P' Proj A B X Y and
    P Q' Proj A B X Y
  shows P' = Q'
  by (metis Proj-def assms(1) assms(2) colx not-par-one-not-par par-id-2 project-id)

lemma project-col-eq:
  assumes P  $\neq$  P' and
    Col P Q P' and
    P P' Proj A B X Y and
    Q Q' Proj A B X Y
  shows P' = Q'
  by (metis Proj-def assms(1) assms(2) assms(3) assms(4) not-par-not-col
    par-not-par project-par project-uniqueness)

lemma project-preserves-bet:
  assumes Bet P Q R and
    P P' Proj A B X Y and
    Q Q' Proj A B X Y and
    R R' Proj A B X Y
  shows Bet P' Q' R'

proof –
  have Col P' Q' R'
    by (metis Proj-def assms(2) assms(3) assms(4) col3)
  have P = Q  $\longrightarrow$  ?thesis
    using assms(2) assms(3) not-bet-distincts project-uniqueness by blast
  moreover
  {
    assume P  $\neq$  Q
    have R = Q  $\longrightarrow$  ?thesis
      using assms(3) assms(4) not-bet-distincts project-uniqueness by blast
    moreover
    {
      assume R  $\neq$  Q
      have P' = Q'  $\longrightarrow$  ?thesis
        by (simp add: between-trivial2)
      moreover
      {
        assume P'  $\neq$  Q'
        have R' = Q'  $\longrightarrow$  ?thesis
          by (simp add: between-trivial)
        moreover
        {
          assume R'  $\neq$  Q'
          {
            assume P P' Par X Y
            {
              assume Q Q' Par X Y
              hence P P' Par Q Q'
                using  $\langle P P' Par X Y \rangle$  not-par-one-not-par by blast
              hence Q Q' ParStrict P P'
                by (metis Par-cases  $\langle P P' Par X Y \rangle$   $\langle P \neq Q \rangle$   $\langle P' \neq Q' \rangle$ )
            }
          }
        }
      }
    }
  }

```



```

    all-one-side-par-strict assms(2) assms(3) not-strict-par
    par-col-par-2 par-neq1 par-not-col-strict project-par)
hence  $Q \ Q' \ OS \ P \ P'$ 
  by (simp add: l12-6)
{
  assume  $R \ R' \ Par \ X \ Y$ 
  hence  $R \ R' \ Par \ Q \ Q'$ 
    using  $\langle Q \ Q' \ Par \ X \ Y \rangle$  not-par-one-not-par by blast
  hence  $Q \ Q' \ ParStrict \ R \ R'$ 
    by (metis Par-cases  $\langle Col \ P' \ Q' \ R' \rangle \langle Q \ Q' \ ParStrict \ P \ P' \rangle \langle R' \neq Q' \rangle$ 
        col-trivial-2 l6-16-1 par-not-col-strict par-strict-not-col-4)
  hence  $Q \ Q' \ OS \ R \ R'$ 
    by (simp add: l12-6)
  hence  $Q \ Q' \ TS \ P \ R$ 
    using  $\langle Q \ Q' \ ParStrict \ P \ P' \rangle$  assms(1) bet--ts os-distincts
        par-strict-not-col-1 by force
  hence  $Q \ Q' \ TS \ P' \ R'$ 
    using  $\langle Q \ Q' \ OS \ P \ P' \rangle \langle Q \ Q' \ OS \ R \ R' \rangle$  l9-2 l9-8-2 by blast
  then obtain  $QQ$  where  $Col \ QQ \ Q \ Q'$  and  $Bet \ P' \ QQ \ R'$ 
    using TS-def by fastforce
  hence  $QQ = Q'$ 
    by (meson  $\langle Col \ P' \ Q' \ R' \rangle \langle Q \ Q' \ TS \ P' \ R' \rangle$  bet-col
        col-permutation-5 colx not-col-permutation-4 ts--ncol)
  hence ?thesis
    using  $\langle Bet \ P' \ QQ \ R' \rangle$  by blast
}
moreover
{
  assume  $R = R'$ 
  have  $Q \ Q' \ TS \ P' \ R$ 
    by (metis  $\langle Q \ Q' \ OS \ P \ P' \rangle \langle Q \ Q' \ ParStrict \ P \ P' \rangle \langle R \neq Q \rangle$ 
        assms(1) bet--ts l9-8-2 par-strict-not-col-1)
  then obtain  $QQ$  where  $Col \ QQ \ Q \ Q'$  and  $Bet \ P' \ QQ \ R$ 
    using TS-def by blast
  hence  $Col \ P' \ QQ \ R$ 
    by (simp add: Col-def)
  have  $QQ = Q'$ 
  proof (rule l6-21 [where ?A=  $Q$  and ?B= $Q'$  and ?C= $P'$  and ?D= $R$ ])
    show  $\neg Col \ Q \ Q' \ P'$ 
      using  $\langle Q \ Q' \ ParStrict \ P \ P' \rangle$  par-strict-not-col-4 by auto
    show  $P' \neq R$ 
      using  $\langle Q \ Q' \ TS \ P' \ R \rangle$  ts-distincts by blast
    show  $Col \ Q \ Q' \ QQ$ 
      using  $\langle Col \ QQ \ Q \ Q' \rangle$  not-col-permutation-2 by blast
    show  $Col \ Q \ Q' \ Q'$ 
      using not-col-distincts by blast
    show  $Col \ P' \ R \ QQ$ 
      using  $\langle Col \ P' \ QQ \ R \rangle$  not-col-permutation-5 by blast
    show  $Col \ P' \ R \ Q'$ 
      by (simp add:  $\langle Col \ P' \ Q' \ R' \rangle \langle R = R' \rangle$  col-permutation-5)
  qed
  hence ?thesis
    using  $\langle Bet \ P' \ QQ \ R \rangle \langle R = R' \rangle$  by auto
}
ultimately have ?thesis
  using Proj-def assms(4) by force
}
moreover
{
  assume  $Q = Q'$ 
  {
    assume  $R \ R' \ Par \ X \ Y$ 
    then obtain  $Qx \ Qy$  where  $Qx \neq Qy$  and  $X \ Y \ Par \ Qx \ Qy$  and  $Col \ Q \ Qx \ Qy$ 
      using  $\langle P \ P' \ Par \ X \ Y \rangle$  par-neq2 parallel-existence by blast
    hence  $Qx \ Qy \ Par \ P \ P'$ 
      using Par-perm  $\langle P \ P' \ Par \ X \ Y \rangle$  par-trans by blast
  }
}

```

```

hence Qx Qy ParStrict P P'
  by (metis NCol-perm Par-perm Par-strict-cases ⟨Col Q Qx Qy⟩
      ⟨P P' Par X Y⟩ ⟨P ≠ Q⟩ ⟨P' ≠ Q'⟩ assms(2) assms(3) col-trivial-3
      par-col2-par par-not-col-strict project-par)
hence Qx Qy OS P P'
  using l12-6 by auto
have Qx Qy Par R R'
  using ⟨P P' Par X Y⟩ ⟨Qx Qy Par P P'⟩ ⟨R R' Par X Y⟩
      not-par-one-not-par by blast
hence Qx Qy ParStrict R R'
  by (metis Par-def ⟨Col Q Qx Qy⟩ ⟨R R' Par X Y⟩ ⟨R ≠ Q⟩ ⟨R' ≠ Q'⟩
      ⟨X Y Par Qx Qy⟩ assms(3) assms(4) parallel-uniqueness
      postulate-of-transitivity-of-parallelism-def
      postulate-of-transitivity-of-parallelism-thm project-par)
hence Qx Qy OS R R'
  using l12-6 by force
hence Qx Qy TS P R
  using OS-def TS-def ⟨Col Q Qx Qy⟩ ⟨Qx Qy OS P P'⟩ assms(1) by auto
hence Qx Qy TS P' R'
  using ⟨Qx Qy OS P P'⟩ ⟨Qx Qy OS R R'⟩ l9-2 l9-8-2 by blast
then obtain QQ where Col QQ Qx Qy and Bet P' QQ R'
  using TS-def by blast
have QQ = Q
proof (rule l6-21 [where ?A=Qx and ?B=Qy and ?C=P' and ?D=R'])
  show ¬ Col Qx Qy P'
    using ⟨Qx Qy OS P P'⟩ col124--nos by blast
  show P' ≠ R'
    using ⟨Qx Qy TS P' R'⟩ not-two-sides-id by force
  show Col Qx Qy QQ
    using Col-cases ⟨Col QQ Qx Qy⟩ by blast
  show Col Qx Qy Q
    using ⟨Col Q Qx Qy⟩ not-col-permutation-2 by blast
  show Col P' R' QQ
    using ⟨Bet P' QQ R'⟩ bet-col not-col-permutation-5 by blast
  show Col P' R' Q
    by (simp add: ⟨Col P' Q' R'⟩ ⟨Q = Q'⟩ col-permutation-5)
qed
hence ?thesis
  using ⟨Bet P' QQ R'⟩ ⟨Q = Q'⟩ by auto
}
moreover
have ¬ Col A B P
  using ⟨P P' Par X Y⟩ assms(2) par-neq1 project-not-col by auto
{
  assume R = R'
  hence Col A B P
    by (metis Proj-def ⟨Q = Q'⟩ ⟨R ≠ Q⟩ assms(1) assms(3) assms(4)
        bet-col bet-col1 l6-21)
  hence False
    using ⟨¬ Col A B P⟩ by blast
}
ultimately have ?thesis
  using Proj-def assms(4) by force
}
ultimately have ?thesis
  using Proj-def assms(3) by force
}
moreover
{
  assume P = P'
  {
    assume Q Q' Par X Y
    {
      assume R R' Par X Y
      hence Q Q' Par R R'
        using ⟨Q Q' Par X Y⟩ not-par-one-not-par by blast
    }
  }
}

```

```

hence  $Q \ Q' \text{ ParStrict } R \ R'$ 
  by (metis Par-def  $\langle R \ R' \text{ Par } X \ Y \rangle \langle R \neq Q \rangle \langle R' \neq Q' \rangle$  assms(3)
      assms(4) ker-col postulate-of-transitivity-of-parallelism-def
      postulate-of-transitivity-of-parallelism-thm project-par)
hence  $Q \ Q' \text{ OS } R \ R'$ 
  using l12-6 by blast
hence  $Q \ Q' \text{ TS } P \ R'$ 
  by (metis  $\langle P \neq Q \rangle$  assms(1) bet--ts between-symmetry l9-2 l9-8-2
      one-side-not-col123)
then obtain  $QQ$  where  $Col \ QQ \ Q \ Q'$  and  $Bet \ P \ QQ \ R'$ 
  using TS-def by blast
have  $QQ = Q'$ 
proof (rule l6-21 [where ?A=Q and ?B=Q' and ?C=P and ?D=R'])
  show  $\neg Col \ Q \ Q' \ P$ 
    using TS-def  $\langle Q \ Q' \text{ TS } P \ R' \rangle$  col-permutation-2 by blast
  show  $P \neq R'$ 
    using  $\langle Q \ Q' \text{ TS } P \ R' \rangle$  ts-distincts by blast
  show  $Col \ Q \ Q' \ QQ$ 
    using Col-cases  $\langle Col \ QQ \ Q \ Q' \rangle$  by blast
  show  $Col \ Q \ Q' \ Q'$ 
    using not-col-distincts by blast
  show  $Col \ P \ R' \ QQ$ 
    using Bet-cases Col-def  $\langle Bet \ P \ QQ \ R' \rangle$  by blast
  show  $Col \ P \ R' \ Q'$ 
    using Col-cases  $\langle Col \ P' \ Q' \ R' \rangle \langle P = P' \rangle$  by auto
qed
hence ?thesis
  using  $\langle Bet \ P \ QQ \ R' \rangle \langle P = P' \rangle$  by blast
}
moreover have  $R = R' \longrightarrow ?thesis$ 
  by (metis  $\langle Col \ P' \ Q' \ R' \rangle \langle P = P' \rangle \langle R' \neq Q' \rangle$  assms(1) assms(3) assms(4)
      not-col-permutation-2 or-bet-out out-bet-out-2 out-col project-col-eq)
ultimately have ?thesis
  using Proj-def assms(4) by force
}
moreover have  $Q = Q' \longrightarrow ?thesis$ 
  by (metis Proj-def  $\langle P = P' \rangle \langle P' \neq Q' \rangle$  assms(1) assms(2) assms(3)
      assms(4) bet-col colx project-not-col)
ultimately have ?thesis
  using Proj-def assms(3) by force
}
ultimately have ?thesis
  using Proj-def assms(2) by force
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

lemma triangle-par:
  assumes  $\neg Col \ A \ B \ C$  and
     $A \ B \text{ Par } A' \ B'$  and
     $B \ C \text{ Par } B' \ C'$  and
     $A \ C \text{ Par } A' \ C'$ 
  shows  $A \ B \ C \text{ Cong } A' \ B' \ C'$ 
proof -
  obtain  $M$  where  $M \text{ Midpoint } B \ B'$ 
    using MidR-uniq-aux by blast

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obtain  $A''$  where  $Bet\ A'\ M\ A''$  and  $Cong\ M\ A''\ A'\ M$ 
  using segment-construction by blast
obtain  $C''$  where  $Bet\ C'\ M\ C''$  and  $Cong\ M\ C''\ C'\ M$ 
  using segment-construction by blast
have  $M\ Midpoint\ A'\ A''$ 
  using  $\langle Bet\ A'\ M\ A'' \rangle \langle Cong\ M\ A''\ A'\ M \rangle$  midpoint-def not-cong-3412 by blast
have  $M\ Midpoint\ C'\ C''$ 
  using  $\langle Bet\ C'\ M\ C'' \rangle \langle Cong\ M\ C''\ C'\ M \rangle$  midpoint-def not-cong-3412 by blast
have  $A' \neq B'$ 
  using assms(2) par-distinct by blast
have  $C' \neq B'$ 
  using assms(3) par-distinct by blast
have  $B'\ C'\ Par\ B\ C''$ 
  using  $\langle C' \neq B' \rangle \langle M\ Midpoint\ B\ B' \rangle \langle M\ Midpoint\ C'\ C'' \rangle$  mid-par-cong2
    par-comm by presburger
hence  $Col\ B\ B\ C \wedge Col\ C''\ B\ C$ 
  using assms(3) not-col-distincts not-par-one-not-par par-id-1 par-symmetry by blast
hence  $Col\ C''\ B\ C$ 
  by blast
have  $B'\ A'\ Par\ B\ A''$ 
  using  $\langle A' \neq B' \rangle \langle M\ Midpoint\ A'\ A'' \rangle \langle M\ Midpoint\ B\ B' \rangle$  mid-par-cong2
    par-comm by presburger
hence  $Col\ A''\ B\ A$ 
  by (meson assms(2) par-comm par-id-5 par-not-par)
have  $A'\ B'\ C'\ CongA\ A''\ B\ C''$ 
  by (meson  $\langle A' \neq B' \rangle \langle C' \neq B' \rangle \langle M\ Midpoint\ A'\ A'' \rangle \langle M\ Midpoint\ B\ B' \rangle$ 
     $\langle M\ Midpoint\ C'\ C'' \rangle$  l7-2 symmetry-preserves-conga)
have  $A\ B\ C\ CongA\ A''\ B\ C''$ 
proof -
  have  $B \neq A''$ 
    using  $\langle B'\ A'\ Par\ B\ A'' \rangle$  par-distinct by auto
  have  $B \neq C''$ 
    using  $\langle B'\ C'\ Par\ B\ C'' \rangle$  par-distinct by auto
  have  $A \neq B$ 
    using  $\langle Col\ B\ B\ C \wedge Col\ C''\ B\ C \rangle$  assms(1) by blast
  have  $C \neq B$ 
    using assms(3) par-distinct by auto
  have  $A\ C\ Par\ A''\ C''$ 
    using  $\langle M\ Midpoint\ A'\ A'' \rangle \langle M\ Midpoint\ C'\ C'' \rangle$  assms(4) l12-17
      par-neq2 par-not-par by blast
{
  assume  $Bet\ C''\ B\ C$ 
  have  $A\ C\ ParStrict\ A''\ C''$ 
    by (metis Col-perm  $\langle A\ C\ Par\ A''\ C'' \rangle \langle Bet\ C''\ B\ C \rangle \langle Col\ C''\ B\ C \rangle$  assms(1)
      between-identity col3 col-trivial-2 par-not-col-strict)
  have  $Bet\ A''\ B\ A \longrightarrow ?thesis$ 
    by (metis Bet-cases  $\langle A \neq B \rangle \langle B \neq A'' \rangle \langle B \neq C'' \rangle \langle Bet\ C''\ B\ C \rangle \langle C \neq B \rangle$  l11-14)
  moreover {
    assume  $Bet\ B\ A\ A'' \vee Bet\ A\ A''\ B$ 
    have  $A = A'' \longrightarrow False$ 
      using  $\langle A\ C\ ParStrict\ A''\ C'' \rangle$  not-par-strict-id by blast
    moreover {
      assume  $A \neq A''$ 
      {
        assume  $Bet\ B\ A\ A''$ 
        have  $A\ C\ TS\ A''\ B$ 
          by (metis Col-cases  $\langle Bet\ B\ A\ A'' \rangle$  assms(1) bet--ts calculation l9-2)
        moreover have  $A\ C\ OS\ B\ C''$ 
          by (metis  $\langle Bet\ C''\ B\ C \rangle \langle C \neq B \rangle$  assms(1) bet-out-1 invert-one-side
            not-col-permutation-2 out-one-side)
        ultimately
        have  $False$ 
          by (meson  $\langle A\ C\ ParStrict\ A''\ C'' \rangle$  l12-6 l9-9 one-side-symmetry
            one-side-transitivity)
      }
    }
  }
}
moreover

```

```

{
  assume Bet  $A A'' B$ 
  have  $A'' C'' TS A B$ 
  using  $\langle A C ParStrict A'' C'' \rangle \langle B \neq A'' \rangle \langle Bet A A'' B \rangle$ 
    bet--ts par-strict-not-col-3 by force
  moreover have  $A'' C'' OS B C$ 
    by (meson Col-cases TS-def  $\langle B \neq C'' \rangle \langle Bet C'' B C \rangle$  bet-out
      calculation invert-one-side out-one-side)
  ultimately have False
    using  $\langle A C ParStrict A'' C'' \rangle$  l9-9 one-side-symmetry
      one-side-transitivity pars--os3412 by blast
}
ultimately have False
  using  $\langle Bet B A A'' \vee Bet A A'' B \rangle$  by fastforce
}
ultimately have False
  by blast
}
ultimately have ?thesis
  using Col-def  $\langle Col A'' B A \rangle$  by blast
}
moreover {
  assume Bet  $B C C''$ 
  {
    assume Bet  $A'' B A$ 
    have  $A C ParStrict A'' C''$ 
    by (metis Par-def  $\langle A C Par A'' C'' \rangle \langle B \neq C'' \rangle \langle Bet A'' B A \rangle$ 
       $\langle Col A'' B A \rangle \langle Col B B C \wedge Col C'' B C \rangle$  assms(1) bet-neq21--neq
      l6-21 not-col-permutation-2)
    have  $C = C'' \longrightarrow ?thesis$ 
    using Par-strict-cases  $\langle A C ParStrict A'' C'' \rangle$  not-par-strict-id by blast
  moreover
  {
    assume  $C \neq C''$ 
    have  $A C TS C'' B$ 
    by (meson  $\langle A C ParStrict A'' C'' \rangle \langle Bet B C C'' \rangle \langle C \neq B \rangle$  bet--ts
      between-symmetry invert-two-sides par-strict-comm par-strict-not-col-1)
    have  $A C OS B A''$ 
    by (metis  $\langle A \neq B \rangle \langle Bet A'' B A \rangle$  assms(1) bet-out-1 col-permutation-5 out-one-side)
    have  $A C TS A'' C''$ 
    using  $\langle A C OS B A'' \rangle \langle A C ParStrict A'' C'' \rangle \langle A C TS C'' B \rangle$  l12-6
      l9-9 one-side-symmetry one-side-transitivity by blast
    hence False
    by (simp add:  $\langle A C ParStrict A'' C'' \rangle$  l12-6 l9-9-bis)
  }
  ultimately have ?thesis
    by blast
}
moreover have  $Bet B A A'' \longrightarrow ?thesis$ 
  using out2--conga  $\langle Bet C'' B C \implies A B C CongA A'' B C'' \rangle \langle Col A'' B A \rangle$ 
     $\langle Col C'' B C \rangle$  calculation l6-4-2 by blast
moreover have  $Bet A A'' B \longrightarrow ?thesis$ 
  using out2--conga Out-def  $\langle B \neq A'' \rangle \langle Bet B C C'' \rangle \langle C \neq B \rangle$  bet-out-1 by metis
ultimately have ?thesis
  using Col-def  $\langle Col A'' B A \rangle$  by blast
}
moreover {
  assume Bet  $C C'' B$ 
  {
    assume Bet  $A'' B A$ 
    have  $A C ParStrict A'' C''$ 
    by (metis Par-def  $\langle A C Par A'' C'' \rangle \langle B \neq C'' \rangle \langle Bet A'' B A \rangle \langle Col A'' B A \rangle$ 
       $\langle Col B B C \wedge Col C'' B C \rangle$  assms(1) bet-neq21--neq col-permutation-2 l6-21)
    have  $C = C'' \longrightarrow ?thesis$ 
    using between-trivial calculation(2) by force
  moreover {

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    assume  $C \neq C''$ 
    have  $A'' C'' TS C B$ 
    by (metis Col-cases  $\langle A C ParStrict A'' C'' \rangle \langle B \neq C'' \rangle \langle Bet C C'' B \rangle$ 
        bet--ts invert-two-sides par-strict-not-col-2)
    moreover have  $A'' C'' OS B A$ 
    using  $\langle A C ParStrict A'' C'' \rangle \langle B \neq A'' \rangle \langle Bet A'' B A \rangle$ 
        bet-out out-one-side par-strict-not-col-3 by presburger
    ultimately have False
    using  $\langle A C ParStrict A'' C'' \rangle$  l9-2 l9-8-2 l9-9 pars--os3412 by blast
  }
  ultimately have ?thesis
  by blast
}
moreover have  $Bet B A A'' \longrightarrow ?thesis$ 
using  $\langle Bet C'' B C \implies A B C CongA A'' B C'' \rangle \langle Col A'' B A \rangle \langle Col C'' B C \rangle$ 
    calculation or-bet-out out2--conga by blast
moreover have  $Bet A A'' B \longrightarrow ?thesis$ 
using  $\langle B \neq A'' \rangle \langle B \neq C'' \rangle \langle Bet C C'' B \rangle$  bet-out-1 out2--conga by presburger
ultimately have ?thesis
using Col-def  $\langle Col A'' B A \rangle$  by blast
}
ultimately show ?thesis
using Col-def  $\langle Col C'' B C \rangle$  by blast
qed
thus ?thesis
using  $\langle A' B' C' CongA A'' B C'' \rangle$  conga-sym-equiv not-conga by blast
qed

```

```

lemma par3-conga3 :
  assumes  $\neg Col A B C$  and
     $A B Par A' B'$  and
     $B C Par B' C'$  and
     $A C Par A' C'$ 
  shows  $A B C CongA3 A' B' C'$ 
proof -
  have  $A B C CongA A' B' C'$ 
  by (simp add: assms(1) assms(2) assms(3) assms(4) triangle-par)
  moreover have  $B C A CongA B' C' A'$ 
  by (simp add: assms(1) assms(2) assms(3) assms(4) not-col-permutation-1
      par-comm triangle-par)
  moreover have  $C A B CongA C' A' B'$ 
  by (metis Col-cases Par-cases assms(1) assms(2) assms(3) assms(4) triangle-par)
  ultimately show ?thesis
  by (simp add: CongA3-def)
qed

```

```

lemma project-par-equiv:
  assumes  $P P' Proj A B X Y$  and
     $Q Q' Proj A B X Y$  and
     $P Q Par A B$ 
  shows  $P Q EqV P' Q'$ 
proof -
  {
    assume  $P Q ParStrict A B$ 
    {
      assume  $Q Q' Par X Y$ 
      hence  $P P' Par Q Q'$ 
      by (metis Proj-def  $\langle P Q ParStrict A B \rangle$  assms(1) not-par-one-not-par
          par-strict-not-col-3)
      have  $P' Q' Par A B$ 
      by (metis Col-cases Par-def Proj-def  $\langle P Q ParStrict A B \rangle$  assms(1) assms(2)
          ker-par not-col-distincts par-not-par par-strict-not-col-1 par-symmetry)
      have  $P P' ParStrict Q P' \longrightarrow Parallelogram P Q Q' P'$ 
      using Par-strict-cases not-par-strict-id by blast
      hence Parallelogram  $P Q Q' P'$ 
      by (metis Col-cases Par-cases Par-def  $\langle P P' Par Q Q' \rangle \langle P' Q' Par A B \rangle$  assms(1))
    }
  }

```

```

    assms(2) assms(3) par-2-plg par-strict-distinct
    postulate-of-transitivity-of-parallelism-def
    postulate-of-transitivity-of-parallelism-thm project-col-eq
  }
  hence Parallelogram P Q Q' P'
    by (metis Col-cases Proj-def ⟨P Q ParStrict A B⟩ assms(2) par-strict-not-col-2)
}
hence Parallelogram P Q Q' P'
  by (metis Col-cases Par-def assms(1) assms(2) assms(3) plg-trivial project-id)
thus ?thesis
  using EqV-def by blast
qed

lemma eqv-project-eq-eq:
  assumes P Q EqV R S and
    P P' Proj A B X Y and
    Q Q' Proj A B X Y and
    R P' Proj A B X Y and
    S S' Proj A B X Y
  shows Q' = S'
proof (cases P = Q)
  case True
  thus ?thesis
    by (metis assms(1) assms(2) assms(3) assms(4) assms(5) null-vector project-uniqueness)
next
  case False
  hence P ≠ Q
    by simp
  have R ≠ S
    using False assms(1) eqv-par par-neq2 by blast
  have P R EqV Q S
    by (simp add: assms(1) eqv-permut)
  have P = R ⟶ ?thesis
    using assms(1) assms(3) assms(5) project-uniqueness vector-uniqueness by blast
  moreover {
    assume P ≠ R
    have Q ≠ S
      using ⟨P R EqV Q S⟩ ⟨P ≠ R⟩ eqv-par par-neq2 by blast
    have P R Par Q S
      using ⟨P R EqV Q S⟩ ⟨P ≠ R⟩ eqv-par by blast
    moreover have P R Par X Y
      using ⟨P ≠ R⟩ assms(2) assms(4) ker-par by blast
    ultimately have ?thesis
      using project-par by (meson assms(3) assms(5) par-symmetry par-trans)
  }
  ultimately show ?thesis
    by blast
qed

lemma eqv-eq-project:
  assumes P Q EqV R S and
    P P' Proj A B X Y and
    Q Q' Proj A B X Y and
    R P' Proj A B X Y
  shows S Q' Proj A B X Y
proof (cases S = Q')
  case True
  thus ?thesis
    using Proj-def assms(3) by auto
next
  case False
  hence S ≠ Q'
    by simp
  {
    assume P ≠ R
    hence P R Par Q S

```

```

    by (simp add: assms(1) eqv-par eqv-permut)
  have Col P R P'
    using assms(2) assms(4) ker-col by force
  {
    assume Q Q' Par X Y
    have P R Par X Y
      using ⟨P ≠ R⟩ assms(2) assms(4) ker-par by auto
    {
      assume Q Q' Par X Y
      have Col S Q Q'
        by (metis Par-cases ⟨P R Par Q S⟩ ⟨P R Par X Y⟩
          ⟨Q Q' Par X Y⟩ par-id-1 par-not-par)
      have Q' Q Par X Y
        using Par-perm ⟨Q Q' Par X Y⟩ by blast
      hence S Q' Par X Y
        by (metis Col-cases False ⟨Col S Q Q'⟩ col-par par-neq1 par-trans)
    }
    hence S Q' Par X Y
      using ⟨Q Q' Par X Y⟩ by auto
  }
  hence S Q' Par X Y
    by (metis Proj-def ⟨P R Par Q S⟩ ⟨P ≠ R⟩ assms(2) assms(3) assms(4)
      ker-par par-left-comm par-not-par par-symmetry)
  hence ?thesis
    using Proj-def assms(3) by force
}
thus ?thesis
  using assms(1) assms(3) vector-uniqueness by blast
qed

lemma eqv-cong:
  assumes A B EqV C D
  shows Cong A B C D
proof -
  have Parallelogram A B C D ⟶ ?thesis
    using plg-cong by auto
  thus ?thesis
    using EqV-def assms cong-trivial-identity not-cong-1243 plg-cong-1 by blast
qed

lemma project-preserves-equiv:
  assumes P Q EqV R S and
    P P' Proj A B X Y and
    Q Q' Proj A B X Y and
    R R' Proj A B X Y and
    S S' Proj A B X Y
  shows P' Q' EqV R' S'
proof (cases P = Q)
  case True
  thus ?thesis
    by (metis assms(1) assms(2) assms(3) assms(4) assms(5)
      eqv-trivial null-vector project-uniqueness)
next
  case False
  hence P ≠ Q
    by simp
  have R ≠ S
    using False assms(1) eqv-par par-neq2 by blast
  {
    assume P Q Par A B
    have P Q EqV P' Q'
      using project-par-equiv [where ?A=A and ?B=B and ?X=X and ?Y=Y]
        assms(2) assms(3) ⟨P Q Par A B⟩ by blast
    have R S EqV R' S'
      by (meson False ⟨P Q Par A B⟩ assms(1) assms(4) assms(5) eqv-par
        par-not-par par-symmetry project-par-equiv)
  }

```



```

hence ?thesis
  by (meson ⟨P Q EqV P' Q'⟩ assms(1) eqv-sym eqv-trans)
}
moreover
{
  assume ¬ P Q Par A B
  hence P Q Par X Y ⟶ ?thesis
    by (metis EqV-def False assms(1) assms(2) assms(3) assms(4) assms(5)
        eqv-par par-symmetry par-trans project-par)
  moreover
  {
    assume ¬ P Q Par X Y
    hence P' ≠ Q'
      using False assms(2) assms(3) ker-par by blast
    have ¬ R S Par X Y
      by (metis False ⟨¬ P Q Par X Y⟩ assms(1) eqv-par par-trans)
    have R' ≠ S'
      using ⟨R ≠ S⟩ ⟨¬ R S Par X Y⟩ assms(4) assms(5) ker-par by force
    obtain Q'' where P Q EqV P' Q''
      using vector-construction by blast
    obtain S'' where R S EqV R' S''
      using vector-construction by blast
    hence P' Q'' EqV R' S''
      by (meson ⟨P Q EqV P' Q'⟩ assms(1) eqv-sym eqv-trans)
    {
      assume Q' = Q''
      have P' ≠ Q'
        by (simp add: ⟨P' ≠ Q'⟩)
      have P' Q' Par A B
        by (metis Col-cases Par-def Proj-def ⟨P' ≠ Q'⟩ assms(2) assms(3))
      hence P Q Par A B
        using False ⟨P Q EqV P' Q'⟩ ⟨Q' = Q''⟩
          eqv-par postulate-of-transitivity-of-parallelism-def
          postulate-of-transitivity-of-parallelism-thm by blast
      hence False
        by (simp add: ⟨¬ P Q Par A B⟩)
    }
    hence Q' ≠ Q''
      by blast
    {
      assume S' = S''
      have P Q Par R S
        by (simp add: False assms(1) eqv-par)
      have P Q Par P' Q''
        by (simp add: False ⟨P Q EqV P' Q'⟩ eqv-par)
      have R S Par R' S'
        by (simp add: ⟨R S EqV R' S'⟩ ⟨R ≠ S⟩ ⟨S' = S''⟩ eqv-par)
      have P' Q'' Par R' S'
        using ⟨P Q Par P' Q'⟩ ⟨P' Q'' EqV R' S'⟩ ⟨S' = S''⟩ eqv-par
          par-neq2 by presburger
      have R' S' Par A B
        by (metis Proj-def ⟨R' ≠ S'⟩ assms(4) assms(5) par-col2-par-bis par-reflexivity)
      hence P Q Par A B
        using ⟨P Q Par R S⟩ ⟨R S Par R' S'⟩ par-not-par by blast
      hence False
        by (simp add: ⟨¬ P Q Par A B⟩)
    }
  }
  hence S' ≠ S''
    by blast
  have P P' EqV Q Q''
    by (simp add: ⟨P Q EqV P' Q'⟩ eqv-permut)
  have R R' EqV S S''
    by (simp add: ⟨R S EqV R' S'⟩ eqv-permut)
  have P' R' EqV Q'' S''
    by (simp add: ⟨P' Q'' EqV R' S'⟩ eqv-permut)
  have Q'' Q' Proj A B X Y

```

```

    using ⟨P Q EqV P' Q'⟩ assms(2) assms(3) eqv-eq-project project-idem by blast
  hence Q'' Q' Par X Y
    using ⟨Q' ≠ Q''⟩ project-par-dir by auto
  have S'' S' Proj A B X Y
    using ⟨R S EqV R' S'⟩ assms(4) assms(5) eqv-eq-project project-idem by blast
  hence S'' S' Par X Y
    using ⟨S' ≠ S''⟩ project-par-dir by auto
  have Q'' Q' Par S'' S'
    using ⟨Q'' Q' Par X Y⟩ ⟨S'' S' Par X Y⟩ not-par-one-not-par by blast
  have ¬ Col A B Q''
    using ⟨Q' ≠ Q''⟩ ⟨Q'' Q' Proj A B X Y⟩ project-not-col by auto
  have ¬ Col P' Q' Q'
    by (metis Proj-def ⟨P' ≠ Q'⟩ ⟨¬ Col A B Q''⟩ assms(2)
        assms(3) col-permutation-2 colx)
  have Cong P' Q'' R' S''
    using ⟨P' Q'' EqV R' S''⟩ eqv-cong by blast
  moreover have P' Q'' Q' CongA3 R' S'' S'
    by (metis Par-def Proj-def ⟨P' Q'' EqV R' S''⟩ ⟨Q'' Q' Par S'' S'⟩
        ⟨R' ≠ S'⟩ ⟨¬ Col P' Q'' Q'⟩ assms(2) assms(3) assms(4) assms(5)
        col3 eqv-par not-col-distincts par3-conga3)
  ultimately have P' Q'' Q' Cong3 R' S'' S'
    using cong-conga3-cong3 ⟨¬ Col P' Q'' Q'⟩ by blast
{
  assume Q' = S'
  hence P' = R'
    using assms(1) assms(2) assms(3) assms(4) assms(5) eqv-comm
        eqv-project-eq-eq by blast
  hence Q'' = S''
    using ⟨P' R' EqV Q'' S''⟩ null-vector by blast
  hence Parallelogram Q'' Q' S' S''
    using ⟨Q' = S'⟩ ⟨Q' ≠ Q''⟩ plg-trivial by presburger
}
moreover
{
  assume Q' ≠ S'
  have P' = R' ⟶ Parallelogram Q'' Q' S' S''
    using assms(1) assms(2) assms(3) assms(4) assms(5) calculation
        eqv-project-eq-eq by blast
  moreover
  {
    assume P' ≠ R'
    have Q'' ≠ S' ∨ Q' ≠ S''
      using Proj-def ⟨¬ Col A B Q''⟩ assms(5) by force
    moreover have ∃ M . M Midpoint Q'' S' ∧ M Midpoint Q' S''
    proof (rule par-cong-mid-os)
      show Q'' Q' ParStrict S'' S'
        by (metis ⟨Q'' Q' Par S'' S'⟩ Par-def ⟨Q' ≠ S'⟩ ⟨Q'' Q' Proj A B X Y⟩
            ⟨S'' S' Proj A B X Y⟩ col2--eq project-col-eq)
      show Cong Q'' Q' S'' S'
        using Cong3-def ⟨P' Q'' Q' Cong3 R' S'' S'⟩ by auto
    {
      assume Parallelogram P' Q'' S'' R'
      hence ParallelogramStrict R' P' Q'' S''
        by (metis Plg-perm Proj-def ⟨P' ≠ R'⟩ ⟨¬ Col A B Q''⟩ assms(2)
            assms(4) colx ncol123-plg-plgs)
      have Q'' S'' ParStrict Q' S'
      proof (rule par-strict-col2-par-strict [where ?C = R' and ?D = P'])
        show Q' ≠ S'
          by (simp add: ⟨Q' ≠ S'⟩)
        show Q'' S'' ParStrict R' P'
          using ⟨ParallelogramStrict R' P' Q'' S''⟩ par-strict-symmetry plgs-pars-1 by blast
        show Col R' P' Q'
          by (metis Proj-def assms(2) assms(3) assms(4) col3)
        show Col R' P' S'
          by (metis Proj-def assms(2) assms(4) assms(5) col3)
      qed
    }
  }
}
qed

```

```

    hence  $Q'' S'' OS Q' S'$ 
    using l12-6 by blast
  }
  thus  $Q'' S'' OS Q' S'$ 
  using EqV-def  $\langle P' Q'' EqV R' S'' \rangle \neg Col P' Q'' Q' \rangle$  not-col-distincts by blast
qed
ultimately have Plg  $Q'' Q' S' S''$ 
  by (simp add: Plg-def)
hence Parallelogram  $Q'' Q' S' S''$ 
  by (simp add: plg-to-parallelogram)
}
ultimately have Parallelogram  $Q'' Q' S' S''$ 
  by blast
}
ultimately have Parallelogram  $Q'' Q' S' S''$ 
  by blast
hence  $Q'' Q' EqV S'' S'$ 
  by (simp add: EqV-def)
hence ?thesis
  using  $\langle P' Q'' EqV R' S'' \rangle$  eqv-sum by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma cop-par--perp2:
  assumes Coplanar  $A B C P$  and
     $A B Par C D$ 
  shows  $P Perp2 A B C D$ 
proof -
  obtain  $Q$  where  $A B Perp Q P$  and Coplanar  $A B C Q$ 
  using assms(2) ex-perp-cop par-neq1 by blast
  have  $Col P P Q$ 
  by (simp add: col-trivial-1)
  moreover have  $P Q Perp A B$ 
  using Perp-perm  $\langle A B Perp Q P \rangle$  by blast
  moreover
  have  $C D Perp P Q$ 
  proof (rule cop-par-perp--perp [where  $?A = A$  and  $?B = B$ ])
    show  $A B Par C D$ 
    by (simp add: assms(2))
    show  $A B Perp P Q$ 
    by (simp add:  $\langle A B Perp Q P \rangle$  perp-right-comm)
  {
    assume  $C D ParStrict A B$ 
    have Coplanar  $C D P Q$ 
    proof (rule coplanar-pseudo-trans [where  $?P = A$  and  $?Q = B$  and  $?R = C$ ])
      show  $\neg Col A B C$ 
      using  $\langle C D ParStrict A B \rangle$  par-strict-not-col-3 by blast
      show Coplanar  $A B C C$ 
      using ncop-distincts by blast
      show Coplanar  $A B C D$ 
      using assms(2) par-cong-cop by blast
      show Coplanar  $A B C P$ 
      by (simp add: assms(1))
      show Coplanar  $A B C Q$ 
      by (simp add:  $\langle Coplanar A B C Q \rangle$ )
    qed
  }
  moreover
  {
    assume  $C \neq D \wedge A \neq B \wedge Col C A B \wedge Col D A B$ 
    have Coplanar  $P Q C D$ 

```

```

proof (rule col2-cop--cop [where ?C = A and ?D = B])
  show Coplanar P Q A B
    using ⟨P Q Perp A B⟩ perp--coplanar by blast
  show A ≠ B
    using ⟨C ≠ D ∧ A ≠ B ∧ Col C A B ∧ Col D A B⟩ by force
  show Col A B C
    using Col-cases ⟨C ≠ D ∧ A ≠ B ∧ Col C A B ∧ Col D A B⟩ by blast
  show Col A B D
    using Col-cases ⟨C ≠ D ∧ A ≠ B ∧ Col C A B ∧ Col D A B⟩ by blast
qed
hence Coplanar C D P Q
  using coplanar-perm-16 by blast
}
ultimately show Coplanar C D P Q
  using Par-cases Par-def assms(2) by blast
qed
hence P Q Perp C D
  using Perp-cases by blast
ultimately show ?thesis
  using Perp2-def by blast
qed

```

lemma l13-11:

```

assumes ¬ Col PO A A' and
  B ≠ PO and
  C ≠ PO and
  Col PO A B and
  Col PO B C and
  B' ≠ PO and
  C' ≠ PO and
  Col PO A' B' and
  Col PO B' C' and
  B C' Par C B' and
  C A' Par A C'
shows A B' Par B A'
proof –
  have Coplanar B C' C PO
    using Col-cases assms(5) ncop--ncols by blast
  have Coplanar C A' PO A
proof (rule col-cop--cop [where ?D = B])
  show Coplanar C A' PO B
    using Col-cases assms(5) ncop--ncols by blast
  show PO ≠ B
    using assms(2) by blast
  show Col PO B A
    using Col-cases assms(4) by blast
qed
hence Coplanar C A' A PO
  using coplanar-perm-1 by blast
have PO Perp2 B C' C B'
  by (simp add: ⟨Coplanar B C' C PO⟩ assms(10) cop-par--perp2)
have PO Perp2 C A' A C'
  by (simp add: ⟨Coplanar C A' A PO⟩ assms(11) cop-par--perp2)
have PO Perp2 A B' B A'
  using ⟨PO Perp2 B C' C B'⟩ ⟨PO Perp2 C A' A C'⟩ assms(1) assms(2) assms(3)
    assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) l13-10 by blast
then obtain X Y where Col PO X Y and X Y Perp A B' and X Y Perp B A'
  using Perp2-def by blast
have Coplanar A X PO A'
proof (rule col-cop--cop [where ?D=B'])
  have Coplanar A B' X PO
proof (rule col-cop--cop [where ?D = Y])
  show Coplanar A B' X Y
    using ⟨X Y Perp A B'⟩ coplanar-perm-16 perp--coplanar by blast
  show X ≠ Y
    using ⟨X Y Perp B A'⟩ perp-distinct by blast

```

```

  show Col X Y PO
    using Col-cases ⟨Col PO X Y⟩ by blast
qed
thus Coplanar A X PO B'
  using coplanar-perm-3 by blast
show PO ≠ B'
  using assms(6) by blast
show Col PO B' A'
  using assms(8) not-col-permutation-5 by blast
qed
hence Coplanar PO A A' X
  using coplanar-perm-13 by blast
have Coplanar A Y PO A'
proof (rule col-cop--cop [where ?D = B'])
  have Coplanar A B' Y PO
  proof (rule col-cop--cop [where ?D = X])
    show Coplanar A B' Y X
      using ⟨X Y Perp A B'⟩ coplanar-perm-17 perp--coplanar by blast
    show Y ≠ X
      using ⟨X Y Perp A B'⟩ perp-not-eq-1 by blast
    show Col Y X PO
      using ⟨Col PO X Y⟩ not-col-permutation-3 by blast
  qed
  thus Coplanar A Y PO B'
    using coplanar-perm-3 by blast
  show PO ≠ B'
    using assms(6) by blast
  show Col PO B' A'
    using assms(8) not-col-permutation-5 by blast
qed
hence Coplanar PO A A' Y
  using coplanar-perm-13 by blast
show ?thesis
proof (rule l12-9 [where ?C1.0=X and ?C2.0=Y])
  show Coplanar X Y A B
    using Coplanar-def ⟨Col PO X Y⟩ assms(4) col-permutation-1 by blast
  show Coplanar X Y A A'
    by (meson ⟨Coplanar PO A A' X⟩ ⟨Coplanar PO A A' Y⟩ assms(1)
      coplanar-pseudo-trans ncop-distincts)
  show Coplanar X Y B' B
  proof (rule coplanar-pseudo-trans [where ?P = PO and ?Q = A and ?R = A'],
    simp-all add: assms(1) ⟨Coplanar PO A A' X⟩ ⟨Coplanar PO A A' Y⟩)
    show Coplanar PO A A' B'
      using assms(8) ncop--ncols by auto
    show Coplanar PO A A' B
      using assms(4) ncop--ncols by blast
  qed
  show Coplanar X Y B' A'
  proof (rule coplanar-pseudo-trans [where ?P = PO and ?Q = A and ?R = A'],
    simp-all add: assms(1) ⟨Coplanar PO A A' X⟩ ⟨Coplanar PO A A' Y⟩)
    show Coplanar PO A A' B'
      using assms(8) ncop--ncols by blast
    show Coplanar PO A A' A'
      using ncop-distincts by blast
  qed
  show A B' Perp X Y
    using Perp-perm ⟨X Y Perp A B'⟩ by blast
  show B A' Perp X Y
    using Perp-perm ⟨X Y Perp B A'⟩ by blast
qed
qed

lemma l13-14:
  assumes PO A ParStrict O' A' and
    Col PO A B and
    Col PO B C and

```

```

Col PO A C and
Col O' A' B' and
Col O' B' C' and
Col O' A' C' and
A C' Par A' C and
B C' Par B' C
shows A B' Par A' B
proof -
have PO ≠ A
  using assms(1) par-strict-distinct by blast
have O' ≠ A'
  using assms(1) par-strict-neq2 by auto
have ¬ Col PO A A'
  using assms(1) par-strict-not-col-4 by presburger
{
  assume A = C
  have A C' ParStrict A' A ⟶ ?thesis
    using Par-strict-cases not-par-strict-id by blast
  hence ?thesis
    by (metis Par-def Par-perm Par-strict-perm ⟨A = C⟩ assms(1) assms(7)
        assms(8) assms(9) col-trivial-2 colx par-strict-not-col-3)
}
moreover
{
  assume A ≠ C
  hence A' ≠ C'
    using Par-cases ⟨¬ Col PO A A'⟩ assms(4) assms(8) col2--eq par-id by blast
  have A C ParStrict A' C'
    by (meson ⟨A ≠ C⟩ ⟨A' ≠ C'⟩ assms(1) assms(4) assms(7) col-trivial-2
        par-strict-col4--par-strict)
  have Plg A C A' C'
    by (metis Col-cases Par-cases ⟨A C ParStrict A' C'⟩ assms(8) l12-19
        not-cong-4321 par-id-5 par-par-cong-cong-parallelogram par-strict-not-col-3
        par-strict-par parallelogram-to-plg)
  {
    assume B = C
    hence A C' Par A' B
      using assms(8) by force
    have B' = C'
      proof -
        have ¬ Col B C' A'
          using ⟨A C ParStrict A' C'⟩ ⟨B = C⟩ col-permutation-5 par-strict-not-col-2 by blast
        moreover have A' ≠ C'
          by (simp add: ⟨A' ≠ C'⟩)
        moreover have Col B C' B'
          using ⟨B = C⟩ assms(9) par-id par-right-comm by blast
        moreover have Col B C' C'
          by (meson not-col-distincts)
        moreover have Col A' C' B'
          using ⟨O' ≠ A'⟩ assms(5) assms(7) col-transitivity-2 by blast
        moreover have Col A' C' C'
          by (simp add: col-trivial-2)
        ultimately show ?thesis
          using l6-21 by blast
      qed
    hence A B' Par A' B
      using ⟨A C' Par A' B⟩ by blast
    hence ?thesis
      by simp
  }
}
moreover
{
  assume B ≠ C
  {
    assume B' = C'
    have B B' ParStrict B' C ⟶ False

```

```

    using Par-strict-cases not-par-strict-id by blast
  moreover
  {
    assume  $B \neq B'$  and  $B' \neq C$  and  $\text{Col } B \ B' \ C$  and  $\text{Col } B' \ B' \ C$ 
    have  $B = C$ 
    proof -
      have  $\neg \text{Col } A \ C \ B'$ 
        using  $\langle A \ C \ \text{ParStrict } A' \ C' \rangle \langle B' = C' \rangle$  par-strict-not-col-4 by auto
      moreover have  $B' \neq B$ 
        using  $\langle B \neq B' \rangle$  by blast
      moreover have  $\text{Col } A \ C \ B$ 
        using  $\langle PO \neq A \rangle$  assms(2) assms(4) col-transitivity-2 by blast
      moreover have  $\text{Col } A \ C \ C$ 
        using not-col-distincts by blast
      moreover have  $\text{Col } B' \ B \ B$ 
        using not-col-distincts by auto
      moreover have  $\text{Col } B' \ B \ C$ 
        using  $\langle \text{Col } B \ B' \ C \rangle$  not-col-permutation-4 by blast
      ultimately show ?thesis
        using l6-21 by blast
    qed
    hence False
      by (simp add:  $\langle B \neq C \rangle$ )
  }
  ultimately have False
    using Par-def  $\langle B' = C' \rangle$  assms(9) by force
}
hence  $B' \neq C'$ 
  by blast
hence  $B \ C \ \text{ParStrict } B' \ C'$ 
  using  $\langle B \neq C \rangle$  assms(1) assms(2) assms(4) assms(5) assms(7)
    par-strict-col4--par-strict by blast
have  $\text{Plg } B \ C \ B' \ C'$ 
  using pars-par-plg  $\langle B \ C \ \text{ParStrict } B' \ C' \rangle$  assms(9) par-right-comm by blast
have Parallelogram  $A \ C \ A' \ C'$ 
  by (simp add:  $\langle \text{Plg } A \ C \ A' \ C' \rangle$  parallelogram-equiv-plg)
have Parallelogram  $B \ C \ B' \ C'$ 
  by (simp add:  $\langle \text{Plg } B \ C \ B' \ C' \rangle$  parallelogram-equiv-plg)
obtain  $M$  where  $M$  Midpoint  $A \ A'$  and  $M$  Midpoint  $C \ C'$ 
  using Plg-def  $\langle \text{Plg } A \ C \ A' \ C' \rangle$  by blast
obtain  $N$  where  $N$  Midpoint  $B \ B'$  and  $N$  Midpoint  $C \ C'$ 
  using Plg-def  $\langle \text{Plg } B \ C \ B' \ C' \rangle$  by blast
hence  $M = N$ 
  using  $\langle M \ \text{Midpoint } C \ C' \rangle$  l7-17 by auto
have Parallelogram  $A \ B \ A' \ B'$ 
  using mid-plg by (metis  $\langle A \ C \ \text{ParStrict } A' \ C' \rangle \langle M = N \rangle \langle M \ \text{Midpoint } A \ A' \rangle$ 
     $\langle N \ \text{Midpoint } B \ B' \rangle$  not-par-strict-id)
have  $A \ B' \ \text{Par } B \ A'$ 
proof (cases  $A = B$ )
  case True
    hence  $A' = B'$ 
      using symmetric-point-uniqueness  $\langle M = N \rangle \langle M \ \text{Midpoint } A \ A' \rangle$ 
         $\langle N \ \text{Midpoint } B \ B' \rangle$  by blast
    thus ?thesis
      by (metis True  $\langle \neg \text{Col } PO \ A \ A' \rangle$  assms(2) par-reflexivity)
  next
    case False
      hence  $B \neq A'$ 
        using  $\langle \neg \text{Col } PO \ A \ A' \rangle$  assms(2) by auto
      thus ?thesis using plg-par
        by (simp add: False  $\langle \text{Parallelogram } A \ B \ A' \ B' \rangle$ )
    qed
  hence ?thesis
    using par-right-comm by presburger
}
ultimately have ?thesis

```

```

    by blast
  }
  ultimately show ?thesis
    by blast
qed

lemma l13-15-1:
  assumes  $\neg \text{Col } A \ B \ C$  and
     $\neg \text{PO } B \ \text{Par } A \ C$  and
     $\text{Coplanar } \text{PO } B \ A \ C$  and
     $A \ B \ \text{ParStrict } A' \ B'$  and
     $A \ C \ \text{ParStrict } A' \ C'$  and
     $\text{Col } \text{PO } A \ A'$  and
     $\text{Col } \text{PO } B \ B'$  and
     $\text{Col } \text{PO } C \ C'$ 
  shows  $B \ C \ \text{Par } B' \ C'$ 
proof -
  {
    assume  $\text{Col } B \ A' \ B'$ 
    hence False
      using assms(4) par-strict-not-col-2 by auto
  }
  {
    assume  $\text{Col } A \ A' \ B'$ 
    hence False
      using assms(4) col-permutation-1 par-strict-not-col-3 by blast
  }
  have  $B \neq \text{PO}$ 
    using assms(4) assms(6) col-permutation-4 par-strict-not-col-1 by blast
  have  $A \neq \text{PO}$ 
    using assms(5) assms(8) par-strict-not-col-4 by blast
  have  $\neg \text{Col } A' \ A \ C$ 
    using assms(5) col-permutation-1 par-strict-not-col-1 by blast
  have  $C \neq \text{PO}$ 
    using Col-cases  $\langle \neg \text{Col } A' \ A \ C \rangle$  assms(6) by blast
  have  $\neg \text{Col } \text{PO } A \ B$ 
    by (metis  $\langle A \neq \text{PO} \rangle \langle B \neq \text{PO} \rangle \langle \text{Col } B \ A' \ B' \implies \text{False} \rangle$  assms(6) assms(7) col3 col-trivial-3)
  {
    assume  $\text{Col } A' \ B' \ C'$ 
    hence  $A' \ C' \ \text{Par } A \ B$ 
      by (metis assms(1) assms(4) assms(5) par-id par-strict-col-par-strict
        par-strict-neq2 par-strict-symmetry par-strict-trans)
    hence  $A \ C \ \text{Par } A \ B$ 
      using  $\langle A' \ C' \ \text{Par } A \ B \rangle$  assms(5) par-not-par par-strict-par by blast
    hence False
      using assms(1) par-id-3 by auto
  }
  show ?thesis
proof (cases  $\text{Col } \text{PO } B \ C$ )
  case True
    thus ?thesis
      by (metis Par-def  $\langle B \neq \text{PO} \rangle \langle C \neq \text{PO} \rangle \langle \text{Col } A' \ B' \ C' \implies \text{False} \rangle$  assms(1) assms(7)
        assms(8) col-transitivity-1 col-trivial-2 not-col-permutation-2)
  next
  case False
    hence  $\neg \text{Col } \text{PO } B \ C$ 
      by auto
    have  $B' \neq \text{PO}$ 
      using Col-cases  $\langle \text{Col } A \ A' \ B' \implies \text{False} \rangle$  assms(6) by blast
    have  $A' \neq \text{PO}$ 
      using Col-cases  $\langle \text{Col } B \ A' \ B' \implies \text{False} \rangle$  assms(7) by blast
    have  $\neg \text{Col } A \ A' \ C'$ 
      using assms(5) col-permutation-1 par-strict-not-col-3 by blast
    have  $C' \neq \text{PO}$ 
      using Col-cases  $\langle \neg \text{Col } A \ A' \ C' \rangle$  assms(6) by blast
    have  $\neg \text{Col } \text{PO } A' \ B'$ 

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  by (meson ⟨B' ≠ PO⟩ ⟨Col B A' B' ⟹ False⟩ assms(7) col2--eq col-permutation-2)
have ¬ Col PO A' C'
  by (metis ⟨A' ≠ PO⟩ ⟨¬ Col A A' C'⟩ assms(6) col2--eq col-permutation-1)
have ¬ Col B' A B
  using assms(4) col-permutation-1 par-strict-not-col-4 by blast
have ¬ Col A' A B
  using assms(4) col-permutation-1 par-strict-not-col-1 by blast
have ∃ C D. C ≠ D ∧ PO B Par C D ∧ Col A C D
  by (metis ⟨B ≠ PO⟩ col-trivial-1 par-distinct parallel-existence1)
then obtain X Y where X ≠ Y and PO B Par X Y and Col A X Y
  by blast
have Coplanar PO B X A
  using ⟨Col A X Y⟩ ⟨PO B Par X Y⟩ ⟨X ≠ Y⟩ col-cop--cop
    col-permutation-1 par-cong-cop by blast
have Coplanar PO B Y A
  using NCol-perm ⟨Col A X Y⟩ ⟨PO B Par X Y⟩ ⟨X ≠ Y⟩ col-cop--cop
    ncoplanar-perm-1 par-cong-cop by blast
have ∃ L. Col L X Y ∧ Col L A' C'
proof -
  have Coplanar X Y A' C'
proof (rule coplanar-pseudo-trans [where ?P=PO and ?Q=A and ?R=B])
  show ¬ Col PO A B
    by (simp add: ⟨¬ Col PO A B⟩)
  show Coplanar PO A B X
    using ⟨Coplanar PO B X A⟩ coplanar-perm-4 by presburger
  show Coplanar PO A B Y
    using ⟨Coplanar PO B Y A⟩ coplanar-perm-4 by blast
  show Coplanar PO A B A'
    using assms(6) ncop--ncols by blast
  show Coplanar PO A B C'
    by (metis ⟨C ≠ PO⟩ assms(3) assms(8) col-cop2--cop coplanar-perm-2 ncop-distincts)
qed
moreover {
  assume X Y Par A' C'
  have PO B Par X Y
    using ⟨PO B Par X Y⟩ by auto
  moreover have X Y Par A C
    using ⟨X Y Par A' C'⟩ assms(5) not-par-one-not-par par-strict-par by blast
  ultimately have False
    using assms(2) par-not-par by blast
}
ultimately show ?thesis
  using cop-npar--inter-exists by blast
qed
then obtain L where Col L X Y and Col L A' C'
  by blast
moreover have B C Par B' C'
proof -
  have A ≠ L
    using ⟨Col L A' C'⟩ ⟨¬ Col A A' C'⟩ by auto
  have A L Par PO B'
    by (metis Par-def ⟨A ≠ L⟩ ⟨B ≠ PO⟩ ⟨B' ≠ PO⟩ ⟨Col A X Y⟩
      ⟨Col L X Y⟩ ⟨PO B Par X Y⟩ ⟨X ≠ Y⟩ assms(7) not-par-not-col
      not-par-one-not-par par-trans)
  have L A Par PO B
proof (cases X = L)
  case True
  show ?thesis
proof (rule par-col-par-2 [where ?B=Y])
  show L ≠ A
    using ⟨A ≠ L⟩ by auto
  show Col L Y A
    using Col-cases True ⟨Col A X Y⟩ by blast
  show L Y Par PO B
    using Par-cases True ⟨PO B Par X Y⟩ by blast
qed

```

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next
  case False
  then show ?thesis
    by (metis ⟨A L Par PO B'⟩ ⟨B ≠ PO⟩ assms(7) not-col-distincts
        par-col2-par-bis par-left-comm)
qed
{
  assume X Y Par PO C
  hence PO B Par PO C
    using ⟨PO B Par X Y⟩ par-trans by blast
  hence False
    using False par-id by auto
}
have Coplanar X Y PO C
  using coplanar-pseudo-trans
  by (meson ⟨Coplanar PO B X A⟩ ⟨Coplanar PO B Y A⟩ ⟨¬ Col PO A B⟩ assms(3)
      coplanar-perm-1 ncop-distincts not-col-permutation-5)
hence ∃ X0. Col X0 X Y ∧ Col X0 PO C
  using ⟨X Y Par PO C ⟹ False⟩ cop-npar--inter-exists by blast
then obtain M where Col M X Y and Col M PO C
  by blast
hence A ≠ M
  by (metis Col-cases ⟨A ≠ PO⟩ ⟨¬ Col A' A C⟩ assms(6) colx not-col-distincts)
have PO B Par A M
  by (meson Col-cases ⟨A ≠ M⟩ ⟨Col A X Y⟩ ⟨Col M X Y⟩ ⟨PO B Par X Y⟩ par-col2-par)
have L ≠ A'
  by (metis ⟨A L Par PO B'⟩ ⟨Col A A' B' ⟹ False⟩ assms(6) not-col-distincts
      not-par-not-col not-strict-par par-id-2)
{
  assume L B' Par A' B'
  {
    assume L B' ParStrict A' B'
    hence False
      using col-trivial-3 par-strict-not-col-2 by blast
  }
  hence False
    using Par-def ⟨Col A' B' C' ⟹ False⟩ ⟨L B' Par A' B'⟩ ⟨L ≠ A'⟩ calculation(2)
    col-transitivity-2 by blast
}
{
  assume A B Par L B'
  hence False using par-not-par
    using ⟨L B' Par A' B' ⟹ False⟩ assms(4) par-strict-par par-symmetry by blast
}
have Coplanar A B L B'
  by (metis ⟨B ≠ PO⟩ ⟨L A Par PO B⟩ assms(7) col-par col-trivial-3 coplanar-perm-8
      ncop-ncols par-cong-cop par-not-par)
then obtain N where Col N A B and Col N L B'
  using ⟨A B Par L B' ⟹ False⟩ cop-npar--inter-exists by blast
have A L ParStrict PO B'
  by (metis NCol-cases Par-def ⟨A L Par PO B'⟩ ⟨A ≠ PO⟩ assms(4) assms(6)
      col-transitivity-2 par-strict-comm par-strict-not-col-2)
hence A ≠ N
  using ⟨Col N L B'⟩ par-strict-not-col-4 by blast
hence A N Par A' B'
  by (metis ⟨B ≠ PO⟩ ⟨Col N A B⟩ ⟨¬ Col PO A B⟩ assms(4)
      col-par par-id-3 par-left-comm par-not-par par-reflexivity par-strict-par)
have PO N Par L A'
proof (cases A PO Par N L)
  case True
  hence A PO ParStrict N L
    using Par-strict-perm ⟨A L ParStrict PO B'⟩ ⟨Col N L B'⟩
    par-not-col-strict par-strict-not-col-2 by blast
  show ?thesis
proof (rule l13-14 [where ?PO=A and ?O'=N and ?C=A' and ?C'=N])
  show A PO ParStrict N L

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    using ⟨A PO ParStrict N L⟩ by auto
  show Col A PO A'
    using Col-cases assms(6) by blast
  show Col A A' A'
    using not-col-distincts by blast
  show Col A PO A'
    using ⟨Col A PO A'⟩ by blast
  show Col N L N
    using not-col-distincts by auto
  show Col N N N
    using not-col-distincts by auto
  show Col N L N
    using not-col-distincts by auto
  show PO N Par L A'
    by (meson ⟨A L Par PO B'⟩ ⟨A N Par A' B'⟩ ⟨A PO ParStrict N L⟩ ⟨Col A PO A'⟩
      ⟨Col N L B'⟩ l13-14 not-col-distincts par-left-comm par-symmetry)
  show A' N Par N A'
    by (metis Par-cases ⟨Col N A B⟩ ⟨¬ Col A' A B⟩ par-reflexivity)
qed
next
case False
hence ¬ A PO Par N L
  by blast
have N ≠ L
  using ⟨A L ParStrict PO B'⟩ ⟨Col N A B⟩ assms(7)
    not-col-permutation-3 not-col-permutation-4 par-not-col by blast
have L ≠ B'
  using Col-cases ⟨Col A' B' C' ⟹ False⟩ calculation(2) by blast
have Col L B' N
  using Col-cases ⟨Col N L B'⟩ by blast
have Coplanar A PO L B'
  using ⟨A L Par PO B'⟩ coplanar-perm-2 par-cong-cop by blast
hence Coplanar A PO N L
  using col-cop--cop ⟨Col L B' N⟩ ⟨L ≠ B'⟩ coplanar-perm-1 by blast
then obtain P where Col P A PO and Col P N L
  using False cop-npar--inter-exists by blast
have PO N Par A' L
proof -
  have P ≠ L
    using ⟨A L ParStrict PO B'⟩ ⟨Col P A PO⟩ col-permutation-4
      par-strict-not-col-1 by blast
  {
    assume P = PO
    hence Col L A L
      using col-trivial-3 by blast
    moreover have Col L PO B'
    proof -
      have Col L N PO
        using Col-cases ⟨Col P N L⟩ ⟨P = PO⟩ by blast
      moreover have Col L N B'
        using Col-cases ⟨Col L B' N⟩ by blast
      ultimately show ?thesis
        by (metis ⟨N ≠ L⟩ col-transitivity-1)
    qed
    ultimately have False
      using ⟨A L ParStrict PO B'⟩ par-strict-not-col-2 by auto
  }
have A ≠ P
  by (metis Col-cases ⟨A L ParStrict PO B'⟩ ⟨A ≠ N⟩ ⟨Col N A B⟩ ⟨Col P N L⟩
    assms(7) col-transitivity-1 par-not-col)
have P ≠ N
  by (metis ⟨A ≠ N⟩ ⟨Col N A B⟩ ⟨Col P A PO⟩ ⟨¬ Col PO A B⟩ col3 col-trivial-2)
have A' ≠ P
  by (metis Par-def ⟨A N Par A' B'⟩ ⟨Col A A' B' ⟹ False⟩ ⟨Col N L B'⟩
    ⟨Col P N L⟩ ⟨N ≠ L⟩ col-transitivity-1 not-col-permutation-2 par-strict-not-col-2)
have B' ≠ P

```

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    using ⟨A L ParStrict PO B'⟩ ⟨Col P A PO⟩ col-permutation-2
    par-strict-not-col-3 by blast
  have ¬ Col P PO L
  by (metis Col-cases ⟨A L ParStrict PO B'⟩ ⟨Col P A PO⟩ ⟨P = PO ⟹ False⟩
    col-transitivity-2 par-strict-not-col-1)
  show ?thesis
  proof (rule l13-11 [where ?PO=P and ?C=A and ?C'=B'],
    simp-all add: ⟨¬ Col P PO L⟩ ⟨A ≠ P⟩ ⟨A' ≠ P⟩ ⟨B' ≠ P⟩ ⟨A L Par PO B'⟩)
    show Col P PO A'
    using Col-cases ⟨A ≠ PO⟩ ⟨Col P A PO⟩ assms(6) col2--eq by blast
    show Col P A' A
    using Col-cases ⟨Col P A PO⟩ ⟨Col P PO A'⟩ ⟨P = PO ⟹ False⟩
      col-transitivity-1 by blast
    show N ≠ P
    using ⟨P ≠ N⟩ by auto
    show Col P L N
    using Col-cases ⟨Col P N L⟩ by auto
    show Col P N B'
    by (meson ⟨Col N L B'⟩ ⟨Col P L N⟩ ⟨Col P N L⟩ ⟨N ≠ L⟩ ⟨P ≠ L⟩
      col-transitivity-1 colx)
    show A' B' Par A N
    by (simp add: ⟨A N Par A' B'⟩ par-symmetry)
  qed
qed
then show ?thesis
  using Par-cases by blast
qed
have A' C' Par PO N
  by (metis Col-def ⟨L ≠ A'⟩ ⟨PO N Par L A'⟩ assms(5) between-trivial2
    calculation(2) col-par not-par-one-not-par par-strict-not-col-2)
hence PO N Par A C
  using assms(5) par-strict-par par-symmetry par-trans by blast
have N M Par B C
proof (cases A N Par PO C)
  case True
  have A N ParStrict PO C
  by (metis Col-cases Par-def Par-strict-cases True assms(5) assms(8)
    col-transitivity-2 par-strict-not-col-3)
  have N M Par C B
proof (rule l13-14 [where ?PO=A and ?C=A and ?O'=PO and ?C'=PO],
  simp-all add: ⟨A N ParStrict PO C⟩)
  show Col A N B
  using Col-cases ⟨Col N A B⟩ by blast
  show Col A B A
  using not-col-distincts by blast
  show Col A N A
  using not-col-distincts by blast
  show Col PO C M
  using Col-cases ⟨Col M PO C⟩ by blast
  show Col PO M PO
  using not-col-distincts by blast
  show Col PO C PO
  by (simp add: col-trivial-3)
  show N PO Par C A
  using Par-cases ⟨PO N Par A C⟩ by blast
  show B PO Par M A
  using Par-cases ⟨PO B Par A M⟩ by blast
qed
thus ?thesis
  using Par-cases by blast
next
case False
hence ¬ A N Par PO C
  by blast
have Coplanar A N PO C
  using ⟨PO N Par A C⟩ coplanar-perm-14 par-cong-cop by blast

```

then obtain P where $Col\ P\ A\ N$ and $Col\ P\ PO\ C$
using $False\ cop-npar--inter-exists$ by blast
have $B \neq P$
using $Col-cases\ \langle Col\ P\ PO\ C \rangle\ \langle \neg\ Col\ PO\ B\ C \rangle$ by blast
have $A \neq P$
by (metis $Col-cases\ \langle A \neq PO \rangle\ \langle Col\ P\ PO\ C \rangle\ \langle \neg\ Col\ A'\ A\ C \rangle$ assms(6)
colx not-col-distincts)
have $M \neq P$
by (metis $\langle A \neq N \rangle\ \langle A \neq PO \rangle\ \langle B \neq PO \rangle\ \langle Col\ N\ A\ B \rangle\ \langle Col\ P\ A\ N \rangle$
 $\langle PO\ B\ Par\ A\ M \rangle\ \langle \neg\ Col\ PO\ A'\ B' \rangle$ assms(6) assms(7) col-trivial-2 col-trivial-3 colx
inter-uniqueness-not-par not-col-permutation-3)
have $PO \neq P$
using $\langle A \neq N \rangle\ \langle Col\ N\ A\ B \rangle\ \langle Col\ P\ A\ N \rangle\ \langle \neg\ Col\ PO\ A\ B \rangle$ l6-16-1
not-col-permutation-3 by blast
have $P \neq N$
by (metis $\langle A \neq N \rangle\ \langle Col\ N\ A\ B \rangle\ \langle Col\ P\ PO\ C \rangle\ \langle PO\ N\ Par\ A\ C \rangle$
 $\langle \neg\ Col\ PO\ A\ B \rangle$ col-trivial-2 colx inter-uniqueness-not-par
not-col-permutation-4 not-col-permutation-5)
show ?thesis
proof (rule l13-11 [where ?PO=P and ?C=A and ?C'=PO], simp-all add:
 $\langle B \neq P \rangle\ \langle A \neq P \rangle\ \langle M \neq P \rangle\ \langle PO \neq P \rangle$)
show $\neg\ Col\ P\ N\ C$
by (metis $\langle A \neq N \rangle\ \langle Col\ N\ A\ B \rangle\ \langle Col\ P\ A\ N \rangle\ \langle P \neq N \rangle$ assms(1) col3 not-col-distincts)
show $Col\ P\ N\ B$
by (metis $\langle A \neq N \rangle\ \langle Col\ N\ A\ B \rangle\ \langle Col\ P\ A\ N \rangle$ col2--eq col-permutation-2)
thus $Col\ P\ B\ A$
using $\langle Col\ P\ A\ N \rangle\ \langle P \neq N \rangle$ col-permutation-5 col-trivial-3 colx by blast
show $Col\ P\ C\ M$
by (metis $Col-cases\ \langle C \neq PO \rangle\ \langle Col\ M\ PO\ C \rangle\ \langle Col\ P\ PO\ C \rangle$ col2--eq)
show $Col\ P\ M\ PO$
using $\langle C \neq PO \rangle\ \langle Col\ M\ PO\ C \rangle\ \langle Col\ P\ PO\ C \rangle$ col2--eq col-permutation-5 by blast
show $B\ PO\ Par\ A\ M$
using $Par-cases\ \langle PO\ B\ Par\ A\ M \rangle$ by blast
show $A\ C\ Par\ N\ PO$
using $Par-cases\ \langle PO\ N\ Par\ A\ C \rangle$ by blast
qed
qed
have $N\ M\ Par\ B'\ C'$
proof (cases $N\ B'\ Par\ PO\ C'$)
case True
have $N\ B'\ ParStrict\ PO\ C'$
by (metis $Par-cases\ Par-def\ True\ \langle A'\ C'\ Par\ PO\ N \rangle\ \langle \neg\ Col\ PO\ A'\ C' \rangle$
par-strict-not-col-4)
have $M \neq L$
by (metis $\langle C \neq PO \rangle\ \langle Col\ M\ PO\ C \rangle\ \langle Col\ N\ L\ B' \rangle\ \langle N\ B'\ ParStrict\ PO\ C' \rangle$
assms(8) colx not-col-distincts par-not-col)
have $L \neq C'$
using $\langle Col\ N\ L\ B' \rangle\ \langle N\ B'\ ParStrict\ PO\ C' \rangle$ col-permutation-5
par-strict-not-col-4 by blast
have $L\ M\ Par\ PO\ B'$
proof (rule par-col-par-2 [where ?B=A])
show $L \neq M$
using $\langle M \neq L \rangle$ by auto
show $Col\ L\ A\ M$
using $Col-cases\ \langle Col\ A\ X\ Y \rangle\ \langle Col\ M\ X\ Y \rangle\ \langle X \neq Y \rangle$ calculation(1) col3 by blast
show $L\ A\ Par\ PO\ B'$
using $Par-cases\ \langle A\ L\ Par\ PO\ B' \rangle$ by auto
qed
have $L\ C'\ Par\ PO\ N$
using par-col-par-2 [where ?B=A] $\langle L \neq C' \rangle$ calculation(2)
 $\langle PO\ N\ Par\ L\ A' \rangle$ par-symmetry by meson
have $N\ M\ Par\ C'\ B'$
proof (rule l13-14 [where ?PO=B' and ?O'=PO and ?C=L and ?C'=PO])
show $B'\ N\ ParStrict\ PO\ C'$
using $Par-strict-cases\ \langle N\ B'\ ParStrict\ PO\ C' \rangle$ by blast
show $Col\ B'\ N\ B'$

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    by (simp add: col-trivial-3)
  show Col B' B' L
    by (simp add: col-trivial-1)
  show Col B' N L
    using Col-cases ⟨Col N L B'⟩ by blast
  show Col PO C' M
    by (metis ⟨C ≠ PO⟩ ⟨Col M PO C⟩ assms(8) col-permutation-1 col-trivial-2 colx)
  show Col PO M PO
    by (simp add: col-trivial-3)
  show Col PO C' PO
    by (simp add: col-trivial-3)
  show N PO Par C' L
    using Par-cases ⟨L C' Par PO N⟩ by auto
  show B' PO Par M L
    using Par-cases ⟨L M Par PO B'⟩ by blast
qed
thus ?thesis
  using Par-cases by blast
next
case False
hence ¬ N B' Par PO C'
  by blast
have Coplanar N B' PO C'
proof (rule coplanar-pseudo-trans [where ?P=PO and ?Q=A and ?R=B],
  simp-all add: ⟨¬ Col PO A B⟩)
  show Coplanar PO A B N
    using Col-cases ⟨Col N A B⟩ ncop--ncols by blast
  show Coplanar PO A B B'
    using assms(7) ncop--ncols by blast
  show Coplanar PO A B PO
    using ncop--distincts by blast
  show Coplanar PO A B C'
    by (metis ⟨C ≠ PO⟩ ⟨Coplanar PO A B PO⟩ assms(3) assms(8)
      col-cop2--cop coplanar-perm-2)
qed
then obtain P where Col P N B' and Col P PO C'
  using False cop-npar--inter-exists by blast
have B' ≠ P using ⟨¬ Col PO B C⟩
  by (metis ⟨B' ≠ PO⟩ ⟨C' ≠ PO⟩ ⟨Col P PO C'⟩ assms(7) assms(8)
    col-transitivity-2 not-col-distincts)
show ?thesis
proof (cases C' = L)
  case True
  hence C' = L
    by blast
  have C' = M
  proof (cases Col PO X C)
    case True
    show ?thesis
  proof (rule l6-21 [where ?A=PO and ?B=C and ?C=Y and ?D=X],
    simp-all add: assms(8))
    show ¬ Col PO C Y
      by (metis True ⟨C ≠ PO⟩ ⟨PO B Par X Y⟩ ⟨¬ Col PO B C⟩
        col-permutation-5 par-col2-par-bis par-id)
    show Y ≠ X
      using ⟨X ≠ Y⟩ by auto
    show Col PO C M
      using Col-cases ⟨Col M PO C⟩ by blast
    show Col Y X C'
      using Col-cases ⟨C' = L⟩ calculation(1) by blast
    show Col Y X M
      using Col-cases ⟨Col M X Y⟩ by auto
  qed
  qed
next
case False
then show ?thesis

```

```

    using True ⟨Col M PO C⟩ ⟨Col M X Y⟩ ⟨X ≠ Y⟩ assms(8)
    calculation(1) l6-21 not-col-permutation-2 not-col-permutation-3 by blast
qed
then show ?thesis
  by (metis True ⟨A' C' Par PO N⟩ ⟨Col A' B' C' ⟹ False⟩ ⟨Col N L B'⟩
    ⟨¬ Col PO A' C'⟩ calculation(2) col-par not-col-permutation-2
    par-comm par-id-4 par-right-comm)
next
case False
hence C' ≠ L
  by blast
{
  assume L = P
  hence Col C' PO A'
    by (metis Col-cases False ⟨Col P PO C'⟩ calculation(2) col-transitivity-1)
  hence Col PO A' C'
    using Col-cases by blast
  hence False
    by (simp add: ⟨¬ Col PO A' C'⟩)
}
show ?thesis
proof (cases L = M)
  case True
  have C' = M
  proof (rule l6-21 [where ?A=PO and ?B=C and ?C=A' and ?D=C'],
    simp-all add: assms(8))
    show ¬ Col PO C A'
      by (metis ⟨C ≠ PO⟩ ⟨¬ Col PO A' C'⟩ assms(8) col-transitivity-1)
    show A' ≠ C'
      using ⟨¬ Col PO C A'⟩ assms(8) by auto
    show Col PO C M
      using Col-cases ⟨Col M PO C⟩ by blast
    show Col A' C' C'
      using not-col-distincts by blast
    show Col A' C' M
      using True calculation(2) not-col-permutation-2 by blast
  qed
  then show ?thesis
    using False True by auto
next
case False
hence L ≠ M
  by blast
have L M Par PO B'
proof (rule par-col-par-2 [where ?B=A], simp-all add: False)
  show Col L A M
    using Col-cases ⟨Col A X Y⟩ ⟨Col M X Y⟩ ⟨X ≠ Y⟩ calculation(1) col3 by blast
  show L A Par PO B'
    using Par-cases ⟨A L Par PO B'⟩ by auto
qed
have L C' Par N PO
  by (metis Par-cases ⟨C' ≠ L⟩ ⟨PO N Par L A'⟩ calculation(2) par-col-par)
have B' ≠ N
  using ⟨Col N A B⟩ ⟨¬ Col B' A B⟩ by blast
show ?thesis
proof (rule l13-11 [where ?PO=P and ?C=L and ?C'=PO], simp-all add: ⟨B' ≠ P⟩
  ⟨Col P N B'⟩ ⟨L C' Par N PO⟩)
{
  assume Col P N C'
  hence Col P B' C'
    by (metis ⟨Col P N B'⟩ ⟨Col P PO C'⟩ ⟨PO N Par A C'⟩ ⟨¬ Col PO A' C'⟩
      assms(5) calculation(2) col-permutation-3 col-transitivity-1 par-strict-par
      par-symmetry parallel-uniqueness)
  have Col C' PO B'
  proof (rule col-transitivity-1 [where ?Q=P])
    show C' ≠ P

```

```

    by (metis  $\langle B' \neq N \rangle \langle C' \neq L \rangle \langle \text{Col } A' B' C' \implies \text{False} \rangle \langle \text{Col } N L B' \rangle$ 
         $\langle \text{Col } P N B' \rangle$  calculation(2) col-transitivity-1 not-col-permutation-1)
  show Col C' P PO
    using Col-cases  $\langle \text{Col } P PO C' \rangle$  by blast
  show Col C' P B'
    using Col-cases  $\langle \text{Col } P B' C' \rangle$  by auto
qed
hence Col PO B C'
  by (metis  $\langle B' \neq PO \rangle$  assms(7) col-trivial-2 col-trivial-3 l6-21)
hence False
  by (metis  $\langle C \neq PO \rangle \langle C' \neq PO \rangle \langle PO B \text{ Par } X Y \rangle \langle X Y \text{ Par } PO C \implies \text{False} \rangle$ 
      assms(8) not-par-not-col par-col-par-2 par-symmetry par-trans)
}
thus  $\neg \text{Col } P N C'$ 
  by blast
show  $L \neq P$ 
  using  $\langle L = P \implies \text{False} \rangle$  by blast
show Col P B' L
  by (metis Col-cases  $\langle B' \neq N \rangle \langle \text{Col } N L B' \rangle \langle \text{Col } P N B' \rangle$  l6-16-1)
show  $M \neq P$ 
  by (metis Col-cases Par-def Par-strict-cases  $\langle A L \text{ ParStrict } PO B' \rangle$ 
       $\langle \text{Col } P B' L \rangle \langle L M \text{ Par } PO B' \rangle$  par-strict-not-col-4)
show  $PO \neq P$ 
  using  $\langle A L \text{ ParStrict } PO B' \rangle \langle \text{Col } P B' L \rangle$  col-permutation-2
  par-strict-not-col-2 by blast
show Col P C' M
  by (metis  $\langle C \neq PO \rangle \langle C' \neq PO \rangle \langle \text{Col } M PO C \rangle \langle \text{Col } P PO C' \rangle$ 
      assms(8) l6-16-1 not-col-permutation-1)
show Col P M PO
  by (metis  $\langle C \neq PO \rangle \langle C' \neq PO \rangle \langle \text{Col } M PO C \rangle \langle \text{Col } P PO C' \rangle$ 
      assms(8) col-permutation-2 l6-16-1)
show  $B' PO \text{ Par } L M$ 
  using Par-cases  $\langle L M \text{ Par } PO B' \rangle$  by auto
qed
qed
qed
qed
thus ?thesis
  using Par-cases  $\langle N M \text{ Par } B C \rangle$  par-trans by blast
qed
ultimately show ?thesis
  using cop-npar--inter-exists by blast
qed
qed

```

lemma l13-15-2-aux:

```

  assumes  $\neg \text{Col } A B C$  and
     $\neg PO A \text{ Par } B C$  and
     $PO B \text{ Par } A C$  and
     $A B \text{ ParStrict } A' B'$  and
     $A C \text{ ParStrict } A' C'$  and
     $\text{Col } PO A A'$  and
     $\text{Col } PO B B'$  and
     $\text{Col } PO C C'$ 
  shows  $B C \text{ Par } B' C'$ 
proof -
  have  $\neg \text{Col } PO A B$ 
    by (metis assms(4) assms(6) assms(7) col3 not-col-distincts par-not-col)
  have  $\neg \text{Col } PO B C$ 
    using  $\langle \neg \text{Col } PO A B \rangle$  assms(3) col-permutation-5 col-trivial-2 not-strict-par by blast
  have  $\neg \text{Col } PO A C$ 
    using  $\langle \neg \text{Col } PO B C \rangle$  assms(3) col-trivial-3 not-col-permutation-2 not-strict-par2 by blast
  have  $PO \neq A$ 
    using  $\langle \neg \text{Col } PO A B \rangle$  col-trivial-1 by blast
  have  $PO \neq B$ 
    using  $\langle \neg \text{Col } PO B C \rangle$  col-trivial-1 by auto

```



```

have  $PO \neq C$ 
  using  $\langle \neg \text{Col } PO \ B \ C \rangle$  not-col-distincts by presburger
have  $PO \neq A'$ 
  using  $\text{assms}(5) \ \text{assms}(8) \ \text{col-permutation-4} \ \text{par-strict-not-col-2}$  by blast
have  $PO \neq B'$ 
  using  $\text{assms}(4) \ \text{assms}(6) \ \text{col-permutation-2} \ \text{par-strict-not-col-3}$  by blast
have  $PO \neq C'$ 
  using  $\text{assms}(5) \ \text{assms}(6) \ \text{col-permutation-2} \ \text{par-strict-not-col-3}$  by blast
have  $\neg \text{Col } PO \ A' \ B'$ 
  by (meson  $\langle PO \neq A' \rangle \ \text{assms}(4) \ \text{assms}(6) \ \text{col-transitivity-2} \ \text{not-col-permutation-5} \ \text{par-strict-not-col-3}$ )
have  $\neg \text{Col } PO \ B' \ C'$ 
  by (metis  $\langle PO \neq B' \rangle \ \langle PO \neq C' \rangle \ \langle \neg \text{Col } PO \ B \ C \rangle \ \text{assms}(7) \ \text{assms}(8) \ \text{col2--eq} \ \text{col-permutation-4}$ )
have  $\neg \text{Col } PO \ A' \ C'$ 
  by (metis Col-cases  $\langle PO \neq A' \rangle \ \text{assms}(5) \ \text{assms}(6) \ \text{col2--eq} \ \text{par-strict-not-col-3}$ )
have  $A \neq A'$ 
  using  $\text{assms}(5) \ \text{not-par-strict-id}$  by blast
have  $B \neq B'$ 
  using  $\text{assms}(4) \ \text{col-trivial-2} \ \text{par-strict-not-col-4}$  by blast
have  $C \neq C'$ 
  using  $\text{assms}(5) \ \text{col-trivial-2} \ \text{par-strict-not-col-4}$  by blast
show ?thesis
proof (cases  $B \ C \ \text{Par} \ B' \ C'$ )
  case True
  then show ?thesis
    by blast
next
case False
hence  $B \neq C$ 
  using  $\text{assms}(1) \ \text{not-col-distincts}$  by presburger
hence  $\exists \ C0 \ D. \ C0 \neq D \wedge B \ C \ \text{Par} \ C0 \ D \wedge \text{Col} \ B' \ C0 \ D$ 
  using parallel-existence by auto
then obtain  $X \ Y$  where  $X \neq Y$  and  $B \ C \ \text{Par} \ X \ Y$  and  $\text{Col} \ B' \ X \ Y$ 
  by blast
{
  assume  $X \ Y \ \text{Par} \ PO \ C$ 
  hence  $PO \ C \ \text{Par} \ B \ C$ 
  using Par-cases  $\langle B \ C \ \text{Par} \ X \ Y \rangle$  not-par-one-not-par by blast
  hence False
  using  $\langle \neg \text{Col } PO \ B \ C \rangle$  par-comm par-id-2 by blast
}
have Coplanar  $X \ Y \ PO \ C$ 
proof -
{
  assume  $B \ C \ \text{ParStrict} \ X \ Y$ 
  have Coplanar  $X \ Y \ PO \ C$ 
  proof (rule coplanar-pseudo-trans [where ?P=B and ?Q=C and ?R=B'])
    show  $\neg \text{Col} \ B \ C \ B'$ 
      using  $\langle B \ C \ \text{ParStrict} \ X \ Y \rangle \ \langle \text{Col} \ B' \ X \ Y \rangle \ \text{col-permutation-2} \ \text{par-not-col}$  by blast
    show Coplanar  $B \ C \ B' \ X$ 
      using  $\langle B \ C \ \text{Par} \ X \ Y \rangle \ \langle \text{Col} \ B' \ X \ Y \rangle \ \text{ncop-distincts} \ \text{not-col-distincts} \ \text{par-col2-par-bis} \ \text{par-cong-cop}$  by blast
    show Coplanar  $B \ C \ B' \ Y$ 
      using ParStrict-def  $\langle B \ C \ \text{ParStrict} \ X \ Y \rangle \ \langle \text{Col} \ B' \ X \ Y \rangle \ \langle \text{Coplanar} \ B \ C \ B' \ X \rangle$ 
        col-cop--cop by blast
    show Coplanar  $B \ C \ B' \ PO$ 
      using Col-cases  $\text{assms}(7) \ \text{ncop--ncols}$  by blast
    show Coplanar  $B \ C \ B' \ C$ 
      using ncop-distincts by blast
  qed
}
moreover have  $B \neq C \wedge X \neq Y \wedge \text{Col} \ B \ X \ Y \wedge \text{Col} \ C \ X \ Y \longrightarrow \text{Coplanar} \ X \ Y \ PO \ C$ 
  using col-permutation-1 ncop--ncols by blast
ultimately show ?thesis
  using Par-def  $\langle B \ C \ \text{Par} \ X \ Y \rangle$  by force
qed

```

```

obtain  $C''$  where  $\text{Col } C'' X Y$  and  $\text{Col } C'' PO C$ 
  using  $\langle \text{Coplanar } X Y PO C \rangle \langle X Y \text{ Par } PO C \implies \text{False} \rangle$  cop-npar--inter-exists by blast
hence  $B' \neq C''$ 
  using  $\langle PO \neq C \rangle \langle \neg \text{Col } PO B' C' \rangle$  assms(8) col-permutation-4 col-trivial-2 colx by blast
have  $B' C'' \text{ Par } B C$ 
  by (metis  $\langle B C \text{ Par } X Y \rangle \langle B' \neq C'' \rangle \langle \text{Col } B' X Y \rangle \langle \text{Col } C'' X Y \rangle \langle X \neq Y \rangle$ 
    col2--eq col-par not-par-one-not-par par-col-par-2 par-left-comm)
have  $A C \text{ Par } A' C''$ 
proof (rule l13-15-1 [where  $?PO=PO$  and  $?A=B$  and  $?A'=B'$ ],
  simp-all add: assms(2) assms(6) assms(7))
  show  $\neg \text{Col } B A C$ 
    using Col-cases assms(1) by auto
  show Coplanar  $PO A B C$ 
    using assms(3) coplanar-perm-2 par-cong-cop by blast
  show  $B A \text{ ParStrict } B' A'$ 
    using Par-strict-cases assms(4) by blast
  show  $B C \text{ ParStrict } B' C''$ 
    by (metis Col-cases Par-def Par-strict-cases  $\langle B \neq B' \rangle \langle B' C'' \text{ Par } B C \rangle$ 
       $\langle \neg \text{Col } PO B C \rangle$  assms(7) l6-16-1)
  show  $\text{Col } PO C C''$ 
    using Col-cases  $\langle \text{Col } C'' PO C \rangle$  by blast
qed
have  $C' \neq C''$ 
  using False  $\langle B' C'' \text{ Par } B C \rangle$  par-symmetry by blast
have  $A' C' \text{ Par } A' C''$ 
  using  $\langle A C \text{ Par } A' C'' \rangle$  assms(5) par-strict-par par-symmetry
    postulate-of-transitivity-of-parallelism-def
    postulate-of-transitivity-of-parallelism-thm by blast
hence  $C' = C''$ 
  by (metis  $\langle \text{Col } C'' PO C \rangle \langle PO \neq C \rangle$  assms(5) assms(8) col-transitivity-2
    not-col-permutation-2 par-id-4 par-strict-not-col-2)
thus ?thesis
  by (simp add:  $\langle C' \neq C'' \rangle$ )
qed
qed

```

lemma *l13-15-2*:

```

assumes  $\neg \text{Col } A B C$  and
   $PO B \text{ Par } A C$  and
   $A B \text{ ParStrict } A' B'$  and
   $A C \text{ ParStrict } A' C'$  and
   $\text{Col } PO A A'$  and
   $\text{Col } PO B B'$  and
   $\text{Col } PO C C'$ 
shows  $B C \text{ Par } B' C'$ 
proof (cases  $B C \text{ Par } B' C'$ )
  case True
    then show ?thesis
      by blast
  next
    case False
      have  $\neg B C \text{ Par } PO A \vee \neg B' C' \text{ Par } PO A$ 
        using False not-par-one-not-par by blast
      moreover {
        assume  $\neg B C \text{ Par } PO A$ 
        hence  $\neg PO A \text{ Par } B C$ 
          using Par-cases by blast
        hence False
          using False assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7)
          l13-15-2-aux by blast
      }
      moreover {
        assume  $\neg B' C' \text{ Par } PO A$ 
        have  $\neg \text{Col } A' B' C'$ 
          by (metis assms(1) assms(3) assms(4) par-id par-strict-col-par-strict
            par-strict-neq2 par-strict-symmetry par-strict-trans)
      }

```

```

moreover have  $B' \neq PO$ 
  using  $assms(3)$   $assms(5)$   $col-permutation-2$   $par-strict-not-col-3$  by  $blast$ 
have  $PO \ B \ Par \ PO \ B'$ 
  by  $(metis \langle B' \neq PO \rangle assms(2) assms(6) not-par-not-col par-distinct)$ 
have  $\neg PO \ A' \ Par \ B' \ C'$ 
  by  $(metis \langle \neg B \ C \ Par \ PO \ A \implies False \rangle \langle \neg B' \ C' \ Par \ PO \ A \rangle assms(5)$ 
     $not-par-not-col not-par-one-not-par par-distinct par-symmetry)$ 
moreover have  $PO \ B' \ Par \ A' \ C'$ 
  by  $(metis Par-def Par-perm \langle PO \ B \ Par \ PO \ B' \rangle assms(2) assms(4) par-trans)$ 
ultimately have  $B' \ C' \ Par \ B \ C$ 
  using  $l13-15-2-aux$   $assms(3)$   $assms(4)$   $assms(5)$   $assms(6)$   $assms(7)$ 
     $col-permutation-5$   $par-strict-symmetry$  by  $blast$ 
hence  $False$ 
  using  $False \ Par-cases$  by  $auto$ 
}
ultimately show  $?thesis$ 
  by  $blast$ 
qed

lemma  $l13-15$ :
assumes  $\neg Col \ A \ B \ C$  and
   $Coplanar \ PO \ B \ A \ C$  and
   $A \ B \ ParStrict \ A' \ B'$  and
   $A \ C \ ParStrict \ A' \ C'$  and
   $Col \ PO \ A \ A'$  and
   $Col \ PO \ B \ B'$  and
   $Col \ PO \ C \ C'$ 
shows  $B \ C \ Par \ B' \ C'$ 
using  $assms(1)$   $assms(2)$   $assms(3)$   $assms(4)$   $assms(5)$   $assms(6)$   $assms(7)$   $l13-15-1$   $l13-15-2$  by  $blast$ 

lemma  $l13-15-par$ :
assumes  $\neg Col \ A \ B \ C$  and
   $A \ B \ ParStrict \ A' \ B'$  and
   $A \ C \ ParStrict \ A' \ C'$  and
   $A \ A' \ Par \ B \ B'$  and
   $A \ A' \ Par \ C \ C'$ 
shows  $B \ C \ Par \ B' \ C'$ 
proof –
have  $Plg \ B' \ A' \ A \ B$ 
  by  $(meson \ Par-cases \ Par-strict-cases \ assms(2) \ assms(4) \ pars-par-plg)$ 
hence  $Parallelogram \ B' \ A' \ A \ B$ 
  using  $parallelogram-equiv-plg$  by  $force$ 
hence  $Parallelogram \ A' \ A \ B \ B'$ 
  using  $Plg-perm$  by  $blast$ 
hence  $Parallelogram \ B \ B' \ A' \ A$ 
  using  $plg-sym$  by  $blast$ 
moreover have  $Plg \ A' \ C' \ C \ A$ 
  using  $Par-cases \ Par-strict-cases \ assms(3) \ assms(5) \ pars-par-plg$  by  $blast$ 
hence  $Parallelogram \ A' \ C' \ C \ A$ 
  using  $parallelogram-equiv-plg$  by  $force$ 
hence  $Parallelogram \ C' \ C \ A \ A'$ 
  using  $Plg-perm$  by  $blast$ 
hence  $Parallelogram \ A' \ A \ C \ C'$ 
  using  $plg-comm2 \ plg-sym$  by  $blast$ 
ultimately have  $Parallelogram \ B \ B' \ C' \ C \vee (B = B' \wedge A' = A \wedge C' = C \wedge B = C)$ 
  using  $plg-pseudo-trans$  by  $blast$ 
have  $B \ B' \ Par \ C \ C'$ 
  using  $assms(4)$   $assms(5)$   $not-par-one-not-par par-symmetry$  by  $blast$ 
moreover {
  assume  $B \ B' \ ParStrict \ C \ C'$ 
  hence  $B \ C \ Par \ B' \ C'$ 
  using  $\langle Parallelogram \ B \ B' \ C' \ C \vee B = B' \wedge A' = A \wedge C' = C \wedge B = C \rangle$ 
     $ncol124-plg-pars1423 \ par-strict-not-col-1 \ par-strict-par$  by  $blast$ 
}
ultimately show  $?thesis$ 
  by  $(metis \ Col-def \ \langle Parallelogram \ B \ B' \ C' \ C \vee B = B' \wedge A' = A \wedge C' = C \wedge B = C \rangle$ 

```

```

    assms(1) between-trivial par-neq1 plg-par-2 plg-trivial plg-uniqueness)
qed

lemma l13-18-2:
  assumes  $\neg$  Col A B C and
    A B ParStrict A' B' and
    A C ParStrict A' C' and
    B C ParStrict B' C' and
    Col PO A A' and
    Col PO B B'
  shows Col PO C C'
proof -
  have  $\neg$  Col PO A B
  by (metis assms(2) assms(5) assms(6) col-transitivity-2 par-strict-not-col-1
    par-strict-not-col-4)
  have Coplanar B' C' PO C
  by (metis NCol-cases assms(4) assms(6) coplanar-trans-1 ncop--ncols
    ncoplanar-perm-4 par-strict-left-comm par-strict-not-col-2 pars--coplanar)
  moreover have  $\neg$  B' C' Par PO C
  by (metis Par-perm Par-strict-perm assms(2) assms(4) assms(5) assms(6)
    col-transitivity-2 not-par-one-not-par par-id-2 par-strict-not-col-3 par-strict-par)
  ultimately have  $\exists X. \text{Col } X B' C' \wedge \text{Col } X PO C$ 
  using cop-npar--inter-exists by blast
  then obtain C'' where Col C'' B' C' and Col C'' PO C
  by blast
  show ?thesis
proof (cases Col PO C C')
  case True
  then show ?thesis
  by simp
next
  case False
  hence C' C'' Par B C
  by (metis NCol-perm Par-def Par-strict-perm  $\langle \text{Col } C'' B' C' \rangle \langle \text{Col } C'' PO C \rangle$ 
    assms(4) par-col-par-2)
  have C' C'' ParStrict B C
  using Par-def  $\langle C' C'' Par B C \rangle \langle \text{Col } C'' B' C' \rangle$  assms(4) par-not-col by blast
  have  $\neg$  Col PO B C
  by (metis  $\langle \neg \text{Col } PO A B \rangle$  assms(4) assms(6) col-transitivity-2 col-trivial-3
    par-strict-not-col-1)
  have B' C'' Par B C
  by (metis Par-cases  $\langle C' C'' Par B C \rangle \langle \text{Col } C'' B' C' \rangle \langle \text{Col } C'' PO C \rangle$ 
     $\langle \neg \text{Col } PO B C \rangle$  assms(2) assms(4) assms(5) assms(6) col-par l6-16-1
    not-col-permutation-1 not-par-one-not-par par-strict-not-col-3 par-strict-par)
  have B' C'' ParStrict B C
  using Par-def  $\langle B' C'' Par B C \rangle \langle C' C'' ParStrict B C \rangle$  par-strict-not-col-2 by blast
  have A C Par A' C''
proof (rule l13-15 [where ?A=B and ?PO=PO and ?A'=B'], simp-all add: assms(6) assms(5))
  show  $\neg$  Col B A C
  using Col-cases assms(1) by blast
  show Coplanar PO A B C
proof -
  have  $\neg$  Col PO A' B'
  by (metis Col-cases assms(2) assms(5) assms(6) colx not-col-distincts
    par-strict-not-col-2 par-strict-not-col-3)
  moreover have Coplanar PO A' B' PO
  using ncop-distincts by blast
  moreover have Coplanar PO A' B' A
  using Col-cases assms(5) ncop--ncols by blast
  moreover have Coplanar PO A' B' B
  using Col-cases assms(6) ncop--ncols by blast
  moreover have Coplanar PO A' B' C
proof -
  have  $\neg$  Col PO A C'
  by (metis  $\langle \neg \text{Col } PO A B \rangle$  assms(3) assms(5) col-transitivity-2 col-trivial-1
    not-col-permutation-2 par-strict-not-col-3)

```

```

moreover have Coplanar PO A C' PO
  using ncop-distincts by blast
moreover have Coplanar PO A C' A'
  using assms(5) ncop--ncols by blast
moreover have Coplanar PO A C' B'
  by (smt (verit) Col-def False ⟨Coplanar B' C' PO C⟩ assms(3) calculation(3)
    coplanar-perm-21 coplanar-perm-8 coplanar-trans-1
    par-strict-not-col-3 pars--coplanar)
moreover have Coplanar PO A C' C
  by (metis ⟨Coplanar B' C' PO C⟩ ⟨¬ Col PO A' B'⟩ assms(4) assms(6)
    calculation(4) col-transitivity-2 coplanar-perm-22
    coplanar-perm-8 coplanar-trans-1 not-col-distincts par-strict-not-col-3)
ultimately show ?thesis
  using coplanar-pseudo-trans by blast
qed
ultimately show ?thesis
  using coplanar-pseudo-trans by blast
qed
show B A ParStrict B' A'
  using Par-strict-cases assms(2) by auto
show B C ParStrict B' C''
  using Par-strict-cases ⟨B' C'' ParStrict B C⟩ by blast
show Col PO C C''
  using Col-cases ⟨Col C'' PO C⟩ by blast
qed
hence A' C' Par A' C''
  using assms(3) par-strict-par par-symmetry
    postulate-of-transitivity-of-parallelism-def
    postulate-of-transitivity-of-parallelism-thm by blast
hence ¬ Col A' B' C'
  by (metis ⟨A C Par A' C''⟩ assms(1) assms(2) col2--eq not-par-one-not-par
    par-col2-par-bis par-id par-strict-neq2 par-strict-par)
hence C = C''
  by (metis False ⟨A' C' Par A' C''⟩ ⟨Col C'' B' C'⟩ ⟨Col C'' PO C⟩
    col2--eq col-permutation-4 col-permutation-5 par-id)
then show ?thesis
  using ⟨Col C'' B' C'⟩ assms(4) par-strict-not-col-2 by blast
qed
qed

lemma l13-18-3-R1:
assumes ¬ Col A B C and
  A B ParStrict A' B' and
  A C ParStrict A' C' and
  B C ParStrict B' C' and
  A A' Par B B'
shows C C' Par A A'
proof –
have A ≠ A'
  using assms(5) par-neq1 by auto
have B ≠ B'
  using assms(4) not-par-strict-id by blast
then obtain P where B B' Par C P
  using parallel-existence1 by blast
show ?thesis
proof (cases C P Par B C)
case True
{
  assume C P ParStrict B C
  hence ?thesis
  using Par-strict-cases not-par-strict-id by blast
}
then show ?thesis
  by (metis Par-def Par-perm True ⟨B B' Par C P⟩ assms(4) col-trivial-1
    par-not-col par-strict-not-col-3)
next

```

```

case False
have  $\neg C P \text{ Par } B' C'$ 
  using False assms(4) not-par-one-not-par par-strict-par by blast
have  $\neg \text{Col } B C B'$ 
  using assms(4) par-strict-not-col-1 by auto
have Coplanar C P B' C'
proof -
  have  $\neg \text{Col } B C B'$ 
    using assms(4) par-strict-not-col-1 by auto
  moreover have Coplanar B C B' B
    using ncop-distincts by blast
  moreover have Coplanar B C B' C
    using ncop-distincts by auto
  moreover have Coplanar B C B' B'
    using ncop-distincts by blast
  moreover have Coplanar B C B' C'
    using assms(4) pars--coplanar by blast
  ultimately show ?thesis
    by (metis  $\langle B B' \text{ Par } C P \rangle$  coplanar-perm-2 coplanar-trans-1 par-cong-cop)
qed
then obtain C'' where  $\text{Col } C'' C P$  and  $\text{Col } C'' B' C'$ 
  using  $\langle \neg C P \text{ Par } B' C' \rangle$  cop-npar--inter-exists by blast
show ?thesis
proof (cases  $B' = C''$ )
case True
  then show ?thesis
    by (metis  $\langle B B' \text{ Par } C P \rangle \langle \text{Col } C'' C P \rangle \langle \neg \text{Col } B C B' \rangle$ 
        col-par not-col-distincts not-par-one-not-par par-distinct par-id-2)
next
case False
  have  $C C'' \text{ Par } B B'$ 
    by (metis  $\langle B B' \text{ Par } C P \rangle \langle \text{Col } C'' B' C' \rangle \langle \text{Col } C'' C P \rangle$  assms(4)
        col-permutation-4 not-par-not-col not-par-one-not-par
        par-distinct par-strict-not-col-2)
  have  $B' C'' \text{ Par } B C$ 
    by (metis False  $\langle \text{Col } C'' B' C' \rangle$  assms(4) col-par not-par-one-not-par
        par-left-comm par-strict-neq2 par-strict-par)
  {
    assume  $\text{Col } A' B' C'$ 
    hence  $C' A' \text{ Par } B C$ 
      by (metis Par-cases assms(3) assms(4) col-transitivity-1 not-col-permutation-5
          par-col2-par-bis par-strict-neq2 par-strict-par)
    hence  $B C \text{ Par } A C$ 
      using Par-cases assms(3) not-par-one-not-par par-strict-par by blast
    {
      assume  $B C \text{ ParStrict } A C$ 
      hence  $\text{Col } A B C$ 
        using col-trivial-2 par-strict-not-col-4 by blast
    }
    hence  $\text{Col } A B C$ 
      using Par-cases  $\langle B C \text{ Par } A C \rangle$  par-id-2 by blast
    hence False
      by (simp add: assms(1))
  }
  have  $A C \text{ Par } A' C''$ 
proof (rule l13-15-par [where ?A=B and ?A'=B'])
  show  $\neg \text{Col } B A C$ 
    using Col-cases assms(1) by blast
  show  $B A \text{ ParStrict } B' A'$ 
    using Par-strict-cases assms(2) by blast
  show  $B C \text{ ParStrict } B' C''$ 
    using Par-def  $\langle B' C'' \text{ Par } B C \rangle \langle \neg \text{Col } B C B' \rangle$  not-col-permutation-2
        par-strict-symmetry by blast
  show  $B B' \text{ Par } A A'$ 
    using Par-cases assms(5) by auto
  show  $B B' \text{ Par } C C''$ 

```

```

    using Par-cases ⟨C C'' Par B B'⟩ by blast
qed
have C' = C''
proof (rule l6-21 [where ?A=C' and ?B=A' and ?C=B' and ?D=C'])
  show ¬ Col C' A' B'
    using Col-cases ⟨Col A' B' C' ⟹ False⟩ by blast
  show B' ≠ C'
    using ⟨Col A' B' C' ⟹ False⟩ col-trivial-2 by auto
  show Col C' A' C'
    by (simp add: col-trivial-3)
  show Col C' A' C''
    by (meson Par-def ⟨A C Par A' C''⟩ assms(3) col-trivial-1 parallel-uniqueness)
  show Col B' C' C'
    by (simp add: col-trivial-2)
  show Col B' C' C''
    using Col-cases ⟨Col C'' B' C'⟩ by auto
qed
thus ?thesis
  using ⟨C C'' Par B B'⟩ assms(5) not-par-one-not-par by blast
qed
qed
qed

```

lemma l13-18-3-R2:

```

assumes ¬ Col A B C and
  A B ParStrict A' B' and
  A C ParStrict A' C' and
  B C ParStrict B' C' and
  A A' Par B B'
shows C C' Par B B'
using assms(1) assms(2) assms(3) assms(4) assms(5) l13-18-3-R1 par-trans by blast

```

lemma l13-18-3:

```

assumes ¬ Col A B C and
  A B ParStrict A' B' and
  A C ParStrict A' C' and
  B C ParStrict B' C' and
  A A' Par B B'
shows C C' Par A A' ∧ C C' Par B B'
using assms(1) assms(2) assms(3) assms(4) assms(5) l13-18-3-R1 l13-18-3-R2 by blast

```

lemma l13-18:

```

assumes ¬ Col A B C and
  A B ParStrict A' B' and
  A C ParStrict A' C'
shows (B C ParStrict B' C' ∧ Col PO A A' ∧ Col PO B B' ⟹ Col PO C C')
∧ ((B C ParStrict B' C' ∧ A A' Par B B') ⟹ (C C' Par A A' ∧ C C' Par B B'))
∧ (A A' Par B B' ∧ A A' Par C C' ⟹ B C Par B' C')
by (meson assms(1) assms(2) assms(3) l13-15-par l13-18-2 l13-18-3-R1 l13-18-3-R2)

```

lemma l13-19-aux:

```

assumes ¬ Col PO A B and
  A ≠ A' and
  A ≠ C and
  PO ≠ A and
  PO ≠ A' and
  PO ≠ C and
  PO ≠ C' and
  PO ≠ B and
  PO ≠ B' and
  PO ≠ D and
  PO ≠ D' and
  Col PO A C and
  Col PO A A' and
  Col PO A C' and
  Col PO B D and

```

```

Col PO B B' and
Col PO B D' and
¬ A B Par C D and
A B Par A' B' and
A D Par A' D' and
B C Par B' C'
shows C D Par C' D'
proof -
have Coplanar A B C D
  using Col-cases Coplanar-def assms(12) assms(15) by auto
then obtain E where Col E A B and Col E C D
  using ⟨Coplanar A B C D⟩ assms(18) cop-npar--inter-exists by blast
hence ¬ A' B' Par PO E
  by (meson assms(1) assms(19) col-trivial-3 par-symmetry parallel-uniqueness)
have Coplanar A' B' PO E
proof (rule coplanar-pseudo-trans [where ?P=PO and ?Q=A and ?R=B], simp-all add:assms(1))
  show Coplanar PO A B A'
    using assms(13) ncop--ncols by blast
  show Coplanar PO A B B'
    using assms(16) ncop--ncols by blast
  show Coplanar PO A B PO
    using ncop-distincts by blast
  show Coplanar PO A B E
    using ⟨Col E A B⟩ ncop--ncols not-col-permutation-2 by blast
qed
then obtain E' where Col E' A' B' and Col E' PO E
  using ⟨¬ A' B' Par PO E⟩ cop-npar--inter-exists by blast
have C ≠ E
  using ⟨Col E A B⟩ assms(1) assms(12) assms(3) l6-16-1 not-col-permutation-3 by blast
show ?thesis
proof (cases Col A D E)
  case True
  hence B = D
    by (metis ⟨C ≠ E⟩ ⟨Col E A B⟩ ⟨Col E C D⟩ assms(1) assms(10) assms(12) assms(15)
        col2--eq col-permutation-2)
  hence A' B' Par A' D'
    using assms(19) assms(20) par-symmetry par-trans by blast
  moreover {
    assume A' B' ParStrict A' D'
    hence ?thesis
      by (simp add: not-par-strict-id)
  }
  ultimately show ?thesis
    by (metis NCol-perm Par-def ⟨B = D⟩ assms(1) assms(13) assms(16)
        assms(17) assms(21) assms(5) col-trivial-3 colx par-comm)
next
case False
have B ≠ B'
  using Par-perm assms(1) assms(13) assms(19) assms(2) l6-16-1 par-id-3 by blast
have D E Par D' E'
proof (rule l13-15 [where ?A=A and ?PO=PO and ?A'=A'], simp-all add: False assms(13))
  show Coplanar PO D A E
    using Coplanar-def NCol-cases ⟨Col E A B⟩ assms(15) by blast
  show A D ParStrict A' D'
    by (metis NCol-cases Par-def assms(1) assms(11) assms(13) assms(17) assms(2)
        assms(20) col-trivial-3 colx)
  show A E ParStrict A' E'
  proof -
    have A E Par A' E'
      by (metis False NCol-perm ⟨C ≠ E⟩ ⟨Col E A B⟩ ⟨Col E C D⟩ ⟨Col E' A' B'⟩
          ⟨Col E' PO E⟩ assms(12) assms(13) assms(19) assms(4) assms(5)
          col-transitivity-1 par-col-par par-col-par-2)
    thus ?thesis
      by (metis NCol-cases Par-def ⟨Col E' A' B'⟩ assms(1) assms(13)
          assms(16) assms(2) assms(9) col-transitivity-2)
  qed
qed

```



```

show Col PO D D'
  using assms(15) assms(17) assms(8) col-transitivity-1 by blast
show Col PO E E'
  using Col-cases ⟨Col E' PO E⟩ by auto
qed
have D C Par D' C'
proof (cases Col B C E)
  case True
  hence B = D
  by (metis ⟨C ≠ E⟩ ⟨Col E C D⟩ assms(1) assms(12) assms(15) assms(6)
    col2--eq col-permutation-4 col-permutation-5)
  thus ?thesis
  using False ⟨Col E A B⟩ not-col-permutation-2 by blast
next
  case False
  have C E Par C' E'
  proof (rule l13-15 [where ?A=B and ?PO=PO and ?A'=B'], simp-all add: False assms(16))
    show Coplanar PO C B E
    using Coplanar-def NCol-perm ⟨Col E A B⟩ assms(12) by blast
    show B C ParStrict B' C'
    by (metis NCol-cases ⟨B ≠ B'⟩ assms(1) assms(12) assms(16) assms(21) assms(6)
      col-trivial-3 colx par-not-col-strict)
    show B E ParStrict B' E'
  proof -
    have B E Par B' E'
    proof (rule par-col-par-2 [where ?B=A])
      show B ≠ E
      using False col-trivial-3 by blast
      show Col B A E
      using Col-cases ⟨Col E A B⟩ by blast
      have B' E' Par B A
      proof (rule par-col-par-2 [where ?B=A'])
        show B' ≠ E'
        by (metis ⟨B ≠ E⟩ ⟨Col E A B⟩ ⟨Col E' PO E⟩ assms(1) assms(16)
          assms(9) col2--eq col-permutation-2)
        show Col B' A' E'
        using Col-cases ⟨Col E' A' B'⟩ by auto
        show B' A' Par B A
        using Par-cases assms(19) by blast
      qed
      thus B A Par B' E'
      using Par-cases by blast
    qed
    thus ?thesis
    by (metis Par-def ⟨B ≠ B'⟩ ⟨Col E A B⟩ assms(1) assms(16) l6-16-1
      not-col-permutation-2)
  qed
  show Col PO C C'
  using assms(12) assms(14) assms(4) col-transitivity-1 by blast
  show Col PO E E'
  using Col-cases ⟨Col E' PO E⟩ by auto
qed
show ?thesis
proof (rule par-col-par-2 [where ?B = E])
  show D ≠ C
  by (metis assms(1) assms(12) assms(15) assms(6) col-trivial-2 colx not-col-permutation-2)
  show Col D E C
  using Col-cases ⟨Col E C D⟩ by auto
  have D' C' Par D E
  proof (rule par-col-par-2 [where ?B = E'])
    show D' ≠ C'
    using assms(1) assms(14) assms(17) assms(7) col2--eq col-permutation-4 by blast
    show Col D' E' C'
    using Par-perm ⟨C E Par C' E'⟩ ⟨Col D E C⟩ ⟨D E Par D' E'⟩ col-par-par-col by blast
    show D' E' Par D E
    using Par-cases ⟨D E Par D' E'⟩ by blast
  qed

```

```

qed
thus  $D \ E \ \text{Par} \ D' \ C'$ 
using Par-cases by blast
qed
qed
thus ?thesis
using Par-cases by blast
qed
qed

```

lemma *l13-19*:

```

assumes  $\neg \text{Col } PO \ A \ B$  and
   $PO \neq A$  and
   $PO \neq A'$  and
   $PO \neq C$  and
   $PO \neq C'$  and
   $PO \neq B$  and
   $PO \neq B'$  and
   $PO \neq D$  and
   $PO \neq D'$  and
   $\text{Col } PO \ A \ C$  and
   $\text{Col } PO \ A \ A'$  and
   $\text{Col } PO \ A \ C'$  and
   $\text{Col } PO \ B \ D$  and
   $\text{Col } PO \ B \ B'$  and
   $\text{Col } PO \ B \ D'$  and
   $A \ B \ \text{Par} \ A' \ B'$  and
   $A \ D \ \text{Par} \ A' \ D'$  and
   $B \ C \ \text{Par} \ B' \ C'$ 
shows  $C \ D \ \text{Par} \ C' \ D'$ 
proof (cases  $A = A'$ )
case True
hence  $B = B'$ 
using assms(1) assms(14) assms(16) col2--eq col-permutation-5 par-id by blast
have  $D = D'$ 
by (metis True assms(1) assms(13) assms(15) assms(17) colx not-col-permutation-2
not-col-permutation-4 par-id-4)
have  $C = C'$ 
using  $\langle B = B' \rangle$  assms(1) assms(10) assms(12) assms(18) colx par-id-2 by blast
then show ?thesis
by (metis NCol-cases  $\langle D = D' \rangle$  assms(1) assms(10) assms(13) assms(4)
col-transitivity-2 par-reflexivity)
next
case False
hence  $A \neq A'$ 
by blast
show ?thesis
proof (cases  $A = C$ )
case True
have  $A' = C'$ 
by (metis True assms(1) assms(11) assms(12) assms(14) assms(16)
assms(18) assms(7) col-par-par-col col-trivial-3 colx not-col-permutation-1)
thus ?thesis
using True assms(17) by auto
next
case False
hence  $A \neq C$ 
by blast
then show ?thesis
proof (cases  $A' = C'$ )
case True
have  $A = C$ 
by (metis True assms(1) assms(10) assms(16) assms(18) l6-16-1
not-par-one-not-par par-comm par-id)
thus ?thesis
using False by blast

```

```

next
  case False
  hence  $A' \neq C'$ 
    by blast
  show ?thesis
  proof (cases  $C \ D \ \text{Par} \ C' \ D'$ )
    case True
    thus ?thesis
      by simp
  next
  case False
  hence  $\neg C \ D \ \text{Par} \ C' \ D'$ 
    by blast
  have  $\neg C \ D \ \text{Par} \ A' \ B' \vee \neg C' \ D' \ \text{Par} \ A' \ B'$ 
    using False not-par-one-not-par by blast
  moreover {
    assume  $\neg C \ D \ \text{Par} \ A' \ B'$ 
    hence  $\neg C \ D \ \text{Par} \ A \ B$ 
      using assms(16) par-trans by blast
    hence  $\neg A \ B \ \text{Par} \ C \ D$ 
      using Par-cases by blast
    hence ?thesis
      using l13-19-aux
      using  $\langle A \neq A' \rangle \langle A \neq C \rangle$  assms(1) assms(10) assms(11) assms(12) assms(13)
        assms(14) assms(15) assms(16) assms(17) assms(18) assms(2) assms(3) assms(4)
        assms(5) assms(6) assms(7) assms(8) assms(9) by force
  }
  moreover {
    assume  $\neg C' \ D' \ \text{Par} \ A' \ B'$ 
    hence  $C' \ D' \ \text{Par} \ C \ D$ 
    proof -
      have  $\neg \text{Col} \ PO \ A' \ B'$ 
        by (meson assms(1) assms(11) assms(14) assms(3) assms(7) col2--eq col-permutation-4)
      moreover have  $A' \neq A$ 
        using  $\langle A \neq A' \rangle$  by auto
      moreover have  $A' \neq C'$ 
        using  $\langle A' \neq C' \rangle$  by auto
      moreover have  $\text{Col} \ PO \ A' \ C'$ 
        using assms(11) assms(12) assms(2) col-transitivity-1 by blast
      moreover have  $\text{Col} \ PO \ A' \ A$ 
        using Col-cases assms(11) by auto
      moreover have  $\text{Col} \ PO \ A' \ C$ 
        using assms(10) assms(11) assms(2) col-transitivity-1 by blast
      moreover have  $\text{Col} \ PO \ B' \ D'$ 
        using assms(14) assms(15) assms(6) col-transitivity-1 by blast
      moreover have  $\text{Col} \ PO \ B' \ B$ 
        using Col-cases assms(14) by blast
      moreover have  $\text{Col} \ PO \ B' \ D$ 
        using assms(13) assms(14) assms(6) col-transitivity-1 by blast
      moreover have  $\neg A' \ B' \ \text{Par} \ C' \ D'$ 
        using Par-cases  $\langle \neg C' \ D' \ \text{Par} \ A' \ B' \rangle$  by blast
      moreover have  $A' \ B' \ \text{Par} \ A \ B$ 
        using Par-cases assms(16) by blast
      moreover have  $A' \ D' \ \text{Par} \ A \ D$ 
        using Par-cases assms(17) by blast
      moreover have  $B' \ C' \ \text{Par} \ B \ C$ 
        using Par-cases assms(18) by auto
      ultimately show ?thesis
        using l13-19-aux assms(2) assms(3) assms(4) assms(5) assms(6) assms(7)
          assms(8) assms(9) by force
    qed
    hence ?thesis
      using Par-cases by blast
  }
  ultimately show ?thesis
    by blast

```

qed
 qed
 qed
 qed

lemma l13-19-par-aux:
 assumes $X \neq A$ and

$Y \neq B$ and

$Col\ X\ A\ C$ and
 $Col\ X\ A\ A'$ and
 $Col\ X\ A\ C'$ and
 $Col\ Y\ B\ D$ and
 $Col\ Y\ B\ B'$ and
 $Col\ Y\ B\ D'$ and
 $A \neq C$ and
 $B \neq D$ and
 $A \neq A'$ and
 $X\ A\ ParStrict\ Y\ B$ and
 $\neg A\ B\ Par\ C\ D$ and
 $A\ B\ Par\ A'\ B'$ and
 $A\ D\ Par\ A'\ D'$ and
 $B\ C\ Par\ B'\ C'$

shows $C\ D\ Par\ C'\ D'$

proof –

have $Coplanar\ B\ Y\ A\ X$
 using $assms(12)\ coplanar-perm-23\ pars--coplanar$ by blast
 hence $Coplanar\ A\ C\ B\ Y$
 by ($meson\ assms(1)\ assms(3)\ col-cop2--cop\ coplanar-perm-16\ ncop-distincts$)
 hence $Coplanar\ A\ B\ C\ D$
 by ($metis\ assms(6)\ assms(2)\ col2-cop2--eq\ col-trivial-2\ coplanar-perm-2\ ncop-distincts\ not-col-permutation-3$)
 then obtain E where $Col\ E\ A\ B$ and $Col\ E\ C\ D$
 using $assms(13)\ cop-npar--inter-exists$ by blast
 obtain Z where $X\ A\ Par\ E\ Z$
 using $assms(1)\ parallel-existence1$ by blast
 hence $\neg A\ B\ Par\ E\ Z$
 using $assms(12)\ not-par-one-not-par\ par-id-1\ par-left-comm\ par-strict-not-col-4$ by blast
 hence $\neg A'\ B'\ Par\ E\ Z$
 using $assms(14)\ postulate-of-transitivity-of-parallelism-def\ postulate-of-transitivity-of-parallelism-thm$ by blast
 have $Coplanar\ A'\ B'\ E\ Z$

proof –

{
 assume $X\ A\ ParStrict\ E\ Z$
 hence $Coplanar\ X\ A\ E\ Z$
 using $\langle X\ A\ Par\ E\ Z \rangle\ par-cong-cop$ by auto
 have $Coplanar\ X\ A\ E\ A'$
 using $assms(4)\ ncop--ncols$ by blast
 have $Coplanar\ X\ A\ E\ E$
 using $ncop-distincts$ by blast
 have $Coplanar\ X\ A\ E\ B'$
 proof (cases $Col\ A\ B\ A'$)
 case True
 thus ?thesis
 by ($metis\ \langle Col\ E\ A\ B \rangle\ assms(4)\ assms(11)\ assms(14)\ l6-16-1\ ncop--ncol\ not-col-permutation-2\ par-distincts$)

next

case False
 moreover have $Coplanar\ A\ B\ A'\ B'$
 by ($simp\ add:\ assms(14)\ par-cong-cop$)
 moreover have $Coplanar\ A\ B\ A'\ X$
 using $assms(4)\ ncop--ncols\ not-col-permutation-2$ by blast
 moreover have $Coplanar\ A\ B\ A'\ E$
 using $\langle Col\ E\ A\ B \rangle\ ncop--ncols\ not-col-permutation-2$ by blast

```

    moreover have Coplanar A B A' A
      using ncop-distincts by blast
    ultimately show ?thesis
      using coplanar-pseudo-trans by presburger
  qed
  have  $\neg$  Col X A E
    using  $\langle X A \text{ ParStrict } E Z \rangle$  par-strict-not-col-1 by auto
  hence ?thesis
    by (simp add:  $\langle \text{Coplanar } X A E A' \rangle \langle \text{Coplanar } X A E B' \rangle \langle \text{Coplanar } X A E E \rangle$ 
       $\langle \text{Coplanar } X A E Z \rangle$  coplanar-pseudo-trans)
}
thus ?thesis
  by (metis NCol-perm Par-def  $\langle X A \text{ Par } E Z \rangle$  assms(4) colx ncop--ncols)
qed
then obtain E' where Col E' A' B' and Col E' E Z
  using  $\langle \neg A' B' \text{ Par } E Z \rangle$  cop-npar--inter-exists by blast
{
  assume Col A D E
  have False
  proof (cases A = E)
    case True
    then show ?thesis
      by (metis  $\langle \text{Col } E C D \rangle$  assms(1) assms(6) assms(9) assms(12) assms(3)
        col-par not-col-permutation-1 par-strict-not-col-2 par-strict-par
        parallel-uniqueness)
  next
    case False
    have Col A B D
      using Col-cases False  $\langle \text{Col } A D E \rangle \langle \text{Col } E A B \rangle$  col-transitivity-1 by blast
    have Col A X A  $\wedge$  Col A Y B
      by (meson  $\langle \text{Col } A B D \rangle$  assms(6) assms(10) assms(12) col2--eq one-side-not-col124 pars--os3412)
    thus ?thesis
      using assms(12) par-strict-not-col-2 by blast
  qed
}
have X A ParStrict E Z
  by (metis  $\langle X A \text{ Par } E Z \rangle$  Col-def Par-def  $\langle \text{Col } A D E \implies \text{False} \rangle \langle \text{Col } E A B \rangle$  assms(12)
    between-trivial l6-16-1 par-strict-not-col-4)
have Y B Par E Z
  by (metis Par-perm  $\langle X A \text{ Par } E Z \rangle$  assms(12) not-par-one-not-par par-strict-par)
have Y B ParStrict E Z
proof -
{
  assume Y  $\neq$  B and E  $\neq$  Z and Col Y E Z and Col B E Z
  have Col E Y B
    by (metis  $\langle \text{Col } B E Z \rangle \langle \text{Col } Y E Z \rangle \langle Y B \text{ Par } E Z \rangle$  assms(8) col2--eq
      col-permutation-2 not-strict-par)
  have Col B D E
    using  $\langle \text{Col } Y E Z \rangle \langle Y B \text{ Par } E Z \rangle$  assms(6) assms(10) assms(2) col-par
      col-permutation-1 parallel-uniqueness by blast
  have B = D
  proof (rule l6-21 [where ?A=B and ?B=E and ?C=C and ?D=D])
    {
      assume Col B E C
      hence Col C X A  $\wedge$  Col C Y B
        by (metis  $\langle \text{Col } E A B \rangle \langle \text{Col } E C D \rangle$  assms(6) assms(9) assms(10)
          assms(12) assms(3) col-permutation-4 col-permutation-5
          col-trivial-3 colx par-not-col)
      hence False
        using assms(12) par-not-col by blast
    }
  thus  $\neg$  Col B E C
    by blast
  show C  $\neq$  D
    using Col-cases  $\langle \text{Col } B D E \rangle \langle \neg \text{Col } B E C \rangle$  by blast
  show Col B E B

```

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    using not-col-distincts by auto
  show Col B E D
    using Col-cases ⟨Col B D E⟩ by blast
  show Col C D B
    by (metis ⟨Col A D E ⟹ False⟩ ⟨Col B D E⟩ ⟨Col E C D⟩ col-permutation-1
        col-trivial-2 colx)
  show Col C D D
    using col-trivial-2 by auto
qed
hence False
  by (simp add: assms(10))
}
thus ?thesis
  using Par-def ⟨Y B Par E Z⟩ by auto
qed
{
  assume Col A' D' E'
  have Col A' B' D'
    by (metis (full-types) ⟨Col A' D' E'⟩ ⟨Col E' A' B'⟩ ⟨Col E' E Z⟩
        ⟨X A Par E Z⟩ ⟨X A ParStrict E Z⟩ assms(4) col2--eq col-permutation-3
        col-permutation-4 not-strict-par2 par-strict-not-col-4)
  have Col A' X A
    using Col-cases assms(4) by blast
  moreover have Col A' Y B
  proof -
    have Col Y B' D'
      using assms(7) assms(8) assms(2) col-transitivity-1 by blast
    have A D Par A B
      using ⟨Col A' B' D'⟩ assms(14) assms(15) not-par-one-not-par par-col-par
        par-distinct by blast
    hence False
      by (meson assms(6) assms(10) assms(12) l6-16-1 one-side-not-col124
          par-id-3 pars--os3412)
    thus ?thesis
      by blast
  qed
  ultimately have False
    using assms(12) par-not-col by blast
}
have ¬ Col X A B
  using assms(12) par-strict-not-col-4 by auto
hence B ≠ B'
  by (metis Col-cases assms(4) assms(11) assms(14)
      col2--eq inter--npar inter-trivial)
hence C ≠ C'
  by (metis Col-cases Par-cases assms(5) assms(7) assms(12) assms(16) colx
      not-col-distincts par-id-5 par-not-col)
have ¬ Col Y A B
  using assms(12) col-permutation-4 par-strict-not-col-2 by blast
have D ≠ D'
  by (metis Col-cases assms(4) assms(8) assms(11) assms(12) assms(15) colx
      inter--npar inter-trivial not-col-distincts par-not-col)
have A' ≠ C'
  using ⟨¬ Col X A B⟩ assms(9) assms(14) assms(16) assms(3) l6-16-1
      not-par-one-not-par par-comm par-id by blast
have B' ≠ D'
  using Col-cases ⟨Col A' D' E' ⟹ False⟩ ⟨Col E' A' B'⟩ by blast
have A ≠ E
  using ⟨Col A D E ⟹ False⟩ col-trivial-3 by force
have A' ≠ E'
  using ⟨Col A' D' E' ⟹ False⟩ col-trivial-3 by blast
have B ≠ E
  using ⟨Y B ParStrict E Z⟩ col-trivial-2 par-strict-not-col-1 by blast
have B' ≠ E'
  using ⟨Col E' E Z⟩ ⟨Y B ParStrict E Z⟩ assms(7) col-permutation-2 par-not-col by blast
have A E Par A' E'

```

```

by (metis  $\langle A \neq E \rangle \langle A' \neq E' \rangle \langle \text{Col } E \ A \ B \rangle \langle \text{Col } E' \ A' \ B' \rangle$  assms(14) col-par
not-par-one-not-par par-left-comm par-neq2 par-trans)
have A E ParStrict A' E'
by (metis NCol-perm  $\langle A \ E \ \text{Par } A' \ E' \rangle \langle X \ A \ \text{ParStrict } E \ Z \rangle$  assms(4)
assms(11) col-trivial-3 l6-16-1 par-not-col-strict
par-strict-not-col-1)
have D E Par D' E'
proof (rule l13-15-par [where ?A=A and ?A'=A'])
show  $\neg \text{Col } A \ D \ E$ 
using  $\langle \text{Col } A \ D \ E \implies \text{False} \rangle$  by auto
show A D ParStrict A' D'
by (metis NCol-perm Par-def  $\langle D \neq D' \rangle$  assms(6) assms(8) assms(12) assms(15)
colx par-strict-comm par-strict-not-col-3)
show A E ParStrict A' E'
using  $\langle A \ E \ \text{ParStrict } A' \ E' \rangle$  by auto
show A A' Par D D'
by (meson  $\langle D \neq D' \rangle \langle X \ A \ \text{Par } E \ Z \rangle \langle Y \ B \ \text{Par } E \ Z \rangle$  assms(4) assms(6) assms(8)
assms(11) not-col-distincts not-par-one-not-par par-col4--par)
have A X Par E E'
by (metis  $\langle A \ E \ \text{ParStrict } A' \ E' \rangle \langle \text{Col } E' \ E \ Z \rangle \langle X \ A \ \text{Par } E \ Z \rangle$  not-col-distincts
par-col2-par-bis par-comm par-strict-not-col-2)
thus A A' Par E E'
using Col-cases assms(4) assms(11) par-col-par-2 by blast
qed
have C E Par C' E'
proof (rule l13-15-par [where ?A=B and ?A'=B'])
show  $\neg \text{Col } B \ C \ E$ 
by (metis  $\langle B \neq E \rangle \langle \text{Col } E \ A \ B \rangle \langle \neg \text{Col } X \ A \ B \rangle$  assms(9) assms(3)
col-permutation-5 col-trivial-3 colx l6-16-1)
show B C ParStrict B' C'
by (metis NCol-perm Par-def  $\langle B \neq B' \rangle$  assms(5) assms(7) assms(12) assms(16)
l6-16-1 par-not-col)
show B E ParStrict B' E'
proof -
have B A Par B' E'
by (metis Col-cases Par-cases  $\langle B' \neq E' \rangle \langle \text{Col } E' \ A' \ B' \rangle$  assms(14) par-col-par)
hence B E Par B' E'
using Col-cases  $\langle B \neq E \rangle \langle \text{Col } E \ A \ B \rangle$  par-col-par-2 by blast
thus ?thesis
by (metis NCol-cases Par-def  $\langle A \ E \ \text{ParStrict } A' \ E' \rangle \langle \text{Col } E' \ A' \ B' \rangle$ 
l6-16-1 par-strict-not-col-2)
qed
show B B' Par C C'
by (metis Par-def Par-perm  $\langle B \neq B' \rangle \langle C \neq C' \rangle$  assms(1) assms(5) assms(7)
assms(12) assms(2) assms(3) col-par not-col-permutation-2 par-trans)
have B Y Par E E'
by (metis  $\langle A \ E \ \text{ParStrict } A' \ E' \rangle \langle \text{Col } E' \ E \ Z \rangle \langle Y \ B \ \text{Par } E \ Z \rangle$  not-col-distincts
par-col2-par-bis par-comm par-strict-not-col-2)
thus B B' Par E E'
using Col-cases  $\langle B \neq B' \rangle$  assms(7) par-col-par-2 by blast
qed
have C  $\neq$  D
using Col-cases assms(6) assms(12) assms(3) par-not-col by blast
moreover have C E Par C' D'
proof -
have C'  $\neq$  D'
using Col-cases assms(5) assms(8) assms(12) par-not-col by blast
moreover have Col C' E' D'
proof (rule col-par-par-col [where ?A=C and ?B=E and ?C=D])
show Col C E D
using Col-cases  $\langle \text{Col } E \ C \ D \rangle$  by auto
show C E Par C' E'
using  $\langle C \ E \ \text{Par } C' \ E' \rangle$  by blast
show E D Par E' D'
using Par-cases  $\langle D \ E \ \text{Par } D' \ E' \rangle$  by blast
qed

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moreover have  $C' E' \text{ Par } C E$ 
  using  $\text{Par-cases } \langle C E \text{ Par } C' E' \rangle$  by blast
ultimately show ?thesis
  using  $\langle C E \text{ Par } C' E' \rangle$  par-col-par by blast
qed
ultimately show ?thesis
  using par-col-par-2  $\langle \text{Col } E C D \rangle$  col-permutation-4 by blast
qed

lemma l13-19-par:
assumes  $X \neq A$  and
   $X \neq A'$  and

   $Y \neq B$  and
   $Y \neq B'$  and

   $\text{Col } X A C$  and
   $\text{Col } X A A'$  and
   $\text{Col } X A C'$  and
   $\text{Col } Y B D$  and
   $\text{Col } Y B B'$  and
   $\text{Col } Y B D'$  and
   $X A \text{ ParStrict } Y B$  and
   $A B \text{ Par } A' B'$  and
   $A D \text{ Par } A' D'$  and
   $B C \text{ Par } B' C'$ 
shows  $C D \text{ Par } C' D'$ 
proof (cases  $A = C$ )
  case True
have  $A' B' \text{ Par } B' C'$ 
  using Par-cases True assms(12) assms(14) par-trans by blast
moreover have  $A' B' \text{ ParStrict } B' C' \longrightarrow ?thesis$ 
  using Par-strict-cases not-par-strict-id by blast
ultimately show ?thesis
  by (metis Par-def True assms(6) assms(7) assms(9) assms(11) assms(13)
    colx not-col-permutation-1 par-not-col)
next
case False
hence  $A \neq C$ 
  by simp
show ?thesis
proof (cases  $B = D$ )
  case True
have  $A' B' \text{ Par } A' D'$ 
  using True assms(12) assms(13) not-par-one-not-par par-symmetry by blast
moreover have  $A' B' \text{ ParStrict } A' D' \longrightarrow ?thesis$ 
  using not-par-strict-id by blast
ultimately show ?thesis
  by (metis True assms(6) assms(9) assms(10) assms(11) assms(14) colx
    not-col-permutation-1 par-id-4 par-left-comm par-not-col par-right-comm)
next
case False
hence  $B \neq D$ 
  by simp
show ?thesis
proof (cases  $A = A'$ )
  case True
hence  $A B \text{ ParStrict } A' B' \longrightarrow ?thesis$ 
  by (simp add: not-par-strict-id)
moreover {
  assume  $A \neq B$  and  $A \neq B'$  and  $\text{Col } A A B'$  and  $\text{Col } B A B'$ 
  have  $B = B'$ 
  by (metis Par-strict-cases True assms(9) assms(11) assms(12)
    col2--eq par-id par-strict-not-col-2)
  hence  $C = C'$ 
  by (metis assms(7) assms(11) assms(14) assms(5) colx par-id-4)

```



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    par-strict-not-col-4)
  moreover have  $D = D'$ 
  by (metis Par-strict-cases True assms(8) assms(10) assms(11)
      assms(13) colx par-id-4 par-strict-not-col-1)
  ultimately have ?thesis
  by (metis assms(7) assms(10) assms(11) not-col-permutation-1
      par-not-col par-reflexivity)
}
ultimately show ?thesis
using True assms(12) col-trivial-1 par-distinct par-id-1 by force
next
case False
hence  $A \neq A'$ 
by simp
show ?thesis
proof (cases  $C \ D \ Par \ C' \ D'$ )
case True
then show ?thesis
by simp
next
case False
hence  $\neg C \ D \ Par \ C' \ D'$ 
by simp
have  $\neg C \ D \ Par \ A \ B \vee \neg C' \ D' \ Par \ A \ B$ 
using False not-par-one-not-par by blast
moreover have  $\neg C \ D \ Par \ A \ B \longrightarrow ?thesis$ 
by (meson  $\langle A \neq A' \rangle \langle A \neq C \rangle \langle B \neq D \rangle$  assms(1) assms(6) assms(7)
    assms(8) assms(9) assms(10) assms(11) assms(12) assms(13) assms(14) assms(3)
    assms(5) l13-19-par-aux par-symmetry)
moreover {
  assume  $\neg C' \ D' \ Par \ A \ B$ 
  have  $A' \neq B'$ 
  using assms(12) par-neq2 by auto
  {
    assume  $A' = C'$ 
    have  $\neg Col \ X \ A \ B$ 
    using assms(11) par-strict-not-col-4 by auto
    moreover have  $Col \ B \ A \ C$ 
    using Par-cases  $\langle A' = C' \rangle$  assms(12) assms(14) par-id-3
        postulate-of-transitivity-of-parallelism-def
        postulate-of-transitivity-of-parallelism-thm by blast
    ultimately have  $A = C$ 
    using assms(5) l6-16-1 by blast
    hence False
    using  $\langle A \neq C \rangle$  by auto
  }
}
moreover {
  assume  $B' = D'$ 
  hence  $A \ D \ Par \ A \ B$ 
  using  $\langle B' = D' \rangle$  assms(12) assms(13) not-par-one-not-par by blast
  have  $\neg Col \ Y \ B \ A$ 
  using assms(11) col-permutation-2 par-strict-not-col-2 by blast
  moreover have  $Col \ A \ B \ D$ 
  using  $\langle A \ D \ Par \ A \ B \rangle$  par-id-3 by auto
  ultimately have False
  using  $\langle B \neq D \rangle$  assms(8) col2--eq by blast
}
}
moreover have  $A' \neq A$ 
using  $\langle A \neq A' \rangle$  by auto
moreover have  $Col \ X \ A' \ C'$ 
using assms(1) assms(6) assms(7) col-transitivity-1 by blast
moreover have  $Col \ X \ A' \ A$ 
using Col-cases assms(6) by auto
moreover have  $Col \ X \ A' \ C$ 
using assms(1) assms(6) assms(5) col-transitivity-1 by blast
moreover have  $Col \ Y \ B' \ D'$ 

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    using assms(9) assms(10) assms(3) col-transitivity-1 by blast
  moreover have Col Y B' B
    using Col-cases assms(9) by auto
  moreover have Col Y B' D
    using assms(8) assms(9) assms(3) col-transitivity-1 by blast
  moreover have X A' ParStrict Y B'
    by (meson assms(4) assms(9) assms(11) assms(2) assms(6)
        not-col-distincts par-strict-col4--par-strict)
  moreover have ¬ A' B' Par C' D'
    using Par-cases ⟨¬ C' D' Par A B⟩ assms(12) par-trans by blast
  moreover have A' B' Par A B
    using Par-cases assms(12) by blast
  moreover have A' D' Par A D
    using Par-cases assms(13) by blast
  moreover have B' C' Par B C
    using Par-cases assms(14) by blast
  ultimately have C' D' Par C D
    using l13-19-par-aux assms(1) assms(6) assms(7) assms(8)
        assms(9) assms(10) assms(2) assms(3) assms(4) assms(5) by presburger
  hence ?thesis
    using Par-cases by blast
}
ultimately show ?thesis
  by blast
qed
qed
qed
qed

lemma sum-to-sump:
  assumes Sum PO E E' A B C
  shows Sump PO E E' A B C
proof -
  have Ar2 PO E E' A B C
    using Sum-def assms by blast
  obtain A' C' where E E' Pj A A' and Col PO E' A' and
    PO E Pj A' C' and PO E' Pj B C' and E' E Pj C' C
    using Sum-def assms by blast
  have A A' Proj PO E' E E'
    by (metis Ar2-def Pj-def Proj-def ⟨Ar2 PO E E' A B C⟩ ⟨Col PO E' A'⟩ ⟨E E' Pj A A'⟩
        not-col-distincts par-comm par-id-2 par-symmetry)
  moreover
  have PO ≠ E
    using Ar2-def ⟨Ar2 PO E E' A B C⟩ col-trivial-1 by force
  then obtain P' where PO E Par A' P'
    using parallel-existence1 by blast
  moreover have B C' Proj A' P' PO E'
    by (smt (verit) Ar2-def Par-perm Pj-def Proj-def ⟨Ar2 PO E E' A B C⟩ ⟨PO E Pj A' C'⟩
        ⟨PO E' Pj B C'⟩ calculation(1) calculation(2) par-col2-par-bis par-id par-id-1 par-trans)
  moreover have E ≠ E'
    using Proj-def calculation(1) by presburger
  hence C' C Proj PO E E E'
    by (metis Ar2-def Par-cases Pj-def Proj-def ⟨Ar2 PO E E' A B C⟩ ⟨E' E Pj C' C⟩
        ⟨PO ≠ E⟩ not-col-permutation-5 par-id-2)
  ultimately show ?thesis
    using Ar2-def Sum-def Sump-def assms by auto
qed

lemma sump-to-sum:
  assumes Sump PO E E' A B C
  shows Sum PO E E' A B C
proof -
  have Col PO E A
    using Sump-def assms by blast
  have Col PO E B
    using Sump-def assms by blast

```

obtain $A' C' P'$ **where** $A A' \text{ Proj } PO E' E E'$ **and** $PO E \text{ Par } A' P'$ **and**
 $B C' \text{ Proj } A' P' PO E'$ **and** $C' C \text{ Proj } PO E E E'$
using *Sump-def* *assms* **by** *blast*
have $Ar2 PO E E' A B C$
by (*metis* $Ar2\text{-def}$ $Proj\text{-def}$ $\langle B C' \text{ Proj } A' P' PO E' \rangle$ $\langle C' C \text{ Proj } PO E E E' \rangle$ $\langle Col PO E A \rangle$
 $\langle Col PO E B \rangle$ $\langle PO E \text{ Par } A' P' \rangle$ *par-col-par* *par-symmetry*)
moreover have $E E' Pj A A'$
using $Pj\text{-def}$ $\langle A A' \text{ Proj } PO E' E E' \rangle$ *par-symmetry* *project-par-dir* **by** *blast*
moreover have $Col PO E' A'$
using $Proj\text{-def}$ $\langle A A' \text{ Proj } PO E' E E' \rangle$ **by** *presburger*
moreover have $PO E Pj A' C'$
by (*metis* $Pj\text{-def}$ $Proj\text{-def}$ $\langle B C' \text{ Proj } A' P' PO E' \rangle$ $\langle PO E \text{ Par } A' P' \rangle$ *par-col-par*)
moreover have $PO E' Pj B C'$
using $Pj\text{-def}$ $\langle B C' \text{ Proj } A' P' PO E' \rangle$ *par-symmetry* *project-par-dir* **by** *blast*
moreover have $E' E Pj C' C$
using *Par-perm* $Pj\text{-def}$ $\langle C' C \text{ Proj } PO E E E' \rangle$ *project-par-dir* **by** *blast*
ultimately show *?thesis*
using *Sum-def* **by** *blast*
qed

lemma *project-col-project*:
assumes $A \neq C$
and $Col A B C$
and $P P' \text{ Proj } A B X Y$
shows $P P' \text{ Proj } A C X Y$
by (*metis* (*full-types*) $Proj\text{-def}$ *assms*(1) *assms*(2) *assms*(3) *col3* *col-trivial-3*
not-par-not-col *par-not-par*)

lemma *pj-uniqueness*:
assumes $\neg Col PO E E'$
and $Col PO E A$
and $Col PO E' A'$
and $Col PO E' A''$
and $E E' Pj A A'$
and $E E' Pj A A''$
shows $A' = A''$
proof (*cases*)
assume $A = PO$
thus *?thesis*
by (*metis* *NCol-cases* *Par-cases* *Par-def* $Pj\text{-def}$ *assms*(1) *assms*(3) *assms*(4)
assms(5) *assms*(6) *par-strict-not-col-4*)

next
assume $A \neq PO$
thus *?thesis*
by (*smt* (*verit*) *Par-cases* $Pj\text{-def}$ *assms*(1) *assms*(3) *assms*(4) *assms*(5) *assms*(6)
col-trivial-3 *par-col2-par-bis* *par-id-2* *par-not-par*)
qed

lemma *pj-right-comm*:
assumes $A B Pj C D$
shows $A B Pj D C$
using *Par-cases* $Pj\text{-def}$ *assms* **by** *auto*

lemma *pj-left-comm*:
assumes $A B Pj C D$
shows $B A Pj C D$
using *Par-cases* $Pj\text{-def}$ *assms* **by** *auto*

lemma *pj-comm*:
assumes $A B Pj C D$
shows $B A Pj D C$
using *assms* *pj-left-comm* *pj-right-comm* **by** *blast*

lemma *grid-not-par-1*:
assumes *grid-ok*: $\neg Col PO E E'$
shows $\neg PO E \text{ Par } E E'$

using *grid-ok par-id-1 par-left-comm* **by** *blast*

lemma *grid-not-par-2*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

shows $\neg PO \ E \ \text{Par } PO \ E'$

using *grid-ok par-id* **by** *blast*

lemma *grid-not-par-3*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

shows $\neg PO \ E' \ \text{Par } E \ E'$

using *grid-ok par-comm par-id-2* **by** *blast*

lemma *grid-not-par-4*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

shows $PO \neq E$

using *col-trivial-1 grid-ok* **by** *blast*

lemma *grid-not-par-5*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

shows $PO \neq E'$

using *grid-ok not-col-distincts* **by** *blast*

lemma *grid-not-par-6*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

shows $E \neq E'$

using *grid-ok not-col-distincts* **by** *presburger*

lemma *grid-not-par*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

shows $\neg PO \ E \ \text{Par } E \ E' \wedge \neg PO \ E \ \text{Par } PO \ E' \wedge$

$\neg PO \ E' \ \text{Par } E \ E' \wedge PO \neq E \wedge PO \neq E' \wedge E \neq E'$

using *grid-ok grid-not-par-1 grid-not-par-2 grid-not-par-3 grid-not-par-4*
grid-not-par-5 grid-not-par-6 **by** *blast*

lemma *proj-id*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

and $A \ A' \ \text{Proj } PO \ E' \ E \ E'$

and $\text{Col } PO \ E \ A$

and $\text{Col } PO \ E \ A'$

shows $A = PO$

proof –

have $A \ A' \ \text{Par } E \ E' \vee A = A'$

using *Proj-def assms(2)* **by** *presburger*

moreover {

assume $A \ A' \ \text{Par } E \ E'$

hence *False*

using *grid-ok assms(3) assms(4) grid-not-par-1 grid-not-par-4*
par-col2-par-bis par-symmetry **by** *blast*

}

moreover {

assume $A = A'$

hence *?thesis*

by (*metis NCol-cases Proj-def assms(2) assms(3) grid-ok l6-16-1*)

}

ultimately show *?thesis*

by *blast*

qed

lemma *sum-uniqueness*:

assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$

and $\text{Sum } PO \ E \ E' \ A \ B \ C1$

and $\text{Sum } PO \ E \ E' \ A \ B \ C2$

shows $C1 = C2$

proof –

have $\text{Sump } PO \ E \ E' \ A \ B \ C1$

using *assms(2) sum-to-sump* **by** *blast*

then obtain $A' C' P'$ **where** $A A' \text{ Proj } PO E' E E'$ **and** $PO E \text{ Par } A' P'$ **and**
 $B C' \text{ Proj } A' P' PO E'$ **and** $C' C1 \text{ Proj } PO E E E'$
using *Sump-def* **by** *blast*
have $\text{Sump } PO E E' A B C2$
using *assms(3)* *sum-to-sump* **by** *blast*
then obtain $A'' C'' P''$ **where** $A A'' \text{ Proj } PO E' E E'$ **and** $PO E \text{ Par } A'' P''$ **and**
 $B C'' \text{ Proj } A'' P'' PO E'$ **and** $C'' C2 \text{ Proj } PO E E E'$
using *Sump-def* **by** *blast*
have $A' = A''$
using $\langle A A' \text{ Proj } PO E' E E' \rangle \langle A A'' \text{ Proj } PO E' E E' \rangle$ *project-uniqueness* **by** *auto*
hence $Col A' P' P''$
using $\langle PO E \text{ Par } A' P' \rangle \langle PO E \text{ Par } A'' P'' \rangle$ *col-trivial-1*
parallel-uniqueness **by** *blast*
thus *?thesis*
by (*metis* *par-distincts* $\langle A' = A'' \rangle \langle B C' \text{ Proj } A' P' PO E' \rangle$
 $\langle B C'' \text{ Proj } A'' P'' PO E' \rangle \langle C' C1 \text{ Proj } PO E E E' \rangle \langle C'' C2 \text{ Proj } PO E E E' \rangle$
 $\langle PO E \text{ Par } A'' P'' \rangle$ *project-col-project* *project-uniqueness*)
qed

lemma *opp0*:
assumes *grid-ok*: $\neg Col PO E E'$
shows $Opp PO E E' PO PO$
using *Ar2-def* *Opp-def* *Pj-def* *Sum-def* *col-trivial-3* *grid-ok* **by** *auto*

lemma *sum-O-O*:
assumes *grid-ok*: $\neg Col PO E E'$
shows $Sum PO E E' PO PO PO$
using *Opp-def* *opp0* *grid-ok* **by** *blast*

lemma *sum-par-strict-a*:
assumes *grid-ok*: $\neg Col PO E E'$
and $Ar2 PO E E' A B C$
and $A \neq PO$
and $E E' Pj A A'$
shows $A' \neq PO$
by (*metis* *Ar2-def* *Pj-def* *grid-ok* *assms(2)* *assms(3)* *assms(4)* *col-trivial-3*
grid-not-par-1 *grid-not-par-4* *par-col2-par-bis* *par-symmetry*)

lemma *sum-par-strict-b*:
assumes *grid-ok*: $\neg Col PO E E'$
and $Ar2 PO E E' A B C$
and $A \neq PO$
and $E E' Pj A A'$
and $Col PO E' A'$
and $PO E Pj A' C'$
and $PO E' Pj B C'$
and $E' E Pj C' C$
shows $(PO E \text{ ParStrict } A' C' \vee B = PO)$

proof –
have $\text{Sum } PO E E' A B C$
using *Sum-def* *assms(2)* *assms(4)* *assms(5)* *assms(6)* *assms(7)* *assms(8)* **by** *blast*
have $A' \neq PO$
using *grid-ok* *assms(2)* *assms(3)* *assms(4)* *sum-par-strict-a* **by** *blast*
show *?thesis*
proof (*cases*)
assume $B = PO$
thus *?thesis*
by *blast*
next
assume $B \neq PO$
have $PO E \text{ ParStrict } A' C'$
proof –
have $PO E \text{ Par } A' C' \vee A' = C'$
using *Pj-def* *assms(6)* **by** *blast*
moreover {

```

    assume  $PO\ E\ Par\ A'\ C'$ 
    hence ?thesis
      by (metis Col-def Par-def  $\langle A' \neq PO \rangle$  assms(5) grid-ok l6-16-1)
  }
  moreover {
    assume  $A' = C'$ 
    hence ?thesis
      by (metis Ar2-def Pj-def  $\langle B \neq PO \rangle$  assms(2) assms(5) assms(7)
          l6-16-1 not-col-distincts not-col-permutation-4 not-strict-par)
  }
  ultimately show ?thesis
    by blast
qed
thus ?thesis
  by blast
qed
qed

```

```

lemma sum-par-strict:
  assumes grid-ok:  $\neg Col\ PO\ E\ E'$ 
  and Ar2  $PO\ E\ E'\ A\ B\ C$ 
  and  $A \neq PO$ 
  and  $E\ E'\ Pj\ A\ A'$ 
  and  $Col\ PO\ E'\ A'$ 
  and  $PO\ E\ Pj\ A'\ C'$ 
  and  $PO\ E'\ Pj\ B\ C'$ 
  and  $E'\ E\ Pj\ C'\ C$ 
  shows  $A' \neq PO \wedge (PO\ E\ ParStrict\ A'\ C' \vee B = PO)$ 
  using assms sum-par-strict-a sum-par-strict-b by blast

end
end

```

theory Tarski-Euclidean-2D

```

imports
  Tarski-Neutral
  Tarski-Postulate-Parallels
  Tarski-Euclidean
  Tarski-Neutral-2D

```

begin

15 Tarski Euclidean 2D

15.1 Tarski's axiom system for Euclidean 2D

```

locale Tarski-Euclidean-2D = Tarski-Euclidean +
  assumes upper-dim:  $\forall\ a\ b\ c\ p\ q.$ 
     $p \neq q \wedge$ 
     $Cong\ a\ p\ a\ q \wedge$ 
     $Cong\ b\ p\ b\ q \wedge$ 
     $Cong\ c\ p\ c\ q$ 
     $\longrightarrow$ 
     $(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$ 

sublocale Tarski-Euclidean-2D  $\subseteq$  Tarski-neutral-2D
proof
  show  $\forall\ a\ b\ c\ p\ q.$ 
     $p \neq q \wedge$ 
     $Cong\ a\ p\ a\ q \wedge$ 
     $Cong\ b\ p\ b\ q \wedge$ 
     $Cong\ c\ p\ c\ q$ 
     $\longrightarrow$ 
     $(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$ 

```

using upper-dim by blast
qed

context Tarski-Euclidean-2D

begin

15.2 Definitions

15.3 Propositions

lemma l12-16-2D:

assumes $A1\ A2\ Par\ B1\ B2$ and
 $X\ Inter\ A1\ A2\ C1\ C2$
shows $\exists\ Y.\ Y\ Inter\ B1\ B2\ C1\ C2$

proof –

have Coplanar $B1\ B2\ C1\ C2$
using all-coplanar by blast
thus ?thesis
using l12-16 using assms(1) assms(2) by blast

qed

lemma par-inter:

assumes $A\ B\ Par\ C\ D$ and
 $\neg\ A\ B\ Par\ P\ Q$
shows $\exists\ Y.\ Col\ P\ Q\ Y \wedge Col\ C\ D\ Y$
using all-coplanar assms(1) assms(2) cop-par--inter by blast

lemma not-par-inter:

assumes $\neg\ A\ B\ Par\ A'\ B'$
shows $\exists\ P.\ Col\ P\ X\ Y \wedge (Col\ P\ A\ B \vee Col\ P\ A'\ B')$

proof –

have Coplanar $A\ B\ X\ Y$
by (simp add: all-coplanar)
moreover
have Coplanar $A'\ B'\ X\ Y$
by (simp add: all-coplanar)
ultimately
show ?thesis
by (simp add: assms cop2-npar--inter)

qed

lemma par-perp--perp:

assumes $A\ B\ Par\ C\ D$ and
 $A\ B\ Perp\ P\ Q$
shows $C\ D\ Perp\ P\ Q$
using all-coplanar assms(1) assms(2) cop-par-perp--perp by blast

lemma par-perp2--par:

assumes $A\ B\ Par\ C\ D$ and
 $A\ B\ Perp\ E\ F$ and
 $C\ D\ Perp\ G\ H$
shows $E\ F\ Par\ G\ H$
using all-coplanar assms(1) assms(2) assms(3) cop4-par-perp2--par by blast

lemma perp3--perp:

assumes $A\ B\ Perp\ B\ C$ and
 $B\ C\ Perp\ C\ D$ and
 $C\ D\ Perp\ D\ A$
shows $D\ A\ Perp\ A\ B$

proof –

have Coplanar $A\ B\ C\ D$
by (simp add: all-coplanar)
thus ?thesis
using assms(1) assms(2) assms(3) cop-perp3--perp by blast

qed

```

lemma perp3--rect:
  assumes  $A B \text{ Perp } B C$  and
     $B C \text{ Perp } C D$  and
     $C D \text{ Perp } D A$ 
  shows  $\text{Rectangle } A B C D$ 
proof -
  have  $\text{Coplanar } A B C D$ 
    by (simp add: all-coplanar)
  thus ?thesis
    using assms(1) assms(2) assms(3) cop-perp3--rect by blast
qed

lemma projp-is-project:
  assumes  $P P' \text{ Projp } A B$ 
  shows  $\exists X Y. P P' \text{ Proj } A B X Y$ 
proof -
  have  $A \neq B$ 
    using Projp-def assms by blast
  moreover
  then obtain  $X Y$  where  $A B \text{ Perp } X Y$ 
    using perp-vector by blast
  have  $X \neq Y$ 
    using  $\langle A B \text{ Perp } X Y \rangle$  perp-not-eq-2 by auto
  moreover have  $\neg A B \text{ Par } X Y$ 
    by (simp add:  $\langle A B \text{ Perp } X Y \rangle$  perp-not-par)
  moreover have  $\text{Col } A B P'$ 
    using Projp-def assms by fastforce
  moreover have  $(P P' \text{ Par } X Y) \vee P = P'$ 
    by (metis Projp-def  $\langle A B \text{ Perp } X Y \rangle$  assms par-perp2--par par-reflexivity)
  ultimately show ?thesis
    using Proj-def by blast
qed

lemma projp-is-project-perp:
  assumes  $P P' \text{ Projp } A B$ 
  shows  $\exists X Y. P P' \text{ Proj } A B X Y \wedge A B \text{ Perp } X Y$ 
proof -
  have  $A \neq B$ 
    using Projp-def assms by blast
  moreover
  then obtain  $X Y$  where  $A B \text{ Perp } X Y$ 
    using perp-vector by blast
  moreover
  have  $X \neq Y$ 
    using  $\langle A B \text{ Perp } X Y \rangle$  perp-not-eq-2 by auto
  moreover have  $\neg A B \text{ Par } X Y$ 
    by (simp add:  $\langle A B \text{ Perp } X Y \rangle$  perp-not-par)
  moreover have  $\text{Col } A B P'$ 
    using Projp-def assms by fastforce
  moreover have  $(P P' \text{ Par } X Y) \vee P = P'$ 
    by (metis Projp-def  $\langle A B \text{ Perp } X Y \rangle$  assms par-perp2--par par-reflexivity)
  ultimately show ?thesis
    using Proj-def by blast
qed

lemma projp-to-project:
  assumes  $A B \text{ Perp } X Y$  and
     $P P' \text{ Projp } A B$ 
  shows  $P P' \text{ Proj } A B X Y$ 
proof -
  have  $A \neq B$ 
    using assms(1) perp-distinct by auto
  moreover have  $X \neq Y$ 
    using assms(1) perp-distinct by blast
  moreover have  $\neg A B \text{ Par } X Y$ 

```



```

    by (simp add:  $\langle A \ B \ \text{Perp} \ X \ Y \rangle \ \text{perp-not-par}$ )
  moreover have  $\text{Col } A \ B \ P'$ 
    using  $\text{Projp-def } \text{assms}(2)$  by fastforce
  moreover have  $(P \ P' \ \text{Par} \ X \ Y) \vee P = P'$ 
    by (metis  $\text{Projp-def } \langle A \ B \ \text{Perp} \ X \ Y \rangle \ \text{assms}(2) \ \text{par-perp2--par} \ \text{par-reflexivity}$ )
  ultimately show ?thesis
    using  $\text{Proj-def}$  by blast
qed

lemma project-to-projp:
  assumes  $P \ P' \ \text{Proj} \ A \ B \ X \ Y$  and
     $A \ B \ \text{Perp} \ X \ Y$ 
  shows  $P \ P' \ \text{Projp} \ A \ B$ 
proof -
  have  $A \neq B$ 
    using  $\text{assms}(2) \ \text{perp-not-eq-1}$  by auto
  moreover have  $\text{Col } A \ B \ P'$ 
    using  $\text{Proj-def } \text{assms}(1)$  by force
  moreover
  {
    assume  $P \ P' \ \text{Par} \ X \ Y$ 
    have  $\text{Col } A \ B \ P'$ 
      using  $\text{calculation}(2)$  by blast
    moreover have  $A \ B \ \text{Perp} \ P \ P'$ 
      using  $\text{Perp-cases } \langle P \ P' \ \text{Par} \ X \ Y \rangle \ \text{assms}(2) \ \text{par-perp--perp} \ \text{par-symmetry}$  by blast
    ultimately have  $\text{Col } A \ B \ P' \wedge A \ B \ \text{Perp} \ P \ P'$ 
      by blast
  }
  ultimately show ?thesis
    using  $\text{Proj-def } \text{Projp-def } \text{assms}(1)$  by auto
qed

```

```

lemma projp-project-to-perp:
  assumes  $P \neq P'$  and
     $P \ P' \ \text{Projp} \ A \ B$  and
     $P \ P' \ \text{Proj} \ A \ B \ X \ Y$ 
  shows  $A \ B \ \text{Perp} \ X \ Y$ 
  by (metis  $\text{Perp-perm} \ \text{Projp-def } \text{assms}(1) \ \text{assms}(2) \ \text{assms}(3) \ \text{par-perp--perp} \ \text{project-par-dir}$ )

```

```

lemma project-par-project:
  assumes  $P \ P' \ \text{Proj} \ A \ B \ X \ Y$  and
     $X \ Y \ \text{Par} \ X' \ Y'$ 
  shows  $P \ P' \ \text{Proj} \ A \ B \ X' \ Y'$ 
  by (metis  $\text{Proj-def } \text{assms}(1) \ \text{assms}(2) \ \text{not-par-one-not-par} \ \text{par-neq2} \ \text{par-symmetry}$ )

```

```

lemma project-project-par :
  assumes  $P \neq P'$  and
     $P \ P' \ \text{Proj} \ A \ B \ X \ Y$  and
     $P \ P' \ \text{Proj} \ A \ B \ X' \ Y'$ 
  shows  $X \ Y \ \text{Par} \ X' \ Y'$ 
proof -
  have  $P \ P' \ \text{Par} \ X \ Y \longrightarrow ?thesis$ 
    using  $\text{assms}(1) \ \text{assms}(3) \ \text{par-not-par} \ \text{par-symmetry} \ \text{project-par-dir}$  by blast
  thus ?thesis
    using  $\text{assms}(1) \ \text{assms}(2) \ \text{project-par-dir}$  by blast
qed

```

```

lemma projp-preserves-bet:
  assumes  $\text{Bet} \ A \ B \ C$  and
     $A \ A' \ \text{Projp} \ X \ Y$  and
     $B \ B' \ \text{Projp} \ X \ Y$  and
     $C \ C' \ \text{Projp} \ X \ Y$ 
  shows  $\text{Bet} \ A' \ B' \ C'$ 
proof -
  obtain  $T \ U$  where  $A \ A' \ \text{Proj} \ X \ Y \ T \ U$  and  $X \ Y \ \text{Perp} \ T \ U$ 

```

```

    using assms(2) projp-is-project-perp by blast
  thus ?thesis
    using assms(1) assms(3) assms(4) project-preserves-bet projp-to-project by blast
qed

lemma projp-preserves-equiv:
  assumes A B EqV C D and
    A A' Projp X Y and
    B B' Projp X Y and
    C C' Projp X Y and
    D D' Projp X Y
  shows A' B' EqV C' D'
proof -
  obtain T U where A A' Proj X Y T U and X Y Perp T U
    using assms(2) projp-is-project-perp by blast
  thus ?thesis
    using assms(1) assms(3) assms(4) assms(5) project-preserves-equiv projp-to-project by blast
qed

lemma projp2-col:
  assumes P A Projp B C and
    Q A Projp B C
  shows Col A P Q
proof -
  {
    assume Col B C A and B C Perp P A
    {
      assume Col B C A and B C Perp Q A
      have ?thesis
        proof (rule perp2--col [where ?A=B and ?B=C])
          show A P Perp B C
            using Perp-cases ⟨B C Perp P A⟩ by auto
          show A Q Perp B C
            using Perp-perm ⟨B C Perp Q A⟩ by blast
        qed
      qed
    }
    moreover
    have Col B C Q ∧ Q = A ⟶ ?thesis
      by (simp add: col-trivial-3)
    ultimately have ?thesis
      using Projp-def assms(2) by force
  }
  moreover have Col B C P ∧ P = A ⟶ ?thesis
    by (simp add: col-trivial-1)
  ultimately show ?thesis
    using Projp-def assms(1) by force
qed

lemma projp-projp-perp:
  assumes P1 ≠ P2 and
    P1 P Projp Q1 Q2 and
    P2 P Projp Q1 Q2
  shows P1 P2 Perp Q1 Q2
proof -
  have Col P1 P P2
    by (metis Col-cases Perp-cases Projp-def assms(2) assms(3) not-col-distincts perp2--col)
  thus ?thesis
    by (metis Perp-cases Projp-def assms(1) assms(2) assms(3) not-col-distincts perp-col2-bis)
qed

lemma perp-projp2-eq:
  assumes A A' Projp C D and
    B B' Projp C D and
    A B Perp C D
  shows A' = B'
proof -

```

```

{
  assume Col C D A' and C D Perp A A'
  {
    assume Col C D B' and C D Perp B B'
    {
      assume ¬ Col A B C
      have Col A B A'
        using Perp-cases ⟨C D Perp A A'⟩ assms(3) perp2--col by blast
      moreover have Col A B B'
        using Col-cases Perp-cases ⟨C D Perp B B'⟩ assms(3) perp2--col by meson
      ultimately have A' = B'
        using ⟨Col C D A'⟩ ⟨Col C D B'⟩ assms(3) l8-14-1 perp-col4 by blast
    }
    moreover {
      assume ¬ Col A B D
      have Col A B A'
        using Perp-cases ⟨C D Perp A A'⟩ assms(3) perp2--col by blast
      moreover have Col A B B'
        using Col-cases Perp-cases ⟨C D Perp B B'⟩ assms(3) perp2--col by meson
      ultimately have A' = B'
        using ⟨Col C D A'⟩ ⟨Col C D B'⟩ assms(3) perp-col0 perp-not-col2 by blast
    }
    ultimately have ?thesis
      using assms(3) perp-not-col2 by blast
  }
  moreover have Col C D B ∧ B = B' ⟶ ?thesis
    using Perp-cases ⟨C D Perp A A'⟩ ⟨Col C D A'⟩ assms(3) l8-18-uniqueness by blast
  ultimately have ?thesis
    using Projp-def assms(2) by force
}
moreover have Col C D A ∧ A = A' ⟶ ?thesis
  by (metis Perp-cases Projp-def assms(2) assms(3) l8-18-uniqueness perp-not-col2)
ultimately show ?thesis
  using Projp-def assms(1) by force
qed

```

```

lemma col-par-projp2-eq:
  assumes Col L11 L12 P and
    L11 L12 Par L21 L22 and
    P P' Projp L21 L22 and
    P' P'' Projp L11 L12
  shows P = P''
proof -
  {
    assume Col L21 L22 P' and L21 L22 Perp P P' and
      Col L11 L12 P'' and L11 L12 Perp P' P''
    have P P' Par P' P''
      using Par-cases ⟨L11 L12 Perp P' P''⟩ ⟨L21 L22 Perp P P'⟩ assms(2)
      par-perp2--par by blast
    have ¬ Col L11 L12 P' ⟶ ?thesis
      by (metis Par-perm Perp-perm ⟨Col L11 L12 P''⟩ ⟨L11 L12 Perp P' P''⟩
        ⟨L21 L22 Perp P P'⟩ assms(1) assms(2) l8-18-uniqueness par-perp--perp)
    moreover have ¬ Col L11 L12 P'' ⟶ ?thesis
      using ⟨Col L11 L12 P''⟩ by blast
    ultimately have P = P''
      using perp-not-col2 ⟨L11 L12 Perp P' P''⟩ by blast
  }
  thus ?thesis
    by (metis Projp-def assms(1) assms(2) assms(3) assms(4) col-not-col-not-par
      col-projp-eq par-symmetry)
qed

```

```

lemma col-2-par-projp2-cong:
  assumes Col L11 L12 A' and
    Col L11 L12 B' and
    L11 L12 Par L21 L22 and

```

```

  A' A Projp L21 L22 and
  B' B Projp L21 L22
shows Cong A B A' B'
proof -
{
  assume L11 L12 ParStrict L21 L22
  {
    assume A' ≠ B'
    {
      assume A ≠ B
      have L11 L12 ParStrict A B
        using ⟨A ≠ B⟩ ⟨L11 L12 ParStrict L21 L22⟩ assms(4) assms(5)
        par-strict-col2-par-strict projp-col by blast
      hence A B ParStrict B' A'
        by (metis Par-strict-cases ⟨A' ≠ B'⟩ assms(1) assms(2) par-strict-col2-par-strict)
      moreover have A A' Par B B'
        by (metis Par-perm Projp-def ⟨L11 L12 ParStrict L21 L22⟩
            assms(1) assms(2) assms(3) assms(4) assms(5) not-strict-par2
            par-perp2--par par-perp--perp par-strict-not-col-4)
      ultimately have Plg A B B' A'
        by (simp add: pars-par-plg)
      hence ?thesis
        using Plg-perm plg-cong-2 plg-to-parallelogram by blast
    }
    moreover {
      assume A = B
      hence Col A A' B'
        using assms(4) assms(5) projp2-col by auto
      moreover have Col L21 L22 A
        using assms(4) projp-col by blast
      ultimately have False using l6-21
        by (metis ⟨A' ≠ B'⟩ ⟨L11 L12 ParStrict L21 L22⟩
            assms(1) assms(2) col-trivial-2 not-col-permutation-1 par-not-col)
    }
    ultimately have ?thesis
      by blast
  }
  hence ?thesis
    using assms(4) assms(5) cong-trivial-identity projp-id by blast
}
thus ?thesis
  by (metis Par-cases assms(1) assms(2) assms(3) assms(4) assms(5) col-projp-eq
      cong-reflexivity par-not-col-strict par-strict-symmetry)
qed

```

lemma project-existence:

```

  assumes X ≠ Y
and A ≠ B
and ¬ X Y Par A B
shows ∃ P'. P P' Proj A B X Y
proof -
  obtain x x0 where x ≠ x0 and X Y Par x x0 and Col P x x0
    by (metis assms(1) not-col-distincts par-distincts parallel-existence1)
  have ¬ x x0 Par A B
    using ⟨X Y Par x x0⟩ assms(3) par-not-par by blast
  then obtain P' where Col P' x x0 and Col P' A B
    using not-par-inter-exists ⟨¬ x x0 Par A B⟩ by blast
  hence P P' Proj A B X Y
    by (metis Proj-def ⟨Col P x x0⟩ ⟨X Y Par x x0⟩ assms(1) assms(2) assms(3)
        not-col-permutation-2 par-col2-par par-symmetry)
  thus ?thesis
    by auto
qed

```

lemma sum-exists:

```

  assumes grid-ok: ¬ Col PO E E'

```

```

    and Col PO E A
    and Col PO E B
shows  $\exists C. \text{Sum } PO E E' A B C$ 
proof -
  have  $PO \neq E$ 
    using col-trivial-1 grid-ok by blast
  have  $PO \neq E'$ 
    using col-trivial-3 grid-ok by auto
  show ?thesis
proof (cases)
  assume  $PO = A$ 
  thus ?thesis
    by (metis Ar2-def Pj-def Sum-def  $\langle PO \neq E \rangle$  assms(3) col-trivial-3 grid-ok not-par-not-col)
next
  assume  $PO \neq A$ 
  obtain A' where A A' Proj PO E' E E'
    using project-existence by (metis Par-cases  $\langle PO \neq E' \rangle$  col-trivial-2 grid-ok par-id-2)
  obtain P where PO E Par A' P
    using  $\langle PO \neq E \rangle$  parallel-existence1 by presburger
  have  $A \neq A'$ 
    by (metis NCol-cases Proj-def  $\langle A A' \text{Proj } PO E' E E' \rangle$   $\langle PO \neq A \rangle$  assms(2)
      col-transitivity-1 grid-ok)
  show ?thesis
proof (cases)
  assume  $B = PO$ 
  have Sum PO E E' A PO A
  proof (unfold Sum-def, standard)
    show Ar2 PO E E' A PO A
      using Ar2-def  $\langle B = PO \rangle$  assms(3) assms(2) grid-ok by blast
    show  $\exists A' C'. E E' Pj A A' \wedge \text{Col } PO E' A' \wedge PO E Pj A' C' \wedge PO E' Pj PO C' \wedge E' E Pj C' A$ 
  proof -
    have  $E E' Pj A A'$ 
      using Pj-def  $\langle A A' \text{Proj } PO E' E E' \rangle$  par-symmetry project-par-dir by blast
    moreover have Col PO E' A'
      using Proj-def  $\langle A A' \text{Proj } PO E' E E' \rangle$  by auto
    moreover have  $PO E Pj A' A'$ 
      by (simp add: Pj-def)
    moreover have  $PO E' Pj PO A'$ 
      using Pj-def  $\langle PO \neq E' \rangle$  calculation(2) not-par-not-col by blast
    moreover have  $E' E Pj A' A$ 
      using Par-cases Pj-def calculation(1) by auto
    ultimately show ?thesis
      by blast
  qed
  qed
  thus ?thesis
    using  $\langle B = PO \rangle$  by auto
next
  assume  $B \neq PO$ 
  have  $\exists C'. B C' \text{Proj } A' P PO E'$ 
    by (metis  $\langle PO E \text{Par } A' P \rangle$   $\langle PO \neq E' \rangle$  grid-ok not-par-one-not-par par-distincts
      par-id-3 project-existence)
  then obtain C' where B C' Proj A' P PO E'
    by blast
  have  $\exists C. C' C \text{Proj } PO E A A'$ 
    by (metis Proj-def  $\langle A A' \text{Proj } PO E' E E' \rangle$   $\langle A \neq A' \rangle$   $\langle PO \neq E \rangle$  col-par-par-col
      col-trivial-3 grid-ok par-left-comm project-existence)
  then obtain C where C' C Proj PO E A A'
    by blast
  have Sum PO E E' A B C
  proof (unfold Sum-def, rule+)
    show Ar2 PO E E' A B C
      using Ar2-def Proj-def  $\langle C' C \text{Proj } PO E A A' \rangle$  assms(3) assms(2) grid-ok by presburger
    show  $\exists A' C'. E E' Pj A A' \wedge \text{Col } PO E' A' \wedge PO E Pj A' C' \wedge PO E' Pj B C' \wedge E' E Pj C' C$ 
  proof (intro exI conjI)
    let ?A' = A' and ?C'1 = C'

```

```

show E E' Pj A ?A'
  using Pj-def ⟨A A' Proj PO E' E E'⟩ par-symmetry project-par-dir by blast
show Col PO E' ?A'
  using Proj-def ⟨A A' Proj PO E' E E'⟩ by presburger
show PO E Pj ?A' ?C'1
  by (metis Pj-def Proj-def ⟨B C' Proj A' P PO E'⟩ ⟨PO E Par A' P⟩ par-col-par)
show PO E' Pj B ?C'1
  using Pj-def ⟨B C' Proj A' P PO E'⟩ par-symmetry project-par-dir by blast
show E' E Pj ?C'1 C
  by (metis Par-perm Pj-def Proj-def ⟨C' C Proj PO E A A'⟩ ⟨E E' Pj A A'⟩ par-trans)
qed
qed
thus ?thesis
  by blast
qed
qed
qed
qed

lemma opp-exists:
  assumes grid-ok: ¬ Col PO E E'
  and Col PO E A
  shows ∃ MA. Opp PO E E' A MA
proof (cases)
  assume A = PO
  thus ?thesis
    by (metis Ar2-def Opp-def Pj-def Sum-def col-trivial-3 grid-ok)
next
  assume A ≠ PO
  obtain MA where Bet A PO MA and Cong PO MA A PO
  using segment-construction by blast
show ?thesis
proof (unfold Opp-def, rule exI[of - MA], intro sump-to-sum, unfold Sump-def, rule conjI)
  show Col PO E MA
    by (metis Col-def not-col-distincts ⟨A - PO - MA⟩ ⟨A ≠ PO⟩ assms
      between-symmetry not-par-not-col par-col-par par-id)
  have ∃ A' C' P'. MA A' Proj PO E' E E' ∧ PO E Par A' P' ∧ A C' Proj A' P' PO E' ∧
    C' PO Proj PO E E E'
  proof -
    obtain A' where MA A' Proj PO E' E E'
    by (metis col-trivial-2 col-trivial-3 grid-ok par-comm par-id-4 project-existence)
    have PO ≠ E
      using col-trivial-1 grid-ok by force
    then obtain P' where PO E Par A' P'
      using parallel-existence1 by presburger
    obtain C' where A C' Proj A' P' PO E'
      by (metis Col-def ⟨PO E Par A' P'⟩ between-trivial grid-ok
        not-par-one-not-par par-distincts par-id project-existence)
    show ?thesis
  proof (rule exI[of - A'], rule exI[of - C'], rule exI[of - P'],
    intro conjI, simp add: ⟨MA A' Proj PO E' E E'⟩, simp add: ⟨PO E Par A' P'⟩,
    simp add: ⟨A C' Proj A' P' PO E'⟩)
    show C' PO Proj PO E E E'
  proof (unfold Proj-def, intro conjI, simp add: ⟨PO ≠ E⟩)
    show E ≠ E'
      using grid-ok not-col-distincts by presburger
    show ¬ PO E Par E E'
      using grid-ok not-col-distincts not-strict-par by blast
    show Col PO E PO
      using not-col-distincts by blast
    have C' PO Par E E'
  proof -
    have PO E' Par PO A'
      by (metis (full-types) Par-cases Proj-def ⟨A - PO - MA⟩ ⟨A ≠ PO⟩
        ⟨Col PO E MA⟩ ⟨Cong PO MA A PO⟩ ⟨MA A' Proj PO E' E E'⟩ ⟨¬ PO E Par E E'⟩
        bet-cong-eq not-par-not-col par-not-par)
    have Plg A C' A' PO

```

```

proof (rule pars-par-plg)
show A C' ParStrict A' PO
proof -
{
  assume A C' Par PO E'
  hence A C' Par A' PO
    by (meson ⟨PO E' Par PO A'⟩ par-right-comm par-trans)
  {
    assume A C' ParStrict A' PO'
    hence ?thesis
      by (metis Col-cases Par-def ⟨A C' Par A' PO⟩ col2--eq par-strict-not-col-1)
  }
  moreover {
    assume A ≠ C' ∧ A' ≠ PO ∧ Col A A' PO ∧ Col C' A' PO
    hence False
      by (metis ⟨A ≠ PO⟩ ⟨PO E' Par PO A'⟩ assms col2--eq
        col-permutation-4 col-permutation-5 grid-ok par-id-1)
  }
  ultimately have ?thesis
    using Par-def ⟨A C' Par A' PO⟩ by blast
}
moreover {
  assume A = C'
  hence False
    by (metis ⟨A = C'⟩ ⟨A C' Proj A' P' PO E'⟩ ⟨Col PO E PO⟩
      ⟨PO E Par A' P'⟩ ⟨PO E' Par PO A'⟩ ⟨PO ≠ E⟩ assms grid-ok not-strict-par1
      par-col2-par-bis par-id-3 project-col)
}
ultimately show ?thesis
  using Proj-def ⟨A C' Proj A' P' PO E'⟩ by fastforce
qed
have PO A Par A' C'
proof (rule par-col-par [of - - - P'])
show A' ≠ C'
  using ⟨A C' ParStrict A' PO⟩ col-trivial-1 par-strict-not-col-2 by blast
show PO A Par A' P'
  using ⟨A ≠ PO⟩ ⟨PO E Par A' P'⟩ assms par-col-par-2 by auto
show Col A' P' C'
  using Proj-def ⟨A C' Proj A' P' PO E'⟩ by auto
qed
thus A PO Par C' A'
  using par-comm by blast
qed
have Parallelogram A PO MA PO
  by (metis Bet-cases Cong-cases ⟨A - PO - MA⟩ ⟨A ≠ PO⟩ ⟨Cong PO MA A PO⟩
    bet-col1 between-cong-3 between-equality-2 between-trivial col-par
    par-cong-plg-2 plg-bet1 point-construction-different)
hence Parallelogram C' A' MA PO
  by (metis Plg-perm ⟨Plg A C' A' PO⟩ parallelogram-equiv-plg plg-pseudo-trans)
hence C' A' Par MA PO ∧ C' PO Par A' MA
  by (metis par-distincts ⟨A ≠ PO⟩ ⟨Col PO E MA⟩ ⟨PO E' Par PO A'⟩
    ⟨PO ≠ E⟩ ⟨Parallelogram A PO MA PO⟩ grid-ok not-par-not-col par-col-par
    par-id-3 plg-par plg-permut)
thus ?thesis
  by (metis Par-cases par-distincts ⟨MA A' Proj PO E' E E'⟩
    not-par-one-not-par project-par-dir)
qed
thus C' PO Par E E' ∨ C' = PO
  by simp
qed
qed
qed
thus Col PO E A ∧
  (∃ A' C' P'. MA A' Proj PO E' E E' ∧ PO E Par A' P' ∧ A C' Proj A' P' PO E' ∧
    C' PO Proj PO E E E')
using assms by blast

```

qed
qed

lemma *sum-A-O*:
 assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$
 and $\text{Col } PO \ E \ A$
 shows $\text{Sum } PO \ E \ E' \ A \ PO \ A$
proof (*unfold Sum-def, rule conjI*)
 show $\text{Ar2 } PO \ E \ E' \ A \ PO \ A$
 by (*simp add: Ar2-def assms(2) col-trivial-3 grid-ok*)
 have $PO \neq E$
 using *grid-ok not-col-distincts* **by** *auto*
 have $E \neq E'$
 using *col-trivial-2 grid-ok* **by** *blast*
 show $\exists A' \ C'. \ E \ E' \ Pj \ A \ A' \wedge \text{Col } PO \ E' \ A' \wedge PO \ E \ Pj \ A' \ C' \wedge PO \ E' \ Pj \ PO \ C' \wedge E' \ E \ Pj \ C' \ A$
proof (*cases*)
 assume $PO = A$
 thus ?thesis
 using *not-col-distincts pj-trivial* **by** *blast*
next
 assume $PO \neq A$
 have $\neg E \ E' \ Par \ PO \ E'$
 using *grid-ok par-comm par-id-4* **by** *blast*
 have $\neg A \ PO \ Par \ E' \ E$
 by (*meson Par-cases assms col-trivial-1 grid-ok not-col-permutation-4 par-trans parallel-uniqueness*)
 then obtain A' **where** $A \ A' \ Proj \ PO \ E' \ E \ E'$
 using *project-existence [of A PO E' E E']* **by** (*metis <E ≠ E'> <¬ E E' Par PO E'> col-trivial-3 grid-ok project-existence*)
 hence $E \ E' \ Pj \ A \ A'$
 using *Pj-def par-symmetry project-par-dir* **by** *blast*
 moreover have $\text{Col } PO \ E' \ A'$
 using *Proj-def <A A' Proj PO E' E E'>* **by** *presburger*
 moreover have $PO \ E \ Pj \ A' \ A'$
 by (*simp add: pj-trivial*)
 moreover have $PO \ E' \ Pj \ PO \ A'$
 using *Pj-def calculation(2) grid-ok not-col-distincts not-par-not-col* **by** *blast*
 moreover have $E' \ E \ Pj \ A' \ A$
 using *Pj-def calculation(1) par-comm* **by** *blast*
 ultimately show ?thesis
 by *blast*
 qed
 qed

lemma *sum-O-B*:
 assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$
 and $\text{Col } PO \ E \ B$
 shows $\text{Sum } PO \ E \ E' \ PO \ B \ B$
 by (*metis Ar2-def Pj-def Sum-def assms grid-ok not-col-distincts not-par-not-col*)

lemma *opp0-uniqueness*:
 assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$
 and $\text{Opp } PO \ E \ E' \ PO \ M$
 shows $M = PO$
 by (*meson Ar2-def Opp-def Sum-def assms sum-A-O sum-uniqueness*)

lemma *proj-pars*:
 assumes *grid-ok*: $\neg \text{Col } PO \ E \ E'$
 and $A \neq PO$
 and $\text{Col } PO \ E \ A$
 and $PO \ E \ Par \ A' \ C'$
 and $A \ A' \ Proj \ PO \ E' \ E \ E'$
 shows $PO \ E \ ParStrict \ A' \ C'$
proof (*unfold ParStrict-def, rule+*)
 show $\text{Coplanar } PO \ E \ A' \ C'$
 by (*simp add: all-coplanar*)


```

have  $\neg \text{Col } PO \ E \ E'$ 
  by (simp add: grid-ok)
show  $\nexists X. \text{Col } X \ PO \ E \wedge \text{Col } X \ A' \ C'$ 
proof (rule ccontr, safe)
  fix X
  assume  $\neg \text{False}$  and  $\text{Col } X \ PO \ E$  and  $\text{Col } X \ A' \ C'$ 
  show False
  proof (cases)
    assume  $PO = A'$ 
    thus ?thesis
      by (metis Par-def Proj-def assms(2) assms(3) assms(5) col-permutation-2
          grid-ok par-strict-not-col-1)
  next
    assume  $PO \neq A'$ 
    hence False
      by (metis Proj-def  $\langle \text{Col } X \ A' \ C' \rangle \langle \text{Col } X \ PO \ E \rangle$  assms(4) assms(5)
          col-permutation-1 col-transitivity-2 grid-ok not-strict-par)
    thus ?thesis
      by simp
  qed
qed
qed

lemma sum-O-B-eq:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
  and Sum  $PO \ E \ E' \ PO \ B \ C$ 
  shows  $B = C$ 
  by (metis Ar2-def Sum-def assms sum-O-B sum-uniqueness)

lemma sum-A-O-eq:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
  and Sum  $PO \ E \ E' \ A \ PO \ C$ 
  shows  $A = C$ 
  by (metis Ar2-def Sum-def assms sum-A-O sum-uniqueness)

lemma sum-A-B-A:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
  and Sum  $PO \ E \ E' \ A \ B \ A$ 
  shows  $B = PO$ 
proof -
  obtain  $A' \ C'$  where  $E \ E' \ Pj \ A \ A'$  and  $\text{Col } PO \ E' \ A' \ A'$  and  $PO \ E \ Pj \ A' \ C'$  and
     $PO \ E' \ Pj \ B \ C'$  and  $E' \ E \ Pj \ C' \ A$ 
  using Sum-def assms by blast
  show ?thesis
  proof (cases)
    assume  $A = PO$ 
    thus ?thesis
      using assms sum-O-B-eq by blast
  next
    assume  $A \neq PO$ 
    hence  $A' \neq PO \wedge (PO \ E \ \text{ParStrict } A' \ C' \vee B = PO)$ 
    using Sum-def  $\langle \text{Col } PO \ E' \ A' \rangle \langle E \ E' \ Pj \ A \ A' \rangle \langle E' \ E \ Pj \ C' \ A \rangle \langle PO \ E \ Pj \ A' \ C' \rangle$ 
       $\langle PO \ E' \ Pj \ B \ C' \rangle$  assms(2) grid-ok sum-par-strict by blast
    moreover {
      assume  $PO \ E \ \text{ParStrict } A' \ C'$  and  $B \neq PO$ 
      hence  $A \ A' \ \text{Par } A \ C'$ 
      by (metis Ar2-def NCol-perm Par-cases Par-strict-cases Pj-def Sum-def
           $\langle E \ E' \ Pj \ A \ A' \rangle \langle E' \ E \ Pj \ C' \ A \rangle$  assms(2) par-strict-not-col-2 par-trans)
      hence False
      by (metis Ar2-def Sum-def  $\langle PO \ E \ \text{ParStrict } A' \ C' \rangle$  assms(2) not-strict-par1
          par-id-2 par-strict-not-col-1 par-strict-par)
    }
    ultimately show ?thesis
      by blast
  qed
qed

```

```

lemma sum-A-B-B:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Sum  $PO \ E \ E' \ A \ B \ B$ 
  shows  $A = PO$ 
proof -
  obtain  $A' \ C'$  where  $E \ E' \ Pj \ A \ A'$  and  $\text{Col } PO \ E' \ A'$  and  $PO \ E \ Pj \ A' \ C'$  and
     $PO \ E' \ Pj \ B \ C'$  and  $E' \ E \ Pj \ C' \ B$ 
    using Sum-def assms by blast
  show ?thesis
proof (cases)
  assume  $A = PO$ 
  thus ?thesis
    using assms sum-O-B-eq by blast
next
  assume  $A \neq PO$ 
  hence  $A' \neq PO \wedge (PO \ E' \ \text{ParStrict } B \ C' \vee B = C')$ 
    by (metis NCol-perm Par-def Pj-def Sum-def  $\langle E \ E' \ Pj \ A \ A' \rangle \langle E' \ E \ Pj \ C' \ B \rangle$ 
       $\langle PO \ E' \ Pj \ B \ C' \rangle$  assms par-not-col-strict par-strict-not-col-3
      sum-par-strict-a)
  moreover {
    assume  $PO \ E' \ \text{ParStrict } B \ C'$ 
    hence False
      by (metis Par-cases Pj-def  $\langle E' \ E \ Pj \ C' \ B \rangle \langle PO \ E' \ Pj \ B \ C' \rangle$  grid-ok
        not-col-distincts not-par-one-not-par par-id-2 par-strict-not-col-2)
  }
  ultimately show ?thesis
    by (smt (verit, best) Ar2-def NCol-perm Pj-def Sum-def  $\langle \text{Col } PO \ E' \ A' \rangle$ 
       $\langle PO \ E \ Pj \ A' \ C' \rangle$  assms col-trivial-3 grid-ok not-col-permutation-4
      not-par-not-col parallel-uniqueness)
qed
qed

lemma sum-uniquenessB:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Sum  $PO \ E \ E' \ A \ X \ C$ 
    and Sum  $PO \ E \ E' \ A \ Y \ C$ 
  shows  $X = Y$ 
proof (cases)
  assume  $A = PO$ 
  thus ?thesis
    using grid-ok assms(2) assms(3) sum-O-B-eq by blast
next
  assume  $A \neq PO$ 
  obtain  $A'' \ C''$  where  $E \ E' \ Pj \ A \ A''$  and  $\text{Col } PO \ E' \ A''$  and  $PO \ E \ Pj \ A'' \ C''$  and
     $PO \ E' \ Pj \ Y \ C''$  and  $E' \ E \ Pj \ C'' \ C$ 
    using Sum-def assms(3) by auto
  obtain  $A' \ C'$  where  $E \ E' \ Pj \ A \ A'$  and  $\text{Col } PO \ E' \ A'$  and  $PO \ E \ Pj \ A' \ C'$  and
     $PO \ E' \ Pj \ X \ C'$  and  $E' \ E \ Pj \ C' \ C$ 
    using Sum-def assms(2) by blast
  have  $A' \neq PO \wedge (PO \ E \ \text{ParStrict } A' \ C' \vee X = PO)$ 
    using Sum-def  $\langle A \neq PO \rangle \langle \text{Col } PO \ E' \ A' \rangle \langle E \ E' \ Pj \ A \ A' \rangle \langle E' \ E \ Pj \ C' \ C \rangle$ 
       $\langle PO \ E \ Pj \ A' \ C' \rangle \langle PO \ E' \ Pj \ X \ C' \rangle$  assms(2) grid-ok sum-par-strict by blast
  have  $A'' \neq PO \wedge (PO \ E \ \text{ParStrict } A'' \ C'' \vee Y = PO)$ 
    using Sum-def  $\langle A \neq PO \rangle \langle \text{Col } PO \ E' \ A'' \rangle \langle E \ E' \ Pj \ A \ A'' \rangle \langle E' \ E \ Pj \ C'' \ C \rangle$ 
       $\langle PO \ E \ Pj \ A'' \ C'' \rangle \langle PO \ E' \ Pj \ Y \ C'' \rangle$  assms(3) grid-ok sum-par-strict by blast
  show ?thesis
proof (cases)
  assume  $X = PO$ 
  thus ?thesis
    using grid-ok assms(2) assms(3) sum-A-B-A sum-A-O-eq by blast
next
  assume  $X \neq PO$ 
  {
    assume  $E \ E' \ \text{Par } A \ A''$ 
    {

```

```

assume E E' Par A A'
have A A' Par A A''
by (meson ⟨E E' Par A A'⟩ ⟨E E' Par A A'⟩ par-symmetry par-trans)
{
  assume A A' ParStrict A A''
  hence ?thesis
  using not-par-strict-id by auto
}
moreover
{
  assume A ≠ A' and A ≠ A'' and Col A A A'' and Col A' A A''
  have ¬ Col PO E' A
  using grid-ok ⟨Col PO E' A'⟩ ⟨E E' Par A A'⟩ grid-not-par
    par-col2-par-bis par-symmetry by blast
  hence A' = A''
  using Col-perm ⟨Col A' A A'⟩ ⟨Col PO E' A'⟩ ⟨Col PO E' A'⟩ colx by blast
  {
    assume PO E ParStrict A' C''
    {
      assume PO E ParStrict A' C'
      have A' C' Par A' C''
      by (meson Par-strict-cases ⟨PO E ParStrict A' C'⟩
        ⟨PO E ParStrict A' C'⟩ par-strict-trans)
      moreover {
        assume A' C' ParStrict A' C''
        hence ?thesis
        using ⟨A' C' ParStrict A' C'⟩ not-par-strict-id by auto
      }
    }
    moreover {
      assume A' ≠ C' and A' ≠ C'' and Col A' A' C'' and Col C' A' C''
      {
        assume E' E Par C' C
        have C C' Par C C''
        by (metis Ar2-def Par-strict-cases Pj-def Sum-def
          ⟨E' E Par C' C⟩ ⟨E' E Pj C'' C⟩ ⟨PO E ParStrict A' C''⟩ assms(2)
          par-not-par par-right-comm par-strict-not-col-1 par-symmetry)
        moreover {
          assume C C' ParStrict C C''
          hence ?thesis
          by (simp add: not-par-strict-id)
        }
      }
    }
    moreover {
      assume C ≠ C' and C ≠ C'' and Col C C C'' and Col C' C C''
      hence C' = C''
      by (metis Pj-def ⟨A' C' Par A' C''⟩ ⟨E' E Pj C'' C⟩
        ⟨PO E ParStrict A' C'⟩ calculation(1) col2-eq col-par-par-col grid-ok
        par-id par-right-comm par-strict-par par-symmetry par-trans)
      {
        assume PO E' Par Y C'
        {
          assume PO E' Par X C'
          hence Y C' Par X C'
          using Par-cases ⟨PO E' Par Y C'⟩ par-trans by blast
          {
            assume Y C' ParStrict X C'
            hence ?thesis
            using not-par-strict-id par-strict-comm by blast
          }
        }
        moreover {
          assume Y ≠ C' and X ≠ C' and Col Y X C' and Col C' X C'
          have ¬ Col PO E C'
          using ⟨PO E ParStrict A' C'⟩ par-strict-not-col-4 by blast
          hence ?thesis
          by (metis Ar2-def Sum-def ⟨Col Y X C'⟩ grid-ok assms(2)
            assms(3) col3 not-col-distincts)
        }
      }
    }
  }
}

```

```

      ultimately have ?thesis
      using Par-def ⟨Y C' Par X C'⟩ by blast
    }
    moreover {
      assume X = C'
      hence ?thesis
      using Ar2-def Sum-def ⟨A' ≠ PO ∧ (PO E ParStrict A' C' ∨ X = PO)⟩
      ⟨X ≠ PO⟩ assms(2) par-strict-not-col-4 by blast
    }
    ultimately have ?thesis
    using Pj-def ⟨PO E' Pj X C'⟩ by blast
  }
  moreover {
    assume Y = C'
    hence False
    using Ar2-def Sum-def ⟨A' ≠ PO ∧ (PO E ParStrict A' C' ∨ X = PO)⟩
    ⟨X ≠ PO⟩ assms(3) par-strict-not-col-4 by blast
  }
  ultimately have ?thesis
  using Pj-def ⟨C' = C''⟩ ⟨PO E' Pj Y C''⟩ by blast
}
ultimately have ?thesis
using Par-def by blast
}
moreover {
  assume C' = C
  hence ?thesis
  using Ar2-def Sum-def ⟨A' ≠ PO ∧ (PO E ParStrict A' C' ∨ X = PO)⟩
  ⟨X ≠ PO⟩ assms(2) par-strict-not-col-4 by force
}
ultimately have ?thesis
using Pj-def ⟨E' E Pj C' C⟩ by blast
}
ultimately have ?thesis
using Par-def by blast
}
moreover {
  assume X = PO
  hence ?thesis
  by (simp add: ⟨X ≠ PO⟩)
}
ultimately have ?thesis
using ⟨A' ≠ PO ∧ (PO E ParStrict A' C' ∨ X = PO)⟩ by blast
}
moreover {
  assume Y = PO
  hence ?thesis
  using grid-ok assms(2) assms(3) sum-A-B-A sum-A-O-eq by blast
}
ultimately have ?thesis
using ⟨A' = A''⟩ ⟨A'' ≠ PO ∧ (PO E ParStrict A'' C'' ∨ Y = PO)⟩ by blast
}
ultimately have ?thesis
using Par-def ⟨A A' Par A A''⟩ by blast
}
hence ?thesis
by (metis grid-ok Pj-def ⟨Col PO E' A''⟩ ⟨Col PO E' A'⟩ ⟨E E' Par A A''⟩
  ⟨E E' Pj A A'⟩ grid-not-par par-col2-par-bis par-symmetry)
}
thus ?thesis
by (metis Ar2-def Pj-def Sum-def ⟨A'' ≠ PO ∧ (PO E ParStrict A'' C'' ∨ Y = PO)⟩
  ⟨E E' Pj A A''⟩ assms(2) assms(3) par-strict-not-col-1 sum-A-B-A sum-A-O-eq)
qed
qed

```

lemma sum-uniquenessA:

```

assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
  and Sum  $PO \ E \ E' \ X \ B \ C$ 
  and Sum  $PO \ E \ E' \ Y \ B \ C$ 
shows  $X = Y$ 
proof (cases)
  assume  $B = PO$ 
  thus ?thesis
    using grid-ok assms(2) assms(3) sum-A-O-eq by blast
next
assume  $B \neq PO$ 
obtain  $A'' \ C''$  where  $E \ E' \ Pj \ Y \ A''$  and  $\text{Col } PO \ E' \ A''$  and  $PO \ E \ Pj \ A'' \ C''$  and
   $PO \ E' \ Pj \ B \ C''$  and  $E' \ E \ Pj \ C'' \ C$ 
  using Sum-def assms(3) by blast
obtain  $A' \ C'$  where  $E \ E' \ Pj \ X \ A'$  and  $\text{Col } PO \ E' \ A'$  and  $PO \ E \ Pj \ A' \ C'$  and
   $PO \ E' \ Pj \ B \ C'$  and  $E' \ E \ Pj \ C' \ C$ 
  using Sum-def assms(2) by blast
show ?thesis
proof (cases)
  assume  $X = PO$ 
  thus ?thesis
    using grid-ok assms(2) assms(3) sum-A-B-B sum-O-B-eq by blast
next
assume  $X \neq PO$ 
show ?thesis
proof (cases)
  assume  $Y = PO$ 
  thus ?thesis
    using grid-ok assms(2) assms(3) sum-A-B-B sum-O-B-eq by blast
next
assume  $Y \neq PO$ 
have  $A' \neq PO \vee (PO \ E \ \text{ParStrict} \ A' \ C' \vee B = PO)$ 
  using Sum-def  $\langle E \ E' \ Pj \ X \ A' \rangle \langle X \neq PO \rangle$  grid-ok assms(2) sum-par-strict-a by blast
have  $A'' \neq PO \vee (PO \ E \ \text{ParStrict} \ A'' \ C'' \vee B = PO)$ 
  using Sum-def  $\langle E \ E' \ Pj \ Y \ A'' \rangle \langle Y \neq PO \rangle$  grid-ok assms(3) sum-par-strict-a by blast
{
  assume  $PO \ E' \ \text{Par} \ B \ C''$ 
  {
    assume  $PO \ E' \ \text{Par} \ B \ C'$ 
    {
      assume  $B \ C' \ \text{ParStrict} \ B \ C''$ 
      hence ?thesis
      by (simp add: not-par-strict-id)
    }
  }
  moreover {
    assume  $B \neq C' \ \text{and} \ B \neq C'' \ \text{and} \ \text{Col } B \ B \ C'' \ \text{and} \ \text{Col } C' \ B \ C''$ 
    {
      assume  $E' \ E \ \text{Par} \ C'' \ C$ 
      {
        assume  $E' \ E \ \text{Par} \ C' \ C$ 
        have  $C \ C' \ \text{Par} \ C \ C''$ 
        by (meson  $\langle E' \ E \ \text{Par} \ C' \ C \rangle \langle E' \ E \ \text{Par} \ C'' \ C \rangle$  par-left-comm par-symmetry par-trans)
        {
          assume  $C \ C' \ \text{ParStrict} \ C \ C''$ 
          hence ?thesis
          by (simp add: not-par-strict-id)
        }
      }
      moreover {
        assume  $C \neq C' \ \text{and} \ C \neq C'' \ \text{and} \ \text{Col } C \ C \ C'' \ \text{and} \ \text{Col } C' \ C \ C''$ 
        have  $\neg \text{Col } C \ C' \ B$ 
        by (metis grid-ok  $\langle B \neq C' \rangle \langle C \neq C' \rangle \langle E' \ E \ \text{Par} \ C' \ C \rangle$ 
           $\langle PO \ E' \ \text{Par} \ B \ C' \rangle$  col-par col-permutation-3 grid-not-par
          not-par-one-not-par par-left-comm)
        hence  $C' = C''$ 
        using Col-cases  $\langle C \ C' \ \text{Par} \ C \ C'' \rangle \langle \text{Col } C' \ B \ C'' \rangle$ 
          col2--eq par-id by blast
      }
    }
  }

```

```

assume  $PO\ E\ ParStrict\ A'\ C'$ 
{
  assume  $PO\ E\ ParStrict\ A''\ C'$ 
  have  $A'\ C'\ Par\ A''\ C'$ 
    using  $Par\text{-}strict\text{-}perm\ \langle PO\ E\ ParStrict\ A'\ C'\rangle$ 
       $\langle PO\ E\ ParStrict\ A''\ C'\rangle\ par\text{-}strict\text{-}trans$  by blast
  {
    assume  $A'\ C'\ ParStrict\ A''\ C'$ 
    hence ?thesis
    using  $not\text{-}par\text{-}strict\text{-}id\ par\text{-}strict\text{-}comm$  by blast
  }
moreover {
  assume  $A' \neq C'$  and  $A'' \neq C'$  and  $Col\ A'\ A''\ C'$  and  $Col\ C'\ A''\ C'$ 
  have  $\neg\ Col\ PO\ E'\ C'$ 
    by (metis  $grid\text{-}ok\ \langle Col\ PO\ E'\ A'\rangle\ \langle PO\ E\ ParStrict\ A'\ C'\rangle$ 
       $col2\text{-}eq\ col\text{-}permutation\text{-}2\ grid\text{-}not\text{-}par\text{-}5$ 
       $par\text{-}strict\text{-}not\text{-}col\text{-}3$ )
  hence  $A' = A''$ 
    using  $\langle Col\ A'\ A''\ C'\rangle\ \langle Col\ PO\ E'\ A''\rangle\ \langle Col\ PO\ E'\ A'\rangle\ colx$  by blast
  {
    assume  $E\ E'\ Par\ Y\ A'$ 
    {
      assume  $E\ E'\ Par\ X\ A'$ 
      have  $Y\ A'\ Par\ X\ A'$ 
      by (meson  $\langle E\ E'\ Par\ X\ A'\rangle\ \langle E\ E'\ Par\ Y\ A'\rangle\ par\text{-}symmetry\ par\text{-}trans$ )
      {
        assume  $Y\ A'\ ParStrict\ X\ A'$ 
        hence ?thesis
        using  $not\text{-}col\text{-}distincts\ par\text{-}strict\text{-}not\text{-}col\text{-}2$  by blast
      }
    }
    moreover {
      assume  $Y \neq A'$  and  $X \neq A'$  and  $Col\ Y\ X\ A'$  and  $Col\ A'\ X\ A'$ 
      have  $\neg\ Col\ PO\ E\ A'$ 
      using  $\langle PO\ E\ ParStrict\ A'\ C'\rangle\ par\text{-}strict\text{-}not\text{-}col\text{-}1$  by auto
      hence ?thesis
      using  $Sum\text{-}def\ Ar2\text{-}def\ \langle Col\ Y\ X\ A'\rangle\ \langle E\ E'\ Par\ X\ A'\rangle$ 
         $grid\text{-}not\text{-}par\ not\text{-}col\text{-}distincts\ par\text{-}col2\text{-}par\text{-}bis\ par\text{-}symmetry$ 
        by (smt (verit, ccfu-threshold) assms(2) assms(3))
    }
    ultimately have ?thesis
    using  $Par\text{-}def\ \langle Y\ A'\ Par\ X\ A'\rangle$  by blast
  }
moreover {
  assume  $X = A'$ 
  hence ?thesis
  using  $Ar2\text{-}def\ Sum\text{-}def\ \langle PO\ E\ ParStrict\ A'\ C'\rangle$ 
     $par\text{-}strict\text{-}not\text{-}col\text{-}1$  by (metis assms(2))
  }
  ultimately have ?thesis
  using  $Pj\text{-}def\ \langle E\ E'\ Pj\ X\ A'\rangle$  by blast
}
moreover {
  assume  $Y = A'$ 
  hence ?thesis
  using  $Ar2\text{-}def\ Sum\text{-}def\ \langle PO\ E\ ParStrict\ A'\ C'\rangle\ assms(3)$ 
     $par\text{-}strict\text{-}not\text{-}col\text{-}1$  by auto
}
ultimately have ?thesis
using  $Pj\text{-}def\ \langle A' = A''\rangle\ \langle E\ E'\ Pj\ Y\ A''\rangle$  by blast
}
ultimately have ?thesis
using  $Par\text{-}def\ \langle A'\ C'\ Par\ A''\ C'\rangle$  by presburger
}
moreover {
  assume  $B = PO$ 
  hence ?thesis

```

```

    by (simp add:  $\langle B \neq PO \rangle$ )
  }
  ultimately have ?thesis
    using Sum-def  $\langle C' = C'' \rangle$  Col PO E' A''  $\langle E E' Pj Y A'' \rangle$ 
       $\langle E' E Pj C'' C \rangle$  PO E Pj A'' C''  $\langle PO E' Pj B C'' \rangle$   $\langle Y \neq PO \rangle$ 
      sum-par-strict-b
      assms(3) grid-ok by blast
  }
  moreover {
    assume B = PO
    hence ?thesis
      by (simp add:  $\langle B \neq PO \rangle$ )
  }
  ultimately have ?thesis
    using Sum-def  $\langle Col PO E' A' \rangle$   $\langle E E' Pj X A' \rangle$   $\langle E' E Pj C' C \rangle$ 
       $\langle PO E Pj A' C' \rangle$   $\langle PO E' Pj B C' \rangle$   $\langle X \neq PO \rangle$  sum-par-strict-b
      assms(2) grid-ok by blast
  }
  ultimately have ?thesis
    using Par-def  $\langle C C' Par C C'' \rangle$  by blast
  }
  moreover {
    assume C' = C
    hence False
      by (metis Ar2-def Sum-def  $\langle PO E' Par B C' \rangle$  assms(2)
        grid-not-par par-col2-par-bis par-symmetry)
  }
  ultimately have ?thesis
    using Pj-def  $\langle E' E Pj C' C \rangle$  by blast
  }
  moreover {
    assume C'' = C
    hence ?thesis
      by (metis Ar2-def Sum-def  $\langle PO E' Par B C'' \rangle$  assms(2)
        grid-not-par par-col2-par-bis par-symmetry)
  }
  ultimately have ?thesis
    using Pj-def  $\langle E' E Pj C'' C \rangle$  by blast
  }
  ultimately have ?thesis
    by (metis not-col-distincts  $\langle PO E' Par B C'' \rangle$   $\langle PO E' Par B C' \rangle$ 
      par-distinct par-id-1 par-symmetry par-trans)
  }
  moreover {
    assume B = C'
    hence ?thesis
      using Ar2-def Par-cases Pj-def Sum-def  $\langle E' E Pj C' C \rangle$  grid-ok
        assms(2) assms(3) grid-not-par par-col2-par-bis sum-A-B-B
      by (smt (verit, ccfv-threshold))
  }
  ultimately have ?thesis
    using Pj-def  $\langle PO E' Pj B C' \rangle$  by blast
  }
  moreover {
    assume B = C''
    hence ?thesis
      using Ar2-def Par-cases Pj-def Sum-def  $\langle E' E Pj C'' C \rangle$   $\langle X \neq PO \rangle$ 
        assms(1) grid-not-par par-col2-par-bis sum-A-B-B
      by (smt (verit, del-insts) assms(2))
  }
  ultimately show ?thesis
    using Pj-def  $\langle PO E' Pj B C'' \rangle$  by blast
qed
qed
qed

```

```

lemma sum-B-null:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Sum  $PO \ E \ E' \ A \ B \ A$ 
  shows  $B = PO$ 
  using assms sum-A-B-A by auto

lemma sum-A-null:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Sum  $PO \ E \ E' \ A \ B \ B$ 
  shows  $A = PO$ 
  using assms sum-A-B-B by force

lemma sum-plg:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Sum  $PO \ E \ E' \ A \ B \ C$ 
    and  $(A \neq PO) \vee (B \neq PO)$ 
  shows  $\exists A' \ C'. \text{Plg } PO \ B \ C' \ A' \wedge \text{Plg } C' \ A' \ A \ C$ 
proof -
  obtain  $A' \ C'$  where  $E \ E' \text{Pj } A \ A'$ 
    and  $\text{Col } PO \ E' \ A'$ 
    and  $PO \ E \text{Pj } A' \ C'$ 
    and  $PO \ E' \text{Pj } B \ C'$ 
    and  $E' \ E \text{Pj } C' \ C$ 
  using Sum-def assms(2) by auto
  show ?thesis
proof (cases)
  assume  $B = PO$ 
  thus ?thesis
    by (metis Plg-triv assms(2) bet-col between-trivial between-trivial2
        grid-not-par-5 grid-ok sum-A-O-eq)
next
  assume  $B \neq PO$ 
  show ?thesis
proof (cases)
  assume  $A = PO$ 
  thus ?thesis
    by (metis grid-ok Plg-triv  $\langle B \neq PO \rangle$  assms(2) parallelogram-equiv-plg
        plg-permut sum-O-B-eq)
next
  assume  $A \neq PO$ 
  have  $A' \neq PO$ 
  using Sum-def  $\langle A \neq PO \rangle \langle E \ E' \text{Pj } A \ A' \rangle$  grid-ok assms(2) sum-par-strict-a by blast
  have  $PO \ E \text{ParStrict } A' \ C' \vee B = PO$ 
  using Sum-def  $\langle A \neq PO \rangle \langle \text{Col } PO \ E' \ A' \rangle \langle E \ E' \text{Pj } A \ A' \rangle \langle E' \ E \text{Pj } C' \ C \rangle$ 
     $\langle PO \ E \text{Pj } A' \ C' \rangle \langle PO \ E' \text{Pj } B \ C' \rangle$  grid-ok assms(2) sum-par-strict-b by blast
  moreover {
    assume  $PO \ E \text{ParStrict } A' \ C'$ 
    have  $PO \ B \text{Par } C' \ A'$ 
    by (metis Sump-def par-col-par-2  $\langle B \neq PO \rangle \langle PO \ E \text{ParStrict } A' \ C' \rangle$ 
        assms(2) par-right-comm par-strict-par sum-to-sump)
    hence  $PO \ B \text{ParStrict } C' \ A'$ 
    using Par-def Par-strict-cases  $\langle PO \ E \text{ParStrict } A' \ C' \rangle$  par-strict-not-col-2 by blast
  }
  {
    assume  $PO \ E' \text{Par } B \ C'$ 
    hence  $PO \ A' \text{Par } B \ C'$ 
    using  $\langle A' \neq PO \rangle \langle \text{Col } PO \ E' \ A' \rangle$  par-col-par-2 by auto
    hence  $\text{Plg } PO \ B \ C' \ A'$ 
    by (simp add:  $\langle PO \ B \text{ParStrict } C' \ A' \rangle$  pars-par-plg)
    have  $C' \ A' \text{Par } A \ C$ 
    proof (rule par-col-par [of - - -  $PO$ ])
    show  $A \neq C$ 
    using  $\langle B \neq PO \rangle$  assms(2) sum-B-null grid-ok by blast
    show  $C' \ A' \text{Par } A \ PO$ 
    by (metis Ar2-def Par-cases Sum-def  $\langle A \neq PO \rangle \langle PO \ E \text{ParStrict } A' \ C' \rangle$ 
        assms(2) par-col-par-2 par-strict-par)
    show  $\text{Col } A \ PO \ C$ 
  }

```



```

    by (metis Ar2-def Sum-def grid-ok assms(2) col3 grid-not-par)
qed
have C' A' ParStrict A C
  by (metis Ar2-def Par-def Sum-def ⟨C' A' Par A C⟩ ⟨PO E ParStrict A' C'⟩
      assms(2) colx not-col-permutation-2 par-strict-not-col-1)
{
  assume E E' Par A A'
  hence ?thesis
    by (metis Par-cases Pj-def ⟨C' A' ParStrict A C⟩ ⟨E' E Pj C' C⟩
        ⟨Plg PO B C' A'⟩ not-col-distincts par-not-par
        par-strict-not-col-4 pars-par-plg)
}
moreover {
  assume A = A'
  hence ?thesis
    using ⟨C' A' ParStrict A C⟩ col-trivial-2 par-strict-not-col-1 by blast
}
ultimately have ?thesis
  using Pj-def ⟨E E' Pj A A'⟩ by blast
}
moreover {
  assume B = C'
  hence ?thesis
    using ⟨PO B ParStrict C' A'⟩ not-col-distincts par-strict-not-col-2 by blast
}
ultimately have ?thesis
  using Pj-def ⟨PO E' Pj B C'⟩ by blast
}
ultimately show ?thesis
  using ⟨B ≠ PO⟩ by blast
qed
qed
qed

```

lemma sum-cong:

```

assumes grid-ok: ¬ Col PO E E'
  and Sum PO E E' A B C
  and (A ≠ PO ∨ B ≠ PO)
shows ParallelogramFlat PO A C B
proof (cases)
  assume A = PO
  thus ?thesis
    using grid-ok assms(2) assms(3) plgf-permut plgf-trivial1 sum-O-B-eq by blast
next
  assume A ≠ PO
  obtain A' C' where Plg PO B C' A' and Plg C' A' A C
    using ⟨A ≠ PO⟩ grid-ok assms(2) sum-plg by blast
  have Parallelogram PO B C' A'
    by (simp add: ⟨Plg PO B C' A'⟩ parallelogram-equiv-plg)
  have Parallelogram C' A' A C
    using ⟨Plg C' A' A C⟩ plg-to-parallelogram by auto
  hence Parallelogram PO B C A ∨ PO = B ∧ C' = A' ∧ A = C ∧ PO = A
    by (simp add: ⟨Parallelogram PO B C' A'⟩ plg-pseudo-trans)
  moreover {
    assume Parallelogram PO B C A
    moreover {
      assume ParallelogramStrict PO B C A
      hence ?thesis
        by (metis Ar2-def Plg-perm Sum-def col-transitivity-1 ⟨A ≠ PO⟩
            ⟨Parallelogram PO B C A ∨ PO = B ∧ C' = A' ∧ A = C ∧ PO = A⟩ assms(2)
            grid-not-par plg-col-plgf)
    }
  }
  moreover {
    assume ParallelogramFlat PO B C A
    hence ?thesis
      using plgf-comm2 plgf-permut by blast
  }

```

```

}
ultimately have ?thesis
  using Parallelogram-def by blast
}
moreover {
  assume  $PO = B \wedge C' = A' \wedge A = C \wedge PO = A$ 
  hence ?thesis
    using  $\langle A \neq PO \rangle$  by auto
}
ultimately show ?thesis
  by blast
qed

lemma sum-cong2:
  assumes grid-ok:  $\neg Col PO E E'$ 
    and Sum  $PO E E' A B C$ 
    and  $(A \neq PO \vee B \neq PO)$ 
  shows  $Cong PO A B C \wedge Cong PO B A C$ 
  using grid-ok assms(3) assms(2) not-cong-1243 plgf-cong sum-cong by blast

lemma sum-comm:
  assumes grid-ok:  $\neg Col PO E E'$ 
    and Sum  $PO E E' A B C$ 
  shows  $Sum PO E E' B A C$ 
proof (cases  $B = PO$ )
  show  $B = PO \implies Sum PO E E' B A C$ 
    by (metis Ar2-def Sum-def assms sum-A-O-eq sum-O-B)
  show  $B \neq PO \implies Sum PO E E' B A C$ 
proof (cases  $A = PO$ )
  show  $B \neq PO \implies A = PO \implies Sum PO E E' B A C$ 
    by (metis Ar2-def Sum-def assms sum-A-O sum-O-B-eq)
  show  $B \neq PO \implies A \neq PO \implies Sum PO E E' B A C$ 
proof -
  obtain  $A' C'$  where  $Plg PO B C' A'$  and  $Plg C' A' A C$ 
    by (metis Plg-triv assms grid-not-par sum-A-O-eq sum-plg)
  obtain  $B'$  where  $B B' Proj PO E' E E'$ 
    by (metis grid-ok Par-cases grid-not-par project-existence)
  obtain  $P'$  where  $PO E Par B' P'$ 
    using grid-ok grid-not-par parallel-existence1 by presburger
  obtain  $D'$  where  $A D' Proj B' P' PO E'$ 
    by (metis  $\langle PO E Par B' P' \rangle$  grid-ok grid-not-par not-par-one-not-par
      par-neq2 project-existence)
  have  $Ar2 PO E E' A B C$ 
    using Sum-def assms by blast
  moreover have  $D' C Proj PO E E'$ 
  proof (unfold Proj-def, intro conjI)
    show  $PO \neq E$ 
      using  $\langle PO E Par B' P' \rangle$  par-neq1 by auto
    show  $E \neq E'$ 
      using grid-ok not-col-distincts by blast
    show  $\neg PO E Par E E'$ 
      using grid-ok grid-not-par by auto
    show  $Col PO E C$ 
      using Ar2-def calculation by blast
    show  $D' C Par E E' \vee D' = C$ 
  proof -
    have  $B' \neq P' \wedge PO \neq E' \wedge \neg B' P' Par PO E' \wedge (A D' Par PO E' \vee A = D')$ 
      using Proj-def  $\langle A D' Proj B' P' PO E' \rangle$  by presburger
    moreover {
      assume  $A D' Par PO E'$ 
      hence  $A D' ParStrict PO E' \vee (A \neq D' \wedge PO \neq E' \wedge Col A PO E' \wedge Col D' PO E')$ 
        by (simp add: Par-def)
    }
    moreover {
      assume  $A D' ParStrict PO E'$ 
      have  $Col B' P' D'$ 
        using Proj-def  $\langle A D' Proj B' P' PO E' \rangle$  by presburger
    }
  end
end

```

```

have D' C Proj PO E E E'
proof -
{
  assume PO A ParStrict B' D'
  hence PO B' Par A D'
    by (metis Par-cases Proj-def ⟨A D' Par PO E'⟩ ⟨B B' Proj PO E' E E'⟩
        col-trivial-3 par-col-par-2 par-strict-not-col-3)
  hence Plg PO A D' B'
    by (simp add: ⟨PO A ParStrict B' D'⟩ par-strict-right-comm pars-par-plg)
  hence Parallelogram D' B' B C ∨ D' = B' ∧ PO = A ∧ C = B ∧ D' = C
    by (metis Plg-perm ⟨Plg C' A' A C⟩ ⟨Plg PO B C' A'⟩
        plg-pseudo-trans plg-to-parallelogram)
  moreover {
    assume Parallelogram D' B' B C
    hence B' B Par C D'
      by (metis Ar2-def Par-cases par-distincts ⟨Ar2 PO E E' A B C⟩
          ⟨B B' Proj PO E' E E'⟩ ⟨PO A ParStrict B' D'⟩ ⟨PO B' Par A D'⟩
          par-strict-distinct plg-par-2 proj-id)
    hence D' C Par E E' ∨ D' = C
      by (metis Par-cases par-distincts
          ⟨B B' Proj PO E' E E'⟩ par-not-par project-par-dir)
    hence ?thesis
      using ⟨B' B Par C D'⟩ ⟨Col PO E C⟩ ⟨PO ≠ E⟩ ⟨¬ PO E Par E E'⟩
          par-col-project par-distinct by blast
  }
  moreover {
    assume D' = B' ∧ PO = A ∧ C = B ∧ D' = C
    hence ?thesis
      using ⟨A D' ParStrict PO E'⟩ not-par-strict-id by force
  }
  ultimately have ?thesis
    by blast
}
moreover {
  assume PO ≠ A and B' ≠ D' and Col PO B' D' and Col A B' D'
  hence False
    by (meson Col-perm Proj-def ⟨A D' ParStrict PO E'⟩
        ⟨B B' Proj PO E' E E'⟩ par-not-col)
}
ultimately show ?thesis
  by (metis Ar2-def Par-def Proj-def ⟨A D' ParStrict PO E'⟩
      ⟨Ar2 PO E E' A B C⟩ ⟨B B' Proj PO E' E E'⟩ ⟨Col B' P' D'⟩ ⟨PO E Par B' P'⟩
      col-trivial-2 not-par-strict-id not-strict-par2 par-col-par par-col-par-2
      par-strict-not-col-4)
qed
hence ?thesis
  using Proj-def by auto
}
moreover {
  assume A ≠ D' and PO ≠ E' and Col A PO E' and Col D' PO E'
  hence ?thesis
    by (smt (verit, ccfv-threshold) Ar2-def NCol-cases Par-cases Pj-def
        Proj-def Sum-def ⟨A D' Proj B' P' PO E'⟩ ⟨Ar2 PO E E' A B C⟩ ⟨B B' Proj PO E' E E'⟩
        assms l6-16-1 par-col2-par-bis par-id)
}
ultimately have ?thesis
  by blast
}
moreover {
  assume A = D'
  hence ?thesis
    by (metis Ar2-def Proj-def ⟨A D' Proj B' P' PO E'⟩ ⟨Ar2 PO E E' A B C⟩
        ⟨B B' Proj PO E' E E'⟩ ⟨PO E Par B' P'⟩ assms inter-uniqueness-not-par
        proj-id sum-A-O-eq)
}
ultimately show ?thesis

```

```

    by blast
  qed
qed
ultimately show ?thesis
  using ⟨B B' Proj PO E' E E'⟩ ⟨PO E Par B' P'⟩ ⟨A D' Proj B' P' PO E'⟩
    Ar2-def Sump-def sump-to-sum by blast
  qed
qed
qed
lemma cong-sum:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and  $PO \neq C \vee B \neq A$ 
    and Ar2 PO E E' A B C
    and Cong PO A B C
    and Cong PO B A C
  shows Sum PO E E' A B C
proof (cases)
  assume A = PO
  thus ?thesis
    using Ar2-def assms(3) assms(4) cong-reverse-identity sum-O-B by blast
next
  assume A  $\neq$  PO
  show ?thesis
  proof (cases)
    assume B = PO
    thus ?thesis
      using Ar2-def assms(3) assms(5) cong-reverse-identity sum-A-O by blast
  next
    assume B  $\neq$  PO
    have  $\neg \text{Col } PO \ E \ E'$ 
      by (simp add: grid-ok)
    have Col PO E A
      using Ar2-def assms(3) by blast
    have Col PO E B
      using Ar2-def assms(3) by force
    have Col PO E C
      using Ar2-def assms(3) by blast
    have  $\neg PO \ E \ \text{Par } E \ E' \wedge \neg PO \ E \ \text{Par } PO \ E' \wedge \neg PO \ E' \ \text{Par } E \ E' \wedge PO \neq E \wedge PO \neq E' \wedge E \neq E'$ 
      using grid-ok grid-not-par by blast
    hence  $\neg E \ E' \ \text{Par } PO \ E'$ 
      using Par-cases by auto
    then obtain A' where A A' Proj PO E' E E'
      using project-existence [of A PO E' E E'] grid-ok grid-not-par
      project-existence by presburger
    hence Col PO E' A'  $\wedge (A \ A' \ \text{Par } E \ E' \vee A = A')$ 
      using Proj-def by simp
    then obtain P' where PO E Par A' P'
      using grid-ok grid-not-par parallel-existence1 by presburger
    obtain C' where B C' Proj A' P' PO E'
      by (metis ⟨PO E Par A' P'⟩ grid-ok grid-not-par not-par-one-not-par
        par-neq2 project-existence)
    have E E' Pj A A'
      using Pj-def ⟨Col PO E' A'  $\wedge (A \ A' \ \text{Par } E \ E' \vee A = A')$ ⟩ par-symmetry by blast
    moreover have Col PO E' A'
      by (simp add: ⟨Col PO E' A'  $\wedge (A \ A' \ \text{Par } E \ E' \vee A = A')$ ⟩)
    moreover have PO E Pj A' C'
      by (metis Pj-def Proj-def ⟨B C' Proj A' P' PO E'⟩ ⟨PO E Par A' P'⟩ par-col-par)
    moreover have PO E' Pj B C'
      by (metis Pj-def Proj-def ⟨B C' Proj A' P' PO E'⟩ par-symmetry)
    moreover have E' E Pj C' C
  proof -
    {
      assume A' = C'
    }
    {
      assume B A' Par PO E'
    }
  
```

```

moreover {
  assume  $B \ A' \text{ ParStrict } PO \ E'$ 
  hence False
  using  $\langle Col \ PO \ E' \ A' \rangle \text{ not-col-permutation-1 par-strict-not-col-2}$  by blast
}
moreover {
  assume  $B \neq A' \text{ and } PO \neq E' \text{ and } Col \ B \ PO \ E' \text{ and } Col \ A' \ PO \ E'$ 
  hence False
  using  $\langle B \neq PO \rangle \langle Col \ PO \ E \ B \rangle \text{ col-transitivity-2 grid-ok}$ 
     $\text{not-col-permutation-1}$  by blast
}
ultimately have False
using Par-def by blast
}
moreover {
  assume  $B = A'$ 
  hence False
  using  $\langle A \ A' \text{ Proj } PO \ E' \ E \ E' \rangle \langle A \neq PO \rangle \langle Col \ PO \ E \ A \rangle \langle Col \ PO \ E \ B \rangle$ 
    grid-ok proj-id by blast
}
ultimately have False
using Proj-def  $\langle A' = C' \rangle \langle B \ C' \text{ Proj } A' \ P' \ PO \ E' \rangle$  by force
}
have  $PO \ B \text{ ParStrict } A' \ C'$ 
by (metis NCol-perm Par-def Pj-def par-col-par-2  $\langle A \neq PO \rangle \langle A' = C' \implies \text{False} \rangle$ 
   $\langle B \neq PO \rangle \langle Col \ PO \ E \ B \rangle \text{ assms}(3) \text{ calculation}(1) \text{ calculation}(2) \text{ calculation}(3)$ 
  col-trivial-3 colx grid-ok sum-par-strict-a)
hence  $PO \ B \text{ ParStrict } C' \ A'$ 
using par-strict-right-comm by blast
moreover have  $PO \ A' \text{ Par } B \ C'$ 
by (metis Pj-def Proj-def  $\langle A \ A' \text{ Proj } PO \ E' \ E \ E' \rangle \langle A \neq PO \rangle \langle B \ C' \text{ Proj } A' \ P' \ PO \ E' \rangle$ 
   $\langle Col \ PO \ E \ A \rangle \langle Col \ PO \ E \ B \rangle \text{ grid-ok } \langle PO \ E \text{ Par } A' \ P' \rangle \langle PO \ E' \text{ Pj } B \ C' \rangle \text{ col-trivial-3}$ 
  not-strict-par1 par-col-par-2 proj-id)
ultimately have  $Plg \ PO \ B \ C' \ A'$ 
using pars-par-plg by auto
have ParallelogramFlat  $PO \ B \ C \ A$ 
by (metis Ar2-def ParallelogramFlat-def col-transitivity-1 assms(2) assms(3)
  assms(4) assms(5) grid-not-par not-cong-1243)
hence Parallelogram  $PO \ B \ C \ A$ 
using Parallelogram-def by blast
hence  $Plg \ PO \ B \ C \ A$ 
by (simp add: parallelogram-equiv-plg)
have Parallelogram  $A \ C \ C' \ A'$ 
using plg-to-parallelogram  $\langle B \neq PO \rangle \langle \text{Parallelogram } PO \ B \ C \ A \rangle \langle Plg \ PO \ B \ C' \ A' \rangle$ 
  plg-pseudo-trans plg-sym by blast
have  $A \ C \text{ Par } C' \ A'$ 
by (metis ParallelogramFlat-def  $\langle B \neq PO \rangle \langle PO \ B \text{ ParStrict } A' \ C' \rangle$ 
   $\langle \text{Parallelogram } A \ C \ C' \ A' \rangle \langle \text{Parallelogram } PO \ B \ C \ A \rangle \langle \text{ParallelogramFlat } PO \ B \ C \ A \rangle$ 
  par-strict-not-col-4 plg-not-comm-R1 plg-par-1)
hence  $A \ A' \text{ Par } C \ C'$ 
by (metis ParallelogramFlat-def  $\langle PO \ B \text{ ParStrict } A' \ C' \rangle \langle \text{Parallelogram } A \ C \ C' \ A' \rangle$ 
   $\langle \text{ParallelogramFlat } PO \ B \ C \ A \rangle \text{ par-neq1 par-strict-not-col-4 plg-par}$ )
{
  assume  $A \ A' \text{ Par } E \ E'$ 
  hence ?thesis
  by (metis Par-perm Pj-def  $\langle A \ A' \text{ Par } C \ C' \rangle \text{ par-trans}$ )
}
moreover {
  assume  $A = A'$ 
  hence ?thesis
  using  $\langle A \ A' \text{ Par } C \ C' \rangle \text{ par-neq1}$  by blast
}
ultimately show ?thesis
using  $\langle Col \ PO \ E' \ A' \wedge (A \ A' \text{ Par } E \ E' \vee A = A') \rangle$  by blast
qed
ultimately show ?thesis

```

```

    unfolding Sum-def using assms(3) by blast
qed
qed

lemma sum-iff-cong-a:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Ar2  $PO \ E \ E' \ A \ B \ C$ 
    and  $PO \neq C \vee B \neq A$ 
    and Cong  $PO \ A \ B \ C$ 
    and Cong  $PO \ B \ A \ C$ 
  shows Sum  $PO \ E \ E' \ A \ B \ C$ 
  using grid-ok assms(5) assms(2) assms(3) assms(4) cong-sum by blast

lemma sum-iff-cong-b:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and  $PO \neq C \vee B \neq A$ 
    and Sum  $PO \ E \ E' \ A \ B \ C$ 
  shows Cong  $PO \ A \ B \ C \wedge \text{Cong } PO \ B \ A \ C$ 
  using grid-ok assms(3) assms(2) sum-A-O-eq sum-cong2 by blast

lemma sum-iff-cong:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Ar2  $PO \ E \ E' \ A \ B \ C$ 
    and  $PO \neq C \vee B \neq A$ 
  shows  $(\text{Cong } PO \ A \ B \ C \wedge \text{Cong } PO \ B \ A \ C) \longleftrightarrow \text{Sum } PO \ E \ E' \ A \ B \ C$ 
  using grid-ok assms(3) assms(2) cong-sum sum-iff-cong-b by blast

lemma opp-comm:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Opp  $PO \ E \ E' \ X \ Y$ 
  shows Opp  $PO \ E \ E' \ Y \ X$ 
  using Opp-def assms sum-comm by presburger

lemma opp-uniqueness:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Opp  $PO \ E \ E' \ A \ MA1$ 
    and Opp  $PO \ E \ E' \ A \ MA2$ 
  shows  $MA1 = MA2$ 
  using grid-ok Opp-def assms(3) assms(2) sum-uniquenessA by presburger

lemma proj-preserves-sum:
  assumes grid-ok:  $\neg \text{Col } PO \ E \ E'$ 
    and Sum  $PO \ E \ E' \ A \ B \ C$ 
    and Ar1  $PO \ E' \ A' \ B' \ C'$ 
    and  $E \ E' \ Pj \ A \ A'$ 
    and  $E \ E' \ Pj \ B \ B'$ 
    and  $E \ E' \ Pj \ C \ C'$ 
  shows Sum  $PO \ E' \ E \ A' \ B' \ C'$ 
proof -
  obtain A0 C0 where  $E \ E' \ Pj \ A \ A0$  and  $\text{Col } PO \ E' \ A0$  and  $PO \ E \ Pj \ A0 \ C0$  and
     $PO \ E' \ Pj \ B \ C0$  and  $E' \ E \ Pj \ C0 \ C$ 
  using Sum-def assms(2) by blast
  have  $\text{Col } PO \ E \ A$ 
  using Ar2-def Sum-def assms(2) by presburger
  have  $\text{Col } PO \ E \ B$ 
  using Ar2-def Sum-def assms(2) by presburger
  have  $\neg PO \ E' \ \text{Par } E \ E'$ 
  using grid-not-par-3 grid-ok by auto
  have  $PO \neq E$ 
  using grid-ok not-col-distincts by blast
  have  $PO \neq E'$ 
  using grid-ok not-col-distincts by blast
  show ?thesis

```

```

proof (cases)
  assume  $A = PO$ 
  {
    assume  $E E' \text{ Par } PO A'$ 
    hence False
    by (metis Ar1-def  $\langle \neg PO E' \text{ Par } E E' \rangle$  assms(3) l6-16-1 par-col2-par-bis par-symmetry)
  }
  moreover {
    assume  $PO = A'$ 
    have  $B = C$ 
    proof –
      have  $\neg \text{Col } PO E E'$ 
      by (simp add: grid-ok)
      moreover have  $\text{Sum } PO E E' PO B C$ 
      using  $\langle A = PO \rangle$  assms(2) by blast
      ultimately show ?thesis
      using sum-O-B-eq by blast
    qed
    have  $B' = C'$ 
    using Ar1-def  $\langle B = C \rangle \langle \text{Col } PO E B \rangle$  assms(3) assms(5) assms(6) grid-ok
      pj-uniqueness by presburger
    hence  $\text{Sum } PO E' E PO B' B'$ 
    by (meson Ar1-def assms(3) grid-ok not-col-permutation-5 sum-O-B)
    hence  $\text{Sum } PO E' E PO B' C'$ 
    by (simp add:  $\langle B' = C' \rangle$ )
    hence ?thesis
    using  $\langle PO = A' \rangle$  by auto
  }
  ultimately show ?thesis
  using Pj-def  $\langle A = PO \rangle$  assms(4) by blast
next
assume  $A \neq PO$ 
show ?thesis
proof (cases)
  assume  $B' = PO$ 
  thus ?thesis
  by (smt (verit, ccfv-threshold) Ar1-def Ar2-def Par-cases Pj-def
    Sum-def assms(2) assms(3) assms(4) assms(5) assms(6) not-col-distincts
    not-col-permutation-5 pj-uniqueness sum-A-O sum-A-O-eq)
next
assume  $B' \neq PO$ 
obtain  $D$  where Parallelogram  $A PO B' D$ 
using  $\langle A \neq PO \rangle$  plg-existence by blast
hence  $A PO \text{ Par } B' D$ 
using  $\langle A \neq PO \rangle \langle B' \neq PO \rangle$  plg-par by auto
have  $\text{Ar2 } PO E' E A' B' C'$ 
using Ar1-def Ar2-def assms(3) grid-ok not-col-permutation-5 by presburger
moreover have  $E' E \text{ Pj } A' A$ 
using assms(4) pj-comm by blast
moreover have  $A D \text{ Par } PO E'$ 
proof (rule par-col-par [of  $PO E' A D B'$ ])
  show  $PO \neq E'$ 
  by (simp add:  $\langle PO \neq E' \rangle$ )
  show  $A D \text{ Par } PO B'$ 
  using  $\langle A \neq PO \rangle \langle B' \neq PO \rangle \langle \text{Parallelogram } A PO B' D \rangle$  plg-par by force
  show  $\text{Col } PO B' E'$ 
  using Ar2-def calculation(1) col-permutation-5 by presburger
qed
hence  $PO E' \text{ Par } A D$ 
using Par-cases by blast
hence  $PO E' \text{ Pj } A D$ 
using Pj-def by auto
moreover have  $PO E \text{ Pj } B' D$ 
by (metis Pj-def  $\langle A PO \text{ Par } B' D \rangle \langle A \neq PO \rangle \langle \text{Col } PO E A \rangle \langle PO \neq E \rangle$ 
  not-par-not-col par-left-comm par-trans)
moreover have  $E E' \text{ Pj } D C'$ 

```

```

proof –
  have ParallelogramFlat PO A C B
  using ⟨A ≠ PO⟩ assms(2) grid-ok sum-cong by blast
  have Parallelogram B' D C B ∨ (B' = D ∧ A = PO ∧ B = C ∧ B' = B)
  by (metis Parallelogram-def ⟨Parallelogram A PO B' D⟩
    ⟨ParallelogramFlat PO A C B⟩ plg-pseudo-trans plg-sym)
  moreover
  {
    assume Parallelogram B' D C B
    have B' D Par C B
    by (metis Par-cases Plg-perm ⟨A PO Par B' D⟩ ⟨Parallelogram B' D C B⟩
      par-neq2 par-reflexivity plg-par-1)
    have B' B Par D C
    by (metis Ar1-def NCol-perm ParallelogramFlat-def Plg-perm ⟨A ≠ PO⟩
      ⟨B' D Par C B⟩ ⟨B' ≠ PO⟩ ⟨Col PO E A⟩ ⟨Parallelogram B' D C B⟩
      ⟨ParallelogramFlat PO A C B⟩ assms(3) col-transitivity-1 grid-ok par-neq1 plg-par-2)
    {
      assume E E' Par B B'
      {
        assume E E' Par C C'
        hence ?thesis
        by (smt (verit, best) Par-perm Pj-def ⟨B' B Par D C⟩ ⟨E E' Par B B'⟩
          col-par-par-col col-trivial-3 not-par-one-not-par par-col2-par)
      }
      moreover {
        assume C = C'
        hence ?thesis
        by (metis Par-cases Pj-def ⟨B' B Par D C⟩ ⟨E E' Par B B'⟩ not-par-one-not-par)
      }
      ultimately have ?thesis
      using Pj-def assms(6) by blast
    }
    moreover {
      assume B = B'
      hence ?thesis
      using ⟨B' B Par D C⟩ par-neq1 by blast
    }
    ultimately have ?thesis
    using Pj-def assms(5) by blast
  }
  moreover {
    assume B' = D ∧ A = PO ∧ B = C ∧ B' = B
    hence False
    using ⟨A ≠ PO⟩ by blast
  }
  ultimately show ?thesis
  by blast
qed
ultimately show ?thesis
using Sum-def ⟨Col PO E A⟩ by blast
qed
qed
qed

```

```

lemma sum-assoc-1:
  assumes Sum PO E E' A B AB
  and Sum PO E E' B C BC
  and Sum PO E E' A BC ABC
  shows Sum PO E E' AB C ABC

```

```

proof –
  have ¬ Col PO E E'
  using Ar2-def Sum-def assms(2) by presburger
  have Col PO E AB
  using Ar2-def Sum-def assms(1) by presburger
  have Col PO E C
  using Ar2-def Sum-def assms(2) by presburger

```



```

have Col PO E ABC
  using Ar2-def Sum-def assms(3) by presburger
{
  assume A = PO
  hence ?thesis
    using  $\langle \neg \text{Col PO E E}' \rangle$  assms(1) assms(2) assms(3) sum-O-B-eq by blast
}
moreover {
  assume A  $\neq$  PO
  {
    assume B = PO
    hence ?thesis
      using  $\langle \neg \text{Col PO E E}' \rangle$  assms(1) assms(2) assms(3) sum-A-O-eq sum-O-B-eq by blast
  }
  moreover {
    assume B  $\neq$  PO
    {
      assume C = PO
      hence ?thesis
        using  $\langle \text{Col PO E AB} \rangle \langle \neg \text{Col PO E E}' \rangle$  assms(1) assms(2) assms(3) sum-A-O
        sum-A-O-eq sum-uniqueness by blast
    }
    moreover {
      assume C  $\neq$  PO
      have PO  $\neq$  E
        using  $\langle \neg \text{Col PO E E}' \rangle$  col-trivial-1 by auto
      have PO  $\neq$  E'
        using  $\langle \neg \text{Col PO E E}' \rangle$  not-col-distincts by blast
      obtain AB2' where AB AB2' Proj PO E' E E'
        using Sump-def  $\langle \text{Col PO E AB} \rangle \langle \neg \text{Col PO E E}' \rangle$  sum-A-O sum-to-sum by blast
      hence Col PO E' AB2'
        by (simp add: Proj-def)
      have AB AB2' Par E E'  $\vee$  AB = AB2'
        using Proj-def  $\langle \text{AB AB2}' \text{ Proj PO E}' \text{ E E}' \rangle$  by presburger
      moreover have A  $\neq$  AB
        using  $\langle B \neq PO \rangle \langle \neg \text{Col PO E E}' \rangle$  assms(1) sum-B-null by force
      have ABC  $\neq$  AB
        using  $\langle C \neq PO \rangle \langle \neg \text{Col PO E E}' \rangle$  assms(1) assms(2) assms(3) sum-B-null
        sum-uniquenessB by blast
      obtain C2 where Parallelogram C PO AB2' C2
        using  $\langle C \neq PO \rangle$  plg-existence by presburger
      {
        assume AB AB2' Par E E'
        hence AB  $\neq$  AB2'
          using par-distinct by blast
        have PO  $\neq$  AB2'
          using  $\langle \text{AB AB2}' \text{ Proj PO E}' \text{ E E}' \rangle \langle \text{AB} \neq \text{AB2}' \rangle \langle \text{Col PO E AB} \rangle \langle \neg \text{Col PO E E}' \rangle$ 
          grid-not-par-5 proj-id by blast
        have C PO Par AB2' C2
          by (simp add:  $\langle C \neq PO \rangle \langle PO \neq \text{AB2}' \rangle \langle \text{Parallelogram C PO AB2}' \text{ C2} \rangle$  plg-par)
        have C C2 Par PO AB2'
          by (simp add:  $\langle C \neq PO \rangle \langle PO \neq \text{AB2}' \rangle \langle \text{Parallelogram C PO AB2}' \text{ C2} \rangle$  plg-par)
        have E E' Pj AB AB2'
          using Par-perm Pj-def  $\langle \text{AB AB2}' \text{ Par E E}' \rangle$  by blast
        moreover have PO E Pj AB2' C2
          by (metis Pj-def  $\langle \text{C PO Par AB2}' \text{ C2} \rangle \langle C \neq PO \rangle \langle \text{Col PO E C} \rangle \langle PO \neq E \rangle$ 
          not-par-not-col par-left-comm par-trans)
        moreover have PO E' Pj C C2
          by (meson Pj-def  $\langle \text{C C2 Par PO AB2}' \rangle \langle \text{Col PO E}' \text{ AB2}' \rangle \langle PO \neq \text{AB2}' \rangle$ 
           $\langle PO \neq E' \rangle$  not-par-not-col not-par-one-not-par)
        moreover have E' E Pj C2 ABC
          proof -
            have Parallelogram PO BC ABC A
              using Parallelogram-def  $\langle A \neq PO \rangle \langle \neg \text{Col PO E E}' \rangle$  assms(3)
              sum-comm sum-cong by blast
            have Parallelogram PO B AB A

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    using Parallelogram-def  $\langle B \neq PO \rangle \neg \text{Col } PO \ E \ E' \rangle$  assms(1)
    sum-comm sum-cong by blast
  have Parallelogram  $PO \ B \ BC \ C$ 
    using Parallelogram-def  $\langle B \neq PO \rangle \neg \text{Col } PO \ E \ E' \rangle$  assms(2) sum-cong by blast
  have Parallelogram  $B \ AB \ ABC \ BC \vee (B = AB \wedge A = PO \wedge BC = ABC \wedge B = BC)$ 
    by (simp add:  $\langle \text{Parallelogram } PO \ B \ AB \ A \rangle \langle \text{Parallelogram } PO \ BC \ ABC \ A \rangle$ 
    plg-permut plg-pseudo-trans)
  hence Parallelogram  $B \ AB \ ABC \ BC$ 
    using  $\langle A \neq PO \rangle$  by auto
  have Parallelogram  $PO \ C \ ABC \ AB$ 
    by (metis Plg-perm  $\langle ABC \neq AB \rangle$ 
     $\langle \text{Parallelogram } B \ AB \ ABC \ BC \vee B = AB \wedge A = PO \wedge BC = ABC \wedge B = BC \rangle$ 
     $\langle \text{Parallelogram } PO \ B \ BC \ C \rangle$  plg-pseudo-trans)
  hence Parallelogram  $ABC \ AB \ AB2' \ C2$ 
    by (metis par-distinct  $\langle C \ PO \ Par \ AB2' \ C2 \rangle \langle \text{Parallelogram } C \ PO \ AB2' \ C2 \rangle$ 
    plg-pseudo-trans plg-sym)
  thus ?thesis
    by (metis Par-perm Pj-def  $\langle AB \ AB2' \ Par \ E \ E' \rangle \langle AB \neq AB2' \rangle \langle ABC \neq AB \rangle$ 
    par-not-par plg-par-2)
qed
ultimately have ?thesis
  using Ar2-def Sum-def  $\langle \text{Col } PO \ E' \ AB2' \rangle \langle \text{Col } PO \ E \ ABC \rangle \langle \text{Col } PO \ E \ AB \rangle$ 
   $\langle \text{Col } PO \ E \ C \rangle \neg \text{Col } PO \ E \ E' \rangle$  by auto
}
moreover {
  assume  $AB = AB2'$ 
  have  $AB = PO$ 
    using  $\langle AB = AB2' \rangle \langle AB \ AB2' \ Proj \ PO \ E' \ E \ E' \rangle \langle \text{Col } PO \ E \ AB \rangle$ 
     $\neg \text{Col } PO \ E \ E' \rangle$  proj-id by blast
  hence  $C = C2$ 
    using  $\langle AB = AB2' \rangle \langle \text{Parallelogram } C \ PO \ AB2' \ C2 \rangle$  cong-identity-inv plg-cong by blast
  have ParallelogramFlat  $PO \ B \ BC \ C$ 
    using  $\langle B \neq PO \rangle \neg \text{Col } PO \ E \ E' \rangle$  assms(2) sum-cong by blast
  have ParallelogramFlat  $PO \ BC \ ABC \ A$ 
    using  $\langle A \neq PO \rangle \neg \text{Col } PO \ E \ E' \rangle$  assms(3) sum-comm sum-cong by blast
  have ParallelogramFlat  $PO \ B \ PO \ A$ 
    using  $\langle A \neq PO \rangle \langle AB = PO \rangle \neg \text{Col } PO \ E \ E' \rangle$  assms(1) plgf-sym sum-cong by blast
  have Parallelogram  $BC \ C \ A \ PO \vee (BC = C \wedge PO = B \wedge PO = A \wedge BC = PO)$ 
    by (metis Parallelogram-def  $\langle \text{ParallelogramFlat } PO \ B \ BC \ C \rangle$ 
     $\langle \text{ParallelogramFlat } PO \ B \ PO \ A \rangle \neg \text{Col } PO \ E \ E' \rangle$  assms(2) plgf-plgf-plgf
    plgf-sym sum-A-B-B)
  hence Parallelogram  $BC \ C \ A \ PO$ 
    using  $\langle B \neq PO \rangle$  by auto
  have Parallelogram  $PO \ BC \ ABC \ A$ 
    using Parallelogram-def  $\langle \text{ParallelogramFlat } PO \ BC \ ABC \ A \rangle$  by auto
  hence ?thesis
    by (metis Plg-perm  $\langle AB = PO \rangle \langle \text{Col } PO \ E \ C \rangle \langle \text{Parallelogram } BC \ C \ A \ PO \rangle$ 
     $\neg \text{Col } PO \ E \ E' \rangle$  plg-uniqueness sum-O-B)
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed
lemma sum-assoc-2:
  assumes Sum  $PO \ E \ E' \ A \ B \ AB$ 
  and Sum  $PO \ E \ E' \ B \ C \ BC$ 
  and Sum  $PO \ E \ E' \ AB \ C \ ABC$ 

```

```

shows Sum PO E E' A BC ABC
by (smt (verit, best) Ar2-def Sum-def assms(1) assms(2) assms(3) sum-assoc-1 sum-comm)

lemma sum-assoc:
  assumes Sum PO E E' A B AB
    and Sum PO E E' B C BC
  shows Sum PO E E' A BC ABC  $\equiv$  Sum PO E E' AB C ABC
  by (smt (verit, ccfv-SIG) assms(1) assms(2) sum-assoc-1 sum-assoc-2)

lemma sum-y-axis-change:
  assumes Sum PO E E' A B C
    and  $\neg$  Col PO E E''
  shows Sum PO E E'' A B C
  by (smt (verit) Ar2-def ParallelogramFlat-def Sum-def assms(1)
      assms(2) sum-A-O sum-cong sum-iff-cong)

lemma sum-x-axis-unit-change:
  assumes Sum PO E E' A B C
    and Col PO E U
    and  $U \neq PO$ 
  shows Sum PO U E' A B C
proof (cases)
  assume  $U = E$ 
  thus ?thesis
    using assms(1) by blast
next
  assume  $U \neq E$ 
  have Ar2 PO E E' A B C
    using Sum-def assms(1) by blast
  have  $\neg$  Col PO E E'
    using Ar2-def  $\langle$ Ar2 PO E E' A B C $\rangle$  by blast
  have Col PO E A
    using Ar2-def  $\langle$ Ar2 PO E E' A B C $\rangle$  by blast
  have Col PO E B
    using Ar2-def  $\langle$ Ar2 PO E E' A B C $\rangle$  by blast
  have Col PO E C
    using Ar2-def  $\langle$ Ar2 PO E E' A B C $\rangle$  by blast
  have  $PO \neq E$ 
    using  $\langle$  $\neg$  Col PO E E' $\rangle$  grid-not-par by blast
  have  $PO \neq E'$ 
    using  $\langle$  $\neg$  Col PO E E' $\rangle$  grid-not-par by blast
  have  $\neg$  Col PO U E'
    by (metis  $\langle$  $\neg$  Col PO E E' $\rangle$  assms(2) assms(3) col-trivial-3 colx)
  {
    assume  $A = PO$ 
    hence ?thesis
      by (metis  $\langle$ Col PO E C $\rangle$   $\langle$  $PO \neq E$  $\rangle$   $\langle$  $\neg$  Col PO U E' $\rangle$  assms(1) assms(2)
          col-transitivity-1 sum-O-B sum-O-B-eq)
  }
  moreover {
    assume  $A \neq PO$ 
    have ?thesis
    proof (cases)
      assume  $B = PO$ 
      thus ?thesis
        by (metis  $\langle$ Col PO E C $\rangle$   $\langle$  $PO \neq E$  $\rangle$   $\langle$  $\neg$  Col PO U E' $\rangle$  assms(1) assms(2)
            col-transitivity-1 sum-A-O sum-A-O-eq)
    next
      assume  $B \neq PO$ 
      have Ar2 PO U E' A B C
        using Ar2-def  $\langle$ Col PO E A $\rangle$   $\langle$ Col PO E B $\rangle$   $\langle$ Col PO E C $\rangle$   $\langle$  $PO \neq E$  $\rangle$ 
             $\langle$  $\neg$  Col PO U E' $\rangle$  assms(2) col-transitivity-1 by presburger
    end
  }

```

```

have Col PO U A
  using ⟨Col PO E A⟩ ⟨PO ≠ E⟩ assms(2) col-transitivity-1 by presburger
have Col PO U B
  using ⟨Col PO E B⟩ ⟨PO ≠ E⟩ assms(2) col-transitivity-1 by blast
obtain A'' where A A'' Proj PO E' U E'
  by (meson Sump-def ⟨Col PO U A⟩ ⟨¬ Col PO U E'⟩ sum-A-O sum-to-sump)
hence Col PO E' A''
  using Proj-def by blast
have A A'' Par U E' ∨ A = A''
  using Proj-def ⟨A A'' Proj PO E' U E'⟩ by presburger
hence A A'' Par U E'
  using ⟨A A'' Proj PO E' U E'⟩ ⟨A ≠ PO⟩ ⟨Col PO U A⟩ ⟨¬ Col PO U E'⟩ proj-id by blast
obtain C'' where Parallelogram B PO A'' C''
  using ⟨B ≠ PO⟩ plg-existence by presburger
have PO ≠ A''
  using ⟨A A'' Proj PO E' U E'⟩ ⟨A ≠ PO⟩ ⟨Col PO U A⟩ ⟨¬ Col PO U E'⟩
    not-col-distincts proj-id by blast
have B PO Par A'' C''
  by (simp add: ⟨B ≠ PO⟩ ⟨PO ≠ A''⟩ ⟨Parallelogram B PO A'' C''⟩ plg-par)
have B C'' Par PO A''
  by (simp add: ⟨B ≠ PO⟩ ⟨PO ≠ A''⟩ ⟨Parallelogram B PO A'' C''⟩ plg-par)
have U E' Pj A A''
  using Par-perm Pj-def ⟨A A'' Par U E'⟩ by blast
moreover have PO U Pj A'' C''
  by (metis Pj-def ⟨B PO Par A'' C''⟩ ⟨B ≠ PO⟩ ⟨Col PO U B⟩ assms(3)
    not-par-not-col par-left-comm par-trans)
moreover have PO E' Pj B C''
  by (meson Pj-def ⟨B C'' Par PO A''⟩ ⟨Col PO E' A''⟩ ⟨PO ≠ E'⟩
    not-col-distincts par-col2-par-bis par-symmetry)
moreover have E' U Par C'' C
proof –
  have ParallelogramFlat PO A C B
    using ⟨A ≠ PO⟩ ⟨¬ Col PO E E'⟩ assms(1) sum-cong by blast
  hence Parallelogram PO A C B
    by (simp add: Parallelogram-def)
  hence Parallelogram A C C'' A''
    using Plg-perm ⟨B ≠ PO⟩ ⟨Parallelogram B PO A'' C''⟩ plg-pseudo-trans by blast
  hence A C Par C'' A''
    by (metis Tarski-neutral-dimensionless.par-distincts
      Tarski-neutral-dimensionless-axioms ⟨A A'' Par U E'⟩ ⟨B PO Par A'' C''⟩
      par-symmetry plg-par-1 plg-sym)
  hence A A'' Par C C''
    by (metis Plg-perm Tarski-neutral-dimensionless.par-distincts
      Tarski-neutral-dimensionless-axioms ⟨A A'' Par U E'⟩ ⟨Parallelogram A C C'' A''⟩ plg-par)
  thus ?thesis
    by (metis Par-perm ⟨A A'' Par U E'⟩ par-trans)
qed
hence E' U Pj C'' C
  by (simp add: Pj-def)
ultimately show ?thesis
  using Sum-def ⟨Col PO E' A''⟩ ⟨Ar2 PO U E' A B C⟩ by blast
qed
}
ultimately show ?thesis
  by blast
qed

lemma change-grid-sum-0:
  assumes PO E ParStrict O' E'
  and Ar1 PO E A B C
  and Ar1 O' E' A' B' C'
  and PO O' Pj E E'
  and PO O' Pj A A'
  and PO O' Pj B B'
  and PO O' Pj C C'
  and Sum PO E E' A B C

```

```

    and  $A = PO$ 
  shows  $Sum\ O'\ E'\ E\ A'\ B'\ C'$ 
proof -
{
  assume  $Ar2\ PO\ E\ E'\ A\ B\ C$ 
  obtain  $A1\ C1$  where  $E\ E'\ Pj\ A\ A1$  and
     $Col\ PO\ E'\ A1$  and
     $PO\ E\ Pj\ A1\ C1$  and
     $PO\ E'\ Pj\ B\ C1$  and
     $E'\ E\ Pj\ C1\ C$ 
  using  $Sum-def\ assms(8)$  by blast
  have  $A' = O'$ 
proof (rule l6-21 [of  $O'\ E'\ PO\ O'$ ])
  show  $\neg Col\ O'\ E'\ PO$ 
  using  $assms(1)\ par-strict-not-col-3$  by auto
  show  $PO \neq O'$ 
  using  $\langle \neg Col\ O'\ E'\ PO \rangle\ col-trivial-3$  by blast
  show  $Col\ O'\ E'\ A'$ 
  using  $assms(3)\ Ar1-def$  by auto
  show  $Col\ O'\ E'\ O'$ 
  using  $grid-not-par-5$  by auto
  show  $Col\ PO\ O'\ A'$ 
  using  $Pj-def\ not-col-distincts\ assms(5)\ assms(9)\ par-id$  by blast
  show  $Col\ PO\ O'\ O'$ 
  by (simp add:  $col-trivial-2$ )
qed
  have  $Sum\ PO\ E\ E'\ PO\ B\ B$ 
  using  $Ar2-def\ \langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle\ sum-O-B$  by presburger
  have  $B = C$ 
  using  $assms(1)\ assms(8)\ assms(9)\ par-strict-not-col-4\ sum-O-B-eq$  by blast
  have  $B' = C'$ 
proof (rule l6-21 [of  $O'\ E'\ B\ B'$ ])
  show  $\neg Col\ O'\ E'\ B$ 
  by (metis  $Ar1-def\ assms(1)\ assms(2)\ par-strict-col-par-strict\ par-strict-not-col-1\ par-strict-not-col-4$ )
  show  $B \neq B'$ 
  using  $Ar1-def\ \langle \neg Col\ O'\ E'\ B \rangle\ assms(3)$  by blast
  show  $Col\ O'\ E'\ B'$ 
  using  $Ar1-def\ assms(3)$  by presburger
  show  $Col\ O'\ E'\ C'$ 
  using  $Ar1-def\ assms(3)$  by presburger
  show  $Col\ B\ B'\ B'$ 
  by (simp add:  $col-trivial-2$ )
  show  $Col\ B\ B'\ C'$ 
  by (metis  $Col-def\ Par-perm\ Pj-def\ \langle B = C \rangle\ \langle B \neq B' \rangle\ assms(6)\ assms(7)\ grid-not-par-1\ par-trans$ )
qed
  hence ?thesis
  by (metis  $Ar1-def\ NCol-cases\ \langle A' = O' \rangle\ assms(1)\ assms(3)\ par-strict-not-col-2\ sum-O-B$ )
}
thus ?thesis
  using  $Sum-def\ assms(8)$  by blast
qed

lemma change-grid-sum:
  assumes  $PO\ E\ ParStrict\ O'\ E'$ 
  and  $Ar1\ PO\ E\ A\ B\ C$ 
  and  $Ar1\ O'\ E'\ A'\ B'\ C'$ 
  and  $PO\ O'\ Pj\ E\ E'$ 
  and  $PO\ O'\ Pj\ A\ A'$ 
  and  $PO\ O'\ Pj\ B\ B'$ 
  and  $PO\ O'\ Pj\ C\ C'$ 
  and  $Sum\ PO\ E\ E'\ A\ B\ C$ 
  shows  $Sum\ O'\ E'\ E\ A'\ B'\ C'$ 
proof (cases)
  assume  $A = PO$ 

```

```

thus ?thesis
  using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6) assms(7) assms(8)
    change-grid-sum-0 by blast
next
  assume  $A \neq PO$ 
  {
    assume  $Ar2\ PO\ E\ E'\ A\ B\ C$ 
    obtain  $A1\ C1$  where  $E\ E'\ Pj\ A\ A1$  and  $Col\ PO\ E'\ A1$  and  $PO\ E\ Pj\ A1\ C1$  and
       $PO\ E'\ Pj\ B\ C1$  and  $E'\ E\ Pj\ C1\ C$ 
    using Sum-def assms(8) by blast
    have  $\neg PO\ E\ Par\ E\ E' \wedge \neg PO\ E\ Par\ PO\ E' \wedge \neg PO\ E'\ Par\ E\ E' \wedge PO \neq E \wedge PO \neq E' \wedge E \neq E'$ 
      using assms(1) grid-not-par par-strict-not-col-4 by blast
    have  $\neg Col\ O'\ E'\ E$ 
      using assms(1) col-permutation-2 par-strict-not-col-2 by blast
    hence  $\neg O'\ E'\ Par\ E'\ E \wedge \neg O'\ E'\ Par\ O'\ E \wedge \neg O'\ E\ Par\ E'\ E \wedge O' \neq E' \wedge O' \neq E \wedge E' \neq E$ 
      using grid-not-par by presburger
    have ?thesis
    proof (cases)
      assume  $B = PO$ 
      thus ?thesis
        using Ar1-def Ar2-def  $\langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle \langle \neg Col\ O'\ E'\ E \rangle$  assms(1) assms(2) assms(3)
          assms(4) assms(5) assms(6) assms(7) assms(8) change-grid-sum-0 sum-comm by auto
    next
      assume  $B \neq PO$ 
      have  $A' \neq PO$ 
        using Ar1-def assms(1) assms(3) par-strict-not-col-3 by force
      have  $\neg Col\ PO\ A\ A'$ 
        by (metis Ar1-def Col-cases  $\langle A \neq PO \rangle$  assms(1) assms(2) assms(3) l6-16-1 par-not-col)
      have  $A' \neq O'$ 
        using Pj-def  $\langle \neg Col\ PO\ A\ A' \rangle$  assms(5) grid-not-par by blast
      have ParallelogramFlat  $PO\ A\ C\ B$ 
        using  $\langle A \neq PO \rangle$  assms(1) assms(8) par-strict-not-col-4 sum-cong by blast
      have  $PO\ O'\ Proj\ O'\ E'\ E\ E'$ 
        by (metis Par-perm Pj-def
           $\langle \neg O'\ E'\ Par\ E'\ E \wedge \neg O'\ E'\ Par\ O'\ E \wedge \neg O'\ E\ Par\ E'\ E \wedge O' \neq E' \wedge O' \neq E \wedge E' \neq E \rangle$ 
          assms(4) grid-not-par-4 pj-col-project)
      have  $A\ A'\ Proj\ O'\ E'\ E\ E'$ 
        by (metis Ar1-def Pj-def Proj-def
           $\langle \neg O'\ E'\ Par\ E'\ E \wedge \neg O'\ E'\ Par\ O'\ E \wedge \neg O'\ E\ Par\ E'\ E \wedge O' \neq E' \wedge O' \neq E \wedge E' \neq E \rangle$ 
          assms(3) assms(4) assms(5) par-right-comm par-symmetry par-trans)
      have  $C\ C'\ Proj\ O'\ E'\ E\ E'$ 
      proof –
        have  $\neg O'\ E'\ Par\ E\ E'$ 
          by (meson  $\langle \neg Col\ O'\ E'\ E \rangle$  grid-not-par not-strict-par1)
        moreover have  $C\ C'\ Par\ E\ E' \vee C = C'$ 
          by (metis Pj-def
             $\langle \neg PO\ E\ Par\ E\ E' \wedge \neg PO\ E\ Par\ PO\ E' \wedge \neg PO\ E'\ Par\ E\ E' \wedge PO \neq E \wedge PO \neq E' \wedge E \neq E' \rangle$ 
            assms(4) assms(7) par-symmetry par-trans)
        ultimately show ?thesis
          using Ar1-def Proj-def
             $\langle \neg O'\ E'\ Par\ E'\ E \wedge \neg O'\ E'\ Par\ O'\ E \wedge \neg O'\ E\ Par\ E'\ E \wedge O' \neq E' \wedge O' \neq E \wedge E' \neq E \rangle$ 
            assms(3) by auto
      qed
      have  $B\ B'\ Proj\ O'\ E'\ E\ E'$ 
      proof –
        have  $\neg O'\ E'\ Par\ E\ E'$ 
          using  $\langle \neg Col\ O'\ E'\ E \rangle$  col-trivial-2 inter-uniqueness-not-par by blast
        moreover have  $B\ B'\ Par\ E\ E' \vee B = B'$ 
          by (metis Par-def Pj-def
             $\langle \neg O'\ E'\ Par\ E'\ E \wedge \neg O'\ E'\ Par\ O'\ E \wedge \neg O'\ E\ Par\ E'\ E \wedge O' \neq E' \wedge O' \neq E \wedge E' \neq E \rangle$ 
            assms(4) assms(6) not-par-inter-exists parallel-uniqueness)
        ultimately show ?thesis
          using Ar1-def Proj-def
             $\langle \neg O'\ E'\ Par\ E'\ E \wedge \neg O'\ E'\ Par\ O'\ E \wedge \neg O'\ E\ Par\ E'\ E \wedge O' \neq E' \wedge O' \neq E \wedge E' \neq E \rangle$ 
            assms(3) by auto
      qed

```

```

have PO A EqV B C
  by (simp add: EqV-def Parallelogram-def ⟨ParallelogramFlat PO A C B⟩)
hence O' A' EqV B' C'
  using ⟨A A' Proj O' E' E E'⟩ ⟨B B' Proj O' E' E E'⟩ ⟨C C' Proj O' E' E E'⟩
  ⟨PO O' Proj O' E' E E'⟩ project-preserves-equiv by blast
moreover {
  assume Parallelogram O' A' C' B'
  hence ?thesis
    by (metis Ar1-def Ar2-def Plg-perm ⟨¬ Col O' E' E⟩ assms(3) plg-cong-2
        plg-irreflexive sum-iff-cong)
}
moreover {
  assume O' = A' and B' = C'
  hence ?thesis
    using ⟨A' ≠ O'⟩ by auto
}
ultimately show ?thesis
  using EqV-def by blast
qed
}
thus ?thesis
  using Sum-def assms(8) by blast

```

qed

```

lemma double-null-null:
  assumes Sum PO E E' A A PO
  shows A = PO
proof (rule ccontr)
  assume A ≠ PO
  hence ParallelogramFlat PO A PO A
    using Ar2-def Sum-def assms sum-cong by blast
  thus False
    using plgf-irreflexive by blast
qed

```

```

lemma not-null-double-not-null:
  assumes Sum PO E E' A A C
  and A ≠ PO
  shows C ≠ PO
  using assms(1) assms(2) double-null-null by blast

```

```

lemma double-not-null-not-nul:
  assumes Sum PO E E' A A C
  and C ≠ PO
  shows A ≠ PO
  using Ar2-def Sum-def assms(1) assms(2) sum-O-B-eq by blast

```

```

lemma diff-ar2:
  assumes Diff PO E E' A B AMB
  shows Ar2 PO E E' A B AMB
proof -
  obtain MA where Opp PO E E' B MA and Sum PO E E' A MA AMB
    using Diff-def assms by blast
  thus ?thesis
    by (simp add: Ar2-def Opp-def Sum-def)
qed

```

```

lemma diff-null:
  assumes grid-ok: ¬ Col PO E E'
  and Col PO E A
  shows Diff PO E E' A A PO
  by (meson Diff-def Opp-def grid-ok assms(2) opp-exists sum-comm)

```

```

lemma diff-exists:
  assumes grid-ok: ¬ Col PO E E'

```

```

    and Col PO E A
    and Col PO E B
  shows  $\exists D. \text{Diff } PO E E' A B D$ 
proof -
  obtain MB where Opp PO E E' B MB
  using grid-ok assms(3) opp-exists by blast
  then obtain C where Sum PO E E' A MB C
  by (meson Ar2-def Opp-def Sum-def assms(2) sum-exists)
  thus ?thesis
  using Diff-def  $\langle \text{Opp } PO E E' B MB \rangle$  by blast
qed

lemma diff-uniqueness:
  assumes Diff PO E E' A B D1
  and Diff PO E E' A B D2
  shows  $D1 = D2$ 
  by (metis Ar2-def Diff-def assms(1) assms(2) diff-ar2 opp-uniqueness sum-uniqueness)

lemma sum-ar2:
  assumes Sum PO E E' A B C
  shows Ar2 PO E E' A B C
  using Sum-def assms by blast

lemma diff-A-O:
  assumes grid-ok:  $\neg \text{Col } PO E E'$ 
  and Col PO E A
  shows Diff PO E E' A PO A
  using Diff-def grid-ok assms(2) opp0 sum-A-O by blast

lemma diff-O-A:
  assumes Opp PO E E' A mA
  shows Diff PO E E' PO A mA
  by (meson Ar2-def Diff-def Opp-def assms sum-O-B sum-ar2)

lemma diff-O-A-opp:
  assumes Diff PO E E' PO A mA
  shows Opp PO E E' A mA
  by (metis Ar2-def Diff-def assms diff-ar2 sum-O-B-eq)

lemma diff-uniquenessA:
  assumes Diff PO E E' A B C
  and Diff PO E E' A' B C
  shows  $A = A'$ 
  by (metis Ar2-def Diff-def assms(1) assms(2) diff-ar2 opp-uniqueness sum-uniquenessA)

lemma diff-uniquenessB:
  assumes Diff PO E E' A B C
  and Diff PO E E' A B' C
  shows  $B = B'$ 
  by (metis Ar2-def Diff-def assms(1) assms(2) diff-ar2 opp-comm opp-uniqueness sum-uniquenessB)

lemma diff-null-eq:
  assumes Diff PO E E' A B PO
  shows  $A = B$ 
  by (meson Ar2-def assms diff-ar2 diff-null diff-uniquenessB)

lemma midpoint-opp:
  assumes Ar2 PO E E' PO A B
  and PO Midpoint A B
  shows Opp PO E E' A B
proof (cases)
  assume  $A = B$ 
  thus ?thesis
  using Ar2-def assms(1) assms(2) l8-20-2 opp0 by blast
next
  assume  $A \neq B$ 

```



```

moreover have Ar2 PO E E' B A PO
  using Ar2-def assms(1) by force
moreover have Cong PO B A PO
  using assms(2) cong-symmetry midpoint-cong by blast
moreover have Cong PO A B PO
  using calculation(3) cong-4321 by blast
ultimately show ?thesis
  by (simp add: Ar2-def Opp-def cong-sum)
qed

lemma sum-diff:
  assumes Sum PO E E' A B S
  shows Diff PO E E' S A B
proof –
  have Ar2 PO E E' A B S
    using assms sum-ar2 by auto
  then obtain mA where Opp PO E E' A mA
    by (meson Ar2-def opp-exists)
  moreover have Sum PO E E' S mA B
    by (meson Ar2-def Opp-def ⟨Ar2 PO E E' A B S⟩ assms calculation sum-O-B sum-assoc-2 sum-comm)
  ultimately show ?thesis
    using Diff-def by auto
qed

lemma diff-sum:
  assumes Diff PO E E' S A B
  shows Sum PO E E' A B S
  by (meson Ar2-def Diff-def Opp-def assms diff-ar2 sum-A-O sum-assoc-1 sum-comm)

lemma diff-opp:
  assumes Diff PO E E' A B AmB
  and Diff PO E E' B A BmA
  shows Opp PO E E' AmB BmA
  by (meson Ar2-def Diff-def Opp-def assms(1) assms(2) diff-ar2 diff-sum sum-assoc sum-comm)

lemma sum-stable:
  assumes A = B
  and Sum PO E E' A C S1
  and Sum PO E E' B C S2
  shows S1 = S2
  using assms(1) assms(2) assms(3) diff-uniquenessA sum-diff by blast

lemma diff-stable:
  assumes A = B
  and Diff PO E E' A C D1
  and Diff PO E E' B C D2
  shows D1 = D2
  using assms(1) assms(2) assms(3) diff-uniqueness by blast

lemma plg-to-sum:
  assumes Ar2 PO E E' A B C
  and ParallelogramFlat PO A C B
  shows Sum PO E E' A B C
  by (metis Ar2-def assms(1) assms(2) not-cong-1243 plgf-cong plgf-irreflexive sum-iff-cong-a)

lemma opp-midpoint:
  assumes Opp PO E E' A MA
  shows PO Midpoint A MA
proof (cases)
  assume A = PO
  hence MA = PO
  by (metis Opp-def assms diff-null-eq local.sum-diff)
  thus ?thesis
    by (simp add: ⟨A = PO⟩ l7-3-2)
next
  assume A ≠ PO

```

thus ?thesis
 by (meson Ar2-def Mid-cases Opp-def assms plgf3-mid sum-ar2 sum-cong)
 qed

lemma *diff-to-plg*:
 assumes $A \neq PO \vee B \neq PO$
 and $Diff\ PO\ E\ E'\ B\ A\ dBA$
 shows $ParallelogramFlat\ PO\ A\ B\ dBA$
 by (metis Ar2-def assms(1) assms(2) diff-null-eq diff-sum sum-ar2 sum-cong)

lemma *sum3-col*:
 assumes $sum3\ PO\ E\ E'\ A\ B\ C\ S$
 shows $\neg Col\ PO\ E\ E' \wedge Col\ PO\ E\ A \wedge Col\ PO\ E\ B \wedge Col\ PO\ E\ C \wedge Col\ PO\ E\ S$
 by (meson Ar2-def assms sum3-def sum-ar2)

lemma *sum3-permut*:
 assumes $sum3\ PO\ E\ E'\ A\ B\ C\ S$
 shows $sum3\ PO\ E\ E'\ C\ A\ B\ S$
 by (smt (verit, ccfv-SIG) assms sum3-col sum3-def sum-assoc sum-comm sum-exists)

lemma *sum3-comm-1-2*:
 assumes $sum3\ PO\ E\ E'\ A\ B\ C\ S$
 shows $sum3\ PO\ E\ E'\ B\ A\ C\ S$
 by (meson assms sum3-col sum3-def sum-comm)

lemma *sum3-comm-2-3*:
 assumes $sum3\ PO\ E\ E'\ A\ B\ C\ S$
 shows $sum3\ PO\ E\ E'\ A\ C\ B\ S$
 using assms sum3-comm-1-2 sum3-permut by blast

lemma *sum3-exists*:
 assumes $Ar2\ PO\ E\ E'\ A\ B\ C$
 shows $\exists S. sum3\ PO\ E\ E'\ A\ B\ C\ S$
 by (metis Ar2-def Sum-def assms sum3-def sum-exists)

lemma *sum3-uniqueness*:
 assumes $sum3\ PO\ E\ E'\ A\ B\ C\ S1$
 and $sum3\ PO\ E\ E'\ A\ B\ C\ S2$
 shows $S1 = S2$
 by (metis assms(1) assms(2) sum3-def sum-stable)

lemma *sum4-col*:
 assumes $Sum4\ PO\ E\ E'\ A\ B\ C\ D\ S$
 shows $\neg Col\ PO\ E\ E' \wedge Col\ PO\ E\ A \wedge Col\ PO\ E\ B \wedge Col\ PO\ E\ C \wedge Col\ PO\ E\ D \wedge Col\ PO\ E\ S$
 by (meson assms sum3-col sum3-def sum4-def)

lemma *sum22-col*:
 assumes $sum22\ PO\ E\ E'\ A\ B\ C\ D\ S$
 shows $\neg Col\ PO\ E\ E' \wedge Col\ PO\ E\ A \wedge Col\ PO\ E\ B \wedge Col\ PO\ E\ C \wedge Col\ PO\ E\ D \wedge Col\ PO\ E\ S$
 by (meson Ar2-def assms sum22-def sum-ar2)

lemma *sum-to-sum3*:
 assumes $Sum\ PO\ E\ E'\ A\ B\ AB$
 and $Sum\ PO\ E\ E'\ AB\ X\ S$
 shows $sum3\ PO\ E\ E'\ A\ B\ X\ S$
 using assms(1) assms(2) sum3-def by blast

lemma *sum3-to-sum4*:
 assumes $sum3\ PO\ E\ E'\ A\ B\ C\ ABC$
 and $Sum\ PO\ E\ E'\ ABC\ X\ S$
 shows $Sum4\ PO\ E\ E'\ A\ B\ C\ X\ S$
 using assms(1) assms(2) sum4-def by blast

lemma *sum-A-exists*:
 assumes $Ar2\ PO\ E\ E'\ A\ AB\ PO$
 shows $\exists B. Sum\ PO\ E\ E'\ A\ B\ AB$

by (meson Ar2-def assms diff-exists diff-sum)

lemma sum-B-exists:

assumes Ar2 PO E E' B AB PO
 shows $\exists A. \text{Sum PO E E' A B AB}$
 using Ar2-def assms sum-A-exists sum-comm by blast

lemma sum4-equiv-a:

assumes Sum4 PO E E' A B C D S
 shows sum22 PO E E' A B C D S

proof -

obtain AB where Sum PO E E' A B AB
 using assms sum4-col sum-exists by blast
 obtain CD where Sum PO E E' C D CD
 using assms sum4-col sum-exists by blast
 have Sum PO E E' AB CD S
 proof -
 have Ar2 PO E E' A B AB
 by (simp add: $\langle \text{Sum PO E E' A B AB} \rangle$ sum-ar2)
 moreover have Ar2 PO E E' C D CD
 by (simp add: $\langle \text{Sum PO E E' C D CD} \rangle$ sum-ar2)
 ultimately show ?thesis
 by (smt (verit, best) $\langle \text{Sum PO E E' A B AB} \rangle \langle \text{Sum PO E E' C D CD} \rangle$ assms
 sum4-col sum3-def sum4-def sum-assoc-2 sum-uniqueness)

qed

thus ?thesis

using $\langle \text{Sum PO E E' A B AB} \rangle \langle \text{Sum PO E E' C D CD} \rangle$ sum22-def by blast

qed

lemma sum4-equiv-b:

assumes sum22 PO E E' A B C D S
 shows Sum4 PO E E' A B C D S

proof -

obtain AB CD where Sum PO E E' A B AB and Sum PO E E' C D CD and Sum PO E E' AB CD S
 using assms sum22-def by blast
 have Ar2 PO E E' A B AB
 using Sum-def $\langle \text{Sum PO E E' A B AB} \rangle$ by blast
 have Ar2 PO E E' C D CD
 by (simp add: $\langle \text{Sum PO E E' C D CD} \rangle$ sum-ar2)
 obtain ABC where Sum PO E E' AB C ABC
 using Ar2-def $\langle \text{Ar2 PO E E' A B AB} \rangle \langle \text{Ar2 PO E E' C D CD} \rangle$ sum-exists by presburger
 thus ?thesis
 by (meson $\langle \text{Sum PO E E' A B AB} \rangle \langle \text{Sum PO E E' AB CD S} \rangle \langle \text{Sum PO E E' C D CD} \rangle$
 sum3-to-sum4 sum-assoc-1 sum-to-sum3)

qed

lemma sum4-equiv:

shows Sum4 PO E E' A B C D S $\longleftrightarrow$ sum22 PO E E' A B C D S
 using sum4-equiv-a sum4-equiv-b by blast

lemma sum4-permut:

assumes Sum4 PO E E' A B C D S
 shows Sum4 PO E E' D A B C S

proof -

have sum22 PO E E' A B C D S
 using assms sum4-equiv by auto
 then obtain AB CD where Sum PO E E' A B AB and Sum PO E E' C D CD and
 Sum PO E E' AB CD S
 using sum22-def by blast
 obtain ABC where sum3 PO E E' A B C ABC and Sum PO E E' ABC D S
 using assms sum4-def by blast
 then obtain AC where Sum PO E E' C A AC and Sum PO E E' AC B ABC
 using sum3-def sum3-permut by blast
 obtain AD where Sum PO E E' D A AD
 using assms sum4-col sum-exists by blast
 then obtain ABD where Sum PO E E' AD B ABD

using *Ar2-def* $\langle \text{Sum } PO \ E \ E' \ AC \ B \ ABC \rangle$ *sum-ar2* *sum-exists* **by** *presburger*
 thus *?thesis*
 by (*smt* (*verit*, *ccfv-threshold*) $\langle \text{Sum } PO \ E \ E' \ ABC \ D \ S \rangle$ $\langle \text{Sum } PO \ E \ E' \ D \ A \ AD \rangle$
 $\langle \text{sum3 } PO \ E \ E' \ A \ B \ C \ ABC \rangle$ *sum3-col* *sum3-def* *sum4-def* *sum-assoc-2* *sum-comm*)
 qed

lemma *sum22-permut*:
 assumes *sum22* $PO \ E \ E' \ A \ B \ C \ D \ S$
 shows *sum22* $PO \ E \ E' \ D \ A \ B \ C \ S$
 using *assms* *sum4-equiv* *sum4-permut* **by** *presburger*

lemma *sum4-comm*:
 assumes *Sum4* $PO \ E \ E' \ A \ B \ C \ D \ S$
 shows *Sum4* $PO \ E \ E' \ B \ A \ C \ D \ S$
by (*meson* *assms* *sum3-comm-1-2* *sum4-def*)

lemma *sum22-comm*:
 assumes *sum22* $PO \ E \ E' \ A \ B \ C \ D \ S$
 shows *sum22* $PO \ E \ E' \ B \ A \ C \ D \ S$
by (*meson* *assms* *sum4-comm* *sum4-equiv-a* *sum4-equiv-b*)

lemma *sum-abcd*:
 assumes *Sum* $PO \ E \ E' \ A \ B \ AB$
 and *Sum* $PO \ E \ E' \ C \ D \ CD$
 and *Sum* $PO \ E \ E' \ B \ C \ BC$
 and *Sum* $PO \ E \ E' \ A \ D \ AD$
 and *Sum* $PO \ E \ E' \ AB \ CD \ S$
 shows *Sum* $PO \ E \ E' \ BC \ AD \ S$
proof –
 have *sum22* $PO \ E \ E' \ A \ B \ C \ D \ S$
 using *assms*(1) *assms*(2) *assms*(5) *sum22-def* **by** *blast*
 then obtain $AD' \ BC'$ **where** *Sum* $PO \ E \ E' \ D \ A \ AD'$ and *Sum* $PO \ E \ E' \ B \ C \ BC'$ and
 $Sum \ PO \ E \ E' \ AD' \ BC' \ S$
 using *sum22-def* *sum22-permut* **by** *blast*
 thus *?thesis*
 by (*metis* $\langle \text{sum22 } PO \ E \ E' \ A \ B \ C \ D \ S \rangle$ *assms*(3) *assms*(4) *sum22-col* *sum-comm* *sum-stable*)
 qed

lemma *sum-diff-diff-a*:
 assumes *Diff* $PO \ E \ E' \ B \ A \ dBA$
 and *Diff* $PO \ E \ E' \ C \ B \ dCB$
 and *Diff* $PO \ E \ E' \ C \ A \ dCA$
 shows *Sum* $PO \ E \ E' \ dCB \ dBA \ dCA$
by (*smt* (*verit*) *Ar2-def* *Diff-def* *assms*(1) *assms*(2) *assms*(3) *diff-ar2* *diff-sum*
opp-uniqueness *sum-assoc* *sum-comm*)

lemma *sum-diff-diff-b*:
 assumes *Diff* $PO \ E \ E' \ B \ A \ dBA$
 and *Diff* $PO \ E \ E' \ C \ B \ dCB$
 and *Sum* $PO \ E \ E' \ dCB \ dBA \ dCA$
 shows *Diff* $PO \ E \ E' \ C \ A \ dCA$
by (*meson* *Ar2-def* *assms*(1) *assms*(2) *assms*(3) *diff-sum* *sum-diff* *sum-ar2* *sum-assoc-2* *sum-comm*)

lemma *sum-diff2-diff-sum2-a*:
 assumes *Sum* $PO \ E \ E' \ A \ B \ C$
 and *Sum* $PO \ E \ E' \ X \ Y \ Z$
 and *Diff* $PO \ E \ E' \ X \ A \ dXA$
 and *Diff* $PO \ E \ E' \ Y \ B \ dYB$
 and *Sum* $PO \ E \ E' \ dXA \ dYB \ dZC$
 shows *Diff* $PO \ E \ E' \ Z \ C \ dZC$
proof –

```

obtain  $Z'$  where  $\text{Sum } PO \ E \ E' \ C \ dZC \ Z'$ 
  using  $\text{Ar2-def Sum-def assms}(1) \ \text{assms}(5) \ \text{sum-exists}$  by presburger
then obtain  $Y' \ X'$  where  $\text{Sum } PO \ E \ E' \ B \ dYB \ Y'$  and  $\text{Sum } PO \ E \ E' \ A \ dXA \ X'$  and
 $\text{Sum } PO \ E \ E' \ Y' \ X' \ Z'$ 
  by (smt (verit, ccfv-threshold)  $\text{Ar2-def assms}(1) \ \text{assms}(3) \ \text{assms}(4) \ \text{assms}(5) \ \text{diff-sum}$ 
 $\text{sum-abcd sum-ar2 sum-comm}$ )
thus ?thesis
  by (metis  $\langle \text{Sum } PO \ E \ E' \ C \ dZC \ Z' \rangle \ \text{assms}(2) \ \text{assms}(3) \ \text{assms}(4) \ \text{diff-sum sum-diff sum3-col}$ 
 $\text{sum-comm sum-stable sum-to-sum3}$ )
qed

lemma sum-diff2-diff-sum2-b:
assumes  $\text{Sum } PO \ E \ E' \ A \ B \ C$ 
  and  $\text{Sum } PO \ E \ E' \ X \ Y \ Z$ 
  and  $\text{Diff } PO \ E \ E' \ X \ A \ dXA$ 
  and  $\text{Diff } PO \ E \ E' \ Y \ B \ dYB$ 
  and  $\text{Diff } PO \ E \ E' \ Z \ C \ dZC$ 
shows  $\text{Sum } PO \ E \ E' \ dXA \ dYB \ dZC$ 
proof –
obtain  $dZC'$  where  $\text{Sum } PO \ E \ E' \ dXA \ dYB \ dZC'$ 
  by (meson  $\text{Ar2-def diff-sum Tarski-Euclidean-2D-axioms assms}(3) \ \text{assms}(4) \ \text{sum-ar2 sum-exists}$ )
thus ?thesis
  by (smt (verit, ccfv-SIG)  $\text{assms}(1) \ \text{assms}(2) \ \text{assms}(3) \ \text{assms}(4) \ \text{assms}(5) \ \text{diff-stable}$ 
 $\text{sum-diff2-diff-sum2-a}$ )
qed

lemma sum-opp:
assumes  $\text{Sum } PO \ E \ E' \ X \ MX \ PO$ 
shows  $\text{Opp } PO \ E \ E' \ X \ MX$ 
by (simp add:  $\text{Opp-def assms diff-O-A diff-sum}$ )

lemma sum-diff-diff:
assumes  $\text{Diff } PO \ E \ E' \ AX \ BX \ AXMBX$ 
  and  $\text{Diff } PO \ E \ E' \ AX \ CX \ AXMCX$ 
  and  $\text{Diff } PO \ E \ E' \ BX \ CX \ BXMCX$ 
shows  $\text{Sum } PO \ E \ E' \ AXMBX \ BXMCX \ AXMCX$ 
using  $\text{assms}(1) \ \text{assms}(2) \ \text{assms}(3) \ \text{sum-diff-diff-a}$  by blast

lemma prod-to-prodp:
assumes  $\text{Prod } PO \ E \ E' \ A \ B \ C$ 
shows  $\text{Prodp } PO \ E \ E' \ A \ B \ C$ 
proof –
obtain  $B'$  where  $E \ E' \ Pj \ B \ B'$  and  $\text{Col } PO \ E' \ B'$  and  $E' \ A \ Pj \ B' \ C$ 
  using  $\text{Prod-def assms}$  by blast
hence  $B \ B' \ \text{Proj } PO \ E' \ E \ E'$ 
  by (metis  $\text{Ar2-def Prod-def assms grid-not-par not-col-permutation-1 pj-col-project}$ )
moreover have  $B' \ C \ \text{Proj } PO \ E \ A \ E'$ 
proof (cases)
  assume  $A = PO$ 
  thus ?thesis
  by (metis  $\text{Ar2-def Prod-def } \langle E' \ A \ Pj \ B' \ C \rangle \ \text{assms grid-not-par}$ 
 $\text{not-col-permutation-1 pj-col-project pj-left-comm}$ )
next
assume  $A \neq PO$ 
thus ?thesis
  by (metis  $\text{Ar2-def Col-cases Par-cases Par-def Pj-def Prod-def } \langle E' \ A \ Pj \ B' \ C \rangle$ 
 $\text{assms col-trivial-1 par-col-project par-strict-not-col-2 project-trivial}$ )
qed
ultimately show ?thesis
  using  $\text{Ar2-def Prod-def Prodp-def assms}$  by auto
qed

lemma project-pj:
assumes  $P \ P' \ \text{Proj } A \ B \ X \ Y$ 
shows  $X \ Y \ Pj \ P \ P'$ 
using  $\text{Par-cases Pj-def Proj-def assms}$  by auto

```

```

lemma prodp-to-prod:
  assumes Prodp PO E E' A B C
  shows Prod PO E E' A B C
proof -
  obtain B' where B B' Proj PO E' E E' and B' C Proj PO E A E'
    using Prodp-def assms by blast
  have E E' Pj B B'
    using ⟨B B' Proj PO E' E E'⟩ project-pj by auto
  moreover have Col PO E' B'
    using Proj-def ⟨B B' Proj PO E' E E'⟩ by presburger
  moreover have E' A Pj B' C
    using ⟨B' C Proj PO E A E'⟩ pj-left-comm project-pj by blast
  ultimately show ?thesis
    by (metis Ar2-def Par-def Prod-def Prodp-def Proj-def assms grid-not-par-5)
qed

lemma prod-exists:
  assumes grid-ok: ¬ Col PO E E'
    and Col PO E A
    and Col PO E B
  shows ∃ C. Prod PO E E' A B C
  by (metis Prodp-def Par-def assms(2) assms(3) grid-not-par grid-ok not-col-permutation-2
    par-strict-not-col-3 prodp-to-prod project-existence)

lemma prod-uniqueness:
  assumes grid-ok: ¬ Col PO E E'
    and Prod PO E E' A B C1
    and Prod PO E E' A B C2
  shows C1 = C2
proof -
  obtain B' where B B' Proj PO E' E E' and B' C1 Proj PO E A E'
    using Prodp-def assms(2) prod-to-prodp by blast
  obtain B'' where B B'' Proj PO E' E E' and B'' C2 Proj PO E A E'
    using Prodp-def assms(3) prod-to-prodp by blast
  thus ?thesis
    by (metis ⟨B B' Proj PO E' E E'⟩ ⟨B' C1 Proj PO E A E'⟩ project-uniqueness)
qed

lemma prod-0-l:
  assumes ¬ Col PO E E'
    and Col PO E A
  shows Prod PO E E' PO A PO
proof (cases)
  assume A = PO
  thus ?thesis
    by (metis Ar2-def Pj-def Prod-def assms(1) grid-not-par-5)
next
  assume A ≠ PO
  obtain B' where A B' Proj PO E' E E'
    by (metis Par-cases assms(1) grid-not-par project-existence)
  hence E E' Pj A B'
    using project-pj by auto
  moreover have Col PO E' B'
    using Proj-def ⟨A B' Proj PO E' E E'⟩ by presburger
  moreover have E' PO Pj B' PO
    by (metis Par-cases Pj-def not-col-distincts assms(1) calculation(2) not-par-not-col)
  ultimately show ?thesis
    using Ar2-def Prod-def assms(1) assms(2) col-trivial-3 by auto
qed

lemma prod-0-r:
  assumes ¬ Col PO E E'
    and Col PO E A
  shows Prod PO E E' A PO PO

```

using *Ar2-def Pj-def Prod-def assms(1,2) col-trivial-3* by *auto*

lemma *prod-1-l*:

assumes $\neg \text{Col } PO \ E \ E'$

and $\text{Col } PO \ E \ A$

shows $\text{Prod } PO \ E \ E' \ E \ A \ A$

proof –

obtain B' where $A \ B' \ \text{Proj } PO \ E' \ E \ E'$

by (*metis assms(1) grid-not-par par-symmetry project-existence*)

moreover

{

assume $A \ B' \ \text{Par } E \ E'$

hence $\text{Prod } PO \ E \ E' \ E \ A \ A$

by (*metis Par-cases Prodp-def assms(1,2) calculation not-col-distincts par-col-project par-id-5 prodp-to-prod*)

}

moreover

{

assume $A = B'$

hence $\text{Prod } PO \ E \ E' \ E \ A \ A$

by (*metis assms(1,2) calculation(1) not-col-distincts prod-0-r proj-id*)

}

ultimately show *?thesis*

using *project-par-dir* by *blast*

qed

lemma *prod-1-r*:

assumes $\neg \text{Col } PO \ E \ E'$

and $\text{Col } PO \ E \ A$

shows $\text{Prod } PO \ E \ E' \ A \ E \ A$

by (*metis Ar2-def Pj-def Prod-def assms(1,2) grid-not-par not-par-not-col*)

lemma *inv-exists*:

assumes $\neg \text{Col } PO \ E \ E'$

and $\text{Col } PO \ E \ A$

and $A \neq PO$

shows $\exists \ IA. \ \text{Prod } PO \ E \ E' \ IA \ A \ E$

proof –

obtain B' where $A \ B' \ \text{Proj } PO \ E' \ E \ E'$

by (*meson Prodp-def assms(1,2) prod-0-l prod-to-prodp*)

have $B' \neq PO$

using $\langle A \ B' \ \text{Proj } PO \ E' \ E \ E' \rangle \ \text{assms}(1,2,3) \ \text{grid-not-par-5} \ \text{proj-id}$ by *blast*

obtain IA where $E' \ IA \ \text{Proj } PO \ E \ B' \ E$

by (*metis NCol-cases $\langle A \ B' \ \text{Proj } PO \ E' \ E \ E' \rangle \ \text{assms}(1,2,3) \ \text{grid-not-par proj-id project-existence}$*)

moreover have $E \ E' \ Pj \ A \ B'$

using $\langle A \ B' \ \text{Proj } PO \ E' \ E \ E' \rangle \ \text{project-pj}$ by *auto*

moreover have $\text{Col } PO \ E' \ B'$

using *Proj-def* $\langle A \ B' \ \text{Proj } PO \ E' \ E \ E' \rangle$ by *auto*

moreover have $E' \ IA \ Pj \ B' \ E$

using *Pj-def Proj-def assms(1) calculation(1)* by *auto*

ultimately show *?thesis*

using *Ar2-def Prod-def Proj-def assms(1,2) col-trivial-2* by *auto*

qed

lemma *prod-null*:

assumes $\text{Prod } PO \ E \ E' \ A \ B \ PO$

shows $A = PO \vee B = PO$

proof –

obtain B' where $E \ E' \ Pj \ B \ B'$ and $\text{Col } PO \ E' \ B'$ and $E' \ A \ Pj \ B' \ PO$

using *Prod-def assms* by *blast*

show *?thesis*

```

proof (cases)
  assume  $B = PO$ 
  thus ?thesis
    by simp
next
  assume  $B \neq PO$ 
  {
    assume  $E E' \text{ Par } B B'$ 
    {
      assume  $E' A \text{ Par } B' PO$ 
      hence  $A = PO$ 
      by (metis Ar2-def Col-cases Par-def Prod-def  $\langle Col PO E' B' \rangle$  assms col-trivial-3
        not-strict-par2)
    }
    moreover
    {
      assume  $B' = PO$ 
      hence  $A = PO$ 
      by (metis Ar2-def Prod-def  $\langle E E' \text{ Par } B B' \rangle$  assms col-permutation-2 grid-not-par
        par-col2-par-bis)
    }
    ultimately have  $A = PO$ 
    using Pj-def  $\langle E' A \text{ Pj } B' PO \rangle$  by blast
  }
  moreover {
    assume  $B = B'$ 
    hence  $A = PO$ 
    by (metis Ar2-def NCol-cases Prod-def  $\langle B \neq PO \rangle \langle Col PO E' B' \rangle$  assms
      colx)
  }
  ultimately have  $A = PO$ 
  using Pj-def  $\langle E E' \text{ Pj } B B' \rangle$  by blast
thus ?thesis
  by simp
qed
qed

```

```

lemma prod-y-axis-change:
  assumes Prod PO E E' A B C
  and  $\neg Col PO E E''$ 
  shows Prod PO E E'' A B C
proof (cases  $B = PO$ )
  show  $B = PO \implies Prod PO E E'' A B C$ 
  by (metis Ar2-def Prod-def assms(1,2) prod-0-r
    prod-uniqueness)
  {
    assume  $B \neq PO$ 
    obtain  $B'$  where  $E E' \text{ Pj } B B'$  and  $Col PO E' B'$  and  $E' A \text{ Pj } B' C$ 
    using Prod-def assms(1) by blast
    show Prod PO E E'' A B C
    proof (cases  $A = PO$ )
    show  $A = PO \implies Prod PO E E'' A B C$ 
    by (metis Ar2-def Prod-def assms(1,2) prod-0-l
      prod-uniqueness)
    {
      assume  $A \neq PO$ 
      have  $C \neq PO$ 
      using  $\langle A \neq PO \rangle \langle B \neq PO \rangle$  assms(1) prod-null by blast
      obtain  $B''$  where  $B B'' \text{ Proj } PO E'' E''$ 
      by (metis Par-cases assms(2) col-trivial-2 col-trivial-3
        grid-not-par-3 project-existence)
      hence  $B B'' \text{ Par } E E''$ 
      by (metis Prodp-def  $\langle B \neq PO \rangle$  assms(1,2) prod-to-prodp proj-id
        project-par-dir)
      have  $E E'' \text{ Pj } B B''$ 
      using  $\langle B B'' \text{ Proj } PO E'' E'' \rangle$  project-pj by auto
    }
  }

```



```

moreover have Col PO E'' B''
  using Proj-def ⟨B B'' Proj PO E'' E E''⟩ by force
moreover have E'' A Pj B'' C
proof (cases B = E)
  show B = E ⇒ E'' A Pj B'' C
    by (metis Ar2-def Col-cases Par-def Pj-def Prod-def
      ⟨B B'' Par E E''⟩ assms(1,2) calculation(2) col2--eq
      par-id-3 prod-1-r prod-uniqueness)
  {
    assume B ≠ E
    have E' A Par B' C
      by (metis Ar2-def NCol-cases Pj-def Prod-def ⟨C ≠ PO⟩
        ⟨Col PO E' B'⟩ ⟨E' A Pj B' C⟩ assms(1) col-trivial-3
        colx)
    have B B'' ParStrict E E''
      by (meson Par-def Prodp-def ⟨B B'' Par E E''⟩ ⟨B ≠ E⟩
        assms(1,2) col-trivial-2 colx prod-to-prodp)
    have E E' Par B B'
      by (metis Ar2-def NCol-cases Pj-def Prod-def ⟨B ≠ PO⟩
        ⟨Col PO E' B'⟩ ⟨E E' Pj B B'⟩ assms(1) col-trivial-3
        colx)
    hence E E' ParStrict B B'
      by (metis Ar2-def NCol-cases Par-def Prod-def ⟨B ≠ E⟩
        assms(1) col-trivial-3 colx)
    have E' A ParStrict B' C
      by (metis (no-types, lifting) Ar2-def Par-def Prod-def
        ⟨C ≠ PO⟩ ⟨Col PO E' B'⟩ ⟨E E' ParStrict B B'⟩
        ⟨E' A Par B' C⟩ assms(1) grid-not-par l6-21
        par-strict-not-col-1)
    show E'' A Pj B'' C
    proof (cases A = E)
      show A = E ⇒ E'' A Pj B'' C
        by (metis Ar2-def Prod-def assms(1) calculation(1) pj-comm
          prod-uniqueness)
      {
        assume A ≠ E
        have B' ≠ PO
          by (metis NCol-cases Prodp-def ⟨E E' ParStrict B B'⟩ assms(1)
            par-strict-not-col-3 prod-to-prodp)
        show E'' A Pj B'' C
        proof (cases E' = E'')
          show E' = E'' ⇒ E'' A Pj B'' C
            by (metis Prodp-def ⟨Col PO E' B'⟩ ⟨E E' Pj B B'⟩
              ⟨E' A Pj B' C⟩ assms(1,2) calculation(1,2) pj-uniqueness
              prod-to-prodp)
          {
            assume E' ≠ E''
            show E'' A Pj B'' C
            proof (cases Col E E' E'')
              {
                assume Col E E' E''
                hence ¬ Col E' E'' A
                  by (metis Prodp-def ⟨A ≠ E⟩ ⟨E' ≠ E''⟩
                    assms(1,2) col2--eq col-permutation-1
                    prod-to-prodp)
                have B' ≠ B''
                  by (metis ⟨B' ≠ PO⟩ ⟨Col E E' E''⟩ ⟨Col PO E' B'⟩ ⟨E' ≠ E''⟩
                    assms(2) calculation(2) col2--eq col-permutation-5
                    colx)
                have E' E'' Par B' B''
                  by (smt (verit, ccfv-threshold) Par-perm ⟨B B'' Par E E''⟩
                    ⟨B' ≠ B''⟩ ⟨Col E E' E''⟩ ⟨E E' Par B B'⟩ ⟨E' ≠ E''⟩
                    col-par-par-col col-trivial-2 not-col-permutation-2
                    par-col4--par)
                have B' ≠ E'
                  using ⟨E' A ParStrict B' C⟩ not-par-strict-id by auto

```

```

have E' E'' ParStrict B' B''
  by (metis NCol-cases Par-def ⟨Col E E' E''⟩
      ⟨E E' ParStrict B B'⟩ ⟨E' E'' Par B' B''⟩ l6-16-1
      par-strict-not-col-4)
have Coplanar PO E'' E' A
  by (simp add: all-coplanar)
moreover have Col PO A C
  by (metis Ar2-def Col-cases Prod-def assms(1) col3
      col-trivial-3)
ultimately have E'' A Par B'' C
  using l13-15 [of E' - - PO B'] ⟨¬ Col E' E'' A⟩
      ⟨E' E'' ParStrict B' B''⟩ ⟨E' A ParStrict B' C⟩
      ⟨Col PO E' B'⟩ ⟨Col PO E'' B''⟩ by blast
thus E'' A Pj B'' C
  by (simp add: Pj-def)
}
{
  assume ¬ Col E E' E''
  have E'' E' Par B'' B'
    using l13-15 [of E E'' - PO B]
    by (meson Par-strict-cases Prodp-def ⟨B B'' ParStrict E E''⟩
        ⟨Col PO E' B'⟩ ⟨E E' ParStrict B B'⟩ ⟨¬ Col E E' E''⟩
        all-coplanar assms(1) calculation(2) col-permutation-5
        prod-to-prodp)
  show E'' A Pj B'' C
  proof (cases Col A E' E'')
  {
    assume Col A E' E''
    hence B' C Par E' E''
      by (metis Par-cases ⟨E' A Par B' C⟩ ⟨E' ≠ E''⟩
          col-par par-distinct par-not-par)
    have B' C Par B' B''
    proof (rule par-trans [of B' C E' E'' B' B''])
      show B' C Par E' E''
        by (simp add: ⟨B' C Par E' E''⟩)
      show E' E'' Par B' B''
        by (simp add: ⟨E'' E' Par B'' B'⟩ par-comm)
    qed
    show E'' A Pj B'' C
      by (smt (verit, best) Ar2-def Pj-def Prod-def
          ⟨B' C Par B' B''⟩ ⟨Col A E' E''⟩ ⟨E' A Par B' C⟩ assms(1,2)
          col-par-par-col col-permutation-4 l6-16-1 par-col4--par
          par-comm par-right-comm par-trans)
  }
  {
    assume ¬ Col A E' E''
    have B' ≠ E'
      using ⟨E' A ParStrict B' C⟩ not-par-strict-id by blast
    have B'' ≠ E''
      using ⟨B B'' ParStrict E E''⟩ col-trivial-2
      par-strict-not-col-4 by blast
    {
      assume E'' E' ParStrict B'' B'
      have E'' A Par B'' C
        using l13-15 [of E' E'' A PO B' B'' C]
        by (metis Ar2-def NCol-cases Par-strict-cases Prod-def
            ⟨Col PO E' B'⟩ ⟨E' A ParStrict B' C⟩
            ⟨E'' E' ParStrict B'' B'⟩ ⟨¬ Col A E' E''⟩
            all-coplanar assms(1) calculation(2)
            col-transitivity-1 col-trivial-3)
      hence E'' A Pj B'' C
        by (simp add: Pj-def)
    }
  }
  moreover
  {
    assume E'' ≠ E' and B'' ≠ B' and Col E'' B'' B' and

```

```

    Col E' B'' B'
  have E'' A Pj B'' C
  proof (cases Col PO E' E'')
  {
    assume Col PO E' E''
    have A E'' Par C B''
    proof (rule l13-19 [of PO E E' B])
      show  $\neg$  Col PO E E'
        using Ar2-def Prod-def assms(1) by presburger
      show PO  $\neq$  E
        using  $\langle$ Col PO E' E'' $\rangle$   $\langle$  $\neg$  Col E E' E'' $\rangle$  by auto
      show PO  $\neq$  B
        using  $\langle$ B  $\neq$  PO $\rangle$  by auto
      show PO  $\neq$  A
        using  $\langle$ A  $\neq$  PO $\rangle$  by auto
      show PO  $\neq$  C
        using  $\langle$ C  $\neq$  PO $\rangle$  by auto
      show PO  $\neq$  E'
        using  $\langle$  $\neg$  Col PO E E' $\rangle$  grid-not-par-5 by blast
      show PO  $\neq$  B'
        using  $\langle$ B'  $\neq$  PO $\rangle$  by auto
      show PO  $\neq$  E''
        using assms(2) grid-not-par-5 by auto
      show PO  $\neq$  B''
        by (metis Prodp-def  $\langle$ B B'' Proj PO E'' E E'' $\rangle$ 
            $\langle$ B  $\neq$  PO $\rangle$  assms(1,2)
           col-trivial-3 prod-to-prodp proj-id)
      show Col PO E A
        using Prodp-def assms(1) prod-to-prodp by blast
      show Col PO E B
        using Prodp-def assms(1) prod-to-prodp by blast
      show Col PO E C
        using Ar2-def Prod-def assms(1) by presburger
      show Col PO E' E''
        by (simp add:  $\langle$ Col PO E' E'' $\rangle$ )
      show Col PO E' B'
        by (simp add:  $\langle$ Col PO E' B' $\rangle$ )
      show Col PO E' B''
        by (metis  $\langle$ Col PO E' E'' $\rangle$   $\langle$ Col PO E'' B'' $\rangle$  assms(2)
           col-trivial-3
           colx)
      show E E' Par B B'
        by (simp add:  $\langle$ E E' Par B B' $\rangle$ )
      show E E'' Par B B''
        by (simp add:  $\langle$ B B'' Par E E'' $\rangle$  par-symmetry)
      show E' A Par B' C
        by (simp add:  $\langle$ E' A Par B' C $\rangle$ )
    qed
    hence E'' A Par B'' C
      using par-comm by blast
    thus E'' A Pj B'' C
      by (simp add: Pj-def)
  }
  show  $\neg$  Col PO E' E''  $\implies$  E'' A Pj B'' C
    by (metis  $\langle$ B''  $\neq$  B' $\rangle$   $\langle$ B''  $\neq$  E'' $\rangle$   $\langle$ Col E' B'' B' $\rangle$ 
         $\langle$ Col E'' B'' B' $\rangle$   $\langle$ Col PO E'' B'' $\rangle$ 
        col2--eq col-permutation-2)
  qed
}
ultimately show E'' A Pj B'' C
  using Par-def  $\langle$ E'' E' Par B'' B' $\rangle$  by presburger
}
qed
}
qed

```

```

    }
  qed
}
qed
}
qed
show Prod PO E E'' A B C
  using Ar2-def Prod-def ⟨E'' A Pj B'' C⟩ assms(1,2)
  calculation(1,2) by auto
}
qed
}
qed

```

```

lemma proj-preserves-prod:
  assumes Prod PO E E' A B C
    and Ar1 PO E' A' B' C'
    and E E' Pj A A'
    and E E' Pj B B'
    and E E' Pj C C'
  shows Prod PO E' E A' B' C'
proof (cases A = PO)
  have Ar2 PO E E' A B C
    using Prod-def assms(1) by blast
  show A = PO ⟹ Prod PO E' E A' B' C'
    by (metis Ar1-def Ar2-def Col-cases Prod-def assms(1,2,3,5) pj-comm
      prod-0-l prod-uniqueness sum-par-strict-a)
{
  assume A ≠ PO
  show Prod PO E' E A' B' C'
  proof (cases B = PO)
    show B = PO ⟹ Prod PO E' E A' B' C'
      by (metis Ar1-def Ar2-def Col-cases ⟨Ar2 PO E E' A B C⟩ assms(1,2,4,5)
        col-trivial-3 pj-trivial pj-uniqueness prod-0-r
        prod-uniqueness)
    {
      assume B ≠ PO
      show Prod PO E' E A' B' C'
      proof (cases C = PO)
        show C = PO ⟹ Prod PO E' E A' B' C'
          using ⟨A ≠ PO⟩ ⟨B ≠ PO⟩ assms(1) prod-null by blast
        {
          assume C ≠ PO
          show Prod PO E' E A' B' C'
          proof (cases A' = PO)
            show A' = PO ⟹ Prod PO E' E A' B' C'
              using Ar2-def ⟨A ≠ PO⟩ ⟨Ar2 PO E E' A B C⟩ assms(3) sum-par-strict-a
              by blast
            {
              assume A' ≠ PO
              show Prod PO E' E A' B' C'
              proof (cases B' = PO)
                show B' = PO ⟹ Prod PO E' E A' B' C'
                  by (metis Ar2-def NCol-cases ⟨Ar2 PO E E' A B C⟩ ⟨B ≠ PO⟩ assms(4)
                    col-trivial-3 pj-comm pj-trivial pj-uniqueness)
                {
                  assume B' ≠ PO
                  show Prod PO E' E A' B' C'
                  proof (cases C' = PO)
                    show C' = PO ⟹ Prod PO E' E A' B' C'
                      by (metis Ar2-def Col-cases Par-cases Par-def Pj-def ⟨Ar2 PO E E' A B C⟩
                        ⟨C ≠ PO⟩ assms(5) par-strict-not-col-1)
                    {
                      assume C' ≠ PO

```

```

obtain  $B''$  where  $E E' Pj B B''$  and  $Col PO E' B''$  and  $E' A Pj B'' C$ 
  using  $Prod-def\ assms(1)$  by blast
hence  $B' = B''$ 
  by (meson  $Ar1-def\ Ar2-def\ Prod-def\ assms(1,2,4)\ pj-uniqueness$ )
have  $E' E Pj B' B$ 
  by (simp add:  $\langle B' = B'' \rangle \langle E E' Pj B B'' \rangle\ pj-comm$ )
moreover have  $Col PO E B$ 
  using  $Ar2-def\ \langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle$  by blast
moreover have  $A' E Par C' B$ 
proof (rule  $l13-19$  [of  $PO E' A B$ ])
  show  $\neg Col PO E' A$ 
    by (metis  $Ar2-def\ Col-cases\ \langle A \neq PO \rangle$ 
       $\langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle\ col2--eq$ )
  show  $PO \neq E'$ 
    using  $\langle \neg Col PO E' A \rangle\ col-trivial-1$  by blast
  show  $PO \neq B'$ 
    using  $\langle B' \neq PO \rangle$  by auto
  show  $PO \neq A'$ 
    using  $\langle A' \neq PO \rangle$  by auto
  show  $PO \neq C'$ 
    using  $\langle C' \neq PO \rangle$  by auto
  show  $PO \neq A$ 
    using  $\langle A \neq PO \rangle$  by auto
  show  $PO \neq C$ 
    using  $\langle C \neq PO \rangle$  by blast
  show  $PO \neq E$ 
    using  $Ar2-def\ \langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle\ col-trivial-1$  by force
  show  $PO \neq B$ 
    using  $\langle B \neq PO \rangle$  by auto
  show  $Col PO E' A'$ 
    using  $Ar1-def\ assms(2)$  by force
  show  $Col PO E' B'$ 
    by (simp add:  $\langle B' = B'' \rangle \langle Col PO E' B'' \rangle$ )
  show  $Col PO E' C'$ 
    using  $Ar1-def\ assms(2)$  by auto
  show  $Col PO A E$ 
    by (meson  $Ar2-def\ Col-cases\ \langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle$ )
  show  $Col PO A C$ 
    by (metis  $Ar2-def\ Col-cases\ \langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle \langle PO \neq E \rangle$ 
       $col2--eq$ )
  show  $Col PO A B$ 
    using  $\langle Col PO A E \rangle \langle PO \neq E \rangle\ calculation(2)\ col-trivial-3\ colx$ 
    by blast
  show  $E' A Par B' C$ 
    by (smt (verit, ccfv-threshold)  $Pj-def\ \langle B' = B'' \rangle \langle Col PO A C \rangle$ 
       $\langle Col PO E' B'' \rangle \langle E' A Pj B'' C \rangle \langle PO \neq B' \rangle$ 
       $\langle \neg Col PO E' A \rangle\ grid-not-par$ 
      not-par-not-col not-par-one-not-par)
  show  $E' E Par B' B$ 
    by (smt (verit, ccfv-threshold)  $Pj-def\ \langle B' = B'' \rangle \langle Col PO A B \rangle$ 
       $\langle Col PO E' B'' \rangle \langle PO \neq B' \rangle \langle \neg Col PO E' A \rangle$ 
       $calculation(1)\ grid-not-par$ 
      not-par-not-col not-par-one-not-par)
  show  $A A' Par C C'$ 
    by (smt (verit, best)  $Ar1-def\ Par-cases\ Pj-def\ \langle E' A Par B' C \rangle$ 
       $\langle \neg Col PO E' A \rangle\ assms(2,3,5)\ grid-not-par\ not-par-one-not-par$ 
      par-col2-par-bis)
qed
hence  $E A' Par B C'$ 
  using par-comm by blast
hence  $E A' Pj B C'$ 
  by (simp add:  $Pj-def$ )
ultimately show  $Prod PO E' E A' B' C'$ 
  by (metis  $Ar1-def\ Ar2-def\ Col-cases\ Prod-def\ assms(1,2)$ )
}
qed

```

```

    }
  qed
}
qed
}
qed
}
qed
}
qed
}
qed
}
qed

```

```

lemma prod-assoc1:
  assumes Prod PO E E' A B AB
    and Prod PO E E' B C BC
    and Prod PO E E' A BC ABC
  shows Prod PO E E' AB C ABC
proof (cases A = PO)
  have Ar2 PO E E' A B AB
    using Prod-def assms(1) by blast
  have Ar2 PO E E' B C BC
    using Prod-def assms(2) by blast
  have Ar2 PO E E' A BC ABC
    using Prod-def assms(3) by auto
  show A = PO  $\implies$  Prod PO E E' AB C ABC
    by (metis Ar2-def  $\langle$ Ar2 PO E E' B C BC $\rangle$  assms(1,3) prod-0-l
        prod-uniqueness)
{
  assume A  $\neq$  PO
  show Prod PO E E' AB C ABC
  proof (cases B = PO)
    show B = PO  $\implies$  Prod PO E E' AB C ABC
      by (metis Ar2-def  $\langle$ Ar2 PO E E' A B AB $\rangle$   $\langle$ Ar2 PO E E' B C BC $\rangle$  assms(1,2,3)
          prod-0-l prod-0-r prod-uniqueness)
    {
      assume B  $\neq$  PO
      show Prod PO E E' AB C ABC
      proof (cases C = PO)
        show C = PO  $\implies$  Prod PO E E' AB C ABC
          by (metis Ar2-def  $\langle$ Ar2 PO E E' A B AB $\rangle$  assms(2,3) prod-0-r
              prod-uniqueness)
        {
          assume C  $\neq$  PO
          have  $\neg$  PO E Par E E'
            using Ar2-def  $\langle$ Ar2 PO E E' B C BC $\rangle$  grid-not-par
            by blast
          have  $\neg$  PO E Par PO E'
            using Ar2-def  $\langle$ Ar2 PO E E' B C BC $\rangle$  grid-not-par-2 by blast
          have PO  $\neq$  E
            using Ar2-def  $\langle$ Ar2 PO E E' A B AB $\rangle$  not-col-distincts by blast
          hence PO  $\neq$  E'
            using Par-cases  $\langle$  $\neg$  PO E Par E E' $\rangle$  par-reflexivity by blast
          hence E  $\neq$  E'
            using  $\langle$  $\neg$  PO E Par PO E' $\rangle$  par-reflexivity by blast
          obtain C' where C C' Proj PO E' E E'
            using Prodp-def assms(2) prod-to-prodp by blast
          hence C C' Par E E'
            by (metis Ar2-def  $\langle$ Ar2 PO E E' B C BC $\rangle$   $\langle$ C  $\neq$  PO $\rangle$  proj-id
                project-par-dir)
          hence E E' Pj C C'
            by (simp add: Pj-def par-symmetry)
          moreover have Col PO E' C'
            using Proj-def  $\langle$ C C' Proj PO E' E E' $\rangle$  by presburger
          moreover have E' AB Pj C' ABC
          proof -

```

obtain B' **where** $E E' Pj B B'$ **and** $Col PO E' B'$ **and** $E' A Pj B' AB$
using $Prod-def\ assms(1)$ **by** *blast*
obtain C'' **where** $E E' Pj C C''$ **and** $Col PO E' C''$ **and** $E' B Pj C'' BC$
using $Prod-def\ assms(2)$ **by** *blast*
hence $C' = C''$
using $Ar2-def\ \langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$ $calculation(1,2)$ $pj-uniqueness$ **by** *auto*
obtain BC' **where** $E E' Pj BC BC'$ **and** $Col PO E' BC'$ **and** $E' A Pj BC' ABC$
using $Prod-def\ assms(3)$ **by** *blast*
have $B' \neq PO$
using $Ar2-def\ \langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$ $\langle B \neq PO \rangle$ $\langle E E' Pj B B' \rangle$
 $sum-par-strict-a$ **by** *blast*
have $BC \neq PO$
using $\langle B \neq PO \rangle$ $\langle C \neq PO \rangle$ $assms(2)$ $prod-null$ **by** *blast*
have $E' AB\ Par\ C' ABC$
proof (*rule l13-19 [of PO B' B BC']*)
show $\neg Col PO B' B$
by (*metis Ar2-def* $\langle Ar2\ PO\ E\ E'\ A\ B\ AB \rangle$ $\langle B \neq PO \rangle$
 $\langle B' \neq PO \rangle$ $\langle Col PO E' B' \rangle$ *l6-16-1*
 $not-col-permutation-4$)
show $PO \neq B'$
using $\langle B' \neq PO \rangle$ **by** *auto*
show $PO \neq BC'$
by (*metis Ar2-def Col-cases Par-cases Par-def Pj-def*
 $\langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$ $\langle BC \neq PO \rangle$ $\langle E E' Pj BC BC' \rangle$
 $par-strict-not-col-1$)
show $PO \neq E'$
by (*simp add:* $\langle PO \neq E' \rangle$)
show $PO \neq C'$
by (*metis Ar2-def* $\langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$
 $\langle C C' Proj PO E' E E' \rangle$ $\langle C \neq PO \rangle$ $grid-not-par$
 $proj-id$)
show $PO \neq B$
using $\langle B \neq PO \rangle$ **by** *auto*
show $PO \neq BC$
using $\langle BC \neq PO \rangle$ **by** *auto*
show $PO \neq AB$
using $\langle A \neq PO \rangle$ $\langle PO \neq B \rangle$ $assms(1)$ $prod-null$
by *blast*
show $PO \neq ABC$
using $\langle A \neq PO \rangle$ $\langle BC \neq PO \rangle$ $assms(3)$ $prod-null$
by *blast*
show $Col PO B' E'$
by (*simp add:* $\langle Col PO E' B' \rangle$ $col-permutation-5$)
show $Col PO B' BC'$
using $\langle Col PO E' B' \rangle$ $\langle Col PO E' BC' \rangle$ $\langle PO \neq E' \rangle$
 $col-transitivity-1$ **by** *blast*
show $Col PO B' C'$
using $\langle C' = C'' \rangle$ $\langle Col PO E' B' \rangle$ $\langle Col PO E' C'' \rangle$
 $\langle PO \neq E' \rangle$ $col-transitivity-1$ **by** *blast*
show $Col PO B AB$
using $Ar2-def\ Prod-def\ \langle PO \neq E \rangle$ $assms(1)$
 $col-transitivity-1$ **by** *presburger*
show $Col PO B BC$
by (*metis Ar2-def* $\langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$ $\langle PO \neq E \rangle$ *l6-16-1*
 $not-col-permutation-2$)
show $Col PO B ABC$
by (*meson Ar2-def Prod-def* $\langle Col PO B BC \rangle$ $\langle PO \neq BC \rangle$
 $\langle PO \neq E \rangle$ $assms(3)$ $col-permutation-5$
 $col-transitivity-1$)
show $B' B\ Par\ BC' BC$
by (*smt (verit, best) Ar2-def Par-cases Pj-def*
 $\langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$ $\langle B \neq PO \rangle$ $\langle Col PO B' BC' \rangle$
 $\langle Col PO E' BC' \rangle$ $\langle E E' Pj B B' \rangle$ $\langle E E' Pj BC BC' \rangle$
 $\langle PO \neq BC' \rangle$ $\langle PO \neq E' \rangle$ $col-transitivity-2$ *l6-16-1*
 $par-trans$)
show $B' AB\ Par\ BC' ABC$

```

    by (smt (verit) Par-def Pj-def ⟨B ≠ PO⟩ ⟨Col PO B ABC⟩
        ⟨Col PO B AB⟩ ⟨Col PO E' B'⟩ ⟨Col PO E' BC'⟩
        ⟨E' A Pj B' AB⟩ ⟨E' A Pj BC' ABC⟩ ⟨PO ≠ ABC⟩ ⟨PO ≠ E'⟩
        ⟨¬ Col PO B' B⟩ col-permutation-4
        not-col-permutation-2 not-par-inter-exists
        not-par-not-col parallel-uniqueness)
  show B E' Par BC C'
  by (metis Ar2-def Par-cases Pj-def ⟨Ar2 PO E E' B C BC⟩
      ⟨C C' Proj PO E' E E'⟩ ⟨C ≠ PO⟩ ⟨C' = C''⟩
      ⟨E' B Pj C'' BC⟩ proj-id)
qed
thus ?thesis
  by (simp add: Pj-def)
qed
ultimately show Prod PO E E' AB C ABC
  using Ar2-def Prod-def ⟨Ar2 PO E E' A B AB⟩ Ar2 PO E E' A BC ABC
    ⟨Ar2 PO E E' B C BC⟩ by auto
}
qed
}
qed
}
qed

```

```

lemma prod-assoc2:
  assumes Prod PO E E' A B AB and
    Prod PO E E' B C BC
  and Prod PO E E' AB C ABC
  shows Prod PO E E' A BC ABC
proof (cases A = PO)
  have Ar2 PO E E' A B AB
    using Prod-def assms(1) by blast
  have Ar2 PO E E' B C BC
    using Prod-def assms(2) by blast
  have Ar2 PO E E' AB C ABC
    using Prod-def assms(3) by blast
  show A = PO ⟹ Prod PO E E' A BC ABC
    by (metis Ar2-def ⟨Ar2 PO E E' B C BC⟩ assms(1,3)
        prod-0-l prod-uniqueness)
  show A ≠ PO ⟹ Prod PO E E' A BC ABC
proof (cases B = PO)
  show A ≠ PO ⟹ B = PO ⟹ Prod PO E E' A BC ABC
    by (metis Ar2-def ⟨Ar2 PO E E' A B AB⟩
        ⟨Ar2 PO E E' B C BC⟩ assms(1,2,3) prod-0-l prod-0-r
        prod-uniqueness)
  show A ≠ PO ⟹ B ≠ PO ⟹ Prod PO E E' A BC ABC
proof (cases C = PO)
  show A ≠ PO ⟹ B ≠ PO ⟹ C = PO ⟹ Prod PO E E' A BC ABC
    by (metis Ar2-def ⟨Ar2 PO E E' A B AB⟩ assms(2,3)
        prod-0-r prod-uniqueness)
  {
    assume A ≠ PO and B ≠ PO and C ≠ PO
    show Prod PO E E' A BC ABC
    proof -
      have ¬ PO E Par E E'
        using Ar2-def ⟨Ar2 PO E E' AB C ABC⟩ grid-not-par by blast
      have ¬ PO E Par PO E'
        using Ar2-def ⟨Ar2 PO E E' B C BC⟩ grid-not-par by auto
      have ¬ PO E' Par E E'
        using Ar2-def ⟨Ar2 PO E E' B C BC⟩ grid-not-par-3 by blast
      have PO ≠ E
        using Ar2-def ⟨Ar2 PO E E' AB C ABC⟩ col-trivial-1 by blast
      hence PO ≠ E'
        using Par-cases ⟨¬ PO E Par E E'⟩ par-reflexivity by blast
      hence E ≠ E'
        using ⟨¬ PO E Par PO E'⟩ par-reflexivity by blast
    }
  }
}

```


have $BC \neq PO$
using $\langle B \neq PO \rangle \langle C \neq PO \rangle$ *assms(2) prod-null* **by** *blast*
obtain BC' **where** $BC BC' Proj PO E' E E'$
using $\langle E \neq E' \rangle \langle PO \neq E' \rangle \langle \neg PO E' Par E E' \rangle$
par-symmetry project-existence **by** *blast*
hence $BC BC' Par E E' \vee BC = BC'$
using *Proj-def* **by** *auto*
moreover have $BC = BC' \implies BC BC' Par E E'$
using *Ar2-def* $\langle Ar2 PO E E' B C BC \rangle$
 $\langle BC BC' Proj PO E' E E' \rangle \langle BC \neq PO \rangle$ *proj-id*
by *blast*
ultimately have $BC BC' Par E E'$ **by** *blast*
have $E E' Pj BC BC'$
by (*simp add: Pj-def* $\langle BC BC' Par E E' \rangle$
par-symmetry)
moreover have $Col PO E' BC'$
using *Proj-def* $\langle BC BC' Proj PO E' E E' \rangle$
by *presburger*
moreover have $E' A Pj BC' ABC$
proof –
obtain B' **where** $E E' Pj B B'$ **and**
 $Col PO E' B'$ **and** $E' A Pj B' AB$
using *Prod-def assms(1)* **by** *blast*
obtain C' **where** $E E' Pj C C'$ **and**
 $Col PO E' C'$ **and** $E' B Pj C' BC$
using *Prod-def assms(2)* **by** *blast*
obtain C'' **where** $E E' Pj C C''$ **and**
 $Col PO E' C''$ **and** $E' AB Pj C'' ABC$
using *Prod-def assms(3)* **by** *blast*
hence $C' = C''$
using *Ar2-def* $\langle Ar2 PO E E' AB C ABC \rangle \langle Col PO E' C' \rangle$
 $\langle E E' Pj C C' \rangle$ *pj-uniqueness* **by** *force*
have $E' A Par BC' ABC$
proof –
have $B' \neq PO$
by (*metis Ar2-def Prod-def* $\langle BC \neq PO \rangle \langle E E' Pj B B' \rangle$
assms(2) prod-0-l prod-uniqueness
sum-par-strict-a)
hence $E' A Par B' AB$
by (*metis Ar2-def Pj-def* $\langle Ar2 PO E E' AB C ABC \rangle$
 $\langle Col PO E' B' \rangle \langle E \neq E' \rangle \langle E' A Pj B' AB \rangle$
not-col-permutation-1 par-col2-par-bis par-reflexivity
sum-par-strict-a)
moreover have $B' AB Par BC' ABC$
proof (*rule l13-19 [of PO E' B C'], insert* $\langle PO \neq E' \rangle \langle Col PO E' B' \rangle$
 $\langle Col PO E' C' \rangle \langle Col PO E' BC' \rangle$)
show $\neg Col PO E' B$
by (*metis Ar2-def Col-cases Par-cases Par-def Pj-def*
 $\langle Ar2 PO E E' A B AB \rangle \langle B' \neq PO \rangle \langle Col PO E' B' \rangle$
 $\langle E E' Pj B B' \rangle$ *col2--eq par-strict-not-col-4*)
show $PO \neq C'$
by (*smt (verit, ccfv-threshold) Ar2-def Par-cases Pj-def*
 $\langle Ar2 PO E E' B C BC \rangle \langle BC \neq PO \rangle \langle E' B Pj C' BC \rangle$
 $\langle PO \neq E \rangle$ *col-par col-trivial-3 not-strict-par*
par-not-par)
show $PO \neq B'$
using $\langle B' \neq PO \rangle$ **by** *auto*
show $PO \neq BC'$
by (*metis Ar2-def* $\langle Ar2 PO E E' B C BC \rangle$
 $\langle BC BC' Proj PO E' E E' \rangle \langle BC \neq PO \rangle$ *not-col-distincts*
proj-id)
show $PO \neq B$
using $\langle \neg Col PO E' B \rangle$ *grid-not-par-5* **by** *auto*
show $PO \neq BC$
using $\langle BC \neq PO \rangle$ **by** *auto*
show $PO \neq AB$

```

    using  $\langle A \neq PO \rangle \langle B \neq PO \rangle$  assms(1) prod-null by blast
  show  $PO \neq ABC$ 
    using  $\langle C \neq PO \rangle \langle PO \neq AB \rangle$  assms(3) prod-null by blast
  show  $Col\ PO\ B\ AB$ 
    using Ar2-def Prod-def  $\langle PO \neq E \rangle$  assms(1)
      col-transitivity-1 by presburger
  show  $Col\ PO\ B\ BC$ 
    using Ar2-def Prod-def  $\langle PO \neq E \rangle$  assms(2)
      col-transitivity-1 by presburger
  show  $Col\ PO\ B\ ABC$ 
    using Ar2-def Prod-def  $\langle PO \neq E \rangle$  assms(2,3)
      col-transitivity-1 by presburger
  show  $E' B\ Par\ C' BC$ 
    by (metis Pj-def  $\langle BC = BC' \implies BC\ BC'\ Par\ E\ E' \rangle$ 
       $\langle BC\ BC'\ Proj\ PO\ E'\ E\ E' \rangle \langle C' = C'' \rangle \langle Col\ PO\ E'\ C'' \rangle$ 
       $\langle E' B\ Pj\ C' BC \rangle$  par-distinct project-not-col)
  show  $E' AB\ Par\ C' ABC$ 
    by (metis Col-cases Pj-def  $\langle C' = C'' \rangle \langle Col\ PO\ B\ ABC \rangle$ 
       $\langle Col\ PO\ E'\ C' \rangle \langle E' AB\ Pj\ C'' ABC \rangle \langle PO \neq C' \rangle$ 
       $\langle \neg Col\ PO\ E' B \rangle$  col2--eq)
  show  $B\ B'\ Par\ BC\ BC'$ 
    by (metis Ar2-def Par-cases Pj-def  $\langle Ar2\ PO\ E\ E'\ B\ C\ BC \rangle$ 
       $\langle BC\ BC'\ Proj\ PO\ E'\ E\ E' \rangle \langle BC \neq PO \rangle \langle Col\ PO\ E'\ B' \rangle$ 
       $\langle E\ E'\ Pj\ B\ B' \rangle \langle E\ E'\ Pj\ BC\ BC' \rangle \langle \neg Col\ PO\ E' B \rangle$ 
      par-trans proj-id)
qed
ultimately show ?thesis
  using postulate-of-transitivity-of-parallelism-def
    postulate-of-transitivity-of-parallelism-thm
  by blast
qed
thus ?thesis
  by (simp add: Pj-def)
qed
ultimately show ?thesis
  using Ar2-def Prod-def assms(1,2,3) by auto
qed
}
qed
qed
qed

```

```

lemma prod-assoc:
  assumes Prod  $PO\ E\ E'\ A\ B\ AB$ 
    and Prod  $PO\ E\ E'\ B\ C\ BC$ 
  shows Prod  $PO\ E\ E'\ A\ BC\ ABC \longleftrightarrow Prod\ PO\ E\ E'\ AB\ C\ ABC$ 
  by (meson assms(1,2) prod-assoc1 prod-assoc2)

```

```

lemma prod-comm:
  assumes Prod  $PO\ E\ E'\ A\ B\ C$ 
  shows Prod  $PO\ E\ E'\ B\ A\ C$ 
proof (cases  $A = PO$ )
  have Ar2  $PO\ E\ E'\ A\ B\ C$ 
    using Prod-def assms by blast
  thus  $A = PO \implies Prod\ PO\ E\ E'\ B\ A\ C$ 
    by (metis Ar2-def assms prod-0-l
      prod-0-r prod-uniqueness)
  show  $A \neq PO \implies Prod\ PO\ E\ E'\ B\ A\ C$ 
proof (cases  $B = PO$ )
  show  $A \neq PO \implies B = PO \implies Prod\ PO\ E\ E'\ B\ A\ C$ 
    by (metis Ar2-def  $\langle Ar2\ PO\ E\ E'\ A\ B\ C \rangle$  assms prod-0-l
      prod-0-r prod-uniqueness)
  show  $A \neq PO \implies B \neq PO \implies Prod\ PO\ E\ E'\ B\ A\ C$ 
proof (cases  $C = PO$ )
  show  $A \neq PO \implies B \neq PO \implies C = PO \implies Prod\ PO\ E\ E'\ B\ A\ C$ 
    using assms prod-null by blast

```

```

{
  assume  $A \neq PO$  and  $B \neq PO$  and  $C \neq PO$ 
  obtain  $B'$  where  $E E' Pj B B'$  and  $Col PO E' B'$  and  $E' A Pj B' C$ 
    using Prod-def assms by blast
  have  $\neg PO E Par E E'$ 
    using Ar2-def  $\langle Ar2 PO E E' A B C \rangle$  grid-not-par by blast
  have  $\neg PO E Par PO E'$ 
    using Ar2-def  $\langle Ar2 PO E E' A B C \rangle$  par-id by blast
  have  $\neg PO E' Par E E'$ 
    using Ar2-def  $\langle Ar2 PO E E' A B C \rangle$  grid-not-par by blast
  have  $PO \neq E$ 
    using Ar2-def  $\langle Ar2 PO E E' A B C \rangle$  grid-not-par by blast
  hence  $PO \neq E'$ 
    using Par-cases  $\langle \neg PO E Par E E' \rangle$  par-reflexivity by blast
  hence  $E \neq E'$ 
    using  $\langle \neg PO E Par PO E' \rangle$  par-reflexivity by blast
  then obtain  $A'$  where  $A A' Proj PO E' E E'$ 
    using  $\langle PO \neq E' \rangle \langle \neg PO E' Par E E' \rangle$ 
      par-symmetry project-existence by blast
  hence  $A A' Par E E' \vee A = A'$ 
    using project-par-dir by blast
  moreover have  $A = A' \implies A A' Par E E'$ 
    using Ar2-def  $\langle A A' Proj PO E' E E' \rangle \langle A \neq PO \rangle \langle Ar2 PO E E' A B C \rangle$  proj-id by blast
  ultimately have  $A A' Par E E'$ 
    by blast
  have  $E E' Pj A A'$ 
    using  $\langle A A' Proj PO E' E E' \rangle$  project-pj by auto
  moreover have  $Col PO E' A'$ 
    using Proj-def  $\langle A A' Proj PO E' E E' \rangle$  by presburger
  moreover
  have  $C A' Par B E'$ 
  proof (rule l13-11 [of  $PO$ ], insert  $\langle A \neq PO \rangle \langle B \neq PO \rangle$ )
    show  $\neg Col PO C E'$ 
      by (metis Ar2-def Col-cases  $\langle Ar2 PO E E' A B C \rangle$ 
         $\langle C \neq PO \rangle$  col3 col-trivial-3)
    show  $Col PO C B$ 
      by (metis Ar2-def Col-cases  $\langle Ar2 PO E E' A B C \rangle$ 
        col-transitivity-1 col-trivial-3)
    show  $Col PO B A$ 
      by (metis Ar2-def Col-cases  $\langle Ar2 PO E E' A B C \rangle$ 
        col-transitivity-1 col-trivial-3)
    show  $A' \neq PO$ 
      using Ar2-def  $\langle A \neq PO \rangle \langle Ar2 PO E E' A B C \rangle$ 
        calculation(1) sum-par-strict-a by blast
    show  $B' \neq PO$ 
      by (metis Ar2-def Col-cases Par-cases Par-def Pj-def
         $\langle Ar2 PO E E' A B C \rangle \langle B \neq PO \rangle \langle E E' Pj B B' \rangle$ 
        par-strict-not-col-1)
    show  $Col PO E' A'$ 
      using calculation(2) by auto
    show  $Col PO A' B'$ 
      using  $\langle Col PO E' B' \rangle \langle PO \neq E' \rangle$  calculation(2)
        col-transitivity-1 by blast
    show  $B B' Par A A'$ 
      by (metis Ar2-def Col-cases Par-cases Pj-def
         $\langle A A' Par E E' \rangle \langle Ar2 PO E E' A B C \rangle \langle B \neq PO \rangle$ 
         $\langle Col PO E' B' \rangle \langle E E' Pj B B' \rangle$  col-transitivity-1
        not-par-one-not-par)
    show  $A E' Par C B'$ 
      by (metis NCol-cases Par-cases Pj-def  $\langle Col PO E' B' \rangle$ 
         $\langle E' A Pj B' C \rangle \langle \neg Col PO C E' \rangle$ )
  qed
  hence  $E' B Pj A' C$ 
    using Par-cases Pj-def by auto
  ultimately show Prod  $PO E E' B A C$ 
    using Ar2-def Prod-def  $\langle Ar2 PO E E' A B C \rangle$  by auto

```

```

    }
  qed
qed
qed

```

```

lemma prod-O-l-eq:
  assumes Prod PO E E' PO B C
  shows C = PO
  by (metis Ar2-def Prod-def assms prod-0-l prod-uniqueness)

```

```

lemma prod-O-r-eq:
  assumes Prod PO E E' A PO C
  shows C = PO
  using assms prod-O-l-eq prod-comm by blast

```

```

lemma prod-uniquenessA:
  assumes B ≠ PO
    and Prod PO E E' A B C
    and Prod PO E E' A' B C
  shows A = A'
proof (cases A' = PO)
  show A' = PO ⇒ A = A'
    using assms(1,2,3) prod-O-l-eq prod-null by blast
  {
    assume A' ≠ PO
    obtain B' where E E' Pj B B' and Col PO E' B' and E' A Pj B' C
      using Prod-def assms(2) by blast
    obtain B'' where E E' Pj B B'' and Col PO E' B'' and E' A' Pj B'' C
      using Prod-def assms(3) by blast
    hence B' = B''
      using Ar2-def Prod-def ⟨Col PO E' B'⟩ ⟨E E' Pj B B'⟩
        assms(3) pj-uniqueness by auto
    {
      assume E' A Par B' C and E' A' Par B' C
      hence E' A' Par E' A
        using not-par-one-not-par by blast
      have E' A' ParStrict E' A ⇒ A = A'
        by (simp add: not-par-strict-id)
      hence A = A'
        by (metis Ar2-def Prod-def ⟨E' A' Par E' A⟩ assms(2,3) colx par-id-4)
    }
    thus A = A'
      by (metis Ar2-def NCol-perm Pj-def Prod-def ⟨B' = B''⟩
        ⟨Col PO E' B'⟩ ⟨E' A Pj B' C⟩ ⟨E' A' Pj B'' C⟩
        assms(1,2,3) col-transitivity-2 prod-null)
  }
}
qed

```

```

lemma prod-uniquenessB:
  assumes A ≠ PO
    and Prod PO E E' A B C
    and Prod PO E E' A B' C
  shows B = B'
  by (meson assms(1,2,3) prod-comm prod-uniquenessA)

```

```

lemma distr-l:
  assumes Sum PO E E' B C D
    and Prod PO E E' A B AB
    and Prod PO E E' A C AC
    and Prod PO E E' A D AD
  shows Sum PO E E' AB AC AD
proof (cases A = PO)

```

```

obtain  $B' C1$  where  $E E' Pj B B'$  and  $Col PO E' B'$  and  $PO E Pj B' C1$  and
 $PO E' Pj C C1$  and  $E' E Pj C1 D$ 
using Sum-def assms(1) by blast
obtain  $B''$  where  $E E' Pj B B''$  and  $Col PO E' B''$  and  $E' A Pj B'' AB$ 
using Prod-def assms(2) by blast
have  $\neg Col PO E E'$ 
using Ar2-def assms(1) sum-ar2 by presburger
have  $Col PO E A$ 
using Ar2-def Prod-def assms(3) by auto
have  $Col PO E B$ 
using Ar2-def assms(1) sum-ar2 by presburger
have  $PO \neq E$ 
using  $\langle \neg Col PO E E' \rangle$  grid-not-par-4 by auto
have  $PO \neq E'$ 
using  $\langle \neg Col PO E E' \rangle$  grid-not-par-5 by presburger
have  $B' = B''$ 
using  $\langle Col PO E B \rangle \langle Col PO E' B'' \rangle \langle Col PO E' B' \rangle$ 
 $\langle E E' Pj B B'' \rangle \langle E E' Pj B B' \rangle \langle \neg Col PO E E' \rangle$ 
pj-uniqueness by blast
obtain  $C'$  where  $E E' Pj C C'$  and  $Col PO E' C'$  and  $E' A Pj C' AC$ 
using Prod-def assms(3) by blast
obtain  $D'$  where  $E E' Pj D D'$  and  $Col PO E' D'$  and  $E' A Pj D' AD$ 
using Prod-def assms(4) by blast
have  $Sum PO E' E B' C' D'$ 
using Ar1-def  $\langle Col PO E' B' \rangle \langle Col PO E' C' \rangle$ 
 $\langle Col PO E' D' \rangle \langle E E' Pj B B' \rangle \langle E E' Pj C C' \rangle$ 
 $\langle E E' Pj D D' \rangle \langle PO \neq E' \rangle \langle \neg Col PO E E' \rangle$  assms(1)
proj-preserves-sum by force
show  $A = PO \implies Sum PO E E' AB AC AD$ 
by (metis  $\langle \neg Col PO E E' \rangle$  assms(2,3,4) prod-O-l-eq sum-O-O)
{
assume  $A \neq PO$ 
have  $Sum PO E' A B' C' D'$ 
by (metis  $\langle A \neq PO \rangle \langle Col PO E A \rangle \langle Sum PO E' E B' C' D' \rangle$ 
 $\langle \neg Col PO E E' \rangle$  col-trivial-3 colx
not-col-permutation-5 sum-y-axis-change)
have  $Sum PO A E' AB AC AD$ 
proof (rule proj-preserves-sum [of - -  $B' C' D'$ ], insert  $\langle E' A Pj C' AC \rangle$ 
 $\langle E' A Pj D' AD \rangle \langle Sum PO E' A B' C' D' \rangle$ )
show  $\neg Col PO E' A$ 
using Col-perm  $\langle A \neq PO \rangle \langle Col PO E A \rangle \langle \neg Col PO E E' \rangle$  col-trivial-3 colx by blast
show Ar1  $PO A AB AC AD$ 
using Ar1-def Ar2-def Prod-def  $\langle A \neq PO \rangle \langle PO \neq E \rangle$  assms(2,3,4)
col-transitivity-1 by presburger
show  $E' A Pj B' AB$ 
by (simp add:  $\langle B' = B'' \rangle \langle E' A Pj B'' AB \rangle$ )
qed
thus  $Sum PO E E' AB AC AD$ 
using  $\langle Col PO E A \rangle \langle PO \neq E \rangle$  not-col-permutation-5 sum-x-axis-unit-change by blast
}
qed

```

```

lemma distr-r:
assumes  $Sum PO E E' A B D$ 
and  $Prod PO E E' A C AC$ 
and  $Prod PO E E' B C BC$ 
and  $Prod PO E E' D C DC$ 
shows  $Sum PO E E' AC BC DC$ 
using assms(1,2,3,4) distr-l prod-comm by blast

```

```

lemma prod-1-l-eq:
  assumes Prod PO E E' A B B
  shows  $A = E \vee B = PO$ 
  by (meson Ar2-def Prod-def assms prod-1-r prod-comm prod-uniquenessB)

lemma prod-1-r-eq:
  assumes Prod PO E E' A B A
  shows  $B = E \vee A = PO$ 
  using assms prod-1-l-eq prod-comm by blast

lemma change-grid-prod-l-O:
  assumes PO E ParStrict O' E'
    and Ar1 PO E PO B C
    and Ar1 O' E' A' B' C'

    and PO O' Pj PO A'

    and PO O' Pj C C'
    and Prod PO E E' PO B C
  shows Prod O' E' E A' B' C'
  by (smt (verit) Ar1-def NCol-cases Pj-def
      assms(1,2,3,4,5,6) grid-not-par not-par-not-col
      par-strict-not-col-1 par-strict-par
      parallel-uniqueness prod-0-r prod-O-l-eq
      prod-comm)

lemma change-grid-prod1:
  assumes PO E ParStrict O' E'
    and Ar1 PO E E B C
    and Ar1 O' E' A' B' C'
    and PO O' Pj E E'
    and PO O' Pj E A'
    and PO O' Pj B B'
    and PO O' Pj C C'
    and Prod PO E E' E B C
  shows Prod O' E' E A' B' C'
  proof (cases B = PO)
    show B = PO  $\implies$  Prod O' E' E A' B' C'
    by (smt (verit, ccfv-SIG) Ar1-def Par-strict-cases
        Pj-def assms(1,3,6,7,8) grid-not-par
        not-col-permutation-2 not-par-not-col
        par-strict-not-col-1 parallel-uniqueness prod-0-r
        prod-O-r-eq)
    show B  $\neq$  PO  $\implies$  Prod O' E' E A' B' C'
  proof (cases C = PO)
    show B  $\neq$  PO  $\implies$  C = PO  $\implies$  Prod O' E' E A' B' C'
    by (metis Ar1-def assms(2,8) prod-null)
    {
      assume B  $\neq$  PO and C  $\neq$  PO
      have Prod PO E E' E B B
        using Ar1-def assms(1,2) par-strict-not-col-4 prod-1-l by presburger
      have B = C
        using  $\langle \text{Prod PO E E' E B B} \rangle$  assms(1,8) par-strict-not-col-4 prod-uniqueness by blast
      have A' = E'
      proof (rule l6-21 [of E E' O' E'])
        show  $\neg \text{Col E E' O'}$ 
        using Col-cases assms(1) par-strict-not-col-2 by blast
        show O'  $\neq$  E'
        using assms(1) par-strict-distinct by blast
        show Col E E' A'
        by (metis Col-cases Pj-def assms(4,5) col-trivial-3 parallel-uniqueness)
        show Col E E' E'
        by (simp add: col-trivial-2)
        show Col O' E' A'
        using Ar1-def assms(3) by presburger
      }
    }
  end
end

```

```

    show Col O' E' E'
      using not-col-distincts by auto
qed
have C' = B'
proof (rule l6-21 [of B' B O' E'])
  show  $\neg$  Col B' B O'
    by (metis Ar1-def Par-def Pj-def  $\langle B = C \rangle \langle B \neq PO \rangle$ 
        assms(1,2,3,6) l6-16-1 not-col-permutation-3
        not-col-permutation-4 not-par-strict-id par-not-col
        pj-comm)
  show O'  $\neq$  E'
    using assms(1) par-strict-distinct by blast
  show Col B' B C'
    by (metis Col-cases Pj-def  $\langle B = C \rangle$  assms(6,7) col-trivial-3 parallel-uniqueness)
  show Col B' B B'
    using grid-not-par-5 by blast
  show Col O' E' C'
    using Ar1-def assms(3) by auto
  show Col O' E' B'
    using Ar1-def assms(3) by auto
qed
thus Prod O' E' E A' B' C'
  by (metis Ar1-def Col-cases  $\langle A' = E' \rangle$  assms(1,3) par-strict-not-col-2 prod-1-l)
}
qed
qed

lemma change-grid-prod:
  assumes PO E ParStrict O' E'
    and Ar1 PO E A B C
    and Ar1 O' E' A' B' C'
    and PO O' Pj E E'
    and PO O' Pj A A'
    and PO O' Pj B B'
    and PO O' Pj C C'
    and Prod PO E E' A B C
  shows Prod O' E' E A' B' C'
proof (cases A = PO)
  show A = PO  $\implies$  Prod O' E' E A' B' C'
    using assms(1,2,3,5,7,8) change-grid-prod-l-O by blast
  show A  $\neq$  PO  $\implies$  Prod O' E' E A' B' C'
proof (cases B = PO)
  show A  $\neq$  PO  $\implies$  B = PO  $\implies$  Prod O' E' E A' B' C'
    by (smt (verit) Ar1-def Col-cases Par-strict-cases
        Pj-def assms(1,3,6,7,8) grid-not-par not-par-not-col
        par-strict-not-col-1 parallel-uniqueness prod-0-r
        prod-O-r-eq)
  show A  $\neq$  PO  $\implies$  B  $\neq$  PO  $\implies$  Prod O' E' E A' B' C'
proof (cases C = PO)
  show A  $\neq$  PO  $\implies$  B  $\neq$  PO  $\implies$  C = PO  $\implies$  Prod O' E' E A' B' C'
    using assms(8) prod-null by blast
  show A  $\neq$  PO  $\implies$  B  $\neq$  PO  $\implies$  C  $\neq$  PO  $\implies$  Prod O' E' E A' B' C'
proof (cases A = E)
  show A  $\neq$  PO  $\implies$  B  $\neq$  PO  $\implies$  C  $\neq$  PO  $\implies$  A = E  $\implies$  Prod O' E' E A' B' C'
    using assms(1,2,3,4,5,6,7,8) change-grid-prod1 by blast
  {
    assume A  $\neq$  PO and B  $\neq$  PO and C  $\neq$  PO and A  $\neq$  E
    obtain E'' where Bet PO O' E'' and Cong O' E'' PO O'
      using segment-construction by blast
    hence E''  $\neq$  PO
      using assms(1) bet-neq12--neq not-par-strict-id by blast
    have  $\neg$  Col PO E E''
      by (metis Col-cases  $\langle E'' \neq PO \rangle \langle PO - O' - E'' \rangle$ 
          assms(1) bet-col col2--eq par-strict-not-col-1)
    have Prod PO E E'' A B C
      using  $\langle \neg$  Col PO E E''  $\rangle$  assms(8) prod-y-axis-change by blast
  }

```

then obtain B'' where $E E'' Pj B B''$ and $Col PO E'' B''$ and $E'' A Pj B'' C$
 using *Prod-def* by *blast*
 obtain $C2$ where $Sum PO E'' E E'' O' C2$
 by (*metis Col-cases Col-def* $\langle PO - O' - E'' \rangle \langle \neg Col PO E E'' \rangle$
col-trivial-1 sum-exists)
 have $\neg Col PO E'' A$
 by (*metis Ar1-def Col-cases* $\langle A \neq PO \rangle \langle \neg Col PO E E'' \rangle$ *assms(2) col-trivial-3 colx*)
 have $Sum PO E'' A E'' O' C2$
 using $\langle Sum PO E'' E E'' O' C2 \rangle \langle \neg Col PO E'' A \rangle$ *sum-y-axis-change* by *blast*
 obtain $A0 A0'$ where $E'' A Pj E'' A0$ and $Col PO A A0$ and
 $PO E'' Pj A0 A0'$ and $PO A Pj O' A0'$ and $A E'' Pj A0' C2$
 using *Sum-def* $\langle Sum PO E'' A E'' O' C2 \rangle$ by *blast*
 hence $A = A0$
 by (*metis Col-cases Pj-def* $\langle \neg Col PO E'' A \rangle$ *col2-eq grid-not-par*)
 have $PO O' Par E E'$
 using *Pj-def* *assms(1,4) grid-not-par par-strict-not-col-2* by *blast*
 have $PO O' Par A A'$
 by (*metis Ar1-def Pj-def* *assms(1,2,3,5) not-col-permutation-1 par-not-col*)
 have $PO O' Par B B'$
 by (*metis Ar1-def Pj-def* *assms(1,2,3,6) not-strict-par2*
par-strict-not-col-4 par-strict-par)
 have $PO O' Par C C'$
 by (*metis Ar1-def Pj-def* *assms(1,2,3,7) not-strict-par2*
par-strict-not-col-4 par-strict-par)
 have $PO \neq O'$
 using *assms(1) not-par-strict-id* by *blast*
 have $A0' = A'$
 proof (rule *l6-21* [of $O' E' A A'$])
 show $\neg Col O' E' A$
 by (*meson Ar1-def Col-cases* *assms(1,2) par-not-col*)
 show $A \neq A'$
 using $\langle PO O' Par A A' \rangle$ *par-distincts* by *auto*
 show $Col O' E' A0'$
 proof –
 {
 assume $PO A Par O' A0'$
 hence $O' E' Par O' A0'$
 by (*metis Ar1-def Col-cases Par-cases Par-def* *assms(1,2)*
col-par-par-col col-trivial-1 par-neq2)
 hence $Col O' E' A0'$
 using *grid-not-par* by *blast*
 }
 moreover have $O' = A0' \implies Col O' E' A0'$
 by (*simp add: col-trivial-3*)
 ultimately show *?thesis*
 using $\langle PO A Pj O' A0' \rangle$ *Pj-def* by *blast*
 qed
 show $Col O' E' A'$
 using *Ar1-def* *assms(3)* by *auto*
 show $Col A A' A0'$
 proof –
 have $PO E'' Par A A0' \implies Col A A' A0'$
 using *Col-def* *Par-cases* $\langle PO O' Par A A' \rangle \langle PO - O' - E'' \rangle$ *col-par-par-col* by *blast*
 moreover have $A = A0' \implies Col A A' A0'$
 using *col-trivial-3* by *blast*
 ultimately show *?thesis*
 using *Pj-def* $\langle A = A0 \rangle \langle PO E'' Pj A0 A0' \rangle$ by *blast*
 qed
 show $Col A A' A'$
 by (*simp add: col-trivial-2*)
 qed
 hence $PO E'' Par A A'$
 using *Pj-def* $\langle A = A0 \rangle \langle PO E'' Pj A0 A0' \rangle \langle PO O' Par A A' \rangle$ *par-distincts* by *blast*
 hence $PO A Par O' A'$
 by (*metis Par-cases Pj-def* $\langle A0' = A' \rangle \langle PO A Pj O' A0' \rangle \langle PO O' Par A A' \rangle$
 $\langle \neg Col PO E'' A \rangle$ *col-trivial-3 not-strict-par2*)


```

have A E'' Par A' C2
proof -
  have A' = C2  $\implies$  A E'' Par A' C2
  by (metis Ar2-def Par-cases  $\langle PO\ E''\ Par\ A\ A' \rangle$   $\langle Sum\ PO\ E''\ A\ E''\ O'\ C2 \rangle$ 
    col-trivial-3 not-strict-par2 sum-ar2)
  thus ?thesis
  using Pj-def  $\langle A\ E''\ Pj\ A0'\ C2 \rangle$   $\langle A0' = A' \rangle$  by blast
qed
obtain E0 E0' where E'' E Pj E'' E0 and Col PO E E0 and
  PO E'' Pj E0 E0' and PO E Pj O' E0' and E E'' Pj E0' C2
  using Sum-def  $\langle Sum\ PO\ E''\ E\ E''\ O'\ C2 \rangle$  by blast
hence E0 = E
  by (metis Pj-def  $\langle \neg\ Col\ PO\ E\ E'' \rangle$  col2-eq grid-not-par)
have E0' = E'
proof (rule l6-21 [of O' E' E E'])
  show  $\neg\ Col\ O'\ E'\ E$ 
  using Col-cases assms(1) par-strict-not-col-2 by blast
  show E  $\neq$  E'
  using  $\langle PO\ O'\ Par\ E\ E' \rangle$  par-distinct by auto
  show Col O' E' E0'
proof -
  have PO E Par O' E0'  $\implies$  Col O' E' E0'
  using Par-cases assms(1) par-id-3 par-not-par par-strict-par by blast
  moreover have O' = E0'  $\implies$  Col O' E' E0'
  using not-col-distincts by blast
  ultimately show ?thesis
  using Pj-def  $\langle PO\ E\ Pj\ O'\ E0' \rangle$  by blast
qed
show Col O' E' E'
  by (simp add: col-trivial-2)
show Col E E' E0'
  by (smt (verit, ccfv-threshold) Par-cases Pj-def
     $\langle E0 = E \rangle$   $\langle PO\ E''\ Pj\ E0\ E0' \rangle$   $\langle PO\ O'\ Par\ E\ E' \rangle$ 
     $\langle PO - O' - E'' \rangle$  bet-col col-permutation-4
    not-par-inter-exists par-col-par-2 par-id-5
    par-trans-implies-playfair playfair-s-postulate-def
    postulate-of-transitivity-of-parallelism-thm)
show Col E E' E'
  by (simp add: col-trivial-2)
qed
have E E'' Par E' C2
  by (metis Ar1-def Pj-def  $\langle A\ E''\ Par\ A'\ C2 \rangle$   $\langle E\ E''\ Pj\ E0'\ C2 \rangle$   $\langle E0' = E' \rangle$ 
     $\langle PO\ A\ Par\ O'\ A' \rangle$   $\langle \neg\ Col\ PO\ E''\ A \rangle$  assms(3) col-par-par-col
    col-permutation-5 par-symmetry)
have E E'' Par B B''
  by (metis Ar1-def Col-cases Pj-def  $\langle B \neq PO \rangle$   $\langle Col\ PO\ E\ E0 \rangle$   $\langle Col\ PO\ E''\ B'' \rangle$ 
     $\langle E\ E''\ Pj\ B\ B'' \rangle$   $\langle E0 = E \rangle$   $\langle \neg\ Col\ PO\ E\ E'' \rangle$  assms(2) colx)
obtain C3 where Sum PO E'' E B'' O' C3
  by (metis Col-def NCol-cases  $\langle Col\ PO\ E''\ B'' \rangle$   $\langle PO - O' - E'' \rangle$   $\langle \neg\ Col\ PO\ E\ E'' \rangle$ 
    sum-exists)
then obtain B0 B0' where E'' E Pj B'' B0 and Col PO E B0 and
  PO E'' Pj B0 B0' and PO E Pj O' B0' and E E'' Pj B0' C3
  using Sum-def by blast
have B''  $\neq$  PO
  by (metis Ar1-def Col-cases Par-cases Par-def  $\langle E\ E''\ Par\ B\ B'' \rangle$   $\langle \neg\ Col\ PO\ E\ E'' \rangle$ 
    assms(2) par-strict-not-col-1)
have B0 = B
  by (meson Ar1-def  $\langle Col\ PO\ E\ B0 \rangle$   $\langle Col\ PO\ E''\ B'' \rangle$   $\langle E\ E''\ Pj\ B\ B'' \rangle$   $\langle E''\ E\ Pj\ B''\ B0 \rangle$ 
     $\langle \neg\ Col\ PO\ E\ E'' \rangle$  assms(2) col-permutation-5 pj-comm pj-uniqueness)
have B0' = B'
proof (rule l6-21 [of O' E' B B'])
  show  $\neg\ Col\ O'\ E'\ B$ 
  using Col-cases  $\langle B0 = B \rangle$   $\langle Col\ PO\ E\ B0 \rangle$  assms(1) par-not-col by blast
  show B  $\neq$  B'
  using  $\langle PO\ O'\ Par\ B\ B' \rangle$  par-distinct by auto
  show Col O' E' B0'

```

by (metis Par-cases Pj-def $\langle PO\ E\ Pj\ O'\ B0' \rangle$ assms(1) grid-not-par
 par-not-par par-strict-par)
 show $Col\ O'\ E'\ B'$
 proof –
 have $PO\ E''\ Par\ B\ B0' \implies Col\ O'\ E'\ B'$
 using Ar1-def assms(3) by auto
 moreover have $B = B0' \implies Col\ O'\ E'\ B'$
 using $\langle Col\ O'\ E'\ B0' \rangle \langle \neg\ Col\ O'\ E'\ B \rangle$ by auto
 ultimately show ?thesis
 using Ar1-def assms(3) by blast
 qed
 show $Col\ B\ B'\ B0'$
 proof –
 have $PO\ E''\ Par\ B\ B0' \implies Col\ B\ B'\ B0'$
 using Col-def Par-perm $\langle PO\ O'\ Par\ B\ B' \rangle \langle PO - O' - E'' \rangle$ col-par-par-col by blast
 moreover have $B = B0' \implies Col\ B\ B'\ B0'$
 using $\langle Col\ O'\ E'\ B0' \rangle \langle \neg\ Col\ O'\ E'\ B \rangle$ by auto
 ultimately show ?thesis
 using Pj-def $\langle B0 = B \rangle \langle PO\ E''\ Pj\ B0\ B0' \rangle$ by blast
 qed
 show $Col\ B\ B'\ B'$
 by (simp add: col-trivial-2)
 qed
 have $B' \neq O'$
 by (metis $\langle B \neq PO \rangle \langle B0 = B \rangle \langle Col\ PO\ E\ B0 \rangle \langle PO\ O'\ Par\ B\ B' \rangle$ assms(1)
 col2--eq col-permutation-4 grid-not-par par-strict-not-col-1)
 hence $E\ E''\ Par\ B'\ C3$
 by (smt (verit, ccfv-threshold) Ar2-def Col-def Pj-def $\langle B0' = B' \rangle$
 $\langle E\ E''\ Pj\ B0'\ C3 \rangle \langle PO\ E\ Pj\ O'\ B0' \rangle \langle PO - O' - E'' \rangle \langle Sum\ PO\ E''\ E\ B''\ O'\ C3 \rangle$
 assms(1) bet3--bet between-inner-transitivity col-trivial-3
 inter-uniqueness-not-par l5-3 par-strict-not-col-1 sum-ar2
 third-point)
 have $E''\ E\ Par\ B''\ B$
 using Par-perm $\langle E\ E''\ Par\ B\ B'' \rangle$ by blast
 have $Sum\ PO\ E''\ A\ B''\ O'\ C3$
 using $\langle Sum\ PO\ E''\ E\ B''\ O'\ C3 \rangle \langle \neg\ Col\ PO\ E''\ A \rangle$ sum-y-axis-change by blast
 then obtain $C0\ C0'$ where $E''\ A\ Pj\ B''\ C0$ and $Col\ PO\ A\ C0$ and
 $PO\ E''\ Pj\ C0\ C0'$ and $PO\ A\ Pj\ O'\ C0'$ and $A\ E''\ Pj\ C0'\ C3$
 using Sum-def by blast
 have $E''\ A\ Par\ B''\ C$
 by (metis Ar1-def Par-cases Pj-def $\langle E''\ A\ Pj\ B''\ C \rangle \langle E''\ E\ Par\ B''\ B \rangle$
 $\langle \neg\ Col\ PO\ E\ E'' \rangle$ assms(2) grid-not-par-1 par-col2-par-bis)
 have $C0 = C$
 proof (rule l6-21 [of $PO\ E\ B''\ C$])
 show $\neg\ Col\ PO\ E\ B''$
 using $\langle B'' \neq PO \rangle \langle Col\ PO\ E''\ B'' \rangle \langle \neg\ Col\ PO\ E\ E'' \rangle$ col2--eq
 col-permutation-4 by blast
 show $B'' \neq C$
 using $\langle E''\ A\ Par\ B''\ C \rangle$ par-distinct by blast
 show $Col\ PO\ E\ C0$
 by (metis Ar1-def $\langle A \neq PO \rangle \langle Col\ PO\ A\ C0 \rangle$ assms(2) col-trivial-3 colx)
 show $Col\ PO\ E\ C$
 using Ar1-def assms(2) by auto
 show $Col\ B''\ C\ C0$
 by (metis Pj-def $\langle E''\ A\ Par\ B''\ C \rangle \langle E''\ A\ Pj\ B''\ C0 \rangle$
 col-par-par-col grid-not-par-5 not-col-permutation-2 pj-comm)
 show $Col\ B''\ C\ C$
 using grid-not-par-6 by blast
 qed
 have $C0' = C'$
 proof (rule l6-21 [of $O'\ E'\ C\ C'$])
 show $\neg\ Col\ O'\ E'\ C$
 by (meson Ar1-def Col-cases assms(1,2) par-not-col)
 show $C \neq C'$
 using $\langle PO\ O'\ Par\ C\ C' \rangle$ par-distinct by auto
 show $Col\ O'\ E'\ C0'$

```

proof -
{
  assume PO A Par O' C0'
  have O' C' Par O' C0'
  proof -
    have PO E Par O' C'
    proof (rule par-col-par [of - - - E'])
      show O' ≠ C'
      by (metis Ar1-def Col-cases ⟨C ≠ PO⟩ ⟨PO O' Par C C'⟩ assms(1,2)
          col2--eq grid-not-par par-strict-not-col-1)
      show PO E Par O' E'
      by (simp add: Par-def assms(1))
      show Col O' E' C'
      using Ar1-def assms(3) by auto
    qed
    hence O' C' Par PO E
    using Par-cases by blast
    moreover have PO E Par O' C0'
    by (metis Ar1-def Col-cases ⟨PO A Par O' C0'⟩ assms(2) par-col-par-2)
    ultimately show ?thesis
    using par-not-par by blast
  qed
  hence Col O' E' C0'
  by (metis Ar1-def Col-cases Par-cases assms(3) not-par-strict-id
      par-not-col-strict)
}
moreover have O' = C0' ⟹ Col O' E' C0'
by (simp add: col-trivial-3)
ultimately show ?thesis
using Pj-def ⟨PO A Pj O' C0'⟩ by blast
qed
show Col O' E' C'
using Ar1-def assms(3) by auto
show Col C C' C0'
proof -
  have PO A Par O' C0' ⟹ Col C C' C0'
  by (smt (verit, ccfv-SIG) Pj-def ⟨A = A0⟩ ⟨C0 = C⟩
      ⟨PO E'' Pj C0 C0'⟩ ⟨PO O' Par C C'⟩ ⟨PO - O' - E''⟩ bet-col
      col-permutation-4 grid-not-par par-col-par-2
      parallel-uniqueness)
  moreover have O' = C0' ⟹ Col C C' C0'
  by (metis Pj-def ⟨C0 = C⟩ ⟨PO E'' Pj C0 C0'⟩ ⟨PO O' Par C C'⟩
      ⟨PO - O' - E''⟩ bet-col col-par-par-col grid-not-par-5
      not-col-permutation-4 par-comm)
  ultimately show ?thesis
  using Pj-def ⟨PO A Pj O' C0'⟩ by blast
qed
show Col C C' C'
by (simp add: col-trivial-2)
qed
have B B'' Par B' C3
using Par-cases ⟨E E'' Par B B''⟩ ⟨E E'' Par B' C3⟩ not-par-one-not-par by blast
have A E'' Par C' C3
proof -
{
  assume C' = C3
  have E'' E Par C' O'
  proof (rule par-col-par [of - - - B'])
    show C' ≠ O'
    by (metis Ar1-def Col-cases ⟨C ≠ PO⟩ ⟨PO O' Par C C'⟩ assms(1,2)
        col2--eq grid-not-par par-strict-not-col-1)
    show E'' E Par C' B'
    using Par-cases ⟨C' = C3⟩ ⟨E E'' Par B' C3⟩ by blast
    show Col C' B' O'
    by (metis Ar1-def Col-cases assms(3) col-transitivity-1)
  qed

```

```

    hence  $E E'' \text{ Par } O' C'$ 
      using Par-cases by auto
    hence  $E E'' \text{ Par } O' E'$ 
      by (metis Ar1-def Col-cases assms(3) par-col-par)
    hence  $E E'' \text{ Par } PO E$ 
      using assms(1) not-par-one-not-par par-strict-par by blast
    hence  $A E'' \text{ Par } C' C3$ 
      using Par-cases  $\langle \neg \text{ Col } PO E E'' \rangle$  par-id-1 by blast
  }
  thus ?thesis
    using Pj-def  $\langle A E'' \text{ Pj } C0' C3 \rangle \langle C0' = C' \rangle$  by blast
  hence  $C B'' \text{ Par } C' C3$ 
    by (metis Par-cases  $\langle E'' A \text{ Par } B'' C \rangle$  par-trans)
qed

have  $\text{Prod } O' E' C2 A' B' C'$ 
proof -
{
  assume  $\text{Col } O' E' C2$ 
  have  $C2 \neq O'$ 
    using  $\langle E'' \neq PO \rangle \langle \text{Sum } PO E'' A E'' O' C2 \rangle \langle \neg \text{ Col } PO E'' A \rangle$  sum-A-null by blast
  have  $\text{Col } PO O' C2$ 
    by (metis Ar2-def NCol-perm  $\langle E'' \neq PO \rangle \langle \text{Sum } PO E'' A E'' O' C2 \rangle$ 
      col-transitivity-2 sum-ar2)
  have  $\text{Col } PO O' E'$ 
    using  $\langle C2 \neq O' \rangle \langle \text{Col } O' E' C2 \rangle \langle \text{Col } PO O' C2 \rangle$  col2--eq col-permutation-4 by blast
  hence False
    using Col-cases assms(1) par-strict-not-col-3 by blast
}
moreover have  $E' C2 \text{ Pj } B' C3$ 
  by (metis Par-perm Pj-def  $\langle E E'' \text{ Par } B' C3 \rangle \langle E E'' \text{ Par } E' C2 \rangle$  par-not-par)
moreover have  $\text{Col } O' C2 C3$ 
  by (metis Ar2-def  $\langle E'' \neq PO \rangle \langle \text{Sum } PO E'' A E'' O' C2 \rangle$ 
     $\langle \text{Sum } PO E'' E B'' O' C3 \rangle$  col-transitivity-2 sum-ar2)
moreover have  $C2 A' \text{ Pj } C3 C'$ 
  by (metis Par-perm Pj-def  $\langle A E'' \text{ Par } A' C2 \rangle \langle A E'' \text{ Par } C' C3 \rangle$  par-trans)
ultimately show ?thesis
  using Ar1-def Ar2-def Prod-def assms(3) by auto
qed
thus  $\text{Prod } O' E' E A' B' C'$ 
  using assms(1) not-col-permutation-1 par-strict-not-col-2 prod-y-axis-change by blast
}
qed
qed
qed
qed

```

lemma *prod-sym*:

```

  assumes  $\text{Prod } PO E E' A B C$ 
  shows  $\text{Prod } PO E E' B A C$ 
  by (simp add: assms prod-comm)

```

lemma *l14-31-1*:

```

  assumes  $\text{Ar2p4 } PO E E' A B C D$ 
  and  $C \neq PO$ 
  and  $\exists X. \text{Prod } PO E E' A B X \wedge \text{Prod } PO E E' C D X$ 

```

```

shows Prod PO C E' A B D
proof (cases A = PO)
obtain X where Prod PO E E' A B X and Prod PO E E' C D X
  using assms(3) by blast
show A = PO  $\implies$  Prod PO C E' A B D
  by (metis Ar2-def Prod-def  $\langle$ Prod PO E E' A B X $\rangle$ 
     $\langle$ Prod PO E E' C D X $\rangle$  assms(2) col-transitivity-1 prod-0-l
    prod-1-r prod-0-l-eq prod-null)
show A  $\neq$  PO  $\implies$  Prod PO C E' A B D
proof (cases B = PO)
  show A  $\neq$  PO  $\implies$  B = PO  $\implies$  Prod PO C E' A B D
    by (metis Ar2-def Prod-def assms(2,3) col-transitivity-1 prod-0-r
      prod-1-r prod-0-r-eq prod-null)
  show A  $\neq$  PO  $\implies$  B  $\neq$  PO  $\implies$  Prod PO C E' A B D
  proof (cases D = PO)
    show A  $\neq$  PO  $\implies$  B  $\neq$  PO  $\implies$  D = PO  $\implies$  Prod PO C E' A B D
      using assms(3) prod-0-r-eq prod-null by blast
    {
      assume A  $\neq$  PO and B  $\neq$  PO and D  $\neq$  PO
      have  $\neg$  PO E Par E E'
        using Ar2-def Prod-def  $\langle$ Prod PO E E' A B X $\rangle$  grid-not-par by auto
      have  $\neg$  PO E Par PO E'
        using Ar2-def Prod-def  $\langle$ Prod PO E E' C D X $\rangle$  par-id by blast
      have PO  $\neq$  E
        using Ar2-def Prod-def assms(3) grid-not-par by blast
      hence PO  $\neq$  E'
        using Par-cases  $\langle$  $\neg$  PO E Par E E' $\rangle$  par-reflexivity by blast
      obtain B'' where B B'' Proj PO E' E' C
        by (metis Ar2-def Col-def Par-cases Prod-def assms(2,3) col2--eq
          grid-not-par project-existence)
      hence B B'' Par E' C
        by (metis Ar2-def Prod-def Proj-def  $\langle$ B  $\neq$  PO $\rangle$  assms(3) col-permutation-2
          col-transitivity-2)
      have C E' Par B B''
        using Par-cases  $\langle$ B B'' Par E' C $\rangle$  by blast
      hence C E' Pj B B''
        using Pj-def by blast
      moreover have Col PO E' B''
        using Proj-def  $\langle$ B B'' Proj PO E' E' C $\rangle$  by auto
      moreover have E' A Pj B'' D
      proof -
        obtain B' where E E' Pj B B' and Col PO E' B' and E' A Pj B' X
          using Prod-def  $\langle$ Prod PO E E' A B X $\rangle$  by blast
        obtain D' where E E' Pj D D' and Col PO E' D' and E' C Pj D' X
          using Prod-def  $\langle$ Prod PO E E' C D X $\rangle$  by blast
        have B'  $\neq$  PO
          by (metis Ar2-def Prod-def  $\langle$ B  $\neq$  PO $\rangle$   $\langle$ E E' Pj B B' $\rangle$  assms(3) sum-par-strict-a)
        have E B'' Par C B'
        proof (rule l13-11 [of PO - - B - E'], insert  $\langle$ C E' Par B B'' $\rangle$   $\langle$ B  $\neq$  PO $\rangle$  assms(2))
          show  $\neg$  Col PO E B'
            by (metis Col-cases Par-def  $\langle$ B'  $\neq$  PO $\rangle$   $\langle$ Col PO E' B' $\rangle$   $\langle$ PO  $\neq$  E' $\rangle$ 
               $\langle$ PO  $\neq$  E $\rangle$   $\langle$  $\neg$  PO E Par PO E' $\rangle$  col2--eq col-trivial-3)
          show Col PO E C
            using Ar2-def Prod-def assms(3) by auto
          show Col PO C B
            by (meson Ar2-def Prod-def  $\langle$ PO  $\neq$  E $\rangle$  assms(3) col3 col-trivial-3)
          show B''  $\neq$  PO
            by (metis Par-cases Proj-def  $\langle$ B B'' Proj PO E' E' C $\rangle$   $\langle$ B  $\neq$  PO $\rangle$ 
               $\langle$ Col PO C B $\rangle$  grid-not-par not-par-not-col par-trans)
          show Col PO B' B''
            using  $\langle$ Col PO E' B' $\rangle$   $\langle$ PO  $\neq$  E' $\rangle$  calculation(2) col-transitivity-1 by blast
          show Col PO B'' E'
            using Col-cases calculation(2) by blast
          show B B' Par E E'
            by (metis Par-cases Pj-def  $\langle$ B B'' Proj PO E' E' C $\rangle$   $\langle$ C E' Par B B'' $\rangle$ 
               $\langle$ Col PO E' B' $\rangle$   $\langle$ E E' Pj B B' $\rangle$  par-distinct project-not-col)
        qed
      qed
    }
  qed

```

```

show  $E' \neq PO$ 
  using  $\langle PO \neq E' \rangle$  by auto
qed
have  $X \neq PO$ 
  using  $\langle D \neq PO \rangle \langle \text{Prod } PO \ E \ E' \ C \ D \ X \rangle \text{assms}(2) \text{prod-null}$  by blast
have  $A \ D' \ \text{Par} \ C \ B'$ 
proof (rule l13-11 [of  $PO \ - \ - \ X \ - \ E'$ ], insert  $\langle X \neq PO \rangle \text{assms}(2)$ )
  show  $\neg \text{Col } PO \ A \ B'$ 
    by (metis Ar2-def Prod-def  $\langle A \neq PO \rangle \langle B' \neq PO \rangle \langle \text{Col } PO \ E' \ B' \rangle$ 
        assms(3) col-permutation-5 col-transitivity-1)
  show  $\text{Col } PO \ A \ C$ 
    by (meson Ar2-def Prod-def  $\langle PO \neq E \rangle \text{assms}(3) \text{col-transitivity-1}$ )
  show  $\text{Col } PO \ C \ X$ 
    by (meson Ar2-def Prod-def  $\langle PO \neq E \rangle \langle \text{Prod } PO \ E \ E' \ A \ B \ X \rangle \text{assms}(3)$ 
        col-transitivity-1)
  show  $D' \neq PO$ 
    by (metis Ar2-def Col-def Par-cases Par-def Pj-def Prod-def
         $\langle A \neq PO \rangle \langle B \neq PO \rangle \langle E \ E' \ Pj \ D \ D' \rangle \text{assms}(3) \text{par-strict-not-col-1}$ 
        prod-O-r-eq prod-null)
  show  $E' \neq PO$ 
    using  $\langle PO \neq E' \rangle$  by auto
  show  $\text{Col } PO \ B' \ D'$ 
    using  $\langle \text{Col } PO \ E' \ B' \rangle \langle \text{Col } PO \ E' \ D' \rangle \langle PO \neq E' \rangle \text{col-transitivity-1}$  by blast
  show  $\text{Col } PO \ D' \ E'$ 
    using Col-cases  $\langle \text{Col } PO \ E' \ D' \rangle$  by blast
  show  $C \ E' \ \text{Par} \ X \ D'$ 
    by (metis Pj-def  $\langle \text{Col } PO \ A \ C \rangle \langle \text{Col } PO \ C \ X \rangle \langle \text{Col } PO \ D' \ E' \rangle$ 
         $\langle \text{Col } PO \ E' \ B' \rangle \langle E' \ C \ Pj \ D' \ X \rangle \langle PO \neq E' \rangle \langle X \neq PO \rangle$ 
         $\langle \neg \text{Col } PO \ A \ B' \rangle \text{assms}(2) \text{col-trivial-3} \text{colx pj-comm}$ )
  show  $X \ B' \ \text{Par} \ A \ E'$ 
    by (metis Par-cases Pj-def  $\langle \text{Col } PO \ A \ C \rangle \langle \text{Col } PO \ C \ X \rangle \langle E' \ A \ Pj \ B' \ X \rangle$ 
         $\langle \neg \text{Col } PO \ A \ B' \rangle \text{assms}(2) \text{col-trivial-3} \text{colx}$ )
qed
have  $D \ B'' \ \text{Par} \ A \ E'$ 
proof (rule l13-11 [of  $PO \ - \ - \ E \ - \ D'$ ], insert  $\langle A \neq PO \rangle$ )
  show  $\neg \text{Col } PO \ D \ E'$ 
    by (metis Ar2-def Prod-def Proj-def  $\langle B \ B'' \ \text{Proj} \ PO \ E' \ E' \ C \rangle$ 
         $\langle D \neq PO \rangle \text{assms}(3) \text{col-par} \text{col-transitivity-1} \text{col-trivial-3}$ )
  show  $E \neq PO$ 
    using  $\langle PO \neq E \rangle$  by auto
  show  $\text{Col } PO \ D \ A$ 
    by (metis Ar2-def Col-cases Prod-def assms(3) col-transitivity-1
        col-trivial-3)
  show  $\text{Col } PO \ A \ E$ 
    by (meson Col-cases Prodp-def assms(3) prod-to-prodp)
  show  $B'' \neq PO$ 
    by (smt (verit, ccfv-SIG) Ar2-def NCol-cases Par-def Prod-def
         $\langle B' \neq PO \rangle \langle \text{Col } PO \ E' \ B' \rangle \langle E \ B'' \ \text{Par} \ C \ B' \rangle \text{assms}(3) \text{colx}$ 
        par-strict-not-col-1)
  show  $D' \neq PO$ 
    by (metis Col-cases Par-cases Pj-def  $\langle A \neq PO \rangle \langle \text{Col } PO \ A \ E \rangle$ 
         $\langle \text{Col } PO \ D \ A \rangle \langle E \ E' \ Pj \ D \ D' \rangle \langle \neg \text{Col } PO \ D \ E' \rangle \text{col2--eq}$ 
        col-trivial-3 not-strict-par2)
  show  $\text{Col } PO \ E' \ B''$ 
    by (simp add: calculation(2))
  show  $\text{Col } PO \ B'' \ D'$ 
    using  $\langle \text{Col } PO \ E' \ D' \rangle \langle PO \neq E' \rangle \text{calculation}(2) \text{col-transitivity-1}$ 
    by blast
  show  $A \ D' \ \text{Par} \ E \ B''$ 
    using  $\langle A \ D' \ \text{Par} \ C \ B' \rangle \langle E \ B'' \ \text{Par} \ C \ B' \rangle \text{not-par-one-not-par}$  by blast
  show  $E \ E' \ \text{Par} \ D \ D'$ 
    using Col-cases Pj-def  $\langle \text{Col } PO \ E' \ D' \rangle \langle E \ E' \ Pj \ D \ D' \rangle \langle \neg \text{Col } PO \ D \ E' \rangle$  by blast
qed
thus ?thesis
  using Par-perm Pj-def by blast
qed

```

```

ultimately show Prod PO C E' A B D
  by (smt (verit, ccfv-threshold) Ar2-def Prod-def Proj-def
    ⟨B B' Proj PO E' E' C⟩ ⟨PO ≠ E⟩ assms(3) col-par
    col-permutation-1 col-transitivity-2)
}
qed
qed
qed

lemma l14-31-2:
  assumes Ar2p4 PO E E' A B C D
    and C ≠ PO
    and Prod PO C E' A B D
  shows ∃ X. Prod PO E E' A B X ∧ Prod PO E E' C D X
proof (cases A = PO)
  show A = PO ⟹ ∃ X. Prod PO E E' A B X ∧ Prod PO E E' C D X
  by (metis Ar2p4-def assms(1,3) prod-0-l prod-0-r prod-O-l-eq)
  show A ≠ PO ⟹ ∃ X. Prod PO E E' A B X ∧ Prod PO E E' C D X
proof (cases B = PO)
  show A ≠ PO ⟹ B = PO ⟹ ∃ X. Prod PO E E' A B X ∧ Prod PO E E' C D X
  by (metis Ar2p4-def assms(1,3) prod-0-r prod-O-r-eq)
  show A ≠ PO ⟹ B ≠ PO ⟹ ∃ X. Prod PO E E' A B X ∧ Prod PO E E' C D X
proof (cases D = PO)
  show A ≠ PO ⟹ B ≠ PO ⟹ D = PO ⟹ ∃ X. Prod PO E E' A B X ∧ Prod PO E E' C D X
  using assms(3) prod-null by blast
  {
    assume A ≠ PO and B ≠ PO and D ≠ PO
    obtain B' where B B' Proj PO E' E E'
      by (metis Ar2p4-def Col-perm assms(1) grid-not-par project-existence)
    hence B B' Par E E'
      by (metis Ar2p4-def ⟨B ≠ PO⟩ assms(1) proj-id project-par-dir)
    obtain X where B' X Proj PO E E' A
      by (metis Ar2p4-def Par-cases assms(1) col-trivial-1 col-trivial-2
        not-strict-par1 project-existence)
    have X ≠ PO
      by (metis Ar2p4-def Proj-def ⟨A ≠ PO⟩ ⟨B B' Proj PO E' E E'⟩
        ⟨B ≠ PO⟩ ⟨B' X Proj PO E E' A⟩ assms(1) col-permutation-2
        not-col-distincts not-strict-par project-col-project
        project-id)
    have B' X Par E E' A
      by (metis Ar2p4-def Proj-def ⟨B B' Proj PO E' E E'⟩ ⟨B ≠ PO⟩
        ⟨B' X Proj PO E E' A⟩ assms(1) proj-id)
    obtain B'' where C E' Pj B B'' and Col PO E' B'' and E' A Pj B'' D
      using Prod-def assms(3) by blast
    have E E' Pj B B'
      using ⟨B B' Proj PO E' E E'⟩ project-pj by auto
    moreover have Col PO E' B'
      using Proj-def ⟨B B' Proj PO E' E E'⟩ by force
    moreover have E' A Pj B' X
      using ⟨B' X Proj PO E E' A⟩ project-pj by auto
    ultimately have Prod PO E E' A B X
      using Ar2-def Ar2p4-def Prod-def Proj-def ⟨B' X Proj PO E E' A⟩ assms(1) by auto
    obtain D' where D D' Proj PO E' E E'
      by (meson Ar2p4-def Prodp-def assms(1) prod-exists prod-to-prodp)
    hence D D' Par E E'
      by (metis Ar2p4-def ⟨D ≠ PO⟩ assms(1) proj-id project-par-dir)
    moreover have Prod PO E E' C D X
    proof -
      have E E' Pj D D'
        using ⟨D D' Proj PO E' E E'⟩ project-pj by auto
      moreover have Col PO E' D'
        using Proj-def ⟨D D' Proj PO E' E E'⟩ by presburger
      moreover have E' C Pj D' X
    proof -
      have E' C Par D' X
    proof -

```

```

have  $X \neq PO \text{ Par } B \neq B''$ 
proof (rule l13-11 [of  $PO \neq D \neq B'$ ])
  show  $\neg \text{Col } PO \neq X \neq B''$ 
    by (metis Ar2-def Col-cases Prod-def  $\langle B \neq PO \rangle \langle C \neq E' \text{ Pj } B \neq B'' \rangle$ 
       $\langle \text{Col } PO \neq E' \neq B'' \rangle \langle \text{Prod } PO \neq E' \neq A \neq B \neq X \rangle \langle X \neq PO \rangle$  assms(3)
      col-trivial-3 colx sum-par-strict-a)
  show  $B \neq PO$ 
    using  $\langle B \neq PO \rangle$  by auto
  show  $D \neq PO$ 
    by (simp add:  $\langle D \neq PO \rangle$ )
  show  $\text{Col } PO \neq X \neq B$ 
    by (metis Ar2p4-def Proj-def  $\langle B' \neq X \text{ Proj } PO \neq E' \neq A \rangle$  assms(1)
      col-transitivity-1)
  show  $\text{Col } PO \neq B \neq D$ 
    by (metis Ar2p4-def Proj-def  $\langle B' \neq X \text{ Proj } PO \neq E' \neq A \rangle$  assms(1)
      col-transitivity-1)
  show  $D' \neq PO$ 
    using Ar2p4-def  $\langle D \neq D' \text{ Proj } PO \neq E' \neq E' \rangle \langle D \neq PO \rangle$  assms(1)
      col-trivial-3 proj-id by auto
  show  $B' \neq PO$ 
    using  $\langle B' \neq X \text{ Proj } PO \neq E' \neq A \rangle \langle X \neq PO \rangle$  not-col-distincts project-not-col by blast
  show  $\text{Col } PO \neq B'' \neq D'$ 
    by (metis Proj-def  $\langle B \neq B' \text{ Proj } PO \neq E' \neq E' \rangle \langle \text{Col } PO \neq E' \neq B'' \rangle$ 
      calculation(2) col-transitivity-1)
  show  $\text{Col } PO \neq D' \neq B'$ 
    by (metis Proj-def  $\langle B \neq B' \text{ Proj } PO \neq E' \neq E' \rangle$  calculation(2) col-transitivity-1)
  show  $B \neq B' \text{ Par } D \neq D'$ 
    using  $\langle B \neq B' \text{ Par } E \neq E' \rangle \langle D \neq D' \text{ Par } E \neq E' \rangle$  not-par-one-not-par by blast
  show  $D \neq B'' \text{ Par } X \neq B'$ 
    by (metis Par-cases Pj-def Proj-def  $\langle B \neq B' \text{ Proj } PO \neq E' \neq E' \rangle$ 
       $\langle B' \neq X \text{ Par } E' \neq A \rangle \langle \text{Col } PO \neq E' \neq B'' \rangle \langle D \neq D' \text{ Par } E \neq E' \rangle \langle E' \neq A \text{ Pj } B'' \neq D \rangle$ 
      calculation(2) par-distinct par-not-par project-id)
qed
thus ?thesis
  by (metis Par-cases Pj-def  $\langle C \neq E' \text{ Pj } B \neq B'' \rangle$  not-par-one-not-par par-distinct)
qed
thus ?thesis
  using Pj-def by blast
qed
ultimately show ?thesis
  using Ar2-def Ar2p4-def Prod-def Proj-def  $\langle B' \neq X \text{ Proj } PO \neq E' \neq A \rangle$  assms(1) by auto
qed
ultimately show  $\exists X. \text{Prod } PO \neq E \neq E' \neq A \neq B \neq X \wedge \text{Prod } PO \neq E \neq E' \neq C \neq D \neq X$ 
  using  $\langle \text{Prod } PO \neq E \neq E' \neq A \neq B \neq X \rangle$  by blast
}
qed
qed
qed

```

lemma prod-x-axis-unit-change:

```

assumes Ar2p4  $PO \neq E \neq E' \neq A \neq B \neq C \neq D$ 
  and  $\text{Col } PO \neq E \neq U$ 
  and  $U \neq PO$ 
  and  $\exists X. \text{Prod } PO \neq E \neq E' \neq A \neq B \neq X \wedge \text{Prod } PO \neq E \neq E' \neq C \neq D \neq X$ 
shows  $\exists Y. \text{Prod } PO \neq U \neq E' \neq A \neq B \neq Y \wedge \text{Prod } PO \neq U \neq E' \neq C \neq D \neq Y$ 
proof (cases  $A = PO$ )
  obtain  $X$  where  $\text{Prod } PO \neq E \neq E' \neq A \neq B \neq X$  and  $\text{Prod } PO \neq E \neq E' \neq C \neq D \neq X$ 
    using assms(4) by auto
  {
    assume  $A = PO$ 
    hence  $X = PO$ 
    using  $\langle \text{Prod } PO \neq E \neq E' \neq A \neq B \neq X \rangle$  prod-O-l-eq by blast
    have  $C = PO \vee D = PO$ 
    using  $\langle \text{Prod } PO \neq E \neq E' \neq C \neq D \neq X \rangle \langle X = PO \rangle$  prod-null by blast
    moreover {
      assume  $C = PO$ 

```



```

have Prod PO U E' PO B PO
  by (metis Ar2-def Prod-def ⟨Prod PO E E' A B X⟩ assms(2,3) col-transitivity-1
      grid-not-par prod-0-l)
moreover have Prod PO U E' PO D PO
  by (metis Ar2-def Prod-def ⟨Prod PO E E' C D X⟩ assms(2) calculation
      col-transitivity-1 grid-not-par prod-0-l)
ultimately have  $\exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
  using ⟨A = PO⟩ ⟨C = PO⟩ by auto
}
moreover {
  assume D = PO
  have Prod PO U E' PO B PO
    by (metis Ar2-def Prod-def ⟨Prod PO E E' A B X⟩ assms(2,3) col-transitivity-1
        grid-not-par prod-0-l)
  moreover have Prod PO U E' C PO PO
    by (metis Ar2-def Prod-def ⟨Prod PO E E' C D X⟩ assms(2) calculation
        col-transitivity-1 grid-not-par prod-0-r)
  ultimately have  $\exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
    using ⟨A = PO⟩ ⟨D = PO⟩ by auto
}
ultimately show  $\exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
  by blast
}
show  $A \neq PO \implies \exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
proof (cases B = PO)
{
  assume A ≠ PO and B = PO
  have X = PO
    using ⟨B = PO⟩ ⟨Prod PO E E' A B X⟩ prod-O-r-eq by auto
  have C = PO  $\vee$  D = PO
    using ⟨Prod PO E E' C D X⟩ ⟨X = PO⟩ prod-null by auto
  moreover have C = PO  $\implies$  Prod PO U E' A B PO  $\wedge$  Prod PO U E' C D PO
    by (metis Ar2-def Prod-def ⟨B = PO⟩ ⟨Prod PO E E' A B X⟩ ⟨Prod PO E E' C D X⟩
        assms(2,3) col-transitivity-1 grid-not-par prod-0-l prod-0-r)
  moreover have D = PO  $\implies$  Prod PO U E' A B PO  $\wedge$  Prod PO U E' C D PO
    by (metis Ar2-def Prod-def ⟨B = PO⟩ ⟨Prod PO E E' A B X⟩ ⟨Prod PO E E' C D X⟩
        assms(2,3) col-transitivity-1 not-col-permutation-5 prod-0-r)
  ultimately show  $\exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
    by blast
}
show  $A \neq PO \implies B \neq PO \implies \exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
proof (cases C = PO)
show  $A \neq PO \implies B \neq PO \implies C = PO \implies \exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
  using assms(4) prod-O-l-eq prod-null by blast
show  $A \neq PO \implies B \neq PO \implies C \neq PO \implies \exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
proof (cases D = PO)
show  $A \neq PO \implies B \neq PO \implies C \neq PO \implies D = PO \implies$ 
 $\exists Y. \text{Prod PO U E' A B Y} \wedge \text{Prod PO U E' C D Y}$ 
  using assms(4) prod-O-r-eq prod-null by blast
{
  assume A ≠ PO and B ≠ PO and C ≠ PO and D ≠ PO
  obtain Bu where B Bu Proj PO E' U E'
    by (metis Ar2-def Col-def Prod-def ⟨Prod PO E E' A B X⟩ assms(2,3)
        col-transitivity-1 grid-not-par project-existence)
  hence B Bu Par U E'
    by (metis Par-def Prodp-def Proj-def ⟨B ≠ PO⟩ ⟨Prod PO E E' A B X⟩
        not-col-distincts prod-to-prodp proj-id)
  obtain Xu where Bu Xu Proj PO E A E'
    by (metis Prodp-def Proj-def ⟨Prod PO E E' A B X⟩ prod-to-prodp project-existence)
  hence Bu Xu Par A E'
    by (metis Ar2-def Prod-def Proj-def ⟨B Bu Proj PO E' U E'⟩ ⟨B ≠ PO⟩ assms(2,3,4)
        col-permutation-5 col-transitivity-1 proj-id)
  have Prod PO U E' A B Xu
    by (smt (verit) Prodp-def Proj-def ⟨B Bu Proj PO E' U E'⟩ ⟨Bu Xu Proj PO E A E'⟩
        assms(2,3,4) col2--eq col3 prod-to-prodp prodp-to-prod
        project-col-project)

```

```

moreover have  $\text{Prod } PO \ U \ E' \ C \ D \ Xu$ 
proof –
  obtain  $Du$  where  $D \ Du \ \text{Proj } PO \ E' \ U \ E'$ 
  using  $\text{Proj-def } \langle B \ Bu \ \text{Proj } PO \ E' \ U \ E' \rangle$  project-existence by presburger
  hence  $D \ Du \ \text{Par } U \ E'$ 
  by (metis Ar2-def Prod-def Proj-def  $\langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle \langle D \neq PO \rangle$  assms(2,4)
    calculation col-transitivity-1 proj-id)
  have  $E' \ C \ Pj \ Du \ Xu$ 
  proof –
    have  $\text{Prod } PO \ C \ E' \ A \ B \ D$ 
    using  $\langle C \neq PO \rangle \langle \text{Prod } PO \ E \ E' \ A \ B \ X \rangle \langle \text{Prod } PO \ E \ E' \ C \ D \ X \rangle$  assms(1) l14-31-1
    by blast
    obtain  $B'$  where  $E \ E' \ Pj \ B \ B'$  and  $\text{Col } PO \ E' \ B'$  and  $E' \ A \ Pj \ B' \ X$ 
    using  $\text{Prod-def } \langle \text{Prod } PO \ E \ E' \ A \ B \ X \rangle$  by auto
    obtain  $D'$  where  $E \ E' \ Pj \ D \ D'$  and  $\text{Col } PO \ E' \ D'$  and  $E' \ C \ Pj \ D' \ X$ 
    using  $\text{Prod-def } \langle \text{Prod } PO \ E \ E' \ C \ D \ X \rangle$  by auto
    obtain  $D''$  where  $C \ E' \ Pj \ B \ D''$  and  $\text{Col } PO \ E' \ D''$  and  $E' \ A \ Pj \ D'' \ D$ 
    using  $\text{Prod-def } \langle \text{Prod } PO \ C \ E' \ A \ B \ D \rangle$  by blast
    have  $Xu \neq PO$ 
    using  $\langle A \neq PO \rangle \langle B \neq PO \rangle$  calculation prod-null by blast
    have  $D'' \neq PO$ 
    by (metis Ar2-def Par-cases Par-def Pj-def Prod-def  $\langle B \neq PO \rangle \langle C \ E' \ Pj \ B \ D'' \rangle$ 
       $\langle \text{Prod } PO \ C \ E' \ A \ B \ D \rangle$  col-permutation-5 par-strict-not-col-1)
    have  $Xu \ Du \ \text{Par } B \ D''$ 
    proof (rule l13-11 [of PO - - - D - Bu], insert  $\langle D \neq PO \rangle \langle B \neq PO \rangle$ )
    show  $\neg \text{Col } PO \ Xu \ D''$ 
    by (metis Proj-def  $\langle B \ Bu \ \text{Proj } PO \ E' \ U \ E' \rangle \langle Bu \ Xu \ \text{Par } A \ E' \rangle$ 
       $\langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle \langle \text{Col } PO \ E' \ D'' \rangle$ 
       $\langle D'' \neq PO \rangle \langle Xu \neq PO \rangle$  col-permutation-5 col-transitivity-1
      par-distinct project-id)
    show  $\text{Col } PO \ Xu \ B$ 
    by (metis Ar2-def Prod-def Proj-def  $\langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle$  assms(4)
      col-transitivity-1)
    show  $\text{Col } PO \ B \ D$ 
    by (metis Ar2-def Prod-def Proj-def  $\langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle$  assms(4)
      col-transitivity-1)
    show  $Du \neq PO$ 
    by (metis Proj-def  $\langle B \ Bu \ \text{Proj } PO \ E' \ U \ E' \rangle \langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle \langle \text{Col } PO \ B \ D \rangle$ 
       $\langle D \ Du \ \text{Proj } PO \ E' \ U \ E' \rangle \langle D \neq PO \rangle \langle Xu \neq PO \rangle$  col2--eq col-permutation-3
      project-col-eq project-id)
    show  $Bu \neq PO$ 
    using  $\langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle \langle Xu \neq PO \rangle$  not-col-distincts project-id
    by blast
    show  $\text{Col } PO \ D'' \ Du$ 
    by (metis Proj-def  $\langle \text{Col } PO \ E' \ D'' \rangle \langle D \ Du \ \text{Proj } PO \ E' \ U \ E' \rangle$ 
      col3 not-col-distincts)
    show  $\text{Col } PO \ Du \ Bu$ 
    by (metis Proj-def  $\langle B \ Bu \ \text{Proj } PO \ E' \ U \ E' \rangle \langle D \ Du \ \text{Proj } PO \ E' \ U \ E' \rangle$  l6-16-1
      not-col-permutation-3 not-col-permutation-5)
    show  $B \ Bu \ \text{Par } D \ Du$ 
    using  $\langle B \ Bu \ \text{Par } U \ E' \rangle \langle D \ Du \ \text{Par } U \ E' \rangle$  not-par-one-not-par by blast
    show  $D \ D'' \ \text{Par } Xu \ Bu$ 
    by (metis Par-cases Pj-def  $\langle B \ Bu \ \text{Par } D \ Du \rangle \langle Bu \ Xu \ \text{Par } A \ E' \rangle \langle \text{Col } PO \ E' \ D'' \rangle$ 
       $\langle D \ Du \ \text{Proj } PO \ E' \ U \ E' \rangle \langle E' \ A \ Pj \ D'' \ D \rangle$  par-distinct par-not-par
      project-id)
  qed
  thus ?thesis
  by (metis Par-cases Pj-def  $\langle C \ E' \ Pj \ B \ D'' \rangle$  par-neq2 par-trans)
qed
moreover have  $U \ E' \ Pj \ D \ Du$ 
using  $\langle D \ Du \ \text{Proj } PO \ E' \ U \ E' \rangle$  project-pj by auto
moreover have  $\text{Col } PO \ E' \ Du$ 
using  $\text{Proj-def } \langle D \ Du \ \text{Proj } PO \ E' \ U \ E' \rangle$  by auto
moreover have  $\text{Ar2 } PO \ U \ E' \ C \ D \ Xu$ 
by (metis Ar2-def Prod-def Proj-def  $\langle Bu \ Xu \ \text{Proj } PO \ E \ A \ E' \rangle \langle \text{Prod } PO \ U \ E' \ A \ B \ Xu \rangle$ 
  assms(2,4) col-transitivity-1)

```

```

    ultimately show ?thesis
    using Prod-def by blast
  qed
  ultimately show  $\exists Y. \text{Prod } PO \ U \ E' \ A \ B \ Y \wedge \text{Prod } PO \ U \ E' \ C \ D \ Y$ 
  by blast
}
qed
qed
qed
qed

lemma opp-prod:
  assumes Opp PO E E' E ME
  and Opp PO E E' X MX
  shows Prod PO E E' X ME MX
proof -
  obtain EPME where Sum PO E E' E ME EPME
  using Opp-def assms(1) sum-opp by blast
  hence PO = EPME
  by (meson Opp-def assms(1) sum-opp sum-stable)
  obtain X' where Prod PO E E' X E X'
  by (meson Ar2-def Opp-def assms(2) prod-1-r sum-ar2)
  hence X = X'
  by (meson Ar2-def Prod-def prod-1-r prod-uniqueness)
  obtain O' where Prod PO E E' X PO O'
  by (meson Ar2-def Prodp-def  $\langle \text{Prod } PO \ E \ E' \ X \ E \ X' \rangle \langle \text{Sum } PO \ E \ E' \ E \ ME \ EPME \rangle$  prod-0-r
    prod-to-prodp sum-ar2)
  hence PO = O'
  using prod-O-r-eq by blast
  obtain MX' where Prod PO E E' X ME MX'
  by (metis Ar2-def Prodp-def  $\langle \text{Prod } PO \ E \ E' \ X \ E \ X' \rangle \langle \text{Sum } PO \ E \ E' \ E \ ME \ EPME \rangle$ 
    prod-exists prod-to-prodp sum-ar2)
  hence Sum PO E E' X MX' PO
  using  $\langle PO = EPME \rangle \langle PO = O' \rangle \langle \text{Prod } PO \ E \ E' \ X \ E \ X' \rangle \langle \text{Prod } PO \ E \ E' \ X \ PO \ O' \rangle$ 
     $\langle \text{Sum } PO \ E \ E' \ E \ ME \ EPME \rangle \langle X = X' \rangle$  distr-l by blast
  thus ?thesis
  by (metis  $\langle \text{Prod } PO \ E \ E' \ X \ ME \ MX' \rangle$  assms(2) opp-uniqueness sum3-col sum3-exists
    sum-ar2 sum-opp)
qed

lemma distr-l-diff:
  assumes Diff PO E E' B C BMC
  and Prod PO E E' A B AB
  and Prod PO E E' A C AC
  and Prod PO E E' A BMC ABMC
  shows Diff PO E E' AB AC ABMC
  by (meson assms(1,2,3,4) diff-sum distr-l sum-diff)

lemma diff-of-squares:
  assumes Prod PO E E' A A A2
  and Prod PO E E' B B B2
  and Diff PO E E' A2 B2 A2MB2
  and Sum PO E E' A B APB
  and Diff PO E E' A B AMB
  and Prod PO E E' APB AMB F
  shows A2MB2 = F
proof -
  obtain F1 where Prod PO E E' A AMB F1
  by (metis Ar2-def Prodp-def assms(4,6) prod-exists prod-to-prodp sum-ar2)
  moreover obtain F2 where Prod PO E E' B AMB F2
  by (meson Ar2-def Prod-def assms(2,6) prod-exists)
  ultimately obtain F' where Sum PO E E' F1 F2 F'
  using assms(4,6) distr-r by blast
  hence F = F'
  using  $\langle \text{Prod } PO \ E \ E' \ A \ AMB \ F1 \rangle \langle \text{Prod } PO \ E \ E' \ B \ AMB \ F2 \rangle$  assms(4,6) distr-r sum-stable
  by blast

```

obtain AB **where** $Prod\ PO\ E\ E'\ A\ B\ AB$
by (*meson* $Ar2-def\ assms(4)\ prod-exists\ sum-ar2$)
then obtain $A2MAB$ **where** $Diff\ PO\ E\ E'\ A2\ AB\ A2MAB$
by (*metis* $Ar2-def\ Prod-def\ assms(3)\ col-trivial-3\ diff-ar2\ sum-diff\ sum-A-exists$)
hence $A2MAB = F1$
using $\langle Prod\ PO\ E\ E'\ A\ AMB\ F1 \rangle \langle Prod\ PO\ E\ E'\ A\ B\ AB \rangle\ assms(1,5)\ diff-uniqueness$
distr-l-diff **by** *blast*
obtain BA **where** $Prod\ PO\ E\ E'\ B\ A\ BA$
using $\langle Prod\ PO\ E\ E'\ A\ B\ AB \rangle\ prod-sym$ **by** *blast*
obtain $BAMB2$ **where** $Diff\ PO\ E\ E'\ BA\ B2\ BAMB2$
using $Ar2-def\ Prod-def\ \langle Prod\ PO\ E\ E'\ B\ A\ BA \rangle\ assms(2)\ diff-exists$ **by** *presburger*
hence $BAMB2 = F2$
using $\langle Prod\ PO\ E\ E'\ B\ A\ BA \rangle \langle Prod\ PO\ E\ E'\ B\ AMB\ F2 \rangle\ assms(2,5)\ diff-uniqueness$
distr-l-diff **by** *blast*
have $BA = AB$
by (*meson* $Ar2-def\ \langle Prod\ PO\ E\ E'\ A\ B\ AB \rangle \langle Prod\ PO\ E\ E'\ B\ A\ BA \rangle \langle Sum\ PO\ E\ E'\ F1\ F2\ F' \rangle$
 $prod-sym\ prod-uniqueness\ sum-ar2$)
thus *?thesis*
by (*metis* $A2MAB = F1 \rangle \langle BAMB2 = F2 \rangle \langle Diff\ PO\ E\ E'\ A2\ AB\ A2MAB \rangle$
 $\langle Diff\ PO\ E\ E'\ BA\ B2\ BAMB2 \rangle \langle F = F' \rangle \langle Sum\ PO\ E\ E'\ F1\ F2\ F' \rangle\ assms(3)\ diff-stable$
 $sum-diff-diff-b$)

qed

lemma *eq-squares-eq-or-opp*:

assumes $Prod\ PO\ E\ E'\ A\ A\ A2$
and $Prod\ PO\ E\ E'\ B\ B\ A2$
shows $A = B \vee Opp\ PO\ E\ E'\ A\ B$

proof –

obtain O' **where** $Diff\ PO\ E\ E'\ A\ A\ O'$
by (*meson* $Ar2-def\ Prod-def\ assms(1)\ diff-null$)
hence $PO = O'$
by (*metis* $Ar2-def\ Prod-def\ assms(1)\ diff-sum\ sum-B-null$)
obtain APB **where** $Sum\ PO\ E\ E'\ A\ B\ APB$
by (*meson* $Ar2-def\ Prodp-def\ \langle Diff\ PO\ E\ E'\ A\ A\ O' \rangle\ assms(2)\ diff-sum\ prod-to-prodp$
 $sum-ar2\ sum-exists$)
then obtain AMB **where** $Diff\ PO\ E\ E'\ A\ B\ AMB$
by (*meson* $Ar2-def\ diff-exists\ sum-ar2$)
then obtain PO' **where** $Prod\ PO\ E\ E'\ APB\ AMB\ PO'$
by (*meson* $Ar2-def\ \langle Sum\ PO\ E\ E'\ A\ B\ APB \rangle\ diff-ar2\ prod-exists\ sum-ar2$)
hence $PO = PO'$
using $diff-of-squares\ [of\ PO\ E\ E'\ A\ A2\ B\ A2 - APB\ AMB]\ Ar2-def\ Prod-def$
 $\langle Diff\ PO\ E\ E'\ A\ B\ AMB \rangle \langle Sum\ PO\ E\ E'\ A\ B\ APB \rangle\ assms(1,2)$
 $diff-null$ **by** *force*
thus *?thesis*
using $\langle Diff\ PO\ E\ E'\ A\ B\ AMB \rangle \langle Prod\ PO\ E\ E'\ APB\ AMB\ PO' \rangle$
 $\langle Sum\ PO\ E\ E'\ A\ B\ APB \rangle\ diff-null-eq\ prod-null\ sum-opp$
by *blast*

qed

lemma *diff-2-prod*:

assumes $Opp\ PO\ E\ E'\ E\ ME$
and $Diff\ PO\ E\ E'\ A\ B\ AMB$
and $Diff\ PO\ E\ E'\ B\ A\ BMA$
shows $Prod\ PO\ E\ E'\ AMB\ ME\ BMA$
using $assms(1,2,3)\ diff-opp\ opp-prod$ **by** *blast*

lemma *l14-36-a*:

assumes $Sum\ PO\ E\ E'\ A\ B\ C$ **and**
 $PO\ Out\ A\ B$
shows $Bet\ PO\ A\ C$

proof (*cases* $A = B$)

have $ParallelogramFlat\ PO\ A\ C\ B$
by (*metis* $assms(1,2)\ between-trivial\ diff-to-plg\ l6-6\ sum-diff$
 $not-bet-and-out$)
hence $Parallelogram\ PO\ A\ C\ B$
by (*simp* $add: Parallelogram-def$)

show $A = B \implies (PO - A - C)$
by (*metis Out-def ParallelogramFlat-def* $\langle \text{ParallelogramFlat } PO \ A \ C \ B \rangle$
between-cong l6-4-2 plgf-permut)
show $A \neq B \implies (PO - A - C)$
by (*smt (z3) Ar2-def ParallelogramFlat-def* $\langle \text{ParallelogramFlat } PO \ A \ C \ B \rangle$
assms(1,2) bet-cong-eq between-trivial2 col-cong2-bet2 l6-4-2 l6-6
not-bet-and-out out-col outer-transitivity-between2 plgf-permut sum-ar2
sum-cong2 third-point)

qed

lemma *l14-36-b*:

assumes $\text{Sum } PO \ E \ E' \ A \ B \ C$
and $PO \text{ Out } A \ B$
shows $PO \neq A \wedge PO \neq C \wedge A \neq C$
by (*metis Ar2-def Out-def assms(1,2) between-equality-2 between-trivial2*
cong3-id l14-36-a out-col sum-ar2 sum-iff-cong-b)

lemma *pos-null-neg*:

assumes $\text{Opp } PO \ E \ E' \ A \ MA$
shows $P_s \ PO \ E \ A \vee PO = A \vee P_s \ PO \ E \ MA$
by (*metis Ar2-def Col-def Opp-def Ps-def assms between-trivial l14-36-b l6-3-2*
l6-4-2 sum-B-null sum-ar2)

lemma *sum-pos-pos*:

assumes $P_s \ PO \ E \ A$
and $P_s \ PO \ E \ B$
and $\text{Sum } PO \ E \ E' \ A \ B \ AB$
shows $P_s \ PO \ E \ AB$

proof –

have $PO \text{ Out } A \ B$
using *Out-cases Ps-def assms(1,2) l6-7* **by** *presburger*
hence $PO \neq A \wedge PO \neq AB \wedge A \neq AB$
using *assms(3) l14-36-b* **by** *blast*
thus *?thesis*
by (*metis Out-def Ps-def* $\langle PO \text{ Out } A \ B \rangle$ *assms(1,3) l14-36-a l6-7*)

qed

lemma *prod-pos-pos*:

assumes $P_s \ PO \ E \ A$
and $P_s \ PO \ E \ B$
and $\text{Prod } PO \ E \ E' \ A \ B \ AB$
shows $P_s \ PO \ E \ AB$

proof –

obtain B' **where** $E \ E' \ P_j \ B \ B'$ **and** $\text{Col } PO \ E' \ B'$ **and** $E' \ A \ P_j \ B' \ AB$
using *Prod-def assms(3)* **by** *auto*
have $E' \neq A$
using *Ar2-def Prod-def assms(3)* **by** *blast*
have $\neg \text{Col } PO \ E' \ A$
by (*metis Ar2-def Col-cases O-not-positive Prod-def assms(1,3) col-trivial-3 colx*)
hence $\neg PO \ E \ \text{Par } E' \ A$
by (*meson Par-def Prodp-def assms(3) par-strict-not-col-4 prod-to-prodp*)
have $E \ E' \ \text{Proj } PO \ E' \ E \ E'$
by (*metis Ar2-def Prod-def Proj-def assms(3) grid-not-par par-reflexivity*)
have $B \ B' \ \text{Proj } PO \ E' \ E \ E'$
by (*metis Proj-def* $\langle \text{Col } PO \ E' \ B' \rangle \langle E \ E' \ P_j \ B \ B' \rangle \langle E \ E' \ \text{Proj } PO \ E' \ E \ E' \rangle$ *not-col-permutation-4*
not-col-permutation-5 pj-col-project)
have $PO \ PO \ \text{Proj } PO \ E' \ E \ E'$
by (*metis Proj-def* $\langle E \ E' \ \text{Proj } PO \ E' \ E \ E' \rangle$ *not-col-distincts*)
have $E' \ A \ \text{Proj } PO \ E \ E' \ A$
by (*metis Par-def Prodp-def Proj-def* $\langle \neg PO \ E \ \text{Par } E' \ A \rangle$ *assms(3) not-col-distincts*
prod-to-prodp)
hence $B' \ AB \ \text{Proj } PO \ E \ E' \ A$
using *Ar2-def Pj-def Prod-def Proj-def* $\langle E' \ A \ P_j \ B' \ AB \rangle$ *assms(3) par-symmetry* **by** *metis*
hence $PO \ PO \ \text{Proj } PO \ E \ E' \ A$
by (*metis Proj-def not-col-distincts*)
have $AB \neq PO$

```

  using O-not-positive  $\langle \neg \text{Col } PO \ E' \ A \rangle$  assms(2,3) col-trivial-3 prod-null by blast
{
  assume Bet PO B E
  hence Bet PO B' E'
    using  $\langle B \ B' \ \text{Proj } PO \ E' \ E \ E' \rangle \langle E \ E' \ \text{Proj } PO \ E' \ E \ E' \rangle \langle PO \ PO \ \text{Proj } PO \ E' \ E \ E' \rangle$ 
      project-preserves-bet by blast
  hence Bet PO AB A
    using  $\langle B' \ AB \ \text{Proj } PO \ E \ E' \ A \rangle \langle E' \ A \ \text{Proj } PO \ E \ E' \ A \rangle \langle PO \ PO \ \text{Proj } PO \ E \ E' \ A \rangle$ 
      project-preserves-bet by blast
  hence Ps PO E AB
    using Ps-def  $\langle AB \neq PO \rangle$  assms(1) bet-out l6-7 by blast
}
moreover
{
  assume Bet PO E B
  hence Bet PO E' B'
    using  $\langle B \ B' \ \text{Proj } PO \ E' \ E \ E' \rangle \langle E \ E' \ \text{Proj } PO \ E' \ E \ E' \rangle \langle PO \ PO \ \text{Proj } PO \ E' \ E \ E' \rangle$ 
      project-preserves-bet
    by blast
  hence Bet PO A AB
    using  $\langle B' \ AB \ \text{Proj } PO \ E \ E' \ A \rangle \langle E' \ A \ \text{Proj } PO \ E \ E' \ A \rangle \langle PO \ PO \ \text{Proj } PO \ E \ E' \ A \rangle$ 
      project-preserves-bet
    by blast
  hence Ps PO E AB
    by (metis O-not-positive Out-cases Ps-def assms(1) bet-out l6-7)
}
ultimately show ?thesis
  using Out-def Ps-def assms(2) by auto
qed

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lemma pos-not-neg:
  assumes Ps PO E A
  shows  $\neg Ng \ PO \ E \ A$ 
  by (metis Ng-def Ps-def assms not-bet-and-out)

```

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lemma neg-not-pos:
  assumes Ng PO E A
  shows  $\neg Ps \ PO \ E \ A$ 
  using assms pos-not-neg by blast

```

```

lemma opp-pos-neg:
  assumes Ps PO E A
    and Opp PO E E' A MA
  shows Ng PO E MA
  by (metis (full-types) Bet-cases Ng-def Out-def Ps-def assms(1,2)
    bet-out--bet is-midpoint-id-2 midpoint-bet opp-midpoint)

```

```

lemma opp-neg-pos:
  assumes Ng PO E A
    and Opp PO E E' A MA
  shows Ps PO E MA
  by (metis Ng-def assms(1,2) neg-not-pos pos-null-neg)

```

```

lemma ltP-ar2:
  assumes LtP PO E E' A B
  shows Ar2 PO E E' A B A
  by (meson Ar2-def LtP-def assms diff-ar2)

```

```

lemma ltP-neg:
  assumes LtP PO E E' A B
  shows  $A \neq B$ 
  using LtP-def assms diff-opp opp-pos-neg pos-not-neg by blast

```

```

lemma leP-refl:
  shows LeP PO E E' A A
  using LeP-def by auto

```

```

lemma ltP-sum-pos:
  assumes Ps PO E B
  and Sum PO E E' A B C
  shows LtP PO E E' A C
  using LtP-def assms(1,2) local.sum-diff by blast

lemma pos-opp-neg:
  assumes Ps PO E A
  and Opp PO E E' A mA
  shows Ng PO E mA
  using assms(1,2) opp-pos-neg by auto

lemma diff-pos-diff-neg:
  assumes Diff PO E E' A B AmB
  and Diff PO E E' B A BmA
  and Ps PO E AmB
  shows Ng PO E BmA
  using assms(1,2,3) diff-opp pos-opp-neg by blast

lemma not-pos-and-neg:
  shows  $\neg(Ps PO E A \wedge Ng PO E A)$ 
  using neg-not-pos by auto

lemma leP-asym:
  assumes LeP PO E E' A B
  and LeP PO E E' B A
  shows A = B
proof -
  have LtP PO E E' A B  $\implies$  A = B
  by (metis LeP-def LtP-def assms(2) diff-pos-diff-neg pos-not-neg)
  moreover have LtP PO E E' B A  $\implies$  A = B
  using LeP-def assms(1) calculation by auto
  ultimately show ?thesis
  using LeP-def assms(1) by auto
qed

lemma leP-trans:
  assumes LeP PO E E' A B
  and LeP PO E E' B C
  shows LeP PO E E' A C
proof -
  {
    assume LtP PO E E' A B
    have LeP PO E E' A C
    proof -
      {
        assume LtP PO E E' B C
        obtain dBA where Diff PO E E' B A dBA and Ps PO E dBA
        using LtP-def  $\langle LtP PO E E' A B \rangle$  by auto
        obtain dCB where Diff PO E E' C B dCB and Ps PO E dCB
        using LtP-def  $\langle LtP PO E E' B C \rangle$  by auto
        have Ar2 PO E E' B A dBA
        by (simp add:  $\langle Diff PO E E' B A dBA \rangle$  diff-ar2)
        have Ar2 PO E E' C B dCB
        by (simp add:  $\langle Diff PO E E' C B dCB \rangle$  diff-ar2)
        then obtain dCA where Sum PO E E' dBA dCB dCA
        using Ar2-def  $\langle Ar2 PO E E' B A dBA \rangle$  sum-exists by blast
        hence LeP PO E E' A C
        by (meson LeP-def  $\langle Diff PO E E' B A dBA \rangle$   $\langle Diff PO E E' C B dCB \rangle$   $\langle Ps PO E dBA \rangle$ 
           $\langle Ps PO E dCB \rangle$  diff-sum ltP-sum-pos sum-assoc-2 sum-pos-pos)
      }
    moreover have B = C  $\implies$  LeP PO E E' A C
    using assms(1) by auto
    ultimately show ?thesis
    using LeP-def assms(2) by auto
  }

```

```

    qed
  }
  moreover have  $A = B \implies \text{LeP } PO \ E \ E' \ A \ C$ 
    by (simp add: assms(2))
  ultimately show ?thesis
    using LeP-def assms(1) by blast
qed

```

lemma leP-sum-leP:

```

  assumes LeP PO E E' A X
    and LeP PO E E' B Y
    and Sum PO E E' A B C
    and Sum PO E E' X Y Z
  shows LeP PO E E' C Z
proof -
  have Ar2 PO E E' A B C
    by (simp add: assms(3) sum-ar2)
  have Ar2 PO E E' X Y Z
    by (simp add: assms(4) sum-ar2)
  {
    assume LtP PO E E' A X
    have LeP PO E E' C Z
    proof -
      {
        assume LtP PO E E' B Y
        obtain dXA where Diff PO E E' X A dXA
          using LtP-def ⟨LtP PO E E' A X⟩ by auto
        obtain dYB where Diff PO E E' Y B dYB
          using LtP-def ⟨LtP PO E E' B Y⟩ by auto
        obtain dZC where Diff PO E E' Z C dZC
          using Ar2-def ⟨Ar2 PO E E' A B C⟩ ⟨Ar2 PO E E' X Y Z⟩ diff-exists
            by presburger
        have Ps PO E dZC
        proof -
          have Sum PO E E' dXA dYB dZC
            using ⟨Diff PO E E' X A dXA⟩ ⟨Diff PO E E' Y B dYB⟩ ⟨Diff PO E E' Z C dZC⟩
              assms(3,4) sum-diff2-diff-sum2-b by blast
          thus ?thesis
            by (metis LtP-def ⟨Diff PO E E' X A dXA⟩ ⟨Diff PO E E' Y B dYB⟩
              ⟨LtP PO E E' A X⟩ ⟨LtP PO E E' B Y⟩ diff-uniqueness sum-pos-pos)
        qed
        hence LeP PO E E' C Z
          using LeP-def LtP-def ⟨Diff PO E E' Z C dZC⟩ by blast
      }
    }
  moreover
  {
    assume B = Y
    hence LtP PO E E' C Z
      by (smt (verit, best) Ar2-def LtP-def ⟨Ar2 PO E E' X Y Z⟩ ⟨LtP PO E E' A X⟩
        assms(3,4) diff-sum local.sum-diff sum-assoc-1 sum-comm)
    hence LeP PO E E' C Z
      by (simp add: LeP-def)
  }
  ultimately show ?thesis
    using LeP-def assms(2) by auto
qed
}
moreover {
  assume A = X
  hence LeP PO E E' C Z
  proof -
    {
      assume LtP PO E E' B Y
      have LtP PO E E' C Z

```



```

    by (smt (z3) LtP-def ⟨A = X⟩ ⟨LtP PO E E' B Y⟩ assms(3,4) diff-sum
       local.sum-diff sum-assoc-1)
  }
  hence LtP PO E E' B Y  $\implies$  LeP PO E E' C Z
  by (simp add: LeP-def)
  moreover have B = Y  $\implies$  LeP PO E E' C Z
  using ⟨A = X⟩ assms(3,4) leP-refl sum-stable by blast
  ultimately show ?thesis
  using LeP-def assms(2) by blast
qed
}
ultimately show ?thesis
using LeP-def assms(1) by blast
qed

lemma square-pos:
  assumes PO  $\neq$  A
  and Prod PO E E' A A A2
  shows Ps PO E A2
proof -
  have  $\neg$  Col PO E E'
  using Ar2-def Prod-def assms(2) by blast
  have Col PO E A
  using Ar2-def Prod-def assms(2) by force
  then obtain MA where Opp PO E E' A MA
  using  $\langle \neg$  Col PO E E'  $\rangle$  opp-exists by blast
  hence Ps PO E A  $\vee$  PO = A  $\vee$  Ps PO E MA
  by (simp add: pos-null-neg)
  moreover have Ps PO E A  $\implies$  Ps PO E A2
  using assms(2) prod-pos-pos by blast
  moreover have PO = A  $\implies$  Ps PO E A2
  by (simp add: assms(1))
  moreover
  {
    assume Ps PO E MA
    obtain ME where Opp PO E E' E ME
    using  $\langle \neg$  Col PO E E'  $\rangle$  col-trivial-2 opp-exists by blast
    have Prod PO E E' MA MA A2
    by (meson ⟨Opp PO E E' A MA⟩ ⟨Opp PO E E' E ME⟩  $\langle \neg$  Col PO E E'  $\rangle$  assms(2)
       opp-comm opp-prod prod-assoc prod-sym)
    hence Ps PO E A2
    using ⟨Ps PO E MA⟩ prod-pos-pos by blast
  }
  ultimately show ?thesis
  by blast
qed

lemma ltP-neg:
  assumes LtP PO E E' A PO
  shows Ng PO E A
proof -
  obtain MA where Diff PO E E' PO A MA
  using LtP-def assms by blast
  hence Opp PO E E' MA A
  by (simp add: Opp-def diff-sum)
  have  $\neg$  Col PO E E'
  using Ar2-def assms ltP-ar2 by blast
  thus ?thesis
  using LtP-def ⟨Diff PO E E' PO A MA⟩ ⟨Opp PO E E' MA A⟩ assms diff-uniqueness
       pos-opp-neg by blast
qed

lemma ps-le:
  assumes  $\neg$  Col PO E E'
  and Bet PO X E  $\vee$  Bet PO E X
  shows LeP PO E E' PO X

```

```

proof (cases  $PO = X$ )
  show  $PO = X \implies LeP\ PO\ E\ E'\ PO\ X$ 
  by (simp add: leP-refl)
  {
    assume  $PO \neq X$ 
    have  $Diff\ PO\ E\ E'\ X\ PO\ X$ 
    using assms(1,2) bet-col diff-A-O not-col-permutation-5 by blast
    moreover have  $Ps\ PO\ E\ X$ 
    by (metis Bet-cases Col-def Ps-def  $\langle PO \neq X \rangle$  assms(1,2) between-equality-2
      col-trivial-1 or-bet-out)
    ultimately show  $LeP\ PO\ E\ E'\ PO\ X$ 
    using LeP-def LtP-def by auto
  }
qed

```

```

lemma lt-diff-ps:
  assumes  $LtP\ PO\ E\ E'\ Y\ X$ 
  and  $Diff\ PO\ E\ E'\ X\ Y\ XMY$ 
  shows  $Ps\ PO\ E\ XMY$ 
  using LtP-def assms(1,2) diff-stable by blast

```

```

lemma col-2-le-or-ge:
  assumes  $\neg\ Col\ PO\ E\ E'$ 
  and  $Col\ PO\ E\ A$ 
  and  $Col\ PO\ E\ B$ 
  shows  $LeP\ PO\ E\ E'\ A\ B \vee LeP\ PO\ E\ E'\ B\ A$ 
proof (cases  $A = B$ )
  have  $PO \neq E$ 
  using assms(1) col-trivial-1 by auto
  show  $A = B \implies LeP\ PO\ E\ E'\ A\ B \vee LeP\ PO\ E\ E'\ B\ A$ 
  by (simp add: leP-refl)
  {
    assume  $A \neq B$ 
    obtain  $D$  where  $Diff\ PO\ E\ E'\ B\ A\ D$ 
    using assms(1,2,3) diff-exists by blast
    hence  $PO \neq D$ 
    using  $\langle A \neq B \rangle$  diff-null-eq by blast
    hence  $Ps\ PO\ E\ D \vee Ng\ PO\ E\ D$ 
    using Ar2-def  $\langle Diff\ PO\ E\ E'\ B\ A\ D \rangle \langle PO \neq E \rangle$  col-pos-or-neg diff-ar2
    by presburger
    moreover have  $Ps\ PO\ E\ D \implies LeP\ PO\ E\ E'\ A\ B$ 
    using LeP-def LtP-def  $\langle Diff\ PO\ E\ E'\ B\ A\ D \rangle$  by blast
    moreover
    {
      assume  $Ng\ PO\ E\ D$ 
      obtain  $MD$  where  $Diff\ PO\ E\ E'\ A\ B\ MD$ 
      using assms(1,2,3) diff-exists by blast
      hence  $Opp\ PO\ E\ E'\ D\ MD$ 
      using  $\langle Diff\ PO\ E\ E'\ B\ A\ D \rangle$  diff-opp by auto
      hence  $LeP\ PO\ E\ E'\ B\ A$ 
      using LeP-def LtP-def  $\langle Diff\ PO\ E\ E'\ A\ B\ MD \rangle \langle Ng\ PO\ E\ D \rangle$  opp-neg-pos
      by blast
    }
    ultimately show  $LeP\ PO\ E\ E'\ A\ B \vee LeP\ PO\ E\ E'\ B\ A$ 
    by blast
  }
qed

```

```

lemma compatibility-of-sum-with-order:
  assumes  $LeP\ PO\ E\ E'\ A\ B$ 
  and  $Sum\ PO\ E\ E'\ A\ C\ APC$ 
  and  $Sum\ PO\ E\ E'\ B\ C\ BPC$ 
  shows  $LeP\ PO\ E\ E'\ APC\ BPC$ 
  using assms(1,2,3) leP-refl leP-sum-leP by blast

```

```

lemma compatibility-of-prod-with-order:

```

```

assumes  $LeP\ PO\ E\ E'\ PO\ A$ 
and  $LeP\ PO\ E\ E'\ PO\ B$ 
and  $Prod\ PO\ E\ E'\ A\ B\ AB$ 
shows  $LeP\ PO\ E\ E'\ PO\ AB$ 
proof –
have  $PO = A \implies ?thesis$ 
using  $assms(3)\ leP-refl\ prod-O-l-eq$  by  $blast$ 
moreover {
assume  $LtP\ PO\ E\ E'\ PO\ A$ 
have  $PO = B \implies LeP\ PO\ E\ E'\ PO\ AB$ 
using  $assms(3)\ leP-refl\ prod-O-r-eq$  by  $blast$ 
moreover
{
assume  $LtP\ PO\ E\ E'\ PO\ B$ 
have  $Ps\ PO\ E\ AB$ 
by  $(metis\ Ar2-def\ \langle LtP\ PO\ E\ E'\ PO\ A\rangle\ \langle LtP\ PO\ E\ E'\ PO\ B\rangle\ assms(3)\ diff-A-O\ ltP-ar2\ lt-diff-ps\ prod-pos-pos)$ 
hence  $LeP\ PO\ E\ E'\ PO\ AB$ 
by  $(metis\ Ar2-def\ LeP-def\ Prod-def\ assms(3)\ col-2-le-or-ge\ col-trivial-3\ ltP-neg\ pos-not-neg)$ 
}
ultimately have  $LeP\ PO\ E\ E'\ PO\ AB$ 
using  $LeP-def\ assms(2)$  by  $auto$ 
}
ultimately show  $?thesis$ 
using  $LeP-def\ assms(1)$  by  $blast$ 
qed

```

```

lemma  $pos-inv-pos$ :
assumes  $PO \neq A$ 
and  $LeP\ PO\ E\ E'\ PO\ A$ 
and  $Prod\ PO\ E\ E'\ IA\ A\ E$ 
shows  $LeP\ PO\ E\ E'\ PO\ IA$ 
proof –
have  $PO = A \implies LeP\ PO\ E\ E'\ PO\ IA$ 
by  $(simp\ add:\ assms(1))$ 
moreover
{
assume  $LtP\ PO\ E\ E'\ PO\ A$ 
then obtain  $IA'$  where  $Diff\ PO\ E\ E'\ IA\ PO\ IA'$ 
by  $(meson\ Ar2-def\ Prodp-def\ assms(3)\ diff-A-O\ ltP-ar2\ prod-to-prodp)$ 
hence  $IA = IA'$ 
by  $(meson\ Ar2-def\ diff-A-O\ diff-ar2\ diff-stable)$ 
obtain  $A'$  where  $Diff\ PO\ E\ E'\ A\ PO\ A'$  and  $Ps\ PO\ E\ A'$ 
using  $LtP-def\ \langle LtP\ PO\ E\ E'\ PO\ A\rangle$  by  $blast$ 
hence  $A = A'$ 
by  $(metis\ Ar2-def\ diff-ar2\ diff-sum\ sum-O-B-eq)$ 
obtain  $MIA$  where  $Opp\ PO\ E\ E'\ IA\ MIA$ 
by  $(metis\ Ar2-def\ \langle Diff\ PO\ E\ E'\ IA\ PO\ IA'\rangle\ diff-ar2\ opp-exists)$ 
have  $Ps\ PO\ E\ IA \vee PO = IA \vee Ps\ PO\ E\ MIA$ 
using  $\langle Opp\ PO\ E\ E'\ IA\ MIA\rangle\ pos-null-neg$  by  $auto$ 
moreover have  $PO = IA \implies LeP\ PO\ E\ E'\ PO\ IA$ 
by  $(simp\ add:\ leP-refl)$ 
moreover {
assume  $Ps\ PO\ E\ MIA$ 
obtain  $ME$  where  $Opp\ PO\ E\ E'\ E\ ME$ 
by  $(meson\ Ar2-def\ \langle Diff\ PO\ E\ E'\ A\ PO\ A'\rangle\ diff-ar2\ not-col-distincts\ opp-exists)$ 
moreover have  $Prod\ PO\ E\ E'\ IA\ ME\ MIA$ 
by  $(simp\ add:\ \langle Opp\ PO\ E\ E'\ IA\ MIA\rangle\ calculation\ opp-prod)$ 
moreover have  $Prod\ PO\ E\ E'\ MIA\ A\ ME$ 
by  $(meson\ assms(3)\ calculation(1,2)\ opp-prod\ prod-assoc\ prod-sym)$ 
ultimately have  $False$ 
by  $(metis\ Ar2-def\ Ng-def\ \langle A = A'\rangle\ \langle Diff\ PO\ E\ E'\ IA\ PO\ IA'\rangle\ \langle Ps\ PO\ E\ A'\rangle\ \langle Ps\ PO\ E\ MIA\rangle\ bet-neq12--neq\ diff-ar2\ opp-comm\ pos-opp-neg\ prod-pos-pos)$ 
}
}

```

```

}
moreover have  $Ps\ PO\ E\ IA \implies LeP\ PO\ E\ E'\ PO\ IA$ 
  using  $LeP\text{-}def\ LtP\text{-}def\ \langle Diff\ PO\ E\ E'\ IA\ PO\ IA \rangle\ \langle IA = IA' \rangle$  by auto
ultimately have  $LeP\ PO\ E\ E'\ PO\ IA$ 
  by blast
}
ultimately show ?thesis
  using  $LeP\text{-}def\ assms(2)$  by auto
qed

```

lemma *le-pos-prod-le*:

```

assumes  $LeP\ PO\ E\ E'\ A\ B$ 
  and  $LeP\ PO\ E\ E'\ PO\ C$ 
  and  $Prod\ PO\ E\ E'\ A\ C\ AC$ 
  and  $Prod\ PO\ E\ E'\ B\ C\ BC$ 
shows  $LeP\ PO\ E\ E'\ AC\ BC$ 
proof –
  obtain  $BCMAC$  where  $Diff\ PO\ E\ E'\ BC\ AC\ BCMAC$ 
    using  $Ar2\text{-}def\ Prod\text{-}def\ assms(3,4)\ diff\text{-}exists$  by presburger
obtain  $BMA$  where  $Diff\ PO\ E\ E'\ B\ A\ BMA$ 
  by (metis  $Ar2\text{-}def\ LeP\text{-}def\ LtP\text{-}def\ Prod\text{-}def\ assms(1,3)\ diff\text{-}null)
then obtain  $BCMAC'$  where  $Prod\ PO\ E\ E'\ BMA\ C\ BCMAC'$ 
  using  $Ar2\text{-}def\ Prodp\text{-}def\ assms(4)\ diff\text{-}ar2\ prod\text{-}exists$ 
     $prod\text{-}to\text{-}prodp$  by presburger
hence  $Diff\ PO\ E\ E'\ BC\ AC\ BCMAC'$ 
  using  $\langle Diff\ PO\ E\ E'\ B\ A\ BMA \rangle\ assms(3,4)\ distr\text{-}l\text{-}diff\ prod\text{-}sym$  by blast
hence  $BCMAC = BCMAC'$ 
  using  $\langle Diff\ PO\ E\ E'\ BC\ AC\ BCMAC \rangle\ diff\text{-}uniqueness$ 
    by auto
hence  $LeP\ PO\ E\ E'\ PO\ BCMAC$ 
  by (metis  $Ar2\text{-}def\ LeP\text{-}def\ Out\text{-}def\ Ps\text{-}def$ 
     $\langle Diff\ PO\ E\ E'\ B\ A\ BMA \rangle\ \langle Prod\ PO\ E\ E'\ BMA\ C\ BCMAC' \rangle\ assms(1,2)$ 
    compatibility-of-prod-with-order\ cong-identity-inv\ diff-ar2\ diff-sum
    lt-diff-ps\ ps-le\ sum-iff-cong-b)
thus ?thesis
  by (metis  $Ar2\text{-}def\ LeP\text{-}def\ LtP\text{-}def\ \langle Diff\ PO\ E\ E'\ BC\ AC\ BCMAC \rangle\ diff\text{-}A\text{-}O$ 
     $diff\text{-}null\text{-}eq\ ltP\text{-}ar2\ lt\text{-}diff\text{-}ps$ )
qed$ 
```

lemma *bet-lt12-le23*:

```

assumes  $Bet\ A\ B\ C$ 
  and  $LtP\ PO\ E\ E'\ A\ B$ 
shows  $LeP\ PO\ E\ E'\ B\ C$ 
proof (cases  $B = C$ )
show  $B = C \implies LeP\ PO\ E\ E'\ B\ C$ 
  by (simp add: leP-refl)
{
  assume  $B \neq C$ 
  have  $A \neq B$ 
    using  $assms(2)\ ltP\text{-}neq$  by auto
  have  $Col\ PO\ E\ A$ 
    using  $Ar2\text{-}def\ assms(2)\ ltP\text{-}ar2$  by blast
  have  $Col\ PO\ E\ B$ 
    using  $Ar2\text{-}def\ assms(2)\ ltP\text{-}ar2$  by blast
  then obtain  $CMB$  where  $Diff\ PO\ E\ E'\ C\ B\ CMB$ 
    using  $diff\text{-}exists$  by (metis  $Ar2\text{-}def\ Col\text{-}def\ \langle A \neq B \rangle\ assms(1,2)\ col\text{-}transitivity\text{-}1$ 
      col-trivial-1\ ltP-ar2)
  show  $LeP\ PO\ E\ E'\ B\ C$ 
proof (cases  $PO = A$ )
    show  $PO = A \implies LeP\ PO\ E\ E'\ B\ C$ 
proof (cases  $PO = B$ )
    show  $PO = A \implies PO = B \implies LeP\ PO\ E\ E'\ B\ C$ 
      using  $\langle A \neq B \rangle$  by fastforce
    show  $PO = A \implies PO \neq B \implies LeP\ PO\ E\ E'\ B\ C$ 
proof (cases  $PO = C$ )
    show  $PO = A \implies PO \neq B \implies PO = C \implies LeP\ PO\ E\ E'\ B\ C$ 

```

```

using assms(1) bet-neq12--neg by auto
{
  assume  $PO = A$  and  $PO \neq B$  and  $PO \neq C$ 
  obtain  $B'$  where  $\text{Diff } PO \ E \ E' \ B \ PO \ B'$  and  $Ps \ PO \ E \ B'$ 
  using LtP-def  $\langle PO = A \rangle$  assms(2) by auto
  hence  $B = B'$ 
  by (metis Ar2-def diff-ar2 diff-sum sum-O-B-eq)
  have  $E \neq PO$ 
  using Ar2-def  $\langle \text{Diff } PO \ E \ E' \ B \ PO \ B' \rangle$  col-trivial-1 diff-ar2 by blast
  moreover
  have ParallelogramFlat  $PO \ B \ C \ CMB$ 
  using  $\langle \text{Diff } PO \ E \ E' \ C \ B \ CMB \rangle \langle PO \neq C \rangle$  diff-to-plg by force
  hence  $\text{Bet } CMB \ C \ PO \wedge \text{Bet } C \ PO \ B \vee \text{Bet } CMB \ PO \ C \wedge \text{Bet } PO \ C \ B \vee$ 
     $\text{Bet } PO \ CMB \ B \wedge \text{Bet } CMB \ B \ C \vee \text{Bet } PO \ B \ CMB \wedge \text{Bet } B \ CMB \ C$ 
  by (simp add: plgf-bet)
  moreover have  $\text{Bet } CMB \ C \ PO \wedge \text{Bet } C \ PO \ B \implies \text{Bet } PO \ CMB \ E \vee \text{Bet } PO \ E \ CMB$ 
  using Bet-cases  $\langle PO = A \rangle \langle PO \neq B \rangle$  assms(1) between-equality-2
  by blast
  moreover have  $\text{Bet } CMB \ PO \ C \wedge \text{Bet } PO \ C \ B \implies \text{Bet } PO \ CMB \ E \vee \text{Bet } PO \ E \ CMB$ 
  using  $\langle B \neq C \rangle \langle PO = A \rangle$  assms(1) between-equality-2 by auto
  moreover have  $\text{Bet } PO \ CMB \ B \wedge \text{Bet } CMB \ B \ C \implies \text{Bet } PO \ CMB \ E \vee \text{Bet } PO \ E \ CMB$ 
  by (metis Ar2-def Bet-cases Ps-def  $\langle B = B' \rangle \langle \text{Diff } PO \ E \ E' \ C \ B \ CMB \rangle$ 
     $\langle Ps \ PO \ E \ B' \rangle$  bet-out--bet between-equality-2 diff-sum l6-6 sum-ar2
    third-point)
  moreover have  $\text{Bet } PO \ B \ CMB \wedge \text{Bet } B \ CMB \ C \implies \text{Bet } PO \ CMB \ E \vee \text{Bet } PO \ E \ CMB$ 
  by (metis Bet-cases Col-def Ng-def  $\langle B = B' \rangle \langle \text{Col } PO \ E \ B \rangle \langle PO \neq B \rangle$ 
     $\langle Ps \ PO \ E \ B' \rangle$  between-inner-transitivity col2--eq col-permutation-4
    pos-not-neg)
  ultimately have  $\text{Bet } PO \ CMB \ E \vee \text{Bet } PO \ E \ CMB$ 
  by blast
  have  $E \neq PO \wedge (\text{Bet } PO \ CMB \ E \vee \text{Bet } PO \ E \ CMB)$ 
  by (simp add:  $\langle E \neq PO \rangle \langle (PO - CMB - E) \vee (PO - E - CMB) \rangle$ )
  hence  $Ps \ PO \ E \ CMB$ 
  by (metis Out-cases Ps-def  $\langle B \neq C \rangle \langle \text{Diff } PO \ E \ E' \ C \ B \ CMB \rangle$  bet-out-1
    between-symmetry diff-null-eq)
  thus  $LeP \ PO \ E \ E' \ B \ C$ 
  using LeP-def LtP-def  $\langle \text{Diff } PO \ E \ E' \ C \ B \ CMB \rangle$  by blast
}
qed
qed
show  $PO \neq A \implies LeP \ PO \ E \ E' \ B \ C$ 
proof (cases  $PO = B$ )
  show  $PO \neq A \implies PO = B \implies LeP \ PO \ E \ E' \ B \ C$ 
  by (metis Ar2-def Ng-def assms(1,2) l5-2 ltP-ar2 ltP-neg ps-le)
  show  $PO \neq A \implies PO \neq B \implies LeP \ PO \ E \ E' \ B \ C$ 
proof (cases  $PO = C$ )
  {
    assume  $PO \neq A$  and  $PO \neq B$  and  $PO = C$ 
    have  $Ps \ PO \ E \ B \vee Ng \ PO \ E \ B$ 
    using LtP-def  $\langle \text{Diff } PO \ E \ E' \ C \ B \ CMB \rangle \langle PO = C \rangle \langle PO \neq B \rangle$  diff-O-A-opp ltP-neg
      pos-null-neg by blast
    {
      assume  $Ps \ PO \ E \ B$ 
      have  $Ps \ PO \ E \ A \vee Ng \ PO \ E \ A$ 
      by (metis NCol-cases Ng-def Ps-def  $\langle \text{Col } PO \ E \ A \rangle \langle PO \neq A \rangle \langle Ps \ PO \ E \ B \rangle$ 
        or-bet-out out-distinct)
      moreover {
        assume  $Ps \ PO \ E \ A$ 
        obtain  $BMA$  where  $\text{Sum } PO \ E \ E' \ A \ BMA \ B$  and  $Ps \ PO \ E \ BMA$ 
        using LtP-def assms(2) diff-sum by blast
        have False
        proof (cases  $A = BMA$ )
          show  $A = BMA \implies False$ 
          by (metis Bet-cases  $\langle A \neq B \rangle \langle PO = C \rangle \langle \text{Sum } PO \ E \ E' \ A \ BMA \ B \rangle$  assms(1) bet-out
            between-equality-2 between-trivial l14-36-a)
        }
      }
    }
  }

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    assume  $A \neq BMA$ 
    hence ParallelogramFlat  $PO\ A\ B\ BMA$ 
      by (metis Ar2-def  $\langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$  sum-ar2 sum-cong)
    hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
       $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
      using plgf-bet by simp
    moreover have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies False$ 
      using Bet-cases  $\langle PO = C \rangle \langle PO \neq B \rangle$  assms(1) between-equality
      by blast
    moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies False$ 
      by (meson Ps-def  $\langle Ps\ PO\ E\ BMA \rangle \langle Ps\ PO\ E\ B \rangle$  bet-out--bet l6-6
        not-bet-and-out)
    moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies False$ 
      by (metis Bet-cases  $\langle A \neq BMA \rangle \langle A \neq B \rangle \langle PO = C \rangle$  assms(1) between-equality-2
        outer-transitivity-between2)
    moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies False$ 
      by (metis Bet-cases  $\langle A \neq BMA \rangle \langle PO = C \rangle$  assms(1) between-equality-2
        outer-transitivity-between)
    ultimately show False
      by blast
  }
qed
}
moreover have  $Ng\ PO\ E\ A \implies False$ 
  by (metis Ng-def  $\langle PO = C \rangle \langle PO \neq B \rangle \langle Ps\ PO\ E\ B \rangle$  assms(1) between-exchange3
    pos-not-neg)
ultimately have False
  by auto
}
hence  $Ng\ PO\ E\ B$ 
  using  $\langle Ps\ PO\ E\ B \vee Ng\ PO\ E\ B \rangle$  by auto
thus  $LeP\ PO\ E\ E'\ B\ C$ 
  by (metis LeP-def LtP-def  $\langle Diff\ PO\ E\ E'\ C\ B\ CMB \rangle \langle PO = C \rangle$  diff-O-A-opp
    opp-neg-pos)
}
{
  assume  $PO \neq A$  and  $PO \neq B$  and  $PO \neq C$ 
  have  $Sum\ PO\ E\ E'\ B\ CMB\ C$ 
    by (simp add:  $\langle Diff\ PO\ E\ E'\ C\ B\ CMB \rangle$  diff-sum)
  have ParallelogramFlat  $PO\ B\ C\ CMB$ 
    using  $\langle Diff\ PO\ E\ E'\ C\ B\ CMB \rangle \langle PO \neq C \rangle$  diff-to-plg by presburger
  hence  $Bet\ CMB\ C\ PO \wedge Bet\ C\ PO\ B \vee Bet\ CMB\ PO\ C \wedge Bet\ PO\ C\ B \vee$ 
     $Bet\ PO\ CMB\ B \wedge Bet\ CMB\ B\ C \vee Bet\ PO\ B\ CMB \wedge Bet\ B\ CMB\ C$ 
    using plgf-bet by simp
  have  $E \neq PO$ 
    using Ar2-def  $\langle Diff\ PO\ E\ E'\ C\ B\ CMB \rangle$  col-trivial-1 diff-ar2 by blast
  moreover have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  proof -
    have  $Ps\ PO\ E\ A \vee Ng\ PO\ E\ A$ 
      using  $\langle Col\ PO\ E\ A \rangle \langle PO \neq A \rangle$  calculation col-pos-or-neg by auto
    moreover have  $Ps\ PO\ E\ B \vee Ng\ PO\ E\ B$ 
      using  $\langle Col\ PO\ E\ B \rangle \langle E \neq PO \rangle \langle PO \neq B \rangle$  col-pos-or-neg by auto
    moreover
    {
      assume  $Ps\ PO\ E\ A$  and  $Ps\ PO\ E\ B$ 
      hence  $Bet\ PO\ A\ E \vee Bet\ PO\ E\ A$ 
        by (metis Ng-def  $\langle Col\ PO\ E\ A \rangle \langle E \neq PO \rangle$  pos-not-neg third-point)
      have  $Bet\ PO\ B\ E \vee Bet\ PO\ E\ B$ 
        by (metis Bet-cases Col-def Ng-def  $\langle Col\ PO\ E\ B \rangle \langle Ps\ PO\ E\ B \rangle$  pos-not-neg)
    }
    {
      assume  $Bet\ CMB\ C\ PO \wedge Bet\ C\ PO\ B$ 
    }
    {
      assume  $Ps\ PO\ E\ A$ 
    }
    {
      assume  $Bet\ PO\ A\ E$ 
    }
  }
}

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assume  $Bet\ PO\ B\ E$ 
hence  $Bet\ PO\ A\ B \vee Bet\ PO\ B\ A$ 
  using  $\langle PO - A - E \rangle$  l5-3 by auto
moreover {
  assume  $Bet\ PO\ A\ B \vee Bet\ PO\ B\ A$ 
  moreover have  $Bet\ PO\ A\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
    by (metis  $Bet$ -cases  $\langle A \neq B \rangle \langle (CMB - C - PO) \wedge (C - PO - B) \rangle$ 
      assms(1) between-equality-2 between-exchange2)
  moreover {
    assume  $Bet\ PO\ B\ A$ 
    obtain  $BMA$  where  $Sum\ PO\ E\ E'\ A\ BMA\ B$  and  $Ps\ PO\ E\ BMA$ 
      using LtP-def assms(2) diff-sum by blast
      have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      proof (cases  $A = BMA$ )
        show  $A = BMA \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
          by (metis  $\langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$  bet-out-1 between-trivial2
            calculation(2) l14-36-a)
        have ParallelogramFlat  $PO\ A\ B\ BMA$ 
          by (metis  $\langle PO \neq A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$ 
            diff-to-plg local.sum-diff)
        hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
           $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
          by (simp add: plgf-bet)
        {
          assume  $A \neq BMA$ 
          have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies$ 
             $(PO - CMB - E) \vee (PO - E - CMB)$ 
          using  $\langle PO - B - A \rangle \langle PO \neq B \rangle$  between-equality by auto
          moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies$ 
             $(PO - CMB - E) \vee (PO - E - CMB)$ 
          by (meson  $Bet$ -cases  $Ps$ -def  $\langle Ps\ PO\ E\ BMA \rangle \langle Ps\ PO\ E\ B \rangle$ 
            bet-out--bet not-bet-and-out)
          moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies$ 
             $(PO - CMB - E) \vee (PO - E - CMB)$ 
          using  $\langle A \neq BMA \rangle$ 
             $\langle (PO - A - B) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
            outer-transitivity-between2 by blast
          moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies$ 
             $(PO - CMB - E) \vee (PO - E - CMB)$ 
          using  $\langle A \neq BMA \rangle$ 
             $\langle (PO - A - B) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
            outer-transitivity-between by blast
          ultimately show  $(PO - CMB - E) \vee (PO - E - CMB)$ 
          using  $\langle Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee$ 
             $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
             $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee$ 
             $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \rangle$  by blast
        }
      }
    }
  }
qed
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  by auto
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  by simp
}
moreover have  $Bet\ PO\ E\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  by (metis  $Bet$ -cases  $\langle (CMB - C - PO) \wedge (C - PO - B) \rangle$ 
     $\langle PO - A - E \rangle$  assms(1) between-exchange3 calculation)
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle (PO - B - E) \vee (PO - E - B) \rangle$  by blast
}
moreover {
  assume  $Bet\ PO\ E\ A$ 
  hence  $Bet\ PO\ A\ B \vee Bet\ PO\ B\ A$ 
    by (metis  $\langle E \neq PO \rangle \langle (PO - B - E) \vee (PO - E - B) \rangle$ 

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    between-exchange4 l5-1)
moreover have  $Bet\ PO\ A\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  by (meson  $\langle A \neq B \rangle \langle CMB - C - PO \rangle \wedge \langle C - PO - B \rangle \langle PO \neq B \rangle$ 
    assms(1) between-equality between-symmetry
    outer-transitivity-between2)
moreover {
  assume  $Bet\ PO\ B\ A$ 
  obtain  $BMA$  where  $Sum\ PO\ E\ E'\ A\ BMA\ B$  and  $Ps\ PO\ E\ BMA$ 
    using LtP-def assms(2) diff-sum by blast
  have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  proof (cases  $A = BMA$ )
    show  $A = BMA \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
      by (metis  $\langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$  bet-out-1 between-trivial2
        calculation(2) l14-36-a)
    {
      assume  $A \neq BMA$ 
      have ParallelogramFlat  $PO\ A\ B\ BMA$ 
        by (metis  $\langle PO \neq A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$  diff-to-plg
          local.sum-diff)
      hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
         $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
        by (simp add: plgf-bet)
      moreover have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies$ 
         $(PO - CMB - E) \vee (PO - E - CMB)$ 
        using  $\langle PO - B - A \rangle \langle PO \neq B \rangle$  between-equality by auto
      moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies$ 
         $(PO - CMB - E) \vee (PO - E - CMB)$ 
        using Out-cases Ps-def  $\langle Ps\ PO\ E\ BMA \rangle \langle Ps\ PO\ E\ B \rangle$ 
        bet-out--bet not-bet-and-out by blast
      moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies$ 
         $(PO - CMB - E) \vee (PO - E - CMB)$ 
        using  $\langle A \neq BMA \rangle$ 
         $\langle (PO - A - B) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
        outer-transitivity-between2 by blast
      moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies$ 
         $(PO - CMB - E) \vee (PO - E - CMB)$ 
        using  $\langle A \neq BMA \rangle$ 
         $\langle (PO - A - B) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
        outer-transitivity-between by blast
      ultimately show  $(PO - CMB - E) \vee (PO - E - CMB)$ 
        by blast
    }
  }
qed
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
by blast
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
using  $\langle (PO - A - E) \vee (PO - E - A) \rangle$  by blast
}
moreover have  $Ng\ PO\ E\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
by (simp add:  $\langle Ps\ PO\ E\ A \rangle$  calculation)
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
by (simp add:  $\langle Ps\ PO\ E\ A \rangle$ )
}
moreover {
  assume  $Bet\ CMB\ PO\ C$  and  $Bet\ PO\ C\ B$ 
  {
    assume  $Bet\ PO\ A\ E$  and  $Bet\ PO\ B\ E$ 
    have  $Bet\ PO\ A\ B \vee Bet\ PO\ B\ A$ 
      using  $\langle PO - A - E \rangle \langle PO - B - E \rangle$  l5-3 by auto
    moreover have  $Bet\ PO\ A\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle A \neq B \rangle \langle B \neq C \rangle \langle PO - C - B \rangle$  assms(1) between-equality-2
      outer-transitivity-between2
      by blast
    moreover {

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assume  $Bet\ PO\ B\ A$ 
obtain  $BMA$  where  $Sum\ PO\ E\ E'\ A\ BMA\ B$  and  $Ps\ PO\ E\ BMA$ 
  using  $LtP-def\ assms(2)\ diff-sum$  by  $blast$ 
have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
proof ( $cases\ A = BMA$ )
  show  $A = BMA \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
    using  $\langle PO \neq A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$   $bet-out\ between-trivial$ 
       $calculation(2)\ l14-36-a$ 
    by  $auto$ 
  {
    assume  $A \neq BMA$ 
    have  $ParallelogramFlat\ PO\ A\ B\ BMA$ 
      by ( $metis\ \langle PO \neq A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$ 
         $diff-to-plg\ local.sum-diff$ )
    hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
       $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
    using  $plgf-bet$  by  $blast$ 
    moreover have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies$ 
       $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle PO - B - A \rangle \langle PO \neq B \rangle$   $between-equality$  by  $auto$ 
    moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies$ 
       $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $Ps-def\ \langle PO - B - E \rangle \langle PO \neq B \rangle \langle Ps\ PO\ E\ BMA \rangle$ 
       $not-bet-and-out\ outer-transitivity-between$  by  $blast$ 
    moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies$ 
       $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle A \neq BMA \rangle \langle (PO - A - B) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
       $outer-transitivity-between2$  by  $blast$ 
    moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies$ 
       $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle A \neq BMA \rangle \langle (PO - A - B) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
       $outer-transitivity-between$  by  $blast$ 
    ultimately show  $(PO - CMB - E) \vee (PO - E - CMB)$ 
      by  $blast$ 
  }
qed
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
by  $blast$ 
}
moreover have  $Bet\ PO\ A\ E \wedge Bet\ PO\ E\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
by ( $metis\ Bet-cases\ \langle A \neq B \rangle \langle B \neq C \rangle \langle PO - C - B \rangle\ assms(1)$ 
   $between-equality\ between-exchange2\ outer-transitivity-between$ )
moreover {
  assume  $Bet\ PO\ E\ A$  and  $Bet\ PO\ B\ E$ 
  obtain  $BMA$  where  $Sum\ PO\ E\ E'\ A\ BMA\ B$  and  $Ps\ PO\ E\ BMA$ 
    using  $LtP-def\ assms(2)\ diff-sum$  by  $blast$ 
  have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  proof ( $cases\ A = BMA$ )
    show  $A = BMA \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
      by ( $metis\ \langle A \neq B \rangle \langle PO - B - E \rangle \langle PO - E - A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$ 
         $bet-out-1\ between-equality-2\ between-exchange4$ 
         $between-trivial2\ l14-36-a$ )
    {
      assume  $A \neq BMA$ 
      have  $ParallelogramFlat\ PO\ A\ B\ BMA$ 
        by ( $metis\ \langle PO \neq A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle\ diff-to-plg\ local.sum-diff$ )
      hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
         $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
      using  $plgf-bet$  by  $blast$ 
      moreover have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle PO - B - E \rangle \langle PO - E - A \rangle \langle PO \neq B \rangle$   $between-equality$ 
       $between-inner-transitivity$  by  $blast$ 
      moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by ( $metis\ Bet-cases\ Ng-def\ O-not-positive\ \langle E \neq PO \rangle \langle PO - B - E \rangle$ 
         $\langle PO \neq B \rangle \langle Ps\ PO\ E\ BMA \rangle$ )
    }
  }

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      neg-not-pos outer-transitivity-between2)
    moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle A \neq BMA \rangle$ 
       $\langle (PO - A - E) \implies (PO - B - E) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
       $\langle PO - B - E \rangle$  between-exchange4 outer-transitivity-between2 by blast
    moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle A \neq BMA \rangle$ 
       $\langle (PO - A - E) \implies (PO - B - E) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
       $\langle PO - B - E \rangle$ 
      between-exchange4 outer-transitivity-between2 by blast
    ultimately show  $(PO - CMB - E) \vee (PO - E - CMB)$ 
      by blast
  }
qed
}
moreover {
  assume  $Bet\ PO\ E\ A$  and  $Bet\ PO\ E\ B$ 
  obtain  $BMA$  where  $Sum\ PO\ E\ E'\ A\ BMA\ B$  and  $Ps\ PO\ E\ BMA$ 
    by (meson LtP-def assms(2) diff-sum)
  have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  proof (cases  $A = BMA$ )
    show  $A = BMA \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
      by (metis  $\langle A \neq B \rangle \langle B \neq C \rangle \langle PO - C - B \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$ 
        assms(1) bet-out between-equality-2
        between-exchange2 between-trivial l14-36-a outer-transitivity-between)
  {
    assume  $A \neq BMA$ 
    have ParallelogramFlat  $PO\ A\ B\ BMA$ 
      by (metis  $\langle PO \neq A \rangle \langle Sum\ PO\ E\ E'\ A\ BMA\ B \rangle$  diff-to-plg local.sum-diff)
    hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
       $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
      using plgf-bet by blast
    moreover have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (meson Bet-cases  $\langle B \neq C \rangle \langle PO - C - B \rangle$  assms(1)
        between-equality-2 between-exchange3)
    moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (metis Bet-cases Ps-def  $\langle PO - E - B \rangle \langle Ps\ PO\ E\ BMA \rangle$ 
        between-exchange3 not-bet-and-out)
    moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (metis  $\langle A \neq BMA \rangle \langle A \neq B \rangle \langle B \neq C \rangle \langle PO - C - B \rangle$  assms(1)
        between-equality-2 outer-transitivity-between2)
    moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (metis  $\langle A \neq BMA \rangle \langle B \neq C \rangle \langle PO - C - B \rangle$  assms(1)
        between-equality-2 between-exchange3 outer-transitivity-between2)
    ultimately show  $(PO - CMB - E) \vee (PO - E - CMB)$ 
      by blast
  }
}
qed
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle (PO - A - E) \vee (PO - E - A) \rangle \langle (PO - B - E) \vee (PO - E - B) \rangle$  by auto
}
moreover have  $Bet\ PO\ CMB\ B \wedge Bet\ CMB\ B\ C \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  by (meson  $\langle (PO - B - E) \vee (PO - E - B) \rangle$  between-exchange4 l5-3)
moreover have  $Bet\ PO\ B\ CMB \wedge Bet\ B\ CMB\ C \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle (PO - B - E) \vee (PO - E - B) \rangle \langle PO \neq B \rangle$  between-exchange4 l5-1 by blast
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle Bet\ CMB\ C\ PO \wedge Bet\ C\ PO\ B \vee Bet\ CMB\ PO\ C \wedge Bet\ PO\ C\ B \vee$ 
     $Bet\ PO\ CMB\ B \wedge Bet\ CMB\ B\ C \vee Bet\ PO\ B\ CMB \wedge Bet\ B\ CMB\ C \rangle$  by blast
}
moreover have  $Ps\ PO\ E\ A \wedge Ng\ PO\ E\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using LtP-def assms(2) calculation(3) diff-sum sum-pos-pos by blast
moreover {
  assume  $Ng\ PO\ E\ A$  and  $Ps\ PO\ E\ B$ 
  have  $Bet\ CMB\ C\ PO \wedge Bet\ C\ PO\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
    by (meson Bet-cases Ng-def Out-cases Ps-def  $\langle Ng\ PO\ E\ A \rangle \langle PO \neq B \rangle \langle Ps\ PO\ E\ B \rangle$ 

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    assms(1)
    bet-out--bet between-equality between-inner-transitivity)
moreover have Bet CMB PO C  $\wedge$  Bet PO C B  $\implies$  Bet PO CMB E  $\vee$  Bet PO E CMB
by (smt (verit, ccfv-threshold) Bet-cases Ng-def  $\langle B \neq C \rangle$   $\langle Ng PO E A \rangle$   $\langle PO \neq B \rangle$ 
     $\langle Ps PO E B \rangle$ 
    assms(1) between-inner-transitivity outer-transitivity-between2 pos-not-neg)
moreover have Bet PO CMB B  $\wedge$  Bet CMB B C  $\implies$  Bet PO CMB E  $\vee$  Bet PO E CMB
by (meson Bet-cases Col-def Ps-def  $\langle Col PO E B \rangle$   $\langle Ps PO E B \rangle$  between-exchange2
    l5-3 not-bet-and-out)
moreover have Bet PO B CMB  $\wedge$  Bet B CMB C  $\implies$  Bet PO CMB E  $\vee$  Bet PO E CMB
by (meson Ar2-def Bet-cases Ps-def  $\langle Ps PO E B \rangle$   $\langle Sum PO E E' B CMB C \rangle$ 
    between-inner-transitivity
    not-bet-and-out sum-ar2 third-point)
ultimately have Bet PO CMB E  $\vee$  Bet PO E CMB
using  $\langle Bet CMB C PO \wedge Bet C PO B \vee Bet CMB PO C \wedge Bet PO C B \vee$ 
     $Bet PO CMB B \wedge Bet CMB B C \vee Bet PO B CMB \wedge Bet B CMB C \rangle$ 
by blast
}
moreover
{
  assume Ng PO E A and Ng PO E B
  have Bet CMB C PO  $\wedge$  Bet C PO B  $\implies$  Bet PO CMB E  $\vee$  Bet PO E CMB
  by (metis Ng-def  $\langle Ng PO E B \rangle$   $\langle PO \neq C \rangle$  between-symmetry l5-2
    outer-transitivity-between2)
  moreover have Bet CMB PO C  $\wedge$  Bet PO C B  $\implies$  Bet PO CMB E  $\vee$  Bet PO E CMB
  by (smt (verit, ccfv-SIG) Bet-cases Col-def Ng-def  $\langle Ng PO E B \rangle$   $\langle PO \neq C \rangle$ 
    between-equality-2 between-inner-transitivity l6-16-1)
  moreover {
    assume Bet PO CMB B and Bet CMB B C
    {
      assume Bet A PO E
      {
        assume Bet B PO E
        have Bet PO A B  $\vee$  Bet PO B A
        by (meson  $\langle A - PO - E \rangle$   $\langle B - PO - E \rangle$   $\langle E \neq PO \rangle$  between-symmetry l5-2)
        moreover {
          assume Bet PO A B
          obtain BMA where Sum PO E E' A BMA B and Ps PO E BMA
          using LtP-def assms(2) diff-sum by blast
          hence ParallelogramFlat PO A B BMA
          by (metis  $\langle PO \neq A \rangle$  diff-to-plg local.sum-diff)
          hence Bet BMA B PO  $\wedge$  Bet B PO A  $\vee$  Bet BMA PO B  $\wedge$  Bet PO B A  $\vee$ 
            Bet PO BMA A  $\wedge$  Bet BMA A B  $\vee$  Bet PO A BMA  $\wedge$  Bet A BMA B
          using plgf-bet by blast
          have Bet PO CMB E  $\vee$  Bet PO E CMB
          proof (cases A = BMA)
          show A = BMA  $\implies$  (PO - CMB - E)  $\vee$  (PO - E - CMB)
          using  $\langle Ng PO E B \rangle$ 
             $\langle Ps PO E A \wedge Ng PO E B \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
             $\langle Ps PO E BMA \rangle$  by auto
          {
            assume A  $\neq$  BMA
            have Bet BMA B PO  $\wedge$  Bet B PO A  $\implies$  Bet PO CMB E  $\vee$  Bet PO E CMB
            using Bet-cases  $\langle PO - A - B \rangle$   $\langle PO \neq A \rangle$  between-equality by blast
            moreover have Bet BMA PO B  $\wedge$  Bet PO B A  $\implies$ 
              Bet PO CMB E  $\vee$  Bet PO E CMB
            using  $\langle A \neq B \rangle$   $\langle PO - A - B \rangle$  between-equality-2 by auto
            moreover have Bet PO BMA A  $\wedge$  Bet BMA A B  $\implies$ 
              Bet PO CMB E  $\vee$  Bet PO E CMB
            using Bet-cases Ps-def  $\langle A - PO - E \rangle$   $\langle Ps PO E BMA \rangle$ 
              between-exchange3 not-bet-and-out by blast
            moreover have Bet PO A BMA  $\wedge$  Bet A BMA B  $\implies$ 
              Bet PO CMB E  $\vee$  Bet PO E CMB
            by (metis Bet-cases Ng-def  $\langle A - PO - E \rangle$   $\langle E \neq PO \rangle$   $\langle PO \neq A \rangle$ 
               $\langle Ps PO E BMA \rangle$  between-equality neg-not-pos
              outer-transitivity-between2)
          }
        }
      }
    }
  }
}

```

```

    ultimately show  $(PO - CMB - E) \vee (PO - E - CMB)$ 
    using  $\langle Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
       $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \rangle$ 
    by blast
  }
qed
}
moreover {
  assume  $Bet\ PO\ B\ A$ 
  have  $Bet\ PO\ B\ C$ 
  by (metis Out-def  $\langle CMB - B - C \rangle \langle PO - CMB - B \rangle \langle PO \neq B \rangle$ 
     $\langle Sum\ PO\ E\ E'\ B\ CMB\ C \rangle$  l14-36-a)
  have  $Bet\ B\ A\ C \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle A \neq B \rangle$  assms(1) between-equality by blast
  hence  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  by (meson Bet-cases  $\langle B \neq C \rangle \langle PO - B - A \rangle \langle PO - B - C \rangle \langle PO \neq B \rangle$ 
    assms(1) between-equality l5-2)
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
by blast
}
hence  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
using Ng-def  $\langle Ng\ PO\ E\ B \rangle$  by blast
}
hence  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
using Ng-def  $\langle Ng\ PO\ E\ A \rangle$  by auto
}
moreover {
  assume  $Bet\ PO\ B\ CMB$  and  $Bet\ B\ CMB\ C$ 
  {
    assume  $Bet\ A\ PO\ E$ 
    {
      assume  $Bet\ B\ PO\ E$ 
      obtain  $BMA$  where  $Sum\ PO\ E\ E'\ A\ BMA\ B$  and  $Ps\ PO\ E\ BMA$ 
      using LtP-def assms(2) diff-sum by blast
      hence ParallelogramFlat  $PO\ A\ B\ BMA$ 
      by (metis  $\langle PO \neq A \rangle$  diff-to-plg local.sum-diff)
      hence  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
         $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B$ 
      using plgf-bet by blast
    }
    {
      assume  $Bet\ PO\ A\ B$ 
      have  $Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (meson Bet-cases  $\langle PO - A - B \rangle \langle PO \neq A \rangle$  between-equality)
      moreover have  $Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle A \neq B \rangle \langle PO - A - B \rangle$  between-equality-2 by auto
      moreover have  $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (metis Bet-cases Ng-def O-not-positive  $\langle A - PO - E \rangle \langle E \neq PO \rangle$ 
         $\langle Ps\ PO\ E\ BMA \rangle$  between-exchange3 pos-not-neg)
      moreover have  $Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \implies Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      by (meson Bet-cases Ps-def  $\langle A - PO - E \rangle \langle PO \neq A \rangle \langle Ps\ PO\ E\ BMA \rangle$ 
        not-bet-and-out outer-transitivity-between)
      ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
      using  $\langle Bet\ BMA\ B\ PO \wedge Bet\ B\ PO\ A \vee Bet\ BMA\ PO\ B \wedge Bet\ PO\ B\ A \vee$ 
         $Bet\ PO\ BMA\ A \wedge Bet\ BMA\ A\ B \vee Bet\ PO\ A\ BMA \wedge Bet\ A\ BMA\ B \rangle$ 
      by blast
    }
  }
}
moreover {
  assume  $Bet\ PO\ B\ A$ 
  have  $Bet\ PO\ B\ C$ 
  using  $\langle B - CMB - C \rangle \langle B - PO - E \rangle \langle E \neq PO \rangle \langle PO - B - CMB \rangle$ 
     $\langle (PO - CMB - B) \implies (CMB - B - C) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
    between-equality
    outer-transitivity-between by blast
  have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  proof (cases  $B = CMB$ )

```

```

    show  $B = CMB \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
    using  $\langle B - CMB - C \rangle \langle PO - B - CMB \rangle$ 
     $\langle (PO - CMB - B) \implies (CMB - B - C) \implies (PO - CMB - E) \vee (PO - E - CMB) \rangle$ 
    by auto
    show  $B \neq CMB \implies (PO - CMB - E) \vee (PO - E - CMB)$ 
    by (metis  $\langle B \neq C \rangle \langle PO - B - A \rangle \langle PO - B - C \rangle \langle PO \neq B \rangle$  assms(1)
        between-equality between-symmetry calculation l5-2)
  qed
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle A - PO - E \rangle \langle B - PO - E \rangle \langle E \neq PO \rangle$  between-symmetry l5-2 by blast
}
hence  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using Ng-def  $\langle Ng\ PO\ E\ B \rangle$  by blast
}
hence  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using Ng-def  $\langle Ng\ PO\ E\ A \rangle$  by auto
}
ultimately have  $Bet\ PO\ CMB\ E \vee Bet\ PO\ E\ CMB$ 
  using  $\langle Bet\ CMB\ C\ PO \wedge Bet\ C\ PO\ B \vee Bet\ CMB\ PO\ C \wedge Bet\ PO\ C\ B \vee$ 
     $Bet\ PO\ CMB\ B \wedge Bet\ CMB\ B\ C \vee Bet\ PO\ B\ CMB \wedge Bet\ B\ CMB\ C \rangle$ 
  by blast
}
ultimately show ?thesis
  by blast
qed
ultimately have  $Ps\ PO\ E\ CMB$ 
  by (metis Out-cases Ps-def  $\langle B \neq C \rangle \langle Diff\ PO\ E\ E'\ C\ B\ CMB \rangle$  bet-out-1
    between-symmetry diff-null-eq)
thus  $LeP\ PO\ E\ E'\ B\ C$ 
  using LeP-def LtP-def  $\langle Diff\ PO\ E\ E'\ C\ B\ CMB \rangle$  by auto
}
qed
qed
qed
}
qed

```

lemma bet-lt12-le13:
 assumes $Bet\ A\ B\ C$
 and $LtP\ PO\ E\ E'\ A\ B$
 shows $LeP\ PO\ E\ E'\ A\ C$
 using LeP-def assms(1,2) bet-lt12-le23 leP-trans by blast

lemma bet-lt21-le32:
 assumes $Bet\ A\ B\ C$
 and $LtP\ PO\ E\ E'\ B\ A$
 shows $LeP\ PO\ E\ E'\ C\ B$
 proof (cases $B = C$)
 show $B = C \implies LeP\ PO\ E\ E'\ C\ B$
 by (simp add: leP-refl)
 {
 assume $B \neq C$
 have $\neg Col\ PO\ E\ E'$
 using Ar2-def assms(2) ltP-ar2 by blast
 moreover have $Col\ PO\ E\ B$
 using Ar2-def assms(2) ltP-ar2 by auto
 moreover have $Col\ PO\ E\ C$
 by (metis Ar2-def assms(1,2) bet-col colx ltP-ar2 ltP-neq)
 ultimately obtain BMC where $Diff\ PO\ E\ E'\ B\ C\ BMC$
 using diff-exists by blast
 obtain MA where $Opp\ PO\ E\ E'\ A\ MA$
 using Ar2-def assms(2) ltP-ar2 opp-exists by presburger
 obtain MB where $Opp\ PO\ E\ E'\ B\ MB$
 using $\langle Col\ PO\ E\ B \rangle \langle \neg Col\ PO\ E\ E' \rangle$ opp-exists by blast
 obtain MC where $Opp\ PO\ E\ E'\ C\ MC$

```

    using Diff-def ⟨Diff PO E E' B C BMC⟩ by auto
have Col PO E MA
  by (meson Ar2-def ⟨Opp PO E E' A MA⟩ diff-O-A diff-ar2)
have Col PO E MB
  by (meson Ar2-def ⟨Opp PO E E' B MB⟩ diff-O-A diff-ar2)
have Col PO E MC
  by (meson Ar2-def ⟨Opp PO E E' C MC⟩ diff-O-A diff-ar2)
have Col PO E BMC
  using Ar2-def ⟨Diff PO E E' B C BMC⟩ diff-ar2 by force
obtain AMB where Diff PO E E' A B AMB
  using LtP-def assms(2) by auto
obtain MAMMB where Diff PO E E' MA MB MAMMB
  using ⟨Col PO E MA⟩ ⟨Col PO E MB⟩ ⟨¬ Col PO E E'⟩ diff-exists by blast
have Col PO E MAMMB
  by (meson Ar2-def ⟨Diff PO E E' MA MB MAMMB⟩ diff-ar2)
have Opp PO E E' AMB MAMMB
proof —
  have Sum PO E E' MB A AMB
    using Diff-def ⟨Diff PO E E' A B AMB⟩ ⟨Opp PO E E' B MB⟩ ⟨¬ Col PO E E'⟩
      opp-uniqueness sum-comm by blast
  moreover have Sum PO E E' A MAMMB B
    by (metis Diff-def ⟨Diff PO E E' MA MB MAMMB⟩ ⟨Opp PO E E' A MA⟩ ⟨Opp PO E E' B MB⟩
      ⟨¬ Col PO E E'⟩ diff-sum sum-comm)
  ultimately show ?thesis
    using ⟨Diff PO E E' A B AMB⟩ diff-opp local.sum-diff by presburger
qed
obtain MBMMA where Diff PO E E' MB MA MBMMA
  using ⟨Col PO E MA⟩ ⟨Col PO E MB⟩ ⟨¬ Col PO E E'⟩ diff-exists by blast
hence Col PO E MBMMA
  by (meson Ar2-def diff-ar2)
have AMB = MBMMA
  by (meson ⟨Diff PO E E' MA MB MAMMB⟩ ⟨Diff PO E E' MB MA MBMMA⟩ ⟨Opp PO E E' AMB MAMMB⟩
    ⟨¬ Col PO E E'⟩ diff-opp opp-comm opp-uniqueness)
have Bet MA MB MC
proof (rule l7-15 [of PO A - B - C], insert assms(1))
  show PO Midpoint A MA
    using opp-midpoint[of PO E E' A MA] by (simp add: ⟨Opp PO E E' A MA⟩)
  show PO Midpoint B MB
    using opp-midpoint [of PO E E' B MB] by (simp add: ⟨Opp PO E E' B MB⟩)
  show PO Midpoint C MC
    using opp-midpoint [of PO E E' C MC] by (simp add: ⟨Opp PO E E' C MC⟩)
qed
obtain MBMMC where Diff PO E E' MB MC MBMMC
  using ⟨Col PO E MB⟩ ⟨Col PO E MC⟩ ⟨¬ Col PO E E'⟩ diff-exists by blast
hence Col PO E MBMMC
  using Ar2-def diff-ar2 by force
have Opp PO E E' BMC MBMMC
proof —
  have Sum PO E E' MC B BMC
    using Diff-def ⟨Diff PO E E' B C BMC⟩ ⟨Opp PO E E' C MC⟩ ⟨¬ Col PO E E'⟩ opp-uniqueness
      sum-comm by blast
  moreover have Sum PO E E' B MBMMC C
proof —
  have Sum PO E E' MB C MBMMC
    by (meson ⟨Col PO E MB⟩ ⟨Diff PO E E' MB MC MBMMC⟩ ⟨Opp PO E E' C MC⟩ ⟨¬ Col PO E E'⟩
      diff-A-O diff-O-A opp-comm sum-diff-diff)
  moreover have Sum PO E E' B MB PO
    using Opp-def ⟨Opp PO E E' B MB⟩ sum-opp by blast
  moreover have Sum PO E E' PO C C
    by (simp add: ⟨Col PO E C⟩ ⟨¬ Col PO E E'⟩ sum-O-B)
  ultimately show ?thesis
    using sum-assoc-2 by blast
qed
ultimately show ?thesis
  using ⟨Diff PO E E' B C BMC⟩ diff-opp local.sum-diff by auto
qed

```

```

obtain MCMMB where Diff PO E E' MC MB MCMMB
  using  $\langle \text{Col } PO \ E \ MB \rangle \ \langle \text{Col } PO \ E \ MC \rangle \ \langle \neg \ \text{Col } PO \ E \ E' \rangle \ \text{diff-exists}$  by blast
hence Opp PO E E' MBMMC MCMMB
  using  $\langle \text{Diff } PO \ E \ E' \ MB \ MC \ MBMMC \rangle \ \text{diff-opp}$  by auto
hence BMC = MCMMB
  using  $\langle \text{Opp } PO \ E \ E' \ BMC \ MBMMC \rangle \ \langle \neg \ \text{Col } PO \ E \ E' \rangle \ \text{opp-comm opp-uniqueness}$  by blast
have LtP PO E E' MA MB
  using LtP-def  $\langle \text{AMB} = \text{MBMMA} \rangle \ \langle \text{Diff } PO \ E \ E' \ A \ B \ \text{AMB} \rangle \ \langle \text{Diff } PO \ E \ E' \ MB \ MA \ \text{MBMMA} \rangle \ \text{assms}(2)$ 
  lt-diff-ps by blast
have LeP PO E E' MB MC
  using  $\langle \text{LtP } PO \ E \ E' \ MA \ MB \rangle \ \langle \text{MA} - \text{MB} - \text{MC} \rangle \ \text{bet-lt12-le23}$  by auto
thus LeP PO E E' C B
  by (metis LeP-def LtP-def  $\langle \text{BMC} = \text{MCMMB} \rangle \ \langle \text{Diff } PO \ E \ E' \ B \ C \ \text{BMC} \rangle$ 
     $\langle \text{Diff } PO \ E \ E' \ MB \ MC \ MBMMC \rangle \ \langle \text{Diff } PO \ E \ E' \ MC \ MB \ \text{MCMMB} \rangle$ 
     $\langle \text{Opp } PO \ E \ E' \ BMC \ MBMMC \rangle \ \text{diff-null-eq lt-diff-ps pos-null-neg}$ )
}
qed

```

```

lemma bet-lt21-le31:
  assumes Bet A B C
  and LtP PO E E' B A
  shows LeP PO E E' C A
  using LeP-def assms(1,2) bet-lt21-le32 leP-trans by blast

```

```

lemma opp-2-le-le:
  assumes Opp PO E E' A MA
  and Opp PO E E' B MB
  and LeP PO E E' A B
  shows LeP PO E E' MB MA
proof –
  obtain MAMB where Sum PO E E' MA MB MAMB
    by (meson Ar2-def Opp-def assms(1,2) sum-ar2 sum-exists)
  have Sum PO E E' B MAMB MA
    using Diff-def  $\langle \text{Sum } PO \ E \ E' \ MA \ MB \ \text{MAMB} \rangle \ \text{assms}(2) \ \text{diff-sum}$  by blast
  moreover have Sum PO E E' A MAMB MB
    by (meson Diff-def  $\langle \text{Sum } PO \ E \ E' \ MA \ MB \ \text{MAMB} \rangle \ \text{assms}(1) \ \text{calculation diff-sum sum3-col sum-comm}$ 
      sum-to-sum3)
  ultimately show ?thesis
    using assms(3) compatibility-of-sum-with-order by blast
qed

```

```

lemma diff-2-le-le:
  assumes Diff PO E E' A C AMC
  and Diff PO E E' B C BMC
  and LeP PO E E' A B
  shows LeP PO E E' AMC BMC
  by (metis Ar2-def Diff-def assms(1,2,3) compatibility-of-sum-with-order diff-ar2
    opp-uniqueness)

```

```

lemma length-pos:
  assumes Length PO E E' A B L
  shows LeP PO E E' PO L
  using Length-def assms by force

```

```

lemma length-id-1:
  assumes Length PO E E' A B PO
  shows A = B
  by (meson Length-def assms cong-reverse-identity)

```

```

lemma length-id-2:
  assumes PO ≠ E
  shows Length PO E E' A A PO

```

by (meson Length-def assms cong-trivial-identity leP-refl
not-col-distincts)

lemma length-id:
shows Length PO E E' A B PO $\longleftrightarrow$ (A = B $\wedge$ PO $\neq$ E)
by (metis Length-def length-id-1 length-id-2)

lemma length-eq-cong-1:
assumes Length PO E E' A B AB
and Length PO E E' C D AB
shows Cong A B C D
by (meson Length-def assms(1,2) cong-inner-transitivity)

lemma length-eq-cong-2:
assumes Length PO E E' A B AB
and Cong A B C D
shows Length PO E E' C D AB
by (metis Length-def assms(1,2) cong-transitivity)

lemma ltP-pos:
assumes LtP PO E E' PO A
shows Ps PO E A
using Ar2-def assms diff-A-O ltP-ar2 lt-diff-ps by presburger

lemma bet-leP:
assumes Bet PO AB CD
and LeP PO E E' PO AB
and LeP PO E E' PO CD
shows LeP PO E E' AB CD
using LeP-def assms(1,2,3) bet-lt12-le23 by blast

lemma leP-bet:
assumes LeP PO E E' AB CD
and LeP PO E E' PO AB
and LeP PO E E' PO CD
shows Bet PO AB CD

proof –
{
 assume LtP PO E E' AB CD
 then obtain X where Diff PO E E' CD AB X and Ps PO E X
 using LtP-def by auto
 hence PO Out AB X $\vee$ AB = PO
 by (metis LeP-def Ps-def assms(2) l6-6 l6-7 ltP-pos)
 moreover have PO Out AB X $\implies$ Bet PO AB CD
 using $\langle$ Diff PO E E' CD AB X $\rangle$ diff-sum l14-36-a by blast
 moreover have AB = PO $\implies$ Bet PO AB CD
 by (simp add: between-trivial2)
 ultimately have Bet PO AB CD
 by blast
}
moreover have AB = CD $\implies$ Bet PO AB CD
by (simp add: between-trivial)
ultimately show ?thesis
using LeP-def assms(1) by blast
qed

lemma length-Ar2:
assumes Length PO E E' A B AB
shows (Col PO E AB $\wedge$ $\neg$ Col PO E E') $\vee$ AB = PO
proof –
 have LtP PO E E' PO AB $\vee$ PO = AB
 using LeP-def assms length-pos by auto
 moreover have LtP PO E E' PO AB $\implies$ (Col PO E AB $\wedge$ $\neg$ Col PO E E')
 using Ar2-def ltP-ar2 by presburger
 ultimately show ?thesis by blast
qed


```

lemma length-leP-le-1:
  assumes Length PO E E' A B AB
    and Length PO E E' C D CD
    and LeP PO E E' AB CD
  shows A B Le C D
proof -
  have Bet PO AB CD
    using assms(1,2,3) leP-bet length-pos by blast
  obtain M' where Bet D C M' and Cong C M' A B
    using segment-construction by blast
  then obtain M where C Midpoint M' M
    using symmetric-point-construction by auto
  hence Cong A B C M
    using Cong-cases Midpoint-def ⟨Cong C M' A B⟩ cong-inner-transitivity by blast
  show ?thesis
proof (rule le-transitivity [of - - C M])
  show A B Le C M
    by (simp add: ⟨Cong A B C M⟩ cong--le)
  have PO AB Le PO CD
    by (simp add: ⟨PO - AB - CD⟩ bet--le1213)
  thus C M Le C D
    by (meson Length-def ⟨Cong A B C M⟩ assms(1,2) cong-transitivity l5-6)
qed
qed

lemma length-leP-le-2:
  assumes Length PO E E' A B AB
    and Length PO E E' C D CD
    and A B Le C D
  shows LeP PO E E' AB CD
proof -
  have Col PO E AB ∧ ¬ Col PO E E' ∨ AB = PO
    using assms(1) length-Ar2 by auto
  have Col PO E CD ∧ ¬ Col PO E E' ∨ CD = PO
    using assms(2) length-Ar2 by auto
  have Bet PO AB CD
  proof (cases PO = CD)
    {
      assume PO = CD
      hence C = D
        using assms(2) length-id-1 by blast
      obtain X where Bet C X C and Cong A B C X
        using Le-def ⟨C = D⟩ assms(3) by blast
      hence C = X
        by (simp add: between-identity)
      hence A = B
        using ⟨Cong A B C X⟩ cong-diff by blast
      hence PO = AB
        by (metis Length-def assms(1) cong-identity)
      thus PO - AB - CD
        using between-trivial2 by auto
    }
    {
      assume PO ≠ CD
      have PO AB Le PO CD
        by (meson Length-def assms(1,2,3) cong-symmetry l5-6)
      then obtain M where Bet PO M CD and Cong PO AB PO M
        using Le-def by blast
      {
        assume Col PO E AB and ¬ Col PO E E' and Col PO E CD and ¬ Col PO E E'
        obtain P where Bet C P D and Cong A B C P
          using Le-def assms(3) by blast
        {
          assume LtP PO E E' PO AB

```

```

{
  assume LtP PO E E' PO CD
  obtain X where Diff PO E E' AB PO X and Ps PO E X
    using LtP-def  $\langle \text{LtP PO E E' PO AB} \rangle$  by blast
  obtain Y where Diff PO E E' CD PO Y and Ps PO E Y
    using LtP-def  $\langle \text{LtP PO E E' PO CD} \rangle$  by blast
  have ParallelogramFlat PO PO AB X
    by (metis O-not-positive ParallelogramFlat-def  $\langle \text{Col PO E AB} \rangle$ 
       $\langle \text{Diff PO E E' AB PO X} \rangle$   $\langle \text{Ps PO E X} \rangle$   $\langle \neg \text{Col PO E E'} \rangle$  col-trivial-1
      cong-pseudo-reflexivity cong-trivial-identity diff-A-O
      diff-uniqueness)
  have ParallelogramFlat PO PO CD Y
    using  $\langle \text{Diff PO E E' CD PO Y} \rangle$   $\langle \text{PO} \neq \text{CD} \rangle$  diff-to-plg by auto
  have  $\text{PO} \neq \text{AB} \vee \text{PO} \neq \text{X}$ 
    using O-not-positive  $\langle \text{Ps PO E X} \rangle$  by blast
  have  $\text{PO} \neq \text{CD} \vee \text{PO} \neq \text{Y}$ 
    by (simp add:  $\langle \text{PO} \neq \text{CD} \rangle$ )
  have Col AB PO M
  proof -
    have PO Out AB CD
      by (metis Bet-cases[of CD M PO] Bet-cases[of - PO AB]
        Bet-cases[of - PO CD] Out-cases[of PO E CD] Ps-def[of PO E CD]
        Ps-def[of PO E AB]  $\langle \text{Cong PO AB PO M} \rangle$   $\langle \text{PO} - \text{M} - \text{CD} \rangle$ 
         $\langle \text{ParallelogramFlat PO PO AB X} \rangle$   $\langle \text{ParallelogramFlat PO PO CD Y} \rangle$ 
         $\langle \text{Ps PO E X} \rangle$   $\langle \text{Ps PO E Y} \rangle$  bet-out-1[of AB PO CD]
        bet-out--bet[of E PO - CD] bet-out--bet[of AB PO - E]
        between-cong-3[of - PO AB M]
        between-inner-transitivity[of - PO CD M]
        plgf-trivial-neq[of PO CD Y] plgf-trivial-neq[of PO AB X]
        point-construction-different[of AB PO])
      thus ?thesis
        using Col-def  $\langle \text{Cong PO AB PO M} \rangle$   $\langle \text{PO AB Le PO CD} \rangle$   $\langle \text{PO} - \text{M} - \text{CD} \rangle$  between-cong
          l5-3 l6-13-1 by blast
    qed
  hence  $\text{AB} = \text{M} \vee \text{PO}$  Midpoint AB M
    using l7-20  $\langle \text{Cong PO AB PO M} \rangle$  by simp
  hence  $\text{PO} - \text{AB} - \text{CD}$ 
    by (metis Bet-cases Cong-cases Midpoint-def Out-cases Ps-def  $\langle \text{PO} - \text{M} - \text{CD} \rangle$ 
       $\langle \text{ParallelogramFlat PO PO AB X} \rangle$   $\langle \text{ParallelogramFlat PO PO CD Y} \rangle$   $\langle \text{Ps PO E X} \rangle$ 
       $\langle \text{Ps PO E Y} \rangle$  bet-cong-eq bet-out--bet between-equality-2
      plgf-trivial-neq)
}
moreover have  $\text{PO} = \text{CD} \implies (\text{PO} - \text{AB} - \text{CD})$ 
  by (simp add:  $\langle \text{PO} \neq \text{CD} \rangle$ )
ultimately have  $\text{PO} - \text{AB} - \text{CD}$ 
  using LeP-def assms(2) length-pos by blast
}
moreover have  $\text{PO} = \text{AB} \implies (\text{PO} - \text{AB} - \text{CD})$ 
  using between-trivial2 by auto
ultimately have  $\text{PO} - \text{AB} - \text{CD}$ 
  using LeP-def assms(1) length-pos by blast
}
moreover have  $\text{Col PO E AB} \wedge \neg \text{Col PO E E'} \wedge \text{CD} = \text{PO} \implies (\text{PO} - \text{AB} - \text{CD})$ 
  using  $\langle \text{PO} \neq \text{CD} \rangle$  by blast
moreover have  $\text{AB} = \text{PO} \wedge \text{Col PO E CD} \wedge \neg \text{Col PO E E'} \implies (\text{PO} - \text{AB} - \text{CD})$ 
  using between-trivial2 by blast
moreover have  $\text{AB} = \text{PO} \wedge \text{CD} = \text{PO} \implies (\text{PO} - \text{AB} - \text{CD})$ 
  using  $\langle \text{PO} \neq \text{CD} \rangle$  by blast
ultimately show  $\text{PO} - \text{AB} - \text{CD}$ 
  using  $\langle \text{Col PO E AB} \wedge \neg \text{Col PO E E'} \vee \text{AB} = \text{PO} \rangle$ 
     $\langle \text{Col PO E CD} \wedge \neg \text{Col PO E E'} \vee \text{CD} = \text{PO} \rangle$  by presburger
}
qed
thus ?thesis
  using assms(1,2) bet-leP length-pos by force
qed

```

lemma *l15-3*:
assumes *Sum PO E E' A B C*
shows *Cong PO B A C*
by (*metis Ar2-def assms cong-reflexivity sum-ar2 sum-cong2 sum-iff-cong-b*)

lemma *length-uniqueness*:
assumes *Length PO E E' A B AB*
and *Length PO E E' A B AB'*
shows $AB = AB'$
by (*meson assms(1,2) leP-asymp leP-refl length-leP-le-1 length-leP-le-2*)

lemma *length-Ps*:
assumes $AB \neq PO$
and *Length PO E E' A B AB*
shows *Ps PO E AB*
using *LeP-def assms(1,2) length-pos ltP-pos* **by** *blast*

lemma *length-not-col-null*:
assumes *Col PO E E'*
and *Length PO E E' A B AB*
shows $AB = PO$
using *assms(1,2) length-Ar2* **by** *blast*

lemma *not-triangular-equality1*:
assumes $PO \neq E$
shows $\neg (\forall E' B C AB BC AC. \text{Bet } A B C \wedge \text{Length } PO E E' A B AB \wedge \text{Length } PO E E' B C BC \wedge$
 $\text{Length } PO E E' A C AC \longrightarrow \text{Sum } PO E E' AB BC AC)$
by (*metis Ar2-def Sum-def assms between-trivial length-id*)

lemma *triangular-equality*:
assumes $PO \neq E$
and *Bet A B C*
and *IsLength PO E E' A B AB*
and *IsLength PO E E' B C BC*
and *IsLength PO E E' A C AC*
shows *Sumg PO E E' AB BC AC*

proof –
have *Length PO E E' A B AB $\vee$ PO = E $\wedge$ PO = AB*
using *IsLength-def assms(3)* **by** *auto*
moreover
{
assume *Length PO E E' A B AB*
have *Length PO E E' B C BC $\vee$ PO = E $\wedge$ PO = BC*
using *IsLength-def assms(4)* **by** *presburger*
moreover {
assume *Length PO E E' B C BC*
have *Length PO E E' A C AC $\vee$ PO = E $\wedge$ PO = AC*
using *IsLength-def assms(5)* **by** *auto*
moreover {
assume *Length PO E E' A C AC*
have *LtP PO E E' PO AB $\vee$ PO = AB*
using *LeP-def Length-def <Length PO E E' A B AB>* **by** *presburger*
moreover
{
assume *LtP PO E E' PO AB*
have *LtP PO E E' PO BC $\vee$ PO = BC*
using *LeP-def Length-def <Length PO E E' B C BC>* **by** *presburger*
moreover {
assume *LtP PO E E' PO BC*
have *LtP PO E E' PO AC $\vee$ PO = AC*
using *LeP-def Length-def <Length PO E E' A C AC>* **by** *presburger*
moreover
{

```

assume  $LtP\ PO\ E\ E'\ PO\ AC$ 
obtain  $X$  where  $Diff\ PO\ E\ E'\ AB\ PO\ X$  and  $Ps\ PO\ E\ X$ 
  using  $LtP\text{-}def\ \langle LtP\ PO\ E\ E'\ PO\ AB \rangle$  by blast
obtain  $Y$  where  $Diff\ PO\ E\ E'\ BC\ PO\ Y$  and  $Ps\ PO\ E\ Y$ 
  using  $LtP\text{-}def\ \langle LtP\ PO\ E\ E'\ PO\ BC \rangle$  by blast
obtain  $Z$  where  $Diff\ PO\ E\ E'\ AC\ PO\ Z$  and  $Ps\ PO\ E\ Z$ 
  using  $LtP\text{-}def\ \langle LtP\ PO\ E\ E'\ PO\ AC \rangle$  by blast
have  $AB = X$ 
  by (meson  $Ar2\text{-}def\ \langle Diff\ PO\ E\ E'\ AB\ PO\ X \rangle\ \langle LtP\ PO\ E\ E'\ PO\ AB \rangle\ diff\text{-}A\text{-}O$ 
     $diff\text{-}uniqueness\ ltP\text{-}ar2$ )
have  $BC = Y$ 
  by (meson  $Ar2\text{-}def\ \langle Diff\ PO\ E\ E'\ BC\ PO\ Y \rangle\ \langle LtP\ PO\ E\ E'\ PO\ BC \rangle\ diff\text{-}A\text{-}O$ 
     $diff\text{-}uniqueness\ ltP\text{-}ar2$ )
have  $AC = Z$ 
  by (meson  $Ar2\text{-}def\ \langle Diff\ PO\ E\ E'\ AC\ PO\ Z \rangle\ \langle LtP\ PO\ E\ E'\ PO\ AC \rangle\ diff\text{-}A\text{-}O$ 
     $diff\text{-}uniqueness\ ltP\text{-}ar2$ )
obtain  $AC'$  where  $Sum\ PO\ E\ E'\ AB\ BC\ AC'$ 
  by (metis  $Ar2\text{-}def\ Ps\text{-}def\ \langle BC = Y \rangle\ \langle LtP\ PO\ E\ E'\ PO\ AB \rangle\ \langle Ps\ PO\ E\ Y \rangle$ 
     $col\text{-}permutation\text{-}2\ col\text{-}permutation\text{-}3\ ltP\text{-}ar2\ out\text{-}col\ sum\text{-}exists$ )
have  $Bet\ PO\ AB\ AC'$ 
  by (metis  $Out\text{-}cases\ Ps\text{-}def\ \langle AB = X \rangle\ \langle BC = Y \rangle\ \langle Ps\ PO\ E\ X \rangle\ \langle Ps\ PO\ E\ Y \rangle$ 
     $\langle Sum\ PO\ E\ E'\ AB\ BC\ AC' \rangle\ l14\text{-}36\text{-}a\ l6\text{-}7$ )
have  $Cong\ PO\ BC\ AB\ AC'$ 
  using  $\langle Sum\ PO\ E\ E'\ AB\ BC\ AC' \rangle\ l15\text{-}3$  by auto
have  $Cong\ PO\ AC'\ A\ C$ 
  by (smt (verit, ccfv-threshold)  $Cong\text{-}cases\ \langle Cong\ PO\ BC\ AB\ AC' \rangle$ 
     $\langle Length\ PO\ E\ E'\ A\ B\ AB \rangle\ \langle Length\ PO\ E\ E'\ B\ C\ BC \rangle\ \langle PO - AB - AC' \rangle\ assms(2)$ 
     $cong\text{-}transitivity\ l2\text{-}11\text{-}b\ length\text{-}cong$ )
have  $Ps\ PO\ E\ AC'$ 
  using  $\langle AB = X \rangle\ \langle BC = Y \rangle\ \langle Ps\ PO\ E\ X \rangle\ \langle Ps\ PO\ E\ Y \rangle\ \langle Sum\ PO\ E\ E'\ AB\ BC\ AC' \rangle$ 
     $sum\text{-}pos\text{-}pos$  by blast
hence  $AC = AC'$ 
  by (smt (verit, ccfv-SIG)  $Cong\text{-}cases\ Out\text{-}def\ Ps\text{-}def\ \langle Cong\ PO\ AC'\ A\ C \rangle$ 
     $\langle Length\ PO\ E\ E'\ A\ C\ AC' \rangle\ \langle LtP\ PO\ E\ E'\ PO\ AC \rangle\ between\text{-}cong$ 
     $cong\text{-}inner\text{-}transitivity\ l6\text{-}7\ length\text{-}cong\ ltP\text{-}pos$ )
hence  $Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by (simp add:  $Sumg\text{-}def\ \langle Sum\ PO\ E\ E'\ AB\ BC\ AC' \rangle$ )
}
moreover have  $PO = AC \implies Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by (metis  $O\text{-}not\text{-}positive\ \langle Length\ PO\ E\ E'\ A\ B\ AB \vee PO = E \wedge PO = AB \rangle$ 
     $\langle Length\ PO\ E\ E'\ A\ C\ AC' \rangle\ \langle LtP\ PO\ E\ E'\ PO\ AB \rangle\ assms(2)\ between\text{-}identity$ 
     $cong\text{-}reverse\text{-}identity\ length\text{-}cong\ length\text{-}id\text{-}1\ ltP\text{-}pos$ )
ultimately have  $Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by blast
}
moreover have  $PO = BC \implies Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by (metis  $Ar2\text{-}def\ Sumg\text{-}def\ \langle Length\ PO\ E\ E'\ A\ B\ AB \rangle\ \langle Length\ PO\ E\ E'\ A\ C\ AC' \rangle$ 
     $\langle Length\ PO\ E\ E'\ B\ C\ BC \rangle\ \langle LtP\ PO\ E\ E'\ PO\ AB \rangle\ length\text{-}id\ length\text{-}uniqueness$ 
     $ltP\text{-}ar2\ sum\text{-}A\text{-}O$ )
ultimately have  $Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by blast
}
moreover have  $PO = AB \implies Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by (metis  $Ar2\text{-}def\ Length\text{-}def\ Sumg\text{-}def\ \langle Length\ PO\ E\ E'\ A\ B\ AB \rangle$ 
     $\langle Length\ PO\ E\ E'\ A\ C\ AC' \rangle\ \langle Length\ PO\ E\ E'\ B\ C\ BC \rangle\ length\text{-}id$ 
     $length\text{-}not\text{-}col\text{-}null\ length\text{-}uniqueness\ sum\text{-}O\text{-}B$ )
ultimately have  $Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by blast
}
moreover have  $PO = E \wedge PO = AC \implies Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  using  $assms(1)$  by blast
ultimately have  $Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by blast
}
moreover have  $PO = E \wedge PO = BC \implies Sumg\ PO\ E\ E'\ AB\ BC\ AC$ 
  by (simp add:  $assms(1)$ )

```

```

ultimately have  $\text{Sumg } PO \ E \ E' \ AB \ BC \ AC$ 
  by blast
}
moreover have  $PO = E \wedge PO = AB \implies \text{Sumg } PO \ E \ E' \ AB \ BC \ AC$ 
  using assms(1) by auto
ultimately show  $\text{Sumg } PO \ E \ E' \ AB \ BC \ AC$ 
  by blast
qed

lemma length-O:
  assumes  $PO \neq E$ 
  shows  $\text{Length } PO \ E \ E' \ PO \ PO \ PO$ 
  by (simp add: assms length-id)

lemma triangular-equality-bis:
  assumes  $A \neq B \vee A \neq C \vee B \neq C$ 
  and  $PO \neq E$ 
  and  $\text{Bet } A \ B \ C$ 
  and  $\text{Length } PO \ E \ E' \ A \ B \ AB$ 
  and  $\text{Length } PO \ E \ E' \ B \ C \ BC$ 
  and  $\text{Length } PO \ E \ E' \ A \ C \ AC$ 
  shows  $\text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
proof -
  have  $\text{LtP } PO \ E \ E' \ PO \ AB \vee PO = AB$ 
    using LeP-def Length-def assms(4) by presburger
  moreover have  $\text{LtP } PO \ E \ E' \ PO \ BC \vee PO = BC$ 
    using LeP-def Length-def assms(5) by presburger
  moreover have  $\text{LtP } PO \ E \ E' \ PO \ AC \vee PO = AC$ 
    using LeP-def Length-def assms(6) by presburger
  moreover {
    assume  $\text{LtP } PO \ E \ E' \ PO \ AB$  and  $\text{LtP } PO \ E \ E' \ PO \ BC$  and  $\text{LtP } PO \ E \ E' \ PO \ AC$ 
    obtain  $X$  where  $\text{Diff } PO \ E \ E' \ AB \ PO \ X$  and  $\text{Ps } PO \ E \ X$ 
      using LtP-def  $\langle \text{LtP } PO \ E \ E' \ PO \ AB \rangle$  by blast
    obtain  $Y$  where  $\text{Diff } PO \ E \ E' \ BC \ PO \ Y$  and  $\text{Ps } PO \ E \ Y$ 
      using LtP-def  $\langle \text{LtP } PO \ E \ E' \ PO \ BC \rangle$  by blast
    obtain  $Z$  where  $\text{Diff } PO \ E \ E' \ AC \ PO \ Z$  and  $\text{Ps } PO \ E \ Z$ 
      using LtP-def  $\langle \text{LtP } PO \ E \ E' \ PO \ AC \rangle$  by blast
    have  $AB = X$ 
      by (meson Ar2-def  $\langle \text{Diff } PO \ E \ E' \ AB \ PO \ X \rangle \langle \text{LtP } PO \ E \ E' \ PO \ AB \rangle$  diff-A-O
        diff-stable ltP-ar2)
    have  $BC = Y$ 
      by (meson Ar2-def  $\langle \text{Diff } PO \ E \ E' \ BC \ PO \ Y \rangle \langle \text{LtP } PO \ E \ E' \ PO \ BC \rangle$  diff-A-O
        diff-stable ltP-ar2)
    have  $AC = Z$ 
      by (meson Ar2-def  $\langle \text{Diff } PO \ E \ E' \ AC \ PO \ Z \rangle \langle \text{LtP } PO \ E \ E' \ PO \ AC \rangle$  diff-A-O
        diff-stable ltP-ar2)
    obtain  $AC'$  where  $\text{Sum } PO \ E \ E' \ AB \ BC \ AC'$ 
      by (metis Ar2-def Ps-def  $\langle BC = Y \rangle \langle \text{LtP } PO \ E \ E' \ PO \ AB \rangle \langle \text{Ps } PO \ E \ Y \rangle$ 
        col-permutation-5 ltP-ar2 out-col sum-exists)
    hence  $\text{Bet } PO \ AB \ AC'$ 
      by (metis Ps-def  $\langle AB = X \rangle \langle BC = Y \rangle \langle \text{Ps } PO \ E \ X \rangle \langle \text{Ps } PO \ E \ Y \rangle$  l14-36-a l6-6
        l6-7)
    have  $\text{Cong } PO \ BC \ AB \ AC'$ 
      using  $\langle \text{Sum } PO \ E \ E' \ AB \ BC \ AC' \rangle$  l15-3 by auto
    have  $\text{Cong } PO \ AC' \ A \ C$ 
      by (smt (verit, ccfv-threshold) Cong-cases  $\langle \text{Cong } PO \ BC \ AB \ AC' \rangle$ 
         $\langle PO - AB - AC' \rangle$  assms(3,4,5) cong-transitivity l2-11-b
        length-cong)
    have  $\text{Ps } PO \ E \ AC'$ 
      using  $\langle AB = X \rangle \langle BC = Y \rangle \langle \text{Ps } PO \ E \ X \rangle \langle \text{Ps } PO \ E \ Y \rangle \langle \text{Sum } PO \ E \ E' \ AB \ BC \ AC' \rangle$ 
        sum-pos-pos by blast
    have  $AC = AC'$ 
      by (smt (verit, ccfv-threshold) Col-cases Cong-cases Out-cases Out-def
        Ps-def  $\langle AC = Z \rangle \langle \text{Cong } PO \ AC' \ A \ C \rangle \langle \text{Ps } PO \ E \ AC' \rangle \langle \text{Ps } PO \ E \ Z \rangle$  assms(6)
        bet-cong-eq bet-out--bet col-permutation-2 col-transitivity-2
        cong-inner-transitivity l6-4-2 length-cong not-bet-and-out)
  }

```

```

    out-col)
  hence  $\text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
    by (simp add:  $\langle \text{Sum } PO \ E \ E' \ AB \ BC \ AC \rangle$ )
}
moreover have  $LtP \ PO \ E \ E' \ PO \ AB \wedge LtP \ PO \ E \ E' \ PO \ BC \wedge PO = AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  using assms(1,3,6) between-equality between-trivial length-id by force
moreover have  $LtP \ PO \ E \ E' \ PO \ AB \wedge PO = BC \wedge LtP \ PO \ E \ E' \ PO \ AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  by (metis assms(4,5,6) calculation(4) length-Ar2 length-id-1 length-uniqueness sum-A-O)
moreover have  $PO = AB \wedge LtP \ PO \ E \ E' \ PO \ BC \wedge LtP \ PO \ E \ E' \ PO \ AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  by (metis Ar2-def assms(4,5,6) length-id-1 length-uniqueness ltP-ar2
    sum-O-B)
moreover have  $PO = AB \wedge PO = BC \wedge LtP \ PO \ E \ E' \ PO \ AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  by (metis Ps-def assms(4,5,6) between-trivial2 cong-reverse-identity
    length-cong length-id-1 ltP-pos not-bet-and-out)
moreover have  $PO = AB \wedge LtP \ PO \ E \ E' \ PO \ BC \wedge PO = AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  by (metis Ps-def assms(4,5,6) between-trivial2 cong-reverse-identity
    length-cong length-id-1 ltP-pos not-bet-and-out)
moreover have  $LtP \ PO \ E \ E' \ PO \ AB \wedge PO = BC \wedge PO = AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  using assms(1,3,6) between-equality between-trivial length-id-1
  by blast
moreover have  $PO = AB \wedge PO = BC \wedge PO = AC \implies \text{Sum } PO \ E \ E' \ AB \ BC \ AC$ 
  using assms(1,4,6) length-id-1 by blast
ultimately show ?thesis
  by blast
qed

```

```

lemma length-out:
  assumes  $A \neq B$ 
    and  $C \neq D$ 
    and  $\text{Length } PO \ E \ E' \ A \ B \ AB$ 
    and  $\text{Length } PO \ E \ E' \ C \ D \ CD$ 
  shows  $PO \ \text{Out } AB \ CD$ 
  by (metis (full-types) Length-def Ps-def assms(1,2,3,4) bet-out--bet
    cong-reverse-identity l6-3-2 length-Ps
    point-construction-different)

```

```

lemma image-preserves-bet1:
  assumes  $\text{Bet } A \ B \ C$ 
    and  $A \ A' \ \text{Reflect } X \ Y$ 
    and  $B \ B' \ \text{Reflect } X \ Y$ 
    and  $C \ C' \ \text{Reflect } X \ Y$ 
  shows  $\text{Bet } A' \ B' \ C'$ 
  using assms(1,2,3,4) image-gen-preserves-bet by blast

```

```

lemma image-preserves-col:
  assumes  $\text{Col } A \ B \ C$ 
    and  $A \ A' \ \text{Reflect } X \ Y$ 
    and  $B \ B' \ \text{Reflect } X \ Y$ 
    and  $C \ C' \ \text{Reflect } X \ Y$ 
  shows  $\text{Col } A' \ B' \ C'$ 
  using Col-def assms(1,2,3,4) image-gen-preserves-bet by presburger

```

```

lemma image-preserves-out:
  assumes  $A \ \text{Out } B \ C$ 
    and  $A \ A' \ \text{Reflect } X \ Y$ 
    and  $B \ B' \ \text{Reflect } X \ Y$ 
    and  $C \ C' \ \text{Reflect } X \ Y$ 
  shows  $A' \ \text{Out } B' \ C'$ 
proof -
  have  $B' \neq A'$ 
    using assms(1,2,3) l10-2-uniqueness out-distinct by blast
  moreover have  $C' \neq A'$ 
    using assms(1,2,4) l10-2-uniqueness out-diff2 by blast
  moreover have  $\text{Bet } A' \ B' \ C' \vee \text{Bet } A' \ C' \ B'$ 

```

by (meson Out-def assms(1,2,3,4) image-preserves-bet1)
ultimately show ?thesis
using Out-def by blast
qed

lemma project-preserves-out:

assumes A Out B C

and $\neg A$ B Par X Y

and A A' Proj P Q X Y

and B B' Proj P Q X Y

and C C' Proj P Q X Y

shows A' Out B' C'

proof –

have $B' \neq A'$

by (metis assms(1,2,3,4) ker-par out-distinct)

moreover have $C' \neq A'$

by (metis Out-def assms(1,2,3,5) ker-par not-par-not-col out-col
par-trans)

moreover have $Bet A' B' C' \vee Bet A' C' B'$

by (meson Out-def assms(1,3,4,5) project-preserves-bet)

ultimately show ?thesis

using Out-def by blast

qed

lemma conga-bet-conga:

assumes A B C CongA D E F

and $A' \neq B$

and $C' \neq B$

and $D' \neq E$

and $F' \neq E$

and $Bet A$ B A'

and $Bet C$ B C'

and $Bet D$ E D'

and $Bet F$ E F'

shows $A' B C' CongA D' E F'$

by (meson assms(1,2,3,4,5,6,7,8,9) conga-comm l11-13)

lemma thales:

assumes $PO \neq E$

and Col P A B

and Col P C D

and $\neg Col P$ A C

and A C Pj B D

and Length PO E E' P A $A1$

and Length PO E E' P B $B1$

and Length PO E E' P C $C1$

and Length PO E E' P D $D1$

and Prod PO E E' $A1$ $D1$ AD

shows Prod PO E E' $C1$ $B1$ AD

proof (cases Col PO E E')

show Col PO E E' $\implies$ Prod PO E E' $C1$ $B1$ AD

using Ar2-def Prod-def Prod-def assms(10) by presburger

{

assume $\neg Col PO E E'$

have Prod $PO E E' A1 D1 AD \vee (\neg Ar2 PO E E' A1 D1 D1 \wedge AD = PO)$

using Prod-def assms(10) by blast

moreover

{

assume Prod $PO E E' A1 D1 AD$

have Prod $PO E E' C1 B1 AD$

proof (cases $P = B$)

show $P = B \implies Prod PO E E' C1 B1 AD$

by (metis (mono-tags, lifting) Length-def Prod-def

$\langle Prod PO E E' A1 D1 AD \rangle \langle \neg Col PO E E' \rangle$

assms(3,4,5,7,8,9) length-id-2 length-uniqueness

not-col-distincts pj-trivial pj-uniqueness prod-0-r

```

    prod-O-l-eq prod-sym)
{
  assume  $P \neq B$ 
  show  $\text{Prodg } PO \ E \ E' \ C1 \ B1 \ AD$ 
  proof (cases  $A = B$ )
  {
    assume  $A = B$ 
    have  $A \ C \ \text{ParStrict } A \ D \implies \text{Prodg } PO \ E \ E' \ C1 \ B1 \ AD$ 
      by (simp add: not-par-strict-id)
    hence  $A \ C \ \text{Par } A \ D \implies \text{Prodg } PO \ E \ E' \ C1 \ B1 \ AD$ 
      by (metis Col-def Prod-def  $\langle A = B \rangle$ 
         $\langle \text{Prod } PO \ E \ E' \ A1 \ D1 \ AD \rangle$  assms(3,4,6,7,8,9) col2--eq
        length-uniqueness par-id prod-sym)
    thus  $\text{Prodg } PO \ E \ E' \ C1 \ B1 \ AD$ 
      using NCol-perm Pj-def  $\langle A = B \rangle$  assms(3,4,5) by blast
  }
  {
    assume  $A \neq B$ 
    have  $\neg \text{Col } PO \ A1 \ E'$ 
      by (metis assms(4,6) col-permutation-5 col-trivial-3 colx length-Ar2
        length-id-1)
    then obtain  $C1'$  where  $P \ A \ C \ \text{Cong3 } PO \ A1 \ C1' \text{ and } PO \ A1 \ OS \ E' \ C1'$ 
      using assms(4,6) l10-16 length-cong by meson
    have  $P \ A \ C \ \text{CongA } PO \ A1 \ C1'$ 
      by (metis  $\langle P \ A \ C \ \text{Cong3 } PO \ A1 \ C1' \rangle$  assms(4) col-trivial-1 col-trivial-2
        cong3-conga)
    have  $\neg \text{Col } PO \ C1 \ C1'$ 
      by (metis  $\langle PO \ A1 \ OS \ E' \ C1' \rangle$  assms(1,4,6,8) col-transitivity-1 col-trivial-3
        length-Ar2 length-id-1 one-side-not-col124)
    obtain  $M$  where  $M \ \text{Midpoint } C1 \ C1'$ 
      using MidR-uniq-aux by blast
    obtain  $D1'$  where  $D1' \ D1 \ \text{Reflect } PO \ M$ 
      using l10-2-existence by fastforce
    moreover {
      assume  $PO \neq M$  and  $D1' \ D1 \ \text{ReflectL } PO \ M$ 
      then obtain  $N$  where  $N \ \text{Midpoint } D1 \ D1' \text{ and } \text{Col } PO \ M \ N$ 
        using ReflectL-def by blast
      have  $PO \ \text{Out } C1 \ D1$ 
      proof (rule length-out [of  $P \ C \ P \ D - E \ E'$ ])
      show  $P \neq C$ 
        using assms(4) not-col-distincts by blast
      {
        assume  $P = D$ 
        have  $PO = AD$ 
          using  $\langle P = D \rangle \langle \text{Prod } PO \ E \ E' \ A1 \ D1 \ AD \rangle$  assms(9)
            cong-reverse-identity length-cong prod-O-r-eq
          by blast
        have  $A \ C \ \text{Par } B \ P \implies \text{False}$ 
          by (metis assms(2,4) not-col-distincts par-col2-par-bis par-id-5)
        hence  $\text{False}$ 
          using Pj-def  $\langle P = D \rangle \langle P \neq B \rangle$  assms(5) by blast
      }
    }
    thus  $P \neq D$ 
      by blast
    show  $\text{Length } PO \ E \ E' \ P \ C \ C1$ 
      by (simp add: assms(8))
    show  $\text{Length } PO \ E \ E' \ P \ D \ D1$ 
      by (simp add: assms(9))
  }
  qed
  have  $PO \ \text{Out } A1 \ C1$ 
    by (metis assms(4,6,8) col-trivial-1 col-trivial-3 length-out)
  have  $M \neq C1$ 
    using  $\langle M \ \text{Midpoint } C1 \ C1' \rangle \langle \neg \text{Col } PO \ C1 \ C1' \rangle$  col-trivial-2 is-midpoint-id
    by blast
  have  $\text{Per } PO \ M \ C1$ 
  proof -

```



```

have Cong PO C1 PO C1'
  by (meson Cong3-def ⟨P A C Cong3 PO A1 C1'⟩ assms(8)
      cong-inner-transitivity length-cong)
thus ?thesis
  using Per-def ⟨M Midpoint C1 C1'⟩ by blast
qed
have M PO Perp C1 M ∨ M M Perp C1 M
  using Perp-cases ⟨M ≠ C1⟩ ⟨PO ≠ M⟩ ⟨Per PO M C1⟩
      col-per-perp col-trivial-3 by blast
have PO Out C1' D1'
proof (rule image-preserves-out [of PO C1 D1 - PO M],
      insert ⟨PO Out C1 D1⟩ image-triv)
  show C1 C1' Reflect PO M
    by (meson Cong3-def ⟨M Midpoint C1 C1'⟩
        ⟨P A C Cong3 PO A1 C1'⟩ assms(8) cong-inner-transitivity
        cong-midpoint-image l7-2 length-cong)
  show D1 D1' Reflect PO M
    by (simp add: calculation l10-4)
qed
have PO N Perp D1 N
proof (rule perp-col [of - - M], insert ⟨Col PO M N⟩)
  show PO ≠ N
    by (metis Out-def ⟨N Midpoint D1 D1'⟩ ⟨PO Out C1 D1⟩
        ⟨PO Out C1' D1'⟩ ⟨¬ Col PO C1 C1'⟩ col-transitivity-1
        not-col-sym-not-col out-col)
  show PO M Perp D1 N
  proof -
    have D1 ≠ N
      using ⟨N Midpoint D1 D1'⟩ ⟨PO Out C1 D1⟩ ⟨PO Out C1' D1'⟩
          ⟨¬ Col PO C1 C1'⟩ is-midpoint-id l6-6 l6-7 out-col
      by blast
    moreover have D1 D1' Perp PO M
      by (metis Perp-perm ReflectL-def ⟨D1' D1 ReflectL PO M⟩
          ⟨N Midpoint D1 D1'⟩ calculation l7-3)
    moreover have Col D1 D1' N
      using ⟨N Midpoint D1 D1'⟩ bet-col1 between-trivial midpoint-bet by blast
    ultimately show ?thesis
      by (metis Perp-cases perp-col)
  qed
qed
have Cong PO D1 PO D1'
  using calculation image-triv l10-10 by blast
have C1 C1' Par D1 D1'
proof -
  have PO M Perp D1 D1' ∨ D1 = D1'
    using ReflectL-def ⟨D1' D1 ReflectL PO M⟩ by blast
  moreover {
    assume PO M Perp D1 D1'
    have C1 C1' Par D1 D1'
    proof -
      {
        assume M PO Perp C1 M
        have C1 C1' Par D1 D1'
        proof (rule l12-9-2D [of - - PO M])
          show C1 C1' Perp PO M
            by (metis ⟨M Midpoint C1 C1'⟩ ⟨M ≠ C1⟩ ⟨PO ≠ M⟩
                ⟨Per PO M C1⟩ bet-cong-eq between-trivial2 col-per-perp
                midpoint-bet midpoint-col midpoint-cong
                not-cong-3412)
          show D1 D1' Perp PO M
            using Perp-cases ⟨PO M Perp D1 D1'⟩ by blast
        qed
      }
    }
  moreover have M M Perp C1 M ⇒ C1 C1' Par D1 D1'
    using perp-distinct by blast
  ultimately show ?thesis

```

```

    using ⟨M PO Perp C1 M ∨ M M Perp C1 M⟩ by blast
  qed
}
moreover have D1 = D1' ⟹ C1 C1' Par D1 D1'
  using ⟨PO Out C1 D1⟩ ⟨PO Out C1' D1'⟩ ⟨¬ Col PO C1 C1'⟩ l6-6
  l6-7 out-col by blast
ultimately show ?thesis
  by blast
qed
hence C1 C1' Pj D1 D1'
  by (simp add: Pj-def)
have P C A Cong3 PO C1' A1
  by (simp add: ⟨P A C Cong3 PO A1 C1'⟩ cong-3-swap-2)
have P C A CongA PO C1' A1
  using ⟨P C A Cong3 PO C1' A1⟩ assms(4) cong3-conga
  not-col-distincts by auto
have Cong P A PO A1 ∧ (P ≠ A ⟹ C P A CongA C1' PO A1 ∧ C A P CongA C1' A1 PO)
  using l11-49 Cong3-def ⟨P C A Cong3 PO C1' A1⟩
  ⟨P C A CongA PO C1' A1⟩ cong-3-swap by blast
moreover have P ≠ A
  using assms(4) col-trivial-1 by blast
ultimately have C P A CongA C1' PO A1 ∧ C A P CongA C1' A1 PO
  by blast
have C A P CongA D B P
proof -
{
  assume Bet C P D
  have Bet A P B
  proof (rule project-preserves-bet [of C P D - A P A C], insert ⟨C - P - D⟩)
    show C A Proj A P A C
      by (metis Par-def assms(4) grid-not-par par-col-project)
    show P P Proj A P A C
      by (metis ⟨P ≠ A⟩ assms(4) col-trivial-2 par-id-1 project-trivial)
    show D B Proj A P A C
      by (metis NCol-perm assms(2,4,5) grid-not-par
        pj-col-project pj-right-comm)
  qed
  have A B TS C D
  proof -
    have ¬ Col C A B
      using ⟨A ≠ B⟩ assms(2,4) col2--eq by blast
    moreover have ¬ Col D A B
      by (metis ⟨P ≠ B⟩ assms(2,3,4,5) calculation col-permutation-2
        colx sum-ar2 sum-exists sum-par-strict-a)
    moreover have ∃ T. Col T A B ∧ Bet C T D
    proof -
      have Col P A B
        using assms(2) by auto
      moreover have Bet C P D
        by (simp add: ⟨C - P - D⟩)
      ultimately show ?thesis
        by fast
    qed
    ultimately show ?thesis
      using TS-def by blast
  qed
  hence C A B CongA D B A ⟷ A C Par B D
    using l12-21 by simp
  have A C Par B D ⟹ C A P CongA D B P
    by (metis NCol-perm Par-cases TS-def ⟨A B TS C D⟩ ⟨P ≠ A⟩ ⟨P ≠ B⟩
      assms(2,3,4) not-par-not-col triangle-par)
  moreover have B = D ⟹ C A P CongA D B P
    using TS-def ⟨A B TS C D⟩ not-col-distincts by blast
  ultimately have C A P CongA D B P
    using Pj-def assms(5) by blast
}

```

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moreover have  $\neg \text{Bet } C \ P \ D \implies C \ A \ P \ \text{CongA } D \ B \ P$ 
  by (metis Pj-def  $\langle P \neq B \rangle$  assms(2,3,4,5) between-trivial col-transitivity-2
    conga-comm not-col-distincts not-par-not-col triangle-par)
ultimately show ?thesis
  by blast
qed

have  $A \neq C$ 
  using assms(4) col-trivial-2 by auto
have  $\neg P \ A \ \text{Par } C \ A$ 
  by (simp add: assms(4) grid-not-par-3 not-col-permutation-5)
have  $C \ P \ A \ \text{CongA } D \ P \ B$ 
  by (metis Pj-def  $\langle C \ A \ P \ \text{CongA } D \ B \ P \rangle$  assms(2,3,4,5) ncol-conga-ncol
    not-col-distincts not-col-permutation-2 not-col-permutation-3
    not-par-not-col par-comm triangle-par)
have  $C1 \neq C1'$ 
  using  $\langle \neg \text{Col } PO \ C1 \ C1' \rangle$  col-trivial-2 by auto
have  $PO \neq C1'$ 
  using  $\langle \neg \text{Col } PO \ C1 \ C1' \rangle$  col-trivial-3 by auto
have  $\neg PO \ C1' \ \text{Par } C1 \ C1'$ 
  by (simp add:  $\langle \neg \text{Col } PO \ C1 \ C1' \rangle$  grid-not-par-3)
have  $\neg PO \ C1 \ \text{Par } C1 \ C1'$ 
  by (simp add:  $\langle \neg \text{Col } PO \ C1 \ C1' \rangle$  grid-not-par-1)
have  $C1' \ PO \ A1 \ \text{CongA } D1' \ PO \ B1$ 
proof (rule out2--conga)
  show  $PO \ \text{Out } D1' \ C1'$ 
  using Out-cases  $\langle PO \ \text{Out } C1' \ D1' \rangle$  by auto
  show  $PO \ \text{Out } B1 \ A1$ 
  using  $\langle P \neq A \rangle \langle P \neq B \rangle$  assms(6,7) length-out by auto
qed
have  $D1' \ PO \ B1 \ \text{CongA } D \ P \ B$ 
  using  $\langle C \ P \ A \ \text{CongA } D \ P \ B \rangle \langle C1' \ PO \ A1 \ \text{CongA } D1' \ PO \ B1 \rangle$ 
   $\langle \text{Cong } P \ A \ PO \ A1 \wedge (P \neq A \implies C \ P \ A \ \text{CongA } C1' \ PO \ A1 \wedge C \ A \ P \ \text{CongA } C1' \ A1 \ PO) \rangle$ 
   $\langle P \neq A \rangle$  conga-sym conga-trans by blast
have  $D1' \neq B1 \implies PO \ D1' \ B1 \ \text{CongA } P \ D \ B \wedge PO \ B1 \ D1' \ \text{CongA } P \ B \ D$ 
  by (meson  $\langle \text{Cong } PO \ D1 \ PO \ D1' \rangle \langle D1' \ PO \ B1 \ \text{CongA } D \ P \ B \rangle$  assms(7,9)
    cong-transitivity l11-49 length-cong not-cong-3412)
moreover have  $D1' \neq B1$ 
  by (metis Col-def  $\langle C1' \ PO \ A1 \ \text{CongA } D1' \ PO \ B1 \rangle \langle PO \ A1 \ OS \ E' \ C1' \rangle$  col124--nos
    grid-not-par-5 ncol-conga-ncol)
ultimately have  $PO \ D1' \ B1 \ \text{CongA } P \ D \ B \wedge PO \ B1 \ D1' \ \text{CongA } P \ B \ D$ 
  by simp
have  $C1' \ A1 \ PO \ \text{CongA } D1' \ B1 \ PO \longleftrightarrow A1 \ C1' \ \text{Par } B1 \ D1'$ 
proof (rule l12-22)
  show  $PO \ \text{Out } A1 \ B1$ 
  using  $\langle P \neq A \rangle \langle P \neq B \rangle$  assms(6,7) length-out by auto
  show  $PO \ A1 \ OS \ C1' \ D1'$ 
  using  $\langle PO \ A1 \ OS \ E' \ C1' \rangle \langle PO \ \text{Out } C1' \ D1' \rangle$  col124--nos out-one-side by blast
qed
moreover have  $C1' \ A1 \ PO \ \text{CongA } D1' \ B1 \ PO$ 
proof (rule conga-trans [of - -  $D \ B \ P$ ])
  show  $C1' \ A1 \ PO \ \text{CongA } D \ B \ P$ 
  using  $\langle C \ A \ P \ \text{CongA } D \ B \ P \rangle \langle C \ P \ A \ \text{CongA } C1' \ PO \ A1 \wedge C \ A \ P \ \text{CongA } C1' \ A1 \ PO \rangle$ 
  not-conga not-conga-sym by blast
  show  $D \ B \ P \ \text{CongA } D1' \ B1 \ PO$ 
  by (simp add:  $\langle PO \ D1' \ B1 \ \text{CongA } P \ D \ B \wedge PO \ B1 \ D1' \ \text{CongA } P \ B \ D \rangle$  conga-comm
    conga-sym)
qed
ultimately have  $A1 \ C1' \ \text{Par } B1 \ D1'$ 
  by blast
have  $\text{Prod } PO \ C1 \ C1' \ A1 \ D1 \ B1$ 
proof (unfold Prod-def, intro strip conjI, unfold Ar2-def, intro strip conjI)
  show  $\neg \text{Col } PO \ C1 \ C1'$ 
  by (simp add:  $\langle \neg \text{Col } PO \ C1 \ C1' \rangle$ )
  show  $\text{Col } PO \ C1 \ A1$ 
  by (simp add:  $\langle PO \ \text{Out } A1 \ C1 \rangle$  l6-6 out-col)

```

```

show Col PO C1 D1
  by (simp add: ⟨PO Out C1 D1⟩ out-col)
show Col PO C1 B1
  by (metis ⟨Col PO C1 A1⟩ ⟨¬ Col PO A1 E'⟩ assms(6,7) col-trivial-3
    l6-16-1 length-Ar2 not-col-permutation-4)
show ∃ B'. C1 C1' Pj D1 B' ∧ Col PO C1' B' ∧ C1' A1 Pj B' B1
  by (meson Pj-def ⟨A1 C1' Par B1 D1'⟩ ⟨C1 C1' Pj D1 D1'⟩ ⟨PO Out C1' D1'⟩
    out-col par-comm)
qed
have ∃ Y. Prod PO E C1' A1 D1 Y ∧ Prod PO E C1' C1 B1 Y
proof (rule prod-x-axis-unit-change [of - C1], insert assms(1))
  show Ar2p4 PO C1 C1' A1 D1 C1 B1
    by (metis Ar2-def Ar2p4-def Prod-def ⟨Prod PO C1 C1' A1 D1 B1⟩
      not-col-distincts)
  show Col PO C1 E
    using Col-cases Col-def assms(8) between-trivial length-Ar2
    by blast
  show E ≠ PO
    using assms(1) by auto
  show ∃ X. Prod PO C1 C1' A1 D1 X ∧ Prod PO C1 C1' C1 B1 X
    using Ar2p4-def ⟨Ar2p4 PO C1 C1' A1 D1 C1 B1⟩ ⟨Prod PO C1 C1' A1 D1 B1⟩
      prod-1-l by blast
qed
hence Prod PO E E' C1 B1 AD
  by (metis Prod-def ⟨Prod PO E E' A1 D1 AD⟩ ⟨¬ Col PO E E'⟩
    prod-uniqueness prod-y-axis-change)
}
moreover have PO = M ∧ PO Midpoint D1 D1' ⇒ Prod PO E E' C1 B1 AD
  using ⟨M Midpoint C1 C1'⟩ ⟨¬ Col PO C1 C1'⟩ midpoint-col by auto
ultimately show Prod PO E E' C1 B1 AD
  using Reflect-def by auto
}
qed
}
qed
}
}
moreover have ¬ Ar2 PO E E' A1 D1 D1 ∧ AD = PO ⇒ Prod PO E E' C1 B1 AD
  by (meson Ar2-def Length-def ⟨¬ Col PO E E'⟩ assms(6,9))
ultimately show Prod PO E E' C1 B1 AD
  by blast
}
qed

lemma length-existence:
  assumes ¬ Col PO E E'
  shows ∃ AB. Length PO E E' A B AB
proof -
  have E ≠ PO
    using assms col-trivial-1 by auto
  obtain AB where Bet PO E AB ∨ Bet PO AB E and Cong PO AB A B
    using ⟨E ≠ PO⟩ segment-construction-2 by blast
  hence AB = PO ∨ PO Out E AB
    by (simp add: Out-def ⟨E ≠ PO⟩)
  moreover have AB = PO ⇒ ?thesis
    using ⟨Cong PO AB A B⟩ ⟨E ≠ PO⟩ cong-diff-3 length-id by blast
  moreover {
    assume PO Out E AB
    hence Col PO E AB
      by (simp add: out-col)
    moreover have LeP PO E E' PO AB
      using ⟨Bet PO E AB ∨ Bet PO AB E⟩ assms ps-le by blast
    ultimately have ?thesis
      using Length-def ⟨Cong PO AB A B⟩ ⟨E ≠ PO⟩ by auto
  }
  ultimately show ?thesis
    by blast

```

qed

lemma l15-7:

```

assumes PO ≠ E
and Per A C B
and H PerpAt C H A B
and Length PO E E' A B AB
and Length PO E E' A C AC
and Length PO E E' A H AH
shows Prod PO E E' AC AC AC2 ⟷ Prod PO E E' AB AH AC2
proof (cases AB = PO)
{
  assume AB = PO
  hence A = B
  using assms(4) length-id by auto
  hence C ≠ H ∧ A ≠ A
  using perp-in-distinct assms(3) by auto
  thus Prod PO E E' AC AC AC2 ⟷ Prod PO E E' AB AH AC2
  by simp
}
{
  assume AB ≠ PO
  have LtP PO E E' PO AB ⟹ ¬ Col PO E E' ∧ Col PO E AB
  using Ar2-def LtP-ar2 by blast
  moreover have PO = AB ⟹ ¬ Col PO E E' ∧ Col PO E AB
  using ⟨AB ≠ PO⟩ by auto
  ultimately have ¬ Col PO E E' ∧ Col PO E AB
  using assms(4) length-Ar2 by blast
  show Prod PO E E' AC AC AC2 ⟷ Prod PO E E' AB AH AC2
  proof (cases H = A)
  show H = A ⟹ (Prod PO E E' AC AC AC2) = (Prod PO E E' AB AH AC2)
  using assms(2,3) l11-47 by blast
  {
    assume H ≠ A
    hence C ≠ A
    using assms(3) l8-8 perp-in-per-1 by blast
    obtain C' where Bet A H C' ∨ Bet A C' H and Cong A C' A C
    using ⟨H ≠ A⟩ segment-construction-2 by blast
    have A Out H C'
    by (metis Out-def ⟨(A - H - C') ∨ (A - C' - H)⟩ ⟨C ≠ A⟩ ⟨Cong A C' A C⟩ ⟨H ≠ A⟩
    cong-reverse-identity)
    obtain H' where Bet A C H' ∨ Bet A H' C and Cong A H' A H
    using ⟨C ≠ A⟩ segment-construction-2 by blast
    have H ≠ C
    using assms(3) perp-in-distinct by blast
    moreover have Cong H C H' C' ∧ (H ≠ C ⟹ A H C CongA A H' C' ∧ A C H CongA A C' H')
    proof (rule l11-49)
    show H A C CongA H' A C'
    by (metis Out-cases ⟨A Out H C'⟩ ⟨(A - C - H') ∨ (A - H' - C)⟩ ⟨C ≠ A⟩
    ⟨Cong A H' A H⟩ ⟨H ≠ A⟩ bet-out
    between-cong between-trivial2 conga-left-comm out2--conga)
    show Cong A H A H'
    using ⟨Cong A H' A H⟩ not-cong-3412 by blast
    show Cong A C A C'
    by (simp add: ⟨Cong A C' A C⟩ cong-symmetry)
    qed
    ultimately have A H C CongA A H' C' ∧ A C H CongA A C' H'
    by blast
    have A H Perp H C
    by (metis Per-cases ⟨H ≠ A⟩ ⟨H ≠ C⟩ assms(3) per-perp perp-in-per-1)
    hence H PerpAt A H H C
    using l8-14-2-1b-bis not-col-distincts by blast
    hence Per A H C
    by (simp add: perp-in-per)
    have Per A H' C'

```

```

    using  $\langle A \ H \ C \ CongA \ A \ H' \ C' \wedge A \ C \ H \ CongA \ A \ C' \ H' \rangle \langle Per \ A \ H \ C \rangle$  l11-17 by blast
  have  $C \ B \ Par \ H' \ C'$ 
  proof (rule l12-9-2D [of - - A C])
    show  $C \ B \ Perp \ A \ C$ 
    by (metis Perp-cases  $\langle C \neq A \rangle \langle H \neq C \rangle$  assms(2,3) per-perp perp-in-id)
    show  $H' \ C' \ Perp \ A \ C$ 
    by (metis Col-def Per-perm Perp-cases  $\langle A \ H \ C \ CongA \ A \ H' \ C' \wedge A \ C \ H \ CongA \ A \ C' \ H' \rangle$ 
         $\langle A \ H \ Perp \ H \ C \rangle \langle (A - C - H') \vee (A - H' - C) \rangle \langle Per \ A \ H' \ C' \rangle$ 
        bet-conga--bet between-symmetry between-trivial col-per-perp
        not-conga-sym perp-not-col2)
  qed
  obtain  $AH'$  where Length  $PO \ E \ E' \ A \ H' \ AH'$ 
    using  $\langle \neg \ Col \ PO \ E \ E' \wedge \ Col \ PO \ E \ AB \rangle$  length-existence by presburger
  obtain  $AC'$  where Length  $PO \ E \ E' \ A \ C' \ AC'$ 
    using  $\langle \neg \ Col \ PO \ E \ E' \wedge \ Col \ PO \ E \ AB \rangle$  length-existence by blast
  obtain  $P$  where Prod  $PO \ E \ E' \ AC' \ AC \ P$ 
    by (metis  $\langle Length \ PO \ E \ E' \ A \ C' \ AC' \rangle \langle \neg \ Col \ PO \ E \ E' \wedge \ Col \ PO \ E \ AB \rangle$  assms(5)
        col-trivial-3 length-Ar2 prod-exists)
  have  $C \ H \ Perp \ A \ B$ 
    using assms(3) perp-in-perp by auto
  have Prod  $PO \ E \ E' \ AH' \ AB \ P$ 
  proof (rule thales [of - - A C' B H' C - AC' - - AC])
    show  $PO \neq E$ 
    using assms(1) by blast
    show  $Col \ A \ C' \ B$ 
    by (metis Col-def Perp-cases Perp-in-cases  $\langle A \ Out \ H \ C' \rangle \langle C \ H \ Perp \ A \ B \rangle$  assms(2,3)
        col-out2-col out-bet--out out-trivial perp-in-col perp-not-col2
        perp-per-bet)
    show  $Col \ A \ H' \ C$ 
    using Col-def  $\langle (A - C - H') \vee (A - H' - C) \rangle$  between-symmetry by blast
    show  $\neg \ Col \ A \ C' \ H'$ 
    by (metis  $\langle A \ H \ C \ CongA \ A \ H' \ C' \wedge A \ C \ H \ CongA \ A \ C' \ H' \rangle \langle Per \ A \ H' \ C' \rangle$  conga-diff45
        conga-diff56 l8-9 not-col-permutation-1 not-col-permutation-3)
    show  $C' \ H' \ Pj \ B \ C$ 
    using Par-cases Pj-def  $\langle C \ B \ Par \ H' \ C' \rangle$  by blast
    show Length  $PO \ E \ E' \ A \ C' \ AC'$ 
    using  $\langle Length \ PO \ E \ E' \ A \ C' \ AC' \rangle$  by auto
    show Length  $PO \ E \ E' \ A \ B \ AB$ 
    by (simp add: assms(4))
    show Length  $PO \ E \ E' \ A \ H' \ AH'$ 
    by (simp add:  $\langle Length \ PO \ E \ E' \ A \ H' \ AH' \rangle$ )
    show Length  $PO \ E \ E' \ A \ C \ AC$ 
    by (simp add: assms(5))
    show Prod  $PO \ E \ E' \ AC' \ AC \ P$ 
    by (simp add: Prod-def  $\langle Prod \ PO \ E \ E' \ AC' \ AC \ P \rangle$ )
  qed
  hence Prod  $PO \ E \ E' \ AH' \ AB \ P$ 
    by (metis Ar2-def Prod-def  $\langle Length \ PO \ E \ E' \ A \ H' \ AH' \rangle$ 
         $\langle \neg \ Col \ PO \ E \ E' \wedge \ Col \ PO \ E \ AB \rangle$  col-trivial-3 length-Ar2)
  have Length  $PO \ E \ E' \ A \ H' \ AH$ 
    using  $\langle Cong \ A \ H' \ A \ H \rangle$  assms(6) length-eq-cong-2 not-cong-3412 by blast
  have Length  $PO \ E \ E' \ A \ C' \ AC$ 
    using Cong-cases  $\langle Cong \ A \ C' \ A \ C \rangle$  assms(5) length-eq-cong-2 by blast
  hence  $(Prod \ PO \ E \ E' \ AC \ AC \ AC2) \longrightarrow (Prod \ PO \ E \ E' \ AB \ AH \ AC2)$ 
    by (metis  $\langle Length \ PO \ E \ E' \ A \ C' \ AC' \rangle \langle Length \ PO \ E \ E' \ A \ H' \ AH' \rangle$ 
         $\langle Length \ PO \ E \ E' \ A \ H' \ AH \rangle \langle Prod \ PO \ E \ E' \ AC' \ AC \ P \rangle$ 
         $\langle Prod \ PO \ E \ E' \ AH' \ AB \ P \rangle \langle \neg \ Col \ PO \ E \ E' \wedge \ Col \ PO \ E \ AB \rangle$  length-uniqueness
        prod-sym prod-uniqueness)
  moreover have  $(Prod \ PO \ E \ E' \ AB \ AH \ AC2) \longrightarrow (Prod \ PO \ E \ E' \ AC \ AC \ AC2)$ 
    by (metis  $\langle Length \ PO \ E \ E' \ A \ C' \ AC' \rangle \langle Length \ PO \ E \ E' \ A \ C' \ AC \rangle$ 
         $\langle Length \ PO \ E \ E' \ A \ H' \ AH' \rangle \langle Length \ PO \ E \ E' \ A \ H' \ AH \rangle \langle Prod \ PO \ E \ E' \ AC' \ AC \ P \rangle$ 
         $\langle Prod \ PO \ E \ E' \ AH' \ AB \ P \rangle \langle \neg \ Col \ PO \ E \ E' \wedge \ Col \ PO \ E \ AB \rangle$  length-uniqueness
        prod-sym prod-uniqueness)
  ultimately show  $(Prod \ PO \ E \ E' \ AC \ AC \ AC2) = (Prod \ PO \ E \ E' \ AB \ AH \ AC2)$ 
    by blast
}

```

qed
 }
 qed

lemma *l15-7-1*:
 assumes $PO \neq E$
 and $Per\ A\ C\ B$
 and $H\ PerpAt\ C\ H\ A\ B$
 and $Length\ PO\ E\ E'\ A\ B\ AB$
 and $Length\ PO\ E\ E'\ A\ C\ AC$
 and $Length\ PO\ E\ E'\ A\ H\ AH$
 and $Prod\ PO\ E\ E'\ AC\ AC\ AC2$
 shows $Prod\ PO\ E\ E'\ AB\ AH\ AC2$
 using *l15-7* [of $PO\ E\ A\ C\ B\ H\ E'\ AB\ AC\ AH\ AC2$] *assms*(1,2,3,4,5,6,7) **by** *blast*

lemma *l15-7-2*:
 assumes $PO \neq E$
 and $Per\ A\ C\ B$
 and $H\ PerpAt\ C\ H\ A\ B$
 and $Length\ PO\ E\ E'\ A\ B\ AB$
 and $Length\ PO\ E\ E'\ A\ C\ AC$
 and $Length\ PO\ E\ E'\ A\ H\ AH$
 and $Prod\ PO\ E\ E'\ AB\ AH\ AC2$
 shows $Prod\ PO\ E\ E'\ AC\ AC\ AC2$
 using *l15-7* [of $PO\ E\ A\ C\ B\ H\ E'\ AB\ AC\ AH\ AC2$] *assms*(1,2,3,4,5,6,7) **by** *blast*

lemma *length-sym*:
 assumes $Length\ PO\ E\ E'\ A\ B\ AB$
 shows $Length\ PO\ E\ E'\ B\ A\ AB$
 using *assms* *cong-pseudo-reflexivity* *length-eq-cong-2* **by** *blast*

lemma *pythagoras*:
 assumes $PO \neq E$
 and $Per\ A\ C\ B$
 and $Length\ PO\ E\ E'\ A\ B\ AB$
 and $Length\ PO\ E\ E'\ A\ C\ AC$
 and $Length\ PO\ E\ E'\ B\ C\ BC$
 and $Prod\ PO\ E\ E'\ AC\ AC\ AC2$
 and $Prod\ PO\ E\ E'\ BC\ BC\ BC2$
 and $Prod\ PO\ E\ E'\ AB\ AB\ AB2$
 shows $Sum\ PO\ E\ E'\ AC2\ BC2\ AB2$
proof (cases $Col\ A\ C\ B$)
 {
 assume $Col\ A\ C\ B$
 moreover have $A = C \longrightarrow Sum\ PO\ E\ E'\ AC2\ BC2\ AB2$
 by (metis *Ar2-def* *Prod-def* *assms*(3,4,5,6,7,8) *cong-reverse-identity* *length-cong*
length-sym *length-uniqueness* *prod-O-r-eq* *prod-uniqueness* *sum-O-B*)
 moreover have $B = C \longrightarrow Sum\ PO\ E\ E'\ AC2\ BC2\ AB2$
 by (metis *Ar2-def* *Prod-def* *assms*(1,3,4,5,6,7,8) *length-id* *length-uniqueness*
prod-O-r-eq *prod-uniqueness* *sum-A-O*)
 ultimately show $Sum\ PO\ E\ E'\ AC2\ BC2\ AB2$
 using *assms*(2) *per-not-col* **by** *force*
 }
 {
 assume $\neg Col\ A\ C\ B$
 then obtain P where $Col\ A\ B\ P$ and $A\ B\ Perp\ C\ P$
 using *col-permutation-5* *l8-18-existence* **by** *blast*
 hence $P\ PerpAt\ A\ B\ C\ P$
 by (*simp* add: *l8-15-1*)
 obtain AP where $Length\ PO\ E\ E'\ A\ P\ AP$
 by (metis $\langle \neg Col\ A\ C\ B \rangle$ *assms*(3,5) *col-trivial-1* *length-existence* *length-id-1*
length-not-col-null)
 obtain BP where $Length\ PO\ E\ E'\ B\ P\ BP$
 by (metis $\langle Length\ PO\ E\ E'\ A\ P\ AP \rangle$ *assms*(3) *length-Ar2*
length-existence *length-id-1*)
 have $Sum\ PO\ E\ E'\ AP\ BP\ AB$

```

    using triangular-equality-bis [of PO E E']
  by (metis Perp-in-cases ‹Length PO E E' A P AP›
    ‹Length PO E E' B P BP› ‹P PerpAt A B C P› ‹¬ Col A C B›
    assms(1,2,3) grid-not-par length-sym perp-per-bet
    triangular-equality-bis)
moreover have Prod PO E E' AB AP AC2
  using Perp-in-cases ‹Length PO E E' A P AP› ‹P PerpAt A B C P›
    assms(1,2,3,4,6) l15-7 by blast
moreover have Prod PO E E' AB BP BC2
  by (meson Per-cases Perp-in-cases ‹Length PO E E' B P BP›
    ‹P PerpAt A B C P› assms(1,2,3,5,7) l15-7 length-sym)
ultimately show Sum PO E E' AC2 BC2 AB2
  using assms(8) distr-l by blast
}
qed

```

```

lemma is-length-exists:
  assumes ¬ Col PO E E'
  shows ∃ XY. IsLength PO E E' X Y XY
proof(cases X = Y)
{
  assume X = Y
  hence IsLength PO E E' X Y PO
    by (simp add: IsLength-def length-id)
  thus ∃ XY. IsLength PO E E' X Y XY
    by auto
}
show X ≠ Y ⟹ ∃ XY. IsLength PO E E' X Y XY
  using length-existence IsLength-def assms by blast
qed

```

```

lemma lt-to-ltp:
  assumes Length PO E E' A B L
    and Length PO E E' C D M
    and A B Lt C D
  shows LtP PO E E' L M
  by (metis LeP-def Lt-def assms(1,2,3) length-eq-cong-1 length-leP-le-2)

```

```

lemma ltp-to-lep:
  assumes LtP PO E E' L M
  shows LeP PO E E' L M
  by (simp add: LeP-def assms)

```

```

lemma ltp-to-lt:
  assumes Length PO E E' A B L
    and Length PO E E' C D M
    and LtP PO E E' L M
  shows A B Lt C D
proof -
  have LeP PO E E' L M
    by (simp add: LeP-def assms(3))
  have A B Le C D
    using ‹LeP PO E E' L M› assms(1,2) length-leP-le-1 by auto
  moreover {
    assume Cong A B C D
    hence Length PO E E' C D L
      using assms(1) length-eq-cong-2 by blast
    hence L = M
      using length-uniqueness [of PO E E' C D L M] assms(2) by auto
    hence False
      using assms(3) ltp-neq by auto
  }
ultimately show ?thesis
  using Lt-def by auto

```


qed

lemma *prod-col*:

assumes $Ar2\ PO\ E\ E'\ A\ B\ A$
 and $Prod\ PO\ E\ E'\ A\ B\ AB$
 shows $Col\ PO\ E\ AB$
 using $Ar2-def\ Prod-def\ assms(2)$ by auto

lemma *square-increase-strict*:

assumes $Ar2\ PO\ E\ E'\ A\ B\ A$
 and $Ps\ PO\ E\ A$
 and $Ps\ PO\ E\ B$
 and $LtP\ PO\ E\ E'\ A\ B$
 and $Prod\ PO\ E\ E'\ A\ A\ A2$
 and $Prod\ PO\ E\ E'\ B\ B\ B2$
 shows $LtP\ PO\ E\ E'\ A2\ B2$

proof –

obtain BmA where $Diff\ PO\ E\ E'\ B\ A\ BmA$
 using $LtP-def\ assms(4)$ by blast
 obtain BpA where $Sum\ PO\ E\ E'\ B\ A\ BpA$
 by (meson $\langle Diff\ PO\ E\ E'\ B\ A\ BmA \rangle\ diff-ar2\ sum3-def\ sum3-exists$)
 obtain $B2mA2$ where $Diff\ PO\ E\ E'\ B2\ A2\ B2mA2$
 by (metis $Ar2-def\ assms(1,5,6)\ diff-exists\ prod-col$)
 obtain F where $Prod\ PO\ E\ E'\ BpA\ BmA\ F$
 by (metis $Ar2-def\ Ps-def\ \langle Diff\ PO\ E\ E'\ B\ A\ BmA \rangle\ \langle Sum\ PO\ E\ E'\ B\ A\ BpA \rangle$
 $assms(1,2,3,4)\ col-permutation-5\ lt-diff-ps\ out-col\ prod-exists$
 $sum-pos-pos$)
 thus ?thesis
 by (metis $LtP-def\ \langle Diff\ PO\ E\ E'\ B\ A\ BmA \rangle\ \langle Diff\ PO\ E\ E'\ B2\ A2\ B2mA2 \rangle$
 $\langle Sum\ PO\ E\ E'\ B\ A\ BpA \rangle\ assms(2,3,4,5,6)\ diff-of-squares\ lt-diff-ps\ prod-pos-pos$
 $sum-pos-pos$)

qed

lemma *square-increase*:

assumes $Ar2\ PO\ E\ E'\ A\ B\ A$
 and $Ps\ PO\ E\ A$
 and $Ps\ PO\ E\ B$
 and $LeP\ PO\ E\ E'\ A\ B$
 and $Prod\ PO\ E\ E'\ A\ A\ A2$
 and $Prod\ PO\ E\ E'\ B\ B\ B2$
 shows $LeP\ PO\ E\ E'\ A2\ B2$
 by (metis $Ar2-def\ LeP-def\ assms(1,2,3,4,5,6)\ prod-uniqueness$
 $square-increase-strict$)

lemma *signeq-prod-pos*:

assumes $SignEq\ PO\ E\ A\ B$
 and $Prod\ PO\ E\ E'\ A\ B\ C$
 shows $Ps\ PO\ E\ C$

proof –

have $Ps\ PO\ E\ A \wedge Ps\ PO\ E\ B \implies Ps\ PO\ E\ C$
 using $assms(2)\ prod-pos-pos$ by blast
 moreover {
 assume $Ng\ PO\ E\ A$ and $Ng\ PO\ E\ B$
 obtain B' where $B\ B'\ Proj\ PO\ E'\ E\ E'$
 using $Prodp-def\ assms(2)\ prod-to-prodp$ by blast
 have $Bet\ B'\ PO\ E'$
 proof –
 have $Bet\ B\ PO\ E$
 using $Ng-def\ \langle Ng\ PO\ E\ B \rangle$ by auto
 moreover have $PO\ PO\ Proj\ PO\ E'\ E\ E'$
 by (metis $Proj-def\ \langle B\ B'\ Proj\ PO\ E'\ E\ E' \rangle\ not-col-distincts$)
 moreover have $E\ E'\ Proj\ PO\ E'\ E\ E'$
 by (metis $Proj-def\ calculation(2)\ not-col-distincts\ par-reflexivity$)
 ultimately show ?thesis
 using $\langle B\ B'\ Proj\ PO\ E'\ E\ E' \rangle\ project-preserves-bet$ by blast

qed

```

moreover have Bet A PO C
proof (rule project-preserves-bet [of E' PO B' - PO E A E'])
  show  $E' - PO - B'$ 
    using Bet-cases calculation by blast
  show  $E' A \text{ Proj } PO E A E'$ 
  proof -
    have  $PO \neq E$ 
      using Ng-def  $\langle Ng PO E A \rangle$  by force
    moreover have  $A \neq E'$ 
      using Ar2-def Prod-def assms(2) by auto
    moreover have  $\neg PO E \text{ Par } A E'$ 
      by (metis Ar2-def Prod-def assms(2) col-trivial-3 not-strict-par2)
    moreover have  $E' A \text{ Par } A E' \vee E' = A$ 
      using Par-cases par-reflexivity by blast
    ultimately show ?thesis
      using Col-def Ng-def  $\langle Ng PO E A \rangle$  par-col-project by force
  qed
show  $PO PO \text{ Proj } PO E A E'$ 
  by (metis NCol-perm Proj-def  $\langle E' A \text{ Proj } PO E A E' \rangle$  col-trivial-1)
show  $B' C \text{ Proj } PO E A E'$ 
  using Prodp-def  $\langle B B' \text{ Proj } PO E' E E' \rangle$  assms(2) prod-to-prodp project-uniqueness
  by blast
qed
ultimately have  $PO \text{ Out } C E$ 
  using Ng-def Out-def  $\langle Ng PO E A \rangle \langle Ng PO E B \rangle$  assms(2) l5-2 prod-null
  by fastforce
hence  $Ps PO E C$ 
  by (simp add: Ps-def)
}
ultimately show ?thesis
  using SignEq-def assms(1) by blast
qed

```

```

lemma pos-neg--prod-neg:
  assumes  $Ps PO E A$ 
  and  $Ng PO E B$ 
  and  $Prod PO E E' A B C$ 
  shows  $Ng PO E C$ 
proof -
  obtain  $B'$  where  $B B' \text{ Proj } PO E' E E'$  and  $B' C \text{ Proj } PO E A E'$ 
    using Prodp-def assms(3) prod-to-prodp by blast
  show ?thesis
  proof (unfold Ng-def, intro conjI)
    show  $C \neq PO$ 
      by (metis Ng-def O-not-positive assms(1,2,3) prod-null)
    show  $E \neq PO$ 
      using Ng-def assms(2) by auto
    show  $Bet C PO E$ 
  proof -
    have  $Bet A PO C$ 
    proof (rule project-preserves-bet [of E' PO B' - PO E A E'])
      have  $Bet B' PO E'$ 
      proof (rule project-preserves-bet [of B PO E - PO E' E E'])
        show  $B - PO - E$ 
          using Ng-def assms(2) by auto
        show  $B B' \text{ Proj } PO E' E E'$ 
          by (simp add:  $\langle B B' \text{ Proj } PO E' E E' \rangle$ )
        show  $PO PO \text{ Proj } PO E' E E'$ 
          using Proj-def  $\langle B B' \text{ Proj } PO E' E E' \rangle$  col-trivial-3 by presburger
        show  $E E' \text{ Proj } PO E' E E'$ 
          by (metis NCol-perm Proj-def  $\langle PO PO \text{ Proj } PO E' E E' \rangle$  col-trivial-3
            par-reflexivity)
      qed
    thus  $E' - PO - B'$ 
      using Bet-perm by blast
    show  $E' A \text{ Proj } PO E A E'$ 

```

```

    by (metis Par-cases Prodp-def Proj-def assms(3) par-reflexivity
        prod-to-prodp)
  show PO PO Proj PO E A E'
    by (metis NCol-perm Proj-def ⟨E' A Proj PO E A E'⟩ col-trivial-1)
  show B' C Proj PO E A E'
    by (simp add: ⟨B' C Proj PO E A E'⟩)
qed
thus ?thesis
  using Bet-cases Ps-def assms(1) bet-out--bet by blast
qed
qed
qed

```

```

lemma not-signEq-prod-neg:
  assumes A ≠ PO
    and B ≠ PO
    and ¬ SignEq PO E A B
    and Prod PO E E' A B C
  shows Ng PO E C
  by (metis Ar2-def Prod-def SignEq-def assms(1,2,3,4) col-pos-or-neg
      pos-neg--prod-neg prod-1-r prod-O-r-eq prod-sym)

```

```

lemma prod-pos--signeq:
  assumes A ≠ PO
    and B ≠ PO
    and Prod PO E E' A B C
    and Ps PO E C
  shows SignEq PO E A B
  using assms(1,2,3,4) neg-not-pos not-signEq-prod-neg by blast

```

```

lemma prod-ng---not-signeq:
  assumes Prod PO E E' A B C
    and Ng PO E C
  shows ¬ SignEq PO E A B
  using assms(1,2) neg-not-pos signeq--prod-pos by blast

```

```

lemma ltp--diff-pos:
  assumes LtP PO E E' A B
    and Diff PO E E' B A D
  shows Ps PO E D
  using assms(1,2) lt-diff-ps by auto

```

```

lemma diff-pos--ltp:
  assumes Diff PO E E' B A D
    and Ps PO E D
  shows LtP PO E E' A B
  using LtP-def assms(1,2) by blast

```

```

lemma square-increase-rev:
  assumes Ps PO E A
    and Ps PO E B
    and LtP PO E E' A2 B2
    and Prod PO E E' A A A2
    and Prod PO E E' B B B2
  shows LtP PO E E' A B
  by (metis Ar2-def LeP-def Prod-def assms(1,2,3,4,5) col-2-le-or-ge leP-asymp
      ltp-neq square-increase)

```

```

lemma ltp--ltps:
  assumes LtP PO E E' A B
  shows LtPs PO E E' A B
  using LtP-def LtPs-def assms diff-sum by blast

```

```

lemma ltps--ltp:
  assumes LtPs PO E E' A B
  shows LtP PO E E' A B

```

using *LtPs-def assms ltP-sum-pos* **by** *blast*

lemma *ltP--lep-neg*:

assumes *LtP PO E E' A B*
shows *LeP PO E E' A B $\wedge$ A $\neq$ B*
using *assms ltP-neg ltP-to-lep* **by** *auto*

lemma *lep-neg--ltP*:

assumes *LeP PO E E' A B*
and *A $\neq$ B*
shows *LtP PO E E' A B*
using *LeP-def assms(1,2)* **by** *force*

lemma *sum-preserves-ltP*:

assumes *LtP PO E E' A B*
and *Sum PO E E' A C AC*
and *Sum PO E E' B C BC*
shows *LtP PO E E' AC BC*
by (*metis Ar2-def assms(1,2,3) compatibility-of-sum-with-order lep-neg--ltP*
ltP-ar2 ltP-neg ltP-to-lep sum-uniquenessA)

lemma *sum-preserves-lep*:

assumes *LeP PO E E' A B*
and *Sum PO E E' A C AC*
and *Sum PO E E' B C BC*
shows *LeP PO E E' AC BC*
using *assms(1,2,3) compatibility-of-sum-with-order* **by** *blast*

lemma *sum-preserves-ltP-rev*:

assumes *Sum PO E E' A C AC*
and *Sum PO E E' B C BC*
and *LtP PO E E' AC BC*
shows *LtP PO E E' A B*
by (*metis Ar2-def assms(1,2,3) col-2-le-or-ge leP-asym lep-neg--ltP ltP-neg*
ltP-to-lep sum-ar2 sum-preserves-lep)

lemma *sum-preserves-lep-rev*:

assumes *Sum PO E E' A C AC*
and *Sum PO E E' B C BC*
and *LeP PO E E' AC BC*
shows *LeP PO E E' A B*
by (*metis Ar2-def Sum-def assms(1,2,3) col-2-le-or-ge*
compatibility-of-sum-with-order leP-asym sum-uniquenessA)

lemma *cong2-lea--le*:

assumes *Cong A B D E*
and *Cong A C D F*
and *F D E LeA C A B*
shows *E F Le B C*

proof –

have *F D E LtA C A B $\vee$ F D E CongA C A B*
using *LtA-def assms(3)* **by** *auto*
moreover have *F D E LtA C A B $\implies$ E F Le B C*
using *Lt-def assms(1,2) t18-18* **by** *blast*
moreover have *F D E CongA C A B $\implies$ E F Le B C*
by (*meson assms(1,2) cong2-conga-cong cong-4321 cong--le cong-symmetry*)
ultimately show *?thesis*
by *blast*

qed

lemma *lea-out-lea*:

assumes *B Out A A'*
and *B Out C C'*
and *E Out D D'*
and *E Out F F'*
and *A B C LeA D E F*

shows $A' B C' \text{ LeA } D' E F'$
 using *assms*(1,2,3,4,5) *lea-out4--lea* **by** *blast*

lemma *lta-out-lta*:

assumes $B \text{ Out } A A'$
 and $B \text{ Out } C C'$
 and $E \text{ Out } D D'$
 and $E \text{ Out } F F'$
 and $A B C \text{ LtA } D E F$
 shows $A' B C' \text{ LtA } D' E F'$
by (*smt* (*verit*) *Out-def* *assms*(1,2,3,4,5) *lea-out-lea lta--nlea*
nlta--lea)

lemma *pythagoras-obtuse*:

assumes $PO \neq E$
 and *Obtuse* $A C B$
 and *Length* $PO E E' A B AB$
 and *Length* $PO E E' A C AC$
 and *Length* $PO E E' B C BC$
 and *Prod* $PO E E' AC AC AC2$
 and *Prod* $PO E E' BC BC BC2$
 and *Prod* $PO E E' AB AB AB2$
 and *Sum* $PO E E' AC2 BC2 S2$
 shows *LtP* $PO E E' S2 AB2$

proof –

obtain $A' B' C'$ **where** *Per* $A' B' C'$ **and** $A' B' C' \text{ LtA } A C B$
 using *Obtuse-def* *assms*(2) **by** *blast*
obtain AA **where** $B' \text{ Out } AA A'$ **and** *Cong* $B' AA C A$
 by (*metis* $\langle A' B' C' \text{ LtA } A C B \rangle$ *l6-11-existence lta-distincts*)
obtain CC **where** $B' \text{ Out } CC C'$ **and** *Cong* $B' CC C B$
 by (*metis* $\langle A' B' C' \text{ LtA } A C B \rangle$ *l6-11-existence lta-distincts*)
 have $AA CC \text{ Lt } A B$
proof (*rule* *t18-18* [*of* $C A B' AA B CC$])
 show *Cong* $C A B' AA$
 using *Cong-perm* $\langle \text{Cong } B' AA C A \rangle$ **by** *blast*
 show *Cong* $C B B' CC$
 using $\langle \text{Cong } B' CC C B \rangle$ *not-cong-3412* **by** *blast*
 show $CC B' AA \text{ LtA } B C A$
 by (*smt* (*verit*, *best*) *Out-cases* $\langle A' B' C' \text{ LtA } A C B \rangle \langle B' \text{ Out } AA A' \rangle \langle B' \text{ Out } CC C' \rangle$
lta-comm lta-distincts lta-out-lta out-trivial)

qed

obtain $S1$ **where** *Length* $PO E E' AA CC S1$
 by (*metis* $\langle A' B' C' \text{ LtA } A C B \rangle$ *assms*(3) *cong-identity length-Ar2 length-cong*
length-existence lta-distincts)
then obtain SS **where** *Prod* $PO E E' S1 S1 SS$
 using *prod-exists* [*of* $PO E E' S1 S1$] *assms*(4,6) *col-trivial-3 length-Ar2* **by** *blast*
 have $B' \neq C'$
 using *Out-def* $\langle B' \text{ Out } CC C' \rangle$ **by** *force*
 hence *Per* $A' B' CC$
 using *per-col* [*of* $B' C' A' CC$] $\langle B' \text{ Out } CC C' \rangle \langle \text{Per } A' B' C' \rangle$ *col-permutation-5*
out-col **by** *blast*
 have $B' \neq A'$
 using *Out-def* $\langle B' \text{ Out } AA A' \rangle$ **by** *blast*
 hence *Per* $CC B' AA$
 using *per-col* [*of* $B' A' CC AA$] *Out-cases* $\langle B' \text{ Out } AA A' \rangle \langle \text{Per } A' B' CC \rangle$ *l8-2 out-col* **by** *blast*
 have *Sum* $PO E E' AC2 BC2 SS$
 using *pythagoras* [*of* - - $AA B' CC - S1 AC BC$] **by** (*meson* *Per-cases* $\langle \text{Cong } B' AA C A \rangle$
 $\langle \text{Cong } B' CC C B \rangle \langle \text{Length } PO E E' AA CC S1 \rangle \langle \text{Per } CC B' AA \rangle$
 $\langle \text{Prod } PO E E' S1 S1 SS \rangle$ *assms*(1,4,5,6,7) *cong-4321 length-eq-cong-2*)
 have $S2 = SS$
 using $\langle \text{Sum } PO E E' AC2 BC2 SS \rangle$ *assms*(9) *sum-stable* **by** *blast*
moreover have *LtP* $PO E E' S1 AB$
 using *lt-to-ltp* [*of* $PO E E' AA CC S1 A B AB$]
 using $\langle AA CC \text{ Lt } A B \rangle \langle \text{Length } PO E E' AA CC S1 \rangle$ *assms*(3) **by** *blast*
moreover have $Ps PO E S1$
 by (*metis* $\langle A' B' C' \text{ LtA } A C B \rangle \langle \text{Length } PO E E' AA CC S1 \rangle \langle \text{Prod } PO E E' S1 S1 SS \rangle$)

```

    assms(4,5,6,7,9) calculation(1) length-id-1 length-pos lep-neq--ltP ltP-pos
    lta-distincts prod-O-l-eq square-pos sum-pos-pos)
ultimately show ?thesis
  using square-increase-strict [of PO E E' S1 AB S2 AB2]
  by (metis SignEq-def ⟨Prod PO E E' S1 S1 SS⟩ assms(3,8)
    length-pos lep-neq--ltP ltP-ar2 ltP-neg ltP-pos pos-neg--prod-neg prod-ng---not-signeq)
qed

```

lemma *pythagoras-obtuse-or-per*:

```

assumes PO ≠ E
  and Obtuse A C B ∨ Per A C B
  and Length PO E E' A B AB
  and Length PO E E' A C AC
  and Length PO E E' B C BC
  and Prod PO E E' AC AC AC2
  and Prod PO E E' BC BC BC2
  and Prod PO E E' AB AB AB2
  and Sum PO E E' AC2 BC2 S2
shows LeP PO E E' S2 AB2
proof -
  have Obtuse A C B ⇒ LeP PO E E' S2 AB2
    using assms(1,3,4,5,6,7,8,9) ltp-to-lep pythagoras-obtuse by blast
  moreover have Per A C B ⇒ LeP PO E E' S2 AB2
    by (meson assms(1,3,4,5,6,7,8,9) leP-refl pythagoras sum-preserves-lep)
  ultimately show ?thesis
    using assms(2) by blast
qed

```

lemma *pythagoras-acute*:

```

assumes PO ≠ E
  and Acute A C B
  and Length PO E E' A B AB
  and Length PO E E' A C AC
  and Length PO E E' B C BC
  and Prod PO E E' AC AC AC2
  and Prod PO E E' BC BC BC2
  and Prod PO E E' AB AB AB2
  and Sum PO E E' AC2 BC2 S2
shows LtP PO E E' AB2 S2
proof (cases A = B)
  show A = B ⇒ LtP PO E E' AB2 S2
    by (smt (verit, ccfv-SIG) LtPs-def acute-distincts assms(1,2,3,4,5,6,7,8,9) between-trivial2
      distr-r length-Ar2 length-id-1 ltps--ltp prod-0-r prod-sym square-pos sum-A-null
      sum-assoc-2 sum-pos-pos triangular-equality-bis)
  {
    assume A ≠ B
    obtain A' B' C' where Per A' B' C' and A C B LtA A' B' C'
      using Acute-def assms(2) by blast
    have A' ≠ B'
      using ⟨A C B LtA A' B' C'⟩ lta-distincts by blast
    then obtain AA where B' Out AA A' and Cong B' AA C A
      by (metis assms(2) between-trivial2 flat-not-acute l6-11-existence)
    have C' ≠ B'
      using ⟨A C B LtA A' B' C'⟩ lta-distincts by blast
    then obtain CC where B' Out CC C' and Cong B' CC C B
      by (metis Out-def ⟨C' ≠ B'⟩ acute-col-out assms(2) l6-11-existence not-col-distincts)
    obtain S1 where Length PO E E' AA CC S1
      using Ar2-def assms(9) length-existence sum-ar2 by presburger
    obtain SS where Prod PO E E' S1 S1 SS
      using Ar2-def Length-def Prod-def ⟨Length PO E E' AA CC S1⟩ assms(7) prod-exists
      by presburger
    have Per A' B' CC
      proof (rule per-col [of B' C' A' CC ], insert ⟨Per A' B' C'⟩)
        show B' ≠ C'
          using ⟨C' ≠ B'⟩ by auto
        show Col B' C' CC

```

```

    using Col-cases  $\langle B' \text{ Out } CC \ C' \rangle$  out-col by blast
qed
have Per CC B' AA
  using per-col [of - A'] by (metis NCol-perm  $\langle A' \neq B' \rangle \langle B' \text{ Out } AA \ A' \rangle$ 
     $\langle \text{Per } A' \ B' \ CC \rangle$  l8-2 out-col)
hence A B Lt AA CC
  using t18-18 [of B' AA C A CC B] by (metis Out-def  $\langle B' \text{ Out } AA \ A' \rangle \langle B' \text{ Out } CC \ C' \rangle$ 
     $\langle \text{Cong } B' \ AA \ C \ A \rangle \langle \text{Cong } B' \ CC \ C \ B \rangle$  acute-per--lta acute-sym assms(2))
hence LtP PO E E' AB S1
  using lt-to-ltp [of PO E E' A B AB AA CC S1]  $\langle \text{Length } PO \ E \ E' \ AA \ CC \ S1 \rangle$  assms(3) by linarith
show LtP PO E E' AB2 S2
proof (rule square-increase-strict [of PO E E' AB S1 AB2 S2], insert
   $\langle \text{LtP } PO \ E \ E' \ AB \ S1 \rangle$  assms(8))
  show Ar2 PO E E' AB S1 AB
    by (simp add:  $\langle \text{LtP } PO \ E \ E' \ AB \ S1 \rangle$  ltP-ar2)
  show Ps PO E AB
    by (metis  $\langle A \neq B \rangle$  assms(3) between-trivial2 length-out length-pos lep-neq--ltp ltP-pos
      not-bet-and-out)
  show Ps PO E S1
    by (metis  $\langle A \ B \ Lt \ AA \ CC \rangle \langle \text{Length } PO \ E \ E' \ AA \ CC \ S1 \rangle$  length-Ps length-id lt-diff)
  have S2 = SS
  proof (rule sum-uniqueness [of PO E E' AC2 BC2 S2 SS], insert assms(9))
    show  $\neg \text{Col } PO \ E \ E'$ 
      using Ar2-def  $\langle \text{Ar2 } PO \ E \ E' \ AB \ S1 \ AB \rangle$  by blast
    show Sum PO E E' AC2 BC2 SS
      using pythagoras [of - - AA B' CC - S1 AC BC]
      by (meson Per-cases  $\langle \text{Cong } B' \ AA \ C \ A \rangle \langle \text{Cong } B' \ CC \ C \ B \rangle$ 
         $\langle \text{Length } PO \ E \ E' \ AA \ CC \ S1 \rangle \langle \text{Per } CC \ B' \ AA \rangle$ 
         $\langle \text{Prod } PO \ E \ E' \ S1 \ S1 \ SS \rangle$  assms(1,4,5,6,7) cong-4321 length-eq-cong-2)
  qed
  thus Prod PO E E' S1 S1 S2
    by (simp add:  $\langle \text{Prod } PO \ E \ E' \ S1 \ S1 \ SS \rangle$ )
qed
}
qed

```

lemma pyth-context:

```

assumes  $\neg \text{Col } PO \ E \ E'$ 
shows  $\exists \ AB \ BC \ AC \ AB2 \ BC2 \ AC2 \ SS. \ \text{Col } PO \ E \ AB \wedge \text{Col } PO \ E \ BC \wedge \text{Col } PO \ E \ AC \wedge$ 
   $\text{Col } PO \ E \ AB2 \wedge \text{Col } PO \ E \ BC2 \wedge \text{Col } PO \ E \ AC2 \wedge$ 
   $\text{Length } PO \ E \ E' \ A \ B \ AB \wedge \text{Length } PO \ E \ E' \ B \ C \ BC \wedge$ 
   $\text{Length } PO \ E \ E' \ A \ C \ AC \wedge \text{Prod } PO \ E \ E' \ AB \ AB \ AB2 \wedge$ 
   $\text{Prod } PO \ E \ E' \ BC \ BC \ BC2 \wedge \text{Prod } PO \ E \ E' \ AC \ AC \ AC2 \wedge$ 
   $\text{Sum } PO \ E \ E' \ AB2 \ BC2 \ SS$ 

```

proof –

```

obtain AB where Length PO E E' A B AB
  using assms length-existence by blast
obtain BC where Length PO E E' B C BC
  using assms length-existence by blast
obtain AC where Length PO E E' A C AC
  using assms length-existence by blast
have Col PO E AB
  using Length-def  $\langle \text{Length } PO \ E \ E' \ A \ B \ AB \rangle$  by auto
then obtain AB2 where Prod PO E E' AB AB AB2
  using assms prod-exists by blast
hence Col PO E AB2
  by (metis Ar2-def  $\langle \text{Col } PO \ E \ AB \rangle$  assms prod-col)
have Col PO E BC
  using Length-def  $\langle \text{Length } PO \ E \ E' \ B \ C \ BC \rangle$  by auto
then obtain BC2 where Prod PO E E' BC BC BC2
  using assms prod-exists by blast
hence Col PO E BC2
  by (metis Ar2-def  $\langle \text{Col } PO \ E \ BC \rangle$  assms prod-col)
have Col PO E AC
  using Length-def  $\langle \text{Length } PO \ E \ E' \ A \ C \ AC \rangle$  by auto
then obtain AC2 where Prod PO E E' AC AC AC2

```

using *assms prod-exists* by *blast*
 hence *Col PO E AC2*
 by (*metis Ar2-def* $\langle \text{Col } PO \ E \ AC \rangle$ *assms prod-col*)
 obtain *SS where Sum PO E E' AB2 BC2 SS*
 using $\langle \text{Col } PO \ E \ AB2 \rangle \langle \text{Col } PO \ E \ BC2 \rangle$ *assms sum-exists* by *blast*
 thus ?thesis
 by (*metis* $\langle \text{Col } PO \ E \ AB2 \rangle \langle \text{Col } PO \ E \ AB \rangle \langle \text{Col } PO \ E \ AC2 \rangle \langle \text{Col } PO \ E \ AC \rangle$
 $\langle \text{Col } PO \ E \ BC2 \rangle \langle \text{Col } PO \ E \ BC \rangle \langle \text{Length } PO \ E \ E' \ A \ B \ AB \rangle \langle \text{Length } PO \ E \ E' \ A \ C \ AC \rangle$
 $\langle \text{Length } PO \ E \ E' \ B \ C \ BC \rangle \langle \text{Prod } PO \ E \ E' \ AB \ AB \ AB2 \rangle$
 $\langle \text{Prod } PO \ E \ E' \ AC \ AC \ AC2 \rangle \langle \text{Prod } PO \ E \ E' \ BC \ BC \ BC2 \rangle$)
 qed

lemma *length-pos-or-null*:
 assumes *Length PO E E' A B AB*
 shows *Ps PO E AB $\vee$ A = B*
 by (*metis assms cong-identity length-Ps length-cong*)

lemma *sum-pos-null*:
 assumes $\neg Ng \ PO \ E \ A$
 and $\neg Ng \ PO \ E \ B$
 and *Sum PO E E' A B PO*
 shows *A = PO $\wedge$ B = PO*
 by (*metis Ar2-def O-not-positive assms(1,2,3) col-pos-or-neg col-trivial-1 sum-A-null*
sum-B-null sum-ar2 sum-pos-pos)

lemma *length-not-neg*:
 assumes *Length PO E E' A B AB*
 shows $\neg Ng \ PO \ E \ AB$
 by (*metis Ng-def assms cong-reverse-identity length-cong length-pos-or-null neg-not-pos*)

lemma *signEq-refl*:
 assumes *PO $\neq$ E*
 and *Col PO E A*
 shows *A = PO $\vee$ SignEq PO E A A*
 by (*metis SignEq-def assms(1,2) col-pos-or-neg*)

lemma *square-not-neg*:
 assumes *Prod PO E E' A A A2*
 shows $\neg Ng \ PO \ E \ A2$
 by (*metis Ng-def assms pos-not-neg prod-O-r-eq square-pos*)

lemma *root-uniqueness*:
 assumes $\neg Ng \ PO \ E \ A$
 and $\neg Ng \ PO \ E \ B$
 and *Prod PO E E' A A C*
 and *Prod PO E E' B B C*
 shows *A = B*
 by (*metis Ar2-def Prod-def assms(1,2,3,4) col-pos-or-neg eq-squares-eq-or-opp*
opp-pos-neg prod-1-r prod-O-r-eq)

lemma *inter-tangent-circle*:
 assumes *P $\neq$ Q*
 and *Cong P PO Q PO*
 and *Col P PO Q*
 and *P M Le P PO*
 and *Q M Le Q PO*
 shows *M = PO*

proof –
 have *P = Q $\vee$ PO Midpoint P Q*
 using *assms(2,3) cong-commutativity l7-20-bis* by *blast*
 moreover
 {
 assume *PO Midpoint P Q*
 obtain *A where Bet PO A P and Cong P M PO A*
 using *Le-def assms(4) le-right-comm* by *blast*
 obtain *B where Bet PO B Q and Cong Q M PO B*


```

    using Le-cases Le-def assms(5) by blast
  have P Q Le A B
    using triangle-inequality-2 [of A PO B P M Q] by (meson Bet-cases ⟨Cong P M PO A⟩
      ⟨Cong Q M PO B⟩ ⟨PO Midpoint P Q⟩ ⟨PO - A - P⟩ ⟨PO - B - Q⟩
      between-exchange3 midpoint-bet not-cong-4312 not-cong-4321)
  have A B Le P Q
    by (meson ⟨PO Midpoint P Q⟩ ⟨PO - A - P⟩ ⟨PO - B - Q⟩ bet2-le2--le2356-R1
      between-exchange2 between-inner-transitivity between-symmetry
      between-trivial2 l5-12-a midpoint-bet)
  have Cong A B P Q
    by (simp add: ⟨A B Le P Q⟩ ⟨P Q Le A B⟩ le-anti-symmetry)
  hence M = PO
    by (smt (verit) Bet-cases Col-def Cong-cases Midpoint-def ⟨Cong P M PO A⟩ ⟨Cong Q M PO B⟩
      ⟨PO Midpoint P Q⟩ ⟨PO - A - P⟩ ⟨PO - B - Q⟩ assms(1) bet-cong-eq between-exchange3
      between-exchange4 l4-18)
}
ultimately show ?thesis
  using assms(1) by blast
qed

lemma inter-circle-per:
  assumes Cong P A Q A
    and P M Le P A
    and Q M Le Q A
    and A T Projp P Q
    and Per P T M
  shows T M Le T A
proof (cases P = T)
  have ¬ (Bet TPA TPB TPC ∨ Bet TPB TPC TPA ∨ Bet TPC TPA TPB)
    by (simp add: lower-dim)
  let ?O = TPA and ?E = TPB and ?E' = TPC
  have ¬ Col ?O ?E ?E'
    by (simp add: Col-def ⟨¬ ((TPA - TPB - TPC) ∨ (TPB - TPC - TPA) ∨ (TPC - TPA - TPB))⟩)
  show P = T ⟹ T M Le T A
    using assms(2) by auto
{
  assume P ≠ T
  have Col P Q T ∧ P Q Perp A T ∨ Col P Q A ∧ A = T
    using Projp-def assms(4) by presburger
  moreover {
    assume Col P Q T and P Q Perp A T
    have Per P T A
      using Per-perm ⟨Col P Q T⟩ ⟨P Q Perp A T⟩ col-trivial-3 l8-16-1 by blast
    obtain PT TA AP PT2 TA2 AP2 SS where Col ?O ?E PT and Col ?O ?E TA and
      Col ?O ?E AP and Col ?O ?E PT2 and Col ?O ?E TA2 and Col ?O ?E AP2 and
      Length ?O ?E ?E' P T PT and Length ?O ?E ?E' T A TA and
      Length ?O ?E ?E' P A AP and Prod ?O ?E ?E' PT PT PT2 and
      Prod ?O ?E ?E' TA TA TA2 and Prod ?O ?E ?E' AP AP AP2 and
      Sum ?O ?E ?E' PT2 TA2 SS
      using pyth-context [of ?O ?E ?E'] ⟨¬ Col ?O ?E ?E'⟩ by metis
    have Sum ?O ?E ?E' PT2 TA2 AP2
      using pythagoras [of ?O ?E P T A ?E' AP PT TA PT2 TA2 AP2]
        ⟨Length TPA TPB TPC P A AP⟩ ⟨Length TPA TPB TPC P T PT⟩
        ⟨Length TPA TPB TPC T A TA⟩ ⟨Per P T A⟩ ⟨Prod TPA TPB TPC AP AP AP2⟩
        ⟨Prod TPA TPB TPC PT PT PT2⟩ ⟨Prod TPA TPB TPC TA TA TA2⟩
        between-trivial2 length-sym lower-dim by blast
    hence SS = AP2
      using ⟨Sum TPA TPB TPC PT2 TA2 SS⟩ sum-stable by blast
    obtain PT' TM PM PT2' TM2 PM2 SS' where Col ?O ?E PT' and Col ?O ?E TM and
      Col ?O ?E PM and Col ?O ?E PT2' and Col ?O ?E TM2 and Col ?O ?E PM2 and
      Length ?O ?E ?E' P T PT' and Length ?O ?E ?E' T M TM and
      Length ?O ?E ?E' P M PM and Prod ?O ?E ?E' PT' PT' PT2' and
      Prod ?O ?E ?E' T M TM2 and Prod ?O ?E ?E' PM PM PM2 and
      Sum ?O ?E ?E' PT2' TM2 SS'
      using pyth-context [of ?O ?E ?E'] ⟨¬ Col ?O ?E ?E'⟩ by metis
    hence PT = PT'

```

```

    using ⟨Length TPA TPB TPC P T PT⟩ length-uniqueness by blast
  hence PT2 = PT2'
    using ⟨Prod TPA TPB TPC PT PT PT2⟩ ⟨Prod TPA TPB TPC PT' PT' PT2'⟩ ⟨¬ Col TPA TPB TPC⟩
      prod-uniqueness by blast
  have Sum ?O ?E ?E' PT2 TM2 PM2
    using pythagoras [of ?O ?E P T M ?E' PM PT TM PT2 TM2 PM2]
      ⟨Length TPA TPB TPC P M PM⟩ ⟨Length TPA TPB TPC P T PT⟩
      ⟨Length TPA TPB TPC T M TM⟩ ⟨Prod TPA TPB TPC PM PM PM2⟩
      ⟨Prod TPA TPB TPC PT PT PT2⟩ ⟨Prod TPA TPB TPC TM TM TM2⟩ assms(5) between-trivial
      length-sym lower-dim by blast
  have SS' = PM2
    using ⟨PT2 = PT2'⟩ ⟨Sum TPA TPB TPC PT2 TM2 PM2⟩ ⟨Sum TPA TPB TPC PT2' TM2 SS'⟩
      ⟨¬ Col TPA TPB TPC⟩ sum-uniqueness by blast
  have T M Le T A
  proof (cases T = M)
    show T = M ⟹ T M Le T A
      by (simp add: le-trivial)
    show T ≠ M ⟹ T M Le T A
      by (meson ⟨Per P T A⟩ assms(2,5) cong-lt-per2--lt cong-reflexivity lt--nle
        nlt--le)
  qed
}
moreover have Col P Q A ∧ A = T ⟹ T M Le T A
  by (metis Col-cases Projp-def assms(1,2,3,4) inter-tangent-circle
    le-reflexivity)
ultimately show T M Le T A
  by blast
}
qed

lemma inter-circle-obtuse:
  assumes Cong P A Q A
  and P M Le P A
  and Q M Le Q A
  and A T Projp P Q
  and Obtuse P T M ∨ Per P T M
  shows T M Le T A
proof (cases P = T)
  have ¬ (Bet TPA TPB TPC ∨ Bet TPB TPC TPA ∨ Bet TPC TPA TPB)
    by (simp add: lower-dim)
  let ?O = TPA and ?E = TPB and ?E' = TPC
  have ¬ Col ?O ?E ?E'
    by (simp add: Col-def ⟨¬ ((TPA - TPB - TPC) ∨ (TPB - TPC - TPA) ∨ (TPC - TPA - TPB))⟩)
  show P = T ⟹ T M Le T A
    using assms(2) by auto
{
  assume P ≠ T
  have ?O ≠ ?E
    using ⟨¬ ((TPA - TPB - TPC) ∨ (TPB - TPC - TPA) ∨ (TPC - TPA - TPB))⟩
      not-bet-distincts by blast
  have Per P T A
    by (metis Per-cases Projp-def assms(4) l8-16-1 l8-5 not-col-distincts)
  obtain PT TA AP PT2 TA2 AP2 SS where Col ?O ?E PT and Col ?O ?E TA and
    Col ?O ?E AP and Col ?O ?E PT2 and Col ?O ?E TA2 and Col ?O ?E AP2 and
    Length ?O ?E ?E' P T PT and Length ?O ?E ?E' T A TA and
    Length ?O ?E ?E' P A AP and Prod ?O ?E ?E' PT PT PT2 and
    Prod ?O ?E ?E' TA TA TA2 and Prod ?O ?E ?E' AP AP AP2 and
    Sum ?O ?E ?E' PT2 TA2 SS
    using pyth-context [of ?O ?E ?E'] ⟨¬ Col ?O ?E ?E'⟩ by metis
  have Sum ?O ?E ?E' PT2 TA2 AP2
    using pythagoras [of ?O ?E P T A ?E' AP PT TA PT2 TA2 AP2]
      ⟨Length TPA TPB TPC P A AP⟩ ⟨Length TPA TPB TPC P T PT⟩
      ⟨Length TPA TPB TPC T A TA⟩ ⟨Per P T A⟩
      ⟨Prod TPA TPB TPC AP AP AP2⟩ ⟨Prod TPA TPB TPC PT PT PT2⟩ ⟨Prod TPA TPB TPC TA TA TA2⟩
      between-trivial2 length-sym lower-dim by blast
  hence SS = AP2

```

```

using ⟨Sum TPA TPB TPC PT2 TA2 SS⟩ sum-stable by blast
obtain PT' TM PM PT2' TM2 PM2 SS' where Col ?O ?E PT' and Col ?O ?E TM and
Col ?O ?E PM and Col ?O ?E PT2' and Col ?O ?E TM2 and Col ?O ?E PM2 and
Length ?O ?E ?E' P T PT' and Length ?O ?E ?E' T M TM and
Length ?O ?E ?E' P M PM and Prod ?O ?E ?E' PT' PT' PT2' and
Prod ?O ?E ?E' TM TM TM2 and Prod ?O ?E ?E' PM PM PM2 and
Sum ?O ?E ?E' PT2' TM2 SS'
using pyth-context [of ?O ?E ?E'] ⟨¬ Col ?O ?E ?E'⟩ by metis
hence PT = PT'
using ⟨Length TPA TPB TPC P T PT⟩ length-uniqueness by blast
hence PT2 = PT2'
using ⟨Prod TPA TPB TPC PT PT PT2⟩ ⟨Prod TPA TPB TPC PT' PT' PT2'⟩ ⟨¬ Col TPA TPB TPC⟩
prod-uniqueness by blast
have LeP ?O ?E ?E' SS' PM2
using pythagoras-obtuse-or-per [of ?O ?E P T M ?E' PM PT TM PT2 TM2 PM2 SS']
by (simp add: ⟨Length TPA TPB TPC P M PM⟩ ⟨Length TPA TPB TPC P T PT'⟩
⟨Length TPA TPB TPC T M TM⟩ ⟨PT = PT'⟩ ⟨PT2 = PT2'⟩ ⟨Prod TPA TPB TPC PM PM PM2⟩
⟨Prod TPA TPB TPC PT' PT' PT2'⟩ ⟨Prod TPA TPB TPC TM TM TM2⟩
⟨Sum TPA TPB TPC PT2' TM2 SS'⟩ ⟨TPA ≠ TPB⟩ assms(5) length-sym)
show T M Le T A
proof (cases T = M)
show T = M ⟹ T M Le T A
by (simp add: le-trivial)
{
assume T ≠ M
show T M Le T A
proof (cases P = M)
show P = M ⟹ T M Le T A
using ⟨T ≠ M⟩ assms(5) l8-8 obtuse-distincts by blast
{
assume P ≠ M
have LeP ?O ?E ?E' PM AP
using length-leP-le-2 [of ?O ?E ?E' P M PM P A AP]
by (simp add: ⟨Length TPA TPB TPC P A AP⟩ ⟨Length TPA TPB TPC P M PM⟩
assms(2))
have Ar2 ?O ?E ?E' PM AP PM
by (simp add: Ar2-def ⟨Col TPA TPB AP⟩ ⟨Col TPA TPB PM⟩
⟨¬ Col TPA TPB TPC⟩)
moreover have Ps ?O ?E PM
proof -
have P = M ⟹ Ps ?O ?E PM
by (simp add: ⟨P ≠ M⟩)
thus ?thesis
by (meson ⟨Length TPA TPB TPC P M PM⟩ length-pos-or-null)
qed
moreover
have P = A ⟹ Ps ?O ?E AP
using ⟨P ≠ T⟩ ⟨Per P T A⟩ l8-8 by blast
hence Ps ?O ?E AP
using ⟨Length TPA TPB TPC P A AP⟩ length-pos-or-null by blast
ultimately have LeP ?O ?E ?E' PM2 AP2
using square-increase [of ?O ?E ?E' PM AP PM PM2] ⟨LeP TPA TPB TPC PM AP⟩
⟨Prod TPA TPB TPC AP AP AP2⟩
⟨Prod TPA TPB TPC PM PM PM2⟩ square-increase by blast
have LeP ?O ?E ?E' SS' AP2
using ⟨LeP TPA TPB TPC PM2 AP2⟩ ⟨LeP TPA TPB TPC SS' PM2⟩ leP-trans by blast
have LeP ?O ?E ?E' TM2 TA2
by (metis ⟨Col TPA TPB TA2⟩ ⟨Col TPA TPB TM2⟩ ⟨LeP TPA TPB TPC SS' AP2⟩
⟨PT2 = PT2'⟩ ⟨SS = AP2⟩ ⟨Sum TPA TPB TPC PT2 TA2 SS⟩
⟨Sum TPA TPB TPC PT2' TM2 SS'⟩
⟨¬ Col TPA TPB TPC⟩ col-2-le-or-ge leP-asym sum-comm sum-preserves-lep
sum-uniquenessB)
have T = M ⟹ Ps ?O ?E TM
using ⟨T ≠ M⟩ by blast
hence Ps ?O ?E TM
using ⟨Length TPA TPB TPC T M TM⟩ length-pos-or-null by blast

```

```

have Ps ?O ?E TA
  by (metis NCol-perm Ng-def Out-def Ps-def ⟨Col TPA TPB TA⟩
      ⟨LeP TPA TPB TPC TM2 TA2⟩
      ⟨Length TPA TPB TPC T A TA⟩ ⟨Prod TPA TPB TPC TA TA TA2⟩
      ⟨Prod TPA TPB TPC TM TM TM2⟩ ⟨Ps TPA TPB TM⟩ ⟨¬ Col TPA TPB TPC⟩ leP-asym
      length-not-neg or-bet-out prod-O-l-eq prod-pos-pos ps-le)
have LeP ?O ?E ?E' TM TA
  by (metis LeP-def ⟨LeP TPA TPB TPC TM2 TA2⟩ ⟨Prod TPA TPB TPC TA TA TA2⟩
      ⟨Prod TPA TPB TPC TM TM TM2⟩ ⟨Ps TPA TPB TA⟩ ⟨Ps TPA TPB TM⟩ pos-not-neg
      root-uniqueness square-increase-rev)
have A T Projp P Q
  by (simp add: assms(4))
hence (Col P Q T ∧ P Q Perp A T) ∨ (Col P Q A ∧ A = T)
  using Projp-def by auto
moreover {
  assume Col P Q T and P Q Perp A T
  have T M Le T A
    using length-leP-le-1 [of ?O ?E ?E' - - TM - - TA]
    using ⟨LeP TPA TPB TPC TM TA⟩ ⟨Length TPA TPB TPC T A TA⟩
    ⟨Length TPA TPB TPC T M TM⟩ by blast
}
moreover {
  assume Col P Q A and A = T
  have T M Le T A
  proof -
    have (Col P A Q ∧ P M Le P A ∧ Q M Le Q A) ⟶ M = A
      by (metis Projp-def assms(1,4) inter-tangent-circle)
    hence M = A
      using Col-cases ⟨Col P Q A⟩ assms(2,3) by blast
    thus ?thesis
      using ⟨A = T⟩ ⟨T ≠ M⟩ by auto
  qed
}
ultimately show T M Le T A
  by blast
}
qed
}
qed
}
qed
}
qed

```

lemma *circle-projp-between*:

```

assumes Cong P A Q A
  and A T Projp P Q
shows Bet P T Q
proof (cases P = T)
  show P = T ⟹ (P - T - Q)
    using not-bet-distincts by blast
  {
    assume P ≠ T
    have Col P Q T ∧ P Q Perp A T ∨ Col P Q A ∧ A = T
      using Projp-def assms(2) by auto
    moreover {
      assume Col P Q T and P Q Perp A T
      hence Per A T P
        using col-trivial-3 l8-16-1 by blast
      obtain T' where T' Midpoint P Q
        using midpoint-existence by blast
      hence Per A T' P
        using Per-def assms(1) cong-commutativity by blast
      have T' = T ∨ ¬ Col T P Q
        by (metis Projp-def ⟨P ≠ T⟩ ⟨Per A T P⟩ ⟨Per A T' P⟩ ⟨T' Midpoint P Q⟩ assms(2)
            col-per2-cases midpoint-col midpoint-distinct-1)
      moreover have T' = T ⟹ Bet P T Q
        using ⟨T' Midpoint P Q⟩ midpoint-bet by blast
    }
  }

```

```

    moreover have  $\neg \text{Col } T \ P \ Q \implies \text{Bet } P \ T \ Q$ 
      using Col-cases  $\langle \text{Col } P \ Q \ T \rangle$  by auto
    ultimately have  $\text{Bet } P \ T \ Q$ 
      by blast
  }
  moreover have  $\text{Col } P \ Q \ A \wedge A = T \implies \text{Bet } P \ T \ Q$ 
    by (metis NCol-perm Projp-def assms(1,2) bet-cong-eq between-trivial col-bet2-cong1
      col-trivial-2 cong-identity third-point)
  ultimately show  $P - T - Q$ 
    by blast
}
qed

```

```

lemma inter-circle:
  assumes Cong  $P \ A \ Q \ A$ 
    and  $P \ M \ \text{Le } P \ A$ 
    and  $Q \ M \ \text{Le } Q \ A$ 
    and  $A \ T \ \text{Projp } P \ Q$ 
  shows  $T \ M \ \text{Le } T \ A$ 
proof (cases  $T = M$ )
  have  $\text{Bet } P \ T \ Q$ 
    using assms(1,4) circle-projp-between by auto
  show  $T = M \implies T \ M \ \text{Le } T \ A$ 
    using le-trivial by blast
  {
    assume  $T \neq M$ 
    show  $T \ M \ \text{Le } T \ A$ 
    proof (cases  $P = T$ )
      show  $P = T \implies T \ M \ \text{Le } T \ A$ 
        using assms(2) by blast
      {
        assume  $P \neq T$ 
        hence  $\text{Acute } P \ T \ M \vee \text{Per } P \ T \ M \vee \text{Obtuse } P \ T \ M$ 
          using  $\langle T \neq M \rangle$  angle-partition by auto
        show  $T \ M \ \text{Le } T \ A$ 
        proof (cases  $Q = T$ )
          show  $Q = T \implies T \ M \ \text{Le } T \ A$ 
            using assms(3) by auto
          {
            assume  $Q \neq T$ 
            {
              assume  $\text{Acute } P \ T \ M$ 
              have  $\text{Obtuse } Q \ T \ M$ 
                using  $\langle \text{Acute } P \ T \ M \rangle \langle P - T - Q \rangle \langle Q \neq T \rangle$  acute-bet--obtuse by blast
              moreover have  $A \ T \ \text{Projp } Q \ P$ 
                by (metis Col-cases Perp-cases Projp-def assms(4))
              ultimately have  $T \ M \ \text{Le } T \ A$ 
                using Cong-cases assms(1,2,3) inter-circle-obtuse by blast
            }
            moreover have  $\text{Per } P \ T \ M \implies T \ M \ \text{Le } T \ A$ 
              using assms(1,2,3,4) inter-circle-per by blast
            moreover have  $\text{Obtuse } P \ T \ M \implies T \ M \ \text{Le } T \ A$ 
              using assms(1,2,3,4) inter-circle-obtuse by blast
            ultimately show  $T \ M \ \text{Le } T \ A$ 
              using  $\langle \text{Acute } P \ T \ M \vee \text{Per } P \ T \ M \vee \text{Obtuse } P \ T \ M \rangle$  by blast
          }
        }
      }
    }
  }
qed
}
qed
}
qed

```

```

lemma projp-lt:
  assumes Cong  $P \ A \ Q \ A$ 
    and  $A \ T \ \text{Projp } P \ Q$ 
  shows  $T \ A \ \text{Lt } P \ A$ 

```

```

proof –
  have  $Col\ P\ Q\ T \wedge P\ Q\ Perp\ A\ T \vee Col\ P\ Q\ A \wedge A = T$ 
    using Projp-def assms(2) by auto
  moreover {
    assume  $Col\ P\ Q\ T$  and  $P\ Q\ Perp\ A\ T$ 
    have ?thesis
    proof (cases  $P = T$ )
      show  $P = T \implies T\ A\ Lt\ P\ A$ 
        by (metis Perp-cases  $\langle P\ Q\ Perp\ A\ T \rangle$  assms(1) cong-nlt l11-46 l8-16-1
          not-col-distincts)
      show  $P \neq T \implies T\ A\ Lt\ P\ A$ 
        by (metis Lt-cases  $\langle Col\ P\ Q\ T \rangle$   $\langle P\ Q\ Perp\ A\ T \rangle$  col-trivial-3 l8-16-1 per-lt)
    qed
  }
  moreover have  $Col\ P\ Q\ A \wedge A = T \implies ?thesis$ 
    by (metis Projp-def assms(1,2) cong-reverse-identity lt1123)
  ultimately show ?thesis
    by blast
qed

lemma Ps-Col:
  assumes  $Ps\ PO\ E\ A$ 
  shows  $Col\ PO\ E\ A$ 
  using Col-cases Ps-def assms out-col by blast

lemma PythRel-exists:
  assumes  $\neg Col\ PO\ E\ E'$ 
  shows  $\forall A\ B. Col\ PO\ E\ A \wedge Col\ PO\ E\ B \longrightarrow (\exists C. PythRel\ PO\ E\ E'\ A\ B\ C)$ 
proof –
  {
    fix  $A\ B$ 
    assume  $Col\ PO\ E\ A$  and  $Col\ PO\ E\ B$ 
    have  $\exists C. PythRel\ PO\ E\ E'\ A\ B\ C$ 
    proof (cases  $B = PO$ )
      show  $B = PO \implies \exists C. PythRel\ PO\ E\ E'\ A\ B\ C$ 
        using Ar2-def PythRel-def  $\langle Col\ PO\ E\ A \rangle$  assms col-trivial-3 by auto
      {
        assume  $B \neq PO$ 
        obtain  $X$  where  $PO\ X\ Perp\ E\ PO$ 
          using Perp-cases assms l8-21-aux by blast
        obtain  $B'$  where  $Bet\ PO\ X\ B' \vee Bet\ PO\ B'\ X$  and  $Cong\ PO\ B'\ PO\ B$ 
          by (metis  $\langle PO\ X\ Perp\ E\ PO \rangle$  perp-distinct segment-construction-2)
        obtain  $C$  where  $Bet\ PO\ E\ C \vee Bet\ PO\ C\ E$  and  $Cong\ PO\ C\ A\ B'$ 
          by (metis segment-construction segment-construction-2)
        have  $Col\ PO\ E\ C$ 
          using Bet-cases Col-def  $\langle (PO - E - C) \vee (PO - C - E) \rangle$  by blast
        moreover have  $PO\ B'\ Perp\ PO\ B \wedge Cong\ PO\ B'\ PO\ B \wedge Cong\ PO\ C\ A\ B'$ 
          by (metis Col-cases Col-def  $\langle B \neq PO \rangle$   $\langle Col\ PO\ E\ B \rangle$   $\langle Cong\ PO\ B'\ PO\ B \rangle$   $\langle Cong\ PO\ C\ A\ B' \rangle$ 
             $\langle PO\ X\ Perp\ E\ PO \rangle$   $\langle (PO - X - B') \vee (PO - B' - X) \rangle$  col-trivial-2
            cong-reverse-identity perp-col0)
        hence  $PO = B \wedge (A = C \vee Opp\ PO\ E\ E'\ A\ C) \vee$ 
           $(\exists B'0. PO\ B'0\ Perp\ PO\ B \wedge Cong\ PO\ B'0\ PO\ B \wedge Cong\ PO\ C\ A\ B'0)$ 
          by blast
        ultimately have  $PythRel\ PO\ E\ E'\ A\ B\ C$ 
          by (simp add: assms  $\langle Col\ PO\ E\ A \rangle$   $\langle Col\ PO\ E\ B \rangle$  Ar2-def PythRel-def)
        thus  $\exists C. PythRel\ PO\ E\ E'\ A\ B\ C$ 
          by fast
      }
    }
  }
  qed
}
thus ?thesis
  by blast
qed

lemma opp-same-square:
  assumes  $Opp\ PO\ E\ E'\ A\ B$ 

```

```

    and Prod PO E E' A A A2
  shows Prod PO E E' B B A2
proof -
  obtain ME where Opp PO E E' E ME
  using Ar2-def Prod-def assms(2) col-trivial-2 opp-exists by presburger
  hence Prod PO E E' A ME B
  using assms(1) opp-prod by blast
  moreover have Prod PO E E' B ME A
  by (meson Ar2-def Prod-def ⟨Opp PO E E' E ME⟩ assms(1) opp-comm opp-prod)
  hence Prod PO E E' ME B A
  by (simp add: prod-sym)
  ultimately show ?thesis
  using assms(2) prod-assoc by blast
qed

```

lemma PythOK:

```

  assumes PythRel PO E E' A B C
    and Prod PO E E' A A A2
    and Prod PO E E' B B B2
    and Prod PO E E' C C C2
  shows Sum PO E E' A2 B2 C2
proof -
  have PO ≠ E
  using Ar2-def PythRel-def assms(1) col-trivial-1 by force
  have PO = B ∧
    (A = C ∨ Opp PO E E' A C) ∨ (∃ B'. PO B' Perp PO B ∧ Cong PO B' PO B ∧ Cong PO C A B')
  using PythRel-def assms(1) by auto
  moreover {
    assume PO = B and A = C ∨ Opp PO E E' A C
    hence PO = B2
    using assms(3) prod-O-l-eq by blast
    have A = C ⟹ A2 = C2
    by (metis Ar2-def Prod-def assms(2,4) prod-uniqueness)
    moreover have Opp PO E E' A C ⟹ A2 = C2
    by (meson Ar2-def PythRel-def assms(1,2,4) opp-same-square prod-uniqueness)
    ultimately have A2 = C2
    using ⟨A = C ∨ Opp PO E E' A C⟩ by blast
    hence Col PO E A2
    by (metis Col-def Ps-Col assms(4) between-trivial prod-O-r-eq
        square-pos)
    hence Sum PO E E' A2 B2 C2
    using Ar2-def PythRel-def ⟨A2 = C2⟩ ⟨PO = B2⟩ assms(1) sum-A-O by auto
  }
  moreover {
    assume ∃ B'. PO B' Perp PO B ∧ Cong PO B' PO B ∧ Cong PO C A B'
    then obtain B' where PO B' Perp PO B and Cong PO B' PO B and Cong PO C A B'
    by blast
    have Per A PO B'
    proof -
      have PO ≠ B
      using ⟨PO B' Perp PO B⟩ perp-distinct by blast
      moreover have Per B' PO B
      by (simp add: ⟨PO B' Perp PO B⟩ perp-per-2)
      moreover have Col PO B A
      using Ar2-def PythRel-def ⟨PO ≠ E⟩ assms(1) col-transitivity-1 by presburger
      ultimately show ?thesis
      using Per-cases per-col by blast
    qed
    have Sum PO E E' A2 B2 C2
    proof (cases A = PO)
      {
        assume A = PO
        hence Cong PO B PO C
        using ⟨∃ B'. PO B' Perp PO B ∧ Cong PO B' PO B ∧ Cong PO C A B'⟩

```

```

    cong-inner-transitivity cong-symmetry by blast
have B = C ∨ PO Midpoint B C
using l7-20 [of B PO C] Ar2-def PythRel-def ⟨Cong PO B PO C⟩ ⟨PO ≠ E⟩
  assms(1) col-permutation-4 col-transitivity-1 by presburger
moreover {
  assume B = C
  have B2 = C2
    using prod-uniqueness [of PO E E' B B2 C2] Ar2-def PythRel-def ⟨B = C⟩
      assms(1,3,4) by presburger
  have A2 = PO
    using prod-uniqueness [of PO E E' PO PO A2 PO] ⟨A = PO⟩ assms(2) prod-O-l-eq by blast
  have Col PO E B2
    by (metis Ps-Col assms(3) grid-not-par-5 prod-O-r-eq square-pos)
  hence Sum PO E E' A2 B2 C2
    using Ar2-def PythRel-def ⟨A2 = PO⟩ ⟨B2 = C2⟩ assms(1) sum-O-B by auto
}
moreover {
  assume PO Midpoint B C
  have A2 = PO
    using prod-uniqueness [of PO E E' PO PO A2 PO] ⟨A = PO⟩ assms(2) prod-O-l-eq by blast
  have Ar2 PO E E' PO B C
    using PythRel-def ⟨A = PO⟩ assms(1) by blast
  hence Opp PO E E' B C
    by (simp add: ⟨PO Midpoint B C⟩ midpoint-opp)
  have C2 = B2
    using prod-uniqueness [of PO E E' C C2 B2] Ar2-def ⟨Ar2 PO E E' PO B C⟩
      ⟨Opp PO E E' B C⟩ assms(3,4) opp-same-square
    by presburger
  have Col PO E B2
    by (metis Col-def Ps-Col ⟨C2 = B2⟩ assms(4) between-trivial2 prod-O-r-eq
      square-pos)
  hence Sum PO E E' A2 B2 C2
    using Ar2-def ⟨A2 = PO⟩ ⟨Ar2 PO E E' PO B C⟩ ⟨C2 = B2⟩ sum-O-B by presburger
}
ultimately show Sum PO E E' A2 B2 C2
  by blast
}
{
  assume A ≠ PO
  show Sum PO E E' A2 B2 C2
  proof (cases PO Out A E)
  {
    assume PO Out A E
    show Sum PO E E' A2 B2 C2
    proof (cases PO Out B E)
    {
      assume PO Out B E
      show Sum PO E E' A2 B2 C2
      proof (cases PO Out C E)
      {
        assume PO Out C E
        show Sum PO E E' A2 B2 C2
        proof (rule pythagoras [of - - A PO B' - C A B], insert ⟨PO ≠ E⟩
          ⟨Per A PO B'⟩ assms(2,3,4))
        have LeP PO E E' PO C
        proof -
          have Diff PO E E' C PO C
            using Ar2-def Prod-def assms(4) diff-A-O by presburger
          moreover have Ps PO E C
            by (simp add: Ps-def ⟨PO Out C E⟩)
          ultimately show ?thesis
            by (simp add: diff-pos--ltp ltp-to-lep)
        qed
        thus Length PO E E' A B' C
          by (meson Length-def NCol-cases ⟨Cong PO C A B'⟩ ⟨PO Out C E⟩ ⟨PO ≠ E⟩
            out-col)
      }
    }
  }
}

```



```

show Length PO E E' A PO A
  by (metis Ar2-def Cong-cases Length-def Out-def PythRel-def
    ⟨PO Out A E⟩ assms(1)
    cong-reflexivity ps-le)
show Length PO E E' B' PO B
  by (metis Length-def Out-def ⟨Cong PO B' PO B⟩ ⟨Length PO E E' A B' C⟩
    ⟨PO Out B E⟩ ⟨PO Out C E⟩ cong-3421 length-Ar2 out-col ps-le)
qed
}
{
  assume ¬ PO Out C E
  obtain OC where Opp PO E E' C OC
  by (meson Ar2-def Diff-def Prod-def assms(4) diff-null)
  have Ng PO E C
  by (metis Col-def Ps-def SignEq-def ⟨A ≠ PO⟩ ⟨Cong PO C A B'⟩ ⟨Per A PO B'⟩
    ⟨¬ PO Out C E⟩ assms(4) between-trivial2 cong-reverse-identity
    not-signEq-prod-neg per-not-col square-not-neg)
  hence Ps PO E OC
  using ⟨Opp PO E E' C OC⟩ opp-neg-pos by auto
  show Sum PO E E' A2 B2 C2
  proof (rule pythagoras [of - - A PO B' - OC A B], insert ⟨PO ≠ E⟩
    ⟨Per A PO B'⟩ assms(2,3))
  show Length PO E E' A B' OC
  proof -
    have Col PO E OC
    by (simp add: Ps-Col ⟨Ps PO E OC⟩)
    moreover have LeP PO E E' PO OC
    by (metis Ar2-def Prod-def Ps-def ⟨Ps PO E OC⟩ assms(2)
      calculation not-bet-and-out ps-le third-point)
    moreover have Cong PO OC A B'
    proof -
      have PO Midpoint C OC
      using opp-midpoint ⟨Opp PO E E' C OC⟩ by auto
      thus ?thesis
      using Cong-cases Midpoint-def ⟨Cong PO C A B'⟩ cong-inner-transitivity
      by blast
    qed
    ultimately show ?thesis
    by (simp add: Length-def ⟨PO ≠ E⟩)
  qed
  show Length PO E E' A PO A
  using Ar2-def Length-def Out-def Prod-def ⟨PO Out A E⟩ assms(2)
  cong-pseudo-reflexivity ps-le by force
  show Length PO E E' B' PO B
  by (metis Ar2-def Length-def Out-def Prod-def ⟨Cong PO B' PO B⟩
    ⟨PO Out B E⟩ assms(3) cong-3421 ps-le)
  show Prod PO E E' OC OC C2
  using ⟨Opp PO E E' C OC⟩ assms(4) opp-same-square by auto
  qed
}
}
{
  assume ¬ PO Out B E
  obtain OB where Opp PO E E' B OB
  using Ar2-def Prod-def assms(3) opp-exists by presburger

  have Ng PO E B
  by (metis Ps-def SignEq-def
    ⟨∃ B'. PO B' Perp PO B ∧ Cong PO B' PO B ∧ Cong PO C A B'⟩
    ⟨¬ PO Out B E⟩ assms(3) not-col-distincts not-signEq-prod-neg
    perp-not-col2 square-not-neg)
  hence Ps PO E OB
  using ⟨Opp PO E E' B OB⟩ opp-neg-pos by auto
  {
    assume PO Out C E

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have Sum PO E E' A2 B2 C2
proof (rule pythagoras [of - - A PO B' - C A OB], insert ⟨PO ≠ E⟩
  ⟨Per A PO B'⟩ assms(2,4))
have LeP PO E E' PO C
proof -
  have PO Out OB E
    using Ps-def ⟨Ps PO E OB⟩ by auto
  have LtP PO E E' PO C
  proof -
    have Diff PO E E' C PO C
      using Ar2-def Prod-def assms(4) diff-A-O by presburger
    moreover have Ps PO E C
      by (simp add: Ps-def ⟨PO Out C E⟩)
    ultimately show ?thesis
      using diff-pos--ltp by blast
  qed
thus ?thesis
  using ltp-to-lep by auto
qed
thus Length PO E E' A B' C
  by (metis Length-def Ps-Col ⟨Cong PO C A B'⟩ ⟨PO ≠ E⟩ length-O lep-neq--ltp
    ltp-pos)
show Length PO E E' A PO A
  by (metis Ar2-def Cong-cases Length-def Out-def PythRel-def
    ⟨PO Out A E⟩ assms(1) cong-reflexivity ps-le)
show Length PO E E' B' PO OB
proof -
  have LeP PO E E' PO OB
    using Ar2-def Out-def Ps-def PythRel-def ⟨Ps PO E OB⟩
    assms(1) ps-le by force
  moreover have PO Midpoint B OB
    using opp-midpoint ⟨Opp PO E E' B OB⟩ by auto
  hence Cong PO OB B' PO
    by (meson Midpoint-def ⟨Cong PO B' PO B⟩ cong-4312 cong-inner-transitivity)
  moreover have Prod PO E E' OB OB B2
    using ⟨Opp PO E E' B OB⟩ assms(3) opp-same-square by blast
  ultimately show ?thesis
    by (simp add: Length-def Ps-Col ⟨PO ≠ E⟩ ⟨Ps PO E OB⟩)
  qed
show Prod PO E E' OB OB B2
  using ⟨Opp PO E E' B OB⟩ assms(3) opp-same-square by blast
qed
}
moreover {
  assume ¬ PO Out C E
  obtain OC where Opp PO E E' C OC
    using Ar2-def Prod-def assms(4) opp-exists by presburger
  have Ng PO E C
    by (metis Ps-def SignEq-def ⟨A ≠ PO⟩ ⟨Cong PO C A B'⟩ ⟨Per A PO B'⟩
      ⟨¬ PO Out C E⟩ assms(4) cong-reverse-identity l8-20-1-R1 l8-7
      not-signEq-prod-neg square-not-neg)
  hence Ps PO E OC
    using ⟨Opp PO E E' C OC⟩ opp-neg-pos by auto
  obtain OB where Opp PO E E' B OB
    using ⟨Opp PO E E' B OB⟩ by auto
  have Ps PO E OB
    using ⟨Ng PO E B⟩ ⟨Opp PO E E' B OB⟩ opp-neg-pos by auto
  have Sum PO E E' A2 B2 C2
  proof (rule pythagoras [of - - A PO B' - OC A OB], insert ⟨PO ≠ E⟩
    ⟨Per A PO B'⟩ assms(2))
    show Length PO E E' A B' OC
  proof -
    have Col PO E OC
      by (simp add: Ps-Col ⟨Ps PO E OC⟩)
    moreover have LeP PO E E' PO OC
      by (meson Ar2-def Ps-def PythRel-def ⟨Ps PO E OC⟩ assms(1) calculation

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      not-bet-and-out ps-le third-point)
    moreover have PO Midpoint C OC
      using opp-midpoint ‹Opp PO E E' C OC› by auto
    hence Cong PO OC A B'
      using Cong-cases Midpoint-def ‹Cong PO C A B'› cong-inner-transitivity
      by blast
    ultimately show ?thesis
      by (simp add: Length-def ‹PO ≠ E›)
  qed
show Length PO E E' A PO A
  using Ar2-def Length-def Out-def Prod-def ‹PO Out A E› assms(2)
  cong-pseudo-reflexivity ps-le by auto
show Length PO E E' B' PO OB
proof -
  have LeP PO E E' PO OB
    using Ar2-def Out-def Prod-def Ps-def ‹Ps PO E OB› assms(2) ps-le by force
  moreover have PO Midpoint B OB
    using opp-midpoint ‹Opp PO E E' B OB› by blast
  hence Cong PO OB B' PO
    by (meson Midpoint-def ‹Cong PO B' PO B› cong-4312 cong-inner-transitivity)
  ultimately show ?thesis
    by (simp add: Length-def Ps-Col ‹PO ≠ E› ‹Ps PO E OB›)
  qed
show Prod PO E E' OB OB B2
  using ‹Opp PO E E' B OB› assms(3) opp-same-square by blast
show Prod PO E E' OC OC C2
  using ‹Opp PO E E' C OC› assms(4) opp-same-square by auto
qed
}
ultimately show Sum PO E E' A2 B2 C2
  by blast
}
qed
}
{
  assume ¬ PO Out A E
  show Sum PO E E' A2 B2 C2
  proof (cases PO Out B E)
    {
      assume PO Out B E
      show Sum PO E E' A2 B2 C2
      proof (cases PO Out C E)
        {
          assume PO Out C E
          obtain OA where Opp PO E E' A OA
            by (meson Ar2-def Diff-def Prod-def assms(2) diff-null)
          have Ng PO E A
            by (metis Ps-def SignEq-def ‹A ≠ PO› ‹¬ PO Out A E› assms(2)
              not-signEq-prod-neg square-not-neg)
          show Sum PO E E' A2 B2 C2
          proof (rule pythagoras [of - - A PO B' - C OA B], insert ‹PO ≠ E›
            ‹Per A PO B'› assms(3,4))
            show Length PO E E' A B' C
              using Ar2-def Length-def Out-def Prod-def ‹Cong PO C A B'›
              ‹PO Out C E› assms(4) ps-le by force
            show Length PO E E' A PO OA
            proof -
              have Col PO E OA
                using Ps-Col ‹Ng PO E A› ‹Opp PO E E' A OA› opp-neg-pos by blast
              moreover have LtP PO E E' PO OA
                by (metis Ar2-def Out-def Ps-def ‹Ng PO E A› ‹Opp PO E E' A OA›
                  diff-O-A diff-ar2 lep-neg--ltp opp-neg-pos ps-le)
              ultimately show ?thesis
                by (meson Length-def Opp-def ‹Opp PO E E' A OA› ‹PO ≠ E› l15-3
                  ltp-to-lep not-cong-4321)
            qed
          qed
        }
      }
    }
  }
}

```

```

show Length PO E E' B' PO B
  by (metis Ar2-def Length-def Out-def Prod-def ‹Cong PO B' PO B›
      ‹PO Out B E› assms(3) cong-3421 ps-le)
show Prod PO E E' OA OA A2
  using ‹Opp PO E E' A OA› assms(2) opp-same-square by force
qed
}
{
  assume ¬ PO Out C E
  obtain OC where Opp PO E E' C OC
  by (meson Diff-def Prod-def assms(4) sum-diff sum3-def sum3-exists)
  have Ng PO E C
  by (metis Ps-def SignEq-def ‹A ≠ PO› ‹Cong PO C A B'› ‹Per A PO B'›
      ‹¬ PO Out C E› assms(4) cong-reverse-identity not-col-distincts
      not-signEq-prod-neg per-not-col square-not-neg)
  hence Ps PO E OC
  using ‹Opp PO E E' C OC› opp-neg-pos by auto
  obtain OA where Opp PO E E' A OA
  by (meson Diff-def Prod-def assms(2) sum-diff sum3-def sum3-exists)
  have Ng PO E A
  by (metis Ps-def SignEq-def ‹A ≠ PO› ‹¬ PO Out A E› assms(2)
      not-signEq-prod-neg square-not-neg)
  show Sum PO E E' A2 B2 C2
  proof (rule pythagoras [of - - A PO B' - OC OA B], insert ‹PO ≠ E›
      ‹Per A PO B'› assms(3))
    show Length PO E E' A B' OC
    proof (unfold Length-def, intro strip conjI, insert ‹PO ≠ E›)
      show Col PO E OC
      by (simp add: Ps-Col ‹Ps PO E OC›)
      show LeP PO E E' PO OC
      by (metis Ar2-def Prod-def Ps-def ‹Col PO E OC› ‹Ps PO E OC›
          assms(2) not-bet-and-out ps-le third-point)
      have PO Midpoint C OC
      using opp-midpoint ‹Opp PO E E' C OC› by blast
      thus Cong PO OC A B'
      using Cong-cases Midpoint-def ‹Cong PO C A B'› cong-inner-transitivity
      by blast
    qed
  show Length PO E E' A PO OA
  proof -
    have LeP PO E E' PO OA
    by (meson Ar2-def Out-def Prod-def Ps-def ‹Ng PO E A›
        ‹Opp PO E E' A OA› assms(2) opp-neg-pos ps-le)
    moreover have Cong PO OA A PO
    using Cong-cases Opp-def ‹Opp PO E E' A OA› l15-3 by blast
    ultimately show ?thesis
    by (metis Length-def Ps-Col ‹A ≠ PO› ‹Ng PO E A› ‹Opp PO E E' A OA›
        ‹PO ≠ E› neg-not-pos pos-null-neg)
  qed
  show Length PO E E' B' PO B
  by (metis Ar2-def Length-def Out-def Prod-def ‹Cong PO B' PO B›
      ‹PO Out B E› assms(3) cong-3421 ps-le)
  show Prod PO E E' OA OA A2
  using ‹Opp PO E E' A OA› assms(2) opp-same-square by auto
  show Prod PO E E' OC OC C2
  using ‹Opp PO E E' C OC› assms(4) opp-same-square by auto
  qed
}
}
{
  assume ¬ PO Out B E
  show Sum PO E E' A2 B2 C2
  proof (cases PO Out C E)
    {
      assume PO Out C E

```

```

obtain  $OB$  where  $Opp\ PO\ E\ E'\ B\ OB$ 
  by (metis Diff-def Prod-def assms(3) sum-diff sum3-def sum3-exists)
have  $Ng\ PO\ E\ B$ 
  by (metis Ps-def SignEq-def  $\langle PO\ B'\ Perp\ PO\ B \rangle \langle \neg\ PO\ Out\ B\ E \rangle$  assms(3)
    not-signEq-prod-neg perp-not-eq-2 square-not-neg)
hence  $Ps\ PO\ E\ OB$ 
  using  $\langle Opp\ PO\ E\ E'\ B\ OB \rangle$  opp-neg-pos by blast
obtain  $OA$  where  $Opp\ PO\ E\ E'\ A\ OA$ 
  by (meson Diff-def Prod-def assms(2) sum-diff sum3-def sum3-exists)
have  $Ng\ PO\ E\ A$ 
  by (metis Ps-def SignEq-def  $\langle A \neq PO \rangle \langle \neg\ PO\ Out\ A\ E \rangle$  assms(2)
    not-signEq-prod-neg square-not-neg)
show  $Sum\ PO\ E\ E'\ A2\ B2\ C2$ 
proof (rule pythagoras [of - -  $A\ PO\ B' - C\ OA\ OB$ ], insert  $\langle PO \neq E \rangle$ 
   $\langle Per\ A\ PO\ B' \rangle$  assms(4))
have  $LtP\ PO\ E\ E'\ PO\ C$ 
  by (metis Ar2-def Out-def PythRel-def  $\langle PO\ Out\ C\ E \rangle$  assms(1) lep-neq--ltP
    ps-le)
thus  $Length\ PO\ E\ E'\ A\ B'\ C$ 
  by (simp add: Length-def Ps-Col  $\langle Cong\ PO\ C\ A\ B' \rangle \langle PO \neq E \rangle$ 
    ltP-pos ltP-to-lep)
show  $Length\ PO\ E\ E'\ A\ PO\ OA$ 
proof -
  have  $LtP\ PO\ E\ E'\ PO\ OA$ 
    by (metis Ar2-def Out-def Ps-def  $\langle LtP\ PO\ E\ E'\ PO\ C \rangle \langle Ng\ PO\ E\ A \rangle$ 
       $\langle Opp\ PO\ E\ E'\ A\ OA \rangle$  lep-neq--ltP ltP-ar2 opp-neg-pos ps-le)
  moreover have  $Cong\ PO\ OA\ A\ PO$ 
    using Cong-cases Opp-def  $\langle Opp\ PO\ E\ E'\ A\ OA \rangle$  l15-3 by blast
  ultimately show ?thesis
    by (simp add: Length-def Ps-Col  $\langle PO \neq E \rangle$  ltP-pos ltP-to-lep)
qed
show  $Length\ PO\ E\ E'\ B'\ PO\ OB$ 
proof -
  have  $LtP\ PO\ E\ E'\ PO\ OB$ 
    by (metis Ar2-def Out-def Prod-def Ps-def  $\langle Ps\ PO\ E\ OB \rangle$  assms(4)
      lep-neq--ltP ps-le)
  moreover have  $Cong\ PO\ OB\ B'\ PO$ 
    by (metis Cong-cases Midpoint-def  $\langle Cong\ PO\ B'\ PO\ B \rangle \langle Opp\ PO\ E\ E'\ B\ OB \rangle$ 
      cong-inner-transitivity opp-midpoint)
  ultimately show ?thesis
    by (simp add: Length-def Ps-Col  $\langle PO \neq E \rangle \langle Ps\ PO\ E\ OB \rangle$  ltP-to-lep)
qed
show  $Prod\ PO\ E\ E'\ OA\ OA\ A2$ 
  using  $\langle Opp\ PO\ E\ E'\ A\ OA \rangle$  assms(2) opp-same-square by blast
show  $Prod\ PO\ E\ E'\ OB\ OB\ B2$ 
  using  $\langle Opp\ PO\ E\ E'\ B\ OB \rangle$  assms(3) opp-same-square by auto
qed
}
{
assume  $\neg\ PO\ Out\ C\ E$ 
obtain  $OC$  where  $Opp\ PO\ E\ E'\ C\ OC$ 
  by (meson Diff-def Prod-def assms(4) sum-diff sum3-def sum3-exists)
have  $Ng\ PO\ E\ C$ 
  by (metis Ar2-def Ng-def Out-def PythRel-def  $\langle A \neq PO \rangle \langle Cong\ PO\ C\ A\ B' \rangle$ 
     $\langle Per\ A\ PO\ B' \rangle \langle \neg\ PO\ Out\ C\ E \rangle$  assms(1) cong-reverse-identity
    not-col-distincts per-not-col third-point)
hence  $Ps\ PO\ E\ OC$ 
  using  $\langle Opp\ PO\ E\ E'\ C\ OC \rangle$  opp-neg-pos by auto
obtain  $OA$  where  $Opp\ PO\ E\ E'\ A\ OA$ 
  by (meson Diff-def Prod-def assms(2) sum-diff sum3-def sum3-exists)
have  $Ng\ PO\ E\ A$ 
  by (metis Ps-def SignEq-def  $\langle A \neq PO \rangle \langle \neg\ PO\ Out\ A\ E \rangle$  assms(2)
    not-signEq-prod-neg square-not-neg)
obtain  $OB$  where  $Opp\ PO\ E\ E'\ B\ OB$ 
  by (meson Ar2-def Diff-def Prod-def assms(3) diff-null)
have  $Ng\ PO\ E\ B$ 

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by (metis Ps-def SignEq-def ⟨PO B' Perp PO B⟩ ⟨¬ PO Out B E⟩ assms(3)
    not-col-distincts not-signEq-prod-neg perp-not-col2 square-not-neg)
hence Ps PO E OB
  using ⟨Opp PO E E' B OB⟩ opp-neg-pos by auto
show Sum PO E E' A2 B2 C2
proof (rule pythagoras [of - - A PO B' - OC OA OB], insert ⟨PO ≠ E⟩
    ⟨Per A PO B'⟩)
show Length PO E E' A B' OC
proof -
  have Col PO E OC
    by (simp add: Ps-Col ⟨Ps PO E OC⟩)
  moreover have LeP PO E E' PO OC
    by (metis Ar2-def Ps-def PythRel-def ⟨Ps PO E OC⟩ assms(1) calculation
        not-bet-and-out ps-le third-point)
  moreover have Cong PO OC A B'
    by (metis Cong-cases Opp-def ⟨Cong PO C A B'⟩ ⟨Opp PO E E' C OC⟩
        cong-transitivity l15-3)
  ultimately show ?thesis
    by (simp add: Length-def ⟨PO ≠ E⟩)
qed
show Length PO E E' A PO OA
proof -
  have LtP PO E E' PO OA
    by (metis Ar2-def Out-def Prod-def Ps-def ⟨Ng PO E A⟩
        ⟨Opp PO E E' A OA⟩ assms(2) lep-neg--ltp
        opp-neg-pos ps-le)
  moreover have Cong PO OA A PO
    using Cong-cases Opp-def ⟨Opp PO E E' A OA⟩ l15-3 by blast
  ultimately show ?thesis
    by (simp add: Length-def Ps-Col ⟨PO ≠ E⟩ ltP-pos ltp-to-lep)
qed
show Length PO E E' B' PO OB
proof -
  have LtP PO E E' PO OB
    by (metis Ar2-def Out-def Ps-def ⟨Opp PO E E' B OB⟩ ⟨Ps PO E OB⟩
        diff-O-A diff-ar2 lep-neg--ltp ps-le)
  moreover have Cong PO OB B' PO
    by (metis Cong-cases Midpoint-def ⟨Cong PO B' PO B⟩ ⟨Opp PO E E' B OB⟩
        cong-inner-transitivity opp-midpoint)
  ultimately show ?thesis
    by (simp add: Length-def Ps-Col ⟨PO ≠ E⟩ ⟨Ps PO E OB⟩ ltp-to-lep)
qed
show Prod PO E E' OA OA A2
  using ⟨Opp PO E E' A OA⟩ assms(2) opp-same-square by auto
show Prod PO E E' OB OB B2
  using ⟨Opp PO E E' B OB⟩ assms(3) opp-same-square by blast
show Prod PO E E' OC OC C2
  using ⟨Opp PO E E' C OC⟩ assms(4) opp-same-square by auto
qed
}
qed
}
qed
}
qed
}
}
ultimately show ?thesis
  by blast
qed

lemma PythRel-uniqueness:
  assumes PythRel PO E E' A B C1
    and PythRel PO E E' A B C2
    and (Ps PO E C1 ∧ Ps PO E C2) ∨ C1 = PO

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shows  $C1 = C2$ 
proof -
  have  $\neg \text{Col } PO \ E \ E'$ 
    using Ar2-def PythRel-def assms(1) by presburger
  have  $\text{Col } PO \ E \ A$ 
    using Ar2-def PythRel-def assms(1) by presburger
  have  $\text{Col } PO \ E \ B$ 
    using Ar2-def PythRel-def assms(1) by presburger
  have  $\text{Col } PO \ E \ C1$ 
    using Ps-Col assms(3) col-trivial-3 by blast
  have  $\text{Col } PO \ E \ C2$ 
    using Ar2-def PythRel-def assms(2) by presburger
  have  $PO = B \wedge$ 
     $(A = C1 \vee \text{Opp } PO \ E \ E' \ A \ C1) \vee (\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C1 \ A \ B')$ 
    using PythRel-def assms(1) by auto
  moreover
  have  $PO = B \wedge$ 
     $(A = C2 \vee \text{Opp } PO \ E \ E' \ A \ C2) \vee (\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C2 \ A \ B')$ 
    using PythRel-def assms(2) by presburger
  moreover
  {
    assume  $PO = B$  and  $A = C1 \vee \text{Opp } PO \ E \ E' \ A \ C1$  and  $A = C2 \vee \text{Opp } PO \ E \ E' \ A \ C2$ 
    moreover have  $A = C1 \wedge A = C2 \implies C1 = C2$ 
      by auto
    moreover have  $A = C1 \wedge \text{Opp } PO \ E \ E' \ A \ C2 \implies C1 = C2$ 
      using  $\langle \neg \text{Col } PO \ E \ E' \rangle \text{ assms(3) neg-not-pos opp0-uniqueness pos-opp-neg}$  by blast
    moreover have  $\text{Opp } PO \ E \ E' \ A \ C1 \wedge A = C2 \implies C1 = C2$ 
      by (metis  $\langle \neg \text{Col } PO \ E \ E' \rangle \text{ assms(3) not-pos-and-neg opp0-uniqueness opp-comm opp-pos-neg}$ )
    moreover have  $\text{Opp } PO \ E \ E' \ A \ C1 \wedge \text{Opp } PO \ E \ E' \ A \ C2 \implies C1 = C2$ 
      using  $\langle \neg \text{Col } PO \ E \ E' \rangle \text{ opp-uniqueness}$  by blast
    ultimately have  $C1 = C2$ 
      by arg0
  }
  moreover {
    assume  $PO = B$  and  $A = C1 \vee \text{Opp } PO \ E \ E' \ A \ C1$  and
       $\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C2 \ A \ B'$ 
    moreover {
      assume  $A = C1$  and  $\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C2 \ A \ B'$ 
      then obtain  $B'$  where  $PO \ B' \text{ Perp } PO \ B$  and  $\text{Cong } PO \ B' \ PO \ B$  and  $\text{Cong } PO \ C2 \ A \ B'$ 
        by blast
      hence  $C1 = C2$ 
        using calculation(1) perp-distinct by blast
    }
    moreover {
      assume  $\text{Opp } PO \ E \ E' \ A \ C1$  and  $\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C2 \ A \ B'$ 
      then obtain  $B'$  where  $PO \ B' \text{ Perp } PO \ B$  and  $\text{Cong } PO \ B' \ PO \ B$  and  $\text{Cong } PO \ C2 \ A \ B'$ 
        by blast
      hence  $C1 = C2$ 
        using calculation(1) perp-distinct by blast
    }
  }
  ultimately have  $C1 = C2$ 
    by arg0
}
moreover {
  assume  $\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C1 \ A \ B'$  and  $PO = B$  and
     $A = C2 \vee \text{Opp } PO \ E \ E' \ A \ C2$ 
  then obtain  $B'$  where  $PO \ B' \text{ Perp } PO \ B$  and  $\text{Cong } PO \ B' \ PO \ B$  and  $\text{Cong } PO \ C1 \ A \ B'$ 
    by blast
  hence  $C1 = C2$ 
    using  $\langle PO = B \rangle \text{ perp-not-eq-2}$  by blast
}
moreover {
  assume  $\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C1 \ A \ B'$  and
     $\exists B'. PO \ B' \text{ Perp } PO \ B \wedge \text{Cong } PO \ B' \ PO \ B \wedge \text{Cong } PO \ C2 \ A \ B'$ 
  then obtain  $B1$  where  $PO \ B1 \text{ Perp } PO \ B$  and  $\text{Cong } PO \ B1 \ PO \ B$  and  $\text{Cong } PO \ C1 \ A \ B1$ 
    by blast
}

```

```

obtain  $B_2$  where  $PO\ B_2\ Perp\ PO\ B$  and  $Cong\ PO\ B_2\ PO\ B$  and  $Cong\ PO\ C_2\ A\ B_2$ 
  using  $\langle \exists B'.\ PO\ B'\ Perp\ PO\ B \wedge Cong\ PO\ B'\ PO\ B \wedge Cong\ PO\ C_2\ A\ B' \rangle$  by blast
have  $Cong\ PO\ B_1\ PO\ B_2$ 
  by (meson  $\langle Cong\ PO\ B_1\ PO\ B \rangle \langle Cong\ PO\ B_2\ PO\ B \rangle$  cong-4312 cong-inner-transitivity)
have  $Col\ PO\ B_1\ B_2$ 
  using  $\langle PO\ B_1\ Perp\ PO\ B \rangle \langle PO\ B_2\ Perp\ PO\ B \rangle$  perp2--col by auto
hence  $B_1 = B_2 \vee PO\ Midpoint\ B_1\ B_2$ 
  using  $\langle Cong\ PO\ B_1\ PO\ B_2 \rangle$  col-permutation-4 l7-20-bis by blast
moreover {
  assume  $B_1 = B_2$ 
  hence  $Cong\ PO\ C_2\ PO\ C_1$ 
    using Cong-cases  $\langle Cong\ PO\ C_1\ A\ B_1 \rangle \langle Cong\ PO\ C_2\ A\ B_2 \rangle$  cong-inner-transitivity by blast
  have  $C_1 = C_2 \vee PO\ Midpoint\ C_1\ C_2$ 
    by (metis Mid-perm  $\langle Col\ PO\ E\ C_1 \rangle \langle Col\ PO\ E\ C_2 \rangle \langle Cong\ PO\ C_2\ PO\ C_1 \rangle \langle \neg Col\ PO\ E\ E' \rangle$ 
      bet-col between-trivial col-transitivity-1 l7-20 not-col-permutation-4)
  moreover {
    assume  $PO\ Midpoint\ C_1\ C_2$ 
    hence  $Ps\ PO\ E\ C_1 \wedge Ps\ PO\ E\ C_2 \implies C_1 = C_2$ 
      by (meson Bet-cases Midpoint-def Ps-def bet-out--bet not-bet-and-out)
    moreover have  $C_1 = PO \implies C_1 = C_2$ 
      using  $\langle Cong\ PO\ C_2\ PO\ C_1 \rangle$  cong-diff by blast
    ultimately have  $C_1 = C_2$ 
      using assms(3) by blast
  }
  ultimately have  $C_1 = C_2$ 
    by arg0
}
moreover {
  assume  $PO\ Midpoint\ B_1\ B_2$ 
  have  $C_1 = C_2$ 
  proof (cases  $A = PO$ )
  {
    assume  $A = PO$ 
    hence  $Cong\ PO\ C_1\ PO\ C_2$ 
      by (metis Cong-cases  $\langle Cong\ PO\ B_1\ PO\ B_2 \rangle \langle Cong\ PO\ C_1\ A\ B_1 \rangle \langle Cong\ PO\ C_2\ A\ B_2 \rangle$ 
        cong-inner-transitivity)
    thus  $C_1 = C_2$ 
      by (metis Bet-cases Ps-def assms(3) bet-out--bet between-cong-3 between-trivial l6-6
        point-construction-different)
  }
  {
    assume  $A \neq PO$ 
    hence  $PO\ A\ Perp\ B_1\ PO$ 
      using perp-col [of  $PO\ A\ B\ B_1\ PO$ ] by (metis Perp-cases  $\langle Col\ PO\ E\ A \rangle \langle Col\ PO\ E\ B \rangle$ 
         $\langle PO\ B_1\ Perp\ PO\ B \rangle \langle \neg Col\ PO\ E\ E' \rangle$  col-transitivity-1
        grid-not-par-4)
    hence  $Per\ A\ PO\ B_1$ 
      using perp-per-1 by auto
    then obtain  $B_2'$  where  $PO\ Midpoint\ B_1\ B_2'$  and  $Cong\ A\ B_1\ A\ B_2'$ 
      using Per-def by blast
    hence  $B_2 = B_2'$ 
      using  $\langle PO\ Midpoint\ B_1\ B_2 \rangle$  symmetric-point-uniqueness by blast
    hence  $Cong\ PO\ C_1\ PO\ C_2$ 
      by (metis  $\langle Cong\ A\ B_1\ A\ B_2' \rangle \langle Cong\ PO\ C_1\ A\ B_1 \rangle \langle Cong\ PO\ C_2\ A\ B_2 \rangle$ 
        cong-transitivity not-cong-3412)
    have  $PO\ Midpoint\ C_1\ C_2 \implies C_1 = C_2$ 
      by (metis Ps-def  $\langle Cong\ PO\ C_1\ PO\ C_2 \rangle$  assms(3) bet-cong-eq bet-out--bet between-trivial
        l6-6 midpoint-bet not-bet-and-out)
    thus  $C_1 = C_2$ 
      by (metis Col-def  $\langle Col\ PO\ E\ C_1 \rangle \langle Col\ PO\ E\ C_2 \rangle \langle Cong\ PO\ C_1\ PO\ C_2 \rangle \langle \neg Col\ PO\ E\ E' \rangle$ 
        between-trivial col-transitivity-1 l7-20)
  }
}
qed
}
ultimately have  $C_1 = C_2$ 
  by arg0

```



```

}
ultimately show ?thesis
  by argo
qed

end
end

```

16 Tarski Euclidean 2D Continuous

```

theory Tarski-Euclidean-2D-Continuous

imports
  Tarski-Euclidean-2D
  Tarski-Neutral-Continuous
begin

locale Tarski-Euclidean-2D-Continuous = Tarski-Euclidean-2D + Tarski-Neutral-Continuous

end

```

```

theory Highschool-Neutral

```

```

imports
  Tarski-Neutral
begin

```

17 Highschool Neutral

```

context Tarski-neutral-dimensionless

```

```

begin

```

17.1 Definitions

17.2 Propositions

```

lemma bisector-existence:
  assumes  $A \neq B$  and  $B \neq C$ 
  shows  $\exists E. E \text{ InAngle } A \ B \ C \wedge A \ B \ E \text{ Cong } A \ E \ B \ C$ 
proof (cases Col A B C)
case True
thus ?thesis
proof (cases B Out A C)
case True
have  $C \text{ InAngle } A \ B \ C$ 
  using assms(1) assms(2) inangle3123 by presburger
moreover
have  $A \ B \ C \text{ Cong } A \ C \ B \ C$ 
  by (metis between-trivial2 True l11-21-b out2-bet-out)
ultimately
show ?thesis
  by blast
next
case False
hence  $Bet \ A \ B \ C$ 
  using True l6-4-2 by blast
have  $A \neq C$ 
  using False assms(1) out-trivial by blast
then obtain E where  $B \ E \text{ Perp } A \ C$ 
  using perp-exists by blast
have  $E \text{ InAngle } A \ B \ C$ 
  by (metis  $\langle B \ E \text{ Perp } A \ C \rangle \langle Bet \ A \ B \ C \rangle$  assms(1) assms(2) in-angle-line perp-not-eq-1)

```

```

moreover
have  $A B E \text{ Cong} A E B C$ 
  by (metis  $\text{True} \langle B E \text{ Perp } A C \rangle \text{ assms}(1) \text{ assms}(2) \text{ col-permutation-5}$ 
     $\text{l11-16 l8-2 not-col-permutation-4 per-col perp-col1 perp-not-eq-1 perp-per-1}$ )
ultimately
show ?thesis
  by blast
qed
next
case False
hence  $\neg \text{Col } A B C$ 
  by simp
obtain  $C'$  where  $\text{Bet } B A C'$  and  $\text{Cong } A C' B C$ 
  using segment-construction by blast
obtain  $A'$  where  $\text{Bet } B C A'$  and  $\text{Cong } C A' B A$ 
  using segment-construction by blast
obtain  $I$  where  $I \text{ Midpoint } A' C'$ 
  using midpoint-existence by blast
hence  $I \neq B$ 
  by (metis  $\text{False bet-col midpoint-bet} \langle \text{Bet } B A C' \rangle \langle \text{Bet } B C A' \rangle$ 
    between-inner-transitivity between-symmetry)
have  $\text{Cong } B C' A' B$ 
  by (meson  $\text{Bet-cases} \langle \text{Bet } B A C' \rangle \langle \text{Cong } A C' B C \rangle \langle \text{Bet } B C A' \rangle \langle \text{Cong } C A' B A \rangle$ 
     $\text{l2-11-b not-cong-2134 not-cong-3421}$ )
hence  $A' C' B \text{ Cong} A C' A' B$ 
  by (metis  $\text{False} \langle \text{Bet } B A C' \rangle \langle \text{Bet } B C A' \rangle \text{ assms}(1) \text{ assms}(2) \text{ bet-col1}$ 
     $\text{bet-neq12--neq cong-3421 cong-reflexivity l11-51 not-col-permutation-4 not-cong-2134}$ )
have  $C' I B \text{ Cong} A A' I B \wedge C' B I \text{ Cong} A A' B I$ 
proof –
  have  $\text{Cong } I B I B$ 
    by (simp add: cong-reflexivity)
  hence  $(I \neq B \longrightarrow (C' I B \text{ Cong} A A' I B \wedge C' B I \text{ Cong} A A' B I))$ 
    by (metis  $\text{Cong-cases conga-distinct l11-51 Tarski-neutral-dimensionless.midpoint-cong}$ 
       $\text{Tarski-neutral-dimensionless-axioms} \langle A' C' B \text{ Cong} A C' A' B \rangle$ 
       $\langle \text{Cong } B C' A' B \rangle \langle I \text{ Midpoint } A' C' \rangle$ )
  thus ?thesis
    using  $\langle I \neq B \rangle$  by blast
qed
have  $I \text{ InAngle } A B C$ 
proof –
  have  $A \neq B$ 
    by (simp add: assms(1))
  moreover
  have  $C \neq B$ 
    using assms(2) by blast
  moreover
  have  $I \neq B$ 
    by (simp add:  $\langle I \neq B \rangle$ )
  moreover
  have  $\exists X. \text{Bet } A X C \wedge (X = B \vee B \text{ Out } X I)$ 
proof –
  have  $\text{Bet } A' I C'$ 
    by (simp add:  $\langle I \text{ Midpoint } A' C' \rangle \text{ midpoint-bet}$ )
  hence  $\exists x1. (\text{Bet } I x1 B \wedge \text{Bet } A x1 A')$ 
    using  $\langle \text{Bet } B A C' \rangle$  inner-pasch by blast
  then obtain  $x1$  where  $\text{Bet } I x1 B$  and  $\text{Bet } A x1 A'$ 
    by blast
  then obtain  $x2$  where  $\text{Bet } x1 x2 B$  and  $\text{Bet } C x2 A$ 
    using  $\langle \text{Bet } B C A' \rangle$  inner-pasch by blast
  hence  $\text{Bet } A x2 C$ 
    using Bet-cases by blast
  moreover
  have  $x2 = B \vee B \text{ Out } x2 I$ 
    by (meson  $\langle \text{Bet } I x1 B \rangle \langle \text{Bet } x1 x2 B \rangle \text{ bet-out-1 between-exchange2}$ )
  ultimately
show ?thesis

```

```

    by blast
qed
ultimately
show ?thesis
  by (simp add: InAngle-def)
qed
moreover
have A B I CongA I B C
proof -
  have C' B I CongA A' B I
    using ⟨C' I B CongA A' I B ∧ C' B I CongA A' B I⟩ by blast
  thus ?thesis
    using assms(1) assms(2) bet-out conga-right-comm l11-10 out-trivial
    using ⟨Bet B A C'⟩ ⟨Bet B C A'⟩ ⟨I ≠ B⟩ by presburger
qed
ultimately
show ?thesis
  by blast
qed

lemma not-col-bfoot-not-equality:
  assumes ¬ Col A B C and
    Coplanar A B C I and
    Col A B H1 and

    A B I CongA I B C and
    A B Perp I H1 and
    B C Perp I H2
  shows H1 ≠ B ∧ H2 ≠ B
proof -
{
  assume H1 = B
  hence Per A B I
    using assms(5) perp-left-comm perp-per-1 by blast
  have Col A C B
  proof -
    have Coplanar I A C B
      using assms(2) ncoplanar-perm-11 by blast
    moreover
    have I ≠ B
      using ⟨H1 = B⟩ assms(5) perp-distinct by blast
    moreover
    have Per I B C
      using ⟨Per A B I⟩ assms(4) l11-17 by blast
    hence Per C B I
      by (simp add: l8-2)
    ultimately
    show ?thesis
      using ⟨Per A B I⟩ cop-per2--col by blast
  qed
  hence False
    using assms(1) col-permutation-5 by blast
}
moreover
{
  assume H2 = B
  hence Per C B I
    using assms(6) perp-per-1 by blast
  hence Per I B C
    using l8-2 by blast
  have Col A C B
  proof -
    have Coplanar I A C B
      by (simp add: assms(2) coplanar-perm-19)
    moreover
    have I ≠ B

```

```

    using ⟨H2 = B⟩ assms(6) perp-distinct by presburger
  moreover
  have Per A B I
    by (meson ⟨Per I B C⟩ assms(4) l11-17 not-conga-sym)
  ultimately
  show ?thesis
    using ⟨Per C B I⟩ cop-per2--col by blast
qed
hence False
  using assms(1) col-permutation-5 by blast
}
ultimately show ?thesis by blast
qed

lemma bisector-foot-out-out:
  assumes ¬ Col A B C and
    Coplanar A B C I and
    Col B C H2 and
    A B I CongA I B C and
    A B Perp I H1 and
    B C Perp I H2 and
    B Out H1 A
  shows B Out H2 C
proof -
  have Col A B H1
    using assms(7) col-permutation-2 out-col by blast
  hence H1 ≠ B ∧ H2 ≠ B
    using not-col-bfoot-not-equality assms(1) assms(2) assms(4) assms(5) assms(6) by blast
  hence Acute H1 B I
    by (metis l8-16-1 ⟨Col A B H1⟩ assms(5) col-trivial-2 l11-43)
  have H1 B I CongA I B C
    by (metis CongA-def Out-def assms(4) assms(7) not-bet-distincts not-conga out2--conga)
  hence Acute I B C
    using ⟨Acute H1 B I⟩ acute-conga--acute by blast
  thus ?thesis
    by (meson acute-col-perp--out assms(3) assms(6))
qed

lemma not-col-efoot-not-equality:
  assumes ¬ Col A B C and
    Coplanar A B C I and
    Col A B H1 and
    Col B C H2 and
    Cong I H1 I H2 and
    A B Perp I H1 and
    B C Perp I H2
  shows H1 ≠ B ∧ H2 ≠ B
proof -
  {
    assume H1 = B
    hence H2 ≠ B
      using assms(1) assms(2) assms(6) assms(7) col-permutation-2 cop-perp2--col
      coplanar-perm-11 perp-comm perp-right-comm by blast
    hence False
      by (metis Cong-cases Perp-cases ⟨H1 = B⟩ acute-not-per assms(4) assms(5)
      assms(7) cong--acute perp-col1 perp-distinct perp-per-1)
  }
  moreover
  {
    assume H2 = B
    have B H1 Perp I H1
      using ⟨H1 = B ⇒ False⟩ assms(3) assms(6) col-trivial-2 perp-col2 by presburger
    hence Per B H1 I
      using Perp-cases perp-per-1 by blast
    have B H1 I CongA H1 B I
      by (metis Cong-cases ⟨H1 = B ⇒ False⟩ ⟨H2 = B⟩ assms(5)

```

```

      cong-identity-inv cong-reflexivity l11-51)
hence Per H1 B I
  using ⟨Per B H1 I⟩ l11-17 by blast
hence False
  using ⟨H1 = B ⟹ False⟩ ⟨Per B H1 I⟩ l8-2 l8-7 by blast
}
ultimately show ?thesis
  by blast
qed

lemma equality-foot-out-out:
  assumes ¬ Col A B C and
    I InAngle A B C and
    Col B C H2 and
    Cong I H1 I H2 and
    A B Perp I H1 and
    B C Perp I H2 and
    B Out H1 A
  shows B Out H2 C
proof -
  have H1 ≠ B ∧ H2 ≠ B
    by (metis Col-cases cong-acute acute-not-per assms(4) assms(5) assms(7)
        col-trivial-2 l8-16-1 out-col out-diff1)
  have Cong H1 B H2 B ∧ H1 B I CongA H2 B I ∧ H1 I B CongA H2 I B
  proof -
    have B H1 I CongA B H2 I
      by (metis Col-cases l11-16 ⟨H1 ≠ B ∧ H2 ≠ B⟩ assms(3) assms(5)
          assms(6) assms(7) col-trivial-2 conga-comm l8-16-1 out-col)
    moreover
    have Per B H1 I
      by (metis Col-cases l8-16-1 l8-2 assms(5) assms(7) col-trivial-2 out-col)
    hence H1 I Le B I
      using l11-46 by (metis perp-distinct ⟨H1 ≠ B ∧ H2 ≠ B⟩ assms(5) lt-le)
    ultimately
    show ?thesis
      using assms(4) cong-reflexivity l11-52 not-cong-2143 by blast
  qed
  have I B TS A C
    by (metis col-permutation-2 col-permutation-5 out-col perp-not-col2
        assms(2) assms(3) assms(5) assms(6) assms(7) in-angle-two-sides)
  have B Out H1 H2 ∨ I B TS H1 H2
proof -
  have Coplanar I B H1 H2
  proof -
    have ¬ Col A I B
      using TS-def ⟨I B TS A C⟩ by blast
    moreover
    have Coplanar A I B H1
      using assms(5) coplanar-perm-2 perp--coplanar by blast
    moreover
    have Coplanar A I B H2
      by (metis not-col-distincts assms(1) assms(2) assms(3) col-cop--cop
          coplanar-perm-6 inangle--coplanar)
    ultimately
    show ?thesis
      using coplanar-trans-1 by blast
  qed
  moreover
  have I B H1 CongA I B H2
    by (simp add: ⟨Cong H1 B H2 B ∧ H1 B I CongA H2 B I ∧ H1 I B CongA H2 I B⟩ conga-comm)
  ultimately
  show ?thesis
    using conga-cop--or-out-ts by blast
qed
moreover
{

```

```

assume  $B \text{ Out } H1 \ H2$ 
hence  $False$ 
  by (metis Col-cases  $\langle B \text{ Out } H1 \ H2 \rangle$  assms(1) assms(3) assms(7)
    col-trivial-3 l9-19 out-diff2 out-one-side)
}
moreover
{
  assume  $I \ B \ TS \ H1 \ H2$ 
  have  $I \ B \ TS \ A \ H2$ 
    by (meson not-col-distincts  $\langle I \ B \ TS \ H1 \ H2 \rangle$  assms(7) l9-5)
  hence  $I \ B \ OS \ H2 \ C$ 
    using OS-def  $\langle I \ B \ TS \ A \ C \rangle$  l9-2 by blast
  hence  $B \ I \ OS \ H2 \ C$ 
    using invert-one-side by blast
  moreover
  have  $Col \ B \ H2 \ C$ 
    using Col-cases assms(3) by blast
  ultimately
  have  $B \text{ Out } H2 \ C$ 
    using col-one-side-out by force
}
ultimately show ?thesis by blast
qed

```

```

lemma bisector-perp-equality:
assumes Coplanar  $A \ B \ C \ I$  and
   $Col \ B \ H1 \ A$  and
   $Col \ B \ C \ H2$  and
   $A \ B \ Perp \ I \ H1$  and
   $B \ C \ Perp \ I \ H2$  and
   $A \ B \ I \ CongA \ I \ B \ C$ 
shows  $Cong \ I \ H1 \ I \ H2$ 
proof (cases  $Col \ A \ B \ C$ )
  case  $True$ 
    hence  $A \ B \ Perp \ I \ H2$ 
      by (metis assms(4) assms(5) col-trivial-3 not-col-permutation-2 perp-col2 perp-not-eq-1)
    have  $H1 = H2$ 
      by (metis col-trivial-2  $True \ \langle A \ B \ Perp \ I \ H2 \rangle$  assms(2) assms(3)
        assms(4) assms(5) colx l8-18-uniqueness not-col-permutation-1 perp-not-eq-1)
    thus ?thesis
      by (simp add: cong-reflexivity)
  next
    case  $False$ 
    hence  $\neg \ Col \ A \ B \ C$ 
      by blast
    have  $Col \ A \ B \ H1$ 
      using Col-cases assms(2) by blast
    have  $Col \ B \ C \ H2$ 
      by (simp add: assms(3))
    hence  $H1 \neq B \wedge H2 \neq B$ 
      using  $False \ \langle Col \ A \ B \ H1 \rangle$  assms(1) assms(4) assms(5) assms(6)
        not-col-bfoot-not-equality by blast
    show ?thesis
    proof (cases  $B \text{ Out } H1 \ A$ )
      case  $True$ 
        hence  $B \text{ Out } H2 \ C$ 
          using  $False$  assms(1) assms(3) assms(4) assms(5) assms(6)
            bisector-foot-out-out by blast
        have  $\neg \ Col \ I \ B \ H1$ 
          by (metis  $\langle Col \ A \ B \ H1 \rangle \ \langle H1 \neq B \wedge H2 \neq B \rangle$  assms(4) l6-16-1 perp-not-col2)
        moreover
        have  $B \ H1 \ I \ CongA \ B \ H2 \ I$ 
        proof –
          have  $Per \ B \ H1 \ I$ 

```

```

    using ⟨Col A B H1⟩ assms(4) col-trivial-2 l8-16-1 l8-2 by blast
  moreover
  have Per B H2 I
    by (meson l8-2 assms(3) assms(5) col-trivial-3 l8-16-1)
  ultimately
  show ?thesis
    by (metis Col-cases Perp-cases ⟨H1 ≠ B ∧ H2 ≠ B⟩ assms(4) assms(5)
        col-trivial-2 l11-16 l8-16-1)
qed
moreover
have H1 B I CongA H2 B I
proof -
  have B Out H1 A
    by (simp add: True)
  moreover
  have B Out H2 C
    using ⟨B Out H2 C⟩ by auto
  moreover
  have A B I CongA I B C
    by (simp add: assms(6))
  have B Out I I
    by (metis ⟨¬ Col I B H1⟩ bet-out-1 not-bet-distincts not-col-distincts)
  ultimately
  show ?thesis
    by (meson conga-left-comm l11-10 assms(6) conga-comm)
qed
hence I B H1 CongA I B H2
  using conga-comm by blast
moreover
have Cong I B I B
  by (simp add: cong-reflexivity)
ultimately
show ?thesis
  using l11-50-2 by blast
next
case False
obtain A' where B Midpoint A A'
  using symmetric-point-construction by presburger
hence Bet A B A'
  by (simp add: midpoint-bet)
hence B Out H1 A'
  by (metis False Out-def midpoint-distinct-1
      Tarski-neutral-dimensionless.perp-distinct
      Tarski-neutral-dimensionless-axioms ⟨B Midpoint A A'⟩
      ⟨Col A B H1⟩ ⟨H1 ≠ B ∧ H2 ≠ B⟩ assms(4) bet-out-1 fourth-point)
obtain C' where B Midpoint C C'
  using symmetric-point-construction by presburger
hence Bet C B C'
  using midpoint-bet by blast
have B Out H2 C'
proof -
  {
    assume Col A' B C'
    hence Col A B C
      by (metis Out-def midpoint-distinct-1 ⟨B Midpoint C C'⟩ ⟨B Out H1 A'⟩
          ⟨Bet A B A'⟩ ⟨Bet C B C'⟩ bet-col col2--eq col-trivial-2)
    hence False
      using ⟨¬ Col A B C⟩ by auto
  }
  hence ¬ Col A' B C'
    by blast
moreover
have Coplanar A' B C' I
proof -
  have Coplanar A B C A'
    by (simp add: ⟨Bet A B A'⟩ bet--coplanar coplanar-perm-1)

```

```

moreover
have Coplanar A B C B
  using ncop-distincts by blast
moreover
have Coplanar A B C C'
  by (metis Bet-cases Col-def  $\langle \text{Bet } C \ B \ C' \rangle$  ncop--ncols)
ultimately
show ?thesis
  by (meson  $\langle \neg \text{Col } A \ B \ C \rangle$  assms(1) coplanar-pseudo-trans)
qed
moreover
have Col B C' H2
  by (metis Col-def  $\langle \text{Bet } C \ B \ C' \rangle$   $\langle \neg \text{Col } A \ B \ C \rangle$  assms(3) col-transitivity-1 col-trivial-2)
moreover
have A' B I CongA I B C'
  by (metis conga-right-comm  $\langle \text{Bet } A \ B \ A' \rangle$   $\langle \text{Bet } C \ B \ C' \rangle$  assms(6)
    calculation(1) l11-13 not-col-distincts)
moreover
have A' B Perp I H1
  by (metis  $\langle \text{Bet } A \ B \ A' \rangle$  assms(4) bet-col calculation(1) not-col-distincts perp-col2)
moreover
have B C' Perp I H2
  by (metis Col-cases Col-def  $\langle \text{Bet } C \ B \ C' \rangle$  assms(5) calculation(1)
    not-col-distincts perp-col)
ultimately
show ?thesis
  using  $\langle B \text{ Out } H1 \ A' \rangle$  bisector-foot-out-out by blast
qed
have  $\neg \text{Col } I \ B \ H1$ 
  by (metis perp-not-col2  $\langle \text{Col } A \ B \ H1 \rangle$   $\langle H1 \neq B \wedge H2 \neq B \rangle$  assms(4) l6-16-1)
moreover
have B H1 I CongA B H2 I
  by (metis  $\langle \text{Col } A \ B \ H1 \rangle$   $\langle H1 \neq B \wedge H2 \neq B \rangle$  assms(3) assms(4) assms(5)
    col-trivial-2 col-trivial-3 conga-comm l11-16 l8-16-1)
moreover
have H1 B I CongA I B H2
proof –
  have A' B I CongA I B C'
    by (metis l11-13 not-col-distincts  $\langle B \text{ Midpoint } A \ A' \rangle$ 
       $\langle B \text{ Midpoint } C \ C' \rangle$   $\langle \text{Bet } A \ B \ A' \rangle$   $\langle \text{Bet } C \ B \ C' \rangle$   $\langle \neg \text{Col } A \ B \ C \rangle$ 
      assms(6) conga-right-comm midpoint-distinct-2)
  moreover
have B Out H1 A'
    by (simp add:  $\langle B \text{ Out } H1 \ A' \rangle$ )
  moreover
have  $I \neq B$ 
    using assms(6) conga-diff45 by auto
  hence B Out I I
    using out-trivial by blast
  ultimately show ?thesis
    using  $\langle B \text{ Out } H2 \ C' \rangle$  l11-10 by blast
qed
hence I B H1 CongA I B H2
  using conga-left-comm by presburger
moreover
have Cong I B I B
  by (simp add: cong-reflexivity)
ultimately
show ?thesis
  using l11-50-2 by blast
qed
qed

```

lemma *perp-equality-bisector*:


```

assumes  $\neg \text{Col } A \ B \ C$  and
   $I \text{ InAngle } A \ B \ C$  and
   $\text{Col } B \ H1 \ A$  and
   $\text{Col } B \ H2 \ C$  and
   $A \ B \ \text{Perp } I \ H1$  and
   $B \ C \ \text{Perp } I \ H2$  and
   $\text{Cong } I \ H1 \ I \ H2$ 
shows  $A \ B \ I \ \text{CongA } I \ B \ C$ 
proof –
  have  $H1 \neq B \wedge H2 \neq B$ 
proof –
    have  $\text{Coplanar } A \ B \ C \ I$ 
      using assms(2) inangle--coplanar ncoplanar-perm-18 by blast
    moreover
    have  $\text{Col } A \ B \ H1$ 
      by (simp add: assms(3) col-permutation-2)
    ultimately
    show ?thesis
      using not-col-efoot-not-equality assms(1) assms(4) assms(5)
        assms(6) assms(7) col-permutation-5 by blast
qed
have  $\text{Cong } H1 \ B \ H2 \ B \wedge H1 \ B \ I \ \text{CongA } H2 \ B \ I \wedge H1 \ I \ B \ \text{CongA } H2 \ I \ B$ 
proof –
  have  $B \ H1 \ I \ \text{CongA } B \ H2 \ I$ 
    by (metis NCol-perm l11-16  $\langle H1 \neq B \wedge H2 \neq B \rangle$  assms(3) assms(4)
      assms(5) assms(6) col-trivial-3 conga-comm l8-16-1)
  moreover
  have  $\text{Cong } B \ I \ B \ I$ 
    by (simp add: cong-reflexivity)
  moreover
  have  $\text{Cong } H1 \ I \ H2 \ I$ 
    using Cong-cases assms(7) by auto
  moreover
  have  $H1 \ I \ \text{Le } B \ I$ 
    by (metis l11-46 assms(4) assms(6) calculation(1) col-trivial-3
      conga-distinct conga-sym l11-17 l8-16-1 l8-2 lt--le not-col-permutation-5)
  ultimately
  show ?thesis
    using l11-52 by blast
qed
show ?thesis
proof (cases B Out A H1)
  case True
  have  $B \ \text{Out } H2 \ C$ 
  proof –
    have  $\text{Col } B \ C \ H2$ 
      using assms(4) col-permutation-5 by blast
    moreover
    have  $B \ \text{Out } H1 \ A$ 
      using Out-cases True by auto
    ultimately
    show ?thesis
      using assms(1) assms(2) assms(5) assms(6) assms(7)
        equality-foot-out-out by blast
  qed
have  $A \ B \ I \ \text{CongA } C \ B \ I$ 
proof –
  have  $H1 \ B \ I \ \text{CongA } H2 \ B \ I$ 
    by (simp add:  $\langle \text{Cong } H1 \ B \ H2 \ B \wedge H1 \ B \ I \ \text{CongA } H2 \ B \ I \wedge H1 \ I \ B \ \text{CongA } H2 \ I \ B \rangle$ )
  moreover
  have  $H1 \ I \ B \ \text{CongA } H2 \ I \ B$ 
    by (simp add:  $\langle \text{Cong } H1 \ B \ H2 \ B \wedge H1 \ B \ I \ \text{CongA } H2 \ B \ I \wedge H1 \ I \ B \ \text{CongA } H2 \ I \ B \rangle$ )
  moreover
  have  $\text{Cong } H1 \ B \ H2 \ B$ 
    by (simp add:  $\langle \text{Cong } H1 \ B \ H2 \ B \wedge H1 \ B \ I \ \text{CongA } H2 \ B \ I \wedge H1 \ I \ B \ \text{CongA } H2 \ I \ B \rangle$ )
  moreover

```

```

have B ≠ I
  using assms(2) inangle-distincts by blast
hence B Out I I
  using out-trivial by auto
moreover
have B Out A H1
  by (simp add: True)
moreover
have B Out C H2
  by (simp add: ⟨B Out H2 C⟩ l6-6)
moreover
have I ≠ H1
  using assms(5) perp-not-eq-2 by auto
moreover
have I ≠ H2
  using assms(6) perp-not-eq-2 by auto
ultimately show ?thesis
  using l11-10 bet-col not-col-distincts assms(1) assms(2) col-inangle2--out
  by blast
qed
thus ?thesis
  using conga-right-comm by blast
next
case False
hence ¬ B Out A H1
  by blast
{
  assume B Out C H2
  have ¬ Col C B A
    by (simp add: assms(1) not-col-permutation-3)
  moreover
  have I InAngle C B A
    by (simp add: assms(2) l11-24)
  moreover
  have Col B A H1
    using Col-cases assms(3) by blast
  moreover
  have Cong I H2 I H1
    using assms(7) not-cong-3412 by blast
  moreover
  have C B Perp I H2
    using Perp-perm assms(6) by blast
  moreover
  have B A Perp I H1
    using assms(5) perp-left-comm by blast
  ultimately
  have B Out H1 A
    using ⟨B Out C H2⟩ equality-foot-out-out l6-6 by blast
  hence False
    using False Out-cases by blast
}
hence Bet C B H2
  using Col-cases assms(4) or-bet-out by blast
moreover
have Bet A B H1
  using False NCol-cases assms(3) or-bet-out by blast
ultimately
have A B I CongA C B I
  by (metis ⟨Cong H1 B H2 B ∧ H1 B I CongA H2 B I ∧ H1 I B CongA H2 I B⟩
    assms(2) between-symmetry inangle-distincts l11-13)
thus ?thesis
  by (meson conga-right-comm)
qed
qed

```

```

lemma col-permut132:
  assumes Col A B C
  shows Col A C B
  using Col-cases assms by blast

lemma col-permut213:
  assumes Col A B C
  shows Col B A C
  using Col-perm assms by blast

lemma col-permut231:
  assumes Col A B C
  shows Col B C A
  using Col-perm assms by blast

lemma col-permut312:
  assumes Col A B C
  shows Col C A B
  using Col-perm assms by blast

lemma col-permut321:
  assumes Col A B C
  shows Col C B A
  using Col-perm assms by blast

lemma col123-124--col134:
  assumes P  $\neq$  Q and
    Col P Q A and
    Col P Q B
  shows Col P A B
  using assms(1) assms(2) assms(3) col-transitivity-1 by blast

lemma col123-124--col234:
  assumes P  $\neq$  Q and
    Col P Q A and
    Col P Q B
  shows Col Q A B
  using assms(1) assms(2) assms(3) col-transitivity-2 by blast

lemma triangle-mid-par-strict:
  assumes  $\neg$  Col A B C and
    P Midpoint B C and
    Q Midpoint A C
  shows A B ParStrict Q P
  using assms(1) assms(2) assms(3) triangle-mid-par by blast

end
end

```

theory *Highschool-Euclidean*

imports

Tarski-Neutral-Continuity
Highschool-Neutral
Tarski-Euclidean

begin

18 Highschool Euclidean dimensionless

context *Tarski-Euclidean*

begin

```

lemma triangle-mid-par-strict-cong-aux:
  assumes  $\neg$  Col A B C and
    P Midpoint B C and
    Q Midpoint A C and
    R Midpoint A B
  shows A B ParStrict Q P  $\wedge$  Cong A R P Q  $\wedge$  Cong B R P Q
proof -
  have Col B P C
    by (simp add: assms(2) bet-col midpoint-bet)
  have Col A Q C
    by (simp add: assms(3) bet-col midpoint-bet)
  have Col A R B
    using Midpoint-def assms(4) bet-col by blast
  have Cong B P C P
    using Midpoint-def assms(2) not-cong-1243 by blast
  have Cong A Q C Q
    using Midpoint-def assms(3) not-cong-1243 by blast
  have Cong A R B R
    using Midpoint-def assms(4) not-cong-1243 by blast
  obtain x where Q Midpoint P x
    using symmetric-point-construction by blast
  hence Cong P Q x Q
    using Cong-perm midpoint-cong by blast
  have Col P Q x
    by (meson  $\langle$ Q Midpoint P x $\rangle$  bet-col midpoint-bet)
  {
    assume Col A P C
    hence Col A B C
      by (metis  $\langle$ Col B P C $\rangle$  assms(2) col2--eq
        col-permutation-5 midpoint-distinct-2)
    hence False
      using assms(1) by blast
  }
  hence  $\neg$  Col A P C
    by blast
  hence A  $\neq$  P
    using col-trivial-1 by force
  have ParallelogramStrict A P C x
    using  $\langle$ Q Midpoint P x $\rangle$   $\langle$  $\neg$  Col A P C $\rangle$  assms(3) mid-plgs by force
  have Cong A P C x
    by (metis l7-13  $\langle$ Q Midpoint P x $\rangle$  assms(3) cong-4321)
  have Cong A x P C
    by (metis l7-13  $\langle$ Q Midpoint P x $\rangle$  assms(3) l7-2 not-cong-2134)
  have Cong A x B P
    by (meson  $\langle$ Cong A x P C $\rangle$   $\langle$ Cong B P C P $\rangle$  cong-4312 cong-inner-transitivity)
  have A x Par B P
    by (metis (full-types) Par-cases  $\langle$ A  $\neq$  P $\rangle$   $\langle$ Col B P C $\rangle$   $\langle$ Cong A P C x $\rangle$ 
       $\langle$ ParallelogramStrict A P C x $\rangle$   $\langle$ Q Midpoint P x $\rangle$  assms(2) assms(3)
      col-par l12-17 l12-20 midpoint-distinct-1 par-not-par plgs-two-sides)
  have Parallelogram A x B P  $\vee$  Parallelogram A x P B
    using par-cong-plg-2 by (simp add:  $\langle$ A x Par B P $\rangle$   $\langle$ Cong A x B P $\rangle$ )
  hence Cong A R P Q
    by (metis Cong-cases  $\langle$ Col A Q C $\rangle$   $\langle$ Col A R B $\rangle$   $\langle$ Q Midpoint P x $\rangle$ 
      assms(1) assms(4) col-transitivity-1 cong-cong-half-1 l7-17-bis
      midpoint-distinct-3 plg-cong-2 plg-mid-2)
  moreover
  have Cong B R P Q
    using  $\langle$ Cong A R B R $\rangle$  calculation cong-inner-transitivity by blast
  ultimately
  show ?thesis
    using assms(1) assms(2) assms(3) triangle-mid-par-strict by blast
qed

```

```

lemma triangle-par-mid:
  assumes  $\neg$  Col A B C and
    P Midpoint B C and

```

```

    A B Par Q P and
    Col Q A C
shows Q Midpoint A C
proof -
have playfair-s-postulate  $\longrightarrow$  midpoint-converse-postulate
using playfair-s-postulate-implies-midpoint-converse-postulate by blast
thus ?thesis
using assms(1) assms(2) assms(3) assms(4) midpoint-converse-postulate-def
not-col-permutation-2 parallel-uniqueness playfair-s-postulate-def by blast
qed

lemma triangle-mid-par-strict-cong-1:
assumes  $\neg$  Col A B C and
P Midpoint B C and
Q Midpoint A C and
R Midpoint A B
shows A B ParStrict Q P  $\wedge$  Cong A R P Q
using assms(1) assms(2) assms(3) assms(4) triangle-mid-par-strict-cong-aux by blast

lemma triangle-mid-par-strict-cong-2:
assumes  $\neg$  Col A B C and
P Midpoint B C and
Q Midpoint A C and
R Midpoint A B
shows A B ParStrict Q P  $\wedge$  Cong B R P Q
using assms(1) assms(2) assms(3) assms(4) triangle-mid-par-strict-cong-aux by blast

lemma triangle-mid-par-strict-cong-simp:
assumes  $\neg$  Col A B C and
P Midpoint B C and
Q Midpoint A C
shows A B ParStrict Q P
using assms(1) assms(2) assms(3) triangle-mid-par-strict by blast

lemma triangle-mid-par-strict-cong:
assumes  $\neg$  Col A B C and
P Midpoint B C and
Q Midpoint A C and
R Midpoint A B
shows A B ParStrict Q P  $\wedge$  A C ParStrict P R  $\wedge$ 
B C ParStrict Q R  $\wedge$  Cong A R P Q  $\wedge$ 
Cong B R P Q  $\wedge$  Cong A Q P R  $\wedge$ 
Cong C Q P R  $\wedge$  Cong B P Q R  $\wedge$  Cong C P Q R
by (meson assms(1) assms(2) assms(3) assms(4) l7-2 not-col-permutation-1
not-col-permutation-5 par-strict-right-comm triangle-mid-par-strict-cong-aux)

lemma triangle-mid-par-flat-cong-aux:
assumes A  $\neq$  B and
Col A B C and
P Midpoint B C and
Q Midpoint A C and
R Midpoint A B
shows A B Par Q P  $\wedge$  Cong A R P Q  $\wedge$  Cong B R P Q
proof (cases A = C)
case True
thus ?thesis
by (metis Cong-cases assms(1) assms(3) assms(4) assms(5)
cong-cong-half-1 l7-17-bis l8-20-2 mid-par-cong2
midpoint-col not-col-permutation-2 not-par-not-col)
next
case False
show ?thesis
proof (cases B = C)
case True
thus ?thesis
by (metis assms(1) assms(2) assms(3) assms(4) assms(5) col-transitivity-1

```

```

    col-trivial-3 cong-inner-transitivity cong-right-commutativity
    l7-17 l7-2 midpoint-col midpoint-cong midpoint-distinct-2
    not-par-inter-uniqueness)
next
case False
hence  $B \neq C$ 
  by blast
show ?thesis
proof (cases  $A = P$ )
  case True
  hence  $Col\ A\ B\ Q$ 
    by (metis assms(2) assms(4) col2--eq midpoint-col
        midpoint-distinct-2 not-col-permutation-4)
  have  $A\ B\ Par\ Q\ A$ 
    by (metis True  $\langle Col\ A\ B\ Q \rangle$  assms(1) assms(3) assms(4) midpoint-distinct-1
        not-par-not-col par-right-comm)
  moreover have  $Cong\ A\ R\ A\ Q$ 
    by (metis True assms(1) assms(3) assms(4) assms(5)
        cong-cong-half-1 l7-3-2 mid-par-cong1)
  moreover have  $Cong\ B\ R\ A\ Q$ 
    by (metis True assms(3) assms(4) assms(5) cong-cong-half-1 l7-2 midpoint-cong)
  ultimately show ?thesis
    using True by force
next
case False
obtain  $x$  where  $Q\ Midpoint\ P\ x$ 
  using symmetric-point-construction by blast
show ?thesis
proof (cases  $P = x$ )
  case True
  thus ?thesis
    by (metis  $\langle Q\ Midpoint\ P\ x \rangle$  assms(1) assms(3) assms(4) l8-20-2 sym-preserve-diff)
next
case False
hence  $Parallelogram\ A\ P\ C\ x$ 
  using  $\langle Q\ Midpoint\ P\ x \rangle$  assms(4) mid-plg by blast
hence  $Cong\ A\ x\ B\ P$ 
  by (meson assms(3) cong-symmetry cong-transitivity midpoint-cong plg-cong-2)
have  $P\ Q\ Par\ A\ B$ 
proof -
  have  $P \neq Q$ 
    using False  $\langle Q\ Midpoint\ P\ x \rangle$  is-midpoint-id by fastforce
  moreover have  $Col\ P\ x\ Q$ 
    using  $\langle Q\ Midpoint\ P\ x \rangle$  col-permutation-1 midpoint-col by blast
  have  $Parallelogram\ A\ x\ B\ P \vee Parallelogram\ A\ x\ P\ B$ 
  proof -
    have  $Cong\ A\ x\ B\ P$ 
      using  $\langle Cong\ A\ x\ B\ P \rangle$  by auto
    moreover
    have  $A\ x\ Par\ B\ P$ 
  proof -
    {
      assume  $B = P$ 
      hence  $B = C$ 
        using Midpoint-def assms(3) cong-diff-3 by blast
      hence False using  $\langle B \neq C \rangle$  by blast
    }
    moreover have  $A\ x\ Par\ P\ C$ 
      by (metis  $\langle B \neq C \rangle$   $\langle Q\ Midpoint\ P\ x \rangle$  assms(3)
          assms(4) mid-par-cong mid-par-cong2
          midpoint-cong-uniqueness midpoint-distinct-1
          not-col-distincts par-right-comm)
    moreover have  $Col\ P\ C\ B$ 
      using assms(3) col-permutation-5 midpoint-col by blast
    moreover have  $Col\ P\ C\ P$ 
      by (simp add: col-trivial-3)

```

```

ultimately show ?thesis
  using par-col2-par by blast
qed
ultimately show ?thesis
  using par-cong-plg-2 by blast
qed
moreover
have  $P \ x \ \text{Par} \ A \ B$ 
proof -
  have  $\text{Parallelogram} \ A \ x \ B \ P \ \vee \ \text{Parallelogram} \ A \ x \ P \ B$ 
    by (simp add: calculation(2))
  moreover
  have  $\text{Parallelogram} \ A \ x \ B \ P \longrightarrow P \ x \ \text{Par} \ A \ B$ 
    by (metis Par-cases  $\langle Q \ \text{Midpoint} \ P \ x \rangle$  assms(1) assms(3) assms(4) assms(5)
        l12-17 l7-17-bis l8-20-2 midpoint-distinct-1 plg-mid-2)
  moreover
  have  $\text{Parallelogram} \ A \ x \ P \ B \longrightarrow P \ x \ \text{Par} \ A \ B$ 
    by (metis False Plg-perm  $\langle \text{Parallelogram} \ A \ x \ B \ P \longrightarrow P \ x \ \text{Par} \ A \ B \rangle$  plg-par)
  ultimately show ?thesis
    by blast
qed
ultimately show ?thesis
  using  $\langle \text{Col} \ P \ x \ Q \rangle$  par-col-par-2 by blast
qed
hence  $A \ B \ \text{Par} \ Q \ P$ 
  using Par-cases by auto
moreover
have  $\text{Cong} \ P \ Q \ A \ R$ 
proof -
  have  $\text{Parallelogram} \ A \ x \ B \ P \longrightarrow \text{Cong} \ P \ Q \ A \ R$ 
    by (metis Plg-perm  $\langle B \neq C \rangle$   $\langle \text{Parallelogram} \ A \ P \ C \ x \rangle$  plg-uniqueness)
  moreover
  have  $\text{Parallelogram} \ A \ x \ P \ B \longrightarrow \text{Cong} \ P \ Q \ A \ R$ 
    by (meson  $\langle Q \ \text{Midpoint} \ P \ x \rangle$  assms(5) cong-cong-half-1 not-cong-3421 plg-cong-2)
  moreover
  have  $\text{Parallelogram} \ A \ x \ B \ P \ \vee \ \text{Parallelogram} \ A \ x \ P \ B$ 
  proof -
    have  $\text{Cong} \ A \ x \ B \ P$ 
      using  $\langle \text{Cong} \ A \ x \ B \ P \rangle$  by auto
    moreover
    have  $A \ x \ \text{Par} \ B \ P$ 
    proof -
      {
        assume  $B = P$ 
        hence  $B = C$ 
          using Midpoint-def assms(3) cong-diff-3 by blast
        hence False using  $\langle B \neq C \rangle$  by blast
      }
    moreover have  $A \ x \ \text{Par} \ P \ C$ 
      by (metis Par-cases  $\langle Q \ \text{Midpoint} \ P \ x \rangle$  assms(3)
          assms(4) calculation is-midpoint-id-2
          l12-17 l7-2)
    moreover have  $\text{Col} \ P \ C \ B$ 
      using assms(3) col-permutation-5 midpoint-col by blast
    moreover have  $\text{Col} \ P \ C \ P$ 
      by (simp add: col-trivial-3)
    ultimately show ?thesis
      using par-col2-par by blast
  qed
  ultimately show ?thesis
    using par-cong-plg-2 by blast
qed
ultimately show ?thesis
  by blast
qed
hence  $\text{Cong} \ A \ R \ P \ Q$ 

```

```

    using cong-symmetry by presburger
  moreover have Cong B R P Q
    using assms(5) calculation(2) cong-transitivity
      midpoint-cong not-cong-3421 by blast
  ultimately show ?thesis
    by blast
qed
qed
qed
qed

```

lemma *triangle-mid-par-flat-cong-1*:

```

  assumes A ≠ B and
    Col A B C and
    P Midpoint B C and
    Q Midpoint A C and
    R Midpoint A B
  shows A B Par Q P ∧ Cong A R P Q
  using assms(1) assms(2) assms(3) assms(4) assms(5)
    triangle-mid-par-flat-cong-aux by blast

```

lemma *triangle-mid-par-flat-cong-2*:

```

  assumes A ≠ B and
    Col A B C and
    P Midpoint B C and
    Q Midpoint A C and
    R Midpoint A B
  shows A B Par Q P ∧ Cong B R P Q
  using assms(1) assms(2) assms(3) assms(4) assms(5)
    triangle-mid-par-flat-cong-aux by blast

```

lemma *triangle-mid-par-flat-cong*:

```

  assumes A ≠ B and
    A ≠ C and
    B ≠ C and
    Col A B C and
    P Midpoint B C and
    Q Midpoint A C and
    R Midpoint A B
  shows A B Par Q P ∧ A C Par P R ∧ B C Par Q R ∧
    Cong A R P Q ∧ Cong B R P Q ∧ Cong A Q P R ∧
    Cong C Q P R ∧ Cong B P Q R ∧ Cong C P Q R
  by (meson assms(1) assms(2) assms(3) assms(4) assms(5)
    assms(6) assms(7) l7-2 not-col-permutation-2
    not-col-permutation-5 par-right-comm triangle-mid-par-flat-cong-aux)

```

lemma *triangle-mid-par-flat*:

```

  assumes A ≠ B and
    Col A B C and
    P Midpoint B C and
    Q Midpoint A C
  shows A B Par Q P
  using assms(1) assms(2) assms(3) assms(4) midpoint-existence
    triangle-mid-par-flat-cong-aux by blast

```

lemma *triangle-mid-par*:

```

  assumes A ≠ B and
    P Midpoint B C and
    Q Midpoint A C
  shows A B Par Q P
proof (cases Col A B C)
case True
  thus ?thesis
    using assms(1) assms(2) assms(3) triangle-mid-par-flat by blast
next
case False

```



```

hence  $A B \text{ ParStrict } Q P$ 
  by (simp add: assms(2) assms(3) triangle-mid-par-strict-cong-simp)
thus ?thesis
  using Par-def by blast
qed

lemma triangle-mid-par-cong:
  assumes  $A \neq B$  and
     $A \neq C$  and
     $B \neq C$  and
     $P \text{ Midpoint } B C$  and
     $Q \text{ Midpoint } A C$  and
     $R \text{ Midpoint } A B$ 
  shows  $A B \text{ Par } Q P \wedge A C \text{ Par } P R \wedge B C \text{ Par } Q R \wedge$ 
     $\text{Cong } A R P Q \wedge \text{Cong } B R P Q \wedge \text{Cong } A Q P R \wedge$ 
     $\text{Cong } C Q P R \wedge \text{Cong } B P Q R \wedge \text{Cong } C P Q R$ 
proof (cases Col A B C)
  case True
  thus ?thesis
    using assms(1) assms(2) assms(3) assms(4) assms(5) assms(6)
      triangle-mid-par-flat-cong by blast
  next
  case False
  have  $A B \text{ ParStrict } Q P \wedge A C \text{ ParStrict } P R \wedge$ 
     $B C \text{ ParStrict } Q R \wedge \text{Cong } A R P Q \wedge \text{Cong } B R P Q \wedge$ 
     $\text{Cong } A Q P R \wedge \text{Cong } C Q P R \wedge \text{Cong } B P Q R \wedge \text{Cong } C P Q R$ 
  using triangle-mid-par-strict-cong False assms(4) assms(5) assms(6) by blast
  thus ?thesis
    using par-strict-par by presburger
qed

lemma triangle-mid-par-cong-1:
  assumes  $B \neq C$  and
     $P \text{ Midpoint } B C$  and
     $Q \text{ Midpoint } A C$  and
     $R \text{ Midpoint } A B$ 
  shows  $B C \text{ Par } Q R \wedge \text{Cong } B P Q R$ 
proof (cases Col A B C)
  case True
  thus ?thesis
    by (meson assms(1) assms(2) assms(3) assms(4) col-permutation-1
      l7-2 par-right-comm triangle-mid-par-flat-cong-aux)
  next
  case False
  thus ?thesis
    using assms(2) assms(3) assms(4) not-col-distincts
      triangle-mid-par-cong by blast
qed

lemma midpoint-thales:
  assumes  $\neg \text{Col } A B C$  and
     $P \text{ Midpoint } A B$  and
     $\text{Cong } P A P C$ 
  shows  $\text{Per } A C B$ 
  using assms(1) assms(2) assms(3) cong-sac--per
    hypothesis-of-right-saccheri-quadrilaterals-def
    plg-cong sac-plg t22-17--rah by blast

lemma midpoint-thales-reci:
  assumes  $\text{Per } A C B$  and
     $P \text{ Midpoint } A B$ 
  shows  $\text{Cong } P A P B \wedge \text{Cong } P B P C$ 
proof (cases Col A B C)
  case True
  thus ?thesis
    by (metis Cong-cases assms(1) assms(2) cong-pseudo-reflexivity

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      l8-9 midpoint-cong not-col-permutation-5)
next
case False
have Cong P A P B
  using Midpoint-def assms(2) not-cong-2134 by blast
moreover
have Cong P B P C
proof -
  obtain X where X Midpoint A C
  using MidR-uniq-aux by blast
  hence C B Par X P
  by (meson False Mid-cases assms(2) col-permutation-3
      par-strict-par triangle-mid-par-strict-cong-simp)
  have C B Perp C A
  by (metis False Perp-perm assms(1) not-col-distincts per-perp)
  have Coplanar X P C A
  by (meson ⟨X Midpoint A C⟩ bet--coplanar coplanar-perm-11 midpoint-bet)
  hence X P Perp C A
  using ⟨C B Par X P⟩ ⟨C B Perp C A⟩ cop-par-perp--perp by blast
  hence C X Perp P X
  by (metis Perp-cases ⟨X Midpoint A C⟩ col-per-perp l8-16-1
      midpoint-col midpoint-distinct-2 not-col-distincts
      not-col-permutation-2)
  hence Per P X C
  using Perp-perm perp-per-2 by blast
  thus ?thesis
  by (meson ⟨X Midpoint A C⟩ calculation cong-symmetry
      cong-transitivity l7-2 per-double-cong)
qed
ultimately
show ?thesis
  by simp
qed

lemma midpoint-thales-reci-1:
  assumes Per A C B and
    P Midpoint A B
  shows Cong P A P B
  using midpoint-thales-reci assms(1) assms(2) by blast

lemma midpoint-thales-reci-2:
  assumes Per A C B and
    P Midpoint A B
  shows Cong P B P C
  using midpoint-thales-reci assms(1) assms(2) by blast

lemma Per-mid-rectangle:
  assumes A ≠ B and
    B ≠ C and
    Per B A C and
    I Midpoint B C and
    J Midpoint A C and
    K Midpoint A B
  shows Rectangle A J I K
proof (cases A = C)
case True
  thus ?thesis
  by (metis Rectangle-triv assms(1) assms(4) assms(5) assms(6) l7-17-bis l8-20-2)
next
case False
  have H1: A B Par J I ∧ A C Par I K ∧ B C Par J K ∧
    Cong A K I J ∧ Cong B K I J ∧ Cong A J I K ∧
    Cong C J I K ∧ Cong B I J K ∧ Cong C I J K
  using False assms(1) assms(2) assms(4) assms(5) assms(6)
    triangle-mid-par-cong by blast
  have Plg A J I K

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```

proof (cases Col A B C)
  case True
  thus ?thesis
    using False assms(1) assms(3) col-permutation-2 l8-2 l8-9 by blast
next
  case False
  hence  $\neg$  Col A B C
    by simp
  have A B ParStrict J I  $\wedge$  A C ParStrict I K  $\wedge$ 
    B C ParStrict J K  $\wedge$  Cong A K I J  $\wedge$  Cong B K I J  $\wedge$ 
    Cong A J I K  $\wedge$  Cong C J I K  $\wedge$  Cong B I J K  $\wedge$  Cong C I J K
    using False assms(4) assms(5) assms(6) triangle-mid-par-strict-cong by blast
  hence A J ParStrict I K
    by (metis Col-perm assms(5) midpoint-col midpoint-distinct-1
      par-strict-col-par-strict par-strict-symmetry)
  moreover
  have A K Par J I
    by (metis Col-def Midpoint-def H1 assms(1) assms(6) cong-diff-3
      not-par-not-col par-trans)
  ultimately
  show ?thesis
    using pars-par-plg by blast
qed
moreover
have Per K A J
proof –
  have Per C A B
    by (simp add: assms(3) l8-2)
  moreover
  have Col A C J
    using assms(5) col-permutation-1 midpoint-col by blast
  moreover
  have Col A B K
    using assms(6) col-permutation-1 midpoint-col by blast
  ultimately
  show ?thesis
    by (metis False assms(1) l8-2 per-col)
qed
hence Per K A J  $\vee$  Per I J A  $\vee$  Per A K I  $\vee$  Per J I K
  by blast
ultimately
show ?thesis
  by (simp add: plg-per-rect)
qed

lemma varignon:
  assumes A  $\neq$  C and
    B  $\neq$  D and
     $\neg$  Col I J K and
    I Midpoint A B and
    J Midpoint B C and
    K Midpoint C D and
    L Midpoint A D
  shows Parallelogram I J K L
proof –
  have I L Par J K
proof –
  have I L Par B D
    by (metis Mid-cases Par-perm assms(2) assms(4) assms(7) triangle-mid-par)
  moreover
  have J K Par B D
    by (meson Mid-cases assms(2) assms(5) assms(6) par-symmetry triangle-mid-par)
  ultimately
  show ?thesis
    using not-par-one-not-par by blast

```

```

qed
moreover
have I J Par K L
proof -
  have I J Par A C
    by (meson Mid-cases assms(1) assms(4) assms(5) par-symmetry triangle-mid-par)
  moreover
  have L K Par A C
    by (meson assms(1) assms(6) assms(7) par-symmetry triangle-mid-par)
  ultimately
  show ?thesis
    by (meson not-par-one-not-par par-left-comm)
qed
ultimately
show ?thesis
  by (simp add: assms(3) par-2-plg)
qed

lemma varignon-aux-aux:
  assumes A  $\neq$  C and
    J  $\neq$  L and
    I Midpoint A B and
    J Midpoint B C and
    K Midpoint C D and
    L Midpoint A D
  shows Parallelogram I J K L
proof (cases B = D)
  case True
  thus ?thesis
    by (metis assms(2) assms(3) assms(4) assms(5) assms(6)
        l7-17 l7-17-bis plg-trivial)
next
  case False
  obtain X where X Midpoint B D
    using MidR-uniq-aux by blast
  have B D Par L I  $\wedge$  Cong B X L I
    using False  $\langle$ X Midpoint B D $\rangle$  assms(3) assms(6)
    triangle-mid-par-cong-1 by blast
  have B D Par K J  $\wedge$  Cong B X K J
    using False  $\langle$ X Midpoint B D $\rangle$  triangle-mid-par-cong-1 assms(4)
    assms(5) l7-2 by blast
  have I L Par J K
    using Par-cases  $\langle$ B D Par K J  $\wedge$  Cong B X K J $\rangle$   $\langle$ B D Par L I  $\wedge$  Cong B X L I $\rangle$ 
    not-par-one-not-par by blast
  have Cong I L J K
    by (meson  $\langle$ B D Par K J  $\wedge$  Cong B X K J $\rangle$   $\langle$ B D Par L I  $\wedge$  Cong B X L I $\rangle$ 
        cong-inner-transitivity cong-right-commutativity)
  obtain X' where X' Midpoint A C
    using MidR-uniq-aux by blast
  have A C Par J I  $\wedge$  Cong A X' J I
    using False  $\langle$ X' Midpoint A C $\rangle$  triangle-mid-par-cong-1 assms(1)
    assms(3) assms(4) l7-2 by blast
  have A C Par K L  $\wedge$  Cong A X' K L
    using False  $\langle$ X' Midpoint A C $\rangle$  triangle-mid-par-cong-1 assms(1)
    assms(5) assms(6) l7-2 by blast
  have I J Par K L
    using Par-cases  $\langle$ A C Par J I  $\wedge$  Cong A X' J I $\rangle$   $\langle$ A C Par K L  $\wedge$  Cong A X' K L $\rangle$ 
    not-par-one-not-par by blast
  have Cong I J K L
    by (meson  $\langle$ A C Par J I  $\wedge$  Cong A X' J I $\rangle$   $\langle$ A C Par K L  $\wedge$  Cong A X' K L $\rangle$ 
        cong-inner-transitivity cong-right-commutativity)
  thus ?thesis
    by (meson Cong-cases  $\langle$ Cong I L J K $\rangle$   $\langle$ I J Par K L $\rangle$   $\langle$ I L Par J K $\rangle$ 
        assms(2) par-par-cong-cong-parallelogram par-symmetry)
qed

```

lemma *varignon-aux*:

assumes

$J \neq L$ **and**
 I Midpoint $A B$ **and**
 J Midpoint $B C$ **and**
 K Midpoint $C D$ **and**
 L Midpoint $A D$

shows *Parallelogram* $I J K L$

by (*metis* *assms*(2) *assms*(3) *assms*(4) *assms*(5) *assms*(1) *l7-17*
l7-17-bis *ply-trivial1* *varignon-aux-aux*)

lemma *varignon-bis*:

assumes $A \neq C \vee B \neq D$ **and**

I Midpoint $A B$ **and**
 J Midpoint $B C$ **and**
 K Midpoint $C D$ **and**
 L Midpoint $A D$

shows *Parallelogram* $I J K L$

proof –

have *Bet* $A I B$

using *assms*(2) *midpoint-bet* **by** *blast*

have *Cong* $A I I B$

by (*simp* *add*: *assms*(2) *midpoint-cong*)

have *Bet* $B J C$

using *assms*(3) *midpoint-bet* **by** *blast*

have *Cong* $B J J C$

by (*simp* *add*: *assms*(3) *midpoint-cong*)

have *Bet* $C K D$

using *assms*(4) *midpoint-bet* **by** *blast*

have *Cong* $C K K D$

by (*simp* *add*: *assms*(4) *midpoint-cong*)

have *Bet* $A L D$

using *assms*(5) *midpoint-bet* **by** *blast*

have *Cong* $A L L D$

by (*simp* *add*: *assms*(5) *midpoint-cong*)

show *Parallelogram* $I J K L$

proof (*cases* $J = L$)

case *True*

hence $J = L$

by *auto*

have *ParallelogramFlat* $I J K L$

proof –

obtain X **where** X Midpoint $B D$

using *MidR-uniq-aux* **by** *blast*

have *Cong* $B X X D$

using $\langle X \text{ Midpoint } B D \rangle$ *midpoint-cong* **by** *auto*

have *Bet* $B X D$

by (*simp* *add*: $\langle X \text{ Midpoint } B D \rangle$ *midpoint-bet*)

have *Cong* $K C K D$

using $\langle \text{Cong } C K K D \rangle$ *not-cong-2134* **by** *blast*

have $\text{Col } I L K \wedge \text{Col } I L L \wedge \text{Cong } I L K L \wedge (I \neq K \vee L \neq L)$

proof (*cases* $A = B$)

case *True*

hence $I = A \wedge I = B$

using *assms*(2) *l8-20-2* **by** *blast*

hence $C = D$

using $\langle J = L \rangle \langle A = B \rangle$ *assms*(3) *assms*(5) *symmetric-point-uniqueness* **by** *blast*

hence $K = C$

using $\langle \text{Bet } C K D \rangle$ *bet-neq12--neq* **by** *blast*

hence $K = D$

using $\langle C = D \rangle$ **by** *auto*

have $X = L$

using *True* $\langle X \text{ Midpoint } B D \rangle$ *assms*(5) *l7-17* **by** *blast*

have X Midpoint $I K$

using *assms*(5) **by** (*simp* *add*: $\langle I = A \wedge I = B \rangle \langle K = D \rangle \langle X = L \rangle$)

hence *Cong* $X I X K$

```

  by (meson midpoint-cong not-cong-2134)
have  $I \neq K \vee L \neq L$ 
  using  $\langle I = A \wedge I = B \rangle \langle K = C \rangle \langle K = D \rangle$  assms(1) by blast
thus ?thesis
  using  $\langle \text{Cong } X \ I \ X \ K \rangle \langle X = L \rangle \langle X \text{ Midpoint } I \ K \rangle$  col-permutation-4
  col-trivial-2 midpoint-col not-cong-2143 by blast
next
case False
hence  $A \neq B$ 
  by auto
show ?thesis
proof (cases  $A = D$ )
  case True
  hence  $X \text{ Midpoint } L \ B$ 
    using True  $\langle X \text{ Midpoint } B \ D \rangle$  assms(2) assms(5) l7-17-bis l8-20-2 by blast
  have  $A = L \wedge L = D$ 
    using True  $\langle \text{Bet } A \ L \ D \rangle$  bet-neq12--neq by blast
  have  $L \neq C \vee B \neq L$ 
    using False True  $\langle \text{Bet } A \ L \ D \rangle$  bet-neq12--neq by blast
  have  $L \text{ Midpoint } B \ C$ 
    using  $\langle J = L \rangle$  assms(3) by auto
  have  $\text{Cong } K \ C \ K \ L$ 
    using  $\langle A = L \wedge L = D \rangle \langle \text{Cong } K \ C \ K \ D \rangle$  by blast
  have  $K \text{ Midpoint } C \ L$ 
    using  $\langle A = L \wedge L = D \rangle$  assms(4) by force
  have  $\text{Cong } X \ L \ X \ B$ 
    using Cong-cases  $\langle A = L \wedge L = D \rangle \langle \text{Cong } B \ X \ X \ D \rangle$  by blast
  have  $L \neq B$ 
    using False  $\langle A = L \wedge L = D \rangle$  by auto
  have  $B \neq C$ 
    using  $\langle \text{Bet } B \ J \ C \rangle \langle J = L \rangle \langle L \neq B \rangle$  bet-neq12--neq by blast
show ?thesis
proof (cases  $X = K$ )
  case True
  hence  $X = K$ 
    by auto
  thus ?thesis
    using  $\langle A = L \wedge L = D \rangle \langle J = L \rangle \langle L \neq B \rangle \langle X \text{ Midpoint } L \ B \rangle$ 
    assms(3) assms(4) l7-9-bis midpoint-distinct-3 by blast
next
case False
hence  $X \neq K$ 
  by auto
have  $C \neq L$ 
  using  $\langle B \neq C \rangle \langle L \text{ Midpoint } B \ C \rangle$  is-midpoint-id-2 by blast
hence  $B \ L \ \text{Par } L \ K \wedge B \ C \ \text{Par } K \ X \wedge L \ C \ \text{Par } L \ X \wedge$ 
  Cong  $B \ X \ K \ L \wedge$  Cong  $L \ X \ K \ L \wedge$  Cong  $B \ L \ K \ X \wedge$ 
  Cong  $C \ L \ K \ X \wedge$  Cong  $L \ K \ L \ X \wedge$  Cong  $C \ K \ L \ X$ 
  by (metis  $\langle B \neq C \rangle \langle K \text{ Midpoint } C \ L \rangle \langle L \text{ Midpoint } B \ C \rangle \langle L \neq B \rangle$ 
   $\langle X \text{ Midpoint } B \ D \rangle$  assms(4) mid-par-cong par-distincts
  symmetric-point-construction triangle-mid-par-cong)
hence Cong  $X \ L \ K \ L$ 
  using Cong-cases by blast
moreover
have Col  $X \ L \ K$ 
  by (metis Bet-cases  $\langle A = L \wedge L = D \rangle$ 
   $\langle B \ L \ \text{Par } L \ K \wedge B \ C \ \text{Par } K \ X \wedge L \ C \ \text{Par } L \ X \wedge$ 
  Cong  $B \ X \ K \ L \wedge$  Cong  $L \ X \ K \ L \wedge$  Cong  $B \ L \ K \ X \wedge$ 
  Cong  $C \ L \ K \ X \wedge$  Cong  $L \ K \ L \ X \wedge$  Cong  $C \ K \ L \ X \rangle$ 
  assms(4) bet-col col-cong2-bet1 midpoint-bet par-id-1)
moreover
have Col  $X \ L \ L$ 
  using col-trivial-2 by auto
have  $X \neq K \vee K \neq K$ 
  by (simp add: False)
moreover

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have  $X = I$ 
  using  $\langle A = L \wedge L = D \rangle \langle X \text{ Midpoint } L \ B \rangle \text{assms}(2) \text{ l7-17 by blast}$ 
hence  $\text{Cong } I \ L \ K \ L$ 
  using  $\text{calculation}(1) \text{ by auto}$ 
ultimately
show  $?thesis$ 
  using  $\langle \text{Col } X \ L \ L \rangle \langle X = I \rangle \text{ by fastforce}$ 
qed
next
case False
hence  $A \neq D$ 
  by simp
show  $?thesis$ 
proof (cases  $B = D$ )
  case True
  hence  $B = D$ 
    by simp
  thus  $?thesis$ 
    using  $\langle J = L \rangle \text{assms}(1) \text{assms}(3) \text{assms}(5) \text{ l7-9-bis by force}$ 
next
case False
hence  $B \neq D$ 
  by simp
have  $H1: A \ B \ \text{Par} \ L \ X \wedge A \ D \ \text{Par} \ X \ I \wedge B \ D \ \text{Par} \ L \ I \wedge$ 
   $\text{Cong } A \ I \ X \ L \wedge \text{Cong } B \ I \ X \ L \wedge \text{Cong } A \ L \ X \ I \wedge$ 
   $\text{Cong } D \ L \ X \ I \wedge \text{Cong } B \ X \ L \ I \wedge \text{Cong } D \ X \ L \ I$ 
  using  $\text{triangle-mid-par-flat-cong } \text{False} \langle A \neq B \rangle \langle A \neq D \rangle$ 
   $\langle X \text{ Midpoint } B \ D \rangle \text{assms}(2) \text{assms}(5) \text{ triangle-mid-par-cong by blast}$ 
hence  $L \text{ Midpoint } I \ K$ 
proof (cases  $C = D$ )
  case True
  thus  $?thesis$ 
    using  $\langle A \neq B \rangle \langle J = L \rangle \text{assms}(3) \text{assms}(5) \text{ l7-9 by blast}$ 
next
case False
hence  $C \neq D$ 
  by simp
show  $?thesis$ 
proof (cases  $B = C$ )
  case True
  show  $?thesis$ 
  proof -
    have  $I \neq X$ 
      using  $H1 \text{ par-neq2 by blast}$ 
    moreover have  $\text{Col } I \ L \ X$ 
      by ( $\text{metis } H1 \ \text{True} \langle \text{Bet } B \ X \ D \rangle \langle J = L \rangle \text{assms}(3) \text{ bet-col}$ 
         $\text{between-symmetry col-cong2-bet1 l8-20-2 par-id-1}$ )
    moreover have  $\text{Cong } I \ L \ L \ X$ 
      by ( $\text{metis } \text{True } H1 \langle J = L \rangle \text{assms}(3)$ 
         $\text{midpoint-distinct-2 not-cong-3421}$ )
    ultimately show  $?thesis$ 
      using  $\text{MidR-uniq-aux } \text{True} \langle X \text{ Midpoint } B \ D \rangle$ 
         $\text{assms}(4) \text{ cong-col-mid by blast}$ 
  qed
next
case False
hence  $B \neq C$ 
  by simp
have  $H2: B \ C \ \text{Par} \ X \ K \wedge B \ D \ \text{Par} \ K \ L \wedge C \ D \ \text{Par} \ X \ L \wedge$ 
   $\text{Cong } B \ L \ K \ X \wedge \text{Cong } C \ L \ K \ X \wedge \text{Cong } B \ X \ K \ L \wedge$ 
   $\text{Cong } D \ X \ K \ L \wedge \text{Cong } C \ K \ X \ L \wedge \text{Cong } D \ K \ X \ L$ 
  using  $\text{triangle-mid-par-flat-cong } \text{False } \text{True} \langle B \neq D \rangle \langle C \neq D \rangle$ 
   $\langle X \text{ Midpoint } B \ D \rangle \text{assms}(3) \text{assms}(4) \text{ triangle-mid-par-cong by blast}$ 
show  $?thesis$ 
proof (cases  $I = K$ )
  case True

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hence  $I = K$ 
  by simp
have Parallelogram  $A D B C$ 
  using mid-plg  $\langle A \neq B \rangle$  assms(2) assms(4) l7-2 by blast
hence Parallelogram  $B D A C$ 
  by (meson Plg-perm)
moreover
have Parallelogram  $A B D C$ 
  using  $\langle A \neq D \rangle$   $\langle J = L \rangle$  assms(3) assms(5) mid-plg by blast
hence Parallelogram  $B D C A$ 
  using Plg-perm by blast
ultimately
have False
  using  $\langle B \neq D \rangle$  plg-not-comm-1 by blast
thus ?thesis
  by blast
next
case False
hence  $I \neq K$ 
  by auto
moreover
have  $I L \text{ Par } K L$ 
proof -
  have  $I L \text{ Par } B D$ 
    using Par-perm H1 by blast
  moreover
  have  $B D \text{ Par } K L$ 
    using H2 by blast
  ultimately
  show ?thesis
    using par-not-par by blast
qed
hence  $Col I L K$ 
  using par-comm par-id-1 by blast
moreover
have  $Cong I L L K$ 
  by (meson H1 H2 cong-inner-transitivity not-cong-4321)
ultimately
show ?thesis
  using cong-col-mid by blast
qed
qed
qed
thus ?thesis
  by (metis Cong-cases False H1 col-permutation-4
    col-trivial-2 cong-diff midpoint-col midpoint-cong
    midpoint-distinct-3)
qed
qed
qed
thus ?thesis
  using ParallelogramFlat-def True by blast
qed
thus ?thesis
  by (metis Mid-cases ParallelogramFlat-def assms(2) assms(3) assms(4)
    assms(5) plg-comm2 varignon-aux)
next
case False
thus ?thesis
  using assms(2) assms(3) assms(4) assms(5) varignon-aux by blast
qed
qed

lemma quadrilateral-midpoints:
  assumes  $\neg Col I J K$  and
     $I \text{ Midpoint } A B$  and

```


J Midpoint *B C* and
K Midpoint *C D* and
L Midpoint *A D* and
X Midpoint *I K* and
Y Midpoint *J L*
shows $X = Y$
proof –
 have *Parallelogram I J K L*
 by (*metis* *assms*(1) *assms*(2) *assms*(3) *assms*(4) *assms*(5) *l7-17*
 not-col-distincts varignon-bis)
 hence *X* Midpoint *J L*
 using *assms*(6) *plg-mid-2* **by** *blast*
 thus *?thesis*
 using *assms*(7) *l7-17* **by** *blast*
qed

lemma *is-circumcenter-coplanar*:
 assumes *G* *IsCircumcenter A B C*
 shows *Coplanar G A B C*
 using *IsCircumcenter-def* *assms* **by** *blast*

lemma *circumcenter-cong-1*:
 assumes *G* *IsCircumcenter A B C*
 shows *Cong A G B G*
 using *IsCircumcenter-def* *assms* **by** *blast*

lemma *circumcenter-cong-2*:
 assumes *G* *IsCircumcenter A B C*
 shows *Cong B G C G*
 using *IsCircumcenter-def* *assms* **by** *blast*

lemma *circumcenter-cong-3*:
 assumes *G* *IsCircumcenter A B C*
 shows *Cong C G A G*
 by (*meson* *assms* *circumcenter-cong-1* *circumcenter-cong-2*
 cong-4312 *cong-transitivity*)

lemma *circumcenter-cong*:
 assumes *G* *IsCircumcenter A B C*
 shows *Cong A G B G* $\wedge$ *Cong B G C G* $\wedge$ *Cong C G A G*
 by (*meson* *assms* *circumcenter-cong-1* *circumcenter-cong-2* *circumcenter-cong-3*)

lemma *is-circumcenter-perm-1*:
 assumes *G* *IsCircumcenter A B C*
 shows *G* *IsCircumcenter A C B*
 by (*meson* *IsCircumcenter-def* *assms* *circumcenter-cong-3*
 cong-transitivity *coplanar-perm-1*)

lemma *is-circumcenter-perm-2*:
 assumes *G* *IsCircumcenter A B C*
 shows *G* *IsCircumcenter B A C*
 using *Cong-cases* *IsCircumcenter-def* *assms* *circumcenter-cong-3*
 coplanar-perm-2 **by** *metis*

lemma *is-circumcenter-perm-3*:
 assumes *G* *IsCircumcenter A B C*
 shows *G* *IsCircumcenter B C A*
 using *assms* *is-circumcenter-perm-1* *is-circumcenter-perm-2* **by** *blast*

lemma *is-circumcenter-perm-4*:
 assumes *G* *IsCircumcenter A B C*
 shows *G* *IsCircumcenter C A B*
 by (*simp* *add*: *assms* *is-circumcenter-perm-3*)

lemma *is-circumcenter-perm-5*:
 assumes *G* *IsCircumcenter A B C*

shows $G \text{ IsCircumcenter } C \ B \ A$
using *assms is-circumcenter-perm-1 is-circumcenter-perm-4* **by** *blast*

lemma *is-circumcenter-perm*:
assumes $G \text{ IsCircumcenter } A \ B \ C$
shows $G \text{ IsCircumcenter } A \ B \ C \wedge G \text{ IsCircumcenter } A \ C \ B \wedge$
 $G \text{ IsCircumcenter } B \ A \ C \wedge G \text{ IsCircumcenter } B \ C \ A \wedge$
 $G \text{ IsCircumcenter } C \ A \ B \wedge G \text{ IsCircumcenter } C \ B \ A$
using *assms is-circumcenter-perm-4 is-circumcenter-perm-5* **by** *blast*

lemma *is-circumcenter-cases*:
assumes $G \text{ IsCircumcenter } A \ B \ C \vee G \text{ IsCircumcenter } A \ C \ B \vee$
 $G \text{ IsCircumcenter } B \ A \ C \vee G \text{ IsCircumcenter } B \ C \ A \vee$
 $G \text{ IsCircumcenter } C \ A \ B \vee G \text{ IsCircumcenter } C \ B \ A$
shows $G \text{ IsCircumcenter } A \ B \ C$
using *assms is-circumcenter-perm* **by** *blast*

lemma *circumcenter-perp*:
assumes
 $B \neq C$ **and**
 $G \neq A'$ **and**
 $G \text{ IsCircumcenter } A \ B \ C$ **and**
 $A' \text{ Midpoint } B \ C$
shows $G \ A' \text{ PerpBisect } B \ C$
using *IsCircumcenter-def assms(2) assms(4) assms(1) assms(3)*
cong-mid-perp-bisect **by** *blast*

lemma *exists-circumcenter*:
assumes $\neg \text{Col } A \ B \ C$
shows $\exists \ G. \ G \text{ IsCircumcenter } A \ B \ C$
proof –
have *triangle-circumscription-principle*
by (*simp add: TarskiPP inter-dec-plus-par-perp-perp-imply-triangle-circumscription*
playfair--universal-posidonius-postulate
tarski-s-euclid-implies-playfair-s-postulate
universal-posidonius-postulate--perpendicular-transversal-postulate)
then obtain CC **where** $\text{Cong } A \ CC \ B \ CC$ **and** $\text{Cong } A \ CC \ C \ CC$
and $\text{Coplanar } A \ B \ C \ CC$
using *assms triangle-circumscription-principle-def* **by** *blast*
thus *?thesis*
by (*meson Cong-cases IsCircumcenter-def coplanar-perm-20*
is-circumcenter-perm-2)

qed

lemma *circumcenter-perp-all*:
assumes $A \neq B$ **and**
 $B \neq C$ **and**
 $A \neq C$ **and**
 $G \neq A'$ **and**
 $G \neq B'$ **and**
 $G \neq C'$ **and**
 $G \text{ IsCircumcenter } A \ B \ C$ **and**
 $A' \text{ Midpoint } B \ C$ **and**
 $B' \text{ Midpoint } A \ C$ **and**
 $C' \text{ Midpoint } A \ B$
shows $G \ A' \text{ PerpBisect } B \ C \wedge G \ B' \text{ PerpBisect } A \ C \wedge G \ C' \text{ PerpBisect } A \ B$
by (*meson assms(1) assms(10) assms(2) assms(3) assms(4) assms(5)*
assms(6) assms(7) assms(8) assms(9) circumcenter-perp
is-circumcenter-perm-2 is-circumcenter-perm-4)

lemma *circumcenter-intersect*:
assumes $A \neq B$ **and**
 $G \neq C'$ **and**
 $C' \text{ Midpoint } A \ B$ **and**
 $G \ A' \text{ PerpBisect } B \ C$ **and**

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    G B' PerpBisect A C
shows G C' Perp A B
proof -
  have Cong B G C G
    using assms(4) perp-bisect-cong-1 by blast
  have Cong A G C G
    using assms(5) perp-bisect-cong-1 by auto
  hence Cong A G B G
    by (meson ⟨Cong B G C G⟩ cong-inner-transitivity cong-symmetry)
  hence G C' Perp C' A
    by (metis Per-def assms(1) assms(2) assms(3) cong-commutativity
        midpoint-distinct-1 per-perp)
  hence G C' Perp A C'
    using Perp-cases by blast
  thus ?thesis
    by (simp add: ⟨Cong A G B G⟩ assms(1) assms(2) assms(3)
        cong-mid-perp-bisect perp-bisect-perp)
qed

lemma is-circumcenter-uniqueness:
  assumes A ≠ B and
    B ≠ C and
    A ≠ C and
    G1 IsCircumcenter A B C and
    G2 IsCircumcenter A B C
  shows G1 = G2
proof (cases Col A B C)
  case True
  hence Col A B C
    by auto
  obtain C' where C' Midpoint A B
    using MidR-uniq-aux by blast
  {
    assume G1 = C'
    hence G1 Midpoint A B
      using ⟨C' Midpoint A B⟩ by auto
    {
      assume Col B G1 C and Cong G1 B G1 C
      hence B = C ∨ G1 Midpoint B C
        by (simp add: l7-20)
    }
    moreover
    hence Col B G1 C
      by (metis NCol-cases True ⟨G1 Midpoint A B⟩ assms(1)
          col-transitivity-2 midpoint-col)
    moreover
    {
      assume G1 Midpoint B C
      hence False
        using ⟨G1 Midpoint A B⟩ assms(3) l7-9-bis by auto
    }
    ultimately
    have False
      using Cong-cases assms(2) assms(4) circumcenter-cong-2 by blast
  }
  hence G1 C' PerpBisect A B
    by (meson ⟨C' Midpoint A B⟩ assms(1) assms(4) circumcenter-cong-1
        cong-mid-perp-bisect)
  obtain A' where A' Midpoint B C
    using MidR-uniq-aux by blast
  {
    assume G1 = A'
    hence G1 Midpoint B C
      using ⟨A' Midpoint B C⟩ by auto
    {
      assume Col A G1 B and Cong G1 A G1 B

```

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    hence  $A = B \vee G1 \text{ Midpoint } A B$ 
      by (simp add: l7-20)
  }
  moreover
  hence  $Col A G1 B$ 
    using True  $\langle G1 \text{ Midpoint } B C \rangle$  assms(2) col2--eq col-permutation-5
      midpoint-col by blast
  moreover
  {
    assume  $G1 \text{ Midpoint } A B$ 
    hence False
      using  $\langle G1 \text{ Midpoint } B C \rangle$  assms(3) l7-9-bis by auto
  }
  ultimately
  have False
    by (metis circumcenter-cong-1 assms(1) assms(4) cong-commutativity)
}
hence  $G1 A' \text{ PerpBisect } B C$ 
  using IsCircumcenter-def  $\langle A' \text{ Midpoint } B C \rangle$  assms(2) assms(4)
    cong-mid-perp-bisect by presburger
have  $G1 A' \text{ ParStrict } G1 C'$ 
proof -
  have  $G1 A' \text{ Par } G1 C'$ 
  proof -
    have Coplanar  $A B G1 G1$ 
      using ncop-distincts by blast
    moreover
    have Coplanar  $A B G1 C'$ 
      by (simp add:  $\langle C' \text{ Midpoint } A B \rangle$  bet--coplanar coplanar-perm-3 midpoint-bet)
    moreover
    have Coplanar  $A B A' C'$ 
      by (simp add:  $\langle C' \text{ Midpoint } A B \rangle$  bet--coplanar coplanar-perm-3 midpoint-bet)
    moreover
    have Coplanar  $A B A' G1$ 
      by (meson IsCircumcenter-def  $\langle A' \text{ Midpoint } B C \rangle$  assms(4)
        bet-cop--cop midpoint-bet ncoplanar-perm-18)
    moreover
    have  $G1 A' \text{ Perp } A B$ 
      by (metis Perp-cases True  $\langle G1 A' \text{ PerpBisect } B C \rangle$  assms(1) assms(2)
        calculation(4) col-permutation-1 cop-par-perp--perp ncoplanar-perm-7
        not-par-not-col perp-bisect-perp)
    moreover
    have  $G1 C' \text{ Perp } A B$ 
      by (simp add:  $\langle G1 C' \text{ PerpBisect } A B \rangle$  perp-bisect-perp)
    ultimately
    show ?thesis
      using l12-9 by blast
  qed
  moreover
  have  $A' \neq C'$ 
    using  $\langle A' \text{ Midpoint } B C \rangle \langle C' \text{ Midpoint } A B \rangle$  assms(3) l7-9-bis by blast
  {
    assume  $Col G1 A' C'$ 
    have  $Col C' A B$ 
      by (simp add:  $\langle C' \text{ Midpoint } A B \rangle$  midpoint-col)
    hence  $Col G1 A B$ 
      by (metis True  $\langle Col G1 A' C' \rangle \langle A' \text{ Midpoint } B C \rangle \langle A' \neq C' \rangle \langle Col C' A B \rangle$ 
        assms(2) col-trivial-2 l6-16-1 midpoint-col not-col-permutation-2)
    have  $Col A G1 B$ 
      using  $\langle Col G1 A B \rangle$  not-col-permutation-4 by blast
    have Cong  $G1 A G1 B$ 
      using IsCircumcenter-def assms(4) not-cong-2143 by blast
  }
  {
    assume  $Col A G1 B$  and Cong  $G1 A G1 B$ 
    have  $A = B \vee G1 \text{ Midpoint } A B$ 
      by (simp add:  $\langle Col A G1 B \rangle \langle Cong G1 A G1 B \rangle$  l7-20)
  }

```

```

}
hence G1 Midpoint A B
  by (simp add: <Col A G1 B> <Cong G1 A G1 B> assms(1))
hence False
  using <C' Midpoint A B> <G1 = C' ==> False> l7-17 by auto
}
ultimately
show ?thesis
  by (simp add: par-id)
qed
thus ?thesis
  using not-par-strict-id by auto
next
case False
hence  $\neg \text{Col } A \ B \ C$ 
  by auto
obtain C' where C' Midpoint A B
  using MidR-uniq-aux by blast
show ?thesis
proof (cases G1 = C')
  case True
  hence G1 = C'
    by auto
  show ?thesis
  proof (cases G2 = C')
    case True
    thus ?thesis
      using <G1 = C'> by auto
  next
  case False
  hence G2 ≠ C'
    by auto
  have Per A C B
    using Cong-cases True <C' Midpoint A B> <¬ Col A B C> assms(4)
      circumcenter-cong-3 midpoint-thales by blast
  obtain B' where B' Midpoint A C
    using MidR-uniq-aux by blast
  have Bet A B' C
    by (simp add: <B' Midpoint A C> midpoint-bet)
  have Cong A B' B' C
    by (simp add: <B' Midpoint A C> midpoint-cong)
  {
    assume G2 = B'
    have Per A B C
    proof -
      have  $\neg \text{Col } A \ C \ B$ 
        using Col-cases <¬ Col A B C> by blast
      moreover
      have G2 Midpoint A C
        using <B' Midpoint A C> <G2 = B'> by blast
      moreover
      have Cong G2 A G2 B
        using assms(5) circumcenter-cong-1 not-cong-2143 by blast
      ultimately
      show ?thesis
        using midpoint-thales by blast
    qed
    hence False
      using <Per A C B> assms(2) l8-7 by blast
  }
  hence G2 ≠ B'
    by auto
  obtain A' where A' Midpoint B C
    using MidR-uniq-aux by blast
  {
    assume G2 = A'

```

```

have Per B A C
proof -
  have  $\neg$  Col B C A
    using Col-cases  $\langle \neg$  Col A B C  $\rangle$  by blast
  moreover
  have G2 Midpoint B C
    using  $\langle A' \text{ Midpoint B C} \rangle \langle G2 = A' \rangle$  by auto
  moreover
  have Cong G2 B G2 A
    using assms(5) circumcenter-cong-1 cong-4321 by blast
  ultimately
  show ?thesis
    using midpoint-thales by blast
qed
hence False
  using  $\langle \text{Per A C B} \rangle$  assms(3) l8-2 l8-7 by blast
}
hence G2 A' PerpBisect B C  $\wedge$  G2 B' PerpBisect A C  $\wedge$  G2 G1 PerpBisect A B
by (metis False True  $\langle A' \text{ Midpoint B C} \rangle \langle B' \text{ Midpoint A C} \rangle$ 
 $\langle C' \text{ Midpoint A B} \rangle \langle G2 \neq B' \rangle$  assms(1) assms(2) assms(3) assms(5)
circumcenter-perp-all)
have G1 A' PerpBisect B C
by (metis True  $\langle A' \text{ Midpoint B C} \rangle \langle C' \text{ Midpoint A B} \rangle$  assms(2)
assms(3) assms(4) circumcenter-perp l7-9-bis)
have G1 B' PerpBisect A C
by (metis Cong-cases circumcenter-cong-3 True  $\langle B' \text{ Midpoint A C} \rangle$ 
 $\langle C' \text{ Midpoint A B} \rangle$  assms(2) assms(3) assms(4)
cong-mid-perp-bisect symmetric-point-uniqueness)
have Rectangle C B' G1 A'
proof -
  have Per B C A
    using Per-cases  $\langle \text{Per A C B} \rangle$  by blast
  moreover
  have G1 Midpoint B A
    using Mid-cases True  $\langle C' \text{ Midpoint A B} \rangle$  by blast
  ultimately
  show ?thesis
    by (metis Per-mid-rectangle  $\langle A' \text{ Midpoint B C} \rangle \langle B' \text{ Midpoint A C} \rangle$ 
assms(1) assms(2) l7-2)
qed
hence Plg C B' G1 A'
  using Rectangle-Plg by blast
hence Parallelogram C B' G1 A'
  using parallelogram-equiv-plg by auto
hence Parallelogram B' C A' G1
  by (simp add: plg-comm2)
{
  assume Col G1 A' B'
  hence ParallelogramFlat B' C A' G1
    by (meson Plg-perm  $\langle \text{Parallelogram B' C A' G1} \rangle$  col-permutation-4
    plg-col-plgf plgf-sym)
  hence False
    by (metis  $\langle A' \text{ Midpoint B C} \rangle \langle B' \text{ Midpoint A C} \rangle \langle \text{Rectangle C B' G1 A'} \rangle$ 
assms(2) assms(3) midpoint-distinct-1 plgf-comm2 plgf-rect-id)
}
moreover
have Col G1 A' G2
proof -
  have Coplanar B C G2 G1
    by (metis IsCircumcenter-def True  $\langle C' \text{ Midpoint A B} \rangle$  assms(5)
    bet-cop--cop between-symmetry coplanar-perm-12
    coplanar-perm-19 midpoint-bet)
  moreover
  have A' G1 Perp B C
    by (meson  $\langle G1 A' \text{ PerpBisect B C} \rangle$  perp-bisect-perp perp-left-comm)
  moreover

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have A' G2 Perp B C
  by (meson ⟨G2 A' PerpBisect B C ∧ G2 B' PerpBisect A C ∧ G2 G1 PerpBisect A B⟩
      perp-bisect-perp perp-left-comm)
ultimately
show ?thesis
  using cop-perp2--col not-col-permutation-1 by blast
qed
moreover
have Col B' G1 G2
proof –
  have Coplanar A C G1 G2
    by (metis (full-types) IsCircumcenter-def True
        ⟨C' Midpoint A B⟩
        assms(5) bet-cop--cop coplanar-perm-13
        coplanar-perm-18 midpoint-bet)
  moreover
  have B' G1 Perp A C
    using ⟨G1 B' PerpBisect A C⟩ perp-bisect-perp perp-left-comm by blast
  moreover
  have B' G2 Perp A C
    by (simp add:
        ⟨G2 A' PerpBisect B C ∧ G2 B' PerpBisect A C ∧ G2 G1 PerpBisect A B⟩
        perp-bisect-perp perp-left-comm)
  ultimately
  show ?thesis
    using cop-perp2--col not-col-permutation-1 by blast
qed
ultimately
have False
  using False True col2--eq col-permutation-4 by blast
thus ?thesis
  by simp
qed
next
case False
hence G1 ≠ C'
  by auto
show ?thesis
proof (cases G2 = C')
case True
hence G2 = C'
  by auto
have Per A C B
proof –
  have Cong G2 A G2 C
    using assms(5) circumcenter-cong not-cong-4321 by blast
  moreover
  have G2 Midpoint A B
    by (simp add: True ⟨C' Midpoint A B⟩)
  ultimately
  show ?thesis
    using ⟨¬ Col A B C⟩ midpoint-thales by blast
qed
obtain B' where B' Midpoint A C
  using MidR-uniq-aux by blast
{
  assume G1 = B'
  have Per A B C
    by (metis circumcenter-cong-1 ⟨B' Midpoint A C⟩ ⟨G1 = B'⟩
        ⟨¬ Col A B C⟩ assms(4) cong-commutativity midpoint-thales
        not-col-permutation-5)
  hence False
    using ⟨Per A C B⟩ assms(2) l8-7 by blast
}
obtain A' where A' Midpoint B C
  using MidR-uniq-aux by blast

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{
  assume  $G1 = A'$ 
  have  $Per\ B\ A\ C$ 
  by (metis NCol-cases midpoint-thales  $\langle A' Midpoint\ B\ C \rangle \langle G1 = A' \rangle$ 
       $\langle Per\ A\ C\ B \rangle$  assms(2) assms(3) assms(4) circumcenter-cong-1
      l8-9 not-cong-4321)
  hence False
  by (meson  $\langle Per\ A\ C\ B \rangle$  assms(3) l8-2 l8-7)
}
have  $H1: G1\ A'\ PerpBisect\ B\ C \wedge G1\ B'\ PerpBisect\ A\ C \wedge G1\ G2\ PerpBisect\ A\ B$ 
  using False True  $\langle A' Midpoint\ B\ C \rangle \langle B' Midpoint\ A\ C \rangle \langle C' Midpoint\ A\ B \rangle$ 
       $\langle G1 = A' \implies False \rangle \langle G1 = B' \implies False \rangle$  assms(1) assms(2) assms(3)
      assms(4) circumcenter-perp-all by blast
have  $G2\ A'\ PerpBisect\ B\ C$ 
  by (metis True  $\langle A' Midpoint\ B\ C \rangle \langle C' Midpoint\ A\ B \rangle$  assms(2)
      assms(3) assms(5) circumcenter-perp l7-9-bis)
have  $G2\ B'\ PerpBisect\ A\ C$ 
  by (metis Cong-cases circumcenter-cong-3 True  $\langle B' Midpoint\ A\ C \rangle$ 
       $\langle C' Midpoint\ A\ B \rangle$  assms(2) assms(3) assms(5) cong-mid-perp-bisect
      symmetric-point-uniqueness)
have Rectangle  $C\ B'\ G2\ A'$ 
  by (metis Per-mid-rectangle True  $\langle A' Midpoint\ B\ C \rangle \langle B' Midpoint\ A\ C \rangle$ 
       $\langle C' Midpoint\ A\ B \rangle \langle Per\ A\ C\ B \rangle$  assms(1) assms(2) l7-2 l8-2)
hence Plg  $C\ B'\ G2\ A'$ 
  using Rectangle-Plg by blast
hence Parallelogram  $C\ B'\ G2\ A'$ 
  using plg-to-parallelogram by auto
hence Parallelogram  $B'\ G2\ A'\ C$ 
  by (simp add: plg-permut)
{
  assume Col  $G2\ A'\ B'$ 
  hence ParallelogramFlat  $B'\ G2\ A'\ C$ 
  using  $\langle Parallelogram\ B'\ G2\ A'\ C \rangle$  not-col-permutation-1
      plg-col-plgf by blast
  hence False
  by (metis  $\langle A' Midpoint\ B\ C \rangle \langle B' Midpoint\ A\ C \rangle \langle Rectangle\ C\ B'\ G2\ A' \rangle$ 
      assms(2) assms(3) midpoint-distinct-1 plgf-rect-id rect-permut)
}
moreover
have  $G1 \neq B'$ 
  using  $\langle G1 = B' \implies False \rangle$  by auto
moreover
have Col  $G2\ A'\ G1$ 
proof –
  have Coplanar  $B\ C\ G1\ G2$ 
  by (metis IsCircumcenter-def True  $\langle C' Midpoint\ A\ B \rangle$  assms(4)
      bet-cop--cop between-symmetry coplanar-perm-12
      coplanar-perm-19 midpoint-bet)
  moreover
  have  $A'\ G1\ Perp\ B\ C$ 
  by (simp add: H1 perp-bisect-perp perp-left-comm)
  moreover
  have  $A'\ G2\ Perp\ B\ C$ 
  by (simp add:  $\langle G2\ A'\ PerpBisect\ B\ C \rangle$  perp-bisect-perp perp-left-comm)
  ultimately
  show ?thesis
  using col-permutation-2 cop-perp2--col by blast
qed
moreover
have Col  $G2\ A'\ G2$ 
  by (simp add: col-trivial-3)
moreover
have Col  $B'\ G1\ G1$ 
  using col-trivial-2 by blast
moreover
have Col  $B'\ G1\ G2$ 

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proof –
  have Coplanar A C G2 G1
    by (metis (full-types) IsCircumcenter-def True ⟨C' Midpoint A B⟩
        assms(4) bet-cop--cop coplanar-perm-13
        coplanar-perm-18 midpoint-bet)
  moreover
  have B' G2 Perp A C
    using ⟨G2 B' PerpBisect A C⟩ perp-bisect-perp perp-left-comm by blast
  moreover
  have B' G1 Perp A C
    by (simp add: H1 perp-bisect-perp perp-left-comm)
  ultimately
  show ?thesis
    using col-permutation-5 cop-perp2--col by blast
qed
ultimately
show ?thesis
  using l6-21 by metis
next
case False
hence G2 ≠ C'
  by blast
obtain B' where B' Midpoint A C
  using MidR-uniq-aux by blast
show ?thesis
proof (cases G1 = B')
  case True
  hence G1 = B'
    by auto
  show ?thesis
proof (cases G2 = B')
  case True
  thus ?thesis
    by (simp add: ⟨G1 = B'⟩)
next
case False
hence G2 ≠ B'
  by auto
show ?thesis
proof –
  have Per A B C
    by (metis midpoint-thales True ⟨B' Midpoint A C⟩ ⟨¬ Col A B C⟩
        assms(4) circumcenter-cong-1 col-permutation-5 not-cong-2143)
  obtain A' where A' Midpoint B C
    using MidR-uniq-aux by blast
  {
    assume G2 = A'
    hence Per B A C
      using ⟨A' Midpoint B C⟩ ⟨¬ Col A B C⟩ assms(5) circumcenter-cong-1
        cong-4321 midpoint-thales not-col-permutation-1 by blast
    hence A C ParStrict B C
      by (meson Per-cases ⟨Per A B C⟩ assms(1) l8-7)
    hence False
      by (meson col-trivial-3 par-strict-not-col-2)
  }
  have H3: G2 A' PerpBisect B C ∧ G2 G1 PerpBisect A C ∧ G2 C' PerpBisect A B
    using False True ⟨A' Midpoint B C⟩ ⟨B' Midpoint A C⟩
      ⟨C' Midpoint A B⟩ ⟨G2 = A' ⟹ False⟩
      ⟨G2 ≠ C'⟩ assms(1) assms(2)
      assms(3) assms(5) circumcenter-perp-all by blast
  have G1 A' PerpBisect B C
    by (metis True ⟨A' Midpoint B C⟩ ⟨B' Midpoint A C⟩ assms(1)
        assms(2) assms(4) circumcenter-perp sym-preserve-diff)
  have G1 C' PerpBisect A B
    by (meson ⟨C' Midpoint A B⟩ ⟨G1 ≠ C'⟩ assms(1) assms(4)
        circumcenter-cong-1 cong-mid-perp-bisect)

```

```

show ?thesis
proof -
  have Rectangle B A' G1 C'
  by (metis Mid-cases Per-mid-rectangle
    True ⟨A' Midpoint B C⟩
    ⟨B' Midpoint A C⟩ ⟨C' Midpoint A B⟩
    ⟨Per A B C⟩ assms(1) assms(3))
  hence Plg B A' G1 C'
  using Rectangle-Plg by blast
  hence Parallelogram A' G1 C' B
  by (meson plg-permut plg-to-parallelogram)
  {
    assume Col G1 A' C'
    hence ParallelogramFlat A' G1 C' B'
    by (metis ⟨C' Midpoint A B⟩ ⟨G1 ≠ C'⟩
      ⟨Parallelogram A' G1 C' B⟩ ⟨Rectangle B A' G1 C'⟩
      assms(1) col-permutation-4 midpoint-distinct-1
      plg-col-plgf plgf-rect-id rect-permut)
    hence Col A' G1 B
    using ⟨Col G1 A' C'⟩ ⟨G1 ≠ C'⟩ ⟨Parallelogram A' G1 C' B⟩
      ⟨Rectangle B A' G1 C'⟩ col-permutation-4
      plg-col-plgf plgf-rect-id
      rect-permut by blast
    hence Col A B C
    proof -
      have A' ≠ B
      using ⟨A' Midpoint B C⟩ assms(2) is-midpoint-id by blast
      moreover
      hence A' ≠ C
      using ⟨A' Midpoint B C⟩ is-midpoint-id-2 by blast
      moreover
      have B' ≠ A
      using ⟨B' Midpoint A C⟩ assms(3) is-midpoint-id by blast
      moreover
      hence B' ≠ C
      using ⟨B' Midpoint A C⟩ is-midpoint-id-2 by blast
      moreover
      have Col A' B C
      by (simp add: ⟨A' Midpoint B C⟩ midpoint-col)
      moreover
      have Col B' A C
      by (simp add: ⟨B' Midpoint A C⟩ midpoint-col)
      ultimately show ?thesis
      by (metis Col-perm True ⟨Col A' G1 B⟩ col-trivial-2 colx)
    qed
    hence False
    using ⟨¬ Col A B C⟩ by blast
  }
  moreover
  have Col G1 A' G2
  proof -
    have Coplanar B C G2 G1
    by (metis IsCircumcenter-def True ⟨B' Midpoint A C⟩ assms(5)
      bet-cop--cop between-symmetry coplanar-perm-13
      coplanar-perm-2 midpoint-bet)
    moreover
    have A' G2 Perp B C
    by (simp add: H3 perp-bisect-perp perp-left-comm)
    moreover
    have A' G1 Perp B C
    using ⟨G1 A' PerpBisect B C⟩ perp-bisect-perp
      perp-left-comm by presburger
    ultimately
    show ?thesis
    using col-permutation-2 cop-perp2--col by blast
  qed

```

```

moreover
have  $Col\ C'\ G2\ G1$ 
proof –
  have  $Coplanar\ B\ C\ G2\ G1$ 
    by (metis  $IsCircumcenter-def\ True\ \langle B'\ Midpoint\ A\ C \rangle\ assms(5)$ 
       $bet-cop--cop\ between-symmetry\ coplanar-perm-13$ 
       $coplanar-perm-2\ midpoint-bet$ )
  hence  $Coplanar\ A\ B\ G1\ G2$ 
    using  $\langle B'\ Midpoint\ A\ C \rangle\ assms(5)\ IsCircumcenter-def$ 
    by (metis  $True\ bet-cop--cop\ midpoint-bet\ ncoplanar-perm-10\ ncoplanar-perm-8$ )
  moreover
  have  $C'\ G1\ Perp\ A\ B$ 
    using  $\langle G1\ C'\ PerpBisect\ A\ B \rangle\ perp-bisect-perp$ 
     $perp-left-comm$  by blast
  moreover
  have  $C'\ G2\ Perp\ A\ B$ 
    by (simp add:  $H3\ perp-bisect-perp\ perp-left-comm$ )
  ultimately
  show ?thesis
    using  $cop-perp2--col\ not-col-permutation-5$  by blast
  qed
ultimately
show ?thesis
  by (metis  $col-trivial-2\ col-trivial-3\ l6-21$ )
qed
qed
qed
next
case False
hence  $G1 \neq B'$ 
  by auto
obtain  $A'$  where  $A'\ Midpoint\ B\ C$ 
  using  $MidR-uniq-aux$  by blast
show ?thesis
proof (cases  $G2 = B'$ )
  case True
  hence  $G2 = B'$ 
  by auto
  {
    assume  $G1 = A'$ 
    hence  $Cong\ G1\ B\ G1\ A$ 
      using  $assms(4)\ circumcenter-cong-1\ cong-4321$  by blast
    hence  $Per\ B\ A\ C$ 
      using  $\langle A'\ Midpoint\ B\ C \rangle\ \langle G1 = A' \rangle\ \langle \neg\ Col\ A\ B\ C \rangle\ midpoint-thales$ 
       $not-col-permutation-1$  by blast
    hence  $A\ C\ ParStrict\ B\ C$ 
      by (metis  $Mid-cases\ True\ \langle B'\ Midpoint\ A\ C \rangle\ \langle \neg\ Col\ A\ B\ C \rangle\ assms(1)$ 
         $assms(5)\ circumcenter-cong-3\ col-permutation-1$ 
         $is-circumcenter-perm-2\ l8-2$ 
         $l8-7\ midpoint-thales\ not-cong-2143$ )
    hence False
      by (meson  $col-trivial-3\ par-strict-not-col-2$ )
  }
hence  $H4: G1\ A'\ PerpBisect\ B\ C \wedge G1\ G2\ PerpBisect\ A\ C \wedge G1\ C'\ PerpBisect\ A\ B$ 
  by (metis  $False\ True\ \langle A'\ Midpoint\ B\ C \rangle\ \langle B'\ Midpoint\ A\ C \rangle$ 
     $\langle C'\ Midpoint\ A\ B \rangle\ \langle G1 \neq C' \rangle\ assms(1)\ assms(2)\ assms(3)\ assms(4)$ 
     $circumcenter-perp-all$ )
have  $G2\ A'\ PerpBisect\ B\ C$ 
  by (metis  $Mid-cases\ True\ \langle A'\ Midpoint\ B\ C \rangle\ \langle B'\ Midpoint\ A\ C \rangle$ 
     $\langle \bigwedge thesis.\ (\bigwedge C'.\ C'\ Midpoint\ A\ B \implies thesis) \implies thesis \rangle\ assms(1)$ 
     $assms(2)\ assms(5)\ circumcenter-perp\ l8-20-2$ )
have  $G2\ C'\ PerpBisect\ A\ B$ 
  by (meson  $\langle C'\ Midpoint\ A\ B \rangle\ \langle G2 \neq C' \rangle\ assms(1)\ assms(5)$ 
     $circumcenter-cong-1\ cong-mid-perp-bisect$ )
show ?thesis
proof –

```

```

{
  assume Col G2 A' C'
  have Rectangle B A' G2 C'
  proof -
    have Per A B C
      by (metis NCol-cases midpoint-thales True ⟨B' Midpoint A C⟩
          ⟨¬ Col A B C⟩ assms(5) circumcenter-cong-1 not-cong-2143)
    moreover
    have G2 Midpoint A C
      using True ⟨B' Midpoint A C⟩ by auto
    moreover
    have A' Midpoint B C
      by (simp add: ⟨A' Midpoint B C⟩)
    moreover
    have C' Midpoint B A
      using Mid-cases ⟨C' Midpoint A B⟩ by blast
    ultimately
    show ?thesis
      by (metis Per-mid-rectangle assms(1) assms(3))
  qed
  hence Parallelogram B A' G2 C'
    using Rectangle-Parallelogram by blast
  hence ParallelogramFlat A' G2 C' B
    by (meson ⟨Col G2 A' C'⟩ col-permutation-4 plg-col-plgf plg-permut)
  hence False
    by (metis ⟨C' Midpoint A B⟩ ⟨G2 ≠ C'⟩ ⟨Rectangle B A' G2 C'⟩
        assms(1) midpoint-distinct-1 plgf-rect-id rect-permut)
}
moreover
have Col G2 A' G1
proof -
  have Coplanar B C G1 G2
    by (metis IsCircumcenter-def True ⟨B' Midpoint A C⟩ assms(4)
        bet-cop--cop between-symmetry coplanar-perm-13
        coplanar-perm-2 midpoint-bet)
  moreover
  have A' G1 Perp B C
    by (simp add: H4 perp-bisect-perp perp-left-comm)
  moreover
  have A' G2 Perp B C
    using ⟨G2 A' PerpBisect B C⟩ perp-bisect-perp perp-left-comm by blast
  ultimately
  show ?thesis
    using cop-perp2--col not-col-permutation-1 by blast
  qed
moreover
have Col C' G1 G2
proof -
  have Coplanar B C G1 G2
    by (metis IsCircumcenter-def True ⟨B' Midpoint A C⟩ assms(4)
        bet-cop--cop between-symmetry coplanar-perm-13
        coplanar-perm-2 midpoint-bet)
  hence Coplanar C B G1 G2
    using coplanar-perm-6 by blast
  hence Coplanar A B G1 G2
    by (metis IsCircumcenter-def True ⟨B' Midpoint A C⟩ assms(4) bet-cop--cop
        coplanar-perm-14 coplanar-perm-21 coplanar-perm-23 midpoint-bet)
  hence Coplanar A B G2 G1
    using coplanar-perm-1 by blast
  moreover
  have C' G1 Perp A B
    by (simp add: H4 perp-bisect-perp perp-left-comm)
  moreover
  have C' G2 Perp A B
    using Perp-perm ⟨G2 C' PerpBisect A B⟩ perp-bisect-perp by blast
  ultimately

```

```

    show ?thesis
    by (meson col-permutation-5 cop-perp2--col)
qed
ultimately
show ?thesis
using Col-cases col-transitivity-2 by blast
qed
next
case False
hence  $G2 \neq B'$ 
by auto
show ?thesis
proof (cases  $G1 = A'$ )
case True
hence  $G1 = A'$ 
by auto
show ?thesis
proof (cases  $G2 = A'$ )
case True
hence  $G2 = A'$ 
by auto
thus ?thesis
using  $\langle G1 = A' \rangle$  by blast
next
case False
hence  $G2 \neq A'$ 
by auto
have Per C A B
by (metis Per-perm midpoint-thales True  $\langle A' \text{ Midpoint } B \ C \rangle$ 
 $\langle \neg \text{ Col } A \ B \ C \rangle$  assms(4) circumcenter-cong-1 col-permutation-2
not-cong-4321)
have  $G2 \ G1 \text{ PerpBisect } B \ C$ 
using False True  $\langle A' \text{ Midpoint } B \ C \rangle$  assms(2) assms(5)
circumcenter-perp by blast
moreover have  $G2 \ B' \text{ PerpBisect } A \ C$ 
by (meson Cong-cases  $\langle B' \text{ Midpoint } A \ C \rangle \langle G2 \neq B' \rangle$  assms(3) assms(5)
circumcenter-cong cong-mid-perp-bisect)
moreover have  $G2 \ C' \text{ PerpBisect } A \ B$ 
using  $\langle C' \text{ Midpoint } A \ B \rangle \langle G2 \neq C' \rangle$  assms(1) assms(5)
circumcenter-cong cong-mid-perp-bisect by blast
have  $G1 \ B' \text{ PerpBisect } A \ C$ 
by (metis Cong-cases circumcenter-cong-3  $\langle B' \text{ Midpoint } A \ C \rangle$ 
 $\langle G1 \neq B' \rangle$  assms(3) assms(4) cong-mid-perp-bisect)
have  $G1 \ C' \text{ PerpBisect } A \ B$ 
by (meson  $\langle C' \text{ Midpoint } A \ B \rangle \langle G1 \neq C' \rangle$  assms(1) assms(4)
circumcenter-cong-1 cong-mid-perp-bisect)
show ?thesis
proof -
{
assume Col G1 B' C'
have Rectangle A B' G1 C'
using Per-cases Per-mid-rectangle True  $\langle A' \text{ Midpoint } B \ C \rangle$ 
 $\langle B' \text{ Midpoint } A \ C \rangle \langle C' \text{ Midpoint } A \ B \rangle \langle \text{Per } C \ A \ B \rangle$ 
assms(1) assms(2) by blast
hence Plg A B' G1 C'
by (meson Rectangle-Plg)
hence Parallelogram B' G1 C' A
using Rectangle-Parallelogram  $\langle \text{Rectangle } A \ B' \ G1 \ C' \rangle$ 
rect-permut by blast
hence ParallelogramFlat B' G1 C' A
by (simp add:  $\langle \text{Col } G1 \ B' \ C' \rangle$  col-permutation-4 plg-col-plgf)
hence False
using  $\langle G1 \neq B' \rangle \langle G1 \neq C' \rangle \langle \text{Rectangle } A \ B' \ G1 \ C' \rangle$ 
plgf-rect-id rect-permut by blast
}
moreover

```

```

have Col G1 B' G2
proof -
  have Coplanar A C G2 G1
    by (metis IsCircumcenter-def True ⟨A' Midpoint B C⟩
      assms(5) bet-cop--cop between-symmetry coplanar-perm-2
      midpoint-bet ncoplanar-perm-7)
  moreover
  have B' G2 Perp A C
    using ⟨G2 B' PerpBisect A C⟩ perp-bisect-perp perp-left-comm by blast
  moreover
  have B' G1 Perp A C
    using ⟨G1 B' PerpBisect A C⟩ perp-bisect-perp perp-left-comm by blast
  ultimately
  show ?thesis
    using col-permutation-2 cop-perp2--col by blast
qed
moreover
have Col C' G2 G1
proof -
  have Coplanar A B G1 G2
    using IsCircumcenter-def True ⟨A' Midpoint B C⟩ assms(5)
      bet-cop--cop coplanar-perm-9 midpoint-bet by blast
  moreover
  have C' G1 Perp A B
    by (simp add: ⟨G1 C' PerpBisect A B⟩ perp-bisect-perp perp-left-comm)
  moreover
  have C' G2 Perp A B
    using Perp-perm ⟨G2 C' PerpBisect A B⟩ perp-bisect-perp by blast
  ultimately
  show ?thesis
    using col-permutation-5 cop-perp2--col by blast
qed
ultimately
show ?thesis
  by (metis col-trivial-2 col-trivial-3 l6-21)
qed
qed
next
case False
hence G1 ≠ A'
  by auto
show ?thesis
proof (cases G2 = A')
case True
hence G2 = A'
  by auto
have Per C A B
proof -
  have ¬ Col B C A
    using Col-cases ⟨¬ Col A B C⟩ by blast
  moreover
  have G2 Midpoint B C
    by (simp add: True ⟨A' Midpoint B C⟩)
  moreover
  have Cong G2 B G2 A
    using assms(5) circumcenter-cong-1 not-cong-4321 by blast
  ultimately
  show ?thesis
    using Per-perm midpoint-thales by blast
qed
have H5: G1 G2 PerpBisect B C ∧ G1 B' PerpBisect A C ∧ G1 C' PerpBisect A B
  using False True ⟨A' Midpoint B C⟩ ⟨B' Midpoint A C⟩
    ⟨C' Midpoint A B⟩ ⟨G1 ≠ B'⟩ ⟨G1 ≠ C'⟩ assms(1) assms(2) assms(3)
    assms(4) circumcenter-perp-all by auto
have G2 B' PerpBisect A C
  by (metis Cong-cases circumcenter-cong-3 True

```

```

    ⟨A' Midpoint B C⟩ ⟨B' Midpoint A C⟩ assms(1) assms(3) assms(5)
    cong-mid-perp-bisect l7-2 symmetric-point-uniqueness)
have G2 C' PerpBisect A B
  by (meson ⟨C' Midpoint A B⟩ ⟨G2 ≠ C'⟩ assms(1) assms(5)
    circumcenter-cong-1 cong-mid-perp-bisect)
show ?thesis
proof -
{
  assume Col G2 B' C'
  have Rectangle A B' G2 C'
    using Per-cases Per-mid-rectangle True ⟨A' Midpoint B C⟩
    ⟨B' Midpoint A C⟩ ⟨C' Midpoint A B⟩ ⟨Per C A B⟩
    assms(1) assms(2) by blast
  hence Parallelogram A B' G2 C'
    using Rectangle-Parallelogram by blast
  hence ParallelogramFlat B' G2 C' A
    by (simp add: ⟨Col G2 B' C'⟩ col-permutation-4
      plg-col-plgf plg-permut)
  hence False
    by (metis ⟨B' Midpoint A C⟩ ⟨C' Midpoint A B⟩
      ⟨Rectangle A B' G2 C'⟩ assms(1) assms(3) midpoint-distinct-1
      plgf-rect-id rect-permut)
}
moreover
have Col G2 B' G1
proof -
  have Coplanar A C G1 G2
    by (metis IsCircumcenter-def True ⟨A' Midpoint B C⟩
      assms(4) bet-cop--cop between-symmetry coplanar-perm-2
      midpoint-bet ncoplanar-perm-7)
  moreover
  have B' G1 Perp A C
    by (simp add: H5 perp-bisect-perp perp-left-comm)
  moreover
  have B' G2 Perp A C
    by (simp add: ⟨G2 B' PerpBisect A C⟩
      perp-bisect-perp perp-bisect-sym-1)
  ultimately
  show ?thesis
    by (meson col-permutation-2 cop-perp2--col)
qed
moreover
have Col C' G1 G2
proof -
  have Coplanar A B G2 G1
    using IsCircumcenter-def True ⟨A' Midpoint B C⟩ assms(4)
    bet-cop--cop coplanar-perm-9 midpoint-bet by blast
  moreover
  have C' G2 Perp A B
    using ⟨G2 C' PerpBisect A B⟩ perp-bisect-perp
    perp-left-comm by blast
  moreover
  have C' G1 Perp A B
    by (simp add: H5 perp-bisect-perp perp-bisect-sym-1)
  ultimately
  show ?thesis
    using col-permutation-5 cop-perp2--col by blast
qed
ultimately
show ?thesis
  using Col-cases col-transitivity-2 by blast
qed
next
case False
hence G2 ≠ A'
  by auto

```

```

have H6:  $G1 \ A' \text{ PerpBisect } B \ C \wedge G1 \ B' \text{ PerpBisect } A \ C \wedge G1 \ C' \text{ PerpBisect } A \ B$ 
  using  $\langle A' \text{ Midpoint } B \ C \rangle \langle B' \text{ Midpoint } A \ C \rangle \langle C' \text{ Midpoint } A \ B \rangle$ 
   $\langle G1 \neq A' \rangle \langle G1 \neq B' \rangle \langle G1 \neq C' \rangle$  assms(1) assms(2) assms(3)
  assms(4) circumcenter-perp-all by force
hence H7:  $G2 \ A' \text{ PerpBisect } B \ C \wedge G2 \ B' \text{ PerpBisect } A \ C \wedge G2 \ C' \text{ PerpBisect } A \ B$ 
  by (metis False circumcenter-perp-all  $\langle A' \text{ Midpoint } B \ C \rangle$ 
   $\langle B' \text{ Midpoint } A \ C \rangle \langle C' \text{ Midpoint } A \ B \rangle \langle G2 \neq B' \rangle \langle G2 \neq C' \rangle$ 
  assms(1) assms(2) assms(3) assms(5))
{
  assume  $Col \ G1 \ A' \ B'$ 
  have  $A \ C \text{ ParStrict } C \ B$ 
  proof –
    have  $A \ C \text{ Par } C \ B$ 
    proof –
      have  $Coplanar \ G1 \ A' \ A \ C$ 
      using  $Coplanar\text{-}def \ \langle B' \text{ Midpoint } A \ C \rangle \langle Col \ G1 \ A' \ B' \rangle$ 
      midpoint-col not-col-permutation-2 by blast
    moreover
      have  $Coplanar \ G1 \ A' \ A \ B$ 
      by (meson IsCircumcenter-def  $\langle A' \text{ Midpoint } B \ C \rangle$  assms(4)
      bet-cop--cop coplanar-perm-4 midpoint-bet)
    moreover
      have  $Coplanar \ G1 \ A' \ C \ C$ 
      using ncop-distincts by blast
    moreover
      have  $Coplanar \ G1 \ A' \ C \ B$ 
      using  $Coplanar\text{-}def \ \langle A' \text{ Midpoint } B \ C \rangle$  col-trivial-2
      midpoint-col by blast
    moreover
      have  $A \ C \text{ Perp } G1 \ A'$ 
      using  $NCol\text{-}perm \text{ Perp}\text{-}perm \ \langle Col \ G1 \ A' \ B' \rangle \ H6 \ \langle G1 \neq A' \rangle$ 
      perp-bisect-perp perp-col by blast
    moreover
      have  $C \ B \text{ Perp } G1 \ A'$ 
      using Perp-perm H6 perp-bisect-perp by blast
    ultimately
    show ?thesis
    using l12-9 by blast
  qed
  moreover
  have  $Col \ C \ B \ B$ 
  by (simp add: col-trivial-2)
  moreover
  have  $\neg \ Col \ A \ C \ B$ 
  by (simp add:  $\neg \ Col \ A \ B \ C$  not-col-permutation-5)
  ultimately
  show ?thesis
  by (simp add: Par-def)
  qed
  hence False
  using col-trivial-2 par-strict-not-col-1 by blast
}
moreover
have  $Col \ G1 \ A' \ G2$ 
proof –
  have  $Coplanar \ B \ C \ G2 \ G1$ 
  by (meson IsCircumcenter-def  $\neg \ Col \ A \ B \ C$ ) assms(4) assms(5)
  coplanar-perm-9 coplanar-trans-1)
  moreover
  have  $A' \ G2 \text{ Perp } B \ C$ 
  by (simp add: H7 perp-bisect-perp perp-left-comm)
  moreover
  have  $A' \ G1 \text{ Perp } B \ C$ 
  by (meson H6 perp-bisect-perp perp-left-comm)
  ultimately
  show ?thesis

```



```

    using Col-perm cop-perp2--col by blast
  qed
  moreover
  have Col B' G1 G2
  proof -
    have Coplanar A C G2 G1
    by (meson IsCircumcenter-def  $\neg$  Col A B C) assms(4) assms(5)
      coplanar-pseudo-trans ncop-distincts ncoplanar-perm-18)
    moreover
    have B' G2 Perp A C
    by (simp add: H7 perp-bisect-perp perp-left-comm)
    moreover
    have B' G1 Perp A C
    by (simp add: H6 perp-bisect-perp perp-left-comm)
    ultimately
    show ?thesis
    using NCol-cases cop-perp2--col by blast
  qed
  ultimately
  show ?thesis
  using  $\langle G2 \neq B' \rangle$  col2--eq col-permutation-4 by blast
qed
qed
qed
qed
qed
qed
qed

```

lemma *midpoint-thales-reci-circum:*

```

  assumes Per A C B and
    P Midpoint A B
  shows P IsCircumcenter A B C
  by (meson Cong-cases IsCircumcenter-def assms(1) assms(2) midpoint-col
    midpoint-thales-reci ncop--ncols)

```

lemma *circumcenter-per:*

```

  assumes A  $\neq$  B and
    B  $\neq$  C and
    Per A B C and
    P IsCircumcenter A B C
  shows P Midpoint A C
  proof -
    obtain P' where P' Midpoint A C
    using MidR-uniq-aux by blast
    hence P' IsCircumcenter A C B
    using assms(3) midpoint-thales-reci-circum by blast
    thus ?thesis
    by (metis  $\langle P' \text{ Midpoint } A \ C \rangle$  assms(1) assms(2) assms(3) assms(4)
      is-circumcenter-perm-1 is-circumcenter-uniqueness per-distinct-1)
  qed

```

lemma *is-orthocenter-coplanar:*

```

  assumes H IsOrthocenter A B C
  shows Coplanar H A B C
  proof -
    have A H Perp B C
    using IsOrthocenter-def assms by blast
    thus ?thesis
    using coplanar-perm-6 perp--coplanar by blast
  qed

```

lemma *construct-intersection:*

```

  assumes  $\neg$  Col A B C and
    A C Par B X1 and
    A B Par C X2 and

```

```

    B C Par A X3
shows  $\exists E. \text{Col } E \text{ A X3} \wedge \text{Col } E \text{ B X1}$ 
proof -
  have Coplanar A X3 B X1
  proof -
    have Coplanar A B C A
      using ncop-distincts by blast
    moreover
    have Coplanar A B C X3
      using assms(4) coplanar-perm-12 par-cong-cop by blast
    moreover
    have Coplanar A B C B
      using ncop-distincts by blast
    moreover
    have Coplanar A B C X1
      by (meson assms(2) coplanar-perm-2 par-cong-cop)
    ultimately
    show ?thesis
      by (meson assms(1) coplanar-pseudo-trans)
  qed
moreover
have  $\neg A \text{ X3 Par B X1}$ 
  by (meson assms(1) assms(2) assms(4) par-comm par-id-4 par-symmetry par-trans)
ultimately
show ?thesis
  using cop-npar--inter-exists by blast
qed

```

lemma not-col-par-col-diff:

```

assumes  $\neg \text{Col } A \text{ B C}$  and
  A B Par C D and
  Col C D E
shows  $A \neq E$ 
using assms(1) assms(2) assms(3) col-trivial-3 not-strict-par1 by blast

```

lemma construct-triangle:

```

assumes  $\neg \text{Col } A \text{ B C}$ 
shows  $\exists D E F. \text{Col } B \text{ D F} \wedge \text{Col } A \text{ E F} \wedge \text{Col } C \text{ D E} \wedge$ 
  A B Par C D  $\wedge$  A C Par B D  $\wedge$  B C Par A E  $\wedge$ 
  A B Par C E  $\wedge$  A C Par B F  $\wedge$  B C Par A F  $\wedge$ 
  D  $\neq$  E  $\wedge$  D  $\neq$  F  $\wedge$  E  $\neq$  F
proof -
  have A  $\neq$  B
    using assms col-trivial-1 by auto
  then obtain X1 where A B Par C X1
    using parallel-existence1 by blast
  have C  $\neq$  B
    using assms col-trivial-2 by blast
  then obtain X3 where B C Par A X3
    using parallel-existence1 by presburger
  have A  $\neq$  C
    using assms col-trivial-3 by auto
  then obtain X2 where A C Par B X2
    using parallel-existence1 by presburger
  obtain D where Col D B X2  $\wedge$  Col D C X1
    using  $\langle A \text{ B Par C X1} \rangle \langle A \text{ C Par B X2} \rangle \langle B \text{ C Par A X3} \rangle$  assms col-permutation-2
      construct-intersection par-left-comm by blast
  obtain E where Col E A X3  $\wedge$  Col E C X1
    using  $\langle A \text{ B Par C X1} \rangle \langle A \text{ C Par B X2} \rangle \langle B \text{ C Par A X3} \rangle$  assms col-permutation-5
      construct-intersection par-left-comm by blast
  obtain F where Col F A X3  $\wedge$  Col F B X2
    using  $\langle A \text{ B Par C X1} \rangle \langle A \text{ C Par B X2} \rangle \langle B \text{ C Par A X3} \rangle$  assms
      construct-intersection by blast
  have A  $\neq$  E

```

by (metis $\langle A \ B \ \text{Par} \ C \ X1 \rangle \langle \text{Col} \ E \ A \ X3 \ \wedge \ \text{Col} \ E \ C \ X1 \rangle$ *assms col-permutation-1 not-col-par-col-diff*)
 have $A \neq F$
 using $\langle A \ C \ \text{Par} \ B \ X2 \rangle \langle A \neq B \rangle \langle \text{Col} \ F \ A \ X3 \ \wedge \ \text{Col} \ F \ B \ X2 \rangle$ *assms col-trivial-2 par-col2-par-bis par-id-3* by blast
 have $B \neq D$
 by (metis $\langle A \ B \ \text{Par} \ C \ X1 \rangle \langle C \neq B \rangle \langle \text{Col} \ D \ B \ X2 \ \wedge \ \text{Col} \ D \ C \ X1 \rangle$ *assms col-par not-par-one-not-par par-distinct par-id-5*)
 have $B \neq F$
 by (metis $\langle B \ C \ \text{Par} \ A \ X3 \rangle \langle \text{Col} \ F \ A \ X3 \ \wedge \ \text{Col} \ F \ B \ X2 \rangle$ *assms not-col-par-col-diff not-col-permutation-1*)
 have $C \neq D$
 by (metis *NCol-perm Par-cases* $\langle A \ C \ \text{Par} \ B \ X2 \rangle \langle \text{Col} \ D \ B \ X2 \ \wedge \ \text{Col} \ D \ C \ X1 \rangle$ *assms not-col-par-col-diff*)
 have $C \neq E$
 by (metis $\langle B \ C \ \text{Par} \ A \ X3 \rangle \langle \text{Col} \ E \ A \ X3 \ \wedge \ \text{Col} \ E \ C \ X1 \rangle$ *assms col-trivial-3 not-col-permutation-4 par-reflexivity parallel-uniqueness*)
 have $A \ B \ \text{Par} \ C \ D$
 by (meson *Col-perm* $\langle A \ B \ \text{Par} \ C \ X1 \rangle \langle C \neq D \rangle \langle \text{Col} \ D \ B \ X2 \ \wedge \ \text{Col} \ D \ C \ X1 \rangle$ *par-col-par*)
 moreover
 have $A \ C \ \text{Par} \ B \ D$
 using *NCol-perm* $\langle A \ C \ \text{Par} \ B \ X2 \rangle \langle B \neq D \rangle \langle \text{Col} \ D \ B \ X2 \ \wedge \ \text{Col} \ D \ C \ X1 \rangle$ *par-col-par* by blast
 moreover
 have $B \ C \ \text{Par} \ A \ E$
 by (meson *NCol-cases* $\langle A \neq E \rangle \langle B \ C \ \text{Par} \ A \ X3 \rangle \langle \text{Col} \ E \ A \ X3 \ \wedge \ \text{Col} \ E \ C \ X1 \rangle$ *par-col-par*)
 moreover
 have $A \ B \ \text{Par} \ C \ E$
 by (meson *Col-cases* $\langle A \ B \ \text{Par} \ C \ X1 \rangle \langle C \neq E \rangle \langle \text{Col} \ E \ A \ X3 \ \wedge \ \text{Col} \ E \ C \ X1 \rangle$ *par-col-par*)
 moreover
 have $A \ C \ \text{Par} \ B \ F$
 by (meson *Col-perm* $\langle A \ C \ \text{Par} \ B \ X2 \rangle \langle B \neq F \rangle \langle \text{Col} \ F \ A \ X3 \ \wedge \ \text{Col} \ F \ B \ X2 \rangle$ *par-col-par*)
 moreover
 have $B \ C \ \text{Par} \ A \ F$
 by (meson $\langle A \neq F \rangle \langle B \ C \ \text{Par} \ A \ X3 \rangle \langle \text{Col} \ F \ A \ X3 \ \wedge \ \text{Col} \ F \ B \ X2 \rangle$ *col-permutation-1 par-col-par*)
 moreover
 have $D \neq E$
 by (metis *Col-perm Par-perm* *assms calculation(1) calculation(2) calculation(3) parallel-2-plg plgs-not-comm-2*)
 moreover
 have $D \neq F$
 by (metis *Par-cases* *assms calculation(1) calculation(5) calculation(6) col-trivial-3 not-strict-par2 parallel-2-plg plgs-not-comm-1*)
 moreover
 have $\text{Col} \ B \ D \ F$
 by (meson $\langle A \ C \ \text{Par} \ B \ D \rangle \langle A \ C \ \text{Par} \ B \ F \rangle$ *not-par-one-not-par par-id-3 par-symmetry*)
 moreover
 have $\text{Col} \ A \ E \ F$
 by (meson $\langle B \ C \ \text{Par} \ A \ E \rangle \langle B \ C \ \text{Par} \ A \ F \rangle$ *not-par-one-not-par par-id-3 par-symmetry*)
 moreover
 have $\text{Col} \ C \ D \ E$
 using $\langle A \ B \ \text{Par} \ C \ X1 \rangle \langle \text{Col} \ D \ B \ X2 \ \wedge \ \text{Col} \ D \ C \ X1 \rangle \langle \text{Col} \ E \ A \ X3 \ \wedge \ \text{Col} \ E \ C \ X1 \rangle$ *l6-16-1 not-col-permutation-2 par-distinct* by blast
 moreover
 have $E \neq F$
 by (metis *assms calculation(11) calculation(2) calculation(4) calculation(7) calculation(9) col2-eq not-par-one-not-par par-col2-par-bis par-id*)
 ultimately
 show ?thesis

by blast

qed

lemma *diff-not-col-col-par4-mid*:

assumes $D \neq E$ and

$\neg \text{Col } A \ B \ C$ and

$\text{Col } C \ D \ E$ and

$A \ B \ \text{Par } C \ D$ and

$A \ B \ \text{Par } C \ E$ and

$A \ E \ \text{Par } B \ C$ and

$A \ C \ \text{Par } B \ D$

shows $C \ \text{Midpoint } D \ E$

proof –

have *ParallelogramStrict* $A \ B \ C \ E$

by (*simp add: assms(2) assms(5) assms(6) parallel-2-plg*)

moreover

have *ParallelogramStrict* $C \ A \ B \ D$

by (*meson assms(2) assms(4) assms(7) not-col-permutation-2
par-left-comm par-symmetry parallel-2-plg*)

ultimately

show ?thesis

by (*meson Col-cases assms(1) assms(3) cong-transitivity l7-20
plgs-cong-1 plgs-cong-2*)

qed

lemma *altitude-is-perp-bisect*:

assumes $A \neq P$ and

$E \neq F$ and

$A \ A1 \ \text{Perp } B \ C$ and

$\text{Col } P \ A1 \ A$ and

$\text{Col } A \ E \ F$ and

$B \ C \ \text{Par } A \ E$ and

$A \ \text{Midpoint } E \ F$

shows $A \ P \ \text{PerpBisect } E \ F$

proof –

have $A \ P \ \text{Perp } E \ F$

proof –

have $B \ C \ \text{Par } E \ F$

by (*metis assms(2) assms(5) assms(6) col-par par-neq2 par-not-par*)

moreover

have $B \ C \ \text{Perp } A \ P$

by (*metis Perp-cases assms(1) assms(3) assms(4)
col-permutation-3 perp-col*)

ultimately

show ?thesis

using *cop-par-perp--perp* by (*metis Perp-cases assms(5)
ncop--ncols not-col-permutation-2*)

qed

thus ?thesis

using *assms(7) perp-mid-perp-bisect* by blast

qed

lemma *altitude-intersect*:

assumes $\neg \text{Col } A \ B \ C$ and

$A \ A1 \ \text{Perp } B \ C$ and

$B \ B1 \ \text{Perp } A \ C$ and

$C \ C1 \ \text{Perp } A \ B$ and

$\text{Col } P \ A \ A1$ and

$\text{Col } P \ B \ B1$

shows $\text{Col } P \ C \ C1$

proof –

obtain $D \ E \ F$ where $\text{Col } B \ D \ F$ and $\text{Col } A \ E \ F$ and $\text{Col } C \ D \ E$ and

$A \ B \ \text{Par } C \ D$ and $A \ C \ \text{Par } B \ D$ and $B \ C \ \text{Par } A \ E$ and

$A \ B \ \text{Par } C \ E$ and $A \ C \ \text{Par } B \ F$ and $B \ C \ \text{Par } A \ F$ and

$D \neq E$ and $D \neq F$ and $E \neq F$

using *assms(1) construct-triangle* by blast

```

have A Midpoint E F
  by (meson NCol-cases Par-cases ⟨A B Par C E⟩ ⟨A C Par B F⟩ ⟨B C Par A E⟩
    ⟨B C Par A F⟩ ⟨Col A E F⟩ ⟨E ≠ F⟩ assms(1) diff-not-col-col-par4-mid)
have B Midpoint D F
  by (metis Col-perm Par-cases ⟨A B Par C D⟩ ⟨A C Par B D⟩ ⟨A C Par B F⟩
    ⟨B C Par A F⟩ ⟨Col B D F⟩ ⟨D ≠ F⟩ assms(1) diff-not-col-col-par4-mid)
have C Midpoint D E
  by (meson diff-not-col-col-par4-mid ⟨A B Par C D⟩ ⟨A B Par C E⟩
    ⟨A C Par B D⟩ ⟨B C Par A E⟩ ⟨Col C D E⟩ ⟨D ≠ E⟩ assms(1) par-symmetry)
show ?thesis
proof (cases A = P)
  case True
  have C A Perp B B1
    using Perp-perm assms(3) by blast
  hence C A Perp A B
    by (metis NCol-perm True assms(4) assms(6) col-trivial-3
      perp-col2-bis perp-not-eq-2)
  moreover
  have Coplanar A B A C1
    using ncop-distincts by blast
  ultimately
  show ?thesis
    using assms(4) True col-permutation-4 cop-perp2--col by blast
next
case False
  hence A ≠ P
    by auto
  show ?thesis
  proof (cases B = P)
    case True
    have Coplanar A B B C1
      by (meson ncop-distincts)
    moreover
    have C B Perp A A1
      using Perp-cases assms(2) by blast
    hence C B Perp A B
      using Col-cases False True assms(5) perp-col1 by blast
    ultimately
    show ?thesis
      using assms(4) True cop-perp2--col not-col-permutation-4 by blast
  next
  case False
  hence B ≠ P
    by auto
  show ?thesis
  proof (cases C = P)
    case True
    thus ?thesis
      by (simp add: col-trivial-1)
  next
  case False
  hence C ≠ P
    by simp
  have A P PerpBisect E F
    by (metis Col-perm ⟨A Midpoint E F⟩ ⟨A ≠ P⟩ ⟨B C Par A E⟩
      ⟨Col A E F⟩ ⟨E ≠ F⟩ altitude-is-perp-bisect assms(2) assms(5))
  have B P PerpBisect D F
    by (meson ⟨A C Par B D⟩ ⟨B Midpoint D F⟩ ⟨B ≠ P⟩
      ⟨Col B D F⟩ ⟨D ≠ F⟩ altitude-is-perp-bisect assms(3) assms(6)
      col-permutation-5)
  have P C Perp D E
  proof -
    have P A PerpBisect E F
      by (meson ⟨A P PerpBisect E F⟩ perp-bisect-sym-1)
    moreover
    have P B PerpBisect D F

```

```

    using ⟨B P PerpBisect D F⟩ perp-bisect-sym-1 by blast
  ultimately
  show ?thesis
    using False ⟨C Midpoint D E⟩ ⟨D ≠ E⟩ circumcenter-intersect by auto
qed
hence C1 C Perp D E
proof -
  have A B Par D E
    by (metis ⟨A B Par C D⟩ ⟨C Midpoint D E⟩ ⟨Col C D E⟩ ⟨D ≠ E⟩
        col-par midpoint-distinct-1 par-not-par)
  moreover
  have Coplanar D E C1 C
    using ⟨Col C D E⟩ col-permutation-1 ncop--ncols by blast
  moreover
  have A B Perp C1 C
    using Perp-perm assms(4) by blast
  ultimately
  show ?thesis
    using Perp-perm cop-par-perp--perp by blast
qed
hence Coplanar D E C1 P
proof -
  have Coplanar A B C D
    by (simp add: ⟨A B Par C D⟩ par-cong-cop)
  moreover
  have Coplanar A B C E
    using ⟨A B Par C E⟩ par-cong-cop by auto
  moreover
  have Coplanar A B C C1
    using assms(4) ncoplanar-perm-16 perp--coplanar by blast
  moreover
  have Coplanar A C B B1
    by (meson assms(3) coplanar-perm-16 perp--coplanar)
  hence Coplanar A B C P
    by (metis Col-cases Perp-cases assms(3) assms(6)
        col-cop--cop ncoplanar-perm-12 ncoplanar-perm-6 perp-not-eq-2)
  ultimately
  show ?thesis
    using assms(1) coplanar-pseudo-trans by presburger
qed
show ?thesis
  by (meson Perp-cases ⟨C1 C Perp D E⟩ ⟨Coplanar D E C1 P⟩
      ⟨P C Perp D E⟩ col-permutation-2 cop-perp2--col)
qed
qed
qed
qed

```

lemma *IsOrthocenter-cases:*

```

  assumes G IsOrthocenter A B C ∨ G IsOrthocenter A C B ∨
    G IsOrthocenter B A C ∨ G IsOrthocenter B C A ∨
    G IsOrthocenter C A B ∨ G IsOrthocenter C B A
  shows G IsOrthocenter A B C
  using Col-cases IsOrthocenter-def Perp-cases assms by blast

```

lemma *IsOrthocenter-perm:*

```

  assumes G IsOrthocenter A B C
  shows G IsOrthocenter A B C ∧ G IsOrthocenter A C B ∧
    G IsOrthocenter B A C ∧ G IsOrthocenter B C A ∧
    G IsOrthocenter C A B ∧ G IsOrthocenter C B A
  using IsOrthocenter-cases assms by blast

```

lemma *IsOrthocenter-perm-1:*

```

  assumes G IsOrthocenter A B C
  shows G IsOrthocenter A C B
  using IsOrthocenter-cases assms by blast

```

```

lemma IsOrthocenter-perm-2:
  assumes  $G$  IsOrthocenter  $B$   $A$   $C$ 
  shows  $G$  IsOrthocenter  $A$   $C$   $B$ 
  using IsOrthocenter-cases assms by blast

lemma IsOrthocenter-perm-3:
  assumes  $G$  IsOrthocenter  $B$   $C$   $A$ 
  shows  $G$  IsOrthocenter  $A$   $C$   $B$ 
  using IsOrthocenter-cases assms by blast

lemma IsOrthocenter-perm-4:
  assumes  $G$  IsOrthocenter  $C$   $A$   $B$ 
  shows  $G$  IsOrthocenter  $A$   $C$   $B$ 
  using IsOrthocenter-cases assms by blast

lemma IsOrthocenter-perm-5:
  assumes  $G$  IsOrthocenter  $C$   $B$   $A$ 
  shows  $G$  IsOrthocenter  $A$   $C$   $B$ 
  using IsOrthocenter-cases assms by blast

lemma orthocenter-per:
  assumes  $Per$   $A$   $B$   $C$  and
     $H$  IsOrthocenter  $A$   $B$   $C$ 
  shows  $H = B$ 
proof -
  have  $A$   $B$  Perp  $B$   $C$ 
  by (metis IsOrthocenter-def assms(1) assms(2) per-perp perp-not-eq-2)
  hence  $A$   $H$  Par  $A$   $B$ 
  using l12-9 by (metis (no-types, lifting) IsOrthocenter-def Par-def
    assms(2) col-perp2-ncol-col cop-npar--inter-exists
    ncop-distincts parallel-uniqueness)
  hence  $Col$   $A$   $B$   $H$ 
  using par-id-3 by auto
  have  $C$   $H$  Par  $B$   $C$ 
proof -
  have  $C$   $H$  Perp  $A$   $B$ 
  using IsOrthocenter-def assms(2) by blast
  moreover
  have  $B$   $C$  Perp  $A$   $B$ 
  using Perp-perm  $\langle A$   $B$  Perp  $B$   $C \rangle$  by blast
  ultimately show ?thesis
  using l12-9
  by (metis Col-cases Par-def  $\langle Col$   $A$   $B$   $H \rangle$  col-perp2-ncol-col
    not-col-distincts perp-not-col)
qed
thus ?thesis
  by (meson IsOrthocenter-def  $\langle A$   $H$  Par  $A$   $B \rangle$  assms(2) l8-16-1
    par-id-4 par-right-comm)
qed

lemma orthocenter-col:
  assumes  $Col$   $H$   $B$   $C$  and
     $H$  IsOrthocenter  $A$   $B$   $C$ 
  shows  $H = B \vee H = C$ 
proof -
  have  $H$  PerpAt  $B$   $C$   $A$   $H$ 
  by (metis IsOrthocenter-def NCol-perm Perp-perm assms(1)
    assms(2) l8-15-1)
  show ?thesis
proof (cases  $B = H$ )
  case True
  thus ?thesis
  by blast
next
  case False

```

hence $B \neq H$
 by *simp*
 have $A \ H \ \text{Perp} \ B \ H$
 by (*meson* *False IsOrthocenter-def* *assms(1)* *assms(2)* *col-trivial-2*
 not-col-permutation-2 perp-col2-bis)
 have *False*
 by (*metis* *Col-cases IsOrthocenter-def* *assms(1)* *assms(2)* *col-trivial-2*
 l8-16-1 l8-7)
 thus *?thesis*
 by *auto*
 qed
 qed

lemma *intersection-two-medians-exist*:

assumes $\neg \text{Col } A \ B \ C$ and
 I *Midpoint* $B \ C$ and
 J *Midpoint* $A \ C$
 shows $\exists \ G. \ \text{Col } G \ A \ I \wedge \text{Col } G \ B \ J$
 proof –
 have *Bet* $A \ J \ C$
 by (*simp* *add: assms(3) midpoint-bet*)
 moreover
 have *Bet* $B \ I \ C$
 by (*simp* *add: assms(2) midpoint-bet*)
 ultimately
 obtain *G* where *Bet* $J \ G \ B \wedge \text{Bet } I \ G \ A$
 using *inner-pasch* by *blast*
 thus *?thesis*
 using *Col-def* by *blast*
 qed

lemma *intersection-two-medians-exist-unique*:

assumes $\neg \text{Col } A \ B \ C$ and
 I *Midpoint* $B \ C$ and
 J *Midpoint* $A \ C$
 shows $\exists! \ G. \ \text{Col } G \ A \ I \wedge \text{Col } G \ B \ J$
 proof –
 obtain *G* where *Col* $G \ A \ I \wedge \text{Col } G \ B \ J$
 using *intersection-two-medians-exist* *assms(1)* *assms(2)* *assms(3)* by *blast*
 moreover
 {
 fix *G1*
 assume *Col* $G1 \ A \ I$ and *Col* $G1 \ B \ J$
 moreover
 have $\neg \text{Col } A \ I \ B$
 by (*metis* *assms(1)* *assms(2)* *col-permutation-1 col-transitivity-2*
 midpoint-col midpoint-distinct-1)
 moreover
 have $B \neq J$
 using *assms(1)* *assms(3)* *bet-col midpoint-bet* by *blast*
 ultimately
 have $G = G1$
 by (*meson* $\langle \text{Col } G \ A \ I \wedge \text{Col } G \ B \ J \rangle$ *col-permutation-1 l6-21*)
 }
 ultimately
 show *?thesis*
 by *blast*
 qed

lemma *intersection-two-medians-unique-R1*:

assumes $\neg \text{Col } A \ B \ C$ and
 I *Midpoint* $B \ C$ and
 J *Midpoint* $A \ C$ and
 Col $G1 \ A \ I$ and
 Col $G1 \ B \ J$ and
 Col $G2 \ A \ I$ and

$Col\ G2\ B\ J$
shows $G1 = G2$
using *intersection-two-medians-exist-unique* *assms(1)* *assms(2)* *assms(3)*
assms(4) *assms(5)* *assms(6)* *assms(7)* **by** *blast*

lemma *is-gravity-center-coplanar*:
assumes $G\ IsGravityCenter\ A\ B\ C$
shows $Coplanar\ G\ A\ B\ C$
proof –
obtain $I\ J$ **where** $I\ Midpoint\ B\ C$ **and** $J\ Midpoint\ A\ C$ **and**
 $Col\ G\ A\ I$ **and** $Col\ G\ B\ J$
using *IsGravityCenter-def* *assms* **by** *blast*
thus *?thesis*
using *Coplanar-def* *midpoint-col* *not-col-permutation-2* **by** *blast*
qed

lemma *is-gravity-center-exist-unique*:
assumes $\neg\ Col\ A\ B\ C$
shows $\exists! G. G\ IsGravityCenter\ A\ B\ C$
proof –
obtain I **where** $I\ Midpoint\ B\ C$
using *MidR-uniq-aux* **by** *blast*
obtain J **where** $J\ Midpoint\ A\ C$
using *MidR-uniq-aux* **by** *blast*
obtain G **where** $Col\ G\ A\ I$ **and** $Col\ G\ B\ J$
using $\langle I\ Midpoint\ B\ C \rangle\ \langle J\ Midpoint\ A\ C \rangle$ *assms* *intersection-two-medians-exist* **by** *blast*
hence $G\ IsGravityCenter\ A\ B\ C$
using *IsGravityCenter-def* $\langle I\ Midpoint\ B\ C \rangle\ \langle J\ Midpoint\ A\ C \rangle$ *assms* **by** *blast*
moreover
{
fix $G1\ G2$
assume $G1\ IsGravityCenter\ A\ B\ C$ **and**
 $G2\ IsGravityCenter\ A\ B\ C$
obtain $I1\ J1$ **where** $I1\ Midpoint\ B\ C$ $J1\ Midpoint\ A\ C$ $Col\ G1\ A\ I1$ $Col\ G1\ B\ J1$
using *IsGravityCenter-def* $\langle G1\ IsGravityCenter\ A\ B\ C \rangle$ **by** *blast*
moreover
obtain $I2\ J2$ **where** $I2\ Midpoint\ B\ C$ $J2\ Midpoint\ A\ C$ $Col\ G2\ A\ I2$ $Col\ G2\ B\ J2$
using *IsGravityCenter-def* $\langle G2\ IsGravityCenter\ A\ B\ C \rangle$ **by** *blast*
ultimately
have $G1 = G2$
using *assms* *intersection-two-medians-unique-R1* *l7-17* **by** *blast*
}
ultimately
show *?thesis*
by *blast*
qed

lemma *three-medians-intersect*:
assumes $\neg\ Col\ A\ B\ C$ **and**
 $I\ Midpoint\ B\ C$ **and**
 $J\ Midpoint\ A\ C$ **and**
 $K\ Midpoint\ A\ B$
shows $\exists G. Col\ G\ A\ I \wedge Col\ G\ B\ J \wedge Col\ G\ C\ K$
proof –
obtain G **where** $Col\ G\ A\ I$ **and** $Col\ G\ B\ J$
using *assms(1)* *assms(2)* *assms(3)* *intersection-two-medians-exist* **by** *blast*
moreover
have $Col\ G\ C\ K$
proof –
obtain D **where** $G\ Midpoint\ C\ D$
using *symmetric-point-construction* **by** *blast*
have $A \neq D$
by (*metis* $\langle G\ Midpoint\ C\ D \rangle$ *assms(1)* *assms(2)* *assms(3)* *calculation(1)*
col-permutation-5 *col-trivial-3* *colx* *l7-17-bis* *midpoint-col*
midpoint-distinct-1 *not-col-par-col-diff* *triangle-mid-par*)
hence $A\ D\ Par\ J\ G$

```

    using ⟨G Midpoint C D⟩ assms(3) l7-2 triangle-mid-par by blast
have B ≠ G
  by (metis (no-types, lifting) assms(1) assms(2) calculation(1)
      col-permutation-2 col-transitivity-2 is-midpoint-id-2 l7-2 midpoint-col)
hence G B Par A D
  by (metis ⟨A D Par J G⟩ calculation(2) not-col-distincts
      par-col2-par-bis par-symmetry)
hence B ≠ D
  by (metis Col-def ⟨B ≠ G⟩ ⟨G Midpoint C D⟩ assms(1) col-par
      col-trivial-3 midpoint-bet parallel-uniqueness)
hence B D Par I G
  by (meson ⟨G Midpoint C D⟩ assms(2) l7-2 triangle-mid-par)
have A ≠ G
  using ⟨A D Par J G⟩ assms(1) assms(3) calculation(2) col-permutation-2
      col-transitivity-2 midpoint-col par-distinct by blast
hence G A Par D B
  by (metis Par-cases ⟨B D Par I G⟩ calculation(1) not-col-distincts
      par-col2-par-bis)
hence D ≠ G
  by (metis ⟨G Midpoint C D⟩ assms(1) midpoint-distinct-1 par-id-2)
{
  assume Col G A D
  hence Col A B D
    by (metis ⟨G B Par A D⟩ col-permutation-4 col-trivial-3
        par-reflexivity parallel-uniqueness)
  hence G B Par A C
    by (metis Col-def ⟨A ≠ D⟩ ⟨Col G A D⟩ ⟨D ≠ G⟩ ⟨G Midpoint C D⟩ assms(1)
        col-trivial-3 colx midpoint-bet)
  hence Col A B C
    by (metis ⟨A ≠ D⟩ ⟨Col A B D⟩ ⟨G B Par A D⟩ col-trivial-3 colx parallel-uniqueness)
  hence False
    using assms(1) by blast
}
hence Parallelogram G A D B
  using ⟨G A Par D B⟩ ⟨G B Par A D⟩ par-2-plg by blast
thus ?thesis
  by (meson ⟨G Midpoint C D⟩ assms(4) col-permutation-1 midpoint-col
      not-col-sym-not-col plg-mid-2 plg-permut)
qed
ultimately
show ?thesis
  by blast
qed

```

```

lemma is-gravity-center-col:
  assumes G IsGravityCenter A B C and
    I Midpoint A B
  shows Col G I C
proof -
  have ¬ Col A B C
    using IsGravityCenter-def assms(1) by blast
  moreover
  obtain J K where J Midpoint B C and K Midpoint A C and
    Col G A J and Col G B K
    using IsGravityCenter-def assms(1) by blast
  moreover
  {
    fix G'
    assume Col G' A J and Col G' B K and Col G' C I
    hence G = G'
      using calculation(1) calculation(2) calculation(3) calculation(4)
          calculation(5) intersection-two-medians-unique-R1 by blast
    hence Col G I C
      using Col-cases ⟨Col G' C I⟩ by blast
  }
ultimately show ?thesis

```

```

    using assms(2) three-medians-intersect
    by blast
qed

lemma is-gravity-center-diff-1:
  assumes  $G$  IsGravityCenter  $A$   $B$   $C$ 
  shows  $G \neq A$ 
proof -
  obtain  $x$   $x0$  where  $x$  Midpoint  $B$   $C$  and  $x0$  Midpoint  $A$   $C$  and
    Col  $G$   $A$   $x$  and Col  $G$   $B$   $x0$ 
    using IsGravityCenter-def assms by auto
  have Col  $x0$   $A$   $C$ 
    using  $\langle x0$  Midpoint  $A$   $C \rangle$  midpoint-col by blast
  have Col  $x$   $B$   $C$ 
    by (simp add:  $\langle x$  Midpoint  $B$   $C \rangle$  midpoint-col)
  {
    assume  $G = A$ 
    hence Col  $A$   $B$   $C$ 
      by (metis  $\langle Col$   $G$   $B$   $x0 \rangle$   $\langle Col$   $x0$   $A$   $C \rangle$   $\langle x0$  Midpoint  $A$   $C \rangle$  colx
        midpoint-distinct-1 not-col-distincts)
    hence False
      using IsGravityCenter-def assms by blast
  }
  thus ?thesis
    by blast
qed

lemma is-gravity-center-diff-2:
  assumes  $G$  IsGravityCenter  $A$   $B$   $C$ 
  shows  $G \neq B$ 
proof -
  obtain  $x$   $x0$  where  $x$  Midpoint  $B$   $C$  and  $x0$  Midpoint  $A$   $C$  and
    Col  $G$   $A$   $x$  and Col  $G$   $B$   $x0$ 
    using IsGravityCenter-def assms by auto
  have Col  $x0$   $A$   $C$ 
    using  $\langle x0$  Midpoint  $A$   $C \rangle$  midpoint-col by blast
  have Col  $x$   $B$   $C$ 
    by (simp add:  $\langle x$  Midpoint  $B$   $C \rangle$  midpoint-col)
  {
    assume  $G = B$ 
    hence Col  $A$   $B$   $C$ 
      by (metis  $\langle Col$   $G$   $A$   $x \rangle$   $\langle Col$   $x$   $B$   $C \rangle$   $\langle x$  Midpoint  $B$   $C \rangle$  col2--eq
        col-permutation-3 col-permutation-4 midpoint-distinct-1)
    hence False
      using IsGravityCenter-def assms by blast
  }
  thus ?thesis
    by blast
qed

lemma is-gravity-center-diff-3:
  assumes  $G$  IsGravityCenter  $A$   $B$   $C$ 
  shows  $G \neq C$ 
proof -
  have  $\neg$  Col  $A$   $B$   $C$ 
    using IsGravityCenter-def assms by auto
  moreover
  obtain  $I$   $J$  where  $I$  Midpoint  $B$   $C$  and  $J$  Midpoint  $A$   $C$  and
    Col  $G$   $A$   $I$  and Col  $G$   $B$   $J$ 
    using IsGravityCenter-def assms by auto
  ultimately
  show ?thesis
    by (metis col-permutation-1 l6-16-1 midpoint-col midpoint-distinct-1)
qed

```

```

lemma is-gravity-center-third-aux-lem:
  assumes  $G$  IsGravityCenter  $A$   $B$   $C$  and
     $I$  Midpoint  $B$   $C$  and  $J$  Midpoint  $A$   $C$  and
    Col  $G$   $A$   $I$  and Col  $G$   $B$   $J$ 
  shows  $G \neq I \wedge G \neq J$ 
proof -
  have  $I \neq J$ 
    using IsGravityCenter-def assms col-trivial-1 l7-9 by blast
  {
    assume  $G = I$ 
    hence Col  $I$   $B$   $J$ 
      using  $\langle \text{Col } G \text{ } B \text{ } J \rangle$  by auto
    hence Col  $B$   $J$   $C$ 
      by (metis  $\langle I \text{ Midpoint } B \text{ } C \rangle$  col3 col-permutation-4 col-trivial-2
        midpoint-col midpoint-distinct-1)
    have Col  $A$   $J$   $C$ 
      using  $\langle J \text{ Midpoint } A \text{ } C \rangle$  col-permutation-4 midpoint-col by blast
    hence Col  $A$   $B$   $C$ 
      by (metis  $\langle \text{Col } B \text{ } J \text{ } C \rangle$   $\langle J \text{ Midpoint } A \text{ } C \rangle$  col-transitivity-1 l6-16-1
        midpoint-distinct-1 not-col-distincts)
    hence False
      using IsGravityCenter-def assms by auto
  }
  moreover
  {
    assume  $G = J$ 
    hence Col  $I$   $A$   $J$ 
      using assms(4) not-col-permutation-3 by blast
    hence Col  $I$   $A$   $C$ 
      by (metis assms(3) colx midpoint-col midpoint-distinct-1 not-col-distincts)
    hence Col  $A$   $B$   $C$ 
      by (metis NCol-cases assms(2) col-trivial-3 colx not-col-sym-not-col)
    hence False
      using IsGravityCenter-def assms(1) by blast
  }
  ultimately show ?thesis by blast
qed

```

```

lemma is-gravity-center-third-aux-1:
  assumes  $G$  IsGravityCenter  $A$   $B$   $C$ 
  shows  $\neg \text{Col } B \text{ } G \text{ } C$ 
proof -
  obtain  $I$   $J$  where  $I$  Midpoint  $B$   $C$  and  $J$  Midpoint  $A$   $C$  and
    Col  $G$   $A$   $I$  and Col  $G$   $B$   $J$ 
    using IsGravityCenter-def assms by auto
  hence  $G \neq I$ 
    using assms is-gravity-center-third-aux-lem by blast
  {
    assume Col  $C$   $G$   $B$ 
    hence False
      by (metis IsGravityCenter-def  $\langle \text{Col } G \text{ } A \text{ } I \rangle$   $\langle G \neq I \rangle$   $\langle I \text{ Midpoint } B \text{ } C \rangle$ 
        assms l6-21 midpoint-col not-col-distincts not-col-permutation-2)
  }
  thus ?thesis
    using not-col-permutation-3 by blast
qed

```

```

lemma is-gravity-center-third-aux-2:
  assumes  $G$  IsGravityCenter  $A$   $B$   $C$ 
  shows  $\neg \text{Col } C \text{ } G \text{ } A$ 
proof -
  obtain  $I$   $J$  where  $I$  Midpoint  $B$   $C$  and  $J$  Midpoint  $A$   $C$  and
    Col  $G$   $A$   $I$  and Col  $G$   $B$   $J$ 
    using IsGravityCenter-def assms by auto
  hence  $G \neq J$ 

```

```

using assms is-gravity-center-third-aux-lem by presburger
{
  assume Col C G A
  hence False
    by (metis IsGravityCenter-def ⟨Col G B J⟩ ⟨G ≠ J⟩ ⟨J Midpoint A C⟩
        assms col-trivial-3 l6-21 midpoint-col not-col-permutation-3
        not-col-permutation-4)
}
thus ?thesis
  using not-col-permutation-3 by blast
qed

```

```

lemma is-gravity-center-third-aux-3:
  assumes G IsGravityCenter A B C
  shows ¬ Col A G B
proof -
  obtain I J where I Midpoint B C and J Midpoint A C and
    Col G A I and Col G B J
  using IsGravityCenter-def assms by auto
  {
    assume Col A G B
    hence False
      by (metis is-gravity-center-third-aux-1 ⟨Col G B J⟩
          ⟨J Midpoint A C⟩ assms col-permutation-3 col-permutation-4
          colx midpoint-col midpoint-distinct-1)
  }
  thus ?thesis
    using not-col-permutation-3 by blast
qed

```

```

lemma is-gravity-center-third:
  assumes G IsGravityCenter A B C and
    G' Midpoint A G and
    A' Midpoint B C
  shows G Midpoint A' G'
proof -
  obtain C' where C' Midpoint A B
  using MidR-uniq-aux by blast
  hence Col G C' C
  using assms(1) is-gravity-center-col by blast
  have ¬ Col A B C
  using IsGravityCenter-def assms by auto
  moreover
  obtain A'' B' where A'' Midpoint B C and B' Midpoint A C and
    Col G A A'' and Col G B B'
  using IsGravityCenter-def assms by auto
  hence A' = A''
  using assms(3) l7-17 by blast
  have A ≠ B
  using calculation not-col-distincts by blast
  have C ≠ B
  using calculation not-col-distincts by blast
  have A' ≠ B
  using ⟨A' = A''⟩ ⟨A'' Midpoint B C⟩ ⟨C ≠ B⟩ is-midpoint-id by blast
  obtain G'' where G'' Midpoint C G
  using MidR-uniq-aux by blast
  have Parallelogram C' A' G'' G'
  using ⟨C' Midpoint A B⟩ ⟨G'' Midpoint C G⟩ assms(2) assms(3)
    calculation not-col-distincts varignon-bis by blast
  then obtain I where I Midpoint C' G'' and I Midpoint A' G'
  using plg-mid by blast
  have C' ≠ G'' ∨ A' ≠ G'
  using ⟨Parallelogram C' A' G'' G'⟩ plg-irreflexive by blast
  have G ≠ A
  by (metis ⟨B' Midpoint A C⟩ ⟨Col G B B'⟩ ⟨G'' Midpoint C G⟩
      calculation col-permutation-2 col-transitivity-2 l7-17-bis)

```

```

    midpoint-col midpoint-distinct-2)
have  $\neg \text{Col } A \ G \ C$ 
using assms(1) is-gravity-center-third-aux-2 not-col-permutation-3 by blast
{
  assume  $C' = G''$ 
  hence  $A \ G \ \text{Par } C \ B$ 
  using Par-perm  $\langle C \neq B \rangle \langle C' \ \text{Midpoint } A \ B \rangle \langle G'' \ \text{Midpoint } C \ G \rangle$ 
    mid-par-cong2 by blast
  hence  $\text{Col } A \ C \ B$ 
  by (metis  $\langle A'' \ \text{Midpoint } B \ C \rangle \langle \text{Col } G \ A \ A'' \rangle \langle \neg \text{Col } A \ G \ C \rangle$ 
    midpoint-col not-col-distincts not-col-permutation-3 par-reflexivity
    par-right-comm parallel-uniqueness)
  hence False
  using calculation not-col-permutation-5 by blast
}
{
  assume  $A' = G'$ 
  hence  $A' = I$ 
  using  $\langle I \ \text{Midpoint } A' \ G' \rangle$  l7-3 by blast
  have  $G' = I$ 
  using  $\langle A' = G' \rangle \langle A' = I \rangle$  by auto
  have  $A' = B'$ 
  proof -
    have f1:  $\text{Bet } A \ B' \ C$ 
    using  $\langle B' \ \text{Midpoint } A \ C \rangle$  midpoint-bet by blast
    have f2:  $\text{Bet } B \ A'' \ C$ 
    using  $\langle A'' \ \text{Midpoint } B \ C \rangle$  midpoint-bet by blast
    have f3:  $\text{Bet } A \ G' \ G$ 
    using assms(2) midpoint-bet by auto
    have f4:  $\neg \text{Col } B \ A \ C$ 
    using Col-perm calculation by blast
    have f5:  $\text{Col } G \ B \ B' \wedge \text{Col } G \ B' \ B \wedge \text{Col } B \ G \ B' \wedge \text{Col } B \ B' \ G \wedge \text{Col } B' \ G \ B \wedge \text{Col } B' \ B \ G$ 
    using Col-perm  $\langle \text{Col } G \ B \ B' \rangle$  by blast
    have f6:  $\text{Col } A \ B \ B$ 
    using col-trivial-2 by blast
    have f7:  $\text{Bet } A \ A'' \ G$ 
    using f3  $\langle A' = A'' \rangle \langle A' = G' \rangle$  by blast
    have  $B \neq C$ 
    using f6 calculation by blast
    then have f8:  $A'' \neq B$ 
    using  $\langle A'' \ \text{Midpoint } B \ C \rangle$  midpoint-distinct-1 by blast
    have f9:  $B' \neq B$ 
    using f4  $\langle B' \ \text{Midpoint } A \ C \rangle$  midpoint-col by blast
    have f10:  $B \neq G$ 
    using f8 f6 f4 by (metis (no-types) Col-perm  $\langle A' = A'' \rangle \langle A' = G' \rangle$ 
       $\langle A'' \ \text{Midpoint } B \ C \rangle$  assms(2) colx midpoint-col)
    then have f11:  $\text{Col } B' \ B \ B'$ 
    using f5 col-trivial-2 colx by blast
    have f12:  $\text{Col } A \ B \ B'$ 
  proof -
    have f1:  $\forall a \ aa \ ab. \text{Col } aa \ ab \ a \longrightarrow \text{Col } a \ aa \ ab$ 
    using not-col-permutation-1 by blast
    have f2:  $\text{Bet } A \ G' \ G$ 
    by (simp add: assms(2) midpoint-bet)
    have f3:  $\text{Bet } B \ A'' \ C$ 
    using  $\langle A'' \ \text{Midpoint } B \ C \rangle$  midpoint-bet by auto
    have f4:  $\text{Col } G' \ A \ G$ 
    using assms(2) midpoint-col by blast
    have f5:  $\text{Col } A'' \ B \ C$ 
    using  $\langle A'' \ \text{Midpoint } B \ C \rangle$  midpoint-col by auto
    have f6:  $B \neq A''$ 
    using  $\langle A' = A'' \rangle \langle A' \neq B \rangle$  by presburger
    have f7:  $\text{Bet } A \ A'' \ G$ 
    using f2  $\langle A' = A'' \rangle \langle A' = G' \rangle$  by blast
    have  $B \neq G$ 
    using f6 f5 f4 f1  $\langle A' = A'' \rangle \langle A' = G' \rangle$  col-transitivity-2 f4

```

```

    not-col-permutation-3 f10 by force
  then show ?thesis
  proof -
    have f1:  $B \in A \ B' \ C$ 
    using  $\langle B' \text{ Midpoint } A \ C \rangle$  midpoint-bet by blast
    have f2:  $B \in B \ A'' \ C$ 
    using  $\langle A'' \text{ Midpoint } B \ C \rangle$  midpoint-bet by blast
    have f3:  $B \in A \ G' \ G$ 
    using  $\text{assms}(2)$  midpoint-bet by blast
    have f4:  $B \neq G \wedge \text{Col } B' \ B \ G \wedge \text{Col } B' \ B \ B \wedge \text{Col } G \ B \ A \longrightarrow \text{Col } B' \ B \ A$ 
    by (metis (no-types) colx)
    have f5:  $B \neq G'$ 
    using  $\langle A' = G' \rangle \langle A' \neq B \rangle$  by presburger
    have  $B \in B \ G' \ C$ 
    using f2  $\langle A' = A'' \rangle \langle A' = G' \rangle$  by blast
    then show ?thesis
    using f5 f4 f3 f1 by (metis NCol-perm  $\langle \text{Col } G \ B \ B' \rangle$   $\text{assms}(2)$ 
      col-trivial-2 f10 impossible-two-sides-not-col midpoint-distinct-1)
  qed
  qed
  then have f13:  $\neg \text{Col } B' \ B \ C$ 
  using f9 f6 calculation colx by blast
  then have  $B' \neq A$ 
  using f11 by (metis (no-types)  $\langle B' \text{ Midpoint } A \ C \rangle$  midpoint-distinct-1)
  then have False
  using f13 f12 f11 by (metis (no-types) Col-perm  $\langle B' \text{ Midpoint } A \ C \rangle$ 
    colx midpoint-col)
  then show ?thesis
  by meson
  qed
  have  $\text{Col } A \ C \ B$ 
  proof -
    have  $A' \text{ Midpoint } B \ C$ 
    by (simp add:  $\text{assms}(3)$ )
    moreover have  $A' \text{ Midpoint } A \ C$ 
    using  $\langle A' = B' \rangle \langle B' \text{ Midpoint } A \ C \rangle$  by blast
    ultimately show ?thesis
    using  $\langle A \neq B \rangle$  l7-9 by blast
  qed
  hence False
  using calculation not-col-permutation-5 by blast
}
hence  $C' \neq G'' \wedge A' \neq G'$ 
using  $\langle C' = G'' \implies \text{False} \rangle$  by blast
have  $G = I$ 
proof -
  have  $\neg \text{Col } A \ G \ C$ 
  by (simp add:  $\langle \neg \text{Col } A \ G \ C \rangle$ )
  moreover have  $C \neq G$ 
  using calculation not-col-distincts by blast
  moreover have  $\text{Col } A \ G \ G$ 
  by (simp add: col-trivial-2)
  moreover
  have  $\text{Col } A \ G \ I$ 
  by (metis Col-cases  $\langle A' = A'' \rangle \langle A' = G' \implies \text{False} \rangle \langle \text{Col } G \ A \ A'' \rangle \langle G \neq A \rangle \langle I \text{ Midpoint } A' \ G' \rangle$ 
     $\text{assms}(2)$  col2-eq midpoint-col)
  moreover have  $\text{Col } C \ G \ G$ 
  by (simp add: col-trivial-2)
  moreover have  $\text{Col } C \ G \ I$ 
  using Col-cases  $\langle \text{Col } G \ C' \ C \rangle \langle G'' \text{ Midpoint } C \ G \rangle$ 
     $\langle I \text{ Midpoint } C' \ G'' \rangle$  l6-16-1 midpoint-col
  by (meson  $\langle C' = G'' \implies \text{False} \rangle$  colx)
  ultimately show ?thesis
  using l6-21 by blast
  qed
  thus ?thesis

```

```

    using ⟨I Midpoint A' G'⟩ by blast
qed

lemma is-gravity-center-third-reci:
  assumes A' Midpoint B C and
    A'' Midpoint A G and
    G Midpoint A' A'' and
    ¬ Col A B C
  shows G IsGravityCenter A B C
proof -
  obtain B' where B' Midpoint A C
    using MidR-uniq-aux by blast
  obtain C' where C' Midpoint A B
    using MidR-uniq-aux by blast
  have ∃ I J. I Midpoint B C ∧ J Midpoint A C ∧ Col G A I ∧ Col G B J
  proof -
    have Col G A A'
      by (meson assms(2) assms(3) midpoint-col not-col-permutation-1
        not-col-permutation-5 not-col-sym-not-col)
    moreover
    have Col G B B'
  proof -
    obtain B'' where B'' Midpoint B G
      using midpoint-existence by blast
    obtain B''' where G Midpoint B'' B'''
      using symmetric-point-construction by auto
    have B A Par A' B'
      by (metis ⟨B' Midpoint A C⟩ assms(1) assms(4) not-col-distincts triangle-mid-par)
    have Cong A C' A' B'
      by (metis triangle-mid-par-cong-1 ⟨B A Par A' B'⟩ ⟨B' Midpoint A C⟩
        ⟨C' Midpoint A B⟩ assms(1) l7-2 par-distinct)
    have A B Par A'' B''
      using ⟨B'' Midpoint B G⟩ assms(2) assms(4) not-col-distincts
        triangle-mid-par by blast
    have Cong A C' A'' B''
      by (metis triangle-mid-par-cong-1 ⟨A B Par A'' B''⟩ ⟨B'' Midpoint B G⟩
        ⟨C' Midpoint A B⟩ assms(2) cong-right-commutativity l7-2 par-distinct)
    have A'' B'' Par A' B'
      using Par-cases ⟨A B Par A'' B''⟩ ⟨B A Par A' B'⟩ par-trans by blast
    have A'' B'' Par A' B'''
      by (metis ⟨A B Par A'' B''⟩ ⟨G Midpoint B'' B'''⟩ assms(3) l7-2
        mid-par-cong1 par-neq2)
    have Cong A'' B'' A' B'''
      by (metis Mid-cases ⟨A'' B'' Par A' B'''⟩ ⟨G Midpoint B'' B'''⟩
        assms(3) mid-par-cong1 par-neq1)
    have Cong A'' B'' A' B'
      using ⟨Cong A C' A' B'⟩ ⟨Cong A C' A'' B''⟩ cong-inner-transitivity by blast
    have Col A' B' B'''
      by (metis Par-cases ⟨A B Par A'' B''⟩ ⟨A'' B'' Par A' B'''⟩
        ⟨B A Par A' B'⟩ par-id par-trans)
  show ?thesis
proof (cases)
  assume B' = B'''
  thus ?thesis
    by (metis Col-cases ⟨B'' Midpoint B G⟩ ⟨G Midpoint B'' B'''⟩ bet-col l7-2
      midpoint-bet not-col-sym-not-col)
next
  assume B' ≠ B'''
  hence A' Midpoint B' B'''
    using ⟨Col A' B' B'''⟩ ⟨Cong A'' B'' A' B'''⟩ ⟨Cong A'' B'' A' B'⟩
      col-permutation-4 cong-inner-transitivity l7-20 by blast
  have C ≠ B'
    using ⟨B' Midpoint A C⟩ assms(4) col-trivial-3 is-midpoint-id-2 by auto
  have A' ≠ B
    using assms(1) assms(4) is-midpoint-id not-col-distincts by blast
  hence A' ≠ C

```



```

    using assms(1) is-midpoint-id-2 by force
have A'' ≠ B''
    using ⟨A B Par A'' B''⟩ par-distincts by blast
have A' ≠ B'''
    using ⟨A'' B'' Par A' B'''⟩ par-distinct by auto
have A' ≠ B'
    using ⟨A'' B'' Par A' B'⟩ par-distincts by blast
have Col G B'' B'''
    by (simp add: ⟨G Midpoint B'' B'''⟩ midpoint-col)
have Col B'' B G
    by (simp add: ⟨B'' Midpoint B G⟩ midpoint-col)
have Col C' A B
    by (simp add: ⟨C' Midpoint A B⟩ midpoint-col)
have Col B' A C
    by (simp add: ⟨B' Midpoint A C⟩ midpoint-col)
have Col G A' A''
    by (simp add: assms(3) midpoint-col)
have Col A'' A G
    by (simp add: assms(2) midpoint-col)
have Col A' B C
    by (simp add: assms(1) midpoint-col)
have A' ≠ G
    by (metis assms(1) assms(2) assms(3) assms(4) midpoint-col midpoint-distinct-1)
have A'' ≠ G
    using ⟨A' ≠ G⟩ assms(3) is-midpoint-id-2 by blast
have B'' ≠ G
    using ⟨A' Midpoint B' B'''⟩ ⟨B'' Midpoint B G⟩ ⟨C ≠ B'⟩
    ⟨G Midpoint B'' B'''⟩ assms(1) l7-9-bis by blast
have A' ≠ B''
    by (metis ⟨B' Midpoint A C⟩ ⟨B' ≠ B'''⟩ ⟨B'' Midpoint B G⟩
    ⟨G Midpoint B'' B'''⟩ assms(1) assms(2) assms(3) l7-17
    symmetric-point-uniqueness)
have Col A' B'' A'
    using col-trivial-3 by blast
have Col G A A'
    by (simp add: calculation)
have B'' ≠ B'''
    using ⟨B'' ≠ G⟩ ⟨G Midpoint B'' B'''⟩ l7-3 by blast
have G ≠ B'''
    using ⟨B'' ≠ B'''⟩ ⟨G Midpoint B'' B'''⟩ midpoint-not-midpoint by blast
have A' ≠ A''
    using ⟨A' ≠ G⟩ assms(3) l7-3 by blast
have B ≠ G
    using ⟨B'' Midpoint B G⟩ ⟨B'' ≠ G⟩ l8-20-2 by blast
have B'' ≠ B
    using ⟨B ≠ G⟩ ⟨B'' Midpoint B G⟩ is-midpoint-id by blast
have A ≠ G
    using ⟨A'' ≠ G⟩ assms(2) l7-3 by blast
have A'' ≠ A
    using ⟨A ≠ G⟩ assms(2) is-midpoint-id by blast
have ¬ Col A B G
    by (metis ⟨A ≠ G⟩ ⟨A' ≠ B⟩ ⟨B ≠ G⟩ ⟨Col A' B C⟩ assms(4) calculation
    colx l6-16-1 not-strict-par2 par-reflexivity)
have A'' ≠ B''
    using ⟨A'' ≠ B''⟩ by blast
have A' ≠ B'''
    using ⟨A' ≠ B'''⟩ by auto
have A' ≠ B'
    by (simp add: ⟨A' ≠ B'⟩)
have B' ≠ B'''
    by (simp add: ⟨B' ≠ B'''⟩)
have Col G B'' B'''
    using ⟨Col G B'' B'''⟩ by auto
have Col B'' B G
    using ⟨Col B'' B G⟩ by auto
have Col C' A B

```

```

    by (simp add: ⟨Col C' A B⟩)
have Col B' A C
    using ⟨Col B' A C⟩ by auto
have Col G A' A''
    using ⟨Col G A' A'⟩ by auto
have Col A'' A G
    using ⟨Col A'' A G⟩ by auto
have Col A' B C
    using ⟨Col A' B C⟩ by auto
have G ≠ A'
    using ⟨A' ≠ G⟩ by auto
have G ≠ A''
    using ⟨A'' ≠ G⟩ by blast
have G ≠ B''
    using ⟨B'' ≠ G⟩ by blast
have A' ≠ B''
    using ⟨A' ≠ B'⟩ by blast
have Col A' B'' A'
    by (simp add: ⟨Col A' B'' A'⟩)
have Col A A'' A'
    by (metis Col-cases ⟨A ≠ G⟩ ⟨Col A'' A G⟩ calculation col-trivial-3 colx)
have A' B'' OS A'' B'''
proof -
    have A' B'' OS G A''
    proof -
        have A' Out G A''
            using ⟨A' ≠ A'⟩ assms(3) midpoint-out by auto
        moreover have ¬ Col A' B'' G
            by (meson ⟨A B Par A'' B'⟩ ⟨Col G A A'⟩ ⟨Col G A' A'⟩ ⟨G ≠ A'⟩
                ⟨¬ Col A B G⟩ not-col-par-col-diff not-col-permutation-1
                not-col-permutation-5 par-col2-par-bis)
        ultimately show ?thesis
            using out-one-side by blast
    qed
    hence A' B'' OS A'' G
        using one-side-symmetry by blast
    moreover have A' B'' OS G B'''
    proof -
        have B'' Out G B'''
            by (simp add: ⟨B'' ≠ B'⟩ ⟨G Midpoint B'' B'⟩ midpoint-out)
        moreover have ¬ Col A' B'' G
            using ⟨A' B'' OS G A'⟩ col123--nos by auto
        ultimately show ?thesis
            using l9-19-R2 not-col-distincts by blast
    qed
    ultimately show ?thesis
        using one-side-transitivity by blast
qed
moreover
have A' B'' TS A'' B'''
proof -
    have A' B'' TS B' B'''
    proof -
        have ¬ Col B''' A' B''
            using calculation col124--nos not-col-permutation-2 by blast
        moreover have ∃ T. Col T A' B'' ∧ Bet B' T B'''
            using Midpoint-def ⟨A' Midpoint B' B'⟩ not-col-distincts by blast
        ultimately show ?thesis
            by (metis Bet-cases Col-cases Midpoint-def ⟨A' Midpoint B' B'⟩
                ⟨A' ≠ B'⟩ bet--ts l9-2)
    qed
    moreover
    have A' B'' OS B' A''
    proof -
        have A' B'' OS A C
        proof -

```

have $A' \text{ Out } G \ A \wedge \neg \text{Col } A' \ B'' \ G$
 proof –
 have $\neg \text{Col } A' \ B'' \ G$
 by (metis $\langle A' \ B'' \ OS \ A'' \ B''' \rangle \langle A' \neq G \rangle \langle \text{Col } A' \ B'' \ A' \rangle$
 $\langle \text{Col } G \ A' \ A'' \rangle \text{colx one-side-not-col123}$)
 moreover have $A' \text{ Out } G \ A$
 by (metis (full-types) Bet-cases $\langle A' \neq G \rangle \langle A'' \neq G \rangle$
 assms(2) assms(3) bet-out midpoint-bet
 outer-transitivity-between2)
 ultimately show ?thesis
 by blast
 qed
 hence $A' \ B'' \ OS \ G \ A$
 using out-one-side by blast
 hence $A' \ B'' \ OS \ A \ G$
 using one-side-symmetry by blast
 moreover have $A' \ B'' \ OS \ G \ C$
 proof –
 have $A' \ B'' \ TS \ G \ B$
 by (metis Col-cases Midpoint-def $\langle A' \text{ Out } G \ A \wedge \neg \text{Col } A' \ B'' \ G \rangle$
 $\langle B'' \text{ Midpoint } B \ G \rangle \langle B'' \neq B \rangle \text{bet--ts between-symmetry}$
 invert-two-sides)
 moreover have $A' \ B'' \ TS \ C \ B$
 by (meson TS-def $\langle A' \neq C \rangle \text{assms(1) bet--ts calculation}$
 l9-2 midpoint-bet not-col-permutation-2)
 ultimately show ?thesis
 using OS-def by blast
 qed
 ultimately show ?thesis
 using one-side-transitivity by blast
 qed
 hence $A' \ B'' \ OS \ B' \ A$
 using $\langle B' \text{ Midpoint } A \ C \rangle \text{l9-17 midpoint-bet one-side-symmetry}$ by blast
 moreover have $A' \ B'' \ OS \ A \ A''$
 proof –
 have $A' \text{ Out } A \ A''$
 by (metis NCol-perm $\langle A \neq G \rangle \langle \text{Col } A \ A'' \ A' \rangle \langle \text{Col } G \ A' \ A'' \rangle \text{assms(2)}$
 assms(3) l6-4-1 l6-4-2 midpoint-bet midpoint-out-1 out-bet-out-2)
 moreover have $\neg \text{Col } A' \ B'' \ A$
 using $\langle A' \ B'' \ OS \ A \ C \rangle \text{one-side-not-col123}$ by auto
 ultimately show ?thesis
 using out-one-side by auto
 qed
 ultimately show ?thesis
 using one-side-transitivity by blast
 qed
 ultimately show ?thesis
 using l9-8-2 by blast
 qed
 ultimately show ?thesis
 using l9-9-bis by blast
 qed
 qed
 ultimately show ?thesis
 using $\langle B' \text{ Midpoint } A \ C \rangle \text{assms(1)}$ by blast
 qed
 thus ?thesis
 using IsGravityCenter-def assms(4) by blast
 qed

 lemma is-gravity-center-perm:
 assumes $G \text{ IsGravityCenter } A \ B \ C$
 shows $G \text{ IsGravityCenter } A \ B \ C \wedge G \text{ IsGravityCenter } A \ C \ B \wedge$
 $G \text{ IsGravityCenter } B \ A \ C \wedge G \text{ IsGravityCenter } B \ C \ A \wedge$
 $G \text{ IsGravityCenter } C \ A \ B \wedge G \text{ IsGravityCenter } C \ B \ A$
 proof –

```

obtain  $I$  where  $I$  Midpoint  $A B$ 
  using MidR-uniq-aux by blast
hence Col  $G I C$ 
  using assms is-gravity-center-col by blast
have  $\neg$  Col  $A B C$ 
  using IsGravityCenter-def assms by blast
obtain  $J K$  where  $J$  Midpoint  $B C$   $K$  Midpoint  $A C$  Col  $G A J$  Col  $G B K$ 
  using IsGravityCenter-def assms by blast
have  $G$  IsGravityCenter  $A B C$ 
  by (simp add: assms)
moreover
have  $G$  IsGravityCenter  $A C B$ 
  using IsGravityCenter-def  $\langle$ Col  $G A J$  $\rangle$   $\langle$ Col  $G I C$  $\rangle$ 
     $\langle$  $I$  Midpoint  $A B$  $\rangle$   $\langle$  $J$  Midpoint  $B C$  $\rangle$   $\langle$  $\neg$  Col  $A B C$  $\rangle$ 
    col-permutation-5 l7-2 by blast
moreover
have  $G$  IsGravityCenter  $B A C$ 
  by (metis IsGravityCenter-def
     $\langle$  $\bigwedge$ thesis.  $(\bigwedge J K. \llbracket J$  Midpoint  $B C; K$  Midpoint  $A C; Col G A J; Col G B K \rrbracket \implies thesis)$ 
     $\implies thesis$  $\rangle$ 
     $\langle$  $\neg$  Col  $A B C$  $\rangle$  col-permutation-4)
moreover
have  $G$  IsGravityCenter  $B C A$ 
  by (meson IsGravityCenter-def  $\langle$ Col  $G B K$  $\rangle$   $\langle$ Col  $G I C$  $\rangle$ 
     $\langle$  $I$  Midpoint  $A B$  $\rangle$   $\langle$  $K$  Midpoint  $A C$  $\rangle$   $\langle$  $\neg$  Col  $A B C$  $\rangle$ 
    col-permutation-2 col-permutation-5 l7-2)
moreover
have  $G$  IsGravityCenter  $C A B$ 
  using IsGravityCenter-def  $\langle$  $\neg$  Col  $A B C$  $\rangle$  calculation(2) col-permutation-1 by blast
moreover
have  $G$  IsGravityCenter  $C B A$ 
  using IsGravityCenter-def  $\langle$  $\neg$  Col  $A B C$  $\rangle$  calculation(4) col-permutation-3 by blast
ultimately
show ?thesis
  by blast
qed

```

```

lemma is-gravity-center-perm-1:
  assumes  $G$  IsGravityCenter  $A B C$ 
  shows  $G$  IsGravityCenter  $A C B$ 
  using assms is-gravity-center-perm by blast

```

```

lemma is-gravity-center-perm-2:
  assumes  $G$  IsGravityCenter  $A B C$ 
  shows  $G$  IsGravityCenter  $B A C$ 
  using assms is-gravity-center-perm by blast

```

```

lemma is-gravity-center-perm-3:
  assumes  $G$  IsGravityCenter  $A B C$ 
  shows  $G$  IsGravityCenter  $B C A$ 
  using assms is-gravity-center-perm by blast

```

```

lemma is-gravity-center-perm-4:
  assumes  $G$  IsGravityCenter  $A B C$ 
  shows  $G$  IsGravityCenter  $C A B$ 
  using assms is-gravity-center-perm by blast

```

```

lemma is-gravity-center-perm-5:
  assumes  $G$  IsGravityCenter  $A B C$ 
  shows  $G$  IsGravityCenter  $C B A$ 
  using assms is-gravity-center-perm by blast

```

```

lemma is-gravity-center-cases:
  assumes  $G$  IsGravityCenter  $A B C \vee G$  IsGravityCenter  $A C B \vee$ 
     $G$  IsGravityCenter  $B A C \vee G$  IsGravityCenter  $B C A \vee$ 
     $G$  IsGravityCenter  $C A B \vee G$  IsGravityCenter  $C B A$ 

```

```

shows  $G$  IsGravityCenter  $A B C$ 
using assms is-gravity-center-perm by blast

lemma concyclic-aux:
  assumes Concyclic2  $A B C D$ 
  shows  $\exists P. \text{Cong } P A P B \wedge \text{Cong } P A P C \wedge \text{Cong } P A P D \wedge \text{Coplanar } A B C P$ 
proof -
  obtain  $O1$  where  $\text{Cong } O1 A O1 B \text{ Cong } O1 A O1 C \text{ Cong } O1 A O1 D$ 
    using assms Concyclic2-def by blast
  have  $\exists P. \text{Cong } P A P B \wedge \text{Cong } P A P C \wedge \text{Cong } P A P D \wedge \text{Coplanar } A B C P$ 
  proof (cases  $\text{Col } A B C$ )
    case True
    thus ?thesis
      using  $\langle \text{Cong } O1 A O1 B \rangle \langle \text{Cong } O1 A O1 C \rangle \langle \text{Cong } O1 A O1 D \rangle \text{ col--coplanar}$  by blast
  next
    case False
    obtain  $P$  where  $\text{Coplanar } A B C P$  and  $\forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E P O1$ 
      using l11-62-existence by blast
    have  $\text{Cong } P A P B$ 
  proof -
    have  $\text{Per } A P O1$ 
      using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E P O1 \rangle \text{ ncop-distincts}$  by blast
    moreover
    have  $\text{Per } B P O1$ 
      using  $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E P O1 \rangle \text{ ncop-distincts}$  by blast
    moreover
    have  $\text{Cong } A O1 B O1$ 
      using  $\langle \text{Cong } O1 A O1 B \rangle \text{ not-cong-2143}$  by blast
    moreover
    have  $\text{Cong } P O1 P O1$ 
      by (simp add: cong-reflexivity)
    ultimately
    show ?thesis
      using cong2-per2--cong
      by blast
  qed
  moreover
  have  $\text{Cong } P A P C$ 
    by (metis cong-reflexivity  $\langle \text{Cong } O1 A O1 C \rangle$ 
       $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E P O1 \rangle \text{ cong2-per2--cong}$ 
      cong-commutativity ncop-distincts)
  moreover
  have  $\text{Cong } P A P D$ 
    by (metis Concyclic2-def  $\langle \text{Cong } O1 A O1 D \rangle$ 
       $\langle \forall E. \text{Coplanar } A B C E \longrightarrow \text{Per } E P O1 \rangle \text{ assms cong2-per2--cong}$ 
      cong-commutativity cong-reflexivity ncop-distincts)
  ultimately
  show ?thesis
    using  $\langle \text{Coplanar } A B C P \rangle$  by blast
  qed
  thus ?thesis
    by blast
qed

lemma concyclic-trans:
  assumes  $\neg \text{Col } A B C$  and
    Concyclic2  $A B C D$  and
    Concyclic2  $A B C E$ 
  shows Concyclic2  $A B D E$ 
proof -
  obtain  $x$  where  $\text{Cong } x A x B \text{ Cong } x A x C \text{ Cong } x A x D \text{ Coplanar } A B C x$ 
    using assms(2) concyclic-aux by blast
  obtain  $x0$  where  $\text{Cong } x0 A x0 B \text{ Cong } x0 A x0 C \text{ Cong } x0 A x0 E \text{ Coplanar } A B C x0$ 
    using assms(3) concyclic-aux by blast
  have  $\text{Cong } x A x B$ 
    using  $\langle \text{Cong } x A x B \rangle$  by auto

```

```

moreover
have Cong  $x\ A\ x\ D$ 
  using  $\langle Cong\ x\ A\ x\ D \rangle$  by auto
moreover
have Cong  $x\ A\ x\ E$ 
  by (metis Col-def  $\langle Cong\ x\ A\ x\ C \rangle \langle Cong\ x0\ A\ x0\ B \rangle$ 
     $\langle Cong\ x0\ A\ x0\ C \rangle \langle Cong\ x0\ A\ x0\ E \rangle \langle Coplanar\ A\ B\ C\ x0 \rangle$ 
     $\langle Coplanar\ A\ B\ C\ x \rangle$  assms(1) between-trivial
    calculation(1) cong4-cop2--eq cong-commutativity)
ultimately
show ?thesis
  by (meson Concyclic2-def assms(1) assms(2) assms(3)
    coplanar-pseudo-trans ncop-distincts)
qed

lemma conconcyclic-perm-1:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $A\ B\ D\ C$ 
  using Concyclic2-def assms coplanar-perm-1 by fastforce

lemma conconcyclic-perm-2:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $A\ C\ B\ D$ 
  by (meson Concyclic2-def assms coplanar-perm-2)

lemma conconcyclic-perm-3:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $A\ C\ D\ B$ 
  by (meson assms conconcyclic-perm-1 conconcyclic-perm-2)

lemma conconcyclic-perm-4:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $A\ D\ B\ C$ 
  using assms conconcyclic-perm-1 conconcyclic-perm-2 by blast

lemma conconcyclic-perm-5:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $A\ D\ C\ B$ 
  by (meson assms conconcyclic-perm-1 conconcyclic-perm-4)

lemma conconcyclic-perm-6:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $B\ A\ C\ D$ 
  by (meson Concyclic2-def assms conconcyclic-perm-4 cong-inner-transitivity
    coplanar-perm-9 not-cong-3412)

lemma conconcyclic-perm-7:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $B\ A\ D\ C$ 
  using assms conconcyclic-perm-1 conconcyclic-perm-6 by blast

lemma conconcyclic-perm-8:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $B\ C\ A\ D$ 
  using assms conconcyclic-perm-4 conconcyclic-perm-7 by blast

lemma conconcyclic-perm-9:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $B\ C\ D\ A$ 
  using assms conconcyclic-perm-3 conconcyclic-perm-6 by blast

lemma conconcyclic-perm-10:
  assumes Concyclic2  $A\ B\ C\ D$ 
  shows Concyclic2  $B\ D\ A\ C$ 
  by (meson assms conconcyclic-perm-3 conconcyclic-perm-9)

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lemma concyclic-perm-11:
  assumes Concyclic2 A B C D
  shows Concyclic2 B D C A
  using assms concyclic-perm-1 concyclic-perm-10 by blast

lemma concyclic-perm-12:
  assumes Concyclic2 A B C D
  shows Concyclic2 C A B D
  using assms concyclic-perm-2 concyclic-perm-6 by blast

lemma concyclic-perm-13:
  assumes Concyclic2 A B C D
  shows Concyclic2 C A D B
  using assms concyclic-perm-3 concyclic-perm-6 by blast

lemma concyclic-perm-14:
  assumes Concyclic2 A B C D
  shows Concyclic2 C B A D
  using assms concyclic-perm-12 concyclic-perm-2 by blast

lemma concyclic-perm-15:
  assumes Concyclic2 A B C D
  shows Concyclic2 C B D A
  by (meson assms concyclic-perm-12 concyclic-perm-3)

lemma concyclic-perm-16:
  assumes Concyclic2 A B C D
  shows Concyclic2 C D A B
  using assms concyclic-perm-15 concyclic-perm-3 by blast

lemma concyclic-perm-17:
  assumes Concyclic2 A B C D
  shows Concyclic2 C D B A
  using assms concyclic-perm-1 concyclic-perm-16 by blast

lemma concyclic-perm-18:
  assumes Concyclic2 A B C D
  shows Concyclic2 D A B C
  using assms concyclic-perm-11 concyclic-perm-17 by blast

lemma concyclic-perm-19:
  assumes Concyclic2 A B C D
  shows Concyclic2 D A C B
  using assms concyclic-perm-10 concyclic-perm-17 by blast

lemma concyclic-perm-20:
  assumes Concyclic2 A B C D
  shows Concyclic2 D B A C
  using assms concyclic-perm-1 concyclic-perm-14 by blast

lemma concyclic-perm-21:
  assumes Concyclic2 A B C D
  shows Concyclic2 D B C A
  using assms concyclic-perm-19 concyclic-perm-5 by blast

lemma concyclic-perm-22:
  assumes Concyclic2 A B C D
  shows Concyclic2 D C A B
  using assms concyclic-perm-21 concyclic-perm-3 by blast

lemma concyclic-perm-23:
  assumes Concyclic2 A B C D
  shows Concyclic2 D C B A
  using assms concyclic-perm-2 concyclic-perm-21 by blast

lemma concyclic-1123:

```

```

assumes  $\neg \text{Col } A \ B \ C$ 
shows  $\text{Concyclic2 } A \ A \ B \ C$ 
proof –
  have  $\text{Coplanar } A \ A \ B \ C$ 
    using coplanar-trivial by blast
  obtain  $G$  where  $G \text{ IsCircumcenter } A \ B \ C$ 
    using assms exists-circumcenter by blast
  have  $\text{Cong } G \ A \ G \ A$ 
    using cong-reflexivity by auto
  moreover
  have  $\text{Cong } G \ A \ G \ B$ 
    using  $\langle G \text{ IsCircumcenter } A \ B \ C \rangle$  circumcenter-cong-1 cong-commutativity by blast
  moreover
  have  $\text{Cong } G \ A \ G \ C$ 
    using  $\langle G \text{ IsCircumcenter } A \ B \ C \rangle$  circumcenter-cong-3 cong-4321 by blast
  ultimately
  show ?thesis
    using Concyclic2-def ncop-distincts by fastforce
qed

```

```

lemma conconcyclic-not-col-or-eq-aux:
  assumes  $\text{Concyclic2 } A \ B \ C \ D$ 
  shows  $A = B \vee A = C \vee B = C \vee \neg \text{Col } A \ B \ C$ 
proof (cases  $A = B$ )
  case True
    thus ?thesis
      by simp
next
  case False
    hence  $A \neq B$ 
      by auto
    show ?thesis
proof (cases  $A = C$ )
  case True
    thus ?thesis
      by simp
next
  case False
    hence  $A \neq C$ 
      by simp
    show ?thesis
proof (cases  $B = C$ )
  case True
    thus ?thesis
      by simp
next
  case False
    hence  $B \neq C$ 
      by simp
    show ?thesis
proof (cases  $\text{Col } A \ B \ C$ )
  case True
    hence  $\text{Col } A \ B \ C$ 
      by simp
    obtain  $P$  where  $\text{Cong } P \ A \ P \ B \ \text{Cong } P \ A \ P \ C \ \text{Cong } P \ A \ P \ D$ 
      using assms concyclic-aux by blast
    obtain  $M1$  where  $M1 \text{ Midpoint } A \ B$ 
      using MidR-uniq-aux by blast
    {
      assume  $P = M1$ 
      hence  $\text{Bet } A \ P \ B$ 
        by (simp add:  $\langle M1 \text{ Midpoint } A \ B \rangle$  midpoint-bet)
      hence  $\text{Col } A \ P \ B$ 
        using Col-def by blast
    }
    {
      assume  $\text{Col } A \ P \ C$  and  $\text{Cong } P \ A \ P \ C$ 

```



```

    hence  $A = C \vee P \text{ Midpoint } A \ C$ 
    using l7-20-bis by blast
  }
  have  $A \neq C$ 
  using  $\langle A \neq C \rangle$  by blast
  moreover
  have  $\neg P \text{ Midpoint } A \ C$ 
  using False  $\langle M1 \text{ Midpoint } A \ B \rangle \langle P = M1 \rangle$  symmetric-point-uniqueness by blast
  moreover
  have  $\text{Col } A \ P \ C$ 
  using True  $\langle A \neq B \rangle \langle \text{Col } A \ P \ B \rangle$  col-trivial-3 colx by blast
  ultimately
  have False
  using  $\langle \text{Cong } P \ A \ P \ C \rangle \langle \llbracket \text{Col } A \ P \ C; \text{Cong } P \ A \ P \ C \rrbracket \implies A = C \vee P \text{ Midpoint } A \ C \rangle$  by blast
}
obtain  $M2$  where  $M2 \text{ Midpoint } A \ C$ 
using MidR-uniq-aux by blast
{
  assume  $P = M2$ 
  {
    assume  $\text{Col } A \ P \ B$  and  $\text{Cong } P \ A \ P \ B$ 
    hence  $A = B \vee P \text{ Midpoint } A \ B$ 
    using l7-20-bis by blast
  }
  have  $A \neq C$ 
  using  $\langle A \neq C \rangle$  by blast
  moreover
  have  $\neg P \text{ Midpoint } A \ B$ 
  using False  $\langle M2 \text{ Midpoint } A \ C \rangle \langle P = M2 \rangle$  symmetric-point-uniqueness by blast
  moreover
  have  $\text{Col } A \ P \ B$ 
  by (meson False True  $\langle A \neq B \rangle \langle \text{Cong } P \ A \ P \ B \rangle \langle \text{Cong } P \ A \ P \ C \rangle$ 
    calculation(1) cong2--ncol not-cong-2143)
  ultimately
  have False
  using  $\langle A \neq B \rangle \langle \text{Cong } P \ A \ P \ B \rangle$ 
     $\langle \llbracket \text{Col } A \ P \ B; \text{Cong } P \ A \ P \ B \rrbracket \implies A = B \vee P \text{ Midpoint } A \ B \rangle$ 
    by fastforce
}
have  $M1 \neq M2$ 
using False  $\langle M1 \text{ Midpoint } A \ B \rangle \langle M2 \text{ Midpoint } A \ C \rangle$ 
  symmetric-point-uniqueness by blast
have  $P \ M1 \text{ PerpBisect } A \ B$ 
by (meson  $\langle A \neq B \rangle \langle \text{Cong } P \ A \ P \ B \rangle \langle M1 \text{ Midpoint } A \ B \rangle \langle P = M1 \implies \text{False} \rangle$ 
  cong-mid-perp-bisect not-cong-2143)
have  $P \ M2 \text{ PerpBisect } A \ C$ 
by (meson  $\langle A \neq C \rangle \langle \text{Cong } P \ A \ P \ C \rangle \langle M2 \text{ Midpoint } A \ C \rangle \langle P = M2 \implies \text{False} \rangle$ 
  cong-mid-perp-bisect not-cong-2143)
{
  assume  $\text{Col } P \ M1 \ M2$ 
  {
    assume  $\text{Col } A \ P \ B$  and  $\text{Cong } P \ A \ P \ B$ 
    hence  $A = B \vee P \text{ Midpoint } A \ B$ 
    using l7-20-bis by blast
  }
  moreover
  have  $A \neq B$ 
  by (simp add:  $\langle A \neq B \rangle$ )
  moreover
  have  $\neg P \text{ Midpoint } A \ B$ 
  using  $\langle M1 \text{ Midpoint } A \ B \rangle \langle P = M1 \implies \text{False} \rangle$  l7-17 by blast
  moreover
  have  $\text{Col } A \ P \ B$ 
  by (meson False True  $\langle A \neq C \rangle \langle \text{Cong } P \ A \ P \ B \rangle \langle \text{Cong } P \ A \ P \ C \rangle$ 
    calculation(2) cong2--ncol cong-commutativity)
  moreover

```

```

    have Cong P A P B
      using ⟨Cong P A P B⟩ by auto
    ultimately
    have False
      by blast
  }
  thus ?thesis
    by (meson Cong-cases ⟨Cong P A P B⟩ ⟨Cong P A P C⟩ cong2--ncol)
next
case False
hence ¬ Col A B C
  by simp
thus ?thesis
  by simp
qed
qed
qed
qed

lemma concyclic-not-col-or-eq:
  assumes Concyclic2 A B C A'
  shows A' = C ∨ A' = B ∨ A = B ∨ A = C ∨ A = A' ∨ (¬ Col A B A' ∧ ¬ Col A C A')
  by (metis Concyclic2-def assms cong2--ncol cong-commutativity)

lemma Euler-line-special-case:
  assumes Per A B C and
    G IsGravityCenter A B C and
    H IsOrthocenter A B C and
    P IsCircumcenter A B C
  shows Col G H P
proof -
  have H = B
    using assms(1) assms(3) orthocenter-per by auto
  have P Midpoint A C
    using IsGravityCenter-def assms(1) assms(2) assms(4) circumcenter-per
    not-col-distincts by blast
  have G IsGravityCenter A C B
    by (simp add: assms(2) is-gravity-center-perm-1)
  thus ?thesis
    by (metis is-gravity-center-col ⟨H = B⟩ ⟨P Midpoint A C⟩ not-col-permutation-5)
qed

lemma gravity-center-change-triangle:
  assumes G IsGravityCenter A B C and
    I Midpoint B C and
    I Midpoint B' C' and
    ¬ Col A B' C'
  shows G IsGravityCenter A B' C'
proof -
  obtain G' where G' Midpoint A G
    using MidR-uniq-aux by blast
  hence G Midpoint I G'
    by (meson assms(1) assms(2) is-gravity-center-third)
  thus ?thesis
    using ⟨G' Midpoint A G⟩ assms(3) assms(4) is-gravity-center-third-reci by force
qed

lemma Euler-line:
  assumes ¬ Col A B C and
    G IsGravityCenter A B C and
    H IsOrthocenter A B C and
    P IsCircumcenter A B C
  shows Col G H P
proof (cases Cong A B A C)
case True
hence Cong A B A C

```

```

  by simp
obtain A' where A' Midpoint B C
  using MidR-uniq-aux by blast
obtain B' where B' Midpoint A C
  using MidR-uniq-aux by blast
obtain C' where C' Midpoint A B
  using MidR-uniq-aux by blast
have A A' PerpBisect B C
  by (metis True ⟨A' Midpoint B C⟩ assms(1) cong-mid-perp-bisect
    midpoint-col not-col-distincts not-cong-2143)
have Col G A' A
  by (meson ⟨A' Midpoint B C⟩ assms(2) is-gravity-center-col
    is-gravity-center-perm-3)
show ?thesis
proof (cases P = G)
  case True
  thus ?thesis
    by (simp add: col-trivial-3)
next
  case False
  hence P ≠ G
    by simp
  show ?thesis
  proof (cases P = H)
    case True
    thus ?thesis
      by (simp add: col-trivial-2)
  next
    case False
    hence P ≠ H
      by simp
    show ?thesis
    proof (cases P = A')
      case True
      have Col A H P
      proof -
        have Coplanar B C H P
          using NCol-perm True ⟨A' Midpoint B C⟩ bet-col
            midpoint-bet ncol--ncols by blast
        moreover
        have A H Perp B C
          using IsOrthocenter-def assms(3) by blast
        moreover
        have A P Perp B C
          by (metis True ⟨A' Midpoint B C⟩ ⟨Cong A B A C⟩ assms(1)
            cong-commutativity cong-mid-perp-bisect midpoint-col not-col-distincts
              perp-bisect-perp)
        ultimately
        show ?thesis
          using cop-perp2--col by blast
      qed
      have Coplanar G A B C
        using Coplanar-def NCol-perm ⟨A' Midpoint B C⟩ ⟨Col G A' A⟩
          midpoint-col by blast
      have Coplanar B C G H
        by (meson IsOrthocenter-def ⟨Coplanar G A B C⟩ assms(3)
          coplanar-perm-21 coplanar-perm-3 coplanar-trans-1 perp--coplanar)
      moreover
      have P G Perp B C
        using NCol-perm True ⟨A A' PerpBisect B C⟩ ⟨Col G A' A⟩ ⟨P ≠ G⟩
          col-trivial-2 perp-bisect-perp perp-col2 by blast
      moreover
      have P H Perp B C
        by (meson False IsOrthocenter-def ⟨Col A H P⟩ assms(3)
          col-trivial-2 perp-col2)
      ultimately

```

```

    show ?thesis
    using col-permutation-1 cop-perp2--col by blast
next
case False
hence  $P \neq A'$ 
  by simp
have  $Col\ A\ A'\ H$ 
proof -
  have Coplanar  $B\ C\ A'\ H$ 
    by (meson  $\langle A'\ Midpoint\ B\ C \rangle\ bet\text{--}coplanar\ coplanar\text{--}perm\text{--}2\ midpoint\text{--}bet$ )
  moreover
  have  $A\ A'\ Perp\ B\ C$ 
    using  $\langle A\ A'\ PerpBisect\ B\ C \rangle\ perp\text{--}bisect\text{--}perp$  by blast
  moreover
  have  $A\ H\ Perp\ B\ C$ 
    using IsOrthocenter-def assms(3) by blast
  ultimately
  show ?thesis
    using cop-perp2--col by blast
qed
have  $P\ A'\ PerpBisect\ B\ C$ 
  by (metis False  $\langle A'\ Midpoint\ B\ C \rangle\ assms(1)\ assms(4)\$ 
    circumcenter-perp not-col-distincts)
have  $Col\ A'\ A\ P$ 
proof -
  have Coplanar  $B\ C\ A\ P$ 
    using IsCircumcenter-def assms(4) coplanar-perm-17 by blast
  moreover
  have  $A'\ A\ Perp\ B\ C$ 
    by (meson  $\langle A\ A'\ PerpBisect\ B\ C \rangle\ perp\text{--}bisect\text{--}perp\ perp\text{--}left\text{--}comm$ )
  moreover
  have  $A'\ P\ Perp\ B\ C$ 
    by (simp add:  $\langle P\ A'\ PerpBisect\ B\ C \rangle\ perp\text{--}bisect\text{--}perp\ perp\text{--}left\text{--}comm$ )
  ultimately
  show ?thesis
    using cop-perp2--col by force
qed
have  $A \neq A'$ 
  using  $\langle A'\ Midpoint\ B\ C \rangle\ assms(1)\ midpoint\text{--}col$  by blast
thus ?thesis
  by (meson Col-cases  $\langle Col\ A\ A'\ H \rangle\ \langle Col\ A'\ A\ P \rangle\ \langle Col\ G\ A'\ A \rangle\ col3$ )
qed
qed
qed
next
case False
hence  $\neg Cong\ A\ B\ A\ C$ 
  by simp
obtain  $A'$  where  $P\ Midpoint\ A\ A'$ 
  using symmetric-point-construction by blast
have  $A \neq B$ 
  using assms(1) not-col-distincts by blast
have  $A \neq C$ 
  using assms(1) not-col-distincts by blast
have  $C \neq B$ 
  using False cong-reflexivity by blast
have Concyclic2  $A\ B\ C\ A'$ 
proof -
  have Coplanar  $A\ B\ C\ A'$ 
  proof (cases  $P = A$ )
  case True
  thus ?thesis
    using  $\langle P\ Midpoint\ A\ A' \rangle\ midpoint\text{--}distinct\text{--}1\ ncop\text{--}distincts$  by blast
  next
  case False
  hence  $P \neq A$ 

```

```

    by simp
  have Coplanar A B C A'
  proof -
    have Coplanar B C A P
    using IsCircumcenter-def assms(4) ncoplanar-perm-22 by blast
  moreover
    have Col A P A'
    using ⟨P Midpoint A A'⟩ col-permutation-4 midpoint-col by blast
  ultimately
    show ?thesis
    using False col-cop--cop coplanar-perm-12 by blast
  qed
  thus ?thesis
  by (simp add: ⟨Coplanar A B C A'⟩)
  qed
  moreover
  have Cong P A P B
  using IsCircumcenter-def assms(4) not-cong-2143 by blast
  moreover
  have Cong P A P C
  using assms(4) circumcenter-cong-3 cong-4321 by blast
  moreover
  have Cong P A P A'
  by (meson ⟨P Midpoint A A'⟩ midpoint-cong not-cong-2134)
  ultimately
  show ?thesis
  using Concyclic2-def by blast
  qed
  have A' = C ∨ A' = B ∨ A = B ∨ A = C ∨ A = A' ∨ ¬ Col A B A' ∧ ¬ Col A C A'
  by (metis Col-cases circumcenter-cong-3 ⟨P Midpoint A A'⟩ assms(4)
    circumcenter-cong-1 col3 col-trivial-3 cong-4321 l7-2 l7-20-bis
    l7-9-bis midpoint-col)
  moreover
  {
    assume A' = C
    have Cong P A P B
    using assms(4) circumcenter-cong-1 cong-commutativity by blast
    hence Per A B C
    using ⟨A' = C⟩ ⟨P Midpoint A A'⟩ assms(1) midpoint-thales
    not-col-permutation-5 by blast
    hence Col G H P
    using Euler-line-special-case assms(2) assms(3) assms(4) by blast
  }
  moreover
  {
    assume A' = B
    have Cong P A P C
    using assms(4) circumcenter-cong-3 not-cong-4321 by blast
    hence Per A C B
    using ⟨A' = B⟩ ⟨P Midpoint A A'⟩ assms(1) midpoint-thales by blast
    hence Col G H P
    by (meson Euler-line-special-case IsOrthocenter-cases
      Per-cases assms(2) assms(3) assms(4) is-circumcenter-perm-3
      is-gravity-center-perm-3)
  }
  moreover
  {
    assume A = B
    hence False
    using ⟨A ≠ B⟩ by auto
  }
  moreover
  {
    assume A = C
    hence False
    using ⟨A ≠ C⟩ by auto
  }

```

```

}
moreover
{
  assume  $A' = A$ 
  hence  $Col\ G\ H\ P$ 
  using Cong-perm False  $\langle P\ Midpoint\ A\ A' \rangle$  assms(4) circumcenter-cong-2
    l8-20-2 by blast
}
moreover
{
  assume  $\neg Col\ A\ B\ A' \wedge \neg Col\ A\ C\ A'$ 
  have  $Per\ A\ B\ A'$ 
  by (meson midpoint-thales  $\langle P\ Midpoint\ A\ A' \rangle$ 
     $\langle \neg Col\ A\ B\ A' \wedge \neg Col\ A\ C\ A' \rangle$  assms(4) circumcenter-cong-1
    not-col-permutation-5 not-cong-2143)
  have  $C\ H\ Perp\ A\ B$ 
  using IsOrthocenter-def assms(3) by blast
  have  $A' B\ Perp\ B\ A$ 
  by (metis  $\langle Per\ A\ B\ A' \rangle$   $\langle \neg Col\ A\ B\ A' \wedge \neg Col\ A\ C\ A' \rangle$  l8-2
    not-col-distincts per-perp)
  hence  $C\ H\ Par\ A' B$ 
  proof –
  have  $Coplanar\ A\ B\ C\ A'$ 
  using Concyclic2-def  $\langle Concyclic2\ A\ B\ C\ A' \rangle$  by blast
  moreover
  have  $Coplanar\ A\ B\ C\ B$ 
  using ncop-distincts by blast
  moreover
  have  $Coplanar\ A\ B\ H\ A'$ 
  proof –
  have  $C\ H\ Perp\ A\ B$ 
  by (simp add:  $\langle C\ H\ Perp\ A\ B \rangle$ )
  hence  $Coplanar\ A\ B\ H\ C$ 
  using coplanar-perm-17 perp--coplanar by blast
  thus ?thesis
  by (meson assms(1) calculation(1) coplanar-perm-12
    coplanar-perm-18 coplanar-trans-1 not-col-permutation-2)
  qed
  moreover
  have  $Coplanar\ A\ B\ H\ B$ 
  using ncop-distincts by blast
  moreover
  have  $A' B\ Perp\ A\ B$ 
  by (simp add:  $\langle A' B\ Perp\ B\ A \rangle$  perp-right-comm)
  ultimately
  show ?thesis
  using  $\langle C\ H\ Perp\ A\ B \rangle$  l12-9 by blast
  qed
  have  $B\ H\ Perp\ A\ C$ 
  using IsOrthocenter-def assms(3) by blast
  have  $Cong\ P\ A\ P\ C$ 
  using assms(4) circumcenter-cong-3 not-cong-4321 by blast
  hence  $Per\ A\ C\ A'$ 
  using  $\langle P\ Midpoint\ A\ A' \rangle$   $\langle \neg Col\ A\ B\ A' \wedge \neg Col\ A\ C\ A' \rangle$  col-permutation-5
    midpoint-thales by blast
  have  $A' C\ Perp\ C\ A$ 
  by (metis  $\langle Per\ A\ C\ A' \rangle$   $\langle \neg Col\ A\ B\ A' \wedge \neg Col\ A\ C\ A' \rangle$  l8-2 not-col-distincts per-perp)
  have  $B\ H\ Par\ C\ A'$ 
  proof –
  have  $Coplanar\ A\ C\ B\ C$ 
  using ncop-distincts by auto
  moreover
  have  $Coplanar\ A\ C\ B\ A'$ 
  using Concyclic2-def  $\langle Concyclic2\ A\ B\ C\ A' \rangle$  coplanar-perm-2 by blast
  moreover
  have  $Coplanar\ A\ C\ H\ C$ 

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    using ncop-distincts by blast
  moreover
  have Coplanar A C H A'
    by (meson  $\langle C \ H \ \text{Perp} \ A \ B \rangle$  assms(1) calculation(2) coplanar-perm-8
        l9-30 ncop-distincts not-col-permutation-5 perp--coplanar)
  moreover
  have B H Perp A C
    using  $\langle B \ H \ \text{Perp} \ A \ C \rangle$  by auto
  moreover
  have C A' Perp A C
    using  $\langle A' \ C \ \text{Perp} \ C \ A \rangle$  perp-comm by blast
  ultimately
  show ?thesis
    using l12-9 by blast
qed
have Col G H P
proof (cases Col B H C)
  case True
  hence Col B H C
    by simp
  hence  $H = B \vee H = C$ 
    by (metis  $\langle C \ H \ \text{Perp} \ A \ B \rangle$  assms(3) col-permutation-1 col-transitivity-2
        l8-16-1 orthocenter-per)
  thus ?thesis
    using  $\langle B \ H \ \text{Par} \ C \ A' \rangle$   $\langle C \ H \ \text{Par} \ A' \ B \rangle$  par-distinct by blast
next
case False
  hence  $\neg \text{Col } B \ H \ C$ 
    by simp
  have Parallelogram B H C A'
    by (meson False Par-cases  $\langle B \ H \ \text{Par} \ C \ A' \rangle$   $\langle C \ H \ \text{Par} \ A' \ B \rangle$  par-2-plg)
  obtain I where I Midpoint B C and I Midpoint H A'
    using  $\langle \text{Parallelogram } B \ H \ C \ A' \rangle$  plg-mid by blast
  {
    assume Per A B C
    hence Col G H P
      using Euler-line-special-case assms(2) assms(3) assms(4) by blast
  }
  moreover
  {
    assume  $\neg \text{Per } A \ B \ C$ 
    have  $A' \neq H$ 
      using False  $\langle B \ H \ \text{Par} \ C \ A' \rangle$  par-comm par-id-1 by blast
    {
      assume Col A H A'
      obtain A'' where A'' Midpoint B C
        using  $\langle I \ \text{Midpoint } B \ C \rangle$  by auto
      hence  $I = A''$ 
        using  $\langle I \ \text{Midpoint } B \ C \rangle$  l7-17 by blast
      {
        assume  $A' = H$ 
        hence False
          by (simp add:  $\langle A' \neq H \rangle$ )
      }
    }
    moreover
    {
      assume  $A' \neq H$ 
      {
        assume  $A'' = P$ 
        hence False
          by (metis IsOrthocenter-def  $\langle A'' = P \rangle$   $\langle \text{Col } A \ H \ A' \rangle$   $\langle I = A'' \rangle$ 
               $\langle I \ \text{Midpoint } H \ A' \rangle$   $\langle P \ \text{Midpoint } A \ A' \rangle$  assms(3)
              midpoint-midpoint-col perp-distinct)
      }
    }
  }
  moreover
  {

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assume  $A'' \neq P$ 
have  $P A'' \text{PerpBisect } B C$ 
  using  $\langle A'' \text{Midpoint } B C \rangle \langle A'' \neq P \rangle \langle C \neq B \rangle \text{assms}(4) \text{circumcenter-perp}$  by force
  {
    assume  $A = A''$ 
    hence False
    using  $\langle A'' \text{Midpoint } B C \rangle \text{assms}(1) \text{midpoint-col}$  by auto
  }
moreover
  {
    assume  $A \neq A''$ 
    hence  $A'' A \text{Perp } B C$ 
    by (metis IsOrthocenter-def  $\langle A' \neq H \rangle \langle \text{Col } A H A' \rangle \langle I = A'' \rangle$ 
       $\langle I \text{Midpoint } H A' \rangle \text{assms}(3) \text{col2--eq midpoint-col perp-col2}$ )
    hence  $A'' A \text{PerpBisect } B C$ 
    using  $\langle A'' \text{Midpoint } B C \rangle \text{perp-mid-perp-bisect}$  by auto
    hence False
    using  $\langle \neg \text{Cong } A B A C \rangle \text{not-cong-2143 perp-bisect-cong-2}$  by blast
  }
ultimately have False
by blast
}
ultimately
have False by blast
}
ultimately have False
by blast
}
hence  $G \text{IsGravityCenter } A H A'$ 
using  $\langle I \text{Midpoint } B C \rangle \langle I \text{Midpoint } H A' \rangle \text{assms}(2)$ 
gravity-center-change-triangle by blast
hence  $G \text{IsGravityCenter } A A' H$ 
using calculation is-gravity-center-perm-1 by blast
hence  $\text{Col } G H P$ 
by (meson is-gravity-center-col  $\langle P \text{Midpoint } A A' \rangle \text{not-col-permutation-5}$ )
}
ultimately
show ?thesis
by blast
qed
}
ultimately
show ?thesis
by fastforce
qed

```

```

lemma sesamath-4ieme-G2-ex35:
  assumes  $\neg \text{Col } G A Z$  and
     $\text{Per } G A Z$  and
     $F \text{Midpoint } A Z$  and
     $E \text{Midpoint } G Z$  and
     $R \text{Midpoint } G A$ 
  shows  $\text{Rectangle } F E R A$ 
by (metis Col-def Per-mid-rectangle assms(1) assms(2) assms(3)
  assms(4) assms(5) between-trivial l7-2 rect-permut)

```

```

lemma sesamath-4ieme-G2-ex36-aux:
  assumes  $\neg \text{Col } A B C$  and

```


I Midpoint A B and
J Midpoint A C and
K Midpoint B C
shows *Plg I J K B*
proof –
have *A B Par J K*
using *assms(1) assms(3) assms(4) not-col-distincts*
triangle-mid-par **by** *blast*
hence *B I Par J K*
using *par-col-par-2* **by** (*metis Par-cases assms(2) midpoint-col*
midpoint-distinct-1 not-col-permutation-3)
moreover
have *B C Par I J*
by (*metis Col-def assms(1) assms(2) assms(3) assms(4) between-trivial*
par-right-comm triangle-mid-par-cong-1)
hence *B K Par I J*
by (*metis Col-cases assms(4) midpoint-col midpoint-distinct-1*
par-col-par par-symmetry)
moreover
{
assume *Col B K J*
hence *Col A B J*
by (*meson <A B Par J K> not-col-distincts not-col-permutation-5*
par-not-col-strict par-strict-not-col-2)
hence *False*
by (*metis assms(1) assms(3) col-transitivity-2 midpoint-col*
midpoint-distinct-1 not-col-permutation-1)
}
ultimately
show *?thesis*
by (*meson col-trivial-3 inter-uniqueness-not-par par-2-plg*
par-right-comm par-symmetry parallelogram-to-plg plg-comm2)
qed

lemma *sesamath-4ieme-G2-ex36:*

assumes *B A C isosceles and*
A H Perp B C and
Col B H C and
I Midpoint A B and
J Midpoint A C
shows *Rhombus A I H J*
proof –
have *Col H B C*
using *assms(3) not-col-permutation-4* **by** *blast*
have *H A Perp B C*
using *Perp-cases assms(2)* **by** *blast*
hence $\neg \text{Col } B A C \wedge B \neq H \wedge C \neq H \wedge H \text{ Midpoint } B C \wedge H A B \text{ Cong } A H A C$
using *isosceles-foot--midpoint-conga <Col H B C> assms(1)* **by** *blast*
have *Plg A I H J*
proof –
have *Plg J H I A*
by (*meson <\neg Col B A C \wedge B \neq H \wedge C \neq H \wedge H Midpoint B C \wedge H A B Cong A H A C>*
assms(4) assms(5) l7-2 not-col-permutation-3 sesamath-4ieme-G2-ex36-aux)
thus *?thesis*
by (*metis Plg-def l7-2*)
qed
hence *A H Perp I J*
proof –
have *I J Par B C*
by (*metis Par-cases <Col H B C> <H A Perp B C>*
 $\neg \text{Col } B A C \wedge B \neq H \wedge C \neq H \wedge H \text{ Midpoint } B C \wedge H A B \text{ Cong } A H A C$
assms(4) assms(5) col-par midpoint-distinct-2 not-col-permutation-5
perp-not-par triangle-mid-par-cong)
hence *B C Par I J*
using *Par-cases* **by** *blast*
moreover

```

have B C Perp A H
  using Perp-perm ⟨H A Perp B C⟩ by blast
moreover
have Coplanar A H I J
  by (metis ⟨¬ Col B A C ∧ B ≠ H ∧ C ≠ H ∧ H Midpoint B C ∧ H A B CongA H A C⟩
    assms(4) assms(5) col--coplanar coplanar-perm-2 midpoint-col
    not-col-distincts not-col-permutation-2 par-col-par-2
    par-cong-cop par-left-comm triangle-mid-par)
ultimately
show ?thesis
  using Perp-cases cop-par-perp--perp coplanar-perm-16 by blast
qed
thus ?thesis
  by (simp add: ⟨Plg A I H J⟩ perp-rmb)
qed

```

```

lemma sesamath-4ieme-G2-ex37:
  assumes ¬ Col S E L and
    I Midpoint L S and
    M Midpoint S E and
    A Midpoint E L and
    A P PerpBisect L E and
    Coplanar S E L P
  shows A P Perp I M
proof -
  have L E Par I M
    by (metis assms(1) assms(2) assms(3) l7-2 not-col-distincts triangle-mid-par)
  have A P Perp L E
    using assms(5) perp-bisect-perp by blast
  have I M Perp P A
  proof -
    have L E Par I M
      using ⟨L E Par I M⟩ by force
    moreover
    have L E Perp P A
      using Perp-perm ⟨A P Perp L E⟩ by blast
    moreover
    have Coplanar I M P A
  proof -
    have Col I L S
      by (simp add: assms(2) midpoint-col)
    moreover
    have Col M S E
      by (simp add: assms(3) midpoint-col)
    moreover
    have Col A L E
      using assms(4) col-permutation-5 midpoint-col by presburger
    ultimately
    show ?thesis
      by (meson assms(1) assms(6) coplanar-perm-9 coplanar-pseudo-trans
        ncop--ncols not-col-permutation-3)
  qed
  ultimately
  show ?thesis
    using cop-par-perp--perp by blast
qed
thus ?thesis
  using Perp-perm by blast
qed

```

```

lemma sesamath-4ieme-G2-ex38 :
  assumes  $\neg \text{Col } E \ A \ U$  and
    N Midpoint E A and
    M Midpoint E U and
    L Midpoint U A
  shows  $\exists P. \text{Col } P \ E \ L \wedge P \text{ Midpoint } M \ N$ 
proof -
  have  $\neg \text{Col } U \ E \ A$ 
    using assms(1) not-col-permutation-2 by presburger
  hence Plg M L N E
    by (meson Mid-perm assms(2) assms(3) assms(4) sesamath-4ieme-G2-ex36-aux)
  hence Parallelogram M L N E
    using parallelogram-equiv-plg by blast
  thus ?thesis
    by (meson midpoint-col not-col-permutation-5 plg-mid)
qed

lemma sesamath-4ieme-G2-ex39:
  assumes  $\neg \text{Col } A \ C \ T$  and
    S IsCircumcenter C T A and
    S Midpoint C T and
    Col C H A and
    S H Perp A C
  shows H Midpoint A C
proof -
  have S H Par T A
proof -
  have Coplanar A C S T
    using IsCircumcenter-def assms(2) ncoplanar-perm-15 by blast
  moreover
  have Coplanar A C S A
    using ncop-distincts by blast
  moreover
  have Coplanar A C H T
    using NCol-cases assms(4) ncop--ncol by blast
  moreover
  have Coplanar A C H A
    using ncop-distincts by blast
  moreover
  have Cong C S T S  $\wedge$  Cong T S A S  $\wedge$  Cong A S C S
    using circumcenter-cong assms(2) by blast
  hence Per C A T
    by (metis Col-cases midpoint-thales assms(1) assms(3) not-cong-4321)
  hence T A Perp A C
    by (metis assms(1) l8-2 not-col-distincts per-perp)
  ultimately
  show ?thesis
    using assms(5) l12-9 by blast
qed
thus ?thesis
  by (metis Mid-cases NCol-cases Par-cases assms(1) assms(3) assms(4) triangle-par-mid)
qed

lemma sesamath-4ieme-G2-ex40:
  assumes  $\neg \text{Col } A \ B \ C$  and
    Bet C R B and
    M Midpoint A B and
    N Midpoint A C and
    S Midpoint B R and
    T Midpoint R C
  shows M S Par N T  $\wedge$  Parallelogram M S T N

```

```

proof –
  have  $A R \text{ Par } M S$ 
    by (metis assms(1) assms(2) assms(3) assms(5) bet-col l7-2
      not-col-permutation-2 triangle-mid-par)
  hence  $A R \text{ Par } N T$ 
    using assms(4) assms(6) par-neq1 triangle-mid-par by blast
  hence  $M S \text{ Par } N T$ 
    by (meson  $\langle A R \text{ Par } M S \rangle$  not-par-one-not-par par-symmetry)
  moreover
  have Parallelogram  $M S T N$ 
  proof (cases  $R = B$ )
    case True
      thus ?thesis
      using  $\langle A R \text{ Par } N T \rangle$  assms(3) assms(4) assms(5) assms(6)
        par-neq1 varignon-bis by blast
    next
      case False
      thus ?thesis
      by (metis assms(2) assms(3) assms(4) assms(5) assms(6) bet-neq12--neq varignon-bis)
  qed
  ultimately
  show ?thesis
    by blast
qed

```

lemma *sesamath-4ieme-G2-ex41* :

```

assumes  $\neg \text{Col } T L H$  and
   $E \text{ Midpoint } T L$  and
   $P \text{ Midpoint } T H$  and
   $A \neq T$  and
   $A \neq P$  and
   $A \neq E$  and
   $A \neq S$  and
   $Bet\ T\ A\ S$  and
   $Bet\ P\ A\ E$  and
   $Bet\ H\ S\ L$  and
   $S \neq H$  and
   $S \neq L$ 
shows  $S A E \text{ CongA } T S H \wedge A \text{ Midpoint } T S$ 

```

```

proof –
  have  $H L \text{ OS } T T$ 
    using Col-cases assms(1) one-side-reflexivity by blast
  have  $H S \text{ OS } T T$ 
    by (metis Bet-cases Col-def  $\langle H L \text{ OS } T T \rangle$  assms(10) assms(11) col-one-side)
  hence  $\neg \text{Col } H S T$ 
    using col123--nos by blast
  hence  $T S \text{ OS } H H$ 
    using invert-one-side one-side-reflexivity by blast
  hence  $T A \text{ OS } H H$ 
    by (metis Bet-cases Col-def assms(4) assms(8) col-one-side)
  have  $Bet\ T\ P\ H$ 
    by (simp add: assms(3) midpoint-bet)
  hence  $T \text{ Out } P H$ 
    using assms(1) assms(3) midpoint-out not-col-distincts by presburger
  have  $T A \text{ OS } H P$ 
    by (meson Out-cases  $\langle T A \text{ OS } H H \rangle \langle T \text{ Out } P H \rangle$  out-out-one-side)
  hence  $T A \text{ OS } P H$ 
    by (simp add: one-side-symmetry)
  have  $H L \text{ Par } P E$ 
    by (metis assms(1) assms(2) assms(3) l7-2 not-col-distincts triangle-mid-par)
  have  $H L \text{ Par } P A$ 
    by (metis  $\langle H L \text{ Par } P E \rangle$  assms(5) assms(9) bet-col not-par-not-col
      not-par-one-not-par par-neq2)
  hence  $H S \text{ Par } P A$ 

```

by (metis assms(1) assms(10) assms(11) bet-col not-col-distincts
 par-col2-par par-reflexivity par-trans)
 have T Out A S
 by (simp add: assms(4) assms(8) bet-out)
 have P A T CongA H S T
 by (meson Par-cases ⟨H S Par P A⟩ ⟨T A OS P H⟩ assms(4) assms(8)
 bet-out l12-22-a)
 have P A T CongA E A S
 using assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) l11-14 by force
 hence H S T CongA E A S
 by (meson ⟨P A T CongA H S T⟩ not-conga not-conga-sym)
 moreover
 have A Midpoint S T
 by (metis NCol-cases ⟨H S Par P A⟩ ⟨¬ Col H S T⟩ assms(3) assms(8)
 bet-col l7-2 par-comm triangle-par-mid)
 ultimately
 show ?thesis
 by (meson conga-comm conga-sym l7-2)
 qed

lemma sesamath-4ieme-G2-ex42:
 assumes ¬ Col A B C and
 G IsGravityCenter A B C and
 I Midpoint A C and
 J Midpoint A B and
 K Midpoint B G and
 L Midpoint C G
 shows Parallelogram I J K L
 proof –
 have G ≠ C
 using assms(2) is-gravity-center-diff-3 by auto
 moreover
 have G Midpoint J L
 proof –
 have J Midpoint B A
 by (simp add: assms(4) l7-2)
 moreover
 have G IsGravityCenter C B A
 using assms(2) is-gravity-center-perm-5 by blast
 ultimately show ?thesis
 using ⟨J Midpoint B A⟩ assms(6) is-gravity-center-third by blast
 qed
 ultimately show ?thesis
 by (metis assms(1) assms(3) assms(4) assms(5) assms(6)
 bet-col between-trivial l7-2 midpoint-distinct-3 varignon-aux-aux)
 qed

lemma sesamath-4ieme-G2-ex44-1:
 assumes ParallelogramStrict A B C D and
 I Midpoint A D and
 J Midpoint B C
 shows Parallelogram B J D I
 by (metis MidR-uniq-aux Mid-cases assms(1) assms(2) assms(3)
 half-plgs-R2 mid-plg plgs-diff plgs-permut)

lemma sesamath-4ieme-G2-ex45:
 assumes ¬ Col A B C and
 I Midpoint B C and

```

  Col I A M and
  A Midpoint I M and
  J Midpoint A I and
  J K Par A C and
  Col K I C and
  Col G C A and
  Col G M K
shows K Midpoint I C  $\wedge$  A K Par M C  $\wedge$  G IsGravityCenter C M I
proof -
  have  $\neg$  Col C I A
    by (metis assms(1) assms(2) col-trivial-2 l6-21 midpoint-col
        midpoint-distinct-1 not-col-permutation-1)
  have K Midpoint I C
    by (meson Par-cases  $\langle \neg$  Col C I A  $\rangle$  assms(5) assms(6) assms(7) l7-2
        not-col-permutation-5 triangle-par-mid)
  moreover
  have A K Par M C  $\wedge$  G IsGravityCenter C M I
  proof (cases I = M)
    case True
      hence A = B
        by (metis  $\langle \neg$  Col C I A  $\rangle$  assms(4) l7-17 l7-3-2 not-col-distincts)
      hence False
        using assms(1) not-col-distincts by auto
      hence A K Par M C
        by blast
    moreover
    have G IsGravityCenter C M I
      using True  $\langle \neg$  Col C I A  $\rangle$  assms(4) l7-17-bis l7-3-2 not-col-distincts by blast
    ultimately
    show ?thesis
      by blast
  next
    case False
      have  $\neg$  Col I M C
        by (meson False  $\langle \neg$  Col C I A  $\rangle$  assms(3) col-permutation-2 col-trivial-3 colx)
      have K Midpoint C I
        by (simp add: calculation l7-2)
      have M C Par A K
        by (metis  $\langle$  K Midpoint C I  $\rangle$   $\langle \neg$  Col I M C  $\rangle$  assms(4) l7-2
            not-col-distincts triangle-mid-par)
      hence A K Par M C
        by (simp add: par-symmetry)
      moreover
      have G IsGravityCenter C M I
        using IsGravityCenter-def Mid-cases  $\langle$  K Midpoint C I  $\rangle$   $\langle \neg$  Col I M C  $\rangle$ 
            assms(4) assms(8) assms(9) col-permutation-3 by blast
      ultimately
      show ?thesis
        by blast
  qed
  ultimately
  show ?thesis
    by blast
qed

```

lemma *sesamath-4ieme-G2-ex47* :

```

  assumes  $\neg$  Col A C C' and
     $\neg$  Col A A' C' and
    Col B' A' C' and
    B Midpoint A C and
    A A' Par B B' and
    A A' Par C C'
  shows B' Midpoint A' C'
proof -

```

```

obtain  $x$  where  $x$  Midpoint  $A$   $C'$ 
  using MidR-uniq-aux by blast
have  $B$   $B'$  Par  $A$   $A'$ 
  using assms(5) par-symmetry by blast
hence  $B$   $B'$  Par  $C$   $C'$ 
  using assms(6) par-trans by blast
have  $C$   $C'$  Par  $B$   $x$ 
  by (metis  $\langle x$  Midpoint  $A$   $C'$   $\rangle$  assms(1) assms(4) l7-2 not-col-distincts
    triangle-mid-par)
hence  $B$   $B'$  Par  $B$   $x$ 
  using  $\langle B$   $B'$  Par  $C$   $C'$   $\rangle$  par-trans by blast
hence Col  $B$   $B'$   $x$ 
  by (simp add: par-id)
have  $A$   $A'$  Par  $B$   $x$ 
  by (meson  $\langle B$   $B'$  Par  $B$   $x$   $\rangle$  assms(5) par-trans)
show ?thesis
proof (cases  $x = B'$ )
  case True
  have Col  $A$   $C'$   $A'$ 
    by (metis True  $\langle x$  Midpoint  $A$   $C'$   $\rangle$  assms(3) col2--eq col-permutation-1
      midpoint-col midpoint-distinct-1)
  thus ?thesis
    using Col-cases assms(2) by blast
next
  case False
  have  $B$   $x$  Par  $B'$   $B$ 
    using Par-perm  $\langle B$   $B'$  Par  $B$   $x$   $\rangle$  by blast
  hence  $B$   $x$  Par  $B'$   $x$ 
    using False  $\langle B$   $B'$  Par  $B$   $x$   $\rangle$  par-col-par par-id-1 by presburger
  hence  $A$   $A'$  Par  $B'$   $x$ 
    using  $\langle A$   $A'$  Par  $B$   $x$   $\rangle$  par-trans by blast
  thus ?thesis
    using  $\langle x$  Midpoint  $A$   $C'$   $\rangle$  assms(2) assms(3) not-col-permutation-4
      par-left-comm triangle-par-mid by blast
qed
qed

end
end

```

theory *Highschool-Euclidean-2D*

imports

Highschool-Euclidean
Tarski-Euclidean-2D

begin

19 Highschool Euclidean 2D

context *Tarski-Euclidean-2D*

begin

19.1 Definitions

19.2 Propositions

lemma *NewEX1*:

assumes *Square* A B C D **and**
Cong K A K C **and**
Cong K A A C
shows *Col* B K D

proof –

obtain M **where** M *Midpoint* A C

```

    using MidR-uniq-aux by blast
have Cong B A B C
    using Cong-cases Square-def assms(1) by auto
hence Per A M B
    using Per-def ⟨M Midpoint A C⟩ l8-2 by blast
hence A M Perp M B
    by (metis Square-def Square-not-triv-3 ⟨M Midpoint A C⟩ assms(1)
        cong-diff-2 is-midpoint-id l8-7 midpoint-thales-reci per-perp rect-per)
have Cong D A D C
    using Square-def assms(1)
    by (meson Square-Rhombus cong-inner-transitivity not-cong-4312 rmb-cong)
hence Per A M D
    using Per-def Per-perm ⟨M Midpoint A C⟩ cong-left-commutativity by blast
hence A M Perp M D
    by (metis Cong-cases Square-def ⟨A M Perp M B⟩ ⟨M Midpoint A C⟩
        assms(1) between-cong between-trivial2 midpoint-cong
        midpoint-thales-reci-2 per-perp rect-per)
moreover
have Per A M K
    using Per-def ⟨M Midpoint A C⟩ assms(2) l8-2 by blast
hence A M Perp M K
    by (metis ⟨A M Perp M B⟩ ⟨M Midpoint A C⟩ assms(3)
        between-cong midpoint-bet midpoint-distinct-2
        not-cong-2134 per-perp perp-distinct)
ultimately
show ?thesis
    by (metis Col-def ⟨Cong B A B C⟩ ⟨Cong D A D C⟩ assms(2) assms(3)
        cong-diff perp2--col perp-not-col upper-dim)
qed

end
end

```

theory Tarski-Non-Euclidean

imports
Tarski-Postulate-Parallels

begin

20 Tarski Non Euclidean

20.1 Definitions

```

locale Tarski-Non-Euclidean = Tarski-neutral-dimensionless +
  fixes A0 B0 C0 D0 T0
  assumes NERule1: Bet A0 D0 T0
    and NERule2: Bet B0 D0 C0
    and NERule3: A0 ≠ D0
    and not-tarski-s-parallel-postulate-aux:
      ∀ X Y. ((Bet A0 B0 X ∧ Bet A0 C0 Y) ⟶ ¬ Bet X T0 Y)

```

context Tarski-Non-Euclidean

begin

20.2 Propositions

```

lemma Tarski-Pre-Non-Euclidean-aux:
  shows ∃ A B C D T. ¬ ((Bet A D T ∧ Bet B D C ∧ A ≠ D)
    ⟶
      (∃ X Y. Bet A B X ∧ Bet A C Y ∧ Bet X T Y))
  using NERule1 NERule2 NERule3 not-tarski-s-parallel-postulate-aux by blast

```



```

lemma NPost01:
  shows  $\neg$  Postulate01
  using Postulate01-def Tarski-Pre-Non-Euclidean-aux tarski-s-parallel-postulate-def by blast

lemma NPost02:
  shows  $\neg$  Postulate02
  using InterAx5 NPost01 by blast

lemma NPost05:
  shows  $\neg$  Postulate05
  using NPost02 Postulate02-def Postulate05-def par-trans-implies-playfair by blast

lemma NPost06:
  shows  $\neg$  Postulate06
  by (simp add: NPost02 equivalent-postulates-without-decidability-of-intersection-of-lines)

lemma NPost07:
  shows  $\neg$  Postulate07
  using NPost02 equivalent-postulates-without-decidability-of-intersection-of-lines by blast

lemma NPost08:
  shows  $\neg$  Postulate08
  using NPost06 equivalent-postulates-without-decidability-of-intersection-of-lines by blast

lemma NPost09:
  shows  $\neg$  Postulate09
  by (simp add: NPost02 equivalent-postulates-without-decidability-of-intersection-of-lines)

lemma NPost10:
  shows  $\neg$  Postulate10
  using NPost05 equivalent-postulates-without-decidability-of-intersection-of-lines by blast

lemma NPost11:
  shows  $\neg$  Postulate11
  using NPost02 equivalent-postulates-without-decidability-of-intersection-of-lines by fastforce

lemma NPost12:
  shows  $\neg$  Postulate12
  using NPost02 Postulate02-def Postulate12-def playfair-bis--playfair by blast

lemma NPost13:
  shows  $\neg$  Postulate13
  using Cycle-2 NPost01 by blast

lemma NPost14:
  shows  $\neg$  Postulate14
  using Cycle-2 NPost13 by blast

lemma NPost15:
  shows  $\neg$  Postulate15
  using Cycle-1 NPost02 by blast

lemma NPost16:
  shows  $\neg$  Postulate16
  using Cycle-2 NPost01 by blast

lemma NPost17:
  shows  $\neg$  Postulate17
  using Cycle-2 NPost01 by blast

lemma NPost18:
  shows  $\neg$  Postulate18
  using Cycle-2 NPost01 by blast

lemma NPost19:

```

shows $\neg \text{Postulate19}$
using *Cycle-2 NPost01* **by** *blast*

lemma *NPost20*:
shows $\neg \text{Postulate20}$
using *Cycle-2 NPost01* **by** *blast*

theorem *not-tarski-s-parallel-postulate-thm*:
shows $\exists A B C D T. (\text{Bet } A D T \wedge \text{Bet } B D C \wedge A \neq D \wedge$
 $(\forall X Y. (\text{Bet } A B X \wedge \text{Bet } A C Y) \longrightarrow \neg \text{Bet } X T Y))$
using *Tarski-Pre-Non-Euclidean-aux* **by** *blast*

theorem *not-playfair-s-postulate-thm*:
shows $\exists A1 A2 B1 B2 C1 C2 P. A1 A2 \text{ Par } B1 B2 \wedge$
 $\text{Col } P B1 B2 \wedge A1 A2 \text{ Par } C1 C2 \wedge \text{Col } P C1 C2 \wedge$
 $(\neg \text{Col } C1 B1 B2 \vee \neg \text{Col } C2 B1 B2)$
using *NPost02 Postulate02-def playfair-s-postulate-def* **by** *blast*

theorem *not-postulate-of-transitivity-of-parallelism-thm*:
shows $\exists A1 A2 B1 B2 C1 C2. A1 A2 \text{ Par } B1 B2 \wedge B1 B2 \text{ Par } C1 C2 \wedge \neg (A1 A2 \text{ Par } C1 C2)$
using *NPost05 Postulate05-def*
postulate-of-transitivity-of-parallelism-def **by** *blast*

theorem *not-midpoint-converse-postulate-thm*:
shows $\exists A B C P Q. \neg \text{Col } A B C \wedge P \text{ Midpoint } B C \wedge A B \text{ Par } Q P \wedge \text{Col } A C Q \wedge \neg Q \text{ Midpoint } A C$
using *NPost06 midpoint-converse-postulate-def*
Postulate06-def **by** *blast*

theorem *not-alternate-interior-angles-postulate-thm*:
shows $\exists A B C D. A C \text{ TS } B D \wedge A B \text{ Par } C D \wedge \neg B A C \text{ Cong } A D C A$
using *NPost07 Postulate07-def alternate-interior-angles-postulate-def* **by** *blast*

theorem *not-consecutive-interior-angles-postulate-thm*:
shows $\exists A B C D. B C \text{ OS } A D \wedge A B \text{ Par } C D \wedge \neg A B C \text{ Supp } A B C D$
using *NPost08 Postulate08-def consecutive-interior-angles-postulate-def* **by** *blast*

theorem *not-perpendicular-transversal-postulate-thm*:
shows $\exists A B C D P Q. A B \text{ Par } C D \wedge A B \text{ Perp } P Q \wedge \text{Coplanar } C D P Q \wedge \neg C D \text{ Perp } P Q$
using *NPost09 Postulate09-def perpendicular-transversal-postulate-def* **by** *blast*

theorem *not-postulate-of-parallelism-of-perpendicular-transversals-thm*:
shows $\exists A1 A2 B1 B2 C1 C2 D1 D2.$
 $A1 A2 \text{ Par } B1 B2 \wedge A1 A2 \text{ Perp } C1 C2 \wedge B1 B2 \text{ Perp } D1 D2 \wedge$
 $\text{Coplanar } A1 A2 C1 D1 \wedge \text{Coplanar } A1 A2 C1 D2 \wedge$
 $\text{Coplanar } A1 A2 C2 D1 \wedge \text{Coplanar } A1 A2 C2 D2 \wedge \neg C1 C2 \text{ Par } D1 D2$
using *NPost10 Postulate10-def*
postulate-of-parallelism-of-perpendicular-transversals-def **by** *blast*

theorem *not-universal-posidonius-postulate*:
shows $\exists A1 A2 A3 A4 B1 B2 B3 B4.$
 $A1 A2 \text{ Par } B1 B2 \wedge \text{Col } A1 A2 A3 \wedge \text{Col } B1 B2 B3 \wedge A1 A2 \text{ Perp } A3 B3 \wedge$
 $\text{Col } A1 A2 A4 \wedge \text{Col } B1 B2 B4 \wedge A1 A2 \text{ Perp } A4 B4 \wedge \neg \text{Cong } A3 B3 A4 B4$
using *NPost11 Postulate11-def universal-posidonius-postulate-def* **by** *blast*

theorem *not-alternative-playfair-s-postulate-thm*:
shows $\exists A1 A2 B1 B2 C1 C2 P.$
 $P \text{ Perp } A1 A2 B1 B2 \wedge \neg \text{Col } A1 A2 P \wedge \text{Col } P B1 B2 \wedge$
 $\text{Coplanar } A1 A2 B1 B2 \wedge A1 A2 \text{ Par } C1 C2 \wedge \text{Col } P C1 C2 \wedge$
 $(\neg \text{Col } C1 B1 B2 \vee \neg \text{Col } C2 B1 B2)$
using *NPost02 Postulate02-def alternative-playfair-s-postulate-def*
playfair-bis--playfair **by** *blast*

theorem *not-proclus-postulate-thm*:
shows $\exists A B C D P Q. (\forall Y. A B \text{ Par } C D \wedge \text{Col } A B P \wedge$
 $\neg \text{Col } A B Q \wedge \text{Coplanar } C D P Q \wedge (\neg \text{Col } P Q Y \vee \neg \text{Col } C D Y))$
using *NPost13 proclus-postulate-def Postulate13-def* **by** *blast*

theorem *not-alternative-proclus-postulate-thm*:
shows $\exists A B C D P Q. (\forall Y. P \text{ Perp2 } A B C D \wedge \neg \text{Col } C D P \wedge \text{Coplanar } A B C D \wedge \text{Col } A B P \wedge$
 $\neg \text{Col } A B Q \wedge \text{Coplanar } C D P Q \wedge (\neg \text{Col } P Q Y \vee \neg \text{Col } C D Y))$
using *NPost14 alternative-proclus-postulate-def Postulate14-def* **by** *blast*

theorem *not-triangle-circumscription-principle-thm*:
shows $\exists A B C. \forall D. (\neg \text{Col } A B C \wedge (\neg \text{Cong } A D B D \vee \neg \text{Cong } A D C D \vee \neg \text{Coplanar } A B C D))$
using *NPost15 Postulate15-def triangle-circumscription-principle-def* **by** *blast*

theorem *not-inverse-projection-postulate-thm*:
shows $\exists A B C P Q. \forall Y.$
 $\text{Acute } A B C \wedge B \text{ Out } A P \wedge P \neq Q \wedge \text{Per } B P Q \wedge \text{Coplanar } A B C Q \wedge$
 $(\neg B \text{ Out } C Y \vee \neg \text{Col } P Q Y)$
using *NPost16 Postulate16-def inverse-projection-postulate-def* **by** *blast*

theorem *not-euclid-5-thm*:
shows $\exists P Q R S T U. \forall I.$
 $\text{BetS } P T Q \wedge \text{BetS } R T S \wedge \text{BetS } Q U R \wedge \neg \text{Col } P Q S \wedge$
 $\text{Cong } P T Q T \wedge \text{Cong } R T S T \wedge (\neg \text{BetS } S Q I \vee \neg \text{BetS } P U I)$
using *NPost17 Postulate17-def euclid-5-def* **by** *blast*

theorem *not-strong-parallel-postulate-thm*:
shows $\exists P Q R S T U. \forall I.$
 $\text{BetS } P T Q \wedge \text{BetS } R T S \wedge \neg \text{Col } P R U \wedge \text{Coplanar } P Q R U \wedge$
 $\text{Cong } P T Q T \wedge \text{Cong } R T S T \wedge (\neg \text{Col } S Q I \vee \neg \text{Col } P U I)$
using *NPost18 Postulate18-def strong-parallel-postulate-def* **by** *blast*

theorem *not-alternative-strong-parallel-postulate-thm*:
shows $\exists A B C D P Q R. \forall Y.$
 $B C \text{ OS } A D \wedge A B C B C D \text{ SumA } P Q R \wedge \neg \text{Bet } P Q R \wedge (\neg \text{Col } B A Y \vee \neg \text{Col } C D Y)$
using *NPost19 Postulate19-def alternative-strong-parallel-postulate-def* **by** *blast*

theorem *not-euclid-s-parallel-postulate-thm*:
shows $\exists A B C D P Q R. \forall Y.$
 $B C \text{ OS } A D \wedge \text{SAMS } A B C B C D \wedge A B C B C D \text{ SumA } P Q R \wedge$
 $\neg \text{Bet } P Q R \wedge (\neg B \text{ Out } A Y \vee \neg C \text{ Out } D Y)$
using *NPost20 Postulate20-def euclid-s-parallel-postulate-def* **by** *blast*

end
end

theory *Tarski-Non-Euclidean-Aristotle*

imports
Tarski-Non-Euclidean

begin

21 Tarski Non Euclidean Aristotle

21.1 Definitions

locale *Tarski-Non-Euclidean-Aristotle* = *Tarski-Non-Euclidean* +
assumes *aristotle: aristotle-s-axiom*

context *Tarski-Non-Euclidean-Aristotle*

begin

21.2 Propositions

lemma *NPost03*:
shows $\neg \text{Postulate03}$

```

using NPost02 Postulate02-def Postulate03-def aristotle aristotle--greenberg
playfair-bis--playfair triangle--playfair-bis by blast

lemma NPost21:
  shows  $\neg$  Postulate21
  using Cycle-3 NPost03 by blast

lemma NPost22:
  shows  $\neg$  Postulate22
  using Cycle-3 NPost21 by blast

lemma NPost23:
  shows  $\neg$  Postulate23
  using Cycle-3 NPost22 by blast

lemma NPost24:
  shows  $\neg$  Postulate24
  using Cycle-3 NPost23 by blast

lemma NPost25:
  shows  $\neg$  Postulate25
  using Cycle-3 NPost24 by blast

lemma NPost26:
  shows  $\neg$  Postulate26
  using Cycle-3 NPost25 by blast

lemma NPost27:
  shows  $\neg$  Postulate27
  using Cycle-3 NPost26 by blast

lemma NPost28:
  shows  $\neg$  Postulate28
  using Cycle-3 NPost27 by blast

lemma NPost29:
  shows  $\neg$  Postulate29
  using Cycle-3 NPost28 by blast

lemma NPost30:
  shows  $\neg$  Postulate30
  using Cycle-3 NPost29 by blast

theorem not-triangle-postulate-thm:
  shows  $\exists A B C D E F. A B C \text{ TriSum} A D E F \wedge \neg \text{Bet } D E F$ 
  using NPost03 Postulate03-def triangle-postulate-def by blast

theorem not-postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-thm:
  shows  $\forall A B C D E F. A B C \text{ TriSum} A D E F \wedge \text{Bet } D E F \longrightarrow \text{Col } A B C$ 
  using NPost21 Postulate21-def
  postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-def by blast

theorem not-posidonius-postulate-thm:
  shows  $\forall A_1 A_2 B_1 B_2. \text{Col } A_1 A_2 B_1 \vee B_1 = B_2 \vee \neg \text{Coplanar } A_1 A_2 B_1 B_2 \vee$ 

$$(\exists A_3 A_4 B_3 B_4. \text{Col } A_1 A_2 A_3 \wedge \text{Col } B_1 B_2 B_3 \wedge A_1 A_2 \text{ Perp } A_3 B_3 \wedge$$


$$\text{Col } A_1 A_2 A_4 \wedge \text{Col } B_1 B_2 B_4 \wedge A_1 A_2 \text{ Perp } A_4 B_4 \wedge$$


$$\neg \text{Cong } A_3 B_3 A_4 B_4)$$

  using NPost22 Postulate22-def posidonius-postulate-def by blast

theorem not-posidonius-postulate-thm-1:
  shows  $\forall A_1 A_2 B. \text{Col } A_1 A_2 B \vee (\exists A_3 A_4 B_3 B_4. \text{Col } A_1 A_2 A_3 \wedge \text{Col } B B B_3 \wedge A_1 A_2 \text{ Perp } A_3 B_3 \wedge$ 

$$\text{Col } A_1 A_2 A_4 \wedge \text{Col } B B B_4 \wedge A_1 A_2 \text{ Perp } A_4 B_4 \wedge$$


$$\neg \text{Cong } A_3 B_3 A_4 B_4)$$


proof —
{
  fix  $A_1 A_2 B$ 

```

```

have Coplanar A1 A2 B B
  using ncop-distincts by blast
assume ¬ Col A1 A2 B
hence (∃ A3 A4 B3 B4. Col A1 A2 A3 ∧ Col B B B3 ∧ A1 A2 Perp A3 B3 ∧
      Col A1 A2 A4 ∧ Col B B B4 ∧ A1 A2 Perp A4 B4 ∧
      ¬ Cong A3 B3 A4 B4)
  by (metis(full-types) col-trivial-1 ncop-distincts not-posidonius-postulate-thm)
}
thus ?thesis
  by blast
qed

theorem not-postulate-of-existence-of-similar-triangles-thm:
shows ∀ A B C D E F. (Col A B C ∨ Cong A B D E ∨ ¬ A B C CongA D E F ∨
    ¬ B C A CongA E F D ∨ ¬ C A B CongA F D E)
using NPost23 Postulate23-def postulate-of-existence-of-similar-triangles-def by blast

theorem not-thales-postulate-thm:
shows ∃ A B C M. M Midpoint A B ∧ Cong M A M C ∧ ¬ Per A C B
using NPost24 Postulate24-def thales-postulate-def by blast

theorem not-thales-converse-postulate-thm:
shows ∃ A B C M. M Midpoint A B ∧ Per A C B ∧ ¬ Cong M A M C
using NPost25 Postulate25-def thales-converse-postulate-def by blast

theorem not-existential-thales-postulate-thm:
shows ∀ A B C M. (M Midpoint A B ∧ Cong M A M C ∧ Per A C B) → Col A B C
using NPost26 Postulate26-def existential-thales-postulate-def by blast

theorem not-postulate-of-right-saccheri-quadrilaterals-thm:
shows ∃ A B C D. Saccheri A B C D ∧ ¬ Per A B C
using Cycle-3 NPost03 Postulate28-def ex-saccheri
    postulate-of-existence-of-a-right-saccheri-quadrilateral-def by blast

theorem not-postulate-of-existence-of-a-right-saccheri-quadrilateral-thm:
shows ∀ A B C D. ¬ (Saccheri A B C D ∧ Per A B C)
using Cycle-3 NPost26 Postulate28-def
    postulate-of-existence-of-a-right-saccheri-quadrilateral-def by blast

theorem not-postulate-of-right-lambert-quadrilaterals-thm:
shows ∃ A B C D. Lambert A B C D ∧ ¬ Per B C D
using Cycle-3 NPost26 Postulate30-def postulate-of-right-lambert-quadrilaterals-def
    rectangle-principle--rectangle-existence by blast

theorem not-postulate-of-existence-of-a-right-lambert-quadrilateral-thm:
shows ∀ A B C D. ¬ (Lambert A B C D ∧ Per B C D)
using Cycle-3 NPost29 Postulate30-def
    postulate-of-existence-of-a-right-lambert-quadrilateral-def by blast

end
end

theory Tarski-Non-Euclidean-Archimedean

imports
  Tarski-Non-Euclidean-Aristotle

begin

```

22 Tarski Non Euclidean

22.1 Definitions

locale *Tarski-Non-Euclidean-Archimedean* = *Tarski-Non-Euclidean-Aristotle* +
assumes *archimedes*: *archimedes-axiom*

context *Tarski-Non-Euclidean-Archimedean*

begin

22.2 Propositions

lemma *NPost04*:
shows $\neg$ *Postulate04*
using *NPost03 Thm10 archimedes* **by** *blast*

lemma *NPost31*:
shows $\neg$ *Postulate31*
by (*simp add: NPost04 P31-P04*)

lemma *NPost32*:
shows $\neg$ *Postulate32*
using *NPost31 P31-P32* **by** *blast*

lemma *NPost33*:
shows $\neg$ *Postulate33*
using *NPost04 P04-P33* **by** *auto*

lemma *NPost34*:
shows $\neg$ *Postulate34*
using *NPost01 Thm10 archimedes* **by** *blast*

lemma *NPost35*:
shows $\neg$ *Postulate35*
using *NPost01 Thm10-4 archimedes* **by** *blast*

theorem *not-bachmann-s-lotschnittaxiom*:
shows $\exists P Q R P1 R1. P \neq Q \wedge Q \neq R \wedge \text{Per } P Q R \wedge \text{Per } Q P P1 \wedge \text{Per } Q R R1 \wedge$
 $\text{Coplanar } P Q R P1 \wedge \text{Coplanar } P Q R R1 \wedge$
 $(\forall S. \neg \text{Col } P P1 S \vee \neg \text{Col } R R1 S)$
using *NPost04 Postulate04-def bachmann-s-lotschnittaxiom-def* **by** *blast*

theorem *not-weak-inverse-projection-postulate*:
shows $\exists A B C D E F P Q Y. \text{Acute } A B C \wedge \text{Per } D E F \wedge A B C A B C \text{Sum } A D E F \wedge$
 $B \text{Out } A P \wedge P \neq Q \wedge \text{Per } B P Q \wedge \text{Coplanar } A B C Q \wedge$
 $(\neg B \text{Out } C Y \vee \neg \text{Col } P Q Y)$
using *NPost31 Postulate31-def weak-inverse-projection-postulate-def* **by** *blast*

theorem *not-weak-tarski-s-parallel-postulate*:
shows $\exists A B C T. \text{Per } A B C \wedge T \text{InAngle } A B C \wedge$
 $(\forall X Y. \neg B \text{Out } A X \vee \neg B \text{Out } C Y \vee \neg \text{Bet } X T Y)$
using *NPost32 Postulate32-def weak-tarski-s-parallel-postulate-def* **by** *blast*

theorem *not-weak-triangle-circumscription-principle*:
shows $\exists A B C A1 A2 B1 B2. \neg \text{Col } A B C \wedge \text{Per } A C B \wedge A1 A2 \text{PerpBisect } B C \wedge$
 $B1 B2 \text{PerpBisect } A C \wedge \text{Coplanar } A B C A1 \wedge \text{Coplanar } A B C A2 \wedge$
 $\text{Coplanar } A B C B1 \wedge \text{Coplanar } A B C B2 \wedge$
 $(\forall I. \neg \text{Col } A1 A2 I \vee \neg \text{Col } B1 B2 I)$
using *NPost33 Postulate33-def weak-triangle-circumscription-principle-def* **by** *blast*

theorem *not-legendre-s-parallel-postulate*:
fixes *A B C*
assumes $\neg \text{Col } A B C$
and *Acute A B C*
shows $\exists T. T \text{InAngle } A B C \wedge (\forall X Y. \neg B \text{Out } A X \vee \neg B \text{Out } C Y \vee \neg \text{Bet } X T Y)$
using *assms(1,2) NPost34 Postulate34-def legendre-s-parallel-postulate-def* **by** *blast*

```

theorem not-existential-playfair-s-postulate:
  assumes  $\neg \text{Col } A1 \ A2 \ P$ 
  shows  $\exists \ B1 \ B2 \ C1 \ C2. \ A1 \ A2 \ \text{Par } B1 \ B2 \wedge \text{Col } P \ B1 \ B2 \wedge \ A1 \ A2 \ \text{Par } C1 \ C2 \wedge \text{Col } P \ C1 \ C2 \wedge$ 
     $(\neg \text{Col } C1 \ B1 \ B2 \vee \neg \text{Col } C2 \ B1 \ B2)$ 
  using assms NPost35 Postulate35-def existential-playfair-s-postulate-def by blast

end
end

```

theory *Gupta-Neutral*

```

imports
  Tarski-Neutral

```

begin

23 Gupta inspired variant of Tarski - Geometry

23.1 Axioms - neutral dimension less

```

locale Gupta-neutral-dimensionless =
  fixes GPA GPB GPC :: 'p
  and BetG :: 'p  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool
  and CongG :: 'p  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  'p  $\Rightarrow$  bool

```

assumes *cong-pseudo-reflexivityG*: $\forall \ a \ b.$

CongG *a b b a*

and *cong-inner-transitivityG*: $\forall \ a \ b \ p \ q \ r \ s.$

```

CongG a b p q  $\wedge$ 
CongG a b r s
 $\longrightarrow$ 
CongG p q r s

```

and *cong-identityG*: $\forall \ a \ b \ c.$

```

CongG a b c c
 $\longrightarrow$ 
a = b

```

and *segment-constructionG*: $\forall \ a \ b \ c \ q.$

$\exists x. (BetG \ q \ a \ x \wedge CongG \ a \ x \ b \ c)$

and *five-segmentG*: $\forall \ a \ b \ c \ d \ a' \ b' \ c' \ d'.$

```

a  $\neq$  b  $\wedge$ 
BetG a b c  $\wedge$ 
BetG a' b' c'  $\wedge$ 
CongG a b a' b'  $\wedge$ 
CongG b c b' c'  $\wedge$ 
CongG a d a' d'  $\wedge$ 
CongG b d b' d'
 $\longrightarrow$ 
CongG c d c' d'

```

and *between-symmetryG*: $\forall \ a \ b \ c.$

```

BetG a b c
 $\longrightarrow$ 
BetG c b a

```

and *between-inner-transitivityG*: $\forall a b c d.$

BetG a b d $\wedge$
BetG b c d
 $\longrightarrow$
BetG a b c

and *inner-paschG*: $\forall a b c p q.$

BetG a p c $\wedge$
BetG b q c $\wedge$
 $a \neq p \wedge$
 $c \neq p \wedge$
 $b \neq q \wedge$
 $c \neq q \wedge$
 $\neg (BetG a b c \vee BetG b c a \vee BetG c a b)$
 $\longrightarrow$
 $(\exists x. BetG p x b \wedge BetG q x a)$

and *lower-dimG*: $\neg (BetG GPA GPB GPC \vee BetG GPB GPC GPA \vee BetG GPC GPA GPB)$

context *Gupta-neutral-dimensionless*

begin

23.2 Definitions

definition *ColG* :: '*p* $\Rightarrow$ '*p* $\Rightarrow$ '*bool* **where**
ColG A B C $\equiv BetG A B C \vee BetG B C A \vee BetG C A B$

23.3 Propositions

lemma *g2-1*:
shows *CongG A B A B*
using *cong-inner-transitivityG cong-pseudo-reflexivityG* **by** *blast*

lemma *g2-2*:
assumes *CongG A B C D*
shows *CongG C D A B*
using *assms cong-inner-transitivityG g2-1* **by** *blast*

lemma *cong-inner-transitivityT*:
assumes *CongG A B C D* **and**
CongG A B E F
shows *CongG C D E F*
using *assms(1) assms(2) cong-inner-transitivityG* **by** *blast*

lemma *g2-3*:
assumes *CongG A B C D*
shows *CongG B A C D*
using *cong-inner-transitivityT assms cong-pseudo-reflexivityG* **by** *blast*

lemma *g2-4*:
assumes *CongG A B C D*
shows *CongG A B D C*
using *assms g2-2 g2-3* **by** *presburger*

lemma *g2-5-a*:
assumes *CongG A B C D* **and**
 $A = B$
shows $C = D$
using *cong-identityG assms(1) assms(2) g2-2* **by** *blast*

lemma *g2-5-b*:
assumes *CongG A B C D* **and**


```

    C = D
  shows A = B
  using assms(1) assms(2) cong-identityG by blast

lemma g2-5:
  assumes CongG A B C D
  shows A = B  $\longleftrightarrow$  C = D
  using assms cong-identityG g2-5-a by blast

lemma g2-6:
  shows BetG A B B  $\wedge$  CongG B B A A
  using cong-identityG segment-constructionG by blast

lemma g2-7:
  assumes CongG A B A' B' and
    CongG B C B' C' and
    BetG A B C and
    BetG A' B' C'
  shows CongG A C A' C'
proof (cases A = B)
  case True
  thus ?thesis
    using assms(1) assms(2) g2-5-a by blast
next
  case False
  have BetG A' A A
    by (simp add: g2-6)
  moreover have CongG A A A' A'
    using g2-6 by auto
  ultimately have CongG B A B' A'
    using assms(1) g2-3 g2-4 by blast
  thus ?thesis
    by (meson False <CongG A A A' A'> assms(2) assms(3) assms(4) five-segmentG g2-3 g2-4)
qed

lemma g2-8:
  assumes BetG A B C and
    BetG A B D and
    CongG B C B D and
    A  $\neq$  B
  shows C = D
  by (meson assms(1) assms(2) assms(3) assms(4) five-segmentG g2-1 g2-5-b)

lemma g2-9:
  assumes BetG A B A
  shows BetG C A B
  using assms between-inner-transitivityG g2-6 by blast

lemma g2-10:
  assumes BetG A B A
  shows BetG C B A
  by (simp add: assms g2-9)

lemma g2-11:
  assumes BetG A B A and
    A  $\neq$  B
  shows  $\exists D. \text{BetG } C D C \wedge \text{BetG } D C D \wedge C \neq D$ 
  by (metis (full-types) assms(1) assms(2) between-symmetryG g2-8 g2-9 segment-constructionG)

lemma g2-12:
  assumes BetG A B A
  shows BetG A B C
  using assms between-symmetryG g2-10 by blast

lemma g2-13:
  assumes BetG A B A and

```

$A \neq B$
shows $BetG\ C\ B\ C$
by (*metis* $assms(1)\ assms(2)\ between-symmetryG\ g2-10\ g2-8\ segment-constructionG$)

lemma $g2-14-1$:
assumes $BetG\ A\ B\ A$ **and**
 $A \neq B$
shows $BetG\ C\ D\ C$
by (*metis* $assms(1)\ assms(2)\ g2-13\ g2-9$)

lemma $g2-14-2$:
assumes $BetG\ A\ B\ A$ **and**
 $A \neq B$
shows $BetG\ D\ C\ D$
using $assms(1)\ assms(2)\ g2-14-1$ **by** *blast*

lemma $g2-14$:
assumes $BetG\ A\ B\ A$ **and**
 $A \neq B$
shows $BetG\ C\ D\ C \wedge BetG\ D\ C\ D$
using $assms(1)\ assms(2)\ g2-14-2$ **by** *blast*

lemma $g2-15$:
assumes $BetG\ A\ B\ A$ **and**
 $A \neq B$
shows $BetG\ C\ D\ E$
using $assms(1)\ assms(2)\ g2-14\ g2-9$ **by** *blast*

lemma $g2-16$:
assumes $\neg BetG\ C\ D\ E$ **and**
 $BetG\ A\ B\ A$
shows $A = B$
by (*meson* $assms(1)\ assms(2)\ g2-15$)

lemma $between-identityT$:
assumes $BetG\ A\ B\ A$
shows $A = B$
using $lower-dimG\ assms\ g2-16$ **by** *blast*

lemma $cong-trivial-identityT$:
shows $CongG\ A\ A\ B\ B$
by (*simp* $add: g2-6$)

lemma $l2-11T$:
assumes $BetG\ A\ B\ C$ **and**
 $BetG\ A'\ B'\ C'$ **and**
 $CongG\ A\ B\ A'\ B'$ **and**
 $CongG\ B\ C\ B'\ C'$
shows $CongG\ A\ C\ A'\ C'$
using $assms(1)\ assms(2)\ assms(3)\ assms(4)\ g2-7$ **by** *blast*

lemma $construction-uniquenessT$:
assumes $Q \neq A$ **and**
 $BetG\ Q\ A\ X$ **and**
 $CongG\ A\ X\ B\ C$ **and**
 $BetG\ Q\ A\ Y$ **and**
 $CongG\ A\ Y\ B\ C$
shows $X = Y$
by (*meson* $assms(1)\ assms(2)\ assms(3)\ assms(4)\ assms(5)\ cong-inner-transitivityG\ g2-2\ g2-8$)

lemma $between-trivialT$:
shows $BetG\ A\ B\ B$
using $g2-6$ **by** *auto*

lemma $bet-decG$:
shows $BetG\ A\ B\ C \vee \neg BetG\ A\ B\ C$

```

by simp

lemma col-decG:
  shows  $\text{ColG } A \ B \ C \vee \neg \text{ColG } A \ B \ C$ 
  by simp

lemma inner-paschT:
  assumes  $\text{BetG } A \ P \ C$  and
     $\text{BetG } B \ Q \ C$ 
  shows  $\exists x. \text{BetG } P \ x \ B \wedge \text{BetG } Q \ x \ A$ 
proof -
{
  assume  $A \neq P$ 
  {
    assume  $P \neq C$ 
    {
      assume  $B \neq Q$ 
      {
        assume  $Q \neq C$ 
        have  $\text{ColG } A \ B \ C \longrightarrow ?thesis$ 
        proof -
          have  $\text{BetG } A \ B \ C \longrightarrow ?thesis$ 
            using assms(2) between-inner-transitivityG between-symmetryG
              between-trivialT by blast
          moreover have  $\text{BetG } B \ C \ A \longrightarrow ?thesis$ 
            using assms(1) assms(2) between-inner-transitivityG between-symmetryG by blast
          moreover have  $\text{BetG } C \ A \ B \longrightarrow ?thesis$ 
            by (meson assms(1) between-inner-transitivityG between-symmetryG
              between-trivialT)
          ultimately show  $?thesis$ 
            using ColG-def by presburger
        qed
        moreover have  $\neg \text{ColG } A \ B \ C \longrightarrow ?thesis$ 
          using ColG-def  $\langle A \neq P \rangle \langle B \neq Q \rangle \langle P \neq C \rangle \langle Q \neq C \rangle$  assms(1) assms(2)
            inner-paschG by blast
          ultimately have  $?thesis$ 
            by blast
        }
      }
    }
  }
  hence  $?thesis$ 
    using assms(1) between-symmetryG between-trivialT by blast
}
  hence  $?thesis$ 
    using between-symmetryG g2-6 by blast
}
  hence  $?thesis$ 
    using assms(2) between-symmetryG g2-6 by blast
}
  thus  $?thesis$ 
    using between-symmetryG g2-6 by blast
qed

end
end

```

```

theory Gupta-Neutral-2D

```

```

imports
  Gupta-Neutral

```

```

begin

```

24 Gupta 2D

24.1 Axioms Gupta 2D

```
locale Gupta-neutral-2D = Gupta-neutral-dimensionless GPA GPB GPC BetG CongG
  for GPA GPB GPC :: 'p
  and BetG :: 'p ⇒ 'p ⇒ 'p ⇒ bool
  and CongG :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool +
  assumes upper-dimG: ∀ a b c p q.
    p ≠ q ∧
    a ≠ b ∧
    a ≠ c ∧
    c ≠ b ∧
    CongG a p a q ∧
    CongG b p b q ∧
    CongG c p c q
    →
    (BetG a b c ∨ BetG b c a ∨ BetG c a b)
```

context Gupta-neutral-2D

begin

24.2 Definitions

24.3 Propositions

```
lemma upper-dimT :
  assumes P ≠ Q and
    CongG A P A Q and
    CongG B P B Q and
    CongG C P C Q
  shows BetG A B C ∨ BetG B C A ∨ BetG C A B
  by (metis Gupta-neutral-2D.upper-dimG Gupta-neutral-2D-axioms assms(1,2,3,4) g2-6)
```

end

end

theory Gupta-Euclidean

imports

Gupta-Neutral

begin

25 Gupta Euclidean

25.1 Axioms Gupta Euclidean

```
locale Gupta-Euclidean = Gupta-neutral-dimensionless GPA GPB GPC BetG CongG
  for GPA GPB GPC :: 'p
  and BetG :: 'p ⇒ 'p ⇒ 'p ⇒ bool
  and CongG :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool +
  assumes euclidG: ∀ A B C D T.
    BetG A D T ∧ BetG B D C ∧ B ≠ D ∧ D ≠ C ∧
    ¬ (BetG A B C ∨ BetG B C A ∨ BetG C A B) →
    (∃ x y. BetG A B x ∧ BetG A C y ∧ BetG x T y)
```

25.2 Definitions

25.3 Propositions

end

theory *Hilbert-Neutral-2D*

imports *Hilbert-Neutral*

begin

26 Hilbert - Geometry - Neutral 2D

26.1 Axioms: Hilbert neutral 2D

locale *Hilbert-neutral-2D* = *Hilbert-neutral-dimensionless IncidL IncidP EqL EqP IsL IsP BetH CongH CongaH*
for

IncidL :: 'p $\Rightarrow$ 'b $\Rightarrow$ bool **and**

IncidP :: 'p $\Rightarrow$ 'c $\Rightarrow$ bool **and**

EqL :: 'b $\Rightarrow$ 'b $\Rightarrow$ bool **and**

EqP :: 'c $\Rightarrow$ 'c $\Rightarrow$ bool **and**

IsL :: 'b $\Rightarrow$ bool **and**

IsP :: 'c $\Rightarrow$ bool **and**

BetH :: 'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ bool **and**

CongH::'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ bool **and**

CongaH::'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ 'p $\Rightarrow$ bool +

assumes *pasch-2D* :

IsL l $\wedge$ $\neg$ *ColH A B C* $\wedge$ $\neg$ *IncidL C l* $\wedge$ *cut l A B* $\longrightarrow$ (*cut l A C* $\vee$ *cut l B C*)

context *Hilbert-neutral-2D*

begin

26.2 Definitions

26.3 Propositions

lemma *plane-separation-2D*:

assumes $\neg$ *ColH A X Y* **and**

$\neg$ *ColH B X Y*

shows *cut' A B X Y* $\vee$ *same-side' A B X Y*

proof –

have *X* $\neq$ *Y*

using *assms(2) colH-trivial122* **by** *blast*

obtain *l* **where** *IsL l* **and** *IncidL X l* **and** *IncidL Y l*

using $\langle X \neq Y \rangle$ *line-existence* **by** *blast*

obtain *C* **where** *cut l A C*

using *ColH-def* $\langle$ *IncidL X l* $\rangle$ $\langle$ *IncidL Y l* $\rangle$ $\langle$ *IsL l* $\rangle$ *assms(1) cut-exists* **by** *blast*

{

assume *A* = *B*

hence *same-side' A B X Y*

using *assms(1) colH-permut-312 same-side-prime-refl* **by** *blast*

hence *?thesis*

by *blast*

}

moreover

{

assume *A* $\neq$ *B*

have *A* $\neq$ *C*

using $\langle$ *local.cut l A C* $\rangle$ *cut-distinct* **by** *blast*

{

assume *B* = *C*

obtain *m* **where** *IsL m* **and** *IncidL X m* **and** *IncidL Y m*

using *Is-line* $\langle$ *IncidL X l* $\rangle$ $\langle$ *IncidL Y l* $\rangle$ **by** *blast*

obtain *I* **where** *IncidL I l* **and** *BetH A I C*

using $\langle$ *local.cut l A C* $\rangle$ *local.cut-def* **by** *blast*

{

fix *m*

assume *IsL m* **and** *IncidL X m* **and** *IncidL Y m*

hence *cut m A B*

by (*metis ColH-def* $\langle$ *B* = *C* $\rangle$ $\langle$ *IncidL X l* $\rangle$ $\langle$ *IncidL Y l* $\rangle$ $\langle$ *X* $\neq$ *Y* $\rangle$

```

    ⟨ $\wedge thesis. (\wedge I. \llbracket IncidL\ I\ l; BetH\ A\ I\ C \rrbracket \implies thesis) \implies thesis$ ⟩ assms(1)
    assms(2) inter-incid-uniquenessH local.cut-def)
  }
  hence cut'  $A\ C\ X\ Y$ 
    using ⟨ $X \neq Y$ ⟩ ⟨ $B = C$ ⟩ cut'-def by auto
  hence ?thesis
    using ⟨ $B = C$ ⟩ by blast
}
moreover
{
  assume  $B \neq C$ 
  {
    assume ColH  $A\ C\ B$ 
    obtain  $I$  where IncidL  $I\ l$  and BetH  $A\ I\ C$ 
      using ⟨local.cut  $l\ A\ C$ ⟩ local.cut-def by blast
    hence ColH  $A\ I\ C$ 
      using betH-colH by blast
    hence ColH  $A\ I\ B$ 
      by (meson colH-permut-132 colH-trans colH-trivial121 ⟨ $A \neq C$ ⟩ ⟨ColH  $A\ C\ B$ ⟩)
    hence BetH  $A\ I\ B \vee BetH\ I\ B\ A \vee BetH\ I\ A\ B$ 
      using ColH-def ⟨ $A \neq B$ ⟩ ⟨BetH  $A\ I\ C$ ⟩ ⟨IncidL  $I\ l$ ⟩ ⟨IncidL  $X\ l$ ⟩
        ⟨IncidL  $Y\ l$ ⟩ ⟨IsL  $l$ ⟩ assms(2) betH-colH between-one by blast
    have  $\neg IncidL\ A\ l$ 
      using ⟨local.cut  $l\ A\ C$ ⟩ local.cut-def by auto
    {
      assume BetH  $A\ I\ B$ 
      {
        fix  $m$ 
        assume IsL  $m$  and IncidL  $X\ m$  and IncidL  $Y\ m$ 
        have  $\neg IncidL\ B\ m$ 
          using ColH-def ⟨IncidL  $X\ m$ ⟩ ⟨IncidL  $Y\ m$ ⟩ ⟨IsL  $m$ ⟩ assms(2) by blast
        moreover have  $\exists I0. IncidL\ I0\ m \wedge BetH\ A\ I0\ B$ 
          using ⟨BetH  $A\ I\ B$ ⟩ ⟨IncidL  $I\ l$ ⟩ ⟨IncidL  $X\ l$ ⟩ ⟨IncidL  $X\ m$ ⟩
            ⟨IncidL  $Y\ l$ ⟩ ⟨IncidL  $Y\ m$ ⟩ ⟨ $X \neq Y$ ⟩ inter-incid-uniquenessH by blast
        ultimately have cut  $m\ A\ B$ 
          using ⟨IncidL  $X\ l$ ⟩ ⟨IncidL  $X\ m$ ⟩ ⟨IncidL  $Y\ l$ ⟩ ⟨IncidL  $Y\ m$ ⟩ ⟨IsL  $m$ ⟩
            ⟨ $X \neq Y$ ⟩ ⟨ $\neg IncidL\ A\ l$ ⟩ inter-incid-uniquenessH local.cut-def by blast
      }
    }
    hence cut'  $A\ B\ X\ Y$ 
      by (simp add: ⟨ $X \neq Y$ ⟩ cut'-def)
  }
  moreover have BetH  $I\ B\ A \longrightarrow same-side'\ A\ B\ X\ Y$ 
    using ⟨IncidL  $I\ l$ ⟩ ⟨IncidL  $X\ l$ ⟩ ⟨IncidL  $Y\ l$ ⟩ ⟨ $X \neq Y$ ⟩ ⟨ $\neg IncidL\ A\ l$ ⟩
      outH-def out-same-side' by auto
  moreover have BetH  $I\ A\ B \longrightarrow same-side'\ A\ B\ X\ Y$ 
    using ⟨IncidL  $I\ l$ ⟩ ⟨IncidL  $X\ l$ ⟩ ⟨IncidL  $Y\ l$ ⟩ ⟨ $X \neq Y$ ⟩ ⟨ $\neg IncidL\ A\ l$ ⟩
      outH-def out-same-side' by auto
  ultimately have ?thesis
    using ⟨BetH  $A\ I\ B \vee BetH\ I\ B\ A \vee BetH\ I\ A\ B$ ⟩ by fastforce
}
moreover
{
  assume  $\neg ColH\ A\ C\ B$ 
  have  $\neg IncidL\ B\ l$ 
    using ColH-def Is-line ⟨IncidL  $X\ l$ ⟩ ⟨IncidL  $Y\ l$ ⟩ assms(2) by blast
  moreover {
    assume cut  $l\ A\ B$ 
    {
      fix  $m$ 
      assume IsL  $m$  and IncidL  $X\ m$  and IncidL  $Y\ m$ 
      hence EqL  $m\ l$ 
        using ⟨IncidL  $X\ l$ ⟩ ⟨IncidL  $Y\ l$ ⟩ ⟨IsL  $l$ ⟩ ⟨IsL  $m$ ⟩ ⟨ $X \neq Y$ ⟩
          line-uniqueness by blast
      hence cut  $m\ A\ B$ 
        by (meson ⟨IsL  $m$ ⟩ ⟨local.cut  $l\ A\ B$ ⟩ local.cut-def morph)
    }
  }
}

```

```

    hence cut' A B X Y
      using ⟨X ≠ Y⟩ cut'-def by auto
  }
  {
    assume cut l C B
    {
      fix m
      assume IsL m and IncidL X m and IncidL Y m
      hence EqL m l
        using ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ ⟨X ≠ Y⟩ line-uniqueness by blast
      have cut m A C
        by (meson ⟨IncidL X l⟩ ⟨IncidL X m⟩ ⟨IncidL Y l⟩ ⟨IncidL Y m⟩
              ⟨IsL m⟩ ⟨X ≠ Y⟩ ⟨local.cut l A C⟩ inter-incid-uniquenessH local.cut-def)
      moreover have cut m B C
        by (meson ⟨EqL m l⟩ ⟨IsL m⟩ ⟨local.cut l C B⟩ cut-comm local.cut-def morph)
      ultimately have same-side A B m
        using ⟨IsL m⟩ same-side-def by blast
    }
    hence same-side' A B X Y
      by (simp add: ⟨X ≠ Y⟩ same-side'-def)
    have ?thesis
      using ⟨IsL l⟩ ⟨¬ ColH A C B⟩ ⟨local.cut l A C⟩ pasch-2D
        ⟨same-side' A B X Y⟩ by blast
  }
  ultimately have ?thesis
    using ⟨IsL l⟩ ⟨¬ ColH A C B⟩ ⟨local.cut l A B ⟹ cut' A B X Y⟩
      ⟨local.cut l A C⟩ pasch-2D by blast
  }
  ultimately have ?thesis
    by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

```

lemma col-upper-dim:

```

  assumes ColH A P Q and
    P ≠ Q and
    A ≠ B and
    A ≠ C and
    B ≠ C and
    A ≠ P and
    A ≠ Q and
    B ≠ P and
    B ≠ Q and
    C ≠ P and
    C ≠ Q and
    CongH A P A Q and
    CongH B P B Q and
    CongH C P C Q
  shows Bet A B C ∨ Bet B C A ∨ Bet C A B
proof -
  {
    assume ColH B P Q
    hence BetH P B Q
      using assms(13) assms(2) assms(8) assms(9) congH-colH-betH by auto
    {
      assume BetH P A B
      hence BetH Q B A
        using ⟨BetH P B Q⟩ betH-trans0 between-comm by blast
      hence False
        using ⟨BetH P A B⟩ assms(12) assms(13) assms(7) assms(9)
          betH-congH2--False congH-permlr by blast
    }
  }

```

```

}
moreover
{
  assume BetH P B A
  hence BetH Q A B
  by (metis assms(1) assms(12) assms(2) assms(6) assms(7)
    betH-trans0 between-comm congH-colH-betH)
  hence False
  using  $\langle \textit{BetH P B A} \rangle$  assms(12) assms(13) assms(7) assms(9)
    betH-congH2--False congH-permlr by blast
}
ultimately have False
by (metis  $\langle \textit{BetH P B Q} \rangle$  assms(1) assms(12) assms(2) assms(3)
  assms(6) betH2-out congH-colH-betH)
}
hence  $\neg \textit{ColH B P Q}$ 
by blast
{
  assume ColH C P Q
  hence BetH P C Q
  using assms(10) assms(11) assms(14) assms(2) congH-colH-betH by force
  {
    assume BetH P A C
    hence BetH Q C A
    using  $\langle \textit{BetH P C Q} \rangle$  betH-trans0 between-comm by blast
    hence False
    using  $\langle \textit{BetH P A C} \rangle$  assms(11) assms(12) assms(14) assms(7)
      betH-congH2--False congH-permlr by blast
  }
  moreover
  {
    assume BetH P C A
    hence BetH Q A C
    by (metis assms(1) assms(12) assms(2) assms(6) assms(7)
      betH-trans0 between-comm congH-colH-betH)
    hence False
    using  $\langle \textit{BetH P C A} \rangle$  assms(10) assms(11) assms(12) assms(14)
      assms(6) assms(7) betH-congH2--False congH-perms by blast
  }
  ultimately have False
  by (metis congH-colH-betH  $\langle \textit{BetH P C Q} \rangle$  assms(1) assms(12) assms(2)
    assms(4) assms(6) betH2-out)
}
hence  $\neg \textit{ColH C P Q}$ 
by blast
{
  assume cut' B C P Q
  obtain l where IsL l and IncidL P l and IncidL Q l
  using assms(2) line-existence by blast
  have cut l B C
  using  $\langle \textit{IncidL P l} \rangle \langle \textit{IncidL Q l} \rangle \langle \textit{IsL l} \rangle \langle \textit{cut' B C P Q} \rangle$  cut'-def by auto
  obtain I where IncidL I l and BetH B I C
  using  $\langle \textit{local.cut l B C} \rangle$  local.cut-def by blast
  {
    assume P = I
    hence ColH B Q C
    by (metis  $\langle \textit{BetH B I C} \rangle$  assms(11) assms(13) assms(14) assms(5)
      congH-permlr congH-refl cong-preserves-col)
    hence False
    by (metis betH2-out  $\langle \textit{BetH B I C} \rangle \langle P = I \rangle \langle \neg \textit{ColH B P Q} \rangle$  assms(13)
      assms(14) assms(2) assms(9) betH-colH bet-cong3-bet colH-permut-132
      congH-permlr congH-refl)
  }
  hence P ≠ I by blast
  {
    assume Q = I

```



```

hence ColH B P C
  by (metis ‹BetH B I C› assms(10) assms(11) assms(13) assms(14)
      assms(5) assms(8) congH-perms congH-refl congH-sym cong-preserves-col)
hence False
  by (metis Hilbert-neutral-dimensionless-pre.Bet-def ‹BetH B I C›
      ‹IncidL P l› ‹Q = I› ‹¬ ColH B P Q› ‹local.cut l B C› betH2-out bet-colH
      between-one colH-permut-132 local.cut-def outH-def out-same-side same-side-not-cut)
}
hence Q ≠ I
  by blast
have ColH P Q I
  using ColH-def ‹IncidL I l› ‹IncidL P l› ‹IncidL Q l› ‹IsL l› by blast
have CongaH I B P I B Q
proof -
  have CongaH C B P C B Q
  proof -
    have ¬ ColH B C P
      by (metis (full-types) ‹BetH B I C› ‹ColH B P Q ⟹ False›
          ‹ColH P Q I› ‹P = I ⟹ False› colH-permut-312 colH-trivial121
          inter-uniquenessH outH-def outH-expand)
    moreover have ¬ ColH B C Q
      by (meson assms(10) assms(11) assms(13) assms(14) assms(5)
          assms(8) assms(9) calculation congH-refl congH-sym cong-preserves-col-stronger)
    moreover have CongH B C B C
      by (simp add: assms(5) congH-refl)
    moreover have CongH B P B Q
      by (simp add: assms(13))
    moreover have CongH C P C Q
      by (simp add: assms(14))
    ultimately show ?thesis
      using th18-aux by blast
  qed
  moreover have outH B C I
    by (simp add: ‹BetH B I C› outH-def)
  moreover have outH B P P
    by (simp add: assms(8) outH-def)
  moreover have outH B Q Q
    by (simp add: assms(9) outH-def)
  ultimately show ?thesis
    using conga-out-conga by blast
qed
have CongH I P I Q
proof -
  have ¬ ColH B I P
    by (metis ‹ColH P Q I› ‹P = I ⟹ False› ‹¬ ColH B P Q›
        colH-permut-312 colH-trivial122 inter-uniquenessH)
  moreover have ¬ ColH B I Q
    using IncidL-not-IncidL--not-colH ‹IncidL I l› ‹IncidL Q l›
        ‹Q = I ⟹ False› ‹local.cut l B C› colH-permut-321 local.cut-def by blast
  moreover have CongH B I B I
    using ‹BetH B I C› betH-distincts congH-refl by auto
  moreover have CongH B P B Q
    by (simp add: assms(13))
  moreover have CongaH I B P I B Q
    by (simp add: ‹CongaH I B P I B Q›)
  ultimately show ?thesis
    using th12 by blast
qed
have A = I ⟹ ?thesis
  using Bet-def ‹BetH B I C› between-comm by blast
moreover
{
  assume A ≠ I
  have ColH I P Q
    by (simp add: ‹ColH P Q I› colH-permut-312)
  hence BetH P I Q

```

```

    by (simp add: ⟨CongH I P I Q⟩ ⟨P ≠ I⟩ ⟨Q ≠ I⟩ assms(2) congH-colH-betH)
  {
    assume BetH P A I
    hence BetH Q I A
      using ⟨BetH P I Q⟩ betH-trans0 between-comm by blast
    hence False
      by (metis ⟨BetH P A I⟩ ⟨CongH I P I Q⟩ ⟨Q ≠ I⟩ assms(12) assms(7)
        betH-congH2--False congH-permlr)
  }
  moreover {
    assume BetH P I A
    hence BetH Q A I
      by (metis assms(1) assms(12) assms(2) assms(6) assms(7) betH-trans0
        between-comm congH-colH-betH)
    hence False
      by (metis ⟨BetH P I A⟩ ⟨CongH I P I Q⟩ ⟨Q = I ⟹ False⟩ assms(12)
        assms(7) betH-congH2--False congH-permlr)
  }
  ultimately have ?thesis
    by (metis ⟨A = I ⟹ Bet A B C ∨ Bet B C A ∨ Bet C A B⟩ ⟨BetH P I Q⟩
      assms(1) assms(12) assms(2) assms(6) betH2-out congH-colH-betH)
}
ultimately have ?thesis
  by blast
}
moreover
{
  assume same-side' B C P Q
  {
    assume ColH P B C
    have ColH C P B
      by (simp add: ⟨ColH P B C⟩ colH-permut-312)
    have ColH B C Q
      by (metis colH-permut-312 ⟨ColH C P B⟩ assms(10) assms(13) assms(14)
        assms(5) assms(8) congH-perm congH-perms cong-preserves-col-stronger)
    have BetH P B Q
      using ColH-def ⟨ColH B C Q⟩ ⟨ColH C P Q ⟹ False⟩ ⟨ColH P B C⟩
        assms(5) colH-IncidL--IncidL by blast
    have BetH P C Q
      using ⟨BetH P B Q⟩ ⟨¬ ColH B P Q⟩ between-col colH-permut-213 by blast
  }
  {
    assume BetH P B C
    hence BetH Q C B
      using ⟨BetH P C Q⟩ betH-trans0 between-comm by blast
    hence False
      using ⟨BetH P B Q⟩ ⟨¬ ColH B P Q⟩ betH-expand colH-permut-213 by blast
  }
  moreover
  {
    assume BetH P C B
    hence BetH B Q C
      using ⟨BetH P B Q⟩ ⟨¬ ColH B P Q⟩ betH-expand colH-permut-213 by blast
    hence False
      using ⟨BetH P C Q⟩ ⟨ColH C P Q ⟹ False⟩ betH-expand colH-permut-213 by blast
  }
  ultimately have False
    using ⟨BetH P B Q⟩ ⟨BetH P C Q⟩ assms(5) betH2-out by blast
}
hence ¬ ColH P B C
  by blast
{
  assume ColH Q B C
  hence ColH B C P
    by (metis (full-types) ⟨¬ ColH P B C⟩ assms(10) assms(11) assms(13)
      assms(14) assms(5) assms(8) assms(9) between-one colH-permut-213 congH-perms
        congH-refl cong-preserves-col)

```

```

hence False
  using  $\langle \neg \text{ColH } P \ B \ C \rangle$  colH-permut-312 by blast
}
hence  $\neg \text{ColH } Q \ B \ C$ 
  by blast
{
  assume cut' P Q B C
  obtain IBC where IsL IBC and IncidL B IBC and IncidL C IBC
    using assms(5) line-existence by blast
  {
    assume  $\neg \text{IncidL } P \ IBC$  and
       $\neg \text{IncidL } Q \ IBC$  and
       $\exists I. \text{IncidL } I \ IBC \wedge \text{BetH } P \ I \ Q$ 
    then obtain I where IncidL I IBC and BetH P I Q
      by blast
    have ColH B I C
      using ColH-def  $\langle \text{IncidL } B \ IBC \rangle \langle \text{IncidL } C \ IBC \rangle \langle \text{IncidL } I \ IBC \rangle \langle \text{IsL } IBC \rangle$  by auto
    {
      assume BetH B I C
      obtain IPQ where IncidL P IPQ and IncidL Q IPQ and  $\neg \text{cut } IPQ \ B \ C$ 
        by (metis same-side' B C P Q) line-existence same-side'-def same-side-not-cut
      have cut IPQ B C
      proof -
        have  $\neg \text{IncidL } C \ IPQ$ 
          using Hilbert-neutral-dimensionless-pre.ColH-def Is-line
             $\langle \text{ColH } C \ P \ Q \implies \text{False} \rangle \langle \text{IncidL } P \ IPQ \rangle \langle \text{IncidL } Q \ IPQ \rangle$  by fastforce
        moreover have IncidL I IPQ
          using  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{IncidL } P \ IPQ \rangle \langle \text{IncidL } Q \ IPQ \rangle$  assms(2) betH-line
            inter-incid-uniquenessH by blast
        ultimately show ?thesis
          using  $\langle \text{BetH } B \ I \ C \rangle$  Is-line  $\langle \text{ColH } B \ I \ C \rangle$  betH-distincts
            colH-IncidL--IncidL local.cut-def by blast
      qed
      hence ?thesis
        using  $\langle \neg \text{local.cut } IPQ \ B \ C \rangle$  by blast
    }
  }
  moreover {
    assume BetH I C B
    have CongaH P C I Q C I
    proof -
      have  $\neg \text{ColH } B \ C \ P$ 
        using  $\langle \neg \text{ColH } P \ B \ C \rangle$  colH-permut-312 by blast
      moreover have  $\neg \text{ColH } B \ C \ Q$ 
        using  $\langle \neg \text{ColH } Q \ B \ C \rangle$  colH-permut-312 by blast
      moreover have CongaH B C P B C Q
        using  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{ColH } B \ I \ C \rangle$  assms(13) assms(14) calculation(1)
          calculation(2) th17 by blast
      moreover have BetH B C I
        by (simp add: BetH I C B) between-comm
      ultimately show ?thesis
        using th14 by blast
    qed
    have CongH I P I Q
    proof -
      have  $\neg \text{ColH } C \ P \ I$ 
        by (metis IncidL-not-IncidL--not-colH BetH P I Q IncidL C IBC)
           $\langle \text{IncidL } I \ IBC \rangle \langle \neg \text{ColH } C \ P \ Q \rangle \langle \neg \text{IncidL } P \ IBC \rangle$  betH-colH colH-permut-132 colH-permut-213
      moreover have  $\neg \text{ColH } C \ Q \ I$ 
        using  $\langle \text{BetH } I \ C \ B \rangle \langle \text{IncidL } C \ IBC \rangle \langle \text{IncidL } I \ IBC \rangle \langle \neg \text{IncidL } Q \ IBC \rangle$ 
          betH-distincts colH-IncidL--IncidL colH-permut-312 by blast
      moreover have CongH C P C Q
        by (simp add: assms(14))
      moreover have CongH C I C I
        by (metis BetH I C B betH-distincts congH-refl)
      ultimately show ?thesis
        using  $\langle \text{CongaH } P \ C \ I \ Q \ C \ I \rangle$ 

```

```

    using congH-permlr ncolH-expand th12 by presburger
qed
have A = I → ?thesis
  using Bet-def ⟨BetH I C B⟩ bet-comm by blast
moreover {
  assume A ≠ I
  have ColH I P Q
    by (simp add: ⟨BetH P I Q⟩ betH-colH colH-permut-213)
  {
    assume BetH P A I
    hence BetH Q I A
      using ⟨BetH P I Q⟩ betH-trans0 between-comm by blast
    hence False
      by (metis betH-expand ⟨BetH P A I⟩ ⟨CongH I P I Q⟩ assms(12)
        betH-congH2--False congH-perms)
  }
  moreover {
    assume BetH P I A
    hence BetH Q A I
      by (metis assms(1) assms(12) assms(2) assms(6) assms(7) betH-trans0
        between-comm congH-colH-betH)
    hence False
      by (metis ⟨BetH P I A⟩ ⟨CongH I P I Q⟩ assms(12) betH-congH2--False
        betH-distincts congH-permlr)
  }
  ultimately have ?thesis
    by (metis ⟨A = I → Bet A B C ∨ Bet B C A ∨ Bet C A B⟩ ⟨BetH P I Q⟩
      assms(1) assms(12) assms(2) assms(6) betH2-out congH-colH-betH)
}
ultimately have ?thesis
  by blast
}
moreover {
  assume BetH I B C
  have CongaH P B I Q B I
  proof -
    have ¬ ColH C B P
      using ⟨¬ ColH P B C⟩ colH-permut-321 by blast
    moreover have ¬ ColH C B Q
      using ⟨¬ ColH Q B C⟩ colH-permut-321 by blast
    moreover have CongaH C B P C B Q
      using colH-permut-321 ⟨BetH P I Q⟩ ⟨ColH B I C⟩ assms(13)
        assms(14) calculation(1) calculation(2) th17 by blast
    moreover have BetH C B I
      by (simp add: ⟨BetH I B C⟩ between-comm)
    ultimately show ?thesis
      using th14 by blast
  qed
  have CongH I P I Q
  proof -
    have ¬ ColH B P I
      using ⟨BetH I B C⟩ ⟨IncidL B lBC⟩ ⟨IncidL I lBC⟩ ⟨¬ IncidL P lBC⟩
        betH-distincts colH-IncidL--IncidL colH-permut-312 by blast
    moreover have ¬ ColH B Q I
      using ⟨BetH I B C⟩ ⟨IncidL B lBC⟩ ⟨IncidL I lBC⟩ ⟨¬ IncidL Q lBC⟩
        betH-distincts colH-IncidL--IncidL colH-permut-312 by blast
    moreover have CongH B P B Q
      using assms(13) by auto
    moreover have CongH B I B I
      by (metis ⟨BetH I B C⟩ betH-distincts congH-refl)
    ultimately show ?thesis
      using ⟨CongaH P B I Q B I⟩
        by (metis ⟨IncidL I lBC⟩ ⟨¬ IncidL Q lBC⟩ congH-permlr th12)
  qed
  have A = I → ?thesis
    using Bet-def ⟨BetH I B C⟩ by blast

```

```

moreover
{
  assume  $A \neq I$ 
  have  $\text{ColH } I \ P \ Q$ 
  using  $\langle \text{BetH } P \ I \ Q \rangle$  betH-expand colH-permut-213 by blast
  {
    assume  $\text{BetH } P \ A \ I$ 
    hence  $\text{BetH } Q \ I \ A$ 
    using  $\langle \text{BetH } P \ I \ Q \rangle$  betH-trans0 between-comm by blast
    hence False
    by (metis betH-expand  $\langle \text{BetH } P \ A \ I \rangle$   $\langle \text{CongH } I \ P \ I \ Q \rangle$  assms(12)
      betH-congH2--False congH-permlr)
  }
  moreover
  {
    assume  $\text{BetH } P \ I \ A$ 
    hence  $\text{BetH } Q \ A \ I$ 
    by (metis assms(1) assms(12) assms(2) assms(6) assms(7)
      betH-trans0 between-comm congH-colH-betH)
    hence False
    by (metis betH-expand  $\langle \text{BetH } P \ I \ A \rangle$   $\langle \text{CongH } I \ P \ I \ Q \rangle$  assms(12)
      betH-congH2--False congH-perms)
  }
  ultimately have ?thesis
  by (metis  $\langle A = I \longrightarrow \text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B \rangle$ 
     $\langle \text{BetH } P \ I \ Q \rangle$  assms(1) assms(12) assms(2) assms(6) betH2-out congH-colH-betH)
  }
  ultimately have ?thesis
  by blast
}
ultimately have ?thesis
by (metis  $\langle \text{BetH } P \ I \ Q \rangle$   $\langle \text{ColH } B \ I \ C \rangle$   $\langle \text{ColH } B \ P \ Q \implies \text{False} \rangle$ 
   $\langle \text{ColH } C \ P \ Q \implies \text{False} \rangle$  assms(5) between-col between-one colH-permut-213)
}
hence ?thesis
using  $\langle \text{IncidL } B \ \text{IBC} \rangle$   $\langle \text{IncidL } C \ \text{IBC} \rangle$   $\langle \text{IsL } \text{IBC} \rangle$   $\langle \text{cut}' \ P \ Q \ B \ C \rangle$  cut'-def
local.cut-def by auto
}
moreover
{
  assume same-side'  $P \ Q \ B \ C$ 
  have  $\text{outH } B \ P \ Q$ 
  proof –
  have  $\neg \text{ColH } P \ B \ C$ 
  by (simp add:  $\langle \neg \text{ColH } P \ B \ C \rangle$ )
  moreover have  $\neg \text{ColH } C \ B \ P$ 
  using calculation colH-permut-321 by blast
  moreover have  $\text{CongaH } C \ B \ P \ C \ B \ P$ 
  by (simp add: calculation(2) conga-refl)
  moreover have  $\text{CongaH } C \ B \ P \ C \ B \ Q$ 
  by (metis  $\langle \neg \text{ColH } Q \ B \ C \rangle$  assms(13) assms(14) assms(5) calculation(2)
    colH-permut-213 colH-permut-312 congH-refl th18-aux)
  moreover have same-side'  $P \ P \ B \ C$ 
  using calculation(2) colH-permut-213 same-side-prime-refl by blast
  ultimately show ?thesis
  using  $\langle \text{same-side}' \ P \ Q \ B \ C \rangle$  cong-4-uniqueness by blast
  qed
  hence ?thesis
  using  $\langle \neg \text{ColH } B \ P \ Q \rangle$  outH-expand by blast
}
ultimately have ?thesis
using  $\langle \neg \text{ColH } P \ B \ C \rangle$   $\langle \neg \text{ColH } Q \ B \ C \rangle$  plane-separation-2D by blast
}
ultimately show ?thesis
using  $\langle \neg \text{ColH } B \ P \ Q \rangle$   $\langle \neg \text{ColH } C \ P \ Q \rangle$  plane-separation-2D by blast
qed

```

```

lemma TS-upper-dim:
  assumes cut' A B P Q and
    P ≠ Q and
    A ≠ B and
    A ≠ C and
    B ≠ C and
    A ≠ P and
    A ≠ Q and
    B ≠ P and
    B ≠ Q and
    C ≠ P and
    C ≠ Q and
    CongH A P A Q and
    CongH B P B Q and
    CongH C P C Q
  shows Bet A B C ∨ Bet B C A ∨ Bet C A B
proof -
  obtain l where IsL l and IncidL P l and IncidL Q l
    using assms(2) line-existence by blast
  have cut l A B
    using ⟨IncidL P l⟩ ⟨IncidL Q l⟩ ⟨IsL l⟩ assms(1) cut'-def by auto
  have ColH A P Q ⟶ ?thesis
    by (simp add: assms(10) assms(11) assms(12) assms(13) assms(14) assms(2)
        assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
  moreover
  {
    assume ¬ ColH A P Q
    have ColH B P Q ⟶ ?thesis
      by (metis assms(10) assms(11) assms(12) assms(13) assms(14) assms(2)
          assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
    moreover
    {
      assume ¬ ColH B P Q
      obtain I where IncidL I l and BetH A I B
        using ⟨local.cut l A B⟩ local.cut-def by blast
      have ColH P Q I
        using ColH-def Is-line ⟨IncidL I l⟩ ⟨IncidL P l⟩ ⟨IncidL Q l⟩ by blast
      {
        assume P = I
        hence ColH A Q B
          by (metis ⟨BetH A I B⟩ assms(12) assms(13) assms(3) assms(9) congH-permlr
              congH-refl cong-preserves-col)
        hence False
          by (metis colH-permut-312 colH-trans colH-trivial121 ⟨BetH A I B⟩
              ⟨P = I⟩ ⟨¬ ColH B P Q⟩ assms(3) between-col)
      }
      hence P ≠ I
        by blast
    }
    {
      assume Q = I
      hence ColH A P B
        by (metis ⟨BetH A I B⟩ assms(12) assms(13) assms(3) assms(6) assms(8)
            assms(9) congH-perms congH-refl congH-sym cong-preserves-col)
      have ColH A P I
        by (metis ⟨BetH A I B⟩ ⟨ColH A P B⟩ assms(3) colH-trans ncolH-expand
            outH-col outH-def)
      hence False
        using ⟨Q = I⟩ ⟨¬ ColH A P Q⟩ by blast
    }
  }
  hence Q ≠ I
    by blast
  have CongaH I A P I A Q
  proof -
    have CongaH B A P B A Q
  proof -

```

```

have  $\neg \text{ColH } A \ B \ P$ 
  by (metis (full-types)  $\langle \text{BetH } A \ I \ B \rangle \langle \text{ColH } P \ Q \ I \rangle \langle P = I \implies \text{False} \rangle$ 
     $\langle \neg \text{ColH } A \ P \ Q \rangle \text{colH-permut-312 colH-trivial121 inter-uniquenessH}$ 
    outH-def outH-expand)
moreover have  $\neg \text{ColH } A \ B \ Q$ 
  by (meson assms(12) assms(13) assms(3) assms(6) assms(7) assms(8)
    assms(9) calculation congH-refl congH-sym cong-preserves-col-stronger)
moreover have CongH A B A B
  using assms(3) congH-refl by blast
moreover have CongH A P A Q
  by (simp add: assms(12))
moreover have CongH B P B Q
  using assms(13) by auto
ultimately show ?thesis
  using th18-aux by blast
qed
moreover have outH A B I
  by (simp add:  $\langle \text{BetH } A \ I \ B \rangle \text{outH-def}$ )
moreover have outH A P P
  by (simp add: assms(6) outH-def)
moreover have outH A B I
  by (simp add: calculation(2))
moreover have outH A Q Q
  by (simp add: assms(7) outH-def)
ultimately show ?thesis
  using conga-out-conga by fastforce
qed
have CongH I P I Q
proof -
  have  $\neg \text{ColH } A \ I \ P$ 
    by (metis  $\langle \text{ColH } P \ Q \ I \rangle \langle P \neq I \rangle \langle \neg \text{ColH } A \ P \ Q \rangle \text{colH-permut-231}$ 
      colH-trans colH-trivial122)
  moreover have  $\neg \text{ColH } A \ I \ Q$ 
    using ColH-def  $\langle \text{IncidL } I \ l \rangle \langle \text{IncidL } P \ l \rangle \langle \text{IncidL } Q \ l \rangle \langle Q = I \implies \text{False} \rangle$ 
      calculation inter-incid-uniquenessH by auto
  moreover have CongH A I A I
    using  $\langle \text{BetH } A \ I \ B \rangle \text{betH-distincts congH-refl}$  by presburger
  ultimately show ?thesis
    using  $\langle \text{CongaH } I \ A \ P \ I \ A \ Q \rangle \text{assms(12)}$ 
    using th12 by blast
qed
have ColH A B C
proof -
  have  $C = I \longrightarrow ?thesis$ 
    by (simp add:  $\langle \text{BetH } A \ I \ B \rangle \text{between-col colH-permut-132}$ )
  moreover
  {
    assume  $C \neq I$ 
    have ColH A C I
      by (metis colH-permut-231  $\langle \text{BetH } A \ I \ B \rangle \langle C \neq I \rangle \langle \text{ColH } P \ Q \ I \rangle$ 
         $\langle \text{CongH } I \ P \ I \ Q \rangle \langle P \neq I \rangle \langle Q \neq I \rangle \text{assms(10) assms(11) assms(12) assms(14)}$ 
        assms(2) assms(4) assms(6) assms(7) betH-to-bet bet-colH col-upper-dim)
    hence ?thesis
      using ColH-def IncidL-not-IncidL--not-colH  $\langle \text{BetH } A \ I \ B \rangle \text{betH-colH}$  by blast
  }
  ultimately show ?thesis
    by blast
qed
have BetH A B C  $\longrightarrow ?thesis$ 
  by (simp add: ColH-bets  $\langle \text{ColH } A \ B \ C \rangle$ )
moreover have BetH B C A  $\longrightarrow ?thesis$ 
  using ColH-bets  $\langle \text{ColH } A \ B \ C \rangle$  by blast
moreover have BetH B A C  $\longrightarrow ?thesis$ 
  by (simp add: ColH-bets  $\langle \text{ColH } A \ B \ C \rangle$ )
ultimately have ?thesis
  using ColH-bets  $\langle \text{ColH } A \ B \ C \rangle$  by blast

```

```

}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma cut'-comm:
  assumes cut' A B X Y
  shows cut' B A X Y
  by (metis assms cut'-def cut-comm)

```

```

lemma TS-upper-dim-bis:
  assumes BetH P I Q and
    BetH I B A and
    P ≠ Q and
    A ≠ B and
    A ≠ C and
    B ≠ C and
    A ≠ P and
    A ≠ Q and
    B ≠ P and
    B ≠ Q and
    C ≠ P and
    C ≠ Q and
    CongH A P A Q and
    CongH B P B Q and
    CongH C P C Q
  shows Bet A B C ∨ Bet B C A ∨ Bet C A B
proof -
  have ColH A P Q ⟶ ?thesis
    by (simp add: assms(10) assms(11) assms(12) assms(13) assms(14) assms(15)
        assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
  moreover
  {
    assume ¬ ColH A P Q
    have ColH B P Q ⟶ ?thesis
      by (metis assms(10) assms(11) assms(12) assms(13) assms(14) assms(15)
          assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
    moreover
    {
      assume ¬ ColH B P Q
      have CongaH P B I Q B I
      proof -
        have P ≠ I
          using assms(1) betH-expand by blast
        have Q ≠ I
          using assms(1) betH-colH by blast
        {
          assume ColH A B P
          have ¬ ColH P Q A
            using ⟨¬ ColH A P Q⟩ colH-permut-312 by blast
          moreover have ColH P Q P
            by (simp add: colH-trivial121)
          moreover have ColH P Q I
            by (simp add: assms(1) betH-colH colH-permut-132)
          moreover have ColH A B I
            by (simp add: assms(2) betH-colH between-comm)
          ultimately
          have P = I
            using assms(4) ⟨ColH A B P⟩ inter-uniquenessH by blast
          hence False
            by (simp add: ⟨P ≠ I⟩)
        }
      }
    }
  }
  hence ¬ ColH A B P

```



```

    by blast
  moreover
  {
    assume ColH A B Q
    have  $\neg$  ColH Q P A
      using  $\langle \neg$  ColH A P Q  $\rangle$  colH-permut-321 by blast
    moreover have ColH Q P Q
      by (simp add: colH-trivial121)
    moreover have ColH Q P I
      using assms(1) betH-colH colH-permut-312 by blast
    moreover have ColH A B I
      using assms(2) between-col colH-permut-321 by blast
    ultimately
    have  $Q = I$ 
      using assms(4)  $\langle$  ColH A B Q  $\rangle$  inter-uniquenessH by blast
    hence False
      by (simp add:  $\langle Q \neq I \rangle$ )
  }
  hence  $\neg$  ColH A B Q
    by blast
  moreover have CongaH A B P A B Q
  proof -
    have  $\neg$  ColH B A P
      using calculation(1) colH-permut-213 by blast
    moreover have  $\neg$  ColH B A Q
      using  $\langle \neg$  ColH A B Q  $\rangle$  colH-permut-213 by blast
    moreover have CongH B A B A
      using assms(4) congH-refl by auto
    ultimately show ?thesis
      using assms(13) assms(14) th18-aux by presburger
  qed
  ultimately show ?thesis
    by (simp add: assms(2) between-comm th14)
  qed
  have CongH I P I Q
    by (metis assms(1) assms(10) assms(13) assms(14) assms(2) assms(7)
      assms(8) assms(9) axiom-five-segmentsH between-comm between-only-one congH-refl)
  have ColH A B C
  proof -
    have ColH A B I
      by (simp add: assms(2) between-col colH-permut-321)
    have  $C = I \longrightarrow ?thesis$ 
      using  $\langle$  ColH A B I  $\rangle$  by blast
    moreover
    {
      assume  $C \neq I$ 
      have ColH I B A
        by (simp add: assms(2) betH-colH)
      hence ColH A C I
        by (metis (mono-tags, lifting) betH-to-bet bet-colH colH-permut-213
          colH-permut-312  $\langle C \neq I \rangle$   $\langle$  CongH I P I Q  $\rangle$  assms(1) assms(11) assms(12)
          assms(13) assms(15) assms(2) assms(5) assms(7) assms(8) col-upper-dim)
      have ColH P I Q
        by (simp add: assms(1) betH-colH)
      hence ?thesis
        by (metis betH-distincts colH-permut-312 colH-trans colH-trivial121
           $\langle$  ColH A C I  $\rangle$   $\langle$  ColH I B A  $\rangle$  assms(2))
    }
    ultimately show ?thesis
      by blast
  qed
  have BetH B A C  $\longrightarrow$  ?thesis
    using ColH-bets  $\langle$  ColH A B C  $\rangle$  by blast
  hence ?thesis
    using ColH-bets  $\langle$  ColH A B C  $\rangle$  by blast
}

```

```

ultimately have ?thesis
  by blast
}
ultimately show ?thesis by blast
qed

```

lemma upper-dim:

```

assumes P ≠ Q and
  A ≠ B and
  A ≠ C and
  B ≠ C and
  Cong A P A Q and
  Cong B P B Q and
  Cong C P C Q
shows Bet A B C ∨ Bet B C A ∨ Bet C A B
proof -
  have C = P ∧ C = Q ⟶ ?thesis
    using assms(1) by blast
  have B = P ∧ B = Q ⟶ ?thesis
    using assms(1) by blast
  have A = P ∧ A = Q ⟶ ?thesis
    using assms(1) by blast
  have CongH A P A Q
    using Cong-def assms(1) assms(5) by auto
  have CongH B P B Q
    using Cong-def assms(1) assms(6) by blast
  have CongH C P C Q
    using Cong-def assms(1) assms(7) by blast
  have A ≠ P
    using Cong-def assms(1) assms(5) by presburger
  have A ≠ Q
    using Cong-def assms(1) assms(5) by presburger
  have B ≠ P
    using Cong-def assms(1) assms(6) by force
  have B ≠ Q
    using Cong-def assms(1) assms(6) by presburger
  have C ≠ P
    using Cong-def assms(1) assms(7) by presburger
  have C ≠ Q
    using Cong-def assms(1) assms(7) by presburger
  have ColH A P Q ⟶ ?thesis
    by (simp add: ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩ ⟨C ≠ Q⟩
      ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1) assms(2) assms(3)
      assms(4) col-upper-dim)
  moreover
  {
    assume ¬ ColH A P Q
    have ColH B P Q ⟶ ?thesis
      by (metis ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩ ⟨C ≠ Q⟩
        ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1) assms(2) assms(3)
        assms(4) col-upper-dim)
    moreover
    {
      assume ¬ ColH B P Q
      have ColH C P Q ⟶ ?thesis
        by (metis ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩ ⟨C ≠ Q⟩
          ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1) assms(2)
          assms(3) assms(4) col-upper-dim)
      moreover
      {
        assume ¬ ColH C P Q
        have cut' A B P Q ⟶ ?thesis
          by (simp add: TS-upper-dim ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩
            ⟨C ≠ Q⟩ ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1)
            assms(2) assms(3) assms(4))
        moreover

```

```

{
  assume same-side' A B P Q
  have cut' A C P Q  $\longrightarrow$  ?thesis
    using TS-upper-dim  $\langle A \neq P \rangle \langle A \neq Q \rangle \langle B \neq P \rangle \langle B \neq Q \rangle \langle C \neq P \rangle \langle C \neq Q \rangle$ 
       $\langle \text{CongH } A \ P \ A \ Q \rangle \langle \text{CongH } B \ P \ B \ Q \rangle \langle \text{CongH } C \ P \ C \ Q \rangle$  assms(1) assms(2) assms(3)
      assms(4) bet-comm by blast
  moreover
  {
    assume same-side' A C P Q
    {
      assume ColH P A B
      hence ColH B P A
        by (simp add: colH-permut-312)
      have ColH A B Q
        by (meson  $\langle A \neq P \rangle \langle B \neq P \rangle \langle \text{ColH } B \ P \ A \rangle \langle \text{CongH } A \ P \ A \ Q \rangle \langle \text{CongH } B \ P \ B \ Q \rangle$ 
          assms(2) colH-permut-312 congH-refl cong-preserves-col-stronger)
      hence False
        by (metis  $\langle \text{ColH } B \ P \ A \rangle \langle \neg \text{ColH } A \ P \ Q \rangle$  assms(2) colH-trans ncolH-distincts)
    }
    hence  $\neg \text{ColH } P \ A \ B$ 
      by blast
    {
      assume ColH Q A B
      hence ColH A B P
        by (meson  $\langle A \neq P \rangle \langle A \neq Q \rangle \langle B \neq P \rangle \langle B \neq Q \rangle \langle \text{CongH } A \ P \ A \ Q \rangle$ 
           $\langle \text{CongH } B \ P \ B \ Q \rangle$  assms(2) colH-permut-231 congH-refl congH-sym
          cong-preserves-col-stronger)
      hence False
        using  $\langle \neg \text{ColH } P \ A \ B \rangle$  colH-permut-312 by blast
    }
  }
  hence  $\neg \text{ColH } Q \ A \ B$ 
    by blast
  {
    assume same-side' P Q A B
    have outH A P Q
    proof -
      have  $\neg \text{ColH } P \ A \ B$ 
        by (simp add:  $\langle \neg \text{ColH } P \ A \ B \rangle$ )
      moreover have  $\neg \text{ColH } B \ A \ P$ 
        using calculation colH-permut-321 by blast
      moreover have CongaH B A P B A P
        using calculation(2) conga-refl by blast
      moreover have CongaH B A P B A Q
        by (metis  $\langle \text{CongH } A \ P \ A \ Q \rangle \langle \text{CongH } B \ P \ B \ Q \rangle \langle \neg \text{ColH } Q \ A \ B \rangle$  assms(2)
          calculation(2) colH-permut-213 colH-permut-312 congH-refl th18-aux)
      moreover have same-side' P P A B
        using calculation(1) colH-permut-312 same-side-prime-refl by blast
      ultimately show ?thesis
        using  $\langle \text{same-side}' \ P \ Q \ A \ B \rangle$ 
        using cong-4-uniqueness by blast
    qed
    hence ?thesis
      using  $\langle \neg \text{ColH } A \ P \ Q \rangle$  outH-expand by blast
  }
  moreover
  {
    assume cut' P Q A B
    obtain lAB where IsL lAB and IncidL A lAB and IncidL B lAB
      using assms(2) line-existence by blast
    then obtain I where IncidL I lAB and BetH P I Q
      using  $\langle \text{cut}' \ P \ Q \ A \ B \rangle$  cut'-def local.cut-def by auto
    hence ColH A I B
      using ColH-def  $\langle \text{IncidL } A \ lAB \rangle \langle \text{IncidL } B \ lAB \rangle \langle \text{IsL } lAB \rangle$  by auto
    {
      assume BetH A I B
      obtain lPQ where IsL lPQ and IncidL P lPQ and IncidL Q lPQ

```

```

    using assms(1) line-existence by blast
  hence  $\neg \text{cut } lPQ \ A \ B$ 
    using  $\langle \text{same-side}' \ A \ B \ P \ Q \rangle$  same-side'-def same-side-not-cut by auto
  have  $\text{cut } lPQ \ A \ B$ 
  proof -
    have  $\neg \text{IncidL } B \ lPQ$ 
      using Hilbert-neutral-dimensionless-pre.ColH-def  $\langle \text{IncidL } P \ lPQ \rangle$ 
         $\langle \text{IncidL } Q \ lPQ \rangle$   $\langle \text{IsL } lPQ \rangle$   $\langle \neg \text{ColH } B \ P \ Q \rangle$  by fastforce
    moreover have  $\text{IncidL } I \ lPQ$ 
      using  $\langle \text{BetH } P \ I \ Q \rangle$   $\langle \text{IncidL } P \ lPQ \rangle$   $\langle \text{IncidL } Q \ lPQ \rangle$  assms(1)
        betH-line inter-incid-uniquenessH by blast
    ultimately show ?thesis
      using  $\langle \text{BetH } A \ I \ B \rangle$ 
        using Is-line  $\langle \text{ColH } A \ I \ B \rangle$  betH-distincts colH-IncidL--IncidL
          local.cut-def by blast
  qed
  hence False
    using  $\langle \neg \text{local.cut } lPQ \ A \ B \rangle$  by blast
  hence ?thesis
    by blast
}
moreover have  $\text{BetH } I \ B \ A \longrightarrow ?thesis$ 
  using TS-upper-dim-bis  $\langle A \neq P \rangle$   $\langle A \neq Q \rangle$   $\langle B \neq P \rangle$   $\langle B \neq Q \rangle$   $\langle \text{BetH } P \ I \ Q \rangle$ 
     $\langle C \neq P \rangle$   $\langle C \neq Q \rangle$   $\langle \text{CongH } A \ P \ A \ Q \rangle$   $\langle \text{CongH } B \ P \ B \ Q \rangle$   $\langle \text{CongH } C \ P \ C \ Q \rangle$ 
    assms(1) assms(2) assms(3) assms(4) by blast
moreover have  $\text{BetH } I \ A \ B \longrightarrow ?thesis$ 
  using TS-upper-dim-bis  $\langle A \neq P \rangle$   $\langle A \neq Q \rangle$   $\langle B \neq P \rangle$   $\langle B \neq Q \rangle$   $\langle \text{BetH } P \ I \ Q \rangle$ 
     $\langle C \neq P \rangle$   $\langle C \neq Q \rangle$   $\langle \text{CongH } A \ P \ A \ Q \rangle$   $\langle \text{CongH } B \ P \ B \ Q \rangle$   $\langle \text{CongH } C \ P \ C \ Q \rangle$ 
    assms(1) assms(2) assms(3) assms(4) bet-comm by blast
ultimately have ?thesis
  by (metis  $\langle \text{BetH } P \ I \ Q \rangle$   $\langle \text{ColH } A \ I \ B \rangle$   $\langle \neg \text{ColH } A \ P \ Q \rangle$   $\langle \neg \text{ColH } B \ P \ Q \rangle$ 
    assms(2) betH-colH between-one colH-permut-213)
}
ultimately have ?thesis
  using  $\langle \neg \text{ColH } P \ A \ B \rangle$   $\langle \neg \text{ColH } Q \ A \ B \rangle$  plane-separation-2D by blast
}
ultimately have ?thesis
  using  $\langle \neg \text{ColH } A \ P \ Q \rangle$   $\langle \neg \text{ColH } C \ P \ Q \rangle$ 
     $\langle \text{cut}' \ A \ C \ P \ Q \longrightarrow \text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B \rangle$ 
     $\langle \text{same-side}' \ A \ C \ P \ Q \implies \text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B \rangle$ 
    plane-separation-2D by blast
}
ultimately have ?thesis
  using  $\langle \neg \text{ColH } A \ P \ Q \rangle$   $\langle \neg \text{ColH } B \ P \ Q \rangle$  plane-separation-2D by blast
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed
end
end

```

theory Hilbert-Neutral-3D

imports Hilbert-Neutral

begin

27 Hilbert - Geometry - Neutral 3D

27.1 Axioms: neutral 3D

locale *Hilbert-neutral-3D* = *Hilbert-neutral-dimensionless* *IncidL* *IncidP* *EqL* *EqP* *IsL* *IsP* *BetH* *CongH* *CongaH*
for
IncidL :: 'p ⇒ 'b ⇒ bool **and**
IncidP :: 'p ⇒ 'c ⇒ bool **and**
EqL :: 'b ⇒ 'b ⇒ bool **and**
EqP :: 'c ⇒ 'c ⇒ bool **and**
IsL :: 'b ⇒ bool **and**
IsP :: 'c ⇒ bool **and**
BetH :: 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongaH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool +
fixes *HS1* *HS2* *HS3* *HS4* :: 'p
assumes *plane-intersection*:
IsP p ∧ *IsP* q ∧ *IncidP* A p ∧ *IncidP* A q
→
(∃ B. *IncidP* B p ∧ *IncidP* B q ∧ A ≠ B)
and *lower-dim-3*:
¬ (∃ p. *IsP* p ∧ *IncidP* *HS1* p ∧ *IncidP* *HS2* p ∧ *IncidP* *HS3* p ∧ *IncidP* *HS4* p)
end

theory *Hilbert-Euclidean*

imports

Hilbert-Neutral
Tarski-Euclidean
Tarski-Neutral-Hilbert

begin

28 Hilbert - Geometry - Euclidean

28.1 Axioms: Euclidean

locale *Hilbert-Euclidean* = *Hilbert-neutral-dimensionless* *IncidL* *IncidP* *EqL* *EqP* *IsL* *IsP* *BetH* *CongH* *CongaH*
for
IncidL :: 'p ⇒ 'b ⇒ bool **and**
IncidP :: 'p ⇒ 'c ⇒ bool **and**
EqL :: 'b ⇒ 'b ⇒ bool **and**
EqP :: 'c ⇒ 'c ⇒ bool **and**
IsL :: 'b ⇒ bool **and**
IsP :: 'c ⇒ bool **and**
BetH :: 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool **and**
CongaH :: 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ 'p ⇒ bool +
assumes *euclid-uniqueness*: ∀ l P m1 m2. *IsL* l ∧ *IsL* m1 ∧ *IsL* m2 ∧
¬ *IncidL* P l ∧ *Para* l m1 ∧ *IncidL* P m1 ∧ *Para* l m2 ∧ *IncidL* P m2
→
EqL m1 m2

context *Tarski-Euclidean*

begin

28.2 Definitions

28.3 Propositions

lemma *Para-Par*:

assumes A ≠ B **and**
C ≠ D **and**

```

  Para-H (Line A B) (Line C D)
shows A B Par C D
proof -
  have isLine (Line A B)
    using assms(3) Para-H-def by blast
  have isLine (Line C D)
    using assms(3) Para-H-def by blast
  have I:  $\neg(\exists X. \text{IncidentL } X \text{ (Line A B)} \wedge \text{IncidentL } X \text{ (Line C D)})$   $\wedge$ 
 $(\exists p. \text{IncidentLP (Line A B) } p \wedge \text{IncidentLP (Line C D) } p)$ 
    using assms(3) Para-H-def by blast
  then obtain p where IncidentLP (Line A B) p and IncidentLP (Line C D) p
    by blast
  have Coplanar A B C D
proof -
  have IncidentP A p
    using IncidentLP-def  $\langle \text{IncidentLP (Line A B) } p \rangle$  assms(1) axiom-line-existence
    eq-incident incident-eq by blast
  moreover have IncidentP B p
    using IncidentLP-def  $\langle \text{IncidentLP (Line A B) } p \rangle$  assms(1) axiom-line-existence
    eq-incident incident-eq by blast
  moreover have IncidentP C p
    using IncidentLP-def  $\langle \text{IncidentLP (Line C D) } p \rangle$  assms(2) axiom-line-existence
    eq-incident incident-eq by blast
  moreover have IncidentP D p
    using IncidentLP-def  $\langle \text{IncidentLP (Line C D) } p \rangle$  assms(2) axiom-line-existence
    eq-incident incident-eq by blast
  ultimately show ?thesis
    by (simp add: plane-cop)
qed
moreover
{
  assume  $\exists X. \text{Col } X \text{ A B} \wedge \text{Col } X \text{ C D}$ 
  then obtain X where Col X A B and Col X C D
    by blast
  hence Col-H X A B
    by (simp add: cols-coincide-2)
  hence IncidentL X (Line A B)
    using assms(1) IncidentL-def Col-H-def by (meson EqTL-def incident-eq)
  moreover
  have Col-H X C D
    using cols-coincide-2 by (simp add:  $\langle \text{Col } X \text{ C D} \rangle$ )
  hence IncidentL X (Line C D)
    using assms(2) IncidentL-def Col-H-def eq-incident incident-eq by blast
  ultimately have False
    using 1 by blast
}
hence  $\neg(\exists X. \text{Col } X \text{ A B} \wedge \text{Col } X \text{ C D})$ 
  by blast
ultimately show ?thesis
  using Par-def by (meson cop-npar--inter-exists)
qed

lemma axiom-euclid-uniqueness:
  assumes  $\neg \text{IncidentL } P \text{ l}$  and
    Para-H l m1 and
    IncidentL P m1 and
    Para-H l m2 and
    IncidentL P m2
  shows  $m1 = l = m2$ 
proof -
  let ?A = fst l
  let ?B = snd l
  let ?C = fst m1
  let ?D = snd m1
  let ?C' = fst m2
  let ?D' = snd m2

```

```

have l = Line ?A ?B
  using Line-def Para-H-def assms(4) isLine-def by auto
have m1 = Line ?C ?D
  using Line-def Para-H-def assms(2) isLine-def by auto
have m2 = Line ?C' ?D'
  using Line-def Para-H-def assms(4) isLine-def by auto
have ¬ Col P ?A ?B
  using IncidentL-def Para-H-def assms(1) assms(4) by blast
have Para-H (Line ?A ?B) (Line ?C ?D)
  using assms(2) ⟨l = Line (fst l) (snd l)⟩ ⟨m1 = Line (fst m1) (snd m1)⟩ by force
have Para-H (Line ?A ?B) (Line ?C' ?D')
  using ⟨l = Line (fst l) (snd l)⟩ ⟨m2 = Line (fst m2) (snd m2)⟩ assms(4) by fastforce
have ?A ?B Par ?C ?D using Para-Par
  using IncidentL-def ⟨Para-H (Line (fst l) (snd l)) (Line (fst m1) (snd m1))⟩
  ⟨¬ Col P (fst l) (snd l)⟩ assms(3) isLine-def not-col-distincts by auto
have Col P ?C ?D
  using IncidentL-def assms(3) by auto
have ?A ?B Par ?C' ?D'
  using IncidentL-def Para-Par ⟨Para-H (Line (fst l) (snd l)) (Line (fst m2) (snd m2))⟩
  ⟨(fst l) (snd l) Par (fst m1) (snd m1)⟩ assms(5)
  isLine-def par-distinct by blast
have Col P ?C' ?D'
  by (simp add: assms(5) incident-col)
have Col ?C' ?C ?D
  using ⟨Col P (fst m1) (snd m1)⟩ ⟨Col P (fst m2) (snd m2)⟩
  ⟨(fst l) (snd l) Par (fst m1) (snd m1)⟩
  ⟨(fst l) (snd l) Par (fst m2) (snd m2)⟩ parallel-uniqueness by blast
moreover
have Col ?D' ?C ?D
  using ⟨Col P (fst m1) (snd m1)⟩ ⟨Col P (fst m2) (snd m2)⟩
  ⟨(fst l) (snd l) Par (fst m1) (snd m1)⟩
  ⟨(fst l) (snd l) Par (fst m2) (snd m2)⟩ parallel-uniqueness by blast
ultimately show ?thesis
  using tarski-s-euclid-implies-playfair-s-postulate
  by (metis IncidentL-def ⟨(fst l) (snd l) Par (fst m2) (snd m2)⟩
    ⟨m2 = Line (fst m2) (snd m2)⟩ assms(3) eq-symmetry incident-eq par-distincts)
qed

end
end

```

theory Tarski-Neutral-Model-Gupta-Neutral

imports Gupta-Neutral

begin

context Gupta-neutral-dimensionless

begin

interpretation Interpretation-Tarski-neutral-dimensionless: Tarski-neutral-dimensionless
where TPA = GPA **and** TPB = GPB **and** TPC = GPC **and** Bet = BetG **and** Cong = CongG
proof
show $\forall a b. CongG a b b a$
by (simp add: cong-pseudo-reflexivityG)
show $\forall a b p q r s. CongG a b p q \wedge CongG a b r s \longrightarrow CongG p q r s$
using cong-inner-transitivityG **by** blast
show $\forall a b c. CongG a b c c \longrightarrow a = b$
by (simp add: cong-identityG)
show $\forall a b c q. \exists x. BetG q a x \wedge CongG a x b c$
by (simp add: segment-constructionG)
show $\forall a b c d a' b' c' d'. a \neq b \wedge BetG a b c \wedge BetG a' b' c' \wedge$
 $CongG a b a' b' \wedge CongG b c b' c' \wedge CongG a d a' d' \wedge CongG b d b' d' \longrightarrow$
 $CongG c d c' d'$

```

using five-segmentG by blast
show  $\forall a b. \text{BetG } a b a \longrightarrow a = b$ 
by (simp add: between-identityT)
show  $\forall a b c p q. \text{BetG } a p c \wedge \text{BetG } b q c \longrightarrow (\exists x. \text{BetG } p x b \wedge \text{BetG } q x a)$ 
using inner-paschT by blast
show  $\neg \text{BetG } GPA GPB GPC \wedge \neg \text{BetG } GPB GPC GPA \wedge \neg \text{BetG } GPC GPA GPB$ 
using lower-dimG by blast
qed

```

28.4 Transport theorem from Tarski Neutral for Gupta Euclidean Model

```

lemma g-l5-2:
assumes  $A \neq B$  and
  BetG A B C and
  BetG A B D
shows  $\text{BetG } B C D \vee \text{BetG } B D C$ 
using Interpretation-Tarski-neutral-dimensionless.l5-2 assms(1) assms(2) assms(3) by blast

```

```

lemma g-l5-3:
assumes BetG A B D and
  BetG A C D
shows  $\text{BetG } A B C \vee \text{BetG } A C B$ 
using Interpretation-Tarski-neutral-dimensionless.l5-3 assms(1) assms(2) by blast

```

```

lemma g-between-exchange4:
assumes BetG A B C and
  BetG A C D
shows BetG A B D
using assms(1) assms(2)
Interpretation-Tarski-neutral-dimensionless.between-exchange4 by blast

```

```

lemma g-between-inner-transitivity:
assumes BetG A B D and
  BetG B C D
shows BetG A B C
using assms(1) assms(2)
Interpretation-Tarski-neutral-dimensionless.between-inner-transitivity by blast

```

```

lemma g-not-bet-distincts:
assumes  $\neg \text{BetG } A B C$ 
shows  $A \neq B \wedge B \neq C$ 
using assms Interpretation-Tarski-neutral-dimensionless.not-bet-distincts by blast

```

```

lemma g-between-symmetry:
assumes BetG A B C
shows BetG C B A
using assms Interpretation-Tarski-neutral-dimensionless.between-symmetry by blast

```

```

lemma g-between-trivial:
shows BetG A B B
using Interpretation-Tarski-neutral-dimensionless.between-trivial by blast

```

```

end
end

```

```

theory Tarski-Neutral-2D-Model-Gupta-Neutral-2D

```

```

imports
  Gupta-Neutral-2D
  Tarski-Neutral-2D

```

```

begin

```

```

context Gupta-neutral-2D

```


begin

interpretation *Interpretation-Tarski-neutral-2D : Tarski-neutral-2D*

where *TPA = GPA and TPB = GPB and TPC = GPC and Bet = BetG and Cong = CongG*

proof

show $\forall a b. \text{CongG } a b b a$

by (*simp add: cong-pseudo-reflexivityG*)

show $\forall a b p q r s. \text{CongG } a b p q \wedge \text{CongG } a b r s \longrightarrow \text{CongG } p q r s$

using *cong-inner-transitivityG* **by** *blast*

show $\forall a b c. \text{CongG } a b c c \longrightarrow a = b$

by (*simp add: cong-identityG*)

show $\forall a b c q. \exists x. \text{BetG } q a x \wedge \text{CongG } a x b c$

by (*simp add: segment-constructionG*)

show $\forall a b c d a' b' c' d'. a \neq b \wedge \text{BetG } a b c \wedge \text{BetG } a' b' c' \wedge$

$\text{CongG } a b a' b' \wedge \text{CongG } b c b' c' \wedge \text{CongG } a d a' d' \wedge \text{CongG } b d b' d' \longrightarrow$

$\text{CongG } c d c' d'$

using *five-segmentG* **by** *blast*

show $\forall a b. \text{BetG } a b a \longrightarrow a = b$

by (*simp add: between-identityT*)

show $\forall a b c p q. \text{BetG } a p c \wedge \text{BetG } b q c \longrightarrow (\exists x. \text{BetG } p x b \wedge \text{BetG } q x a)$

using *inner-paschT* **by** *blast*

show $\neg \text{BetG } \text{GPA } \text{GPB } \text{GPC} \wedge \neg \text{BetG } \text{GPB } \text{GPC } \text{GPA} \wedge \neg \text{BetG } \text{GPC } \text{GPA } \text{GPB}$

using *lower-dimG* **by** *blast*

show $\forall a b c p q.$

$p \neq q \wedge \text{CongG } a p a q \wedge \text{CongG } b p b q \wedge \text{CongG } c p c q \longrightarrow$

$\text{BetG } a b c \vee \text{BetG } b c a \vee \text{BetG } c a b$ **using** *upper-dimT* **by** *blast*

qed

end

end

theory *Tarski-Euclidean-Model-Gupta-Euclidean*

imports

Tarski-Neutral-Model-Gupta-Neutral

Gupta-Euclidean

Tarski-Euclidean

begin

context *Gupta-Euclidean*

begin

28.5 Interpretation Tarski Gupta

lemma *euclidT*:

assumes *BetG A D T and*

BetG B D C and

A $\neq$ D and

$\forall A B C D T.$

BetG A D T $\wedge$ BetG B D C $\wedge$ A $\neq$ D

$\longrightarrow$

($\exists X Y. \text{BetG } A B X \wedge \text{BetG } A C Y \wedge \text{BetG } X T Y$)

shows $\exists X Y. \text{BetG } A B X \wedge \text{BetG } A C Y \wedge \text{BetG } X T Y$

proof –

{

assume *B $\neq$ D*

{

assume *ColG A B C*

{

assume *BetG A B C*

{

assume *A $\neq$ B*

moreover have *BetG A B C*

```

    using ⟨BetG A B C⟩ by blast
  moreover have BetG A B T
    using assms(1) assms(2) g-between-exchange4
      g-between-inner-transitivity calculation(2) by blast
  moreover have BetG B C T  $\longrightarrow$  ?thesis
    using calculation(3) g-not-bet-distincts by blast
  moreover have BetG B T C  $\longrightarrow$  ?thesis
    using g-not-bet-distincts by blast
  ultimately have ?thesis using g-l5-2 by blast
}
hence ?thesis
  using g-between-symmetry g-between-trivial by blast
}
moreover have BetG B C A  $\longrightarrow$  ?thesis
  using assms(1) assms(2) assms(3) assms(4) by presburger
moreover have BetG C A B  $\longrightarrow$  ?thesis
  using assms(1) assms(2) assms(3) assms(4) by presburger
ultimately have ?thesis
  using ColG-def ⟨ColG A B C⟩ by blast
}
moreover
{
  assume  $\neg$  ColG A B C
  {
    {
      assume  $D \neq C$ 
      hence ?thesis
        using assms(1) assms(2) assms(3) assms(4) by blast
    }
    hence ?thesis
      using assms(1) g-not-bet-distincts by blast
  }
  hence ?thesis
    by blast
}
ultimately have ?thesis
  by blast
}
thus ?thesis
  using assms(1) g-between-symmetry g-between-trivial by blast
qed

```

lemma lem-euclidG:

```

assumes BetG A D T and
  BetG B D C and
  A  $\neq$  D
shows  $\exists X Y. \text{BetG } A B X \wedge \text{BetG } A C Y \wedge \text{BetG } X T Y$ 
proof -
{
  assume B  $\neq$  D and D  $\neq$  C and
     $\neg (\text{BetG } A B C \vee \text{BetG } B C A \vee \text{BetG } C A B)$ 
  hence ?thesis
    using assms(1) assms(2) euclidG by presburger
}
moreover
{
  assume B = D
  hence ?thesis
    using assms(1) g-not-bet-distincts by blast
}
moreover
{
  assume D = C
  hence ?thesis
    using assms(1) g-between-trivial by blast
}

```

```

moreover
{
  assume  $BetG\ A\ B\ C \vee BetG\ B\ C\ A \vee BetG\ C\ A\ B$ 
  moreover have  $BetG\ A\ B\ C \longrightarrow ?thesis$ 
    by (meson assms(1) assms(2) between-inner-transitivityG g-between-exchange4
      g-not-bet-distincts)
  moreover have  $BetG\ B\ C\ A \longrightarrow ?thesis$ 
    by (metis assms(1) assms(2) between-inner-transitivityG between-symmetryG
      g-between-exchange4 g-not-bet-distincts segment-constructionG)
  moreover
  {
    assume  $BetG\ C\ A\ B$ 
    hence  $BetG\ B\ A\ D \vee BetG\ B\ D\ A$ 
    using assms(1) assms(2) g-l5-3 g-between-symmetry by blast
    moreover
    {
      assume  $BetG\ B\ A\ D$ 
      hence  $\exists\ X\ Y. BetG\ A\ B\ X \wedge BetG\ A\ C\ Y \wedge BetG\ X\ T\ Y$ 
      using assms(1) assms(2) assms(3) between-symmetryG g-between-exchange4 g-l5-2
        g-not-bet-distincts by metis
    }
    moreover
    {
      assume  $BetG\ B\ D\ A$ 
      hence  $\exists\ X\ Y. BetG\ A\ B\ X \wedge BetG\ A\ C\ Y \wedge BetG\ X\ T\ Y$ 
      using assms(1) assms(2) assms(3) between-symmetryG g-between-exchange4 g-l5-2
        g-not-bet-distincts by metis
    }
    ultimately have  $\exists\ X\ Y. BetG\ A\ B\ X \wedge BetG\ A\ C\ Y \wedge BetG\ X\ T\ Y$ 
    by blast
  }
  ultimately have  $?thesis$ 
  by blast
}
ultimately show  $?thesis$ 
by blast
qed

```

interpretation *Interpretation-Tarski-euclidean : Tarski-Euclidean*

where $TPA = GPA$ **and** $TPB = GPB$ **and** $TPC = GPC$ **and** $Bet = BetG$ **and** $Cong = CongG$

proof

```

show  $\forall a\ b. CongG\ a\ b\ b\ a$ 
  by (simp add: cong-pseudo-reflexivityG)
show  $\forall a\ b\ p\ q\ r\ s. CongG\ a\ b\ p\ q \wedge CongG\ a\ b\ r\ s \longrightarrow CongG\ p\ q\ r\ s$ 
  using cong-inner-transitivityG by blast
show  $\forall a\ b\ c. CongG\ a\ b\ c\ c \longrightarrow a = b$ 
  by (simp add: cong-identityG)
show  $\forall a\ b\ c\ q. \exists x. BetG\ q\ a\ x \wedge CongG\ a\ x\ b\ c$ 
  by (simp add: segment-constructionG)
show  $\forall a\ b\ c\ d\ a'\ b'\ c'\ d'. a \neq b \wedge BetG\ a\ b\ c \wedge BetG\ a'\ b'\ c' \wedge$ 
   $CongG\ a\ b\ a'\ b' \wedge CongG\ b\ c\ b'\ c' \wedge CongG\ a\ d\ a'\ d' \wedge CongG\ b\ d\ b'\ d' \longrightarrow$ 
   $CongG\ c\ d\ c'\ d'$ 
  using five-segmentG by blast
show  $\forall a\ b. BetG\ a\ b\ a \longrightarrow a = b$ 
  by (simp add: between-identityT)
show  $\forall a\ b\ c\ p\ q. BetG\ a\ p\ c \wedge BetG\ b\ q\ c \longrightarrow (\exists x. BetG\ p\ x\ b \wedge BetG\ q\ x\ a)$ 
  using inner-paschT by blast
show  $\neg BetG\ GPA\ GPB\ GPC \wedge \neg BetG\ GPB\ GPC\ GPA \wedge \neg BetG\ GPC\ GPA\ GPB$ 
  using lower-dimG by blast
show  $\forall A\ B\ C\ D\ T.$ 
   $BetG\ A\ D\ T \wedge BetG\ B\ D\ C \wedge A \neq D \longrightarrow$ 
   $(\exists X\ Y. BetG\ A\ B\ X \wedge BetG\ A\ C\ Y \wedge BetG\ X\ T\ Y)$ 
  using lem-euclidG by blast

```

qed

end

end

theory *Gupta-Neutral-Model-Tarski-Neutral*

imports

Tarski-Neutral

Gupta-Neutral

begin

context *Tarski-neutral-dimensionless*

begin

interpretation *Interpretation-Gupta-neutral-dimensionless : Gupta-neutral-dimensionless*

where $GPA = TPA$ **and** $GPB = TPB$ **and** $GPC = TPC$ **and** $BetG = Bet$ **and** $CongG = Cong$

proof

show $\forall a b. Cong\ a\ b\ b\ a$

by (*simp add: cong-pseudo-reflexivity*)

show $\forall a b p q r s. Cong\ a\ b\ p\ q \wedge Cong\ a\ b\ r\ s \longrightarrow Cong\ p\ q\ r\ s$

using *cong-inner-transitivity* **by** *blast*

show $\forall a b c. Cong\ a\ b\ c\ c \longrightarrow a = b$

using *cong-diff* **by** *blast*

show $\forall a b c q. \exists x. Bet\ q\ a\ x \wedge Cong\ a\ x\ b\ c$

using *segment-construction* **by** *force*

show $\forall a b c d a' b' c' d'. a \neq b \wedge Bet\ a\ b\ c \wedge Bet\ a' b' c' \wedge$

$Cong\ a\ b\ a' b' \wedge Cong\ b\ c\ b' c' \wedge Cong\ a\ d\ a' d' \wedge Cong\ b\ d\ b' d' \longrightarrow$

$Cong\ c\ d\ c' d'$

using *five-segment* **by** *blast*

show $\forall a b c. Bet\ a\ b\ c \longrightarrow Bet\ c\ b\ a$

using *between-symmetry* **by** *blast*

show $\forall a b c d. Bet\ a\ b\ d \wedge Bet\ b\ c\ d \longrightarrow Bet\ a\ b\ c$

using *between-inner-transitivity* **by** *blast*

show $\forall a b c p q.$

$Bet\ a\ p\ c \wedge Bet\ b\ q\ c \wedge a \neq p \wedge c \neq p \wedge b \neq q \wedge c \neq q \wedge$

$\neg (Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b) \longrightarrow$

$(\exists x. Bet\ p\ x\ b \wedge Bet\ q\ x\ a)$

using *inner-pasch* **by** *blast*

show $\neg (Bet\ TPA\ TPB\ TPC \vee Bet\ TPB\ TPC\ TPA \vee Bet\ TPC\ TPA\ TPB)$

by (*simp add: lower-dim*)

qed

end

end

theory *Gupta-Neutral-2D-Model-Tarski-Neutral-2D*

imports

Tarski-Neutral-2D

Gupta-Neutral-2D

begin

context *Tarski-neutral-2D*

begin

interpretation *Interpretation-Gupta-neutral-2D : Gupta-neutral-2D*

where $GPA = TPA$ **and** $GPB = TPB$ **and** $GPC = TPC$ **and** $BetG = Bet$ **and** $CongG = Cong$

proof

show $\forall a b. Cong\ a\ b\ b\ a$

by (*simp add: cong-pseudo-reflexivity*)

show $\forall a b p q r s. Cong\ a\ b\ p\ q \wedge Cong\ a\ b\ r\ s \longrightarrow Cong\ p\ q\ r\ s$

using *cong-inner-transitivity* **by** *blast*

```

show  $\forall a b c. \text{Cong } a b c c \longrightarrow a = b$ 
  using cong-diff by blast
show  $\forall a b c q. \exists x. \text{Bet } q a x \wedge \text{Cong } a x b c$ 
  using segment-construction by force
show  $\forall a b c d a' b' c' d'. a \neq b \wedge \text{Bet } a b c \wedge \text{Bet } a' b' c' \wedge$ 
   $\text{Cong } a b a' b' c' \wedge \text{Cong } b c b' c' \wedge \text{Cong } a d a' d' \wedge \text{Cong } b d b' d' \longrightarrow$ 
   $\text{Cong } c d c' d'$ 
  using five-segment by blast
show  $\forall a b c. \text{Bet } a b c \longrightarrow \text{Bet } c b a$ 
  using between-symmetry by blast
show  $\forall a b c d. \text{Bet } a b d \wedge \text{Bet } b c d \longrightarrow \text{Bet } a b c$ 
  using between-inner-transitivity by blast
show  $\forall a b c p q.$ 
   $\text{Bet } a p c \wedge \text{Bet } b q c \wedge a \neq p \wedge c \neq p \wedge b \neq q \wedge c \neq q \wedge$ 
   $\neg (\text{Bet } a b c \vee \text{Bet } b c a \vee \text{Bet } c a b) \longrightarrow$ 
   $(\exists x. \text{Bet } p x b \wedge \text{Bet } q x a)$ 
  using inner-pasch by blast
show  $\neg (\text{Bet } \text{TPA } \text{TPB } \text{TPC} \vee \text{Bet } \text{TPB } \text{TPC } \text{TPA} \vee \text{Bet } \text{TPC } \text{TPA } \text{TPB})$ 
  by (simp add: lower-dim)
show  $\forall a b c p q.$ 
   $p \neq q \wedge a \neq b \wedge a \neq c \wedge c \neq b \wedge \text{Cong } a p a q \wedge \text{Cong } b p b q \wedge \text{Cong } c p c q \longrightarrow$ 
   $\text{Bet } a b c \vee \text{Bet } b c a \vee \text{Bet } c a b$ 
  using upper-dim by blast
qed

end
end

theory Gupta-Euclidean-Model-Tarski-Euclidean

imports
  Tarski-Euclidean
  Gupta-Euclidean

begin

context Tarski-Euclidean

begin

interpretation Interpretation-Gupta-euclidean : Gupta-Euclidean
  where  $\text{GPA} = \text{TPA}$  and  $\text{GPB} = \text{TPB}$  and  $\text{GPC} = \text{TPC}$  and  $\text{BetG} = \text{Bet}$  and  $\text{CongG} = \text{Cong}$ 
proof
show  $\forall a b. \text{Cong } a b b a$ 
  by (simp add: cong-pseudo-reflexivity)
show  $\forall a b p q r s. \text{Cong } a b p q \wedge \text{Cong } a b r s \longrightarrow \text{Cong } p q r s$ 
  using cong-inner-transitivity by blast
show  $\forall a b c. \text{Cong } a b c c \longrightarrow a = b$ 
  using cong-diff by blast
show  $\forall a b c q. \exists x. \text{Bet } q a x \wedge \text{Cong } a x b c$ 
  using segment-construction by force
show  $\forall a b c d a' b' c' d'. a \neq b \wedge \text{Bet } a b c \wedge \text{Bet } a' b' c' \wedge \text{Cong } a b a' b' b' \wedge$ 
   $\text{Cong } b c b' c' \wedge \text{Cong } a d a' d' \wedge \text{Cong } b d b' d' \longrightarrow$ 
   $\text{Cong } c d c' d'$ 
  using five-segment by blast
show  $\forall a b c. \text{Bet } a b c \longrightarrow \text{Bet } c b a$ 
  using between-symmetry by blast
show  $\forall a b c d. \text{Bet } a b d \wedge \text{Bet } b c d \longrightarrow \text{Bet } a b c$ 
  using between-inner-transitivity by blast
show  $\forall a b c p q.$ 
   $\text{Bet } a p c \wedge \text{Bet } b q c \wedge a \neq p \wedge c \neq p \wedge b \neq q \wedge c \neq q \wedge$ 
   $\neg (\text{Bet } a b c \vee \text{Bet } b c a \vee \text{Bet } c a b) \longrightarrow$ 
   $(\exists x. \text{Bet } p x b \wedge \text{Bet } q x a)$ 
  using inner-pasch by blast
show  $\neg (\text{Bet } \text{TPA } \text{TPB } \text{TPC} \vee \text{Bet } \text{TPB } \text{TPC } \text{TPA} \vee \text{Bet } \text{TPC } \text{TPA } \text{TPB})$ 

```

```

    by (simp add: lower-dim)
  show  $\forall A B C D T.$ 
     $Bet A D T \wedge$ 
     $Bet B D C \wedge B \neq D \wedge D \neq C \wedge \neg (Bet A B C \vee Bet B C A \vee Bet C A B) \longrightarrow$ 
     $(\exists x y. Bet A B x \wedge Bet A C y \wedge Bet x T y)$ 
  using Bet-perm tarski-s-parallel-postulate-def tarski-s-parallel-postulate-thm by blast
qed

end
end

```

theory *Hilbert-Neutral-Model-Tarski-Neutral*

imports *Tarski-Neutral-Hilbert*

begin

29 Hilbert Neutral - Tarski Neutral Model

29.1 Interpretation

context *Tarski-neutral-dimensionless*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre: Hilbert-neutral-dimensionless-pre*

where *IncidL = IncidentL and*

IncidP = IncidentP and

EqL = EqTL and

EqP = EqTP and

IsL = isLine and

IsP = isPlane and

BetH = Between-H and

CongH = Cong and

CongaH = CongA-H

proof –

qed

interpretation *Intrepretation-Hilbert-neutral-dimensionless: Hilbert-neutral-dimensionless*

where *IncidL = IncidentL and*

IncidP = IncidentP and

EqL = EqTL and

EqP = EqTP and

IsL = isLine and

IsP = isPlane and

BetH = Between-H and

CongH = Cong and

CongaH = CongA-H and

PP = TPA and

PQ = TPB and

PR = TPC

proof

have $\forall A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C \longleftrightarrow Col-H A B C$

proof –

{

fix $A B C$

assume *Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C*

hence $\exists l. isLine l \wedge IncidentL A l \wedge IncidentL B l \wedge IncidentL C l$

by (simp add: Hilbert-neutral-dimensionless-pre.ColH-def)

hence *Col-H A B C*

using *Col-H-def EqTL-def axiom-line-uniqueness cols-coincide-2*

not-col-distincts **by** blast

}

```

moreover
{
  fix  $A B C$ 
  assume  $Col-H A B C$ 
  hence  $\exists l. isLine\ l \wedge IncidentL\ A\ l \wedge IncidentL\ B\ l \wedge IncidentL\ C\ l$ 
    using  $Col-H-def$  by blast
  have  $Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ A\ B\ C$ 
    using  $Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def$ 
     $\langle \exists l. isLine\ l \wedge IncidentL\ A\ l \wedge IncidentL\ B\ l \wedge IncidentL\ C\ l \rangle$  by blast
}
ultimately show ?thesis
by blast
qed
have  $\forall A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ A\ B\ C \longleftrightarrow Col\ A\ B\ C$ 
  using  $\langle \forall A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ A\ B\ C = Col-H\ A\ B\ C \rangle$ 
  cols-coincide by blast

have  $1: \forall A B l. Interpretation-Hilbert-neutral-dimensionless-pre.same-side\ A\ B\ l$ 
 $\longleftrightarrow same-side-H\ A\ B\ l$ 
  by (simp add: Hilbert-neutral-dimensionless-pre.cut-def
    Interpretation-Hilbert-neutral-dimensionless-pre.same-side-def
    cut-H-def same-side-H-def)
have  $\forall A B C D. Interpretation-Hilbert-neutral-dimensionless-pre.same-side'\ A\ B\ C\ D$ 
 $\longleftrightarrow same-side'-H\ A\ B\ C\ D$ 
  by (simp add: Hilbert-neutral-dimensionless-pre.same-side'-def 1 same-side'-H-def)

show  $\bigwedge l. isLine\ l \longrightarrow l = l = l$ 
  by (simp add: eq-reflexivity)
show  $\bigwedge l1\ l2. isLine\ l1 \wedge isLine\ l2 \wedge l1 = l = l2 \longrightarrow l2 = l = l1$ 
  using eq-symmetry by blast
show  $\bigwedge l1\ l2\ l3. l1 = l = l2 \wedge l2 = l = l3 \longrightarrow l1 = l = l3$ 
  using eq-transitivity by blast
show  $\bigwedge p. isPlane\ p \longrightarrow p = p = p$ 
  by (simp add: eqp-reflexivity)
show  $\bigwedge p1\ p2. p1 = p = p2 \longrightarrow p2 = p = p1$ 
  using eqp-symmetry by blast
show  $\bigwedge p1\ p2\ p3. p1 = p = p2 \wedge p2 = p = p3 \longrightarrow p1 = p = p3$ 
  using eqp-transitivity by blast
show  $\bigwedge A\ B. A \neq B \longrightarrow (\exists l. isLine\ l \wedge IncidentL\ A\ l \wedge IncidentL\ B\ l)$ 
  using axiom-line-existence by blast
show  $\bigwedge A\ B\ l\ m.$ 
   $A \neq B \wedge isLine\ l \wedge isLine\ m \wedge$ 
   $IncidentL\ A\ l \wedge IncidentL\ B\ l \wedge IncidentL\ A\ m \wedge IncidentL\ B\ m \longrightarrow$ 
   $l = l = m$ 
  using axiom-line-uniqueness by blast
show  $\forall l. isLine\ l \longrightarrow (\exists A\ B. IncidentL\ A\ l \wedge IncidentL\ B\ l \wedge A \neq B)$ 
  using axiom-two-points-on-line by blast
show  $TPA \neq TPB \wedge TPB \neq TPC \wedge TPA \neq TPC \wedge$ 
   $\neg Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ TPA\ TPB\ TPC$ 
  using Bet-cases between-trivial lower-dim third-point
   $\langle \forall A\ B\ C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ A\ B\ C = (Col\ A\ B\ C) \rangle$ 
  by blast
show  $\forall A\ B\ C.$ 
   $\neg Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ A\ B\ C \longrightarrow$ 
   $(\exists p. isPlane\ p \wedge IncidentP\ A\ p \wedge IncidentP\ B\ p \wedge IncidentP\ C\ p)$ 
  using bet-coplanar between-trivial2 cop-plane IncidentP-def by blast
show  $\forall p. \exists A. isPlane\ p \longrightarrow IncidentP\ A\ p$ 
  by (simp add: axiom-one-point-on-plane)
show  $\bigwedge A\ B\ C\ p\ q.$ 
   $\neg Interpretation-Hilbert-neutral-dimensionless-pre.ColH\ A\ B\ C \wedge$ 
   $isPlane\ p \wedge isPlane\ q \wedge$ 
   $IncidentP\ A\ p \wedge$ 
   $IncidentP\ B\ p \wedge$ 
   $IncidentP\ C\ p \wedge IncidentP\ A\ q \wedge IncidentP\ B\ q \wedge IncidentP\ C\ q \longrightarrow$ 

```

```

    p = p = q
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    axiom-plane-uniqueness by blast
show  $\forall A B l p.$ 
   $A \neq B \wedge \text{isLine } l \wedge \text{isPlane } p \wedge$ 
   $\text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentP } A p \wedge \text{IncidentP } B p \longrightarrow$ 
   $\text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l p$ 
  by (meson IncidentLP-def Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
    axiom-line-on-plane)
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow A \neq C$ 
  using axiom-between-diff by blast
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$ 
  by (meson Between-H-def Col-H-def Col-def
    Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def cols-coincide-2)
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow \text{Between-H } C B A$ 
  using axiom-between-comm by blast
show  $\bigwedge A B. A \neq B \longrightarrow (\exists C. \text{Between-H } A B C)$ 
  by (simp add: axiom-between-out)
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow \neg \text{Between-H } B C A$ 
  using axiom-between-only-one by blast
{
  fix A B C p l
  assume  $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$  and
    isLine l and
    IncidentP A p and
    IncidentP B p and
    IncidentP C p and
    Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p and
     $\neg \text{IncidentL } C l$  and
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B
  have  $\neg \text{Col } A B C$ 
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
       $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \rangle$  cols-coincide-2)
  moreover have cut-H l A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.cut-def
       $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l A B \rangle$  cut-H-def by blast
  ultimately have cut-H l A C  $\vee$  cut-H l B C
    using axiom-pasch Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
       $\langle \text{IncidentP } A p \rangle \langle \text{IncidentP } B p \rangle \langle \text{IncidentP } C p \rangle$ 
       $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l p \rangle \langle \neg \text{IncidentL } C l \rangle$ 
      axiom-line-on-plane axiom-two-points-on-line cols-coincide-1 by meson
  hence Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C  $\vee$ 
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
    by (simp add: Hilbert-neutral-dimensionless-pre.cut-def cut-H-def)
}
thus  $\bigwedge A B C p l. \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \wedge$ 
  isLine l  $\wedge$  isPlane p  $\wedge$  IncidentP A p  $\wedge$  IncidentP B p  $\wedge$  IncidentP C p  $\wedge$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p  $\wedge$ 
   $\neg \text{IncidentL } C l \wedge \text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l A B \longrightarrow$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C  $\vee$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
  by blast
{
  fix l
  fix A B A' P::'p
  assume  $A \neq B$  and
     $A' \neq P$  and
    isLine l and
    IncidentL A' l and
    IncidentL P l
  then obtain B' where IncidentL B' l  $\wedge$  outH A' P B'  $\wedge$  Cong A' B' A B
    using axiom-hcong-1-existence by presburger
  hence  $\exists B'. \text{IncidentL } B' l \wedge$ 
    Interpretation-Hilbert-neutral-dimensionless-pre.outH A' P B'  $\wedge$  Cong A' B' A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.outH-def outH-def by auto
}

```



```

thus  $\bigwedge l A B A' P :: p. A \neq B \wedge A' \neq P \wedge \text{isLine } l \wedge \text{IncidentL } A' l \wedge \text{IncidentL } P l \longrightarrow$ 
  ( $\exists B'. \text{IncidentL } B' l \wedge$ 
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } A' P B' \wedge \text{Cong } A' B' A B$ )
  by blast
show  $\bigwedge A B C D E F. \text{Cong } A B C D \wedge \text{Cong } A B E F \longrightarrow \text{Cong } C D E F$ 
  using cong-inner-transitivity by blast
{
  fix  $A B C A' B' C'$ 
  assume  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$  and
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C'$  and
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A B B C$  and
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A' B' B' C'$  and
     $\text{Cong } A B A' B'$  and
     $\text{Cong } B C B' C'$ 
  have ( $\exists l. \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$ )
    using  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def}$ 
     $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \rangle$  by force
  then obtain  $l$  where  $\text{IncidentL } A l$  and  $\text{IncidentL } B l$  and  $\text{IncidentL } C l$ 
    by blast
  hence  $\text{Col } A B C$ 
    using  $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \rangle$ 
     $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = (\text{Col } A B C) \rangle$ 
    by auto
  moreover
  have ( $\exists l. \text{IncidentL } A' l \wedge \text{IncidentL } B' l \wedge \text{IncidentL } C' l$ )
    using  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def}$ 
     $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \rangle$  by force
  then obtain  $l'$  where  $\text{IncidentL } A' l'$  and  $\text{IncidentL } B' l'$  and  $\text{IncidentL } C' l'$ 
    by blast
  hence  $\text{Col } A' B' C'$ 
    using  $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \rangle$ 
     $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = (\text{Col } A B C) \rangle$ 
    by blast
  moreover
  have  $\neg (\exists P. \text{Between-H } A P B \wedge \text{Between-H } B P C)$ 
    using  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def}$ 
     $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A B B C \rangle$  by blast
  have  $\neg (\exists P. \text{Between-H } A' P B' \wedge \text{Between-H } B' P C')$ 
    using  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def}$ 
     $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A' B' B' C' \rangle$  by blast
  have  $\text{Col-H } A B C$ 
    using calculation(1) cols-coincide-2 by blast
  hence  $\text{Bet } A B C$ 
    using  $\langle \nexists P. \text{Between-H } A P B \wedge \text{Between-H } B P C \rangle$  col-disjoint-bet disjoint-H-def by blast
  moreover
  have  $\text{Col-H } A' B' C'$ 
    by (simp add: calculation(2) cols-coincide-2)
  moreover
  hence  $\text{Bet } A' B' C'$ 
    by (simp add:  $\langle \nexists P. \text{Between-H } A' P B' \wedge \text{Between-H } B' P C' \rangle$  col-disjoint-bet disjoint-H-def)
  ultimately
  have  $\text{Cong } A C A' C'$ 
    using  $\text{Tarski-neutral-dimensionless.l2-11-b Tarski-neutral-dimensionless-axioms}$ 
     $\langle \text{Cong } A B A' B' \rangle \langle \text{Cong } B C B' C' \rangle$  by fastforce
}
thus  $\bigwedge A B C A' B' C'. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \wedge$ 
   $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \wedge$ 
   $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A B B C \wedge$ 
   $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A' B' B' C' \wedge$ 
   $\text{Cong } A B A' B' \wedge \text{Cong } B C B' C' \longrightarrow$ 
   $\text{Cong } A C A' C'$ 
  by blast
{
  fix  $A B C$ 
  assume  $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$ 

```

```

hence  $\neg \text{Col } A \ B \ C$ 
  using  $\langle \forall A \ B \ C. \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C = (\text{Col } A \ B \ C) \rangle$ 
  by blast
hence  $A \ B \ C \text{ CongA } A \ B \ C$ 
  by (metis conga-refl not-col-distincts)
hence  $\text{CongA-H } A \ B \ C \ A \ B \ C$ 
  by (simp add: CongA-H-def)
}
thus  $\bigwedge A \ B \ C.$ 
   $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \longrightarrow \text{CongA-H } A \ B \ C \ A \ B \ C$ 
  by blast
show  $\bigwedge A \ B \ C.$ 
   $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \longrightarrow \text{CongA-H } A \ B \ C \ C \ B \ A$ 
  using CongA-H-def
   $\langle \bigwedge Ca \ Ba \ A. \neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ Ba \ Ca$ 
     $\longrightarrow \text{CongA-H } A \ Ba \ Ca \ A \ Ba \ Ca \rangle$  conga-right-comm by presburger
show  $\bigwedge A \ B \ C \ D \ E \ F. \text{ CongA-H } A \ B \ C \ D \ E \ F \longrightarrow \text{CongA-H } C \ B \ A \ F \ E \ D$ 
  by (meson CongA-H-def axiom-conga-permlr)
show  $\bigwedge A \ B \ C \ D \ E \ F \ A' \ C' \ D' \ F'.$ 
   $\text{CongA-H } A \ B \ C \ D \ E \ F \wedge$ 
   $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.outH } B \ A \ A' \wedge$ 
   $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.outH } B \ C \ C' \wedge$ 
   $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.outH } E \ D \ D' \wedge$ 
   $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.outH } E \ F \ F' \longrightarrow$ 
   $\text{CongA-H } A' \ B \ C' \ D' \ E \ F'$ 
  using axiom-congaH-outH-congaH CongA-H-def outH-def
   $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.outH-def}$  by force
{
  fix  $P \ PO \ X \ A \ B \ C$ 
  assume  $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } P \ PO \ X$  and
     $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C$ 
  have  $\neg \text{Col } P \ PO \ X$ 
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
       $\langle \neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } P \ PO \ X \rangle$  cols-coincide-2)
  moreover
  have  $\neg \text{Col } A \ B \ C$ 
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
     $\langle \neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \rangle$  cols-coincide-2 by blast
  ultimately
  obtain  $Y$  where  $A \ B \ C \text{ CongA } X \ PO \ Y$  and same-side'-H  $P \ Y \ PO \ X$ 
    using axiom-hcong-4-existence cols-coincide-1 by blast
  hence  $PO \neq X \wedge (\forall l. \text{isLine } l \longrightarrow (\text{IncidentL } PO \ l \wedge \text{IncidentL } X \ l) \longrightarrow \text{same-side-H } P \ Y \ l)$ 
    using same-side'-H-def by auto
  have  $\exists Y. \text{ CongA-H } A \ B \ C \ X \ PO \ Y \wedge$ 
     $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' \ P \ Y \ PO \ X$ 
  proof -
    have  $\text{CongA-H } A \ B \ C \ X \ PO \ Y$ 
      by (simp add: CongA-H-def  $\langle A \ B \ C \text{ CongA } X \ PO \ Y \rangle$ )
    moreover
    have  $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' \ P \ Y \ PO \ X$ 
      by (simp add:
         $\langle \forall A \ B \ C \ D. \text{ Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' \ A \ B \ C \ D$ 
           $= \text{same-side}'\text{-H } A \ B \ C \ D \rangle \langle \text{same-side}'\text{-H } P \ Y \ PO \ X \rangle$ )
    ultimately show ?thesis
      by blast
  qed
}
thus  $\bigwedge P \ PO \ X \ A \ B \ C.$ 
   $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } P \ PO \ X \wedge$ 
   $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \longrightarrow$ 
   $(\exists Y. \text{ CongA-H } A \ B \ C \ X \ PO \ Y \wedge$ 
     $\text{ Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' \ P \ Y \ PO \ X)$ 
  by blast
{
  fix  $P \ PO \ X \ A \ B \ C \ Y \ Y'$ 
  assume  $\neg \text{ Interpretation-Hilbert-neutral-dimensionless-pre.ColH } P \ PO \ X$  and

```

```

  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
  CongA-H A B C X PO Y and
  CongA-H A B C X PO Y' and
  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X and
  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X
have ¬ Col P PO X
  by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X⟩ cols-coincide-2)
moreover
have ¬ Col A B C
  using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
  ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
moreover
have A B C CongA X PO Y
  using CongA-H-def ⟨CongA-H A B C X PO Y⟩ by blast
moreover
have A B C CongA X PO Y'
  using CongA-H-def ⟨CongA-H A B C X PO Y'⟩ by auto
ultimately
have Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
  using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X⟩
  ⟨Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X⟩
  axiom-hcong-4-uniqueness cols-coincide-1
  by (metis Interpretation-Hilbert-neutral-dimensionless-pre.outH-def
      ⟨∀ A B C D. Interpretation-Hilbert-neutral-dimensionless-pre.same-side' A B C D
        = same-side'-H A B C D⟩
      outH-def)
}
thus ∧ P PO X A B C Y Y'.
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X ∧
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
  CongA-H A B C X PO Y ∧
  CongA-H A B C X PO Y' ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X →
  Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
  by blast
{
  fix A B C A' B' C'
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
  Cong A B A' B' and
  Cong A C A' C' and
  CongA-H B A C B' A' C'
  have ¬ Col-H A B C
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = Col-H A B C⟩
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ by auto
  moreover
  have ¬ Col-H A' B' C'
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = Col-H A B C⟩
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ by blast
  have ¬ Col A B C
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
  moreover
  have ¬ Col A' B' C'
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ cols-coincide-2
    by blast
  moreover
  have B A C CongA B' A' C'
    using CongA-H-def ⟨CongA-H B A C B' A' C'⟩ by auto
  ultimately
  have CongA-H A B C A' B' C'
    using CongA-H-def ⟨Cong A B A' B'⟩ ⟨Cong A C A' C'⟩ l11-49 not-col-distincts by blast
}

```

```

thus  $\bigwedge A B C A' B' C'.$ 
   $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \wedge$ 
   $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \wedge$ 
   $\text{Cong } A B A' B' \wedge \text{Cong } A C A' C' \wedge \text{CongA-H } B A C B' A' C' \longrightarrow$ 
   $\text{CongA-H } A B C A' B' C'$ 
by blast
show  $\bigwedge A B C D. \text{Cong } A B C D \longrightarrow \text{Cong } A B D C$ 
using not-cong-1243 by blast
show  $\bigwedge l m P. \text{isLine } l \wedge \text{isLine } m \wedge \text{IncidentL } P l \wedge l = m \longrightarrow \text{IncidentL } P m$ 
using axiom-Incidl-morphism by blast
show  $\bigwedge p q M. \text{isPlane } p \wedge \text{isPlane } q \wedge \text{IncidentP } M p \wedge p = q \longrightarrow \text{IncidentP } M q$ 
using axiom-Incidp-morphism by blast
show  $\bigwedge P l. \text{IncidentL } P l \longrightarrow \text{isLine } l$ 
using IncidentL-def by blast
show  $\bigwedge P p. \text{IncidentP } P p \longrightarrow \text{isPlane } p$ 
using IncidentP-def by blast
qed

end
end

```

theory *Hilbert-Neutral-2D-Model-Tarski-Neutral-2D*

imports

Tarski-Neutral-Hilbert
Tarski-Neutral-2D
Hilbert-Neutral-2D

begin

30 Hilbert Neutral 2D - Tarski Neutral 2D Model

30.1 Interpretation

context *Tarski-neutral-2D*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre: Hilbert-neutral-dimensionless-pre*

where *IncidL* = *IncidentL* **and**

IncidP = *IncidentP* **and**

EqL = *EqTL* **and**

EqP = *EqTP* **and**

IsL = *isLine* **and**

IsP = *isPlane* **and**

BetH = *Between-H* **and**

CongH = *Cong* **and**

CongaH = *CongA-H*

proof –

qed

interpretation *Intrepretation-Hilbert-neutral-dimensionless: Hilbert-neutral-dimensionless*

where *IncidL* = *IncidentL* **and**

IncidP = *IncidentP* **and**

EqL = *EqTL* **and**

EqP = *EqTP* **and**

IsL = *isLine* **and**

IsP = *isPlane* **and**

BetH = *Between-H* **and**

CongH = *Cong* **and**

CongaH = *CongA-H* **and**

PP = *TPA* **and**

PQ = *TPB* **and**

PR = *TPC*

proof

```

have  $\forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longleftrightarrow \text{Col-H } A B C$ 
proof -
{
  fix  $A B C$ 
  assume  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$ 
  hence  $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$ 
    by (simp add:  $\text{Hilbert-neutral-dimensionless-pre.ColH-def}$ )
  hence  $\text{Col-H } A B C$ 
    using  $\text{Col-H-def EqTL-def axiom-line-uniqueness cols-coincide-2 not-col-distincts}$  by blast
}
moreover
{
  fix  $A B C$ 
  assume  $\text{Col-H } A B C$ 
  hence  $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$ 
    using  $\text{Col-H-def}$  by blast
  have  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$ 
    using  $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def}$ 
       $\langle \exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l \rangle$  by blast
}
ultimately show ?thesis
  by blast
qed
have  $\forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longleftrightarrow \text{Col } A B C$ 
  using  $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = \text{Col-H } A B C \rangle$ 
     $\text{cols-coincide}$  by blast
have  $\forall A B l. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side } A B l$ 
   $\longleftrightarrow \text{same-side-H } A B l$ 
  by (simp add:  $\text{Hilbert-neutral-dimensionless-pre.cut-def}$ 
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side-def cut-H-def same-side-H-def}$ )
have  $\forall A B C D. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' A B C D$ 
   $\longleftrightarrow \text{same-side}'\text{-H } A B C D$ 
  by (simp add:  $\text{Hilbert-neutral-dimensionless-pre.same-side}'\text{-def}$ 
     $\langle \forall A B l. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side } A B l$ 
       $= \text{same-side-H } A B l$ 
     $\text{same-side}'\text{-H-def} \rangle$ )

show  $\bigwedge l. \text{isLine } l \longrightarrow l = l$ 
  by (simp add:  $\text{eq-reflexivity}$ )
show  $\bigwedge l1 l2. \text{isLine } l1 \wedge \text{isLine } l2 \wedge l1 = l2 \longrightarrow l2 = l1$ 
  using  $\text{eq-symmetry}$  by blast
show  $\bigwedge l1 l2 l3. l1 = l2 \wedge l2 = l3 \longrightarrow l1 = l3$ 
  using  $\text{eq-transitivity}$  by blast
show  $\bigwedge p. \text{isPlane } p \longrightarrow p = p$ 
  by (simp add:  $\text{eqp-reflexivity}$ )
show  $\bigwedge p1 p2. p1 = p2 \longrightarrow p2 = p1$ 
  using  $\text{eqp-symmetry}$  by blast
show  $\bigwedge p1 p2 p3. p1 = p2 \wedge p2 = p3 \longrightarrow p1 = p3$ 
  using  $\text{eqp-transitivity}$  by blast
show  $\bigwedge A B. A \neq B \longrightarrow (\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l)$ 
  using  $\text{axiom-line-existence}$  by blast
show  $\bigwedge A B l m. A \neq B \wedge \text{isLine } l \wedge \text{isLine } m \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } A m \wedge \text{IncidentL } B m \longrightarrow l = m$ 
  using  $\text{axiom-line-uniqueness}$  by blast
show  $\forall l. \text{isLine } l \longrightarrow (\exists A B. \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge A \neq B)$ 
  using  $\text{axiom-two-points-on-line}$  by blast
show  $\text{TPA} \neq \text{TPB} \wedge \text{TPB} \neq \text{TPC} \wedge \text{TPA} \neq \text{TPC} \wedge$ 
   $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } \text{TPA } \text{TPB } \text{TPC}$ 
  using  $\text{Bet-cases between-trivial lower-dim third-point}$ 
     $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = (\text{Col } A B C) \rangle$ 

```

```

by blast
show  $\forall A B C.$ 
   $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\longrightarrow$ 
     $(\exists p. \text{isPlane } p \wedge \text{IncidentP } A p \wedge \text{IncidentP } B p \wedge \text{IncidentP } C p)$ 
  using bet-coplanar between-trivial2 cop-plane IncidentP-def by blast
show  $\forall p. \exists A. \text{isPlane } p \longrightarrow \text{IncidentP } A p$ 
  by (simp add: axiom-one-point-on-plane)
show  $\bigwedge A B C p q.$ 
   $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\wedge$ 
    isPlane p  $\wedge$  isPlane q  $\wedge$ 
    IncidentP A p  $\wedge$ 
    IncidentP B p  $\wedge$ 
    IncidentP C p  $\wedge$  IncidentP A q  $\wedge$  IncidentP B q  $\wedge$  IncidentP C q  $\longrightarrow$ 
    p = q
  using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    axiom-plane-uniqueness by blast
show  $\forall A B l p.$ 
  A  $\neq$  B  $\wedge$  isLine l  $\wedge$  isPlane p  $\wedge$ 
  IncidentL A l  $\wedge$  IncidentL B l  $\wedge$  IncidentP A p  $\wedge$  IncidentP B p  $\longrightarrow$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p
  by (meson IncidentLP-def Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
    axiom-line-on-plane)
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow A \neq C$ 
  using axiom-between-diff by blast
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$ 
  by (meson Between-H-def Col-H-def Col-def
    Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def cols-coincide-2)
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow \text{Between-H } C B A$ 
  using axiom-between-comm by blast
show  $\bigwedge A B. A \neq B \longrightarrow (\exists C. \text{Between-H } A B C)$ 
  by (simp add: axiom-between-out)
show  $\bigwedge A B C. \text{Between-H } A B C \longrightarrow \neg \text{Between-H } B C A$ 
  using axiom-between-only-one by blast
{
  fix A B C p l
  assume  $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
    isLine l and
    IncidentP A p and
    IncidentP B p and
    IncidentP C p and
    Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p and
     $\neg$  IncidentL C l and
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B
  have  $\neg$  Col A B C
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
       $\langle \neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\rangle$  cols-coincide-2)
  moreover have cut-H l A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.cut-def
       $\langle$  Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B  $\rangle$  cut-H-def by blast
  ultimately have cut-H l A C  $\vee$  cut-H l B C
    using axiom-pasch Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
       $\langle$  IncidentP A p  $\rangle$   $\langle$  IncidentP B p  $\rangle$   $\langle$  IncidentP C p  $\rangle$ 
       $\langle$  Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p  $\rangle$   $\langle \neg$  IncidentL C l  $\rangle$ 
      axiom-line-on-plane axiom-two-points-on-line cols-coincide-1 by meson
  hence Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C  $\vee$ 
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
    by (simp add: Hilbert-neutral-dimensionless-pre.cut-def cut-H-def)
}
thus  $\bigwedge A B C p l.$ 
   $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\wedge$ 
  isLine l  $\wedge$  isPlane p  $\wedge$  IncidentP A p  $\wedge$  IncidentP B p  $\wedge$  IncidentP C p  $\wedge$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p  $\wedge$ 
   $\neg$  IncidentL C l  $\wedge$  Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B  $\longrightarrow$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C  $\vee$ 
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
  by blast

```

```

{
  fix l
  fix A B A' P::'a
  assume A ≠ B and
    A' ≠ P and
    isLine l and
    IncidentL A' l and
    IncidentL P l
  then obtain B' where IncidentL B' l ∧ outH A' P B' ∧ Cong A' B' A B
    using axiom-hcong-1-existence by presburger
  hence ∃ B'. IncidentL B' l ∧
    Interpretation-Hilbert-neutral-dimensionless-pre.outH A' P B' ∧ Cong A' B' A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.outH-def outH-def by auto
}
thus ∧l A B A' P.
  A ≠ B ∧ A' ≠ P ∧ isLine l ∧ IncidentL A' l ∧ IncidentL P l →
  (∃ B'. IncidentL B' l ∧
    Interpretation-Hilbert-neutral-dimensionless-pre.outH A' P B' ∧ Cong A' B' A B)
  by blast
show ∧A B C D E F. Cong A B C D ∧ Cong A B E F → Cong C D E F
  using cong-inner-transitivity by blast
{
  fix A B C A' B' C'
  assume Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
    Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
    Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B C and
    Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C' and
    Cong A B A' B' and
    Cong B C B' C'
  have (∃ l. IncidentL A l ∧ IncidentL B l ∧ IncidentL C l)
    using Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ by force
  then obtain l where IncidentL A l and IncidentL B l and IncidentL C l
    by blast
  hence Col A B C
    using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩
    ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = (Col A B C)⟩
    by auto
  moreover
  have (∃ l. IncidentL A' l ∧ IncidentL B' l ∧ IncidentL C' l)
    using Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ by force
  then obtain l' where IncidentL A' l' and IncidentL B' l' and IncidentL C' l'
    by blast
  hence Col A' B' C'
    using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩
    ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = (Col A B C)⟩
    by blast
  moreover
  have ¬ (∃ P. Between-H A P B ∧ Between-H B P C)
    using Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def
    ⟨Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B C⟩ by blast
  have ¬ (∃ P. Between-H A' P B' ∧ Between-H B' P C')
    using Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def
    ⟨Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C'⟩ by blast
  have Col-H A B C
    using calculation(1) cols-coincide-2 by blast
  hence Bet A B C
    using ⟨∄ P. Between-H A P B ∧ Between-H B P C⟩ col-disjoint-bet disjoint-H-def by blast
  moreover
  have Col-H A' B' C'
    by (simp add: calculation(2) cols-coincide-2)
  moreover
  hence Bet A' B' C'
    by (simp add: ⟨∄ P. Between-H A' P B' ∧ Between-H B' P C'⟩ col-disjoint-bet disjoint-H-def)
  ultimately

```

```

have Cong A C A' C'
  using Tarski-neutral-dimensionless.l2-11-b Tarski-neutral-dimensionless-axioms
    ⟨Cong A B A' B'⟩ ⟨Cong B C B' C'⟩ by fastforce
}
thus ∧ A B C A' B' C'.
  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B C ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C' ∧
  Cong A B A' B' ∧ Cong B C B' C' →
  Cong A C A' C'
by blast
{
  fix A B C
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C
  hence ¬ Col A B C
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = (Col A B C)⟩
    by blast
  hence A B C CongA A B C
    by (metis conga-refl not-col-distincts)
  hence CongA-H A B C A B C
    by (simp add: CongA-H-def)
}
thus ∧ A B C.
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C → CongA-H A B C A B C
by blast
show ∧ A B C.
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C → CongA-H A B C C B A
using CongA-H-def
  ⟨∧ Ca Ba A. ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A Ba Ca
    → CongA-H A Ba Ca A Ba Ca⟩ conga-right-comm by presburger
show ∧ A B C D E F. CongA-H A B C D E F → CongA-H C B A F E D
  by (meson CongA-H-def axiom-conga-permlr)
show ∧ A B C D E F A' C' D' F'.
  CongA-H A B C D E F ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.outH B A A' ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.outH B C C' ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.outH E D D' ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.outH E F F' →
  CongA-H A' B C' D' E F'
using axiom-congaH-outH-congaH CongA-H-def outH-def
  Interpretation-Hilbert-neutral-dimensionless-pre.outH-def by force
{
  fix P PO X A B C
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X and
    ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C
  have ¬ Col P PO X
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X⟩ cols-coincide-2)
  moreover
  have ¬ Col A B C
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
  ultimately
  obtain Y where A B C CongA X PO Y and same-side'-H P Y PO X
    using axiom-hcong-4-existence cols-coincide-1 by blast
  hence PO ≠ X ∧ (∀ l. isLine l → (IncidentL PO l ∧ IncidentL X l) → same-side-H P Y l)
    using same-side'-H-def by auto
  have ∃ Y. CongA-H A B C X PO Y ∧
    Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X
  proof -
    have CongA-H A B C X PO Y
      by (simp add: CongA-H-def ⟨A B C CongA X PO Y⟩)
    moreover
    have Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X
      by (simp add:

```



```

      ⟨∀ A B C D. Interpretation-Hilbert-neutral-dimensionless-pre.same-side' A B C D
        = same-side'-H A B C D⟩ ⟨same-side'-H P Y PO X⟩)
    ultimately show ?thesis
      by blast
  qed
}
thus ∧ P PO X A B C.
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X ∧
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ⟶
  (∃ Y. CongA-H A B C X PO Y ∧
    Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X)
  by blast
{
  fix P PO X A B C Y Y'
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X and
    ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
    CongA-H A B C X PO Y and
    CongA-H A B C X PO Y' and
    Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X and
    Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X
  have ¬ Col P PO X
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X⟩ cols-coincide-2)
  moreover
  have ¬ Col A B C
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
  moreover
  have A B C CongA X PO Y
    using CongA-H-def ⟨CongA-H A B C X PO Y⟩ by blast
  moreover
  have A B C CongA X PO Y'
    using CongA-H-def ⟨CongA-H A B C X PO Y'⟩ by auto
  ultimately
  have Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
    using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X⟩
      ⟨Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X⟩
      axiom-hcong-4-uniqueness cols-coincide-1
    by (metis Interpretation-Hilbert-neutral-dimensionless-pre.outH-def
      ⟨∀ A B C D. Interpretation-Hilbert-neutral-dimensionless-pre.same-side' A B C D
        = same-side'-H A B C D⟩
      outH-def)
}
thus ∧ P PO X A B C Y Y'.
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X ∧
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
  CongA-H A B C X PO Y ∧
  CongA-H A B C X PO Y' ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X ∧
  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X ⟶
  Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
  by blast
{
  fix A B C A' B' C'
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
    ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
    Cong A B A' B' and
    Cong A C A' C' and
    CongA-H B A C B' A' C'
  have ¬ Col-H A B C
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = Col-H A B C⟩
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ by auto
  moreover
  have ¬ Col-H A' B' C'
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = Col-H A B C⟩
      ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ by blast

```

```

have  $\neg \text{Col } A \ B \ C$ 
  using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
   $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \rangle$  cols-coincide-2 by blast
moreover
have  $\neg \text{Col } A' \ B' \ C'$ 
  using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
   $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' \ B' \ C' \rangle$  cols-coincide-2 by blast
moreover
have  $B \ A \ C \text{ CongA } B' \ A' \ C'$ 
  using CongA-H-def  $\langle \text{CongA-H } B \ A \ C \ B' \ A' \ C' \rangle$  by auto
ultimately
have  $\text{CongA-H } A \ B \ C \ A' \ B' \ C'$ 
  using CongA-H-def  $\langle \text{Cong } A \ B \ A' \ B' \rangle \langle \text{Cong } A \ C \ A' \ C' \rangle$  l11-49 not-col-distincts by blast
}
thus  $\bigwedge A \ B \ C \ A' \ B' \ C'.$ 
   $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \wedge$ 
   $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' \ B' \ C' \wedge$ 
   $\text{Cong } A \ B \ A' \ B' \wedge \text{Cong } A \ C \ A' \ C' \wedge \text{CongA-H } B \ A \ C \ B' \ A' \ C' \longrightarrow$ 
   $\text{CongA-H } A \ B \ C \ A' \ B' \ C'$ 
  by blast
show  $\bigwedge A \ B \ C \ D. \text{ Cong } A \ B \ C \ D \longrightarrow \text{Cong } A \ B \ D \ C$ 
  using not-cong-1243 by blast
show  $\bigwedge l \ m \ P. \text{ isLine } l \wedge \text{isLine } m \wedge \text{IncidentL } P \ l \wedge l = m \longrightarrow \text{IncidentL } P \ m$ 
  using axiom-Incidl-morphism by blast
show  $\bigwedge p \ q \ M. \text{ isPlane } p \wedge \text{isPlane } q \wedge \text{IncidentP } M \ p \wedge p = q \longrightarrow \text{IncidentP } M \ q$ 
  using axiom-Incidp-morphism by blast
show  $\bigwedge P \ l. \text{ IncidentL } P \ l \longrightarrow \text{isLine } l$ 
  using IncidentL-def by blast
show  $\bigwedge P \ p. \text{ IncidentP } P \ p \longrightarrow \text{isPlane } p$ 
  using IncidentP-def by blast
qed

```

interpretation *Intrepretation-Hilbert-neutral-2D: Hilbert-neutral-2D*

```

where IncidL = IncidentL and
      IncidP = IncidentP and
      EqL = EqTL and
      EqP = EqTP and
      IsL = isLine and
      IsP = isPlane and
      BetH = Between-H and
      CongH = Cong and
      CongaH = CongA-H and
      PP = TPA and
      PQ = TPB and
      PR = TPC
proof
{
  fix  $l \ A \ B \ C$ 
  assume isLine  $l$  and
     $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C$  and
     $\neg \text{IncidentL } C \ l$  and
    Interpretation-Hilbert-neutral-dimensionless-pre.cut  $l \ A \ B$ 
  have  $\neg \text{Col } A \ B \ C$ 
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
       $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \rangle$  cols-coincide-2)
  have cut-H  $l \ A \ B$ 
    using Interpretation-Hilbert-neutral-dimensionless-pre.cut-def
     $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l \ A \ B \rangle$  cut-H-def by auto
  hence  $\text{isLine } l \wedge \neg \text{IncidentL } A \ l \wedge \neg \text{IncidentL } B \ l \wedge (\exists I. \text{ IncidentL } I \ l \wedge \text{Between-H } A \ I \ B)$ 
    using cut-H-def by blast
  then obtain  $I$  where IncidentL  $I \ l$  and Between-H  $A \ I \ B$ 
    by blast
  hence  $\text{Bet } A \ I \ B \wedge A \neq B \wedge B \neq C \wedge A \neq C$ 
    using Between-H-def  $\langle \neg \text{Col } A \ B \ C \rangle$  col-trivial-2 col-trivial-3 by force
  have Coplanar  $A \ B$  (fst  $l$ ) (snd  $l$ )
    using  $\langle \text{cut-H } l \ A \ B \rangle$  cut-two-sides ncoplanar-perm-16 ts--coplanar by blast
}

```

```

have Col I (fst l) (snd l)
  using IncidentL-def ⟨IncidentL I l⟩ by auto
let ?p = Plane A B C
have isPlane ?p
  using EqTP-def ⟨¬ Col A B C⟩ axiom-plane-existence cols-coincide incidentp-eqp by blast
moreover have IncidentP A ?p
  using IncidentP-def calculation coplanar-trivial by blast
moreover have IncidentP B ?p
  using IncidentP-def calculation(1) ncop-distincts by blast
moreover have IncidentP C ?p
  using IncidentP-def calculation(1) ncop-distincts by blast
moreover have IncidentLP l ?p
proof -
  have isLine l
    by (simp add: ⟨isLine l⟩)
  moreover have isPlane ?p
    using ⟨isPlane (Plane A B C)⟩ by blast
  moreover
  {
    fix AA
    assume IncidentL AA l
    hence Col AA (fst l) (snd l)
      by (simp add: incident-col)
    hence Coplanar AA A (fst l) (snd l)
      using ncop--ncols by blast
    have Coplanar AA B (fst l) (snd l)
      using ⟨Col AA (fst l) (snd l)⟩ ncop--ncols by blast
    have Coplanar AA I (fst l) (snd l)
      using ⟨Col I (fst l) (snd l)⟩ ncop--ncols by blast
    have isPlane ?p
      using ⟨isPlane (Plane A B C)⟩ by blast
    moreover
    have Coplanar AA A B C
      using all-coplanar by blast
    hence ∃ P Q R. ?p = Plane P Q R ∧ Coplanar AA P Q R
      by blast
    ultimately have IncidentP AA ?p
      using IncidentP-def by blast
  }
  hence ∀ A. IncidentL A l ⟶ IncidentP A ?p
    by blast
  ultimately show ?thesis
    using IncidentLP-def by blast
qed
moreover have ¬ IncidentL C l
  using ⟨¬ IncidentL C l⟩ by blast
ultimately have Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C ∨
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
  by (meson IncidentLP-def Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
    Intrepretation-Hilbert-neutral-dimensionless.pasch
    ⟨Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B⟩
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩)
}
thus ⋀ l A B C.
  isLine l ∧
  ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
  ¬ IncidentL C l ∧ Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B ⟶
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C ∨
  Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
  by blast
qed
end
end

```

theory *Hilbert-Neutral-3D-Model-Tarski-Neutral-3D*

imports

Tarski-Neutral-3D-Hilbert

Hilbert-Neutral-3D

begin

31 Hilbert Neutral 3D - Tarski Neutral 3D Model

31.1 Interpretation

context *Tarski-neutral-3D*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre: Hilbert-neutral-dimensionless-pre*

where *IncidL* = *IncidentL* and *IncidP* = *IncidentP* and *EqL* = *EqTL*

and *EqP* = *EqTP* and *IsL* = *isLine* and *IsP* = *isPlane*

and *BetH* = *Between-H* and *CongH* = *Cong* and *CongaH* = *CongA-H*

proof –

qed

interpretation *Intrepretation-Hilbert-neutral-dimensionless: Hilbert-neutral-dimensionless*

where *IncidL* = *IncidentL* and *IncidP* = *IncidentP* and *EqL* = *EqTL*

and *EqP* = *EqTP* and *IsL* = *isLine* and *IsP* = *isPlane*

and *BetH* = *Between-H* and *CongH* = *Cong* and *CongaH* = *CongA-H*

and *PP* = *TPA* and *PQ* = *TPB* and *PR* = *TPC*

proof

have $\forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longleftrightarrow \text{Col-H } A B C$

proof –

{

fix *A B C*

assume *Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C*

hence $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$

by (*simp add: Hilbert-neutral-dimensionless-pre.ColH-def*)

hence *Col-H A B C*

using *Col-H-def EqTL-def axiom-line-uniqueness cols-coincide-2 not-col-distincts* by *blast*

}

moreover

{

fix *A B C*

assume *Col-H A B C*

hence $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$

using *Col-H-def* by *blast*

have *Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C*

using *Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def*

$\langle \exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l \rangle$ by *blast*

}

ultimately show *?thesis*

by *blast*

qed

have $\forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longleftrightarrow \text{Col } A B C$

using $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = \text{Col-H } A B C \rangle$

cols-coincide by *blast*

have $\forall A B l. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side } A B l$

$\longleftrightarrow \text{same-side-H } A B l$

by (*simp add: Hilbert-neutral-dimensionless-pre.cut-def*

Interpretation-Hilbert-neutral-dimensionless-pre.same-side-def cut-H-def same-side-H-def)

have $\forall A B C D. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' A B C D$

$\longleftrightarrow \text{same-side}'\text{-H } A B C D$

by (*simp add: Hilbert-neutral-dimensionless-pre.same-side'\text{-def}*

$\langle \forall A B l. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side } A B l$

$= \text{same-side-H } A B l \rangle$

same-side'\text{-H-def})

show $\bigwedge l. \text{isLine } l \longrightarrow l = l = l$

```

    by (simp add: eq-reflexivity)
show  $\bigwedge l1\ l2. \text{isLine } l1 \wedge \text{isLine } l2 \wedge l1 = l2 \longrightarrow l2 = l1$ 
  using eq-symmetry by blast
show  $\bigwedge l1\ l2\ l3. l1 = l2 \wedge l2 = l3 \longrightarrow l1 = l3$ 
  using eq-transitivity by blast
show  $\bigwedge p. \text{isPlane } p \longrightarrow p = p$ 
  by (simp add: eqp-reflexivity)
show  $\bigwedge p1\ p2. p1 = p2 \longrightarrow p2 = p1$ 
  using eqp-symmetry by blast
show  $\bigwedge p1\ p2\ p3. p1 = p2 \wedge p2 = p3 \longrightarrow p1 = p3$ 
  using eqp-transitivity by blast
show  $\bigwedge A\ B. A \neq B \longrightarrow (\exists l. \text{isLine } l \wedge \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l)$ 
  using axiom-line-existence by blast
show  $\bigwedge A\ B\ l\ m. A \neq B \wedge \text{isLine } l \wedge \text{isLine } m \wedge \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l \wedge \text{IncidentL } A\ m \wedge \text{IncidentL } B\ m \longrightarrow l = m$ 
  using axiom-line-uniqueness by blast
show  $\forall l. \text{isLine } l \longrightarrow (\exists A\ B. \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l \wedge A \neq B)$ 
  using axiom-two-points-on-line by blast
show  $TPA \neq TPB \wedge TPB \neq TPC \wedge TPA \neq TPC \wedge \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } TPA\ TPB\ TPC$ 
  using Bet-cases between-trivial lower-dim third-point
   $\langle \forall A\ B\ C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C = (\text{Col } A\ B\ C) \rangle$ 
  by blast
show  $\forall A\ B\ C. \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C \longrightarrow (\exists p. \text{isPlane } p \wedge \text{IncidentP } A\ p \wedge \text{IncidentP } B\ p \wedge \text{IncidentP } C\ p)$ 
  using bet-coplanar between-trivial2 cop-plane IncidentP-def by blast
show  $\forall p. \exists A. \text{isPlane } p \longrightarrow \text{IncidentP } A\ p$ 
  by (simp add: axiom-one-point-on-plane)
show  $\bigwedge A\ B\ C\ p\ q. \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C \wedge \text{isPlane } p \wedge \text{isPlane } q \wedge \text{IncidentP } A\ p \wedge \text{IncidentP } B\ p \wedge \text{IncidentP } C\ p \wedge \text{IncidentP } A\ q \wedge \text{IncidentP } B\ q \wedge \text{IncidentP } C\ q \longrightarrow p = q$ 
  using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
  axiom-plane-uniqueness by blast
show  $\forall A\ B\ l\ p. A \neq B \wedge \text{isLine } l \wedge \text{isPlane } p \wedge \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l \wedge \text{IncidentP } A\ p \wedge \text{IncidentP } B\ p \longrightarrow \text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l\ p$ 
  by (meson IncidentLP-def Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
  axiom-line-on-plane)
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow A \neq C$ 
  using axiom-between-diff by blast
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C$ 
  by (meson Between-H-def Col-H-def Col-def
  Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def cols-coincide-2)
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow \text{Between-H } C\ B\ A$ 
  using axiom-between-comm by blast
show  $\bigwedge A\ B. A \neq B \longrightarrow (\exists C. \text{Between-H } A\ B\ C)$ 
  by (simp add: axiom-between-out)
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow \neg \text{Between-H } B\ C\ A$ 
  using axiom-between-only-one by blast
{
  fix  $A\ B\ C\ p\ l$ 
  assume  $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C$  and
     $\text{isLine } l$  and
     $\text{IncidentP } A\ p$  and
     $\text{IncidentP } B\ p$  and
     $\text{IncidentP } C\ p$  and
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l\ p$  and
     $\neg \text{IncidentL } C\ l$  and

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    Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B
have  $\neg \text{Col } A \ B \ C$ 
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
         $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \rangle \text{ cols-coincide-2}$ )
moreover have cut-H l A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.cut-def
         $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l \ A \ B \rangle \text{ cut-H-def}$  by blast
ultimately have cut-H l A C  $\vee$  cut-H l B C
    using axiom-pasch Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
         $\langle \text{IncidentP } A \ p \rangle \langle \text{IncidentP } B \ p \rangle \langle \text{IncidentP } C \ p \rangle$ 
         $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l \ p \rangle \langle \neg \text{IncidentL } C \ l \rangle$ 
        axiom-line-on-plane axiom-two-points-on-line cols-coincide-1 by meson
hence Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C  $\vee$ 
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
    by (simp add: Hilbert-neutral-dimensionless-pre.cut-def cut-H-def)
}
thus  $\bigwedge A \ B \ C \ p \ l.$ 
 $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \wedge$ 
 $\text{isLine } l \wedge \text{isPlane } p \wedge$ 
 $\text{IncidentP } A \ p \wedge$ 
 $\text{IncidentP } B \ p \wedge$ 
 $\text{IncidentP } C \ p \wedge$ 
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l \ p \wedge$ 
 $\neg \text{IncidentL } C \ l \wedge \text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l \ A \ B \longrightarrow$ 
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l \ A \ C \vee$ 
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.cut } l \ B \ C$ 
by blast
{
    fix l
    fix A B A' P::'a
    assume  $A \neq B$  and
         $A' \neq P$  and
         $\text{isLine } l$  and
         $\text{IncidentL } A' \ l$  and
         $\text{IncidentL } P \ l$ 
    then obtain B' where  $\text{IncidentL } B' \ l \wedge \text{outH } A' \ P \ B' \wedge \text{Cong } A' \ B' \ A \ B$ 
    using axiom-hcong-1-existence by presburger
    hence  $\exists B'. \text{IncidentL } B' \ l \wedge$ 
     $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } A' \ P \ B' \wedge \text{Cong } A' \ B' \ A \ B$ 
    using Interpretation-Hilbert-neutral-dimensionless-pre.outH-def outH-def by auto
}
thus  $\bigwedge l \ A \ B \ A' \ P. A \neq B \wedge A' \neq P \wedge \text{isLine } l \wedge \text{IncidentL } A' \ l \wedge \text{IncidentL } P \ l \longrightarrow$ 
 $(\exists B'. \text{IncidentL } B' \ l \wedge$ 
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } A' \ P \ B' \wedge \text{Cong } A' \ B' \ A \ B)$ 
by blast
show  $\bigwedge A \ B \ C \ D \ E \ F. \text{Cong } A \ B \ C \ D \wedge \text{Cong } A \ B \ E \ F \longrightarrow \text{Cong } C \ D \ E \ F$ 
using cong-inner-transitivity by blast
{
    fix A B C A' B' C'
    assume Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
        Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
        Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B' C' and
        Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C' and
         $\text{Cong } A \ B \ A' \ B'$  and
         $\text{Cong } B \ C \ B' \ C'$ 
    have  $(\exists l. \text{IncidentL } A \ l \wedge \text{IncidentL } B \ l \wedge \text{IncidentL } C \ l)$ 
    using Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
         $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \rangle$  by force
    then obtain l where  $\text{IncidentL } A \ l$  and  $\text{IncidentL } B \ l$  and  $\text{IncidentL } C \ l$ 
    by blast
    hence Col A B C
    using  $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C \rangle$ 
         $\langle \forall A \ B \ C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A \ B \ C = (\text{Col } A \ B \ C) \rangle$  by auto
moreover
    have  $(\exists l. \text{IncidentL } A' \ l \wedge \text{IncidentL } B' \ l \wedge \text{IncidentL } C' \ l)$ 
    using Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def

```

$\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \rangle$ **by force**
then obtain l' **where** $\text{IncidentL } A' l'$ **and** $\text{IncidentL } B' l'$ **and** $\text{IncidentL } C' l'$
by blast
hence $\text{Col } A' B' C'$
using $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \rangle$
 $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = (\text{Col } A B C) \rangle$ **by auto**
moreover
have $\neg (\exists P. \text{Between-H } A P B \wedge \text{Between-H } B P C)$
using $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def}$
 $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A B B C \rangle$ **by blast**
have $\neg (\exists P. \text{Between-H } A' P B' \wedge \text{Between-H } B' P C')$
using $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def}$
 $\langle \text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A' B' B' C' \rangle$ **by blast**
have $\text{Col-H } A B C$
using $\text{calculation}(1)$ cols-coincide-2 **by blast**
hence $\text{Bet } A B C$
using $\langle \nexists P. \text{Between-H } A P B \wedge \text{Between-H } B P C \rangle$ $\text{col-disjoint-bet disjoint-H-def}$ **by blast**
moreover
have $\text{Col-H } A' B' C'$
by ($\text{simp add: calculation}(2)$ cols-coincide-2)
moreover
hence $\text{Bet } A' B' C'$
by ($\text{simp add: } \langle \nexists P. \text{Between-H } A' P B' \wedge \text{Between-H } B' P C' \rangle$ $\text{col-disjoint-bet disjoint-H-def}$)
ultimately
have $\text{Cong } A C A' C'$
using $\text{Tarski-neutral-dimensionless.l2-11-b Tarski-neutral-dimensionless-axioms}$
 $\langle \text{Cong } A B A' B' \rangle \langle \text{Cong } B C B' C' \rangle$ **by fastforce**

}
thus $\bigwedge A B C A' B' C'.$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A' B' C' \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A B B C \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.disjoint } A' B' B' C' \wedge$
 $\text{Cong } A B A' B' \wedge \text{Cong } B C B' C' \longrightarrow$
 $\text{Cong } A C A' C'$
by blast

{
fix $A B C$
assume $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$
hence $\neg \text{Col } A B C$
using $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = (\text{Col } A B C) \rangle$
by blast
hence $A B C \text{ CongA } A B C$
by ($\text{metis Tarski-neutral-dimensionless.conga-refl}$
 $\text{Tarski-neutral-dimensionless.not-col-distincts Tarski-neutral-dimensionless-axioms}$)
hence $\text{CongA-H } A B C A B C$
by ($\text{simp add: CongA-H-def}$)

}
thus $\bigwedge A B C.$
 $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longrightarrow \text{CongA-H } A B C A B C$
by blast
show $\bigwedge A B C.$
 $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longrightarrow \text{CongA-H } A B C C B A$
using CongA-H-def
 $\langle \bigwedge Ca Ba A. \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A Ba Ca$
 $\longrightarrow \text{CongA-H } A Ba Ca A Ba Ca \rangle$ conga-right-comm **by presburger**

show $\bigwedge A B C D E F. \text{CongA-H } A B C D E F \longrightarrow \text{CongA-H } C B A F E D$
by ($\text{meson CongA-H-def Tarski-neutral-dimensionless.axiom-conga-permlr}$
 $\text{Tarski-neutral-dimensionless-axioms}$)

show $\bigwedge A B C D E F A' C' D' F'. \text{CongA-H } A B C D E F \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } B A A' \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } B C C' \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } E D D' \wedge$
 $\text{Interpretation-Hilbert-neutral-dimensionless-pre.outH } E F F' \longrightarrow$
 $\text{CongA-H } A' B C' D' E F'$
using $\text{axiom-congaH-outH-congaH CongA-H-def outH-def}$

```

    Interpretation-Hilbert-neutral-dimensionless-pre.outH-def by force
  {
    fix P PO X A B C
    assume  $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X and
       $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C
    have  $\neg$  Col P PO X
      by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
         $\langle \neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X  $\rangle$  cols-coincide-2)
    moreover
    have  $\neg$  Col A B C
      using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
         $\langle \neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\rangle$  cols-coincide-2 by blast
    ultimately
    obtain Y where A B C CongA X PO Y and same-side'-H P Y PO X
      using axiom-hcong-4-existence cols-coincide-1 by blast
    hence  $PO \neq X \wedge (\forall l. \text{isLine } l \longrightarrow (\text{IncidentL } PO \ l \wedge \text{IncidentL } X \ l) \longrightarrow \text{same-side-H } P \ Y \ l)$ 
      using same-side'-H-def by auto
    have  $\exists Y. \text{CongA-H } A \ B \ C \ X \ PO \ Y \wedge$ 
      Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X
    proof -
      have CongA-H A B C X PO Y
        by (simp add: CongA-H-def  $\langle A \ B \ C \ \text{CongA } X \ PO \ Y \rangle$ )
      moreover
      have Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X
        by (simp add:
           $\langle \forall A \ B \ C \ D. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' A \ B \ C \ D$ 
             $= \text{same-side}'\text{-H } A \ B \ C \ D \rangle$ 
           $\langle \text{same-side}'\text{-H } P \ Y \ PO \ X \rangle$ )
      ultimately show ?thesis
        by blast
    qed
  }
  thus  $\bigwedge P \ PO \ X \ A \ B \ C.$ 
   $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X  $\wedge$ 
   $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\longrightarrow$ 
  ( $\exists Y. \text{CongA-H } A \ B \ C \ X \ PO \ Y \wedge$ 
    Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X)
  by blast
  {
    fix P PO X A B C Y Y'
    assume  $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X and
       $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
      CongA-H A B C X PO Y and
      CongA-H A B C X PO Y' and
      Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X and
      Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X
    have  $\neg$  Col P PO X
      by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
         $\langle \neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X  $\rangle$  cols-coincide-2)
    moreover
    have  $\neg$  Col A B C
      using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
         $\langle \neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\rangle$  cols-coincide-2 by blast
    moreover
    have A B C CongA X PO Y
      using CongA-H-def  $\langle \text{CongA-H } A \ B \ C \ X \ PO \ Y \rangle$  by blast
    moreover
    have A B C CongA X PO Y'
      using CongA-H-def  $\langle \text{CongA-H } A \ B \ C \ X \ PO \ Y' \rangle$  by auto
    ultimately
    have Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
      using  $\langle$  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X  $\rangle$ 
         $\langle$  Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X  $\rangle$ 
        axiom-hcong-4-uniqueness cols-coincide-1
      by (metis Interpretation-Hilbert-neutral-dimensionless-pre.outH-def
         $\langle \forall A \ B \ C \ D. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' A \ B \ C \ D$ 

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    = same-side'-H A B C D)
  outH-def)
}
thus  $\bigwedge P PO X A B C Y Y'$ .
 $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X  $\wedge$ 
 $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\wedge$ 
CongA-H A B C X PO Y  $\wedge$ 
CongA-H A B C X PO Y'  $\wedge$ 
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X  $\wedge$ 
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X  $\longrightarrow$ 
Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
  by blast
{
  fix A B C A' B' C'
  assume  $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
     $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
    Cong A B A' B' and
    Cong A C A' C' and
    CongA-H B A C B' A' C'
  have  $\neg$  Col-H A B C
    using  $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C} = \text{Col-H A B C} \rangle$ 
     $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C} \rangle$  by auto
  moreover
  have  $\neg$  Col-H A' B' C'
    using  $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C} = \text{Col-H A B C} \rangle$ 
     $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'} \rangle$  by blast
  have  $\neg$  Col A B C
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
     $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C} \rangle$  cols-coincide-2 by blast
  moreover
  have  $\neg$  Col A' B' C'
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
     $\langle \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'} \rangle$ 
    cols-coincide-2 by blast
  moreover
  have B A C CongA B' A' C'
    using CongA-H-def  $\langle \text{CongA-H B A C B' A' C'} \rangle$  by auto
  ultimately
  have CongA-H A B C A' B' C'
    using CongA-H-def  $\langle \text{Cong A B A' B'} \rangle \langle \text{Cong A C A' C'} \rangle$  l11-49 not-col-distincts by blast
}
thus  $\bigwedge A B C A' B' C'$ .
 $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C  $\wedge$ 
 $\neg$  Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'  $\wedge$ 
Cong A B A' B'  $\wedge$  Cong A C A' C'  $\wedge$  CongA-H B A C B' A' C'  $\longrightarrow$ 
CongA-H A B C A' B' C'
  by blast
show  $\bigwedge A B C D. \text{Cong A B C D} \longrightarrow \text{Cong A B D C}$ 
  using not-cong-1243 by blast
show  $\bigwedge l m P. \text{isLine } l \wedge \text{isLine } m \wedge \text{IncidentL } P l \wedge l = l = m \longrightarrow \text{IncidentL } P m$ 
  using axiom-Incidl-morphism by blast
show  $\bigwedge p q M. \text{isPlane } p \wedge \text{isPlane } q \wedge \text{IncidentP } M p \wedge p = p = q \longrightarrow \text{IncidentP } M q$ 
  using axiom-Incidp-morphism by blast
show  $\bigwedge P l. \text{IncidentL } P l \longrightarrow \text{isLine } l$ 
  using IncidentL-def by blast
show  $\bigwedge P p. \text{IncidentP } P p \longrightarrow \text{isPlane } p$ 
  using IncidentP-def by blast
qed

```

interpretation Intrepretation-Hilbert-neutral-3D: Hilbert-neutral-3D
 where IncidL = IncidentL and IncidP = IncidentP and EqL = EqTL
 and EqP = EqTP and IsL = isLine and IsP = isPlane
 and BetH = Between-H and CongH = Cong and CongaH = CongA-H
 and PP = TPA and PQ = TPB and PR = TPC
 and HS1 = TS1 and HS2 = TS2 and HS3 = TS3
 and HS4 = TS4

```

proof
{
  fix p q A
  assume isPlane p and
    isPlane q and
      IncidentP A p and IncidentP A q
  obtain PA PB PC where p = Plane PA PB PC and  $\neg$  Col-H PA PB PC
    by (metis Plane-def  $\langle$ isPlane p $\rangle$  cols-coincide-1 isPlane-def surj-pair)
  hence  $\neg$  Col PA PB PC
    by (simp add: cols-coincide)
  obtain QA QB QC where q = Plane QA QB QC and  $\neg$  Col-H QA QB QC
    by (metis Plane-def  $\langle$ isPlane q $\rangle$  cols-coincide-1 isPlane-def surj-pair)
  hence  $\neg$  Col QA QB QC
    by (simp add: cols-coincide)
  have Coplanar A PA PB PC
  proof (rule plane-cop [where ?p = p])
    show IncidentP A p
      by (simp add:  $\langle$ IncidentP A p $\rangle$ )
    show IncidentP PA p
      using  $\langle$  $\neg$  Col-H PA PB PC $\rangle$   $\langle$ p = Plane PA PB PC $\rangle$  axiom-plane-existence
        eqp-incidentp incidentp-eqp by blast
    show IncidentP PB p
      by (metis IncidentP-def  $\langle$  $\neg$  Col-H PA PB PC $\rangle$   $\langle$ isPlane p $\rangle$   $\langle$ p = Plane PA PB PC $\rangle$ 
        axiom-plane-existence plane-cop)
    show IncidentP PC p
      using IncidentP-def  $\langle$ isPlane p $\rangle$   $\langle$ p = Plane PA PB PC $\rangle$  ncop-distincts by blast
  qed
  have Coplanar A QA QB QC
  proof (rule plane-cop [where ?p = q])
    show IncidentP A q
      by (simp add:  $\langle$ IncidentP A q $\rangle$ )
    show IncidentP QA q
      using  $\langle$  $\neg$  Col-H QA QB QC $\rangle$   $\langle$ q = Plane QA QB QC $\rangle$  axiom-plane-existence
        eqp-incidentp incidentp-eqp by blast
    show IncidentP QB q
      by (metis IncidentP-def  $\langle$  $\neg$  Col-H QA QB QC $\rangle$   $\langle$ isPlane q $\rangle$   $\langle$ q = Plane QA QB QC $\rangle$ 
        axiom-plane-existence plane-cop)
    show IncidentP QC q
      using IncidentP-def  $\langle$ isPlane q $\rangle$   $\langle$ q = Plane QA QB QC $\rangle$  ncop-distincts by blast
  qed
  have plane-intersection-axiom
    using upper-dim-3 upper-dim-3-axiom-def upper-dim-3-equivalent-axioms by blast
  have  $\exists B. \text{Coplanar } PA \ PB \ PC \ B \wedge \text{Coplanar } QA \ QB \ QC \ B \wedge A \neq B$ 
  proof -
    have Coplanar PA PB PC A
      by (simp add:  $\langle$ Coplanar A PA PB PC $\rangle$  coplanar-perm-9)
    moreover have Coplanar QA QB QC A
      using  $\langle$ Coplanar A QA QB QC $\rangle$  coplanar-perm-9 by blast
    ultimately show ?thesis
      using  $\langle$ plane-intersection-axiom $\rangle$  plane-intersection-axiom-def by blast
  qed
  hence  $\exists B. \text{IncidentP } B \ p \wedge \text{IncidentP } B \ q \wedge A \neq B$ 
    using IncidentP-def  $\langle$ isPlane p $\rangle$   $\langle$ isPlane q $\rangle$   $\langle$ p = Plane PA PB PC $\rangle$   $\langle$ q = Plane QA QB QC $\rangle$ 
      coplanar-perm-18 by blast
}
thus  $\bigwedge p \ q \ A. \text{isPlane } p \wedge \text{isPlane } q \wedge \text{IncidentP } A \ p \wedge \text{IncidentP } A \ q \longrightarrow$ 
  ( $\exists B. \text{IncidentP } B \ p \wedge \text{IncidentP } B \ q \wedge A \neq B$ )
  by blast
show  $\nexists p. \text{isPlane } p \wedge \text{IncidentP } TS1 \ p \wedge \text{IncidentP } TS2 \ p \wedge \text{IncidentP } TS3 \ p \wedge \text{IncidentP } TS4 \ p$ 
  using lower-dim-3' by blast
qed

end
end

```

theory *Hilbert-Euclidean-Model-Tarski-Euclidean*

imports *Hilbert-Euclidean*

begin

32 Hilbert Euclidean - Tarski Euclidean Model

32.1 Interpretation

context *Tarski-Euclidean*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre: Hilbert-neutral-dimensionless-pre*

where $\text{IncidL} = \text{IncidentL}$ **and** $\text{IncidP} = \text{IncidentP}$ **and** $\text{EqL} = \text{EqTL}$
and $\text{EqP} = \text{EqTP}$ **and** $\text{IsL} = \text{isLine}$ **and** $\text{IsP} = \text{isPlane}$
and $\text{BetH} = \text{Between-H}$ **and** $\text{CongH} = \text{Cong}$ **and** $\text{CongaH} = \text{CongA-H}$

proof –

qed

interpretation *Intrepretation-Hilbert-neutral-dimensionless: Hilbert-neutral-dimensionless*

where $\text{IncidL} = \text{IncidentL}$ **and** $\text{IncidP} = \text{IncidentP}$ **and** $\text{EqL} = \text{EqTL}$
and $\text{EqP} = \text{EqTP}$ **and** $\text{IsL} = \text{isLine}$ **and** $\text{IsP} = \text{isPlane}$
and $\text{BetH} = \text{Between-H}$ **and** $\text{CongH} = \text{Cong}$ **and** $\text{CongaH} = \text{CongA-H}$
and $\text{PP} = \text{TPA}$ **and** $\text{PQ} = \text{TPB}$ **and** $\text{PR} = \text{TPC}$

proof

have $\forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longleftrightarrow \text{Col-H } A B C$

proof –

{
 fix $A B C$
 assume $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$
 hence $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$
 by (*simp add: Hilbert-neutral-dimensionless-pre.ColH-def*)
 hence $\text{Col-H } A B C$
 using $\text{Col-H-def EqTL-def axiom-line-uniqueness cols-coincide-2 not-col-distincts}$ **by** *blast*
}

moreover

{
 fix $A B C$
 assume $\text{Col-H } A B C$
 hence $\exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l$
 using Col-H-def **by** *blast*
 have $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C$
 using $\text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def}$
 $\langle \exists l. \text{isLine } l \wedge \text{IncidentL } A l \wedge \text{IncidentL } B l \wedge \text{IncidentL } C l \rangle$ **by** *blast*
}

ultimately show *?thesis*

by *blast*

qed

have $\forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C \longleftrightarrow \text{Col } A B C$

using $\langle \forall A B C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A B C = \text{Col-H } A B C \rangle$
cols-coincide **by** *blast*

have $\forall A B l. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side } A B l$

$\longleftrightarrow \text{same-side-H } A B l$

by (*simp add: Hilbert-neutral-dimensionless-pre.cut-def*
Interpretation-Hilbert-neutral-dimensionless-pre.same-side-def cut-H-def same-side-H-def)

have $\forall A B C D. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side}' A B C D$

$\longleftrightarrow \text{same-side}'\text{-H } A B C D$

by (*simp add: Hilbert-neutral-dimensionless-pre.same-side'\text{-def}*
 $\langle \forall A B l. \text{Interpretation-Hilbert-neutral-dimensionless-pre.same-side } A B l = \text{same-side-H } A B l \rangle$
same-side'\text{-H-def})

```

show  $\bigwedge l. \text{isLine } l \longrightarrow l = l$ 
  by (simp add: eq-reflexivity)
show  $\bigwedge l1\ l2. \text{isLine } l1 \wedge \text{isLine } l2 \wedge l1 = l2 \longrightarrow l2 = l1$ 
  using eq-symmetry by blast
show  $\bigwedge l1\ l2\ l3. l1 = l2 \wedge l2 = l3 \longrightarrow l1 = l3$ 
  using eq-transitivity by blast
show  $\bigwedge p. \text{isPlane } p \longrightarrow p = p$ 
  by (simp add: eqp-reflexivity)
show  $\bigwedge p1\ p2. p1 = p2 \longrightarrow p2 = p1$ 
  using eqp-symmetry by blast
show  $\bigwedge p1\ p2\ p3. p1 = p2 \wedge p2 = p3 \longrightarrow p1 = p3$ 
  using eqp-transitivity by blast
show  $\bigwedge A\ B. A \neq B \longrightarrow (\exists l. \text{isLine } l \wedge \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l)$ 
  using axiom-line-existence by blast
show  $\bigwedge A\ B\ l\ m. A \neq B \wedge \text{isLine } l \wedge \text{isLine } m \wedge \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l \wedge \text{IncidentL } A\ m \wedge \text{IncidentL } B\ m \longrightarrow l = m$ 
  using axiom-line-uniqueness by blast
show  $\forall l. \text{isLine } l \longrightarrow (\exists A\ B. \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l \wedge A \neq B)$ 
  using axiom-two-points-on-line by blast
show  $\text{TPA} \neq \text{TPB} \wedge \text{TPB} \neq \text{TPC} \wedge \text{TPA} \neq \text{TPC} \wedge \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH TPA TPB TPC}$ 
  using Bet-cases between-trivial lower-dim third-point
   $\langle \forall A\ B\ C. \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C = (\text{Col } A\ B\ C) \rangle$ 
  by blast
show  $\forall A\ B\ C. \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C \longrightarrow (\exists p. \text{isPlane } p \wedge \text{IncidentP } A\ p \wedge \text{IncidentP } B\ p \wedge \text{IncidentP } C\ p)$ 
  using bet-coplanar between-trivial2 cop-plane IncidentP-def by blast
show  $\forall p. \exists A. \text{isPlane } p \longrightarrow \text{IncidentP } A\ p$ 
  by (simp add: axiom-one-point-on-plane)
show  $\bigwedge A\ B\ C\ p\ q. \neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C \wedge \text{isPlane } p \wedge \text{isPlane } q \wedge \text{IncidentP } A\ p \wedge \text{IncidentP } B\ p \wedge \text{IncidentP } C\ p \wedge \text{IncidentP } A\ q \wedge \text{IncidentP } B\ q \wedge \text{IncidentP } C\ q \longrightarrow p = q$ 
  using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
  axiom-plane-uniqueness by blast
show  $\forall A\ B\ l\ p. A \neq B \wedge \text{isLine } l \wedge \text{isPlane } p \wedge \text{IncidentL } A\ l \wedge \text{IncidentL } B\ l \wedge \text{IncidentP } A\ p \wedge \text{IncidentP } B\ p \longrightarrow \text{Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP } l\ p$ 
  by (meson IncidentLP-def Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
  axiom-line-on-plane)
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow A \neq C$ 
  using axiom-between-diff by blast
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C$ 
  by (meson Between-H-def Col-H-def Col-def
  Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def cols-coincide-2)
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow \text{Between-H } C\ B\ A$ 
  using axiom-between-comm by blast
show  $\bigwedge A\ B. A \neq B \longrightarrow (\exists C. \text{Between-H } A\ B\ C)$ 
  by (simp add: axiom-between-out)
show  $\bigwedge A\ B\ C. \text{Between-H } A\ B\ C \longrightarrow \neg \text{Between-H } B\ C\ A$ 
  using axiom-between-only-one by blast
{
  fix  $A\ B\ C\ p\ l$ 
  assume  $\neg \text{Interpretation-Hilbert-neutral-dimensionless-pre.ColH } A\ B\ C$  and
     $\text{isLine } l$  and
     $\text{IncidentP } A\ p$  and

```

```

    IncidentP B p and
    IncidentP C p and
    Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p and
    ¬ IncidentL C l and
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B
  have ¬ Col A B C
    by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
        ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2)
  moreover have cut-H l A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.cut-def
        ⟨Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B⟩ cut-H-def by blast
  ultimately have cut-H l A C ∨ cut-H l B C
    using axiom-pasch Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
        ⟨IncidentP A p⟩ ⟨IncidentP B p⟩ ⟨IncidentP C p⟩
        ⟨Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p⟩ ⟨¬ IncidentL C l⟩
        axiom-line-on-plane axiom-two-points-on-line cols-coincide-1 by meson
  hence Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C ∨
    Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
    by (simp add: Hilbert-neutral-dimensionless-pre.cut-def cut-H-def)
}
thus ∧ A B C p l.
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
isLine l ∧ isPlane p ∧
IncidentP A p ∧
IncidentP B p ∧
IncidentP C p ∧
Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP l p ∧
¬ IncidentL C l ∧ Interpretation-Hilbert-neutral-dimensionless-pre.cut l A B →
Interpretation-Hilbert-neutral-dimensionless-pre.cut l A C ∨
Interpretation-Hilbert-neutral-dimensionless-pre.cut l B C
  by blast
{
  fix l
  fix A B A' P::'a
  assume A ≠ B and
    A' ≠ P and
    isLine l and
    IncidentL A' l and
    IncidentL P l
  then obtain B' where IncidentL B' l ∧ outH A' P B' ∧ Cong A' B' A B
    using axiom-hcong-1-existence by presburger
  hence ∃ B'. IncidentL B' l ∧
    Interpretation-Hilbert-neutral-dimensionless-pre.outH A' P B' ∧ Cong A' B' A B
    using Interpretation-Hilbert-neutral-dimensionless-pre.outH-def outH-def by auto
}
thus ∧ l A B A' P. A ≠ B ∧ A' ≠ P ∧ isLine l ∧ IncidentL A' l ∧ IncidentL P l →
(∃ B'. IncidentL B' l ∧
Interpretation-Hilbert-neutral-dimensionless-pre.outH A' P B' ∧ Cong A' B' A B)
  by blast
show ∧ A B C D E F. Cong A B C D ∧ Cong A B E F → Cong C D E F
  using cong-inner-transitivity by blast
{
  fix A B C A' B' C'
  assume Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
    Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
    Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B C and
    Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C' and
    Cong A B A' B' and
    Cong B C B' C'
  have (∃ l. IncidentL A l ∧ IncidentL B l ∧ IncidentL C l)
    using Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
        ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ by force
  then obtain l where IncidentL A l and IncidentL B l and IncidentL C l
    by blast
  hence Col A B C
    using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩

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    ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = (Col A B C)⟩
  by blast
moreover
have (∃ l. IncidentL A' l ∧ IncidentL B' l ∧ IncidentL C' l)
  using Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
  ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ by force
then obtain l' where IncidentL A' l' and IncidentL B' l' and IncidentL C' l'
  by blast
hence Col A' B' C'
  using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩
  ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = (Col A B C)⟩
  by blast
moreover
have ¬ (∃ P. Between-H A P B ∧ Between-H B P C)
  using Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def
  ⟨Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B C⟩ by blast
have ¬ (∃ P. Between-H A' P B' ∧ Between-H B' P C')
  using Interpretation-Hilbert-neutral-dimensionless-pre.disjoint-def
  ⟨Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C'⟩ by blast
have Col-H A B C
  using calculation(1) cols-coincide-2 by blast
hence Bet A B C
  using ⟨⊢ P. Between-H A P B ∧ Between-H B P C⟩ col-disjoint-bet disjoint-H-def by blast
moreover
have Col-H A' B' C'
  by (simp add: calculation(2) cols-coincide-2)
moreover
hence Bet A' B' C'
  by (simp add: ⟨⊢ P. Between-H A' P B' ∧ Between-H B' P C'⟩ col-disjoint-bet disjoint-H-def)
ultimately
have Cong A C A' C'
  using Tarski-neutral-dimensionless.l2-11-b Tarski-neutral-dimensionless-axioms
  ⟨Cong A B A' B'⟩ ⟨Cong B C B' C'⟩ by fastforce
}
thus ∧ A B C A' B' C'.
Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' ∧
Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A B B C ∧
Interpretation-Hilbert-neutral-dimensionless-pre.disjoint A' B' B' C' ∧
Cong A B A' B' ∧ Cong B C B' C' →
Cong A C A' C'
  by blast
{
  fix A B C
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C
  hence ¬ Col A B C
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = (Col A B C)⟩
    by blast
  hence A B C CongA A B C
    by (metis Tarski-neutral-dimensionless.conga-refl
      Tarski-neutral-dimensionless.not-col-distincts Tarski-neutral-dimensionless-axioms)
  hence CongA-H A B C A B C
    by (simp add: CongA-H-def)
}
thus ∧ A B C.
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C → CongA-H A B C A B C
  by blast
show ∧ A B C.
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C → CongA-H A B C C B A
  using CongA-H-def ⟨⊢ Ca Ba A. ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A Ba Ca
    → CongA-H A Ba Ca A Ba Ca⟩ conga-right-comm by presburger
show ∧ A B C D E F. CongA-H A B C D E F → CongA-H C B A F E D
  by (meson CongA-H-def Tarski-neutral-dimensionless.axiom-conga-permlr
    Tarski-neutral-dimensionless-axioms)
show ∧ A B C D E F A' C' D' F'.
CongA-H A B C D E F ∧

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Interpretation-Hilbert-neutral-dimensionless-pre.outH B A A' ∧
Interpretation-Hilbert-neutral-dimensionless-pre.outH B C C' ∧
Interpretation-Hilbert-neutral-dimensionless-pre.outH E D D' ∧
Interpretation-Hilbert-neutral-dimensionless-pre.outH E F F' →
CongA-H A' B C' D' E F'
using axiom-congaH-outH-congaH CongA-H-def outH-def
Interpretation-Hilbert-neutral-dimensionless-pre.outH-def by force
{
fix P PO X A B C
assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X and
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C
have ¬ Col P PO X
by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X⟩ cols-coincide-2)
moreover
have ¬ Col A B C
using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
ultimately
obtain Y where A B C CongA X PO Y and same-side'-H P Y PO X
using axiom-hcong-4-existence cols-coincide-1 by blast
hence PO ≠ X ∧ (∀ l. isLine l → (IncidentL PO l ∧ IncidentL X l) → same-side-H P Y l)
using same-side'-H-def by auto
have ∃ Y. CongA-H A B C X PO Y ∧
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X
proof –
have CongA-H A B C X PO Y
by (simp add: CongA-H-def ⟨A B C CongA X PO Y⟩)
moreover
have Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X
by (simp add:
⟨∀ A B C D. Interpretation-Hilbert-neutral-dimensionless-pre.same-side' A B C D
= same-side'-H A B C D⟩
⟨same-side'-H P Y PO X⟩)
ultimately show ?thesis
by blast
qed
}
thus ∧P PO X A B C.
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X ∧
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C →
(∃ Y. CongA-H A B C X PO Y ∧
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X)
by blast
{
fix P PO X A B C Y Y'
assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X and
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
CongA-H A B C X PO Y and
CongA-H A B C X PO Y' and
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X and
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X
have ¬ Col P PO X
by (meson Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X⟩ cols-coincide-2)
moreover
have ¬ Col A B C
using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
moreover
have A B C CongA X PO Y
using CongA-H-def ⟨CongA-H A B C X PO Y⟩ by blast
moreover
have A B C CongA X PO Y'
using CongA-H-def ⟨CongA-H A B C X PO Y'⟩ by auto
ultimately

```

```

have Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
  using ⟨Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X⟩
    ⟨Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X⟩
    axiom-hcong-4-uniqueness cols-coincide-1
  by (metis Interpretation-Hilbert-neutral-dimensionless-pre.outH-def
    ⟨∀ A B C D. Interpretation-Hilbert-neutral-dimensionless-pre.same-side' A B C D
      = same-side'-H A B C D⟩
    outH-def)
}
thus ∧P PO X A B C Y Y'.
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH P PO X ∧
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
CongA-H A B C X PO Y ∧
CongA-H A B C X PO Y' ∧
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y PO X ∧
Interpretation-Hilbert-neutral-dimensionless-pre.same-side' P Y' PO X →
Interpretation-Hilbert-neutral-dimensionless-pre.outH PO Y Y'
  by blast
{
  fix A B C A' B' C'
  assume ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C and
    ¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' and
    Cong A B A' B' and
    Cong A C A' C' and
    CongA-H B A C B' A' C'
  have ¬ Col-H A B C
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = Col-H A B C⟩
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ by auto
  moreover
  have ¬ Col-H A' B' C'
    using ⟨∀ A B C. Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C = Col-H A B C⟩
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ by blast
  have ¬ Col A B C
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C⟩ cols-coincide-2 by blast
  moreover
  have ¬ Col A' B' C'
    using Col-H-def Interpretation-Hilbert-neutral-dimensionless-pre.ColH-def
    ⟨¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C'⟩ cols-coincide-2
    by blast
  moreover
  have B A C CongA B' A' C'
    using CongA-H-def ⟨CongA-H B A C B' A' C'⟩ by auto
  ultimately
  have CongA-H A B C A' B' C'
    using CongA-H-def ⟨Cong A B A' B'⟩ ⟨Cong A C A' C'⟩ l11-49 not-col-distincts by blast
}
thus ∧A B C A' B' C'.
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A B C ∧
¬ Interpretation-Hilbert-neutral-dimensionless-pre.ColH A' B' C' ∧
Cong A B A' B' ∧ Cong A C A' C' ∧ CongA-H B A C B' A' C' →
CongA-H A B C A' B' C'
  by blast
show ∧A B C D. Cong A B C D → Cong A B D C
  using not-cong-1243 by blast
show ∧l m P. isLine l ∧ isLine m ∧ IncidentL P l ∧ l = l = m → IncidentL P m
  using axiom-Incidl-morphism by blast
show ∧p q M. isPlane p ∧ isPlane q ∧ IncidentP M p ∧ p = p = q → IncidentP M q
  using axiom-Incidp-morphism by blast
show ∧P l. IncidentL P l → isLine l
  using IncidentL-def by blast
show ∧P p. IncidentP P p → isPlane p
  using IncidentP-def by blast
qed

```

interpretation Intrepretation-Hilbert-euclidean: Hilbert-Euclidean


```

where IncidL = IncidentL and IncidP = IncidentP and EqL = EqTL
and EqP = EqTP and IsL = isLine and IsP = isPlane
and BetH = Between-H and CongH = Cong and CongaH = CongA-H
and PP = TPA and PQ = TPB and PR = TPC
proof
show  $\forall l P m1 m2. isLine l \wedge isLine m1 \wedge isLine m2 \wedge$ 
 $\neg IncidentL P l \wedge Interpretation-Hilbert-neutral-dimensionless-pre.Para l m1 \wedge$ 
 $IncidentL P m1 \wedge Interpretation-Hilbert-neutral-dimensionless-pre.Para l m2 \wedge$ 
 $IncidentL P m2 \longrightarrow m1 =l= m2$ 
using IncidentLP-def Interpretation-Hilbert-neutral-dimensionless-pre.IncidLP-def
Interpretation-Hilbert-neutral-dimensionless-pre.Para-def
Para-H-def axiom-euclid-uniqueness by force
qed

end
end

```

theory *Tarski-Neutral-Model-Hilbert-Neutral*

```

imports
  Tarski-Postulate-Parallels
  Hilbert-Neutral
  Gupta-Neutral

```

begin

33 Tarski Neutral Model Hilbert Neutral

context *Hilbert-neutral-dimensionless*

begin

```

interpretation Interp-Gupta-of-Tarski-neutral-dimensionless:
  Gupta-neutral-dimensionless
where BetG = Bet and
  CongG = Cong and
  GPA = PP and
  GPB = PQ and
  GPC = PR
proof
show  $\forall a b. Cong a b b a$ 
by (simp add: cong-permT)
show  $\forall a b p q r s. Cong a b p q \wedge Cong a b r s \longrightarrow Cong p q r s$ 
using cong-inner-transitivity by blast
show  $\forall a b c. Cong a b c c \longrightarrow a = b$ 
by (simp add: cong-identity)
show  $\forall a b c q. \exists x. Bet q a x \wedge Cong a x b c$ 
by (simp add: segment-construction)
show  $\forall a b c d a' b' c' d'. a \neq b \wedge Bet a b c \wedge Bet a' b' c' \wedge$ 
 $Cong a b a' b' \wedge Cong b c b' c' \wedge Cong a d a' d' \wedge Cong b d b' d' \longrightarrow$ 
 $Cong c d c' d'$ 
using five-segment by blast
show  $\forall a b c. Bet a b c \longrightarrow Bet c b a$ 
using bet-comm by blast
show  $\forall a b c d. Bet a b d \wedge Bet b c d \longrightarrow Bet a b c$ 
using bet-trans by blast
show  $\forall a b c p q. Bet a p c \wedge Bet b q c \wedge a \neq p \wedge c \neq p \wedge b \neq q \wedge$ 
 $c \neq q \wedge \neg (Bet a b c \vee Bet b c a \vee Bet c a b) \longrightarrow$ 
 $(\exists x. Bet p x b \wedge Bet q x a)$ 
using pasch-general-case by auto
show  $\neg (Bet PP PQ PR \vee Bet PQ PR PP \vee Bet PR PP PQ)$ 
using lower-dim-l by blast
qed

```

interpretation $H\text{-to-}T : \text{Tarski-neutral-dimensionless}$
where $Bet = Bet$ **and**
 $Cong = Cong$ **and**
 $TPA = PP$ **and**
 $TPB = PQ$ **and**
 $TPC = PR$

proof
show $\forall a\ b. Cong\ a\ b\ b\ a$
by (*simp add:*
 $Interp\text{-Gupta-of-Tarski-neutral-dimensionless.cong-pseudo-reflexivity}G$)
show $\forall a\ b\ p\ q\ r\ s. Cong\ a\ b\ p\ q \wedge Cong\ a\ b\ r\ s \longrightarrow Cong\ p\ q\ r\ s$
using *cong-inner-transitivity* **by** *blast*
show $\forall a\ b\ c. Cong\ a\ b\ c\ c \longrightarrow a = b$
by (*simp add:*
 $Interp\text{-Gupta-of-Tarski-neutral-dimensionless.cong-identity}G$)
show $\forall a\ b\ c\ q. \exists x. Bet\ q\ a\ x \wedge Cong\ a\ x\ b\ c$
by (*simp add:*
 $Interp\text{-Gupta-of-Tarski-neutral-dimensionless.segment-construction}G$)
show $\forall a\ b\ c\ d\ a'\ b'\ c'\ d'. a \neq b \wedge Bet\ a\ b\ c \wedge Bet\ a'\ b'\ c' \wedge$
 $Cong\ a\ b\ a'\ b' \wedge Cong\ b\ c\ b'\ c' \wedge Cong\ a\ d\ a'\ d' \wedge Cong\ b\ d\ b'\ d' \longrightarrow$
 $Cong\ c\ d\ c'\ d'$
using $Interp\text{-Gupta-of-Tarski-neutral-dimensionless.five-segment}G$ **by** *blast*
show $\forall a\ b. Bet\ a\ b\ a \longrightarrow a = b$
by (*simp add:* *bet-identity*)
show $\forall a\ b\ c\ p\ q. Bet\ a\ p\ c \wedge Bet\ b\ q\ c \longrightarrow (\exists x. Bet\ p\ x\ b \wedge Bet\ q\ x\ a)$
using $Interp\text{-Gupta-of-Tarski-neutral-dimensionless.inner-pasch}T$ **by** *blast*
show $\neg Bet\ PP\ PQ\ PR \wedge \neg Bet\ PQ\ PR\ PP \wedge \neg Bet\ PR\ PP\ PQ$
using *lower-dim-l* **by** *blast*

qed

lemma *MidH--Mid*:
assumes $M\ Midpoint\ A\ B$
shows $H\text{-to-}T.Midpoint\ M\ A\ B$

proof –
have $Bet\ A\ M\ B$
using *Midpoint-def assms* **by** *blast*
moreover
have $Cong\ A\ M\ M\ B$
using *Midpoint-def assms* **by** *blast*
ultimately show *?thesis*
using $H\text{-to-}T.Midpoint\text{-def}$ **by** *blast*

qed

lemma *Mid--MidH*:
assumes $H\text{-to-}T.Midpoint\ M\ A\ B$
shows $M\ Midpoint\ A\ B$

proof –
have $Bet\ A\ M\ B$
using $H\text{-to-}T.Midpoint\text{-def assms}$ **by** *blast*
moreover
have $Cong\ A\ M\ M\ B$
using $H\text{-to-}T.Midpoint\text{-def assms}$ **by** *blast*
ultimately show *?thesis*
by (*simp add:* *Midpoint-def*)

qed

lemma *col-colh-1*:
assumes $H\text{-to-}T.Col\ A\ B\ C$
shows $ColH\ A\ B\ C$

proof –
have $Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$
using *assms H-to-T.Col-def* **by** *blast*
thus *?thesis*
using *bet-colH colH-permut-312* **by** *blast*

qed

```

lemma col-colh-2:
  assumes ColH A B C
  shows H-to-T.Col A B C
proof -
  have Bet A B C  $\vee$  Bet B C A  $\vee$  Bet C A B
  by (simp add: ColH-bets assms)
  thus ?thesis
  using H-to-T.Col-def by presburger
qed

lemma col-colh:
  shows H-to-T.Col A B C  $\longleftrightarrow$  ColH A B C
  using col-colh-1 col-colh-2 by blast

lemma line-col:
  assumes IsL l and
    IncidL A l and
    IncidL B l and
    IncidL C l
  shows H-to-T.Col A B C
proof -
  have ColH A B C
  using ColH-def assms(1) assms(2) assms(3) assms(4) by blast
  thus ?thesis
  by (simp add: col-colh-2)
qed

lemma bet--beth:
  assumes A  $\neq$  B and
    B  $\neq$  C and
    Bet A B C
  shows BetH A B C
  using Bet-def assms(1) assms(2) assms(3) by blast

lemma coplanar-plane0:
  assumes ColH A B X and
    ColH C D X
  shows  $\exists p. \text{IncidP } A p \wedge \text{IncidP } B p \wedge \text{IncidP } C p \wedge \text{IncidP } D p$ 
proof -
  obtain l where IsL l and IncidL A l and IncidL B l and IncidL X l
  using ColH-def assms(1) by blast
  obtain m where IsL m and IncidL C m and IncidL D m and IncidL X m
  using ColH-def assms(2) by blast
  obtain Y where X  $\neq$  Y and IncidL Y l
  using  $\langle \text{IncidL } X l \rangle$  other-point-on-line by blast
  {
    assume EqL m l
    obtain Z where  $\neg \text{ColH } X Y Z$ 
    using  $\langle X \neq Y \rangle$  ncolH-exists by blast
    obtain p where IsP p and IncidP X p and IncidP Y p and IncidP Z p
    using  $\langle \neg \text{ColH } X Y Z \rangle$  plan-existence by blast
    hence IncidLP l p
    using  $\langle \text{IncidL } X l \rangle \langle \text{IncidL } Y l \rangle \langle \text{IsL } l \rangle \langle X \neq Y \rangle$  line-on-plane by blast
    hence ?thesis
    using IncidLP-def IncidL-morphism  $\langle \text{EqL } m l \rangle \langle \text{IncidL } A l \rangle \langle \text{IncidL } B l \rangle$ 
       $\langle \text{IncidL } C m \rangle \langle \text{IncidL } D m \rangle \langle \text{IsL } m \rangle$  by blast
  }
  moreover
  {
    assume  $\neg \text{EqL } m l$ 
    obtain Z where X  $\neq$  Z and IncidL Z m
    using  $\langle \text{IncidL } X m \rangle$  other-point-on-line by blast
    have  $\neg \text{ColH } X Y Z$ 
    using IncidL-not-IncidL--not-colH  $\langle \text{IncidL } X l \rangle \langle \text{IncidL } X m \rangle \langle \text{IncidL } Y l \rangle$ 
       $\langle \text{IncidL } Z m \rangle \langle \text{IsL } l \rangle \langle \text{IsL } m \rangle \langle X \neq Y \rangle \langle X \neq Z \rangle \langle \neg \text{EqL } m l \rangle$  line-uniqueness by blast
    then obtain p where IsP p and IncidP X p and IncidP Y p and IncidP Z p
  }

```

```

    using plan-existence by blast
  hence IncidLP l p
    using ⟨IncidL X l⟩ ⟨IncidL Y l⟩ ⟨IsL l⟩ ⟨X ≠ Y⟩ line-on-plane by blast
  moreover have IncidLP m p
    using ⟨IncidL X m⟩ ⟨IncidL Z m⟩ ⟨IncidP X p⟩ ⟨IncidP Z p⟩ ⟨IsL m⟩
      ⟨IsP p⟩ ⟨X ≠ Z⟩ line-on-plane by blast
  ultimately have ?thesis
    using IncidLP-def ⟨IncidL A l⟩ ⟨IncidL B l⟩ ⟨IncidL C m⟩ ⟨IncidL D m⟩ by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma coplanar-plane1:
  assumes Bet A B X ∨ Bet B X A ∨ Bet X A B and
    Bet C D X ∨ Bet D X C ∨ Bet X C D
  shows ∃ p. IsP p ∧ IncidP A p ∧ IncidP B p ∧ IncidP C p ∧ IncidP D p
proof -
  have ColH A B X
    by (meson assms(1) bet-colH colH-permut-312)
  moreover have ColH C D X
    by (simp add: H-to-T.Col-def assms(2) col-colh-1)
  ultimately show ?thesis
    using Is-plane coplanar-plane0 by blast
qed

```

```

lemma coplanar-plane:
  assumes H-to-T.Coplanar A B C D
  shows ∃ p. IsP p ∧ IncidP A p ∧ IncidP B p ∧ IncidP C p ∧ IncidP D p
proof -
  obtain X where H1: (H-to-T.Col A B X ∧ H-to-T.Col C D X) ∨
    (H-to-T.Col A C X ∧ H-to-T.Col B D X) ∨
    (H-to-T.Col A D X ∧ H-to-T.Col B C X)
    using H-to-T.Coplanar-def assms by auto
  moreover have (H-to-T.Col A B X ∧ H-to-T.Col C D X) ⟶ ?thesis
    using H-to-T.Col-def coplanar-plane1 by force
  moreover {
    assume H-to-T.Col A C X and
      H-to-T.Col B D X
    have Bet A C X ∨ Bet C X A ∨ Bet X A C
      using H-to-T.Col-def ⟨H-to-T.Col A C X⟩ by presburger
    moreover have Bet B D X ∨ Bet D X B ∨ Bet X B D
      using H1 H-to-T.Col-def ⟨H-to-T.Col B D X⟩ by blast
    ultimately have ?thesis
      using coplanar-plane1 by blast
  }
  moreover {
    assume H-to-T.Col A D X and
      H-to-T.Col B C X
    have Bet A D X ∨ Bet D X A ∨ Bet X A D
      using H-to-T.Col-def ⟨H-to-T.Col A D X⟩ by auto
    moreover have Bet B C X ∨ Bet C X B ∨ Bet X B C
      using H-to-T.Col-def ⟨H-to-T.Col B C X⟩ by auto
    ultimately have ?thesis
      using coplanar-plane1 by blast
  }
  ultimately show ?thesis
    by blast
qed

```

```

lemma plane-coplanar:
  assumes IncidP A p and
    IncidP B p and
    IncidP C p and
    IncidP D p
  shows H-to-T.Coplanar A B C D

```

```

proof –
{
  assume ColH A B C
  hence H-to-T.Col A B C
    using col-coll-2 by blast
  hence ?thesis
    using H-to-T.ncop--ncols by blast
}
moreover
{
  assume ¬ ColH A B C
  hence A ≠ B
    by (simp add: ncolH-distincts)
  obtain M where M Midpoint A B
    using H-to-T.midpoint-existence Mid-MidH by blast
  have M ≠ A
    using Cong-def Hilbert-neutral-dimensionless-pre.Midpoint-def
      ⟨A ≠ B⟩ ⟨M Midpoint A B⟩ by force
  have M ≠ B
    using Midpoint-def Interp-Gupta-of-Tarski-neutral-dimensionless.cong-identityG
      ⟨A ≠ B⟩ ⟨M Midpoint A B⟩ by blast
  have ColH A B M
    using Midpoint-def ⟨M Midpoint A B⟩ bet-colH colH-permut-132 by blast
  then obtain m where IsL m and IncidL A m and IncidL B m and IncidL M m
    using ColH-def by blast
  {
    assume D = M
    hence ?thesis
      using H-to-T.ncop--ncols col-coll-2 ⟨ColH A B M⟩ by blast
  }
  moreover
  {
    assume D ≠ M
    obtain l where IsL l and IncidL D l and IncidL M l
      using ⟨D ≠ M⟩ line-existence by blast
    {
      assume IncidL A l
      hence EqL m l
        using ⟨IncidL A m⟩ ⟨IncidL M l⟩ ⟨IncidL M m⟩ ⟨IsL l⟩ ⟨IsL m⟩ ⟨M ≠ A⟩
          line-uniqueness by presburger
      have H-to-T.Col A B A ∧ H-to-T.Col C D A ∨ H-to-T.Col A C A ∧ H-to-T.Col B D A ∨
        H-to-T.Col A D A ∧ H-to-T.Col B C A
        by (meson ColH-def H-to-T.not-col-distincts IncidL-morphism Is-line
          ⟨EqL m l⟩ ⟨IncidL A l⟩ ⟨IncidL B m⟩ ⟨IncidL D l⟩ col-coll-2)
      hence ?thesis
        using H-to-T.Coplanar-def by blast
    }
  }
  moreover
  {
    assume ¬ IncidL A l
    {
      assume IncidL B l
      hence EqL l m
        using ⟨IncidL B m⟩ ⟨IncidL M l⟩ ⟨IncidL M m⟩ ⟨IsL l⟩ ⟨IsL m⟩
          ⟨M ≠ B⟩ line-uniqueness by blast
      hence False
        using ⟨IncidL A m⟩ ⟨IncidL B l⟩ ⟨IncidL B m⟩ ⟨IncidL M l⟩
          ⟨IncidL M m⟩ ⟨M ≠ B⟩ ⟨¬ IncidL A l⟩ inter-incid-uniquenessH by blast
    }
  }
  {
    assume IncidL C l
    hence H-to-T.Col A B M ∧ H-to-T.Col C D M ∨ H-to-T.Col A C M ∧ H-to-T.Col B D M ∨
      H-to-T.Col A D M ∧ H-to-T.Col B C M
      using Hilbert-neutral-dimensionless-pre.ColH-def ⟨ColH A B M⟩
        ⟨IncidL D l⟩ ⟨IncidL M l⟩ ⟨IsL l⟩ col-coll by fastforce
    hence ?thesis

```

```

    using H-to-T.Coplanar-def by blast
  }
  moreover
  {
    assume  $\neg \text{IncidL } C \ l$ 
    have IncidP M p
      using  $\langle A \neq B \rangle \langle \text{ColH } A \ B \ M \rangle \text{ assms}(1) \text{ assms}(2) \text{ line-on-plane'}$  by blast
    have
       $\neg \text{ColH } A \ B \ C \wedge \text{IsL } l \wedge \text{IsP } p \wedge \text{IncidP } A \ p \wedge \text{IncidP } B \ p \wedge$ 
       $\text{IncidP } C \ p \wedge \text{IncidLP } l \ p \wedge \neg \text{IncidL } C \ l \wedge (\text{cut } l \ A \ B)$ 
     $\longrightarrow$ 
     $(\text{cut } l \ A \ C) \vee (\text{cut } l \ B \ C)$ 
    using pasch by blast
    moreover
    {
      assume  $\text{cut } l \ A \ C$ 
      then obtain I where IncidL I l and BetH A I C
      using local.cut-def by blast
      {
        assume H-to-T.Col I A B
        have  $A \neq I$ 
          using  $\langle \text{IncidL } I \ l \rangle \langle \neg \text{IncidL } A \ l \rangle$  by auto
        moreover have ColH A I A
          by (simp add: colH-trivial121)
        moreover have ColH A I C
          by (simp add:  $\langle \text{BetH } A \ I \ C \rangle$  between-col)
        ultimately have False
          using  $\langle \neg \text{ColH } A \ B \ C \rangle$  colH-trans  $\langle \text{H-to-T.Col } I \ A \ B \rangle$ 
          colH-permut-213 col-colh-1 by blast
      }
      hence  $\neg \text{H-to-T.Col } I \ A \ B$ 
        by blast
      moreover have H-to-T.Coplanar I A B C
        using H-to-T.bet--coplanar H-to-T.ncoplanar-perm-7  $\langle \text{BetH } A \ I \ C \rangle$ 
        betH-to-bet by blast
      moreover
      have H-to-T.Col I D M
        using ColH-def  $\langle \text{IncidL } D \ l \rangle \langle \text{IncidL } I \ l \rangle \langle \text{IncidL } M \ l \rangle \langle \text{IsL } l \rangle$  col-colh by auto
      have H-to-T.Col A B M
        using  $\langle \text{ColH } A \ B \ M \rangle$  col-colh-2 by fastforce
      hence  $\text{H-to-T.Col } I \ A \ M \wedge \text{H-to-T.Col } B \ D \ M \vee \text{H-to-T.Col } I \ B \ M \wedge \text{H-to-T.Col } A \ D \ M \vee$ 
         $\text{H-to-T.Col } I \ D \ M \wedge \text{H-to-T.Col } A \ B \ M$ 
        using  $\langle \text{H-to-T.Col } I \ D \ M \rangle$  by auto
      hence H-to-T.Coplanar I A B D
        using H-to-T.Coplanar-def by blast
      ultimately have ?thesis
        using H-to-T.coplanar-trans-1 by blast
    }
  }
  moreover
  {
    assume  $\text{cut } l \ B \ C$ 
    then obtain I where IncidL I l and BetH B I C
    using local.cut-def by blast
    {
      assume H-to-T.Col I B A
      have  $B \neq I$ 
        using  $\langle \text{IncidL } B \ l \implies \text{False} \rangle \langle \text{IncidL } I \ l \rangle$  by fastforce
      moreover have ColH B I A
        using  $\langle \text{H-to-T.Col } I \ B \ A \rangle$  colH-permut-213 col-colh by blast
      moreover have ColH B I B
        by (simp add: colH-trivial121)
      ultimately have False
        using  $\langle \text{BetH } B \ I \ C \rangle \langle \neg \text{ColH } A \ B \ C \rangle$  between-col colH-trans by blast
    }
  }
  hence  $\neg \text{H-to-T.Col } I \ B \ A$ 
    by blast

```

```

moreover have  $H\text{-to-}T.Coplanar\ I\ B\ A\ C$ 
  using  $H\text{-to-}T.bet\text{-}coplanar\ H\text{-to-}T.ncoplanar\text{-}perm\text{-}7\ \langle BetH\ B\ I\ C \rangle$ 
     $betH\text{-to-}bet$  by  $blast$ 
moreover
have  $H\text{-to-}T.Col\ I\ D\ M$ 
  using  $ColH\text{-}def\ \langle IncidL\ D\ l \rangle\ \langle IncidL\ I\ l \rangle\ \langle IncidL\ M\ l \rangle\ \langle IsL\ l \rangle\ col\text{-}colh$  by  $auto$ 
have  $H\text{-to-}T.Col\ B\ A\ M$ 
  by  $(metis\ Bet\text{-}def\ H\text{-to-}T.Col\text{-}def\ \langle ColH\ A\ B\ M \rangle\ between\text{-}comm\ between\text{-}one)$ 
hence  $H\text{-to-}T.Col\ I\ B\ M \wedge H\text{-to-}T.Col\ A\ D\ M \vee H\text{-to-}T.Col\ I\ A\ M \wedge H\text{-to-}T.Col\ B\ D\ M \vee$ 
   $H\text{-to-}T.Col\ I\ D\ M \wedge H\text{-to-}T.Col\ B\ A\ M$ 
  using  $\langle H\text{-to-}T.Col\ I\ D\ M \rangle$  by  $auto$ 
hence  $H\text{-to-}T.Coplanar\ I\ B\ A\ D$ 
  using  $H\text{-to-}T.Coplanar\text{-}def$  by  $blast$ 
ultimately have  $?thesis$ 
  using  $H\text{-to-}T.coplanar\text{-}trans\text{-}1\ H\text{-to-}T.ncoplanar\text{-}perm\text{-}6$  by  $blast$ 
}
ultimately have  $?thesis$ 
by  $(metis\ (mono\text{-}tags,\ lifting)\ Bet\text{-}def\ line\text{-}on\text{-}plane\ Is\text{-}plane\ Midpoint\text{-}def$ 
   $\langle ColH\ A\ B\ C \implies H\text{-to-}T.Coplanar\ A\ B\ C\ D \rangle\ \langle D = M \implies H\text{-to-}T.Coplanar\ A\ B\ C\ D \rangle$ 
   $\langle IncidL\ B\ l \implies False \rangle\ \langle IncidL\ D\ l \rangle\ \langle IncidL\ M\ l \rangle\ \langle IncidP\ M\ p \rangle\ \langle IsL\ l \rangle$ 
   $\langle M\ Midpoint\ A\ B \rangle\ \langle \neg\ IncidL\ A\ l \rangle\ \langle \neg\ IncidL\ C\ l \rangle\ assms(1)\ assms(2)$ 
   $assms(3)\ assms(4)\ local.cut\text{-}def)$ 
}
ultimately have  $?thesis$ 
by  $blast$ 
}
ultimately have  $?thesis$ 
by  $blast$ 
}
ultimately have  $?thesis$ 
by  $blast$ 
}
ultimately show  $?thesis$ 
by  $blast$ 
qed

lemma  $pars\text{-}para$ :
assumes  $IncidL\ A\ l$  and
   $IncidL\ B\ l$  and
   $IncidL\ C\ m$  and
   $IncidL\ D\ m$  and
   $H\text{-to-}T.ParStrict\ A\ B\ C\ D$ 
shows  $Para\ l\ m$ 
proof –
have  $A \neq B$ 
  using  $H\text{-to-}T.par\text{-}strict\text{-}neq1\ assms(5)$  by  $auto$ 
have  $C \neq D$ 
  using  $H\text{-to-}T.par\text{-}strict\text{-}distinct\ assms(5)$  by  $presburger$ 
have  $IsL\ l$ 
  using  $Is\text{-}line\ assms(1)$  by  $blast$ 
moreover have  $IsL\ m$ 
  using  $Is\text{-}line\ assms(4)$  by  $blast$ 
moreover
{
  fix  $X$ 
  assume  $IncidL\ X\ l$  and  $IncidL\ X\ m$ 
  hence  $H\text{-to-}T.Col\ X\ A\ B$ 
    using  $assms(1)\ assms(2)\ Is\text{-}line\ line\text{-}col$  by  $blast$ 
  moreover have  $H\text{-to-}T.Col\ X\ C\ D$ 
    using  $assms(1)\ assms(2)\ Is\text{-}line\ line\text{-}col\ \langle IncidL\ X\ m \rangle\ assms(3)\ assms(4)$  by  $auto$ 
  ultimately have  $False$ 
    using  $H\text{-to-}T.par\text{-}not\text{-}col\ assms(5)$  by  $blast$ 
}
moreover
obtain  $p$  where  $IsP\ p$  and  $IncidP\ A\ p$  and  $IncidP\ B\ p$  and  $IncidP\ C\ p$  and  $IncidP\ D\ p$ 
  using  $H\text{-to-}T.pars\text{-}coplanar\ assms(5)\ coplanar\text{-}plane$  by  $blast$ 

```

```

have  $\exists p. \text{IncidLP } l \ p \wedge \text{IncidLP } m \ p$ 
proof -
  have  $\text{IncidLP } l \ p$ 
    using  $\langle A \neq B \rangle \langle \text{IncidP } A \ p \rangle \langle \text{IncidP } B \ p \rangle \langle \text{IsP } p \rangle \text{assms}(1) \text{assms}(2)$ 
    calculation(1) line-on-plane by blast
  moreover have  $\text{IncidLP } m \ p$ 
    using  $\langle C \neq D \rangle \langle \text{IncidP } C \ p \rangle \langle \text{IncidP } D \ p \rangle \langle \text{IsP } p \rangle \text{assms}(3) \text{assms}(4)$ 
    line-on-plane  $\langle \text{IsL } m \rangle$  by blast
  ultimately show ?thesis
    by blast
qed
ultimately show ?thesis
  using Para-def by blast
qed

lemma par--or-eq-para:
  assumes  $\text{IncidL } A \ l$  and
     $\text{IncidL } B \ l$  and
     $\text{IncidL } C \ m$  and
     $\text{IncidL } D \ m$  and
     $H\text{-to-}T.\text{Par } A \ B \ C \ D$ 
  shows  $\text{Para } l \ m \vee \text{EqL } l \ m$ 
proof -
  {
    assume  $H\text{-to-}T.\text{ParStrict } A \ B \ C \ D$ 
    hence ?thesis
      using assms(1) assms(2) assms(3) assms(4) pars--para by blast
  }
  moreover
  {
    assume  $A \neq B$  and
       $C \neq D$  and
       $H\text{-to-}T.\text{Col } A \ C \ D$  and
       $H\text{-to-}T.\text{Col } B \ C \ D$ 
    hence ?thesis
      by (meson IncidL-not-IncidL--not-colH Is-line assms(1) assms(2) assms(3)
        assms(4) colH-permut-231 col-colh-1 line-uniqueness)
  }
  ultimately show ?thesis
    using  $H\text{-to-}T.\text{Par-def}$  assms(5) by presburger
qed

lemma tarski-upper-dim:
  assumes
    plane-intersection-assms:  $\forall A \ p \ q. \text{IsP } p \wedge \text{IsP } q \wedge \text{IncidP } A \ p \wedge \text{IncidP } A \ q$ 
       $\longrightarrow (\exists B. \text{IncidP } B \ p \wedge \text{IncidP } B \ q \wedge A \neq B)$  and
    lower-dim-3-assms:  $\neg (\exists p. \text{IsP } p \wedge \text{IncidP } HS1 \ p \wedge \text{IncidP } HS2 \ p \wedge$ 
       $\text{IncidP } HS3 \ p \wedge \text{IncidP } HS4 \ p)$  and
     $P \neq Q$  and
     $Q \neq R$  and
     $P \neq R$  and
     $\text{Cong } A \ P \ A \ Q$  and
     $\text{Cong } B \ P \ B \ Q$  and
     $\text{Cong } C \ P \ C \ Q$  and
     $\text{Cong } A \ P \ A \ R$  and
     $\text{Cong } B \ P \ B \ R$  and
     $\text{Cong } C \ P \ C \ R$ 
  shows  $\text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B$ 
proof -
  have  $H\text{-to-}T.\text{upper-dim-3-axiom} \longrightarrow ?thesis$ 
    using  $H\text{-to-}T.\text{upper-dim-3-axiom-def}$  assms(10) assms(11) assms(3) assms(4)
    assms(5) assms(6) assms(7) assms(8) assms(9) by blast
  moreover
  have  $H\text{-to-}T.\text{plane-intersection-axiom} \longrightarrow H\text{-to-}T.\text{upper-dim-3-axiom}$ 
    using  $H\text{-to-}T.\text{upper-dim-3-equivalent-axioms}$  by force
  moreover

```



```

{
  fix A B C D E F P
  assume H-to-T.Coplanar A B C P and
    H-to-T.Coplanar D E F P
  obtain p where IsP p and IncidP A p and IncidP B p and IncidP C p and IncidP P p
    using ⟨H-to-T.Coplanar A B C P⟩ coplanar-plane by blast
  obtain q where IsP q and IncidP D q and IncidP E q and IncidP F q and IncidP P q
    using ⟨H-to-T.Coplanar D E F P⟩ coplanar-plane by blast
  obtain Q where IncidP Q p and IncidP Q q and P ≠ Q
    using plane-intersection-assms Is-plane ⟨IncidP P p⟩ ⟨IncidP P q⟩ by blast
  have H-to-T.Coplanar A B C Q
    using ⟨IncidP A p⟩ ⟨IncidP B p⟩ ⟨IncidP C p⟩ ⟨IncidP Q p⟩ plane-coplanar by force
  moreover have H-to-T.Coplanar D E F Q
    using ⟨IncidP D q⟩ ⟨IncidP E q⟩ ⟨IncidP F q⟩ ⟨IncidP Q q⟩ plane-coplanar by force
  ultimately have ∃ Q. H-to-T.Coplanar A B C Q ∧ H-to-T.Coplanar D E F Q ∧ P ≠ Q
    using ⟨P ≠ Q⟩ by blast
}
hence H-to-T.plane-intersection-axiom
  using H-to-T.plane-intersection-axiom-def by force
ultimately show ?thesis
  by simp
qed

```

```

lemma Col--ColH:
  assumes H-to-T.Col A B C
  shows ColH A B C
  using assms col-colh by blast

```

```

lemma ColH--Col:
  assumes ColH A B C
  shows H-to-T.Col A B C
  using assms ColH-bets col-colh by blast

```

```

lemma playfair-s-postulateH:
  assumes euclid-uniqueness: ∀ l P m1 m2. IsL l ∧ IsL m1 ∧ IsL m2 ∧
    ¬ IncidL P l ∧ Para l m1 ∧ IncidL P m1 ∧ Para l m2 ∧ IncidL P m2 ⟶
      EqL m1 m2
  shows H-to-T.playfair-s-postulate

```

```

proof -
{
  fix A1 A2 B1 B2 C1 C2 P
  assume H-to-T.Par A1 A2 B1 B2 and
    H-to-T.Col P B1 B2 and
    H-to-T.Par A1 A2 C1 C2 and
    H-to-T.Col P C1 C2
  have A1 ≠ A2
    using H-to-T.par-distinct ⟨H-to-T.Par A1 A2 B1 B2⟩ by blast
  then obtain l where IsL l and IncidL A1 l and IncidL A2 l
    using line-existence by fastforce
  have B1 ≠ B2
    using H-to-T.par-distinct ⟨H-to-T.Par A1 A2 B1 B2⟩ by blast
  then obtain m1 where IsL m1 and IncidL B1 m1 and IncidL B2 m1
    using line-existence by fastforce
  have C1 ≠ C2
    using H-to-T.par-distinct ⟨H-to-T.Par A1 A2 C1 C2⟩ by blast
  then obtain m2 where IsL m2 and IncidL C1 m2 and IncidL C2 m2
    using line-existence by fastforce
{
  assume IncidL P l
  have ColH A1 A2 P
    using ColH-def Is-line ⟨IncidL A1 l⟩ ⟨IncidL A2 l⟩ ⟨IncidL P l⟩ by blast
  hence H-to-T.Col A1 A2 P
    by (simp add: col-colh)
  have H-to-T.Col A1 B1 B2
    by (metis H-to-T.col3 H-to-T.not-col-distincts H-to-T.not-strict-par1
      H-to-T.not-strict-par2 ⟨A1 ≠ A2⟩ ⟨H-to-T.Col A1 A2 P⟩ ⟨H-to-T.Col P B1 B2⟩

```

```

    ⟨H-to-T.Par A1 A2 B1 B2⟩)
have H-to-T.Col A2 B1 B2
  using H-to-T.Par-def H-to-T.not-col-permutation-2 H-to-T.par-strict-not-col-3
  ⟨H-to-T.Col A1 B1 B2⟩ ⟨H-to-T.Par A1 A2 B1 B2⟩ by blast
have H-to-T.ParStrict A1 A2 C1 C2 ⟶ H-to-T.Col C1 B1 B2 ∧ H-to-T.Col C2 B1 B2
  using H-to-T.Col-cases H-to-T.par-not-col ⟨H-to-T.Col A1 A2 P⟩
  ⟨H-to-T.Col P C1 C2⟩ by blast
moreover have A1 ≠ A2 ∧ C1 ≠ C2 ∧ H-to-T.Col A1 C1 C2 ∧ H-to-T.Col A2 C1 C2
  ⟶ H-to-T.Col C1 B1 B2 ∧ H-to-T.Col C2 B1 B2
proof -
  have A1 ≠ A2 ∧ C1 ≠ C2 ∧ H-to-T.Col A1 C1 C2 ∧ H-to-T.Col A2 C1 C2
    ⟶ H-to-T.Col C1 B1 B2
  by (meson H-to-T.col-permutation-1 H-to-T.l6-21 ⟨H-to-T.Col A1 B1 B2⟩
    ⟨H-to-T.Col A2 B1 B2⟩)
moreover have A1 ≠ A2 ∧ C1 ≠ C2 ∧ H-to-T.Col A1 C1 C2 ∧ H-to-T.Col A2 C1 C2
  ⟶ H-to-T.Col C2 B1 B2
  using H-to-T.col-permutation-1 H-to-T.l6-21
  by (metis H-to-T.col-permutation-1 H-to-T.colx ⟨H-to-T.Col A1 B1 B2⟩
    ⟨H-to-T.Col A2 B1 B2⟩ calculation)
ultimately show ?thesis
  by blast
qed
ultimately have H-to-T.Col C1 B1 B2 ∧ H-to-T.Col C2 B1 B2
  using H-to-T.Par-def ⟨H-to-T.Par A1 A2 C1 C2⟩ by force
}
moreover
{
  assume ¬ IncidL P l
  have ¬ ColH A1 A2 P
    using IncidL-not-IncidL--not-colH ⟨A1 ≠ A2⟩ ⟨IncidL A1 l⟩ ⟨IncidL A2 l⟩
    ⟨¬ IncidL P l⟩ by blast
  hence ¬ H-to-T.Col A1 A2 P
    using col-colh by blast
  have Para l m1
  proof -
    have H-to-T.ParStrict A1 A2 B1 B2
      using H-to-T.col-permutation-2 H-to-T.par-not-col-strict
      ⟨H-to-T.Col P B1 B2⟩ ⟨H-to-T.Par A1 A2 B1 B2⟩ ⟨¬ H-to-T.Col A1 A2 P⟩ by blast
    moreover have H-to-T.ParStrict A1 A2 B1 B2 ⟶ Para l m1
      using ⟨IncidL A1 l⟩ ⟨IncidL A2 l⟩ ⟨IncidL B1 m1⟩ ⟨IncidL B2 m1⟩ pars--para by blast
    ultimately show ?thesis
      by blast
  qed
  have Para l m2
  proof -
    have H-to-T.ParStrict A1 A2 C1 C2
      using H-to-T.not-col-permutation-2 H-to-T.par-not-col-strict
      ⟨H-to-T.Col P C1 C2⟩ ⟨H-to-T.Par A1 A2 C1 C2⟩ ⟨¬ H-to-T.Col A1 A2 P⟩ by blast
    moreover have H-to-T.ParStrict A1 A2 C1 C2 ⟶ Para l m2
      using ⟨IncidL A1 l⟩ ⟨IncidL A2 l⟩ ⟨IncidL C1 m2⟩ ⟨IncidL C2 m2⟩ pars--para by auto
    ultimately show ?thesis
      by blast
  qed
  have IncidL P m1
    using H-to-T.not-col-permutation-2 IncidL-not-IncidL--not-colH ⟨B1 ≠ B2⟩
    ⟨H-to-T.Col P B1 B2⟩ ⟨IncidL B1 m1⟩ ⟨IncidL B2 m1⟩ col-colh by blast
  have IncidL P m2
    using H-to-T.not-col-permutation-2 IncidL-not-IncidL--not-colH ⟨C1 ≠ C2⟩
    ⟨H-to-T.Col P C1 C2⟩ ⟨IncidL C1 m2⟩ ⟨IncidL C2 m2⟩ col-colh by blast
  have EqL m1 m2
    using ⟨IncidL P m1⟩ ⟨IncidL P m2⟩ ⟨IsL l⟩ ⟨IsL m1⟩ ⟨IsL m2⟩ ⟨Para l m1⟩
    ⟨Para l m2⟩ ⟨¬ IncidL P l⟩ assms by blast
  hence H-to-T.Col C1 B1 B2 ∧ H-to-T.Col C2 B1 B2
    by (meson IncidL-morphism ⟨IncidL B1 m1⟩ ⟨IncidL B2 m1⟩ ⟨IncidL C1 m2⟩
      ⟨IncidL C2 m2⟩ ⟨IsL m1⟩ ⟨IsL m2⟩ line-col)
}

```

```

    ultimately have  $H\text{-to-}T.\text{Col } C1 \ B1 \ B2 \wedge H\text{-to-}T.\text{Col } C2 \ B1 \ B2$ 
      by blast
  }
  thus ?thesis
    using  $H\text{-to-}T.\text{playfair-s-postulate-def}$  by blast
qed

lemma tarski-s-euclid-aux:
  assumes euclid-uniqueness:  $\forall l \ P \ m1 \ m2. \text{IsL } l \wedge \text{IsL } m1 \wedge \text{IsL } m2 \wedge$ 
     $\neg \text{IncidL } P \ l \wedge \text{Para } l \ m1 \wedge \text{IncidL } P \ m1 \wedge \text{Para } l \ m2 \wedge \text{IncidL } P \ m2 \longrightarrow$ 
     $\text{EqL } m1 \ m2$ 
  shows  $H\text{-to-}T.\text{tarski-s-parallel-postulate}$ 
proof -
  have  $H\text{-to-}T.\text{playfair-s-postulate} \longrightarrow H\text{-to-}T.\text{tarski-s-parallel-postulate}$ 
    using  $H\text{-to-}T.\text{InterAx5}$   $H\text{-to-}T.\text{Postulate01-def}$   $H\text{-to-}T.\text{Postulate02-def}$  by blast
  moreover
  have  $H\text{-to-}T.\text{playfair-s-postulate}$ 
    using playfair-s-postulateH assms(1) by blast
  ultimately show ?thesis
    by blast
qed

lemma tarski-s-euclid:
  assumes euclid-uniqueness:  $\forall l \ P \ m1 \ m2. \text{IsL } l \wedge \text{IsL } m1 \wedge \text{IsL } m2 \wedge$ 
     $\neg \text{IncidL } P \ l \wedge \text{Para } l \ m1 \wedge \text{IncidL } P \ m1 \wedge \text{Para } l \ m2 \wedge \text{IncidL } P \ m2 \longrightarrow$ 
     $\text{EqL } m1 \ m2$ 
  shows
     $\forall A \ B \ C \ D \ T. \text{Bet } A \ D \ T \wedge \text{Bet } B \ D \ C \wedge A \neq D \longrightarrow (\exists X \ Y. \text{Bet } A \ B \ X \wedge \text{Bet } A \ C \ Y \wedge \text{Bet } X \ T \ Y)$ 
  using assms(1) tarski-s-euclid-aux  $H\text{-to-}T.\text{tarski-s-parallel-postulate-def}$  by blast

end
end

theory Tarski-Neutral-2D-Model-Hilbert-Neutral-2D

imports
  Tarski-Neutral-2D
  Hilbert-Neutral-2D

begin

```

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```

context Hilbert-neutral-2D

```

```

begin

```

```

lemma plane-separation-2D:
  assumes  $\neg \text{ColH } A \ X \ Y$  and
     $\neg \text{ColH } B \ X \ Y$ 
  shows  $\text{cut}' A \ B \ X \ Y \vee \text{same-side}' A \ B \ X \ Y$ 
proof -
  have  $X \neq Y$ 
    using assms(2) colH-trivial122 by blast
  obtain  $l$  where  $\text{IsL } l$  and  $\text{IncidL } X \ l$  and  $\text{IncidL } Y \ l$ 
    using  $\langle X \neq Y \rangle$  line-existence by blast
  obtain  $C$  where  $\text{cut } l \ A \ C$ 
    using ColH-def  $\langle \text{IncidL } X \ l \rangle \langle \text{IncidL } Y \ l \rangle \langle \text{IsL } l \rangle$  assms(1) cut-exists by blast
  {
    assume  $A = B$ 
    hence  $\text{same-side}' A \ B \ X \ Y$ 
      using assms(1) colH-permut-312 same-side-prime-refl by blast
    hence ?thesis
      by blast
  }

```

```

}
moreover
{
  assume  $A \neq B$ 
  have  $A \neq C$ 
  using  $\langle \text{local.cut } l \ A \ C \rangle$  cut-distinct by blast
  {
    assume  $B = C$ 
    obtain  $m$  where  $IsL \ m$  and  $IncidL \ X \ m$  and  $IncidL \ Y \ m$ 
    using  $Is\text{-}line \ \langle IncidL \ X \ l \ \langle IncidL \ Y \ l \ \rangle \rangle$  by blast
    obtain  $I$  where  $IncidL \ I \ l$  and  $BetH \ A \ I \ C$ 
    using  $\langle \text{local.cut } l \ A \ C \rangle$  local.cut-def by blast
    {
      fix  $m$ 
      assume  $IsL \ m$  and  $IncidL \ X \ m$  and  $IncidL \ Y \ m$ 
      hence cut  $m \ A \ B$ 
      by (metis ColH-def  $\langle B = C \rangle \langle IncidL \ X \ l \ \langle IncidL \ Y \ l \ \langle X \neq Y \rangle$ 
         $\langle \bigwedge thesis. (\bigwedge I. \llbracket IncidL \ I \ l; \ BetH \ A \ I \ C \rrbracket \implies thesis) \implies thesis \rangle$  assms(1)
        assms(2) inter-incid-uniquenessH local.cut-def)
    }
    hence cut'  $A \ C \ X \ Y$ 
    using  $\langle X \neq Y \rangle \langle B = C \rangle$  cut'-def by auto
    hence ?thesis
    using  $\langle B = C \rangle$  by blast
  }
}
moreover
{
  assume  $B \neq C$ 
  {
    assume ColH  $A \ C \ B$ 
    obtain  $I$  where  $IncidL \ I \ l$  and  $BetH \ A \ I \ C$ 
    using  $\langle \text{local.cut } l \ A \ C \rangle$  local.cut-def by blast
    hence ColH  $A \ I \ C$ 
    using betH-colH by blast
    hence ColH  $A \ I \ B$ 
    by (meson colH-permut-132 colH-trans colH-trivial121  $\langle A \neq C \rangle \langle ColH \ A \ C \ B \rangle$ )
    hence  $BetH \ A \ I \ B \vee BetH \ I \ B \ A \vee BetH \ I \ A \ B$ 
    using ColH-def  $\langle A \neq B \rangle \langle BetH \ A \ I \ C \rangle \langle IncidL \ I \ l \ \langle IncidL \ X \ l \ \langle IncidL \ Y \ l \ \langle IsL \ l \ \rangle \rangle \rangle$ 
      assms(2) betH-colH between-one by blast
    have  $\neg IncidL \ A \ l$ 
    using  $\langle \text{local.cut } l \ A \ C \rangle$  local.cut-def by auto
    {
      assume  $BetH \ A \ I \ B$ 
      {
        fix  $m$ 
        assume  $IsL \ m$  and  $IncidL \ X \ m$  and  $IncidL \ Y \ m$ 
        have  $\neg IncidL \ B \ m$ 
        using ColH-def  $\langle IncidL \ X \ m \rangle \langle IncidL \ Y \ m \rangle \langle IsL \ m \rangle$  assms(2) by blast
        moreover have  $\exists I0. IncidL \ I0 \ m \wedge BetH \ A \ I0 \ B$ 
        using  $\langle BetH \ A \ I \ B \rangle \langle IncidL \ I \ l \ \langle IncidL \ X \ l \ \langle IncidL \ Y \ l \ \langle IncidL \ X \ m \rangle \langle IncidL \ Y \ m \rangle \langle X \neq Y \rangle$ 
          inter-incid-uniquenessH by blast
        ultimately have cut  $m \ A \ B$ 
        using  $\langle IncidL \ X \ l \ \langle IncidL \ X \ m \rangle \langle IncidL \ Y \ l \ \langle IncidL \ Y \ m \rangle \langle IsL \ m \rangle \langle X \neq Y \rangle \langle \neg IncidL \ A \ l \rangle$ 
          inter-incid-uniquenessH local.cut-def by blast
      }
    }
    hence cut'  $A \ B \ X \ Y$ 
    by (simp add:  $\langle X \neq Y \rangle$  cut'-def)
  }
}
moreover have  $BetH \ I \ B \ A \longrightarrow same\text{-}side' \ A \ B \ X \ Y$ 
  using  $\langle IncidL \ I \ l \ \langle IncidL \ X \ l \ \langle IncidL \ Y \ l \ \langle X \neq Y \rangle \langle \neg IncidL \ A \ l \rangle$ 
    outH-def out-same-side' by auto
moreover have  $BetH \ I \ A \ B \longrightarrow same\text{-}side' \ A \ B \ X \ Y$ 
  using  $\langle IncidL \ I \ l \ \langle IncidL \ X \ l \ \langle IncidL \ Y \ l \ \langle X \neq Y \rangle \langle \neg IncidL \ A \ l \rangle$ 
    outH-def out-same-side' by auto
ultimately have ?thesis
  using  $\langle BetH \ A \ I \ B \vee BetH \ I \ B \ A \vee BetH \ I \ A \ B \rangle$  by fastforce

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}
moreover
{
  assume  $\neg \text{ColH } A \ C \ B$ 
  have  $\neg \text{IncidL } B \ l$ 
  using  $\text{ColH-def Is-line } \langle \text{IncidL } X \ l \rangle \langle \text{IncidL } Y \ l \rangle \text{ asms}(2)$  by blast
  moreover {
    assume  $\text{cut } l \ A \ B$ 
    {
      fix  $m$ 
      assume  $\text{IsL } m$  and  $\text{IncidL } X \ m$  and  $\text{IncidL } Y \ m$ 
      hence  $\text{EqL } m \ l$ 
      using  $\langle \text{IncidL } X \ l \rangle \langle \text{IncidL } Y \ l \rangle \langle \text{IsL } l \rangle \langle \text{IsL } m \rangle \langle X \neq Y \rangle$ 
      line-uniqueness by blast
      hence  $\text{cut } m \ A \ B$ 
      by  $(\text{meson } \langle \text{IsL } m \rangle \langle \text{local.cut } l \ A \ B \rangle \text{ local.cut-def morph})$ 
    }
    hence  $\text{cut}' \ A \ B \ X \ Y$ 
    using  $\langle X \neq Y \rangle \text{ cut'-def}$  by auto
  }
}
{
  assume  $\text{cut } l \ C \ B$ 
  {
    fix  $m$ 
    assume  $\text{IsL } m$  and  $\text{IncidL } X \ m$  and  $\text{IncidL } Y \ m$ 
    hence  $\text{EqL } m \ l$ 
    using  $\langle \text{IncidL } X \ l \rangle \langle \text{IncidL } Y \ l \rangle \langle \text{IsL } l \rangle \langle X \neq Y \rangle$  line-uniqueness by blast
    have  $\text{cut } m \ A \ C$ 
    by  $(\text{meson } \langle \text{IncidL } X \ l \rangle \langle \text{IncidL } X \ m \rangle \langle \text{IncidL } Y \ l \rangle \langle \text{IncidL } Y \ m \rangle$ 
     $\langle \text{IsL } m \rangle \langle X \neq Y \rangle \langle \text{local.cut } l \ A \ C \rangle \text{ inter-incid-uniquenessH local.cut-def})$ 
    moreover have  $\text{cut } m \ B \ C$ 
    by  $(\text{meson } \langle \text{EqL } m \ l \rangle \langle \text{IsL } m \rangle \langle \text{local.cut } l \ C \ B \rangle \text{ cut-comm local.cut-def morph})$ 
    ultimately have same-side  $A \ B \ m$ 
    using  $\langle \text{IsL } m \rangle$  same-side-def by blast
  }
  hence same-side'  $A \ B \ X \ Y$ 
  by  $(\text{simp add: } \langle X \neq Y \rangle \text{ same-side'-def})$ 
  have ?thesis
  using  $\langle \text{IsL } l \rangle \langle \neg \text{ColH } A \ C \ B \rangle \langle \text{local.cut } l \ A \ C \rangle$  pasch-2D
   $\langle \text{same-side}' \ A \ B \ X \ Y \rangle$  by blast
}
ultimately have ?thesis
using  $\langle \text{IsL } l \rangle \langle \neg \text{ColH } A \ C \ B \rangle \langle \text{local.cut } l \ A \ B \implies \text{cut}' \ A \ B \ X \ Y \rangle$ 
 $\langle \text{local.cut } l \ A \ C \rangle$  pasch-2D by blast
}
ultimately have ?thesis
by blast
}
ultimately have ?thesis
by blast
}
ultimately show ?thesis
by blast
qed

```

```

lemma col-upper-dim:
assumes  $\text{ColH } A \ P \ Q$  and
   $P \neq Q$  and
   $A \neq B$  and
   $A \neq C$  and
   $B \neq C$  and
   $A \neq P$  and
   $A \neq Q$  and
   $B \neq P$  and
   $B \neq Q$  and
   $C \neq P$  and

```

```

C ≠ Q and
CongH A P A Q and
CongH B P B Q and
CongH C P C Q
shows Bet A B C ∨ Bet B C A ∨ Bet C A B
proof -
{
  assume ColH B P Q
  hence BetH P B Q
    using assms(13) assms(2) assms(8) assms(9) congH-colH-betH by auto
  {
    assume BetH P A B
    hence BetH Q B A
      using ⟨BetH P B Q⟩ betH-trans0 between-comm by blast
    hence False
      using ⟨BetH P A B⟩ assms(12) assms(13) assms(7) assms(9)
        betH-congH2--False congH-permlr by blast
  }
  moreover
  {
    assume BetH P B A
    hence BetH Q A B
      by (metis assms(1) assms(12) assms(2) assms(6) assms(7)
        betH-trans0 between-comm congH-colH-betH)
    hence False
      using ⟨BetH P B A⟩ assms(12) assms(13) assms(7) assms(9)
        betH-congH2--False congH-permlr by blast
  }
  ultimately have False
    by (metis ⟨BetH P B Q⟩ assms(1) assms(12) assms(2) assms(3)
      assms(6) betH2-out congH-colH-betH)
}
hence ¬ ColH B P Q
  by blast
{
  assume ColH C P Q
  hence BetH P C Q
    using assms(10) assms(11) assms(14) assms(2) congH-colH-betH by force
  {
    assume BetH P A C
    hence BetH Q C A
      using ⟨BetH P C Q⟩ betH-trans0 between-comm by blast
    hence False
      using ⟨BetH P A C⟩ assms(11) assms(12) assms(14) assms(7)
        betH-congH2--False congH-permlr by blast
  }
  moreover
  {
    assume BetH P C A
    hence BetH Q A C
      by (metis assms(1) assms(12) assms(2) assms(6) assms(7)
        betH-trans0 between-comm congH-colH-betH)
    hence False
      using ⟨BetH P C A⟩ assms(10) assms(11) assms(12) assms(14)
        assms(6) assms(7) betH-congH2--False congH-perms by blast
  }
  ultimately have False
    by (metis congH-colH-betH ⟨BetH P C Q⟩ assms(1) assms(12) assms(2)
      assms(4) assms(6) betH2-out)
}
hence ¬ ColH C P Q
  by blast
{
  assume cut' B C P Q
  obtain l where IsL l and IncidL P l and IncidL Q l
    using assms(2) line-existence by blast

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have cut l B C
  using ⟨IncidL P l⟩ ⟨IncidL Q l⟩ ⟨IsL l⟩ ⟨cut' B C P Q⟩ cut'-def by auto
obtain I where IncidL I l and BetH B I C
  using ⟨local.cut l B C⟩ local.cut-def by blast
{
  assume P = I
  hence ColH B Q C
    by (metis ⟨BetH B I C⟩ assms(11) assms(13) assms(14) assms(5)
        congH-permlr congH-refl cong-preserves-col)
  hence False
    by (metis betH2-out ⟨BetH B I C⟩ ⟨P = I⟩ ⟨¬ ColH B P Q⟩ assms(13)
        assms(14) assms(2) assms(9) betH-colH bet-cong3-bet colH-permut-132
        congH-permlr congH-refl)
}
hence P ≠ I by blast
{
  assume Q = I
  hence ColH B P C
    by (metis ⟨BetH B I C⟩ assms(10) assms(11) assms(13) assms(14)
        assms(5) assms(8) congH-perms congH-refl congH-sym cong-preserves-col)
  hence False
    by (metis Hilbert-neutral-dimensionless-pre.Bet-def ⟨BetH B I C⟩
        ⟨IncidL P l⟩ ⟨Q = I⟩ ⟨¬ ColH B P Q⟩ ⟨local.cut l B C⟩ betH2-out bet-colH
        between-one colH-permut-132 local.cut-def outH-def out-same-side same-side-not-cut)
}
hence Q ≠ I
  by blast
have ColH P Q I
  using ColH-def ⟨IncidL I l⟩ ⟨IncidL P l⟩ ⟨IncidL Q l⟩ ⟨IsL l⟩ by blast
have CongaH I B P I B Q
proof -
  have CongaH C B P C B Q
  proof -
    have ¬ ColH B C P
      by (metis (full-types) ⟨BetH B I C⟩ ⟨ColH B P Q ⟹ False⟩
          ⟨ColH P Q I⟩ ⟨P = I ⟹ False⟩ colH-permut-312 colH-trivial121
          inter-uniquenessH outH-def outH-expand)
    moreover have ¬ ColH B C Q
      by (meson assms(10) assms(11) assms(13) assms(14) assms(5)
          assms(8) assms(9) calculation congH-refl congH-sym cong-preserves-col-stronger)
    moreover have CongH B C B C
      by (simp add: assms(5) congH-refl)
    moreover have CongH B P B Q
      by (simp add: assms(13))
    moreover have CongH C P C Q
      by (simp add: assms(14))
    ultimately show ?thesis
      using th18-aux by blast
  qed
  moreover have outH B C I
    by (simp add: ⟨BetH B I C⟩ outH-def)
  moreover have outH B P P
    by (simp add: assms(8) outH-def)
  moreover have outH B Q Q
    by (simp add: assms(9) outH-def)
  ultimately show ?thesis
    using conga-out-conga by blast
qed
have CongH I P I Q
proof -
  have ¬ ColH B I P
    by (metis ⟨ColH P Q I⟩ ⟨P = I ⟹ False⟩ ⟨¬ ColH B P Q⟩
        colH-permut-312 colH-trivial122 inter-uniquenessH)
  moreover have ¬ ColH B I Q
    using IncidL-not-IncidL--not-colH ⟨IncidL I l⟩ ⟨IncidL Q l⟩
        ⟨Q = I ⟹ False⟩ ⟨local.cut l B C⟩ colH-permut-321 local.cut-def by blast

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moreover have CongH B I B I
  using ⟨BetH B I C⟩ betH-distincts congH-refl by auto
moreover have CongH B P B Q
  by (simp add: assms(13))
moreover have CongaH I B P I B Q
  by (simp add: ⟨CongaH I B P I B Q⟩)
ultimately show ?thesis
  using th12 by blast
qed
have A = I  $\longrightarrow$  ?thesis
  using Bet-def ⟨BetH B I C⟩ between-comm by blast
moreover
{
  assume A  $\neq$  I
  have ColH I P Q
    by (simp add: ⟨ColH P Q I⟩ colH-permut-312)
  hence BetH P I Q
    by (simp add: ⟨CongH I P I Q⟩ ⟨P  $\neq$  I⟩ ⟨Q  $\neq$  I⟩ assms(2) congH-colH-betH)
  {
    assume BetH P A I
    hence BetH Q I A
      using ⟨BetH P I Q⟩ betH-trans0 between-comm by blast
    hence False
      by (metis ⟨BetH P A I⟩ ⟨CongH I P I Q⟩ ⟨Q  $\neq$  I⟩ assms(12) assms(7)
        betH-congH2--False congH-permlr)
  }
  moreover {
    assume BetH P I A
    hence BetH Q A I
      by (metis assms(1) assms(12) assms(2) assms(6) assms(7) betH-trans0
        between-comm congH-colH-betH)
    hence False
      by (metis ⟨BetH P I A⟩ ⟨CongH I P I Q⟩ ⟨Q = I  $\implies$  False⟩ assms(12)
        assms(7) betH-congH2--False congH-permlr)
  }
  ultimately have ?thesis
    by (metis ⟨A = I  $\longrightarrow$  Bet A B C  $\vee$  Bet B C A  $\vee$  Bet C A B⟩ ⟨BetH P I Q⟩
      assms(1) assms(12) assms(2) assms(6) betH2-out congH-colH-betH)
}
ultimately have ?thesis
  by blast
}
moreover
{
  assume same-side' B C P Q
  {
    assume ColH P B C
    have ColH C P B
      by (simp add: ⟨ColH P B C⟩ colH-permut-312)
    have ColH B C Q
      by (metis colH-permut-312 ⟨ColH C P B⟩ assms(10) assms(13) assms(14)
        assms(5) assms(8) congH-perm congH-perms cong-preserves-col-stronger)
    have BetH P B Q
      using ColH-def ⟨ColH B C Q⟩ ⟨ColH C P Q  $\implies$  False⟩ ⟨ColH P B C⟩
        assms(5) colH-IncidL--IncidL by blast
    have BetH P C Q
      using ⟨BetH P B Q⟩  $\neg$  ColH B P Q between-col colH-permut-213 by blast
    {
      assume BetH P B C
      hence BetH Q C B
        using ⟨BetH P C Q⟩ betH-trans0 between-comm by blast
      hence False
        using ⟨BetH P B Q⟩  $\neg$  ColH B P Q betH-expand colH-permut-213 by blast
    }
  }
  moreover
  {

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    assume BetH P C B
    hence BetH B Q C
    using  $\langle \text{BetH } P \ B \ Q \rangle \langle \neg \text{ColH } B \ P \ Q \rangle$  betH-expand colH-permut-213 by blast
    hence False
    using  $\langle \text{BetH } P \ C \ Q \rangle \langle \text{ColH } C \ P \ Q \implies \text{False} \rangle$  betH-expand colH-permut-213 by blast
  }
  ultimately have False
  using  $\langle \text{BetH } P \ B \ Q \rangle \langle \text{BetH } P \ C \ Q \rangle$  assms(5) betH2-out by blast
}
hence  $\neg \text{ColH } P \ B \ C$ 
  by blast
{
  assume ColH Q B C
  hence ColH B C P
  by (metis (full-types)  $\langle \neg \text{ColH } P \ B \ C \rangle$  assms(10) assms(11) assms(13)
    assms(14) assms(5) assms(8) assms(9) between-one colH-permut-213 congH-perms
    congH-refl cong-preserves-col)
  hence False
  using  $\langle \neg \text{ColH } P \ B \ C \rangle$  colH-permut-312 by blast
}
hence  $\neg \text{ColH } Q \ B \ C$ 
  by blast
{
  assume cut' P Q B C
  obtain lBC where IsL lBC and IncidL B lBC and IncidL C lBC
  using assms(5) line-existence by blast
  {
    assume  $\neg \text{IncidL } P \ lBC$  and
       $\neg \text{IncidL } Q \ lBC$  and
       $\exists I. \text{IncidL } I \ lBC \wedge \text{BetH } P \ I \ Q$ 
    then obtain I where IncidL I lBC and BetH P I Q
    by blast
    have ColH B I C
    using ColH-def  $\langle \text{IncidL } B \ lBC \rangle \langle \text{IncidL } C \ lBC \rangle \langle \text{IncidL } I \ lBC \rangle \langle \text{IsL } lBC \rangle$  by auto
    {
      assume BetH B I C
      obtain lPQ where IncidL P lPQ and IncidL Q lPQ and  $\neg \text{cut } lPQ \ B \ C$ 
      by (metis  $\langle \text{same-side}' \ B \ C \ P \ Q \rangle$  line-existence same-side'-def same-side-not-cut)
      have cut lPQ B C
      proof –
      have  $\neg \text{IncidL } C \ lPQ$ 
      using Hilbert-neutral-dimensionless-pre.ColH-def Is-line
         $\langle \text{ColH } C \ P \ Q \implies \text{False} \rangle \langle \text{IncidL } P \ lPQ \rangle \langle \text{IncidL } Q \ lPQ \rangle$  by fastforce
      moreover have IncidL I lPQ
      using  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{IncidL } P \ lPQ \rangle \langle \text{IncidL } Q \ lPQ \rangle$  assms(2) betH-line
        inter-incid-uniquenessH by blast
      ultimately show ?thesis
      using  $\langle \text{BetH } B \ I \ C \rangle$  Is-line  $\langle \text{ColH } B \ I \ C \rangle$  betH-distincts
        colH-IncidL--IncidL local.cut-def by blast
    qed
    hence ?thesis
    using  $\langle \neg \text{local.cut } lPQ \ B \ C \rangle$  by blast
  }
}
moreover {
  assume BetH I C B
  have CongaH P C I Q C I
  proof –
  have  $\neg \text{ColH } B \ C \ P$ 
  using  $\langle \neg \text{ColH } P \ B \ C \rangle$  colH-permut-312 by blast
  moreover have  $\neg \text{ColH } B \ C \ Q$ 
  using  $\langle \neg \text{ColH } Q \ B \ C \rangle$  colH-permut-312 by blast
  moreover have CongaH B C P B C Q
  using  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{ColH } B \ I \ C \rangle$  assms(13) assms(14) calculation(1)
    calculation(2) th17 by blast
  moreover have BetH B C I
  by (simp add:  $\langle \text{BetH } I \ C \ B \rangle$  between-comm)
}

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ultimately show ?thesis
  using th14 by blast
qed
have CongH I P I Q
proof -
  have  $\neg$  ColH C P I
  by (metis IncidL-not-IncidL--not-colH  $\langle$ BetH P I Q $\rangle$   $\langle$ IncidL C lBC $\rangle$ 
     $\langle$ IncidL I lBC $\rangle$   $\langle$  $\neg$  ColH C P Q $\rangle$   $\langle$  $\neg$  IncidL P lBC $\rangle$  betH-colH colH-permut-132 colH-permut-213)
  moreover have  $\neg$  ColH C Q I
  using  $\langle$ BetH I C B $\rangle$   $\langle$ IncidL C lBC $\rangle$   $\langle$ IncidL I lBC $\rangle$   $\langle$  $\neg$  IncidL Q lBC $\rangle$ 
    betH-distincts colH-IncidL--IncidL colH-permut-312 by blast
  moreover have CongH C P C Q
  by (simp add: assms(14))
  moreover have CongH C I C I
  by (metis  $\langle$ BetH I C B $\rangle$  betH-distincts congH-refl)
  ultimately show ?thesis
  using  $\langle$ CongH P C I Q C I $\rangle$ 
  using congH-permlr ncolH-expand th12 by presburger
qed
have A = I  $\longrightarrow$  ?thesis
  using Bet-def  $\langle$ BetH I C B $\rangle$  bet-comm by blast
moreover {
  assume A  $\neq$  I
  have ColH I P Q
  by (simp add:  $\langle$ BetH P I Q $\rangle$  betH-colH colH-permut-213)
  {
    assume BetH P A I
    hence BetH Q I A
    using  $\langle$ BetH P I Q $\rangle$  betH-trans0 between-comm by blast
    hence False
    by (metis betH-expand  $\langle$ BetH P A I $\rangle$   $\langle$ CongH I P I Q $\rangle$  assms(12)
      betH-congH2--False congH-perms)
  }
  moreover {
    assume BetH P I A
    hence BetH Q A I
    by (metis assms(1) assms(12) assms(2) assms(6) assms(7) betH-trans0
      between-comm congH-colH-betH)
    hence False
    by (metis  $\langle$ BetH P I A $\rangle$   $\langle$ CongH I P I Q $\rangle$  assms(12) betH-congH2--False
      betH-distincts congH-permlr)
  }
  ultimately have ?thesis
  by (metis  $\langle$ A = I  $\longrightarrow$  Bet A B C  $\vee$  Bet B C A  $\vee$  Bet C A B $\rangle$   $\langle$ BetH P I Q $\rangle$ 
    assms(1) assms(12) assms(2) assms(6) betH2-out congH-colH-betH)
}
ultimately have ?thesis
  by blast
}
moreover {
  assume BetH I B C
  have CongaH P B I Q B I
  proof -
    have  $\neg$  ColH C B P
    using  $\langle$  $\neg$  ColH P B C $\rangle$  colH-permut-321 by blast
    moreover have  $\neg$  ColH C B Q
    using  $\langle$  $\neg$  ColH Q B C $\rangle$  colH-permut-321 by blast
    moreover have CongaH C B P C B Q
    using colH-permut-321  $\langle$ BetH P I Q $\rangle$   $\langle$ ColH B I C $\rangle$  assms(13)
      assms(14) calculation(1) calculation(2) th17 by blast
    moreover have BetH C B I
    by (simp add:  $\langle$ BetH I B C $\rangle$  between-comm)
    ultimately show ?thesis
    using th14 by blast
  qed
  have CongH I P I Q

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proof -
  have  $\neg \text{ColH } B \ P \ I$ 
    using  $\langle \text{BetH } I \ B \ C \rangle \langle \text{IncidL } B \ \text{IBC} \rangle \langle \text{IncidL } I \ \text{IBC} \rangle \langle \neg \text{IncidL } P \ \text{IBC} \rangle$ 
       $\text{betH-distincts colH-IncidL--IncidL colH-permut-312}$  by blast
  moreover have  $\neg \text{ColH } B \ Q \ I$ 
    using  $\langle \text{BetH } I \ B \ C \rangle \langle \text{IncidL } B \ \text{IBC} \rangle \langle \text{IncidL } I \ \text{IBC} \rangle \langle \neg \text{IncidL } Q \ \text{IBC} \rangle$ 
       $\text{betH-distincts colH-IncidL--IncidL colH-permut-312}$  by blast
  moreover have  $\text{CongH } B \ P \ B \ Q$ 
    using  $\text{assms}(13)$  by auto
  moreover have  $\text{CongH } B \ I \ B \ I$ 
    by (metis  $\langle \text{BetH } I \ B \ C \rangle \text{betH-distincts congH-refl}$ )
  ultimately show ?thesis
    using  $\langle \text{CongaH } P \ B \ I \ Q \ B \ I \rangle$ 
    by (metis  $\langle \text{IncidL } I \ \text{IBC} \rangle \langle \neg \text{IncidL } Q \ \text{IBC} \rangle \text{congH-permlr th12}$ )
qed
have  $A = I \longrightarrow ?thesis$ 
  using  $\text{Bet-def } \langle \text{BetH } I \ B \ C \rangle$  by blast
moreover
{
  assume  $A \neq I$ 
  have  $\text{ColH } I \ P \ Q$ 
    using  $\langle \text{BetH } P \ I \ Q \rangle \text{betH-expand colH-permut-213}$  by blast
  {
    assume  $\text{BetH } P \ A \ I$ 
    hence  $\text{BetH } Q \ I \ A$ 
      using  $\langle \text{BetH } P \ I \ Q \rangle \text{betH-trans0 between-comm}$  by blast
    hence False
      by (metis  $\text{betH-expand } \langle \text{BetH } P \ A \ I \rangle \langle \text{CongH } I \ P \ I \ Q \rangle \text{assms}(12)$ 
         $\text{betH-congH2--False congH-permlr}$ )
  }
  moreover
  {
    assume  $\text{BetH } P \ I \ A$ 
    hence  $\text{BetH } Q \ A \ I$ 
      by (metis  $\text{assms}(1) \text{assms}(12) \text{assms}(2) \text{assms}(6) \text{assms}(7)$ 
         $\text{betH-trans0 between-comm congH-colH-betH}$ )
    hence False
      by (metis  $\text{betH-expand } \langle \text{BetH } P \ I \ A \rangle \langle \text{CongH } I \ P \ I \ Q \rangle \text{assms}(12)$ 
         $\text{betH-congH2--False congH-perms}$ )
  }
  ultimately have ?thesis
    by (metis  $\langle A = I \longrightarrow \text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B \rangle$ 
       $\langle \text{BetH } P \ I \ Q \rangle \text{assms}(1) \text{assms}(12) \text{assms}(2) \text{assms}(6) \text{betH2-out congH-colH-betH}$ )
  }
  ultimately have ?thesis
    by blast
}
ultimately have ?thesis
  by (metis  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{ColH } B \ I \ C \rangle \langle \text{ColH } B \ P \ Q \implies \text{False} \rangle$ 
     $\langle \text{ColH } C \ P \ Q \implies \text{False} \rangle \text{assms}(5) \text{between-col between-one colH-permut-213}$ )
}
hence ?thesis
  using  $\langle \text{IncidL } B \ \text{IBC} \rangle \langle \text{IncidL } C \ \text{IBC} \rangle \langle \text{IsL } \text{IBC} \rangle \langle \text{cut}' \ P \ Q \ B \ C \rangle \text{cut}'\text{-def}$ 
     $\text{local.cut-def}$  by auto
}
moreover
{
  assume  $\text{same-side}' \ P \ Q \ B \ C$ 
  have  $\text{outH } B \ P \ Q$ 
  proof -
    have  $\neg \text{ColH } P \ B \ C$ 
      by (simp add:  $\langle \neg \text{ColH } P \ B \ C \rangle$ )
    moreover have  $\neg \text{ColH } C \ B \ P$ 
      using  $\text{calculation colH-permut-321}$  by blast
    moreover have  $\text{CongaH } C \ B \ P \ C \ B \ P$ 
      by (simp add:  $\text{calculation}(2) \text{conga-refl}$ )
  }
}

```

```

moreover have CongaH C B P C B Q
  by (metis  $\neg$  ColH Q B C) assms(13) assms(14) assms(5) calculation(2)
    colH-permut-213 colH-permut-312 congH-refl th18-aux)
moreover have same-side' P P B C
  using calculation(2) colH-permut-213 same-side-prime-refl by blast
ultimately show ?thesis
  using <same-side' P Q B C> cong-4-uniqueness by blast
qed
hence ?thesis
  using  $\neg$  ColH B P Q> outH-expand by blast
}
ultimately have ?thesis
  using  $\neg$  ColH P B C>  $\neg$  ColH Q B C> plane-separation-2D by blast
}
ultimately show ?thesis
  using  $\neg$  ColH B P Q>  $\neg$  ColH C P Q> plane-separation-2D by blast
qed

lemma TS-upper-dim:
assumes cut' A B P Q and
  P  $\neq$  Q and
  A  $\neq$  B and
  A  $\neq$  C and
  B  $\neq$  C and
  A  $\neq$  P and
  A  $\neq$  Q and
  B  $\neq$  P and
  B  $\neq$  Q and
  C  $\neq$  P and
  C  $\neq$  Q and
  CongH A P A Q and
  CongH B P B Q and
  CongH C P C Q
shows Bet A B C  $\vee$  Bet B C A  $\vee$  Bet C A B
proof –
obtain l where IsL l and IncidL P l and IncidL Q l
  using assms(2) line-existence by blast
have cut l A B
  using <IncidL P l> <IncidL Q l> <IsL l> assms(1) cut'-def by auto
have ColH A P Q  $\longrightarrow$  ?thesis
  by (simp add: assms(10) assms(11) assms(12) assms(13) assms(14) assms(2)
    assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
moreover
{
  assume  $\neg$  ColH A P Q
  have ColH B P Q  $\longrightarrow$  ?thesis
    by (metis assms(10) assms(11) assms(12) assms(13) assms(14) assms(2)
      assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
  moreover
  {
    assume  $\neg$  ColH B P Q
    obtain I where IncidL I l and BetH A I B
      using <local.cut l A B> local.cut-def by blast
    have ColH P Q I
      using ColH-def Is-line <IncidL I l> <IncidL P l> <IncidL Q l> by blast
    {
      assume P = I
      hence ColH A Q B
        by (metis <BetH A I B> assms(12) assms(13) assms(3) assms(9) congH-permlr
          congH-refl cong-preserves-col)
      hence False
        by (metis colH-permut-312 colH-trans colH-trivial121 <BetH A I B>
          <P = I>  $\neg$  ColH B P Q> assms(3) between-col)
    }
    hence P  $\neq$  I
      by blast
  }
}

```

```

{
  assume  $Q = I$ 
  hence  $ColH\ A\ P\ B$ 
  by (metis  $\langle BetH\ A\ I\ B \rangle$   $assms(12)$   $assms(13)$   $assms(3)$   $assms(6)$   $assms(8)$ 
       $assms(9)$   $congH$ -perms  $congH$ -refl  $congH$ -sym  $cong$ -preserves-col)
  have  $ColH\ A\ P\ I$ 
  by (metis  $\langle BetH\ A\ I\ B \rangle$   $\langle ColH\ A\ P\ B \rangle$   $assms(3)$   $colH$ -trans  $ncolH$ -expand
       $outH$ -col  $outH$ -def)
  hence  $False$ 
  using  $\langle Q = I \rangle$   $\langle \neg\ ColH\ A\ P\ Q \rangle$  by blast
}
hence  $Q \neq I$ 
  by blast
have  $CongaH\ I\ A\ P\ I\ A\ Q$ 
proof -
  have  $CongaH\ B\ A\ P\ B\ A\ Q$ 
  proof -
    have  $\neg\ ColH\ A\ B\ P$ 
    by (metis (full-types)  $\langle BetH\ A\ I\ B \rangle$   $\langle ColH\ P\ Q\ I \rangle$   $\langle P = I \implies False \rangle$ 
         $\langle \neg\ ColH\ A\ P\ Q \rangle$   $colH$ -permut-312  $colH$ -trivial121  $inter$ -uniquenessH
         $outH$ -def  $outH$ -expand)
    moreover have  $\neg\ ColH\ A\ B\ Q$ 
    by (meson  $assms(12)$   $assms(13)$   $assms(3)$   $assms(6)$   $assms(7)$   $assms(8)$ 
         $assms(9)$   $calculation$   $congH$ -refl  $congH$ -sym  $cong$ -preserves-col-stronger)
    moreover have  $CongH\ A\ B\ A\ B$ 
    using  $assms(3)$   $congH$ -refl by blast
    moreover have  $CongH\ A\ P\ A\ Q$ 
    by (simp add:  $assms(12)$ )
    moreover have  $CongH\ B\ P\ B\ Q$ 
    using  $assms(13)$  by auto
    ultimately show ?thesis
    using  $th18$ -aux by blast
  qed
  moreover have  $outH\ A\ B\ I$ 
  by (simp add:  $\langle BetH\ A\ I\ B \rangle$   $outH$ -def)
  moreover have  $outH\ A\ P\ P$ 
  by (simp add:  $assms(6)$   $outH$ -def)
  moreover have  $outH\ A\ B\ I$ 
  by (simp add:  $calculation(2)$ )
  moreover have  $outH\ A\ Q\ Q$ 
  by (simp add:  $assms(7)$   $outH$ -def)
  ultimately show ?thesis
  using  $conga$ -out-conga by fastforce
qed
have  $CongH\ I\ P\ I\ Q$ 
proof -
  have  $\neg\ ColH\ A\ I\ P$ 
  by (metis  $\langle ColH\ P\ Q\ I \rangle$   $\langle P \neq I \rangle$   $\langle \neg\ ColH\ A\ P\ Q \rangle$   $colH$ -permut-231
       $colH$ -trans  $colH$ -trivial122)
  moreover have  $\neg\ ColH\ A\ I\ Q$ 
  using  $ColH$ -def  $\langle IncidL\ I\ l \rangle$   $\langle IncidL\ P\ l \rangle$   $\langle IncidL\ Q\ l \rangle$   $\langle Q = I \implies False \rangle$ 
       $calculation$   $inter$ -incid-uniquenessH by auto
  moreover have  $CongH\ A\ I\ A\ I$ 
  using  $\langle BetH\ A\ I\ B \rangle$   $betH$ -distincts  $congH$ -refl by presburger
  ultimately show ?thesis
  using  $\langle CongaH\ I\ A\ P\ I\ A\ Q \rangle$   $assms(12)$ 
  using  $th12$  by blast
qed
have  $ColH\ A\ B\ C$ 
proof -
  have  $C = I \longrightarrow ?thesis$ 
  by (simp add:  $\langle BetH\ A\ I\ B \rangle$   $between$ -col  $colH$ -permut-132)
  moreover
  {
    assume  $C \neq I$ 
    have  $ColH\ A\ C\ I$ 

```

```

    by (metis colH-permut-231 <BetH A I B> <C ≠ I> <ColH P Q I>
      <CongH I P I Q> <P ≠ I> <Q ≠ I> assms(10) assms(11) assms(12) assms(14)
      assms(2) assms(4) assms(6) assms(7) betH-to-bet bet-colH col-upper-dim)
  hence ?thesis
    using ColH-def IncidL-not-IncidL--not-colH <BetH A I B> betH-colH by blast
}
ultimately show ?thesis
  by blast
qed
have BetH A B C ⟶ ?thesis
  by (simp add: ColH-bets <ColH A B C>)
moreover have BetH B C A ⟶ ?thesis
  using ColH-bets <ColH A B C> by blast
moreover have BetH B A C ⟶ ?thesis
  by (simp add: ColH-bets <ColH A B C>)
ultimately have ?thesis
  using ColH-bets <ColH A B C> by blast
}
ultimately have ?thesis
  by blast
}
ultimately show ?thesis
  by blast
qed

```

```

lemma cut'-comm:
  assumes cut' A B X Y
  shows cut' B A X Y
  by (metis assms cut'-def cut-comm)

```

```

lemma TS-upper-dim-bis:
  assumes BetH P I Q and
    BetH I B A and
    P ≠ Q and
    A ≠ B and
    A ≠ C and
    B ≠ C and
    A ≠ P and
    A ≠ Q and
    B ≠ P and
    B ≠ Q and
    C ≠ P and
    C ≠ Q and
    CongH A P A Q and
    CongH B P B Q and
    CongH C P C Q
  shows Bet A B C ∨ Bet B C A ∨ Bet C A B
proof -
  have ColH A P Q ⟶ ?thesis
    by (simp add: assms(10) assms(11) assms(12) assms(13) assms(14) assms(15)
      assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
  moreover
  {
    assume ¬ ColH A P Q
    have ColH B P Q ⟶ ?thesis
      by (metis assms(10) assms(11) assms(12) assms(13) assms(14) assms(15)
        assms(3) assms(4) assms(5) assms(6) assms(7) assms(8) assms(9) col-upper-dim)
    moreover
    {
      assume ¬ ColH B P Q
      have CongaH P B I Q B I
      proof -
        have P ≠ I
          using assms(1) betH-expand by blast
        have Q ≠ I
          using assms(1) betH-colH by blast

```

```

{
  assume ColH A B P
  have  $\neg$  ColH P Q A
    using  $\langle \neg$  ColH A P Q  $\rangle$  colH-permut-312 by blast
  moreover have ColH P Q P
    by (simp add: colH-trivial121)
  moreover have ColH P Q I
    by (simp add: assms(1) betH-colH colH-permut-132)
  moreover have ColH A B I
    by (simp add: assms(2) betH-colH between-comm)
  ultimately
  have  $P = I$ 
    using assms(4)  $\langle$  ColH A B P  $\rangle$  inter-uniquenessH by blast
  hence False
    by (simp add:  $\langle P \neq I \rangle$ )
}
hence  $\neg$  ColH A B P
  by blast
moreover
{
  assume ColH A B Q
  have  $\neg$  ColH Q P A
    using  $\langle \neg$  ColH A P Q  $\rangle$  colH-permut-321 by blast
  moreover have ColH Q P Q
    by (simp add: colH-trivial121)
  moreover have ColH Q P I
    using assms(1) betH-colH colH-permut-312 by blast
  moreover have ColH A B I
    using assms(2) between-col colH-permut-321 by blast
  ultimately
  have  $Q = I$ 
    using assms(4)  $\langle$  ColH A B Q  $\rangle$  inter-uniquenessH by blast
  hence False
    by (simp add:  $\langle Q \neq I \rangle$ )
}
hence  $\neg$  ColH A B Q
  by blast
moreover have CongaH A B P A B Q
proof -
  have  $\neg$  ColH B A P
    using calculation(1) colH-permut-213 by blast
  moreover have  $\neg$  ColH B A Q
    using  $\langle \neg$  ColH A B Q  $\rangle$  colH-permut-213 by blast
  moreover have CongH B A B A
    using assms(4) congH-refl by auto
  ultimately show ?thesis
    using assms(13) assms(14) th18-aux by presburger
qed
ultimately show ?thesis
  by (simp add: assms(2) between-comm th14)
qed
have CongH I P I Q
  by (metis assms(1) assms(10) assms(13) assms(14) assms(2) assms(7)
    assms(8) assms(9) axiom-five-segmentsH between-comm between-only-one congH-refl)
have ColH A B C
proof -
  have ColH A B I
    by (simp add: assms(2) between-col colH-permut-321)
  have  $C = I \longrightarrow ?thesis$ 
    using  $\langle$  ColH A B I  $\rangle$  by blast
  moreover
  {
    assume  $C \neq I$ 
    have ColH I B A
      by (simp add: assms(2) betH-colH)
    hence ColH A C I

```

```

    by (metis (mono-tags, lifting) betH-to-bet bet-colH colH-permut-213
        colH-permut-312  $\langle C \neq I \rangle$   $\langle CongH\ I\ P\ I\ Q \rangle$  assms(1) assms(11) assms(12)
        assms(13) assms(15) assms(2) assms(5) assms(7) assms(8) col-upper-dim)
  have ColH P I Q
  by (simp add: assms(1) betH-colH)
  hence ?thesis
  by (metis betH-distincts colH-permut-312 colH-trans colH-trivial121
       $\langle ColH\ A\ C\ I \rangle$   $\langle ColH\ I\ B\ A \rangle$  assms(2))
}
ultimately show ?thesis
by blast
qed
have BetH B A C  $\longrightarrow$  ?thesis
using ColH-bets  $\langle ColH\ A\ B\ C \rangle$  by blast
hence ?thesis
using ColH-bets  $\langle ColH\ A\ B\ C \rangle$  by blast
}
ultimately have ?thesis
by blast
}
ultimately show ?thesis by blast
qed

lemma upper-dim:
  assumes  $P \neq Q$  and
     $A \neq B$  and
     $A \neq C$  and
     $B \neq C$  and
     $Cong\ A\ P\ A\ Q$  and
     $Cong\ B\ P\ B\ Q$  and
     $Cong\ C\ P\ C\ Q$ 
  shows  $Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$ 
proof -
  have  $C = P \wedge C = Q \longrightarrow ?thesis$ 
  using assms(1) by blast
  have  $B = P \wedge B = Q \longrightarrow ?thesis$ 
  using assms(1) by blast
  have  $A = P \wedge A = Q \longrightarrow ?thesis$ 
  using assms(1) by blast
  have  $CongH\ A\ P\ A\ Q$ 
  using Cong-def assms(1) assms(5) by auto
  have  $CongH\ B\ P\ B\ Q$ 
  using Cong-def assms(1) assms(6) by blast
  have  $CongH\ C\ P\ C\ Q$ 
  using Cong-def assms(1) assms(7) by blast
  have  $A \neq P$ 
  using Cong-def assms(1) assms(5) by presburger
  have  $A \neq Q$ 
  using Cong-def assms(1) assms(5) by presburger
  have  $B \neq P$ 
  using Cong-def assms(1) assms(6) by force
  have  $B \neq Q$ 
  using Cong-def assms(1) assms(6) by presburger
  have  $C \neq P$ 
  using Cong-def assms(1) assms(7) by presburger
  have  $C \neq Q$ 
  using Cong-def assms(1) assms(7) by presburger
  have  $ColH\ A\ P\ Q \longrightarrow ?thesis$ 
  by (simp add:  $\langle A \neq P \rangle$   $\langle A \neq Q \rangle$   $\langle B \neq P \rangle$   $\langle B \neq Q \rangle$   $\langle C \neq P \rangle$   $\langle C \neq Q \rangle$ 
       $\langle CongH\ A\ P\ A\ Q \rangle$   $\langle CongH\ B\ P\ B\ Q \rangle$   $\langle CongH\ C\ P\ C\ Q \rangle$  assms(1) assms(2) assms(3)
      assms(4) col-upper-dim)
  moreover
  {
    assume  $\neg ColH\ A\ P\ Q$ 
    have  $ColH\ B\ P\ Q \longrightarrow ?thesis$ 
    by (metis  $\langle A \neq P \rangle$   $\langle A \neq Q \rangle$   $\langle B \neq P \rangle$   $\langle B \neq Q \rangle$   $\langle C \neq P \rangle$   $\langle C \neq Q \rangle$ 

```



```

    ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1) assms(2) assms(3)
    assms(4) col-upper-dim)
moreover
{
  assume ¬ ColH B P Q
  have ColH C P Q ⟶ ?thesis
  by (metis ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩ ⟨C ≠ Q⟩
    ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1) assms(2)
    assms(3) assms(4) col-upper-dim)
moreover
{
  assume ¬ ColH C P Q
  have cut' A B P Q ⟶ ?thesis
  by (simp add: TS-upper-dim ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩
    ⟨C ≠ Q⟩ ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1)
    assms(2) assms(3) assms(4))
moreover
{
  assume same-side' A B P Q
  have cut' A C P Q ⟶ ?thesis
  using TS-upper-dim ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨C ≠ P⟩ ⟨C ≠ Q⟩
    ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨CongH C P C Q⟩ assms(1) assms(2) assms(3)
    assms(4) bet-comm by blast
moreover
{
  assume same-side' A C P Q
  {
    assume ColH P A B
    hence ColH B P A
    by (simp add: colH-permut-312)
    have ColH A B Q
    by (meson ⟨A ≠ P⟩ ⟨B ≠ P⟩ ⟨ColH B P A⟩ ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩
      assms(2) colH-permut-312 congH-refl cong-preserves-col-stronger)
    hence False
    by (metis ⟨ColH B P A⟩ ⟨¬ ColH A P Q⟩ assms(2) colH-trans ncolH-distincts)
  }
  hence ¬ ColH P A B
  by blast
  {
    assume ColH Q A B
    hence ColH A B P
    by (meson ⟨A ≠ P⟩ ⟨A ≠ Q⟩ ⟨B ≠ P⟩ ⟨B ≠ Q⟩ ⟨CongH A P A Q⟩
      ⟨CongH B P B Q⟩ assms(2) colH-permut-231 congH-refl congH-sym
      cong-preserves-col-stronger)
    hence False
    using ⟨¬ ColH P A B⟩ colH-permut-312 by blast
  }
  hence ¬ ColH Q A B
  by blast
  {
    assume same-side' P Q A B
    have outH A P Q
    proof -
      have ¬ ColH P A B
      by (simp add: ⟨¬ ColH P A B⟩)
      moreover have ¬ ColH B A P
      using calculation colH-permut-321 by blast
      moreover have CongaH B A P B A P
      using calculation(2) conga-refl by blast
      moreover have CongaH B A P B A Q
      by (metis ⟨CongH A P A Q⟩ ⟨CongH B P B Q⟩ ⟨¬ ColH Q A B⟩ assms(2)
        calculation(2) colH-permut-213 colH-permut-312 congH-refl th18-aux)
      moreover have same-side' P P A B
      using calculation(1) colH-permut-312 same-side-prime-refl by blast
      ultimately show ?thesis
      using ⟨same-side' P Q A B⟩

```

```

    using cong-4-uniqueness by blast
qed
hence ?thesis
  using  $\langle \neg \text{ColH } A \ P \ Q \rangle$  outH-expand by blast
}
moreover
{
  assume cut' P Q A B
  obtain lAB where IsL lAB and IncidL A lAB and IncidL B lAB
    using assms(2) line-existence by blast
  then obtain I where IncidL I lAB and BetH P I Q
    using  $\langle \text{cut}' \ P \ Q \ A \ B \rangle$  cut'-def local.cut-def by auto
  hence ColH A I B
    using ColH-def  $\langle \text{IncidL } A \ lAB \rangle \langle \text{IncidL } B \ lAB \rangle \langle \text{IsL } lAB \rangle$  by auto
  {
    assume BetH A I B
    obtain lPQ where IsL lPQ and IncidL P lPQ and IncidL Q lPQ
      using assms(1) line-existence by blast
    hence  $\neg \text{cut } lPQ \ A \ B$ 
      using  $\langle \text{same-side}' \ A \ B \ P \ Q \rangle$  same-side'-def same-side-not-cut by auto
    have cut lPQ A B
      proof -
        have  $\neg \text{IncidL } B \ lPQ$ 
          using Hilbert-neutral-dimensionless-pre.ColH-def  $\langle \text{IncidL } P \ lPQ \rangle$ 
             $\langle \text{IncidL } Q \ lPQ \rangle \langle \text{IsL } lPQ \rangle \langle \neg \text{ColH } B \ P \ Q \rangle$  by fastforce
        moreover have IncidL I lPQ
          using  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{IncidL } P \ lPQ \rangle \langle \text{IncidL } Q \ lPQ \rangle$  assms(1)
            betH-line inter-incid-uniquenessH by blast
        ultimately show ?thesis
          using  $\langle \text{BetH } A \ I \ B \rangle$ 
            using Is-line  $\langle \text{ColH } A \ I \ B \rangle$  betH-distincts colH-IncidL--IncidL
              local.cut-def by blast
      qed
    hence False
      using  $\langle \neg \text{local.cut } lPQ \ A \ B \rangle$  by blast
    hence ?thesis
      by blast
  }
}
moreover have BetH I B A  $\longrightarrow$  ?thesis
  using TS-upper-dim-bis  $\langle A \neq P \rangle \langle A \neq Q \rangle \langle B \neq P \rangle \langle B \neq Q \rangle \langle \text{BetH } P \ I \ Q \rangle$ 
     $\langle C \neq P \rangle \langle C \neq Q \rangle \langle \text{CongH } A \ P \ A \ Q \rangle \langle \text{CongH } B \ P \ B \ Q \rangle \langle \text{CongH } C \ P \ C \ Q \rangle$ 
    assms(1) assms(2) assms(3) assms(4) by blast
moreover have BetH I A B  $\longrightarrow$  ?thesis
  using TS-upper-dim-bis  $\langle A \neq P \rangle \langle A \neq Q \rangle \langle B \neq P \rangle \langle B \neq Q \rangle \langle \text{BetH } P \ I \ Q \rangle$ 
     $\langle C \neq P \rangle \langle C \neq Q \rangle \langle \text{CongH } A \ P \ A \ Q \rangle \langle \text{CongH } B \ P \ B \ Q \rangle \langle \text{CongH } C \ P \ C \ Q \rangle$ 
    assms(1) assms(2) assms(3) assms(4) bet-comm by blast
ultimately have ?thesis
  by (metis  $\langle \text{BetH } P \ I \ Q \rangle \langle \text{ColH } A \ I \ B \rangle \langle \neg \text{ColH } A \ P \ Q \rangle \langle \neg \text{ColH } B \ P \ Q \rangle$ 
    assms(2) betH-colH between-one colH-permut-213)
}
ultimately have ?thesis
  using  $\langle \neg \text{ColH } P \ A \ B \rangle \langle \neg \text{ColH } Q \ A \ B \rangle$  plane-separation-2D by blast
}
ultimately have ?thesis
  using  $\langle \neg \text{ColH } A \ P \ Q \rangle \langle \neg \text{ColH } C \ P \ Q \rangle$ 
     $\langle \text{cut}' \ A \ C \ P \ Q \longrightarrow \text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B \rangle$ 
     $\langle \text{same-side}' \ A \ C \ P \ Q \implies \text{Bet } A \ B \ C \vee \text{Bet } B \ C \ A \vee \text{Bet } C \ A \ B \rangle$ 
    plane-separation-2D by blast
}
ultimately have ?thesis
  using  $\langle \neg \text{ColH } A \ P \ Q \rangle \langle \neg \text{ColH } B \ P \ Q \rangle$  plane-separation-2D by blast
}
ultimately have ?thesis
  by blast
}
ultimately have ?thesis

```

```

    by blast
  }
  ultimately show ?thesis
    by blast
qed

interpretation H2D-to-T2D : Tarski-neutral-2D
  where Bet = Bet and
    Cong = Cong and
    TPA = PP and
    TPB = PQ and
    TPC = PR
proof
  show  $\forall a\ b. \text{Cong } a\ b\ b\ a$ 
    by (simp add: cong-permT)
  show  $\forall a\ b\ p\ q\ r\ s. \text{Cong } a\ b\ p\ q \wedge \text{Cong } a\ b\ r\ s \longrightarrow \text{Cong } p\ q\ r\ s$ 
    using cong-inner-transitivity by blast
  show  $\forall a\ b\ c. \text{Cong } a\ b\ c\ c \longrightarrow a = b$ 
    by (simp add: cong-identity)
  show  $\forall a\ b\ c\ q. \exists x. \text{Bet } q\ a\ x \wedge \text{Cong } a\ x\ b\ c$ 
    by (simp add: segment-construction)
  show  $\forall a\ b\ c\ d\ a'\ b'\ c'\ d'. a \neq b \wedge \text{Bet } a\ b\ c \wedge \text{Bet } a'\ b'\ c' \wedge$ 
 $\text{Cong } a\ b\ a'\ b' \wedge \text{Cong } b\ c\ b'\ c' \wedge \text{Cong } a\ d\ a'\ d' \wedge \text{Cong } b\ d\ b'\ d' \longrightarrow$ 
 $\text{Cong } c\ d\ c'\ d'$ 
    using five-segment by blast
  show  $\forall a\ b. \text{Bet } a\ b\ a \longrightarrow a = b$ 
    by (simp add: bet-identity)
  show  $\forall a\ b\ c\ p\ q. \text{Bet } a\ p\ c \wedge \text{Bet } b\ q\ c \longrightarrow (\exists x. \text{Bet } p\ x\ b \wedge \text{Bet } q\ x\ a)$ 
proof -
  {
    fix a b c p q
    assume Bet a p c and Bet b q c and
      Bet a b c  $\vee$  Bet b c a  $\vee$  Bet c a b
    hence  $\exists x. \text{Bet } p\ x\ b \wedge \text{Bet } q\ x\ a$ 
      using Bet-def bet-comm bet-trans by metis
  }
  moreover
  {
    fix a b c p q
    assume Bet a p c and Bet b q c and
       $\neg (\text{Bet } a\ b\ c \vee \text{Bet } b\ c\ a \vee \text{Bet } c\ a\ b)$  and
       $a = p \vee p = c \vee b = q \vee q = c$ 
    hence  $\exists x. \text{Bet } p\ x\ b \wedge \text{Bet } q\ x\ a$ 
      by (metis Hilbert-neutral-dimensionless-pre.Bet-def bet-comm)
  }
  moreover
  {
    fix a b c p q
    assume Bet a p c and Bet b q c and
       $\neg (\text{Bet } a\ b\ c \vee \text{Bet } b\ c\ a \vee \text{Bet } c\ a\ b)$  and
       $a \neq p$  and  $p \neq c$  and  $b \neq q$  and  $q \neq c$ 
    hence  $\exists x. \text{Bet } p\ x\ b \wedge \text{Bet } q\ x\ a$ 
      using pasch-general-case by blast
  }
  ultimately show ?thesis
    by blast
qed
show  $\neg \text{Bet } PP\ PQ\ PR \wedge \neg \text{Bet } PQ\ PR\ PP \wedge \neg \text{Bet } PR\ PP\ PQ$ 
  using lower-dim-1 by blast
show  $\forall a\ b\ c\ p\ q. p \neq q \wedge \text{Cong } a\ p\ a\ q \wedge \text{Cong } b\ p\ b\ q \wedge \text{Cong } c\ p\ c\ q \longrightarrow$ 
 $\text{Bet } a\ b\ c \vee \text{Bet } b\ c\ a \vee \text{Bet } c\ a\ b$ 
  by (metis Bet-def upper-dim)
qed
end

```

end

theory *Tarski-Neutral-3D-Model-Hilbert-Neutral-3D*

imports

Tarski-Neutral-3D

Hilbert-Neutral-3D

Tarski-Neutral-Model-Hilbert-Neutral

begin

35 Tarski Neutral 3D - Hilbert Neutral 3D Model

context *Hilbert-neutral-3D*

begin

interpretation *H3D-to-T3D* : *Tarski-neutral-3D*

where *Bet* = *Bet* and

Cong = *Cong* and

TPA = *PP* and

TPB = *PQ* and

TPC = *PR* and

TS1 = *HS1* and *TS2* = *HS2* and *TS3* = *HS3* and *TS4* = *HS4*

proof

show $\forall a b. \text{Cong } a b b a$

using *cong-permT* by *blast*

show $\forall a b p q r s. \text{Cong } a b p q \wedge \text{Cong } a b r s \longrightarrow \text{Cong } p q r s$

using *cong-inner-transitivity* by *blast*

show $\forall a b c. \text{Cong } a b c c \longrightarrow a = b$

by (*simp add: cong-identity*)

show $\forall a b c q. \exists x. \text{Bet } q a x \wedge \text{Cong } a x b c$

by (*simp add: segment-construction*)

show $\forall a b c d a' b' c' d'. a \neq b \wedge \text{Bet } a b c \wedge \text{Bet } a' b' c' \wedge$

$\text{Cong } a b a' b' \wedge \text{Cong } b c b' c' \wedge \text{Cong } a d a' d' \wedge \text{Cong } b d b' d' \longrightarrow$
 $\text{Cong } c d c' d'$

using *five-segment* by *blast*

show $\forall a b. \text{Bet } a b a \longrightarrow a = b$

using *bet-identity* by *blast*

show $\forall a b c p q. \text{Bet } a p c \wedge \text{Bet } b q c \longrightarrow (\exists x. \text{Bet } p x b \wedge \text{Bet } q x a)$

proof –

{

fix *a b c p q*

assume *Bet a p c* and *Bet b q c* and

Bet a b c $\vee$ *Bet b c a* $\vee$ *Bet c a b*

hence $\exists x. \text{Bet } p x b \wedge \text{Bet } q x a$

using *Bet-def bet-comm bet-trans* by *metis*

}

moreover

{

fix *a b c p q*

assume *Bet a p c* and *Bet b q c* and

$\neg (\text{Bet } a b c \vee \text{Bet } b c a \vee \text{Bet } c a b)$ and

$a = p \vee p = c \vee b = q \vee q = c$

hence $\exists x. \text{Bet } p x b \wedge \text{Bet } q x a$

by (*metis Hilbert-neutral-dimensionless-pre.Bet-def bet-comm*)

}

moreover

{

fix *a b c p q*

assume *Bet a p c* and *Bet b q c* and

$\neg (\text{Bet } a b c \vee \text{Bet } b c a \vee \text{Bet } c a b)$ and

$a \neq p$ and $p \neq c$ and $b \neq q$ and $q \neq c$

hence $\exists x. \text{Bet } p x b \wedge \text{Bet } q x a$

```

    using pasch-general-case by blast
  }
  ultimately show ?thesis
    by blast
qed
show  $\neg \text{Bet } PP \text{ } PQ \text{ } PR \wedge \neg \text{Bet } PQ \text{ } PR \text{ } PP \wedge \neg \text{Bet } PR \text{ } PP \text{ } PQ$ 
  using lower-dim-1 by blast
show  $\# X. (\text{Bet } HS1 \text{ } HS2 \text{ } X \vee \text{Bet } HS2 \text{ } X \text{ } HS1 \vee \text{Bet } X \text{ } HS1 \text{ } HS2) \wedge$ 
   $(\text{Bet } HS3 \text{ } HS4 \text{ } X \vee \text{Bet } HS4 \text{ } X \text{ } HS3 \vee \text{Bet } X \text{ } HS3 \text{ } HS4) \vee$ 
   $(\text{Bet } HS1 \text{ } HS3 \text{ } X \vee \text{Bet } HS3 \text{ } X \text{ } HS1 \vee \text{Bet } X \text{ } HS1 \text{ } HS3) \wedge$ 
   $(\text{Bet } HS2 \text{ } HS4 \text{ } X \vee \text{Bet } HS4 \text{ } X \text{ } HS2 \vee \text{Bet } X \text{ } HS2 \text{ } HS4) \vee$ 
   $(\text{Bet } HS1 \text{ } HS4 \text{ } X \vee \text{Bet } HS4 \text{ } X \text{ } HS1 \vee \text{Bet } X \text{ } HS1 \text{ } HS4) \wedge$ 
   $(\text{Bet } HS2 \text{ } HS3 \text{ } X \vee \text{Bet } HS3 \text{ } X \text{ } HS2 \vee \text{Bet } X \text{ } HS2 \text{ } HS3)$ 
  using coplanar-plane1 lower-dim-3 by blast
show  $\forall A \text{ } B \text{ } C \text{ } P \text{ } Q \text{ } R. P \neq Q \wedge Q \neq R \wedge P \neq R \wedge \text{Cong } A \text{ } P \text{ } A \text{ } Q \wedge$ 
   $\text{Cong } B \text{ } P \text{ } B \text{ } Q \wedge \text{Cong } C \text{ } P \text{ } C \text{ } Q \wedge \text{Cong } A \text{ } P \text{ } A \text{ } R \wedge \text{Cong } B \text{ } P \text{ } B \text{ } R \wedge \text{Cong } C \text{ } P \text{ } C \text{ } R \longrightarrow$ 
   $\text{Bet } A \text{ } B \text{ } C \vee \text{Bet } B \text{ } C \text{ } A \vee \text{Bet } C \text{ } A \text{ } B$ 
  using tarski-upper-dim-plane-intersection lower-dim-3 by blast
qed

end
end

```

theory *Tarski-Euclidean-Model-Hilbert-Euclidean*

imports

Hilbert-Euclidean

Tarski-Neutral-3D-Model-Hilbert-Neutral-3D

begin

36 Tarski Euclidean Model Hilbert Euclidean

context *Hilbert-Euclidean*

begin

interpretation *Heucl-to-Teucl* : *Tarski-Euclidean*

where *Bet* = *Bet* **and**

Cong = *Cong* **and**

TPA = *PP* **and**

TPB = *PQ* **and**

TPC = *PR*

proof

show $\forall a \text{ } b. \text{Cong } a \text{ } b \text{ } b \text{ } a$

using *cong-permT* by blast

show $\forall a \text{ } b \text{ } p \text{ } q \text{ } r \text{ } s. \text{Cong } a \text{ } b \text{ } p \text{ } q \wedge \text{Cong } a \text{ } b \text{ } r \text{ } s \longrightarrow \text{Cong } p \text{ } q \text{ } r \text{ } s$

using *cong-inner-transitivity* by blast

show $\forall a \text{ } b \text{ } c. \text{Cong } a \text{ } b \text{ } c \text{ } c \longrightarrow a = b$

by (*simp add: cong-identity*)

show $\forall a \text{ } b \text{ } c \text{ } q. \exists x. \text{Bet } q \text{ } a \text{ } x \wedge \text{Cong } a \text{ } x \text{ } b \text{ } c$

by (*simp add: segment-construction*)

show $\forall a \text{ } b \text{ } c \text{ } d \text{ } a' \text{ } b' \text{ } c' \text{ } d'. a \neq b \wedge \text{Bet } a \text{ } b \text{ } c \wedge \text{Bet } a' \text{ } b' \text{ } c' \wedge$

$\text{Cong } a \text{ } b \text{ } a' \text{ } b' \wedge \text{Cong } b \text{ } c \text{ } b' \text{ } c' \wedge \text{Cong } a \text{ } d \text{ } a' \text{ } d' \wedge \text{Cong } b \text{ } d \text{ } b' \text{ } d' \longrightarrow$

$\text{Cong } c \text{ } d \text{ } c' \text{ } d'$

using *five-segment* by blast

show $\forall a \text{ } b. \text{Bet } a \text{ } b \text{ } a \longrightarrow a = b$

using *bet-identity* by blast

show $\forall a \text{ } b \text{ } c \text{ } p \text{ } q. \text{Bet } a \text{ } p \text{ } c \wedge \text{Bet } b \text{ } q \text{ } c \longrightarrow (\exists x. \text{Bet } p \text{ } x \text{ } b \wedge \text{Bet } q \text{ } x \text{ } a)$

proof —

{

fix *a b c p q*

assume *Bet a p c and Bet b q c and*

Bet a b c $\vee$ Bet b c a $\vee$ Bet c a b

```

    hence  $\exists x. \text{Bet } p \ x \ b \wedge \text{Bet } q \ x \ a$ 
    using Bet-def bet-comm bet-trans by metis
  }
  moreover
  {
    fix  $a \ b \ c \ p \ q$ 
    assume Bet  $a \ p \ c$  and Bet  $b \ q \ c$  and
       $\neg (\text{Bet } a \ b \ c \vee \text{Bet } b \ c \ a \vee \text{Bet } c \ a \ b)$  and
       $a = p \vee p = c \vee b = q \vee q = c$ 
    hence  $\exists x. \text{Bet } p \ x \ b \wedge \text{Bet } q \ x \ a$ 
    by (metis Hilbert-neutral-dimensionless-pre.Bet-def bet-comm)
  }
  moreover
  {
    fix  $a \ b \ c \ p \ q$ 
    assume Bet  $a \ p \ c$  and Bet  $b \ q \ c$  and
       $\neg (\text{Bet } a \ b \ c \vee \text{Bet } b \ c \ a \vee \text{Bet } c \ a \ b)$  and
       $a \neq p$  and  $p \neq c$  and  $b \neq q$  and  $q \neq c$ 
    hence  $\exists x. \text{Bet } p \ x \ b \wedge \text{Bet } q \ x \ a$ 
    using pasch-general-case by blast
  }
  ultimately show ?thesis
    by blast
qed
show  $\neg \text{Bet } PP \ PQ \ PR \wedge \neg \text{Bet } PQ \ PR \ PP \wedge \neg \text{Bet } PR \ PP \ PQ$ 
  using lower-dim-l by blast
show  $\forall A \ B \ C \ D \ T. \text{Bet } A \ D \ T \wedge \text{Bet } B \ D \ C \wedge A \neq D$ 
   $\longrightarrow (\exists X \ Y. \text{Bet } A \ B \ X \wedge \text{Bet } A \ C \ Y \wedge \text{Bet } X \ T \ Y)$ 
  using tarski-s-euclid euclid-uniqueness by blast
qed
end
end

```

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