Irrational Rapidly Convergent Series

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Abstract

We formalize with Isabelle/HOL a proof of a theorem by J. Hančl asserting the irrationality of the sum of a series consisting of rational numbers, built up by sequences that fulfill certain properties. Even though the criterion is a number theoretic result, the proof makes use only of analytical arguments. We also formalize a corollary of the theorem for a specific series fulfilling the assumptions of the theorem.

Contents

1 Main Theorem and Sketch of the Proof

We formalize the proof of the following theorem by J. Hančl (Theorem 3 in [1]) :

Theorem 1. *(Theorem 3 in [1]) Let $A \in \mathbb{R}$ with $A > 1$. Let $\{d_n\}_{n=1}^{\infty} \in \mathbb{R}$ with $d_n > 1$ for all $n \in \mathbb{N}$. Let $\{a_n\}_{n=1}^{\infty} \in \mathbb{Z}^+$, $\{b_n\}_{n=1}^{\infty} \in \mathbb{Z}^+$ such that :*

$$(1)\ \lim_{n\to\infty} a_n^{\frac{1}{2^n}} = A,$$

for all sufficiently large $n \in \mathbb{N}$:

$$(2)\ \frac{A}{a_n^{\frac{1}{2^n}}} > \prod_{j=n}^{\infty} d_j$$

and

$$(3)\ \lim_{n\to\infty} \frac{d_n^{2^n}}{b_n} = \infty.$$

Then the series $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is an irrational number.*

The first step is to show that the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ converges to some $\alpha \in \mathbb{R}$. To show that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we argue by proof by contradiction (to this end several auxiliary lemmas are firstly shown). In particular, assuming that $\alpha \in \mathbb{Q}$, i.e. that there exist $p, q \in \mathbb{Z}^+$ such that $\alpha = \frac{p}{q}$, we show that a quantity $\mathcal{A}(n) \geq 1$ for all $n \in \mathbb{N}$. At the same time, we find $n \in \mathbb{N}$ for which $\mathcal{A}(n) < 1$, yielding a contradiction from which we deduce the irrationality of the sum of the series.

For the proof see [1]. We note that the proof involves only elementary Analysis (criteria for convergence/divergence for sequences and series and several inequalities) and not any arithmetical/number theoretic arguments. Obviously for the formal proof we had to make many intermediate arguments explicit. Proofs of length of roughly 2 A4 pages in the original paper by J. Hančl were formalized in almost 1100 lines of code.

2 Corollary

We moreover formalize the following corollary that asserts the irrationality of the sum of an instance of a series that fulfills the assumptions of the theorem :

Corollary 1. *(Corollary 2 in [1]) Let* $A \in \mathbb{R}$ *with* $A > 1$*. Let* $\{a_n\}_{n=1}^{\infty} \in \mathbb{Z}^+$, $\{b_n\}_{n=1}^{\infty} \in \mathbb{Z}^+$ *such that :*

$$\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = A$$

and for all sufficiently large $n \in \mathbb{N}$ *(in particular: for* $n \geq 6$*)*

$$a_n^{\frac{1}{2^n}} (1 + 4(2/3)^n) \leq A$$

and

$$b_n \leq 2^{(4/3)^{n-1}}.$$

Then the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is an irrational number.*

The above corollary is an immediate consequence of the theorem by setting $d_n = 1 + (2/3)^n$. For the formalized proof of the corollary one more auxiliary lemma was required.

3 Irrational Rapidly Convergent Series

theory *Irrationality-J-Hancl*
imports *HOL−Analysis.Analysis HOL−Decision-Procs.Approximation*
begin

This is the formalisation of a proof by J. Hanl, in particular of the proof of his Theorem 3 in the paper: Irrational Rapidly Convergent Series, Rend. Sem. Mat. Univ. Padova, Vol 107 (2002).

The statement asserts the irrationality of the sum of a series consisting of rational numbers defined using sequences that fulfill certain properties. Even though the statement is number-theoretic, the proof uses only arguments from introductory Analysis.

We prove the central result (theorem Hancl3) by contradiction, by making use of some of the auxiliary lemmas. To this end, assuming that the sum is a rational number, for a quantity ALPHA(n) we show that ALPHA(n) ≥ 1 for all $n \in \mathbb{N}$. After that we show that we can find an $n \in \mathbb{N}$ for which ALPHA(n) < 1 which yields a contradiction and we thus conclude that the sum of the series is rational. We finally give an immediate application of theorem Hancl3 for a specific series (corollary Hancl3corollary, requiring lemma summable_ln_plus) which corresponds to Corollary 2 in the original paper by J. Hanl.

hide-const *floatarith.Max*

3.1 Misc

lemma *filterlim-sequentially-iff*:
filterlim f F1 sequentially $\longleftrightarrow$ *filterlim* $(\lambda x.\ f\ (x{+}k))$ *F1 sequentially*
⟨*proof*⟩

lemma *filterlim-realpow-sequentially-at-top*:
$(x{::}real) > 1 \Longrightarrow$ *filterlim* (*power x*) *at-top sequentially*
⟨*proof*⟩

lemma *filterlim-at-top-powr-real*:
fixes $g{::}'b \Rightarrow real$
assumes *filterlim f at-top F* $(g \longrightarrow g')$ *F* $g'{>}1$
shows *LIM x F. g x powr f x :> at-top*
⟨*proof*⟩

lemma *powrfinitesum*:
fixes $a{::}real$ **and** $s{::}nat$ **assumes** $s \leq n$
shows $(\prod j{=}s..(n{::}nat).(a\ powr\ (2\hat{}j))) = a\ powr\ (\sum j{=}s..(n{::}nat).(2\hat{}j))$
⟨*proof*⟩

lemma *summable-ratio-test-tendsto*:
fixes $f :: nat \Rightarrow 'a{::}banach$

assumes $c < 1$ **and** $\forall n.\ f\ n \neq 0$ **and** $(\lambda n.\ norm\ (f\ (Suc\ n))\ /\ norm\ (f\ n)) \longrightarrow c$
shows *summable f*
⟨*proof*⟩

lemma *summable-ln-plus*:
fixes $f::nat \Rightarrow real$
assumes *summable f* $\forall n.\ f\ n > 0$
shows *summable* $(\lambda n.\ ln\ (1+f\ n))$
⟨*proof*⟩

lemma *suminf-real-offset-le*:
fixes $f :: nat \Rightarrow real$
assumes f: $\bigwedge i.\ 0 \leq f\ i$ **and** *summable f*
shows $(\sum i.\ f\ (i + k)) \leq suminf\ f$
⟨*proof*⟩

lemma *factt*:
fixes $s\ n :: nat$ **assumes** $s \leq n$
shows $(\sum i=s..n.\ 2\hat{\ }i) < (2\hat{\ }(n+1) :: real)$ ⟨*proof*⟩

3.2 Auxiliary lemmas and the main proof

lemma *showpre7*:
fixes $a\ b :: nat \Rightarrow int$ **and** $q\ p::int$
assumes $q>0$ **and** $p>0$**and** a: $\forall n.\ a\ n>0$ **and** $\forall n.\ b\ n>0$ **and**
assumerational:$(\lambda\ n.\ b\ (n+1)\ /\ a\ (n+1)\ )$ *sums* (p/q)
shows $q*((\prod j=1..n.\ of\text{-}int(\ a\ j)))*(suminf\ (\lambda(j::nat).\ (b\ (j+n+1)/\ a\ (j+n+1)))))$
$= ((\prod j=1..n.\ of\text{-}int(\ a\ j)))*(p\ -q*\ (\sum j=1..n.\ b\ j\ /\ a\ j))$
⟨*proof*⟩

lemma *show7*:
fixes $d::nat \Rightarrow real$ **and** $a\ b::nat \Rightarrow int$ **and** $q\ p::int$
assumes $q \geq 1$ **and** $p \geq 1$ **and** a: $\forall n.\ a\ n \geq 1$ **and** $\forall n.\ b\ n \geq 1$
and *assumerational*:$(\lambda\ n.\ b\ (n+1)\ /\ a\ (n+1)\ )$ *sums* (p/q)
shows $q*((\prod j=1..n.\ of\text{-}int(\ a\ j)))*(\ suminf\ (\lambda\ (j::nat).\ (b\ (j+n+1)/\ a\ (j+n+1)))))) \geq 1$
(**is** $?L \geq$ -)
⟨*proof*⟩

lemma *show8*:
fixes $d :: nat \Rightarrow real$ **and** $a :: nat \Rightarrow int$ **and** $s\ k::nat$
assumes $A > 1$ **and** d: $\forall n.\ d\ n > 1$ **and** a:$\forall n.\ a\ n>0$ **and** $s>0$
and *convergent-prod d*
and *assu2*: $\forall n \geq s.\ A\ /\ of\text{-}int\ (a\ n)\ powr\ (1\ /\ of\text{-}int\ (2\hat{\ }n)) > (\prod j.\ d\ (n + j))$

shows $\forall\, n \geq s.\ (\prod j.\ d\ (j+n)) < A\ /\ (MAX\ j \in \{s..n\}.\ \textit{of-int}\ (a\ j)\ \textit{powr}\ (1\ /\ \textit{of-int}\ (2\ \hat{}\ j)))$
⟨*proof*⟩

lemma *auxiliary1-9*:
fixes d ::*nat*⇒*real* **and** a::*nat*⇒*int* **and** $s\ m$::*nat*
assumes d: $\forall\, n.\ d\ n > 1$ **and** a: $\forall\, n.\ a\ n > 0$ **and** $s > 0$ **and** $n \geq m$ **and** $m \geq s$
and *auxifalse-assu*: $\forall\, n \geq m.\ (\textit{of-int}\ (a\ (n+1)))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}(n+1))) <$
$(d\ (n+1)) * (Max\ ((\lambda\ (j{::}nat).\ (\textit{of-int}\ (a\ j))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}j)))\ `\ \{s..n\}\ ))$
shows $(\textit{of-int}\ (a\ (n+1)))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}(n+1))) <$
$(\prod j=(m+1)..(n+1).\ d\ j) * (Max\ ((\lambda\ (j{::}nat).\ (\textit{of-int}\ (a\ j))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}j)))\ `\ \{s..m\}))$
⟨*proof*⟩

lemma *show9*:
fixes d ::*nat*⇒*real* **and** a :: *nat*⇒*int* **and** s ::*nat* **and** A::*real*
assumes $A > 1$ **and** d: $\forall\, n.\ d\ n > 1$ **and** a: $\forall\, n.\ a\ n > 0$ **and** $s > 0$
and *assu1*: $((\ \lambda\ n.\ (\textit{of-int}\ (a\ n))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}n))) \longrightarrow A)$ *sequentially*
and *convergent-prod d*
and *8*: $\forall\, n \geq s.\ \textit{prodinf}\ (\lambda j.\ d(\ n+j))$
$< A/(Max\ ((\lambda(j{::}nat).\ (\textit{of-int}\ (a\ j))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}j)))\ `\ \{s..n\}))$

shows $\forall\, m \geq s.\ \exists\, n \geq m.\ (\ (\textit{of-int}\ (a\ (n+1)))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}(n+1))) \geq$
$(d\ (n+1)) * (Max\ (\ (\ \lambda\ (j{::}nat).\ (\textit{of-int}\ (a\ j))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}j)))\ `\ \{s..n\}\ )))$
⟨*proof*⟩

lemma *show10*:
fixes d ::*nat*⇒*real* **and** a ::*nat*⇒*int* **and** s::*nat*
assumes d [*rule-format*]: $\forall\, n.\ d\ n > 1$
and a [*rule-format*]: $\forall\, n.\ a\ n > 0$ **and** $s > 0$
and *9*: $\forall\, m \geq s.\ \exists\, n \geq m.\ a\ (n+1)\ \textit{powr}(1\ /\ \textit{of-int}\ (2\hat{}(n+1))) \geq$
$d\ (n+1) * (Max\ ((\lambda j.\ (\textit{of-int}\ (a\ j))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}j)))\ `\ \{s..n\}\ ))$
shows $\forall\, m \geq s.\ \exists\, n \geq m.\ d\ (n+1)\ \textit{powr}(2\hat{}(n+1)) * (\prod j=1..n.\ \textit{of-int}(\ a\ j)) *$
$(1\ /\ (\prod j=1..s-1.\ \textit{of-int}(\ a\ j))) \leq a\ (n+1)$
⟨*proof*⟩

lemma *lasttoshow*:
fixes d ::*nat*⇒*real* **and** $a\ b$ ::*nat*⇒*int* **and** q::*int* **and** s::*nat*
assumes d: $\forall\, n.\ d\ n > 1$
and a:$\forall\, n.\ a\ n > 0$ **and** $s > 0$ **and** $q > 0$
and $A > 1$ **and** b:$\forall\, n.\ b\ n > 0$ **and** *9*:
$\forall\, m \geq s.\ \exists\, n \geq m.\ ((\textit{of-int}\ (a\ (n+1)))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}(n+1))) \geq$
$(d\ (n+1)) * (Max\ ((\lambda(j{::}nat).\ (\textit{of-int}\ (a\ j))\ \textit{powr}(1\ /\textit{of-int}\ (2\hat{}j)))\ `\ \{s..n\}\ )))$
and *assu3*: *filterlim*$(\ \lambda\ n.\ (d\ n)\hat{}(2\hat{}n)/\ b\ n)$ *at-top sequentially*
and *5*: $\forall_F\ n$ *in sequentially*. $(\sum j.\ (b\ (n+j))\ /\ \ (a\ (n+j))) \leq 2 * b\ n\ /\ a\ n$
shows $\exists\, n.\ q*((\prod j=1..n.\ \textit{real-of-int}(a\ j))) * \textit{suminf}\ (\lambda(j{::}nat).\ (b\ (j+n+1)/\ a$

(j+n+1)))<1
⟨*proof*⟩

lemma
  fixes *d ::nat⇒real* **and** *a b ::nat⇒int* **and** *A::real*
  assumes *A > 1* **and** *d*: ∀ *n. d n> 1* **and** *a*: ∀ *n. a n>0* **and** *b*:∀ *n. b n>0*
    and *assu1*: ((λ *n.* (*of-int* (*a n*)) *powr*(*1* /*of-int* (*2^n*)))⟶ *A*) *sequentially*
    and *assu3*: *filterlim* (λ *n.* (*d n*)^(*2^n*)/ *b n*) *at-top sequentially*
    and *convergent-prod d*
  shows *issummable*: *summable* (λ*j. b j / a j*)
    and *show5*: $\forall_F$ *n in sequentially.* (∑ *j.* (*b* (*n* + *j*)) / (*a* (*n* + *j*))) ≤ *2* ∗ *b n* / *a n*
⟨*proof*⟩

theorem *Hancl3*:
  fixes *d ::nat⇒real* **and** *a b :: nat⇒int*
  assumes *A > 1* **and** *d*: ∀ *n. d n > 1* **and** *a*: ∀ *n. a n>0* **and** *b*: ∀ *n. b n > 0* **and** *s>0*
    and *assu1*: (λ*n.* (*a n*) *powr*(*1* / *of-int*(*2^n*))) ⟶ *A*
    and *assu2*: ∀ *n* ≥ *s. A* / (*a n*) *powr* (*1* / *of-int*(*2^n*)) > (∏ *j. d* (*n+j*))
    and *assu3*: *LIM n sequentially. d n ^ 2 ^ n / b n :> at-top*
    and *convergent-prod d*
  shows (∑ *n. b n / a n*) ∉ ℚ
⟨*proof*⟩

corollary *Hancl3corollary*:
  fixes *A::real* **and** *a b :: nat⇒int*
  assumes *A > 1* **and** *a*: ∀ *n. a n>0* **and** *b*: ∀ *n. b n>0*
    and *assu1*: (λ*n.* (*a n*) *powr*(*1* / *of-int*(*2^n*))) ⟶ *A*
    and *asscor2*: ∀ *n* ≥ *6. a n powr*(*1* / *of-int* (*2^n*)) ∗ (*1* + *4*∗(*2/3*)^*n*) ≤ *A*
            ∧ *b n* ≤ *2 powr* (*4/3*)^(*n*−*1*)
  shows (∑ *n. b n / a n*) ∉ ℚ
⟨*proof*⟩

end

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References

[1] J. Hančl. Irrational rapidly convergent series. *Rendiconti del Seminario Matematico della Università di Padova*, 107:225–231, 2002.