Irrationality Criteria for Series by Erdős and Straus

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Abstract

We formalise certain irrationality criteria for infinite series of the form:

$$\sum_n \frac{b_n}{\prod_{i \le n} a_i}$$

where b_n, a_i are integers. The result is due to P. Erdős and E.G. Straus [1], and in particular we formalise Theorem 2.1, Corollary 2.10 and Theorem 3.1. The latter is an application of Theorem 2.1 involving the prime numbers.

Contents

theory *Irrational-Series-Erdos-Straus* **imports**
  Prime-Number-Theorem.Prime-Number-Theorem
  Prime-Distribution-Elementary.PNT-Consequences
begin

1 Miscellaneous

lemma *suminf-comparison*:
  assumes *summable f* **and** *gf*: $\bigwedge n.\ norm\ (g\ n) \le f\ n$
  shows *suminf g* $\le$ *suminf f*
⟨*proof*⟩

lemma *tendsto-of-int-diff-0*:
assumes $(\lambda n.\ f\ n\ -\ \textit{of-int}(g\ n)) \longrightarrow (0::real)$ $\forall_F\ n\ \textit{in sequentially}.\ f\ n > 0$
shows $\forall_F\ n\ \textit{in sequentially}.\ 0 \le g\ n$
⟨*proof*⟩

lemma *eventually-mono-sequentially*:
assumes *eventually P sequentially*
assumes $\bigwedge x.\ P\ (x+k) \Longrightarrow Q\ (x+k)$
shows *eventually Q sequentially*
⟨*proof*⟩

lemma *frequently-eventually-at-top*:
fixes $P\ Q::'a::linorder \Rightarrow bool$
assumes *frequently P at-top eventually Q at-top*
shows *frequently* $(\lambda x.\ P\ x \wedge (\forall y \ge x.\ Q\ y)\ )$ *at-top*
⟨*proof*⟩

lemma *eventually-at-top-mono*:
fixes $P\ Q::'a::linorder \Rightarrow bool$
assumes *event-P*:*eventually P at-top*
assumes *PQ-imp*:$\bigwedge x.\ x \ge z \Longrightarrow \forall y \ge x.\ P\ y \Longrightarrow Q\ x$
shows *eventually Q at-top*
⟨*proof*⟩

lemma *frequently-at-top-elim*:
fixes $P\ Q::'a::linorder \Rightarrow bool$
assumes $\exists_F x\ \textit{in at-top}.\ P\ x$
assumes $\bigwedge i.\ P\ i \Longrightarrow \exists j > i.\ Q\ j$
shows $\exists_F x\ \textit{in at-top}.\ Q\ x$
⟨*proof*⟩

lemma *less-Liminf-iff*:
fixes X :: - $\Rightarrow$ - :: *complete-linorder*
shows *Liminf F X* $< C \longleftrightarrow (\exists y < C.\ \textit{frequently}\ (\lambda x.\ y \ge X\ x)\ F)$
⟨*proof*⟩

lemma *sequentially-even-odd-imp*:
assumes $\forall_F\ N\ \textit{in sequentially}.\ P\ (2*N)$ $\forall_F\ N\ \textit{in sequentially}.\ P\ (2*N+1)$
shows $\forall_F\ n\ \textit{in sequentially}.\ P\ n$
⟨*proof*⟩

2 Theorem 2.1 and Corollary 2.10

context
fixes $a\ b\ ::nat \Rightarrow int$
assumes *a-pos*: $\forall\ n.\ a\ n > 0$ **and** *a-large*: $\forall_F\ n\ \textit{in sequentially}.\ a\ n > 1$
and *ab-tendsto*: $(\lambda n.\ |b\ n|\ /\ (a\ (n-1)\ *\ a\ n)) \longrightarrow 0$
begin

private lemma *aux-series-summable*: *summable* ($\lambda n.\ b\ n\ /\ (\prod k{\le}n.\ a\ k)$)
⟨*proof*⟩ **fun** *get-c*::(*nat* ⇒ *int*) ⇒ (*nat* ⇒ *int*) ⇒ *int* ⇒ *nat* ⇒ (*nat* ⇒ *int*) **where**
  get-c a′ b′ B N 0 = round (B ∗ b′ N / a′ N)|
  get-c a′ b′ B N (Suc n) = get-c a′ b′ B N n ∗ a′ (n+N) − B ∗ b′ (n+N)

lemma *ab-rationality-imp*:
  assumes *ab-rational*:($\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \in \mathbb{Q}$
  shows ∃ (*B*::*int*)>*0*. ∃ *c*::*nat*⇒ *int*.
      (∀$_F$ *n in sequentially. B∗b n = c n ∗ a n − c(n+1) ∧ |c(n+1)|<a n/2*)
      ∧ ($\lambda n.\ c\ (Suc\ n)\ /\ a\ n) \longrightarrow 0$
⟨*proof*⟩ **lemma** *imp-ab-rational*:
  assumes ∃ (*B*::*int*)>*0*. ∃ *c*::*nat*⇒ *int*.
      (∀$_F$ *n in sequentially. B∗b n = c n ∗ a n − c(n+1) ∧ |c(n+1)|<a n/2*)
  shows ($\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \in \mathbb{Q}$
⟨*proof*⟩

theorem *theorem-2-1-Erdos-Straus* :
  ($\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \in \mathbb{Q} \longleftrightarrow$ (∃ (*B*::*int*)>*0*. ∃ *c*::*nat*⇒ *int*.
      (∀$_F$ *n in sequentially. B∗b n = c n ∗ a n − c(n+1) ∧ |c(n+1)|<a n/2*))
  ⟨*proof*⟩

The following is a Corollary to Theorem 2.1.

corollary *corollary-2-10-Erdos-Straus*:
  assumes *ab-event*:∀$_F$ *n in sequentially. b n > 0 ∧ a (n+1) ≥ a n*
    and *ba-lim-leq*:*lim* ($\lambda n.\ (b(n+1) - b\ n\ )/a\ n) \le 0$
    and *ba-lim-exist*:*convergent* ($\lambda n.\ (b(n+1) - b\ n\ )/a\ n$)
    and *liminf* ($\lambda n.\ a\ n\ /\ b\ n) = 0$
  shows ($\sum n.\ (b\ n\ /\ (\prod i \le n.\ a\ i))) \notin \mathbb{Q}$
⟨*proof*⟩

end

3 Some auxiliary results on the prime numbers.

lemma *nth-prime-nonzero*[*simp*]:*nth-prime n ≠ 0*
  ⟨*proof*⟩

lemma *nth-prime-gt-zero*[*simp*]:*nth-prime n >0*
  ⟨*proof*⟩

lemma *ratio-of-consecutive-primes*:
  ($\lambda n.\ nth\text{-}prime\ (n+1)/nth\text{-}prime\ n) \longrightarrow 1$
⟨*proof*⟩

lemma *nth-prime-double-sqrt-less*:
  assumes $\varepsilon > 0$
  shows ∀$_F$ *n in sequentially. (nth-prime (2∗n) − nth-prime n)*
      / sqrt (nth-prime n) < n powr (1/2+ε)

⟨*proof*⟩

4 Theorem 3.1

Theorem 3.1 is an application of Theorem 2.1 with the sequences considered involving the prime numbers.

theorem *theorem-3-10-Erdos-Straus*:
  fixes *a*::*nat* ⇒ *int*
  assumes *a-pos*:∀ *n. a n* >*0* **and** *mono a*
    and *nth-1*:(λ*n. nth-prime n* / (*a n*)^*2*) ⟶ *0*
    and *nth-2*:*liminf* (λ*n. a n* / *nth-prime n*) = *0*
  shows (∑ *n.* (*nth-prime n* / (∏ *i* ≤ *n. a i*))) ∉ ℚ
⟨*proof*⟩

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References

[1] P. Erdős and E. Straus. On the irrationality of certain series. *Pacific journal of mathematics*, 55(1):85–92, 1974.