

# **(Extended) Interval Analysis**

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## Abstract

Interval analysis (also called interval arithmetic) is a well known mathematical technique to analyse or mitigate rounding errors or measurement errors. Thus, it is promising to integrate interval analysis into program verification environments. Such an integration is not only useful for the verification of numerical algorithms: the need to ensure that computations stay within certain bounds is common. For example to show that computations stay within the hardware bounds of a given number representation.

Another application is the verification of cyber-physical systems, where a discretised implementation approximates a system described in physical quantities expressed using perfect mathematical reals, and perfect ordinary differential equations.

In this AFP entry, we formalise extended interval analysis, including the concept of inclusion isotone (or inclusion isotonic) (extended) interval analysis. The main result is the formal proof that interval-splitting converges for Lipschitz-continuous interval isotone functions. From pragmatic perspective, we provide the datatypes and theory required for integrating interval analysis into other formalisations and applications.

**Keywords:** Extended Interval Analysis, Formalising Mathematics, Isabelle/HOL



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# 1 Introduction

Interval analysis [4] in general and, in particular, inclusion isotone extended interval arithmetic [5] are well known mathematical techniques to analyse or mitigate rounding errors or measurement errors. Thus, it is promising to integrate interval analysis into program verification environments. Such an integration is not only useful for the verification of numerical algorithms: the need to ensure that computations stay within certain bounds is common. For example to show that computations stay within the hardware bounds of a given number representation.

Another application is the verification of cyber-physical systems, where a discretised implementation approximates a system described in physical quantities expressed using perfect mathematical reals, and perfect ordinary differential equations. Moreover, first applications of interval analysis to the security analysis of neural networks are promising [6, 3] and, when combined with existing works of verifying neural networks in Isabelle [2] provide a verification approach using a “correct-by-construction” verification tool for neural networks.

In this AFP entry, we formalise extended interval analysis, including the concept of inclusion isotone (extended) interval analysis. The main result is the formal proof that interval-splitting converges for Lipschitz-continuous inclusion isotone functions. From pragmatic perspective, we provide the datatypes and theory required for integrating interval analysis into other formalisations and applications. In more detail, our contributions are:

1. a conservative formalisation of (extended) interval arithmetic in Isabelle/HOL, including inclusion isotonicity;
2. we formally prove that interval-splitting converges for Lipschitz-continuous inclusion isotone functions;

From an end-user’s perspective, the main entry points into this session are the following three theories:

- `Interval_Analysis` (Chapter 10): This theory provides interval analysis over standard types such as real or integer. All operations work over (closed) intervals.
- `Extended_Interval_Analysis` (Chapter 12): This theory provides extended interval analysis over the type extended reals. All operations work over (closed) intervals.
- `Extended_Multi_Interval_Analysis` (Chapter 14): This theory provides extended multi-interval analysis over the type extended reals. All operations work over multi-intervals, i.e., lists of (closed) intervals.

The following publication [1] gives a high-level overview of this AFP entry:

A. D. Brucker, T. Cameron-Burke, and A. Stell. Formally verified interval arithmetic and its application to program verification. In 13th IEEE/ACM International Conference on Formal Methods in Software Engineering (FormaliSE 2024). IEEE, 2024.

The rest of this document is automatically generated from the formalisation in Isabelle/HOL, i.e., all content is checked by Isabelle. Overall, the structure of this document follows the theory dependencies (see Figure 1.1).

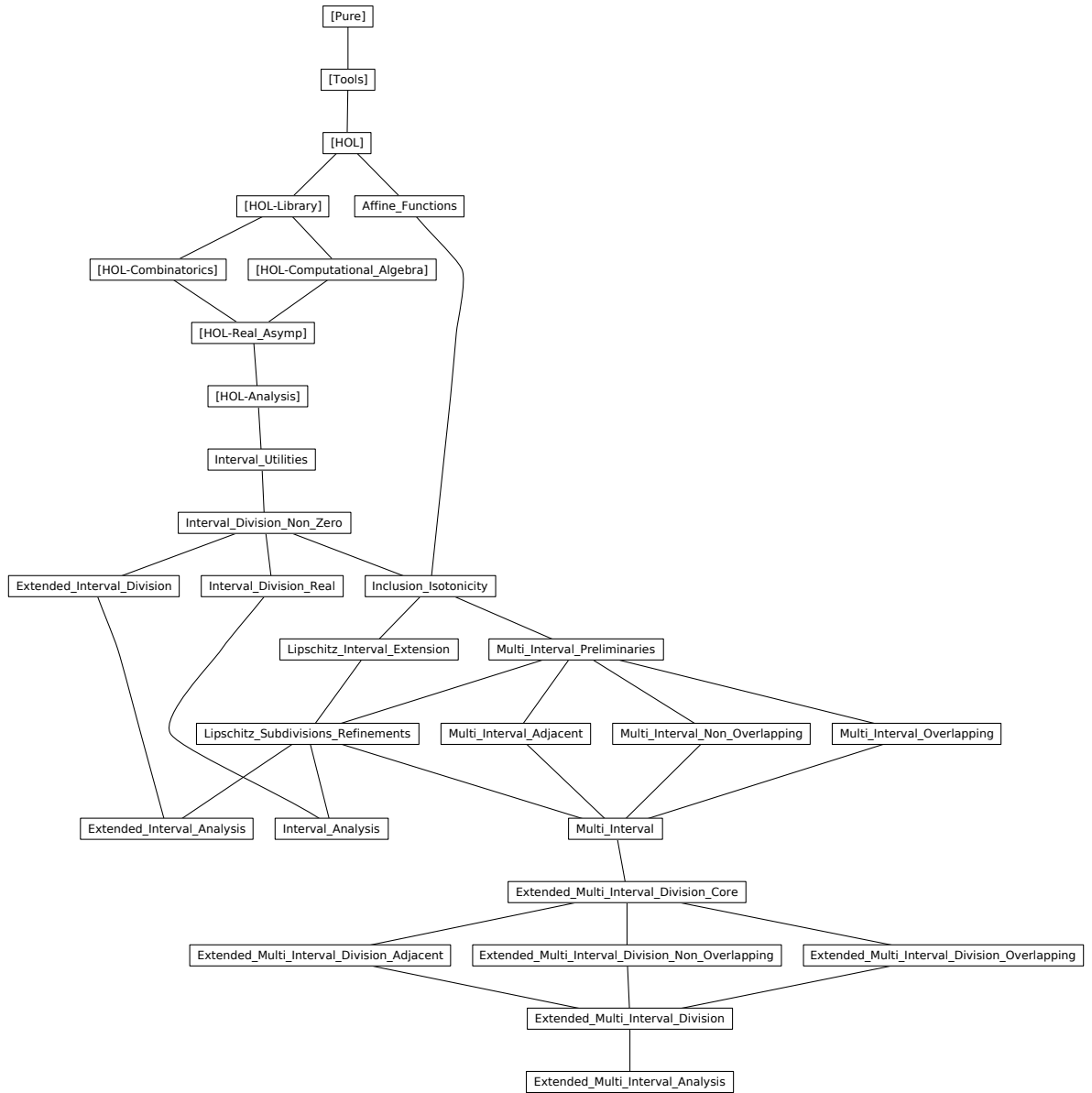


Figure 1.1: The Dependency Graph of the Isabelle Theories.

## Generated Sessions



## 2 Interval Utilities (Interval\_Utilities)

theory

Interval\_Utilities

imports

HOL—Library.Interval

HOL—Analysis.Analysis

HOL—Library.Interval\_Float

begin

### 2.1 Preliminaries

lemma compact\_set\_of:

fixes  $X :: \langle 'a :: \{preorder, topological\_space, ordered\_euclidean\_space\} \text{ interval} \rangle$

shows  $\langle compact \ (set\_of\ X) \rangle$

by(simp add:set\_of\_eq compact\_interval[of lower X upper X])

lemma bounded\_set\_of:

fixes  $X :: \langle 'a :: \{preorder, topological\_space, ordered\_euclidean\_space\} \text{ interval} \rangle$

shows  $\langle bounded \ (set\_of\ X) \rangle$

by(simp add:set\_of\_eq compact\_interval[of lower X upper X])

lemma compact\_img\_set\_of:

fixes  $X :: \langle real \ interval \rangle$  and  $f :: \langle real \Rightarrow real \rangle$

assumes  $\langle continuous\_on \ (set\_of\ X) \ f \rangle$

shows  $\langle compact \ (f \ ' \ set\_of\ X) \rangle$

using compact\_continuous\_image[of set\_of X f, simplified assms] compact\_interval

unfolding set\_of\_eq

by simp

lemma sup\_inf\_dist\_bounded:

fixes  $X :: \langle real \ set \rangle$

shows  $\langle bdd\_below\ X \implies bdd\_above\ X \implies \forall x \in X. \forall x' \in X. dist\ x\ x' \leq Sup\ X - Inf\ X \rangle$

using cInf\_lower[of \_ X] cSup\_upper[of \_ X]

apply(auto simp add:dist\_real\_def)[1]

by (smt (z3))

lemma set\_of\_nonempty[simp]:  $\langle set\_of\ X \neq \{\} \rangle$

by(simp add:set\_of\_eq)

lemma lower\_in\_interval[simp]:  $\langle lower\ X \in_i\ X \rangle$

by(simp add: in\_intervalI)

lemma upper\_in\_interval[simp]:  $\langle upper\ X \in_i\ X \rangle$

by(simp add: in\_intervalI)

lemma bdd\_below\_set\_of:  $\langle bdd\_below \ (set\_of\ X) \rangle$

by (metis atLeastAtMost\_iff bdd\_below.unfold set\_of\_eq)

```

lemma bdd_above_set_of: <bdd_above (set_of X)>
  by (metis atLeastAtMost_iff bdd_above.unfold set_of_eq)

lemma closed_set_of: <closed (set_of (X::real interval))>
  by (metis closed_real_atLeastAtMost set_of_eq)

lemma set_f_nonempty: <f ' set_of X ≠ {}>
  apply(simp add:image_def set_of_eq)
  by fastforce

lemma interval_linorder_case_split[case_names LeftOf Including RightOf]:
assumes <(upper x < c ⇒ P (x::('a::linorder interval)))>
  <(c ∈i x ⇒ P x)>
  <(c < lower x ⇒ P x)>
shows< P x>
proof(insert assms, cases upper x < c)
  case True
  then show ?thesis
    using assms(1) by blast
  next
  case False
  then show ?thesis
  proof(cases c ∈i x)
    case True
    then show ?thesis
      using assms(2) by blast
  next
  case False
  then show ?thesis
    by (meson assms(1) assms(3) in_intervall not_le_imp_less)
  qed
qed

lemma foldl_conj_True:
<foldl (∧) x xs = list_all (λ e. e = True) (x#xs)>
  by(induction xs rule:rev_induct, auto)

lemma foldl_conj_set_True:
<foldl (∧) x xs = (∀ e ∈ set (x#xs) . e = True)>
  by(induction xs rule:rev_induct, auto)

```

## 2.2 Interval Bounds and Set Conversion

```

lemma sup_set_of:
  fixes X :: 'a::{conditionally_complete_lattice} interval
  shows Sup (set_of X) = upper X
  unfolding set_of_def upper_def
  using cSup_atLeastAtMost lower_le_upper[of X]
  by (simp add: upper_def lower_def)

```

```

lemma inf_set_of:
  fixes X :: 'a::{conditionally_complete_lattice} interval

```

```

shows Inf (set_of X) = lower X
unfolding set_of_def lower_def
using cInf_atLeastAtMost lower_le_upper[of X]
by (simp add: upper_def lower_def)

```

```

lemma inf_le_sup_set_of:
  fixes X :: 'a::{conditionally_complete_lattice} interval
  shows Inf (set_of X) ≤ Sup (set_of X)
  using sup_set_of inf_set_of lower_le_upper
  by metis

```

```

lemma in_bounds: ⟨x ∈i X ⟹ lower X ≤ x ∧ x ≤ upper X⟩
  by (simp add: set_of_eq)

```

```

lemma lower_bounds[simp]:
  assumes ⟨L ≤ U⟩
  shows ⟨lower (Interval(L,U)) = L⟩
  using assms
  apply (simp add: lower.rep_eq)
  by (simp add: bounds_of_interval_eq_lower_upper lower_Interval upper_Interval)

```

```

lemma upper_bounds [simp]:
  assumes ⟨L ≤ U⟩
  shows ⟨upper (Interval(L,U)) = U⟩
  using assms
  apply (simp add: upper.rep_eq)
  by (simp add: bounds_of_interval_eq_lower_upper lower_Interval upper_Interval)

```

```

lemma lower_point_interval[simp]: ⟨lower (Interval (x,x)) = x⟩
  by (simp)

```

```

lemma upper_point_interval[simp]: ⟨upper (Interval (x,x)) = x⟩
  by (simp)

```

```

lemma map2_nth:
  assumes ⟨length xs = length ys⟩
  and ⟨n < length xs⟩
  shows ⟨(map2 f xs ys)!n = f (xs!n) (ys!n)⟩
  using assms by simp

```

```

lemma map_set: ⟨a ∈ set (map f X) ⟹ (∃ x ∈ set X . f x = a)⟩
  by auto

```

```

lemma map_pair_set_left: ⟨(a,b) ∈ set (zip (map f X) (map f Y)) ⟹ (∃ x ∈ set X . f x = a)⟩
  by (meson map_set set_zip_leftD)

```

```

lemma map_pair_set_right: ⟨(a,b) ∈ set (zip (map f X) (map f Y)) ⟹ (∃ y ∈ set Y . f y = b)⟩
  by (meson map_set set_zip_rightD)

```

```

lemma map_pair_set: ⟨(a,b) ∈ set (zip (map f X) (map f Y)) ⟹ (∃ x ∈ set X . f x = a) ∧ (∃ y ∈ set Y . f y = b)⟩
  by (meson map_pair_set_left set_zip_rightD zip_same)

```

```

lemma map_pair_f_all:
  assumes ⟨length X = length Y⟩
  shows ⟨(∀ (x,y) ∈ set (zip (map f X) (map f Y)). x ≤ y) = (∀ (x,y) ∈ set (zip X Y). f x ≤ f y)⟩
  by (insert assms(1), induction X Y rule: list_induct2, auto)

```

```

definition map_interval_swap :: ⟨('a::linorder × 'a) list ⟹ 'a interval list⟩ where

```

$\langle \text{map\_interval\_swap} = \text{map } (\lambda (x,y). \text{Interval } (\text{if } x \leq y \text{ then } (x,y) \text{ else } (y,x))) \rangle$

**definition**  $\text{mk\_interval} :: \langle 'a::\text{linorder} \times 'a \rangle \Rightarrow 'a \text{ interval} \rangle$  **where**  
 $\langle \text{mk\_interval} = (\lambda (x,y). \text{Interval } (\text{if } x \leq y \text{ then } (x,y) \text{ else } (y,x))) \rangle$

**definition**  $\text{mk\_interval}' :: \langle 'a::\text{linorder} \times 'a \rangle \Rightarrow 'a \text{ interval} \rangle$  **where**  
 $\langle \text{mk\_interval}' = (\lambda (x,y). (\text{if } x \leq y \text{ then } \text{Interval}(x,y) \text{ else } \text{Interval}(y,x))) \rangle$

**lemma**  $\text{map\_interval\_swap\_code}[\text{code}]$ :  
 $\langle \text{map\_interval\_swap} = \text{map } (\lambda (x,y). \text{the } (\text{if } x \leq y \text{ then } \text{Interval}' x y \text{ else } \text{Interval}' y x)) \rangle$   
**unfolding**  $\text{map\_interval\_swap\_def}$  **by**  $(\text{rule ext, rule arg\_cong, auto simp add: Interval'.abs\_eq})$

**lemma**  $\text{mk\_interval\_code}[\text{code}]$ :  
 $\langle \text{mk\_interval} = (\lambda (x,y). \text{the } (\text{if } x \leq y \text{ then } \text{Interval}' x y \text{ else } \text{Interval}' y x)) \rangle$   
**unfolding**  $\text{mk\_interval\_def}$  **by**  $(\text{rule ext, rule arg\_cong, auto simp add: Interval'.abs\_eq})$

**lemma**  $\text{mk\_interval}'$ :  
 $\langle \text{mk\_interval} = (\lambda (x,y). (\text{if } x \leq y \text{ then } \text{Interval}(x,y) \text{ else } \text{Interval}(y,x))) \rangle$   
**unfolding**  $\text{mk\_interval\_def}$  **by**  $(\text{rule ext, rule arg\_cong, auto})$

**lemma**  $\text{mk\_interval\_lower}[\text{simp}]$ :  $\langle \text{lower } (\text{mk\_interval } (x,y)) = (\text{if } x \leq y \text{ then } x \text{ else } y) \rangle$   
**by**  $(\text{simp add: lower\_def Interval\_inverse mk\_interval\_def})$

**lemma**  $\text{mk\_interval\_upper}[\text{simp}]$ :  $\langle \text{upper } (\text{mk\_interval } (x,y)) = (\text{if } x \leq y \text{ then } y \text{ else } x) \rangle$   
**by**  $(\text{simp add: upper\_def Interval\_inverse mk\_interval\_def})$

## 2.3 Linear Order on List of Intervals

**definition**  
 $\text{le\_interval\_list} :: \langle 'a::\text{linorder} \rangle \text{ interval list} \Rightarrow 'a \text{ interval list} \Rightarrow \text{bool} \langle \langle \_ / \leq_I \_ \rangle [51, 51] 50 \rangle$   
**where**  
 $\langle \text{le\_interval\_list } Xs Ys \equiv (\text{length } Xs = \text{length } Ys) \wedge (\text{foldl } (\wedge) \text{True } (\text{map2 } (\leq) Xs Ys)) \rangle$

**lemma**  $\text{le\_interval\_single}$ :  $\langle (x \leq y) = ([x] \leq_I [y]) \rangle$   
**unfolding**  $\text{le\_interval\_list\_def}$  **by**  $\text{simp}$

**lemma**  $\text{le\_interval\_empty}[\text{simp}]$ :  $\langle [] \leq_I [] \rangle$   
**unfolding**  $\text{le\_interval\_list\_def}$  **by**  $\text{simp}$

**lemma**  $\text{le\_interval\_list\_rev}$ :  $\langle (is \leq_I js) = (\text{rev } is \leq_I \text{rev } js) \rangle$   
**unfolding**  $\text{le\_interval\_list\_def}$   
**by**  $(\text{safe, simp\_all add: foldl\_conj\_set\_True zip\_rev})$

**lemma**  $\text{le\_interval\_list\_imp\_length}$ :  
**assumes**  $\langle Xs \leq_I Ys \rangle$  **shows**  $\langle \text{length } Xs = \text{length } Ys \rangle$   
**using**  $\text{assms}$  **unfolding**  $\text{le\_interval\_list\_def}$   
**by**  $\text{simp}$

**lemma**  $\text{lsplit\_left}$ : **assumes**  $\langle \text{length } (xs) = \text{length } (ys) \rangle$   
**and**  $\langle (\forall n < \text{length } (x \# xs). (x \# xs) ! n \leq (y \# ys) ! n) \rangle$  **shows**  $\langle$   
 $((\forall n < \text{length } xs. xs ! n \leq ys ! n) \wedge x \leq y) \rangle$   
**using**  $\text{assms}$  **by**  $\text{auto}$

```

lemma lsplit_right: assumes  $\langle \text{length } xs = \text{length } ys \rangle$ 
  and  $\langle (\forall n < \text{length } xs. xs!n \leq ys!n) \wedge x \leq y \rangle$ 
shows  $\langle n < \text{length } (x \# xs) \longrightarrow (x \# xs)!n \leq (y \# ys)!n \rangle$ 
proof(cases  $n=0$ )
  case True
  then show ?thesis using assms by(simp)
next
  case False
  then show ?thesis using assms by simp
qed

```

```

lemma lsplit: assumes  $\langle \text{length } xs = \text{length } ys \rangle$ 
shows  $\langle (\forall n < \text{length } (x \# xs). (x \# xs)!n \leq (y \# ys)!n) =$ 
   $(\langle (\forall n < \text{length } xs. xs!n \leq ys!n) \wedge x \leq y \rangle)$ 
using assms lsplit_left lsplit_right by metis

```

```

lemma le_interval_list_all':
  assumes  $\langle \text{length } Xs = \text{length } Ys \rangle$  and  $\langle Xs \leq_I Ys \rangle$  shows  $\langle \forall n < \text{length } Xs. Xs!n \leq Ys!n \rangle$ 
proof(insert assms, induction rule:list_induct2)
  case Nil
  then show ?case by simp
next
  case (Cons  $x$   $xs$   $y$   $ys$ )
  then show ?case
    using lsplit[of  $xs$   $ys$   $x$   $y$ ] le_interval_list_def
    unfolding le_interval_list_def
    by(auto simp add: foldl_conj_True)
qed

```

```

lemma le_interval_list_all2:
  assumes  $\langle \text{length } Xs = \text{length } Ys \rangle$ 
  and  $\langle \forall n < \text{length } Xs. (Xs!n \leq Ys!n) \rangle$ 
shows  $\langle Xs \leq_I Ys \rangle$ 
proof(insert assms, induction rule:list_induct2)
  case Nil
  then show ?case by simp
next
  case (Cons  $x$   $xs$   $y$   $ys$ )
  then show ?case
    using lsplit[of  $xs$   $ys$   $x$   $y$ ] le_interval_list_def
    unfolding le_interval_list_def
    by(auto simp add: foldl_conj_True)
qed

```

```

lemma le_interval_list_all:
  assumes  $\langle Xs \leq_I Ys \rangle$  shows  $\langle \forall n < \text{length } Xs. Xs!n \leq Ys!n \rangle$ 
using assms le_interval_list_all' le_interval_list_imp_length
by auto

```

```

lemma le_interval_list_imp:
  assumes  $\langle Xs \leq_I Ys \rangle$  shows  $\langle n < \text{length } Xs \longrightarrow Xs!n \leq Ys!n \rangle$ 
using assms le_interval_list_all' le_interval_list_imp_length
by auto

```

```

lemma interval_set_leq_eq:  $\langle X \leq Y \rangle = (\text{set\_of } X \subseteq \text{set\_of } Y)$ 
for  $X :: \langle 'a :: \text{linordered\_ring interval} \rangle$ 
by (simp add: less_eq_interval_def set_of_eq)

```

```

lemma times_interval_right:
fixes  $X Y C :: \langle 'a :: \text{linordered\_ring interval} \rangle$ 
assumes  $\langle X \leq Y \rangle$ 
shows  $\langle C * X \leq C * Y \rangle$ 
using assms[simplified interval_set_leq_eq]
apply(subst interval_set_leq_eq)
by (simp add: set_of_mul_inc_right)

```

```

lemma times_interval_left:
fixes  $X Y C :: \langle 'a :: \{\text{real\_normed\_algebra, linordered\_ring, linear\_continuum\_topology}\} \text{ interval} \rangle$ 
assumes  $\langle X \leq Y \rangle$ 
shows  $\langle X * C \leq Y * C \rangle$ 
using assms[simplified interval_set_leq_eq]
apply(subst interval_set_leq_eq)
by (simp add: set_of_mul_inc_left)

```

## 2.4 Support for Lists of Intervals

```

abbreviation in_interval_list:: $\langle ('a :: \text{preorder}) \text{ list} \Rightarrow 'a \text{ interval list} \Rightarrow \text{bool} \rangle$   $\langle (\_ / \in_I \_) \rangle$  [51, 51] 50
where  $\langle \text{in\_interval\_list } xs \ Xs \equiv \text{foldl } (\wedge) \ \text{True } (\text{map2 } (\text{in\_interval}) \ xs \ Xs) \rangle$ 

```

```

lemma interval_of_in_interval_list[simp]:  $\langle xs \in_I \text{ map interval\_of } xs \rangle$ 
proof(induction xs)
case Nil
then show ?case by simp
next
case (Cons a xs)
then show ?case
by (simp add: in_interval_to_interval)
qed

```

```

lemma interval_of_in_eq:  $\langle \text{interval\_of } x \leq X = (x \in_i X) \rangle$ 
by (simp add: less_eq_interval_def set_of_eq)

```

```

lemma interval_of_list_in:
assumes  $\langle \text{length inputs} = \text{length Inputs} \rangle$ 
shows  $\langle (\text{map interval\_of inputs} \leq_I \text{ Inputs}) = (\text{inputs} \in_I \text{ Inputs}) \rangle$ 
unfolding le_interval_list_def
proof(insert assms, induction inputs Inputs rule:list_induct2)
case Nil
then show ?case by simp
next
case (Cons x xs y ys)
then show ?case
proof(cases interval_of x  $\leq$  y)
case True
then show ?thesis

```

```

    using Cons by (simp add: interval_of_in_eq)
next
case False
then show ?thesis using foldl_conj_True interval_of_in_eq by auto
qed
qed

```

## 2.5 Interval Width and Arithmetic Operations

```

lemma interval_width_addition:
fixes A :: 'a::{linordered_ring} interval
shows <width (A + B) = width A + width B>
by (simp add: width_def)

```

```

lemma interval_width_times:
fixes a :: 'a::{linordered_ring} and A :: 'a interval
shows width (interval_of a * A) = |a| * width A
proof -
have width (interval_of a * A) = upper (interval_of a * A) - lower (interval_of a * A)
by (simp add: width_def)
also have ... = (if a > 0 then a * upper A - a * lower A else a * lower A - a * upper A)
proof -
have upper (interval_of a * A) = max (a * lower A) (a * upper A)
by (simp add: upper_times)
moreover have lower (interval_of a * A) = min (a * lower A) (a * upper A)
by (simp add: lower_times)
ultimately show ?thesis
by (simp add: mult_left_mono mult_left_mono_neg)
qed
also have ... = |a| * (upper A - lower A)
by (simp add: right_diff_distrib)
finally show ?thesis
by (simp add: width_def)
qed

```

```

lemma interval_sup_width:
fixes X Y :: 'a::{linordered_ring, lattice} interval
shows width (sup X Y) = max (upper X) (upper Y) - min (lower X) (lower Y)
proof -
have a0:  $\forall i \text{ ia. lower (sup i ia) = min (lower i::real) (lower ia)}$ 
by (simp add: inf_min)
have a1:  $\forall i \text{ ia. upper (sup i ia) = max (upper i::real) (upper ia)}$ 
by (simp add: sup_max)
show ?thesis
using a0 a1 by (simp add: inf_min sup_max width_def)
qed

```

```

lemma width_expanded: <interval_of (width Y) = Interval(upper Y - lower Y, upper Y - lower Y)>
unfolding width_def interval_of_def by simp

```

```

lemma interval_width_positive:
fixes X :: 'a::{linordered_ring} interval
shows <0 ≤ width X>

```

```
using lower_le_upper
by (simp add: width_def)
```

## 2.6 Interval Multiplication

```
lemma interval_interval_times:
  ⟨X * Y = Interval (Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)},
    Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)})⟩
  by (simp add: times_interval_def bounds_of_interval_eq_lower_upper Let_def min_def)
```

```
lemma interval_times_scalar: ⟨interval_of a * A = mk_interval(a * lower A, a * upper A)⟩
  proof –
    have upper (interval_of a * A) = max (a * lower A) (a * upper A)
      by (simp add: upper_times)
    moreover have lower (interval_of a * A) = min (a * lower A) (a * upper A)
      by (simp add: lower_times)
    ultimately show ?thesis
      by (simp add: interval_eq_iff)
  qed
```

```
lemma interval_times_scalar_pos_l:
  fixes a :: 'a::{ordered_semiring,times,linorder}
  assumes ‹0 ≤ a›
  shows ⟨interval_of a * A = Interval(a * lower A, a * upper A)⟩
  proof –
    have upper (interval_of a * A) = a * upper A
      using assms
      by (simp add: upper_times mult_left_mono)
    moreover have lower (interval_of a * A) = a * lower A
      using assms
      by (simp add: lower_times mult_left_mono)
    ultimately show ?thesis
      using assms
      by (metis bounds_of_interval_eq_lower_upper bounds_of_interval_inverse lower_le_upper)
  qed
```

```
lemma interval_times_scalar_pos_r:
  fixes a :: 'a::{linordered_idom}
  assumes ‹0 ≤ a›
  shows ⟨A * interval_of a = Interval(a * lower A, a * upper A)⟩
  proof –
    have upper (A * interval_of a) = a * upper A
      using assms
      by (simp add: mult commute mult_left_mono upper_times)
    moreover have lower (A * interval_of a) = a * lower A
      using assms
      by (simp add: lower_times mult_right_mono)
    ultimately show ?thesis
      using assms
      by (metis bounds_of_interval_eq_lower_upper bounds_of_interval_inverse lower_le_upper)
  qed
```

## 2.7 Distance-based Properties of Intervals

Given two real intervals  $X$  and  $Y$ , and two real numbers  $a$  and  $b$ , the width of the sum of the scaled intervals is equivalent to the width of the two individual intervals.

**lemma** *width\_of\_scaled\_interval\_sum*:

**fixes**  $X :: 'a::\{\text{linordered\_ring}\}$  interval

**shows**  $\langle \text{width} (\text{interval\_of } a * X + \text{interval\_of } b * Y) = |a| * \text{width } X + |b| * \text{width } Y \rangle$

**using** *interval\_width\_addition interval\_width\_times* **by** *metis*

**lemma** *width\_of\_product\_interval\_bound\_real*:

**fixes**  $X :: \text{real interval}$

**shows**  $\langle \text{interval\_of} (\text{width } (X * Y)) \leq \text{abs\_interval}(X) * \text{interval\_of} (\text{width } Y) + \text{abs\_interval}(Y) * \text{interval\_of} (\text{width } X) \rangle$

**proof** –

**have**  $a_0: \text{lower } X \leq \text{upper } X \langle \text{lower } Y \leq \text{upper } Y \rangle$

**using** *lower\_le\_upper* **by** *simp\_all*

**have**  $a_1: \langle \text{width } Y \geq 0 \rangle$  **and**  $a_1': \langle \text{width } X \geq 0 \rangle$

**using**  $a_0$  *interval\_width\_positive* **by** *simp\_all*

**have**  $a_2: \langle \text{abs\_interval}(X) * \text{interval\_of} (\text{width } Y) = \text{Interval} (\text{width } Y * \text{lower} (\text{abs\_interval } X), \text{width } Y * \text{upper} (\text{abs\_interval } X)) \rangle$

**using** *interval\_times\_scalar\_pos\_r*  $a_0$   $a_1$  **by** *blast*

**have**  $a_3: \langle \text{abs\_interval}(Y) * \text{interval\_of} (\text{width } X) = \text{Interval} (\text{width } X * \text{lower} (\text{abs\_interval } Y), \text{width } X * \text{upper} (\text{abs\_interval } Y)) \rangle$

**using** *interval\_times\_scalar\_pos\_r*  $a_0$  *interval\_width\_positive* **by** *blast*

**have**  $a_4: \langle \text{abs\_interval}(X) * \text{interval\_of} (\text{width } Y) + \text{abs\_interval}(Y) * \text{interval\_of} (\text{width } X) = \text{Interval} (\text{width } Y * \text{lower} (\text{abs\_interval } X) + \text{width } X * \text{lower} (\text{abs\_interval } Y), \text{width } Y * \text{upper} (\text{abs\_interval } X) + \text{width } X * \text{upper} (\text{abs\_interval } Y)) \rangle$

**using**  $a_0$   $a_2$   $a_3$  *unfolding plus\_interval\_def*

**apply** (*simp add: bounds\_of\_interval\_eq\_lower\_upper mult\_left\_mono interval\_width\_positive*)

**by** (*auto simp add: bounds\_of\_interval\_eq\_lower\_upper mult\_left\_mono interval\_width\_positive*)

**have**  $a_5: \langle \text{width}(X * Y) = \text{Max} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} - \text{Min} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} \rangle$

**using** *lower\_times upper\_times unfolding width\_def* **by** *metis*

**have**  $a_6: \langle \text{Max} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} - \text{Min} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} \leq \text{width } Y * \text{upper} (\text{abs\_interval } X) + \text{width } X * \text{upper} (\text{abs\_interval } Y) \rangle$

**using**  $a_0$   $a_1$   $a_1'$  *unfolding width\_def*

**by** (*simp, smt (verit) a\_0 mult.commute mult\_less\_cancel\_left\_pos mult\_minus\_left right\_diff\_distrib*)

**have**  $a_7: \langle \text{width } Y * \text{lower} (\text{abs\_interval } X) + \text{width } X * \text{lower} (\text{abs\_interval } Y) \leq \text{Max} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} - \text{Min} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} \rangle$

**using**  $a_0$   $a_1$   $a_1'$  *unfolding width\_def*

**by** (*simp, smt (verit) a\_0 mult.commute mult\_less\_cancel\_left\_pos mult\_minus\_left right\_diff\_distrib*)

**have**  $\langle \text{interval\_of} (\text{Max} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\} - \text{Min} \{(\text{lower } X * \text{lower } Y), (\text{lower } X * \text{upper } Y), (\text{upper } X * \text{lower } Y), (\text{upper } X * \text{upper } Y)\}) \leq$

$\text{Interval} (\text{width } Y * \text{lower} (\text{abs\_interval } X) + \text{width } X * \text{lower} (\text{abs\_interval } Y),$

$\text{width } Y * \text{upper} (\text{abs\_interval } X) + \text{width } X * \text{upper} (\text{abs\_interval } Y)) \rangle$

**using**  $a_6$   $a_7$  *unfolding less\_eq\_interval\_def*

**by** (*smt (verit, ccfv\_threshold) lower\_bounds lower\_interval\_of upper\_bounds upper\_interval\_of*)

**then show** *?thesis* **using**  $a_4$   $a_5$   $a_6$  **by** *presburger*

**qed**

**lemma** *width\_of\_product\_interval\_bound\_int*:

```

fixes X :: int interval
shows <interval_of (width (X * Y)) ≤ abs_interval(X) * interval_of (width Y) + abs_interval(Y) * interval_of (width X)>
proof –
  have a0: lower X ≤ upper X <lower Y ≤ upper Y>
    using lower_le_upper by simp_all
  have a1: <width Y ≥ 0> and <width X ≥ 0>
    using a0 interval_width_positive by simp_all
  have a2: <abs_interval(X) * interval_of (width Y) = Interval (width Y * lower (abs_interval X), width Y * upper
(abs_interval X))>
    using interval_times_scalar_pos_r a0 a1 by blast
  have a3: <abs_interval(Y) * interval_of (width X) = Interval (width X * lower (abs_interval Y), width X * upper
(abs_interval Y))>
    using interval_times_scalar_pos_r a0 interval_width_positive by blast
  have a4: <abs_interval(X) * interval_of (width Y) + abs_interval(Y) * interval_of (width X) = Interval (width Y * lower
(abs_interval X) + width X * lower (abs_interval Y), width Y * upper (abs_interval X) + width X * upper (abs_interval
Y))>
    using a0 a2 a3 unfolding plus_interval_def
    by (auto simp add: bounds_of_interval_eq_lower_upper mult_left_mono interval_width_positive)
  have a5: <width(X * Y) = Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} –
Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)}>
    using lower_times_upper_times unfolding width_def by metis
  have a6: <Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} – Min {(lower X *
lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} ≤ width Y * upper (abs_interval X) + width X
* upper (abs_interval Y)>
    using a0 unfolding width_def
    by (simp, smt (verit) a0 mult_commute mult_less_cancel_left_pos mult_minus_left right_diff_distrib)
  have a7: <width Y * lower (abs_interval X) + width X * lower (abs_interval Y) ≤ Max {(lower X * lower Y), (lower X *
upper Y), (upper X * lower Y), (upper X * upper Y)} – Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower
Y), (upper X * upper Y)}>
    using a0 unfolding width_def
    apply (auto)[1]
    by (smt (verit) a0(1) a0(2) left_diff_distrib mult_less_cancel_left_pos int_distrib(3)
mult_less_cancel_left mult_minus_right mult_commute mult_le_o_iff right_diff_distrib'
mult_left_mono mult_le_cancel_right right_diff_distrib zero_less_mult_iff )+
  have <interval_of (Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} –
Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)}) ≤
Interval (width Y * lower (abs_interval X) + width X * lower (abs_interval Y),
width Y * upper (abs_interval X) + width X * upper (abs_interval Y))>
    using a6 a7 unfolding less_eq_interval_def
    by (smt (verit, ccfv_threshold) lower_bounds lower_interval_of upper_bounds upper_interval_of)
  then show ?thesis using a4 a5 a6 by presburger
qed

end

```

## 3 Basic Properties of Interval Division

### (Interval\_Division\_Non\_Zero)

theory

Interval\_Division\_Non\_Zero

imports

Interval\_Utilities

begin

The theory *HOL—Library.Interval* does not define a division operation on intervals. In the following we define a locale capturing the core properties of division by an interval that does not contain zero.

### 3.1 Preliminaries

lemma *division\_leq\_neg*:

fixes  $x :: 'a::\{linordered\_field\}$

assumes  $0 < x$  and  $y < 0$  and  $z < 0$  and  $y \leq z$

shows  $x / z \leq x / y$

proof —

have  $x * y \leq x * z$  using *assms* by *simp*

hence  $(x * y) / (y * z) \leq (x * z) / (y * z)$

using *assms*

by (*simp add: divide\_right\_mono zero\_less\_mult\_iff neg\_divide\_le\_eq*)

thus ?thesis using *assms* by *auto*[1]

qed

lemma *division\_leq*:

fixes  $x :: 'a::\{linordered\_field\}$

assumes  $0 < x$  and  $y \leq z$  and  $\langle y \neq 0 \wedge z \neq 0 \rangle$  and  $\langle (y < 0 \wedge z < 0) \vee (0 < y \wedge 0 < z) \rangle$

shows  $x / z \leq x / y$

proof (cases  $\langle (y < 0 \wedge z < 0) \rangle$ )

case *True*

then show ?thesis using *assms* *division\_leq\_neg* by *blast*

next

case *False*

have  $\langle (0 < y \wedge 0 < z) \rangle$  using *assms* *False* by *blast*

then show ?thesis

using *assms*

by (*simp add: frac\_le*)

qed

lemma *upper\_leq\_lower\_div*:

fixes  $Y :: 'a::\{linordered\_field\}$  interval

assumes  $\langle \text{lower } Y \leq \text{upper } Y \rangle$  and  $\langle \neg 0 \in_i Y \rangle$

shows  $\langle 1 / \text{upper } Y \leq 1 / \text{lower } Y \rangle$

using *assms* *division\_leq* *frac\_le*

by (*metis atLeastAtMost\_iff inverse\_eq\_divide le\_imp\_inverse\_le*)

`le_imp_inverse_le_neg linorder_not_less set_of_eq)`

### 3.2 A Locale for Interval Division Where the Quotient-Interval does not Contain Zero

```

locale interval_division = inverse +
  constrains inverse :: <'a::{linordered_field, real_normed_algebra, linear_continuum_topology} interval  $\Rightarrow$  'a interval>
    and divide :: <'a::{linordered_field, real_normed_algebra, linear_continuum_topology} interval  $\Rightarrow$  'a interval  $\Rightarrow$  'a interval>
  assumes inverse_left: < $\neg 0 \in_i x \Longrightarrow 1 \leq (\text{inverse } x) * x$ >
    and divide: < $\neg 0 \in_i y \Longrightarrow x \leq (\text{divide } x y) * y$ >
begin
end

```

```

lemma interval_non_zero_eq:
  < $\neg 0 \in_i (i::'a::\{\text{linorder, zero}\} \text{interval}) = (\text{lower } i < 0 \wedge \text{upper } i < 0) \vee (\text{lower } i > 0 \wedge \text{upper } i > 0)$ >
by (metis in_bounds in_interval linorder_not_less lower_le_upper order_le_less_trans)

```

```

lemma inverse_includes_one:
  assumes < $\neg 0 \in_i (i::'a::\{\text{division\_ring, linordered\_ring}\} \text{interval})$ >
  shows < $1 \in_i (\text{mk\_interval } (1 / \text{upper } i, 1 / \text{lower } i)) * i$ >
  using assms interval_non_zero_eq[of i]
  apply(simp add: set_of_eq)
  apply(safe, simp_all)
by (metis in_bounds lower_in_interval mk_interval_upper nonzero_eq_divide_eq times_in_interval upper_in_interval)+

```

```

lemma inverse_includes_one':
  assumes < $\neg 0 \in_i (i::'a::\{\text{division\_ring, linordered\_ring}\} \text{interval})$ >
  shows < $1 \leq (\text{mk\_interval } (1 / \text{upper } i, 1 / \text{lower } i)) * i$ >
by (simp add: assms in_bounds inverse_includes_one less_eq_interval_def)

```

```

locale interval_division_inverse = inverse +
  constrains inverse :: <'a::{linordered_field, real_normed_algebra, linear_continuum_topology} interval  $\Rightarrow$  'a interval>
    and divide :: <'a::{linordered_field, real_normed_algebra, linear_continuum_topology} interval  $\Rightarrow$  'a interval  $\Rightarrow$  'a interval>
  assumes inverse_non_zero_def: < $\neg 0 \in_i x \Longrightarrow (\text{inverse } x) = \text{mk\_interval}(1 / (\text{upper } x), 1 / (\text{lower } x))$ >
    and divide_non_zero_def: < $\neg 0 \in_i y \Longrightarrow (\text{divide } x y) = \text{inverse } y * x$ >
begin

```

```

sublocale interval_division divide inverse
  apply(standard)
  subgoal
    by (simp add: inverse_includes_one' inverse_non_zero_def)
  subgoal
    by (metis (no_types, opaque_lifting) divide_non_zero_def interval_mul_commute inverse_includes_one' inverse_non_zero_def mult.assoc one_times_ivl_right times_interval_right)
  done

```

```

lemma inverse_left_ge_one:
  assumes < $\neg 0 \in_i x$ >
  shows < $1 \leq (\text{inverse } x) * x$ >

```

```

proof –
  have lower_ne_zero: ⟨lower x ≠ 0⟩
    using assms lower_in_interval by metis
  have upper_ne_zero: ⟨lower x ≠ 0⟩
    using assms lower_in_interval by metis
  have ⟨1 ≤ (mk_interval (1 / (upper x), 1 / (lower x))) * x⟩
  proof(cases 1 / upper x ≤ 1 / lower x)
  case True note * = this
  then show ?thesis
  proof(cases upper x = lower x)
  case True
  then show ?thesis
    using upper_times[of mk_interval (1 / upper x, 1 / lower x) x]
      lower_times[of mk_interval (1 / upper x, 1 / lower x) x]
      interval_eq_iff[of mk_interval (1 / upper x, 1 / lower x) * x 1]
      lower_ne_zero upper_ne_zero
    unfolding mk_interval'
    by simp
  next
  case False
  then show ?thesis
    using interval_eq_iff[of mk_interval (1 / upper x, 1 / lower x) * x 1]
      upper_times[of mk_interval (1 / upper x, 1 / lower x) x]
      lower_times[of mk_interval (1 / upper x, 1 / lower x) x]
  proof –
  have 1 / lower x = upper (mk_interval (1 / upper x, 1 / lower x))
    by (simp add: *)
  then show ?thesis
    by (metis (no_types) in_bounds less_eq_interval_def lower_in_interval lower_one
      nonzero_divide_eq_eq times_in_interval upper_in_interval upper_ne_zero upper_one)
  qed
qed
next
  case False
  then show ?thesis
    using interval_eq_iff[of mk_interval (1 / upper x, 1 / lower x) * x 1]
      upper_times[of mk_interval (1 / upper x, 1 / lower x) x]
      lower_times[of mk_interval (1 / upper x, 1 / lower x) x]
    using assms lower_le_upper upper_leq_lower_div by blast
  qed
  then show ?thesis
by (simp add: assms inverse_non_zero_def)
qed

lemma division_right_ge_refl:
assumes ⟨¬ 0 ∈i y⟩
shows ⟨x ≤ x * ((inverse y) * y)⟩
proof –
have a1: ⟨set_of 1 ⊆ set_of ((inverse y) * y)⟩
  using inverse_left_ge_one[of y, simplified assms, simplified]
  by (simp add: interval_set_leq_eq)
show ?thesis
  using set_of_mul_inc_right[of 1 mk_interval (1 / upper y, 1 / lower y) * y x,
    simplified one_times_ivl_right[of x] a1, simplified]

```

by (metis a1 interval\_set\_leq\_eq one\_times\_ivl\_right times\_interval\_right)  
qed

lemma division\_right\_ge\_refl':  
assumes  $\langle \neg 0 \in_i y \rangle$   
shows  $\langle x \leq x * \text{inverse } y * y \rangle$   
by (simp add: assms division\_right\_ge\_refl mult.assoc)

lemma interval\_div\_constant:  
assumes  $\langle 0 \notin \text{set\_of } Y \rangle$  and  $\langle 0 \leq x \rangle$   
shows  $\langle \text{divide } (\text{interval\_of } x) \ Y = \text{Interval}(x / \text{upper } Y, x / \text{lower } Y) \rangle$   
proof –  
have l:  $\langle \text{lower } Y \leq \text{upper } Y \rangle$  using lower\_le\_upper by simp  
have  $\langle 1 / \text{upper } Y \leq 1 / \text{lower } Y \rangle$  using assms l  
by (metis divide\_left\_mono frac\_le in\_intervall linorder\_not\_less mult\_neg\_neg order\_less\_le zero\_less\_one\_class.zero\_le\_one)  
then show ?thesis  
using  
interval\_of.abs\_eq[of x]  
assms divide\_non\_zero\_def[of Y interval\_of x, simplified] assms, simplified  
inverse\_non\_zero\_def[of Y, simplified] assms, simplified  
interval\_times\_scalar\_pos\_l interval\_times\_scalar\_pos\_r by fastforce  
qed

lemma interval\_of\_width:  
assumes  $\langle \neg 0 \in_i Y \rangle$   
shows  $\langle \text{interval\_of}(\text{width } (\text{divide } (\text{interval\_of } 1) \ Y)) = \text{Interval}(1 / \text{lower } Y - 1 / \text{upper } Y, 1 / \text{lower } Y - 1 / \text{upper } Y) \rangle$   
proof(cases Y rule:interval\_linorder\_case\_split[of \_ 0  $\lambda$  Y. interval\_of(width (divide (interval\_of 1) Y))  
= Interval(1 / lower Y - 1 / upper Y, 1 / lower Y - 1 / upper Y) ])  
case LeftOf  
have  $\langle 1 / \text{upper } Y \leq 1 / \text{lower } Y \rangle$   
using assms division\_leq\_neg LeftOf  
by (simp add: le\_divide\_eq)  
then show ?case  
using interval\_div\_constant upper\_bounds lower\_bounds assms  
unfolding width\_def interval\_of\_def by fastforce  
next  
case Including  
then show ?case using assms by simp  
next  
case RightOf  
have  $\langle 1 / \text{upper } Y \leq 1 / \text{lower } Y \rangle$   
by (simp add: assms upper\_leq\_lower\_div)  
then show ?case  
using interval\_div\_constant upper\_bounds lower\_bounds assms  
unfolding width\_def interval\_of\_def by fastforce  
qed

lemma abs\_pos:  
assumes  $\langle 0 < \text{lower } Y \rangle$  and  $\langle \neg 0 \in_i Y \rangle$   
shows  $\langle \text{abs\_interval}(\text{divide } (\text{interval\_of } 1) \ Y) = \text{Interval}(1 / \text{upper } Y, 1 / \text{lower } Y) \rangle$   
proof –  
have l:  $\langle \text{lower } Y \leq \text{upper } Y \rangle$  using lower\_le\_upper by simp  
have  $\langle 0 < 1 / \text{upper } Y \rangle$

```

    by (metis assms(1) l dual_order.strict_trans1 zero_less_divide_1_iff)
  moreover have < 0 < 1 / lower Y >
    by (metis assms(1) zero_less_divide_1_iff)
  moreover have < 1 / upper Y ≤ 1 / lower Y >
    using assms by (simp add: frac_le)
  moreover have < divide (interval_of 1) Y = Interval( 1 / upper Y, 1 / lower Y) >
    using assms interval_div_constant[of Y 1] by simp
  ultimately show ?thesis
    unfolding abs_interval_def by (simp add: bounds_of_interval_eq_lower_upper)
qed

```

lemma abs\_neg:

```

  assumes < upper Y < 0 > and < ¬ 0 ∈i Y >
  shows < abs_interval(divide (interval_of 1) Y) = Interval(1 / |lower Y|, 1 / |upper Y|) >
proof -
  have l: < lower Y ≤ upper Y > using lower_le_upper by simp
  have i0: < 1 / upper Y < 0 > and i1: < 1 / lower Y < 0 >
    using assms by (simp, meson assms(1) divide_less_0_1_iff lower_le_upper order_le_less_trans)
  moreover have i2: < |upper Y| ≤ |lower Y| >
    using assms l by linarith
  then have i3: < |1 / lower Y| ≤ |1 / upper Y| >
    using assms division_leq_neg i1
    by (simp add: division_leq)
  moreover have < divide (interval_of 1) Y = Interval( 1 / upper Y, 1 / lower Y) >
    using assms interval_div_constant[of Y 1] by simp
  moreover have < abs_interval(Interval( 1 / upper Y, 1 / lower Y)) = Interval(|1 / lower Y|, |1 / upper Y|) >
    using assms i0 i1 i2 i3 unfolding abs_interval_def min_def max_def
    by (simp add: bounds_of_interval_eq_lower_upper)
  moreover have < ... = Interval(1 / |lower Y|, 1 / |upper Y|) >
    by auto[1]
  ultimately show ?thesis
    using assms interval_div_constant by force
qed

```

end

end



## 4 A Naive Interval Division for Real Intervals

### (Interval\_Division\_Real)

theory

Interval\_Division\_Real

imports

Interval\_Division\_Non\_Zero

begin

The theory *HOL—Library.Interval* does not define a division operation on intervals. Actually, in the following we define division in a straight forward way. This is possible, as in HOL, the property  $?a / o = o$  holds. Therefore, we do not need to use, in the first instance, extended interval analysis (e.g., based on the type *ereal*). As a consequence, results obtained using this definition might differ from results obtained using definitions of divisions using extended reals (e.g., [4]).

instantiation interval :: (*{linordered\_field, real\_normed\_algebra, linear\_continuum\_topology}*) inverse

begin

definition inverse\_interval :: 'a interval  $\Rightarrow$  'a interval

where inverse\_interval a = mk\_interval (1 / (upper a), 1 / (lower a))

definition divide\_interval :: 'a interval  $\Rightarrow$  'a interval  $\Rightarrow$  'a interval

where divide\_interval a b = inverse b \* a

instance ..

end

interpretation interval\_division\_inverse divide inverse

apply (unfold\_locales)

subgoal by (simp add: inverse\_interval\_def)

subgoal by (simp add: divide\_interval\_def)

done

lemma width\_of\_reciprocal\_interval\_bound\_real:

fixes Y :: real interval

assumes  $\langle \neg 0 \in_i Y \rangle$

shows  $\langle \text{interval\_of}(\text{width}((\text{interval\_of } 1) / Y)) \leq$

$(\text{abs\_interval}((\text{interval\_of } 1) / Y) * \text{abs\_interval}((\text{interval\_of } 1) / Y)) * \text{interval\_of}(\text{width } Y) \rangle$

proof (cases Y rule: interval\_linorder\_case\_split[of \_ 0  $\lambda$  Y. interval\_of(width((interval\_of 1) / Y))  $\leq$

$(\text{abs\_interval}((\text{interval\_of } 1) / Y)) * (\text{abs\_interval}((\text{interval\_of } 1) / Y)) * \text{interval\_of}(\text{width } Y) ])$

case LeftOf

have a0:  $\langle \text{lower } Y \leq \text{upper } Y \rangle$  using lower\_le\_upper by simp

have a1:  $\langle \text{interval\_of}(\text{width}((\text{interval\_of } 1) / Y)) = \text{Interval}(1 / \text{lower } Y - 1 / \text{upper } Y, 1 / \text{lower } Y - 1 / \text{upper } Y) \rangle$

using assms interval\_of\_width by blast

have a2:  $\langle (\text{abs\_interval}((\text{interval\_of } 1) / Y) * \text{abs\_interval}((\text{interval\_of } 1) / Y)) * \text{interval\_of}(\text{width } Y) = \text{Interval}((\text{upper } Y - \text{lower } Y) * 1 / |\text{lower } Y| * 1 / |\text{lower } Y|, (\text{upper } Y - \text{lower } Y) * 1 / |\text{upper } Y| * 1 / |\text{upper } Y|) \rangle$

proof –

have b0:  $\langle 1 / |\text{lower } Y| \leq 1 / |\text{upper } Y| \rangle$  using assms a0 LeftOf

by (smt (verit, best) frac\_le)

have b1:  $\langle 0 \leq \text{upper } Y - \text{lower } Y \rangle$

using assms LeftOf by simp

```

moreover have b2: ⟨(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) =
Interval(1/|lower Y|, 1/|upper Y|) * Interval(1/|lower Y|, 1/|upper Y|) * Interval(upper Y - lower Y, upper Y - lower Y)⟩
using assms LeftOf abs_neg[of Y] width_expanded[of Y] by simp
moreover have b3: ⟨... = Interval(Min { (1 / |lower Y| * 1 / |lower Y|), (1 / |lower Y| * 1 / |upper Y|), (1 / |upper Y| * 1
/ |lower Y|), (1 / |upper Y| * 1 / |upper Y|) }, Max { (1 / |lower Y| * 1 / |lower Y|), (1 / |lower Y| * 1 / |upper Y|), (1 / |upper
Y| * 1 / |lower Y|), (1 / |upper Y| * 1 / |upper Y|) }) * Interval(upper Y - lower Y, upper Y - lower Y)⟩
using assms ao LeftOf interval_interval_times[of Interval(1/|lower Y|, 1/|upper Y|) Interval(1/|lower Y|, 1/|upper Y|)]
using lower_bounds upper_bounds bo by simp
have b4: ⟨Min { (1 / |lower Y| * 1 / |lower Y|), (1 / |lower Y| * 1 / |upper Y|), (1 / |upper Y| * 1 / |lower Y|), (1 / |upper Y|
* 1 / |upper Y|) } = (1 / |lower Y| * 1 / |lower Y|)⟩
using assms ao LeftOf upper_leq_lower_div
apply (simp add: min_def, safe)
apply (smt (z3) divide_divide_eq_left' divide_less_cancel divide_minus_right nonzero_minus_divide_divide)
apply argo+
by (smt (z3) ao divide_divide_eq_left' divide_le_cancel divide_nonneg_neg frac_le minus_divide_right mult commute
mult_left_mono mult_minus_left)
have b5: ⟨Max { (1 / |lower Y| * 1 / |lower Y|), (1 / |lower Y| * 1 / |upper Y|), (1 / |upper Y| * 1 / |lower Y|), (1 / |upper Y|
* 1 / |upper Y|) } = (1 / |upper Y| * 1 / |upper Y|)⟩
using assms ao LeftOf upper_leq_lower_div
apply (simp add: max_def, safe)
apply (smt (verit, del_insts) frac_le mult_mono_nonpos_nonpos mult_neg_neg)
apply argo
apply argo
apply (smt (z3) bo divide_divide_eq_left' divide_less_cancel minus_divide_right nonzero_minus_divide_divide)
apply (smt (verit, best) division_leq mult_mono_nonpos_nonpos not_real_square_gt_zero)
by argo+
moreover have ⟨(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) = Inter-
val(1 / |lower Y| * 1 / |lower Y|, 1 / |upper Y| * 1 / |upper Y|) * Interval(upper Y - lower Y, upper Y - lower Y)⟩
using b2 b3 b4 b5 by presburger
moreover have ⟨... = Interval((upper Y - lower Y) * 1 / |lower Y| * 1 / |lower Y|, (upper Y - lower Y) * 1 / |upper Y| *
1 / |upper Y|)⟩
using assms ao LeftOf b4 b5 interval_times_scalar_pos_r[of upper Y - lower Y Interval(1 / |lower Y| * 1 / |lower Y|, 1
/ |upper Y| * 1 / |upper Y|)]
unfolding interval_of_def by simp
ultimately show ?thesis by metis
qed
have a3: ⟨(upper Y - lower Y) * 1 / |lower Y| * 1 / |lower Y| ≤ 1 / lower Y - 1 / upper Y⟩
proof -
have (1 / lower Y) - (1 / upper Y) = (upper Y - lower Y) / (lower Y * upper Y)
using assms ao LeftOf
by (smt (verit) diff_divide_distrib nonzero_divide_mult_cancel_left nonzero_divide_mult_cancel_right)
moreover have ⟨(upper Y - lower Y) * 1 / |lower Y| * 1 / |lower Y| = (upper Y - lower Y) / |lower Y|^2⟩
using assms LeftOf by (simp add: power2_eq_square)
moreover have ⟨((upper Y - lower Y) / |lower Y|^2) ≤ (upper Y - lower Y) / (lower Y * upper Y)⟩
using assms ao LeftOf
by (smt (verit) frac_le minus_mult_minus mult_left_mono_neg mult_neg_neg power2_eq_square)
ultimately show ?thesis by metis
qed
have a4: ⟨1 / lower Y - 1 / upper Y ≤ (upper Y - lower Y) * 1 / |upper Y| * 1 / |upper Y|⟩
proof -
have (1 / lower Y) - (1 / upper Y) = (upper Y - lower Y) / (lower Y * upper Y)
using assms ao LeftOf
by (smt (verit) diff_divide_distrib nonzero_divide_mult_cancel_left nonzero_divide_mult_cancel_right)
moreover have ⟨(upper Y - lower Y) * 1 / |upper Y| * 1 / |upper Y| = (upper Y - lower Y) / |upper Y|^2⟩

```

```

    using assms LeftOf by (simp add: power2_eq_square)
    moreover have  $\langle (upper\ Y - lower\ Y) / (lower\ Y * upper\ Y) \leq ((upper\ Y - lower\ Y) / |upper\ Y|^2) \rangle$ 
    using assms LeftOf
    by (simp add: frac_le power2_eq_square zero_less_mult_iff)
    ultimately show ?thesis by metis
qed
have  $\langle Interval(1 / lower\ Y - 1 / upper\ Y, 1 / lower\ Y - 1 / upper\ Y) \leq Interval((upper\ Y - lower\ Y) * 1 / |lower\ Y| * 1 / |lower\ Y|, (upper\ Y - lower\ Y) * 1 / |upper\ Y| * 1 / |upper\ Y|) \rangle$ 
    using a3 a4 unfolding less_eq_interval_def by simp
then show ?case using a1 a2 a3 a4 by simp
next
case Including
then show ?case using assms by simp
next
case RightOf
have ao:  $\langle lower\ Y \leq upper\ Y \rangle$  using lower_le_upper by simp
have a1:  $\langle interval\_of(width\ ((interval\_of\ 1) / Y)) = Interval(1 / lower\ Y - 1 / upper\ Y, 1 / lower\ Y - 1 / upper\ Y) \rangle$ 
    using assms interval_of_width by blast
have a2:  $\langle (abs\_interval((interval\_of\ 1) / Y) * abs\_interval((interval\_of\ 1) / Y)) * interval\_of(width\ Y) = Interval((upper\ Y - lower\ Y) * 1 / upper\ Y * 1 / upper\ Y, (upper\ Y - lower\ Y) * 1 / lower\ Y * 1 / lower\ Y) \rangle$ 
    proof -
    have bo:  $\langle 0 \leq upper\ Y - lower\ Y \rangle$ 
    using assms RightOf by simp
    moreover have b1:  $\langle (abs\_interval((interval\_of\ 1) / Y) * abs\_interval((interval\_of\ 1) / Y)) * interval\_of(width\ Y) = Interval(1 / upper\ Y, 1 / lower\ Y) * Interval(1 / upper\ Y, 1 / lower\ Y) * Interval(upper\ Y - lower\ Y, upper\ Y - lower\ Y) \rangle$ 
    using assms RightOf abs_pos[of Y] width_expanded[of Y] by simp
    moreover have b2:  $\langle \dots = Interval(\text{Min } \{ (1 / upper\ Y * 1 / upper\ Y), (1 / upper\ Y * 1 / lower\ Y), (1 / lower\ Y * 1 / upper\ Y), (1 / lower\ Y * 1 / lower\ Y) \}, \text{Max } \{ (1 / upper\ Y * 1 / upper\ Y), (1 / upper\ Y * 1 / lower\ Y), (1 / lower\ Y * 1 / upper\ Y), (1 / lower\ Y * 1 / lower\ Y) \}) * Interval(upper\ Y - lower\ Y, upper\ Y - lower\ Y) \rangle$ 
    using assms RightOf upper_leq_lower_div interval_interval_times[of Interval(1 / upper\ Y, 1 / lower\ Y) Interval(1 / upper\ Y, 1 / lower\ Y)]
    by fastforce
    have b3:  $\langle \text{Min } \{ (1 / upper\ Y * 1 / upper\ Y), (1 / upper\ Y * 1 / lower\ Y), (1 / lower\ Y * 1 / upper\ Y), (1 / lower\ Y * 1 / lower\ Y) \} = (1 / upper\ Y * 1 / upper\ Y) \rangle$ 
    using assms ao RightOf upper_leq_lower_div
    apply (simp add: min_def, safe)
    apply (smt (verit, ccfv_SIG) frac_le mult_less_cancel_left_pos zero_less_mult_iff)
    apply argo+
    by (simp add: frac_le)
    have b4:  $\langle \text{Max } \{ (1 / upper\ Y * 1 / upper\ Y), (1 / upper\ Y * 1 / lower\ Y), (1 / lower\ Y * 1 / upper\ Y), (1 / lower\ Y * 1 / lower\ Y) \} = (1 / lower\ Y * 1 / lower\ Y) \rangle$ 
    using assms ao RightOf upper_leq_lower_div
    apply (simp add: max_def, safe)
    subgoal
    using frac_le le_numeral_extra(4) less_numeral_extra(3) zero_le_one
    mult_mono' not_real_square_gt_zero order_less_imp_le by metis
    apply argo+
    apply (simp add: frac_le)
    subgoal
    using frac_le zero_le_one mult_mono' mult_pos_pos
    order.refl order_less_imp_le by meson
    by argo+
    moreover have  $\langle (abs\_interval((interval\_of\ 1) / Y) * abs\_interval((interval\_of\ 1) / Y)) * interval\_of(width\ Y) = Interval(1 / upper\ Y * 1 / upper\ Y, 1 / lower\ Y * 1 / lower\ Y) * Interval(upper\ Y - lower\ Y, upper\ Y - lower\ Y) \rangle$ 

```

```

using b1 b2 b3 b4 by presburger
moreover have <... = Interval((upper Y - lower Y) * 1 / upper Y * 1 / upper Y, (upper Y - lower Y) * 1 / lower Y * 1 / lower Y)>
using assms RightOf bo interval_times_scalar_pos_r[of upper Y - lower Y Interval(1 / upper Y * 1 / upper Y, 1 / lower Y * 1 / lower Y)]
unfolding interval_of_def
by (smt (verit, del_insts) divide_cancel_left divide_nonneg_nonneg frac_le interval_times_scalar_pos_r lower_bounds nonzero_divide_mult_cancel_right times_divide_eq_right upper_bounds width_def width_expanded)
ultimately show ?thesis by simp
qed
have a3: <1 / upper Y * 1 / upper Y * (upper Y - lower Y) ≤ 1 / lower Y - 1 / upper Y>
proof -
have bo: <0 < upper Y> using assms ao RightOf by argo
then have (1 / lower Y) - (1 / upper Y) = (upper Y - lower Y) / (lower Y * upper Y)
using assms RightOf bo by (simp add: diff_divide_distrib)
then have b1: <((1 / upper Y) * (1 / upper Y) * (upper Y - lower Y) * (lower Y * upper Y) ≤ (upper Y - lower Y)) = ((1 / upper Y) * (1 / upper Y) * (upper Y - lower Y) ≤ 1 / lower Y - 1 / upper Y)>
using assms RightOf bo by (simp add: pos_le_divide_eq)
then have b2: <((upper Y - lower Y) * lower Y / upper Y) ≤ upper Y - lower Y>
using assms ao RightOf bo by (smt (verit) mult_left_less_imp_less pos_less_divide_eq)
moreover have <(upper Y - lower Y) * lower Y ≤ (upper Y - lower Y) * upper Y>
using assms RightOf bo calculation pos_divide_le_eq by blast
show ?thesis using assms bo b1 b2 by force
qed
have a4: <1 / lower Y - 1 / upper Y ≤ 1 / lower Y * 1 / lower Y * (upper Y - lower Y)>
proof -
have bo: <0 < upper Y> using assms ao RightOf by argo
then have (1 / lower Y) - (1 / upper Y) = (upper Y - lower Y) / (lower Y * upper Y)
using assms RightOf bo by (simp add: diff_divide_distrib)
then have b1: <(upper Y - lower Y ≤ (1 / lower Y) * (1 / lower Y) * (upper Y - lower Y) * (lower Y * upper Y)) = (1 / lower Y - 1 / upper Y ≤ ((1 / lower Y) * (1 / lower Y) * (upper Y - lower Y)))>
using assms RightOf bo by (simp add: pos_divide_le_eq)
then have b2: <upper Y - lower Y ≤ ((upper Y - lower Y) * upper Y / lower Y)>
using assms ao RightOf bo by (smt (verit) division_leq mult_pos_pos nonzero_mult_div_cancel_right)
moreover have <(upper Y - lower Y) * lower Y ≤ (upper Y - lower Y) * upper Y>
using calculation pos_divide_le_eq by (simp add: RightOf pos_le_divide_eq)
show ?thesis using assms bo b1 b2 RightOf by simp
qed
have <Interval(1 / lower Y - 1 / upper Y, 1 / lower Y - 1 / upper Y) ≤ Interval((upper Y - lower Y) * 1 / upper Y * 1 / upper Y, (upper Y - lower Y) * 1 / lower Y * 1 / lower Y)>
using a3 a4 unfolding less_eq_interval_def by simp
then show ?case using a1 a2 a3 a4 by simp
qed
end

```

## 5 Affine Functions (Affine\_Functions)

In this theory, we provide formalisation of affine functions (sometimes also called linear polynomial functions).

```
theory
  Affine_Functions
imports
  Complex_Main
begin
```

### 5.1 Definition of Affine Functions, Alternative Definitions, and Special Cases

```
locale affine_fun =
  fixes f
  assumes  $\langle \exists b. \text{linear } (\lambda x. f x - b) \rangle$ 

lemma affine_fun_alt:
 $\langle \text{affine\_fun } f = (\exists c g. (f = (\lambda x. g x + c)) \wedge \text{linear } g) \rangle$ 
  unfolding affine_fun_def
  by (safe, force, simp add: Real_Vector_Spaces.linear_iff)

lemma affine_fun_real_linfun:
 $\langle \text{affine\_fun } (f :: \text{real} \Rightarrow \text{real}) = (\exists a b. f = (\lambda x. a * x + b)) \rangle$ 
  by (simp add: affine_fun_alt, metis real_linearD linear_times)

lemma linear_is_affine_fun:  $\langle \text{linear } f \Longrightarrow \text{affine\_fun } f \rangle$ 
  by (standard, simp add: Real_Vector_Spaces.linear_iff)

lemma affine_zero_is_linear: assumes  $\langle \text{affine\_fun } f \rangle$  and  $\langle f 0 = 0 \rangle$  shows  $\langle \text{linear } f \rangle$ 
  apply (insert assms)
  unfolding affine_fun_def
  using linear_0 by fastforce

lemma affine_add:
  assumes  $\langle \text{affine\_fun } f \rangle$  and  $\langle \text{affine\_fun } g \rangle$ 
  shows  $\langle \text{affine\_fun } (\lambda x. f x + g x) \rangle$ 
  using assms linear_compose_add
  by (auto simp add: affine_fun_alt, force)

lemma scaleR:
  assumes  $\langle \text{affine\_fun } f \rangle$  shows  $\langle \text{affine\_fun } (\lambda x. a *_R (f x)) \rangle$ 
  using assms
  apply (simp add: affine_fun_alt)
  by (metis linear_compose_scale_right scaleR_right_distrib)

lemma real_affine_funD:
  fixes  $f :: \text{real} \Rightarrow \text{real}$ 
  assumes  $\text{affine\_fun } f$  obtains  $c b$  where  $f = (\lambda x. c * x + b)$ 
  using assms
```

```

apply(simp add: affine_fun_alt)
by (metis real_linearD)

```

## 5.2 Common Linear Polynomial Functions

```

lemma affine_fun_const[simp]: <affine_fun (λ x. c)>

```

```

apply(simp add: affine_fun_alt)
using module_hom_zero by force

```

```

lemma affine_fun_id[simp]: <affine_fun (λ x. x)>

```

```

by (simp add: linear_is_affine_fun module_hom_ident)

```

```

lemma affine_fun_mult[simp]: <affine_fun (λ x. (c::'a::real_algebra) * x)>

```

```

by(simp add: linear_is_affine_fun)

```

```

lemma affine_fun_scaled[simp]: <affine_fun (λ x. x / c)>

```

```

for c :: 'a::real_normed_field

```

```

using bounded_linear_divide[of c] linear_is_affine_fun[of (λ x. x / c)] bounded_linear.linear
by blast

```

```

lemma affine_fun_add[simp]: <affine_fun (λ x. x + c)>

```

```

apply(simp add: affine_fun_alt)
using module_hom_ident by force

```

```

lemma affine_fun_diff[simp]: <affine_fun (λ x. x - c)>

```

```

apply(simp add: affine_fun_alt)
using module_hom_ident by force

```

```

lemma affine_fun_triv[simp]: <affine_fun (λ x. a *R x + c)>

```

```

apply(simp add: affine_fun_alt)
by fastforce

```

```

lemma affine_fun_add_const[simp]: assumes <affine_fun f> shows <affine_fun (λ x. (f x) + c)>

```

```

using assms apply(simp add:affine_fun_alt)
by (metis add.commute add.left_commute)

```

```

lemma affine_fun_diff_const[simp]: assumes <affine_fun f> shows <affine_fun (λ x. (f x) - c)>

```

```

using assms apply(simp add:affine_fun_alt) by force

```

```

lemma affine_fun_comp[simp]: assumes <affine_fun (f)>

```

```

and <affine_fun (g)> shows <affine_fun (f ∘ g)>

```

```

using assms
unfolding affine_fun_alt
apply(simp add:o_def)
using linear_add linear_compose[simplified o_def] add.assoc
by metis

```

```

lemma affine_fun_linear[simp]: assumes <affine_fun f> shows <affine_fun (λ x. a *R (f x) + c)>

```

```

by(rule affine_fun_comp[of λ x. a *R x + c f, simplified o_def], simp_all add: assms)

```

## 5.3 Linear Polynomial Functions and Orderings

```

lemma affine_fun_leq_pos:

```

```

assumes <affine_fun (f::real ⇒ real)> and <affine_fun g>
and <x∈{0..u}> and <(f o ≤ g o)> and <(f u ≤ g u)>
shows <f x ≤ g x>
using assms
apply(auto simp add: linear_o affine_fun_real_linfun)[1]
by (smt (verit) left_diff_distrib mult_left_mono_neg mult_right_mono)

```

```

lemma affine_fun_leq_neg:
assumes <affine_fun (f::real ⇒ real)> and <affine_fun g>
and <x∈{l..o}> and <(f l ≤ g l)> and <(f o ≤ g o)>
shows <f x ≤ g x>
using assms
apply(auto simp add: linear_o affine_fun_real_linfun)[1]
using Groups.mult_ac(2) left_diff_distrib mult_right_mono
by (smt (verit, ccfv_SIG))

```

```

lemma affine_fun_leq:
assumes <affine_fun (f::real ⇒ real)> and <affine_fun g>
and <x∈{l..u}> and <(f l ≤ g l)> and <(f u ≤ g u)>
shows <f x ≤ g (x::real)>
using assms affine_fun_leq_neg[of f g x l] affine_fun_leq_pos[of f g x u]
apply(simp add: linear_o affine_fun_real_linfun)
by (smt (verit, ccfv_SIG) left_diff_distrib mult_left_mono_neg)

```

```

lemma affine_fun_le_pos:
assumes <affine_fun (f::real ⇒ real)> and <affine_fun g>
and <x∈{0..u}> and <(f o < g o)> and <(f u < g u)>
shows <f x < g (x::real)>
using assms
apply(auto simp add: linear_o affine_fun_real_linfun)[1]
using Groups.mult_ac(2) left_diff_distrib mult_right_mono
by (smt (verit, best))

```

```

lemma affine_fun_le_neg:
assumes <affine_fun (f::real ⇒ real)> and <affine_fun g>
and <x∈{l..o}> and <(f l < g l)> and <(f o < g o)>
shows <f x < g (x::real)>
using assms
apply(auto simp add: linear_o affine_fun_real_linfun)[1]
using Groups.mult_ac(2) left_diff_distrib mult_right_mono
by (smt (verit, ccfv_SIG))

```

```

lemma affine_fun_le:
assumes <affine_fun (f::real ⇒ real)> and <affine_fun g>
and <x∈{l..u}> and <(f l < g l)> and <(f u < g u)>
shows <f x < g (x::real)>
using assms affine_fun_le_neg[of f g x l] affine_fun_le_pos[of f g x u]
apply(simp add: linear_o affine_fun_real_linfun)
by (smt (verit, ccfv_SIG) left_diff_distrib mult_left_mono_neg)

```

**end**



## 6 Interval Inclusion Isotonicity (📄 Inclusion\_Isotonicity)

**theory**

*Inclusion\_Isotonicity*

**imports**

*Interval\_Utilities*

*Affine\_Functions*

*Interval\_Division\_Non\_Zero*

**begin**

### 6.1 Interval Extension

#### 6.1.1 Textbook Definition of Interval Extension

**definition**

*is\_interval\_extension\_of* ::  $\langle ('a::preorder\ interval \Rightarrow 'b::preorder\ interval) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool \rangle$   
(**infix**  $\langle ('is\_interval\_extension\_of) \rangle 50$ )

**where**

$\langle ((F\ is\_interval\_extension\_of\ f)) = (\forall x. (F\ (interval\_of\ x)) = interval\_of\ (f\ x)) \rangle$

**definition**  $\langle extend\_natural\ f = (\lambda X. mk\_interval\ (f\ (lower\ X), f\ (upper\ X))) \rangle$

**lemma** *interval\_extension\_comp*:

**assumes** *interval\_ext\_F*:  $\langle F\ is\_interval\_extension\_of\ f \rangle$

**and** *interval\_ext\_G*:  $\langle G\ is\_interval\_extension\_of\ g \rangle$

**shows**  $\langle (F\ o\ G)\ is\_interval\_extension\_of\ (f\ o\ g) \rangle$

**using** *assms* **unfolding** *is\_interval\_extension\_of\_def*

**by** *simp*

**lemma** *interval\_extension\_const*:  $\langle (\lambda x. interval\_of\ c)\ is\_interval\_extension\_of\ (\lambda x. c) \rangle$

**unfolding** *is\_interval\_extension\_of\_def*

**by** (*simp* *add: interval\_eq\_iff*)

**lemma** *interval\_extension\_id*:  $\langle (\lambda x. x)\ is\_interval\_extension\_of\ (\lambda x. x) \rangle$

**unfolding** *is\_interval\_extension\_of\_def*

**by** (*simp* *add: interval\_eq\_iff*)

#### 6.1.2 A Stronger Definition of Interval Extension

**definition**

*is\_natural\_interval\_extension\_of* ::  $\langle ('a::linorder\ interval \Rightarrow 'b::linorder\ interval) \Rightarrow ('a \Rightarrow 'b) \Rightarrow bool \rangle$   
(**infix**  $\langle ('is\_natural\_interval\_extension\_of) \rangle 50$ )

**where**

$\langle ((F\ is\_natural\_interval\_extension\_of\ f)) = (\forall l\ u. (F\ (mk\_interval\ (l,u))) = mk\_interval\ (f\ l, f\ u)) \rangle$

**lemma**  $\langle (extend\_natural\ f)\ is\_interval\_extension\_of\ f \rangle$

**unfolding** *is\_interval\_extension\_of\_def* *extend\_natural\_def*

**by** (*auto* *simp* *add: mk\_interval' interval\_of\_def*)

**lemma**  $\langle \text{extend\_natural } f \rangle \text{ is\_natural\_interval\_extension\_of } f \rangle$   
**unfolding**  $\text{is\_natural\_interval\_extension\_of\_def } \text{extend\_natural\_def}$   
**by**  $(\text{auto simp add: mk\_interval' interval\_of\_def})$

**lemma**  $\text{natural\_interval\_extension\_implies\_interval\_extension}$ :  
**assumes**  $\langle F \text{ is\_natural\_interval\_extension\_of } f \rangle$  **shows**  $\langle F \text{ is\_interval\_extension\_of } f \rangle$   
**using**  $\text{assms unfolding is\_natural\_interval\_extension\_of\_def is\_interval\_extension\_of\_def}$   
 $\text{mk\_interval\_def interval\_of\_def}$   
**by**  $(\text{auto split:if\_splits})$

**lemma**  $\text{const\_add\_is\_natural\_interval\_extensions}$ :  
 $\langle (\lambda x. (\text{interval\_of } c) + x) \text{ is\_natural\_interval\_extension\_of } (\lambda x. c + x) \rangle$   
**unfolding**  $\text{is\_natural\_interval\_extension\_of\_def mk\_interval\_def}$   
**by**  $(\text{simp add: add\_mono\_thms\_linordered\_semiring}(1) \text{ antisym interval\_eq\_iff add\_mono}$   
 $\text{split:if\_splits})$

**lemma**  $\text{const\_times\_is\_natural\_interval\_extensions}$ :  
 $\langle (\lambda x. (\text{interval\_of } c) * x) \text{ is\_natural\_interval\_extension\_of } (\lambda x. c * x) \rangle$   
**unfolding**  $\text{is\_natural\_interval\_extension\_of\_def mk\_interval\_def}$   
**unfolding**  $\text{times\_interval\_def Let\_def}$   
**by**  $(\text{simp add: interval\_eq\_iff Interval\_inverse interval\_of.rep\_eq add\_mono}$   
 $\text{split:if\_splits})$

**lemma**  $\text{const\_is\_interval\_extension}$ :  $\langle F \text{ is\_natural\_interval\_extension\_of } (\lambda x. b) \implies F = (\lambda x. (\text{interval\_of } b)) \rangle$   
**unfolding**  $\text{is\_natural\_interval\_extension\_of\_def}$   
**apply**  $(\text{auto simp add:mk\_interval' interval\_of\_def split:if\_splits})[1]$   
**by**  $(\text{metis bounds\_of\_interval\_eq\_lower\_upper bounds\_of\_interval\_inverse lower\_le\_upper})$

**lemma**  $\text{id\_is\_interval\_extension}$ :  $\langle F \text{ is\_natural\_interval\_extension\_of } (\lambda x. x) \implies F = (\lambda x. x) \rangle$   
**unfolding**  $\text{is\_natural\_interval\_extension\_of\_def}$   
**apply**  $(\text{auto simp add:mk\_interval' interval\_of\_def split:if\_splits})[1]$   
**by**  $(\text{metis bounds\_of\_interval\_eq\_lower\_upper bounds\_of\_interval\_inverse lower\_le\_upper})$

**lemma**  $\text{extend\_natural\_is\_interval\_extension}$ :  
 $\langle \text{extend\_natural } f \rangle \text{ is\_natural\_interval\_extension\_of } f \rangle$   
**unfolding**  $\text{extend\_natural\_def is\_natural\_interval\_extension\_of\_def mk\_interval'\_def}$   
**by**  $(\text{smt } (z3) \text{ case\_prod\_conv mk\_interval\_def lower\_bounds nle\_le upper\_bounds})$

**lemma**  $\text{is\_natural\_interval\_extension\_implies\_bounds}$ :  
**fixes**  $F :: \langle \text{real interval} \Rightarrow \text{real interval} \rangle$   
**assumes**  $\langle F \text{ is\_natural\_interval\_extension\_of } f \rangle$  **and**  $\langle F x \leq F x' \rangle$   
**shows**  
 $\langle (f (\text{lower } x')) \leq (f (\text{lower } x)) \rangle \vee \langle (f (\text{upper } x')) \leq (f (\text{upper } x)) \rangle \rangle$   
**by**  $(\text{smt } (\text{verit}) \text{ assms interval\_eq1 is\_natural\_interval\_extension\_of\_def less\_eq\_interval\_def}$   
 $\text{lower\_le\_upper mk\_interval\_lower mk\_interval\_upper})$

**lemma**  $\text{interval\_extension\_lower}$ :  
 $\langle (F \text{ is\_interval\_extension\_of } f) \implies \text{lower } (F (\text{interval\_of } x)) = (f x) \rangle$   
**unfolding**  $\text{is\_interval\_extension\_of\_def}$  **by**  $\text{simp}$

**lemma**  $\text{interval\_extension\_upper}$ :  
 $\langle (F \text{ is\_interval\_extension\_of } f) \implies \text{upper } (F (\text{interval\_of } x)) = (f x) \rangle$   
**unfolding**  $\text{is\_interval\_extension\_of\_def}$  **by**  $\text{simp}$

```

lemma is_interval_extension_eq_upper_and_lower:
  <((F is_interval_extension_of f))
    = (∀ x. lower (F (interval_of x)) = (f x) ∧ upper (F (interval_of x)) = (f x))>
unfolding is_interval_extension_of_def
by (simp add: interval_eq_iff)

```

```

lemma interval_extension_lower_simp[simp]:
  assumes <F is_interval_extension_of f> and <X = Interval(x,x)>
  shows <lower (F X) = f x >
by (metis assms interval_extension_lower interval_of.abs_eq)

```

```

lemma interval_extension_upper_simp[simp]:
  assumes <F is_interval_extension_of f> and <X = Interval(x,x)>
  shows <upper (F X) = f x >
by (metis assms interval_extension_upper interval_of.abs_eq)

```

## 6.2 Interval Inclusion Isotonicity

In this section, we introduce the concept of inclusion isotonicity. The formalization in this theory generalises the definitions from [5]:

```

definition
  inclusion_isotonic :: <('a::preorder interval ⇒ 'b::preorder interval) ⇒ bool>
  where
  <inclusion_isotonic f = (∀ x x'. x ≤ x' ⟶ (f x) ≤ (f x'))>

```

We can immediately show that any natural extension of an affine function of from type *real* to *real* is interval inclusion isotonic:

```

lemma real_affine_fun_is_inclusion_isotonicM:
  assumes <affine_fun (f::real => real)>
  shows <inclusion_isotonic (extend_natural f)>
using assms
unfolding inclusion_isotonic_def extend_natural_def Interval.less_eq_interval_def affine_fun_real_linfun
by(auto, (metis lower_le_upper mult_le_cancel_left nle_le)+)

```

```

definition
  inclusion_isotonic_on :: <('a interval ⇒ bool) ⇒ ('a::preorder interval ⇒ 'b::preorder interval) ⇒ bool>
  where
  <inclusion_isotonic_on P f = (∀ x x'. P x ∧ P x' ∧ x ≤ x' ⟶ (f x) ≤ (f x'))>

```

```

lemma inclusion_isotonic_eq: <inclusion_isotonic_on (λ x . True) = inclusion_isotonic>
unfolding inclusion_isotonic_on_def inclusion_isotonic_def
by simp

```

```

definition inclusion_isotonic_binary :: <('a::preorder interval ⇒ 'a interval ⇒ 'b::preorder interval) ⇒ bool>
  where
  <inclusion_isotonic_binary f = (∀ x x' y y'. x ≤ x' ∧ y ≤ y' ⟶ (f x y) ≤ (f x' y'))>

```

```

definition inclusion_isotonic_binary_on :: <('a interval ⇒ bool) ⇒ ('a::preorder interval ⇒ 'a interval ⇒ 'b::preorder interval) ⇒ bool>
  where
  <inclusion_isotonic_binary_on P f = (∀ x x' y y'. P x ∧ P x' ∧ P y ∧ P y' ∧ x ≤ x' ∧ y ≤ y' ⟶ (f x y) ≤ (f x' y'))>

```

**lemma** *inclusion\_isotonic\_binary\_eq*:  $\langle \text{inclusion\_isotonic\_binary\_on } (\lambda x . \text{True}) = \text{inclusion\_isotonic\_binary} \rangle$   
**unfolding** *inclusion\_isotonic\_binary\_on\_def inclusion\_isotonic\_binary\_def*  
**by** *simp*

**definition** *inclusion\_isotonicM\_n* ::  $\langle \text{nat} \Rightarrow ('a::\text{linorder interval list} \Rightarrow 'a::\text{linorder interval}) \Rightarrow \text{bool} \rangle$  **where**  
 $\langle \text{inclusion\_isotonicM\_n } n f = (\forall \text{ is js. } (\text{length is} = n \wedge \text{length js} = n \wedge (\text{is} \leq_I \text{js})) \longrightarrow f \text{ is} \leq f \text{ js}) \rangle$

**definition** *inclusion\_isotonicM\_n\_on* ::  $\langle ('a \text{ interval} \Rightarrow \text{bool}) \Rightarrow \text{nat} \Rightarrow ('a::\text{linorder interval list} \Rightarrow 'a::\text{linorder interval}) \Rightarrow \text{bool} \rangle$  **where**  
 $\langle \text{inclusion\_isotonicM\_n\_on } P n f = (\forall \text{ is js. } (\forall i \in \text{set is. } P i) \wedge (\forall j \in \text{set js. } P j) \wedge (\text{length is} = n \wedge \text{length js} = n \wedge (\text{is} \leq_I \text{js})) \longrightarrow f \text{ is} \leq f \text{ js}) \rangle$

**lemma** *inclusion\_isotonicM\_n\_eq*:  $\langle \text{inclusion\_isotonicM\_n\_on } (\lambda x . \text{True}) = \text{inclusion\_isotonicM\_n} \rangle$   
**unfolding** *inclusion\_isotonicM\_n\_on\_def inclusion\_isotonicM\_n\_def*  
**by** *simp*

Finally, we extend the definition to functions over lists and show that the three definitions of interval inclusion isotonicity are, for their corresponding types, equivalent:

**locale** *inclusion\_isotonicM* =  
**fixes** *n* :: *nat*  
**and** *f* ::  $\langle ('a::\text{linorder}) \text{ interval list} \Rightarrow 'a \text{ interval list} \rangle$   
**assumes** *inclusion\_isotonicM* :  
 $\langle (\forall \text{ is js. } (\text{length is} = n) \wedge (\text{length js} = n) \wedge (\text{is} \leq_I \text{js}) \longrightarrow f \text{ is} \leq_I f \text{ js}) \rangle$   
**begin**  
**end**

**locale** *inclusion\_isotonicM\_on* =  
**fixes** *P* ::  $\langle 'a::\text{linorder interval} \Rightarrow \text{bool} \rangle$   
**and** *n* :: *nat*  
**and** *f* ::  $\langle ('a::\text{linorder}) \text{ interval list} \Rightarrow 'a \text{ interval list} \rangle$   
**assumes** *inclusion\_isotonicM* :  
 $\langle (\forall \text{ is js. } (\forall i \in \text{set is. } P i) \wedge (\forall j \in \text{set js. } P j) \wedge (\text{length is} = n) \wedge (\text{length js} = n) \wedge (\text{is} \leq_I \text{js}) \longrightarrow f \text{ is} \leq_I f \text{ js}) \rangle$   
**begin**  
**end**

**lemma** *inclusion\_isotonicM\_on\_eq*:  $\langle \text{inclusion\_isotonicM\_on } (\lambda x . \text{True}) = \text{inclusion\_isotonicM} \rangle$   
**unfolding** *inclusion\_isotonicM\_on\_def inclusion\_isotonicM\_def*  
**by** *simp*

**lemma** *inclusion\_isotonic\_conv*:  
 $\langle \text{inclusion\_isotonic } g = \text{inclusion\_isotonicM } 1 (\lambda xs . \text{case } xs \text{ of } [x] \Rightarrow [g x]) \rangle$   
**unfolding** *inclusion\_isotonic\_def inclusion\_isotonicM\_def*  
**by**(*auto simp add: le\_interval\_single split:list.split*)

**lemma** *inclusion\_isotonicM\_n\_conv1*:  
 $\langle \text{inclusion\_isotonicM\_n } n f = \text{inclusion\_isotonicM } n (\lambda xs . [f xs]) \rangle$   
**unfolding** *inclusion\_isotonicM\_n\_def inclusion\_isotonicM\_def le\_interval\_list\_def*  
**by**(*auto*)

**lemma** *inclusion\_isotonicM\_conv2*:  
**assumes**  $\langle \text{inclusion\_isotonicM } n f \rangle$   
**and**  $\langle \forall xs . (\text{length } xs = n) \longrightarrow N = (\text{length } (f xs)) \rangle$   
**shows**  $\langle \text{inclusion\_isotonicM } n (\lambda xs . \text{if } n' < N \text{ then } [(f xs)!n'] \text{ else } []) \rangle$

```

using assms unfolding inclusion_isotonicM_def
apply(simp split:if_splits, safe)
apply(subst le_interval_single[symmetric])
apply(subst le_interval_list_all)
subgoal by blast
subgoal by blast
subgoal by(rule Truel)
done

```

```

lemma inclusion_isotonicM_n_on_conv1:
  <inclusion_isotonicM_n_on P n f = inclusion_isotonicM_on P n (λ xs. [f xs])>
  unfolding inclusion_isotonicM_n_on_def inclusion_isotonicM_on_def le_interval_list_def
  by(auto)

```

```

lemma inclusion_isotonicM_on_conv2:
  assumes <inclusion_isotonicM_on P n f>
  and < $\forall xs. (length\ xs = n) \longrightarrow N = (length\ (f\ xs))$ >
  shows <inclusion_isotonicM_on P n (λ xs. if n' < N then [(f xs)!n'] else [])>
  using assms unfolding inclusion_isotonicM_on_def
  apply(simp split:if_splits, safe)
  apply(subst le_interval_single[symmetric])
  apply(subst le_interval_list_all)
  subgoal by blast
  subgoal by blast
  subgoal by(rule Truel)
done

```

```

lemma inclusion_isotonic_binary_conv1:
  <inclusion_isotonic_binary f = inclusion_isotonicM_n 2 (λ xs . case xs of [x,y] => f x y)>
  unfolding inclusion_isotonic_binary_def inclusion_isotonicM_n_def le_interval_list_def
  by(auto simp add: le_interval_list_def split:list.split)

```

```

lemma inclusion_isotonic_binary_conv2:
  <inclusion_isotonic_binary f = inclusion_isotonicM 2 (λ xs . case xs of [x,y] => [f x y])>
  apply(simp add: inclusion_isotonic_binary_conv1)
  apply(simp add: inclusion_isotonicM_n_conv1)
  unfolding inclusion_isotonicM_def
  by(auto split:list.split)

```

```

lemma inclusion_isotonic_binary_on_conv1:
  <inclusion_isotonic_binary_on P f = inclusion_isotonicM_n_on P 2 (λ xs . case xs of [x,y] => f x y)>
  unfolding inclusion_isotonic_binary_on_def inclusion_isotonicM_n_on_def le_interval_list_def
  by(auto simp add: le_interval_list_def split:list.split)

```

```

lemma inclusion_isotonic_binary_on_conv2:
  <inclusion_isotonic_binary_on P f = inclusion_isotonicM_on P 2 (λ xs . case xs of [x,y] => [f x y])>
  apply(simp add: inclusion_isotonic_binary_on_conv1)
  apply(simp add: inclusion_isotonicM_n_on_conv1)
  unfolding inclusion_isotonicM_on_def
  by(auto split:list.split)

```

## 6.2.1 Compositionality of Interval Inclusion Isotonicity

```

lemma inclusion_isotonicM_comp:

```

```

assumes <inclusion_isotonicM n f>
  and <inclusion_isotonicM m g>
  and < $\forall xs. \text{length } xs = n \longrightarrow \text{length } (f \text{ } xs) = m$ >
shows <inclusion_isotonicM n (g o f)>
using assms unfolding inclusion_isotonicM_def
by(simp)

```

```

lemma inclusion_isotonicM_on_comp:
assumes <inclusion_isotonicM_on P n f>
  and <inclusion_isotonicM_on Q m g>
  and < $\forall xs. \text{length } xs = n \longrightarrow \text{length } (f \text{ } xs) = m$ >
  and < $\forall is. (\forall i \in \text{set } is. P \ i) \longrightarrow (\forall x \in \text{set } (f \text{ } is). Q \ x)$ >
shows <inclusion_isotonicM_on P n (g o f)>
using assms unfolding inclusion_isotonicM_on_def o_def
by(auto)

```

```

lemma inclusion_isotonic_comp:
assumes <inclusion_isotonic f>
  and <inclusion_isotonic g>
shows <inclusion_isotonic (g o f)>
using assms unfolding inclusion_isotonic_def
by(simp)

```

```

lemma inclusion_isotonic_on_comp:
assumes <inclusion_isotonic_on P f>
  and <inclusion_isotonic_on Q g>
  and < $\forall x. P \ x \longrightarrow Q \ (f \ x)$ >
shows <inclusion_isotonic_on P (g o f)>
using assms unfolding inclusion_isotonic_on_def
by(simp)

```

## 6.2.2 Interval Inclusion Isotonicity of the Core Operator

### Unary Minus (Negation)

```

lemma inclusion_isotonicM_uminus[simp]: <inclusion_isotonic uminus>
by (simp add: inclusion_isotonic_def less_eq_interval_def)

```

### Addition

```

lemma inclusion_isotonicM_plus[simp]: <inclusion_isotonic_binary (+)>
by (simp add: inclusion_isotonic_binary_def less_eq_interval_def add_mono)

```

### Substraction

```

lemma inclusion_isotonicM_minus[simp]: <inclusion_isotonic_binary (-)>
by (simp add: inclusion_isotonic_binary_def less_eq_interval_def diff_mono)

```

### Multiplication

```

lemma inclusion_isotonicM_times[simp]:
  <inclusion_isotonic_binary ( $\lambda x \ y. (x :: 'a :: \{\text{linordered\_ring, real\_normed\_algebra, linear\_continuum\_topology}\}$  interval) * y)>
unfolding inclusion_isotonic_binary_def using set_of_mul_inc interval_set_leq_eq
by metis

```

### 6.2.3 Interval Inclusion Isotonicity of Various Functions

**lemma** *inclusion\_isotonicM\_n\_empty*[simp]:  $\langle \text{inclusion\_isotonicM } n \ (\lambda \text{ xs. } []) \rangle$   
**unfolding** *inclusion\_isotonicM\_def* **by**(simp)

**lemma** *inclusion\_isotonic\_id*[simp]:  $\langle \text{inclusion\_isotonic } \text{id} \rangle$   
**unfolding** *inclusion\_isotonic\_def le\_interval\_list\_def*  
**by**(simp)

**lemma** *inclusion\_isotonicM\_id*[simp]:  $\langle \text{inclusion\_isotonicM } n \ \text{id} \rangle$   
**unfolding** *inclusion\_isotonicM\_def le\_interval\_list\_def*  
**by**(simp)

**lemma** *inclusion\_isotonicM\_hd*[simp]:  
**assumes**  $\langle 0 < n \rangle$   
**shows**  $\langle \text{inclusion\_isotonicM\_n } n \ \text{hd} \rangle$   
**unfolding** *inclusion\_isotonicM\_n\_def le\_interval\_list\_def*  
**proof**(*insert assms, induction n rule:nat\_induct\_non\_zero*)  
**case 1**  
**then show** ?case  
**by** (*auto, metis (no\_types, lifting) Nil\_eq\_zip\_iff case\_proxD foldl\_conj\_set\_True hd\_zip insert\_iff length\_o\_conv list.map\_disc\_iff list.map\_sel(1) list.set(2) list.set\_sel(1) nat.distinct(1)*)  
**next**  
**case** (*Suc n*)  
**then show** ?case  
**by** (*auto, metis (no\_types, lifting) Nil\_eq\_zip\_iff Zero\_not\_Suc case\_proxD foldl\_conj\_set\_True hd\_zip insert\_iff length\_o\_conv list.map\_disc\_iff list.map\_sel(1) list.set(2) list.set\_sel(1)*)  
**qed**

**lemma** *inclusion\_isotonic\_add\_const1*[simp]:  
 $\langle \text{inclusion\_isotonic } (\lambda x. x + c) \rangle$   
**unfolding** *inclusion\_isotonic\_def*  
**by** (*simp add: add\_mono\_thms\_linordered\_semiring(3) less\_eq\_interval\_def*)

**lemma** *inclusion\_isotonicM\_1\_add\_const2*[simp]:  
 $\langle \text{inclusion\_isotonic } (\lambda x. c + x) \rangle$   
**unfolding** *inclusion\_isotonic\_def*  
**by** (*simp add: add\_mono\_thms\_linordered\_semiring(2) less\_eq\_interval\_def*)

**lemma** *inclusion\_isotonic\_times\_right*[simp]:  
 $\langle \text{inclusion\_isotonic } (\lambda x. C * (x :: 'a :: \text{linordered\_ring } \text{interval})) \rangle$   
**unfolding** *inclusion\_isotonic\_def*  
**using** *times\_interval\_right* **by** *auto*

**lemma** *inclusion\_isotonic\_times\_left*[simp]:  
 $\langle \text{inclusion\_isotonic } (\lambda x. (x :: 'a :: \{\text{real\_normed\_algebra, linordered\_ring, linear\_continuum\_topology}\} \text{interval}) * C) \rangle$   
**unfolding** *inclusion\_isotonic\_def*  
**using** *times\_interval\_left* **by** *auto*

**lemma** *map\_inclusion\_isotonicM*:  
**assumes**  $\langle \text{inclusion\_isotonic } f \rangle$   
**shows**  $\langle \text{inclusion\_isotonicM } n \ (\text{map } f) \rangle$   
**using** *assms map\_set map\_pair\_set\_left map\_pair\_set\_right map\_pair\_set*  
**unfolding** *inclusion\_isotonicM\_def le\_interval\_list\_def inclusion\_isotonic\_def*

**by**(simp add: foldl\_conj\_set\_True map\_pair\_f\_all, blast)

**lemma** inclusion\_isotonicM\_fun\_plus:  
**assumes**  $\langle \text{inclusion\_isotonic } f \rangle$  **and**  $\langle \text{inclusion\_isotonic } g \rangle$   
**shows**  $\langle \text{inclusion\_isotonic } (\lambda x. f x + g x) \rangle$   
**using** assms **unfolding** inclusion\_isotonic\_def  
**by** (simp add: add\_mono\_thms\_linordered\_semiring(1) less\_eq\_interval\_def)

**lemma** inclusion\_isotonic\_plus\_const:  
**assumes**  $\langle \text{inclusion\_isotonic } f \rangle$  **and**  $\langle \text{inclusion\_isotonic } g \rangle$   
**shows**  $\langle \text{inclusion\_isotonic } (\lambda x. g x + c) \rangle$   
**using** assms **unfolding** inclusion\_isotonic\_def  
**by** (simp add: add\_mono\_thms\_linordered\_semiring(1) less\_eq\_interval\_def)

**lemma** inclusion\_isotonic\_times\_const\_real:  
**assumes**  $\langle \text{inclusion\_isotonic } f \rangle$   
**shows**  $\langle \text{inclusion\_isotonic } (\lambda x. (c::\text{real interval}) * (f x)) \rangle$   
**using** inclusion\_isotonic\_comp assms  
**unfolding** inclusion\_isotonic\_def  
**by** (simp add: times\_interval\_right)

**lemma** interval\_le\_fldr:  
**assumes**  $\langle \text{inclusion\_isotonic\_binary } f \rangle$   
**shows**  $\langle \text{length } js = \text{length } is \implies is \leq_I js \implies (\text{foldr } f \text{ is } e) \leq (\text{foldr } f \text{ js } e) \rangle$   
**by** (induction rule: list\_induct2)  
(use assms in  $\langle \text{simp\_all add: less\_eq\_interval\_def le\_interval\_list\_def inclusion\_isotonic\_binary\_def foldl\_conj\_True} \rangle$ )

**lemma** interval\_le\_fldr\_map:  
**assumes**  $\langle \text{inclusion\_isotonic\_binary } f \rangle$   
**and**  $\langle \text{inclusion\_isotonic } g \rangle$   
**shows**  $\langle \text{length } js = \text{length } is \implies is \leq_I js \implies (\text{foldr } f (\text{map } g \text{ is}) e) \leq (\text{foldr } f (\text{map } g \text{ js}) e) \rangle$   
**proof** (induction rule: list\_induct2)  
**case** Nil  
**then show** ?case  
**by** (simp add: less\_eq\_interval\_def)  
**next**  
**case** (Cons x xs y ys)  
**then show** ?case  
**using** assms **unfolding** inclusion\_isotonicM\_n\_def le\_interval\_list\_def  
**by**(simp add: foldl\_conj\_True inclusion\_isotonic\_def inclusion\_isotonic\_binary\_def)  
**qed**

**lemma** interval\_le\_fldr\_e:  
**assumes**  $\langle \text{inclusion\_isotonic\_binary } f \rangle$   
**and**  $\langle is \leq js \rangle$   
**shows**  $\langle (\text{foldr } f \text{ xs } is) \leq (\text{foldr } f \text{ xs } js) \rangle$   
**proof** (induction xs)  
**case** Nil  
**then show** ?case **using** assms **by**(simp)  
**next**  
**case** (Cons x)  
**then show** ?case  
**using** assms **unfolding** le\_interval\_list\_def inclusion\_isotonic\_binary\_def  
**by** (simp add: less\_eq\_interval\_def)

qed

```
lemma foldr_inclusion_isotonicM_e:  
  assumes <inclusion_isotonic_binary f>  
  shows < $\forall x. inclusion\_isotonic (foldr f x)$ >  
  using assms  
  unfolding inclusion_isotonic_def  
  by(simp add: intervall_le_foldr_e)
```

```
lemma foldr_inclusion_isotonicM:  
  assumes <inclusion_isotonic_binary f>  
  shows <inclusion_isotonicM_n n ( $\lambda x. foldr f x e$ )>  
  using assms  
  unfolding inclusion_isotonicM_n_def using intervall_le_foldr  
  by auto
```

```
lemma foldr_inclusion_isotonicM_g:  
  assumes <inclusion_isotonic_binary f>  
  and <inclusion_isotonicM n g>  
  shows <inclusion_isotonicM_n n ( $\lambda x. foldr f ((g x)) e$ )>  
  using assms(2)  
  unfolding inclusion_isotonicM_n_def inclusion_isotonicM_def  
  by (metis assms(1) intervall_le_foldr le_interval_list_imp_length)
```

```
lemma foldr_map_inclusion_isotonicM_g:  
  assumes <inclusion_isotonic_binary f>  
  and <inclusion_isotonic g>  
  and <inclusion_isotonicM n P>  
  shows <inclusion_isotonicM_n n ( $\lambda x. foldr f (map g (P x)) e$ )>  
  using intervall_le_foldr_map assms(3)  
  unfolding inclusion_isotonicM_n_def inclusion_isotonicM_def  
  by (metis (no_types, lifting) assms(1) assms(2) intervall_le_foldr_map le_interval_list_imp_length)
```

```
lemma foldl_inclusion_isotonicM:  
  assumes <inclusion_isotonic_binary f>  
  shows <inclusion_isotonicM_n n (foldl f e)>  
  unfolding inclusion_isotonicM_n_def  
  apply(subst foldl_conv_foldr)  
  apply(subst foldl_conv_foldr)  
  using assms foldr_inclusion_isotonicM[simplified inclusion_isotonicM_def]  
  le_interval_list_rev length_rev  
  unfolding inclusion_isotonicM_n_def inclusion_isotonic_binary_def  
  by (metis)
```

```
lemma fold_inclusion_isotonicM:  
  assumes <inclusion_isotonic_binary f>  
  shows <inclusion_isotonicM_n n ( $\lambda x. fold f x e$ )>  
  unfolding inclusion_isotonicM_n_def  
  apply(subst foldl_conv_fold[symmetric])  
  apply(subst foldl_conv_fold[symmetric])  
  using assms foldl_inclusion_isotonicM[simplified inclusion_isotonicM_def]  
  unfolding inclusion_isotonic_binary_def inclusion_isotonicM_n_def  
  by (metis)
```

**lemma** *map2\_inclusion\_isotonicM\_left*: **assumes**  $\langle \text{inclusion\_isotonic\_binary } f \rangle$   
**shows**  $\langle \text{inclusion\_isotonicM } n \text{ (map2 } f \text{ xs)} \rangle$   
**unfolding** *inclusion\_isotonicM\_def inclusion\_isotonic\_binary\_def*  
**apply**(*safe,subst le\_interval\_list\_all2, simp\_all*)  
**using** *le\_interval\_list\_imp le\_interval\_list\_all*  
**by** (*metis assms dual\_order.refl inclusion\_isotonic\_binary\_def less\_eq\_interval\_def*)

**lemma** *map2\_inclusion\_isotonicM\_right*: **assumes**  $\langle \text{inclusion\_isotonic\_binary } f \rangle$   
**shows**  $\langle \text{inclusion\_isotonicM } n \text{ (}\lambda \text{ ys. map2 } f \text{ ys xs)} \rangle$   
**unfolding** *inclusion\_isotonicM\_def inclusion\_isotonic\_binary\_def*  
**apply**(*safe,subst le\_interval\_list\_all2, simp\_all*)  
**using** *le\_interval\_list\_imp le\_interval\_list\_all*  
**by** (*metis assms dual\_order.refl inclusion\_isotonic\_binary\_def less\_eq\_interval\_def*)

**lemma** *map2\_inclusion\_isotonicM\_right\_g*:  
**assumes**  $\langle \text{inclusion\_isotonic\_binary } f \rangle$   
**and**  $\langle \forall \text{ xs. length (g xs) = length xs} \rangle$   
**and**  $\langle \text{inclusion\_isotonicM } n \text{ g} \rangle$   
**and**  $\langle \text{length xs} = n \rangle$   
**and**  $\langle \text{length is} = n \rangle$   
**and**  $\langle \text{length js} = n \rangle$   
**and**  $\langle \text{is} \leq_I \text{ js} \rangle$   
**shows**  $\langle \text{map2 } f \text{ (g is) (h xs)} \leq_I \text{ map2 } f \text{ (g js) (h xs)} \rangle$   
**using** *assms unfolding inclusion\_isotonic\_binary\_def*  
**apply**(*subst le\_interval\_list\_all2, simp\_all*)  
**using** *assms unfolding inclusion\_isotonic\_binary\_def*  
**by** (*metis dual\_order.refl inclusion\_isotonicM\_def le\_interval\_list\_imp less\_eq\_interval\_def*)

**lemma** *inclusion\_isotonicM\_map*:  
**assumes**  $\langle \forall x \in \text{set xs} . g x \leq h x \rangle$   
**shows**  $\langle \text{map } g \text{ xs} \leq_I \text{map } h \text{ xs} \rangle$   
**using** *assms by(subst le\_interval\_list\_all2, simp\_all)*

### 6.3 Interval Extension and Inclusion Properties

**lemma** *interval\_extension\_inclusion*:  
**assumes**  $\langle F \text{ is\_interval\_extension\_of } f \rangle$   
**shows**  $\langle \forall X . \text{interval\_of } (f X) \leq F \text{ (interval\_of } X) \rangle$   
**using** *assms*  
**unfolding** *is\_interval\_extension\_of\_def*  
**by** (*simp add: in\_interval\_to\_interval interval\_of\_in\_eq*)

**lemma** *interval\_extension\_subset\_const*:  
**assumes** *interval\_ext\_F*:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$   
**shows**  $\langle \forall X . \text{set\_of } (\text{interval\_of } (f X)) \subseteq \text{set\_of } (F \text{ (interval\_of } X)) \rangle$   
**using** *assms*  
**unfolding** *is\_interval\_extension\_of\_def set\_of\_def* **by** *auto*

**lemma** *fundamental\_theorem\_of\_interval\_analysis*:  
**fixes** *F* ::  $\langle \text{real interval} \Rightarrow \text{real interval} \rangle$

```

assumes interval_ext_F:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
and inc_isotonic_F:  $\langle \text{inclusion\_isotonic } F \rangle$ 
shows  $\langle \forall X. f'(\text{set\_of } X) \subseteq \text{set\_of } (F X) \rangle$ 
proof
  fix X
  show  $\langle f'(\text{set\_of } X) \subseteq \text{set\_of } (F X) \rangle$ 
  proof
    fix y
    assume  $\langle y \in f'(\text{set\_of } X) \rangle$ 
    then obtain x where i:  $\langle x \in \text{set\_of } X \rangle$  and  $\langle y = f x \rangle$  by auto
    then have  $\langle \text{interval\_of } (f x) = F(\text{interval\_of } x) \rangle$  using assms unfolding is_interval_extension_of_def by simp
    then have  $\langle y \in \text{set\_of } (F(\text{interval\_of } x)) \rangle$  using  $\langle y = f x \rangle$  by (metis in_interval_to_interval)
    then have  $\langle \text{interval\_of } x \leq X \rangle$  using  $\langle x \in \text{set\_of } X \rangle$  using interval_of_in_eq by blast
    then have  $\langle F(\text{interval\_of } x) \leq F X \rangle$  using inc_isotonic_F unfolding inclusion_isotonic_def by simp
    then show  $\langle y \in \text{set\_of } (F X) \rangle$  using  $\langle y \in \text{set\_of } (F(\text{interval\_of } x)) \rangle$  using set_of_subset_iff' by auto
  qed
qed

```

```

lemma interval_extension_bounds:
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows  $\langle \forall x \in \text{set\_of } X. \text{lower } (F X) \leq f x \rangle \vee \langle \forall x \in \text{set\_of } X. f x \leq \text{lower } (F X) \rangle$ 
  using assms
  by (metis (no_types, lifting) in_bounds inclusion_isotonic_def interval_of_in_eq
    is_interval_extension_of_def)

```

```

lemma inclusion_isotonic_extension_subset:
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows  $\langle f'(\text{set\_of } X) \subseteq \text{set\_of } (F X) \rangle$ 
  using assms interval_of_in_eq
  unfolding inclusion_isotonic_def set_of_eq is_interval_extension_of_def
  by (metis (mono_tags, lifting) image_subsetI)

```

```

lemma inclusion_isotonic_extension_includes:
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows  $\langle \forall x \in \text{set\_of } X. f x \in \text{set\_of } (F X) \rangle$ 
  using assms inclusion_isotonic_extension_subset
  by blast

```

```

lemma inclusion_isotonic_extension_lower_bound:
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows  $\langle \forall x \in \text{set\_of } X. \text{lower } (F X) \leq f x \rangle$ 
  using assms inclusion_isotonic_extension_includes
  using in_bounds by blast

```

```

lemma inclusion_isotonic_extension_upper_bound:
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows  $\langle \forall x \in \text{set\_of } X. f x \leq \text{upper } (F X) \rangle$ 
  using assms inclusion_isotonic_extension_includes
  using in_bounds by blast

```

```

lemma inclusion_isotonic_inf:
  fixes F::⟨real interval ⇒ real interval⟩
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows ⟨lower (F (X::real interval)) ≤ Inf (f' set_of X)⟩
  using inclusion_isotonic_extension_subset[of F f X, simplified assms]
  cInf_superset_mono[of f' set_of X set_of (F X)]
  by (simp add: bdd_below_set_of inf_set_of)

lemma inclusion_isotonic_sup:
  fixes F::⟨real interval ⇒ real interval⟩
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows ⟨Sup (f' set_of X) ≤ upper (F X)⟩
  using inclusion_isotonic_extension_subset[of F f X, simplified assms]
  cSup_subset_mono[of f' set_of X set_of (F X)]
  by (simp add: bdd_above_set_of sup_set_of)

lemma lower_inf:
  fixes F::⟨real interval ⇒ real interval⟩
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows Inf (f' set_of X) ≤ f (lower X)
  using cInf_superset_mono[of {f (lower X)} f' set_of X]
  by (metis assms(1) assms(2) bdd_below_mono bdd_below_set_of bot.extremum
      cInf_singleton imageI insert_not_empty insert_subsetI
      fundamental_theorem_of_interval_analysis lower_in_interval)

lemma upper_sup:
  fixes F::⟨real interval ⇒ real interval⟩
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows f (upper X) ≤ Sup (f' set_of X)
  using cSup_subset_mono[of {f (upper X)} f' set_of X]
  by (meson assms(1) assms(2) bdd_above_mono bdd_above_set_of cSup_upper
      imageI fundamental_theorem_of_interval_analysis upper_in_interval)

lemma lower_F_f:
  fixes F::⟨real interval ⇒ real interval⟩
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows lower (F X) ≤ f (lower X)
  by (simp add: assms(1) assms(2) inclusion_isotonic_extension_lower_bound)

lemma upper_F_f:
  fixes F::⟨real interval ⇒ real interval⟩
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows f (upper X) ≤ upper (F X)
  by (simp add: assms(1) assms(2) inclusion_isotonic_extension_upper_bound)

lemma inclusion_isotonic_interval_extension_le:
  assumes inclusion_isotonic: ⟨inclusion_isotonic F⟩

```

```

and interval_extension: ⟨F is_interval_extension_of f⟩
and adjacent: ⟨upper a = lower b⟩
shows ⟨lower (F b) ≤ upper (F a)⟩
using inclusion_isotonic_extension_lower_bound[of F f, simplified assms, simplified]
      inclusion_isotonic_extension_upper_bound[of F f, simplified assms, simplified]
      le_left_mono lower_in_interval upper_in_interval
by(metis adjacent)

```

## 6.4 Division

```

context interval_division_inverse
begin
lemma inclusion_isotonic_on_inverse[simp]:
  ⟨inclusion_isotonic_on (λ x . ¬ 0 ∈i x) ((inverse::('a interval ⇒ 'a interval)))⟩
using inverse_non_zero_def
unfolding inclusion_isotonic_on_def less_eq_interval_def
by (smt (z3) dual_order.trans in_bounds in_intervall lower_le_upper mk_interval_lower
      mk_interval_upper upper_leq_lower_div)

lemma inclusion_isotonic_on_division[simp]:
  ⟨inclusion_isotonic_binary_on (λ x . ¬ 0 ∈i x) (λ x y. divide x y)⟩
using inclusion_isotonicM_times divide_non_zero_def inclusion_isotonic_on_inverse
unfolding o_def inclusion_isotonic_binary_on_def
      inclusion_isotonic_on_def inclusion_isotonic_binary_def
by metis

end

end

```



## 7 Lipschitz Continuity of Intervals

### ( Lipschitz\_Interval\_Extension)

An extension of of Lipschitz Continuity, developed for the Lipschitz Continuity of intervals.

**theory**

*Lipschitz\_Interval\_Extension*

**imports**

*Inclusion\_Isotonicity*

*HOL—Analysis.Lipschitz*

*Interval\_Utilities*

**begin**

#### 7.1 Definition of Lipschitz Continuity on Intervals

An interval extension  $F$  is said to be lipschitz in  $X$  if there is a constant  $L$  such that  $w(F X) \leq L w X$  for every  $X \subseteq X$ . Hence the width of  $F X$  approaches 0 at least linearly with the width of  $X$ .

**definition** *lipschitzl\_on* :: 'a::{zero,minus,preorder,times}  $\Rightarrow$  'a interval set  $\Rightarrow$  ('a interval  $\Rightarrow$  'a interval)  $\Rightarrow$  bool  
**where** *lipschitzl\_on*  $C U F \longleftrightarrow (0 \leq C \wedge (\forall x \in U. \text{width}(F x) \leq C * \text{width } x))$

The definition *lipschitzl\_on* refers to definition 6.1 in [4]

**open\_bundle** *lipschitzl\_syntax*

**begin**

**notation** *lipschitzl\_on* ( $\leftarrow$ —*lipschitzl'\_on*) [1000]

**end**

**lemma** *lipschitzl\_onI*:  $C \text{—} \text{lipschitzl\_on } U F$

**if**  $\bigwedge x. x \in U \implies \text{width}(F x) \leq C * \text{width } x$   $0 \leq C$   
**using that by** (auto simp: *lipschitzl\_on\_def*)

**lemma** *lipschitzl\_onD*:

$\text{width}(F x) \leq C * \text{width } x$   
**if**  $C \text{—} \text{lipschitzl\_on } U F$   $x \in U$   
**using that by** (auto simp: *lipschitzl\_on\_def*)

**lemma** *lipschitzl\_on\_nonneg*:

$0 \leq C$  **if**  $C \text{—} \text{lipschitzl\_on } U F$   
**using that by** (auto simp: *lipschitzl\_on\_def*)

**lemma** *lipschitz\_on\_normD*:

$\text{norm}(\text{width}(F x)) \leq C * \text{norm}(\text{width } x)$   
**if**  $C \text{—} \text{lipschitzl\_on } U F$   $x \in U$   
**using** *lipschitzl\_onD*[OF that]  
**by** (simp add: *width\_def dist\_norm*)

**lemma** *lipschitzl\_on\_mono*:  $L \text{—} \text{lipschitzl\_on } D (F:: 'a::\{\text{linordered\_ring}\} \text{ interval} \Rightarrow 'a \text{ interval})$

**if**  $M \text{—} \text{lipschitzl\_on } E F M \leq L D \subseteq E$

```

using that diff_ge_o_iff_ge lower_le_upper
order_trans[OF _ mult_right_mono]
unfolding lipschitzl_on_def width_def
by (metis (no_types, lifting) order_trans subsetD width_def)

```

```

lemmas lipschitzl_on_subset
= lipschitzl_on_mono[OF _ order_refl]
and lipschitzl_on_le = lipschitzl_on_mono[OF _ order_refl]

```

```

lemma lipschitzl_on_lel:
  C—lipschitzl_on U F
  if  $\bigwedge x. x \in U \implies \text{width}(F x) \leq C * \text{width } x$ 
  o  $\leq C$ 
  for F::'a::{\minus,preorder,times,zero} interval  $\implies$  'a interval
proof (rule lipschitzl_onl)
fix x assume x: x  $\in U$ 
consider upper x  $\leq$  lower x | lower x  $\leq$  upper x
  by simp
then show width (F x)  $\leq$  C * width x
proof cases
  case 1
  then have width (F x)  $\leq$  C * width x
  by (auto intro!: that x)
  then show ?thesis
  by (simp add: dist_commute)
qed (auto intro!: that x)
qed fact

```

### 7.1.1 Lipschitz Continuity of Operations

Let  $F$  and  $G$  be inclusion isotonic interval extensions with  $F$  Lipschitz in  $Y$  and  $G$  Lipschitz in  $X$  and  $G X \subseteq Y$ . Then the composition  $H X = F (G X)$  is Lipschitz in  $X$  and is inclusion isotonic

```

lemma lipschitzl_on_compose:
  fixes U :: <'a::{\linordered_ring} interval set>
  assumes f: C—lipschitzl_on U F and g: D—lipschitzl_on (F'U) G
  shows (D * C)—lipschitzl_on U (G o F)
proof (rule lipschitzl_onl)
  show D * C  $\geq$  0 using lipschitzl_on_nonneg[OF f] lipschitzl_on_nonneg[OF g] by simp
  fix x assume H: x  $\in U$ 
  have width (G(F x))  $\leq$  D * C * width x
  using lipschitzl_onD[OF g] H assms
  unfolding width_def lipschitzl_on_def apply simp
  by (smt (verit, ccfv_SIG) mult.assoc mult_left_mono order_trans)
  then show width ((G o F) x)  $\leq$  D * C * width x
  unfolding comp_def by simp
qed

```

The definition  $\llbracket ?C—lipschitzl_on ?U ?F; ?D—lipschitzl_on (?F' ?U) ?G \rrbracket \implies (?D * ?C)—lipschitzl_on ?U (?G o ?F)$  refers to lemma 6.3 in [4]

```

lemma lipschitzl_on_compose2:
  fixes U :: <'a::{\linordered_ring} interval set>
  assumes F: C—lipschitzl_on U F and G: D—lipschitzl_on (F'U) G
  shows (D * C)—lipschitzl_on U ( $\lambda x. G (F x)$ )

```

using lipschitzl\_on\_compose F G unfolding o\_def by blast

lemma lipschitzl\_on\_cong:

$C \text{--} \text{lipschitzl\_on } U \ G \longleftrightarrow D \text{--} \text{lipschitzl\_on } V \ F$

if  $C = D \ U = V \ \bigwedge x. x \in V \implies G \ x = F \ x$

using that by (auto simp: lipschitzl\_on\_def)

lemma lipschitzl\_on\_transform:

$C \text{--} \text{lipschitzl\_on } U \ G$

if  $C \text{--} \text{lipschitzl\_on } U \ F$

$\bigwedge x. x \in U \implies G \ x = F \ x$

using that

unfolding lipschitzl\_on\_def width\_def

by simp

lemma lipschitz\_on\_empty\_iff:  $C \text{--} \text{lipschitzl\_on } \{\} \ f \longleftrightarrow C \geq 0$

by (auto simp: lipschitzl\_on\_def)

lemma lipschitz\_on\_id:  $(1::\text{real}) \text{--} \text{lipschitzl\_on } U \ (\lambda x. x)$

by (auto simp: lipschitzl\_on\_def)

lemma lipschitz\_on\_constant:

assumes  $\langle \text{lower } c = \text{upper } c \rangle$

shows  $(0::\text{real}) \text{--} \text{lipschitzl\_on } U \ (\lambda x. c)$

using assms by (auto simp: lipschitzl\_on\_def width\_def)

lemma lipschitzl\_on\_mult:

fixes  $X :: 'a :: \{\text{linordered\_idom}\}$

assumes lipschitzl\_on C U f

and  $1 \leq X$

shows lipschitzl\_on  $(X * C) \ U \ f$

using assms interval\_width\_positive lower\_le\_upper mult\_le\_cancel\_right1

unfolding lipschitzl\_on\_def width\_def

by (smt (verit) diff\_ge\_o\_iff\_ge linorder\_not\_le mult.assoc order\_trans)

## 7.1.2 Interval bounds on reals

lemma bounded\_image\_real:

fixes  $X :: \text{real interval}$  and  $f :: \text{real} \Rightarrow \text{real}$

assumes  $\forall x \in \{\text{lower } X .. \text{upper } X\}.$

$f (\text{lower } X) - L * (\text{upper } X - \text{lower } X) \leq f \ x \wedge f \ x \leq f (\text{lower } X) + L * (\text{upper } X - \text{lower } X)$

shows  $\exists x \ e. \forall y \in f' \{\text{lower } X .. \text{upper } X\}. \text{dist } x \ y \leq e$

proof -

let  $?x = f (\text{lower } X)$

let  $?e = L * (\text{upper } X - \text{lower } X)$

have  $\forall y \in f' \{\text{lower } X .. \text{upper } X\}. \text{dist } ?x \ y \leq ?e$

proof

fix y assume  $y \in f' \{\text{lower } X .. \text{upper } X\}$

then obtain x where  $x \in \{\text{lower } X .. \text{upper } X\}$  and  $y = f \ x$  by auto

then have  $?x - ?e \leq y \wedge y \leq ?x + ?e$  using assms by auto

then show  $\text{dist } ?x \ y \leq ?e$

unfolding dist\_real\_def by force

qed

then show ?thesis by auto

qed

**lemma** *lipschitz\_bounded\_image\_real*:  
fixes  $X :: \text{real set}$  and  $f :: \text{real} \Rightarrow \text{real}$   
assumes *bounded*  $X$  and *L-lipschitz\_on*  $X$   $f$   
shows *bounded*  $(f' X)$   
using *assms(1)* *assms(2)* *bounded\_uniformly\_continuous\_image* *lipschitz\_on\_uniformly\_continuous* **by** *blast*

**lemma** *inf\_le\_sup\_image\_real*:  
fixes  $X :: \text{real interval}$  and  $f :: \text{real} \Rightarrow \text{real}$   
assumes *L-lipschitz\_on*  $(\text{set_of } X)$   $f$   
shows  $\text{Inf } (f' \text{ set_of } X) \leq \text{Sup } (f' \text{ set_of } X)$

**proof** –

have *bounded*  $(f' \text{ set_of } X)$   
using *assms* *bounded\_set\_of* *lipschitz\_bounded\_image\_real* **by** *blast*  
then have *bdd\_below*  $(f' \text{ set_of } X)$  and *bdd\_above*  $(f' \text{ set_of } X)$   
using *bounded\_imp\_bdd\_below* *bounded\_imp\_bdd\_above* **by** *auto*  
then have  $\text{Inf } (f' \text{ set_of } X) \leq \text{Sup } (f' \text{ set_of } X)$   
**by**  $(\text{metis } \text{set_of\_nonempty } \text{cInf\_le\_cSup } \text{empty\_is\_image})$   
then show *?thesis* **by** *simp*

qed

**lemma** *sup\_image\_le\_real*:  
fixes  $f :: \text{real} \Rightarrow \text{real}$  and  $F :: \text{real interval} \Rightarrow \text{real interval}$  and  $X :: \text{real interval}$   
assumes  $f' \text{ set_of } X \subseteq \text{set_of } (F X)$   
and *L-lipschitz\_on*  $(\text{set_of } X)$   $f$   
shows  $\text{Sup } (f' \text{ set_of } X) \leq \text{Sup } (\text{set_of } (F X))$

**proof** –

have *ao*: *bounded*  $(f' \text{ set_of } X)$   
using *lipschitz\_bounded\_image\_real*[*of set\_of X*] *assms* *bounded\_set\_of*[*of X*] **by** *simp*  
have *a1*: *bdd\_above*  $(f' \text{ set_of } X)$   
using *assms*  
using *ao* *bounded\_imp\_bdd\_above* **by** *presburger*  
then have *a2*:  $\forall y \in f' \text{ set_of } X. y \leq \text{Sup } (f' \text{ set_of } X)$  using *bounded\_has\_Sup(1)* *ao*  
**by** *blast*  
then have *a3*:  $\text{Sup } (f' \text{ set_of } X) \leq \text{Sup } (\text{set_of } (F X))$   
using *set\_of\_nonempty* *assms* *ao* *a1* *a2* *atLeastAtMost\_iff* *closed\_real\_atLeastAtMost* *closed\_subset\_contains\_Sup*  
*empty\_is\_image* *set\_of\_eq\_sup\_set\_of*  
**by** *metis*  
then show *?thesis* **by** *simp*

qed

**lemma** *inf\_image\_le\_real*:  
fixes  $f :: \text{real} \Rightarrow \text{real}$  and  $F :: \text{real interval} \Rightarrow \text{real interval}$  and  $X :: \text{real interval}$   
assumes  $f' \text{ set_of } X \subseteq \text{set_of } (F X)$   
and *L-lipschitz\_on*  $(\text{set_of } X)$   $f$   
shows  $\text{Inf } (\text{set_of } (F X)) \leq \text{Inf } (f' (\text{set_of } X))$

**proof** –

have *ao*: *bounded*  $(f' \text{ set_of } X)$   
using *lipschitz\_bounded\_image\_real*[*of set\_of X*] *assms* *bounded\_set\_of*[*of X*] **by** *simp*  
have *a1*: *bdd\_above*  $(f' \text{ set_of } X)$   
using *assms*  
**by**  $(\text{metis } \text{atLeastAtMost\_iff } \text{bdd\_above.unfold } \text{set\_of\_eq } \text{subset\_eq})$   
then have *a2*:  $\forall y \in f' \text{ set_of } X. y \geq \text{Inf } (f' \text{ set_of } X)$

```
using bounded_has_Inf(1) a0
by blast
then have a3:  $\text{Inf}(\text{set\_of}(FX)) \leq \text{Inf}(f'(\text{set\_of}X))$  using assms
  by (metis set_of_nonempty a0 bounded_imp_bdd_below closed_real_atLeastAtMost closed_subset_contains_Inf
empty_is_image in_bounds inf_set_of set_of_eq)
then show ?thesis by simp
qed

end
```



## 8 Multi-Intervals (Multi\_Interval\_Preliminaries)

### 8.1 Preliminaries

**theory**

*Multi\_Interval\_Preliminaries*

**imports**

*HOL-Library.Interval*

*HOL-Analysis.Analysis*

*Inclusion\_Isotonicity*

**begin**

#### 8.1.1 A Class for Capturing Monotonicity of Minus

We try to keep our formalisation of interval arithmetic as generic as possible. In particular, we want to support intervals of type *nat*, *int*, *real*, and *ereal*. For all these types, minus (subtraction) is monotonous. Sadly, Isabelle lacks a type class capturing this. Luckily, it is very easy to define our own:

```
class minus_mono = minus + linorder +  
  assumes minus_mono:  $A \leq B \implies D \leq C \implies A - C \leq B - D$   
begin end
```

```
instance nat::minus_mono
```

```
  by(standard, simp)
```

```
instance int::minus_mono
```

```
  by(standard, simp)
```

```
instance real::minus_mono
```

```
  by(standard, simp)
```

```
instance integer::minus_mono
```

```
  by(standard, simp)
```

```
instance ereal::minus_mono
```

```
  by(standard, simp add: ereal_minus_mono)
```

#### 8.1.2 Infrastructure for Lifting Interval Operations to Lists of Intervals

```
definition un_op_interval_list:: $\langle 'a \text{ interval} \Rightarrow 'a \text{ interval} \rangle$   
   $\Rightarrow 'a \text{ interval list} \Rightarrow 'a \text{ interval list}$ 
```

```
  where
```

```
   $\langle \text{un\_op\_interval\_list } op \text{ } xs = \text{map } op \text{ } xs \rangle$ 
```

```
definition bin_op_interval_list:: $\langle 'a \text{ interval} \Rightarrow 'a \text{ interval} \Rightarrow 'a \text{ interval} \rangle$   
   $\Rightarrow 'a \text{ interval list} \Rightarrow 'a \text{ interval list} \Rightarrow 'a \text{ interval list}$ 
```

```
  where
```

```
   $\langle \text{bin\_op\_interval\_list } op \text{ } xs \text{ } ys = \text{concat } (\text{map } (\lambda x . \text{map } (op \text{ } x) \text{ } xs) \text{ } ys) \rangle$ 
```

```
lemma bin_op_interval_list_non_empty:  $\langle xs \neq [] \wedge ys \neq [] \rangle = (\text{bin\_op\_interval\_list } op \text{ } xs \text{ } ys \neq [])$ 
```

```
  unfolding bin_op_interval_list_def
```

```
  by(auto simp add: ex_in_conv)
```

```

lemma bin_op_interval_list_empty: <(xs = [] ∨ ys = []) = (bin_op_interval_list op xs ys = [])>
  unfolding bin_op_interval_list_def
  by(auto simp add: ex_in_conv)

```

```

lemma bin_op_interval_unroll: <bin_op_interval_list op (xs) (y#ys) = (map (op y) xs)@(bin_op_interval_list op xs ys)>
  unfolding bin_op_interval_list_def by(simp)

```

```

lemma bin_op_interval_list_commute:
  assumes op_commute: <∧ x y. op x y = op y x>
  shows <set (bin_op_interval_list (op) xs ys) = set (bin_op_interval_list (op) ys xs)>
  unfolding bin_op_interval_list_def
  proof(induction xs ys rule:list_induct2')
  case 1
  then show ?case by simp
  next
  case (2 x xs)
  then show ?case by simp
  next
  case (3 y ys)
  then show ?case by simp
  next
  case (4 x xs y ys)
  then show ?case
  by(auto simp add: op_commute, blast)
  qed

```

```

lemma bin_op_interval_list_assoc:
  assumes op_assoc: <∧ x y z. op (op x y) z = op x (op y z)>
  shows <set (bin_op_interval_list (op) ((bin_op_interval_list (op) xs ys) zs) = set (bin_op_interval_list (op) xs
  ((bin_op_interval_list (op) ys zs)))>
  unfolding bin_op_interval_list_def
  proof(induction xs ys rule:list_induct2')
  case 1
  then show ?case by simp
  next
  case (2 x xs)
  then show ?case by simp
  next
  case (3 y ys)
  then show ?case by simp
  next
  case (4 x xs y ys)
  then show ?case
  proof(induction zs)
  case Nil
  then show ?case by simp
  next
  case (Cons a zs)
  then show ?case
  apply(auto simp add: 4 list.set_map op_assoc)[1]
  subgoal
  by (meson imagel)
  subgoal

```

```

  by blast
subgoal
  by (meson UnionI imagel insertI1)
subgoal
  by (meson UN_I imagel insertCI)
done
qed
qed

```

## Lifting Unary Minus, Addition, and Multiplication

```

definition <iList_uminus = un_op_interval_list (λ x. - x)>

```

```

definition <iList_plus = bin_op_interval_list (+)>

```

```

definition <iList_times = bin_op_interval_list (*)>

```

```

lemma iList_plus_lower:

```

```

  assumes <A ≠ []> and <B ≠ []>

```

```

shows <lower (hd (iList_plus A B)) = lower (hd A) + lower (hd B)>

```

```

proof(insert assms, induction B)

```

```

  case Nil

```

```

  then show ?case by simp

```

```

next

```

```

  case (Cons a B)

```

```

  then show ?case

```

```

    unfolding iList_plus_def

```

```

    apply(simp add: hd_map bin_op_interval_unroll[of (+) A a B])

```

```

    using add.commute by blast

```

```

qed

```

```

lemma iList_plus_upper:

```

```

  assumes <A ≠ []> and <B ≠ []>

```

```

shows <upper (hd (iList_plus A B)) = upper (hd A) + upper (hd B)>

```

```

proof(insert assms, induction B)

```

```

  case Nil

```

```

  then show ?case by simp

```

```

next

```

```

  case (Cons a B)

```

```

  then show ?case

```

```

    unfolding iList_plus_def

```

```

    apply(simp add: hd_map bin_op_interval_unroll[of (+) A a B])

```

```

    using add.commute by blast

```

```

qed

```

```

lemma iList_plus_unroll:

```

```

<iList_plus ys (x # xs) = map ((+) x) ys @ iList_plus ys xs>

```

```

by (simp add: iList_plus_def bin_op_interval_unroll[of (+) ys x xs])

```

```

lemma a ≠ [] ⇒ (iList_plus [Interval (0, 0)] a) = (a::'a::{ordered_ab_group_add,zero} interval list)

```

```

proof(induction a rule:list_nonempty_induct)

```

```

  case (single x)

```

```

  then show ?case

```

```

    by (metis add.right_neutral append.right_neutral

```

```

        bin_op_interval_list_non_empty iList_plus_def iList_plus_unroll list.map(2) zero_interval_def)

```

```

next

```

```

case (cons x xs)
then show ?case
by (metis add.right_neutral append_Cons append_Nil iList_plus_unroll list.map(1) list.map(2) zero_interval_def)
qed

```

```

lemma iList_plus_commute:
<set (iList_plus xs ys) = set (iList_plus ys xs)>
using bin_op_interval_list_commute[of (+) xs ys] add.commute
unfolding iList_plus_def
by auto

```

```

lemma iList_plus_assoc:
<set (iList_plus xs (iList_plus ys zs)) = set (iList_plus (iList_plus xs ys) zs)>
using bin_op_interval_list_assoc[of (+) xs ys zs] add.assoc
unfolding iList_plus_def
by auto

```

```

lemma remdups_append1:
remdups (remdups xs @ ys) = remdups (xs @ ys)
by(induction xs) auto

```

```

lemma bin_op_interval_remdups_left:
<remdups (bin_op_interval_list op (remdups x) y) = remdups (bin_op_interval_list op x y)>
proof(induction y)
case Nil
then show ?case
by (simp add: bin_op_interval_list_def)
next
case (Cons y' ys')
then show ?case
by (metis (no_types, opaque_lifting) bin_op_interval_list_non_empty bin_op_interval_unroll
remdups_append1 remdups_append2 remdups_map_remdups)
qed

```

```

lemma iList_plus_remdups_left:
remdups (iList_plus (remdups a) b) = remdups (iList_plus a b)
for a::'a::{minus_mono,ordered_ab_semigroup_add,linorder,linordered_field} interval list
proof(induction b)
case Nil
then show ?case
by (metis bin_op_interval_list_empty iList_plus_def)
next
case (Cons a b)
then show ?case
by (metis bin_op_interval_remdups_left iList_plus_def list.discr remdups.simps(1))
qed

```

```

lemma interval_add_eq: <a + b = Interval(lower a + lower b, upper a + upper b)>
apply(subst interval_eq_iff[of a+b Interval (lower a + lower b, upper a + upper b)])
using upper_plus[of a b] lower_plus[of a b]
by (simp add: add_mono)

```

```

lemma iList_plus_lower_upper_eq:
  <iList_plus = bin_op_interval_list (λ a b. Interval(lower a + lower b, upper a + upper b))>
  unfolding iList_plus_def using interval_add_eq
  by metis

```

### A Locale for Multi-Interval Division Where the Quotient does not Contain 0

```

context interval_division
begin

```

```

  definition <iList_inverse = un_op_interval_list inverse>
  definition <iList_divide = bin_op_interval_list divide>

```

```

end

```

### 8.1.3 Utilities for (Sorted) Lists of Intervals

```

definition <cmp_lower_width = (λ x y. if lower x = lower y then width x ≤ width y else lower x < lower y)>

```

```

definition <sorted_wrt_lower = sorted_wrt cmp_lower_width>

```

```

lemma interval_lower_width_eq:
  <(lower x = lower y ∧ width x = width y) = (x = (y::'a::{minus_mono, linordered_field} interval))>
  by (metis interval_eq le_add_diff_inverse lower_le_upper width_def)

```

```

lemma sorted_wrt_lower_distinct_lists_eq:
  assumes <set xs = set (ys::'a::{minus_mono, linordered_field} interval list)>
  and <distinct xs> and <distinct ys>
  and <sorted_wrt_lower xs> and <sorted_wrt_lower ys>
  shows <xs = ys>

```

```

proof(insert assms, unfold sorted_wrt_lower_def cmp_lower_width_def, induct xs ys rule:list_induct2')
  case 1
  then show ?case by simp
next
  case (2 x xs)
  then show ?case by simp
next
  case (3 y ys)
  then show ?case by simp
next
  case (4 x xs y ys)
  then show ?case
  using interval_lower_width_eq
  apply(auto)[1]
  subgoal by (smt (verit) insert_iff interval_lower_width_eq order_asym order_le_imp_less_or_eq)
  subgoal by (smt (verit, ccfv_SIG) antisym insert1 insert_eq_iff interval_lower_width_eq not_less_iff_gr_or_eq)
  done
qed

```

```

definition <sorted_wrt_upper = sorted_wrt (λ x y. upper x ≤ upper y)>

```

```

definition <cmp_non_overlapping = (λ x y. upper x ≤ lower y)>

```

```

lemma cmp_non_overlapping_lower: <cmp_non_overlapping x y ⇒ lower x ≤ lower y>

```

```
using lower_le_upper unfolding cmp_non_overlapping_def
by(metis dual_order.trans)
```

```
lemma cmp_non_overlapping_upper: <cmp_non_overlapping x y  $\implies$  upper x  $\leq$  upper y>
using lower_le_upper unfolding cmp_non_overlapping_def
by(metis dual_order.trans)
```

```
definition <non_overlapping_sorted = sorted_wrt cmp_non_overlapping>
definition <contiguous xs = ( $\forall$  i < length xs - 1 . upper (xs ! i) = lower (xs ! (i + 1)))>
```

```
lemma non_overlapping_sorted_empty: <non_overlapping_sorted []>
by (simp add: non_overlapping_sorted_def cmp_non_overlapping_def)
```

```
lemma non_overlapping_sorted_unroll:
  assumes xs  $\neq$  [] shows non_overlapping_sorted (x # xs) = (upper x  $\leq$  lower (hd xs)  $\wedge$  non_overlapping_sorted xs)
proof(cases xs)
  case Nil
  then show ?thesis using assms by(simp)
next
  case (Cons y ys)
  then show ?thesis
  apply(simp add: non_overlapping_sorted_def cmp_non_overlapping_def cmp_lower_width_def)
  using lower_le_upper
  by (metis dual_order.trans)
qed
```

```
lemma contiguous_non_overlapping: <contiguous (is::'a::{preorder} interval list)  $\implies$  non_overlapping_sorted is>
proof(induction isrule:induct_listo12)
  case 1
  then show ?case
  unfolding contiguous_def non_overlapping_sorted_def cmp_non_overlapping_def
  using lower_le_upper
  by simp
next
  case (2 x)
  then show ?case
  unfolding contiguous_def non_overlapping_sorted_def cmp_non_overlapping_def
  using lower_le_upper
  by simp
next
  case (3 x y zs)
  then show ?case
  apply(subst non_overlapping_sorted_unroll[of (y#zs) x])
  subgoal by(simp)
  subgoal apply(simp add: 3) using 3
  unfolding contiguous_def non_overlapping_sorted_def cmp_non_overlapping_def
  using lower_le_upper
  by fastforce
  done
qed
```

```
definition <cmp_non_adjacent = ( $\lambda$  x y. upper x < lower y)>
```

```

lemma cmp_non_adjacent_lower: <cmp_non_adjacent x y  $\implies$  lower x < lower y>
using lower_le_upper unfolding cmp_non_adjacent_def
by(metis dual_order.strict_trans2)

```

```

lemma cmp_non_adjacent_upper: <cmp_non_adjacent x y  $\implies$  upper x < upper y>
using lower_le_upper unfolding cmp_non_adjacent_def
by(metis dual_order.strict_trans1)

```

```

definition <non_adjacent_sorted_wrt_lower = sorted_wrt cmp_non_adjacent>

```

```

lemma non_adjacent_sorted_wrt_lower_unroll:
assumes xs  $\neq$  []
shows non_adjacent_sorted_wrt_lower (x # xs) =
  ((upper x < lower (hd xs))  $\wedge$  non_adjacent_sorted_wrt_lower xs)
proof(cases xs)
case Nil
then show ?thesis using assms by(simp)
next
case (Cons y ys)
then show ?thesis
  apply(simp add:non_adjacent_sorted_wrt_lower_def cmp_non_adjacent_def)
  by (meson cmp_non_adjacent_def cmp_non_adjacent_lower dual_order.strict_trans)
qed

```

```

lemma non_adjacent_implies_non_overlapping:
assumes <non_adjacent_sorted_wrt_lower is> shows <non_overlapping_sorted is>
proof(insert assms, induction is)
case Nil
then show ?case
  unfolding non_overlapping_sorted_def cmp_non_overlapping_def
    non_adjacent_sorted_wrt_lower_def cmp_non_adjacent_def
  by(simp)
next
case (Cons a is)
then show ?case
  unfolding non_overlapping_sorted_def cmp_non_overlapping_def
    non_adjacent_sorted_wrt_lower_def cmp_non_adjacent_def
  using order.strict_implies_order by auto
qed

```

```

lemma non_overlapping_implies_sorted_wrt_lower:
assumes <non_overlapping_sorted (is::'a::{minus_mono} interval list)>
shows <sorted_wrt_lower is>
proof(insert assms, induction is)
case Nil
then show ?case unfolding sorted_wrt_lower_def by simp
next
case (Cons a is)
then show ?case
  unfolding non_overlapping_sorted_def sorted_wrt_lower_def
    cmp_lower_width_def cmp_non_overlapping_def
  by (simp, metis (no_types, lifting) cmp_non_adjacent_def cmp_non_adjacent_lower minus_mono
    dual_order.strict_iff_order lower_le_upper width_def)

```

qed

**lemma** *non\_overlapping\_implies\_sorted\_wrt\_upper*:

**assumes**  $\langle \text{non\_overlapping\_sorted } is \rangle$

**shows**  $\langle \text{sorted\_wrt\_upper } is \rangle$

**proof**(*insert assms, induction is rule:induct\_list012*)

**case** 1

**then show** ?case **by**(*simp add: leD sorted\_wrt\_upper\_def*)

**next**

**case** (2 x)

**then show** ?case **by**(*simp add: leD sorted\_wrt\_upper\_def*)

**next**

**case** (3 x y zs)

**then show** ?case

**by**(*auto simp add: cmp\_non\_overlapping\_upper non\_overlapping\_sorted\_def sorted\_wrt\_upper\_def leD*)[1]

qed

**lemma** *non\_adjacent\_implies\_sorted\_wrt\_lower*:

**assumes**  $\langle \text{non\_adjacent\_sorted\_wrt\_lower } is \rangle$

**shows**  $\langle \text{sorted\_wrt\_lower } is \rangle$

**proof**(*insert assms, induction is*)

**case** Nil

**then show** ?case

**unfolding** *sorted\_wrt\_lower\_def*

**by** *simp*

**next**

**case** (Cons a is)

**then show** ?case

**unfolding** *sorted\_wrt\_lower\_def non\_adjacent\_sorted\_wrt\_lower\_def*

*cmp\_non\_adjacent\_def cmp\_lower\_width\_def width\_def*

**by**(*simp split:if\_splits, metis dual\_order.strict\_trans2 lower\_le\_upper order\_less\_irrefl*)

qed

**lemma** *non\_adjacent\_implies\_distinct*:

**assumes**  $\langle \text{non\_adjacent\_sorted\_wrt\_lower } is \rangle$

**shows**  $\langle \text{distinct } is \rangle$

**proof**(*insert assms, induction is*)

**case** Nil

**then show** ?case

**unfolding** *sorted\_wrt\_lower\_def*

**by** *simp*

**next**

**case** (Cons a is)

**then show** ?case

**unfolding** *sorted\_wrt\_lower\_def non\_adjacent\_sorted\_wrt\_lower\_def*

*cmp\_non\_adjacent\_def cmp\_lower\_width\_def width\_def*

**by**(*simp split:if\_splits, metis dual\_order.strict\_trans2 lower\_le\_upper order\_less\_irrefl*)

qed

**fun** *merge\_overlapping\_intervals\_sorted\_wrt\_lower* :: 'a::linorder interval list  $\Rightarrow$  'a interval list **where**

*merge\_overlapping\_intervals\_sorted\_wrt\_lower* [] = [] |

*merge\_overlapping\_intervals\_sorted\_wrt\_lower* [x] = [x] |

*merge\_overlapping\_intervals\_sorted\_wrt\_lower* (x#y#ys) =

( if upper x  $\leq$  lower y

```

then x#(merge_overlapping_intervals_sorted_wrt_lower (y#ys))
else merge_overlapping_intervals_sorted_wrt_lower ((mk_interval(lower x, max (upper x) (upper y)))#ys)
)

```

**lemma sorted\_wrt\_lower\_unroll:**

**assumes**  $xs \neq []$

**shows**  $sorted\_wrt\_lower (x \# xs) = ((if\ lower\ x \neq\ lower\ (hd\ xs)$   
 $\quad then\ lower\ x < lower\ (hd\ xs)$   
 $\quad else\ width\ x \leq width\ (hd\ xs)) \wedge sorted\_wrt\_lower (xs))$

**proof**(cases xs)

**case** Nil

**then show** ?thesis **using** *assms* **by** (*simp*)

**next**

**case** (Cons y ys)

**then show** ?thesis

**apply** (*simp* *add: sorted\_wrt\_lower\_def cmp\_lower\_width\_def*)

**by** (*smt* (*verit*) *dual\_order.irrefl dual\_order.strict\_trans2 dual\_order.trans order.strict\_implies\_order*)

**qed**

**lemma sorted\_wrt\_upper\_unroll:**

**assumes**  $xs \neq []$

**shows**  $sorted\_wrt\_upper (x \# xs) = ((upper\ x \leq upper\ (hd\ xs)) \wedge sorted\_wrt\_upper (xs))$

**proof**(cases xs)

**case** Nil

**then show** ?thesis **using** *assms* **by** (*simp*)

**next**

**case** (Cons y ys)

**then show** ?thesis

**apply** (*simp* *add: sorted\_wrt\_upper\_def cmp\_lower\_width\_def*)

**using** *dual\_order.trans* **by** *blast*

**qed**

**lemma sorted\_wrt\_lower\_tail:**  $sorted\_wrt\_lower (x \# xs) \implies sorted\_wrt\_lower (xs)$

**unfolding** *sorted\_wrt\_lower\_def*

**by** *simp*

**lemma sorted\_wrt\_lower\_tail':**  $sorted\_wrt\_lower (x \# y \# ys) \implies sorted\_wrt\_lower (x \# ys)$

**unfolding** *sorted\_wrt\_lower\_def* **by** *simp*

**lemma iList\_plus\_leq\_B:**

**assumes** *sorted\_wrt\_lower A* **and** *sorted\_wrt\_lower B* **and**  $1 < length\ B$

**shows**  $hd (map\ lower (iList\_plus\ A\ B)) \leq hd (map\ lower (iList\_plus\ A (tl\ B)))$

**proof**(*insert assms, induction B*)

**case** Nil

**then show** ?case **by** *simp*

**next**

**case** (Cons b bs) **note** \* = *this*

**then show** ?case

**proof**(cases bs)

**case** Nil

**then show** ?thesis

**using** \* **by** *simp*

```

next
  case (Cons b' bs^) note ** = this
  then have a: <hd (map lower (iList_plus A (b#bs))) ≤ hd (map lower (iList_plus A bs))>
  proof(induction bs)
    case Nil
    then show ?case by simp
  next
  case (Cons b'' bs'')
  then show ?case
    using * **
    by (smt (z3) add.commute add_right_mono dual_order.eq_iff hd_append iList_plus_lower
        iList_plus_unroll list.map_disc_iff list.map_sel(1) list.sel(1) list.simps(3)
        map_append order_less_imp_le sorted_wrt_lower_unroll)
  qed
  then show ?thesis
  using * ** a by simp
qed
qed
qed

lemma iList_plus_leq_A:
  assumes sorted_wrt_lower A and sorted_wrt_lower B and 1 < length A
  shows hd (map lower (iList_plus A B)) ≤ hd (map lower (iList_plus (tl A) B))
  proof(insert assms, induction A)
    case Nil
    then show ?case by simp
  next
  case (Cons a as) note * = this
  then show ?case
  proof(cases as)
    case Nil
    then show ?thesis
    using * by simp
  next
  case (Cons a' as') note ** = this
  then have a: <hd (map lower (iList_plus (a#as) B)) ≤ hd (map lower (iList_plus as B))>
  proof(induction as)
    case Nil
    then show ?case by simp
  next
  case (Cons a'' as'')
  then show ?case
    using * **
    by (smt (verit, best) add_right_mono bin_op_interval_list_empty dual_order.eq_iff
        iList_plus_def iList_plus_lower list.map_sel(1) list.sel(1) list.simps(3)
        order_less_imp_le sorted_wrt_lower_unroll)
  qed
  then show ?thesis
  using * ** a by simp
qed
qed

```

**value** <merge\_overlapping\_intervals\_sorted\_wrt\_lower [mk\_interval(1::int,2), mk\_interval(2,3), mk\_interval(5,7), mk\_interval(6,10)]>

```

lemma merge_overlapping_intervals_sorted_wrt_lower_non_nil:
  assumes  $\langle xs \neq [] \rangle$ 
  shows  $\langle \text{merge\_overlapping\_intervals\_sorted\_wrt\_lower } xs \rangle \neq []$ 
  using assms
  by(induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct, simp_all)

lemma merge_overlapping_intervals_sorted_hd_lower:
  assumes  $\langle xs \neq [] \rangle$ 
  shows  $\text{lower } (\text{hd } (\text{merge\_overlapping\_intervals\_sorted\_wrt\_lower } (xs))) = \text{lower } (\text{hd } xs)$ 
  proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case by(simp)
  next
  case (2 x)
  then show ?case by(simp)
  next
  case (3 x y ys)
  then show ?case
    using 3.IH(2) list.sel(1) lower_le_upper max.coboundedI2 mk_interval_lower
    by (metis list.discI max.commute merge_overlapping_intervals_sorted_wrt_lower.simps(3))
  qed

lemma merge_overlapping_intervals_sorted_hd_upper:
  assumes  $\langle xs \neq [] \rangle$ 
  shows  $\text{upper } (\text{hd } xs) \leq \text{upper } (\text{hd } (\text{merge\_overlapping\_intervals\_sorted\_wrt\_lower } xs))$ 
  proof(insert assms(1), induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case by(simp)
  next
  case (2 x)
  then show ?case by(simp)
  next
  case (3 x y ys)
  then show ?case
    proof(cases upper x ≤ lower y)
    case True note non_overlapping = this
    then show ?thesis by simp
  next
  case False note overlapping_or_included = this
  then show ?thesis proof(cases upper x ≤ upper y)
    case True note overlapping = this
    then show ?thesis
      proof(cases lower x ≤ upper y)
      case True
      then show ?thesis
        using 3.IH(2)[simplified overlapping_or_included]
        apply(simp add: overlapping_or_included overlapping True)
        using overlapping by simp
    next
    case False
    then show ?thesis
      using overlapping_or_included overlapping False
      by (metis lower_le_upper max.coboundedI1 max_def)
  qed

```

```

next
case False note included = this
then show ?thesis
proof(cases lower x ≤ upper y)
case True
then show ?thesis
using 3.IH(2)[simplified overlapping_or_included]
using included by simp
next
case False
then show ?thesis
using overlapping_or_included included False
lower_le_upper[of x] lower_le_upper[of y]
using 3.IH(2) by fastforce
qed
qed
qed
qed

lemma Interval_id[simp]: ⟨Interval (lower x, upper x) = x ⟩
by (simp add: interval_eq_iff)

lemma mk_interval_id[simp]: ⟨(mk_interval (lower x, upper x)) = x⟩
using lower_le_upper[of x] unfolding mk_interval' by(auto)

lemma merge_overlapping_intervals_sorted_hd_width:
assumes ⟨xs ≠ []⟩
shows width (hd xs) ≤ width (hd (merge_overlapping_intervals_sorted_wrt_lower (xs::'a::{minus_mono} interval
list)))
proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
case 1
then show ?case by(simp)
next
case (2 x)
then show ?case by(simp)
next
case (3 x y ys)
then show ?case
proof(cases upper x ≤ lower y)
case True note non_overlapping = this
then show ?thesis by simp
next
case False note overlapping_or_included = this
then show ?thesis proof(cases upper x ≤ upper y)
case True note overlapping = this
then show ?thesis
proof(cases lower x ≤ upper y)
case True
then show ?thesis
using 3.IH(2)[simplified overlapping_or_included]
apply(simp add: width_def overlapping_or_included overlapping True)
using True diff_right_mono list.sel(1) list.simps(3)
merge_overlapping_intervals_sorted_hd_lower merge_overlapping_intervals_sorted_hd_upper
mk_interval_lower mk_interval_upper order.trans overlapping minus_mono

```

```

    by (smt (verit, best) dual_order.refl)
  next
  case False
  then show ?thesis
  using overlapping_or_included overlapping False
  by (metis lower_le_upper max.coboundedl1 max_def)
qed
next
case False note included = this
then show ?thesis
proof(cases lower x ≤ upper y)
  case True
  then show ?thesis
  using 3.IH(2)[simplified overlapping_or_included] included
  by (metis list.discr list.sel(1) max_def
      merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id
      overlapping_or_included)
  next
  case False
  then show ?thesis
  using overlapping_or_included included False
  lower_le_upper[of x] lower_le_upper[of y]
  using 3.IH(2)[simplified overlapping_or_included]
  by (metis list.discr list.sel(1) max_def merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id)
qed
qed
qed
qed

```

```

lemma merge_overlapping_intervals_sorted_wrt_lower_sorted_lower:
  assumes ‹sorted_wrt_lower (xs::'a::{minus_mono} interval list)›
  shows ‹sorted_wrt_lower (merge_overlapping_intervals_sorted_wrt_lower xs)›
proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case
  unfolding sorted_wrt_lower_def
  by(simp)
next
case (2 x)
  then show ?case
  unfolding sorted_wrt_lower_def
  by(simp)
next
case (3 x y ys)
  then show ?case
  proof(cases upper x ≤ lower y)
    case True note non_overlapping = this
    then show ?thesis
    by (smt (verit) 3.IH(1) 3.prem1 dual_order.trans list.discr merge_overlapping_intervals_sorted_hd_lower
        merge_overlapping_intervals_sorted_hd_width merge_overlapping_intervals_sorted_wrt_lower.simps(3)
        merge_overlapping_intervals_sorted_wrt_lower_non_nil sorted_wrt_lower_unroll)
  next

```

```

case False note overlapping_or_included = this
then show ?thesis proof(cases upper x ≤ upper y)
  case True note overlapping = this
  then show ?thesis
  proof(cases lower x ≤ upper y)
    case True
    then show ?thesis
    using 3.IH(2)[simplified overlapping_or_included]
    by (smt (verit, ccfv_threshold) 3.prem list.sel(1) max.absorb2
      merge_overlapping_intervals_sorted_wrt_lower.simps(3)
      mk_interval_id mk_interval_lower overlapping overlapping_or_included sorted_wrt1
      sorted_wrt_lower_def sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
    next
    case False
    then show ?thesis
    using overlapping_or_included overlapping False
    using lower_le_upper order_trans by blast
    qed
  next
  case False note included = this
  then show ?thesis
  proof(cases lower x ≤ upper y)
    case True
    then show ?thesis
    using 3.IH(2)[simplified overlapping_or_included] included
      3.prem overlapping_or_included sorted_wrt_lower_tail'
    by (metis max_def merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id)
    next
    case False
    then show ?thesis
    using overlapping_or_included included False
      lower_le_upper[of x] lower_le_upper[of y]
    using 3.IH(2)[simplified overlapping_or_included]
      3.prem sorted_wrt_lower_tail'
    by (metis max_def merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id)
    qed
  qed
qed
qed

```

```

lemma merge_overlapping_intervals_sorted_sorted_non_non_overlapping:
assumes <sorted_wrt_lower (xs::'a::{minus_mono} interval list)>
shows <non_overlapping_sorted (merge_overlapping_intervals_sorted_wrt_lower xs)>
proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case unfolding non_overlapping_sorted_def
    by (metis merge_overlapping_intervals_sorted_wrt_lower.simps(1) sorted_wrt.simps(1))
  next
  case (2 x)
  then show ?case unfolding non_overlapping_sorted_def
    by (metis merge_overlapping_intervals_sorted_wrt_lower.simps(2) sorted_wrt1)
  next
  case (3 x y ys)
  then show ?case

```

```

proof(cases upper x ≤ lower y)
case True note non_overlapping = this
then show ?thesis
  using 3dual_order.trans list.disc1 merge_overlapping_intervals_sorted_hd_lower
    merge_overlapping_intervals_sorted_hd_width merge_overlapping_intervals_sorted_wrt_lower.simps
    merge_overlapping_intervals_sorted_wrt_lower_non_nil non_overlapping_sorted_unroll
  by (smt (verit, ccfv_threshold) list.sel(1) sorted_wrt_lower_tail)
next
case False note overlapping_or_included = this
then show ?thesis proof(cases upper x ≤ upper y)
  case True note overlapping = this
  then show ?thesis
  proof(cases lower x ≤ upper y)
  case True
  then show ?thesis
  using 3.IH(2)[simplified overlapping_or_included]
  by (smt (verit) 3.prem1 list.disc1 list.sel(1)
    max.absorb2 merge_overlapping_intervals_sorted_wrt_lower.simps(3)
    mk_interval_id mk_interval_lower overlapping_or_included sorted_wrt_lower_tail'
    sorted_wrt_lower_unroll)
  next
  case False
  then show ?thesis
  using overlapping_or_included overlapping False
  using lower_le_upper order_trans by blast
  qed
next
case False note included = this
then show ?thesis
proof(cases lower x ≤ upper y)
  case True
  then show ?thesis
  using 3.IH(2)[simplified overlapping_or_included] included
  using 3.prem1 overlapping_or_included sorted_wrt_lower_tail'
  by (metis max.orderE merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id nle_le)
  next
  case False
  then show ?thesis
  using overlapping_or_included included False
    lower_le_upper[of x] lower_le_upper[of y]
  using 3.IH(2)[simplified overlapping_or_included]
  using 3.prem1 sorted_wrt_lower_tail'
  by (metis max.orderE merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id nle_le)
  qed
qed
qed
qed

```

```

fun merge_adjacent_intervals_sorted_wrt_lower :: 'a::linorder interval list ⇒ 'a interval list where
merge_adjacent_intervals_sorted_wrt_lower [] = [] |
merge_adjacent_intervals_sorted_wrt_lower [x] = [x] |
merge_adjacent_intervals_sorted_wrt_lower (x#y#ys) =

```

```

( if upper x < lower y
  then x#(merge_adjacent_intervals_sorted_wrt_lower (y#ys))
  else merge_adjacent_intervals_sorted_wrt_lower ((mk_interval(lower x, max (upper y) (upper x)))#ys)
)

```

```

value <merge_adjacent_intervals_sorted_wrt_lower [mk_interval(1::int,2), mk_interval(2,3), mk_interval(5,7),
mk_interval(6,10)]>

```

```

lemma merge_adjacent_intervals_sorted_wrt_lower_non_nil:
  assumes <xs ≠ []>
  shows <(merge_adjacent_intervals_sorted_wrt_lower xs) ≠ []>
  using assms by(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct, simp_all)

```

```

lemma merge_adjacent_intervals_sorted_wrt_lower_non_nil':
  shows <(merge_adjacent_intervals_sorted_wrt_lower (x#xs)) ≠ []>
  by(simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil)

```

```

lemma merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd:
  assumes <sorted_wrt_lower xs>
  shows <lower (hd (merge_adjacent_intervals_sorted_wrt_lower xs)) = lower (hd xs)>
  using assms
proof(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case
    by (metis merge_adjacent_intervals_sorted_wrt_lower.simps(1))
next
  case (2 x)
  then show ?case by simp
next
  case (3 x y ys)
  then show ?case
proof(cases (sorted_wrt_lower (mk_interval (lower x, max (upper y) (upper x)) # ys)))
  case True
  then show ?thesis
    by (simp, metis 3.IH(2) list.sel(1) lower_le_upper max.coboundedI2 mk_interval_lower)
next
  case False
  then show ?thesis
    apply(simp, subst 3 (2))
    apply (smt (verit) 3 dual_order.strict_trans leD list.discI list.sel(1) lower_le_upper max.coboundedI2
      mk_interval_lower sorted_wrt1 sorted_wrt_lower_def sorted_wrt_lower_unroll)
    apply (smt (verit) 3.prem1 dual_order.trans interval_eqI list.discI list.sel(1)
      lower_le_upper max.strict_order_iff max_def mk_interval_lower mk_interval_upper
      sorted_wrt1 sorted_wrt_lower_def sorted_wrt_lower_unroll)
    by (simp add: 3 max.coboundedI2 sorted_wrt_lower_def)
qed
qed

```

```

lemma merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_subset:
  <set (map lower (merge_adjacent_intervals_sorted_wrt_lower xs)) ⊆ set (map lower xs)>
  apply(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
  by(auto simp add: max.coboundedI2 mk_interval' sorted_wrt_lower_def image_def)

```

```

lemma merge_adjacent_intervals_sorted_wrt_lower_set_eq:

```

```

assumes <set (xs::real interval list) = set ys>
  and <distinct xs> and <distinct ys>
  and <sorted_wrt_lower xs> and <sorted_wrt_lower ys>
shows <merge_adjacent_intervals_sorted_wrt_lower xs = merge_adjacent_intervals_sorted_wrt_lower ys>
using sorted_wrt_lower_distinct_lists_eq[of xs ys, simplified assms, simplified]
by(auto)

```

```

lemma merge_adjacent_intervals_sorted_wrt_lower_lower_upper:
  assumes sorted_wrt_lower xs
  shows  $x \in \text{set } (\text{merge\_adjacent\_intervals\_sorted\_wrt\_lower } xs) \implies \exists l \in \text{set } xs. \exists u \in \text{set } xs. \text{lower } l = \text{lower } x \wedge \text{upper } u = \text{upper } x$ 
proof(insert assms, induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case
  by simp
next
  case (2 x)
  then show ?case
  by simp
next
  case (3 x y ys)
  then show ?case
  proof(cases ys)
  case Nil
  then show ?thesis
  using 3
  by (simp, smt (z3) empty_iff_empty_set list.sel(1) list.sel(3) list.set_cases
    max_def merge_adjacent_intervals_sorted_wrt_lower.simps(2)
    merge_adjacent_intervals_sorted_wrt_lower.simps(3)
    merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd mk_interval_lower
    mk_interval_upper)
  next
  case (Cons a list)
  then show ?thesis
  apply(auto simp add: mk_interval' max_def)[1]
  subgoal
  using 3
  by (smt (z3) insert_iff interval_eq_iff list.sel(1) list.simps(15) list.simps(3) lower_le_upper
    max.cobounded1 max_def_raw merge_adjacent_intervals_sorted_wrt_lower.simps(3)
    mk_interval_lower mk_interval_upper sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
  subgoal
  using 3
  by (smt (verit) dual_order.trans list.discr list.inject list.sel(1) lower_le_upper max_def
    merge_adjacent_intervals_sorted_wrt_lower.elims mk_interval_id mk_interval_lower
    mk_interval_upper order.asym set_ConsD sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
  done
  qed
qed

```

```

primrec interval_insert_sort_lower_width :: ('a::{linorder,minus}) interval  $\Rightarrow$  'a interval list  $\Rightarrow$  'a interval list where
  interval_insert_sort_lower_width x [] = [x] |

```

```
interval_insert_sort_lower_width x (y#ys) =
  (if cmp_lower_width x y then (x#y#ys) else y#(interval_insert_sort_lower_width x ys))
```

```
lemma interval_insert_sort_lower_width_length:
<length (interval_insert_sort_lower_width x xs) = 1 + length xs >
by(induction xs, auto)
```

```
lemma interval_insert_sort_lower_width_nonempty:
<interval_insert_sort_lower_width x xs ≠ [] >
by(induction xs, auto)
```

```
lemma interval_insert_sort_wrt_lower:
<sorted_wrt_lower xs ⇒ sorted_wrt_lower (interval_insert_sort_lower_width x xs)>
proof(induction xs)
  case Nil
  then show ?case by(simp add: sorted_wrt_lower_def cmp_lower_width_def)
next
  case (Cons a xs)
  then show ?case
  apply(auto)[1]
  subgoal
  by (metis cmp_lower_width_def list.discr list.sel(1) sorted_wrt_lower_unroll)
  subgoal
  apply(subst sorted_wrt_lower_unroll)
  apply(simp add: interval_insert_sort_lower_width_nonempty)
  apply(simp add: cmp_lower_width_def sorted_wrt_lower_def split:if_split)
  using interval_insert_sort_lower_width.simps(1)[of a]
  interval_insert_sort_lower_width.simps(2)
  list.sel(1) list.set_cases list.set_sel(1)
  by (smt (verit) interval_insert_sort_lower_width.simps(1) less_le_not_le nle_le)
  done
qed
```

```
lemma interval_isort_elements: set (interval_insert_sort_lower_width x xs) = {x} ∪ set xs
proof(induction xs)
  case Nil
  then show ?case by simp
next
  case (Cons a xs)
  then show ?case by(auto)
qed
```

```
lemma foldr_isort_elements: set (foldr interval_insert_sort_lower_width xs []) = set xs
proof(induction xs)
  case Nil
  then show ?case by simp
next
  case (Cons a xs)
  then show ?case
  using interval_isort_elements by(auto)
qed
```

**definition** *interval\_sort\_lower\_width* :: ('a::{linorder,minus}) interval list  $\Rightarrow$  'a interval list **where**  
*interval\_sort\_lower\_width* xs = *foldr interval\_insert\_sort\_lower\_width* xs []

**lemma** *interval\_sort\_lower\_width\_length*:  $\langle \text{length } (\text{interval\_sort\_lower\_width } xs) = (\text{length } xs) \rangle$

**proof**(*induction* xs)

**case** Nil

**then show** ?case **unfolding** *interval\_sort\_lower\_width\_def* **by** *simp*

**next**

**case** (Cons a xs)

**then show** ?case

**unfolding** *interval\_sort\_lower\_width\_def*

**by** (*simp add: interval\_insert\_sort\_lower\_width\_length*)

**qed**

**lemma** *interval\_sort\_lower\_width\_sorted*:  $\langle \text{sorted\_wrt\_lower } (\text{interval\_sort\_lower\_width } xs) \rangle$

**proof**(*induction* xs)

**case** Nil

**then show** ?case

**unfolding** *interval\_sort\_lower\_width\_def* *sorted\_wrt\_lower\_def*

**by** *simp*

**next**

**case** (Cons a xs)

**then show** ?case

**unfolding** *interval\_sort\_lower\_width\_def*

**using** *interval\_insert\_sort\_wrt\_lower*

**by** *auto*

**qed**

**lemma** *interval\_sort\_lower\_width\_set\_eq*:

$\langle \text{set } (\text{interval\_sort\_lower\_width } x) = \text{set } x \rangle$

**by** (*simp add: foldr\_isort\_elements interval\_sort\_lower\_width\_def*)

**lemma** *interval\_sort\_lower\_width\_remdups*:

$\langle \text{remdups } (\text{interval\_sort\_lower\_width } (\text{remdups } xs)) = \text{interval\_sort\_lower\_width } (\text{remdups } xs) \rangle$

**unfolding** *interval\_sort\_lower\_width\_def*

**proof**(*induction* xs)

**case** Nil

**then show** ?case **by** *simp*

**next**

**case** (Cons a xs)

**then show** ?case

**by** (*metis (no\_types, opaque\_lifting) foldr\_isort\_elements interval\_sort\_lower\_width\_def*

*interval\_sort\_lower\_width\_length length\_remdups\_card\_conv length\_remdups\_eq set\_remdups*)

**qed**

**lemma** *interval\_sort\_lower\_width\_distinct*:

**assumes**  $\langle \text{distinct } xs \rangle$  **shows**

$\langle \text{distinct } (\text{interval\_sort\_lower\_width } (\text{remdups } xs)) \rangle$

**using** *interval\_sort\_lower\_width\_remdups* **by** *auto*

**lemma** *foldr\_interval\_insert\_sort\_lower\_width\_distinct*:

**assumes**  $\langle \text{distinct } zs \rangle$

**shows**  $\langle \text{distinct } (\text{foldr } \text{interval\_insert\_sort\_lower\_width } zs [] \rangle$

```

proof(insert assms, induction zs)
  case Nil
  then show ?case
  by simp
next
  case (Cons a zs)
  then show ?case
  by (simp, metis (no_types, opaque_lifting) Cons.prem distinct_remdups_id foldr.simps(2)
    interval_sort_lower_width_def interval_sort_lower_width_distinct o_apply)
qed

```

```

lemma non_overlapping_sorted_remdups:
  non_overlapping_sorted xs  $\implies$  non_overlapping_sorted (remdups xs)
proof(induction xs)
  case Nil
  then show ?case
  by(simp)
next
  case (Cons a xs)
  then show ?case
  unfolding non_overlapping_sorted_def
  by(auto)
qed

```

```

lemma insert_in_lower_width:  $x \in \text{set } (\text{interval\_insert\_sort\_lower\_width } a \text{ list}) = (x = a \vee x \in \text{set list})$ 
proof(induction list)
  case Nil
  then show ?case by(simp)
next
  case (Cons a list)
  then show ?case
  unfolding interval_insert_sort_lower_width_def cmp_lower_width_def
  by auto
qed

```

```

lemma remdups_set_eq:
  assumes <set xs = set ys>
  shows <set (remdups xs) = set (remdups ys)>
  using assms by simp

```

```

lemma remdups_lower_hd:
  assumes  $xs \neq []$  and sorted_wrt_lower xs
  shows  $(\text{lower } \circ \text{hd}) (\text{remdups } xs) = (\text{lower } \circ \text{hd}) xs$ 
  using assms proof(induction xs rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:interval_sort_lower_width_def)
next
  case (cons x xs)
  then show ?case
  by (simp, metis cmp_lower_width_def list.collapse order.asym set_ConsD sorted_wrt.simps(2)
    sorted_wrt_lower_def sorted_wrt_lower_unroll)
qed

```

## 8.1.4 Various Notions of Validity of Sorted Lists of Intervals

### Validity Tests

**definition**  $\langle \text{valid\_mInterval\_ovl } is = (\text{sorted\_wrt\_lower } is \wedge \text{distinct } is \wedge is \neq []) \rangle$

The predicate `valid_mInterval_ovl` requires that a list of intervals is distinct and sorted with respect to the lower bound of each interval.

**definition** `valid_mInterval_adj :: 'a::minus_mono interval list  $\Rightarrow$  bool`  
**where**  $\langle \text{valid\_mInterval\_adj } is = (\text{non\_overlapping\_sorted } is \wedge \text{distinct } is \wedge is \neq []) \rangle$

The predicate `valid_mInterval_adj` is strictly stronger than `valid_mInterval_ovl`:

**lemma** `valid_adj_imp_ovl:  $\langle \text{valid\_mInterval\_adj } x \Longrightarrow \text{valid\_mInterval\_ovl } x \rangle$`   
**by** (`simp add: non_overlapping_implies_sorted_wrt_lower valid_mInterval_adj_def valid_mInterval_ovl_def`)

Informally, `valid_mInterval_ovl` further limits the list of intervals to be non-overlapping. Note that adjacent intervals (i.e., intervals that share the same bounds) are allowed. For example:

**lemma** `valid_mInterval_adj [Interval(1::int,2), Interval(2,3)]`  
**apply** (`simp add: valid_mInterval_adj_def non_overlapping_sorted_def cmp_non_overlapping_def`)  
**by** (`smt (verit) lower_bounds`)

**definition**  $\langle \text{valid\_mInterval\_non\_ovl } is = (\text{valid\_mInterval\_ovl } is \wedge \text{non\_adjacent\_sorted\_wrt\_lower } is) \rangle$

Informally, `valid_mInterval_non_ovl` further limits the list of intervals to also forbid adjacent intervals (i.e., intervals that share the same bounds) are allowed. It is strictly stronger than the other two predicates:

**lemma** `valid_non_ovl_imp_ovl:  $\langle \text{valid\_mInterval\_non\_ovl } x \Longrightarrow \text{valid\_mInterval\_ovl } x \rangle$`   
**using** `valid_mInterval_non_ovl_def` **by** `blast`  
**lemma** `valid_non_ovl_imp_adj:  $\langle \text{valid\_mInterval\_non\_ovl } x \Longrightarrow \text{valid\_mInterval\_adj } x \rangle$`   
**by** (`simp add: non_adjacent_implies_non_overlapping valid_mInterval_adj_def valid_mInterval_non_ovl_def valid_mInterval_ovl_def`)

**lemma** `valid_mInterval_non_ovl_sorted:  $\text{valid\_mInterval\_non\_ovl } xs \Longrightarrow \text{sorted\_wrt\_lower } xs$`   
**by** (`metis (no_types, lifting) valid_mInterval_non_ovl_def valid_mInterval_ovl_def`)

**lemma** `valid_mInterval_non_ovl_unroll:`  
 $\langle \text{ys} \neq [] \Longrightarrow \text{valid\_mInterval\_non\_ovl } (y \# \text{ys}) \Longrightarrow \text{valid\_mInterval\_non\_ovl } \text{ys} \rangle$   
**unfolding** `valid_mInterval_non_ovl_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def`  
**by** (`auto simp add: sorted_wrt_lower_tail`)

**lemma** `valid_mInterval_non_ovl_eq1:`  
**assumes**  $\langle \text{valid\_mInterval\_non\_ovl } xs \rangle$   
**and**  $\langle \text{valid\_mInterval\_non\_ovl } ys \rangle$   
**and**  $\langle \text{set } xs = \text{set } ys \rangle$   
**shows**  $\langle xs = ys \rangle$   
**proof** (`insert assms, induction xs ys rule: list_induct2'`)  
**case** 1  
**then show** ?case  
**by** (`simp`)  
**next**  
**case** (2 x xs)  
**then show** ?case  
**by** `simp`  
**next**

```

case (3 y ys)
then show ?case
  by simp
next
case (4 x xs y ys)
then show ?case
proof(cases xs)
  case Nil note xsNil = this
  then show ?thesis
  proof(cases ys)
    case Nil
    then show ?thesis
    using xsNil 4 by simp
  next
  case (Cons a list)
  then show ?thesis
  using xsNil 4
  by (simp add: valid_mInterval_non_ovl_def valid_mInterval_ovl_def)
qed
next
case (Cons x' xs') note xCons = this
then show ?thesis
proof(cases ys)
  case Nil
  then show ?thesis
  using xCons 4
  by (simp add: valid_mInterval_non_ovl_def valid_mInterval_ovl_def)
next
case (Cons y' ys')
then show ?thesis
  using xCons 4 valid_mInterval_non_ovl_unroll[of xs x] valid_mInterval_non_ovl_unroll[of ys y]
  unfolding cmp_non_adjacent_def non_adjacent_sorted_wrt_lower_def valid_mInterval_non_ovl_def
  valid_mInterval_ovl_def sorted_wrt_lower_def cmp_lower_width_def
  by (metis (no_types, lifting) 4.premis(1) 4.premis(2) cmp_non_adjacent_lower insert_eq_iff
  list.distinct(1) list.set_intros(1) list.simps(15) non_adjacent_sorted_wrt_lower_def
  order.asym set_ConsD sorted_wrt.simps(2) valid_mInterval_non_ovl_def)
qed
qed
qed

```

## Constructors

**Overlapping Intervals** `definition < mk_mInterval_ovl = remdups o interval_sort_lower_width >`

```

lemma mk_mInterval_ovl_non_empty: < is ≠ [] ⇒ (mk_mInterval_ovl is) ≠ [] >
  unfolding mk_mInterval_ovl_def o_def interval_sort_lower_width_def
proof(induction is rule: list_nonempty_induct)
  case (single x)
  then show ?case by simp
next
  case (cons x xs)
  then show ?case
  apply(simp)
  by (metis (no_types, lifting) interval_insert_sort_lower_width.simps(2) list.collapse list.simps(3))

```

qed

```
lemma mk_mInterval_ovl_empty[simp]:  
  mk_mInterval_ovl [] = []  
  by(simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)
```

```
lemma mk_mInterval_ovl_distinct: <distinct (mk_mInterval_ovl is)>  
  unfolding mk_mInterval_ovl_def by simp
```

```
lemma sorted_wrt_lower_remdups:  
  sorted_wrt_lower xs  $\implies$  sorted_wrt_lower (remdups xs)  
  proof(induction xs)  
    case Nil  
    then show ?case  
      by(simp)  
  next  
    case (Cons a xs)  
    then show ?case  
      unfolding sorted_wrt_lower_def  
      by(auto)  
  qed
```

```
lemma interval_sort_lower_width_swap_remdups:  
  <remdups (interval_sort_lower_width xs) = interval_sort_lower_width (remdups xs)>  
  for xs::'a::{minus_mono, linordered_field} interval list  
  proof(induction xs)  
    case Nil  
    then show ?case  
      by (metis interval_sort_lower_width_remdups remdups.simps(1))  
  next  
    case (Cons a xs)  
    then show ?case  
      by (metis (mono_tags, lifting) distinct_remdups interval_sort_lower_width_remdups  
        interval_sort_lower_width_set_eq interval_sort_lower_width_sorted set_remdups  
        sorted_wrt_lower_distinct_lists_eq sorted_wrt_lower_remdups)  
  qed
```

```
lemma mk_mInterval_ovl_sorted: <sorted_wrt_lower (mk_mInterval_ovl is)>  
  proof(induction is)  
    case Nil  
    then show ?case  
      unfolding mk_mInterval_ovl_def sorted_wrt_lower_def interval_sort_lower_width_def  
      by simp  
  next  
    case (Cons a is)  
    then show ?case  
      using interval_sort_lower_width_sorted sorted_wrt_lower_remdups  
      unfolding mk_mInterval_ovl_def sorted_wrt_lower_def interval_sort_lower_width_def o_def  
      by blast  
  qed
```

```
theorem mk_mInterval_ovl_valid: <is  $\neq$  []  $\implies$  valid_mInterval_ovl (mk_mInterval_ovl is)>  
  unfolding valid_mInterval_ovl_def
```

by (simp add: mk\_mInterval\_ovl\_distinct mk\_mInterval\_ovl\_non\_empty mk\_mInterval\_ovl\_sorted)

**lemma** valid\_mk\_mInterval\_ovl\_id:

assumes <valid\_mInterval\_ovl xs>

shows <mk\_mInterval\_ovl xs = xs>

**proof**(insert assms, induction xs)

case Nil

then show ?case

by (simp add: valid\_mInterval\_ovl\_def)

next

case (Cons a xs)

then show ?case

apply (simp add: mk\_mInterval\_ovl\_def interval\_sort\_lower\_width\_def valid\_mInterval\_ovl\_def  
sorted\_wrt\_lower\_def cmp\_lower\_width\_def)

by (metis (no\_types, opaque\_lifting) cmp\_lower\_width\_def distinct\_singleton foldr\_isort\_elements  
insertI1 interval\_insert\_sort\_lower\_width.simps(1) interval\_insert\_sort\_lower\_width.simps(2)  
interval\_sort\_lower\_width\_def list.distinct(1) list.simps(15) min\_list.cases remdups.simps(2)  
remdups\_id\_iff\_distinct)

qed

**lemma** mk\_mInterval\_ovl\_eq:

assumes <set xs = set (ys::'a::{minus\_mono, linordered\_field} interval list)>

shows <mk\_mInterval\_ovl xs = mk\_mInterval\_ovl ys >

by (metis (no\_types, lifting) assms comp\_def interval\_sort\_lower\_width\_set\_eq mk\_mInterval\_ovl\_def  
mk\_mInterval\_ovl\_def mk\_mInterval\_ovl\_distinct mk\_mInterval\_ovl\_sorted set\_remdups  
sorted\_wrt\_lower\_distinct\_lists\_eq)

**Adjacent Intervals** **definition** mk\_mInterval\_adj :: ('a::{minus, linorder, linorder}) interval list  $\Rightarrow$  'a interval list  
where <mk\_mInterval\_adj = remdups o merge\_overlapping\_intervals\_sorted\_wrt\_lower o mk\_mInterval\_ovl>

**lemma** mk\_mInterval\_adj\_non\_overlapping\_sorted: <non\_overlapping\_sorted (mk\_mInterval\_adj  
(is::'a::{minus\_mono} interval list))>

using interval\_sort\_lower\_width\_sorted sorted\_wrt\_lower\_remdups mk\_mInterval\_ovl\_sorted[of is]  
merge\_overlapping\_intervals\_sorted\_wrt\_lower\_sorted\_lower[of (mk\_mInterval\_ovl is)]  
merge\_overlapping\_intervals\_sorted\_sorted\_non\_non\_overlapping non\_overlapping\_sorted\_remdups

unfolding mk\_mInterval\_adj\_def o\_def

by blast

**lemma** mk\_mInterval\_adj\_sorted: <sorted\_wrt\_lower (mk\_mInterval\_adj (is::'a::minus\_mono interval list))>

using interval\_sort\_lower\_width\_sorted sorted\_wrt\_lower\_remdups mk\_mInterval\_ovl\_sorted[of is]  
merge\_overlapping\_intervals\_sorted\_wrt\_lower\_sorted\_lower[of (mk\_mInterval\_ovl is)]

unfolding mk\_mInterval\_adj\_def o\_def

by auto

**lemma** mk\_mInterval\_adj\_non\_empty: <is  $\neq$  []  $\implies$  (mk\_mInterval\_adj is)  $\neq$  []>

unfolding mk\_mInterval\_adj\_def o\_def

using mk\_mInterval\_ovl\_non\_empty merge\_overlapping\_intervals\_sorted\_wrt\_lower\_non\_nil

by (metis remdups\_eq\_nil\_right\_iff)

**lemma** mk\_mInterval\_adj\_empty[simp]:

mk\_mInterval\_adj [] = []

by (simp add: mk\_mInterval\_adj\_def)

**lemma** mk\_mInterval\_adj\_distinct: <distinct (mk\_mInterval\_adj is)>

**unfolding** *mk\_mInterval\_adj\_def* **by** *simp*

**theorem** *mk\_mInterval\_adj\_valid*:  $\langle is \neq [] \implies valid\_mInterval\_adj (mk\_mInterval\_adj\ is) \rangle$

**unfolding** *valid\_mInterval\_adj\_def*

**using** *mk\_mInterval\_adj\_non\_overlapping\_sorted mk\_mInterval\_adj\_distinct mk\_mInterval\_adj\_non\_empty*  
**by** *auto*

**lemma** *valid\_mk\_mInterval\_adj\_id*:

**assumes**  $\langle valid\_mInterval\_adj\ xs \rangle$

**shows**  $\langle mk\_mInterval\_adj\ xs = xs \rangle$

**proof**(*insert assms, induction xs*)

**case** *Nil*

**then show** *?case*

**by**(*simp add: valid\_mInterval\_adj\_def*)

**next**

**case** (*Cons a xs*)

**then show** *?case*

**using** *valid\_mk\_mInterval\_ovl\_id[of a#xs, simplified valid\_adj\_imp\_ovl[of a#xs, simplified Cons, simplified], simplified]*

*valid\_mk\_mInterval\_ovl\_id[of xs, simplified valid\_adj\_imp\_ovl[of xs, simplified Cons, simplified], simplified]*

**unfolding** *valid\_mInterval\_adj\_def mk\_mInterval\_adj\_def o\_def*

**apply**(*simp add: mk\_mInterval\_ovl\_def interval\_sort\_lower\_width\_def valid\_mInterval\_ovl\_def*

*sorted\_wrt\_lower\_def cmp\_lower\_width\_def non\_overlapping\_sorted\_def cmp\_non\_overlapping\_def*)

**by** (*smt (verit) One\_nat\_def add.commute foldr\_interval\_insert\_sort\_lower\_width\_distinct foldr\_isort\_elements*

*interval\_insert\_sort\_lower\_width\_simps(2) interval\_insert\_sort\_lower\_width\_length length\_remdups\_card\_conv*

*length\_remdups\_eq list.sel(1) list.sel(3) list.set\_intros(1) list.size(4)*

*merge\_overlapping\_intervals\_sorted\_wrt\_lower.elims*

*merge\_overlapping\_intervals\_sorted\_wrt\_lower\_simps(2) remdups\_simps(2) remdups\_id\_iff\_distinct set\_remdups*)

**qed**

**lemma** *mk\_mInterval\_adj\_eq*:

**assumes**  $\langle set\ xs = set\ (ys::\{minus\_mono, linordered\_field\}\ interval\ list) \rangle$

**shows**  $\langle mk\_mInterval\_adj\ xs = mk\_mInterval\_adj\ ys \rangle$

**by** (*metis (no\_types, lifting) assms comp\_def interval\_sort\_lower\_width\_set\_eq mk\_mInterval\_adj\_def*

*mk\_mInterval\_ovl\_def mk\_mInterval\_ovl\_distinct mk\_mInterval\_ovl\_sorted set\_remdups*

*sorted\_wrt\_lower\_distinct\_lists\_eq*)

**lemma** *mk\_mInterval\_ovl\_id*:

*mk\_mInterval\_ovl (mk\_mInterval\_ovl x) = mk\_mInterval\_ovl x*

**unfolding** *mk\_mInterval\_ovl\_def o\_def*

**by** (*metis comp\_def mk\_mInterval\_ovl\_def mk\_mInterval\_ovl\_empty mk\_mInterval\_ovl\_valid valid\_mk\_mInterval\_ovl\_id*)

**value** *valid\_mInterval\_adj (mk\_mInterval\_adj ([lvl (1::int) 2, lvl 1 3, lvl 1 1]))*

**value** *valid\_mInterval\_adj (mk\_mInterval\_adj ([lvl (1::int) 1, lvl 1 2, lvl 2 3]))*

**Non-Overlapping Intervals definition**  $\langle mk\_mInterval\_non\_ovl = remdups\ o\ merge\_adjacent\_intervals\_sorted\_wrt\_lower\ o\ mk\_mInterval\_ovl \rangle$

**lemma** *mk\_mInterval\_non\_ovl\_distinct*:

*distinct (mk\_mInterval\_non\_ovl is)*

**by** (*simp add: mk\_mInterval\_non\_ovl\_def*)

```

lemma mk_mInterval_non_ovl_non_empty:
  is ≠ [] ⇒ mk_mInterval_non_ovl is ≠ []
    by (simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil mk_mInterval_ovl_non_empty
mk_mInterval_non_ovl_def)

```

```

lemma mk_mInterval_non_ovl_empty[simp]:
  mk_mInterval_non_ovl [] = []
by (simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil mk_mInterval_non_ovl_def)

```

```

lemma mk_mInterval_non_ovl_eq:
  assumes <set xs = set (ys::'a::{minus_mono, linordered_field} interval list)>
  shows <mk_mInterval_non_ovl xs = mk_mInterval_non_ovl ys >
  unfolding mk_mInterval_non_ovl_def o_def
by (metis (no_types, opaque_lifting) assms comp_apply foldr_isort_elements
interval_sort_lower_width_def mk_mInterval_ovl_def mk_mInterval_ovl_distinct
mk_mInterval_ovl_sorted set_remdups sorted_wrt_lower_distinct_lists_eq)

```

```

lemma sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower:
  sorted_wrt_lower xs ⇒ sorted_wrt_lower (merge_adjacent_intervals_sorted_wrt_lower xs)
proof(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
  case 1
  then show ?case by simp
next
  case (2 x)
  then show ?case by simp
next
  case (3 x y ys)
  then show ?case
  proof(cases upper x < lower y)
  case True
  then show ?thesis
    using 3
    by (smt (verit, del_insts) leD list.discl list.sel(1) lower_le_upper
merge_adjacent_intervals_sorted_wrt_lower.simps(3)
merge_adjacent_intervals_sorted_wrt_lower_non_nil
merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd
sorted_wrt_lower_unroll)
  next
  case False
  then show ?thesis
    using 3
    by (smt (verit, del_insts) list.distinct(1) list.sel(1) max.absorb3 max_absorb2
merge_adjacent_intervals_sorted_wrt_lower.simps(3) mk_interval_id mk_interval_lower
not_le_imp_less sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
  qed
qed

```

```

lemma mk_mInterval_non_ovl_sorted_wrt_lower:
  is ≠ [] ⇒ sorted_wrt_lower (mk_mInterval_non_ovl (is::int interval list))
  unfolding mk_mInterval_non_ovl_def o_def
    using mk_mInterval_ovl_sorted sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower

```

*sorted\_wrt\_lower\_remdups* by blast

**lemma** *valid\_ovl\_mkInterval\_non\_ovl*:  $is \neq [] \implies \text{valid\_mInterval\_ovl } (mk\_mInterval\_non\_ovl \ is)$

**unfolding** *valid\_mInterval\_ovl\_def*

**by** (*simp add: merge\_adjacent\_intervals\_sorted\_wrt\_lower\_non\_nil mk\_mInterval\_non\_ovl\_def*  
*mk\_mInterval\_ovl\_non\_empty mk\_mInterval\_ovl\_sorted*)

*sorted\_wrt\_lower\_merge\_adjacent\_intervals\_sorted\_wrt\_lower*  
*sorted\_wrt\_lower\_remdups*)

**lemma** *non\_adj\_sorted\_mkInterval\_non\_ovl*:

*sorted\_wrt\_lower xs*

$\implies \text{non\_adjacent\_sorted\_wrt\_lower } (merge\_adjacent\_intervals\_sorted\_wrt\_lower \ xs)$

**proof**(*induction xs rule:merge\_adjacent\_intervals\_sorted\_wrt\_lower.induct*)

**case 1**

**then show** ?*case*

**unfolding** *non\_adjacent\_sorted\_wrt\_lower\_def* by(*simp*)

**next**

**case** (2 *x*)

**then show** ?*case*

**unfolding** *non\_adjacent\_sorted\_wrt\_lower\_def* by(*simp*)

**next**

**case** (3 *x y ys*)

**then show** ?*case*

**proof**(*cases upper x < lower y*)

**case True**

**then show** ?*thesis*

**apply**(*simp*)

**apply**(*subst non\_adjacent\_sorted\_wrt\_lower\_unroll*)

**subgoal by** (*simp add: merge\_adjacent\_intervals\_sorted\_wrt\_lower\_non\_nil*)

**subgoal using** 3 *True*

**by** (*metis list.sel(1) merge\_adjacent\_intervals\_sorted\_wrt\_lower\_sorted\_lower\_lower\_hd sorted\_wrt\_lower\_tail*)

**done**

**next**

**case False**

**then show** ?*thesis*

**proof**(*cases ys*)

**case Nil**

**then show** ?*thesis*

**unfolding** *non\_adjacent\_sorted\_wrt\_lower\_def cmp\_non\_adjacent\_def*

**by**(*simp add: max\_def 3 False split:if\_splits*)

**next**

**case** (*Cons a list*)

**then have** *a*: *sorted\_wrt\_lower* (*mk\_interval* (*lower x*, *max* (*upper y*) (*upper x*))) # *ys*

**proof**(*cases upper y ≤ upper x*)

**case True note** \* = *this*

**then show** ?*thesis*

**proof** (*cases upper y = upper x*)

**case True**

**then show** ?*thesis*

**using** *False True* **apply**(*simp*)

**using** 3 \* *False sorted\_wrt\_lower\_unroll*[*of y#ys x, simplified*]

*sorted\_wrt\_lower\_tail*[*of y ys*]

*sorted\_wrt\_lower\_tail'*[*of x y ys*]

**by auto**

```

next
case False
then show ?thesis
using False True apply(simp)
using 3 * False sorted_wrt_lower_unroll[of y#ys x, simplified]
sorted_wrt_lower_tail[of y ys]
sorted_wrt_lower_tail'[of x y ys]
by auto
qed
next
case False
then show ?thesis
using Cons False apply(simp)
using 3 False sorted_wrt_lower_unroll[of y#ys x, simplified]
sorted_wrt_lower_tail[of y ys]
sorted_wrt_lower_tail'[of x y ys]
apply(simp)
by (smt (verit, ccfv_threshold) less_le_not_le list.distinct(1) list.sel(1) max.absorb1
max.assoc max.cobounded2 mk_interval_id mk_interval_lower sorted_wrt_lower_unroll)
qed
then show ?thesis
using False 3[simplified a False , simplified]
by simp
qed
qed
qed

```

**lemma** *bin\_op\_mInterval\_commute:*

```

assumes op_commute:  $\langle \bigwedge x y. op\ x\ y = op\ y\ x \rangle$ 
shows  $\langle mk\_mInterval\_non\_ovl\ (bin\_op\_interval\_list\ op\ x\ y) = mk\_mInterval\_non\_ovl\ (bin\_op\_interval\_list\ op\ y\ x) \rangle$ 
using bin_op_interval_list_commute mk_mInterval_non_ovl_eq
by (metis op_commute)

```

**lemma** *iList\_plus\_mInterval\_ovl\_assoc:*

```

 $\langle mk\_mInterval\_ovl\ (iList\_plus\ x\ (iList\_plus\ y\ z)) = mk\_mInterval\_ovl\ (iList\_plus\ (iList\_plus\ x\ (y::'a::\{minus\_mono,\ linordered\_field\}\ interval\ list))\ z) \rangle$ 
by (meson iList_plus_assoc mk_mInterval_ovl_eq)

```

**lemma** *iList\_plus\_mInterval\_adj\_commute:*

```

 $\langle mk\_mInterval\_adj\ (iList\_plus\ x\ y) = mk\_mInterval\_adj\ (iList\_plus\ y\ (x::'a::\{minus\_mono,\ linordered\_field\}\ interval\ list)) \rangle$ 
by (meson iList_plus_commute mk_mInterval_adj_eq)

```

**lemma** *iList\_plus\_mInterval\_non\_ovl\_assoc:*

```

 $\langle mk\_mInterval\_non\_ovl\ (iList\_plus\ x\ (iList\_plus\ y\ z)) = mk\_mInterval\_non\_ovl\ (iList\_plus\ (iList\_plus\ x\ (y::'a::\{minus\_mono,\ linordered\_field\}\ interval\ list))\ z) \rangle$ 
by (meson iList_plus_assoc mk_mInterval_non_ovl_eq)

```

**lemma** *iList\_plus\_mInterval\_non\_ovl\_commute:*

```

 $\langle mk\_mInterval\_non\_ovl\ (iList\_plus\ x\ y) = mk\_mInterval\_non\_ovl\ (iList\_plus\ y\ (x::'a::\{minus\_mono,\ linordered\_field\}\ interval\ list)) \rangle$ 

```

**by** (meson iList\_plus\_commute mk\_mInterval\_non\_ovl\_eq)

**lemma** iList\_plus\_mInterval\_adj\_assoc:

⟨mk\_mInterval\_non\_ovl (iList\_plus x (iList\_plus y z)) = mk\_mInterval\_non\_ovl (iList\_plus (iList\_plus x (y::'a::{minus\_mono, linordered\_field} interval list)) z)⟩

**by** (meson iList\_plus\_assoc mk\_mInterval\_non\_ovl\_eq)

**lemma** sorted\_wrt\_lower\_mk\_mInterval\_non\_ovl: sorted\_wrt\_lower (mk\_mInterval\_non\_ovl xs)

**unfolding** mk\_mInterval\_non\_ovl\_def

**by** (simp add: mk\_mInterval\_ovl\_sorted sorted\_wrt\_lower\_merge\_adjacent\_intervals\_sorted\_wrt\_lower sorted\_wrt\_lower\_remdups)

**theorem** mk\_mInterval\_non\_ovl\_valid: ⟨sorted\_wrt\_lower is  $\implies$  is  $\neq$  []  $\implies$  valid\_mInterval\_non\_ovl (mk\_mInterval\_non\_ovl is)⟩

**using** non\_adj\_sorted\_mkInterval\_non\_ovl[of is] valid\_ovl\_mkInterval\_non\_ovl[of is]

**unfolding** valid\_mInterval\_non\_ovl\_def mk\_mInterval\_non\_ovl\_def o\_def

**by** (simp add: distinct\_remdups\_id mk\_mInterval\_ovl\_sorted non\_adj\_sorted\_mkInterval\_non\_ovl non\_adjacent\_implies\_distinct)

**lemma** valid\_mk\_mInterval\_non\_ovl\_id:

**assumes** ⟨valid\_mInterval\_non\_ovl xs⟩

**shows** ⟨mk\_mInterval\_non\_ovl xs = xs⟩

**proof**(insert assms, induction xs)

**case** Nil

**then show** ?case

**by**(simp add: valid\_mInterval\_non\_ovl\_def valid\_mInterval\_ovl\_def)

**next**

**case** (Cons a xs)

**then show** ?case

**using** valid\_mk\_mInterval\_ovl\_id[of a#xs, simplified] valid\_non\_ovl\_imp\_ovl[of a#xs, simplified] Cons, simplified], simplified]

valid\_mk\_mInterval\_ovl\_id[of xs, simplified] valid\_non\_ovl\_imp\_ovl[of xs, simplified] Cons, simplified], simplified]

**unfolding** valid\_mInterval\_non\_ovl\_def mk\_mInterval\_non\_ovl\_def o\_def

**apply**(simp)

**apply**(simp add: mk\_mInterval\_ovl\_def interval\_sort\_lower\_width\_def valid\_mInterval\_ovl\_def

sorted\_wrt\_lower\_def cmp\_lower\_width\_def non\_overlapping\_sorted\_def cmp\_non\_overlapping\_def)

**by** (smt (verit) Cons.prem1 list.sel(1) list.sel(3) merge\_adjacent\_intervals\_sorted\_wrt\_lower.elims

merge\_adjacent\_intervals\_sorted\_wrt\_lower.simps(2) non\_adj\_sorted\_mkInterval\_non\_ovl

non\_adjacent\_implies\_distinct

non\_adjacent\_sorted\_wrt\_lower\_def non\_adjacent\_sorted\_wrt\_lower\_unroll remdups\_id\_iff\_distinct

sorted\_wrt\_lower.simps(2)

valid\_mInterval\_non\_ovl\_sorted)

**qed**

**lemma** mk\_mInterval\_non\_ovl\_single:

mk\_mInterval\_non\_ovl [x] = [x]

**by** (simp add: non\_adjacent\_implies\_sorted\_wrt\_lower non\_adjacent\_sorted\_wrt\_lower\_def

valid\_mInterval\_non\_ovl\_def valid\_mInterval\_ovl\_def valid\_mk\_mInterval\_non\_ovl\_id)

**lemma** mk\_mInterval\_non\_ovl\_id:

mk\_mInterval\_non\_ovl (mk\_mInterval\_non\_ovl x) = mk\_mInterval\_non\_ovl x

**unfolding** mk\_mInterval\_non\_ovl\_def o\_def

**by** (metis (no\_types, opaque\_lifting) comp\_apply distinct\_remdups\_id mk\_mInterval\_non\_ovl\_def mk\_mInterval\_non\_ovl\_empty)

```
mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct
valid_mInterval_non_ovl_def valid_mk_mInterval_non_ovl_id valid_ovl_mkInterval_non_ovl)
```

```
value valid_mInterval_non_ovl (mk_mInterval_non_ovl ([lvl (1::int) 2, lvl 1 3, lvl 1 1]))
value valid_mInterval_non_ovl (mk_mInterval_non_ovl ([lvl (1::int) 1, lvl 1 2, lvl 2 3]))
```

```
value<mk_mInterval_ovl [mk_interval ((1::int), 4), mk_interval (0,2), mk_interval (3,5), mk_interval (5,7),
mk_interval (7,7), mk_interval (8,8)]>
value<mk_mInterval_adj [mk_interval ((1::int), 4), mk_interval (0,2), mk_interval (3,5), mk_interval (5,7),
mk_interval (7,7), mk_interval (8,8)]>
value<mk_mInterval_non_ovl [mk_interval ((1::int), 4), mk_interval (0,2), mk_interval (3,5), mk_interval (5,7),
mk_interval (7,7), mk_interval (8,8)]>
```

### 8.1.5 Union over a List of Intervals

```
definition <set_of_interval_list XS = foldr (λx a. set_of x ∪ a) XS {}>
```

```
lemma set_of_interval_list_nonempty:
  assumes non_empty: <XS ≠ ([]::real interval list)>
  shows <set_of_interval_list XS ≠ {}>
  using assms proof(induction XS rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:set_of_interval_list_def set_of_eq)
  next
  case (cons x xs)
  then show ?case
    by(simp add:set_of_interval_list_def)
qed
```

```
lemma set_of_interval_list_bdd_below:
  assumes non_empty: <XS ≠ ([]::real interval list)>
  shows <bdd_below (set_of_interval_list XS)>
  using assms proof(induction XS rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:set_of_interval_list_def set_of_eq)
  next
  case (cons x xs)
  then show ?case
    by(simp add:set_of_eq set_of_interval_list_def sorted_wrt_lower_def)
qed
```

```
lemma set_of_interval_list_bdd_above:
  assumes non_empty: <XS ≠ ([]::real interval list)>
  shows <bdd_above (set_of_interval_list XS)>
  using assms proof(induction XS rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:set_of_interval_list_def set_of_eq)
  next
  case (cons x xs)
  then show ?case
    by(simp add:set_of_eq set_of_interval_list_def contiguous_def)
```

qed

```
lemma inf_set_of_interval_list_lower:
  assumes non_empty: <math>XS \neq []</math>::real interval list>
  and sorted: <math>\text{sorted\_wrt\_lower } XS</math>
  shows <math>\text{Inf } (\text{set\_of\_interval\_list } XS) = \text{lower } (\text{hd } XS)</math>
  using assms proof(induction XS rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:set_of_interval_list_def set_of_eq)
next
case (cons x xs)
then show ?case
  apply(simp add: set_of_interval_list_def)
  apply(subst cInf_union_distrib)
  subgoal by simp
  subgoal by(simp add:set_of_eq)
  subgoal by(fold set_of_interval_list_def , simp add: set_of_interval_list_nonempty)
  subgoal
    apply (fold set_of_interval_list_def)
    using sorted_wrt_lower_unroll[of xs x, simplified cons, simplified]
      set_of_interval_list_bdd_below
    by(simp)
  subgoal
    using sorted_wrt_lower_unroll[of xs x, simplified cons, simplified]
    by (metis inf.orderE interval_bounds_real(2) lower_le_upper nle_le order_less_le set_of_eq)
  done
```

qed

```
lemma contiguous_sorted_wrt_upper:
  assumes contiguous (xs:: real interval list)
  shows sorted_wrt_upper xs
  unfolding sorted_wrt_upper_def
  using assms unfolding contiguous_def lower_le_upper
  by (metis assms contiguous_non_overlapping non_overlapping_implies_sorted_wrt_upper sorted_wrt_upper_def)
```

```
lemma contiguous_sorted_wrt_lower:
  assumes contiguous (XS:: real interval list)
  shows sorted_wrt_lower XS
  unfolding sorted_wrt_lower_def
  using assms contiguous_non_overlapping[of XS] non_overlapping_implies_sorted_wrt_lower
  sorted_wrt_lower_def by metis
```

```
lemma max_last_sorted_wrt_upper:
  assumes  $XS \neq []$  sorted_wrt_upper (XS:: 'a:: {linorder} interval list)
  shows  $\text{Max } (\text{set } (\text{map upper } XS)) = \text{upper } (\text{last } XS)$ 
  using assms
  proof (induction XS rule: list_nonempty_induct)
  case (single x)
  then show ?case by simp
next
case (cons x xs)
then have a0: sorted_wrt ( $\lambda x y. \text{upper } x \leq \text{upper } y$ ) (x # xs) by (simp add: sorted_wrt_upper_def)
then have a1:  $\text{Max } (\text{set } (\text{map upper } (x \# xs))) = \text{upper } (\text{last } (x \# xs))$  using cons unfolding sorted_wrt_upper_def by
```

```

simp
then show ?case using ao a1 cons unfolding sorted_wrt_upper_def by simp
qed

lemma min_hd_sorted_wrt_lower:
  assumes  $XS \neq []$  sorted_wrt_lower ( $XS :: 'a :: \{linorder, minus, preorder\}$  interval list)
  shows  $Min (set (map lower XS)) = lower (hd XS)$ 
  using assms
proof (induction XS rule: list_nonempty_induct)
  case (single x)
  then show ?case by simp
next
  case (cons x xs)
  then have ao: sorted_wrt ( $\lambda x y. \text{if } lower\ x = lower\ y \text{ then } width\ x \leq width\ y \text{ else } lower\ x < lower\ y$ ) ( $x \# xs$ )
    unfolding sorted_wrt_lower_def cmp_lower_width_def by simp
  then have a1:  $Min (set (map lower (x \# xs))) = lower (hd (x \# xs))$  using cons unfolding sorted_wrt_lower_def
    cmp_lower_width_def
    by (smt (verit, del_insts) arg_min_list.simps(2) cons.IH cons.prem1 f_arg_min_list_f image_set list.distinct(1)
      list.exhaust_sel list.sel(1) order_less_imp_le sorted_wrt_lower_unroll)
  then show ?case using ao a1 cons unfolding sorted_wrt_upper_def by simp
qed

lemma lower_isort:
  assumes  $\langle xs \neq [] \rangle$  and  $\langle (lower \circ hd) xs = Min (lower ` (set xs)) \rangle$ 
  shows  $\langle (lower \circ hd) (interval\_sort\_lower\_width\ xs) = (lower \circ hd) xs \rangle$ 
  proof (insert assms, induction xs rule: list_nonempty_induct)
  case (single x)
  then show ?case by (simp add: interval_sort_lower_width_def)
next
  case (cons x xs)
  then show ?case
    apply (simp add: interval_sort_lower_width_def)
    by (metis (no_types, opaque_lifting) comp_eq_dest_lhs cons.prem1 foldr.simps(2) foldr_isort_elements
      interval_insert_sort_lower_width_nonempty interval_sort_lower_width_def
      interval_sort_lower_width_sorted list.sel(1) list.set_map min_hd_sorted_wrt_lower)
qed

lemma min_sort:
   $Min (set (map lower (foldr interval\_insert\_sort\_lower\_width\ xs\ []))) = Min (set (map lower\ xs))$ 
  using foldr_isort_elements
  by (metis list.set_map)

lemma mk_mInterval_lower:
  assumes  $xs \neq []$ 
  shows  $Min (set (map lower (mk\_mInterval\_non\_ovl\ xs))) = Min (set (map lower\ xs))$ 
proof (induction xs)
  case Nil
  then show ?case
    unfolding mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def
    by simp
next
  case (Cons a xs)
  have a:  $Min (set (map lower (foldr interval\_insert\_sort\_lower\_width\ xs\ []))) = Min (set (map lower\ xs))$ 
    using foldr_isort_elements

```

```

by (metis list.set_map)
then show ?case
unfolding mk_mInterval_non_ovl_def mk_mInterval_ovl_def o_def
using Cons
by (smt (verit, ccfv_SIG) foldr_isort_elements interval_sort_lower_width_def interval_sort_lower_width_sorted
list.discl list.set_map merge_adjacent_intervals_sorted_wrt_lower_non_nil set_empty
merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd min_hd_sorted_wrt_lower
set_remdups sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower sorted_wrt_lower_remdups)
qed

lemma sup_set_of_interval_list_upper:
assumes non_empty:  $\langle XS \neq [] :: \text{real interval list} \rangle$ 
and sorted:  $\langle \text{sorted\_wrt\_upper } XS \rangle$ 
shows  $\langle \text{Sup (set\_of\_interval\_list } XS) = \text{upper (last } XS) \rangle$ 
using assms proof (induction XS rule: list_nonempty_induct)
case (single x)
then show ?case by (simp add: set_of_interval_list_def set_of_eq)
next
case (cons x xs)
have Max (set (map upper (x # xs))) = upper (last (x # xs)) using max_last_sorted_wrt_upper assms cons by blast
then show ?case
apply (simp add: set_of_interval_list_def)
apply (subst cSup_union_distrib)
apply (fold set_of_interval_list_def)
subgoal by simp
subgoal using bdd_above_set_of by metis
subgoal using cons.hyps set_of_interval_list_nonempty by presburger
subgoal using cons.hyps set_of_interval_list_bdd_above by presburger
subgoal
proof —
assume a1:  $\text{Max (insert (upper } x) (\text{upper ' set } xs)) = \text{upper (if } xs = [] \text{ then } x \text{ else last } xs)$ 
have f2:  $\forall i. \text{Sup (set\_of } i) = \text{upper } i :: \text{real}$ 
by (simp add: set_of_eq)
have [] =  $xs \vee \text{sorted\_wrt\_upper } xs$ 
by (metis cons.prem1 sorted_wrt_upper_unroll)
then show ?thesis
using f2 a1 cons.IH sup_real_def by force
qed
done
qed

lemma compact_set_of_interval_list:
 $\langle \text{compact (set\_of\_interval\_list (XS :: ('a :: \{preorder, ordered\_euclidean\_space, topological\_space\} interval list)))} \rangle$ 
proof (induction XS)
case Nil
then show ?case by (simp add: set_of_interval_list_def)
next
case (Cons a XS)
then show ?case
by (simp add: set_of_interval_list_def, subst compact_Un, simp_all add: compact_set_of Cons set_of_eq)
qed

lemma lower_le_upper_aux:  $\langle xs \neq [] \implies \text{non\_overlapping\_sorted } xs \implies \text{lower (hd } xs) \leq \text{upper (last } xs) \rangle$ 
proof (induction xs rule: induct_list012)

```

```

case 1
then show ?case by(simp)
next
case (2 x)
then show ?case by(simp)
next
case (3 x y zs)
then show ?case
  proof(cases zs)
    case Nil
      then show ?thesis
        using 3 dual_order.trans
        unfolding non_overlapping_sorted_def cmp_non_overlapping_def
        by (simp, smt (verit, best) dual_order.trans lower_le_upper)
    next
      case (Cons a list)
        then show ?thesis
          using 3 dual_order.trans
          unfolding non_overlapping_sorted_def cmp_non_overlapping_def
          by (simp, smt (verit, best) dual_order.trans lower_le_upper)
    qed
  qed

```

```

lemma contiguous_lower_le_upper:
  assumes non_empty:  $\langle XS \neq ([] :: \text{real interval list}) \rangle$ 
  and contiguous:  $\langle \text{contiguous } XS \rangle$ 
shows  $\langle (\text{lower } (\text{hd } XS)) \leq (\text{upper } (\text{last } XS)) \rangle$ 
by (simp add: contiguous contiguous_non_overlapping lower_le_upper_aux non_empty)

```

```

lemma diameter_Sup_Inf:
  assumes  $\langle \text{compact } X \rangle \langle X \neq \{\} \rangle$ 
shows  $\langle \text{diameter } X \leq \text{Sup } X - \text{Inf } X \rangle$ 
using assms diameter_compact_attained[of X]
by (metis bounded_imp_bdd_above bounded_imp_bdd_below compact_imp_bounded sup_inf_dist_bounded)

```

```

lemma diameter_width_compact:
  assumes  $\langle \text{compact } X \rangle \langle \text{bdd\_below } X \rangle \langle \text{bdd\_above } X \rangle \langle X \neq \{\} \rangle$ 
shows  $\langle \text{diameter } X = \text{Sup } X - \text{Inf } X \rangle$ 
using assms diameter_Sup_Inf[of X, simplified assms, simplified]
  closed_contains_Inf[of X, simplified assms, simplified]
  closed_contains_Sup[of X, simplified assms, simplified]
by (smt (verit, best) compact_imp_bounded compact_imp_closed diameter_bounded_bound dist_real_def)

```

```

lemma diameter_contiguous:
  assumes non_empty:  $\langle XS \neq ([] :: \text{real interval list}) \rangle$ 
  and contiguous:  $\langle \text{contiguous } XS \rangle$ 
shows  $\langle \text{diameter } (\text{set\_of\_interval\_list } XS) = \text{dist } (\text{lower } (\text{hd } XS)) (\text{upper } (\text{last } XS)) \rangle$ 
apply(subst diameter_width_compact)
subgoal
  by (simp add: compact_set_of_interval_list contiguous non_empty)
subgoal
  by (simp add:  $\langle \text{compact } (\text{set\_of\_interval\_list } XS) \rangle$  bounded_imp_bdd_below compact_imp_bounded)
subgoal

```

```

  by (simp add: contiguous set_of_interval_list_bdd_above non_empty)
subgoal
  by (simp add: set_of_interval_list_nonempty non_empty)
subgoal
  using sup_set_of_interval_list_upper[of XS, simplified assms, simplified]
    inf_set_of_interval_list_lower[of XS, simplified assms, simplified]
    contiguous_lower_le_upper[of XS, simplified assms, simplified]
  unfolding dist_real_def abs_real_def
    by (simp add: contiguous contiguous_non_overlapping contiguous_sorted_wrt_upper
non_overlapping_implies_sorted_wrt_lower)
done

```

```

lemma interval_list_union_contiguous_lower:
  assumes non_empty: <XS ≠ []>
  and sorted: <sorted_wrt_lower XS>
  shows <lower (interval_list_union XS) = lower (hd XS)>
  using assms proof(induction XS rule:interval_list_union.induct)
  case 1
  then show ?case by simp
next
  case (2 l)
  then show ?case by simp
next
  case (3 l v va)
  then show ?case
  unfolding sorted_wrt_lower_def cmp_lower_width_def
  apply (simp)
  by (metis inf.idem inf.strict_order_iff)
qed

```

```

lemma interval_list_union_contiguous_upper:
  assumes non_empty: <XS ≠ []>
  and sorted: <sorted_wrt_upper XS>
  shows <upper (interval_list_union XS) = upper (last XS)>
  using assms proof(induction XS rule:interval_list_union.induct)
  case 1
  then show ?case by simp
next
  case (2 l)
  then show ?case by simp
next
  case (3 l v va)
  then show ?case
  unfolding sorted_wrt_upper_def cmp_lower_width_def
  apply (simp)
  by (metis last_in_set sup.absorb2)
qed

```

```

lemma interval_list_union_contiguous:
  assumes non_empty: <XS ≠ []>
  and sorted_lower: <sorted_wrt_lower XS>
  and sorted_upper: <sorted_wrt_upper XS>

```

**shows**  $\langle \text{interval\_list\_union } XS = \text{Interval } (\text{lower } (\text{hd } XS), \text{upper } (\text{last } XS)) \rangle$   
**by**  $(\text{metis Interval\_id assms interval\_list\_union\_contiguous\_lower interval\_list\_union\_contiguous\_upper non\_empty})$

**lemma contiguous\_bounds\_lower:**  
**assumes**  $\text{non\_empty}: \langle XS \neq [] \rangle$   
**and**  $\text{contiguous}: \langle \text{contiguous } (XS::\text{real interval list}) \rangle$   
**shows**  $\text{lower } (\text{hd } XS) = \text{Min } (\text{set } (\text{map } \text{lower } XS))$   
**using**  $\text{min\_hd\_sorted\_wrt\_lower}[\text{of } XS, \text{simplified assms contiguous\_sorted\_wrt\_lower}[\text{of } XS, \text{simplified assms}]]$   
**by**  $\text{auto}[1]$

**lemma contiguous\_bounds\_upper:**  
**assumes**  $\text{non\_empty}: \langle XS \neq [] \rangle$   
**and**  $\text{contiguous}: \langle \text{contiguous } (XS::\text{real interval list}) \rangle$   
**shows**  $\text{upper } (\text{last } XS) = \text{Max } (\text{set } (\text{map } \text{upper } XS))$   
**using**  $\text{max\_last\_sorted\_wrt\_upper}[\text{of } XS, \text{simplified assms contiguous\_sorted\_wrt\_upper}[\text{of } XS, \text{simplified assms}]]$   
**by**  $\text{auto}[1]$

**lemma set\_of\_interval\_list\_contiguous:**  
**assumes**  $\text{non\_empty}: \langle XS \neq ([]::\text{real interval list}) \rangle$   
**and**  $\text{contiguous}: \langle \text{contiguous } XS \rangle$   
**shows**  $\langle \text{set\_of\_interval\_list } XS = \{ \text{lower } (\text{hd } XS) .. \text{upper } (\text{last } XS) \} \rangle$   
**using**  $\text{assms}$   
**proof** $(\text{induction } XS \text{ rule: list\_nonempty\_induct})$   
**case**  $(\text{single } x)$   
**then show**  $?case$   
**using**  $\text{set\_of\_eq}$   
**unfolding**  $\text{contiguous\_def set\_of\_interval\_list\_def}$  **by**  $\text{auto}[1]$   
**next**  
**case**  $(\text{cons } x \text{ xs})$   
**then show**  $?case$   
**using**  $\text{set\_of\_eq}$   
 $\text{sup\_set\_of\_interval\_list\_upper}[\text{of } (x \# \text{xs}), \text{simplified cons contiguous\_sorted\_wrt\_upper}[\text{of } (x \# \text{xs}), \text{simplified assms}], \text{simplified}]$   
 $\text{inf\_set\_of\_interval\_list\_lower}[\text{of } (x \# \text{xs}), \text{simplified cons contiguous\_sorted\_wrt\_lower}[\text{of } (x \# \text{xs}), \text{simplified assms}], \text{simplified}]$   
**unfolding**  $\text{set\_of\_interval\_list\_def contiguous\_def set\_of\_eq}$   
**apply** $(\text{auto})[1]$   
**subgoal**  
**using**  $\text{cons.premis contiguous\_non\_overlapping lower\_le\_upper\_aux non\_overlapping\_sorted\_unroll}$   
**by**  $\text{fastforce}$   
**subgoal**  
**by**  $(\text{smt } (\text{verit}, \text{ccfv\_threshold}) \text{Suc\_less\_eq Suc\_pred atLeastAtMost\_iff length\_greater\_o\_conv list.sel}(1) \text{lower\_le\_upper neq\_Nil\_conv nth\_Cons\_o nth\_Cons\_Suc})$   
**subgoal**  
**using**  $\text{less\_diff\_conv}$  **by**  $\text{force}$   
**subgoal**  
**by**  $(\text{smt } (\text{verit}, \text{del\_insts}) \text{One\_nat\_def Suc\_eq\_plus1 atLeastAtMost\_iff hd\_conv\_nth length\_greater\_o\_conv less\_diff\_conv nth\_Cons\_o nth\_Cons\_Suc})$   
**done**  
**qed**

**lemma set\_of\_interval\_list\_set\_eq\_interval\_list\_union\_contiguous:**  
**assumes**  $\text{non\_empty}: \langle XS \neq ([]::\text{real interval list}) \rangle$   
**and**  $\text{contiguous}: \langle \text{contiguous } XS \rangle$

```

shows <set_of_interval_list XS = set_of (interval_list_union XS)>
using interval_list_union_contiguous[of XS, simplified assms, simplified]
      set_of_interval_list_contiguous[of XS, simplified assms, simplified]
      contiguous_non_overlapping[of XS, simplified assms, simplified]
      non_overlapping_implies_sorted_wrt_upper[of XS]
      non_overlapping_implies_sorted_wrt_lower[of XS]
      interval_list_union_contiguous_lower[of XS]
      interval_list_union_contiguous_upper[of XS]
apply(simp add:set_of_eq)
by (metis non_empty)

```

```

lemma mInterval_ovl_lower_hd_min:
  <valid_mInterval_ovl x  $\implies$  Min (set (map lower x)) = (lower o hd) x>
proof(induction x rule:induct_list012)
case 1
then show ?case unfolding valid_mInterval_ovl_def by simp
next
case (2 x)
then show ?case unfolding valid_mInterval_ovl_def by simp
next
case (3 x y zs)
then show ?case
  apply(simp add: valid_mInterval_ovl_def non_overlapping_sorted_def cmp_non_overlapping_def image_def)
  by (metis (no_types, lifting) list.sel(1) list.simps(3) min.absorb3 min_def sorted_wrt_lower_unroll)
qed

```

```

lemma mInterval_adj_lower_hd_min:
  <valid_mInterval_adj x  $\implies$  Min (set (map lower x)) = (lower o hd) x>
proof(induction x rule:induct_list012)
case 1
then show ?case unfolding valid_mInterval_adj_def by simp
next
case (2 x)
then show ?case by simp
next
case (3 x y zs)
then show ?case
  apply(simp add: valid_mInterval_adj_def non_overlapping_sorted_def cmp_non_overlapping_def image_def)
  by (metis lower_le_upper min.absorb1 min.bounded_iff)
qed

```

```

lemma mInterval_non_ovl_lower_hd_min:
  <valid_mInterval_non_ovl x  $\implies$  Min (set (map lower x)) = (lower o hd) x>
unfolding valid_mInterval_non_ovl_def
using mInterval_ovl_lower_hd_min
by(auto)

```

```

lemma mInterval_ovl_lower_last_max:
  <valid_mInterval_ovl x  $\implies$  (Max (set (map lower x))) = (lower o last) x>
proof(induction x rule:induct_list012)
case 1
then show ?case unfolding valid_mInterval_ovl_def by simp

```

```

next
  case (2 x)
  then show ?case unfolding valid_mInterval_ovl_def by simp
next
  case (3 x y zs)
  then show ?case
    by(auto simp add: sorted_wrt_lower_def cmp_lower_width_def valid_mInterval_ovl_def
      non_overlapping_sorted_def max_def cmp_non_overlapping_def image_def
      split: if_splits)
qed

lemma mInterval_adj_upper_hd_min:
  ⟨ valid_mInterval_adj x ⟹ Min (set (map upper x)) = (upper o hd) x ⟩
proof(induction x rule:induct_list012)
  case 1
  then show ?case unfolding valid_mInterval_adj_def by simp
next
  case (2 x)
  then show ?case unfolding valid_mInterval_adj_def by simp
next
  case (3 x y zs)
  then show ?case
    apply(simp add: sorted_wrt_lower_def cmp_lower_width_def valid_mInterval_adj_def
      non_overlapping_sorted_def max_def cmp_non_overlapping_def image_def
      split: if_splits)
    by (metis (no_types, opaque_lifting) dual_order.trans lower_le_upper min.absorb2 min commute)
qed

lemma mInterval_adj_upper_last_max:
  ⟨ valid_mInterval_adj x ⟹ Max (set (map upper x)) = (upper o last) x ⟩
proof(induction x rule:induct_list012)
  case 1
  then show ?case unfolding valid_mInterval_adj_def by simp
next
  case (2 x)
  then show ?case unfolding valid_mInterval_adj_def by simp
next
  case (3 x y zs)
  then show ?case
    apply(simp add: sorted_wrt_lower_def cmp_lower_width_def valid_mInterval_adj_def
      non_overlapping_sorted_def max_def cmp_non_overlapping_def image_def
      split: if_splits)
    subgoal using le_left_mono lower_le_upper by blast
    subgoal using last_in_set lower_le_upper order_trans by blast
  done
qed

lemma set_of_subeq_aux:
  ⟨ (⋃ x∈set is. {lower x..upper x}) ⊆ {Min (lower ' (set is)) .. Max (upper ' (set is))} ⟩
proof(induction is)
  case Nil
  then show ?case by simp
next

```

```

case (Cons a is)
then show ?case
  apply(auto)[1]
  apply (meson List.finite_set Min.coboundedI finite_imageI finite_insert imageI insertCI order.trans)
  by (meson List.finite_set Max.coboundedI finite_imageI finite_insert imageI insert_iff order_trans)+
qed

```

```

lemma lower_merge_adjacent_intervals:
assumes xs ≠ []
  and ⟨sorted_wrt_lower xs⟩
shows (lower ∘ hd) (merge_adjacent_intervals_sorted_wrt_lower xs) = (lower ∘ hd) xs
using assms proof(induction xs rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:interval_sort_lower_width_def)
next
  case (cons x xs)
  then show ?case
    apply(simp)
    by (metis list.sel(1) merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd)
qed

```

```

lemma sorted_wrt_lower_hd_min:
  ⟨x ≠ [] ⟹ sorted_wrt_lower x ⟹ Min (set (map lower x)) = (lower ∘ hd) x⟩
proof(induction x rule:induct_list012)
  case 1
  then show ?case by simp
next
  case (2 x)
  then show ?case by simp
next
  case (3 x y zs)
  then show ?case
    apply(simp add: valid_mInterval_ovl_def non_overlapping_sorted_def cmp_non_overlapping_def image_def)
    by (metis (no_types, lifting) list.sel(1) list.simps(3) min.absorb3 min_def sorted_wrt_lower_unroll)
qed

```

```

lemma lower_hd_min_over_mk_mInterval_non_ovl:
  xs ≠ [] ⟹ (lower ∘ hd) xs = Min (lower ` (set xs)) ⟹ (lower ∘ hd) (mk_mInterval_non_ovl xs) = (lower ∘ hd) xs
unfolding mk_mInterval_non_ovl_def mk_mInterval_ovl_def
  by (metis list.set_map mInterval_ovl_lower_hd_min mk_mInterval_lower mk_mInterval_non_ovl_def
mk_mInterval_ovl_def valid_ovl_mkInterval_non_ovl)

```

```

theorem valid_mInterval_non_ovl_empty: valid_mInterval_non_ovl x ⟹ x ≠ []
unfolding valid_mInterval_non_ovl_def valid_mInterval_ovl_def
by simp

```

**end**



## 9 Subdivisions and Refinements

### ( Lipschitz\_Subdivisions\_Refinements)

**theory**

*Lipschitz\_Subdivisions\_Refinements*

**imports**

*Lipschitz\_Interval\_Extension*

*Multi\_Interval\_Preliminaries*

**begin**

#### 9.1 Subdivisions

A uniform subdivision of an interval  $X$  splits  $X$  into a vector of equal length, contiguous intervals.

**definition** *uniform\_subdivision* :: 'a::linordered\_field interval  $\Rightarrow$  nat  $\Rightarrow$  'a interval list **where**  
*uniform\_subdivision* A n = map ( $\lambda i$ . let  $i' = \text{of\_nat } i$  in  
 mk\_interval (lower A + (upper A - lower A) \*  $i' / \text{of\_nat } n$ ,  
 lower A + (upper A - lower A) \* ( $i' + 1$ ) /  $\text{of\_nat } n$ )) [0.. $n$ ]

The definition *uniform\_subdivision* refers to definition 6.2 in [4]

**definition** *overlapping\_ordered* :: 'a::{preorder} interval list  $\Rightarrow$  bool **where**  
*overlapping\_ordered* xs = ( $\forall i$ .  $i < \text{length } xs - 1 \longrightarrow \text{lower } (xs ! (i + 1)) \leq \text{upper } (xs ! i)$ )

**definition** *overlapping\_non\_zero\_width* :: 'a::{preorder, minus, zero, ord} interval list  $\Rightarrow$  bool **where**  
*overlapping\_non\_zero\_width* xs = ( $\forall i < \text{length } xs - 1$ .  $\exists e$ .  $e \in_i (xs ! (i + 1)) \wedge e \in_i (xs ! i) \wedge 0 < \text{width } (xs ! (i + 1)) \wedge 0 < \text{width } (xs ! i)$ )

**definition** *overlapping* :: 'a::{preorder} interval list  $\Rightarrow$  bool **where**  
*overlapping* xs = ( $\forall i < \text{length } xs - 1$ .  $\exists e$ .  $e \in_i (xs ! (i + 1)) \wedge e \in_i (xs ! i)$ )

**definition** *check\_is\_uniform\_subdivision* :: 'a::linordered\_field interval  $\Rightarrow$  'a interval list  $\Rightarrow$  bool **where**  
*check\_is\_uniform\_subdivision* A xs = (let  $n = \text{length } xs$  in  
 if  $n = 0$  then True  
 else  
 let  $d = \text{width } A / \text{of\_nat } n$  in  
 list\_all ( $\lambda x$ .  $\text{width } x = d$ ) xs  $\wedge$   
 contiguous xs  $\wedge$   
 lower (hd xs) = lower A  $\wedge$   
 upper (last xs) = upper A)

**lemma** *non\_empty\_subdivision*:

**assumes**  $0 < n$

**shows** *uniform\_subdivision* A n  $\neq []$

**unfolding** *uniform\_subdivision\_def* **using** *assms* **by** *simp*

**lemma** *uniform\_subdivision\_id*: *uniform\_subdivision* X 1 = [X]

**unfolding** *uniform\_subdivision\_def* **by** *simp*

```

lemma subdivision_length_n:
  assumes  $0 < n$ 
  shows  $\text{length}(\text{uniform\_subdivision } A \ n) = n$ 
  using assms
proof(induction n rule:nat_induct_non_zero)
  case 1
  then show ?case unfolding uniform_subdivision_def by simp
next
  case (Suc n)
  then show ?case unfolding uniform_subdivision_def by simp
qed

```

```

lemma contiguous_uniform_subdivision: contiguous (uniform_subdivision A n)
proof -
  have a0:  $\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
     $\text{upper}(\text{uniform\_subdivision } A \ n \ ! \ i) = \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n$ 
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
  have a1:  $\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
     $\text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n = \text{lower}(\text{uniform\_subdivision } A \ n \ ! \ (i + 1))$ 
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
  have a2:  $\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
     $\text{upper}(\text{uniform\_subdivision } A \ n \ ! \ i) = \text{lower}(\text{uniform\_subdivision } A \ n \ ! \ (i + 1))$ 
  using a0 a1 by simp
  have a3:  $\text{contiguous}(\text{uniform\_subdivision } A \ n) =$ 
     $(\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
       $\text{upper}(\text{uniform\_subdivision } A \ n \ ! \ i) = \text{lower}(\text{uniform\_subdivision } A \ n \ ! \ (i + 1)))$ 
  unfolding contiguous_def by simp
  show ?thesis using a0 a1 a2 a3 by simp
qed

```

```

lemma overlapping_ordered_uniform_subdivision:
  assumes  $0 < n$ 
  shows overlapping_ordered (uniform_subdivision A n)
proof -
  have a0:  $\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
     $\text{upper}(\text{uniform\_subdivision } A \ n \ ! \ i) \geq \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n$ 
  using assms
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
  have a1:  $\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
     $\text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n \geq \text{lower}(\text{uniform\_subdivision } A \ n \ ! \ (i + 1))$ 
  using assms
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
  have a2:  $\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
     $\text{upper}(\text{uniform\_subdivision } A \ n \ ! \ i) \geq \text{lower}(\text{uniform\_subdivision } A \ n \ ! \ (i + 1))$ 
  using a0 a1 by force
  have a3:  $\text{overlapping\_ordered}(\text{uniform\_subdivision } A \ n) =$ 
     $(\forall i < \text{length}(\text{uniform\_subdivision } A \ n) - 1.$ 
       $\text{upper}(\text{uniform\_subdivision } A \ n \ ! \ i) \geq \text{lower}(\text{uniform\_subdivision } A \ n \ ! \ (i + 1)))$ 
  unfolding overlapping_ordered_def by simp
  show ?thesis using a0 a1 a2 a3 by simp
qed

```

**lemma overlapping\_uniform\_subdivision:**  
**assumes**  $o < N$   
**shows**  $\text{overlapping (uniform\_subdivision } X N)$   
**using** *assms*  
**proof** –  
**let**  $?n = \text{length (uniform\_subdivision } X N) - 1$   
**have**  $a0: \forall i < ?n. \text{lower (uniform\_subdivision } X N! (i + 1)) = \text{upper (uniform\_subdivision } X N! i)$   
**using** *assms contiguous\\_uniform\\_subdivision unfolding contiguous\\_def by metis*  
**have**  $a1: \forall i < ?n. \text{upper (uniform\_subdivision } X N! i) \in_i \text{uniform\_subdivision } X N! i$   
 $\wedge \text{upper (uniform\_subdivision } X N! i) \in_i (\text{uniform\_subdivision } X N! (i + 1))$   
**using** *ao in\\_intervall lower\\_le\\_upper order.refl*  
**by** *metis*  
**have**  $a2: \forall i < ?n. \exists e. e \in_i \text{uniform\_subdivision } X N! (i + 1) \wedge e \in_i \text{uniform\_subdivision } X N! i$   
**using** *a1 by auto[1]*  
**show**  $?thesis$  **using** *a2 unfolding overlapping\\_def by simp*  
**qed**

**lemma hd\_lower\_uniform\_subdivision:**  
**assumes**  $o < n$   
**shows**  $\text{lower (hd (uniform\_subdivision } A n)) = \text{lower } A$   
**proof** –  
**have**  $\text{lower (hd (uniform\_subdivision } A n)) = \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } o / \text{of\_nat } n$   
**using** *assms*  
**by** (*simp add: uniform\\_subdivision\\_def mk\\_interval' hd\\_map*)  
**also have**  $\dots = \text{lower } A$   
**by** *simp*  
**finally show**  $?thesis$  .  
**qed**

**lemma last\_upper\_uniform\_subdivision:**  
**assumes**  $o < n$   
**shows**  $\text{upper (last (uniform\_subdivision } A n)) = \text{upper } A$   
**using** *assms*  
**by** (*simp add: uniform\\_subdivision\\_def mk\\_interval' last\\_map Let\\_def field\\_simps*)

**lemma uniform\_subdivisions\_width\_single:**  
**fixes**  $A :: 'a::\text{linordered\_field interval}$   
**shows**  $\langle \text{width (Interval (lower } A + (\text{upper } A - \text{lower } A) * x / \text{of\_nat } n,$   
 $\text{lower } A + (\text{upper } A - \text{lower } A) * (x + 1) / \text{of\_nat } n)) = \text{width } A / \text{of\_nat } n \rangle$   
**proof** –  
**have**  $\text{lower } A \leq \text{upper } A$  **using** *lower\\_le\\_upper by simp*  
**then have**  $\text{leq: lower } A + (\text{upper } A - \text{lower } A) * x / \text{of\_nat } n \leq$   
 $\text{lower } A + (\text{upper } A - \text{lower } A) * (x + 1) / \text{of\_nat } n$   
**by** (*simp add: divide\\_le\\_cancel linorder\\_not\\_less mult\\_le\\_cancel\\_left*)  
**have**  $U: \langle \text{upper (Interval (lower } A + (\text{upper } A - \text{lower } A) * x / \text{of\_nat } n,$   
 $\text{lower } A + (\text{upper } A - \text{lower } A) * (x + 1) / \text{of\_nat } n)) =$   
 $\text{lower } A + (\text{upper } A - \text{lower } A) * (x + 1) / \text{of\_nat } n \rangle$   
**using** *upper\\_bounds leq by blast*  
**have**  $L: \langle \text{lower (Interval (lower } A + (\text{upper } A - \text{lower } A) * x / \text{of\_nat } n,$   
 $\text{lower } A + (\text{upper } A - \text{lower } A) * (x + 1) / \text{of\_nat } n)) =$   
 $\text{lower } A + (\text{upper } A - \text{lower } A) * x / \text{of\_nat } n \rangle$   
**using** *lower\\_bounds leq by blast*  
**then show**  $?thesis$  **using** *U L add\\_diff\\_cancel\\_left add\\_diff\\_cancel\\_left' diff\\_divide\\_distrib*  
 $\text{mult.comm\_neutral vector\_space\_over\_itself.scale\_right\_diff\_distrib unfolding width\_def}$

by metis  
qed

lemma uniform\_subdivisions\_width:

assumes  $0 < n$

shows  $\langle \forall A. A \in \text{set } (\text{uniform\_subdivision } X \ n) \longrightarrow \text{width } A = \text{width } X / \text{of\_nat } n \rangle$

apply (simp add: uniform\_subdivision\_def mk\_interval' o\_def image\_def width\_def Let\_def split: if\_split)

apply auto[1]

apply (metis add\_diff\_cancel\_left' diff\_divide\_distrib mult.right\_neutral right\_diff\_distrib)

using assms uniform\_subdivisions\_width\_single[simplified width\_def]

by (simp add: divide\_right\_mono mult\_left\_mono)

lemma uniform\_subdivision\_sum\_width:

assumes  $0 < n$

shows  $\langle \text{sum\_list } (\text{map width } (\text{uniform\_subdivision } X \ n)) = \text{width } X \rangle$

proof –

have  $\langle \forall a. a \in \text{set } (\text{uniform\_subdivision } X \ n) \longrightarrow \text{width } a = \text{width } X / \text{of\_nat } n \rangle$

using uniform\_subdivisions\_width using assms by blast

then have width:  $\forall a. a \in \text{set } (\text{map width } (\text{uniform\_subdivision } X \ n)) \longrightarrow a = \text{width } X / \text{of\_nat } n$

unfolding width\_def by auto[1]

then have width\_list:  $\text{list\_all } (\lambda a. a = \text{width } X / \text{of\_nat } n) (\text{map width } (\text{uniform\_subdivision } X \ n))$

unfolding width\_def using list\_all\_iff by blast

then have length:  $\text{length } (\text{map width } (\text{uniform\_subdivision } X \ n)) = n$

unfolding uniform\_subdivision\_def by simp

then have sum\_list:  $\text{sum\_list } (\text{map width } (\text{uniform\_subdivision } X \ n)) = (\text{width } X / \text{of\_nat } n) * \text{of\_nat } n$

using width\_list by (metis list.map\_ident\_strong mult\_of\_nat\_commute sum\_list\_triv width)

then show ?thesis by (simp add: assms)

qed

lemma uniform\_subdivisions\_distinct:

assumes  $0 < n$   $0 < \text{width } A$

shows  $\text{distinct } (\text{uniform\_subdivision } A \ n)$

proof –

have  $\forall i < n. \forall j < n. i \neq j \longrightarrow (\text{uniform\_subdivision } A \ n) ! i \neq (\text{uniform\_subdivision } A \ n) ! j$

proof –

have f1:  $\forall i < n. \forall j < n. i \neq j \longrightarrow (\text{upper } A - \text{lower } A) * \text{of\_nat } i / \text{of\_nat } n \neq (\text{upper } A - \text{lower } A) * \text{of\_nat } j / \text{of\_nat } n$

using assms(1) assms(2) divide\_cancel\_right less\_numeral\_extra(3) mult\_cancel\_left of\_nat\_eq\_o\_iff of\_nat\_eq\_iff width\_def

by metis

have f2:  $\forall i < n. \forall j < n. i \neq j \longrightarrow \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n \neq \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (j + 1) / \text{of\_nat } n$

using assms(1) assms(2) unfolding width\_def by simp

have f3:  $\forall i < n. \text{lower } ((\text{uniform\_subdivision } A \ n) ! i) = \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } i / \text{of\_nat } n$

using assms by (simp add: uniform\_subdivision\_def divide\_right\_mono mult\_left\_mono Let\_def)

have f5:  $\forall i < n - 1. (\text{upper } ((\text{uniform\_subdivision } A \ n) ! i)) = \text{lower } ((\text{uniform\_subdivision } A \ n) ! \text{Suc } i)$

using assms by (simp add: uniform\_subdivision\_def divide\_right\_mono mult\_left\_mono)

have f6:  $\forall j < n - 1. (\text{upper } ((\text{uniform\_subdivision } A \ n) ! j)) = \text{lower } ((\text{uniform\_subdivision } A \ n) ! \text{Suc } j)$

using assms(1) f5 contiguous\_uniform\_subdivision unfolding contiguous\_def subdivision\_length\_n by blast

have  $\forall i < n. \forall j < n. i \neq j \longrightarrow \text{lower } ((\text{uniform\_subdivision } A \ n) ! i) \neq \text{lower } ((\text{uniform\_subdivision } A \ n) ! j)$

using f1 f2 f3 by auto[1]

then show ?thesis by metis

qed

then show ?thesis using assms(1) distinct\_conv\_nth subdivision\_length\_n by metis

qed

```

lemma uniform_subdivisions_non_overlapping:
  assumes  $o < n$ 
  shows non_overlapping_sorted (uniform_subdivision A n)
proof -
  have  $\forall i < n. \forall j < n. i < j \longrightarrow \text{upper } ((\text{uniform\_subdivision } A \ n) \ ! \ i) \leq \text{lower } ((\text{uniform\_subdivision } A \ n) \ ! \ j)$ 
  proof -
    have fo:  $\langle o \leq \text{width } A \rangle$  unfolding width_def lower_le_upper by simp
    have f2:  $\forall i < n. \text{upper } ((\text{uniform\_subdivision } A \ n) \ ! \ i) = \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n$ 
      using assms by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono Let_def)
    have f3:  $\forall j < n. \text{lower } ((\text{uniform\_subdivision } A \ n) \ ! \ j) = \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } j / \text{of\_nat } n$ 
      using assms by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono Let_def)
    have f4:  $\forall i < n. \forall j < n. i < j \longrightarrow \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } (i + 1) / \text{of\_nat } n \leq \text{lower } A + (\text{upper } A - \text{lower } A) * \text{of\_nat } j / \text{of\_nat } n$ 
      using assms divide_right_mono mult_left_mono Suc_eq_plus1 Suc_le1 add_le_cancel_left
        interval_width_positive of_nat_o_le_iff of_nat_le_iff width_def
      by metis
    have  $\forall i < n. \forall j < n. i < j \longrightarrow \text{upper } ((\text{uniform\_subdivision } A \ n) \ ! \ i) \leq \text{lower } ((\text{uniform\_subdivision } A \ n) \ ! \ j)$ 
      using fo f2 f3 f4 by simp
    then show ?thesis by auto[1]
  qed
  then show ?thesis
  unfolding non_overlapping_sorted_def cmp_non_overlapping_def
  by (simp add: assms(1) sorted_wrt_iff_nth_less subdivision_length_n)
qed

```

We prove that our uniform subdivision meets the multi-interval type

```

lemma uniform_subdivisions_valid_ainterval:
  assumes  $o < n$   $o < \text{width } A$ 
  shows valid_mInterval_adj (uniform_subdivision A n)
  using assms
  unfolding valid_mInterval_adj_def
  apply safe
  subgoal using uniform_subdivisions_non_overlapping by blast
  subgoal using uniform_subdivisions_distinct by blast
  subgoal using non_empty_subdivision by blast
  done

lemma uniform_subdivisions_valid:
  assumes  $o < n$ 
  shows check_is_uniform_subdivision A (uniform_subdivision A n)
  unfolding check_is_uniform_subdivision_def Let_def
  apply (simp split: if_split)
  apply safe
  subgoal using assms uniform_subdivisions_width subdivision_length_n
    by (metis (mono_tags, lifting) Ball_set)
  subgoal using assms contiguous_uniform_subdivision by blast
  subgoal using assms hd_lower_uniform_subdivision by blast
  subgoal using assms last_upper_uniform_subdivision by blast
  done

```

## 9.2 Refinement

Let  $F X$  be an inclusion isotonic, Lipschitz, interval extension for  $X \subseteq Y$ . A refinement  $F_N X$  of  $F X$  is the union of interval values of  $X$  over the elements of a uniform subdivision of  $X$

**definition** *refinement* :: ( $'a::\{\text{linordered\_field}, \text{lattice}\}$  interval  $\Rightarrow 'a$  interval)  $\Rightarrow \text{nat} \Rightarrow 'a$  interval  $\Rightarrow 'a$  interval **where**  
 $\langle \text{refinement } F N X = (\text{interval\_list\_union } (\text{map } F (\text{uniform\_subdivision } X N))) \rangle$

**definition** *check\_is\_refinement where*

$\langle \text{check\_is\_refinement } F n A s B = (\text{let } I = \text{refinement } F n A s \text{ in } \text{lower } B \leq \text{lower } I \wedge \text{upper } I \leq \text{upper } B) \rangle$

**definition** *refinement\_set* :: ( $'a::\{\text{linordered\_field}, \text{lattice}\}$  interval  $\Rightarrow 'a$  interval)  $\Rightarrow \text{nat} \Rightarrow 'a$  interval  $\Rightarrow 'a$  set **where**  
 $\langle \text{refinement\_set } F N X = (\text{set\_of\_interval\_list } (\text{map } F (\text{uniform\_subdivision } X N))) \rangle$

The definition *refinement* refers to definition 6.3 in [4].

The excess width of  $F X$  is  $w(E X) = w(F X) - w(f X)$ . The united extension  $f x$  for  $x \in X$  has zero excess width and we can compute  $f x$  as closely as desired by computing refinements of an extension  $F X$ .

**definition** *width\_set*  $s = \text{Sup } s - \text{Inf } s$

**lemma** *width\_set\_bounded*:

**fixes**  $X :: \langle \text{real set} \rangle$   
**assumes**  $\langle \text{bdd\_below } X \rangle \langle \text{bdd\_above } X \rangle$   
**shows**  $\langle \forall x \in X. \forall x' \in X. \text{dist } x x' \leq \text{width\_set } X \rangle$   
**using** *assms sup\_inf\_dist\_bounded*  
**unfolding** *width\_set\_def*  
**by** *(simp)*

**lemma** *width\_inclusion\_isotonic\_approx*:

**fixes**  $F :: \langle \text{real interval} \Rightarrow \text{real interval} \rangle$   
**assumes** *inclusion\_isotonic*  $F$  *F is\_interval\_extension\_of*  $f$   
**shows**  $\langle 0 \leq \text{width } (F X) - \text{width\_set } (f' \text{ set\_of } X) \rangle$   
**by** *(smt (verit, del\_insts) assms(1) assms(2) inclusion\_isotonic\_inf inclusion\_isotonic\_sup width\_def width\_set\_def)*

**lemma** *diameter\_width*:

**assumes**  $\langle a \leq b \rangle$   
**shows**  $\langle \text{diameter } \{a..b\} = \text{width\_set } \{a..b\} \rangle$   
**by** *(simp add: assms linorder\_not\_less diameter\_Sup\_Inf width\_set\_def)*

**lemma** *lipschitz\_dist\_diameter\_limit*:

**fixes**  $S :: \langle 'a::\{\text{metric\_space}, \text{heine\_borel}\} \text{ set} \rangle$   
**and**  $f :: \langle 'a::\{\text{metric\_space}, \text{heine\_borel}\} \Rightarrow 'b::\{\text{metric\_space}, \text{heine\_borel}\} \rangle$   
**assumes**  $\langle C\text{-lipschitz\_on } S f \rangle$  **and**  $\langle \text{bounded } S \rangle$   
**shows**  $\langle x \in (f' S) \Rightarrow y \in (f' S) \Rightarrow \text{dist } x y \leq \text{diameter } (f' S) \rangle$   
**using** *lipschitz\_on\_uniformly\_continuous*[*of C S f, simplified assms*]  
*bounded\_uniformly\_continuous\_image*[*of S f, simplified assms*]  
*diameter\_bounded\_bound*[*of f' S x y*]  
**by** *simp*

**definition** *excess\_width\_diameter* :: ( $'a::\text{preorder interval} \Rightarrow \text{real interval}$ )  $\Rightarrow ('a \Rightarrow 'b::\text{metric\_space}) \Rightarrow 'a$  interval  $\Rightarrow$  real **where**

$\langle \text{excess\_width\_diameter } F f X = \text{width}(F X) - \text{diameter } (f' \text{ set\_of } X) \rangle$

**definition** *excess\_width\_set* :: ( $'a::\{\text{minus}, \text{linorder}, \text{Inf}, \text{Sup}\}$  interval  $\Rightarrow 'a$  set)  $\Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a$  interval  $\Rightarrow 'a$  **where**

$\langle \text{excess\_width\_set } F f X = \text{width\_set}(F X) - \text{width\_set}(f' \text{ set\_of } X) \rangle$

**definition**  $\text{excess\_width} :: ('a :: \{\text{minus}, \text{linorder}, \text{Inf}, \text{Sup}\} \text{ interval} \Rightarrow 'a \text{ interval}) \Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a \text{ interval} \Rightarrow 'a \text{ where}$   
 $\langle \text{excess\_width } F f X = \text{width}(F X) - \text{width\_set}(f' \text{ set\_of } X) \rangle$

The definition  $\text{excess\_width}$  refers to definition 6.4 in [4]

**lemma**  $\text{width\_set\_of}$ : **fixes**  $X :: \text{real interval}$   
**shows**  $\text{width\_set\_upper\_lower}$ :  $\langle \text{width\_set}(\text{set\_of } X) = |(\text{lower } X) - (\text{upper } X)| \rangle$   
**by** ( $\text{simp add: width\_set\_def set\_of\_eq}$ )

**lemma**  $\text{width\_set\_dist}$ :  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**shows**  $\text{width\_set}(\text{set\_of } X) = (\text{dist}(\text{lower } X)(\text{upper } X))$   
**by**( $\text{simp add:set\_of\_eq width\_set\_def dist\_real\_def}$ )

**lemma**  $\text{diameter\_of}$ : **fixes**  $X :: \text{real interval}$   
**shows**  $\text{diameter\_upper\_lower}$ :  $\langle \text{diameter}(\text{set\_of } X) = |(\text{lower } X) - (\text{upper } X)| \rangle$   
**by** ( $\text{simp add: linorder\_not\_less set\_of\_eq}$ )

**lemma**  $\text{diameter\_dist}$ :  
**fixes**  $X :: \text{real interval}$   
**shows**  $\text{diameter}(\text{set\_of } X) = (\text{dist}(\text{lower } X)(\text{upper } X))$   
**unfolding**  $\text{set\_of\_eq dist\_real\_def abs\_real\_def}$   
**using**  $\text{lower\_le\_upper[of } X] \text{diameter\_closed\_interval[of lower } X \text{ upper } X]$   
**by**  $\text{argo}$

**lemma**  $\text{bdd\_below\_f\_set\_of}$ :  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**assumes**  $C\text{-lipschitz\_on } X f$   
**and**  $\langle \text{bounded } X \rangle$  **and**  $\langle X \neq \{\} \rangle$   
**shows**  $\langle \text{bdd\_below}(f' X) \rangle$   
**using**  $\text{assms atLeastAtMost\_iff bdd\_below.unfold bounded\_imp\_bdd\_below image\_def}$   
 $\text{lipschitz\_bounded\_image\_real set\_of\_eq set\_of\_nonempty}$   
**by**  $\text{simp}$

**lemma**  $\text{bdd\_above\_f\_set\_of}$ :  
**fixes**  $f :: \text{real} \Rightarrow \text{real}$   
**assumes**  $C\text{-lipschitz\_on } (X) f$   
**and**  $\langle \text{bounded } X \rangle$  **and**  $\langle X \neq \{\} \rangle$   
**shows**  $\langle \text{bdd\_above}(f' X) \rangle$   
**using**  $\text{assms atLeastAtMost\_iff bdd\_above.unfold bounded\_imp\_bdd\_above image\_def}$   
 $\text{lipschitz\_bounded\_image\_real set\_of\_eq set\_of\_nonempty}$   
**by**  $\text{simp}$

**lemma**  $\text{diameter\_image\_dist}$ :  
**fixes**  $f :: \langle \text{real} \Rightarrow \text{real} \rangle$   
**assumes**  $\langle \text{continuous\_on}(\text{set\_of } X) f \rangle$   
**shows**  $\langle (\exists x \in \text{set\_of } X. \exists x' \in \text{set\_of } X. \text{diameter}(f' \text{ set\_of } X) = \text{dist}(f x)(f x')) \rangle$   
**using**  $\text{assms compact\_continuous\_image[of set\_of } X f, \text{simplified assms compact\_set\_of[of } X]]$   
 $\text{lower\_le\_upper[of } X] \text{diameter\_closed\_interval[of } f(\text{lower } X) f(\text{upper } X), \text{symmetric}]$   
 $\text{diameter\_compact\_attained[of } f' \text{ set\_of } X] \text{set\_f\_nonempty[of } f X]$   
**by**  $\text{fastforce}$

**lemma**  $\text{excess\_width\_inf\_diameter}$ :

```

fixes F::<real interval  $\Rightarrow$  real interval>
assumes inclusion_isotonic F F is_interval_extension_of f < C—lipschitz_on (set_of X) f>
shows <dist (Inf (f' set_of X)) (lower (F X))  $\leq$  excess_width_diameter F f X>
  unfolding dist_real_def abs_real_def excess_width_diameter_def width_def
  using inclusion_isotonic_inf[of F f X, simplified assms]
    inclusion_isotonic_sup[of F f X, simplified assms]
    diameter_Sup_Inf[of f' set_of X, simplified assms lipschitz_on_continuous_on[of C (set_of X) f]
      compact_img_set_of[of X f], simplified]
by simp

```

```

lemma excess_width_inf:
fixes F::<real interval  $\Rightarrow$  real interval>
assumes inclusion_isotonic F F is_interval_extension_of f < C—lipschitz_on (set_of X) f>
shows <dist (Inf (f' set_of X)) (lower (F X))  $\leq$  excess_width F f X>
  unfolding dist_real_def abs_real_def excess_width_def width_def
  using inclusion_isotonic_inf[of F f X, simplified assms]
    inclusion_isotonic_sup[of F f X, simplified assms]
by (simp add: width_set_def)

```

```

lemma excess_width_sup_diameter:
fixes F::<real interval  $\Rightarrow$  real interval>
assumes inclusion_isotonic F F is_interval_extension_of f < C—lipschitz_on (set_of X) f>
shows <dist (Sup (f' set_of X)) (upper (F X))  $\leq$  excess_width F f X>
  unfolding dist_real_def abs_real_def excess_width_diameter_def width_def
  using inclusion_isotonic_inf[of F f X, simplified assms]
    inclusion_isotonic_sup[of F f X, simplified assms]
    diameter_Sup_Inf[of f' set_of X, simplified assms lipschitz_on_continuous_on[of C (set_of X) f]
      compact_img_set_of[of X f], simplified]
by (simp add: excess_width_def width_def width_set_def)

```

```

lemma excess_width_sup:
fixes F::<real interval  $\Rightarrow$  real interval>
assumes inclusion_isotonic F F is_interval_extension_of f < C—lipschitz_on (set_of X) f>
shows <dist (Sup (f' set_of X)) (upper (F X))  $\leq$  excess_width F f X>
  unfolding dist_real_def abs_real_def excess_width_def width_def
  using inclusion_isotonic_inf[of F f X, simplified assms]
    inclusion_isotonic_sup[of F f X, simplified assms]
by (simp add: width_set_def)

```

If  $F X$  is an inclusion isotonic, Lipschitz, interval extension then the excess width of a refinement is of order  $1 / N$

If  $X$  and  $X$  are intervals such that  $X \subseteq Y$ , then there is an interval  $E$  with  $lower E \leq o \wedge o \leq upper E$  such that  $Y = X + E$  and  $w Y = w X + w E$ .

```

lemma interval_subset_width:
fixes X Y :: 'a::{\linordered_ring, lattice} interval
assumes X  $\leq$  Y
and X_def: X = Interval(a, b) and x_valid: a  $\leq$  b
and Y_def: Y = Interval(c, d) and y_valid: c  $\leq$  d
shows  $\exists E. o \in_i E \wedge Y = X + E \wedge width Y = width X + width E$ 
proof —
  have c  $\leq$  a b  $\leq$  d
proof —

```

```

have leq: lower Y ≤ lower X ∧ upper X ≤ upper Y using assms(1) unfolding less_eq_interval_def by simp
have X_bounds: lower X = a upper X = b unfolding X_def by (simp add: x_valid bounds_of_interval_eq_lower_upper)+
have Y_bounds: lower Y = c upper Y = d unfolding Y_def by (simp add: y_valid bounds_of_interval_eq_lower_upper)+
show c ≤ a ≤ b ≤ d using assms(1) X_bounds Y_bounds unfolding less_eq_interval_def by simp_all
qed
define e where e = c - a
define f where f = d - b
define E where E = Interval(e, f)
have o ∈i E unfolding E_def e_def f_def using <c ≤ a> <b ≤ d> set_of_subset_iff
by (smt (verit, ccfv_SIG) diff_ge_o_iff_ge_in_interval le_iff_diff_le_o lower_bounds order.trans upper_bounds)
have Y = X + E
proof -
have X_bounds: lower X = a upper X = b
unfolding X_def by (simp add: x_valid bounds_of_interval_eq_lower_upper)+
have Y_bounds: lower Y = c upper Y = d
unfolding Y_def by (simp add: y_valid bounds_of_interval_eq_lower_upper)+
have E_bound_1: lower E = c - a
unfolding E_def e_def f_def
using <b ≤ d> <c ≤ a> diff_left_mono diff_self lower_bounds order_trans
by metis
have E_bound_2: upper E = d - b
unfolding E_def e_def f_def
using <b ≤ d> <c ≤ a> E_bound_1 <o ∈i E> diff_ge_o_iff_ge dual_order.trans in_bounds upper_bounds
by metis
have add: Interval(a, b) + Interval(c - a, d - b) = Interval(c, d)
using X_bounds Y_bounds E_bound_1 E_bound_2
by (simp add: E_def X_def Y_def e_def f_def interval_eq)
show ?thesis unfolding E_def X_def Y_def e_def f_def using add by simp
qed
have width Y = width X + width E unfolding width_def using E_def X_def Y_def <Y = X + E> e_def f_def by force
from <o ∈i E> <Y = X + E> <width Y = width X + width E> show ?thesis by auto[1]
qed

```

**lemma** excess\_width\_incl:

```

fixes F :: real interval ⇒ real interval and X :: real interval
assumes int: <F is_interval_extension_of f>
and iso: inclusion_isotonic F
and L-lipschitz_on (set_of X) f
shows <∃ E . F X = Interval(Inf (f' set_of X), Sup (f' set_of X)) + E>
proof -
have ao: <f' (set_of X) ≠ {}> using in_interval by fastforce
have a1: Inf (f' set_of X) ≤ Sup (f' set_of X)
using assms ao inf_le_sup_image_real by simp
have a2: <f' (set_of X) ⊆ set_of (F X)>
using assms fundamental_theorem_of_interval_analysis by simp
have max: <Sup (f' (set_of X)) ≤ Sup (set_of (F X))>
using assms(3) ao a2 sup_image_le_real[of f X F] by blast
have max_interval: Sup (f' (set_of X)) ≤ upper (F X)
using max sup_set_of by metis
have min: <Inf (set_of (F X)) ≤ Inf (f' (set_of X))>
using assms(3) ao a2 inf_image_le_real[of f X F] by blast
have min_interval: lower (F X) ≤ Inf (f' (set_of X))
using min inf_set_of by metis
have lower_min: lower (Interval (Inf (f' set_of X), Sup (f' set_of X))) = Inf (f' set_of X)

```

```

using lower_bounds a1 by simp
have upper_max: upper (Interval (Inf (f' set_of X), Sup (f' set_of X))) = Sup (f' set_of X)
using upper_bounds a1 by simp
have a3: Interval(Inf (f' set_of X), Sup (f' set_of X)) ≤ F X
using min_interval max_interval lower_min upper_max unfolding less_eq_interval_def by simp
have a4: ∃ E . Interval(Inf (f' set_of X), Sup (f' set_of X)) + E = F X using a3
by (metis (no_types, opaque_lifting) bounds_of_interval_eq_lower_upper
    bounds_of_interval_inverse interval_subset_width lower_le_upper)
then show ?thesis by metis
qed

```

```

lemma excess_interval_superset_interval:
fixes F :: real interval ⇒ real interval and X :: real interval
assumes int: ⟨F is_interval_extension_of f⟩
and iso: inclusion_isotonic F
and L—lipschitz_on (set_of X) f
and ex: ⟨∃ E . F X = Interval(Inf (f' set_of X), Sup (f' set_of X)) + E⟩
shows ⟨Interval(Inf (f' set_of X), Sup (f' set_of X)) ≤ F X⟩
proof —
have lhs: lower (F X) ≤ lower (Interval(Inf (f' set_of X), Sup (f' set_of X)))
using assms(1,2,3) inf_image_le_real inf_le_sup_image_real inf_set_of
    fundamental_theorem_of_interval_analysis lower_bounds
by metis
have rhs: upper (Interval(Inf (f' set_of X), Sup (f' set_of X))) ≤ upper (F X)
using assms(1,2,3) inf_le_sup_image_real sup_image_le_real sup_set_of
    fundamental_theorem_of_interval_analysis upper_bounds
by metis
show ?thesis using lhs rhs unfolding less_eq_interval_def by simp
qed

```

```

lemma each_subdivision_width_order:
fixes X :: 'a::{linordered_field,lattice,metric_space} interval
assumes inclusion_isotonic F lipschitzL_on C U F F is_interval_extension_of f
and set (uniform_subdivision X N) ⊆ U o < N Xs ∈ set (uniform_subdivision X N)
shows width(F Xs) ≤ C * width (X) / of_nat N
proof —
have a0: ∀ Xs ∈ set (uniform_subdivision X N). width (F Xs) ≤ C * width Xs
using assms(2) assms(4) lipschitzL_onD by blast
have a1: ∀ Xs ∈ set (uniform_subdivision X N). width(F Xs) ≤ C * width (X) / of_nat N
using a0 assms(5) uniform_subdivisions_width[of N X] by simp
show ?thesis using a1 assms by simp
qed

```

```

lemma each_subdivision_excess_width_order:
fixes X :: real interval
assumes inclusion_isotonic F lipschitzL_on C U F F is_interval_extension_of f
and set (uniform_subdivision X N) ⊆ U o < N
and L—lipschitz_on (set_of (interval_list_union (uniform_subdivision X N))) f
shows ∀ Xs ∈ set (uniform_subdivision X N) . excess_width F f Xs ≤ C * width (X) / of_nat N
proof —
have a0: ∀ Xs ∈ set (uniform_subdivision X N). width (F Xs) ≤ C * width Xs
using assms lipschitzL_onD by blast
have a1: ∀ Xs ∈ set (uniform_subdivision X N). width(F Xs) ≤ C * width (X) / of_nat N
using a0 assms uniform_subdivisions_width[of N X] by simp

```

```

have a2:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). \text{excess\_width } F f Xs \leq C * \text{width } (X) / \text{of\_nat } N$ 
proof –
  have b0:  $\text{set } (\text{uniform\_subdivision } X N) \neq \{\}$ 
    using assms non_empty_subdivision by simp
  have b1:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). 0 \leq \text{width\_set } (f' \text{ set\_of } Xs)$ 
proof –
  have c0:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). \text{set\_of } Xs \subseteq \text{set\_of } (\text{interval\_list\_union } (\text{uniform\_subdivision } X N))$ 
    using assms interval_list_union_correct in_set_conv_nth non_empty_subdivision
    by metis
  then have c1:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). \text{set\_of } Xs \neq \{\}$  using in_interval by fastforce
  then have c2:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). \text{Inf } (f' \text{ set\_of } Xs) \leq \text{Sup } (f' \text{ set\_of } Xs)$ 
    using assms c0 c1 inf_le_sup_image_real lipschitz_on_subset
    by (metis inf_le_sup_image_real)
  then have c3:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). 0 \leq \text{width\_set } (f' \text{ set\_of } Xs)$ 
    by (simp add: width_set_def)
  then show ?thesis by simp
qed
have b2:  $\forall Xs \in \text{set } (\text{uniform\_subdivision } X N). \text{width } (F Xs) - \text{width\_set } (f' \text{ set\_of } Xs) \leq \text{width } (F Xs)$ 
    using b0 b1 assms inf_set_of_lower_le_upper sup_set_of_inf_le_sup_image_real image_is_empty
    by (simp add: width_set_def)
  then show ?thesis
    using a1 unfolding excess_width_def width_set_def by fastforce
qed
show ?thesis using assms a0 a1 a2 by simp
qed

```

The theorem  $\llbracket \text{inclusion\_isotonic } ?F; ?C\text{-lipschitz\_on } ?U ?F; ?F \text{ is\_interval\_extension\_of } ?f; \text{set } (\text{uniform\_subdivision } ?X ?N) \subseteq ?U; 0 < ?N; ?Xs \in \text{set } (\text{uniform\_subdivision } ?X ?N) \rrbracket \implies \text{width } (?F ?Xs) \leq ?C * \text{width } ?X / \text{of\_nat } ?N$  refers to Theorem 6.1 in [4].

**lemma** *sup\_interval\_max*:

```

fixes  $X Y :: 'a :: \{\text{linordered\_ring, lattice}\} \text{interval}$ 
shows  $\text{sup } X Y = \text{Interval}(\text{min } (\text{lower } X) (\text{lower } Y), \text{max } (\text{upper } X) (\text{upper } Y))$ 
using Interval.lower_sup Interval.upper_sup Interval_id inf_real_def sup_max
by (metis inf_min)

```

**lemma** *interval\_inf\_sup\_lower*:  $\text{inf } (\text{lower } l1) (\text{lower } l2) = \text{lower } (\text{sup } l1 l2)$

```

unfolding sup_interval_def
by (metis (mono_tags, lifting) Interval.lower_sup sup_interval_def)

```

**lemma** *interval\_sup\_sup\_upper*:  $\text{sup } (\text{upper } l1) (\text{upper } l2) = \text{upper } (\text{sup } l1 l2)$

```

unfolding sup_interval_def
by (metis (mono_tags, lifting) Interval.upper_sup sup_interval_def)

```

**lemma** *interval\_union\_lower*:

```

assumes contiguous Xs Xs  $\neq$  []
shows  $\text{lower } (\text{interval\_list\_union } Xs) = \text{lower } (Xs!0)$ 
using assms
proof(induction Xs rule: interval_list_union.induct)
  case 1
  then show ?case by simp
next
  case (2 l)
  then show ?case by simp
next

```

```

case (3 I v va)
have ao: contiguous (v # va) using 3 unfolding contiguous_def by auto
have a1: (v # va) ≠ [] by simp
then show ?case
  unfolding sorted_wrt_lower_def 3
proof —
have bo: lower ((I # v # va) ! o) = lower I by simp
have b1: lower I ≤ lower v
  using 3.prem1 a1
  unfolding contiguous_def
  by (metis Suc_eq_plus1 add_diff_cancel_left' length_Cons less_Suc_eq_o_disj
    lower_le_upper nth_Cons_o nth_Cons_Suc plus_1_eq_Suc)
show lower (interval_list_union (I # v # va)) = lower ((I # v # va) ! o)
  using bo b1 3(2) interval_list_union.simps(3) 3.IH Interval.lower_sup ao a1 inf.orderE
    nth_Cons_o
  unfolding sorted_wrt_lower_def
  by metis
qed
qed

lemma interval_union_upper:
  assumes contiguous Xs Xs ≠ []
  shows upper (interval_list_union Xs) = upper (last Xs)
  using assms
proof(induction Xs rule:interval_list_union.induct)
  case 1
  then show ?case by simp
next
  case (2 I)
  then show ?case by simp
next
  case (3 I v va)
  have ao: contiguous (v # va) using 3 unfolding contiguous_def by auto
  have a1: (v # va) ≠ [] by simp
  then show ?case
  proof —
  have bo: upper ((I # v # va) ! (length (I # v # va) - 1)) = upper (last (I # v # va))
    using last_conv_nth by fastforce
  have b1: upper I ≤ upper v using 3.prem1 unfolding contiguous_def by auto
  show upper (interval_list_union (I # v # va)) = upper (last (I # v # va))
    using bo b1 interval_list_union.simps(3) 3.IH ao a1 interval_list_union.simps(3)
      Interval.upper_sup last.simps
    by (metis (no_types, opaque_lifting) dual_order.trans min_list.cases sup_absorb2 sup_ge1)
  qed
qed

lemma union_set:
  assumes 0 < n
  shows interval_list_union (uniform_subdivision X n) = X
  using assms
proof (induction n rule:nat_induct_non_zero)
  case 1
  then show ?case using uniform_subdivision_id interval_list_union.simps(2) by metis
next

```

```

case (Suc n)
then show ?case
proof (induction uniform_subdivision X (Suc n) rule: interval_list_union.induct)
  case 1
  then show ?case by (metis less_Suc_eq non_empty_subdivision)
next
  case (2 I)
  then show ?case using One_nat_def interval_list_union.simps(2) subdivision_length_n uniform_subdivision_id
zero_less_Suc
  by metis
next
  case (3 I v va)
  then have a0: lower I = lower X
  using hd_lower_uniform_subdivision assms
  by (metis list.collapse list.simps(3) nth_Cons_o zero_less_Suc)
  then have a1: upper (last (I # v # va)) = upper X
  using last_upper_uniform_subdivision[of Suc n X] assms 3.hyps(2) by simp
  then have a2: contiguous (I # v # va)
  using contiguous_uniform_subdivision assms zero_less_Suc 3.hyps(2) by metis
  then have a3: overlapping_ordered (I # v # va)
  using overlapping_uniform_subdivision assms zero_less_Suc 3.hyps(2)
  by (simp add: overlapping_ordered_uniform_subdivision)
  then have a4:  $\forall i < \text{length } (I \# v \# va) - 1. \text{upper } ((I \# v \# va) ! i) = \text{lower } ((I \# v \# va) ! (i + 1))$ 
  using a0 a2 unfolding contiguous_def by simp
  have a5:  $\forall i < \text{length } (I \# v \# va) - 1. \text{lower } ((I \# v \# va) ! i) \leq \text{lower } ((I \# v \# va) ! (i + 1))$ 
  using a0 a2 contiguous_def lower_le_upper by metis
  have a6:  $\forall i < \text{length } (I \# v \# va) - 1. \text{upper } ((I \# v \# va) ! i) \leq \text{upper } ((I \# v \# va) ! (i + 1))$ 
  using a0 a2 contiguous_def lower_le_upper by metis
  then show ?case
  proof -
  have lower (interval_list_union (I # v # va)) = lower X
  proof -
  have c0: lower (interval_list_union (I # v # va)) = lower (sup I (interval_list_union (v # va)))
  using interval_list_union.simps(3) by metis
  have c1: lower (sup I (interval_list_union (v # va))) = min (lower I) (lower (interval_list_union (v # va)))
  using inf_real_def sup_interval_def sup_interval_max sup_real_def
  by (simp add: inf_min)
  have c2: min (lower I) (lower (interval_list_union (v # va))) = lower I
  using 3 hd_lower_uniform_subdivision[of Suc n X] a0
  contiguous_uniform_subdivision[of X, simplified contiguous_def 3(2)[symmetric]]
  interval_union_lower[of (I # v # va)]
  by (metis a2 c0 c1 list.discr nth_Cons_o)
  show ?thesis using a0 c0 c1 c2 by simp
qed
moreover have upper (interval_list_union (I # v # va)) = upper X
proof -
  have c0: upper (interval_list_union (I # v # va)) = upper (sup I (interval_list_union (v # va)))
  using interval_list_union.simps(3) by metis
  have c1: upper (sup I (interval_list_union (v # va))) = max (upper I) (upper (interval_list_union (v # va)))
  using inf_real_def sup_interval_def sup_interval_max sup_real_def
  by (simp add: sup_max)
  have c2: max (upper I) (upper (interval_list_union (v # va))) = upper (last (I # v # va))
  using contiguous_uniform_subdivision[of X, simplified contiguous_def 3(2)[symmetric]]
  interval_union_upper[of (I # v # va)]

```

```

    by (metis a2 co c1 list.discr)
    show ?thesis using co c1 c2 3.hyps(2) last_upper_uniform_subdivision by auto[1]
  qed
ultimately show ?thesis
using interval_eq1 3 by auto[1]
qed
qed
qed

```

```

lemma sum_list_less:
  assumes list_all ( $\lambda n. n \leq (y::real)$ ) xs
  shows sum_list xs  $\leq y * \text{length } xs$ 
proof -
  have  $\forall x \in \text{set } xs. x \leq y$ 
  using assms list_all_iff by metis
  hence sum_list xs  $\leq \text{sum\_list } (\text{replicate } (\text{length } xs) y)$ 
  using sum_list_mono
  by (simp add: order_less_imp_le sum_list_mono2)
  also have  $\dots = y * \text{length } xs$ 
  by (simp add: sum_list_replicate)
  finally show ?thesis by simp
qed

```

```

lemma in_bounds2:
  fixes X Y :: 'a::{linordered_ring} interval
  shows  $x \in_i X \wedge x \in_i Y \implies$ 
    ( $\text{lower } Y \leq \text{lower } X \wedge \text{upper } Y \leq \text{upper } X \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )  $\vee$ 
    ( $\text{lower } X \leq \text{lower } Y \wedge \text{upper } X \leq \text{upper } Y \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )  $\vee$ 
    ( $\text{lower } Y \leq \text{lower } X \wedge \text{upper } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{lower } X \wedge \text{lower } Y \leq \text{upper } X$ )  $\vee$ 
    ( $\text{lower } X \leq \text{lower } Y \wedge \text{upper } Y \leq \text{upper } X \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )
  apply (clarify)
  by (metis (full_types) in_bounds nle_le order.trans)

```

```

lemma overlapping_width_sum:
  fixes X Y :: 'a::{linordered_ring, lattice} interval
  assumes overlapping [X,Y]
  shows width (sup X Y)  $\leq \text{width } X + \text{width } Y$ 
proof -
  have a0:  $\forall i < \text{length } [X, Y] - 1. \exists e. e \in_i [X, Y] \wedge (i + 1) \wedge e \in_i [X, Y] \wedge ! i$ 
  using assms unfolding overlapping_def by blast
  have a1: ( $\text{lower } Y \leq \text{lower } X \wedge \text{upper } Y \leq \text{upper } X \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )  $\vee$ 
    ( $\text{lower } X \leq \text{lower } Y \wedge \text{upper } X \leq \text{upper } Y \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )  $\vee$ 
    ( $\text{lower } Y \leq \text{lower } X \wedge \text{upper } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{lower } X \wedge \text{lower } Y \leq \text{upper } X$ )  $\vee$ 
    ( $\text{lower } X \leq \text{lower } Y \wedge \text{upper } Y \leq \text{upper } X \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )
  using assms in_bounds2[of _ X Y] unfolding overlapping_def by auto
  have a2: ( $\text{lower } Y \leq \text{lower } X \wedge \text{upper } Y \leq \text{upper } X \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )  $\implies \text{width } (\text{sup } X Y) \leq$ 
width X + width Y
  by (simp add: inf_min width_def)
  have a3: ( $\text{lower } X \leq \text{lower } Y \wedge \text{upper } X \leq \text{upper } Y \wedge \text{lower } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{upper } X$ )  $\implies \text{width } (\text{sup } X Y) \leq$ 
width X + width Y
  by (simp add: inf_min width_def)
  have a4: ( $\text{lower } Y \leq \text{lower } X \wedge \text{upper } X \leq \text{upper } Y \wedge \text{lower } Y \leq \text{lower } X \wedge \text{lower } Y \leq \text{upper } X$ )  $\implies \text{width } (\text{sup } X Y) \leq$ 
width X + width Y
  by (simp add: inf_min sup_max width_def)

```

```

have a5: (lower X ≤ lower Y ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ⇒ width (sup X Y) ≤
width X + width Y
by (metis interval_width_positive le_add_same_cancel1 less_eq_interval_def sup.order_iff)
show ?thesis using assms a0 a1 a2 a3 a4 a5 by blast
qed

```

**lemma** interval\_list\_union\_width:

```

fixes xs :: 'a::{linordered_ring, lattice} interval list
assumes overlapping xs xs ≠ []
shows overlapping xs ⇒ width (interval_list_union xs) ≤ sum_list (map width xs)
using assms unfolding contiguous_def width_def
proof (induction xs rule: interval_list_union.induct)
case 1
then show ?case by simp
next
case (2 l)
then show ?case by simp
next
case (3 l1 l2 l3)
then show ?case unfolding overlapping_def
proof –
have a0: width (interval_list_union (l1 # l2 # l3)) = width (sup l1 (interval_list_union (l2 # l3)))
using interval_list_union.simps by simp
have a1: ... ≤ width l1 + width (interval_list_union (l2 # l3))
proof –
have b0: overlapping [l1, interval_list_union (l2 # l3)]
using 3.prem1 list.simps(3) list.size(3) list.size(4) interval_list_union_correct
unfolding overlapping_def
by (smt (verit, ccfv_threshold) One_nat_def Suc_eq_plus1 diff_add_inverse2
length_greater_o_conv less_one nth_Cons' subsetD)
show ?thesis
using overlapping_width_sum[of l1 (interval_list_union (l2 # l3)), simplified b0]
by blast
qed
have a2: ... ≤ width l1 + sum_list (map width (l2 # l3))
proof –
have b0: (width l1 + width (interval_list_union (l2 # l3)) ≤ width l1 + sum_list (map width (l2 # l3))) =
(width (interval_list_union (l2 # l3)) ≤ sum_list (map width (l2 # l3))) by simp
have b1: overlapping (l2 # l3) using 3.prem1 unfolding overlapping_def by force
have b2: l2 # l3 ≠ [] by simp
have b3: width (interval_list_union (l2 # l3)) ≤ sum_list (map width (l2 # l3))
using 3.IH b1 b2 unfolding width_def by simp
show ?thesis using b0 b3 by simp
qed
have a3: ... = sum_list (map width (l1 # l2 # l3)) by auto[1]
show ?thesis using a0 a1 a2 a3 map_eq_conv width_def
by (smt (verit, ccfv_SIG) dual_order.trans)
qed
qed

```

**lemma** map\_non\_zero\_width:

```

fixes U :: 'a::{linordered_idom} interval set
assumes C—lipschitzl_on U F inclusion_isotonic F set xs ⊆ U
shows ∀ x ∈ set xs. 0 ≤ width x → 0 ≤ width (F x)

```

```

proof
  fix x
  assume a0: x ∈ set xs
  show 0 ≤ width x → 0 ≤ width (F x)
proof
  assume a1: 0 ≤ width x
  have a2: width (F x) ≤ C * width x using assms(1,3) a0 unfolding lipschitzl_on_def by blast
  have a4: width (F x) ≤ C * width x + 1 using assms(1,3) a0 unfolding lipschitzl_on_def by fastforce
  have 0 ≤ C * width x using assms(1) a1 unfolding lipschitzl_on_def by simp
  show 0 ≤ width (F x) using assms(1) a0 interval_width_positive unfolding lipschitzl_on_def interval_width_positive by
blast
qed
qed

```

**lemma** inclusion\_isotonic\_preserves\_overlapping:

**assumes** inclusion\_isotonic F xs ≠ [] F is\_interval\_extension\_of f  
**shows** contiguous xs ⇒ overlapping (map F xs)

**proof** (induct xs rule: induct\_list012)

**case** 1

**then show** ?case

**unfolding** overlapping\_def **by** simp

**next**

**case** (2 x)

**then show** ?case

**unfolding** overlapping\_def **by** simp

**next**

**case** (3 x y zs)

**then show** ?case

**proof** –

**have** a0:  $\forall i < \text{length}(\text{map } F(x \# y \# zs)) - 1. \exists e. e \in_i \text{map } F(x \# y \# zs)!(i+1) \wedge e \in_i \text{map } F(x \# y \# zs)!i$

**proof** –

**have** b0:  $\text{length}(\text{map } F(x \# y \# zs)) \geq 1$  **using** 3.prem1 **by** simp

**have** b2:  $\forall i < \text{length}(x \# y \# zs) - 1. \text{upper}((x \# y \# zs)!i) = \text{lower}((x \# y \# zs)!(i+1))$   
**using** 3.prem2 **unfolding** contiguous\_def **by** auto[1]

**have** b3:  $\forall i < \text{length}(x \# y \# zs) - 1. \text{upper}((x \# y \# zs)!i) \leq \text{upper}((x \# y \# zs)!(i+1))$   
**using** 3.prem3 **unfolding** contiguous\_def **by** auto[1]

**have** b4:  $\forall i < \text{length}(x \# y \# zs) - 1. \text{lower}((x \# y \# zs)!i) \leq \text{lower}((x \# y \# zs)!(i+1))$   
**using** 3.prem4 **unfolding** contiguous\_def **by** metis

**have** b5:  $\forall i < \text{length}(x \# y \# zs) - 1. f(\text{upper}((x \# y \# zs)!i)) = f(\text{lower}((x \# y \# zs)!(i+1)))$   
**using** b2 **by** simp

**have** b6:  $\forall i < \text{length}(x \# y \# zs) - 1. f(\text{upper}((x \# y \# zs)!i)) \in_i F((x \# y \# zs)!i)$   
**using** assms b2 b4 **in\_intervall** **interval\_of\_in\_eq**

**unfolding** is\_interval\_extension\_of\_def inclusion\_isotonic\_def  
**by** (metis lower\_interval\_of\_lower\_le\_upper\_upper\_interval\_of)

**have** b7:  $\forall i < \text{length}(x \# y \# zs) - 1. f(\text{upper}((x \# y \# zs)!i)) \in_i F((x \# y \# zs)!(i+1))$   
**using** assms b2 b3 **in\_intervall** **interval\_of\_in\_eq**

**unfolding** is\_interval\_extension\_of\_def inclusion\_isotonic\_def  
**by** (metis lower\_interval\_of\_lower\_le\_upper\_upper\_interval\_of)

**have** b8:  $\forall i < \text{length}(x \# y \# zs) - 1. f(\text{upper}((x \# y \# zs)!i)) \in_i F((x \# y \# zs)!(i+1)) \wedge f(\text{upper}((x \# y \# zs)!i)) \in_i F((x \# y \# zs)!i)$

**using** b6 b7 **by** blast

**show** ?thesis **using** b8 **apply** simp

**by** (metis One\_nat\_def Suc\_eq\_plus1 le\_simps(2) list.map(2) list.size(4) nth\_map order\_less\_le)

```

qed
show ?thesis using ao unfolding overlapping_def by simp
qed
qed

```

```

lemma bounded_image_of:
  fixes f :: <real ⇒ real>
  assumes <L-lipschitz_on (set_of X) f>
  shows <bounded (f ' set_of X)>
  using lipschitz_bounded_image_real[of set_of X L f] assms
  using bounded_set_of set_of_nonempty by blast

```

```

lemma dist_le_diameter:
  fixes f :: <real ⇒ real>
  assumes <C-lipschitz_on (set_of X) f>
  shows <dist (f (upper X)) (f (lower X)) ≤ diameter (f ' set_of X)>
  using diameter_bounded_bound[of f ' set_of X f (upper X) f (lower X),
    simplified
    bounded_image_of[of C X f, simplified assms]]
  by simp

```

```

lemma excess_width_inf_sup:
  fixes X :: real interval and f :: <real ⇒ real>
  assumes <continuous_on (set_of X) f>
  shows <Inf (f ' set_of X) - lower (F X) + upper (F X) - Sup (f ' set_of X) ≤ excess_width F f X>
  using diameter_Sup_Inf[of f ' set_of X, simplified compact_img_set_of set_of_nonempty assms]
  unfolding excess_width_def width_def width_set_def set_of_eq
  by simp

```

```

lemma excess_width_lower_bound:
  fixes X :: real interval
  assumes inclusion_isotonic F F is_interval_extension_of f <continuous_on (set_of X) f>
  shows <Inf (f ' set_of X) - lower (F X) ≤ excess_width F f X>
  proof -
  have ao: 0 ≤ upper (F X) - Sup (f ' set_of X)
    using assms(1) assms(2) diff_ge_o_iff_ge inclusion_isotonic_sup
    by metis
  show ?thesis using ao excess_width_inf_sup assms
    by (smt (verit, ccfv_threshold))
  qed

```

```

lemma excess_width_upper_bound:
  fixes X :: real interval
  assumes inclusion_isotonic F F is_interval_extension_of f <continuous_on (set_of X) f>
  shows <upper (F X) - Sup (f ' set_of X) ≤ excess_width F f X>
  proof -
  have ao: 0 ≤ Inf (f ' {lower X..upper X}) - lower (F X)
    using assms(1) assms(2) diff_ge_o_iff_ge inclusion_isotonic_sup set_of_eq
    inclusion_isotonic_inf
    by metis
  show ?thesis using ao excess_width_inf_sup assms unfolding set_of_eq
    by (smt (verit, ccfv_threshold))
  qed

```

```

lemma lipschitz_excess_width_lower_bound:
  fixes X :: real interval
  assumes inclusion_isotonic F lipschitzl_on C U F F is_interval_extension_of f
  and set (uniform_subdivision X N)  $\subseteq$  U N = 1
  and L—lipschitz_on (set_of (interval_list_union (uniform_subdivision X N))) f
  shows  $\text{Inf } (f' \text{ set\_of } X) - \text{lower } (F X) \leq C * \text{width } X$ 
proof—
  have ao: excess_width F f X  $\leq C * \text{width } X$ 
  using each_subdivision_excess_width_order[of F C U f X N L, simplified assms]
  assms(4) assms(5) assms(6) div_by_1 insert_iff less_numeral_extra(1) list.set(2) of_nat_1
  uniform_subdivision_id
  by metis
  have a1:  $\text{Inf } (f' \text{ set\_of } X) - \text{lower } (F X) \leq \text{excess\_width } F f X$ 
  using excess_width_lower_bound[of F f X, simplified assms]
  by (metis assms(5) assms(6) lipschitz_on_continuous_on union_set zero_less_one)
  show ?thesis using ao a1 by simp
qed

```

```

lemma lipschitz_excess_width_upper_bound:
  fixes X :: real interval
  assumes inclusion_isotonic F lipschitzl_on C U F F is_interval_extension_of f
  and set (uniform_subdivision X N)  $\subseteq$  U N = 1
  and L—lipschitz_on (set_of (interval_list_union (uniform_subdivision X N))) f
  shows  $\text{upper } (F X) - \text{Sup } (f' \text{ set\_of } X) \leq C * \text{width } X$ 
proof—
  have ao: excess_width F f X  $\leq C * \text{width } X$ 
  using each_subdivision_excess_width_order[of F C U f X N L, simplified assms]
  assms(4) assms(5) assms(6) div_by_1 insert_iff less_numeral_extra(1) list.set(2) of_nat_1
  uniform_subdivision_id
  by metis
  have a1:  $\text{upper } (F X) - \text{Sup } (f' \text{ set\_of } X) \leq \text{excess\_width } F f X$ 
  using excess_width_upper_bound[of F f X, simplified assms]
  by (metis assms(5) assms(6) lipschitz_on_continuous_on union_set zero_less_one)
  show ?thesis using ao a1 by simp
qed

```

```

lemma excess_width_bound_inf:
  fixes X :: real interval
  assumes excess_width_bound:  $\langle \text{excess\_width } F f X \leq k \rangle$ 
  and inclusion_isotonic:  $\langle \text{inclusion\_isotonic } F \rangle$ 
  and interval_extension:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
  shows  $\langle \text{Inf } (f' \text{ set\_of } X) - k \leq \text{lower } (F X) \rangle$ 
  using excess_width_bound[simplified excess_width_def width_def width_set_def]
  inclusion_isotonic interval_extension
  by (smt (verit, best) inclusion_isotonic_sup)

```

```

lemma excess_width_bound_sup:
  fixes X :: real interval
  assumes excess_width_bound:  $\langle \text{excess\_width } F f X \leq k \rangle$ 
  and inclusion_isotonic:  $\langle \text{inclusion\_isotonic } F \rangle$ 
  and interval_extension:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
  shows  $\langle \text{upper } (F X) \leq \text{Sup } (f' \text{ set\_of } X) + k \rangle$ 
  using excess_width_bound[simplified excess_width_def width_def width_set_def]
  inclusion_isotonic interval_extension

```

**by** (smt (verit, best) inclusion\_isotonic\_inf)

**lemma** set\_of\_interval\_list\_subset\_inf\_sup:  
 **assumes** non\_empty:  $\langle XS \neq ([] :: \text{real interval list}) \rangle$   
 **shows**  $\langle \text{set\_of\_interval\_list } XS \subseteq \{ \text{Min}(\text{set}(\text{map lower } XS)) .. \text{Max}(\text{set}(\text{map upper } XS)) \} \rangle$   
 **using** assms **unfolding** set\_of\_interval\_list\_def  
 **proof**(induction XS rule:list\_nonempty\_induct)  
 **case** (single x)  
 **then show** ?case **by**(simp add:set\_of\_eq)  
**next**  
 **case** (cons x xs)  
 **then show** ?case  
 **apply**(simp add:set\_of\_eq)  
 **by** fastforce  
**qed**

**lemma** lower\_bound\_F\_inf:  
 **assumes** non\_empty:  $\langle XS \neq ([] :: \text{real interval list}) \rangle$   
 **and** inclusion\_isotonic:  $\langle \text{inclusion\_isotonic } F \rangle$   
 **and** interval\_extension:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$   
 **and** sorted\_lower:  $\langle \text{sorted\_wrt\_lower } XS \rangle$   
 **and** lipschitz:  $\langle 0 \leq C \rangle \langle C - \text{lipschitz\_on}((\text{set\_of\_interval\_list } XS)) f \rangle$   
 **and** excess\_width\_bounded:  $\langle \text{Max}(\text{set}((\text{map}(\text{excess\_width } F f)) XS)) \leq k \rangle$   
 **shows**  $\langle (\text{Inf}(f'(\text{set\_of\_interval\_list } XS))) - k \leq \text{Inf}(\text{set\_of\_interval\_list}(\text{map } F XS)) \rangle$   
 **using** assms **proof**(induction XS rule:list\_nonempty\_induct)  
 **case** (single x)  
 **then show** ?case  
 **using** excess\_width\_bound\_inf[of F f x k]  
 **unfolding** set\_of\_interval\_list\_def  
 **by** (simp add: inf\_set\_of)  
**next**  
 **case** (cons x xs)  
 **then show** ?case  
 **using** excess\_width\_bound\_inf[of F f x]  
 **unfolding** set\_of\_interval\_list\_def  
 **apply**(simp)  
 **apply**(fold set\_of\_interval\_list\_def)  
 **apply**(subst image\_Un)  
 **apply**(subst cInf\_union\_distrib)  
 **subgoal**  
 **by** simp  
 **subgoal**  
 **apply**(simp add: set\_of\_eq)  
 **by** (meson bounded\_imp\_bdd\_below compact\_lcc compact\_continuous\_image compact\_imp\_bounded  
 lipschitz\_on\_continuous\_on lipschitz\_on\_subset sup\_ge1)  
 **subgoal**  
 **unfolding** set\_of\_interval\_list\_def  
 **by** (metis image\_is\_empty set\_of\_interval\_list\_def set\_of\_interval\_list\_nonempty)  
 **subgoal**  
 **using** compact\_set\_of\_interval\_list[of XS]  
 compact\_imp\_bounded[of (set\_of\_interval\_list XS)]  
 **by** (meson bounded\_imp\_bdd\_below compact\_continuous\_image compact\_imp\_bounded  
 compact\_set\_of\_interval\_list lipschitz\_on\_continuous\_on lipschitz\_on\_mono sup\_ge2)  
 **subgoal**

```

apply(subst cInf_union_distrib)
subgoal
  by simp
subgoal
  by (meson bdd_below_set_of)
subgoal
  by(simp add: set_of_interval_list_nonempty)
subgoal
  using set_of_interval_list_bdd_below by simp
subgoal
proof –
  assume a1: xs ≠ []
  assume a2: [sorted_wrt_lower xs; C–lipschitz_on (set_of_interval_list xs) f] ⇒ Inf (f' set_of_interval_list xs) – k
≤ Inf (set_of_interval_list (map F xs))
  assume a3: sorted_wrt_lower (x # xs)
  assume a4: C–lipschitz_on (set_of x ∪ set_of_interval_list xs) f
  assume a5: excess_width F f x ≤ k ∧ (∀ a ∈ set xs. excess_width F f a ≤ k)
  assume a6: ∧ k. excess_width F f x ≤ k ⇒ Inf (f' set_of x) – k ≤ lower (F x)
  have f7: sorted_wrt_lower xs
  using a3 a1 sorted_wrt_lower_unroll by blast
  have f8: Inf (set_of (F x)) = lower (F x)
  using inf_set_of by blast
  have f9: C–lipschitz_on (set_of_interval_list xs) f
  using a4 lipschitz_on_subset by blast
  have f10: inf (Inf (f' set_of x)) (Inf (f' set_of_interval_list xs)) ≤ Inf (f' set_of x)
  using inf_le1 by blast
  have f11: inf (Inf (f' set_of x)) (Inf (f' set_of_interval_list xs)) ≤ Inf (f' set_of_interval_list xs)
  using inf_le2 by blast
  have Inf (f' set_of x) – excess_width F f x ≤ lower (F x)
  using a6 by blast
  have inf (Inf (f' set_of x)) (Inf (f' set_of_interval_list xs)) – k ≤ Inf (set_of (F x))
  ∧ inf (Inf (f' set_of x)) (Inf (f' set_of_interval_list xs)) – k ≤ Inf (set_of_interval_list (map F xs))
  using f11 f10 f9 f8 f7 a5 a2
  using <Inf (f' set_of x) – excess_width F f x ≤ lower (F x)> by linarith
  then show ?thesis by simp
qed
done
done
qed

```

```

lemma upper_bound_F_sup:
assumes non_empty: <XS ≠ ([]::real interval list)>
and inclusion_isotonic: <inclusion_isotonic F>
and interval_extension: <F is_interval_extension_of f>
and sorted_upper: <sorted_wrt_upper XS>
and lipschitz: <0 ≤ C> <C–lipschitz_on ((set_of_interval_list XS)) f>
and excess_width_bounded: <(Max (set ((map (excess_width F f)) XS))) ≤ k>
shows <Sup (set_of_interval_list (map F XS)) ≤ (Sup (f' (set_of_interval_list XS))) + k>
using assms proof(induction XS rule:list_nonempty_induct)
case (single x)
then show ?case
  using excess_width_bound_sup[of F f x k]
  unfolding set_of_interval_list_def
  by (simp add: sup_set_of)

```

```

next
case (cons x xs)
then show ?case
  using excess_width_bound_inf[of F f x]
  unfolding set_of_interval_list_def
  apply(simp)
  apply(fold set_of_interval_list_def)
  apply(subst image_Un)
  apply(subst cSup_union_distrib)
  subgoal
    by simp
  subgoal
    by(simp add: set_of_eq)
  subgoal
    by(simp add: set_of_interval_list_nonempty)
  subgoal
    using set_of_interval_list_bdd_above by simp
  subgoal
    apply(subst cSup_union_distrib)
    subgoal
      by simp
    subgoal
      by (meson bdd_above_mono bdd_above_set_of fundamental_theorem_of_interval_analysis)
    subgoal
      by(simp add: set_of_interval_list_nonempty)
    subgoal
      using set_of_interval_list_bdd_above
      by (meson bounded_imp_bdd_above compact_continuous_image compact_imp_bounded
        compact_set_of_interval_list lipschitz_on_continuous_on lipschitz_on_subset sup_ge2)
    subgoal
      apply(auto)[1]
    subgoal
      by (smt (verit) excess_width_def inclusion_isotonic_inf sup_ge1 sup_set_of width_def width_set_def)
    subgoal
      by (smt (verit, ccfv_threshold) lipschitz_on_subset sorted_wrt_upper_unroll sup_ge2)
    done
  done
done
qed

```

```

lemma Inf_interval_list_approx: assumes non_empty:  $\langle XS \neq ([ ]::\text{real interval list}) \rangle$ 
and inclusion_isotonic:  $\langle \text{inclusion\_isotonic } F \rangle$ 
and interval_extension:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
and sorted_upper:  $\langle \text{sorted\_wrt\_upper } XS \rangle$ 
and lipschitz:  $\langle 0 \leq C \rangle \langle C - \text{lipschitz\_on } ((\text{set\_of\_interval\_list } XS)) f \rangle$ 
and excess_width_bounded:  $\langle (\text{Max } (\text{set } ((\text{map } (\text{excess\_width } F f)) XS))) \leq k \rangle$ 
shows  $\text{Inf } (\text{set\_of\_interval\_list } (\text{map } F XS)) \leq \text{Inf } (f' \text{ set\_of\_interval\_list } XS)$ 
using assms proof(induction XS rule:list_nonempty_induct)
case (single x)
then show ?case
  apply(simp add: set_of_interval_list_def set_of_eq)
  by (metis inclusion_isotonic_inf set_of_eq)
next
case (cons x xs)

```

```

then show ?case
  apply(simp add: set_of_interval_list_def set_of_eq)
  apply(subst image_Un)
  apply(subst cInf_union_distrib)
  subgoal
    by simp
  subgoal
    by simp
  subgoal
    proof —
      assume  $xs \neq []$ 
      then have set_of_interval_list (map F xs)  $\neq \{\}$ 
        by (simp add: set_of_interval_list_nonempty)
      then show ?thesis
        by (simp add: set_of_eq set_of_interval_list_def)
    qed
  subgoal
    proof —
      have  $\forall is. bdd\_below (set\_of\_interval\_list is::real set)$ 
        by (simp add: bounded_imp_bdd_below compact_imp_bounded compact_set_of_interval_list)
      then show ?thesis
        by (simp add: set_of_eq set_of_interval_list_def)
    qed
  subgoal
    apply(subst cInf_union_distrib)
    subgoal
      by simp
    subgoal
      by (meson bounded_imp_bdd_below compact_lcc compact_continuous_image compact_imp_bounded
        lipschitz_on_continuous_on lipschitz_on_subset sup_ge1)
    subgoal
      proof —
        assume  $xs \neq []$ 
        then have set_of_interval_list xs  $\neq \{\}$ 
          by (simp add: set_of_interval_list_nonempty)
        then show ?thesis
          by (simp add: set_of_eq set_of_interval_list_def)
        qed
    subgoal
      proof —
        assume  $C \text{--}lipschitz\_on (\{lower\ x..upper\ x\} \cup foldr (\lambda x. (\cup) \{lower\ x..upper\ x\}) xs \{\}) f$ 
        then have  $C \text{--}lipschitz\_on (\{lower\ x..upper\ x\} \cup set\_of\_interval\_list\ xs) f$ 
          by (simp add: set_of_eq set_of_interval_list_def)
        then have bounded (f' set_of_interval_list xs)
          by (metis compact_imp_bounded compact_set_of_interval_list lipschitz_bounded_image_real lipschitz_on_subset
            sup_ge2)
        then show ?thesis
          by (simp add: bounded_imp_bdd_below set_of_eq set_of_interval_list_def)
        qed
    subgoal
      apply(simp, rule conjI)
    subgoal
      using inclusion_isotonic_inf[of F f x, simplified assms set_of_eq, simplified]
      by (smt (verit, ccfv_threshold) inclusion_isotonic_inf le_inf1 set_of_eq)

```

```

subgoal
proof –
  assume a1: xs ≠ []
  assume a2:  $\llbracket \text{sorted\_wrt\_upper } xs; C\text{-lipschitz\_on } (\text{foldr } (\lambda x. (\cup) \{ \text{lower } x.. \text{upper } x \}) xs \{ \}) f \rrbracket \implies \text{Inf } (\text{foldr } (\lambda x. (\cup) \{ \text{lower } x.. \text{upper } x \}) (\text{map } F xs) \{ \}) \leq \text{Inf } (f' \text{ foldr } (\lambda x. (\cup) \{ \text{lower } x.. \text{upper } x \}) xs \{ \})$ 
  assume a3: sorted_wrt_upper (x # xs)
  assume C–lipschitz_on ( $\{ \text{lower } x.. \text{upper } x \} \cup \text{foldr } (\lambda x. (\cup) \{ \text{lower } x.. \text{upper } x \}) xs \{ \}) f$ )
  then have C–lipschitz_on ( $\text{foldr } (\lambda i. (\cup) \{ \text{lower } i.. \text{upper } i \}) xs \{ \}) f$ )
    using lipschitz_on_mono by blast
  then show ?thesis
    using a3 a2 a1 by (meson le_infl2 sorted_wrt_upper_unroll)
qed
done
done
done
qed

```

**lemma** *Sup\_interval\_list\_approx*: **assumes** *non\_empty*:  $\langle XS \neq ([ ]::\text{real interval list}) \rangle$

```

and inclusion_isotonic:  $\langle \text{inclusion\_isotonic } F \rangle$ 
and interval_extension:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
and sorted_lower:  $\langle \text{sorted\_wrt\_lower } XS \rangle$ 
and lipschitz:  $\langle 0 \leq C \rangle \langle C\text{-lipschitz\_on } ((\text{set\_of\_interval\_list } XS)) f \rangle$ 
and excess_width_bounded:  $\langle \text{Max } (\text{set } ((\text{map } (\text{excess\_width } F f)) XS))) \leq k \rangle$ 
shows  $\text{Sup } (f' \text{ set\_of\_interval\_list } XS) \leq \text{Sup } (\text{set\_of\_interval\_list } (\text{map } F XS))$ 
using assms proof(induction XS rule:list_nonempty_induct)
case (single x)
then show ?case
  apply(simp add: set_of_interval_list_def set_of_eq)
  by (metis inclusion_isotonic_sup set_of_eq)

```

**next**

**case** (*cons x xs*)

**then show** ?*case*

```

apply(simp add: set_of_interval_list_def set_of_eq)
apply(subst image_Un)
apply(subst cSup_union_distrib)

```

**subgoal**

**by** *simp*

**subgoal**

**by** (*meson bounded\_imp\_bdd\_above compact\_lcc compact\_continuous\_image compact\_imp\_bounded lipschitz\_on\_continuous\_on lipschitz\_on\_subset sup\_ge1*)

**subgoal**

**proof** –

**assume** *a1*:  $xs \neq [ ]$

**have** *f2*:  $\forall R Ra \text{ is } isa \text{ f } fa. ((R::\text{real set}) \neq Ra \vee \text{is } \neq \text{isa} \vee (\exists R i. (i::\text{real interval}) \in \text{set } is \wedge f i R \neq fa i R)) \vee \text{foldr } f \text{ is } R = \text{foldr } fa \text{ isa } Ra$

**by** (*meson foldr\_cong*)

**obtain** *ii* ::  $(\text{real interval} \Rightarrow \text{real set} \Rightarrow \text{real set}) \Rightarrow (\text{real interval} \Rightarrow \text{real set} \Rightarrow \text{real set}) \Rightarrow \text{real interval list} \Rightarrow \text{real interval}$  and *RR* ::  $(\text{real interval} \Rightarrow \text{real set} \Rightarrow \text{real set}) \Rightarrow (\text{real interval} \Rightarrow \text{real set} \Rightarrow \text{real set}) \Rightarrow \text{real interval list} \Rightarrow \text{real set}$  **where**

$\forall x_0 x_1 x_3. (\exists v_6 v_7. v_7 \in \text{set } x_3 \wedge x_1 v_7 v_6 \neq x_0 v_7 v_6) = (ii \ x_0 \ x_1 \ x_3 \in \text{set } x_3 \wedge x_1 (ii \ x_0 \ x_1 \ x_3) (RR \ x_0 \ x_1 \ x_3) \neq x_0 (ii \ x_0 \ x_1 \ x_3) (RR \ x_0 \ x_1 \ x_3))$

**by** *moura*

**then have**  $\forall R Ra \text{ is } isa \text{ f } fa. (R \neq Ra \vee \text{is } \neq \text{isa} \vee ii \text{ f } fa \text{ f } is \in \text{set } is \wedge f (ii \text{ f } fa \text{ f } is) (RR \text{ f } fa \text{ f } is) \neq fa (ii \text{ f } fa \text{ f } is) (RR \text{ f } fa \text{ f } is)) \vee \text{foldr } f \text{ is } R = \text{foldr } fa \text{ isa } Ra$

```

    using f2 by presburger
  then show ?thesis
    using a1 by (smt (z3) image_is_empty set_of_eq set_of_interval_list_def set_of_interval_list_nonempty)
qed
subgoal
proof —
  assume C—lipschitz_on ({lower x..upper x} ∪ foldr (λx. (∪) {lower x..upper x}) xs {}) f
  then have C—lipschitz_on ({lower x..upper x} ∪ set_of_interval_list xs) f
    by (simp add: set_of_eq set_of_interval_list_def)
  then have bounded (f' set_of_interval_list xs)
    by (metis (no_types) compact_imp_bounded compact_set_of_interval_list lipschitz_bounded_image_real lips-
chitz_on_subset sup_ge2)
  then show ?thesis
    by (simp add: bounded_imp_bdd_above set_of_eq set_of_interval_list_def)
qed
subgoal
apply(subst cSup_union_distrib)
subgoal
by simp
subgoal
by (meson bdd_above_lcc)
subgoal
proof —
  assume xs ≠ []
  then have set_of_interval_list (map F xs) ≠ {}
    by (simp add: set_of_interval_list_nonempty)
  then show ?thesis
    by (simp add: set_of_eq set_of_interval_list_def)
qed
subgoal
proof —
  have ∀ is. bdd_above (set_of_interval_list is::real set)
    by (simp add: bounded_imp_bdd_above compact_imp_bounded compact_set_of_interval_list)
  then show ?thesis
    by (simp add: set_of_eq set_of_interval_list_def)
qed
subgoal
apply(simp, rule conjI)
subgoal
using inclusion_isotonic_sup[of F f x, simplified assms set_of_eq, simplified] le_supI1
by auto
subgoal
proof —
  assume a1: xs ≠ []
  assume a2: [sorted_wrt_lower xs; C—lipschitz_on (foldr (λx. (∪) {lower x..upper x}) xs {}) f] ⇒ Sup (f' foldr
(λx. (∪) {lower x..upper x}) xs {}) ≤ Sup (foldr (λx. (∪) {lower x..upper x}) (map F xs) {})
  assume a3: sorted_wrt_lower (x # xs)
  assume a4: C—lipschitz_on ({lower x..upper x} ∪ foldr (λx. (∪) {lower x..upper x}) xs {}) f
  have f5: sorted_wrt_lower xs
    using a3 a1 sorted_wrt_lower_unroll by blast
  have f6: Sup (foldr (λi. (∪) {lower i..upper i}) (map F xs) {}) ≤ sup (upper (F x)) (Sup (foldr (λi. (∪) {lower
i..upper i}) (map F xs) {}))
    using sup_ge2 by blast
  have C—lipschitz_on (foldr (λi. (∪) {lower i..upper i}) xs {}) f

```

```

    using a4 lipschitz_on_mono by blast
  then show ?thesis
    using f6 f5 a2 by linarith
  qed
done
done
done
qed

```

**lemma** *map\_inclusion\_isotonic\_excess\_width\_bounded*:

```

assumes non_empty:  $\langle XS \neq ([ ] :: \text{real interval list}) \rangle$ 
and inclusion_isotonic:  $\langle \text{inclusion\_isotonic } F \rangle$ 
and interval_extension:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
and sorted_lower:  $\langle \text{sorted\_wrt\_lower } XS \rangle$ 
and sorted_upper:  $\langle \text{sorted\_wrt\_upper } XS \rangle$ 
and lipschitz:  $\langle C - \text{lipschitz\_on } ((\text{set\_of\_interval\_list } XS)) f \rangle$ 
and excess_width_bounded:  $\langle (\text{Max } (\text{set } ((\text{map } (\text{excess\_width } F f)) XS))) \leq k \rangle$ 
shows  $\langle \text{width\_set } (\text{set\_of\_interval\_list } (\text{map } F XS)) - \text{width\_set } (f' (\text{set\_of\_interval\_list } XS)) \leq 2 * k \rangle$ 
and  $\langle \text{width\_set } (\text{set\_of\_interval\_list } (\text{map } F XS)) - \text{width\_set } (f' (\text{set\_of\_interval\_list } XS)) \geq 0 \rangle$ 
subgoal
  unfolding width_set_def
  using lipschitz_on_nonneg[of  $C ((\text{set\_of\_interval\_list } XS)) f$ , simplified assms, simplified]
    lower_bound_F_inf[of  $XS F f C k$ , simplified assms, simplified]
    upper_bound_F_sup[of  $XS F f C k$ , simplified assms, simplified]
  by(simp)
subgoal
  unfolding width_set_def
  using lipschitz_on_nonneg[of  $C ((\text{set\_of\_interval\_list } XS)) f$ , simplified assms, simplified]
    Inf_interval_list_approx[of  $XS F f C k$ , simplified assms, simplified]
    Sup_interval_list_approx[of  $XS F f C k$ , simplified assms, simplified]
  by simp
done

```

**lemma** *max\_subdivision\_excess\_width\_order*:

```

fixes X :: real interval
assumes inclusion_isotonic F lipschitz_l_on C U F F is_interval_extension_of f
and set (uniform_subdivision X N)  $\subseteq U$   $0 < N$ 
and  $L - \text{lipschitz\_on } (\text{set\_of\_interval\_list } (\text{uniform\_subdivision } X N)) f$ 
shows  $\langle \text{Max } (\text{set } (\text{map } (\text{excess\_width } F f) (\text{uniform\_subdivision } X N))) \leq C * \text{width } X / \text{real } N \rangle$ 
proof(cases (set (map (excess_width F f) (uniform_subdivision X N))) = { })
  case True
  then show ?thesis
    using non_empty_subdivision[of  $N X$ , simplified assms, simplified]
    by simp
  next
  case False
  then show ?thesis
    apply(subst set_map)
    using each_subdivision_excess_width_order[of  $F C U f X N L$ , simplified assms, simplified]
      set_of_interval_list_set_eq_interval_list_union_contiguous[of (uniform_subdivision X N), simplified contiguous]
      non_empty_subdivision[of  $X N$ ] non_empty_subdivision[of  $N X$ , simplified assms, simplified], simplified]
    using assms(6) by auto[1]
qed

```

**lemma** *set\_of\_interval\_list\_set\_eq\_interval\_list\_union\_contiguous*:

**assumes** *non\_empty*:  $\langle XS \neq ([ ] :: \text{real interval list}) \rangle$

**and** *contiguous*:  $\langle \text{contiguous } XS \rangle$

**shows**  $\langle \text{set\_of\_interval\_list } XS = \text{set\_of } (\text{interval\_list\_union } XS) \rangle$

**using** *interval\_list\_union\_contiguous*[of *XS*, *simplified assms*, *simplified*]

*set\_of\_interval\_list\_contiguous*[of *XS*, *simplified assms*, *simplified*]

*contiguous\_non\_overlapping*[of *XS*, *simplified assms*, *simplified*]

*non\_overlapping\_implies\_sorted\_wrt\_upper*[of *XS*]

*non\_overlapping\_implies\_sorted\_wrt\_lower*[of *XS*]

*interval\_list\_union\_contiguous\_lower*[of *XS*]

*interval\_list\_union\_contiguous\_upper*[of *XS*]

**using** *set\_of\_eq\_non\_empty* **by** *metis*

**lemma** *width\_eq\_wdith\_set*:

**fixes** *X* ::  $\langle 'a :: \{\text{conditionally\_complete\_lattice, minus, preorder}\} \text{ interval} \rangle$

**shows**  $\langle \text{width } X = \text{width\_set } (\text{set\_of } X) \rangle$

**unfolding** *width\_def* *set\_of\_eq* *width\_set\_def* **by** (*simp*)

**lemma** *width\_zero\_lower\_upper*:

**fixes** *X* :: *real interval*

**assumes**  $\langle \text{width } X = 0 \rangle$

**shows**  $\langle \text{lower } X = \text{upper } X \rangle$

**using** *assms* *width\_def*[of *X*]

**by** *simp*

**lemma** *width\_zero\_mk\_interval*:

**fixes** *X* :: *real interval*

**assumes**  $\langle \text{width } X = 0 \rangle$

**shows**  $\langle \exists x. X = \text{mk\_interval}(x, x) \rangle$

**using** *assms* *width\_def*[of *X*]

**unfolding** *mk\_interval'*

**by** (*auto*, *metis* *Interval\_id*)

**lemma** *width\_zero\_interval\_of*:

**fixes** *X* :: *real interval*

**assumes**  $\langle \text{width } X = 0 \rangle$

**shows**  $\langle \exists x. X = \text{interval\_of } x \rangle$

**using** *assms* *width\_def*[of *X*]

**by** (*metis* *eq\_iff\_diff\_eq\_0* *interval\_eqI* *lower\_interval\_of* *upper\_interval\_of*)

**lemma** *width\_zero\_interval\_extension*:

**fixes** *F* :: *real interval*  $\Rightarrow$  *real interval*

**assumes**  $\langle F \text{ is\_interval\_extension\_of } f \rangle$

**and**  $\langle \text{width } X = 0 \rangle$

**shows**  $\langle \text{width } (F X) = 0 \rangle$

**using** *assms* *width\_zero\_interval\_of*[of *X*, *simplified assms*, *simplified*]

**unfolding** *is\_interval\_extension\_of\_def*

**by** (*metis* *add\_0* *add\_diff\_cancel* *lower\_interval\_of* *upper\_interval\_of* *width\_def*)

### 9.3 Lipschitz Interval Inclusive

If  $F$  is a natural interval extension of a real valued rational function with  $F X$  defined for  $X \subseteq Y$  where  $X$  and  $Y$  are intervals or  $n$ -dimensional interval vectors then  $F$  is Lipschitz in  $Y$

```

lemma interval_extension_bounded:
  fixes  $F :: \text{real interval} \Rightarrow \text{real interval}$ 
  assumes  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
  and  $\langle (\text{width } (F X)) / (\text{width } X) \leq L \rangle$ 
shows  $\text{width } (F X) \leq L * \text{width } X$ 
proof(cases  $\text{width } X = 0$ )
  case True
  then show ?thesis
    using width_zero_interval_extension[of F f X, simplified True assms, simplified]
    by auto
next
  case False
  then show ?thesis
    using assms interval_width_positive[of X]
    by (metis mult.commute mult_right_mono nonzero_mult_div_cancel_left times_divide_eq_right)
qed

```

```

lemma lipschitz_on_implies_lipschitzl_on:
  fixes  $F :: \text{real interval} \Rightarrow \text{real interval}$ 
  assumes  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
  and  $\langle C\text{-lipschitz\_on } X f \rangle$ 
  and  $\langle \bigcup (\text{set\_of } 'Y) \subseteq X \rangle$ 
  and  $\langle 0 \leq L \rangle$ 
  and  $\langle \forall y \in Y. (\text{width } (F y)) / (\text{width } y) \leq L \rangle$ 
shows  $L\text{-lipschitzl\_on } Y F$ 
unfolding lipschitzl_on_def
using assms interval_extension_bounded
by(auto)

```

```

lemma lipschitz_on_implies_lipschitzl_on2:
  fixes  $f :: \text{real} \Rightarrow \text{real}$ 
  assumes  $\langle S \neq [] \rangle$  and  $\langle 0 \leq C \rangle$ 
  and  $\langle F \text{ is\_interval\_extension\_of } f \rangle$ 
  and  $\langle 0 \leq L \rangle$ 
  and  $\langle \forall y \in (\text{set } S). (\text{width } (F y)) / (\text{width } y) \leq L \rangle$ 
  and  $\langle C\text{-lipschitz\_on } (\text{set\_of } (\text{interval\_list\_union } (S))) f \rangle$ 
shows  $\langle L\text{-lipschitzl\_on } (\text{set } (S)) F \rangle$ 
apply(rule lipschitzl_on1)
subgoal using assms interval_extension_bounded by blast
subgoal using assms by blast
done

```

```

lemma width_img_Max:
  assumes  $\langle \text{finite } S \rangle$ 
  shows  $\langle \forall x \in S. \text{width } (F x) \leq \text{Max } (\text{width } 'F 'S) \rangle$ 
using assms by auto
lemma width_Min:
  assumes  $\langle \text{finite } S \rangle$ 

```

**shows**  $\langle \forall x \in S. \text{Min}(\text{width } 'S) \leq \text{width } x \rangle$   
**using** *assms by auto*

**lemma** *lipschitzl\_on\_le\_interval*:  
**fixes**  $F :: \langle \text{real interval} \Rightarrow \text{real interval} \rangle$   
**assumes** *inc\_isotonic\_F*:  $\langle \text{inclusion\_isotonic } F \rangle$   
**and** *lipschitzl\_F*:  $\langle C\text{-lipschitzl\_on } \{X\} F \rangle$   
**and** *interval\_inc*:  $\langle x \leq X \rangle$   
**shows**  $\langle \text{width } (F x) \leq C * \text{width } X \rangle$   
**using** *assms*  
**unfolding** *inclusion\_isotonic\_def lipschitzl\_on\_def*  
*width\_def less\_eq\_interval\_def*  
**by** *fastforce*

**lemma** *lipschitzl\_on\_le\_lipschitzl\_on*:  
**fixes**  $F :: \langle \text{real interval} \Rightarrow \text{real interval} \rangle$   
**assumes** *inc\_isotonic\_F*:  $\langle \text{inclusion\_isotonic } F \rangle$   
**and** *lipschitzl\_F*:  $\langle C\text{-lipschitzl\_on } \{X\} F \rangle$   
**and** *interval\_inc*:  $\langle x \leq X \rangle$   
**and** *interval\_ext*:  $\langle F \text{ is\_interval\_extension\_of } f \rangle$   
**shows**  $\langle \exists L. L\text{-lipschitzl\_on } \{x\} F \rangle$   
**apply**(*rule exI[of\_ C \* (width X) / (width x)]*)  
**apply**(*subst lipschitzl\_onI, simp\_all add: assms*)  
**subgoal**  
**apply**(*rule conjI*)  
**subgoal**  
**using** *width\_zero\_interval\_extension*[*of F f x, simplified assms, simplified*]  
**by** *simp*  
**subgoal**  
**using** *inc\_isotonic\_F interval\_inc lipschitzl\_F lipschitzl\_on\_le\_interval* **by** *blast*  
**done**  
**using** *lipschitzl\_on\_le\_interval*[*of F C X x, simplified assms, simplified*]  
**subgoal**  
**using** *lipschitzl\_on\_nonneg assms*  
**by** (*metis divide\_nonneg\_nonneg interval\_width\_positive mult\_nonneg\_nonneg*)  
**done**

**lemma** *uniform\_subdivision\_le*:  
**fixes**  $X :: \langle \text{real interval} \rangle$   
**assumes**  $\langle N > 0 \rangle$   
**shows**  $\langle \forall x \in \text{set}(\text{uniform\_subdivision } X N). x \leq X \rangle$   
**using** *last\_upper\_uniform\_subdivision*[*of N X, simplified assms, simplified*]  
*hd\_lower\_uniform\_subdivision*[*of N X, simplified assms, simplified*]  
*non\_overlapping\_implies\_sorted\_wrt\_upper*[*of (uniform\_subdivision X N), simplified uni-*  
*form\_subdivisions\_non\_overlapping assms, simplified*]  
*non\_overlapping\_implies\_sorted\_wrt\_lower*[*of (uniform\_subdivision X N), simplified uni-*  
*form\_subdivisions\_non\_overlapping assms, simplified*]  
**unfolding** *sorted\_wrt\_upper\_def sorted\_wrt\_lower\_def cmp\_lower\_width\_def less\_eq\_interval\_def*  
**by** (*metis (no\_types, lifting) assms in\_set\_conv\_nth interval\_list\_union\_correct non\_empty\_subdivision*  
*set\_of\_subset\_iff union\_set*)

**lemma** *lipschitzl\_on\_uniform\_subdivision*:  
**fixes**  $F :: \langle \text{real interval} \Rightarrow \text{real interval} \rangle$   
**assumes** *inc\_isotonic\_F*:  $\langle \text{inclusion\_isotonic } F \rangle$

```

and lipschitzl_F: <C—lipschitzl_on ({X}) F>
and <N>0>
shows < $\forall x \in (\text{set } (\text{uniform\_subdivision } X N)). \text{width } (F x) \leq C * \text{width } X$ >
using lipschitzl_on_le_interval[of F C X, simplified assms, simplified]
uniform_subdivision_le[of N X, simplified assms, simplified ]
by simp

```

```

lemma division_leq_pos:
fixes x :: 'a::{linordered_field}
assumes x > 0 and y > 0 and z > 0 and y ≤ z
shows x / z ≤ x / y
proof —
have x * y ≤ x * z using assms by simp
hence (x * y) / (y * z) ≤ (x * z) / (y * z)
using assms
by (simp add: frac_le)
thus ?thesis using assms by auto[1]
qed

```

```

lemma each_subdivision_width_order':
fixes X :: real interval
assumes F is_interval_extension_of f
and 0 < N
and Xs ∈ set (uniform_subdivision X N)
shows  $\exists L. \text{width}(F Xs) \leq L * \text{width}(X) / \text{of\_nat } N$ 
by (metis assms less_numeral_extra(3) mult_eq_o_iff nonzero_divide_eq_eq
of_nat_le_o_iff order_refl uniform_subdivisions_width width_zero_interval_extension)

```

```

lemma uniform_subdivision_min_nonzero:
assumes <N > 0>
and <width X > 0>
shows <0 < Min (width 'set (uniform_subdivision X N))>
using assms unfolding image_set uniform_subdivision_def Let_def width_def
by (simp, simp add: order_le_less)

```

```

lemma uniform_subdivision_width_zero_replicate_eq:
fixes X::real interval
assumes positive_N: <0 < N>
and zero_width_X: <0 = width X >
shows <replicate N X = (uniform_subdivision X N)>
using assms
proof(induction N)
case 0
then show ?case
by simp
next
case (Suc N)
then show ?case
using width_zero_lower_upper[of X, simplified assms, simplified]
unfolding uniform_subdivision_def
by (simp, metis append.right_neutral map_replicate_trivial mk_interval_id replicate_app_Cons_same)
qed

```

```

lemma set_of_interval_list_zero_width:

```

```

fixes X::⟨real interval⟩
assumes positive_N: ⟨0 < N⟩
and zero_width_X: ⟨0 = width X⟩
shows⟨set_of_interval_list (uniform_subdivision X N) = {lower X..upper X}⟩
proof(insert assms, simp add: uniform_subdivision_width_zero_replicate_eq[of N X, simplified assms, simplified, symmetric], induction N)
case 0
then show ?case
  by(simp)
next
case (Suc N)
then show ?case
  unfolding set_of_interval_list_def set_of_eq
  by(simp, fastforce)
qed

```

```

lemma width_zero_subdivision: width X = (o::real) ==> N > o ==> set (uniform_subdivision X N) = {X}
using width_zero_lower_upper[of X]
unfolding uniform_subdivision_def Let_def mk_interval'
apply(auto simp add: image_def)[1]
apply (metis Interval_id)
apply (metis Interval_id)
done

```

```

lemma lipschitz_on_implies_lipschitzl_on_pre:
fixes f :: ⟨real ⇒ real⟩
and F :: ⟨real interval ⇒ real interval⟩
assumes ⟨finite S⟩
and ⟨0 < Min (width 'S)⟩
shows ⟨let max_F_width = Max (width ' (F ' S));
      min_f_width = Min (width ' S)
      in ∀ x ∈ S. width (F x) ≤ (max_F_width/min_f_width) * width x⟩
unfolding Let_def
by (simp, smt (verit) assms division_leq_pos interval_width_positive mult_eq_o_iff
      nonzero_mult_div_cancel_right width_Min width_img_Max zero_le_divide_iff)

```

```

lemma lipschitz_on_implies_lipschitzl_on':
fixes f :: ⟨real ⇒ real⟩
and F :: ⟨real interval ⇒ real interval⟩
assumes non_empty: ⟨S ≠ {}⟩
and finite: ⟨finite S⟩
and non_zero_width: ⟨0 < Min (width 'S)⟩
and interval_ext_F: ⟨F is_interval_extension_of f⟩
shows ⟨∃ L. L—lipschitzl_on S F⟩
unfolding lipschitzl_on_def
apply(rule exI[of_ (Max (width ' (F ' S)))/(Min (width ' S))])
apply(rule conjI)
subgoal
apply(rule divide_nonneg_nonneg)
subgoal
  using assms interval_width_positive
  by (metis (mono_tags, lifting) Max_in finite_imageI imageE image_is_empty)
subgoal
  using assms interval_width_positive

```

```

    by(auto)[1]
  done
subgoal
  by (meson assms lipschitz_on_implies_lipschitzl_on_pre)
done

```

**lemma** *natural\_extension\_transfer\_lipschitz*:

```

  assumes positive_N: <0 < N>
  and inc_isotonic_F: <inclusion_isotonic F>
  and interval_ext_F: <F is_natural_interval_extension_of f>
  and lipschitz_f: <C-lipschitz_on (set_of X) f>
shows <C-lipschitzl_on (set (uniform_subdivision X N)) F>
proof(cases o < width X)
  case True
  then show ?thesis
  apply(subst lipschitzl_onl)
  subgoal
    using assms
    each_subdivision_width_order'[of F f N X ]
    each_subdivision_width_order[of F C set (uniform_subdivision X N) f X N ]
    uniform_subdivision_le[of N X ]
    unfolding lipschitz_on_def is_natural_interval_extension_of_def
      inclusion_isotonic_def
    dist_real_def abs_real_def width_def mk_interval'
    apply(auto split:if_splits)[1]
    by (smt (z3) Interval_id interval_of.abs_eq interval_of_in_eq less_eq_interval_def lower_bounds lower_in_interval up-
per_bounds)
  subgoal
    using assms lipschitz_on_nonneg by auto
  subgoal by simp
  done
next
  case False
  have width_zero: <width X = 0>
  using False
  by (meson interval_width_positive linorder_not_le nle_le)
  have us_X: <set (uniform_subdivision X N) = {X}>
  using width_zero width_zero_subdivision
  using positive_N by blast
  then show ?thesis
  using width_zero
  apply(simp add:assms)
  apply(subst lipschitzl_onl)
  subgoal using assms
    each_subdivision_width_order'[of F f N X ]
    each_subdivision_width_order[of F C set (uniform_subdivision X N) f X N ]
    uniform_subdivision_le[of N X ]
    unfolding lipschitz_on_def is_natural_interval_extension_of_def
      inclusion_isotonic_def
    dist_real_def abs_real_def width_def mk_interval'
    apply(auto split:if_splits)[1]
    by (meson interval_ext_F natural_interval_extension_implies_interval_extension)
  subgoal

```

```

    using assms lipschitz_on_nonneg by auto
  subgoal by simp
done
qed

```

```

lemma lipschitz_on_division_lipschitz_on:
  assumes lipschitz_f: C-lipschitz_on (set_of X) f
    and non_empty: uniform_subdivision X N ≠ []
    and subdivision: Xs ∈ set(uniform_subdivision (X::real interval) N)
  shows ∃ L . L-lipschitz_on (set_of Xs) f

```

```

proof -
  fix L
  have subset: Xs ≤ X
  using assms(3) uniform_subdivision_le[of N X]
  by (metis (no_types, lifting) bot_nat_o_extremum_strict
    bot_nat_o_not_eq_extremum in_set_conv_nth length_map list.size(3) map_nth
    uniform_subdivision_def)
  show ?thesis
  by (meson assms(1) interval_set_leq_eq lipschitz_on_subset subset)
qed

```

```

lemma lipschitz_on_lipschitz_on_subdivisions:
  assumes lipschitz_f: C-lipschitz_on (set_of X) f
    and non_empty: uniform_subdivision X N ≠ []
    and non_zero: 0 < N
  shows ∃ L . ∀ Xs ∈ set(uniform_subdivision (X::real interval) N). L-lipschitz_on (set_of Xs) f

```

```

proof -
  have subset: ∀ Xs ∈ set(uniform_subdivision (X::real interval) N) . Xs ≤ X
  using assms uniform_subdivision_le[of N X] by blast
  show ∃ L . ∀ Xs ∈ set (uniform_subdivision X N). L-lipschitz_on (set_of Xs) f
  proof -
    obtain L where L: L = C using lipschitz_f by auto
    have L-lipschitz_on (set_of Xs) f if Xs ∈ set (uniform_subdivision X N) for Xs
    proof -
      show ?thesis using lipschitz_on_division_lipschitz_on[of C X f N Xs, simplified assms, simplified] L
      by (meson interval_set_leq_eq lipschitz_f lipschitz_on_subset subset that)
    qed
  thus ?thesis by auto
qed
qed

```

```

lemma lipschitz_on_lipschitz_on_subdivisions_n:
  assumes lipschitz_f: C-lipschitz_on (set_of X) f
    and non_empty: uniform_subdivision X N ≠ []
    and non_zero: 0 < N
  shows ∃ L . ∀ N > 0 . ∀ Xs ∈ set(uniform_subdivision (X::real interval) N). L-lipschitz_on (set_of Xs) f

```

```

proof -
  have subset: ∀ Xs ∈ set(uniform_subdivision (X::real interval) N) . Xs ≤ X
  using assms uniform_subdivision_le[of N X] by blast
  show ∃ L . ∀ N > 0 . ∀ Xs ∈ set (uniform_subdivision X N). L-lipschitz_on (set_of Xs) f
  proof -
    obtain L where L: L = C using lipschitz_f by auto
    have L-lipschitz_on (set_of Xs) f if Xs ∈ set (uniform_subdivision X N) for Xs

```

```

proof –
  show ?thesis using lipschitz_on_division_lipschitz_on[of C X f N Xs, simplified assms, simplified] L
  by (meson interval_set_leq_eq lipschitz_f lipschitz_on_subset subset that)
qed
thus ?thesis
by (meson interval_set_leq_eq lipschitz_f lipschitz_on_subset uniform_subdivision_le)
qed
qed

```

```

lemma lipschitzl_on_division_lipschitzl_on:
  assumes lipschitz_f: C – lipschitzl_on (set(uniform_subdivision X N)) F
  and non_empty: uniform_subdivision X N ≠ []
  and subdivision: Xs ∈ set(uniform_subdivision (X::real interval) N)
  shows ∃ L . L – lipschitzl_on {Xs} F

```

```

proof –
  fix L
  have subset: Xs ≤ X
  using assms(3) uniform_subdivision_le[of N X]
  by (smt (verit, del_insts) gr_zero1 list.map_disc_iff list.size(3) map_nth non_empty uniform_subdivision_def)
  show ?thesis
  by (meson empty_subset1 insert_subset lipschitzl_on_le lipschitz_f subdivision)
qed

```

```

lemma lipschitzl_on_lipschitzl_on_subdivisions:

```

```

  fixes X :: real interval
  assumes lipschitz_f: C – lipschitzl_on (set(uniform_subdivision X N)) F
  and non_zero: 0 < N
  shows ∃ L . ∀ Xs ∈ set(uniform_subdivision X N). L – lipschitzl_on {Xs} F

```

```

proof –
  have subset: ∀ Xs ∈ set(uniform_subdivision (X::real interval) N) . Xs ≤ X
  using assms uniform_subdivision_le[of N X] by blast
  show ∃ L . ∀ Xs ∈ set (uniform_subdivision X N). L – lipschitzl_on {Xs} F
  proof –
    obtain L where L: L = C using lipschitz_f by auto
    have L – lipschitzl_on {Xs} F if Xs ∈ set (uniform_subdivision X N) for Xs
    proof –
      show ?thesis using assms
      by (simp add: L lipschitzl_on_def that)
    qed
  thus ?thesis by auto
qed
qed

```

## 9.4 Lipschitz Convergence

```

lemma isotonic_lipschitz_refinement':

```

```

  assumes positive_N: <0 < N>
  and inc_isotonic_F: <inclusion_isotonic F>
  and interval_ext_F: <F is_interval_extension_of f>
  and lipschitz_f: <C – lipschitz_on (set_of X) f>
  shows <∃ L . width_set (set_of_interval_list (map F (uniform_subdivision X N))) – width_set (f' (set_of X)) ≤ 2 * (L * width X / real N)>
  proof (cases 0 < width X)

```

```

case True
have us_eq_set_of_X: ⟨(set_of_interval_list (uniform_subdivision X N)) = set_of X⟩
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
    contiguous_uniform_subdivision non_empty_subdivision positive_N union_set)
have lipschitz_f': ⟨C—lipschitz_on (set_of_interval_list (uniform_subdivision X N)) f⟩
  by (simp add: lipschitz_f us_eq_set_of_X)
have us_nonempty: ⟨uniform_subdivision X N ≠ []⟩
  by (simp add: assms(1) non_empty_subdivision)
have us_nonempty_set: ⟨set (uniform_subdivision X N) ≠ {}⟩
  by (simp add: us_nonempty)
have us_finite: ⟨finite (set (uniform_subdivision X N))⟩
  by simp
have sorted_lower: ⟨sorted_wrt_lower (uniform_subdivision X N)⟩
  by (simp add: contiguous_sorted_wrt_lower contiguous_uniform_subdivision)
have sorted_upper: ⟨sorted_wrt_upper (uniform_subdivision X N)⟩
  by (simp add: contiguous_sorted_wrt_upper contiguous_uniform_subdivision)
have lipschitzl: ⟨∃ L. L—lipschitzl_on (set (uniform_subdivision X N)) F⟩
  using lipschitz_on_implies_lipschitzl_on'[of set (uniform_subdivision X N) F f, simplified
    us_nonempty_set us_finite interval_ext_F]
    uniform_subdivision_min_nonzero[of N X, simplified positive_N True, simplified]
  by(simp)
  have set_of_interval_eq: ⟨(set_of_interval_list (uniform_subdivision X N)) = (set_of (interval_list_union
    (uniform_subdivision X N)))⟩
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
    contiguous_uniform_subdivision us_nonempty)
have width_bounded: ⟨∃ L. Max (excess_width F f' set (uniform_subdivision X N)) ≤ 2 * (L * width X / real N)⟩
  using
    each_subdivision_excess_width_order[of F _ (set (uniform_subdivision X N)) f X N C,
      simplified inc_isotonic_F interval_ext_F positive_N lipschitz_f'[simplified set_of_interval_eq], simplified]
    max_subdivision_excess_width_order[of F _ (set (uniform_subdivision X N)) f X N C,
      simplified inc_isotonic_F interval_ext_F positive_N lipschitz_f', simplified]
proof —
  assume a1: ∧ C. C—lipschitzl_on (set (uniform_subdivision X N)) F ⇒
    Max (excess_width F f' set (uniform_subdivision X N)) ≤ C * width X / real N
  have ∀ r ra rb. (r::real) / rb * ra = r * ra / rb
  by fastforce
  then show ?thesis
  using a1 by (metis (∃ L. L—lipschitzl_on (set (uniform_subdivision X N)) F)
    divide_numeral_1 le_divide_eq_numerals(1) mult_numeral_1 order_antisym_conv order_refl)
qed
have width_limit:⟨∃ L. width_set (set_of_interval_list (map F (uniform_subdivision X N)))
  — width_set (f' (set_of X))
  ≤ 2 * (L * width X / real N)⟩
  using width_bounded
    map_inclusion_isotonic_excess_width_bounded(1)[of uniform_subdivision X N F f C,
      simplified us_nonempty interval_ext_F inc_isotonic_F sorted_lower sorted_upper lipschitz_f', simplified]
  by (metis (no_types, lifting) us_eq_set_of_X inc_isotonic_F interval_ext_F lipschitzl lipschitz_f' list.set_map
    max_subdivision_excess_width_order order.refl positive_N)
then show ?thesis
  by simp
next
case False
have width_zero: ⟨width X = 0⟩
  using False

```

```

  by (meson interval_width_positive linorder_not_le nle_le)
  have us_nonempty: ⟨uniform_subdivision X N ≠ []⟩
  by (simp add: assms(1) non_empty_subdivision)
  have set_of_interval_eq: ⟨(set_of (interval_list_union (uniform_subdivision X N))) = (set_of_interval_list
(uniform_subdivision X N))⟩
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
contiguous_uniform_subdivision us_nonempty)
  have w_zero_1: ⟨width_set (set_of_interval_list (map F (uniform_subdivision X N))) = 0⟩
  by (metis (full_types) Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
contiguous_uniform_subdivision interval_ext_F map_replicate non_empty_subdivision positive_N
uniform_subdivision_width_zero_replicate_eq union_set width_eq_wdith_set width_zero
width_zero_interval_extension)
  have w_zero_2: ⟨width_set (f' (set_of X)) = 0⟩
  using set_of_interval_list_zero_width[of N X, simplified positive_N width_zero, simplified]
width_zero_width_zero_lower_upper[of X, simplified width_zero, simplified]
  by (metis diff_ge_o_iff_ge inc_isotonic_F inf_le_sup_image_real interval_ext_F lipschitz_f_nle_le
width_inclusion_isotonic_approx width_set_def width_zero_interval_extension)
  then show ?thesis
  using width_zero w_zero_1 w_zero_2
  by simp
qed

```

**lemma isotonic\_lipschitz\_refinement1:**

```

  assumes positive_N: ⟨0 < N⟩
  and inc_isotonic_F: ⟨inclusion_isotonic F⟩
  and interval_ext_F: ⟨F is_interval_extension_of f⟩
  and lipschitz_f: ⟨L—lipschitz_on (set_of X) f⟩
  and lipschitz_F: ⟨C—lipschitzl_on (set (uniform_subdivision X N)) F⟩
  shows ⟨width_set (set_of_interval_list (map F (uniform_subdivision X N))) — width_set (f' (set_of X)) ≤ 2 * (C * width
X / real N)⟩
  proof (cases 0 < width X)
  case True
  have us_eq_set_of_X: ⟨(set_of_interval_list (uniform_subdivision X N)) = set_of X⟩
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
contiguous_uniform_subdivision non_empty_subdivision positive_N union_set)
  have lipschitz_f': ⟨L—lipschitz_on (set_of_interval_list (uniform_subdivision X N)) f⟩
  by (simp add: lipschitz_f us_eq_set_of_X)
  have us_nonempty: ⟨uniform_subdivision X N ≠ []⟩
  by (simp add: assms(1) non_empty_subdivision)
  have us_nonempty_set: ⟨set (uniform_subdivision X N) ≠ {}⟩
  by (simp add: us_nonempty)
  have us_finite: ⟨finite (set (uniform_subdivision X N))⟩
  by simp
  have sorted_lower: ⟨sorted_wrt_lower (uniform_subdivision X N)⟩
  by (simp add: contiguous_sorted_wrt_lower contiguous_uniform_subdivision)
  have sorted_upper: ⟨sorted_wrt_upper (uniform_subdivision X N)⟩
  by (simp add: contiguous_sorted_wrt_upper contiguous_uniform_subdivision)
  have lipschitzl: ⟨C—lipschitzl_on (set (uniform_subdivision X N)) F⟩
  using lipschitz_F by blast
  have set_of_interval_eq: ⟨(set_of_interval_list (uniform_subdivision X N)) = (set_of (interval_list_union
(uniform_subdivision X N)))⟩
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
contiguous_uniform_subdivision us_nonempty)

```

```

have width_bounded: ⟨Max (excess_width F f' set (uniform_subdivision X N)) ≤ 2 * (C * width X / real N)⟩
using
  each_subdivision_excess_width_order[of F C (set (uniform_subdivision X N)) f X N _,
    simplified inc_isotonic_F interval_ext_F positive_N lipschitz_f'[simplified set_of_interval_eq], simplified]
  max_subdivision_excess_width_order[of F C (set (uniform_subdivision X N)) f X N _,
    simplified inc_isotonic_F interval_ext_F positive_N lipschitz_f', simplified]
  by (smt (verit) divide_nonneg_nonneg interval_width_positive lipschitzl_on_def lipschitz_F lipschitz_f'
of_nat_o_less_iff positive_N split_mult_pos_le)
have width_limit:⟨width_set (set_of_interval_list (map F (uniform_subdivision X N)))
  – width_set (f' (set_of X))
  ≤ 2 * (C * width X / real N)⟩
using width_bounded
  by (metis assms(3) inc_isotonic_F lipschitz_F lipschitz_f' map_inclusion_isotonic_excess_width_bounded(1)
max_subdivision_excess_width_order order_le_less positive_N sorted_lower sorted_upper us_eq_set_of_X us_nonempty)

then show ?thesis
by simp
next
case False
have width_zero: ⟨width X = 0⟩
using False
by (meson interval_width_positive linorder_not_le nle_le)
have us_nonempty: ⟨uniform_subdivision X N ≠ []⟩
by (simp add: assms(1) non_empty_subdivision)
have set_of_interval_eq: ⟨(set_of (interval_list_union (uniform_subdivision X N))) = (set_of_interval_list
(uniform_subdivision X N))⟩
by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
contiguous_uniform_subdivision us_nonempty)
have w_zero_1: ⟨width_set (set_of_interval_list (map F (uniform_subdivision X N))) = 0⟩
by (metis (full_types) Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
contiguous_uniform_subdivision interval_ext_F map_replicate non_empty_subdivision positive_N
uniform_subdivision_width_zero_replicate_eq union_set width_eq_wdith_set width_zero
width_zero_interval_extension)
have w_zero_2: ⟨width_set (f' (set_of X)) = 0⟩
using set_of_interval_list_zero_width[of N X, simplified positive_N width_zero, simplified]
width_zero_width_zero_lower_upper[of X, simplified width_zero, simplified]
by (simp add: set_of_eq width_set_def)
then show ?thesis
using width_zero w_zero_1 w_zero_2
by simp
qed

lemma isotonic_lipschitz_refinement:
assumes positive_N: ⟨0 < N⟩
and inc_isotonic_F: ⟨inclusion_isotonic F⟩
and interval_ext_F: ⟨F is_interval_extension_of f⟩
and lipschitz_f: ⟨L–lipschitz_on (set_of X) f⟩
and lipschitz_F: ⟨C–lipschitzl_on (set (uniform_subdivision X N)) F⟩
shows ⟨excess_width_set (refinement_set F N) f X ≤ 2 * (C * width X / real N)⟩
using isotonic_lipschitz_refinementl[of N F f L X C, simplified assms, simplified]
unfolding excess_width_set_def refinement_set_def by simp

end

```

## 10 Interval Analysis (Interval\_Analysis)

**theory**

*Interval\_Analysis*

**imports**

*Interval\_Division\_Real*

*Lipschitz\_Subdivisions\_Refinements*

**begin**

This theory provides interval analysis over standard types such as real or integer. All operations work over (closed) intervals.

**end**



# 11 Extended Division on Intervals

## ( Extended\_Interval\_Division)

**theory**

*Extended\_Interval\_Division*

**imports**

*Interval\_Division\_Non\_Zero*

**begin**

In this theory, we define an extended division operation on intervals. This definition is inspired by the definition given in [4], but we use an over-approximation for the case in which zero is an element of the divisor interval. By this, we avoid the need for multi-intervals.

**instantiation** *interval* :: (*{infinity, linordered\_field, real\_normed\_algebra, linear\_continuum\_topology}*) *inverse*

**begin**

**definition** *inverse\_interval* :: '*a interval*  $\Rightarrow$  '*a interval*

**where** *inverse\_interval* *a* = (  
     *if* ( $\neg o \in_i a$ ) *then* *mk\_interval* ( $1 / (\text{upper } a), 1 / (\text{lower } a)$ )  
     *else if* *lower* *a* = *o*  $\wedge$  *o* < *upper* *a* *then* *mk\_interval* ( $1 / \text{upper } a, \infty$ )  
     *else if* *lower* *a* < *o*  $\wedge$  *o* < *upper* *a* *then* *mk\_interval* ( $-\infty, \infty$ )  
     *else if* *lower* *a* < *upper* *a*  $\wedge$  *upper* *a* = *o* *then* *mk\_interval* ( $-\infty, 1 / \text{lower } a$ )  
     *else* *undefined*  
 )

**definition** *divide\_interval* :: '*a interval*  $\Rightarrow$  '*a interval*  $\Rightarrow$  '*a interval*

**where** *divide\_interval* *a b* = *inverse* *b* \* *a*

**instance** ..

**end**

**interpretation** *interval\_division\_inverse* *divide* *inverse*

**apply**(*unfold\_locales*)

**subgoal**

**by** (*simp add: inverse\_interval\_def*)

**subgoal**

**by**(*simp add: divide\_interval\_def*)

**done**

**end**



## 12 Extended Interval Analysis

### ( Extended\_Interval\_Analysis)

**theory**

*Extended\_Interval\_Analysis*

**imports**

*Extended\_Interval\_Division*

*Lipschitz\_Subdivisions\_Refinements*

**begin**

This theory provides extended interval analysis over the type extended reals. All operations work over (closed) intervals.

**end**

### 12.1 Overlapping Multi-Intervals ( Multi\_Interval\_Overlapping)

**theory**

*Multi\_Interval\_Overlapping*

**imports**

*Multi\_Interval\_Preliminaries*

**begin**

#### 12.1.1 Type Definition

```
typedef (overloaded) 'a minterval_ovl =
  {is::'a::{minus_mono} interval list. valid_mInterval_ovl is}
morphisms bounds_of_minterval_ovl mInterval_ovl
unfolding valid_mInterval_ovl_def
apply(intro exI[where x=[Interval (l,l) ] for l])
by(auto simp add: sorted_wrt_lower_def non_overlapping_sorted_def)
```

**setup\_lifting** type\_definition\_minterval\_ovl

```
lift_definition mlower_ovl::('a::{minus_mono}) minterval_ovl  $\Rightarrow$  'a is <lower o hd>.
lift_definition mupper_ovl::('a::{minus_mono}) minterval_ovl  $\Rightarrow$  'a is <upper o last>.
lift_definition mlist_ovl::('a::{minus_mono}) minterval_ovl  $\Rightarrow$  'a interval list is <id>.
```

#### 12.1.2 Equality and Orderings

```
lemma minterval_ovl_eq_iff:  $a = b \iff$  mlist_ovl a = mlist_ovl b
by transfer auto
```

```
lemma ainterval_eqI: mlist_ovl a = mlist_ovl b  $\implies$  a = b
by (auto simp: minterval_ovl_eq_iff)
```

```
lemma minterval_ovl_imp_upper_lower_eq :
  a = b  $\implies$  mlower_ovl a = mlower_ovl b  $\wedge$  mupper_ovl a = mupper_ovl b
```

**by** *transfer auto*

**lemma** *valid\_mInterval\_ovl\_lower\_le\_upper*:

*valid\_mInterval\_ovl i*  $\implies$  (*lower*  $\circ$  *hd*) *i*  $\leq$  (*upper*  $\circ$  *last*) *i*

**proof**(*induction i*)

**case** *Nil*

**then show** ?*case*

**unfolding** *valid\_mInterval\_ovl\_def o\_def sorted\_wrt\_lower\_def cmp\_lower\_width\_def*

**by** *simp*

**next**

**case** (*Cons a i*)

**then show** ?*case*

**unfolding** *valid\_mInterval\_ovl\_def o\_def sorted\_wrt\_lower\_def cmp\_lower\_width\_def*

**by** (*metis* (*no\_types*, *lifting*) *Cons.IH Cons.premis comp\_apply distinct.simps(2)*)

*in\_bounds last.simps list.sel(1) lower\_in\_interval order\_less\_imp\_le order\_less\_le\_trans sorted\_wrt\_lower\_unroll valid\_mInterval\_ovl\_def*)

**qed**

**lemma** *mlower\_non\_ovl\_le\_mupper\_non\_ovl*[*simp*]: *mlower\_ovl i*  $\leq$  *mupper\_ovl i*

**by**(*transfer*, *metis valid\_mInterval\_ovl\_lower\_le\_upper*)

**lift\_definition** *set\_of\_ovl* :: '*a*::{*minus\_mono*} *minterval\_ovl*  $\implies$  '*a* *set*

*is*  $\lambda$  *is*.  $\bigcup_{x \in \text{set } is} \{ \text{lower } x.. \text{upper } x \}$ .

**lemma** *not\_in\_ovl\_eq*:

$\langle \neg e \in \text{set\_of\_ovl } xs \rangle = (\forall x \in \text{set } (m\text{list\_ovl } xs). \neg e \in \text{set\_of } x)$

**proof**(*induction* (*mlist\_ovl xs*))

**case** *Nil*

**then show** ?*case*

**by** (*metis UN\_iff empty\_iff empty\_set mlist\_ovl.rep\_eq set\_of\_ovl.rep\_eq*)

**next**

**case** (*Cons a x*)

**then show** ?*case*

**by** (*simp add: mlist\_ovl.rep\_eq set\_of\_eq set\_of\_ovl.rep\_eq*)

**qed**

**lemma** *in\_ovl\_eq*:

$\langle e \in \text{set\_of\_ovl } xs \rangle = (\exists x \in \text{set } (m\text{list\_ovl } xs). e \in \text{set\_of } x)$

**proof**(*induction* (*mlist\_ovl xs*))

**case** *Nil*

**then show** ?*case*

**by** (*metis UN\_iff empty\_iff empty\_set mlist\_ovl.rep\_eq set\_of\_ovl.rep\_eq*)

**next**

**case** (*Cons a x*)

**then show** ?*case*

**by** (*simp add: mlist\_ovl.rep\_eq set\_of\_eq set\_of\_ovl.rep\_eq*)

**qed**

**context notes** [[*typedef\_overloaded*]] **begin**

**lift\_definition**(*code\_dt*) *mInterval\_ovl'* :: '*a*::*minus\_mono* *interval list*  $\implies$  '*a* *minterval\_ovl option*

*is*  $\lambda$  *is*. *if valid\_mInterval\_ovl is then Some is else None*

**by** (*auto simp add: valid\_mInterval\_ovl\_def*)

**lemma** *mInterval\_ovl'\_split*:  
 $P (mInterval\_ovl' is) \longleftrightarrow$   
 $(\forall ivl. valid\_mInterval\_ovl is \longrightarrow mlist\_ovl ivl = is \longrightarrow P (Some ivl)) \wedge (\neg valid\_mInterval\_ovl is \longrightarrow P None)$   
**by** *transfer auto*

**lemma** *mInterval\_ovl'\_split\_asm*:  
 $P (mInterval\_ovl' is) \longleftrightarrow$   
 $\neg((\exists ivl. valid\_mInterval\_ovl is \wedge mlist\_ovl ivl = is \wedge \neg P (Some ivl)) \vee (\neg valid\_mInterval\_ovl is \wedge \neg P None))$   
**unfolding** *mInterval\_ovl'\_split*  
**by** *auto*

**lemmas** *mInterval\_ovl'\_splits = mInterval\_ovl'\_split mInterval\_ovl'\_split\_asm*

**lemma** *mInterval'\_eq\_Some*:  $mInterval\_ovl' is = Some i \implies mlist\_ovl i = is$   
**by** (*simp split: mInterval\_ovl'\_splits*)

**end**

**lemma** *set\_of\_ovl\_non\_zero\_list\_all*:  
 $\langle 0 \notin set\_of\_ovl xs \implies \forall x \in set (mlist\_ovl xs). \neg 0 \in_i x \rangle$   
**proof**(*induction mlist\_ovl xs*)  
**case** *Nil*  
**then show** ?*case*  
**by** *simp*  
**next**  
**case** (*Cons a x*)  
**then show** ?*case*  
**using** *in\_ovl\_eq* **by** *blast*  
**qed**

**instantiation** *minterval\_ovl* :: (*{minus\_mono}*) *equal*  
**begin**

**definition** *equal\_class.equal a b*  $\equiv (mlist\_ovl a = mlist\_ovl b)$

**instance proof** **qed** (*simp add: equal\_minterval\_ovl\_def minterval\_ovl\_eq\_iff*)  
**end**

**instantiation** *minterval\_ovl* :: (*{minus\_mono}*) *ord* **begin**

**definition** *less\_eq\_minterval\_ovl* :: '*a* *minterval\_ovl*  $\Rightarrow$  '*a* *minterval\_ovl*  $\Rightarrow$  *bool*  
**where** *less\_eq\_minterval\_ovl a b*  $\longleftrightarrow mlower\_ovl b \leq mlower\_ovl a \wedge mupper\_ovl a \leq mupper\_ovl b$

**definition** *less\_minterval\_ovl* :: '*a* *minterval\_ovl*  $\Rightarrow$  '*a* *minterval\_ovl*  $\Rightarrow$  *bool*  
**where** *less\_minterval\_ovl x y*  $= (x \leq y \wedge \neg y \leq x)$

**instance proof** **qed**  
**end**

**instantiation** *minterval\_ovl* :: (*{minus\_mono, lattice}*) *sup*  
**begin**

**lift\_definition** *sup\_minterval\_non\_ovl* :: '*a* *minterval\_ovl*  $\Rightarrow$  '*a* *minterval\_ovl*  $\Rightarrow$  '*a* *minterval\_ovl*  
**is**  $\lambda a b. [Interval (inf (lower (hd a)) (lower (hd b))), sup (upper (last a)) (upper (last b))]$

```

by(auto simp: valid_mInterval_ovl_def sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def
    non_overlapping_sorted_def le_inf1 le_sup1 valid_mInterval_non_ovl_def mupper_ovl_def
    mlower_ovl_def)

```

```

instance
  by(standard)
end

```

```

instantiation minterval_ovl :: ({lattice, minus_mono}) preorder

```

```

begin
instance
  apply(standard)
  subgoal
    using less_minterval_ovl_def by auto
  subgoal
    by (simp add: less_eq_minterval_ovl_def)
  subgoal
    by (meson less_eq_minterval_ovl_def order.trans)
  done
end

```

```

lift_definition minterval_ovl_of :: 'a::{minus_mono} ⇒ 'a minterval_ovl is λx. [Interval(x, x)]
unfolding valid_mInterval_ovl_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def
  cmp_non_adjacent_def sorted_wrt_lower_def
by simp

```

```

lemma mlower_ovl_minterval_ovl_of[simp]: mlower_ovl (minterval_ovl_of a) = a
by transfer auto

```

```

lemma mupper_ovl_minterval_ovl_of[simp]: mupper_ovl (minterval_ovl_of a) = a
by transfer auto

```

```

definition width_ovl :: 'a::{minus_mono} minterval_ovl ⇒ 'a
where width_ovl i = mupper_ovl i - mlower_ovl i

```

### 12.1.3 Zero and One

```

instantiation minterval_ovl :: ({minus_mono, zero}) zero
begin

```

```

lift_definition zero_minterval_ovl:: 'a minterval_ovl is mk_mInterval_ovl [Interval (o, o)]
by (simp add: mk_mInterval_ovl_valid)

```

```

lemma mlower_ovl_zero[simp]: mlower_ovl o = o
by (transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

```

```

lemma mupper_ovl_zero[simp]: mupper_ovl o = o
by (transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

```

```

instance proof qed
end

```

```

instantiation minterval_ovl :: ({minus_mono, one}) one
begin

```

```
lift_definition one_minterval_ovl::'a minterval_ovl is mk_mInterval_ovl [Interval (1, 1)]
  by (simp add: mk_mInterval_ovl_valid)
```

```
lemma mlower_ovl_one[simp]: mlower_ovl 1 = 1
  by (transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)
```

```
lemma mupper_ovl_one[simp]: mupper_ovl 1 = 1
  by (transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)
```

```
instance proof qed
end
```

#### 12.1.4 Addition

```
instantiation minterval_ovl :: ({minus_mono,ordered_ab_semigroup_add,linordered_field}) plus
begin
```

```
lift_definition plus_minterval_ovl::'a minterval_ovl  $\Rightarrow$  'a minterval_ovl  $\Rightarrow$  'a minterval_ovl
  is  $\lambda$  a b . mk_mInterval_ovl (iList_plus a b)
  by (metis bin_op_interval_list_non_empty iList_plus_def mk_mInterval_ovl_valid valid_mInterval_ovl_def)
```

```
lemma valid_mk_interval_iList_plus:
  assumes valid_mInterval_ovl a and valid_mInterval_ovl b
  shows valid_mInterval_ovl (mk_mInterval_ovl (iList_plus a b))
  by (metis (no_types, lifting) assms(1) assms(2) bin_op_interval_list_empty iList_plus_lower_upper_eq
      mk_mInterval_ovl_id mk_mInterval_ovl_valid)
```

```
lift_definition plus_minterval_non_ovl::'a minterval_ovl  $\Rightarrow$  'a minterval_ovl  $\Rightarrow$  'a minterval_ovl
  is  $\lambda$  a b . mk_mInterval_ovl (iList_plus a b)
  by (simp add: valid_mk_interval_iList_plus)
```

```
lemma interval_plus_com:
 $\langle a + b = b + a \rangle$  for  $a::'a::\{minus\_mono,ordered\_ab\_semigroup\_add,linordered\_field\}$  minterval_ovl
  apply (transfer)
  using plus_minterval_ovl_def
  by (metis (no_types, opaque_lifting) foldr_isort_elements iList_plus_commute
      interval_sort_lower_width_def mk_mInterval_ovl_def mk_mInterval_ovl_distinct
      mk_mInterval_ovl_sorted_o_def set_remdups sorted_wrt_lower_distinct_lists_eq)
```

```
instance proof qed
end
```

#### 12.1.5 Unary Minus

```
lemma a: ( $x::'a::ordered\_ab\_group\_add$  interval)  $\neq$  y  $\implies$   $-x \neq -y$ 
  apply (simp add: uminus_interval_def)
  by (smt (z3) Pair_inject bounds_of_interval_inverse case_prod_Pair_iden case_prod_unfold neg_equal_iff_equal uminus_interval.rep_eq)
```

```
lemma b: distinct (is::'a::ordered_ab_group_add interval list)  $\implies$  distinct (map ( $\lambda$  i.  $-i$ ) is)
proof (induction is)
  case Nil
  then show ?case by simp
next
```

```

case (Cons a is)
then show ?case using a by force
qed

```

```

instantiation minterval_ovl :: ({minus_mono, ordered_ab_group_add}) uminus
begin

```

```

lift_definition uminus_minterval_ovl::'a minterval_ovl  $\Rightarrow$  'a minterval_ovl
  is  $\lambda$  is . mk_mInterval_ovl (rev (map ( $\lambda$  i.  $-i$ ) is))
  by (metis list.map_disc_iff mk_mInterval_ovl_valid rev_is_Nil_conv valid_mInterval_ovl_def)
instance ..
end

```

### 12.1.6 Subtraction

```

instantiation minterval_ovl :: ({minus_mono, linordered_field, ordered_ab_group_add}) minus
begin

```

```

definition minus_minterval_ovl::'a minterval_ovl  $\Rightarrow$  'a minterval_ovl  $\Rightarrow$  'a minterval_ovl
  where minus_minterval_ovl a b = a +  $-$  b

```

```

instance ..
end

```

### 12.1.7 Multiplication

```

instantiation minterval_ovl :: ({minus_mono, linordered_semiring}) times
begin

```

```

lift_definition times_minterval_ovl :: 'a minterval_ovl  $\Rightarrow$  'a minterval_ovl  $\Rightarrow$  'a minterval_ovl
  is  $\lambda$  a b . mk_mInterval_ovl (iList_times a b)
  by (metis bin_op_interval_list_non_empty iList_times_def mk_mInterval_ovl_empty
    mk_mInterval_ovl_valid)

```

```

instance ..
end

```

### 12.1.8 Multiplicative Inverse and Division

```

locale minterval_ovl_division = inverse +
  constrains inverse ::  $\langle$ 'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}
minterval_ovl  $\Rightarrow$  'a minterval_ovl $\rangle$ 
  and divide ::  $\langle$ 'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}
minterval_ovl  $\Rightarrow$  'a minterval_ovl  $\Rightarrow$  'a minterval_ovl $\rangle$ 
  assumes inverse_left:  $\langle \neg 0 \in \text{set\_of\_ovl } x \implies 1 \leq (\text{inverse } x) * x \rangle$ 
  and divide:  $\langle \neg 0 \in \text{set\_of\_ovl } y \implies x \leq (\text{divide } x y) * y \rangle$ 
begin
end

```

```

locale minterval_ovl_division_inverse = inverse +
  constrains inverse ::  $\langle$ 'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}
minterval_ovl  $\Rightarrow$  'a minterval_ovl $\rangle$ 

```

```

    and divide :: ⟨'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}
minterval_ovl ⇒ 'a minterval_ovl ⇒ 'a minterval_ovl⟩
    assumes inverse_non_zero_def: ⟨¬ 0 ∈ set_of_ovl x ⇒ (inverse x)
      = mInterval_ovl (mk_mInterval_ovl (un_op_interval_list (λ i. mk_interval (1 / (upper i), 1 / (lower i))))
(mlist_ovl x)))⟩
    and divide_non_zero_def: ⟨¬ 0 ∈ set_of_ovl y ⇒ (divide x y) = inverse y * x⟩
begin

end

```

### 12.1.9 Membership

```

abbreviation (in preorder) in_minterval_ovl (⟨_ / ∈no _⟩ [51, 51] 50)
  where in_minterval_ovl x X ≡ x ∈ set_of_ovl X

```

```

lemma in_minterval_ovl_to_minterval_ovl[intro!]: a ∈no minterval_ovl_of a
  by (metis (mono_tags, lifting) UN_iff list.set_intros(1) lower_in_interval lower_point_interval
    minterval_ovl_of.rep_eq set_of_eq set_of_ovl.rep_eq)

```

```

instance minterval_ovl :: ({one, preorder, minus_mono, linordered_semiring}) power
proof qed

```

```

lemma set_of_one_ovl[simp]: set_of_ovl (1::'a::{one, order, minus_mono} minterval_ovl) = {1}
  apply (auto simp: set_of_ovl_def)[1]
  subgoal
  by (simp add: interval_sort_lower_width_set_eq mk_mInterval_ovl_def one_minterval_ovl.rep_eq)
  subgoal
  by (simp add: interval_sort_lower_width_set_eq mk_mInterval_ovl_def one_minterval_ovl.rep_eq)
done

```

```

lifting_update minterval_ovl.lifting
lifting_forget minterval_ovl.lifting

```

```
end
```

## 12.2 Non-Overlapping Multi-Intervals (≡ Multi\_Interval\_Non\_Overlapping)

```

theory
  Multi_Interval_Non_Overlapping
imports
  Multi_Interval_Preliminaries
begin

```

### 12.2.1 Type Definition

```

typedef (overloaded) 'a minterval_non_ovl =
  {is::'a::{minus_mono} interval list. valid_mInterval_non_ovl is}
morphisms bounds_of_minterval_non_ovl mInterval_non_ovl
unfolding valid_mInterval_non_ovl_def
apply(intro ex1[where x=[Interval (l,l) ] for l])
by(auto simp add: valid_mInterval_ovl_def sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def)

```

**setup\_lifting** type\_definition\_minterval\_non\_ovl

**lift\_definition** mlower\_non\_ovl::('a::{minus\_mono}) minterval\_non\_ovl  $\Rightarrow$  'a is <lower o hd> .

**lift\_definition** mupper\_non\_ovl::('a::{minus\_mono}) minterval\_non\_ovl  $\Rightarrow$  'a is <upper o last> .

**lift\_definition** mlist\_non\_ovl::('a::{minus\_mono}) minterval\_non\_ovl  $\Rightarrow$  'a interval list is <id> .

## 12.2.2 Equality and Orderings

**lemma** minterval\_non\_ovl\_eq\_iff:  $a = b \longleftrightarrow$  mlist\_non\_ovl a = mlist\_non\_ovl b

**by** transfer auto

**lemma** ainterval\_eq!: mlist\_non\_ovl a = mlist\_non\_ovl b  $\implies$  a = b

**by** (auto simp: minterval\_non\_ovl\_eq\_iff)

**lemma** minterval\_non\_ovl\_imp\_upper\_lower\_eq :

$a = b \longrightarrow$  mlower\_non\_ovl a = mlower\_non\_ovl b  $\wedge$  mupper\_non\_ovl a = mupper\_non\_ovl b

**by** transfer auto

**lemma** mlower\_non\_ovl\_le\_mupper\_non\_ovl[simp]: mlower\_non\_ovl i  $\leq$  mupper\_non\_ovl i

**by** (transfer, metis comp\_def hd\_Nil\_eq\_last lower\_le\_upper lower\_le\_upper\_aux  
non\_adjacent\_implies\_non\_overlapping valid\_mInterval\_non\_ovl\_def)

**lift\_definition** set\_of\_non\_ovl :: 'a::{minus\_mono} minterval\_non\_ovl  $\Rightarrow$  'a set

**is**  $\lambda$  is.  $\bigcup x \in \text{set is. } \{ \text{lower } x .. \text{upper } x \}$  .

**lemma** set\_non\_ovl\_of\_subset: set\_of\_non\_ovl (x::'a::minus\_mono minterval\_non\_ovl)  $\subseteq$  {mlower\_non\_ovl x .. mupper\_non\_ovl x}

**apply** (transfer, simp)

**using** set\_of\_subeq\_aux

mlInterval\_ovl\_lower\_hd\_min[symmetric, simplified o\_def]

mlInterval\_adj\_upper\_last\_max[symmetric, simplified o\_def]

valid\_non\_ovl\_imp\_adj valid\_non\_ovl\_imp\_ovl

list.set\_map

**by** (metis (mono\_tags, lifting))

**lemma** not\_in\_non\_ovl\_eq:

$\langle \neg e \in \text{set\_of\_non\_ovl } xs \rangle = \langle \forall x \in \text{set } (mlist\_non\_ovl } xs). \neg e \in \text{set\_of } x \rangle$

**proof**(induction (mlist\_non\_ovl xs))

**case** Nil

**then show** ?case

**by** (metis UN\_iff empty\_iff empty\_set mlist\_non\_ovl.rep\_eq set\_of\_non\_ovl.rep\_eq)

**next**

**case** (Cons a x)

**then show** ?case

**by** (simp add: mlist\_non\_ovl.rep\_eq set\_of\_eq set\_of\_non\_ovl.rep\_eq)

**qed**

**lemma** in\_non\_ovl\_eq:

$\langle e \in \text{set\_of\_non\_ovl } xs \rangle = \langle \exists x \in \text{set } (mlist\_non\_ovl } xs). e \in \text{set\_of } x \rangle$

**proof**(induction (mlist\_non\_ovl xs))

**case** Nil

```

then show ?case
  by (metis UN_iff empty_iff empty_set mlist_non_ovl.rep_eq set_of_non_ovl.rep_eq)
next
case (Cons a x)
then show ?case
  by (simp add: mlist_non_ovl.rep_eq set_of_eq set_of_non_ovl.rep_eq)
qed

```

```

lemma set_of_non_ovl_non_zero_list_all:
  <0 ∉ set_of_non_ovl xs ⟹ ∀ x ∈ set (mlist_non_ovl xs). ¬ 0 ∈i x>
proof(induction mlist_non_ovl xs)
case Nil
then show ?case
  by simp
next
case (Cons a x)
then show ?case
  using in_non_ovl_eq by blast
qed

```

```

context notes [[typedef_overloaded]] begin

```

```

lift_definition (code_dt) mInterval_non_ovl' :: 'a::minus_mono interval list ⇒ 'a minterval_non_ovl option
is λ is. if valid_mInterval_non_ovl is then Some is else None
by (auto simp add: valid_mInterval_non_ovl_def)

```

```

lemma mInterval_non_ovl'_split:
  P (mInterval_non_ovl' is) ⟷
  (∀ ivl. valid_mInterval_non_ovl is ⟶ mlist_non_ovl ivl = is ⟶ P (Some ivl)) ∧ (¬ valid_mInterval_non_ovl is ⟶
P None)
by transfer auto

```

```

lemma mInterval_non_ovl'_split_asm:
  P (mInterval_non_ovl' is) ⟷
  ¬((∃ ivl. valid_mInterval_non_ovl is ∧ mlist_non_ovl ivl = is ∧ ¬P (Some ivl)) ∨ (¬ valid_mInterval_non_ovl is ∧ ¬P
None))
unfolding mInterval_non_ovl'_split
by auto

```

```

lemmas mInterval_non_ovl'_splits = mInterval_non_ovl'_split mInterval_non_ovl'_split_asm

```

```

lemma mInterval'_eq_Some: mInterval_non_ovl' is = Some i ⟹ mlist_non_ovl i = is
  by (simp split: mInterval_non_ovl'_splits)

```

```

end

```

```

instantiation minterval_non_ovl :: ({minus_mono}) equal
begin

```

```

definition equal_class.equal a b ≡ (mlist_non_ovl a = mlist_non_ovl b)

```

```

instance proof qed (simp add: equal_minterval_non_ovl_def minterval_non_ovl_eq_iff)
end

```

**instantiation** *minterval\_non\_ovl* :: (*{minus\_mono}*) ord **begin**

**definition** *less\_eq\_minterval\_non\_ovl* :: 'a *minterval\_non\_ovl* ⇒ 'a *minterval\_non\_ovl* ⇒ bool

**where** *less\_eq\_minterval\_non\_ovl* a b  $\longleftrightarrow$  *mlower\_non\_ovl* b  $\leq$  *mlower\_non\_ovl* a  $\wedge$  *mupper\_non\_ovl* a  $\leq$  *mupper\_non\_ovl* b

**definition** *less\_minterval\_non\_ovl* :: 'a *minterval\_non\_ovl* ⇒ 'a *minterval\_non\_ovl* ⇒ bool

**where** *less\_minterval\_non\_ovl* x y = (x  $\leq$  y  $\wedge$   $\neg$  y  $\leq$  x)

**instance proof qed**

**end**

**instantiation** *minterval\_non\_ovl* :: (*{minus\_mono,lattice}*) sup

**begin**

**lift\_definition** *sup\_minterval\_non\_ovl* :: 'a *minterval\_non\_ovl* ⇒ 'a *minterval\_non\_ovl* ⇒ 'a *minterval\_non\_ovl*

**is**  $\lambda$  a b. [*Interval* (*inf* (*lower* (*hd* a)) (*lower* (*hd* b)), *sup* (*upper* (*last* a)) (*upper* (*last* b)))]

**by**(*auto simp: valid\_mInterval\_ovl\_def sorted\_wrt\_lower\_def non\_adjacent\_sorted\_wrt\_lower\_def non\_overlapping\_sorted\_def le\_inf1 le\_sup1 valid\_mInterval\_non\_ovl\_def mupper\_non\_ovl\_def mlower\_non\_ovl\_def*)

**lemma** *mlower\_non\_ovl\_sup[simp]*: *mlower\_non\_ovl* (*sup* A B) = *inf* (*mlower\_non\_ovl* A) (*mlower\_non\_ovl* B)

**apply**(*transfer*)

**by** (*metis comp\_apply le\_supE le\_sup1 list.sel(1) lower\_bounds lower\_le\_upper\_aux sup\_inf\_absorb valid\_mInterval\_adj\_def valid\_non\_ovl\_imp\_adj*)

**lemma** *mupper\_non\_ovl\_sup[simp]*: *mupper\_non\_ovl* (*sup* A B) = *sup* (*mupper\_non\_ovl* A) (*mupper\_non\_ovl* B)

**apply**(*transfer*)

**by** (*metis (no\_types, lifting) comp\_def inf\_sup\_absorb last.simps le\_inf1 le\_inf\_iff lower\_le\_upper\_aux upper\_bounds valid\_mInterval\_adj\_def valid\_non\_ovl\_imp\_adj*)

**instance**

**by**(*standard*)

**end**

**instantiation** *minterval\_non\_ovl* :: (*{lattice,minus\_mono}*) preorder

**begin**

**instance**

**apply**(*standard*)

**subgoal**

**using** *less\_minterval\_non\_ovl\_def* **by** *auto*

**subgoal**

**by** (*simp add: less\_eq\_minterval\_non\_ovl\_def*)

**subgoal**

**by** (*meson less\_eq\_minterval\_non\_ovl\_def order.trans*)

**done**

**end**

**lemma** *set\_of\_minterval\_non\_ovl\_union*: *set\_of\_non\_ovl* A  $\cup$  *set\_of\_non\_ovl* B  $\subseteq$  *set\_of\_non\_ovl* (*sup* A B)

**for** A: 'a::(*{lattice, minus\_mono}*) *minterval\_non\_ovl*

**apply**(*transfer, simp*)

**using** *set\_of\_subeq\_aux*

*mInterval\_ovl\_lower\_hd\_min*[*symmetric, simplified o\_def*]

*mInterval\_adj\_upper\_last\_max*[*symmetric, simplified o\_def*]

```

    valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
    list.set_map
  by (smt (verit) le_sup_iff lower_bounds lower_le_upper_aux lower_sup set_of_eq set_of_subset_iff
      sup commute sup commute sup.order_iff sup_ge1 sup_ge1 sup_inf_absorb upper_bounds
      valid_mInterval_adj_def)

```

**lemma** *mininterval\_non\_ovl\_union\_commute*:  $\sup A B = \sup B A$  **for**  $A :: 'a :: \{\text{minus\_mono}, \text{lattice}\}$  *mininterval\_non\_ovl*  
**apply** (auto simp add: *mininterval\_non\_ovl\_eq\_iff inf commute sup commute*)<sub>[1]</sub>  
**by** (simp add: *mlist\_non\_ovl.rep\_eq inf commute sup\_mininterval\_non\_ovl.rep\_eq sup commute*)

**lemma** *mininterval\_non\_ovl\_union\_mono1*:  $\text{set\_of\_non\_ovl } a \subseteq \text{set\_of\_non\_ovl } (\sup a A)$

```

for  $A :: 'a :: \{\text{minus\_mono}, \text{lattice}\}$  mininterval_non_ovl
apply(transfer, simp)
using set_of_subeq_aux
    mininterval_ovl_lower_hd_min[symmetric, simplified o_def]
    mininterval_adj_upper_last_max[symmetric, simplified o_def]
    valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
    list.set_map
by (smt (verit, del_insts) inf.absorb_iff2 inf_le1 le_inf1 lower_bounds lower_le_upper_aux
    set_of_eq set_of_subset_iff sup_ge1 upper_bounds valid_mInterval_adj_def)

```

**lemma** *mininterval\_non\_ovl\_union\_mono2*:  $\text{set\_of\_non\_ovl } A \subseteq \text{set\_of\_non\_ovl } (\sup a A)$  **for**  $A :: 'a :: \{\text{lattice}, \text{minus\_mono}\}$   
*mininterval\_non\_ovl*

```

apply(transfer, simp)
using set_of_subeq_aux
    mininterval_ovl_lower_hd_min[symmetric, simplified o_def]
    mininterval_adj_upper_last_max[symmetric, simplified o_def]
    valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
    list.set_map
by (smt (verit, del_insts) inf.absorb_iff2 le_sup_iff lower_bounds lower_le_upper_aux nle_le
    set_of_eq set_of_subset_iff sup_inf_absorb upper_bounds valid_mInterval_adj_def)

```

**lift\_definition** *mininterval\_non\_ovl\_of* ::  $'a :: \{\text{minus\_mono}\} \Rightarrow 'a$  *mininterval\_non\_ovl* **is**  $\lambda x. [\text{Interval}(x, x)]$

```

unfolding valid_mInterval_non_ovl_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def
    cmp_non_adjacent_def sorted_wrt_lower_def
by simp

```

**lemma** *mlower\_non\_ovl\_mininterval\_non\_ovl\_of*<sub>[simp]</sub>:  $\text{mlower\_non\_ovl } (\text{mininterval\_non\_ovl\_of } a) = a$   
**by** transfer auto

**lemma** *mupper\_non\_ovl\_mininterval\_non\_ovl\_of*<sub>[simp]</sub>:  $\text{mupper\_non\_ovl } (\text{mininterval\_non\_ovl\_of } a) = a$   
**by** transfer auto

**definition** *width\_non\_ovl* ::  $'a :: \{\text{minus\_mono}\}$  *mininterval\_non\_ovl*  $\Rightarrow 'a$   
**where** *width\_non\_ovl*  $i = \text{mupper\_non\_ovl } i - \text{mlower\_non\_ovl } i$

### 12.2.3 Zero and One

**instantiation** *mininterval\_non\_ovl* ::  $(\{\text{minus\_mono}, \text{zero}\})$  zero  
**begin**

**lift\_definition** *zero\_mininterval\_non\_ovl*:: $'a$  *mininterval\_non\_ovl* **is**  $\text{mk\_mInterval\_non\_ovl } [\text{Interval } (o, o)]$   
**by** (simp add: *mk\_mInterval\_non\_ovl\_valid sorted\_wrt\_lower\_def*)

**lemma** *mlower\_non\_ovl\_zero*[simp]: *mlower\_non\_ovl* 0 = 0  
**by**(transfer, simp add: *mk\_mInterval\_non\_ovl\_def* *mk\_mInterval\_ovl\_def* *interval\_sort\_lower\_width\_def*)

**lemma** *mupper\_non\_ovl\_zero*[simp]: *mupper\_non\_ovl* 0 = 0  
**by**(transfer, simp add: *mk\_mInterval\_non\_ovl\_def* *mk\_mInterval\_ovl\_def* *interval\_sort\_lower\_width\_def*)

**instance proof qed**  
**end**

**instantiation** *minterval\_non\_ovl* :: (*{minus\_mono,one}*) one  
**begin**

**lift\_definition** *one\_minterval\_non\_ovl*:: '*a* *minterval\_non\_ovl* is *mk\_mInterval\_non\_ovl* [*Interval* (1, 1)]  
**by** (*metis* *list.simps(3)* *mk\_mInterval\_non\_ovl\_single* *mk\_mInterval\_non\_ovl\_valid*  
*sorted\_wrt\_lower\_mk\_mInterval\_non\_ovl*)

**lemma** *mlower\_non\_ovl\_one*[simp]: *mlower\_non\_ovl* 1 = 1  
**by**(transfer, simp add: *mk\_mInterval\_non\_ovl\_def* *mk\_mInterval\_ovl\_def* *interval\_sort\_lower\_width\_def*)

**lemma** *mupper\_non\_ovl\_one*[simp]: *mupper\_non\_ovl* 1 = 1  
**by**(transfer, simp add: *mk\_mInterval\_non\_ovl\_def* *mk\_mInterval\_ovl\_def* *interval\_sort\_lower\_width\_def*)

**instance proof qed**  
**end**

## 12.2.4 Addition

**instantiation** *minterval\_non\_ovl* :: (*{minus\_mono,ordered\_ab\_semigroup\_add,linordered\_field}*) plus  
**begin**

**lemma** *valid\_mk\_interval\_iList\_plus*:  
**assumes** *valid\_mInterval\_non\_ovl* *a* **and** *valid\_mInterval\_non\_ovl* *b*  
**shows** *valid\_mInterval\_non\_ovl* (*mk\_mInterval\_non\_ovl* (*iList\_plus* *a* *b*))  
**by** (*metis* (*no\_types*, *lifting*) *assms(1)* *assms(2)* *bin\_op\_interval\_list\_empty* *iList\_plus\_lower\_upper\_eq*  
*mk\_mInterval\_non\_ovl\_id* *mk\_mInterval\_non\_ovl\_non\_empty* *mk\_mInterval\_non\_ovl\_valid*  
*sorted\_wrt\_lower\_mk\_mInterval\_non\_ovl*)

**lift\_definition** *plus\_minterval\_non\_ovl*:: '*a* *minterval\_non\_ovl*  $\Rightarrow$  '*a* *minterval\_non\_ovl*  $\Rightarrow$  '*a* *minterval\_non\_ovl*  
**is**  $\lambda$  *a* *b* . *mk\_mInterval\_non\_ovl* (*iList\_plus* *a* *b*)  
**by** (*simp* add: *valid\_mk\_interval\_iList\_plus*)

**lemma** *interval\_plus\_com*:  
 $\langle a + b = b + a \rangle$  **for** *a*::'*a*::*{minus\_mono,ordered\_ab\_semigroup\_add,linordered\_field}* *minterval\_non\_ovl*  
**apply**(transfer)  
**using** *iList\_plus\_mInterval\_non\_ovl\_commute* *plus\_minterval\_non\_ovl\_def*  
**by** auto

**instance proof qed**

**end**

## 12.2.5 Unary Minus

**lemma** *a*: (*x*::'*a*::*ordered\_ab\_group\_add* *interval*)  $\neq$  *y*  $\implies$   $-x \neq -y$

```

    apply(simp add:uminus_interval_def)
  by (smt (z3) Pair_inject bounds_of_interval_inverse case_prod_Pair_iden case_prod_unfold neg_equal_iff_equal uminus_interval.rep_eq)

```

```

lemma b: distinct (is::'a::ordered_ab_group_add interval list)  $\implies$  distinct (map ( $\lambda$  i.  $-i$ ) is)
proof(induction is)
  case Nil
  then show ?case by simp
next
  case (Cons a is)
  then show ?case using a by force
qed

```

```

instantiation mininterval_non_ovl :: ( $\{$ minus_mono, ordered_ab_group_add $\}$ ) uminus
begin

```

```

lift_definition uminus_mininterval_non_ovl::'a mininterval_non_ovl  $\Rightarrow$  'a mininterval_non_ovl
  is  $\lambda$  is . mk_mInterval_non_ovl (rev (map ( $\lambda$  i.  $-i$ ) is))
by (metis (no_types, lifting) distinct_remdups_id list.map_disc_iff mk_mInterval_non_ovl_def
  mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct
  o_def rev_is_Nil_conv valid_mInterval_non_ovl_def valid_mInterval_ovl_def
  valid_ovl_mkInterval_non_ovl)

```

```

instance ..
end

```

## 12.2.6 Subtraction

```

instantiation mininterval_non_ovl :: ( $\{$ minus_mono, linordered_field, ordered_ab_group_add $\}$ ) minus
begin

```

```

definition minus_mininterval_non_ovl::'a mininterval_non_ovl  $\Rightarrow$  'a mininterval_non_ovl  $\Rightarrow$  'a mininterval_non_ovl
  where minus_mininterval_non_ovl a b = a +  $-$  b

```

```

instance ..
end

```

## 12.2.7 Multiplication

```

instantiation mininterval_non_ovl :: ( $\{$ minus_mono, linordered_field $\}$ ) times
begin

```

```

lift_definition times_mininterval_non_ovl::'a mininterval_non_ovl  $\Rightarrow$  'a mininterval_non_ovl  $\Rightarrow$  'a mininterval_non_ovl
  is  $\lambda$  a b . mk_mInterval_non_ovl (iList_times a b)
by (metis (no_types, lifting) bin_op_interval_list_empty distinct_remdups_id iList_times_def
  mk_mInterval_non_ovl_def mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl
  non_adjacent_implies_distinct o_def valid_mInterval_non_ovl_def valid_mInterval_ovl_def
  valid_ovl_mkInterval_non_ovl)

```

```

instance ..
end

```

## 12.2.8 Multiplicative Inverse and Division

```
locale minterval_non_ovl_division = inverse +
  constrains inverse :: <'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}>
minterval_non_ovl ⇒ 'a minterval_non_ovl
  and divide :: <'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}>
minterval_non_ovl ⇒ 'a minterval_non_ovl ⇒ 'a minterval_non_ovl
  assumes inverse_left: <¬ 0 ∈ set_of_non_ovl x ⇒ 1 ≤ (inverse x) * x>
  and divide: <¬ 0 ∈ set_of_non_ovl y ⇒ x ≤ (divide x y) * y>
begin
end
```

```
locale minterval_non_ovl_division_inverse = inverse +
  constrains inverse :: <'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}>
minterval_non_ovl ⇒ 'a minterval_non_ovl
  and divide :: <'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}>
minterval_non_ovl ⇒ 'a minterval_non_ovl ⇒ 'a minterval_non_ovl
  assumes inverse_non_zero_def: <¬ 0 ∈ set_of_non_ovl x ⇒ (inverse x)
    = mInterval_non_ovl (mk_mInterval_non_ovl (un_op_interval_list (λ i. mk_interval (1 / (upper i), 1 /
(lower i))) (mlist_non_ovl x))))>
  and divide_non_zero_def: <¬ 0 ∈ set_of_non_ovl y ⇒ (divide x y) = inverse y * x>
begin
end
```

## 12.2.9 Membership

```
abbreviation (in preorder) in_minterval_non_ovl (<(_ / ∈no _)> [51, 51] 50)
  where in_minterval_non_ovl x X ≡ x ∈ set_of_non_ovl X
```

```
lemma in_minterval_non_ovl_to_minterval_non_ovl[intro!]: a ∈no minterval_non_ovl_of a
  by (simp add: minterval_non_ovl_of.rep_eq set_of_non_ovl_def)
```

```
instance minterval_non_ovl :: ({one, preorder, minus_mono, linordered_semiring}) power
proof qed
```

```
lemma set_of_one_non_ovl[simp]: set_of_non_ovl (1::'a::{one, minus_mono, order} minterval_non_ovl) = {1}
  apply (transfer)
  by (auto simp: mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def set_of_non_ovl_def)
```

```
lifting_update minterval_non_ovl.lifting
lifting_forget minterval_non_ovl.lifting
```

end

## 12.3 Adjacent Multi-Intervals (Multi\_Interval\_Adjacent)

```
theory
  Multi_Interval_Adjacent
imports
  Multi_Interval_Preliminaries
begin
```

### 12.3.1 A Type For Non Overlapping Multi Intervals

```

typedef (overloaded) 'a minterval_adj =
  {is::'a::{minus_mono,linorder} interval list. valid_mInterval_adj is}
morphisms bounds_of_minterval_adj mInterval_adj
unfolding valid_mInterval_adj_def
apply(intro ex1[where x=[Interval (l,l) ] for l])
by(auto simp add: sorted_wrt_lower_def non_overlapping_sorted_def)

```

```

setup_lifting type_definition_minterval_adj

```

```

lift_definition mlower_adj::('a::{minus_mono}) minterval_adj  $\Rightarrow$  'a is <lower o hd> .
lift_definition mupper_adj::('a::{minus_mono}) minterval_adj  $\Rightarrow$  'a is <upper o last> .
lift_definition mlist_adj::('a::{minus_mono}) minterval_adj  $\Rightarrow$  'a interval list is <id> .

```

### 12.3.2 Equality and Orderings

```

lemma minterval_adj_eq_iff: a = b  $\longleftrightarrow$  mlist_adj a = mlist_adj b
by transfer auto

```

```

lemma ainterval_eq1: mlist_adj a = mlist_adj b  $\implies$  a = b
by (auto simp: minterval_adj_eq_iff)

```

```

lemma minterval_adj_imp_upper_lower_eq :
  a = b  $\longrightarrow$  mlower_adj a = mlower_adj b  $\wedge$  mupper_adj a = mupper_adj b
by transfer auto

```

```

lemma mlower_adj_le_mupper_adj[simp]: mlower_adj i  $\leq$  mupper_adj i
by (transfer, metis comp_def lower_le_upper_aux valid_mInterval_adj_def)

```

```

lift_definition set_of_adj :: 'a::{minus_mono} minterval_adj  $\Rightarrow$  'a set
is  $\lambda$  is.  $\bigcup_{x \in \text{set } is. \{ \text{lower } x .. \text{upper } x \}}$  .

```

```

lemma set_adj_of_subset: set_of_adj (x::'a::{minus_mono} minterval_adj)  $\subseteq$  {mlower_adj x .. mupper_adj x}
apply(transfer, simp)
using set_of_subeq_aux
  mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
  mInterval_adj_upper_last_max[symmetric, simplified o_def]
  valid_adj_imp_ovl
  list.set_map
by (smt (verit, best))

```

```

lemma not_in_adj_eq:
   $\langle \neg e \in \text{set\_of\_adj } xs = (\forall x \in \text{set } (mlist\_adj \text{ } xs). \neg e \in \text{set\_of } x) \rangle$ 
proof(induction (mlist_adj xs))
case Nil
then show ?case
by (metis UN_iff empty_iff empty_set mlist_adj.rep_eq set_of_adj.rep_eq)
next
case (Cons a x)
then show ?case
by (simp add: mlist_adj.rep_eq set_of_eq set_of_adj.rep_eq)
qed

```

```

lemma in_adj_eq:
  <(e ∈ set_of_adj xs) = (∃ x ∈ set (mlist_adj xs). e ∈ set_of x)>
proof(induction (mlist_adj xs))
  case Nil
  then show ?case
    by (metis UN_iff empty_iff empty_set mlist_adj.rep_eq set_of_adj.rep_eq)
next
  case (Cons a x)
  then show ?case
    by (simp add: mlist_adj.rep_eq set_of_eq set_of_adj.rep_eq)
qed

```

```

lemma set_of_adj_non_zero_list_all:
  <0 ∉ set_of_adj xs ⇒ ∀ x ∈ set (mlist_adj xs). ¬ 0 ∈i x>
proof(induction mlist_adj xs)
  case Nil
  then show ?case
    by simp
next
  case (Cons a x)
  then show ?case
    using in_adj_eq by blast
qed

```

**context notes** [[*typedef\_overloaded*]] **begin**

**lift\_definition** (code\_dt) mInterval\_adj' :: 'a::minus\_mono interval list ⇒ 'a minterval\_adj option  
**is** λ is. if valid\_mInterval\_adj is then Some is else None  
**by** (auto simp add: valid\_mInterval\_adj\_def)

```

lemma mInterval_adj'_split:
  P (mInterval_adj' is) ⟷
  (∀ ivl. valid_mInterval_adj is ⟶ mlist_adj ivl = is ⟶ P (Some ivl)) ∧ (¬ valid_mInterval_adj is ⟶ P None)
by transfer auto

```

```

lemma mInterval_adj'_split_asm:
  P (mInterval_adj' is) ⟷
  ¬((∃ ivl. valid_mInterval_adj is ∧ mlist_adj ivl = is ∧ ¬P (Some ivl)) ∨ (¬ valid_mInterval_adj is ∧ ¬P None))
unfolding mInterval_adj'_split
by auto

```

**lemmas** mInterval\_adj'\_splits = mInterval\_adj'\_split mInterval\_adj'\_split\_asm

```

lemma mInterval'_eq_Some: mInterval_adj' is = Some i ⇒ mlist_adj i = is
by (simp split: mInterval_adj'_splits)

```

**end**

**instantiation** minterval\_adj :: ({minus\_mono}) equal  
**begin**

**definition** equal\_class.equal a b ≡ (mlist\_adj a = mlist\_adj b)

**instance proof qed** (*simp add: equal\_minterval\_adj\_def minterval\_adj\_eq\_iff*)  
**end**

**instantiation minterval\_adj** :: (*{minus\_mono}*) **ord begin**

**definition less\_eq\_minterval\_adj** :: '*a minterval\_adj* ⇒ '*a minterval\_adj* ⇒ *bool*  
**where** *less\_eq\_minterval\_adj a b*  $\longleftrightarrow$  *mlower\_adj b*  $\leq$  *mlower\_adj a*  $\wedge$  *mupper\_adj a*  $\leq$  *mupper\_adj b*

**definition less\_minterval\_adj** :: '*a minterval\_adj* ⇒ '*a minterval\_adj* ⇒ *bool*  
**where** *less\_minterval\_adj x y* = (*x*  $\leq$  *y*  $\wedge$   $\neg$  *y*  $\leq$  *x*)

**instance proof qed**  
**end**

**instantiation minterval\_adj** :: (*{minus\_mono,lattice}*) **sup begin**

**lift\_definition sup\_minterval\_adj** :: '*a minterval\_adj* ⇒ '*a minterval\_adj*  
**is**  $\lambda$  *a b*. [*Interval* (*inf* (*lower* (*hd a*)) (*lower* (*hd b*))), *sup* (*upper* (*last a*)) (*upper* (*last b*)))]  
**by** (*auto simp: valid\_minterval\_ovl\_def sorted\_wrt\_lower\_def non\_adjacent\_sorted\_wrt\_lower\_def*  
*non\_overlapping\_sorted\_def le\_inf1 le\_sup1 valid\_minterval\_adj\_def mupper\_adj\_def*  
*mlower\_adj\_def* )

**lemma mlower\_adj\_sup**[*simp*]: *mlower\_adj* (*sup A B*) = *inf* (*mlower\_adj A*) (*mlower\_adj B*)  
**apply** (*transfer*)  
**by** (*metis comp\_apply le\_supE le\_sup1 list.sel(1) lower\_bounds lower\_le\_upper\_aux sup\_inf\_absorb*  
*valid\_minterval\_adj\_def* )

**lemma mupper\_adj\_sup**[*simp*]: *mupper\_adj* (*sup A B*) = *sup* (*mupper\_adj A*) (*mupper\_adj B*)  
**apply** (*transfer*)  
**by** (*metis* (*no\_types, lifting*) *comp\_def inf\_sup\_absorb last.simps le\_inf1 le\_inf\_iff lower\_le\_upper\_aux*  
*upper\_bounds valid\_minterval\_adj\_def* )

**instance**  
**by** (*standard*)  
**end**

**instantiation minterval\_adj** :: (*{lattice,minus\_mono}*) **preorder begin**

**instance**  
**apply** (*standard*)  
**subgoal**  
**using** *less\_minterval\_adj\_def* **by** *auto*  
**subgoal**  
**by** (*simp add: less\_eq\_minterval\_adj\_def*)  
**subgoal**  
**by** (*meson less\_eq\_minterval\_adj\_def order.trans*)  
**done**  
**end**

**lemma set\_of\_minterval\_adj\_union**: *set\_of\_adj A*  $\cup$  *set\_of\_adj B*  $\subseteq$  *set\_of\_adj* (*sup A B*)  
**for** *A*: '*a*::*{lattice, minus\_mono}* *minterval\_adj*  
**apply** (*transfer, simp*)  
**using** *set\_of\_subeq\_aux*

```

mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
mInterval_adj_upper_last_max[symmetric, simplified o_def]
valid_adj_imp_ovl
list.set_map
by (smt (verit) le_sup_iff lower_bounds lower_le_upper_aux lower_sup set_of_eq set_of_subset_iff
sup commute sup commute sup.order_iff sup_ge1 sup_ge1 sup_inf_absorb upper_bounds
valid_mInterval_adj_def)

```

**lemma** *minterval\_adj\_union\_commute*:  $\sup A B = \sup B A$  **for**  $A :: 'a :: \{\text{minus\_mono}, \text{lattice}\}$  *minterval\_adj*  
**apply** (auto simp add: *minterval\_adj\_eq\_iff inf commute sup commute*) [1]  
**by** (simp add: *minterval\_adj\_rep\_eq inf\_commute sup\_minterval\_adj\_rep\_eq sup\_commute*)

**lemma** *minterval\_adj\_union\_mono1*:  $\text{set\_of\_adj } a \subseteq \text{set\_of\_adj } (\sup a A)$

**for**  $A :: 'a :: \{\text{minus\_mono}, \text{lattice}\}$  *minterval\_adj*

**apply** (transfer, simp)

**using** *set\_of\_subeq\_aux*

*mInterval\_ovl\_lower\_hd\_min*[*symmetric, simplified o\_def*]

*mInterval\_adj\_upper\_last\_max*[*symmetric, simplified o\_def*]

*valid\_adj\_imp\_ovl*

*list.set\_map*

**by** (smt (verit, del\_insts) *inf\_absorb\_iff2 inf\_le1 le\_inf1 lower\_bounds lower\_le\_upper\_aux*  
*set\_of\_eq set\_of\_subset\_iff sup\_ge1 upper\_bounds valid\_mInterval\_adj\_def*)

**lemma** *minterval\_adj\_union\_mono2*:  $\text{set\_of\_adj } A \subseteq \text{set\_of\_adj } (\sup a A)$  **for**  $A :: 'a :: \{\text{lattice}, \text{minus\_mono}\}$  *minterval\_adj*

**apply** (transfer, simp)

**using** *set\_of\_subeq\_aux*

*mInterval\_ovl\_lower\_hd\_min*[*symmetric, simplified o\_def*]

*mInterval\_adj\_upper\_last\_max*[*symmetric, simplified o\_def*]

*valid\_adj\_imp\_ovl*

*list.set\_map*

**by** (smt (verit, del\_insts) *inf\_absorb\_iff2 le\_sup\_iff lower\_bounds lower\_le\_upper\_aux nle\_le*  
*set\_of\_eq set\_of\_subset\_iff sup\_inf\_absorb upper\_bounds valid\_mInterval\_adj\_def*)

**lift\_definition** *minterval\_adj\_of* ::  $'a :: \{\text{minus\_mono}\} \Rightarrow 'a$  *minterval\_adj* **is**  $\lambda x. [\text{Interval}(x, x)]$

**unfolding** *valid\_mInterval\_adj\_def valid\_mInterval\_ovl\_def non\_adjacent\_sorted\_wrt\_lower\_def*

*cmp\_non\_adjacent\_def sorted\_wrt\_lower\_def non\_overlapping\_sorted\_def*

**by** *simp*

**lemma** *mlower\_adj\_minterval\_adj\_of*[*simp*]:  $\text{mlower\_adj } (\text{minterval\_adj\_of } a) = a$

**by** *transfer auto*

**lemma** *mupper\_adj\_minterval\_adj\_of*[*simp*]:  $\text{mupper\_adj } (\text{minterval\_adj\_of } a) = a$

**by** *transfer auto*

**definition** *width\_adj* ::  $'a :: \{\text{minus\_mono}\}$  *minterval\_adj*  $\Rightarrow 'a$

**where**  $\text{width\_adj } i = \text{mupper\_adj } i - \text{mlower\_adj } i$

### 12.3.3 Zero and One

**instantiation** *minterval\_adj* ::  $(\{\text{minus\_mono}, \text{zero}\})$  *zero*

**begin**

**lift\_definition** *zero\_minterval\_adj*::  $'a$  *minterval\_adj* **is** *mk\_mInterval\_adj* [*Interval* (*o*, *o*)]

**by** (simp add: *mk\_mInterval\_adj\_valid*)

```

lemma mlower_adj_zero[simp]: mlower_adj 0 = 0
  by(transfer, simp add: mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def)

```

```

lemma mupper_adj_zero[simp]: mupper_adj 0 = 0
  by(transfer, simp add: mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def)

```

```

instance proof qed
end

```

```

instantiation minterval_adj :: ({minus_mono,one}) one
begin

```

```

lift_definition one_minterval_adj::'a minterval_adj is mk_mInterval_adj [Interval (1, 1)]
  by (simp add: mk_mInterval_adj_valid)

```

```

lemma mlower_adj_one[simp]: mlower_adj 1 = 1
  by(transfer, simp add: mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def)

```

```

lemma mupper_adj_one[simp]: mupper_adj 1 = 1
  by(transfer, simp add: mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def)

```

```

instance proof qed
end

```

### 12.3.4 Addition

```

instantiation minterval_adj :: ({minus_mono,ordered_ab_semigroup_add,linordered_field}) plus
begin

```

```

lift_definition plus_minterval_adj::'a minterval_adj  $\Rightarrow$  'a minterval_adj  $\Rightarrow$  'a minterval_adj
  is  $\lambda$  a b . mk_mInterval_adj (iList_plus a b)
  by (metis bin_op_interval_list_empty iList_plus_def mk_mInterval_adj_valid valid_mInterval_adj_def)

```

```

instance proof qed

```

```

lemma interval_plus_com:
   $\langle a + b = b + a \rangle$  for a::'a minterval_adj
  apply(transfer)
  using iList_plus_mInterval_adj_commute plus_minterval_adj_def
  by(auto)

```

```

end

```

### 12.3.5 Unary Minus

```

lemma a: (x::'a::ordered_ab_group_add interval)  $\neq$  y  $\implies$   $-x \neq -y$ 
  apply(simp add:uminus_interval_def)
  by (smt (z3) Pair_inject bounds_of_interval_inverse case_prod_Pair_iden case_prod_unfold neg_equal_iff_equal uminus_interval.rep_eq)

```

```

lemma b: distinct (is::'a::ordered_ab_group_add interval list)  $\implies$  distinct (map ( $\lambda$  i.  $-i$ ) is)
proof(induction is)
  case Nil

```

```

then show ?case by simp
next
case (Cons a is)
then show ?case using a by force
qed

```

```

instantiation minterval_adj :: ({minus_mono, ordered_ab_group_add}) uminus
begin

```

```

lift_definition uminus_minterval_non_ovl::'a minterval_adj  $\Rightarrow$  'a minterval_adj
  is  $\lambda$  is . mk_mInterval_non_ovl (rev (map ( $\lambda$  i .  $-i$ ) is))
by (metis (no_types, opaque_lifting) list.map_disc_iff mk_mInterval_non_ovl_id mk_mInterval_non_ovl_non_empty
  mk_mInterval_non_ovl_valid rev_is_Nil_conv sorted_wrt_lower_mk_mInterval_non_ovl
  valid_non_ovl_imp_adj)

```

```

instance ..
end

```

### 12.3.6 Subtraction

```

instantiation minterval_adj :: ({minus_mono, linordered_field, ordered_ab_group_add}) minus
begin

```

```

definition minus_minterval_non_ovl::'a minterval_adj  $\Rightarrow$  'a minterval_adj  $\Rightarrow$  'a minterval_adj
  where minus_minterval_non_ovl a b = a +  $-$  b

```

```

instance ..
end

```

### 12.3.7 Multiplication

```

instantiation minterval_adj :: ({minus_mono, linordered_field}) times
begin

```

```

lift_definition times_minterval_non_ovl::'a minterval_adj  $\Rightarrow$  'a minterval_adj  $\Rightarrow$  'a minterval_adj
  is  $\lambda$  a b . mk_mInterval_non_ovl (iList_times a b)
by (metis bin_op_interval_list_empty iList_times_def mk_mInterval_non_ovl_id
  mk_mInterval_non_ovl_non_empty mk_mInterval_non_ovl_valid sorted_wrt_lower_mk_mInterval_non_ovl
  valid_non_ovl_imp_adj)

```

```

instance ..
end

```

### 12.3.8 Multiplicative Inverse and Division

```

locale minterval_adj_division = inverse +
  constrains inverse ::  $\langle 'a :: \{linordered\_field, zero, minus, minus\_mono, real\_normed\_algebra, linear\_continuum\_topology\}$ 
  minterval_adj  $\Rightarrow$  'a minterval_adj
    and divide ::  $\langle 'a :: \{linordered\_field, zero, minus, minus\_mono, real\_normed\_algebra, linear\_continuum\_topology\}$ 
  minterval_adj  $\Rightarrow$  'a minterval_adj  $\Rightarrow$  'a minterval_adj
    assumes inverse_left:  $\langle \neg 0 \in \text{set\_of\_adj } x \Longrightarrow 1 \leq (\text{inverse } x) * x \rangle$ 
    and divide:  $\langle \neg 0 \in \text{set\_of\_adj } y \Longrightarrow x \leq (\text{divide } x y) * y \rangle$ 
begin

```

end

```
locale minterval_adj_division_inverse = inverse +
  constrains inverse :: <'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}
  minterval_adj => 'a minterval_adj>
  and divide :: <'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra, linear_continuum_topology}
  minterval_adj => 'a minterval_adj => 'a minterval_adj>
  assumes inverse_non_zero_def: <¬ 0 ∈ set_of_adj x => (inverse x)
    = mInterval_adj (mk_mInterval_adj (un_op_interval_list (λ i. mk_interval (1 / (upper i), 1 / (lower i))))
  (mlist_adj x)))>
  and divide_non_zero_def: <¬ 0 ∈ set_of_adj y => (divide x y) = inverse y * x>
begin
end
```

### 12.3.9 Membership

```
abbreviation (in preorder) in_minterval_adj (<(_ / ∈adj _)> [51, 51] 50)
  where in_minterval_adj x X ≡ x ∈ set_of_adj X
```

```
lemma in_minterval_adj_to_minterval_adj[intro!]: a ∈adj minterval_adj_of a
  by (metis (mono_tags, lifting) UN_iff list.set_intros(1) lower_in_interval lower_point_interval
  minterval_adj_of.rep_eq set_of_eq set_of_adj.rep_eq)
```

```
instance minterval_adj :: ({one, preorder, minus_mono, linordered_semiring}) power
proof qed
```

```
lemma set_of_one_adj[simp]: set_of_adj (1::'a::{one, minus_mono, order} minterval_adj) = {1}
  apply (transfer)
  by (auto simp: mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def set_of_adj_def)
```

```
lifting_update minterval_adj.lifting
lifting_forget minterval_adj.lifting
```

end

## 12.4 Bringing Everything Together (📦 Multi\_Interval)

```
theory
  Multi_Interval
  imports
  Multi_Interval_Overlapping
  Multi_Interval_Non_Overlapping
  Multi_Interval_Adjacent
  Lipschitz_Subdivisions_Refinements
begin
```

For convince, we provide a theory that provides all three variants of multi-intervals. Note that the order in which these theories are imported is important: importing the theory `Multi_Interval_Adjacent` last ensures that it provides the default definitions and lemmas.

end



## 13 Extended Division on Multi-Intervals

### ( Extended\_Multi\_Interval\_Division\_Core)

```
theory
  Extended_Multi_Interval_Division_Core
imports
  Interval_Division_Non_Zero
  Multi_Interval
begin
```

#### 13.1 Division over List of Intervals

In this theory, we define an extended division operation on intervals. This is a formalization of the interval division given in [4].

```
definition inverse_interval :: ('a::{linorder,minus_mono,zero,one,inverse,infinity,uminus}) interval  $\Rightarrow$  ('a interval) list
  where inverse_interval a = (
    if ( $\neg o \in_i a$ ) then [mk_interval (1 / (upper a), 1 / (lower a))]
    else if lower a = o  $\wedge$  o < upper a then [mk_interval (1 / upper a,  $\infty$ )]
    else if lower a < o  $\wedge$  o < upper a then [mk_interval ( $-\infty$ , 1 / lower a), mk_interval (1 / upper a,  $\infty$ )]
    else if lower a < upper a  $\wedge$  upper a = o then [mk_interval( $-\infty$ , 1 / lower a)]
    else undefined
  )
definition <minverse = concat o (map inverse_interval)>
```

#### 13.2 Multi-Interval Division

```
end
```

##### 13.2.1 Overlapping Multi-Intervals ( Extended\_Multi\_Interval\_Division\_Overlapping)

```
theory
  Extended_Multi_Interval_Division_Overlapping
imports
  Extended_Multi_Interval_Division_Core
begin
```

```
definition <mininterval_ovl_inverse x = mInterval_ovl (mk_mInterval_ovl(minverse (mList_ovl x)))>
definition <mininterval_ovl_divide x y = (mininterval_ovl_inverse y) * x>
```

```
lemma set_of_ovl_non_zero_map_inverse:
  assumes <o  $\notin$  set_of_ovl xs>
  shows <concat (map inverse_interval (mList_ovl xs)) = map ( $\lambda i. mk\_interval (1 / upper i, 1 / lower i)$ ) (mList_ovl xs)>
proof(insert assms, induction mList_ovl xs)
  case Nil
  then show ?case
  by simp
```

```

next
case (Cons a x)
then show ?case
  using set_of_ovl_non_zero_list_all[of xs, simplified Cons, simplified]
  by (metis (no_types, lifting) concat_map_singleton inverse_interval_def map_eq_conv)
qed

```

```

interpretation minterval_ovl_division_inverse minterval_ovl_divide minterval_ovl_inverse
apply(unfold_locales)
subgoal
  using set_of_ovl_non_zero_map_inverse
  unfolding minterval_ovl_inverse_def minverse_def o_def un_op_interval_list_def
  by fastforce
subgoal by(metis minterval_ovl_divide_def)
done

```

end

### 13.2.2 Non Overlapping Multi-Intervals (Extended\_Multi\_Interval\_Division\_Non\_Overlapping)

theory

Extended\_Multi\_Interval\_Division\_Non\_Overlapping

imports

Extended\_Multi\_Interval\_Division\_Core

begin

**definition** <mininterval\_non\_ovl\_inverse x = mInterval\_non\_ovl (mk\_mInterval\_non\_ovl(minverse (mList\_non\_ovl x)))>

**definition** <mininterval\_non\_ovl\_divide x y = (mininterval\_non\_ovl\_inverse y) \* x>

**lemma** set\_of\_non\_ovl\_non\_zero\_map\_inverse:

**assumes** <0 ∉ set\_of\_non\_ovl xs>

**shows** <concat (map inverse\_interval (mList\_non\_ovl xs)) = map (λi. mk\_interval (1 / upper i, 1 / lower i)) (mList\_non\_ovl xs)>

**proof**(insert\_assms, induction mList\_non\_ovl xs)

**case** Nil

**then show** ?case

**by simp**

next

**case** (Cons a x)

**then show** ?case

**using** set\_of\_non\_ovl\_non\_zero\_list\_all[of xs, simplified Cons, simplified]

**by** (metis (no\_types, lifting) concat\_map\_singleton inverse\_interval\_def map\_eq\_conv)

qed

**interpretation** mininterval\_non\_ovl\_division\_inverse mininterval\_non\_ovl\_divide mininterval\_non\_ovl\_inverse

**apply**(unfold\_locales)

**subgoal**

**using** set\_of\_non\_ovl\_non\_zero\_map\_inverse

**unfolding** mininterval\_non\_ovl\_inverse\_def minverse\_def o\_def un\_op\_interval\_list\_def

**by fastforce**

**subgoal by**(metis mininterval\_non\_ovl\_divide\_def)

**done**

end

### 13.2.3 Adjacent Multi-Intervals (Extended\_Multi\_Interval\_Division\_Adjacent)

theory

Extended\_Multi\_Interval\_Division\_Adjacent

imports

Extended\_Multi\_Interval\_Division\_Core

begin

**definition** <mininterval\_adj\_inverse x = mInterval\_adj (mk\_mInterval\_adj (minverse (mlist\_adj x)))>

**definition** <mininterval\_adj\_divide x y = (mininterval\_adj\_inverse y) \* x>

**lemma** set\_of\_adj\_non\_zero\_map\_inverse:

**assumes** <0 ∉ set\_of\_adj xs>

**shows** <concat (map inverse\_interval (mlist\_adj xs)) = map (λi. mk\_interval (1 / upper i, 1 / lower i)) (mlist\_adj xs)>

**proof**(insert assms, induction mlist\_adj xs)

**case** Nil

**then show** ?case

**by** simp

**next**

**case** (Cons a x)

**then show** ?case

**using** set\_of\_adj\_non\_zero\_list\_all[of xs, simplified Cons, simplified]

**by** (metis (no\_types, lifting) concat\_map\_singleton inverse\_interval\_def map\_eq\_conv)

qed

**interpretation** mininterval\_adj\_division\_inverse mininterval\_adj\_divide mininterval\_adj\_inverse

apply(unfold\_locales)

**subgoal**

**using** set\_of\_adj\_non\_zero\_map\_inverse

**unfolding** mininterval\_adj\_inverse\_def minverse\_def o\_def un\_op\_interval\_list\_def

**by** fastforce

**subgoal** by(metis mininterval\_adj\_divide\_def)

**done**

end

### 13.3 Bringing Everything Together (Extended\_Multi\_Interval\_Division)

theory

Extended\_Multi\_Interval\_Division

imports

Extended\_Multi\_Interval\_Division\_Overlapping

Extended\_Multi\_Interval\_Division\_Non\_Overlapping

Extended\_Multi\_Interval\_Division\_Adjacent

begin

end



## 14 Extended Multi-Interval Analysis

### ( Extended\_Multi\_Interval\_Analysis)

**theory**

*Extended\_Multi\_Interval\_Analysis*

**imports**

*Extended\_Multi\_Interval\_Division*

**begin**

This theory provides extended multi-interval analysis over the type extended reals. All operations work over multi-intervals, i.e., lists of (closed) intervals.

**end**



## Bibliography

- [1] A. D. Brucker, T. Cameron-Burke, and A. Stell. Formally verified interval arithmetic and its application to program verification. In *13th IEEE/ACM International Conference on Formal Methods in Software Engineering (FormalISE 2024)*. IEEE, 2024.
- [2] A. D. Brucker and A. Stell. Verifying feedforward neural networks for classification in Isabelle/HOL. In M. Chechik, J.-P. Katoen, and M. Leucker, editors, *Formal Methods (FM 2023)*. Lübeck, Germany, 2023. ISBN: 978-3-642-38915-3. URL: <http://www.brucker.ch/bibliography/abstract/brucker.ea-feedforward-nn-verification-2023>.
- [3] A. Harapanahalli, S. Jafarpour, and S. Coogan. A toolbox for fast interval arithmetic in numpy with an application to formal verification of neural network controlled systems. *CoRR*, abs/2306.15340, 2023. DOI: 10.48550/ARXIV.2306.15340. arXiv: 2306.15340. URL: <https://doi.org/10.48550/arXiv.2306.15340>.
- [4] R. E. Moore, R. B. Kearfott, and M. J. Cloud. *Introduction to Interval Analysis*. Society for Industrial and Applied Mathematics, USA, 2009. ISBN: 0898716691.
- [5] D. Ratz. *Inclusion isotone extended interval arithmetic. A toolbox update*. 1997. DOI: 10.5445/IR/67997. Karlsruhe 1996. (Berichte aus dem Forschungsschwerpunkt Computerarithmetik, Intervallrechnung und Numerische Algorithmen mit Ergebnisverifikation. 1996,5.)
- [6] S. Wang, K. Pei, J. Whitehouse, J. Yang, and S. Jana. Formal security analysis of neural networks using symbolic intervals. In *Proceedings of the 27th USENIX Conference on Security Symposium, SEC'18*, pages 1599–1614, Baltimore, MD, USA. USENIX Association, 2018. ISBN: 9781931971461.