The Hales–Jewett Theorem

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September 9, 2026

Abstract

This article is a formalisation of a proof of the Hales–Jewett theorem presented in the textbook *Ramsey Theory* by Graham et al. [1].

The Hales–Jewett theorem is a result in Ramsey Theory which states that, for any non-negative integers r and t, there exists a minimal dimension N, such that any r-coloured N'-dimensional cube over t elements (with $N' \geq N$) contains a monochromatic line. This theorem generalises Van der Waerden's Theorem, which has already been formalised in another AFP entry [2].

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theory *Hales-Jewett*
imports *Main HOL−Library.Disjoint-Sets HOL−Library.FuncSet*
begin

1 Preliminaries

The Hales–Jewett Theorem is at its core a statement about sets of tuples called the n-dimensional cube over t elements (denoted by C_t^n); i.e. the set $\{0,\ldots,t-1\}^n$, where $\{0,\ldots,t-1\}$ is called the base. We represent tuples by functions $f : \{0,\ldots,n-1\} \to \{0,\ldots,t-1\}$ because they're easier to deal with. The set of tuples then becomes the function space $\{0,\ldots,t-1\}^{\{0,\ldots,n-1\}}$. Furthermore, r-colourings of the cube are represented by mappings from the function space to the set $\{0,\ldots,r-1\}$.

1.1 The n-dimensional cube over t elements

Function spaces in Isabelle are supported by the library component FuncSet. In essence, $f \in A \to_E B$ means $a \in A \Longrightarrow f\ a \in B$ and $a \notin A \Longrightarrow f\ a =$ *undefined*

The (canonical) n-dimensional cube over t elements is defined in the following using the variables:

n:	*nat*	dimension
t:	*nat*	number of elements

definition *cube* :: *nat* ⇒ *nat* ⇒ (*nat* ⇒ *nat*) *set*
where *cube n t* ≡ {..<*n*} →$_E$ {..<*t*}

For any function f whose image under a set A is a subset of another set B, there's a unique function g in the function space B^A that equals f everywhere in A. The function g is usually written as $f|_A$ in the mathematical literature.

lemma *PiE-uniqueness*: *f* ' *A* ⊆ *B* ⟹ ∃!*g* ∈ *A*
→$_E$ *B*. ∀ *a*∈*A*. *g a* = *f a*
using *exI*[*of* λ*x*. *x* ∈ *A* →$_E$ *B* ∧ (∀ *a*∈*A*. *x a* = *f a*)
restrict f A] *PiE-ext PiE-iff* **by** *fastforce*

Any prefix of length j of an n-tuple (i.e. element of C_t^n) is a j-tuple (i.e. element of C_t^j).

lemma *cube-restrict*:
assumes *j* < *n*
and *y* ∈ *cube n t*
shows (λ*g* ∈ {..<*j*}. *y g*) ∈ *cube j t* **using** *assms* **unfolding** *cube-def* **by** *force*

Narrowing down the obvious fact $B^A \subseteq C^A$ if $B \subseteq C$ to a specific case for cubes.

lemma *cube-subset*: *cube n t* $\subseteq$ *cube n* (*t* + *1*)
unfolding *cube-def* **using** *PiE-mono*[*of* {..<*n*} $\lambda x.$ {..<*t*} $\lambda x.$ {..<*t*+*1*}]
by *simp*

A simplifying definition for the 0-dimensional cube.

lemma *cube0-alt-def*: *cube 0 t* = {$\lambda x.$ *undefined*}
unfolding *cube-def* **by** *simp*

The cardinality of the n-dimensional over t elements is simply a consequence of the overarching definition of the cardinality of function spaces (over finite sets).

lemma *cube-card*: *card* ({..<*n*::*nat*} $\rightarrow_E$ {..<*t*::*nat*}) = *t* ^ *n*
by (*simp add*: *card-PiE*)

A simplifying definition for the n-dimensional cube over a single element, i.e. the single n-dimensional point (*0*, ..., *0*).

lemma *cube1-alt-def*: *cube n 1* = {$\lambda x \in$ {..<*n*}. *0*} **unfolding** *cube-def* **by** (*simp add*: *lessThan-Suc*)

1.2 Lines

The property of being a line in C_t^n is defined in the following using the variables:

L:	*nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*	line
n:	*nat*	dimension of cube
t:	*nat*	the size of the cube's base

definition *is-line* :: (*nat* $\Rightarrow$ (*nat* $\Rightarrow$ *nat*)) $\Rightarrow$ *nat* $\Rightarrow$
nat $\Rightarrow$ *bool*
where *is-line L n t* $\equiv$ (*L* $\in$ {..<*t*} $\rightarrow_E$ *cube n t* $\wedge$
(($\forall$ *j*<*n*. ($\forall$ *x*<*t*. $\forall$ *y*<*t*. *L x j* = *L y j*) $\vee$ ($\forall$ *s*<*t*. *L s j* = *s*))
$\wedge$ ($\exists$ *j* < *n*. ($\forall$ *s* < *t*. *L s j* = *s*))))

We introduce an elimination rule to relate lines with the more general definition of a subspace (see below).

lemma *is-line-elim-t-1*:
assumes *is-line L n t* **and** *t* = *1*
obtains B_0 B_1
where $B_0 \cup B_1$ = {..<*n*} $\wedge$ $B_0 \cap B_1$ = {} $\wedge$
$B_0 \neq$ {} $\wedge$ ($\forall$ *j* $\in$ B_1. ($\forall$ *x*<*t*. $\forall$ *y*<*t*. *L x j* = *L y*
j)) $\wedge$ ($\forall$ *j* $\in$ B_0. ($\forall$ *s*<*t*. *L s j* = *s*))
proof −
define *B0* **where** *B0* = {..<*n*}
define *B1* **where** *B1* = ({}::*nat set*)
have *B0* $\cup$ *B1* = {..<*n*} **unfolding** *B0-def B1-def* **by** *simp*
moreover have *B0* $\cap$ *B1* = {} **unfolding** *B0-def B1-def* **by** *simp*
moreover have *B0* $\neq$ {} **using** *assms* **unfolding** *B0-def is-line-def* **by** *auto*

moreover have $(\forall j \in B1.\ (\forall x{<}t.\ \forall y{<}t.\ L\ x\ j = L\ y\ j))$ **unfolding** *B1-def* **by** *simp*
moreover have $(\forall j \in B0.\ (\forall s{<}t.\ L\ s\ j = s))$ **using** *assms*(*1*, *2*) *cube1-alt-def*
unfolding *B0-def is-line-def* **by** *auto*
ultimately show *?thesis* **using** *that* **by** *simp*
qed

The next two lemmas are used to simplify proofs by enabling us to use the resulting facts directly. This avoids having to unfold the definition of *is-line* each time.

lemma *line-points-in-cube*:
assumes *is-line L n t*
and $s < t$
shows $L\ s \in cube\ n\ t$
using *assms* **unfolding** *cube-def is-line-def*
by *auto*

lemma *line-points-in-cube-unfolded*:
assumes *is-line L n t*
and $s < t$
and $j < n$
shows $L\ s\ j \in \{..{<}t\}$
using *assms line-points-in-cube* **unfolding** *cube-def* **by** *blast*

The incrementation of all elements of a set is defined in the following using the variables:

n:	*nat*	increment size
S:	*nat set*	set

definition *set-incr* :: *nat* ⇒ *nat set* ⇒ *nat set*
where
set-incr n S ≡ $(\lambda a.\ a + n)$ ‘ *S*

lemma *set-incr-disjnt*:
assumes *disjnt A B*
shows *disjnt* (*set-incr n A*) (*set-incr n B*)
using *assms* **unfolding** *disjnt-def set-incr-def* **by** *force*

lemma *set-incr-disjoint-family*:
assumes *disjoint-family-on B* $\{..k\}$
shows *disjoint-family-on* $(\lambda i.$ *set-incr n* (*B i*)) $\{..k\}$
using *assms set-incr-disjnt* **unfolding** *disjoint-family-on-def* **by** (*meson disjnt-def*)

lemma *set-incr-altdef*: *set-incr n S* = (+) *n* ‘ *S*
by (*auto simp*: *set-incr-def*)

lemma *set-incr-image*:
assumes $(\bigcup i{\in}\{..k\}.\ B\ i) = \{..{<}n\}$

shows $(\bigcup i{\in}\{..k\}.\ \textit{set-incr}\ m\ (B\ i)) = \{m..{<}m{+}n\}$
using *assms* **by** (*simp add*: *set-incr-altdef add.commute flip*: *image-UN atLeast0LessThan*)

Each tuple of dimension $k+1$ can be split into a tuple of dimension 1 (the first entry) and a tuple of dimension k (the remaining entries).

lemma *split-cube*:
assumes $x \in \textit{cube}\ (k{+}1)\ t$
shows $(\lambda y \in \{..{<}1\}.\ x\ y) \in \textit{cube}\ 1\ t$
and $(\lambda y \in \{..{<}k\}.\ x\ (y+1)) \in \textit{cube}\ k\ t$
using *assms* **unfolding** *cube-def* **by** *auto*

1.3 Subspaces

The property of being a k-dimensional subspace of C_t^n is defined in the following using the variables:

S:	$(nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$	the subspace
k:	nat	the dimension of the subspace
n:	nat	the dimension of the cube
t:	nat	the size of the cube's base

definition *is-subspace*
where *is-subspace S k n t* $\equiv (\exists B\ f.$ *disjoint-family-on B* $\{..k\} \wedge \bigcup(B\ `$
$\{..k\}) = \{..{<}n\} \wedge (\{\} \notin B\ `\ \{..{<}k\}) \wedge f \in (B\ k) \to_E \{..{<}t\}$
$\wedge\ S \in (\textit{cube}\ k\ t) \to_E (\textit{cube}\ n\ t) \wedge (\forall\, y \in \textit{cube}\ k\ t.$
$(\forall\, i \in B\ k.\ S\ y\ i = f\ i) \wedge (\forall\, j{<}k.\ \forall\, i \in B\ j.\ (S\ y)\ i = y\ j)))$

A k-dimensional subspace of C_t^n can be thought of as an embedding of the C_t^k into C_t^n, akin to how a k-dimensional vector subspace of $\mathbf{R}^n$ may be thought of as an embedding of $\mathbf{R}^k$ into $\mathbf{R}^n$.

lemma *subspace-inj-on-cube*:
assumes *is-subspace S k n t*
shows *inj-on S* (*cube k t*)
proof
fix *x y*
assume *a*: $x \in \textit{cube}\ k\ t\ y \in \textit{cube}\ k\ t\ S\ x = S\ y$
from *assms* **obtain** *B f* **where** *Bf-props*: *disjoint-family-on B* $\{..k\} \wedge \bigcup(B\ `$
$\{..k\}) =$
$\{..{<}n\} \wedge (\{\} \notin B\ `\ \{..{<}k\}) \wedge f \in (B\ k) \to_E \{..{<}t\} \wedge$
$S \in (\textit{cube}\ k\ t) \to_E (\textit{cube}\ n\ t) \wedge (\forall\, y \in \textit{cube}\ k\ t.$
$(\forall\, i \in B\ k.\ S\ y\ i = f\ i) \wedge (\forall\, j{<}k.\ \forall\, i \in B\ j.\ (S\ y)\ i = y\ j))$
unfolding *is-subspace-def* **by** *auto*
have $\forall\, i{<}k.\ x\ i = y\ i$
proof (*intro allI impI*)
fix *j* **assume** $j < k$
then have $B\ j \neq \{\}$ **using** *Bf-props* **by** *auto*
then obtain *i* **where** *i-prop*: $i \in B\ j$ **by** *blast*
then have $y\ j = S\ y\ i$ **using** *Bf-props a*(2) $\langle j < k \rangle$ **by** *auto*
also have $\ldots = S\ x\ i$ **using** *a* **by** *simp*

also have *... = x j* **using** *Bf-props a(1)* ‹*j* < *k*› *i-prop* **by** *blast*
finally show *x j = y j* **by** *simp*
qed
then show *x = y* **using** *a(1,2)* **unfolding** *cube-def* **by** (*meson PiE-ext lessThan-iff*)
qed

The following is required to handle base cases in the key lemmas.

lemma *dim0-subspace-ex*:
assumes *t > 0*
shows ∃ *S*. *is-subspace S 0 n t*
proof−
define *B* **where** *B* ≡ (λ*x*::*nat*. *undefined*)(*0*:={..<*n*})

have {..<*t*} ≠ {} **using** *assms* **by** *auto*
then have ∃*f*. *f* ∈ (*B 0*) →$_E$ {..<*t*}
by (*meson PiE-eq-empty-iff all-not-in-conv*)
then obtain *f* **where** *f-prop*: *f* ∈ (*B 0*) →$_E$ {..<*t*} **by** *blast*
define *S* **where** *S* ≡ (λ*x*::(*nat* ⇒ *nat*). *undefined*)((λ*x*. *undefined*):=*f*)

have *disjoint-family-on B* {..*0*} **unfolding** *disjoint-family-on-def* **by** *simp*
moreover have ⋃(*B* ‘ {..*0*}) = {..<*n*} **unfolding** *B-def* **by** *simp*
moreover have ({} ∉ *B* ‘ {..<*0*}) **by** *simp*
moreover have *S* ∈ (*cube 0 t*) →$_E$ (*cube n t*)
using *f-prop PiE-I* **unfolding** *B-def cube-def S-def* **by** *auto*
moreover have (∀ *y* ∈ *cube 0 t*. (∀ *i* ∈ *B 0*. *S y i* = *f i*) ∧
(∀ *j*<*0*. ∀ *i* ∈ *B j*. (*S y*) *i* = *y j*)) **unfolding** *cube-def S-def* **by** *force*
ultimately have *is-subspace S 0 n t* **using** *f-prop* **unfolding** *is-subspace-def* **by** *blast*
then show ∃ *S*. *is-subspace S 0 n t* **by** *auto*
qed

1.4 Equivalence classes

Defining the equivalence classes of *cube n* (*t* + *1*): {*classes n t 0*, ..., *classes n t n*}

definition *classes*
where *classes n t* ≡ (λ*i*. {*x* . *x* ∈ (*cube n* (*t* + *1*)) ∧ (∀ *u* ∈
{(*n*−*i*)..<*n*}. *x u* = *t*) ∧ *t* ∉ *x* ‘ {..<(*n* − *i*)}})

lemma *classes-subset-cube*: *classes n t i* ⊆ *cube n* (*t*+*1*) **unfolding** *classes-def* **by** *blast*

definition *layered-subspace*
where *layered-subspace S k n t r* χ ≡ (*is-subspace S k n* (*t* + *1*) ∧ (∀ *i*
∈ {..*k*}. ∃ *c*<*r*. ∀ *x* ∈ *classes k t i*. χ (*S x*) = *c*)) ∧ χ ∈
cube n (*t* + *1*) →$_E$ {..<*r*}

lemma *layered-eq-classes*:

assumes *layered-subspace S k n t r* χ
shows $\forall\, i \in \{..k\}.\ \forall\, x \in$ *classes k t i.* $\forall\, y \in$ *classes k t i.*
$\chi\ (S\ x) = \chi\ (S\ y)$
proof (*safe*)
fix *i x y*
assume *a*: $i \leq k\ x \in$ *classes k t i* $y \in$ *classes k t i*
then obtain *c* **where** $c < r \wedge \chi\ (S\ x) = c \wedge \chi\ (S\ y) = c$ **using** *assms* **unfolding** *layered-subspace-def* **by** *fast*
then show $\chi\ (S\ x) = \chi\ (S\ y)$ **by** *simp*
qed

lemma *dim0-layered-subspace-ex*:
assumes $\chi \in (cube\ n\ (t + 1)) \rightarrow_E \{..<r::nat\}$
shows $\exists\, S.$ *layered-subspace S* (*0*::*nat*) *n t r* χ
proof−
obtain *S* **where** *S-prop*: *is-subspace S* (*0*::*nat*) *n* (*t*+*1*) **using** *dim0-subspace-ex* **by** *auto*
have *classes* (*0*::*nat*) *t 0* = *cube 0* (*t*+*1*) **unfolding** *classes-def* **by** *simp*
moreover have $(\forall\, i \in \{..0::nat\}.\ \exists\, c<r.\ \forall\, x \in$ *classes* (*0*::*nat*) *t i.* $\chi\ (S\ x) = c)$
proof(*safe*)
fix *i*
have $\forall\, x \in$ *classes 0 t 0.* $\chi\ (S\ x) = \chi\ (S\ (\lambda x.$ *undefined*)) **using** *cube0-alt-def*
using ‹*classes 0 t 0* = *cube 0* (*t* + *1*)› **by** *auto*
moreover have *S* ($\lambda x.$ *undefined*) $\in$ *cube n* (*t*+*1*) **using** *S-prop cube0-alt-def*
unfolding *is-subspace-def* **by** *auto*
moreover have $\chi\ (S\ (\lambda x.$ *undefined*)) $< r$ **using** *assms calculation* **by** *auto*
ultimately show $\exists\, c<r.\ \forall\, x \in$ *classes 0 t 0.* $\chi\ (S\ x) = c$ **by** *auto*
qed
ultimately have *layered-subspace S 0 n t r* χ **using** *S-prop assms* **unfolding** *layered-subspace-def* **by** *blast*
then show $\exists\, S.$ *layered-subspace S* (*0*::*nat*) *n t r* χ **by** *auto*
qed

lemma *disjoint-family-onI* [*intro*]:
assumes $\bigwedge m\ n.\ m \in S \Longrightarrow n \in S \Longrightarrow m \neq n$
$\Longrightarrow A\ m \cap A\ n = \{\}$
shows *disjoint-family-on A S*
using *assms* **by** (*auto simp*: *disjoint-family-on-def*)

lemma *fun-ex*: $a \in A \Longrightarrow b \in B \Longrightarrow \exists f \in A$
$\rightarrow_E B.\ f\ a = b$
proof−
assume *assms*: $a \in A\ b \in B$
then obtain *g* **where** *g-def*: $g \in A \rightarrow B \wedge g\ a = b$ **by** *fast*
then have *restrict g A* $\in A \rightarrow_E B \wedge$ (*restrict g A*) *a* = *b* **using** *assms*(*1*) **by** *auto*
then show *?thesis* **by** *blast*
qed

lemma *ex-bij-betw-nat-finite-2*:
assumes *card A = n*
and *n > 0*
shows ∃*f*. *bij-betw f A {..<n}*
using *assms ex-bij-betw-finite-nat[of A] atLeast0LessThan card-ge-0-finite* **by** *auto*

lemma *one-dim-cube-eq-nat-set*: *bij-betw* (λ*f*. *f 0*) (*cube 1 k*) {..<*k*}
proof (*unfold bij-betw-def*)
have ∗: (λ*f*. *f 0*) ' *cube 1 k* = {..<*k*}
proof(*safe*)
fix *x f*
assume *f* ∈ *cube 1 k*
then show *f 0* < *k* **unfolding** *cube-def* **by** *blast*
next
fix *x*
assume *x* < *k*
then have *x* ∈ {..<*k*} **by** *simp*
moreover have *0* ∈ {..<*1*::*nat*} **by** *simp*
ultimately have ∃ *y* ∈ {..<*1*::*nat*} →$_E$ {..<*k*}. *y 0* = *x* **using**
fun-ex[*of 0* {..<*1*::*nat*} *x* {..<*k*}] **by** *auto*
then show *x* ∈ (λ*f*. *f 0*) ' *cube 1 k* **unfolding** *cube-def* **by** *blast*
qed
moreover
{
have *card* (*cube 1 k*) = *k* **using** *cube-card* **by** (*simp add*: *cube-def*)
moreover have *card* {..<*k*} = *k* **by** *simp*
ultimately have *inj-on* (λ*f*. *f 0*) (*cube 1 k*) **using** ∗ *eq-card-imp-inj-on*[*of cube 1 k* λ*f*. *f 0*]
by *force*
}
ultimately show *inj-on* (λ*f*. *f 0*) (*cube 1 k*) ∧ (λ*f*. *f 0*) ' *cube 1 k* = {..<*k*} **by** *simp*
qed

An alternative introduction rule for the ∃!*x* quantifier, which means "there exists exactly one *x*".

lemma *ex1I-alt*: (∃ *x*. *P x* ∧ (∀ *y*. *P y* ⟶ *x* = *y*)) ⟹ (∃ !*x*. *P x*)
by *auto*
lemma *nat-set-eq-one-dim-cube*: *bij-betw* (λ*x*. λ*y*∈{..<*1*::*nat*}. *x*) {..<*k*::*nat*} (*cube 1 k*)
proof (*unfold bij-betw-def*)
have ∗: (λ*x*. λ*y*∈{..<*1*::*nat*}. *x*) ' {..<*k*} = *cube 1 k*
proof (*safe*)
fix *x y*
assume *y* < *k*
then show (λ*z*∈{..<*1*}. *y*) ∈ *cube 1 k* **unfolding** *cube-def* **by** *simp*
next
fix *x*
assume *x* ∈ *cube 1 k*

```
    have x = (λz. λy∈{..<1::nat}. z) (x 0::nat)
    proof
      fix j
      consider j ∈ {..<1} | j ∉ {..<1::nat} by linarith
      then show x j = (λz. λy∈{..<1::nat}. z) (x 0::nat) j using ‹x
      ∈ cube 1 k› unfolding cube-def by auto
    qed
   moreover have x 0 ∈ {..<k} using ‹x ∈ cube 1 k› by (auto simp add: cube-def)
    ultimately show x ∈ (λz. λy∈{..<1}. z) ` {..<k}  by blast
  qed
  moreover
  {
    have card (cube 1 k) = k using cube-card by (simp add: cube-def)
    moreover have card {..<k} = k by simp
    ultimately have  inj-on (λx. λy∈{..<1::nat}. x) {..<k} using ∗
        eq-card-imp-inj-on[of {..<k} λx. λy∈{..<1::nat}. x] by force
  }
  ultimately show inj-on (λx. λy∈{..<1::nat}. x) {..<k} ∧ (λx.
  λy∈{..<1::nat}. x) ` {..<k} = cube 1 k by blast
qed
```

A bijection f between domains A_1 and A_2 creates a correspondence between functions in $A_1 \to B$ and $A_2 \to B$.

```
lemma bij-domain-PiE:
  assumes bij-betw f A1 A2
    and g ∈ A2 →_E B
  shows (restrict (g ∘ f) A1) ∈ A1 →_E B
  using bij-betwE assms by fastforce
```

The following three lemmas relate lines to 1-dimensional subspaces (in the natural way). This is a direct consequence of the elimination rule *is-line-elim* introduced above.

```
lemma line-is-dim1-subspace-t-1:
  assumes n > 0
    and is-line L n 1
  shows is-subspace (restrict (λy. L (y 0)) (cube 1 1)) 1 n 1
proof −
  obtain B_0 B_1 where B-props: B_0 ∪ B_1 = {..<n} ∧ B_0
  ∩ B_1 = {} ∧ B_0 ≠ {} ∧ (∀ j ∈ B_1.
  (∀ x<1. ∀ y<1. L x j = L y j)) ∧ (∀ j ∈ B_0. (∀ s<1. L
  s j = s)) using is-line-elim-t-1[of L n 1] assms by auto
  define B where B ≡ (λi::nat. {}::nat set)(0:=B_0, 1:=B_1)
  define f where f ≡ (λi ∈ B 1. L 0 i)
  have ∗: L 0 ∈ {..<n} →_E {..<1} using assms(2) unfolding cube-def is-line-def
by auto
  have disjoint-family-on B {..1} unfolding B-def using B-props
    by (simp add: Int-commute disjoint-family-onI)
   moreover have ⋃ (B ` {..1}) = {..<n} unfolding B-def using B-props by
auto
```

```
  moreover have {} ∉ B ‘ {..<1} unfolding B-def using B-props by auto
  moreover have  f ∈ B 1 →E {..<1} using ∗ calculation(2) unfolding f-def by
auto
  moreover have (restrict (λy. L (y 0)) (cube 1 1)) ∈ cube 1 1 →E cube n 1
    using assms(2) cube1-alt-def unfolding is-line-def by auto
  moreover have (∀ y∈cube 1 1. (∀ i∈B 1. (restrict (λy. L (y 0)) (cube 1 1)) y i
= f i)
  ∧ (∀ j<1. ∀ i∈B j. (restrict (λy. L (y 0)) (cube 1 1)) y i = y j))
    using cube1-alt-def B-props ∗ unfolding B-def f-def by auto
  ultimately show ?thesis unfolding is-subspace-def by blast
qed

lemma line-is-dim1-subspace-t-ge-1:
  assumes n > 0
    and t > 1
    and is-line L n t
  shows is-subspace (restrict (λy. L (y 0)) (cube 1 t)) 1 n t
proof −
  let ?B1 = {i::nat . i < n ∧ (∀ x<t. ∀ y<t. L x i =  L y i)}
  let ?B0 = {i::nat . i < n ∧ (∀ s < t. L s i = s)}
  define B where B ≡ (λi::nat. {}::nat set)(0:=?B0, 1:=?B1)
  let ?L = (λy ∈ cube 1 t. L (y 0))
  have ?B0 ≠ {} using assms(3) unfolding is-line-def by simp

  have L1: ?B0 ∪ ?B1 = {..<n} using assms(3) unfolding is-line-def by auto
  {
    have (∀ s < t. L s i = s) ⟶ ¬(∀ x<t. ∀ y<t. L x i =
    L y i) if i < n for i using assms(2) less-trans by auto
    then have ∗:i ∉ ?B0 if i ∈ ?B1 for i using that by blast
  }
  moreover
  {
    have (∀ x<t. ∀ y<t. L x i =  L y i) ⟶ ¬(∀ s < t. L s i = s)
      if i < n for i using that calculation by blast
    then have ∗∗: ∀ i ∈ ?B0. i ∉ ?B1
      by blast
  }
  ultimately have L2: ?B0 ∩ ?B1 = {} by blast

  let ?f = (λi. if i ∈ B 1 then L 0 i else undefined)
  {
    have {..1::nat} = {0, 1} by auto
    then have ⋃(B ‘ {..1::nat}) = B 0 ∪ B 1 by simp
    then have ⋃(B ‘ {..1::nat}) = ?B0 ∪ ?B1 unfolding B-def by simp
    then have A1: disjoint-family-on B {..1::nat} using L2
      by (simp add: B-def Int-commute disjoint-family-onI)
  }
  moreover
  {
```

```
  have ⋃(B ` {..1::nat}) = B 0 ∪ B 1 unfolding B-def by auto
  then have ⋃(B ` {..1::nat}) = {..<n} using L1 unfolding B-def by simp
 }
 moreover
 {
  have ∀ i ∈ {..<1::nat}. B i ≠ {}
   using ‹{i. i < n ∧ (∀ s<t. L s i = s)} ≠ {}› fun-upd-same lessThan-iff less-one

    unfolding B-def by auto
  then have {} ∉ B ` {..<1::nat} by blast
 }
 moreover
 {
  have ?f ∈ (B 1) →E {..<t}
  proof
    fix i
    assume asm: i ∈ (B 1)
   have L a b ∈ {..<t} if a < t and b < n for a b using assms(3) that unfolding
is-line-def cube-def by auto
    then have L 0 i ∈ {..<t} using assms(2) asm calculation(2) by blast
    then show ?f i ∈ {..<t} using asm by presburger
  qed (auto)
 }

 moreover
 {
  have L ∈ {..<t} →E (cube n t) using assms(3) by (simp add: is-line-def)
  then have ?L ∈ (cube 1 t) →E (cube n t)
  using bij-domain-PiE[of (λf. f 0) (cube 1 t) {..<t} L cube n t] one-dim-cube-eq-nat-set[of
t]
    by auto
 }
 moreover
 {
  have ∀ y ∈ cube 1 t. (∀ i ∈ B 1. ?L y i = ?f i) ∧ (∀ j < 1.
  ∀ i ∈ B j. (?L y) i = y j)
  proof
    fix y
    assume y ∈ cube 1 t
    then have y 0 ∈ {..<t} unfolding cube-def by blast

    have (∀ i ∈ B 1. ?L y i = ?f i)
    proof
      fix i
      assume i ∈ B 1
      then have ?f i = L 0 i
        by meson
      moreover have ?L y i = L (y 0) i using ‹y ∈ cube 1 t› by simp
      moreover have L (y 0) i = L 0 i
```

```
      proof −
       have i ∈ ?B1 using ‹i ∈ B 1› unfolding B-def fun-upd-def by presburger
        then have (∀ x<t. ∀ y<t. L x i = L y i) by blast
        then show L (y 0) i = L 0 i using ‹y 0 ∈ {..<t}› by blast
      qed
      ultimately show ?L y i = ?f i by simp
    qed

    moreover have (?L y) i = y j if j < 1 and i ∈ B j for i j
    proof−
      have i ∈ B 0 using that by blast
      then have i ∈ ?B0 unfolding B-def by auto
      then have (∀ s < t. L s i = s) by blast
      moreover have y 0 < t using ‹y ∈ cube 1 t› unfolding cube-def by auto
      ultimately have L (y 0) i = y 0 by simp
      then show ?L y i = y j using that using ‹y ∈ cube 1 t› by force
    qed

    ultimately show (∀ i ∈ B 1. ?L y i = ?f i) ∧ (∀ j < 1. ∀ i
    ∈ B j. (?L y) i = y j)
      by blast
  qed
 }
 ultimately show is-subspace ?L 1 n t unfolding is-subspace-def by blast
qed

lemma line-is-dim1-subspace:
 assumes n > 0
   and t > 0
   and is-line L n t
 shows is-subspace (restrict (λy. L (y 0)) (cube 1 t)) 1 n t
 using line-is-dim1-subspace-t-1[of n L] line-is-dim1-subspace-t-ge-1[of n t L] assms
not-less-iff-gr-or-eq by blast
```

The key property of the existence of a minimal dimension N, such that for any r-colouring in $C_t^{N'}$ (for $N' \geq N$) there exists a monochromatic line is defined in the following using the variables:

r:	*nat*	the number of colours
t:	*nat*	the size of of the base

```
definition hj
 where hj r t ≡ (∃ N>0. ∀ N' ≥ N. ∀ χ. χ ∈ (cube N'
 t) →_E {..<r::nat} ⟶ (∃ L. ∃ c<r. is-line L N' t
 ∧ (∀ y ∈ L ` {..<t}. χ y = c)))
```

The key property of the existence of a minimal dimension N, such that for any r-colouring in $C_t^{N'}$ (for $N' \geq N$) there exists a layered subspace of dimension k is defined in the following using the variables:

r: *nat* the number of colours
t: *nat* the size of of the base
k: *nat* the dimension of the subspace

definition *lhj*
where *lhj r t k* ≡ (∃ *N* > *0*. ∀ *N'* ≥ *N*. ∀ χ. χ ∈ (*cube N'* (*t* + *1*)) →$_E$ {..<*r*::*nat*} ⟶ (∃ *S*. *layered-subspace S k N' t r* χ))

We state some useful facts about 1-dimensional subspaces.

lemma *dim1-subspace-elims*:
assumes *disjoint-family-on B* {..*1*::*nat*} **and** ⋃(*B* ' {..*1*::*nat*}) = {..<*n*} **and** ({} ∉ *B* ' {..<*1*::*nat*}) **and** *f* ∈ (*B 1*) →$_E$ {..<*t*} **and** *S* ∈ (*cube 1 t*) →$_E$ (*cube n t*) **and** (∀ *y* ∈ *cube 1 t*. (∀ *i* ∈ *B 1*. *S y i* = *f i*) ∧ (∀ *j*<*1*. ∀ *i* ∈ *B j*. (*S y*) *i* = *y j*))
shows *B 0* ∪ *B 1* = {..<*n*}
and *B 0* ∩ *B 1* = {}
and (∀ *y* ∈ *cube 1 t*. (∀ *i* ∈ *B 1*. *S y i* = *f i*) ∧ (∀ *i* ∈ *B 0*. (*S y*) *i* = *y 0*))
and *B 0* ≠ {}
proof −
have {..*1*} = {*0*::*nat*, *1*} **by** *auto*
then show *B 0* ∪ *B 1* = {..<*n*} **using** *assms*(*2*) **by** *simp*
next
show *B 0* ∩ *B 1* = {} **using** *assms*(*1*) **unfolding** *disjoint-family-on-def* **by** *simp*
next
show (∀ *y* ∈ *cube 1 t*. (∀ *i* ∈ *B 1*. *S y i* = *f i*) ∧ (∀ *i* ∈ *B 0*. (*S y*) *i* = *y 0*))
using *assms*(*6*) **by** *simp*
next
show *B 0* ≠ {} **using** *assms*(*3*) **by** *auto*
qed

We state some properties of cubes.

lemma *cube-props*:
assumes *s* < *t*
shows ∃ *p* ∈ *cube 1 t*. *p 0* = *s*
and (*SOME p*. *p* ∈ *cube 1 t* ∧ *p 0* = *s*) *0* = *s*
and (λ*s*∈{..<*t*}. *S* (*SOME p*. *p*∈*cube 1 t* ∧ *p 0* = *s*)) *s* = (λ*s*∈{..<*t*}. *S* (*SOME p*. *p*∈*cube 1 t* ∧ *p 0* = *s*)) ((*SOME p*. *p* ∈ *cube 1 t* ∧ *p 0* = *s*) *0*)
and (*SOME p*. *p* ∈ *cube 1 t* ∧ *p 0* = *s*) ∈ *cube 1 t*
proof −
show *1*: ∃ *p* ∈ *cube 1 t*. *p 0* = *s* **using** *assms* **unfolding** *cube-def* **by** (*simp add*: *fun-ex*)
show *2*: (*SOME p*. *p* ∈ *cube 1 t* ∧ *p 0* = *s*) *0* = *s* **using** *assms 1 someI-ex*[*of* λ*x*. *x* ∈ *cube 1 t* ∧ *x 0* = *s*] **by** *blast*
show *3*: (λ*s*∈{..<*t*}. *S* (*SOME p*. *p*∈*cube 1 t* ∧ *p 0* = *s*)) *s* = (λ*s*∈{..<*t*}. *S* (*SOME p*. *p*∈*cube 1 t* ∧ *p 0* = *s*)) ((*SOME p*. *p* ∈ *cube 1 t*

```
  ∧ p 0 = s) 0) using 2 by simp
  show 4: (SOME p. p ∈ cube 1 t ∧ p 0 = s) ∈ cube 1 t using 1 someI-ex[of
      λp. p ∈ cube 1 t ∧ p 0 = s] assms by blast
qed
```

The following lemma relates 1-dimensional subspaces to lines, thus establishing a bidirectional correspondence between the two together with *line-is-dim1-subspace*.

```
lemma dim1-subspace-is-line:
  assumes t > 0
    and is-subspace S 1 n t
  shows   is-line (λs∈{..<t}. S (SOME p. p∈cube 1 t ∧ p 0 = s)) n t
proof–
  define L where L ≡ (λs∈{..<t}. S (SOME p. p∈cube 1 t ∧ p 0 = s))
  have {..1} = {0::nat, 1} by auto
  obtain B f where Bf-props: disjoint-family-on B {..1::nat} ∧ ⋃(B ' {..1::nat})
=
  {..<n} ∧ ({} ∉ B ' {..<1::nat}) ∧ f ∈ (B 1) →E {..<t}
  ∧ S ∈ (cube 1 t) →E (cube n t) ∧ (∀ y ∈ cube 1 t.
  (∀ i ∈ B 1. S y i = f i) ∧ (∀ j<1. ∀ i ∈ B j. (S y) i = y j))
    using assms(2) unfolding is-subspace-def by auto
 then have 1: B 0 ∪ B 1 = {..<n} ∧ B 0 ∩ B 1 = {} using dim1-subspace-elims(1,
      2)[of B n f t S] by simp

  have L ∈ {..<t} →E cube n t
  proof
    fix s assume a: s ∈ {..<t}
   then have L s = S (SOME p. p∈cube 1 t ∧ p 0 = s) unfolding L-def by simp
   moreover have (SOME p. p∈cube 1 t ∧ p 0 = s) ∈ cube 1 t using cube-props(1)
a
       someI-ex[of λp. p ∈ cube 1 t ∧ p 0 = s] by blast
    moreover have S (SOME p. p∈cube 1 t ∧ p 0 = s) ∈ cube n t
      using assms(2) calculation(2) is-subspace-def by auto
    ultimately show L s ∈ cube n t by simp
  next
    fix s assume a: s ∉ {..<t}
    then show L s = undefined unfolding L-def by simp
  qed
  moreover have (∀ x<t. ∀ y<t. L x j = L y j) ∨ (∀ s<t. L s j = s) if j < n for j
  proof–
    consider j ∈ B 0 | j ∈ B 1 using ‹j < n› 1 by blast
    then show (∀ x<t. ∀ y<t. L x j = L y j) ∨ (∀ s<t. L s j = s)
    proof (cases)
      case 1
      have L s j = s if s < t for s
      proof–
        have ∀ y ∈ cube 1 t. (S y) j = y 0 using Bf-props 1 by simp
       then show L s j = s using that cube-props(2,4)  unfolding L-def by auto
      qed
      then show ?thesis by blast
```

```
    next
      case 2
      have L x j = L y j if x < t and y < t for x y
      proof−
        have ∗: S y j = f j if y ∈ cube 1 t for y using 2 that Bf-props by simp
      then have L y j = f j using that(2) cube-props(2,4) lessThan-iff restrict-apply
unfolding L-def by fastforce
      moreover from ∗ have L x j = f j using that(1) cube-props(2,4) lessThan-iff
restrict-apply unfolding L-def
          by fastforce
        ultimately show L x j = L y j by simp
      qed
      then show ?thesis by blast
    qed
  qed
  moreover have (∃ j<n. ∀ s<t. (L s j = s))
  proof −
    obtain j where j-prop: j ∈ B 0 ∧ j < n using Bf-props by blast
    then have (S y) j = y 0 if y ∈ cube 1 t for y using that Bf-props by auto
   then have L s j = s if s < t for s using that cube-props(2,4) unfolding L-def
by auto
    then show ∃ j<n. ∀ s<t. (L s j = s) using j-prop by blast
  qed
  ultimately show is-line (λs∈{..<t}. S (SOME p. p∈cube 1 t ∧ p 0 = s)) n t
    unfolding L-def is-line-def by auto
qed

lemma bij-unique-inv:
  assumes bij-betw f A B
    and x ∈ B
  shows ∃!y ∈ A. (the-inv-into A f) x = y
  using assms unfolding bij-betw-def inj-on-def the-inv-into-def
  by blast

lemma inv-into-cube-props:
  assumes s < t
  shows the-inv-into (cube 1 t) (λf. f 0) s ∈ cube 1 t
    and the-inv-into (cube 1 t) (λf. f 0) s 0 = s
  using assms bij-unique-inv one-dim-cube-eq-nat-set f-the-inv-into-f-bij-betw
  by fastforce+

lemma some-inv-into:
  assumes s < t
  shows (SOME p. p∈cube 1 t ∧ p 0 = s) = (the-inv-into (cube 1 t) (λf. f 0) s)
  using inv-into-cube-props[of s t] one-dim-cube-eq-nat-set[of t] assms unfolding
bij-betw-def inj-on-def by auto

lemma some-inv-into-2:
  assumes s < t
```

shows (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) = (*the-inv-into* (*cube 1 t*) (*λf. f 0*) *s*)
proof−
have ∗: (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) ∈ *cube 1* (*t+1*) **using** *cube-props assms* **by** *simp*
then have (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) *0* = *s* **using** *cube-props assms* **by** *simp*
moreover
{
have (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) ' {..<*1*} ⊆ {..<*t*} **using** *calculation assms* **by** *force*
then have (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) ∈ *cube 1 t* **using** ∗ **unfolding** *cube-def* **by** *auto*
}
moreover have *inj-on* (*λf. f 0*) (*cube 1 t*) **using** *one-dim-cube-eq-nat-set*[*of t*]
unfolding *bij-betw-def inj-on-def* **by** *auto*
ultimately show (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) = (*the-inv-into* (*cube 1 t*) (*λf. f 0*) *s*)
using *the-inv-into-f-eq* [*of λf. f 0 cube 1 t* (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*) *s*] **by** *auto*
qed

lemma *dim1-layered-subspace-as-line*:
assumes *t* > *0*
and *layered-subspace S 1 n t r χ*
shows ∃ *c1 c2. c1*<*r* ∧ *c2*<*r* ∧ (∀ *s*<*t. χ* (*S* (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*)) = *c1*) ∧ *χ* (*S* (*SOME p. p∈cube 1 (t+1) ∧ p 0 = t*)) = *c2*
proof −
have *x u* < *t* **if** *x* ∈ *classes 1 t 0* **and** *u* < *1* **for** *x u*
proof −
have *x* ∈ *cube 1* (*t+1*) **using** *that* **unfolding** *classes-def* **by** *blast*
then have *x u* ∈ {..<*t+1*} **using** *that* **unfolding** *cube-def* **by** *blast*
then have *x u* ∈ {..<*t*} **using** *that*
using *that less-Suc-eq* **unfolding** *classes-def* **by** *auto*
then show *x u* < *t* **by** *simp*
qed
then have *classes 1 t 0* ⊆ *cube 1 t* **unfolding** *cube-def classes-def* **by** *auto*
moreover have *cube 1 t* ⊆ *classes 1 t 0* **using** *cube-subset*[*of 1 t*] **unfolding** *cube-def classes-def* **by** *auto*
ultimately have *X*: *classes 1 t 0* = *cube 1 t* **by** *blast*

obtain *c1* **where** *c1-prop*: *c1* < *r* ∧ (∀ *x*∈*classes 1 t 0. χ* (*S x*) = *c1*) **using** *assms*(*2*)
unfolding *layered-subspace-def* **by** *blast*
then have (*χ* (*S x*) = *c1*) **if** *x* ∈ *cube 1 t* **for** *x* **using** *X that* **by** *blast*
then have *χ* (*S* (*the-inv-into* (*cube 1 t*) (*λf. f 0*) *s*)) = *c1* **if** *s* < *t* **for** *s*
using *one-dim-cube-eq-nat-set*[*of t*] **by** (*meson that bij-betwE bij-betw-the-inv-into lessThan-iff*)
then have *K1*: *χ* (*S* (*SOME p. p∈cube 1 (t+1) ∧ p 0 = s*)) = *c1* **if** *s* < *t* **for** *s*

```
    using that some-inv-into-2 by simp

  have ∗: ∃ c<r. ∀ x ∈ classes 1 t 1. χ (S x) = c
    using assms(2) unfolding layered-subspace-def by blast

  have x 0 = t if x ∈ classes 1 t 1 for x using that unfolding classes-def by
simp
  moreover have ∃!x ∈ cube 1 (t+1). x 0 = t using one-dim-cube-eq-nat-set[of
t+1]
   unfolding bij-betw-def inj-on-def using inv-into-cube-props(1) inv-into-cube-props(2)
by force
  moreover have ∗∗: ∃!x. x ∈ classes 1 t 1 unfolding classes-def using calcu-
lation(2) by simp
  ultimately have the-inv-into (cube 1 (t+1)) (λf. f 0) t ∈ classes 1 t 1
    using inv-into-cube-props[of t t+1] unfolding classes-def by simp

  then have ∃ c2. c2 < r ∧ χ (S (the-inv-into (cube 1 (t+1)) (λf. f 0) t)) = c2
    using ∗ ∗∗ by blast
  then have K2: ∃ c2. c2 < r ∧ χ (S (SOME p. p∈cube 1 (t+1) ∧ p 0 = t)) = c2
    using some-inv-into by simp

  from K1 K2 show ?thesis
    using c1-prop by blast
qed

lemma dim1-layered-subspace-mono-line:
  assumes t > 0
    and layered-subspace S 1 n t r χ
  shows ∀ s<t. ∀ l<t. χ (S (SOME p. p∈cube 1 (t+1) ∧ p 0 = s)) =
  χ (S (SOME p. p∈cube 1 (t+1) ∧ p 0 = l)) ∧ χ (S (SOME p. p∈cube 1
  (t+1) ∧ p 0 = s)) < r
  using dim1-layered-subspace-as-line[of t S n r χ] assms by auto

definition join :: (nat ⇒ 'a) ⇒ (nat ⇒ 'a) ⇒ nat
⇒ nat ⇒ (nat ⇒ 'a)
  where
    join f g n m ≡ (λx. if x ∈ {..<n} then f x else (if x ∈ {n..<n+m} then g
    (x − n) else undefined))

lemma join-cubes:
  assumes f ∈ cube n (t+1)
    and g ∈ cube m (t+1)
  shows join f g n m ∈ cube (n+m) (t+1)
proof (unfold cube-def; intro PiE-I)
  fix i
  assume i ∈ {..<n+m}
  then consider i < n | i ≥ n ∧ i < n+m by fastforce
  then show join f g n m i ∈ {..<t + 1}
  proof (cases)
```

```
    case 1
    then have join f g n m i = f i unfolding join-def by simp
   moreover have f i ∈ {..<t+1} using assms(1) 1 unfolding cube-def by blast
    ultimately show ?thesis by simp
  next
    case 2
    then have join f g n m i = g (i − n) unfolding join-def by simp
    moreover have i − n ∈ {..<m} using 2 by auto
   moreover have g (i − n) ∈ {..<t+1} using calculation(2) assms(2) unfolding
cube-def by blast
    ultimately show ?thesis by simp
  qed
next
  fix i
  assume i ∉ {..<n+m}
  then show join f g n m i = undefined unfolding join-def by simp
qed

lemma subspace-elems-embed:
  assumes is-subspace S k n t
  shows S ‘ (cube k t) ⊆ cube n t
  using assms unfolding cube-def is-subspace-def by blast
```

2 Core proofs

The numbering of the theorems has been borrowed from the textbook [1].

2.1 Theorem 4

2.1.1 Base case of Theorem 4

```
lemma hj-imp-lhj-base:
  fixes r t
  assumes t > 0
    and ⋀r'. hj r' t
  shows lhj r t 1
proof−
  from assms(2) obtain N where N-def: N > 0 ∧ (∀ N' ≥ N. ∀ χ. χ
  ∈ (cube N' t) →E {..<r::nat} ⟶ (∃ L. ∃ c<r.
  is-line L N' t ∧ (∀ y ∈ L ‘ {..<t}. χ y = c))) unfolding hj-def by blast

  have (∃ S. is-subspace S 1 N' (t + 1) ∧ (∀ i ∈ {..1}. ∃ c < r.
  (∀ x ∈ classes 1 t i. χ (S x) = c))) if asm: N' ≥ N χ ∈ (cube N'
  (t + 1)) →E {..<r::nat} for N' χ
  proof−
    have N'-props: N' > 0 ∧ (∀ χ. χ ∈ (cube N' t) →E
    {..<r::nat} ⟶ (∃ L. ∃ c<r. is-line L N' t ∧ (∀ y ∈
    L ‘ {..<t}. χ y = c))) using asm N-def by simp
```

let *?chi-t* = $\lambda x \in$ *cube* N' *t*. χ *x*
have *?chi-t* $\in$ *cube* N' *t* $\rightarrow_E$ {..<*r*::*nat*} **using** *cube-subset asm* **by** *auto*
then obtain *L* **where** *L-def*: *is-line L N' t* $\wedge$ ($\exists$ *c*<*r*. ($\forall$ *y* $\in$ *L* ' {..<*t*}. *?chi-t y* = *c*))
using *N'-props* **by** *blast*

have *is-subspace* (*restrict* (λ*y*. *L* (*y* 0)) (*cube* 1 *t*)) 1 *N' t* **using** *line-is-dim1-subspace N'-props L-def*
using *assms*(1) **by** *auto*
then obtain *B f* **where** *Bf-defs*: *disjoint-family-on B* {..1} $\wedge$ $\bigcup$(*B* ' {..1}) = {..<*N'*}
$\wedge$ ({} $\notin$ *B* ' {..<1}) $\wedge$ *f* $\in$ (*B* 1) $\rightarrow_E$ {..<*t*} $\wedge$
(*restrict* (λ*y*. *L* (*y* 0)) (*cube* 1 *t*)) $\in$ (*cube* 1 *t*) $\rightarrow_E$ (*cube N' t*)
$\wedge$ ($\forall$ *y* $\in$ *cube* 1 *t*. ($\forall$ *i* $\in$ *B* 1. (*restrict* (λ*y*. *L* (*y* 0)) (*cube* 1 *t*)) *y i* = *f i*) $\wedge$ ($\forall$ *j*<1. $\forall$ *i* $\in$ *B j*. ((*restrict* (λ*y*. *L* (*y* 0)) (*cube* 1 *t*)) *y*) *i* = *y j*)) **unfolding** *is-subspace-def* **by** *auto*

have {..1::*nat*} = {0, 1} **by** *auto*
then have *B-props*: *B* 0 $\cup$ *B* 1 = {..<*N'*} $\wedge$ (*B* 0 $\cap$ *B* 1 = {})
using *Bf-defs* **unfolding** *disjoint-family-on-def* **by** *auto*
define *L'* **where** *L'* $\equiv$ *L*(*t*:=(λ*j*. *if j* $\in$ *B* 1 *then L* (*t* − 1) *j else* (*if j* $\in$ *B* 0 *then t else undefined*)))

S1 is the corresponding 1-dimensional subspace of L'.

define *S1* **where** *S1* $\equiv$ *restrict* (λ*y*. *L'* (*y* (0::*nat*))) (*cube* 1 (*t*+1))
have *line-prop*: *is-line L' N'* (*t* + 1)
proof−
have *A1*: *L'* $\in$ {..<*t*+1} $\rightarrow_E$ *cube N'* (*t* + 1)
proof
fix *x*
assume *asm*: *x* $\in$ {..<*t* + 1}
then show *L' x* $\in$ *cube N'* (*t* + 1)
proof (*cases x* < *t*)
case *True*
then have *L' x* = *L x* **by** (*simp add*: *L'-def*)
then have *L' x* $\in$ *cube N' t* **using** *L-def True* **unfolding** *is-line-def* **by** *auto*
then show *L' x* $\in$ *cube N'* (*t* + 1) **using** *cube-subset* **by** *blast*
next
case *False*
then have *x* = *t* **using** *asm* **by** *simp*
show *L' x* $\in$ *cube N'* (*t* + 1)
proof(*unfold cube-def*, *intro PiE-I*)
fix *j*
assume *j* $\in$ {..<*N'*}
have *j* $\in$ *B* 1 $\vee$ *j* $\in$ *B* 0 $\vee$ *j* $\notin$ (*B* 0 $\cup$ *B* 1) **by** *blast*
then show *L' x j* $\in$ {..<*t* + 1}
proof (*elim disjE*)
assume *j* $\in$ *B* 1

```
            then have L' x j = L (t − 1) j
              by (simp add: ‹x = t› L'-def)
            have L (t − 1) ∈ cube N' t using line-points-in-cube L-def
              by (meson assms(1) diff-less less-numeral-extra(1))
             then have L (t − 1) j < t using ‹j ∈ {..<N'}› unfolding cube-def
by auto
            then show L' x j ∈ {..<t + 1} using ‹L' x j = L (t − 1) j› by simp
          next
            assume j ∈ B 0
           then have j ∉ B 1 using Bf-defs unfolding disjoint-family-on-def by
auto
            then have L' x j = t  by (simp add: ‹j ∈ B 0› ‹x = t› L'-def)
            then show L' x j ∈ {..<t + 1} by simp
          next
            assume a: j ∉ (B 0 ∪ B 1)
            have {..1::nat} = {0, 1} by auto
            then have B 0 ∪ B 1 = (⋃(B ' {..1::nat})) by simp
         then have B 0 ∪ B 1 = {..<N'} using Bf-defs unfolding partition-on-def
by simp
            then have ¬(j ∈ {..<N'}) using a by simp
            then have False using ‹j ∈ {..<N'}› by simp
            then show ?thesis by simp
          qed
         next
          fix j
          assume j ∉ {..<N'}
         then have j ∉ (B 0) ∧ j ∉ B 1 using Bf-defs unfolding partition-on-def
by auto
          then show L' x j = undefined using ‹x = t› by (simp add: L'-def)
         qed
        qed
       next
        fix x
        assume asm: x ∉ {..<t+1}
        then have x ∉ {..<t} ∧ x ≠ t by simp
        then show L' x = undefined using L-def unfolding L'-def is-line-def by
auto
      qed
      have A2: (∃j<N'. (∀ s < (t + 1). L' s j = s))
      proof (cases t = 1)
        case True
        obtain j where j-prop: j ∈ B 0 ∧ j < N' using Bf-defs by blast
        then have L' s j = L s j if s < t for s using that by (auto simp: L'-def)
         moreover have L s j = 0 if s < t for s  using that True L-def j-prop
line-points-in-cube-unfolded[of L N' t]
          by simp
         moreover have L' s j = s if s < t for s using True calculation that by
simp
        moreover have L' t j = t using j-prop B-props by (auto simp: L'-def)
```

```
      ultimately show ?thesis unfolding L'-def using j-prop by auto
    next
      case False
      then show ?thesis
      proof–
       have (∃ j<N'. (∀ s < t. L' s j = s)) using L-def unfolding is-line-def by
(auto simp: L'-def)
        then obtain j where j-def: j < N' ∧ (∀ s < t. L' s j = s) by blast
        have j ∉ B 1
        proof
          assume a:j ∈ B 1
           then have (restrict (λy. L (y 0)) (cube 1 t)) y j = f j if y ∈ cube 1 t
for y
            using Bf-defs that by simp
          then have L (y 0) j = f j if y ∈ cube 1 t for y using that by simp
          moreover have ∃!i. i < t ∧ y 0 = i if y ∈ cube 1 t for y
          using that one-dim-cube-eq-nat-set[of t] unfolding bij-betw-def by blast
          moreover have ∃!y. y ∈ cube 1 t ∧ y 0 = i if i < t for i
          proof (intro ex1I-alt)
            define y where y ≡ (λx::nat. λy∈{..<1::nat}. x)
            have y i ∈ (cube 1 t) using that unfolding cube-def y-def by simp
            moreover have y i 0 = i unfolding y-def by simp
            moreover have z = y i if z ∈ cube 1 t and z 0 = i for z
            proof (rule ccontr)
              assume z ≠ y i
              then obtain l where l-prop: z l ≠ y i l by blast
              consider l ∈ {..<1::nat} | l ∉ {..<1::nat} by blast
              then show False
              proof cases
                case 1
                then show ?thesis using l-prop that(2) unfolding y-def by auto
              next
                case 2
               then have z l = undefined using that unfolding cube-def by blast
               moreover have y i l = undefined unfolding y-def using 2 by auto
                ultimately show ?thesis using l-prop by presburger
              qed
            qed
            ultimately show ∃ y. (y ∈ cube 1 t ∧ y 0 = i) ∧ (∀ ya. ya
            ∈ cube 1 t ∧ ya 0 = i ⟶ y = ya) by blast
          qed

          moreover have L i j = f j if i < t for i using that calculation by blast
          moreover have (∃ j<N'. (∀ s < t. L s j = s)) using
              ‹(∃ j<N'. (∀ s < t. L' s j = s))› by (auto simp: L'-def)
          ultimately show False using False
           by (metis (no-types, lifting) L'-def assms(1) fun-upd-apply j-def less-one
nat-neq-iff)
        qed
```

```
        then have j ∈ B 0 using ‹j ∉ B 1› j-def B-props by auto

        then have L' t j = t using ‹j ∉ B 1› by (auto simp: L'-def)
        then have L' s j = s if s < t + 1 for s using j-def that by (auto simp:
L'-def)
        then show ?thesis using j-def by blast
       qed
      qed
      have A3: (∀ x<t+1. ∀ y<t+1. L' x j =  L' y j) ∨ (∀ s<t+1. L' s j = s) if j
< N' for j
      proof−
       consider j ∈ B 1 | j ∈ B 0 using ‹j < N'› B-props by auto
       then show (∀ x<t+1. ∀ y<t+1. L' x j =  L' y j) ∨ (∀ s<t+1. L' s j = s)
       proof (cases)
        case 1
       then have (restrict (λy. L (y 0)) (cube 1 t)) y j = f j if y ∈ cube 1 t for y
         using that Bf-defs by simp
        moreover have ∃!i. i < t ∧ y 0 = i if y ∈ cube 1 t for y
         using that one-dim-cube-eq-nat-set[of t] unfolding bij-betw-def by blast
        moreover have ∃!y. y ∈ cube 1 t ∧ y 0 = i if i < t for i
        proof (intro ex1I-alt)
         define y where y ≡ (λx::nat. λy∈{..<1::nat}. x)
         have y i ∈ (cube 1 t) using that unfolding cube-def y-def by simp
         moreover have y i 0 = i unfolding y-def by auto
         moreover have z = y i if z ∈ cube 1 t and z 0 = i for z
         proof (rule ccontr)
          assume z ≠ y i
          then obtain l where l-prop: z l ≠ y i l by blast
          consider l ∈ {..<1::nat} | l ∉ {..<1::nat} by blast
          then show False
          proof cases
           case 1
           then show ?thesis using l-prop that(2) unfolding y-def by auto
          next
           case 2
           then have z l = undefined using that unfolding cube-def by blast
           moreover have y i l = undefined unfolding y-def using 2 by auto
           ultimately show ?thesis using l-prop by presburger
          qed
         qed
         ultimately show ∃ y. (y ∈ cube 1 t ∧ y 0 = i) ∧ (∀ ya. ya
         ∈ cube 1 t ∧ ya 0 = i ⟶ y = ya) by blast

        qed
        moreover have L i j = f j if i < t for i using calculation that by force
       moreover have  L i j = L x j if x < t i < t for x i using that calculation
by simp
        moreover have L' x j = L x j if x < t for x using that fun-upd-other[of
x t L
```

```
      λj. if j ∈ B 1 then L (t − 1) j else if j ∈ B 0 then t else undefined]
    unfolding L'-def by simp
   ultimately have ∗: L' x j = L' y j if x < t y < t for x y using that by
presburger

   have L' t j = L' (t − 1) j using ‹j ∈ B 1› by (auto simp: L'-def)
  also have ... = L' x j if x < t for x using ∗ by (simp add: assms(1) that)
   finally have ∗∗: L' t j = L' x j if x < t for x using that by auto
   have L' x j = L' y j if x < t + 1 y < t + 1 for x y
   proof−
    consider x < t ∧ y = t | y < t ∧ x = t | x = t ∧ y = t | x < t ∧ y < t
      using ‹x < t + 1› ‹y < t + 1› by linarith
    then show L' x j = L' y j
    proof cases
      case 1
      then show ?thesis using ∗∗ by auto
    next
      case 2
      then show ?thesis using ∗∗ by auto
    next
      case 3
      then show ?thesis by simp
    next
      case 4
      then show ?thesis using ∗ by auto
    qed
   qed
   then show ?thesis by blast
  next
   case 2
   then have ∀ y ∈ cube 1 t. ((restrict (λy. L (y 0)) (cube 1 t)) y) j = y 0
    using ‹j ∈ B 0› Bf-defs by auto
   then have ∀ y ∈ cube 1 t. L (y 0) j = y 0  by auto
   moreover have ∃!y. y ∈ cube 1 t ∧ y 0 = i if i < t for i
   proof (intro ex1I-alt)
    define y where y ≡ (λx::nat. λy∈{..<1::nat}. x)
    have y i ∈ (cube 1 t) using that unfolding cube-def y-def by simp
    moreover have y i 0 = i unfolding y-def by auto
    moreover have z = y i if z ∈ cube 1 t and z 0 = i for z
    proof (rule ccontr)
     assume z ≠ y i
     then obtain l where l-prop: z l ≠ y i l by blast
     consider l ∈ {..<1::nat} | l ∉ {..<1::nat} by blast
     then show False
     proof cases
      case 1
      then show ?thesis using l-prop that(2) unfolding y-def by auto
     next
      case 2
```

then have $z\ l = \mathit{undefined}$ **using** *that* **unfolding** *cube-def* **by** *blast*
moreover have $y\ i\ l = \mathit{undefined}$ **unfolding** *y-def* **using** *2* **by** *auto*
ultimately show *?thesis* **using** *l-prop* **by** *presburger*
qed
qed
ultimately show $\exists\, y.\ (y \in \mathit{cube}\ 1\ t \wedge y\ 0 = i) \wedge (\forall\, ya.\ ya \in \mathit{cube}\ 1\ t \wedge ya\ 0 = i \longrightarrow y = ya)$ **by** *blast*

qed
ultimately have $L\ s\ j = s$ **if** $s < t$ **for** s **using** *that* **by** *blast*
then have $L'\ s\ j = s$ **if** $s < t$ **for** s **using** *that* **by** (*auto simp*: *L′-def*)
moreover have $L'\ t\ j = t$ **using** *2 B-props* **by** (*auto simp*: *L′-def*)
ultimately have $L'\ s\ j = s$ **if** $s < t{+}1$ **for** s **using** *that* **by** (*auto simp*: *L′-def*)
then show *?thesis* **by** *blast*
qed
qed
from *A1 A2 A3* **show** *?thesis* **unfolding** *is-line-def* **by** *simp*
qed
then have *F1*: *is-subspace S1 1* $N'\ (t + 1)$ **unfolding** *S1-def*
using *line-is-dim1-subspace*[*of* $N'\ t{+}1$] *N′-props assms*(*1*) **by** *force*
moreover have *F2*: $\exists\, c < r.\ (\forall\, x \in \mathit{classes}\ 1\ t\ i.\ \chi\ (S1\ x) = c)$ **if** $i \leq 1$ **for** i
proof–
have $\exists\, c < r.\ (\forall\, y \in L'\ `\ \{..{<}t\}.\ \mathit{?chi\text{-}t}\ y = c)$ **unfolding** *L′-def* **using** *L-def* **by** *fastforce*
have $\forall\, x \in (L\ `\ \{..{<}t\}).\ x \in \mathit{cube}\ N'\ t$ **using** *L-def*
using *line-points-in-cube* **by** *blast*
then have $\forall\, x \in (L'\ `\ \{..{<}t\}).\ x \in \mathit{cube}\ N'\ t$ **by** (*auto simp*: *L′-def*)
then have $*{:}\forall\, x \in (L'\ `\ \{..{<}t\}).\ \chi\ x = \mathit{?chi\text{-}t}\ x$ **by** *simp*
then have $\mathit{?chi\text{-}t}\ `\ (L'\ `\ \{..{<}t\}) = \chi\ `\ (L'\ `\ \{..{<}t\})$ **by** *force*
then have $\exists\, c < r.\ (\forall\, y \in L'\ `\ \{..{<}t\}.\ \chi\ y = c)$ **using**
‹$\exists\, c < r.\ (\forall\, y \in L'\ `\ \{..{<}t\}.\ \mathit{?chi\text{-}t}\ y = c)$› **by** *fastforce*
then obtain *linecol* **where** *lc-def*: $\mathit{linecol} < r \wedge (\forall\, y \in L'\ `\ \{..{<}t\}.\ \chi\ y = \mathit{linecol})$ **by** *blast*
consider $i = 0 \mid i = 1$ **using** ‹$i \leq 1$› **by** *linarith*
then show $\exists\, c < r.\ (\forall\, x \in \mathit{classes}\ 1\ t\ i.\ \chi\ (S1\ x) = c)$
proof (*cases*)
case *1*
assume $i = 0$
have $*{:}\ \forall\, a\ t.\ a \in \{..{<}t{+}1\} \wedge a \neq t \longleftrightarrow a \in \{..{<}(t{::}\mathit{nat})\}$ **by** *auto*
from ‹$i = 0$› **have** $\mathit{classes}\ 1\ t\ 0 = \{x\ .\ x \in (\mathit{cube}\ 1\ (t + 1)) \wedge (\forall\, u \in \{((1{::}\mathit{nat}) - 0)..{<}1\}.\ x\ u = t) \wedge t \notin x\ `\ \{..{<}(1 - (0{::}\mathit{nat}))\}\}$
using *classes-def* **by** *simp*
also have $... = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge t \notin x\ `\ \{..{<}(1{::}\mathit{nat})\}\}$ **by** *simp*
also have $... = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge (x\ 0 \neq t)\}$ **by** *blast*
also have $... = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge (x\ 0 \in \{..{<}t{+}1\} \wedge x\ 0 \neq t)\}$
unfolding *cube-def* **by** *blast*
also have $... = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge (x\ 0 \in \{..{<}t\})\}$ **using** $*$ **by** *simp*
finally have *redef*: $\mathit{classes}\ 1\ t\ 0 = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge (x\ 0 \in \{..{<}t\})\}$

by *simp*
have $\{x\ 0 \mid x\ .\ x \in \mathit{classes}\ 1\ t\ 0\} \subseteq \{..<t\}$ **using** *redef* **by** *auto*
moreover have $\{..<t\} \subseteq \{x\ 0 \mid x\ .\ x \in \mathit{classes}\ 1\ t\ 0\}$
proof
fix x **assume** x: $x \in \{..<t\}$
hence $\exists\, a{\in}\mathit{cube}\ 1\ t.\ a\ 0 = x$
unfolding *cube-def* **by** (*intro fun-ex*) *auto*
then show $x \in \{x\ 0 \mid x.\ x \in \mathit{classes}\ 1\ t\ 0\}$
using x *cube-subset* **unfolding** *redef* **by** *auto*
qed
ultimately have $**$: $\{x\ 0 \mid x\ .\ x \in \mathit{classes}\ 1\ t\ 0\} = \{..<t\}$ **by** *blast*

have $\chi\ (S1\ x) = \mathit{linecol}$ **if** $x \in \mathit{classes}\ 1\ t\ 0$ **for** x
proof−
have $x \in \mathit{cube}\ 1\ (t{+}1)$ **unfolding** *classes-def* **using** *that redef* **by** *blast*
then have $S1\ x = L'\ (x\ 0)$ **unfolding** *S1-def* **by** *simp*
moreover have $x\ 0 \in \{..<t\}$ **using** $**$ **using** ‹$x \in \mathit{classes}\ 1\ t\ 0$› **by** *blast*
ultimately show $\chi\ (S1\ x) = \mathit{linecol}$ **using** *lc-def* **using** *fun-upd-triv image-eqI* **by** *blast*
qed
then show *?thesis* **using** *lc-def* ‹$i = 0$› **by** *auto*
next
case *2*
assume $i = 1$
have $\mathit{classes}\ 1\ t\ 1 = \{x\ .\ x \in (\mathit{cube}\ 1\ (t + 1)) \wedge (\forall\, u \in \{0{::}\mathit{nat}..<1\}.\ x\ u = t) \wedge t \notin x\ `\ \{..<0\}\}$ **unfolding** *classes-def* **by** *simp*
also have $\ldots = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge (\forall\, u \in \{0\}.\ x\ u = t)\}$ **by** *simp*
finally have *redef*: $\mathit{classes}\ 1\ t\ 1 = \{x\ .\ x \in \mathit{cube}\ 1\ (t{+}1) \wedge (x\ 0 = t)\}$ **by** *auto*
have $\forall\, s \in \{..<t{+}1\}.\ \exists!x \in \mathit{cube}\ 1\ (t{+}1).\ (\lambda p.\ \lambda y{\in}\{..<1{::}\mathit{nat}\}.\ p)\ s = x$ **using** *nat-set-eq-one-dim-cube*[*of* $t{+}1$]
unfolding *bij-betw-def* **by** *blast*
then have $\exists!x \in \mathit{cube}\ 1\ (t{+}1).\ (\lambda p.\ \lambda y{\in}\{..<1{::}\mathit{nat}\}.\ p)\ t = x$ **by** *auto*
then obtain x **where** *x-prop*: $x \in \mathit{cube}\ 1\ (t{+}1)$ **and** $(\lambda p.\ \lambda y{\in}\{..<1{::}\mathit{nat}\}.\ p)\ t = x$ **and** $\forall\, z \in \mathit{cube}\ 1\ (t{+}1).\ (\lambda p.\ \lambda y{\in}\{..<1{::}\mathit{nat}\}.\ p)\ t = z \longrightarrow z = x$ **by** *blast*
then have $(\lambda p.\ \lambda y{\in}\{0\}.\ p)\ \ t\ = x \wedge (\forall\, z \in \mathit{cube}\ 1\ (t{+}1).\ (\lambda p.\ \lambda y{\in}\{0\}.\ p)\ t = z \longrightarrow z = x)$ **by** *force*
then have $*$:$((\lambda p.\ \lambda y{\in}\{0\}.\ p)\ t)\ 0\ = x\ 0 \wedge (\forall\, z \in \mathit{cube}\ 1\ (t{+}1).\ (\lambda p.\ \lambda y{\in}\{0\}.\ p)\ t\ = z\ \longrightarrow z = x)$
using *x-prop* **by** *force*

then have $\exists!y \in \mathit{cube}\ 1\ (t + 1).\ y\ 0 = t$
proof (*intro ex1I-alt*)
define y **where** $y \equiv (\lambda x{::}\mathit{nat}.\ \lambda y{\in}\{..<1{::}\mathit{nat}\}.\ x)$
have $y\ t \in (\mathit{cube}\ 1\ (t + 1))$ **unfolding** *cube-def y-def* **by** *simp*
moreover have $y\ t\ 0 = t$ **unfolding** *y-def* **by** *auto*
moreover have $z = y\ t$ **if** $z \in \mathit{cube}\ 1\ (t + 1)$ **and** $z\ 0 = t$ **for** z
proof (*rule ccontr*)

```
        assume z ≠ y t
        then obtain l where l-prop: z l ≠ y t l by blast
        consider l ∈ {..<1::nat} | l ∉ {..<1::nat} by blast
        then show False
        proof cases
          case 1
          then show ?thesis using l-prop that(2) unfolding y-def by auto
        next
          case 2
          then have z l = undefined using that unfolding cube-def by blast
          moreover have y t l = undefined unfolding y-def using 2 by auto
          ultimately show ?thesis using l-prop by presburger
        qed
      qed
      ultimately show ∃ y. (y ∈ cube 1 (t + 1) ∧ y 0 = t) ∧ (∀ ya.
      ya ∈ cube 1 (t + 1) ∧ ya 0 = t ⟶ y = ya) by blast
    qed
    then have ∃!x ∈ classes 1 t 1. True using redef by simp
    then obtain x where x-def: x ∈ classes 1 t 1 ∧ (∀ y ∈ classes 1 t 1. x =
y) by auto

    have χ (S1 y) < r if y ∈ classes 1 t 1 for y
    proof–
      have y = x using x-def that by auto
      then have χ (S1 y) = χ (S1 x) by auto
      moreover have S1 x ∈ cube N' (t+1) unfolding S1-def is-line-def
        using line-prop line-points-in-cube redef x-def by fastforce
      ultimately show χ (S1 y) < r using asm unfolding cube-def by auto
    qed
    then show ?thesis using lc-def ‹i = 1› using x-def by fast
  qed
 qed
 ultimately show (∃ S. is-subspace S 1 N' (t + 1) ∧ (∀ i ∈ {..1}.
 ∃ c < r. (∀ x ∈ classes 1 t i. χ (S x) = c))) by blast
qed
then show ?thesis using N-def unfolding layered-subspace-def lhj-def by auto
qed
```

2.1.2 Induction step of theorem 4

The proof has four parts:

1. We obtain two layered subspaces of dimension 1 and k (respectively), whose existence is guaranteed by the assumption *lhj* (i.e. the induction hypothesis). Additionally, we prove some useful facts about these.

2. We construct a *k+1*-dimensional subspace with the goal of showing that it is layered.

3. We prove that our construction is a subspace in the first place.

4. We prove that it is a layered subspace.

lemma *hj-imp-lhj-step*:
fixes $r\ k$
assumes $t > 0$
and $k \geq 1$
and *True*
and $(\bigwedge r\ k'.\ k' \leq k \Longrightarrow lhj\ r\ t\ k')$
and $r > 0$
shows $lhj\ r\ t\ (k+1)$
proof–
obtain m **where** *m-props*: $(m > 0 \wedge (\forall M' \geq m.\ \forall \chi.\ \chi \in (cube\ M'\ (t+1)) \rightarrow_E \{..<r::nat\} \longrightarrow (\exists S.\ layered\text{-}subspace\ S\ k\ M'\ t\ r\ \chi)))$ **using** *assms(4)*[*of k r*] **unfolding** *lhj-def* **by** *blast*
define s **where** $s \equiv r\frown((t+1)\frown m)$
obtain n' **where** n'*-props*: $(n' > 0 \wedge (\forall N \geq n'.\ \forall \chi.\ \chi \in (cube\ N\ (t+1)) \rightarrow_E \{..<s::nat\} \longrightarrow (\exists S.\ layered\text{-}subspace\ S\ 1\ N\ t\ s\ \chi)))$ **using** *assms(2) assms(4)*[*of 1 s*] **unfolding** *lhj-def* **by** *auto*

have $(\exists T.\ layered\text{-}subspace\ T\ (k+1)\ (M')\ t\ r\ \chi)$ **if** χ*-prop*: $\chi \in cube\ M'\ (t+1) \rightarrow_E \{..<r\}$ **and** M'*-prop*: $M' \geq n' + m$ **for** $\chi\ M'$
proof –
define d **where** $d \equiv M' - (n' + m)$
define n **where** $n \equiv n' + d$
have $n \geq n'$ **unfolding** *n-def d-def* **by** *simp*
have $n + m = M'$ **unfolding** *n-def d-def* **using** M'*-prop* **by** *simp*
have *line-subspace-s*: $\exists S.\ layered\text{-}subspace\ S\ 1\ n\ t\ s\ \chi \wedge is\text{-}line\ (\lambda s{\in}\{..<t+1\}.\ S\ (SOME\ p.\ p{\in}cube\ 1\ (t+1) \wedge p\ 0 = s))\ n\ (t+1)$ **if** $\chi \in (cube\ n\ (t+1)) \rightarrow_E \{..<s::nat\}$ **for** χ
proof–
have $\exists S.\ layered\text{-}subspace\ S\ 1\ n\ t\ s\ \chi$ **using** *that n'-props* ‹$n \geq n'$› **by** *blast*
then obtain L **where** *layered-subspace L 1 n t s* χ **by** *blast*
then have *is-subspace L 1 n* $(t+1)$ **unfolding** *layered-subspace-def* **by** *simp*
then have *is-line* $(\lambda s{\in}\{..<t+1\}.\ L\ (SOME\ p.\ p{\in}cube\ 1\ (t+1) \wedge p\ 0 = s))\ n\ (t+1)$
using *dim1-subspace-is-line*[*of t+1 L n*] *assms(1)* **by** *simp*
then show $\exists S.\ layered\text{-}subspace\ S\ 1\ n\ t\ s\ \chi \wedge is\text{-}line\ (\lambda s{\in}\{..<t+1\}.\ S\ (SOME\ p.\ p \in cube\ 1\ (t+1) \wedge p\ 0 = s))\ n\ (t+1)$ **using** ‹*layered-subspace L 1 n t s* χ›
by *auto*
qed

Part 1: Obtaining the subspaces L and S

Recall that *lhj* claims the existence of a layered subspace for any colouring (of a fixed size, where the size of a colouring refers to the number of colours). Therefore, the colourings have to be defined first, before the layered subspaces can be obtained. The colouring χL here is χ^* in the book [1], an s-colouring; see the fact *s-coloured* a couple of lines below.

define χL **where** $\chi L \equiv (\lambda x \in cube\ n\ (t+1).\ (\lambda y \in cube\ m$

```
  (t + 1). χ (join x y n m)))
  have A: ∀ x ∈ cube n (t+1). ∀ y ∈ cube m (t+1). χ (join x y n m) ∈ {..<r}
  proof(safe)
    fix x y
    assume x ∈ cube n (t+1) y ∈ cube m (t+1)
    then have join x y n m ∈ cube (n+m) (t+1) using join-cubes[of x n t y m]
by simp
    then show χ (join x y n m) < r using χ-prop ‹n + m = M'› by blast
  qed
  have χL-prop: χL ∈ cube n (t+1) →E cube m (t+1) →E {..<r}
    using A by (auto simp: χL-def)

  have card (cube m (t+1) →E {..<r}) = (card {..<r}) ^ (card (cube m (t+1)))

    using card-PiE[of cube m (t + 1) λ-. {..<r}] by (simp add: cube-def finite-PiE)
  also have ... = r ^ (card (cube m (t+1))) by simp
  also have ... = r ^ ((t+1) ^ m) using cube-card unfolding cube-def by simp
  finally have card (cube m (t+1) →E {..<r}) = r ^ ((t+1) ^ m) .
   then have s-coloured: card (cube m (t+1) →E {..<r}) = s unfolding s-def
by simp
  have s > 0 using assms(5) unfolding s-def by simp
  then obtain φ where φ-prop: bij-betw φ (cube m (t+1) →E {..<r}) {..<s}
   using assms(5) ex-bij-betw-nat-finite-2[of cube m (t+1) →E {..<r} s] s-coloured
by blast
  define χL-s where χL-s ≡ (λx∈cube n (t+1). φ (χL x))
  have χL-s ∈ cube n (t+1) →E {..<s}
  proof
    fix x assume a: x ∈ cube n (t+1)
    then have χL-s x = φ (χL x) unfolding χL-s-def by simp
    moreover have χL x ∈ (cube m (t+1) →E {..<r})
      using a χL-def χL-prop unfolding χL-def by blast
    moreover have φ (χL x) ∈ {..<s} using φ-prop calculation(2) unfolding
bij-betw-def by blast
    ultimately show χL-s x ∈ {..<s} by auto
  qed (auto simp: χL-s-def)
```

L is the layered line which we obtain from the monochromatic line guaranteed to exist by the assumption *hj s t*.

```
then obtain L where L-prop: layered-subspace L 1 n t s χL-s using line-subspace-s
by blast
  define L-line where L-line ≡ (λs∈{..<t+1}. L (SOME p. p∈cube 1 (t+1) ∧ p
0 = s))
  have L-line-base-prop: ∀ s ∈ {..<t+1}. L-line s ∈ cube n (t+1)
   using assms(1) dim1-subspace-is-line[of t+1 L n] L-prop line-points-in-cube[of
L-line n t+1]
    unfolding layered-subspace-def L-line-def by auto
```

Here, χS is χ^{**} in the book [1], an r-colouring.

```
define χS where χS ≡ (λy∈cube m (t+1). χ (join (L-line 0) y n m))
```

have $\chi S \in (cube\ m\ (t\ +\ 1)) \rightarrow_E \{..<r::nat\}$
proof
fix x **assume** a: $x \in cube\ m\ (t+1)$
then have $\chi S\ x = \chi\ (join\ (L\text{-}line\ 0)\ x\ n\ m)$ **unfolding** $\chi S\text{-}def$ **by** *simp*
moreover have $L\text{-}line\ 0 = L\ (SOME\ p.\ p \in cube\ 1\ (t+1) \wedge p\ 0 = 0)$
using $L\text{-}prop\ assms(1)$ **unfolding** $L\text{-}line\text{-}def$ **by** *simp*
moreover have $(SOME\ p.\ p \in cube\ 1\ (t+1) \wedge p\ 0 = 0) \in cube\ 1\ (t+1)$ **using** $cube\text{-}props(4)[of\ 0\ t+1]$
using $assms(1)$ **by** *auto*
moreover have $L \in cube\ 1\ (t+1) \rightarrow_E cube\ n\ (t+1)$
using $L\text{-}prop$ **unfolding** $layered\text{-}subspace\text{-}def\ is\text{-}subspace\text{-}def$ **by** *blast*
moreover have $L\ (SOME\ p.\ p \in cube\ 1\ (t+1) \wedge p\ 0 = 0) \in cube\ n\ (t+1)$
using $calculation\ (3,4)$ **unfolding** $cube\text{-}def$ **by** *auto*
moreover have $join\ (L\text{-}line\ 0)\ x\ n\ m \in cube\ (n\ +\ m)\ (t+1)$ **using** $join\text{-}cubes\ a\ calculation(2,\ 5)$ **by** *auto*
ultimately show $\chi S\ x \in \{..<r\}$ **using** $A\ a$ **by** *fastforce*
qed (*auto simp*: $\chi S\text{-}def$)

S is the k-dimensional layered subspace that arises as a consequence of the induction hypothesis. Note that the colouring is χS, an r-colouring.

then obtain S **where** $S\text{-}prop$: $layered\text{-}subspace\ S\ k\ m\ t\ r\ \chi S$ **using** $assms(4)\ m\text{-}props$ **by** *blast*

Remark: *L-Line i* returns the i-th point of the line.

Part 2: Constructing the $(k+1)$-dimensional subspace T

Below, *Tset* is the set as defined in the book [1]. It represents the $(k+1)$-dimensional subspace. In this construction, subspaces (e.g. T) are functions whose image is a set. See the fact *im-T-eq-Tset* below.

Having obtained our subspaces S and L, we define the $(k+1)$-dimensional subspace very straightforwardly Namely, T = L ×S. Since we represent tuples by function sets, we need an appropriate operator that mirrors the Cartesian product × for these. We call this *join* and define it for elements of a function set.

define $Tset$ **where** $Tset \equiv \{join\ (L\text{-}line\ i)\ s\ n\ m \mid i\ s\ .\ i \in \{..<t+1\} \wedge s \in S\ `\ (cube\ k\ (t+1))\}$
define T' **where** $T' \equiv (\lambda x \in cube\ 1\ (t+1).\ \lambda y \in cube\ k\ (t+1).\ join\ (L\text{-}line\ (x\ 0))\ (S\ y)\ n\ m)$
have $T'\text{-}prop$: $T' \in cube\ 1\ (t+1) \rightarrow_E cube\ k\ (t+1) \rightarrow_E cube\ (n\ +\ m)\ (t+1)$
proof
fix x **assume** a: $x \in cube\ 1\ (t+1)$
show $T'\ x \in cube\ k\ (t\ +\ 1) \rightarrow_E cube\ (n\ +\ m)\ (t\ +\ 1)$
proof
fix y **assume** b: $y \in cube\ k\ (t+1)$
then have $T'\ x\ y = join\ (L\text{-}line\ (x\ 0))\ (S\ y)\ n\ m$ **using** a **unfolding** $T'\text{-}def$ **by** *simp*

moreover have *L-line* $(x\ 0) \in$ *cube* $n\ (t{+}1)$ **using** *a L-line-base-prop* **unfolding** *cube-def* **by** *blast*
moreover have $S\ y \in$ *cube* $m\ (t{+}1)$
using *subspace-elems-embed*[*of* $S\ k\ m\ t{+}1$] *S-prop b* **unfolding** *layered-subspace-def* **by** *blast*
ultimately show $T'\ x\ y \in$ *cube* $(n + m)\ (t + 1)$ **using** *join-cubes* **by** *presburger*
next
qed (*unfold* T'*-def*; *use a* **in** *simp*)
qed (*auto simp*: T'*-def*)

define T **where** $T \equiv (\lambda x \in$ *cube* $(k + 1)\ (t{+}1).\ T'\ (\lambda y \in \{..{<}1\}.\ x\ y)\ (\lambda y \in \{..{<}k\}.\ x\ (y + 1)))$
have *T-prop*: $T \in$ *cube* $(k{+}1)\ (t{+}1) \to_E$ *cube* $(n{+}m)\ (t{+}1)$
proof
fix x **assume** *a*: $x \in$ *cube* $(k{+}1)\ (t{+}1)$
then have $T\ x = T'\ (\lambda y \in \{..{<}1\}.\ x\ y)\ (\lambda y \in \{..{<}k\}.\ x\ (y + 1))$ **unfolding** *T-def* **by** *auto*
moreover have $(\lambda y \in \{..{<}1\}.\ x\ y) \in$ *cube* $1\ (t{+}1)$ **using** *a* **unfolding** *cube-def* **by** *auto*
moreover have $(\lambda y \in \{..{<}k\}.\ x\ (y + 1)) \in$ *cube* $k\ (t{+}1)$ **using** *a* **unfolding** *cube-def* **by** *auto*
moreover have $T'\ (\lambda y \in \{..{<}1\}.\ x\ y)\ (\lambda y \in \{..{<}k\}.\ x\ (y + 1)) \in$ *cube* $(n + m)\ (t{+}1)$
using T'*-prop calculation* **unfolding** T'*-def* **by** *blast*
ultimately show $T\ x \in$ *cube* $(n + m)\ (t{+}1)$ **by** *argo*
qed (*auto simp*: *T-def*)

have *im-T-eq-Tset*: T ‘ *cube* $(k{+}1)\ (t{+}1) =$ *Tset*
proof
show T ‘ *cube* $(k + 1)\ (t + 1) \subseteq$ *Tset*
proof
fix x **assume** $x \in T$ ‘ *cube* $(k{+}1)\ (t{+}1)$
then obtain y **where** *y-prop*: $y \in$ *cube* $(k{+}1)\ (t{+}1) \wedge x = T\ y$ **by** *blast*
then have $T\ y = T'\ (\lambda i \in \{..{<}1\}.\ y\ i)\ (\lambda i \in \{..{<}k\}.\ y\ (i + 1))$ **unfolding** *T-def* **by** *simp*
moreover have $(\lambda i \in \{..{<}1\}.\ y\ i) \in$ *cube* $1\ (t{+}1)$ **using** *y-prop* **unfolding** *cube-def* **by** *auto*
moreover have $(\lambda i \in \{..{<}k\}.\ y\ (i + 1)) \in$ *cube* $k\ (t{+}1)$ **using** *y-prop* **unfolding** *cube-def* **by** *auto*
moreover have $T'\ (\lambda i \in \{..{<}1\}.\ y\ i)\ (\lambda i \in \{..{<}k\}.\ y\ (i + 1)) =$
join (*L-line* $((\lambda i \in \{..{<}1\}.\ y\ i)\ 0))$ $(S\ (\lambda i \in \{..{<}k\}.\ y\ (i + 1)))\ n\ m$
using *calculation* **unfolding** T'*-def* **by** *auto*
ultimately have $*$: $T\ y =$ *join* (*L-line* $((\lambda i \in \{..{<}1\}.\ y\ i)\ 0))$
$(S\ (\lambda i \in \{..{<}k\}.\ y\ (i + 1)))\ n\ m$ **by** *simp*

have $(\lambda i \in \{..{<}1\}.\ y\ i)\ 0 \in \{..{<}t{+}1\}$ **using** *y-prop* **unfolding** *cube-def* **by** *auto*
moreover have $S\ (\lambda i \in \{..{<}k\}.\ y\ (i + 1)) \in S$ ‘ (*cube* $k\ (t{+}1)$)

```
      using ‹(λi∈{..<k}. y (i + 1)) ∈ cube k (t + 1)› by blast
    ultimately have T y ∈ Tset using ∗ unfolding Tset-def by blast
    then show x ∈ Tset using y-prop by simp
  qed

  show Tset ⊆ T ‘ cube (k + 1) (t + 1)
  proof
    fix x assume x ∈ Tset
    then obtain i sx sxinv where isx-prop: x = join (L-line i) sx n m ∧ i ∈
{..<t+1}
    ∧ sx ∈ S ‘ (cube k (t+1)) ∧ sxinv ∈ cube k (t+1) ∧ S sxinv = sx
      unfolding Tset-def by blast
    let ?f1 = (λj ∈ {..<1::nat}. i)
    let ?f2 = sxinv
    have ?f1 ∈ cube 1 (t+1) using isx-prop unfolding cube-def by simp
    moreover have ?f2 ∈ cube k (t+1) using isx-prop by blast
      moreover have x = join (L-line (?f1 0)) (S ?f2) n m by (simp add:
isx-prop)
    ultimately have ∗: x = T' ?f1 ?f2 unfolding T'-def by simp

    define f where f ≡ (λj ∈ {1..<k+1}. ?f2 (j − 1))(0:=i)
    have f ∈ cube (k+1) (t+1)
    proof (unfold cube-def; intro PiE-I)
      fix j assume j ∈ {..<k+1}
      then consider j = 0 | j ∈ {1..<k+1} by fastforce
      then show f j ∈ {..<t+1}
      proof (cases)
        case 1
        then have f j = i unfolding f-def by simp
        then show ?thesis using isx-prop by simp
      next
        case 2
        then have j − 1 ∈ {..<k} by auto
        moreover have f j = ?f2 (j − 1) using 2 unfolding f-def by simp
         moreover have ?f2 (j − 1) ∈ {..<t+1} using calculation(1) isx-prop
unfolding cube-def by blast
        ultimately show ?thesis by simp
      qed
    qed (auto simp: f-def)
    have ?f1 = (λj ∈ {..<1}. f j) unfolding f-def using isx-prop by auto
    moreover have ?f2 = (λj∈{..<k}. f (j+1))
      using calculation isx-prop unfolding cube-def f-def by fastforce
   ultimately have T' ?f1 ?f2 = T f using ‹f ∈ cube (k+1) (t+1)› unfolding
T-def by simp
    then show x ∈ T ‘ cube (k + 1) (t + 1) using ∗
      using ‹f ∈ cube (k + 1) (t + 1)› by blast
  qed
```

qed
have *Tset* $\subseteq$ *cube* $(n + m)$ $(t{+}1)$
proof
fix x **assume** a: $x \in Tset$
then obtain i sx **where** *isx-props*: $x = join\ (L\text{-}line\ i)\ sx\ n\ m \wedge i \in \{..<t{+}1\} \wedge$
$sx \in S\ `\ (cube\ k\ (t{+}1))$ **unfolding** *Tset-def* **by** *blast*
then have *L-line* $i \in cube\ n\ (t{+}1)$ **using** *L-line-base-prop* **by** *blast*
moreover have $sx \in cube\ m\ (t{+}1)$
using *subspace-elems-embed*[*of S k m t+1*] *S-prop isx-props* **unfolding** *layered-subspace-def* **by** *blast*
ultimately show $x \in cube\ (n + m)\ (t{+}1)$ **using** *join-cubes*[*of L-line i n t sx m*] *isx-props* **by** *simp*
qed

Part 3: Proving that T is a subspace

To prove something is a subspace, we have to provide the B and f satisfying the subspace properties. We construct BT and fT from BS, fS and BL, fL, which correspond to the k-dimensional subspace S and the 1-dimensional subspace (i.e. line) L, respectively.

obtain *BS fS* **where** *BfS-props*: *disjoint-family-on BS* $\{..k\}$ $\bigcup(BS\ `\ \{..k\}) = \{..<m\}$ $(\{\}$
$\notin BS\ `\ \{..<k\})$ $fS \in (BS\ k) \to_E \{..<t{+}1\}$ $S \in (cube\ k\ (t{+}1))$
$\to_E$ $(cube\ m\ (t{+}1))$ $(\forall\, y \in cube\ k\ (t{+}1).\ (\forall\, i \in BS\ k.$
$S\ y\ i = fS\ i) \wedge (\forall\, j{<}k.\ \forall\, i \in BS\ j.\ (S\ y)\ i = y\ j))$ **using** *S-prop*
unfolding *layered-subspace-def is-subspace-def* **by** *auto*

obtain *BL fL* **where** *BfL-props*: *disjoint-family-on BL* $\{..1\}$ $\bigcup(BL\ `\ \{..1\}) = \{..<n\}$
$(\{\} \notin BL\ `\ \{..<1\})$ $fL \in (BL\ 1) \to_E \{..<t{+}1\}$ $L \in (cube\ 1$
$(t{+}1)) \to_E (cube\ n\ (t{+}1))$ $(\forall\, y \in cube\ 1\ (t{+}1).\ (\forall\, i \in$
$BL\ 1.\ L\ y\ i = fL\ i) \wedge (\forall\, j{<}1.\ \forall\, i \in BL\ j.\ (L\ y)\ i = y\ j))$ **using** *L-prop*
unfolding *layered-subspace-def is-subspace-def* **by** *auto*

define *Bstat* **where** $Bstat \equiv set\text{-}incr\ n\ (BS\ k) \cup BL\ 1$
define *Bvar* **where** $Bvar \equiv (\lambda i{::}nat.\ (if\ i = 0\ then\ BL\ 0\ else\ set\text{-}incr\ n\ (BS\ (i - 1))))$
define *BT* **where** $BT \equiv (\lambda i \in \{..<k{+}1\}.\ Bvar\ i)((k{+}1){:=}Bstat)$
define *fT* **where** $fT \equiv (\lambda x.\ (if\ x \in BL\ 1\ then\ fL\ x\ else\ (if\ x \in set\text{-}incr\ n$
$(BS\ k)\ then\ fS\ (x - n)\ else\ undefined)))$

have *fact1*: $set\text{-}incr\ n\ (BS\ k) \cap BL\ 1 = \{\}$ **using** *BfL-props BfS-props* **unfolding** *set-incr-def* **by** *auto*
have *fact2*: $BL\ 0 \cap (\bigcup i{\in}\{..<k\}.\ set\text{-}incr\ n\ (BS\ i)) = \{\}$
using *BfL-props BfS-props* **unfolding** *set-incr-def* **by** *auto*
have *fact3*: $\forall\, i \in \{..<k\}.\ BL\ 0 \cap set\text{-}incr\ n\ (BS\ i) = \{\}$
using *BfL-props BfS-props* **unfolding** *set-incr-def* **by** *auto*
have *fact4*: $\forall\, i \in \{..<k{+}1\}.\ \forall\, j \in \{..<k{+}1\}.\ i \neq j$

$\longrightarrow$ *set-incr n (BS i) ∩ set-incr n (BS j) = {}*
using *set-incr-disjoint-family[of BS k] BfS-props* **unfolding** *disjoint-family-on-def*
by *simp*
have *fact5*: $\forall i \in \{..<k+1\}$. *Bvar i ∩ Bstat = {}*
proof
fix *i* **assume** *a*: $i \in \{..<k+1\}$
show *Bvar i ∩ Bstat = {}*
proof (*cases i*)
case *0*
then have *Bvar i = BL 0* **unfolding** *Bvar-def* **by** *simp*
moreover have *BL 0 ∩ BL 1 = {}* **using** *BfL-props* **unfolding** *disjoint-family-on-def* **by** *simp*
moreover have *set-incr n (BS k) ∩ BL 0 = {}* **using** *BfL-props BfS-props* **unfolding** *set-incr-def* **by** *auto*
ultimately show *?thesis* **unfolding** *Bstat-def* **by** *blast*
next
case (*Suc nat*)
then have *Bvar i = set-incr n (BS nat)* **unfolding** *Bvar-def* **by** *simp*
moreover have *set-incr n (BS nat) ∩ BL 1 = {}* **using** *BfS-props BfL-props a Suc* **unfolding** *set-incr-def*
by *auto*
moreover have *set-incr n (BS nat) ∩ set-incr n (BS k) = {}* **using** *a Suc fact4* **by** *simp*
ultimately show *?thesis* **unfolding** *Bstat-def* **by** *blast*
qed
qed

The facts *F1*, ..., *F5* are the disjuncts in the subspace definition.

have *Bvar ‘ {..<k+1} = BL ‘ {..<1} ∪ Bvar ‘ {1..<k+1}* **unfolding** *Bvar-def* **by** *force*
also have *... = BL ‘ {..<1} ∪ {set-incr n (BS i) | i . i ∈ {..<k}}* **unfolding** *Bvar-def* **by** *fastforce*
moreover have *{} ∉ BL ‘ {..<1}* **using** *BfL-props* **by** *auto*
moreover have *{} ∉ {set-incr n (BS i) | i . i ∈ {..<k}}* **using** *BfS-props(2, 3) set-incr-def* **by** *fastforce*
ultimately have *{} ∉ Bvar ‘ {..<k+1}* **by** *simp*
then have *F1*: *{} ∉ BT ‘ {..<k+1}* **unfolding** *BT-def* **by** *simp*
moreover
{
have *F2-aux*: *disjoint-family-on Bvar {..<k+1}*
proof (*unfold disjoint-family-on-def*; *safe*)
fix *m n x* **assume** *a*: $m < k + 1$ $n < k + 1$ $m \neq n$ $x \in Bvar\ m$ $x \in Bvar\ n$
show $x \in \{\}$
proof (*cases n*)
case *0*
then show *?thesis* **using** *a fact3* **unfolding** *Bvar-def* **by** *auto*
next
case (*Suc nnat*)
then have $*$: $n = Suc\ nnat$ **by** *simp*

```
        then show ?thesis
        proof (cases m)
          case 0
          then show ?thesis using a fact3 unfolding Bvar-def by auto
        next
          case (Suc mnat)
          then show ?thesis using a fact4  ∗ unfolding Bvar-def by fastforce
        qed
      qed
    qed

    have F2: disjoint-family-on BT {..k+1}
    proof
      fix m n assume a: m∈{..k+1} n∈{..k+1} m ≠ n
      have ∀ x. x ∈ BT m ∩ BT n ⟶ x ∈ {}
      proof (intro allI impI)
        fix x assume b: x ∈ BT m ∩ BT n
        have m < k + 1 ∧ n < k + 1 ∨ m = k + 1 ∧ n = k + 1 ∨ m < k + 1
        ∧ n = k + 1 ∨ m = k + 1 ∧ n < k + 1 using a le-eq-less-or-eq by auto
        then show x ∈ {}
        proof (elim disjE)
          assume c: m < k + 1 ∧ n < k + 1
           then have BT m = Bvar m ∧ BT n = Bvar n unfolding BT-def by
simp
             then show x ∈ {} using a b c fact4 F2-aux unfolding Bvar-def
disjoint-family-on-def by auto
        qed (use a b fact5 in ‹auto simp: BT-def›)
      qed
      then show BT m ∩ BT n = {} by auto
    qed
  }
  moreover have F3: ⋃(BT ‘ {..k+1}) = {..<n + m}
  proof
    show ⋃ (BT ‘ {..k + 1}) ⊆ {..<n + m}
    proof
      fix x assume x ∈ ⋃ (BT ‘ {..k + 1})
      then obtain i where i-prop: i ∈ {..k+1} ∧ x ∈ BT i by blast
      then consider i = k +1 | i ∈ {..<k+1} by fastforce
      then show x ∈ {..<n + m}
      proof (cases)
        case 1
        then have x ∈ Bstat using i-prop unfolding BT-def by simp
         then have x ∈ BL 1 ∨ x ∈ set-incr n (BS k) unfolding Bstat-def by
blast
        then have x ∈ {..<n} ∨ x ∈ {n..<n+m} using BfL-props BfS-props(2)
set-incr-image[of BS k m n]
          by blast
        then show ?thesis by auto
      next
```

```
        case 2
        then have x ∈ Bvar i using i-prop unfolding BT-def by simp
        then have x ∈ BL 0 ∨ x ∈ set-incr n (BS (i − 1)) unfolding Bvar-def
by presburger
        then show ?thesis
        proof (elim disjE)
          assume x ∈ BL 0
          then have x ∈ {..<n} using BfL-props by auto
          then show x ∈ {..<n + m} by simp
        next
          assume a: x ∈ set-incr n (BS (i − 1))
          then have i − 1 ≤ k
            by (meson atMost-iff i-prop le-diff-conv)
        then have set-incr n (BS (i − 1)) ⊆ {n..<n+m} using set-incr-image[of
BS k m n] BfS-props
            by auto
          then show x ∈ {..<n+m} using a by auto
        qed
      qed
    qed
  next
    show {..<n + m} ⊆ ⋃ (BT ‘ {..k + 1})
    proof
      fix x assume x ∈ {..<n + m}
      then consider x ∈ {..<n} | x ∈ {n..<n+m} by fastforce
      then show x ∈ ⋃ (BT ‘ {..k + 1})
      proof (cases)
        case 1
        have ∗: {..1::nat} = {0, 1::nat} by auto
        from 1 have x ∈ ⋃ (BL ‘ {..1::nat}) using BfL-props by simp
        then have x ∈ BL 0 ∨ x ∈ BL 1 using ∗ by simp
        then show ?thesis
        proof (elim disjE)
          assume x ∈ BL 0
          then have x ∈ Bvar 0 unfolding Bvar-def by simp
          then have x ∈ BT 0 unfolding BT-def by simp
          then show x ∈ ⋃ (BT ‘ {..k + 1}) by auto
        next
          assume x ∈ BL 1
          then have x ∈ Bstat unfolding Bstat-def by simp
          then have x ∈ BT (k+1) unfolding BT-def by simp
          then show x ∈ ⋃ (BT ‘ {..k + 1}) by auto
        qed
      next
        case 2
        then have x ∈ (⋃ i≤k. set-incr n (BS i)) using set-incr-image[of BS k
m n] BfS-props by simp
        then obtain i where i-prop: i ≤ k ∧ x ∈ set-incr n (BS i) by blast
        then consider i = k | i < k by fastforce
```

```
      then show ?thesis
      proof (cases)
        case 1
        then have x ∈ Bstat unfolding Bstat-def using i-prop by auto
        then have x ∈ BT (k+1) unfolding BT-def by simp
        then show ?thesis by auto
      next
        case 2
       then have x ∈ Bvar (i + 1) unfolding Bvar-def using i-prop by simp
        then have x ∈ BT (i + 1) unfolding BT-def using 2 by force
        then show ?thesis using 2 by auto
      qed
    qed
  qed

  moreover have F4: fT ∈ (BT (k+1)) →E {..<t+1}
  proof
    fix x assume x ∈ BT (k+1)
    then have x ∈ Bstat unfolding BT-def by simp
    then have x ∈ BL 1 ∨ x ∈ set-incr n (BS k) unfolding Bstat-def by auto
    then show fT x ∈ {..<t + 1}
    proof (elim disjE)
      assume x ∈ BL 1
      then have fT x = fL x unfolding fT-def by simp
      then show fT x ∈ {..<t+1} using BfL-props ‹x ∈ BL 1› by auto
    next
      assume a: x ∈ set-incr n (BS k)
      then have fT x = fS (x − n) using fact1 unfolding fT-def by auto
      moreover have x − n ∈ BS k using a unfolding set-incr-def by auto
      ultimately show fT x ∈ {..<t+1} using BfS-props by auto
    qed
  qed(auto simp: BT-def Bstat-def fT-def)
  moreover have F5: ((∀ i ∈ BT (k + 1). T y i = fT i) ∧ (∀ j<k+1.
  ∀ i ∈ BT j. (T y) i = y j)) if y ∈ cube (k + 1) (t + 1) for y
  proof(intro conjI allI impI ballI)
    fix i assume i ∈ BT (k + 1)
    then have i ∈ Bstat unfolding BT-def by simp
     then consider i ∈ set-incr n (BS k) | i ∈ BL 1 unfolding Bstat-def by
blast
    then show T y i = fT i
    proof (cases)
      case 1
       then have ∃ s<m. i = n + s unfolding set-incr-def using BfS-props(2)
by auto
      then obtain s where s-prop: s < m ∧ i = n + s by blast
      then have ∗: i ∈ {n..<n+m} by simp
      have i ∉ BL 1 using 1 fact1 by auto
      then have fT i = fS (i − n) using 1 unfolding fT-def by simp
```

then have $**$: *fT i = fS s* **using** *s-prop* **by** *simp*

have *XX*: $(\lambda z \in \{..<k\}.\ y\ (z + 1)) \in$ *cube k (t+1)* **using** *split-cube that* **by** *simp*
have *XY*: $s \in$ *BS k* **using** *s-prop 1* **unfolding** *set-incr-def* **by** *auto*

from *that* **have** $T\ y\ i = (T'\ (\lambda z \in \{..<1\}.\ y\ z)\ (\lambda z \in \{..<k\}.\ y\ (z + 1)))\ i$
unfolding *T-def* **by** *auto*
also have ... = (*join* (*L-line* $((\lambda z \in \{..<1\}.\ y\ z)\ 0)$) (*S* $(\lambda z \in \{..<k\}.\ y\ (z + 1)))\ n\ m)\ i$ **using** *split-cube that* **unfolding** *T'-def* **by** *simp*
also have ... = (*join* (*L-line* $(y\ 0)$) (*S* $(\lambda z \in \{..<k\}.\ y\ (z + 1)))\ n\ m)\ i$ **by** *simp*
also have ... = (*S* $(\lambda z \in \{..<k\}.\ y\ (z + 1)))\ s$ **using** $*$ *s-prop* **unfolding** *join-def* **by** *simp*
also have ... = *fS s* **using** *XX XY BfS-props(6)* **by** *blast*
finally show *?thesis* **using** $**$ **by** *simp*
next
case *2*
have *XZ*: $y\ 0 \in \{..<t+1\}$ **using** *that* **unfolding** *cube-def* **by** *auto*
have *XY*: $i \in \{..<n\}$ **using** *2 BfL-props(2)* **by** *blast*
have *XX*: $(\lambda z \in \{..<1\}.\ y\ z) \in$ *cube 1 (t+1)* **using** *that split-cube* **by** *simp*

have *some-eq-restrict*: (*SOME p. p*$\in$*cube 1 (t+1)* $\wedge$ $p\ 0 = ((\lambda z \in \{..<1\}.\ y\ z)\ 0)) = (\lambda z \in \{..<1\}.\ y\ z)$
proof
show *restrict y* $\{..<1\} \in$ *cube 1* $(t + 1) \wedge$ *restrict y* $\{..<1\}\ 0 =$ *restrict y* $\{..<1\}\ 0$
using *XX* **by** *simp*
next
fix *p*
assume $p \in$ *cube 1 (t+1)* $\wedge\ p\ 0 =$ *restrict y* $\{..<1\}\ 0$
moreover have $p\ u =$ *restrict y* $\{..<1\}\ u$ **if** $u \notin \{..<1\}$ **for** *u*
using *that calculation XX* **unfolding** *cube-def*
using *PiE-arb*[*of restrict y* $\{..<1\}$ $\{..<1\}$ $\lambda x.$ $\{..<t + 1\}$ *u*]
PiE-arb[*of p* $\{..<1\}$ $\lambda x.$ $\{..<t + 1\}$ *u*] **by** *simp*
ultimately show $p =$ *restrict y* $\{..<1\}$ **by** *auto*
qed

from *that* **have** $T\ y\ i = (T'\ (\lambda z \in \{..<1\}.\ y\ z)\ (\lambda z \in \{..<k\}.\ y\ (z + 1)))\ i$
unfolding *T-def* **by** *auto*
also have ... = (*join* (*L-line* $((\lambda z \in \{..<1\}.\ y\ z)\ 0)$) (*S* $(\lambda z \in \{..<k\}.\ y\ (z + 1)))\ n\ m)\ i$
using *split-cube that* **unfolding** *T'-def* **by** *simp*
also have ... = (*L-line* $((\lambda z \in \{..<1\}.\ y\ z)\ 0))\ i$ **using** *XY* **unfolding** *join-def* **by** *simp*
also have ... = *L* (*SOME p. p*$\in$*cube 1 (t+1)* $\wedge\ p\ 0 = ((\lambda z \in \{..<1\}.\ y\ z)\ 0))\ i$
using *XZ* **unfolding** *L-line-def* **by** *auto*
also have ... = *L* $(\lambda z \in \{..<1\}.\ y\ z)\ i$ **using** *some-eq-restrict* **by** *simp*

```
      also have ... = fL i using BfL-props(6) XX 2 by blast
      also have ... = fT i using 2 unfolding fT-def by simp
      finally show ?thesis .
    qed
  next
    fix j i assume j < k + 1 i ∈ BT j
    then have i-prop: i ∈ Bvar j unfolding BT-def by auto
    consider j = 0 | j > 0 by auto
    then show T y i = y j
    proof cases
      case 1
      then have i ∈ BL 0 using i-prop unfolding Bvar-def by auto
      then have XY: i ∈ {..<n} using 1 BfL-props(2) by blast
      have XX: (λz ∈ {..<1}. y z) ∈ cube 1 (t+1) using that split-cube by simp
      have XZ: y 0 ∈ {..<t+1} using that unfolding cube-def by auto

      have some-eq-restrict: (SOME p. p∈cube 1 (t+1) ∧ p 0 = ((λz ∈ {..<1}.
      y z) 0)) = (λz ∈ {..<1}. y z)
      proof
        show restrict y {..<1} ∈ cube 1 (t + 1) ∧ restrict y {..<1} 0 = restrict y
{..<1} 0 using XX by simp
      next
        fix p
        assume p ∈ cube 1 (t+1) ∧ p 0 = restrict y {..<1} 0
        moreover have p u = restrict y {..<1} u if u ∉ {..<1} for u
          using that calculation XX unfolding cube-def
          using PiE-arb[of restrict y {..<1} {..<1} λx. {..<t + 1} u]
            PiE-arb[of p {..<1} λx. {..<t + 1} u] by simp
        ultimately show p = restrict y {..<1} by auto
      qed

      from that have T y i = (T' (λz ∈ {..<1}. y z) (λz ∈ {..<k}. y (z + 1))) i
        unfolding T-def by auto
      also have ... = (join (L-line ((λz ∈ {..<1}. y z) 0)) (S (λz ∈ {..<k}. y (z
+ 1))) n m) i
        using split-cube that unfolding T'-def by simp
        also have ... = (L-line ((λz ∈ {..<1}. y z) 0)) i using XY unfolding
join-def by simp
      also have ... = L (SOME p. p∈cube 1 (t+1) ∧ p 0 = ((λz ∈ {..<1}. y z)
0)) i
        using XZ unfolding L-line-def by auto
      also have ... = L (λz ∈ {..<1}. y z) i using some-eq-restrict by simp
      also have ... =  (λz ∈ {..<1}. y z) j using BfL-props(6) XX 1 ‹i ∈ BL 0›
by blast
      also have ... = (λz ∈ {..<1}. y z) 0 using 1 by blast
      also have ... = y 0 by simp
      also have ... = y j using 1 by simp
      finally show ?thesis .
    next
```

case *2*
then have $i \in$ *set-incr n* (*BS* (*j* − *1*)) **using** *i-prop* **unfolding** *Bvar-def* **by** *simp*
then have $\exists s{<}m.\ n + s = i$ **using** *BfS-props*(*2*) ‹*j* < *k* + *1*› **unfolding** *set-incr-def* **by** *force*
then obtain *s* **where** *s-prop*: $s < m\ i = s + n$ **by** *auto*
then have ∗: $i \in \{n..{<}n{+}m\}$ **by** *simp*

have *XX*: $(\lambda z \in \{..{<}k\}.\ y\ (z + 1)) \in$ *cube k* (*t*+*1*) **using** *split-cube that* **by** *simp*
have *XY*: $s \in$ *BS* (*j* − *1*) **using** *s-prop 2* ‹*i* ∈ *set-incr n* (*BS* (*j* − *1*))›
unfolding *set-incr-def* **by** *force*

from *that* **have** $T\ y\ i = (T'\ (\lambda z \in \{..{<}1\}.\ y\ z)\ (\lambda z \in \{..{<}k\}.\ y\ (z + 1)))\ i$
unfolding *T-def* **by** *auto*
also have ... = (*join* (*L-line* ((λ*z* ∈ {..<*1*}. *y z*) *0*)) (*S* (λ*z* ∈ {..<*k*}. *y* (*z* + *1*))) *n m*) *i*
using *split-cube that* **unfolding** *T′-def* **by** *simp*
also have ... = (*join* (*L-line* (*y 0*)) (*S* (λ*z* ∈ {..<*k*}. *y* (*z* + *1*))) *n m*) *i* **by** *simp*
also have ... = (*S* (λ*z* ∈ {..<*k*}. *y* (*z* + *1*))) *s* **using** ∗ *s-prop* **unfolding** *join-def* **by** *simp*
also have ... = (λ*z* ∈ {..<*k*}. *y* (*z* + *1*)) (*j*−*1*)
using *XX XY BfS-props*(*6*) *2* ‹*j* < *k* + *1*› **by** *auto*
also have ... = *y j* **using** *2* ‹*j* < *k* + *1*› **by** *force*
finally show *?thesis* **.**
qed
qed

ultimately have *subspace-T*: *is-subspace T* (*k*+*1*) (*n*+*m*) (*t*+*1*) **unfolding** *is-subspace-def* **using** *T-prop* **by** *metis*

Part 4: Proving T is layered

The following redefinition of the classes makes proving the layered property easier.

define *T-class* **where** *T-class* ≡ (λ*j*∈{..*k*}. {*join* (*L-line i*) *s n m* | *i s* . *i* ∈ {..<*t*} ∧ *s* ∈ *S* ‘ (*classes k t j*)})(*k*+*1*:= {*join* (*L-line t*) (*SOME s*. *s* ∈ *S* ‘ (*cube m* (*t*+*1*))) *n m*})
have *classprop*: *T-class j* = *T* ‘ *classes* (*k* + *1*) *t j* **if** *j-prop*: $j \le k$ **for** *j*
proof
show *T-class j* ⊆ *T* ‘ *classes* (*k* + *1*) *t j*
proof
fix *x* **assume** *x* ∈ *T-class j*
from *that* **have** *T-class j* = {*join* (*L-line i*) *s n m* | *i s* . *i* ∈ {..<*t*} ∧ *s* ∈ *S* ‘ (*classes k t j*)}
unfolding *T-class-def* **by** *simp*
then obtain *i s* **where** *is-defs*: *x* = *join* (*L-line i*) *s n m* ∧ *i* < *t* ∧ *s* ∈ *S* ‘ (*classes k t j*)

using ‹$x \in$ *T-class j*› **unfolding** *T-class-def* **by** *auto*
moreover have $*$:*classes k t j* $\subseteq$ *cube k* (*t+1*) **unfolding** *classes-def* **by** *simp*
moreover have $\exists!y.\ y \in$ *classes k t j* $\wedge$ $s = S\ y$
using *subspace-inj-on-cube*[*of S k m t+1*] *S-prop inj-onD*[*of S cube k* (*t+1*)] *calculation*
unfolding *layered-subspace-def inj-on-def* **by** *blast*
ultimately obtain *y* **where** *y-prop*: $y \in$ *classes k t j* $\wedge$ $s = S\ y$ $\wedge$ $(\forall z{\in}$*classes k t j*. $s = S\ z \longrightarrow y = z)$ **by** *auto*

define *p* **where** $p \equiv$ *join* $(\lambda g{\in}\{..<1\}.\ i)$ *y 1 k*
have $(\lambda g{\in}\{..<1\}.\ i) \in$ *cube 1* (*t+1*) **using** *is-defs* **unfolding** *cube-def* **by** *simp*
then have *p-in-cube*: $p \in$ *cube* $(k + 1)$ (*t+1*)
using *join-cubes*[*of* $(\lambda g{\in}\{..<1\}.\ i)$ *1 t y k*] *y-prop* $*$ **unfolding** *p-def* **by** *auto*
then have $**$: $p\ 0 = i \wedge (\forall l < k.\ p\ (l + 1) = y\ l)$ **unfolding** *p-def join-def* **by** *simp*

have $t \notin y$ ‘ $\{..<(k - j)\}$ **using** *y-prop* **unfolding** *classes-def* **by** *simp*
then have $\forall u < k - j.\ y\ u \neq t$ **by** *auto*
then have $\forall u < k - j.\ p\ (u + 1) \neq t$ **using** $**$ **by** *simp*
moreover have $p\ 0 \neq t$ **using** *is-defs* $**$ **by** *simp*
moreover have $\forall u < k - j + 1.\ p\ u \neq t$
using *calculation* **by** (*auto simp*: *algebra-simps less-Suc-eq-0-disj*)
ultimately have $\forall u < (k + 1) - j.\ p\ u \neq t$ **using** *that* **by** *auto*
then have *A1*: $t \notin p$ ‘ $\{..<((k+1) - j)\}$ **by** *blast*

have $p\ u = t$ **if** $u \in \{k - j + 1..<k+1\}$ **for** *u*
proof −
from *that* **have** $u - 1 \in \{k - j..<k\}$ **by** *auto*
then have $y\ (u - 1) = t$ **using** *y-prop* **unfolding** *classes-def* **by** *blast*
then show $p\ u = t$ **using** $**$ *that* ‹$u - 1 \in \{k - j..<k\}$› **by** *auto*
qed
then have *A2*: $\forall u{\in}\{(k+1) - j..<k+1\}.\ p\ u = t$ **using** *that* **by** *auto*

from *A1 A2 p-in-cube* **have** $p \in$ *classes* $(k+1)$ *t j* **unfolding** *classes-def* **by** *blast*

moreover have $x = T\ p$
proof−
have *loc-useful*:$(\lambda y \in \{..<k\}.\ p\ (y + 1)) = (\lambda z \in \{..<k\}.\ y\ z)$ **using** $**$ **by** *auto*
have $T\ p = T'\ (\lambda y \in \{..<1\}.\ p\ y)\ (\lambda y \in \{..<k\}.\ p\ (y + 1))$
using *p-in-cube* **unfolding** *T-def* **by** *auto*

have $T'\ (\lambda y \in \{..<1\}.\ p\ y)\ (\lambda y \in \{..<k\}.\ p\ (y + 1))$
$= join\ (L\text{-}line\ ((\lambda y \in \{..<1\}.\ p\ y)\ 0))\ (S\ (\lambda y \in \{..<k\}.\ p\ (y + 1)))\ n$

```
m
          using split-cube p-in-cube unfolding T'-def by simp
         also have ... = join (L-line (p 0)) (S (λy ∈ {..<k}. p (y + 1))) n m by
simp
        also have ... = join (L-line i) (S (λy ∈ {..<k}. p (y + 1))) n m by (simp
add: **)
        also have ... = join (L-line i) (S (λz ∈ {..<k}. y z)) n m using loc-useful
by simp
       also have ... = join (L-line i) (S y) n m using y-prop * unfolding cube-def
by auto
        also have ... = x using is-defs y-prop by simp
        finally show x = T p
        using ‹T p = T' (restrict p {..<1}) (λy∈{..<k}. p (y + 1))› by presburger
       qed
       ultimately show x ∈ T ‘ classes (k + 1) t j by blast
     qed
   next
     show T ‘ classes (k + 1) t j ⊆ T-class j
     proof
       fix x assume x ∈ T ‘ classes (k+1) t j
       then obtain y where y-prop: y ∈ classes (k+1) t j ∧ T y = x by blast
      then have y-props: (∀ u ∈ {((k+1)−j)..<k+1}. y u = t) ∧ t ∉ y ‘ {..<(k+1)
− j }
        unfolding classes-def by blast

       define z where z ≡ (λv ∈ {..<k}. y (v+1))
     have z ∈ cube k (t+1) using  y-prop classes-subset-cube[of k+1 t j] unfolding
z-def cube-def by auto
       moreover
       {
       have z ‘ {..<k − j} = y ‘ ((+) 1 ‘ {..<k−j})   unfolding z-def by fastforce
      also have ... = y ‘ {1..<k−j+1} by (simp add: atLeastLessThanSuc-atLeastAtMost
image-Suc-lessThan)
        also have ... = y ‘ {1..<(k+1)−j} using j-prop by auto
        finally have z ‘ {..<k − j} ⊆ y ‘ {..<(k+1)−j} by auto
        then have t ∉ z ‘ {..<k − j} using y-props by blast

       }
       moreover have ∀ u ∈ {k−j..<k}. z u = t unfolding z-def using y-props
by auto
       ultimately have z-in-classes: z ∈ classes k t j unfolding classes-def by
blast

       have y 0 ≠ t
       proof−
         from that have 0 ∈ {..<k + 1 − j} by simp
         then show y 0 ≠ t using y-props by blast
       qed
     then have tr: y 0 < t using y-prop classes-subset-cube[of k+1 t j] unfolding
```

cube-def **by** *fastforce*

have $(\lambda g \in \{..<1\}.\ y\ g) \in$ *cube 1* $(t+1)$
using *y-prop classes-subset-cube*[*of* $k+1\ t\ j$] *cube-restrict*[*of 1* $(k+1)\ y\ t+1$] *assms*(*2*) **by** *auto*
then have $T\ y = T'\ (\lambda g \in \{..<1\}.\ y\ g)\ z$ **using** *y-prop classes-subset-cube*[*of* $k+1\ t\ j$]
unfolding *T-def z-def* **by** *auto*
also have $\ldots =$ *join* (*L-line* $((\lambda g \in \{..<1\}.\ y\ g)\ 0))$ $(S\ z)\ n\ m$
unfolding *T'-def*
using $\langle(\lambda g \in \{..<1\}.\ y\ g) \in$ *cube 1* $(t+1)\rangle$ $\langle z \in$ *cube k* $(t+1)\rangle$
by *auto*
also have $\ldots =$ *join* (*L-line* $(y\ 0)$) $(S\ z)\ n\ m$ **by** *simp*
also have $\ldots \in$ *T-class j* **using** *tr z-in-classes that* **unfolding** *T-class-def* **by** *force*
finally show $x \in$ *T-class j* **using** *y-prop* **by** *simp*
qed
qed

The core case $i \leq k$. The case $i = k + 1$ is trivial since $k + 1$ has only one point.

have $\chi\ x = \chi\ y \wedge \chi\ x < r$ **if** *a*: $i \leq k\ x \in T$ ‘ *classes* $(k+1)\ t\ i$
$y \in T$ ‘ *classes* $(k+1)\ t\ i$ **for** $i\ x\ y$
proof–
from *a* **have** $*$: T ‘ *classes* $(k+1)\ t\ i =$ *T-class i* **by** (*simp add*: *classprop*)
then have $x \in$ *T-class i* **using** *that* **by** *simp*
moreover have $**$: *T-class* $i = \{$*join* (*L-line l*) $s\ n\ m \mid l\ s\ .\ l \in \{..<t\} \wedge s \in S$ ‘ $($*classes k t i*$)\}$
using *a* **unfolding** *T-class-def* **by** *simp*
ultimately obtain *xs xi* **where** *xdefs*: $x =$ *join* (*L-line xi*) *xs n m* $\wedge$ *xi* $< t \wedge$ *xs* $\in S$ ‘ $($*classes k t i*$)$
by *blast*

from $*$ $**$ **obtain** *ys yi* **where** *ydefs*: $y =$ *join* (*L-line yi*) *ys n m* $\wedge$ *yi* $< t \wedge$ *ys* $\in S$ ‘ $($*classes k t i*$)$
using *a* **by** *auto*

have (*L-line xi*) $\in$ *cube n* $(t+1)$ **using** *L-line-base-prop xdefs* **by** *simp*
moreover have *xs* $\in$ *cube m* $(t+1)$
using *xdefs S-prop subspace-elems-embed imageE image-subset-iff mem-Collect-eq*

unfolding *layered-subspace-def classes-def* **by** *blast*
ultimately have *AA1*: $\chi\ x = \chi L$ (*L-line xi*) *xs* **using** *xdefs* **unfolding** χ*L-def* **by** *simp*

have (*L-line yi*) $\in$ *cube n* $(t+1)$ **using** *L-line-base-prop ydefs* **by** *simp*
moreover have *ys* $\in$ *cube m* $(t+1)$
using *ydefs S-prop subspace-elems-embed imageE image-subset-iff mem-Collect-eq*

unfolding *layered-subspace-def classes-def* **by** *blast*
ultimately have *AA2*: $\chi\ y = \chi L\ (L\text{-}line\ yi)\ ys$ **using** *ydefs* **unfolding** χL*-def* **by** *simp*

have $\forall\, s{<}t.\ \forall\, l < t.\ \chi L\text{-}s\ (L\ (SOME\ p.\ p{\in}cube\ 1\ (t{+}1) \wedge p\ 0 = s))$
$= \chi L\text{-}s\ (L\ (SOME\ p.\ p{\in}cube\ 1\ (t{+}1) \wedge p\ 0 = l))$ **using**
dim1-layered-subspace-mono-line[*of t L n s* χL*-s*] *L-prop assms*(*1*) **by** *blast*
then have *key-aux*: $\chi L\text{-}s\ (L\text{-}line\ s) = \chi L\text{-}s\ (L\text{-}line\ l)$ **if** $s \in \{..{<}t\}\ l \in \{..{<}t\}$ **for** *s l*
using *that* **unfolding** *L-line-def*
by (*metis* (*no-types*, *lifting*) *add.commute*
lessThan-iff less-Suc-eq plus-1-eq-Suc restrict-apply)
have *key*: $\chi L\ (L\text{-}line\ s) = \chi L\ (L\text{-}line\ l)$ **if** $s < t\ l < t$ **for** *s l*
proof−
have *L1*: $\chi L\ (L\text{-}line\ s) \in cube\ m\ (t + 1) \rightarrow_E \{..{<}r\}$ **unfolding** χL*-def*
using *A L-line-base-prop* ‹$s < t$› **by** *simp*
have *L2*: $\chi L\ (L\text{-}line\ l) \in cube\ m\ (t + 1) \rightarrow_E \{..{<}r\}$ **unfolding** χL*-def*
using *A L-line-base-prop* ‹$l < t$› **by** *simp*
have $\varphi\ (\chi L\ (L\text{-}line\ s)) = \chi L\text{-}s\ (L\text{-}line\ s)$ **unfolding** χL*-s-def*
using ‹$s < t$› *L-line-base-prop* **by** *simp*
also have $\ldots = \chi L\text{-}s\ (L\text{-}line\ l)$ **using** *key-aux* ‹$s <t$› ‹$l < t$› **by** *blast*
also have $\ldots = \varphi\ (\chi L\ (L\text{-}line\ l))$ **unfolding** χL*-s-def* **using** *L-line-base-prop* ‹$l{<}t$›
by *simp*
finally have $\varphi\ (\chi L\ (L\text{-}line\ s)) = \varphi\ (\chi L\ (L\text{-}line\ l))$ **by** *simp*
then show $\chi L\ (L\text{-}line\ s) = \chi L\ (L\text{-}line\ l)$
using φ*-prop L-line-base-prop L1 L2* **unfolding** *bij-betw-def inj-on-def* **by** *blast*
qed
then have $\chi L\ (L\text{-}line\ xi)\ xs = \chi L\ (L\text{-}line\ 0)\ xs$ **using** *xdefs assms*(*1*) **by** *metis*
also have $\ldots = \chi S\ xs$ **unfolding** χS*-def* χL*-def* **using** *xdefs L-line-base-prop* **by** *auto*
also have $\ldots = \chi S\ ys$ **using** *xdefs ydefs layered-eq-classes*[*of S k m t r* χS] *S-prop a* **by** *blast*
also have $\ldots = \chi L\ (L\text{-}line\ 0)\ ys$ **unfolding** χS*-def* χL*-def* **using** *xdefs L-line-base-prop*
by *auto*
also have $\ldots = \chi L\ (L\text{-}line\ yi)\ ys$ **using** *ydefs key assms*(*1*) **by** *metis*
finally have *core-prop*: $\chi L\ (L\text{-}line\ xi)\ xs = \chi L\ (L\text{-}line\ yi)\ ys$ **by** *simp*
then have $\chi\ x = \chi\ y$ **using** *AA1 AA2* **by** *simp*
then show $\chi\ x = \chi\ y \wedge \chi\ x < r$
using *xdefs AA1 key assms*(*1*) *A*
‹*L-line xi* $\in$ *cube n* $(t + 1)$› ‹$xs \in cube\ m\ (t + 1)$› **by** *blast*
qed
then have $\exists\, c{<}r.\ \forall\, x \in T$ ‘ *classes* $(k{+}1)\ t\ i.\ \chi\ x = c$ **if** $i \leq k$ **for** *i*
using *that assms*(*5*) **by** *blast*

moreover have $\exists\, c{<}r.\ \forall\, x \in T$ ‘ *classes* $(k{+}1)\ t\ (k{+}1).\ \chi\ x = c$

```
    proof −
     have ∀ x ∈ classes (k+1) t (k+1). ∀ u < k + 1. x u = t unfolding classes-def
by auto
      have (λu. t) ‘ {..<k + 1} ⊆ {..<t + 1} by auto
      then have ∃!y ∈ cube (k+1) (t+1). (∀ u < k + 1. y u = t)
        using PiE-uniqueness[of (λu. t) {..<k+1} {..<t+1}] unfolding cube-def
by auto
      then have ∃!y ∈ classes (k+1) t (k+1). (∀ u < k + 1. y u = t)
       unfolding classes-def using classes-subset-cube[of k+1 t k+1] by auto
      then have ∃!y. y ∈ classes (k+1) t (k+1)
       using ‹∀ x ∈ classes (k+1) t (k+1). ∀ u < k + 1. x u = t› by auto
      have ∃ c<r. ∀ y ∈ classes (k+1) t (k+1). χ (T y) = c
      proof −
       have ∀ y ∈ classes (k+1) t (k+1). T y ∈ cube (n+m) (t+1) using T-prop
classes-subset-cube
         by blast
       then have ∀ y ∈ classes (k+1) t (k+1). χ (T y) < r using χ-prop
         unfolding n-def d-def using M′-prop by auto
       then show ∃ c<r. ∀ y ∈ classes (k+1) t (k+1). χ (T y) = c
         using ‹∃!y. y ∈ classes (k+1) t (k+1)› by blast
      qed
      then show ∃ c<r. ∀ x ∈ T ‘ classes (k+1) t (k+1). χ x = c by blast
    qed
    ultimately have ∃ c<r. ∀ x ∈ T ‘ classes (k+1) t i. χ x = c if i ≤ k + 1 for i
      using that by (metis Suc-eq-plus1 le-Suc-eq)
    then have ∃ c<r. ∀ x ∈ classes (k+1) t i. χ (T x) = c if i ≤ k + 1 for i
      using that by simp
    then have layered-subspace T (k+1) (n + m) t r χ using subspace-T that(1)
‹n + m = M′›
      unfolding layered-subspace-def by blast
   then show ?thesis using ‹n + m = M′› by blast
  qed
  then show ?thesis unfolding lhj-def
    using m-props
      exI[of λM. ∀ M′≥M. ∀ χ. χ ∈ cube M′ (t + 1)
      →E {..<r} ⟶ (∃ S. layered-subspace S (k + 1) M′ t r
      χ) m]
    by blast
qed

theorem hj-imp-lhj:
  fixes k
  assumes ⋀r′. hj r′ t
  shows lhj r t k
proof (induction k arbitrary: r rule: less-induct)
  case (less k)
  consider k = 0 | k = 1 | k ≥ 2 by linarith
  then show ?case
  proof (cases)
```

```
  case 1
  then show ?thesis using dim0-layered-subspace-ex unfolding lhj-def by auto
 next
  case 2
  then show ?thesis
  proof (cases t > 0)
   case True
   then show ?thesis using hj-imp-lhj-base[of t] assms 2 by blast
  next
   case False
   then show ?thesis using assms unfolding hj-def lhj-def cube-def by fastforce
  qed
 next
  case 3
  note less
  then show ?thesis
  proof (cases t > 0 ∧ r > 0)
   case True
   then show ?thesis  using hj-imp-lhj-step[of t k−1 r]
    using assms less.IH 3 One-nat-def Suc-pred by fastforce
  next
   case False
   then consider t = 0 | t > 0 ∧ r = 0 | t = 0 ∧ r = 0 by fastforce
   then show ?thesis
   proof cases
    case 1
     then show ?thesis using assms unfolding hj-def lhj-def cube-def by
fastforce
   next
    case 2
    then obtain N where N-props: N > 0 ∀ N'≥N. ∀ χ ∈ cube N' t
    →E {..<r}. (∃ L c. c < r ∧ is-line L N' t ∧ (∀ y
    ∈ L ‘ {..<t}. χ y = c)) using assms[of r] unfolding hj-def by force
    have cube N' (t + 1) →E {..<r} = {} if N' ≥ N for N'
    proof−
     have cube N' t ≠ {} using N-props(2) that 2 by fastforce
     then have cube N' (t + 1) ≠ {} using cube-subset[of N' t] by blast
     then show ?thesis using 2 by blast
    qed
    then show ?thesis unfolding lhj-def using N-props(1) by blast
   next
    case 3
    then have (∃ L c. c < r ∧ is-line L N' t ∧ (∀ y ∈ L ‘ {..<t}. χ y = c))
    ⟹ False for N' χ by blast
    then have False using assms 3 unfolding hj-def cube-def by fastforce
    then show ?thesis by blast
   qed

  qed
```

qed
qed

2.2 Theorem 5

We provide a way to construct a monochromatic line in C^n_{t+1} from a k-dimensional k-coloured layered subspace S in C^n_{t+1}. The idea is to rely on the fact that there are $k+1$ classes in S, but only k colours. It thus follows from the Pigeonhole Principle that two classes must share the same colour. The way classes are defined allows for a straightforward construction of a line with points only from those two classes. Thus we have our monochromatic line.

theorem *layered-subspace-to-mono-line*:
assumes *layered-subspace S k n t k* χ
and $t > 0$
shows $(\exists\, L.\ \exists\, c{<}k.$ *is-line L n* $(t{+}1) \wedge (\forall\, y \in L$ ‘ $\{..{<}t{+}1\}.\ \chi\ y = c))$
proof−
define *x* **where** $x \equiv (\lambda i{\in}\{..k\}.\ \lambda j{\in}\{..{<}k\}.$ (*if* $j < k - i$ *then 0 else t*))

have *A*: *x i* $\in$ *cube k* $(t + 1)$ **if** $i \leq k$ **for** *i* **using** *that* **unfolding** *cube-def x-def*
by *simp*
then have *S* $(x\ i) \in$ *cube n* $(t{+}1)$ **if** $i \leq k$ **for** *i* **using** *that assms*(*1*)
unfolding *layered-subspace-def is-subspace-def* **by** *fast*

have $\chi \in$ *cube n* $(t + 1) \to_E \{..{<}k\}$ **using** *assms* **unfolding** *layered-subspace-def*
by *linarith*
then have χ ‘ (*cube n* $(t{+}1)) \subseteq \{..{<}k\}$ **by** *blast*
then have *card* $(\chi$ ‘ (*cube n* $(t{+}1))) \leq$ *card* $\{..{<}k\}$
by (*meson card-mono finite-lessThan*)
then have $*$: *card* $(\chi$ ‘ (*cube n* $(t{+}1))) \leq k$ **by** *auto*
have $k > 0$ **using** *assms*(*1*) **unfolding** *layered-subspace-def* **by** *auto*
have *inj-on x* $\{..k\}$
proof −
have $*$:*x i1* $(k - i2) \neq$ *x i2* $(k - i2)$ **if** $i1 \leq k\ i2 \leq k\ i1 \neq i2\ i1 < i2$ **for** *i1 i2*
using *that assms*(*2*) **unfolding** *x-def* **by** *auto*
have $\exists j{<}k.$ *x i1 j* $\neq$ *x i2 j* **if** $i1 \leq k\ i2 \leq k\ i1 \neq i2$ **for** *i1 i2*
proof (*cases i1* $\leq$ *i2*)
case *True*
then have $k - i2 < k$
using ‹$0 < k$› *that*(*3*) **by** *linarith*
then show *?thesis* **using** *that* $*$
by (*meson True nat-less-le*)
next
case *False*
then have *i2* $<$ *i1* **by** *simp*
then show *?thesis* **using** *that* $*$[*of i2 i1*] ‹$k > 0$›
by (*metis diff-less gr-implies-not0 le0 nat-less-le*)
qed

then have *x i1* $\neq$ *x i2* **if** *i1* $\leq$ *k i2* $\leq$ *k i1* $\neq$ *i2 i1* $<$ *i2* **for** *i1 i2* **using** *that*
by *fastforce*
then show *?thesis* **unfolding** *inj-on-def* **by** (*metis atMost-iff linorder-cases*)
qed
then have *card* (*x* ‘ {..*k*}) = *card* {..*k*} **using** *card-image* **by** *blast*
then have *B*: *card* (*x* ‘ {..*k*}) = *k+1* **by** *simp*
have *x* ‘ {..*k*} $\subseteq$ *cube k* (*t+1*) **using** *A* **by** *blast*
then have *S* ‘ *x* ‘ {..*k*} $\subseteq$ *S* ‘ *cube k* (*t+1*) **by** *fast*
also have ... $\subseteq$ *cube n* (*t+1*)
by (*meson assms*(*1*) *layered-subspace-def subspace-elems-embed*)
finally have *S* ‘ *x* ‘ {..*k*} $\subseteq$ *cube n* (*t+1*) **by** *blast*
then have χ ‘ *S* ‘ *x* ‘ {..*k*} $\subseteq$ χ ‘ *cube n* (*t+1*) **by** *auto*
then have *card* (χ ‘ *S* ‘ *x* ‘ {..*k*}) $\leq$ *card* (χ ‘ *cube n* (*t+1*))
by (*simp add*: *card-mono cube-def finite-PiE*)
also have ... $\leq$ *k* **using** $*$ **by** *blast*
also have ... $<$ *k* + *1* **by** *auto*
also have ... = *card* {..*k*} **by** *simp*
also have ... = *card* (*x* ‘ {..*k*}) **using** *B* **by** *auto*
also have ... = *card* (*S* ‘ *x* ‘ {..*k*})
using *subspace-inj-on-cube*[*of S k n t+1*] *card-image*[*of S x* ‘ {..*k*}]
inj-on-subset[*of S cube k* (*t+1*) *x* ‘ {..*k*}] *assms*(*1*) ‹*x* ‘ {..*k*} $\subseteq$ *cube k* (*t* + *1*)›
unfolding *layered-subspace-def* **by** *simp*
finally have *card* (χ ‘ *S* ‘ *x* ‘ {..*k*}) $<$ *card* (*S* ‘ *x* ‘ {..*k*}) **by** *blast*
then have $\neg$*inj-on* χ (*S* ‘ *x* ‘ {..*k*}) **using** *pigeonhole*[*of* χ *S* ‘ *x* ‘ {..*k*}] **by** *blast*
then have $\exists$ *a b*. *a* $\in$ *S* ‘ *x* ‘ {..*k*} $\wedge$ *b* $\in$ *S* ‘ *x* ‘ {..*k*} $\wedge$ *a* $\neq$ *b* $\wedge$ χ *a* =
χ *b* **unfolding** *inj-on-def* **by** *auto*
then obtain *ax bx* **where** *ab-props*: *ax* $\in$ *S* ‘ *x* ‘ {..*k*} $\wedge$ *bx* $\in$ *S* ‘ *x* ‘ {..*k*} $\wedge$ *ax* $\neq$ *bx* $\wedge$
χ *ax* = χ *bx* **by** *blast*
then have $\exists$ *u v*. *u* $\in$ {..*k*} $\wedge$ *v* $\in$ {..*k*} $\wedge$ *u* $\neq$ *v* $\wedge$ χ (*S* (*x u*)) = χ (*S* (*x*
v)) **by** *blast*
then obtain *u v* **where** *uv-props*: *u* $\in$ {..*k*} $\wedge$ *v* $\in$ {..*k*} $\wedge$ *u* $<$ *v* $\wedge$ χ (*S* (*x u*))
= χ (*S* (*x v*)) **by** (*metis linorder-cases*)

let *?f* = λ*s*. (λ*i* $\in$ {..<*k*}. *if i* $<$ *k* $-$ *v then 0 else* (*if i* $<$ *k* $-$ *u then s else t*))
define *y* **where** *y* $\equiv$ (λ*s* $\in$ {..*t*}. *S* (*?f s*))

have *line1*: *?f s* $\in$ *cube k* (*t+1*) **if** *s* $\leq$ *t* **for** *s* **unfolding** *cube-def* **using** *that* **by** *auto*

have *f-cube*: *?f j* $\in$ *cube k* (*t+1*) **if** *j* $<$ *t+1* **for** *j* **using** *line1 that* **by** *simp*
have *f-classes-u*: *?f j* $\in$ *classes k t u* **if** *j-prop*: *j* $<$ *t* **for** *j*
using *that j-prop uv-props f-cube* **unfolding** *classes-def* **by** *auto*
have *f-classes-v*: *?f j* $\in$ *classes k t v* **if** *j-prop*: *j* = *t* **for** *j*
using *that j-prop uv-props assms*(*2*) *f-cube* **unfolding** *classes-def* **by** *auto*

obtain *B f* **where** *Bf-props*: *disjoint-family-on B* {..*k*} $\bigcup$(*B* ‘ {..*k*}) = {..<*n*}
({} $\notin$ *B* ‘ {..<*k*})

```
    f ∈ (B k) →E {..<t+1} S ∈ (cube k (t+1)) →E (cube n (t+1))
    (∀ y ∈ cube k (t+1). (∀ i ∈ B k. S y i = f i) ∧ (∀ j<k. ∀ i ∈ B j.
      (S y) i = y j))
      using assms(1) unfolding layered-subspace-def is-subspace-def by auto

  have y ∈ {..<t+1} →E cube n (t+1) unfolding y-def using line1 ‹S ‘ cube k
(t + 1)
  ⊆ cube n (t + 1)› by auto
  moreover have (∀ u<t+1. ∀ v<t+1. y u j = y v j) ∨ (∀ s<t+1. y s j = s)
    if j-prop: j<n for j
  proof−
    show (∀ u<t+1. ∀ v<t+1. y u j = y v j) ∨ (∀ s<t+1. y s j = s)
    proof −
      consider j ∈ B k | ∃ ii<k. j ∈ B ii using Bf-props(2) j-prop
        by (metis UN-E atMost-iff le-neq-implies-less lessThan-iff)
      then have y a j = y b j ∨ y s j = s if a < t + 1 b < t +1 s < t +1 for a b s
      proof cases
        case 1
        then have y a j = S (?f a) j using that(1) unfolding y-def by auto
        also have ... = f j using Bf-props(6) f-cube 1 that(1) by auto
        also have ... = S (?f b) j using Bf-props(6) f-cube 1 that(2) by auto
        also have ... = y b j using that(2) unfolding y-def by simp
        finally show ?thesis by simp
      next
        case 2
        then obtain ii where ii-prop: ii < k ∧ j ∈ B ii by blast
        then consider ii < k − v | ii ≥ k − v ∧ ii < k − u | ii ≥ k − u ∧ ii < k
using not-less
          by blast
        then show ?thesis
        proof cases
          case 1
          then have y a j = S (?f a) j using that(1) unfolding y-def by auto
          also have ... = (?f a) ii using Bf-props(6) f-cube that(1) ii-prop by auto
          also have ... = 0 using 1 by (simp add: ii-prop)
          also have ... = (?f b) ii using 1 by (simp add: ii-prop)
            also have ... = S (?f b) j using Bf-props(6) f-cube that(2) ii-prop by
auto
          also have ... = y b j using that(2) unfolding y-def by auto
          finally show ?thesis by simp
        next
          case 2
          then have y s j = S (?f s) j using that(3) unfolding y-def by auto
          also have ... = (?f s) ii using Bf-props(6) f-cube that(3) ii-prop by auto
          also have ... = s using 2 by (simp add: ii-prop)
          finally show ?thesis by simp
        next
          case 3
          then have y a j = S (?f a) j using that(1) unfolding y-def by auto
```

```
        also have  ... = (?f a) ii using Bf-props(6) f-cube that(1) ii-prop by auto
        also have  ... = t using 3 uv-props by auto
        also have  ... = (?f b) ii using 3 uv-props by auto
         also have  ... = S (?f b) j using Bf-props(6) f-cube that(2) ii-prop by
auto
        also have  ... = y b j using that(2) unfolding y-def by auto
        finally show ?thesis by simp
       qed
     qed
     then show ?thesis by blast
   qed
 qed
 moreover have ∃ j < n. ∀ s<t+1. y s j = s
 proof −
   have k > 0 using uv-props by simp
   have k − v < k using uv-props by auto
   have k − v < k − u using uv-props by auto
   then have B (k − v) ≠ {} using Bf-props(3) uv-props by auto
  then obtain j where j-prop: j ∈ B (k − v) ∧ j < n using Bf-props(2) uv-props
by force
   then have y s j = s if s<t+1 for s
   proof
     have y s j = S (?f s) j using that unfolding y-def by auto
     also have  ... = (?f s) (k − v) using Bf-props(6) f-cube that j-prop ‹k − v
< k› by fast
     also have  ... = s using that j-prop ‹k − v < k − u› by simp
     finally show ?thesis .
   qed
   then show ∃ j < n. ∀ s<t+1. y s j = s using j-prop by blast
 qed
 ultimately have Z1: is-line y n (t+1) unfolding is-line-def by blast
 moreover
 {
   have k-colour: χ e < k if e ∈ y ‘ {..<t+1} for e
     using ‹y ∈ {..<t+1} →E cube n (t + 1)› ‹χ ∈ cube n (t + 1)
     →E {..<k}› that by auto
   have χ e1 = χ e2 ∧ χ e1 < k if e1 ∈ y ‘ {..<t+1} e2 ∈ y ‘ {..<t+1} for e1 e2
   proof
     from that obtain i1 i2 where i-props: i1 < t + 1 i2 < t + 1 e1 = y i1 e2
= y i2 by blast
     from i-props(1,2) have χ (y i1) = χ (y i2)
     proof (induction i1 i2 rule: linorder-wlog)
       case (le a b)
       then show ?case
       proof (cases a = b)
         case True
         then show ?thesis by blast
       next
         case False
```

```
        then have a < b using le by linarith
        then consider b = t | b < t using le.prems(2) by linarith
        then show ?thesis
        proof cases
          case 1
          then have y b ∈ S ‘ classes k t v
          proof −
            have y b = S (?f b) unfolding y-def using ‹b = t› by auto
            moreover have ?f b ∈ classes k t v using ‹b = t› f-classes-v by blast
            ultimately show y b ∈ S ‘ classes k t v by blast
          qed
          moreover have x u ∈ classes k t u
          proof −
            have x u cord = t if cord ∈ {k − u..<k} for cord using uv-props that
unfolding x-def by simp
            moreover
            {
              have x u cord ≠ t if cord ∈ {..<k − u} for cord
                using uv-props that assms(2) unfolding x-def by auto
              then have t ∉ x u ‘ {..<k − u} by blast
            }
            ultimately show x u ∈ classes k t u unfolding classes-def
              using ‹x ‘ {..k} ⊆ cube k (t + 1)› uv-props by blast
          qed
          moreover have x v ∈ classes k t v
          proof −
            have x v cord = t if cord ∈ {k − v..<k} for cord using uv-props that
unfolding x-def by simp
            moreover
            {
              have x v cord ≠ t if cord ∈ {..<k − v} for cord
                using uv-props that assms(2) unfolding x-def by auto
              then have t ∉ x v ‘ {..<k − v} by blast
            }
            ultimately show x v ∈ classes k t v unfolding classes-def
              using ‹x ‘ {..k} ⊆ cube k (t + 1)› uv-props by blast
          qed
          moreover have χ (y b) = χ (S (x v))
            using assms(1) calculation(1, 3) unfolding layered-subspace-def by
(metis imageE uv-props)
          moreover have y a ∈ S ‘ classes k t u
          proof −
            have y a = S (?f a) unfolding y-def using ‹a < b› 1 by simp
             moreover have ?f a ∈ classes k t u using ‹a < b› 1 f-classes-u by
blast
            ultimately show y a ∈ S ‘ classes k t u by blast
          qed
         moreover have χ (y a) = χ (S (x u)) using assms(1) calculation(2, 5)
            unfolding layered-subspace-def by (metis imageE uv-props)
```

```
          ultimately have χ (y a) = χ (y b) using uv-props by simp
          then show ?thesis by blast
        next
          case 2
          then have a < t using ‹a < b› less-trans by blast
          then have y a ∈ S ‘ classes k t u
          proof −
            have y a = S (?f a) unfolding y-def using ‹a < t› by auto
           moreover have ?f a ∈ classes k t u using ‹a < t› f-classes-u by blast
            ultimately show y a ∈ S ‘ classes k t u by blast
          qed
          moreover have y b ∈ S ‘ classes k t u
          proof −
            have y b = S (?f b) unfolding y-def using ‹b < t› by auto
            moreover have ?f b ∈ classes k t u using ‹b < t› f-classes-u by blast
            ultimately show y b ∈ S ‘ classes k t u by blast
          qed
         ultimately have χ (y a) = χ (y b) using assms(1) uv-props unfolding
layered-subspace-def
            by (metis imageE)
          then show ?thesis by blast
        qed
      qed
    next
      case (sym a b)
      then show ?case by presburger
    qed
    then show χ e1 = χ e2 using i-props(3,4) by blast
  qed (use that(1) k-colour in blast)
  then have Z2: ∃ c < k. ∀ e ∈ y ‘ {..<t+1}. χ e = c
    by (meson image-eqI lessThan-iff less-add-one)
 }
 ultimately show ∃ L c. c < k ∧ is-line L n (t + 1) ∧ (∀ y∈L ‘ {..<t + 1}. χ y
= c)
  by blast

qed
```

2.3 Corollary 6

```
corollary lhj-imp-hj:
 assumes (⋀r k. lhj r t k)
   and t>0
 shows (hj r (t+1))
 using assms(1)[of r r] assms(2) unfolding lhj-def hj-def using layered-subspace-to-mono-line[of
- r - t] by metis
```

2.4 Main result

2.4.1 Edge cases and auxiliary lemmas

lemma *single-point-line*:
assumes $N > 0$
shows *is-line* ($\lambda s{\in}\{..<1\}.\ \lambda a{\in}\{..<N\}.\ 0$) $N\ 1$
using *assms* **unfolding** *is-line-def cube-def* **by** *auto*

lemma *single-point-line-is-monochromatic*:
assumes $\chi \in$ *cube* $N\ 1 \rightarrow_E \{..<r\}$ $N > 0$
shows ($\exists\, c < r.$ *is-line* ($\lambda s{\in}\{..<1\}.\ \lambda a{\in}\{..<N\}.\ 0$) $N\ 1 \wedge (\forall\, i \in$
($\lambda s{\in}\{..<1\}.\ \lambda a{\in}\{..<N\}.\ 0$) ‘ $\{..<1\}.\ \chi\ i = c$))
proof −
have *is-line* ($\lambda s{\in}\{..<1\}.\ \lambda a{\in}\{..<N\}.\ 0$) $N\ 1$ **using** *assms*(*2*) *single-point-line* **by** *blast*
moreover have $\exists\, c < r.\ \chi\ ((\lambda s{\in}\{..<1\}.\ \lambda a{\in}\{..<N\}.\ 0)\ j) = c$
if (j::*nat*) $<$ *1* **for** j **using** *assms line-points-in-cube calculation that* **unfolding** *cube-def* **by** *blast*
ultimately show *?thesis* **by** *auto*
qed

lemma *hj-r-nonzero-t-0*:
assumes $r > 0$
shows *hj r 0*
proof−
have ($\exists\, L\ c.\ c < r \wedge$ *is-line* $L\ N'\ 0 \wedge (\forall\, y \in L$ ‘ $\{..<0$::*nat*$\}.\ \chi\ y = c$))
if $N' \geq 1\ \chi \in$ *cube* $N'\ 0 \rightarrow_E \{..<r\}$ **for** $N'\ \chi$ **using** *assms is-line-def that*(*1*) **by** *fastforce*
then show *?thesis* **unfolding** *hj-def* **by** *auto*
qed

Any cube over 1 element always has a single point, which also forms the only line in the cube. Since it's a single point line, it's trivially monochromatic. We show the result for dimension 1.

lemma *hj-t-1*: *hj r 1*
unfolding *hj-def*
proof−
let *?N = 1*
have $\exists\, L\ c.\ c < r \wedge$ *is-line* $L\ N'\ 1 \wedge (\forall\, y{\in}L$ ‘ $\{..<1\}.\ \chi\ y = c$) **if** $N' \geq$ *?N* $\chi \in$ *cube* $N'\ 1 \rightarrow_E \{..<r\}$ **for** $N'\ \chi$
using *single-point-line-is-monochromatic*[*of* $\chi\ N'\ r$] *that* **by** *force*
then show $\exists\, N{>}0.\ \forall\, N'{\geq}N.\ \forall\, \chi.\ \chi \in$ *cube* $N'\ 1 \rightarrow_E \{..<r\} \longrightarrow (\exists\, L\ c.\ c < r \wedge$ *is-line* $L\ N'\ 1 \wedge (\forall\, y{\in}L$ ‘ $\{..<1\}.\ \chi\ y = c$))
by *blast*
qed

2.4.2 Main theorem

We state the main result *hj r t*. The explanation for the choice of assumption is offered subsequently.

theorem *hales-jewett*:
  assumes $\neg(r = 0 \wedge t = 0)$
  shows *hj r t*
  using *assms*
proof (*induction t arbitrary*: *r*)
  case *0*
  then show *?case* **using** *hj-r-nonzero-t-0*[*of r*] **by** *blast*
next
  case (*Suc t*)
  then show *?case* **using** *hj-t-1*[*of r*] *hj-imp-lhj*[*of t*] *lhj-imp-hj*[*of t r*] **by** *auto*
qed

We offer a justification for having excluded the special case $r = t = 0$ from the statement of the main theorem *hales-jewett*. The exclusion is a consequence of the fact that colourings are defined as members of the function set *cube n t* $\rightarrow_E$ $\{..<r\}$, which for $r = t = 0$ means there's a dummy colouring $\lambda x.$ *undefined*, even though *cube n 0* $= \{\}$ for $n > 0$. Hence, in this case, no line exists at all (let alone one monochromatic under the aforementioned colouring). This means *hj 0 0* $\Longrightarrow$ *False*—but only because of the quirky behaviour of the FuncSet *cube n t* $\rightarrow_E$ $\{..<r\}$. This could have been circumvented by letting colourings χ be arbitrary functions constraint only by χ ' *cube n t* $\subseteq \{..<r\}$. We avoided this in order to have consistency with the cube's definition, for which FuncSets were crucial because the proof heavily relies on arguments about the cardinality of the cube. he constraint x ' $\{..<n\} \subseteq \{..<t\}$ for elements x of C_t^n would not have sufficed there, as there are infinitely many functions over the naturals satisfying it.

end

References

[1] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey Theory, 2nd Edition*. Wiley-Interscience, March 1990.

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