

The Hahn and Jordan Decomposition Theorems

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1 Introduction

Signed measures are a generalization of measures that can map measurable sets to negative values. In this work we formalize the Hahn decomposition theorem for signed measures, namely that if $(\Omega, \mathcal{A}, \mu)$ is a measure space for a signed measure μ , then Ω can be decomposed as $\Omega^+ \cup \Omega^-$, where every measurable subset of Ω^+ has a positive measure, and every measurable subset of Ω^- has a negative measure. We then prove that this decomposition is essentially unique, meaning that if $X^+ \cup X^-$ is another such decomposition, then any measurable subset in $(\Omega^+ \triangle X^+) \cup (\Omega^- \triangle X^-)$ has a zero measure.

We also formalize the Jordan decomposition theorem as a corollary, which states that the signed measure μ admits a unique decomposition into a difference $\mu = \mu^+ - \mu^-$ of two positive measures, at least one of which is finite, and such that for any Hahn decomposition $\Omega^+ \cup \Omega^-$ and measurable set A , if $A \subseteq \Omega^-$ then $\mu^+(A) = 0$ and if $A \subseteq \Omega^+$ then $\mu^-(A) = 0$. The formalization is mostly based on [1], Section 16 of Chapter 4.

2 Signed measures

In this section we define signed measures. These are generalizations of measures that can also take negative values but cannot contain both ∞ and $-\infty$ in their range.

2.1 Basic definitions

theory *Hahn-Jordan-Decomposition* **imports**

HOL-Probability.Probability

Hahn-Jordan-Prelims

begin

definition *signed-measure*:: 'a measure \Rightarrow ('a set \Rightarrow ereal) \Rightarrow bool **where**
signed-measure $M \mu \iff \mu \{ \} = 0 \wedge (-\infty \notin \text{range } \mu \vee \infty \notin \text{range } \mu) \wedge$
 $(\forall A. \text{range } A \subseteq \text{sets } M \longrightarrow \text{disjoint-family } A \longrightarrow \bigcup (\text{range } A) \in \text{sets } M \longrightarrow$
 $(\lambda i. \mu (A i)) \text{ sums } \mu (\bigcup (\text{range } A))) \wedge$
 $(\forall A. \text{range } A \subseteq \text{sets } M \longrightarrow \text{disjoint-family } A \longrightarrow \bigcup (\text{range } A) \in \text{sets } M \longrightarrow$
 $|\mu (\bigcup (\text{range } A))| < \infty \longrightarrow \text{summable } (\lambda i. \text{real-of-ereal } |\mu (A i)|))$

lemma *signed-measure-empty*:

assumes *signed-measure* $M \mu$

shows $\mu \{ \} = 0$ *<proof>*

lemma *signed-measure-sums*:

assumes *signed-measure* $M \mu$

and $\text{range } A \subseteq M$

and *disjoint-family* A

and $\bigcup (\text{range } A) \in \text{sets } M$

shows $(\lambda i. \mu (A i)) \text{ sums } \mu (\bigcup (\text{range } A))$

<proof>

lemma *signed-measure-summable*:

assumes *signed-measure* $M \mu$

and $\text{range } A \subseteq M$

and *disjoint-family* A

and $\bigcup (\text{range } A) \in \text{sets } M$

and $|\mu (\bigcup (\text{range } A))| < \infty$

shows *summable* $(\lambda i. \text{real-of-ereal } |\mu (A i)|)$

<proof>

lemma *signed-measure-inf-sum*:

assumes *signed-measure* $M \mu$

and $\text{range } A \subseteq M$

and *disjoint-family* A

and $\bigcup (\text{range } A) \in \text{sets } M$

shows $(\sum i. \mu (A i)) = \mu (\bigcup (\text{range } A))$ *<proof>*

lemma *signed-measure-abs-convergent*:

assumes *signed-measure* $M \mu$
and *range* $A \subseteq \text{sets } M$
and *disjoint-family* A
and $\bigcup (\text{range } A) \in \text{sets } M$
and $|\mu (\bigcup (\text{range } A))| < \infty$
shows *summable* $(\lambda i. \text{real-of-ereal } |\mu (A i)|)$ *<proof>*

lemma *signed-measure-additive:*

assumes *signed-measure* $M \mu$
shows *additive* $M \mu$
<proof>

lemma *signed-measure-add:*

assumes *signed-measure* $M \mu$
and $a \in \text{sets } M$
and $b \in \text{sets } M$
and $a \cap b = \{\}$
shows $\mu (a \cup b) = \mu a + \mu b$ *<proof>*

lemma *signed-measure-disj-sum:*

shows *finite* $I \implies \text{signed-measure } M \mu \implies \text{disjoint-family-on } A I \implies$
 $(\bigwedge i. i \in I \implies A i \in \text{sets } M) \implies \mu (\bigcup i \in I. A i) = (\sum i \in I. \mu (A i))$
<proof>

lemma *pos-signed-measure-count-additive:*

assumes *signed-measure* $M \mu$
and $\forall E \in \text{sets } M. 0 \leq \mu E$
shows *countably-additive* $(\text{sets } M) (\lambda A. \text{e2ennreal } (\mu A))$
<proof>

lemma *signed-measure-minus:*

assumes *signed-measure* $M \mu$
shows *signed-measure* $M (\lambda A. - \mu A)$ *<proof>*

locale *near-finite-function* =

fixes $\mu:: 'b \text{ set} \Rightarrow \text{ereal}$
assumes *inf-range*: $-\infty \notin \text{range } \mu \vee \infty \notin \text{range } \mu$

lemma **(in** *near-finite-function*) *finite-subset:*

assumes $|\mu E| < \infty$
and $A \subseteq E$
and $\mu E = \mu A + \mu (E - A)$
shows $|\mu A| < \infty$
<proof>

locale *signed-measure-space* =

fixes $M:: 'a \text{ measure}$ **and** μ
assumes *sgn-meas*: *signed-measure* $M \mu$

sublocale *signed-measure-space* \subseteq *near-finite-function*
<proof>

context *signed-measure-space*
begin

lemma *signed-measure-finite-subset*:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

and $A \in \text{sets } M$

and $A \subseteq E$

shows $|\mu A| < \infty$

<proof>

lemma *measure-space-e2ennreal* :

assumes *measure-space* (*space* M) (*sets* M) $m \wedge (\forall E \in \text{sets } M. m E < \infty) \wedge$
 $(\forall E \in \text{sets } M. m E \geq 0)$

shows $\forall E \in \text{sets } M. e2ennreal (m E) < \infty$

<proof>

2.2 Positive and negative subsets

The Hahn decomposition theorem is based on the notions of positive and negative measurable sets. A measurable set is positive (resp. negative) if all its measurable subsets have a positive (resp. negative) measure by μ . The decomposition theorem states that any measure space for a signed measure can be decomposed into a positive and a negative measurable set.

definition *pos-meas-set* **where**

pos-meas-set $E \longleftrightarrow E \in \text{sets } M \wedge (\forall A \in \text{sets } M. A \subseteq E \longrightarrow 0 \leq \mu A)$

definition *neg-meas-set* **where**

neg-meas-set $E \longleftrightarrow E \in \text{sets } M \wedge (\forall A \in \text{sets } M. A \subseteq E \longrightarrow \mu A \leq 0)$

lemma *pos-meas-setI*:

assumes $E \in \text{sets } M$

and $\bigwedge A. A \in \text{sets } M \implies A \subseteq E \implies 0 \leq \mu A$

shows *pos-meas-set* E *<proof>*

lemma *pos-meas-setD1* :

assumes *pos-meas-set* E

shows $E \in \text{sets } M$

<proof>

lemma *neg-meas-setD1* :

assumes *neg-meas-set* E

shows $E \in \text{sets } M$ *<proof>*

lemma *neg-meas-setI*:

assumes $E \in \text{sets } M$

and $\bigwedge A. A \in \text{sets } M \implies A \subseteq E \implies \mu A \leq 0$
shows *neg-meas-set* E $\langle \text{proof} \rangle$

lemma *pos-meas-self*:
assumes *pos-meas-set* E
shows $0 \leq \mu E$ $\langle \text{proof} \rangle$

lemma *empty-pos-meas-set*:
shows *pos-meas-set* $\{\}$
 $\langle \text{proof} \rangle$

lemma *empty-neg-meas-set*:
shows *neg-meas-set* $\{\}$
 $\langle \text{proof} \rangle$

lemma *pos-measure-meas*:
assumes *pos-meas-set* E
and $A \subseteq E$
and $A \in \text{sets } M$
shows $0 \leq \mu A$ $\langle \text{proof} \rangle$

lemma *pos-meas-subset*:
assumes *pos-meas-set* A
and $B \subseteq A$
and $B \in \text{sets } M$
shows *pos-meas-set* B $\langle \text{proof} \rangle$

lemma *neg-meas-subset*:
assumes *neg-meas-set* A
and $B \subseteq A$
and $B \in \text{sets } M$
shows *neg-meas-set* B $\langle \text{proof} \rangle$

lemma *pos-meas-set-Union*:
assumes $\bigwedge (i::\text{nat}). \text{pos-meas-set } (A\ i)$
and $\bigwedge i. A\ i \in \text{sets } M$
and $|\mu (\bigcup i. A\ i)| < \infty$
shows *pos-meas-set* $(\bigcup i. A\ i)$
 $\langle \text{proof} \rangle$

lemma *pos-meas-set-pos-lim*:
assumes $\bigwedge (i::\text{nat}). \text{pos-meas-set } (A\ i)$
and $\bigwedge i. A\ i \in \text{sets } M$
shows $0 \leq \mu (\bigcup i. A\ i)$
 $\langle \text{proof} \rangle$

lemma *pos-meas-disj-union*:
assumes *pos-meas-set* A
and *pos-meas-set* B

```

    and  $A \cap B = \{\}$ 
    shows pos-meas-set ( $A \cup B$ ) <proof>

lemma pos-meas-set-union:
  assumes pos-meas-set  $A$ 
    and pos-meas-set  $B$ 
  shows pos-meas-set ( $A \cup B$ )
<proof>

lemma neg-meas-disj-union:
  assumes neg-meas-set  $A$ 
    and neg-meas-set  $B$ 
    and  $A \cap B = \{\}$ 
  shows neg-meas-set ( $A \cup B$ ) <proof>

lemma neg-meas-set-union:
  assumes neg-meas-set  $A$ 
    and neg-meas-set  $B$ 
  shows neg-meas-set ( $A \cup B$ )
<proof>

lemma neg-meas-self :
  assumes neg-meas-set  $E$ 
  shows  $\mu E \leq 0$  <proof>

lemma pos-meas-set-opp:
  assumes signed-measure-space.pos-meas-set  $M$  ( $\lambda A. - \mu A$ )  $A$ 
  shows neg-meas-set  $A$ 
<proof>

lemma neg-meas-set-opp:
  assumes signed-measure-space.neg-meas-set  $M$  ( $\lambda A. - \mu A$ )  $A$ 
  shows pos-meas-set  $A$ 
<proof>
end

lemma signed-measure-inter:
  assumes signed-measure  $M$   $\mu$ 
    and  $A \in \text{sets } M$ 
  shows signed-measure  $M$  ( $\lambda E. \mu (E \cap A)$ ) <proof>

context signed-measure-space
begin
lemma pos-signed-to-meas-space :
  assumes pos-meas-set  $M1$ 
    and  $m1 = (\lambda A. \mu (A \cap M1))$ 
  shows measure-space (space  $M$ ) (sets  $M$ )  $m1$  <proof>

lemma neg-signed-to-meas-space :

```

assumes *neg-meas-set* $M2$
and $m2 = (\lambda A. -\mu (A \cap M2))$
shows *measure-space* (*space* M) (*sets* M) $m2$ \langle *proof* \rangle

lemma *pos-part-meas-nul-neg-set* :

assumes *pos-meas-set* $M1$
and *neg-meas-set* $M2$
and $m1 = (\lambda A. \mu (A \cap M1))$
and $E \in$ *sets* M
and $E \subseteq M2$
shows $m1 E = 0$
 \langle *proof* \rangle

lemma *neg-part-meas-nul-pos-set* :

assumes *pos-meas-set* $M1$
and *neg-meas-set* $M2$
and $m2 = (\lambda A. -\mu (A \cap M2))$
and $E \in$ *sets* M
and $E \subseteq M1$
shows $m2 E = 0$
 \langle *proof* \rangle

definition *pos-sets* **where**

$pos\text{-sets} = \{A. A \in$ *sets* $M \ \wedge$ *pos-meas-set* $A\}$

definition *pos-img* **where**

$pos\text{-img} = \{\mu A | A. A \in$ *pos-sets}\}*

2.3 Essential uniqueness

In this part, under the assumption that a measure space for a signed measure admits a decomposition into a positive and a negative set, we prove that this decomposition is essentially unique; in other words, that if two such decompositions (P, N) and (X, Y) exist, then any measurable subset of $(P \triangle X) \cup (N \triangle Y)$ has a null measure.

definition *hahn-space-decomp* **where**

$hahn\text{-space-decomp } M1 M2 \equiv (pos\text{-meas-set } M1) \wedge (neg\text{-meas-set } M2) \wedge$
 $(space\ M = M1 \cup M2) \wedge (M1 \cap M2 = \{\})$

lemma *pos-neg-null-set*:

assumes *pos-meas-set* A
and *neg-meas-set* A
shows $\mu A = 0$ \langle *proof* \rangle

lemma *pos-diff-neg-meas-set*:

assumes (*pos-meas-set* $M1$)
and (*neg-meas-set* $N2$)
and (*space* $M = N1 \cup N2$)

and $N1 \in \text{sets } M$
shows $\text{neg-meas-set } ((M1 - N1) \cap \text{space } M)$ $\langle \text{proof} \rangle$

lemma *neg-diff-pos-meas-set*:
assumes $(\text{neg-meas-set } M2)$
and $(\text{pos-meas-set } N1)$
and $(\text{space } M = N1 \cup N2)$
and $N2 \in \text{sets } M$
shows $\text{pos-meas-set } ((M2 - N2) \cap \text{space } M)$
 $\langle \text{proof} \rangle$

lemma *pos-sym-diff-neg-meas-set*:
assumes $\text{hahn-space-decomp } M1 M2$
and $\text{hahn-space-decomp } N1 N2$
shows $\text{neg-meas-set } ((\text{sym-diff } M1 N1) \cap \text{space } M)$ $\langle \text{proof} \rangle$

lemma *neg-sym-diff-pos-meas-set*:
assumes $\text{hahn-space-decomp } M1 M2$
and $\text{hahn-space-decomp } N1 N2$
shows $\text{pos-meas-set } ((\text{sym-diff } M2 N2) \cap \text{space } M)$ $\langle \text{proof} \rangle$

lemma *pos-meas-set-diff*:
assumes $\text{pos-meas-set } A$
and $B \in \text{sets } M$
shows $\text{pos-meas-set } ((A - B) \cap (\text{space } M))$ $\langle \text{proof} \rangle$

lemma *pos-meas-set-sym-diff*:
assumes $\text{pos-meas-set } A$
and $\text{pos-meas-set } B$
shows $\text{pos-meas-set } ((\text{sym-diff } A B) \cap \text{space } M)$ $\langle \text{proof} \rangle$

lemma *neg-meas-set-diff*:
assumes $\text{neg-meas-set } A$
and $B \in \text{sets } M$
shows $\text{neg-meas-set } ((A - B) \cap (\text{space } M))$ $\langle \text{proof} \rangle$

lemma *neg-meas-set-sym-diff*:
assumes $\text{neg-meas-set } A$
and $\text{neg-meas-set } B$
shows $\text{neg-meas-set } ((\text{sym-diff } A B) \cap \text{space } M)$ $\langle \text{proof} \rangle$

lemma *hahn-decomp-space-diff*:
assumes $\text{hahn-space-decomp } M1 M2$
and $\text{hahn-space-decomp } N1 N2$
shows $\text{pos-meas-set } ((\text{sym-diff } M1 N1 \cup \text{sym-diff } M2 N2) \cap \text{space } M)$
 $\text{neg-meas-set } ((\text{sym-diff } M1 N1 \cup \text{sym-diff } M2 N2) \cap \text{space } M)$
 $\langle \text{proof} \rangle$

lemma *hahn-decomp-ess-unique*:

assumes *hahn-space-decomp* $M1\ M2$
and *hahn-space-decomp* $N1\ N2$
and $C \subseteq \text{sym-diff } M1\ N1 \cup \text{sym-diff } M2\ N2$
and $C \in \text{sets } M$
shows $\mu\ C = 0$
 $\langle \text{proof} \rangle$

3 Existence of a positive subset

The goal of this part is to prove that any measurable set of finite and positive measure must contain a positive subset with a strictly positive measure.

3.1 A sequence of negative subsets

definition *inf-neg* **where**

$\text{inf-neg } A = (\text{if } (A \notin \text{sets } M \vee \text{pos-meas-set } A) \text{ then } (0::\text{nat})$
 $\text{else } \text{Inf } \{n | n. (1::\text{nat}) \leq n \wedge (\exists B \in \text{sets } M. B \subseteq A \wedge \mu\ B < \text{ereal}(-1/n))\})$

lemma *inf-neg-ne*:

assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $\{n::\text{nat} | n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq A \wedge \mu\ B < \text{ereal } (-1/n))\} \neq \{\}$
 $\langle \text{proof} \rangle$

lemma *inf-neg-ge-1*:

assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $(1::\text{nat}) \leq \text{inf-neg } A$
 $\langle \text{proof} \rangle$

lemma *inf-neg-pos*:

assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $\exists B \in \text{sets } M. B \subseteq A \wedge \mu\ B < -1/(\text{inf-neg } A)$
 $\langle \text{proof} \rangle$

definition *rep-neg* **where**

$\text{rep-neg } A = (\text{if } (A \notin \text{sets } M \vee \text{pos-meas-set } A) \text{ then } \{\} \text{ else}$
 $\text{SOME } B. B \in \text{sets } M \wedge B \subseteq A \wedge \mu\ B \leq \text{ereal } (-1 / (\text{inf-neg } A)))$

lemma *g-rep-neg*:

assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $\text{rep-neg } A \in \text{sets } M$ $\text{rep-neg } A \subseteq A$
 $\mu\ (\text{rep-neg } A) \leq \text{ereal } (-1 / (\text{inf-neg } A))$
 $\langle \text{proof} \rangle$

lemma *rep-neg-sets*:
shows $\text{rep-neg } A \in \text{sets } M$
 $\langle \text{proof} \rangle$

lemma *rep-neg-subset*:
shows $\text{rep-neg } A \subseteq A$
 $\langle \text{proof} \rangle$

lemma *rep-neg-less*:
assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $\mu (\text{rep-neg } A) \leq \text{ereal } (-1 / (\text{inf-neg } A)) \langle \text{proof} \rangle$

lemma *rep-neg-leq*:
shows $\mu (\text{rep-neg } A) \leq 0$
 $\langle \text{proof} \rangle$

3.2 Construction of the positive subset

fun *pos-wtn*
where
pos-wtn-base: $\text{pos-wtn } E \ 0 = E|$
pos-wtn-step: $\text{pos-wtn } E \ (\text{Suc } n) = \text{pos-wtn } E \ n - \text{rep-neg } (\text{pos-wtn } E \ n)$

lemma *pos-wtn-subset*:
shows $\text{pos-wtn } E \ n \subseteq E$
 $\langle \text{proof} \rangle$

lemma *pos-wtn-sets*:
assumes $E \in \text{sets } M$
shows $\text{pos-wtn } E \ n \in \text{sets } M$
 $\langle \text{proof} \rangle$

definition *neg-wtn* **where**
 $\text{neg-wtn } E \ (n::\text{nat}) = \text{rep-neg } (\text{pos-wtn } E \ n)$

lemma *neg-wtn-neg-meas*:
shows $\mu (\text{neg-wtn } E \ n) \leq 0 \langle \text{proof} \rangle$

lemma *neg-wtn-sets*:
shows $\text{neg-wtn } E \ n \in \text{sets } M \langle \text{proof} \rangle$

lemma *neg-wtn-subset*:
shows $\text{neg-wtn } E \ n \subseteq E \langle \text{proof} \rangle$

lemma *neg-wtn-union-subset*:
shows $(\bigcup i \leq n. \text{neg-wtn } E \ i) \subseteq E \langle \text{proof} \rangle$

lemma *pos-wtn-Suc*:

shows $\text{pos-wtn } E (\text{Suc } n) = E - (\bigcup i \leq n. \text{neg-wtn } E i)$ $\langle \text{proof} \rangle$

definition *pos-sub* **where**

$\text{pos-sub } E = (\bigcap n. \text{pos-wtn } E n)$

lemma *pos-sub-sets*:

assumes $E \in \text{sets } M$

shows $\text{pos-sub } E \in \text{sets } M$ $\langle \text{proof} \rangle$

lemma *pos-sub-subset*:

shows $\text{pos-sub } E \subseteq E$ $\langle \text{proof} \rangle$

lemma *pos-sub-infity*:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

shows $|\mu (\text{pos-sub } E)| < \infty$ $\langle \text{proof} \rangle$

lemma *neg-wtn-djn*:

shows *disjoint-family* $(\lambda n. \text{neg-wtn } E n)$ $\langle \text{proof} \rangle$

end

lemma *disjoint-family-imp-on*:

assumes *disjoint-family* A

shows *disjoint-family-on* $A S$

$\langle \text{proof} \rangle$

context *signed-measure-space*

begin

lemma *neg-wtn-union-neg-meas*:

shows $\mu (\bigcup i \leq n. \text{neg-wtn } E i) \leq 0$
 $\langle \text{proof} \rangle$

lemma *pos-wtn-meas-gt*:

assumes $0 < \mu E$

and $E \in \text{sets } M$

shows $0 < \mu (\text{pos-wtn } E n)$
 $\langle \text{proof} \rangle$

definition *union-wit* **where**

$\text{union-wit } E = (\bigcup n. \text{neg-wtn } E n)$

lemma *union-wit-sets*:

shows $\text{union-wit } E \in \text{sets } M$ $\langle \text{proof} \rangle$

lemma *union-wit-subset*:

shows $\text{union-wit } E \subseteq E$
 $\langle \text{proof} \rangle$

lemma *pos-sub-diff*:

shows $\text{pos-sub } E = E - \text{union-wit } E$
<proof>

definition *num-wtn where*

$\text{num-wtn } E \ n = \text{inf-neg } (\text{pos-wtn } E \ n)$

lemma *num-wtn-geq:*

shows $\mu (\text{neg-wtn } E \ n) \leq \text{ereal } (-1/(\text{num-wtn } E \ n))$
<proof>

lemma *neg-wtn-infity:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows $|\mu (\text{neg-wtn } E \ i)| < \infty$
<proof>

lemma *union-wit-infity:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows $|\mu (\text{union-wit } E)| < \infty$ *<proof>*

lemma *neg-wtn-summable:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows $\text{summable } (\lambda i. - \text{real-of-ereal } (\mu (\text{neg-wtn } E \ i)))$
<proof>

lemma *inv-num-wtn-summable:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows $\text{summable } (\lambda n. 1/(\text{num-wtn } E \ n))$
<proof>

lemma *inv-num-wtn-shift-summable:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows $\text{summable } (\lambda n. 1/(\text{num-wtn } E \ n - 1))$
<proof>

lemma *neg-wtn-meas-sums:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows $(\lambda i. - (\mu (\text{neg-wtn } E \ i))) \text{ sums}$
 $\text{suminf } (\lambda i. - \text{real-of-ereal } (\mu (\text{neg-wtn } E \ i)))$
<proof>

lemma *neg-wtn-meas-suminf-le:*

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$

shows $\text{suminf } (\lambda i. \mu (\text{neg-wtn } E i)) \leq - \text{suminf } (\lambda n. 1/(\text{num-wtn } E n))$
 ⟨proof⟩

lemma *union-wit-meas-le*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
shows $\mu (\text{union-wit } E) \leq - \text{suminf } (\lambda n. 1 / \text{real } (\text{num-wtn } E n))$
 ⟨proof⟩

lemma *pos-sub-pos-meas*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $0 < \mu E$
and $\neg \text{pos-meas-set } E$
shows $0 < \mu (\text{pos-sub } E)$
 ⟨proof⟩

lemma *num-wtn-conv*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
shows $(\lambda n. 1/(\text{num-wtn } E n)) \longrightarrow 0$
 ⟨proof⟩

lemma *num-wtn-shift-conv*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
shows $(\lambda n. 1/(\text{num-wtn } E n - 1)) \longrightarrow 0$
 ⟨proof⟩

lemma *inf-neg-E-set*:

assumes $0 < \text{inf-neg } E$
shows $E \in \text{sets } M$ ⟨proof⟩

lemma *inf-neg-pos-meas*:

assumes $0 < \text{inf-neg } E$
shows $\neg \text{pos-meas-set } E$ ⟨proof⟩

lemma *inf-neg-mem*:

assumes $0 < \text{inf-neg } E$
shows $\text{inf-neg } E \in \{n::\text{nat} \mid n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq E \wedge \mu B < \text{ereal } (-1/n))\}$
 ⟨proof⟩

lemma *prec-inf-neg-pos*:

assumes $0 < \text{inf-neg } E - 1$
and $B \in \text{sets } M$
and $B \subseteq E$
shows $-1/(\text{inf-neg } E - 1) \leq \mu B$
 ⟨proof⟩

lemma *pos-wtn-meas-ge*:
assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $C \in \text{sets } M$
and $\bigwedge n. C \subseteq \text{pos-wtn } E n$
and $\bigwedge n. 0 < \text{num-wtn } E n$
shows $\exists N. \forall n \geq N. -1 / (\text{num-wtn } E n - 1) \leq \mu C$
<proof>

lemma *pos-sub-pos-meas-subset*:
assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $C \in \text{sets } M$
and $C \subseteq (\text{pos-sub } E)$
and $\bigwedge n. 0 < \text{num-wtn } E n$
shows $0 \leq \mu C$
<proof>

lemma *pos-sub-pos-meas'*:
assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $0 < \mu E$
and $\forall n. 0 < \text{num-wtn } E n$
shows $0 < \mu (\text{pos-sub } E)$
<proof>

We obtain the main result of this part on the existence of a positive subset.

lemma *exists-pos-meas-subset*:
assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $0 < \mu E$
shows $\exists A. A \subseteq E \wedge \text{pos-meas-set } A \wedge 0 < \mu A$
<proof>

4 The Hahn decomposition theorem

definition *seq-meas where*
 $\text{seq-meas} = (\text{SOME } f. \text{incseq } f \wedge \text{range } f \subseteq \text{pos-img} \wedge \bigsqcup \text{pos-img} = \bigsqcup \text{range } f)$

lemma *seq-meas-props*:
shows $\text{incseq } \text{seq-meas} \wedge \text{range } \text{seq-meas} \subseteq \text{pos-img} \wedge$
 $\bigsqcup \text{pos-img} = \bigsqcup \text{range } \text{seq-meas}$
<proof>

definition *seq-meas-rep where*
 $\text{seq-meas-rep } n = (\text{SOME } A. A \in \text{pos-sets} \wedge \text{seq-meas } n = \mu A)$

lemma *seq-meas-rep-ex*:

shows $\text{seq-meas-rep } n \in \text{pos-sets} \wedge \mu (\text{seq-meas-rep } n) = \text{seq-meas } n$
<proof>

lemma *seq-meas-rep-pos*:

assumes $\forall E \in \text{sets } M. \mu E < \infty$
shows $\text{pos-meas-set } (\bigcup i. \text{seq-meas-rep } i)$
<proof>

lemma *sup-seq-meas-rep*:

assumes $\forall E \in \text{sets } M. \mu E < \infty$
and $S = (\bigsqcup \text{pos-img})$
and $A = (\bigcup i. \text{seq-meas-rep } i)$
shows $\mu A = S$
<proof>

lemma *seq-meas-rep-compl*:

assumes $\forall E \in \text{sets } M. \mu E < \infty$
and $A = (\bigcup i. \text{seq-meas-rep } i)$
shows $\text{neg-meas-set } ((\text{space } M) - A)$ *<proof>*

lemma *hahn-decomp-finite*:

assumes $\forall E \in \text{sets } M. \mu E < \infty$
shows $\exists M1 M2. \text{hahn-space-decomp } M1 M2$ *<proof>*

theorem *hahn-decomposition*:

shows $\exists M1 M2. \text{hahn-space-decomp } M1 M2$
<proof>

5 The Jordan decomposition theorem

definition *jordan-decomp where*

$\text{jordan-decomp } m1 m2 \longleftrightarrow ((\text{measure-space } (\text{space } M) (\text{sets } M) m1) \wedge$
 $(\text{measure-space } (\text{space } M) (\text{sets } M) m2) \wedge$
 $(\forall A \in \text{sets } M. 0 \leq m1 A) \wedge$
 $(\forall A \in \text{sets } M. 0 \leq m2 A) \wedge$
 $(\forall A \in \text{sets } M. \mu A = (m1 A) - (m2 A)) \wedge$
 $(\forall P N A. \text{hahn-space-decomp } P N \longrightarrow$
 $(A \in \text{sets } M \longrightarrow A \subseteq P \longrightarrow (m2 A) = 0) \wedge$
 $(A \in \text{sets } M \longrightarrow A \subseteq N \longrightarrow (m1 A) = 0)) \wedge$
 $((\forall A \in \text{sets } M. m1 A < \infty) \vee (\forall A \in \text{sets } M. m2 A < \infty)))$

lemma *jordan-decomp-pos-meas*:

assumes $\text{jordan-decomp } m1 m2$
and $\text{hahn-space-decomp } P N$
and $A \in \text{sets } M$
shows $m1 A = \mu (A \cap P)$
<proof>

lemma *jordan-decomp-neg-meas*:
assumes *jordan-decomp* $m1$ $m2$
and *hahn-space-decomp* P N
and $A \in \text{sets } M$
shows $m2 A = -\mu (A \cap N)$
 $\langle \text{proof} \rangle$

lemma *pos-inter-neg-0*:
assumes *hahn-space-decomp* $M1$ $M2$
and *hahn-space-decomp* P N
and $A \in \text{sets } M$
and $A \subseteq N$
shows $\mu (A \cap M1) = 0$
 $\langle \text{proof} \rangle$

lemma *neg-inter-pos-0*:
assumes *hahn-space-decomp* $M1$ $M2$
and *hahn-space-decomp* P N
and $A \in \text{sets } M$
and $A \subseteq P$
shows $\mu (A \cap M2) = 0$
 $\langle \text{proof} \rangle$

lemma *jordan-decomposition* :
shows $\exists m1 m2. \text{jordan-decomp } m1 m2$
 $\langle \text{proof} \rangle$

lemma *jordan-decomposition-unique* :
assumes *jordan-decomp* $m1$ $m2$
and *jordan-decomp* $n1$ $n2$
and $A \in \text{sets } M$
shows $m1 A = n1 A$ $m2 A = n2 A$
 $\langle \text{proof} \rangle$

end

end

References

- [1] E. DiBenedetto. *Real Analysis*. Birkhäuser Advanced Texts. Birkhäuser.