

The Hahn and Jordan Decomposition Theorems

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1 Introduction

Signed measures are a generalization of measures that can map measurable sets to negative values. In this work we formalize the Hahn decomposition theorem for signed measures, namely that if $(\Omega, \mathcal{A}, \mu)$ is a measure space for a signed measure μ , then Ω can be decomposed as $\Omega^+ \cup \Omega^-$, where every measurable subset of Ω^+ has a positive measure, and every measurable subset of Ω^- has a negative measure. We then prove that this decomposition is essentially unique, meaning that if $X^+ \cup X^-$ is another such decomposition, then any measurable subset in $(\Omega^+ \triangle X^+) \cup (\Omega^- \triangle X^-)$ has a zero measure.

We also formalize the Jordan decomposition theorem as a corollary, which states that the signed measure μ admits a unique decomposition into a difference $\mu = \mu^+ - \mu^-$ of two positive measures, at least one of which is finite, and such that for any Hahn decomposition $\Omega^+ \cup \Omega^-$ and measurable set A , if $A \subseteq \Omega^-$ then $\mu^+(A) = 0$ and if $A \subseteq \Omega^+$ then $\mu^-(A) = 0$. The formalization is mostly based on [1], Section 16 of Chapter 4.

2 Signed measures

In this section we define signed measures. These are generalizations of measures that can also take negative values but cannot contain both ∞ and $-\infty$ in their range.

2.1 Basic definitions

theory *Hahn-Jordan-Decomposition* **imports**

HOL-Probability.Probability

Hahn-Jordan-Prelims

begin

definition *signed-measure*:: 'a measure \Rightarrow ('a set \Rightarrow ereal) \Rightarrow bool **where**
signed-measure $M \mu \iff \mu \{ \} = 0 \wedge (-\infty \notin \text{range } \mu \vee \infty \notin \text{range } \mu) \wedge$
 $(\forall A. \text{range } A \subseteq \text{sets } M \longrightarrow \text{disjoint-family } A \longrightarrow \bigcup (\text{range } A) \in \text{sets } M \longrightarrow$
 $(\lambda i. \mu (A i)) \text{ sums } \mu (\bigcup (\text{range } A))) \wedge$
 $(\forall A. \text{range } A \subseteq \text{sets } M \longrightarrow \text{disjoint-family } A \longrightarrow \bigcup (\text{range } A) \in \text{sets } M \longrightarrow$
 $|\mu (\bigcup (\text{range } A))| < \infty \longrightarrow \text{summable } (\lambda i. \text{real-of-ereal } |\mu (A i)|))$

lemma *signed-measure-empty*:

assumes *signed-measure* $M \mu$

shows $\mu \{ \} = 0$ **using** *assms unfolding signed-measure-def by simp*

lemma *signed-measure-sums*:

assumes *signed-measure* $M \mu$

and $\text{range } A \subseteq M$

and *disjoint-family* A

and $\bigcup (\text{range } A) \in \text{sets } M$

shows $(\lambda i. \mu (A i)) \text{ sums } \mu (\bigcup (\text{range } A))$

using *assms unfolding signed-measure-def by simp*

lemma *signed-measure-summable*:

assumes *signed-measure* $M \mu$

and $\text{range } A \subseteq M$

and *disjoint-family* A

and $\bigcup (\text{range } A) \in \text{sets } M$

and $|\mu (\bigcup (\text{range } A))| < \infty$

shows *summable* $(\lambda i. \text{real-of-ereal } |\mu (A i)|)$

using *assms unfolding signed-measure-def by simp*

lemma *signed-measure-inf-sum*:

assumes *signed-measure* $M \mu$

and $\text{range } A \subseteq M$

and *disjoint-family* A

and $\bigcup (\text{range } A) \in \text{sets } M$

shows $(\sum i. \mu (A i)) = \mu (\bigcup (\text{range } A))$ **using** *sums-unique assms*

signed-measure-sums by (metis)

lemma *signed-measure-abs-convergent*:
assumes *signed-measure* $M \mu$
and $\text{range } A \subseteq \text{sets } M$
and *disjoint-family* A
and $\bigcup (\text{range } A) \in \text{sets } M$
and $|\mu (\bigcup (\text{range } A))| < \infty$
shows *summable* $(\lambda i. \text{real-of-ereal } |\mu (A \ i)|)$ **using** *assms*
unfolding *signed-measure-def* **by** *simp*

lemma *signed-measure-additive*:
assumes *signed-measure* $M \mu$
shows *additive* $M \mu$
proof (*auto simp add: additive-def*)
fix $x \ y$
assume $x: x \in M$ **and** $y: y \in M$ **and** $x \cap y = \{\}$
hence *disjoint-family* $(\text{binaryset } x \ y)$
by (*auto simp add: disjoint-family-on-def binaryset-def*)
have $(\lambda i. \mu ((\text{binaryset } x \ y) \ i)) \text{ sums } (\mu \ x + \mu \ y)$ **using** *binaryset-sums*
signed-measure-empty[of M μ] assms **by** *simp*
have $\text{range } (\text{binaryset } x \ y) = \{x, y, \{\}\}$ **using** *range-binaryset-eq* **by** *simp*
moreover $\{x, y, \{\}\} \subseteq M$ **using** $x \ y$ **by** *auto*
moreover $x \cup y \in \text{sets } M$ **using** $x \ y$ **by** *simp*
moreover $(\bigcup (\text{range } (\text{binaryset } x \ y))) = x \cup y$
by (*simp add: calculation(1)*)
ultimately $(\lambda i. \mu ((\text{binaryset } x \ y) \ i)) \text{ sums } \mu (x \cup y)$ **using** *assms x y*
signed-measure-empty[of M μ] signed-measure-sums[of M μ]
 $\langle \text{disjoint-family } (\text{binaryset } x \ y) \rangle$ **by** (*metis*)
then $\text{show } \mu (x \cup y) = \mu \ x + \mu \ y$
using $\langle (\lambda i. \mu ((\text{binaryset } x \ y) \ i)) \text{ sums } (\mu \ x + \mu \ y) \rangle$ *sums-unique2* **by** *force*
qed

lemma *signed-measure-add*:
assumes *signed-measure* $M \mu$
and $a \in \text{sets } M$
and $b \in \text{sets } M$
and $a \cap b = \{\}$
shows $\mu (a \cup b) = \mu \ a + \mu \ b$ **using** *additiveD[OF signed-measure-additive]*
assms **by** *auto*

lemma *signed-measure-disj-sum*:
shows *finite* $I \implies \text{signed-measure } M \mu \implies \text{disjoint-family-on } A \ I \implies$
 $(\bigwedge i. i \in I \implies A \ i \in \text{sets } M) \implies \mu (\bigcup_{i \in I} A \ i) = (\sum_{i \in I} \mu (A \ i))$
proof (*induct rule:finite-induct*)
case *empty*
then *show ?case* **unfolding** *signed-measure-def* **by** *simp*
next
case (*insert x F*)
have $\mu (\bigcup (A \ ' \text{insert } x \ F)) = \mu ((\bigcup (A \ 'F)) \cup A \ x)$
by (*simp add: Un-commute*)

also have $\dots = \mu (\bigcup (A \text{ 'F})) + \mu (A \ x)$
proof –
have $(\bigcup (A \text{ 'F})) \cap (A \ x) = \{\}$ **using** *insert*
by (*metis disjoint-family-on-insert inf-commute*)
moreover have $\bigcup (A \text{ 'F}) \in \text{sets } M$ **using** *insert by auto*
moreover have $A \ x \in \text{sets } M$ **using** *insert by simp*
ultimately show *?thesis* **by** (*meson insert.premis(1) signed-measure-add*)
qed
also have $\dots = (\sum_{i \in F} \mu (A \ i)) + \mu (A \ x)$ **using** *insert*
by (*metis disjoint-family-on-insert insert-iff*)
also have $\dots = (\sum_{i \in \text{insert } x \ F} \mu (A \ i))$
by (*simp add: add commute insert.hyps(1) insert.hyps(2)*)
finally show *?case* .
qed

lemma *pos-signed-measure-count-additive*:
assumes *signed-measure M μ*
and $\forall E \in \text{sets } M. 0 \leq \mu \ E$
shows *countably-additive (sets M) (λA. e2ennreal (μ A))*
unfolding *countably-additive-def*
proof (*intro allI impI*)
fix $A::\text{nat} \Rightarrow \text{'a set}$
assume $\text{range } A \subseteq \text{sets } M$
and *disjoint-family A*
and $\bigcup (\text{range } A) \in \text{sets } M$ **note** *Aprops = this*
have $\text{eq}: \bigwedge i. \mu (A \ i) = \text{enn2ereal} (\text{e2ennreal} (\mu (A \ i)))$
using *assms enn2ereal-e2ennreal Aprops by simp*
have $(\lambda n. \sum_{i \leq n} \mu (A \ i)) \longrightarrow \mu (\bigcup (\text{range } A))$ **using**
sums-def-le[of λi. μ (A i) μ (⋃ (range A))] assms
signed-measure-sums[of M] Aprops by simp
hence $(\lambda n. \text{e2ennreal} (\sum_{i \leq n} \mu (A \ i))) \longrightarrow$
 $\text{e2ennreal} (\mu (\bigcup (\text{range } A)))$ *sequentially*
using *tendsto-e2ennrealI[of (λn. ∑ i≤n. μ (A i) μ (⋃ (range A))]*
by *simp*
moreover have $\bigwedge n. \text{e2ennreal} (\sum_{i \leq n} \mu (A \ i)) = (\sum_{i \leq n} \text{e2ennreal} (\mu (A \ i)))$
i)))
using *e2ennreal-finite-sum by (metis enn2ereal-nonneg eq finite-atMost)*
ultimately have $(\lambda n. (\sum_{i \leq n} \text{e2ennreal} (\mu (A \ i)))) \longrightarrow$
 $\text{e2ennreal} (\mu (\bigcup (\text{range } A)))$ *sequentially by simp*
hence $(\lambda i. \text{e2ennreal} (\mu (A \ i))) \text{ sums } \text{e2ennreal} (\mu (\bigcup (\text{range } A)))$
using *sums-def-le[of λi. e2ennreal (μ (A i)) e2ennreal (μ (⋃ (range A)))]*
by *simp*
thus $(\sum i. \text{e2ennreal} (\mu (A \ i))) = \text{e2ennreal} (\mu (\bigcup (\text{range } A)))$
using *sums-unique assms by (metis)*
qed

lemma *signed-measure-minus*:
assumes *signed-measure M μ*
shows *signed-measure M (λA. - μ A)* **unfolding** *signed-measure-def*

```

proof (intro conjI)
  show  $-\mu \{ \} = 0$  using assms unfolding signed-measure-def by simp
  show  $-\infty \notin \text{range } (\lambda A. -\mu A) \vee \infty \notin \text{range } (\lambda A. -\mu A)$ 
  proof (cases  $\infty \in \text{range } \mu$ )
    case True
      hence  $-\infty \notin \text{range } \mu$  using assms unfolding signed-measure-def by simp
      hence  $\infty \notin \text{range } (\lambda A. -\mu A)$  using ereal-uminus-eq-reorder by blast
      thus  $-\infty \notin \text{range } (\lambda A. -\mu A) \vee \infty \notin \text{range } (\lambda A. -\mu A)$  by simp
    next
      case False
        hence  $-\infty \notin \text{range } (\lambda A. -\mu A)$  using ereal-uminus-eq-reorder
        by (simp add: image-iff)
        thus  $-\infty \notin \text{range } (\lambda A. -\mu A) \vee \infty \notin \text{range } (\lambda A. -\mu A)$  by simp
  qed
  show  $\forall A. \text{range } A \subseteq \text{sets } M \longrightarrow \text{disjoint-family } A \longrightarrow \bigcup (\text{range } A) \in \text{sets } M$ 
   $\longrightarrow$ 
   $|\!-\mu (\bigcup (\text{range } A))| < \infty \longrightarrow \text{summable } (\lambda i. \text{real-of-ereal } |\!-\mu (A\ i)|)$ 
  proof (intro allI impI)
    fix  $A::\text{nat} \Rightarrow 'a \text{ set}$ 
    assume  $\text{range } A \subseteq \text{sets } M$  and disjoint-family A and  $\bigcup (\text{range } A) \in \text{sets } M$ 
    and  $|\!-\mu (\bigcup (\text{range } A))| < \infty$ 
    thus  $\text{summable } (\lambda i. \text{real-of-ereal } |\!-\mu (A\ i)|)$  using assms
    unfolding signed-measure-def by simp
  qed
  show  $\forall A. \text{range } A \subseteq \text{sets } M \longrightarrow \text{disjoint-family } A \longrightarrow \bigcup (\text{range } A) \in \text{sets } M$ 
   $\longrightarrow$ 
   $(\lambda i. -\mu (A\ i)) \text{ sums } -\mu (\bigcup (\text{range } A))$ 
  proof -
  {
    fix  $A::\text{nat} \Rightarrow 'a \text{ set}$ 
    assume  $\text{range } A \subseteq \text{sets } M$  and disjoint-family A and
     $\bigcup (\text{range } A) \in \text{sets } M$  note Aprops = this
    have  $-\infty \notin \text{range } (\lambda i. \mu (A\ i)) \vee \infty \notin \text{range } (\lambda i. \mu (A\ i))$ 
    proof -
      have  $\text{range } (\lambda i. \mu (A\ i)) \subseteq \text{range } \mu$  by auto
      thus ?thesis using assms unfolding signed-measure-def by auto
    qed
    moreover have  $(\lambda i. \mu (A\ i)) \text{ sums } \mu (\bigcup (\text{range } A))$ 
    using signed-measure-sums[of M] Aprops assms by simp
    ultimately have  $(\lambda i. -\mu (A\ i)) \text{ sums } -\mu (\bigcup (\text{range } A))$ 
    using sums-minus[of  $\lambda i. \mu (A\ i)$ ] by simp
  }
  thus ?thesis by auto
qed
qed

locale near-finite-function =
  fixes  $\mu::'b \text{ set} \Rightarrow \text{ereal}$ 
  assumes inf-range:  $-\infty \notin \text{range } \mu \vee \infty \notin \text{range } \mu$ 

```

lemma (in *near-finite-function*) *finite-subset*:
assumes $|\mu E| < \infty$
and $A \subseteq E$
and $\mu E = \mu A + \mu (E - A)$
shows $|\mu A| < \infty$
proof (cases $\infty \in \text{range } \mu$)
case *False*
show *?thesis*
proof (cases $0 < \mu A$)
case *True*
hence $|\mu A| = \mu A$ **by** *simp*
also have $\dots < \infty$ **using** *False* **by** (*metis ereal-less-PInfty rangeI*)
finally show *?thesis* .
next
case *False*
hence $|\mu A| = -\mu A$ **using** *not-less-iff-gr-or-eq* **by** *fastforce*
also have $\dots = \mu (E - A) - \mu E$
proof -
have $\mu E = \mu A + \mu (E - A)$ **using** *assms* **by** *simp*
hence $\mu E - \mu A = \mu (E - A)$
by (*metis abs-ereal-uminus assms(1) calculation ereal-diff-add-inverse*
ereal-inf-ty-less(2) ereal-minus(5) ereal-minus-less-iff
ereal-minus-less-minus ereal-uminus-uminus less-ereal.simps(2)
minus-ereal-def plus-ereal.simps(3))
thus *?thesis* **using** *assms(1) ereal-add-uminus-conv-diff ereal-eq-minus*
by *auto*
qed
also have $\dots \leq \mu (E - A) + |\mu E|$
by (*metis <-\mu A = \mu (E - A) - \mu E> abs-ereal-less0 abs-ereal-pos*
ereal-diff-le-self ereal-le-add-mono1 less-eq-ereal-def
minus-ereal-def not-le-imp-less)
also have $\dots < \infty$ **using** *assms <\infty \notin \text{range } \mu>*
by (*metis UNIV-I ereal-less-PInfty ereal-plus-eq-PInfty image-eqI*)
finally show *?thesis* .
qed
next
case *True*
hence $-\infty \notin \text{range } \mu$ **using** *inf-range* **by** *simp*
hence $-\infty < \mu A$ **by** (*metis ereal-inf-ty-less(2) rangeI*)
show *?thesis*
proof (cases $\mu A < 0$)
case *True*
hence $|\mu A| = -\mu A$ **using** *not-less-iff-gr-or-eq* **by** *fastforce*
also have $\dots < \infty$ **using** $\langle -\infty < \mu A \rangle$ **using** *ereal-uminus-less-reorder*
by *blast*
finally show *?thesis* .
next
case *False*

hence $|\mu A| = \mu A$ by *simp*
 also have $\dots = \mu E - \mu (E - A)$
 proof –
 have $\mu E = \mu A + \mu (E - A)$ using *assms* by *simp*
 thus $\mu A = \mu E - \mu (E - A)$ by (*metis add.right-neutral assms(1)*
 add-diff-eq-ereal calculation ereal-diff-add-eq-diff-diff-swap
 ereal-diff-add-inverse ereal-infty-less(1) ereal-plus-eq-PInfty
 ereal-x-minus-x)
 qed
 also have $\dots \leq |\mu E| - \mu (E - A)$
 by (*metis $|\mu A| = \mu A$ $|\mu A = \mu E - \mu (E - A)$ *abs-ereal-ge0*
 abs-ereal-pos abs-ereal-uminus antisym-conv ereal-0-le-uminus-iff
 ereal-abs-diff ereal-diff-le-mono-left ereal-diff-le-self le-cases
 less-eq-ereal-def minus-ereal-def)
 also have $\dots < \infty$
 proof –
 have $-\infty < \mu (E - A)$ using $\langle -\infty \notin \text{range } \mu \rangle$
 by (*metis ereal-infty-less(2) rangeI*)
 hence $-\mu (E - A) < \infty$ using *ereal-uminus-less-reorder* by *blast*
 thus *?thesis* using *assms* by (*simp add: ereal-minus-eq-PInfty-iff*
 ereal-uminus-eq-reorder)
 qed
 finally show *?thesis* .
 qed
 qed
 locale *signed-measure-space*=
 fixes *M::'a measure* and μ
 assumes *sgn-meas: signed-measure M μ*

 sublocale *signed-measure-space* \subseteq *near-finite-function*
 proof (*unfold-locales*)
 show $-\infty \notin \text{range } \mu \vee \infty \notin \text{range } \mu$ using *sgn-meas*
 unfolding *signed-measure-def* by *simp*
 qed

 context *signed-measure-space*
 begin
 lemma *signed-measure-finite-subset*:
 assumes $E \in \text{sets } M$
 and $|\mu E| < \infty$
 and $A \in \text{sets } M$
 and $A \subseteq E$
 shows $|\mu A| < \infty$
 proof (*rule finite-subset*)
 show $|\mu E| < \infty \wedge A \subseteq E$ using *assms* by *auto*
 show $\mu E = \mu A + \mu (E - A)$ using *assms*
 sgn-meas signed-measure-add[of M μ A E - A]
 by (*metis Diff-disjoint Diff-partition sets.Diff*)*

qed

lemma *measure-space-e2ennreal* :

assumes *measure-space* (*space* M) (*sets* M) $m \wedge (\forall E \in \text{sets } M. m E < \infty) \wedge$
 $(\forall E \in \text{sets } M. m E \geq 0)$

shows $\forall E \in \text{sets } M. e2ennreal (m E) < \infty$

proof

fix E

assume $E \in \text{sets } M$

show $e2ennreal (m E) < \infty$

proof –

have $m E < \infty$ **using** *assms* $\langle E \in \text{sets } M \rangle$

by *blast*

then have $e2ennreal (m E) < \infty$ **using** *e2ennreal-less-top*

using $\langle m E < \infty \rangle$ **by** *auto*

thus *?thesis* **by** *simp*

qed

qed

2.2 Positive and negative subsets

The Hahn decomposition theorem is based on the notions of positive and negative measurable sets. A measurable set is positive (resp. negative) if all its measurable subsets have a positive (resp. negative) measure by μ . The decomposition theorem states that any measure space for a signed measure can be decomposed into a positive and a negative measurable set.

definition *pos-meas-set* **where**

pos-meas-set $E \longleftrightarrow E \in \text{sets } M \wedge (\forall A \in \text{sets } M. A \subseteq E \longrightarrow 0 \leq \mu A)$

definition *neg-meas-set* **where**

neg-meas-set $E \longleftrightarrow E \in \text{sets } M \wedge (\forall A \in \text{sets } M. A \subseteq E \longrightarrow \mu A \leq 0)$

lemma *pos-meas-setI*:

assumes $E \in \text{sets } M$

and $\bigwedge A. A \in \text{sets } M \implies A \subseteq E \implies 0 \leq \mu A$

shows *pos-meas-set* E **unfolding** *pos-meas-set-def* **using** *assms* **by** *simp*

lemma *pos-meas-setD1* :

assumes *pos-meas-set* E

shows $E \in \text{sets } M$

using *assms* **unfolding** *pos-meas-set-def*

by *simp*

lemma *neg-meas-setD1* :

assumes *neg-meas-set* E

shows $E \in \text{sets } M$ **using** *assms* **unfolding** *neg-meas-set-def* **by** *simp*

lemma *neg-meas-setI*:

assumes $E \in \text{sets } M$
and $\bigwedge A. A \in \text{sets } M \implies A \subseteq E \implies \mu A \leq 0$
shows *neg-meas-set* E **unfolding** *neg-meas-set-def* **using** *assms* **by** *simp*

lemma *pos-meas-self*:
assumes *pos-meas-set* E
shows $0 \leq \mu E$ **using** *assms* **unfolding** *pos-meas-set-def* **by** *simp*

lemma *empty-pos-meas-set*:
shows *pos-meas-set* $\{\}$
by (*metis bot.extremum-uniqueI eq-iff pos-meas-set-def sets.empty-sets sgn-meas signed-measure-empty*)

lemma *empty-neg-meas-set*:
shows *neg-meas-set* $\{\}$
by (*metis neg-meas-set-def order-refl sets.empty-sets sgn-meas signed-measure-empty subset-empty*)

lemma *pos-measure-meas*:
assumes *pos-meas-set* E
and $A \subseteq E$
and $A \in \text{sets } M$
shows $0 \leq \mu A$ **using** *assms* **unfolding** *pos-meas-set-def* **by** *simp*

lemma *pos-meas-subset*:
assumes *pos-meas-set* A
and $B \subseteq A$
and $B \in \text{sets } M$
shows *pos-meas-set* B **using** *assms* *pos-meas-set-def* **by** *auto*

lemma *neg-meas-subset*:
assumes *neg-meas-set* A
and $B \subseteq A$
and $B \in \text{sets } M$
shows *neg-meas-set* B **using** *assms* *neg-meas-set-def* **by** *auto*

lemma *pos-meas-set-Union*:
assumes $\bigwedge (i::\text{nat}). \text{pos-meas-set } (A i)$
and $\bigwedge i. A i \in \text{sets } M$
and $|\mu (\bigcup i. A i)| < \infty$
shows *pos-meas-set* $(\bigcup i. A i)$
proof (*rule pos-meas-setI*)
show $\bigcup (\text{range } A) \in \text{sets } M$ **using** *sigma-algebra.countable-UN* *assms* **by** *simp*
obtain B **where** *disjoint-family* B **and** $(\bigcup (i::\text{nat}). B i) = (\bigcup (i::\text{nat}). A i)$
and $\bigwedge i. B i \in \text{sets } M$ **and** $\bigwedge i. B i \subseteq A i$ **using** *disj-Union2* *assms* **by** *auto*
fix C
assume $C \in \text{sets } M$ **and** $C \subseteq (\bigcup i. A i)$
hence $C = C \cap (\bigcup i. A i)$ **by** *auto*
also have $\dots = C \cap (\bigcup i. B i)$ **using** $\langle (\bigcup i. B i) = (\bigcup i. A i) \rangle$ **by** *simp*

also have $\dots = (\bigcup i. C \cap B i)$ **by** *auto*
finally have $C = (\bigcup i. C \cap B i)$.
hence $\mu C = \mu (\bigcup i. C \cap B i)$ **by** *simp*
also have $\dots = (\sum i. \mu (C \cap B i))$
proof (*rule signed-measure-inf-sum[symmetric]*)
 show *signed-measure* $M \mu$ **using** *sgn-meas* **by** *simp*
 show *disjoint-family* $(\lambda i. C \cap B i)$ **using** $\langle \text{disjoint-family } B \rangle$
 by (*meson Int-iff disjoint-family-subset subset-iff*)
 show *range* $(\lambda i. C \cap B i) \subseteq \text{sets } M$ **using** $\langle C \in \text{sets } M \rangle \langle \bigwedge i. B i \in \text{sets } M \rangle$
 by *auto*
 show $(\bigcup i. C \cap B i) \in \text{sets } M$ **using** $\langle C = (\bigcup i. C \cap B i) \rangle \langle C \in \text{sets } M \rangle$
 by *simp*
qed
also have $\dots \geq 0$
proof (*rule suminf-nonneg*)
 show $\bigwedge n. 0 \leq \mu (C \cap B n)$
 proof –
 fix n
 have $C \cap B n \subseteq A n$ **using** $\langle \bigwedge i. B i \subseteq A i \rangle$ **by** *auto*
 moreover have $C \cap B n \in \text{sets } M$ **using** $\langle C \in \text{sets } M \rangle \langle \bigwedge i. B i \in \text{sets } M \rangle$
 by *simp*
 ultimately show $0 \leq \mu (C \cap B n)$ **using** *assms pos-measure-meas[of A n]*
 by *simp*
qed
have *summable* $(\lambda i. \text{real-of-ereal } (\mu (C \cap B i)))$
proof (*rule summable-norm-cancel*)
 have $\bigwedge n. \text{norm } (\text{real-of-ereal } (\mu (C \cap B n))) =$
 $\text{real-of-ereal } |\mu (C \cap B n)|$ **by** *simp*
 moreover have *summable* $(\lambda i. \text{real-of-ereal } |\mu (C \cap B i)|)$
 proof (*rule signed-measure-abs-convergent*)
 show *signed-measure* $M \mu$ **using** *sgn-meas* **by** *simp*
 show *range* $(\lambda i. C \cap B i) \subseteq \text{sets } M$ **using** $\langle C \in \text{sets } M \rangle$
 $\langle \bigwedge i. B i \in \text{sets } M \rangle$ **by** *auto*
 show *disjoint-family* $(\lambda i. C \cap B i)$ **using** $\langle \text{disjoint-family } B \rangle$
 by (*meson Int-iff disjoint-family-subset subset-iff*)
 show $(\bigcup i. C \cap B i) \in \text{sets } M$ **using** $\langle C = (\bigcup i. C \cap B i) \rangle \langle C \in \text{sets } M \rangle$
 by *simp*
 have $|\mu C| < \infty$
 proof (*rule signed-measure-finite-subset*)
 show $(\bigcup i. A i) \in \text{sets } M$ **using** *assms* **by** *simp*
 show $|\mu (\bigcup (\text{range } A))| < \infty$ **using** *assms* **by** *simp*
 show $C \in \text{sets } M$ **using** $\langle C \in \text{sets } M \rangle$.
 show $C \subseteq \bigcup (\text{range } A)$ **using** $\langle C \subseteq \bigcup (\text{range } A) \rangle$.
qed
 thus $|\mu (\bigcup i. C \cap B i)| < \infty$ **using** $\langle C = (\bigcup i. C \cap B i) \rangle$ **by** *simp*
qed
 ultimately show *summable* $(\lambda n. \text{norm } (\text{real-of-ereal } (\mu (C \cap B n))))$
 by *auto*
qed

thus *summable* $(\lambda i. \mu (C \cap B i))$ **by** (*simp add:* $\langle \bigwedge n. 0 \leq \mu (C \cap B n) \rangle$
summable-ereal-pos)

qed

finally show $0 \leq \mu C$.

qed

lemma *pos-meas-set-pos-lim:*

assumes $\bigwedge (i::nat). \text{pos-meas-set } (A i)$
and $\bigwedge i. A i \in \text{sets } M$
shows $0 \leq \mu (\bigcup i. A i)$

proof –

obtain B **where** *disjoint-family* B **and** $(\bigcup (i::nat). B i) = (\bigcup (i::nat). A i)$
and $\bigwedge i. B i \in \text{sets } M$ **and** $\bigwedge i. B i \subseteq A i$ **using** *disj-Union2* *assms* **by** *auto*

note $Bprops = \text{this}$

have *sums:* $(\lambda n. \mu (B n)) \text{ sums } \mu (\bigcup i. B i)$

proof (*rule signed-measure-sums*)

show *signed-measure* $M \mu$ **using** *sgn-meas* .

show $\text{range } B \subseteq \text{sets } M$ **using** $Bprops$ **by** *auto*

show *disjoint-family* B **using** $Bprops$ **by** *simp*

show $\bigcup (\text{range } B) \in \text{sets } M$ **using** $Bprops$ **by** *blast*

qed

hence *summable* $(\lambda n. \mu (B n))$ **using** *sums-summable*[*of* $\lambda n. \mu (B n)$] **by** *simp*

hence *suminf* $(\lambda n. \mu (B n)) = \mu (\bigcup i. B i)$ **using** *sums sums-iff* **by** *auto*

thus *?thesis* **using** *suminf-nonneg*

by (*metis* $Bprops(2)$ $Bprops(3)$ $Bprops(4)$ $\langle \text{summable } (\lambda n. \mu (B n)) \rangle$ *assms(1)*
pos-measure-meas)

qed

lemma *pos-meas-disj-union:*

assumes *pos-meas-set* A
and *pos-meas-set* B
and $A \cap B = \{\}$

shows *pos-meas-set* $(A \cup B)$ **unfolding** *pos-meas-set-def*

proof (*intro conjI ballI impI*)

show $A \cup B \in \text{sets } M$

by (*metis* *assms(1)* *assms(2)* *pos-meas-set-def sets.Un*)

next

fix C

assume $C \in \text{sets } M$ **and** $C \subseteq A \cup B$

define DA **where** $DA = C \cap A$

define DB **where** $DB = C \cap B$

have $DA \in \text{sets } M$ **using** *DA-def* $\langle C \in \text{sets } M \rangle$ *assms(1)* *pos-meas-set-def*
by *blast*

have $DB \in \text{sets } M$ **using** *DB-def* $\langle C \in \text{sets } M \rangle$ *assms(2)* *pos-meas-set-def*
by *blast*

have $DA \cap DB = \{\}$ **unfolding** *DA-def* *DB-def* **using** *assms* **by** *auto*

have $C = DA \cup DB$ **unfolding** *DA-def* *DB-def* **using** $\langle C \subseteq A \cup B \rangle$ **by** *auto*

have $0 \leq \mu DB$ **using** *assms* **unfolding** *DB-def* *pos-meas-set-def*
by (*metis* *DB-def Int-lower2* $\langle DB \in \text{sets } M \rangle$)

also have $\dots \leq \mu DA + \mu DB$ **using** *assms unfolding pos-meas-set-def*
by (*metis DA-def Diff-Diff-Int Diff-subset Int-commute* $\langle DA \in \text{sets } M \rangle$
ereal-le-add-self2)
also have $\dots = \mu C$ **using** *signed-measure-add sgn-meas* $\langle DA \in \text{sets } M \rangle$
 $\langle DB \in \text{sets } M \rangle \langle DA \cap DB = \{\} \rangle \langle C = DA \cup DB \rangle$ **by** *metis*
finally show $0 \leq \mu C$.
qed

lemma *pos-meas-set-union:*

assumes *pos-meas-set A*
and *pos-meas-set B*
shows *pos-meas-set (A \cup B)*

proof –

define *C* **where** $C = B - A$
have $A \cup C = A \cup B$ **unfolding** *C-def* **by** *auto*
moreover have *pos-meas-set (A \cup C)*
proof (*rule pos-meas-disj-union*)
show *pos-meas-set C* **unfolding** *C-def*
by (*meson Diff-subset assms(1) assms(2) sets.Diff*
signed-measure-space.pos-meas-set-def
signed-measure-space.pos-meas-subset signed-measure-space-axioms)
show *pos-meas-set A* **using** *assms* **by** *simp*
show $A \cap C = \{\}$ **unfolding** *C-def* **by** *auto*

qed

ultimately show *?thesis* **by** *simp*

qed

lemma *neg-meas-disj-union:*

assumes *neg-meas-set A*
and *neg-meas-set B*
and $A \cap B = \{\}$
shows *neg-meas-set (A \cup B)* **unfolding** *neg-meas-set-def*

proof (*intro conjI ballI impI*)

show $A \cup B \in \text{sets } M$
by (*metis assms(1) assms(2) neg-meas-set-def sets.Un*)

next

fix *C*

assume $C \in \text{sets } M$ **and** $C \subseteq A \cup B$

define *DA* **where** $DA = C \cap A$

define *DB* **where** $DB = C \cap B$

have $DA \in \text{sets } M$ **using** *DA-def* $\langle C \in \text{sets } M \rangle$ *assms(1) neg-meas-set-def*
by *blast*

have $DB \in \text{sets } M$ **using** *DB-def* $\langle C \in \text{sets } M \rangle$ *assms(2) neg-meas-set-def*
by *blast*

have $DA \cap DB = \{\}$ **unfolding** *DA-def DB-def* **using** *assms* **by** *auto*

have $C = DA \cup DB$ **unfolding** *DA-def DB-def* **using** $\langle C \subseteq A \cup B \rangle$ **by** *auto*

have $\mu C = \mu DA + \mu DB$ **using** *signed-measure-add sgn-meas* $\langle DA \in \text{sets } M \rangle$
 $\langle DB \in \text{sets } M \rangle \langle DA \cap DB = \{\} \rangle \langle C = DA \cup DB \rangle$ **by** *metis*

also have $\dots \leq \mu DB$ **using** *assms unfolding neg-meas-set-def*

by (metis DA-def Int-lower2 ⟨DA ∈ sets M⟩ add-decreasing dual-order.refl)
 also have ... ≤ 0 using assms unfolding DB-def neg-meas-set-def
 by (metis DB-def Int-lower2 ⟨DB ∈ sets M⟩)
 finally show $\mu C \leq 0$.
 qed

lemma *neg-meas-set-union*:
 assumes *neg-meas-set A*
 and *neg-meas-set B*
 shows *neg-meas-set (A ∪ B)*
proof –
 define *C* where $C = B - A$
 have $A \cup C = A \cup B$ **unfolding** *C-def* **by** *auto*
 moreover have *neg-meas-set (A ∪ C)*
proof (*rule neg-meas-disj-union*)
 show *neg-meas-set C* **unfolding** *C-def*
 by (meson *Diff-subset* assms(1) assms(2) *sets.Diff* *neg-meas-set-def*
neg-meas-subset *signed-measure-space-axioms*)
 show *neg-meas-set A* **using** *assms* **by** *simp*
 show $A \cap C = \{\}$ **unfolding** *C-def* **by** *auto*
 qed
 ultimately show *?thesis* **by** *simp*
 qed

lemma *neg-meas-self* :
 assumes *neg-meas-set E*
 shows $\mu E \leq 0$ **using** *assms* **unfolding** *neg-meas-set-def* **by** *simp*

lemma *pos-meas-set-opp*:
 assumes *signed-measure-space.pos-meas-set M (λ A. - μ A) A*
 shows *neg-meas-set A*
proof –
 have *m-meas-pos* : *signed-measure M (λ A. - μ A)*
 using *assms signed-measure-space-def*
 by (*simp add: sgn-meas signed-measure-minus*)
 thus *?thesis*
 by (metis *assms ereal-0-le-uminus-iff neg-meas-setI*
signed-measure-space.intro signed-measure-space.pos-meas-set-def)
 qed

lemma *neg-meas-set-opp*:
 assumes *signed-measure-space.neg-meas-set M (λ A. - μ A) A*
 shows *pos-meas-set A*
proof –
 have *m-meas-neg* : *signed-measure M (λ A. - μ A)*
 using *assms signed-measure-space-def*
 by (*simp add: sgn-meas signed-measure-minus*)
 thus *?thesis*
 by (metis *assms ereal-uminus-le-0-iff m-meas-neg pos-meas-setI*)

signed-measure-space.intro signed-measure-space.neg-meas-set-def)

qed
end

lemma *signed-measure-inter*:
assumes *signed-measure* M μ
and $A \in \text{sets } M$
shows *signed-measure* M $(\lambda E. \mu (E \cap A))$ **unfolding** *signed-measure-def*
proof (*intro conjI*)
show $\mu (\{\} \cap A) = 0$ **using** *assms(1) signed-measure-empty* **by** *auto*
show $-\infty \notin \text{range } (\lambda E. \mu (E \cap A)) \vee \infty \notin \text{range } (\lambda E. \mu (E \cap A))$
proof (*rule ccontr*)
assume $\neg (-\infty \notin \text{range } (\lambda E. \mu (E \cap A)) \vee \infty \notin \text{range } (\lambda E. \mu (E \cap A)))$
hence $-\infty \in \text{range } (\lambda E. \mu (E \cap A)) \wedge \infty \in \text{range } (\lambda E. \mu (E \cap A))$ **by** *simp*
hence $-\infty \in \text{range } \mu \wedge \infty \in \text{range } \mu$ **by** *auto*
thus *False* **using** *assms* **unfolding** *signed-measure-def* **by** *simp*
qed
show $\forall E. \text{range } E \subseteq \text{sets } M \longrightarrow \text{disjoint-family } E \longrightarrow \bigcup (\text{range } E) \in \text{sets } M$
 \longrightarrow
 $(\lambda i. \mu (E i \cap A)) \text{ sums } \mu (\bigcup (\text{range } E) \cap A)$
proof (*intro allI impI*)
fix $E::\text{nat} \Rightarrow 'a \text{ set}$
assume $\text{range } E \subseteq \text{sets } M$ **and** *disjoint-family* E **and** $\bigcup (\text{range } E) \in \text{sets } M$
note $E\text{props} = \text{this}$
define F **where** $F = (\lambda i. E i \cap A)$
have $(\lambda i. \mu (F i)) \text{ sums } \mu (\bigcup (\text{range } F))$
proof (*rule signed-measure-sums*)
show *signed-measure* M μ **using** *assms* **by** *simp*
show $\text{range } F \subseteq \text{sets } M$ **using** $E\text{props}$ $F\text{-def}$ *assms* **by** *blast*
show *disjoint-family* F **using** $E\text{props}$ $F\text{-def}$ *assms*
by (*metis disjoint-family-subset inf.absorb-iff2 inf-commute inf-right-idem*)
show $\bigcup (\text{range } F) \in \text{sets } M$ **using** $E\text{props}$ *assms* **unfolding** $F\text{-def}$
by (*simp add: Eprops assms countable-Un-Int(1) sets.Int*)
qed
moreover **have** $\bigcup (\text{range } F) = A \cap \bigcup (\text{range } E)$ **unfolding** $F\text{-def}$ **by** *auto*
ultimately **show** $(\lambda i. \mu (E i \cap A)) \text{ sums } \mu (\bigcup (\text{range } E) \cap A)$
unfolding $F\text{-def}$ **by** *simp*
qed
show $\forall E. \text{range } E \subseteq \text{sets } M \longrightarrow$
 $\text{disjoint-family } E \longrightarrow$
 $\bigcup (\text{range } E) \in \text{sets } M \longrightarrow |\mu (\bigcup (\text{range } E) \cap A)| < \infty \longrightarrow$
 $\text{summable } (\lambda i. \text{real-of-ereal } |\mu (E i \cap A)|)$
proof (*intro allI impI*)
fix $E::\text{nat} \Rightarrow 'a \text{ set}$
assume $\text{range } E \subseteq \text{sets } M$ **and** *disjoint-family* E **and**
 $\bigcup (\text{range } E) \in \text{sets } M$ **and** $|\mu (\bigcup (\text{range } E) \cap A)| < \infty$ **note** $E\text{props} = \text{this}$
show *summable* $(\lambda i. \text{real-of-ereal } |\mu (E i \cap A)|)$
proof (*rule signed-measure-summable*)

```

    show signed-measure M μ using assms by simp
    show range (λi. E i ∩ A) ⊆ sets M using Eprops assms by blast
    show disjoint-family (λi. E i ∩ A) using Eprops assms
      disjoint-family-subset inf.absorb-iff2 inf-commute inf-right-idem
      by fastforce
    show (∪ i. E i ∩ A) ∈ sets M using Eprops assms
      by (simp add: Eprops assms countable-Un-Int(1) sets.Int)
    show |μ (∪ i. E i ∩ A)| < ∞ using Eprops by auto
  qed
qed
qed

context signed-measure-space
begin
lemma pos-signed-to-meas-space :
  assumes pos-meas-set M1
    and m1 = (λA. μ (A ∩ M1))
  shows measure-space (space M) (sets M) m1 unfolding measure-space-def
proof (intro conjI)
  show sigma-algebra (space M) (sets M)
    by (simp add: sets.sigma-algebra-axioms)
  show positive (sets M) m1 using assms unfolding pos-meas-set-def
    by (metis Sigma-Algebra.positive-def Un-Int-eq(4)
      e2ennreal-neg neg-meas-self sup-bot-right empty-neg-meas-set)
  show countably-additive (sets M) m1
proof (rule pos-signed-measure-count-additive)
  show ∀ E ∈ sets M. 0 ≤ m1 E by (metis assms inf.cobounded2
    pos-meas-set-def sets.Int)
  show signed-measure M m1 using assms pos-meas-set-def
    signed-measure-inter[of M μ M1] sgn-meas by blast
qed
qed

lemma neg-signed-to-meas-space :
  assumes neg-meas-set M2
    and m2 = (λA. -μ (A ∩ M2))
  shows measure-space (space M) (sets M) m2 unfolding measure-space-def
proof (intro conjI)
  show sigma-algebra (space M) (sets M)
    by (simp add: sets.sigma-algebra-axioms)
  show positive (sets M) m2 using assms unfolding neg-meas-set-def
    by (metis Sigma-Algebra.positive-def e2ennreal-neg ereal-uminus-zero
      inf.absorb-iff2 inf.orderE inf-bot-right neg-meas-self pos-meas-self
      empty-neg-meas-set empty-pos-meas-set)
  show countably-additive (sets M) m2
proof (rule pos-signed-measure-count-additive)
  show ∀ E ∈ sets M. 0 ≤ m2 E
    by (metis assms ereal-uminus-eq-reorder ereal-uminus-le-0-iff
      inf.cobounded2 neg-meas-set-def sets.Int)

```

have *signed-measure* M $(\lambda A. \mu (A \cap M2))$ **using** *assms neg-meas-set-def*
signed-measure-inter[of M μ $M2$] *sgn-meas* **by** *blast*
thus *signed-measure* M $m2$ **using** *signed-measure-minus* *assms* **by** *simp*
qed
qed

lemma *pos-part-meas-nul-neg-set* :

assumes *pos-meas-set* $M1$
and *neg-meas-set* $M2$
and $m1 = (\lambda A. \mu (A \cap M1))$
and $E \in \text{sets } M$
and $E \subseteq M2$
shows $m1 E = 0$
proof –
have $m1 E \geq 0$ **using** *assms unfolding pos-meas-set-def*
by (*simp add: $\langle E \in \text{sets } M \rangle \text{sets.Int}$*)
have $\mu E \leq 0$ **using** $\langle E \subseteq M2 \rangle$ *assms unfolding neg-meas-set-def*
using $\langle E \in \text{sets } M \rangle$ **by** *blast*
then have $m1 E \leq 0$ **using** $\langle \mu E \leq 0 \rangle$ *assms*
by (*metis Int-Un-eq(1) Un-subset-iff $\langle E \in \text{sets } M \rangle \langle E \subseteq M2 \rangle$ pos-meas-setD1*
sets.Int signed-measure-space.neg-meas-set-def
signed-measure-space-axioms)
thus $m1 E = 0$ **using** $\langle m1 E \geq 0 \rangle \langle m1 E \leq 0 \rangle$ **by** *auto*
qed

lemma *neg-part-meas-nul-pos-set* :

assumes *pos-meas-set* $M1$
and *neg-meas-set* $M2$
and $m2 = (\lambda A. -\mu (A \cap M2))$
and $E \in \text{sets } M$
and $E \subseteq M1$
shows $m2 E = 0$
proof –
have $m2 E \geq 0$ **using** *assms unfolding neg-meas-set-def*
by (*simp add: $\langle E \in \text{sets } M \rangle \text{sets.Int}$*)
have $\mu E \geq 0$ **using** *assms unfolding pos-meas-set-def* **by** *blast*
then have $m2 E \leq 0$ **using** $\langle \mu E \geq 0 \rangle$ *assms*
by (*metis $\langle E \in \text{sets } M \rangle \langle E \subseteq M1 \rangle$ ereal-0-le-uminus-iff ereal-uminus-uminus*
inf-sup-ord(1) neg-meas-setD1 pos-meas-set-def pos-meas-subset
sets.Int)
thus $m2 E = 0$ **using** $\langle m2 E \geq 0 \rangle \langle m2 E \leq 0 \rangle$ **by** *auto*
qed

definition *pos-sets* **where**

$\text{pos-sets} = \{A. A \in \text{sets } M \ \wedge \ \text{pos-meas-set } A\}$

definition *pos-img* **where**

$\text{pos-img} = \{\mu A \mid A. A \in \text{pos-sets}\}$

2.3 Essential uniqueness

In this part, under the assumption that a measure space for a signed measure admits a decomposition into a positive and a negative set, we prove that this decomposition is essentially unique; in other words, that if two such decompositions (P, N) and (X, Y) exist, then any measurable subset of $(P \triangle X) \cup (N \triangle Y)$ has a null measure.

definition *hahn-space-decomp* **where**

hahn-space-decomp $M1\ M2 \equiv (\text{pos-meas-set } M1) \wedge (\text{neg-meas-set } M2) \wedge$
 $(\text{space } M = M1 \cup M2) \wedge (M1 \cap M2 = \{\})$

lemma *pos-neg-null-set*:

assumes *pos-meas-set* A

and *neg-meas-set* A

shows $\mu A = 0$ **using** *assms pos-meas-self[of A] neg-meas-self[of A]* **by** *simp*

lemma *pos-diff-neg-meas-set*:

assumes (*pos-meas-set* $M1$)

and (*neg-meas-set* $N2$)

and (*space* $M = N1 \cup N2$)

and $N1 \in \text{sets } M$

shows *neg-meas-set* $((M1 - N1) \cap \text{space } M)$ **using** *assms neg-meas-subset*

by (*metis Diff-subset-conv Int-lower2 pos-meas-setD1 sets.Diff*

sets.Int-space-eq2)

lemma *neg-diff-pos-meas-set*:

assumes (*neg-meas-set* $M2$)

and (*pos-meas-set* $N1$)

and (*space* $M = N1 \cup N2$)

and $N2 \in \text{sets } M$

shows *pos-meas-set* $((M2 - N2) \cap \text{space } M)$

proof –

have $(M2 - N2) \cap \text{space } M \subseteq N1$ **using** *assms* **by** *auto*

thus *?thesis* **using** *assms pos-meas-subset neg-meas-setD1* **by** *blast*

qed

lemma *pos-sym-diff-neg-meas-set*:

assumes *hahn-space-decomp* $M1\ M2$

and *hahn-space-decomp* $N1\ N2$

shows *neg-meas-set* $((\text{sym-diff } M1\ N1) \cap \text{space } M)$ **using** *assms*

unfolding *hahn-space-decomp-def*

by (*metis Int-Un-distrib2 neg-meas-set-union pos-meas-setD1*

pos-diff-neg-meas-set)

lemma *neg-sym-diff-pos-meas-set*:

assumes *hahn-space-decomp* $M1\ M2$

and *hahn-space-decomp* $N1\ N2$

shows *pos-meas-set* $((\text{sym-diff } M2\ N2) \cap \text{space } M)$ **using** *assms*

neg-diff-pos-meas-set **unfolding** *hahn-space-decomp-def*

by (metis (no-types, lifting) Int-Un-distrib2 neg-meas-setD1
pos-meas-set-union)

lemma *pos-meas-set-diff*:
assumes *pos-meas-set A*
and *B ∈ sets M*
shows *pos-meas-set ((A - B) ∩ (space M))* **using** *pos-meas-subset*
by (metis *Diff-subset assms(1) assms(2) pos-meas-setD1 sets.Diff*
sets.Int-space-eq2)

lemma *pos-meas-set-sym-diff*:
assumes *pos-meas-set A*
and *pos-meas-set B*
shows *pos-meas-set ((sym-diff A B) ∩ space M)* **using** *pos-meas-set-diff*
by (metis *Int-Un-distrib2 assms(1) assms(2) pos-meas-setD1*
pos-meas-set-union)

lemma *neg-meas-set-diff*:
assumes *neg-meas-set A*
and *B ∈ sets M*
shows *neg-meas-set ((A - B) ∩ (space M))* **using** *neg-meas-subset*
by (metis *Diff-subset assms(1) assms(2) neg-meas-setD1 sets.Diff*
sets.Int-space-eq2)

lemma *neg-meas-set-sym-diff*:
assumes *neg-meas-set A*
and *neg-meas-set B*
shows *neg-meas-set ((sym-diff A B) ∩ space M)* **using** *neg-meas-set-diff*
by (metis *Int-Un-distrib2 assms(1) assms(2) neg-meas-setD1*
neg-meas-set-union)

lemma *hahn-decomp-space-diff*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp N1 N2*
shows *pos-meas-set ((sym-diff M1 N1 ∪ sym-diff M2 N2) ∩ space M)*
neg-meas-set ((sym-diff M1 N1 ∪ sym-diff M2 N2) ∩ space M)
proof –
show *pos-meas-set ((sym-diff M1 N1 ∪ sym-diff M2 N2) ∩ space M)*
by (metis *Int-Un-distrib2 assms(1) assms(2) hahn-space-decomp-def*
neg-sym-diff-pos-meas-set pos-meas-set-sym-diff pos-meas-set-union)
show *neg-meas-set ((sym-diff M1 N1 ∪ sym-diff M2 N2) ∩ space M)*
by (metis *Int-Un-distrib2 assms(1) assms(2) hahn-space-decomp-def*
neg-meas-set-sym-diff neg-meas-set-union pos-sym-diff-neg-meas-set)
qed

lemma *hahn-decomp-ess-unique*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp N1 N2*
and *C ⊆ sym-diff M1 N1 ∪ sym-diff M2 N2*

and $C \in \text{sets } M$
shows $\mu C = 0$
proof –
have $C \subseteq (\text{sym-diff } M1 \ N1 \cup \text{sym-diff } M2 \ N2) \cap \text{space } M$ **using** *assms*
by (*simp add: sets.sets-into-space*)
thus *?thesis* **using** *assms hahn-decomp-space-diff pos-neg-null-set*
by (*meson neg-meas-subset pos-meas-subset*)
qed

3 Existence of a positive subset

The goal of this part is to prove that any measurable set of finite and positive measure must contain a positive subset with a strictly positive measure.

3.1 A sequence of negative subsets

definition *inf-neg* **where**

inf-neg $A = (\text{if } (A \notin \text{sets } M \vee \text{pos-meas-set } A) \text{ then } (0::\text{nat})$
else $\text{Inf } \{n | n. (1::\text{nat}) \leq n \wedge (\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < \text{ereal}(-1/n))\}$)

lemma *inf-neg-ne*:

assumes $A \in \text{sets } M$

and $\neg \text{pos-meas-set } A$

shows $\{n::\text{nat} | n. (1::\text{nat}) \leq n \wedge$

$(\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < \text{ereal}(-1/n))\} \neq \{\}$

proof –

define N **where** $N = \{n::\text{nat} | n. (1::\text{nat}) \leq n \wedge$

$(\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < \text{ereal}(-1/n))\}$

have $\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < 0$ **using** *assms unfolding pos-meas-set-def*

by *auto*

from this obtain B **where** $B \in \text{sets } M$ **and** $B \subseteq A$ **and** $\mu B < 0$ **by** *auto*

hence $\exists n::\text{nat}. (1::\text{nat}) \leq n \wedge \mu B < \text{ereal}(-1/n)$

proof (*cases* $\mu B = -\infty$)

case *True*

hence $\mu B < -1/(2::\text{nat})$ **by** *simp*

thus *?thesis* **using** *numeral-le-real-of-nat-iff one-le-numeral* **by** *blast*

next

case *False*

hence $\text{real-of-ereal } (\mu B) < 0$ **using** $\langle \mu B < 0 \rangle$

by (*metis Infty-neg-0(3) ereal-mult-eq-MInfty ereal-zero-mult*

less-eq-ereal-def less-eq-real-def less-ereal.simps(2)

real-of-ereal-eq-0 real-of-ereal-le-0)

hence $\exists n::\text{nat}. \text{Suc } 0 \leq n \wedge \text{real-of-ereal } (\mu B) < -1/n$

proof –

define nw **where** $nw = \text{Suc } (\text{nat } (\text{floor } (-1/(\text{real-of-ereal } (\mu B))))))$

have $\text{Suc } 0 \leq nw$ **unfolding** *nw-def* **by** *simp*

have $0 < -1/(\text{real-of-ereal } (\mu B))$ **using** $\langle \text{real-of-ereal } (\mu B) < 0 \rangle$

by *simp*

have $-1/(\text{real-of-ereal } (\mu B)) < nw$ **unfolding** *nw-def* **by** *linarith*
hence $1/nw < 1/(-1/(\text{real-of-ereal } (\mu B)))$
using $\langle 0 < -1/(\text{real-of-ereal } (\mu B)) \rangle$ **by** (*metis frac-less2*
le-eq-less-or-eq of-nat-1 of-nat-le-iff zero-less-one)
also have $\dots = -(\text{real-of-ereal } (\mu B))$ **by** *simp*
finally have $1/nw < -(\text{real-of-ereal } (\mu B))$.
hence $\text{real-of-ereal } (\mu B) < -1/nw$ **by** *simp*
thus *?thesis* **using** $\langle \text{Suc } 0 \leq nw \rangle$ **by** *auto*
qed
from *this* **obtain** $n1::\text{nat}$ **where** $\text{Suc } 0 \leq n1$
and $\text{real-of-ereal } (\mu B) < -1/n1$ **by** *auto*
hence $\text{ereal } (\text{real-of-ereal } (\mu B)) < -1/n1$ **using** *real-ereal-leq[of μB]*
 $\langle \mu B < 0 \rangle$ **by** *simp*
moreover have $\mu B = \text{real-of-ereal } (\mu B)$ **using** $\langle \mu B < 0 \rangle$ *False*
by (*metis less-ereal.simps(2) real-of-ereal.elims zero-ereal-def*)
ultimately show *?thesis* **using** $\langle \text{Suc } 0 \leq n1 \rangle$ **by** *auto*
qed
from *this* **obtain** $n0::\text{nat}$ **where** $(1::\text{nat}) \leq n0$ **and** $\mu B < -1/n0$ **by** *auto*
hence $n0 \in \{n::\text{nat} \mid n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < \text{ereal}(-1/n))\}$
using $\langle B \in \text{sets } M \rangle \langle B \subseteq A \rangle$ **by** *auto*
thus *?thesis* **by** *auto*
qed

lemma *inf-neg-ge-1*:
assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $(1::\text{nat}) \leq \text{inf-neg } A$
proof –
define N **where** $N = \{n::\text{nat} \mid n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < \text{ereal}(-1/n))\}$
have $N \neq \{\}$ **unfolding** *N-def* **using** *assms inf-neg-ne* **by** *auto*
moreover have $\bigwedge n. n \in N \implies (1::\text{nat}) \leq n$ **unfolding** *N-def* **by** *simp*
ultimately show $1 \leq \text{inf-neg } A$ **unfolding** *inf-neg-def N-def*
using *Inf-nat-def1 assms(1) assms(2)* **by** *presburger*
qed

lemma *inf-neg-pos*:
assumes $A \in \text{sets } M$
and $\neg \text{pos-meas-set } A$
shows $\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < -1/(\text{inf-neg } A)$
proof –
define N **where** $N = \{n::\text{nat} \mid n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq A \wedge \mu B < \text{ereal}(-1/n))\}$
have $N \neq \{\}$ **unfolding** *N-def* **using** *assms inf-neg-ne* **by** *auto*
hence $\text{Inf } N \in N$ **using** *Inf-nat-def1[of N]* **by** *simp*
hence $\text{inf-neg } A \in N$ **unfolding** *N-def inf-neg-def* **using** *assms* **by** *auto*
thus *?thesis* **unfolding** *N-def* **by** *auto*
qed

definition *rep-neg* **where**

rep-neg $A = (\text{if } (A \notin \text{sets } M \vee \text{pos-meas-set } A) \text{ then } \{\} \text{ else}$
 $\text{SOME } B. B \in \text{sets } M \wedge B \subseteq A \wedge \mu B \leq \text{ereal } (-1 / (\text{inf-neg } A)))$

lemma *g-rep-neg*:

assumes $A \in \text{sets } M$

and $\neg \text{pos-meas-set } A$

shows $\text{rep-neg } A \in \text{sets } M$ $\text{rep-neg } A \subseteq A$

$\mu (\text{rep-neg } A) \leq \text{ereal } (-1 / (\text{inf-neg } A))$

proof –

have $\exists B. B \in \text{sets } M \wedge B \subseteq A \wedge \mu B \leq -1 / (\text{inf-neg } A)$ **using** *assms*
inf-neg-pos[*of* A] **by** *auto*

from *someI-ex*[*OF* *this*] **show** $\text{rep-neg } A \in \text{sets } M$ $\text{rep-neg } A \subseteq A$

$\mu (\text{rep-neg } A) \leq -1 / (\text{inf-neg } A)$

unfolding *rep-neg-def* **using** *assms* **by** *auto*

qed

lemma *rep-neg-sets*:

shows $\text{rep-neg } A \in \text{sets } M$

proof (*cases* $A \notin \text{sets } M \vee \text{pos-meas-set } A$)

case *True*

then show *?thesis* **unfolding** *rep-neg-def* **by** *simp*

next

case *False*

then show *?thesis* **using** *g-rep-neg(1)* **by** *blast*

qed

lemma *rep-neg-subset*:

shows $\text{rep-neg } A \subseteq A$

proof (*cases* $A \notin \text{sets } M \vee \text{pos-meas-set } A$)

case *True*

then show *?thesis* **unfolding** *rep-neg-def* **by** *simp*

next

case *False*

then show *?thesis* **using** *g-rep-neg(2)* **by** *blast*

qed

lemma *rep-neg-less*:

assumes $A \in \text{sets } M$

and $\neg \text{pos-meas-set } A$

shows $\mu (\text{rep-neg } A) \leq \text{ereal } (-1 / (\text{inf-neg } A))$ **using** *assms* *g-rep-neg(3)*

by *simp*

lemma *rep-neg-leq*:

shows $\mu (\text{rep-neg } A) \leq 0$

proof (*cases* $A \notin \text{sets } M \vee \text{pos-meas-set } A$)

case *True*

hence $\text{rep-neg } A = \{\}$ **unfolding** *rep-neg-def* **by** *simp*

```

then show ?thesis using sgn-meas signed-measure-empty by force
next
  case False
  then show ?thesis using rep-neg-less by (metis le-ereal-le minus-divide-left
    neg-le-0-iff-le of-nat-0 of-nat-le-iff zero-ereal-def zero-le
    zero-le-divide-1-iff)
qed

```

3.2 Construction of the positive subset

```

fun pos-wtn
  where
    pos-wtn-base: pos-wtn E 0 = E|
    pos-wtn-step: pos-wtn E (Suc n) = pos-wtn E n - rep-neg (pos-wtn E n)

```

```

lemma pos-wtn-subset:
  shows pos-wtn E n  $\subseteq$  E
proof (induct n)
  case 0
  then show ?case using pos-wtn-base by simp
next
  case (Suc n)
  hence rep-neg (pos-wtn E n)  $\subseteq$  pos-wtn E n using rep-neg-subset by simp
  then show ?case using Suc by auto
qed

```

```

lemma pos-wtn-sets:
  assumes E  $\in$  sets M
  shows pos-wtn E n  $\in$  sets M
proof (induct n)
  case 0
  then show ?case using assms by simp
next
  case (Suc n)
  then show ?case using pos-wtn-step rep-neg-sets by auto
qed

```

```

definition neg-wtn where
  neg-wtn E (n::nat) = rep-neg (pos-wtn E n)

```

```

lemma neg-wtn-neg-meas:
  shows  $\mu$  (neg-wtn E n)  $\leq$  0 unfolding neg-wtn-def using rep-neg-leq by simp

```

```

lemma neg-wtn-sets:
  shows neg-wtn E n  $\in$  sets M unfolding neg-wtn-def using rep-neg-sets by simp

```

```

lemma neg-wtn-subset:
  shows neg-wtn E n  $\subseteq$  E unfolding neg-wtn-def
  using pos-wtn-subset[of E n] rep-neg-subset[of pos-wtn E n] by simp

```

lemma *neg-wtn-union-subset*:

shows $(\bigcup i \leq n. \text{neg-wtn } E i) \subseteq E$ **using** *neg-wtn-subset* **by** *auto*

lemma *pos-wtn-Suc*:

shows $\text{pos-wtn } E (\text{Suc } n) = E - (\bigcup i \leq n. \text{neg-wtn } E i)$ **unfolding** *neg-wtn-def*

proof (*induct n*)

case *0*

then show *?case* **using** *pos-wtn-base pos-wtn-step* **by** *simp*

next

case (*Suc n*)

have $\text{pos-wtn } E (\text{Suc } (\text{Suc } n)) = \text{pos-wtn } E (\text{Suc } n) -$

$\text{rep-neg } (\text{pos-wtn } E (\text{Suc } n))$

using *pos-wtn-step* **by** *simp*

also have $\dots = (E - (\bigcup i \leq n. \text{rep-neg } (\text{pos-wtn } E i))) -$

$\text{rep-neg } (\text{pos-wtn } E (\text{Suc } n))$

using *Suc* **by** *simp*

also have $\dots = E - (\bigcup i \leq (\text{Suc } n). \text{rep-neg } (\text{pos-wtn } E i))$

using *diff-union*[*of E λi. rep-neg (pos-wtn E i) n*] **by** *auto*

finally show $\text{pos-wtn } E (\text{Suc } (\text{Suc } n)) =$

$E - (\bigcup i \leq (\text{Suc } n). \text{rep-neg } (\text{pos-wtn } E i))$.

qed

definition *pos-sub* **where**

$\text{pos-sub } E = (\bigcap n. \text{pos-wtn } E n)$

lemma *pos-sub-sets*:

assumes $E \in \text{sets } M$

shows $\text{pos-sub } E \in \text{sets } M$ **unfolding** *pos-sub-def* **using** *pos-wtn-sets* *assms*

by *auto*

lemma *pos-sub-subset*:

shows $\text{pos-sub } E \subseteq E$ **unfolding** *pos-sub-def* **using** *pos-wtn-subset* **by** *blast*

lemma *pos-sub-infty*:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

shows $|\mu (\text{pos-sub } E)| < \infty$ **using** *signed-measure-finite-subset* *assms*

pos-sub-sets pos-sub-subset **by** *simp*

lemma *neg-wtn-djn*:

shows *disjoint-family* $(\lambda n. \text{neg-wtn } E n)$ **unfolding** *disjoint-family-on-def*

proof –

{

fix *n*

fix *m::nat*

assume $n < m$

hence $\exists p. m = \text{Suc } p$ **using** *old.nat.exhaust* **by** *auto*

from this obtain *p* **where** $m = \text{Suc } p$ **by** *auto*

have $\text{neg-wtn } E m \subseteq \text{pos-wtn } E m$ **unfolding** neg-wtn-def
by $(\text{simp add: rep-neg-subset})$
also have $\dots = E - (\bigcup i \leq p. \text{neg-wtn } E i)$ **using** $\text{pos-wtn-Suc } \langle m = \text{Suc } p \rangle$
by simp
finally have $\text{neg-wtn } E m \subseteq E - (\bigcup i \leq p. \text{neg-wtn } E i)$.
moreover have $\text{neg-wtn } E n \subseteq (\bigcup i \leq p. \text{neg-wtn } E i)$ **using** $\langle n < m \rangle$
 $\langle m = \text{Suc } p \rangle$ **by** $(\text{simp add: UN-upper})$
ultimately have $\text{neg-wtn } E n \cap \text{neg-wtn } E m = \{\}$ **by** auto
}
thus $\forall m \in \text{UNIV}. \forall n \in \text{UNIV}. m \neq n \longrightarrow \text{neg-wtn } E m \cap \text{neg-wtn } E n = \{\}$
by $(\text{metis inf-commute linorder-neqE-nat})$
qed
end

lemma $\text{disjoint-family-imp-on}$:
assumes $\text{disjoint-family } A$
shows $\text{disjoint-family-on } A S$
using $\text{assms disjoint-family-on-mono subset-UNIV}$ **by** blast

context $\text{signed-measure-space}$

begin

lemma $\text{neg-wtn-union-neg-meas}$:

shows $\mu (\bigcup i \leq n. \text{neg-wtn } E i) \leq 0$

proof –

have $\mu (\bigcup i \leq n. \text{neg-wtn } E i) = (\sum i \in \{.. n\}. \mu (\text{neg-wtn } E i))$

proof $(\text{rule signed-measure-disj-sum, simp+})$

show $\text{signed-measure } M \mu$ **using** sgn-meas .

show $\text{disjoint-family-on } (\text{neg-wtn } E) \{..n\}$ **using** neg-wtn-djn

$\text{disjoint-family-imp-on[of neg-wtn } E]$ **by** simp

show $\bigwedge i. i \in \{..n\} \implies \text{neg-wtn } E i \in \text{sets } M$ **using** neg-wtn-sets **by** simp

qed

also have $\dots \leq 0$ **using** neg-wtn-neg-meas **by** $(\text{simp add: sum-nonpos})$

finally show $?thesis$.

qed

lemma pos-wtn-meas-gt :

assumes $0 < \mu E$

and $E \in \text{sets } M$

shows $0 < \mu (\text{pos-wtn } E n)$

proof $(\text{cases } n = 0)$

case True

then show $?thesis$ **using** assms **by** simp

next

case False

hence $\exists m. n = \text{Suc } m$ **by** $(\text{simp add: not0-implies-Suc})$

from this obtain m **where** $n = \text{Suc } m$ **by** auto

hence $\text{eq: pos-wtn } E n = E - (\bigcup i \leq m. \text{neg-wtn } E i)$ **using** pos-wtn-Suc

by simp

hence $\text{pos-wtn } E n \cap (\bigcup i \leq m. \text{neg-wtn } E i) = \{\}$ **by** auto

moreover have $E = \text{pos-wtn } E \ n \cup (\bigcup_{i \leq m}. \text{neg-wtn } E \ i)$
using *eq neg-wtn-union-subset*[of $E \ m$] **by** *auto*
ultimately have $\mu \ E = \mu (\text{pos-wtn } E \ n) + \mu (\bigcup_{i \leq m}. \text{neg-wtn } E \ i)$
using *signed-measure-add*[of $M \ \mu \ \text{pos-wtn } E \ n \ \bigcup_{i \leq m}. \text{neg-wtn } E \ i$]
pos-wtn-sets neg-wtn-sets assms sgn-meas **by** *auto*
hence $0 < \mu (\text{pos-wtn } E \ n) + \mu (\bigcup_{i \leq m}. \text{neg-wtn } E \ i)$ **using** *assms* **by** *simp*
thus *?thesis* **using** *neg-wtn-union-neg-meas*
by (*metis add.right-neutral add-mono not-le*)
qed

definition *union-wit* **where**
 $\text{union-wit } E = (\bigcup_n. \text{neg-wtn } E \ n)$

lemma *union-wit-sets*:
shows $\text{union-wit } E \in \text{sets } M$ **unfolding** *union-wit-def*
proof (*intro sigma-algebra.countable-nat-UN*)
show *sigma-algebra* (*space* M) (*sets* M)
by (*simp add: sets.sigma-algebra-axioms*)
show $\text{range } (\text{neg-wtn } E) \subseteq \text{sets } M$
proof –
{
 fix n
 have $\text{neg-wtn } E \ n \in \text{sets } M$ **unfolding** *neg-wtn-def*
 by (*simp add: rep-neg-sets*)
}
thus *?thesis* **by** *auto*
qed
qed

lemma *union-wit-subset*:
shows $\text{union-wit } E \subseteq E$
proof –
{
 fix n
 have $\text{neg-wtn } E \ n \subseteq E$ **unfolding** *neg-wtn-def* **using** *pos-wtn-subset*
 rep-neg-subset[of $\text{pos-wtn } E \ n$] **by** *auto*
}
thus *?thesis* **unfolding** *union-wit-def* **by** *auto*
qed

lemma *pos-sub-diff*:
shows $\text{pos-sub } E = E - \text{union-wit } E$
proof
show $\text{pos-sub } E \subseteq E - \text{union-wit } E$
proof –
 have $\text{pos-sub } E \subseteq E$ **using** *pos-sub-subset* **by** *simp*
 moreover have $\text{pos-sub } E \cap \text{union-wit } E = \{\}$
 proof (*rule ccontr*)
 assume $\text{pos-sub } E \cap \text{union-wit } E \neq \{\}$

hence $\exists a. a \in \text{pos-sub } E \cap \text{union-wit } E$ **by** *auto*
from this obtain a **where** $a \in \text{pos-sub } E \cap \text{union-wit } E$ **by** *auto*
hence $a \in \text{union-wit } E$ **by** *simp*
hence $\exists n. a \in \text{rep-neg } (\text{pos-wtn } E \ n)$ **unfolding** *union-wit-def neg-wtn-def*
by *auto*
from this obtain n **where** $a \in \text{rep-neg } (\text{pos-wtn } E \ n)$ **by** *auto*
have $a \in \text{pos-wtn } E \ (\text{Suc } n)$ **using** $\langle a \in \text{pos-sub } E \cap \text{union-wit } E \rangle$
unfolding *pos-sub-def* **by** *blast*
hence $a \notin \text{rep-neg } (\text{pos-wtn } E \ n)$ **using** *pos-wtn-step* **by** *simp*
thus *False* **using** $\langle a \in \text{rep-neg } (\text{pos-wtn } E \ n) \rangle$ **by** *simp*
qed
ultimately show *?thesis* **by** *auto*
qed
next
show $E - \text{union-wit } E \subseteq \text{pos-sub } E$
proof
fix a
assume $a \in E - \text{union-wit } E$
show $a \in \text{pos-sub } E$ **unfolding** *pos-sub-def*
proof
fix n
show $a \in \text{pos-wtn } E \ n$
proof (*cases* $n = 0$)
case *True*
thus *?thesis* **using** *pos-wtn-base* $\langle a \in E - \text{union-wit } E \rangle$ **by** *simp*
next
case *False*
hence $\exists m. n = \text{Suc } m$ **by** (*simp add: not0-implies-Suc*)
from this obtain m **where** $n = \text{Suc } m$ **by** *auto*
have $(\bigcup i \leq m. \text{rep-neg } (\text{pos-wtn } E \ i)) \subseteq$
 $(\bigcup n. (\text{rep-neg } (\text{pos-wtn } E \ n)))$ **by** *auto*
hence $a \in E - (\bigcup i \leq m. \text{rep-neg } (\text{pos-wtn } E \ i))$
using $\langle a \in E - \text{union-wit } E \rangle$ **unfolding** *union-wit-def neg-wtn-def*
by *auto*
thus $a \in \text{pos-wtn } E \ n$ **using** *pos-wtn-Suc* $\langle n = \text{Suc } m \rangle$
unfolding *neg-wtn-def* **by** *simp*
qed
qed
qed
qed

definition *num-wtn* **where**

$\text{num-wtn } E \ n = \text{inf-neg } (\text{pos-wtn } E \ n)$

lemma *num-wtn-geq*:

shows $\mu (\text{neg-wtn } E \ n) \leq \text{ereal } (-1/(\text{num-wtn } E \ n))$

proof (*cases* $(\text{pos-wtn } E \ n) \notin \text{sets } M \vee \text{pos-meas-set } (\text{pos-wtn } E \ n)$)

case *True*

hence $\text{neg-wtn } E \ n = \{\}$ **unfolding** *neg-wtn-def rep-neg-def* **by** *simp*

moreover have $\text{num-wtn } E \ n = 0$ **using** *True* **unfolding** *num-wtn-def inf-neg-def*

by *simp*

ultimately show *?thesis* **using** *sgn-meas signed-measure-empty* **by** *force*

next

case *False*

then show *?thesis* **using** *g-rep-neg(3)[of pos-wtn E n]* **unfolding** *neg-wtn-def num-wtn-def* **by** *simp*

qed

lemma *neg-wtn-infnty*:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

shows $|\mu (\text{neg-wtn } E \ i)| < \infty$

proof (*rule signed-measure-finite-subset*)

show $E \in \text{sets } M$ $|\mu E| < \infty$ **using** *assms* **by** *auto*

show $\text{neg-wtn } E \ i \in \text{sets } M$

proof (*cases pos-wtn E i \notin sets M \vee pos-meas-set (pos-wtn E i)*)

case *True*

then show *?thesis* **unfolding** *neg-wtn-def rep-neg-def* **by** *simp*

next

case *False*

then show *?thesis* **unfolding** *neg-wtn-def*

using *g-rep-neg(1)[of pos-wtn E i]* **by** *simp*

qed

show $\text{neg-wtn } E \ i \subseteq E$ **unfolding** *neg-wtn-def* **using** *pos-wtn-subset[of E]*

rep-neg-subset[of pos-wtn E i] **by** *auto*

qed

lemma *union-wit-infnty*:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

shows $|\mu (\text{union-wit } E)| < \infty$ **using** *union-wit-subset union-wit-sets signed-measure-finite-subset assms* **unfolding** *union-wit-def* **by** *simp*

lemma *neg-wtn-summable*:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

shows *summable* ($\lambda i. - \text{real-of-ereal } (\mu (\text{neg-wtn } E \ i))$)

proof –

have *signed-measure* $M \ \mu$ **using** *sgn-meas* .

moreover have $\text{range } (\text{neg-wtn } E) \subseteq \text{sets } M$ **unfolding** *neg-wtn-def*

using *rep-neg-sets* **by** *auto*

moreover have *disjoint-family* ($\text{neg-wtn } E$) **using** *neg-wtn-djn* **by** *simp*

moreover have $\bigcup (\text{range } (\text{neg-wtn } E)) \in \text{sets } M$ **using** *union-wit-sets*

unfolding *union-wit-def* **by** *simp*

moreover have $|\mu (\bigcup (\text{range } (\text{neg-wtn } E)))| < \infty$

using *union-wit-subset signed-measure-finite-subset union-wit-sets assms*

unfolding *union-wit-def* **by** *simp*

ultimately have *summable* ($\lambda i. \text{real-of-ereal } |\mu \text{ (neg-wtn } E \ i)|$)
using *signed-measure-abs-convergent*[of M] **by** *simp*
moreover have $\bigwedge i. |\mu \text{ (neg-wtn } E \ i)| = -(\mu \text{ (neg-wtn } E \ i))$
proof –
fix i
have $\mu \text{ (neg-wtn } E \ i) \leq 0$ **using** *rep-neg-leq*[of *pos-wtn* $E \ i$]
unfolding *neg-wtn-def* .
thus $|\mu \text{ (neg-wtn } E \ i)| = -\mu \text{ (neg-wtn } E \ i)$ **using** *less-eq-ereal-def* **by** *auto*
qed
ultimately show *?thesis* **by** *simp*
qed

lemma *inv-num-wtn-summable*:

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows *summable* ($\lambda n. 1/(\text{num-wtn } E \ n)$)
proof (*rule summable-bounded*)
show $\bigwedge i. 0 \leq 1 / \text{real } (\text{num-wtn } E \ i)$ **by** *simp*
show $\bigwedge i. 1 / \text{real } (\text{num-wtn } E \ i) \leq (\lambda n. -\text{real-of-ereal } (\mu \text{ (neg-wtn } E \ n))) \ i$
proof –
fix i
have $|\mu \text{ (neg-wtn } E \ i)| < \infty$ **using** *assms neg-wtn-infty* **by** *simp*
have $\text{ereal } (1/(\text{num-wtn } E \ i)) \leq -\mu \text{ (neg-wtn } E \ i)$ **using** *num-wtn-geq*[of $E \ i$]
ereal-minus-le-minus **by** *fastforce*
also have $\dots = \text{ereal}(- \text{real-of-ereal } (\mu \text{ (neg-wtn } E \ i)))$
using $\langle |\mu \text{ (neg-wtn } E \ i)| < \infty \rangle$ *ereal-real'* **by** *auto*
finally have $\text{ereal } (1/(\text{num-wtn } E \ i)) \leq$
 $\text{ereal}(- \text{real-of-ereal } (\mu \text{ (neg-wtn } E \ i)))$.
thus $1 / \text{real } (\text{num-wtn } E \ i) \leq -\text{real-of-ereal } (\mu \text{ (neg-wtn } E \ i))$ **by** *simp*
qed
show *summable* ($\lambda i. - \text{real-of-ereal } (\mu \text{ (neg-wtn } E \ i))$)
using *assms neg-wtn-summable* **by** *simp*
qed

lemma *inv-num-wtn-shift-summable*:

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows *summable* ($\lambda n. 1/(\text{num-wtn } E \ n - 1)$)
proof (*rule sum-shift-denum*)
show *summable* ($\lambda n. 1 / \text{real } (\text{num-wtn } E \ n)$) **using** *assms inv-num-wtn-summable*
by *simp*
qed

lemma *neg-wtn-meas-sums*:

assumes $E \in \text{sets } M$
and $|\mu \ E| < \infty$
shows ($\lambda i. - (\mu \text{ (neg-wtn } E \ i))$) *sums*
suminf ($\lambda i. - \text{real-of-ereal } (\mu \text{ (neg-wtn } E \ i))$)
proof –

have $(\lambda i. \text{ereal} (- \text{real-of-ereal} (\mu (\text{neg-wtn } E i)))) \text{ sums}$
 $\text{suminf} (\lambda i. - \text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$
proof $(\text{rule sums-ereal}[\text{THEN iffD2}])$
have $\text{summable} (\lambda i. - \text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$
using $\text{neg-wtn-summable assms by simp}$
thus $(\lambda x. - \text{real-of-ereal} (\mu (\text{neg-wtn } E x)))$
 $\text{sums} (\sum i. - \text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$
by auto
qed
moreover have $\bigwedge i. \mu (\text{neg-wtn } E i) = \text{ereal} (\text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$
proof –
fix i
show $\mu (\text{neg-wtn } E i) = \text{ereal} (\text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$
using $\text{assms}(1) \text{ assms}(2) \text{ ereal-real' neg-wtn-infnty by auto}$
qed
ultimately show $?thesis$
by $(\text{metis (no-types, lifting) sums-cong uminus-ereal.simps}(1))$
qed

lemma $\text{neg-wtn-meas-suminf-le}$:

assumes $E \in \text{sets } M$

and $|\mu E| < \infty$

shows $\text{suminf} (\lambda i. \mu (\text{neg-wtn } E i)) \leq - \text{suminf} (\lambda n. 1/(\text{num-wtn } E n))$

proof –

have $\text{suminf} (\lambda n. 1/(\text{num-wtn } E n)) \leq$

$\text{suminf} (\lambda i. - \text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$

proof (rule suminf-le)

show $\text{summable} (\lambda n. 1 / \text{real} (\text{num-wtn } E n))$ **using** assms

$\text{inv-num-wtn-summable}[of E]$

$\text{summable-minus}[of \lambda n. 1 / \text{real} (\text{num-wtn } E n)]$ **by** simp

show $\text{summable} (\lambda i. - \text{real-of-ereal} (\mu (\text{neg-wtn } E i)))$

using $\text{neg-wtn-summable assms}$

$\text{summable-minus}[of \lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E i))]$

by $(\text{simp add: summable-minus-iff})$

show $\bigwedge n. 1 / \text{real} (\text{num-wtn } E n) \leq - \text{real-of-ereal} (\mu (\text{neg-wtn } E n))$

proof –

fix n

have $\mu (\text{neg-wtn } E n) \leq \text{ereal} (- 1 / \text{real} (\text{num-wtn } E n))$

using $\text{num-wtn-geq by simp}$

hence $\text{ereal} (1 / \text{real} (\text{num-wtn } E n)) \leq - \mu (\text{neg-wtn } E n)$

by $(\text{metis add.inverse-inverse eq-iff ereal-uminus-le-reorder linear}$

$\text{minus-divide-left uminus-ereal.simps}(1))$

have $\text{real-of-ereal} (\text{ereal} (1 / \text{real} (\text{num-wtn } E n))) \leq$

$\text{real-of-ereal} (- \mu (\text{neg-wtn } E n))$

proof $(\text{rule real-of-ereal-positive-mono})$

show $0 \leq \text{ereal} (1 / \text{real} (\text{num-wtn } E n))$ **by** simp

show $\text{ereal} (1 / \text{real} (\text{num-wtn } E n)) \leq - \mu (\text{neg-wtn } E n)$

using $\langle \text{ereal} (1 / \text{real} (\text{num-wtn } E n)) \leq - \mu (\text{neg-wtn } E n) \rangle$.

show $- \mu (\text{neg-wtn } E n) \neq \infty$ **using** $\text{neg-wtn-infnty}[of E n] \text{ assms by auto}$

qed
thus $(1 / \text{real} (\text{num-wtn } E \ n)) \leq -\text{real-of-ereal} (\mu (\text{neg-wtn } E \ n))$
by *simp*
qed
qed
also have $\dots = -\text{suminf} (\lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i)))$
proof (*rule suminf-minus*)
show *summable* $(\lambda n. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ n)))$
using *neg-wtn-summable assms*
summable-minus[*of* $\lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))$]
by (*simp add: summable-minus-iff*)
qed
finally have $\text{suminf} (\lambda n. 1/(\text{num-wtn } E \ n)) \leq$
 $-\text{suminf} (\lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i)))$.
hence a: $\text{suminf} (\lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))) \leq$
 $-\text{suminf} (\lambda n. 1/(\text{num-wtn } E \ n))$ **by** *simp*
show $\text{suminf} (\lambda i. (\mu (\text{neg-wtn } E \ i))) \leq \text{ereal} (-\text{suminf} (\lambda n. 1/(\text{num-wtn } E \ n)))$
proof –
have *sumeq:* $\text{suminf} (\lambda i. \text{ereal} (\text{real-of-ereal} (\mu (\text{neg-wtn } E \ i)))) =$
 $\text{suminf} (\lambda i. (\text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))))$
proof (*rule sums-suminf-ereal*)
have *summable* $(\lambda i. -\text{real-of-ereal} (\mu (\text{neg-wtn } E \ i)))$
using *neg-wtn-summable assms*
summable-minus[*of* $\lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))$]
by (*simp add: summable-minus-iff*)
thus $(\lambda i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i)))$ *sums*
 $(\sum i. \text{real-of-ereal} (\mu (\text{neg-wtn } E \ i)))$
using *neg-wtn-summable*[*of* E] *assms summable-minus-iff* **by** *blast*
qed
hence $\text{suminf} (\lambda i. \mu (\text{neg-wtn } E \ i)) =$
 $\text{suminf} (\lambda i. (\text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))))$
proof –
have $\bigwedge i. \text{ereal} (\text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))) = \mu (\text{neg-wtn } E \ i)$
proof –
fix i
show $\text{ereal} (\text{real-of-ereal} (\mu (\text{neg-wtn } E \ i))) = \mu (\text{neg-wtn } E \ i)$
using *neg-wtn-infty*[*of* E] *assms* **by** (*simp add: ereal-real'*)
qed
thus *?thesis* **using** *sumeq* **by** *auto*
qed
thus *?thesis* **using** a **by** *simp*
qed
qed

lemma *union-wit-meas-le:*
assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
shows $\mu (\text{union-wit } E) \leq -\text{suminf} (\lambda n. 1 / \text{real} (\text{num-wtn } E \ n))$
proof –

have μ (*union-wit* E) = μ (\bigcup (*range* (*neg-wtn* E))) **unfolding** *union-wit-def*
by *simp*
also have ... = (\sum i . μ (*neg-wtn* E i))
proof (*rule signed-measure-inf-sum[symmetric]*)
show *signed-measure* M μ **using** *sgn-meas* .
show *range* (*neg-wtn* E) \subseteq *sets* M
by (*simp add: image-subset-iff neg-wtn-def rep-neg-sets*)
show *disjoint-family* (*neg-wtn* E) **using** *neg-wtn-djn* **by** *simp*
show \bigcup (*range* (*neg-wtn* E)) \in *sets* M **using** *union-wit-sets*
unfolding *union-wit-def* **by** *simp*
qed
also have ... $\leq -$ *suminf* (λn . $1 / \text{real} (\text{num-wtn } E \ n)$)
using *assms neg-wtn-meas-suminf-le* **by** *simp*
finally show *?thesis* .
qed

lemma *pos-sub-pos-meas*:
assumes $E \in$ *sets* M
and $|\mu E| < \infty$
and $0 < \mu E$
and \neg *pos-meas-set* E
shows $0 < \mu$ (*pos-sub* E)
proof –
have $0 < \mu E$ **using** *assms* **by** *simp*
also have ... = μ (*pos-sub* E) + μ (*union-wit* E)
proof –
have $E =$ *pos-sub* $E \cup$ (*union-wit* E)
using *pos-sub-diff[of E] union-wit-subset* **by** *force*
moreover have *pos-sub* $E \cap$ *union-wit* $E = \{\}$
using *pos-sub-diff* **by** *auto*
ultimately show *?thesis*
using *signed-measure-add[of M μ *pos-sub* E *union-wit* E]*
pos-sub-sets union-wit-sets assms sgn-meas **by** *simp*
qed
also have ... $\leq \mu$ (*pos-sub* E) + ($-$ *suminf* (λn . $1 / \text{real} (\text{num-wtn } E \ n)$))
proof –
have μ (*union-wit* E) $\leq -$ *suminf* (λn . $1 / \text{real} (\text{num-wtn } E \ n)$)
using *union-wit-meas-le[of E] assms* **by** *simp*
thus *?thesis* **using** *union-wit-infty assms* **using** *add-left-mono* **by** *blast*
qed
also have ... = μ (*pos-sub* E) – *suminf* (λn . $1 / \text{real} (\text{num-wtn } E \ n)$)
by (*simp add: minus-ereal-def*)
finally have $0 < \mu$ (*pos-sub* E) – *suminf* (λn . $1 / \text{real} (\text{num-wtn } E \ n)$) .
moreover have $0 < \text{suminf}$ (λn . $1 / \text{real} (\text{num-wtn } E \ n)$)
proof (*rule suminf-pos2*)
show $0 < 1 / \text{real} (\text{num-wtn } E \ 0)$
using *inf-neg-ge-1[of E] assms pos-wtn-base* **unfolding** *num-wtn-def* **by** *simp*
show $\bigwedge n$. $0 \leq 1 / \text{real} (\text{num-wtn } E \ n)$ **by** *simp*
show *summable* (λn . $1 / \text{real} (\text{num-wtn } E \ n)$)

using *assms inv-num-wtn-summable* by *simp*
 qed
 ultimately show *?thesis* using *pos-sub-infnty assms* by *fastforce*
 qed

lemma *num-wtn-conv*:
 assumes $E \in \text{sets } M$
 and $|\mu E| < \infty$
 shows $(\lambda n. 1/(\text{num-wtn } E n)) \longrightarrow 0$
proof (*rule summable-LIMSEQ-zero*)
 show *summable* $(\lambda n. 1 / \text{real } (\text{num-wtn } E n))$
 using *assms inv-num-wtn-summable* by *simp*
 qed

lemma *num-wtn-shift-conv*:
 assumes $E \in \text{sets } M$
 and $|\mu E| < \infty$
 shows $(\lambda n. 1/(\text{num-wtn } E n - 1)) \longrightarrow 0$
proof (*rule summable-LIMSEQ-zero*)
 show *summable* $(\lambda n. 1 / \text{real } (\text{num-wtn } E n - 1))$
 using *assms inv-num-wtn-shift-summable* by *simp*
 qed

lemma *inf-neg-E-set*:
 assumes $0 < \text{inf-neg } E$
 shows $E \in \text{sets } M$ using *assms unfolding inf-neg-def* by *presburger*

lemma *inf-neg-pos-meas*:
 assumes $0 < \text{inf-neg } E$
 shows $\neg \text{pos-meas-set } E$ using *assms unfolding inf-neg-def* by *presburger*

lemma *inf-neg-mem*:
 assumes $0 < \text{inf-neg } E$
 shows $\text{inf-neg } E \in \{n::\text{nat} \mid n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq E \wedge \mu B < \text{ereal } (-1/n))\}$
proof –
 have $E \in \text{sets } M$ using *assms unfolding inf-neg-def* by *presburger*
 moreover have $\neg \text{pos-meas-set } E$ using *assms unfolding inf-neg-def*
 by *presburger*
 ultimately have $\{n::\text{nat} \mid n. (1::\text{nat}) \leq n \wedge$
 $(\exists B \in \text{sets } M. B \subseteq E \wedge \mu B < \text{ereal } (-1/n))\} \neq \{\}$
 using *inf-neg-ne[of E]* by *simp*
 thus *?thesis* using *unfolding inf-neg-def*
 by (*meson Inf-nat-def1* $\langle E \in \text{sets } M \rangle \langle \neg \text{pos-meas-set } E \rangle$)
 qed

lemma *prec-inf-neg-pos*:
 assumes $0 < \text{inf-neg } E - 1$
 and $B \in \text{sets } M$

and $B \subseteq E$
shows $-1/(\text{inf-neg } E - 1) \leq \mu B$
proof (rule ccontr)
define S **where** $S = \{p::\text{nat} \mid p. (1::\text{nat}) \leq p \wedge$
 $(\exists B \in \text{sets } M. B \subseteq E \wedge \mu B < \text{ereal } (-1/p))\}$
assume $\neg \text{ereal } (-1 / \text{real } (\text{inf-neg } E - 1)) \leq \mu B$
hence $\mu B < -1/(\text{inf-neg } E - 1)$ **by** *auto*
hence $\text{inf-neg } E - 1 \in S$ **unfolding** $S\text{-def}$ **using** *assms* **by** *auto*
have $\text{Suc } 0 < \text{inf-neg } E$ **using** *assms* **by** *simp*
hence $\text{inf-neg } E \in S$ **unfolding** $S\text{-def}$ **using** $\text{inf-neg-mem}[of E]$ **by** *simp*
hence $S \neq \{\}$ **by** *auto*
have $\text{inf-neg } E = \text{Inf } S$ **unfolding** $S\text{-def}$ inf-neg-def
using *assms* inf-neg-E-set inf-neg-pos-meas **by** *auto*
have $\text{inf-neg } E - 1 < \text{inf-neg } E$ **using** *assms* **by** *simp*
hence $\text{inf-neg } E - 1 \notin S$
using $\text{cInf-less-iff}[of S]$ $\langle S \neq \{\} \rangle$ $\langle \text{inf-neg } E = \text{Inf } S \rangle$ **by** *auto*
thus *False* **using** $\langle \text{inf-neg } E - 1 \in S \rangle$ **by** *simp*
qed

lemma *pos-wtn-meas-ge*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $C \in \text{sets } M$
and $\bigwedge n. C \subseteq \text{pos-wtn } E n$
and $\bigwedge n. 0 < \text{num-wtn } E n$
shows $\exists N. \forall n \geq N. -1/(\text{num-wtn } E n - 1) \leq \mu C$
proof –
have $\exists N. \forall n \geq N. 1/(\text{num-wtn } E n) < 1/2$ **using** $\text{num-wtn-conv}[of E]$
 $\text{conv-0-half}[of \lambda n. 1 / \text{real } (\text{num-wtn } E n)]$ *assms* **by** *simp*
from *this* **obtain** N **where** $\forall n \geq N. 1/(\text{num-wtn } E n) < 1/2$ **by** *auto*
{
fix n
assume $N \leq n$
hence $1/(\text{num-wtn } E n) < 1/2$ **using** $\langle \forall n \geq N. 1/(\text{num-wtn } E n) < 1/2 \rangle$ **by**
simp
have $1/(1/2) < 1/(1/(\text{num-wtn } E n))$
proof (rule *frac-less2*, *auto*)
show $2 / \text{real } (\text{num-wtn } E n) < 1$ **using** $\langle 1/(\text{num-wtn } E n) < 1/2 \rangle$
by *linarith*
show $0 < \text{num-wtn } E n$ **unfolding** num-wtn-def **using** inf-neg-ge-1 *assms*
by (*simp add: num-wtn-def*)
qed
hence $2 < (\text{num-wtn } E n)$ **by** *simp*
hence $\text{Suc } 0 < \text{num-wtn } E n - 1$ **unfolding** num-wtn-def **by** *simp*
hence $-1/(\text{num-wtn } E n - 1) \leq \mu C$ **using** *assms* prec-inf-neg-pos
unfolding num-wtn-def **by** *simp*
}
thus *?thesis* **by** *auto*
qed

lemma *pos-sub-pos-meas-subset*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $C \in \text{sets } M$
and $C \subseteq (\text{pos-sub } E)$
and $\bigwedge n. 0 < \text{num-wtn } E n$
shows $0 \leq \mu C$

proof –

have $\bigwedge n. C \subseteq \text{pos-wtn } E n$ **using** *assms unfolding pos-sub-def* **by** *auto*

hence $\exists N. \forall n \geq N. -1 / (\text{num-wtn } E n - 1) \leq \mu C$ **using** *assms*

pos-wtn-meas-ge[of $E C$] **by** *simp*

from this obtain N **where** *Nprop*: $\forall n \geq N. -1 / (\text{num-wtn } E n - 1) \leq \mu C$

by *auto*

show $0 \leq \mu C$

proof (*rule lim-mono*)

show $\bigwedge n. N \leq n \implies -1 / (\text{num-wtn } E n - 1) \leq (\lambda n. \mu C) n$

using *Nprop* **by** *simp*

have $(\lambda n. (1 / \text{real } (\text{num-wtn } E n - 1))) \longrightarrow 0$

using *assms num-wtn-shift-conv*[of E] **by** *simp*

hence $(\lambda n. (-1 / \text{real } (\text{num-wtn } E n - 1))) \longrightarrow 0$

using *tendsto-minus*[of $\lambda n. 1 / \text{real } (\text{num-wtn } E n - 1)$ 0] **by** *simp*

thus $(\lambda n. \text{ereal } (-1 / \text{real } (\text{num-wtn } E n - 1))) \longrightarrow 0$

by (*simp add: zero-ereal-def*)

show $(\lambda n. \mu C) \longrightarrow \mu C$ **by** *simp*

qed

qed

lemma *pos-sub-pos-meas'*:

assumes $E \in \text{sets } M$
and $|\mu E| < \infty$
and $0 < \mu E$
and $\forall n. 0 < \text{num-wtn } E n$
shows $0 < \mu (\text{pos-sub } E)$

proof –

have $0 < \mu E$ **using** *assms* **by** *simp*

also have $\dots = \mu (\text{pos-sub } E) + \mu (\text{union-wit } E)$

proof –

have $E = \text{pos-sub } E \cup (\text{union-wit } E)$

using *pos-sub-diff*[of E] *union-wit-subset* **by** *force*

moreover have $\text{pos-sub } E \cap \text{union-wit } E = \{\}$

using *pos-sub-diff* **by** *auto*

ultimately show *?thesis*

using *signed-measure-add*[of $M \mu \text{pos-sub } E \text{union-wit } E$]

pos-sub-sets union-wit-sets *assms sgn-meas* **by** *simp*

qed

also have $\dots \leq \mu (\text{pos-sub } E) + (- \text{suminf } (\lambda n. 1 / \text{real } (\text{num-wtn } E n)))$

proof –

have $\mu (\text{union-wit } E) \leq - \text{suminf } (\lambda n. 1 / \text{real } (\text{num-wtn } E n))$

```

    using union-wit-meas-le[of E] assms by simp
    thus ?thesis using union-wit-infty assms using add-left-mono by blast
qed
also have ... =  $\mu$  (pos-sub E) -  $\text{suminf } (\lambda n. 1 / \text{real } (\text{num-wtn } E n))$ 
  by (simp add: minus-ereal-def)
finally have  $0 < \mu$  (pos-sub E) -  $\text{suminf } (\lambda n. 1 / \text{real } (\text{num-wtn } E n))$  .
moreover have  $0 < \text{suminf } (\lambda n. 1 / \text{real } (\text{num-wtn } E n))$ 
proof (rule suminf-pos2)
  show  $0 < 1 / \text{real } (\text{num-wtn } E 0)$  using assms by simp
  show  $\bigwedge n. 0 \leq 1 / \text{real } (\text{num-wtn } E n)$  by simp
  show summable  $(\lambda n. 1 / \text{real } (\text{num-wtn } E n))$ 
    using assms inv-num-wtn-summable by simp
qed
ultimately show ?thesis using pos-sub-infty assms by fastforce
qed

```

We obtain the main result of this part on the existence of a positive subset.

lemma *exists-pos-meas-subset*:

```

assumes E ∈ sets M
  and  $|\mu E| < \infty$ 
  and  $0 < \mu E$ 
shows  $\exists A. A \subseteq E \wedge \text{pos-meas-set } A \wedge 0 < \mu A$ 
proof (cases  $\forall n. 0 < \text{num-wtn } E n$ )
case True
  have pos-meas-set (pos-sub E)
  proof (rule pos-meas-setI)
    show pos-sub E ∈ sets M by (simp add: assms(1) pos-sub-sets)
    fix A
    assume A ∈ sets M and  $A \subseteq \text{pos-sub } E$ 
    thus  $0 \leq \mu A$  using assms True pos-sub-pos-meas-subset[of E] by simp
  qed
  moreover have  $0 < \mu$  (pos-sub E)
    using pos-sub-pos-meas'[of E] True assms by simp
  ultimately show ?thesis using pos-meas-set-def by (metis pos-sub-subset)
next
case False
  hence  $\exists n. \text{num-wtn } E n = 0$  by simp
  from this obtain n where  $\text{num-wtn } E n = 0$  by auto
  hence  $\text{pos-wtn } E n \notin \text{sets } M \vee \text{pos-meas-set } (\text{pos-wtn } E n)$ 
    using inf-neg-ge-1 unfolding num-wtn-def by fastforce
  hence pos-meas-set (pos-wtn E n) using assms
    by (simp add:  $\langle E \in \text{sets } M \rangle$  pos-wtn-sets)
  moreover have  $0 < \mu$  (pos-wtn E n) using pos-wtn-meas-gt assms by simp
  ultimately show ?thesis using pos-meas-set-def by (meson pos-wtn-subset)
qed

```

4 The Hahn decomposition theorem

definition *seq-meas* where

$$\text{seq-meas} = (\text{SOME } f. \text{incseq } f \wedge \text{range } f \subseteq \text{pos-img} \wedge \bigsqcup \text{pos-img} = \bigsqcup \text{range } f)$$

lemma *seq-meas-props*:

shows $\text{incseq } \text{seq-meas} \wedge \text{range } \text{seq-meas} \subseteq \text{pos-img} \wedge$
 $\bigsqcup \text{pos-img} = \bigsqcup \text{range } \text{seq-meas}$

proof –

have $ex: \exists f. \text{incseq } f \wedge \text{range } f \subseteq \text{pos-img} \wedge \bigsqcup \text{pos-img} = \bigsqcup \text{range } f$

proof (rule *Extended-Real.Sup-countable-SUP*)

show $\text{pos-img} \neq \{\}$

proof –

have $\{\} \in \text{pos-sets}$ **using** *empty-pos-meas-set unfolding pos-sets-def*
by *simp*

hence $\mu \{\} \in \text{pos-img}$ **unfolding** *pos-img-def* **by** *auto*

thus *?thesis* **by** *auto*

qed

qed

let $?V = \text{SOME } f. \text{incseq } f \wedge \text{range } f \subseteq \text{pos-img} \wedge \bigsqcup \text{pos-img} = \bigsqcup \text{range } f$

have $vprop: \text{incseq } ?V \wedge \text{range } ?V \subseteq \text{pos-img} \wedge \bigsqcup \text{pos-img} = \bigsqcup \text{range } ?V$

using *someI-ex*[of $\lambda f. \text{incseq } f \wedge \text{range } f \subseteq \text{pos-img} \wedge$

$\bigsqcup \text{pos-img} = \bigsqcup \text{range } f]$ ex **by** *blast*

show *?thesis* **using** *seq-meas-def vprop* **by** *presburger*

qed

definition *seq-meas-rep* where

$$\text{seq-meas-rep } n = (\text{SOME } A. A \in \text{pos-sets} \wedge \text{seq-meas } n = \mu A)$$

lemma *seq-meas-rep-ex*:

shows $\text{seq-meas-rep } n \in \text{pos-sets} \wedge \mu (\text{seq-meas-rep } n) = \text{seq-meas } n$

proof –

have $ex: \exists A. A \in \text{pos-sets} \wedge \text{seq-meas } n = \mu A$ **using** *seq-meas-props*
by (*smt (verit) UNIV-I image-subset-iff mem-Collect-eq pos-img-def*)

let $?V = \text{SOME } A. A \in \text{pos-sets} \wedge \text{seq-meas } n = \mu A$

have $vprop: ?V \in \text{pos-sets} \wedge \text{seq-meas } n = \mu ?V$ **using**

someI-ex[of $\lambda A. A \in \text{pos-sets} \wedge \text{seq-meas } n = \mu A]$ **using** ex **by** *blast*

show *?thesis* **using** *seq-meas-rep-def vprop* **by** *fastforce*

qed

lemma *seq-meas-rep-pos*:

assumes $\forall E \in \text{sets } M. \mu E < \infty$

shows $\text{pos-meas-set } (\bigcup i. \text{seq-meas-rep } i)$

proof (rule *pos-meas-set-Union*)

show $\bigwedge i. \text{pos-meas-set } (\text{seq-meas-rep } i)$

using *seq-meas-rep-ex signed-measure-space.pos-sets-def*

signed-measure-space-axioms **by** *auto*

then show $\bigwedge i. \text{seq-meas-rep } i \in \text{sets } M$

by (*simp add: pos-meas-setD1*)

show $|\mu (\bigcup (\text{range seq-meas-rep}))| < \infty$
proof –
have $(\bigcup (\text{range seq-meas-rep})) \in \text{sets } M$
proof (*rule sigma-algebra.countable-Union*)
show *sigma-algebra (space M) (sets M)*
by (*simp add: sets.sigma-algebra-axioms*)
show *countable (range seq-meas-rep) by simp*
show $\text{range seq-meas-rep} \subseteq \text{sets } M$
by (*simp add: $\langle \bigwedge i. \text{seq-meas-rep } i \in \text{sets } M \rangle \text{ image-subset-iff}$*)
qed
hence $\mu (\bigcup (\text{range seq-meas-rep})) \geq 0$
using $\langle \bigwedge i. \text{pos-meas-set (seq-meas-rep } i) \rangle \langle \bigwedge i. \text{seq-meas-rep } i \in \text{sets } M \rangle$
signed-measure-space.pos-meas-set-pos-lim signed-measure-space-axioms
by *blast*
thus *?thesis using assms $\langle \bigcup (\text{range seq-meas-rep}) \in \text{sets } M \rangle \text{ abs-ereal-ge0}$*
by *simp*
qed
qed

lemma *sup-seq-meas-rep*:
assumes $\forall E \in \text{sets } M. \mu E < \infty$
and $S = (\bigsqcup \text{pos-img})$
and $A = (\bigcup i. \text{seq-meas-rep } i)$
shows $\mu A = S$
proof –
have *pms: pos-meas-set $(\bigcup i. \text{seq-meas-rep } i)$*
using *assms seq-meas-rep-pos by simp*
hence $\mu A \leq S$
by (*metis (mono-tags, lifting) Sup-upper $\langle S = \bigsqcup \text{pos-img} \rangle \text{ mem-Collect-eq}$*
pos-img-def pos-meas-setD1 pos-sets-def assms(2) assms(3))
have $\forall n. (\mu A = \mu (A - \text{seq-meas-rep } n) + \mu (\text{seq-meas-rep } n))$
proof
fix n
have $A = (A - \text{seq-meas-rep } n) \cup \text{seq-meas-rep } n$
using $\langle A = \bigcup (\text{range seq-meas-rep}) \rangle$ **by** *blast*
hence $\mu A = \mu ((A - \text{seq-meas-rep } n) \cup \text{seq-meas-rep } n)$ **by** *simp*
also have $\dots = \mu (A - \text{seq-meas-rep } n) + \mu (\text{seq-meas-rep } n)$
proof (*rule signed-measure-add*)
show *signed-measure M μ using sgn-meas by simp*
show $\text{seq-meas-rep } n \in \text{sets } M$
using *pos-sets-def seq-meas-rep-ex by auto*
then show $A - \text{seq-meas-rep } n \in \text{sets } M$
by (*simp add: assms pms pos-meas-setD1 sets.Diff*)
show $(A - \text{seq-meas-rep } n) \cap \text{seq-meas-rep } n = \{\}$ **by** *auto*
qed
finally show $\mu A = \mu (A - \text{seq-meas-rep } n) + \mu (\text{seq-meas-rep } n)$.
qed
have $\forall n. \mu A \geq \mu (\text{seq-meas-rep } n)$
proof

fix n
have $\mu A \geq 0$ **using** *pms assms unfolding pos-meas-set-def* **by** *auto*
have $(A - \text{seq-meas-rep } n) \subseteq A$ **by** *simp*
hence *pos-meas-set* $(A - \text{seq-meas-rep } n)$
proof –
 have $(A - \text{seq-meas-rep } n) \in \text{sets } M$
 using *pms assms pos-meas-setD1 pos-sets-def seq-meas-rep-ex* **by** *auto*
 thus *?thesis* **using** *pms assms unfolding pos-meas-set-def* **by** *auto*
qed
hence $\mu (A - \text{seq-meas-rep } n) \geq 0$ **unfolding** *pos-meas-set-def* **by** *auto*
thus $\mu (\text{seq-meas-rep } n) \leq \mu A$
 using $\langle \forall n. (\mu A = \mu (A - \text{seq-meas-rep } n) + \mu (\text{seq-meas-rep } n)) \rangle$
 by *(metis ereal-le-add-self2)*
qed
hence $\mu A \geq (\bigsqcup \text{range seq-meas})$ **by** *(simp add: Sup-le-iff seq-meas-rep-ex)*
moreover **have** $S = (\bigsqcup \text{range seq-meas})$
 using *seq-meas-props* $\langle S = (\bigsqcup \text{pos-img}) \rangle$ **by** *simp*
ultimately **have** $\mu A \geq S$ **by** *simp*
thus $\mu A = S$ **using** $\langle \mu A \leq S \rangle$ **by** *simp*
qed

lemma *seq-meas-rep-compl*:
assumes $\forall E \in \text{sets } M. \mu E < \infty$
 and $A = (\bigcup i. \text{seq-meas-rep } i)$
shows *neg-meas-set* $((\text{space } M) - A)$ **unfolding** *neg-meas-set-def*
proof *(rule ccontr)*
 assume *asm*: $\neg (\text{space } M - A \in \text{sets } M \wedge$
 $(\forall Aa \in \text{sets } M. Aa \subseteq \text{space } M - A \longrightarrow \mu Aa \leq 0))$
 define S **where** $S = (\bigsqcup \text{pos-img})$
 have *pos-meas-set* A **using** *assms seq-meas-rep-pos* **by** *simp*
 have $\mu A = S$ **using** *sup-seq-meas-rep assms S-def* **by** *simp*
 hence $S < \infty$ **using** *assms* $\langle \text{pos-meas-set } A \rangle$ *pos-meas-setD1* **by** *blast*
 have $(\text{space } M - A \in \text{sets } M)$
 by *(simp add: pos-meas-set A pos-meas-setD1 sets.compl-sets)*
 hence $\neg (\forall Aa \in \text{sets } M. Aa \subseteq \text{space } M - A \longrightarrow \mu Aa \leq 0)$ **using** *asm* **by** *blast*
 hence $\exists E \in \text{sets } M. E \subseteq ((\text{space } M) - A) \wedge \mu E > 0$
 by *(metis less-eq-ereal-def linear)*
from this obtain E **where** $E \in \text{sets } M$ **and** $E \subseteq ((\text{space } M) - A)$ **and**
 $\mu E > 0$ **by** *auto*
have $\exists A0 \subseteq E. \text{pos-meas-set } A0 \wedge \mu A0 > 0$
proof *(rule exists-pos-meas-subset)*
 show $E \in \text{sets } M$ **using** $\langle E \in \text{sets } M \rangle$ **by** *simp*
 show $0 < \mu E$ **using** $\langle \mu E > 0 \rangle$ **by** *simp*
 show $|\mu E| < \infty$
 proof –
 have $\mu E < \infty$ **using** *assms* $\langle E \in \text{sets } M \rangle$ **by** *simp*
 moreover **have** $-\infty < \mu E$ **using** $\langle 0 < \mu E \rangle$ **by** *simp*
 ultimately **show** *?thesis*
 by *(meson ereal-inf-ty-less(1) not-inf-tyI)*

qed
qed
from *this* **obtain** $A0$ **where** $A0 \subseteq E$ **and** *pos-meas-set* $A0$ **and** $\mu A0 > 0$
by *auto*
have *pos-meas-set* $(A \cup A0)$
using *pos-meas-set-union* \langle *pos-meas-set* $A0\rangle$ \langle *pos-meas-set* $A\rangle$ **by** *simp*
have $\mu (A \cup A0) = \mu A + \mu A0$
proof (*rule signed-measure-add*)
show *signed-measure* M μ **using** *sgn-meas* **by** *simp*
show $A \in$ *sets* M **using** \langle *pos-meas-set* $A\rangle$
unfolding *pos-meas-set-def* **by** *simp*
show $A0 \in$ *sets* M **using** \langle *pos-meas-set* $A0\rangle$
unfolding *pos-meas-set-def* **by** *simp*
show $(A \cap A0) = \{\}$ **using** $\langle A0 \subseteq E\rangle$ $\langle E \subseteq ((\text{space } M) - A)\rangle$ **by** *auto*
qed
then **have** $\mu (A \cup A0) > S$
using $\langle \mu A = S\rangle$ $\langle \mu A0 > 0\rangle$
by (*metis* $\langle S < \infty\rangle$ \langle *pos-meas-set* $(A \cup A0)\rangle$ *abs-ereal-ge0* *ereal-between(2)*
not-infntyI *not-less-iff-gr-or-eq* *pos-meas-self*)
have $(A \cup A0) \in$ *pos-sets*
proof –
have $(A \cup A0) \in$ *sets* M **using** *sigma-algebra.countable-Union*
by (*simp add:* \langle *pos-meas-set* $(A \cup A0)\rangle$ *pos-meas-setD1*)
moreover **have** *pos-meas-set* $(A \cup A0)$ **using** \langle *pos-meas-set* $(A \cup A0)\rangle$ **by**
simp
ultimately **show** *?thesis* **unfolding** *pos-sets-def* **by** *simp*
qed
then **have** $\mu (A \cup A0) \in$ *pos-img* **unfolding** *pos-img-def* **by** *auto*
show *False* **using** $\langle \mu (A \cup A0) > S\rangle$ $\langle \mu (A \cup A0) \in$ *pos-img* \rangle $\langle S = (\bigsqcup$ *pos-img* $\rangle)$
by (*metis* *Sup-upper* *sup.absorb-iff2* *sup.strict-order-iff*)
qed

lemma *hahn-decomp-finite*:
assumes $\forall E \in$ *sets* $M. \mu E < \infty$
shows $\exists M1 M2. \text{hahn-space-decomp } M1 M2$ **unfolding** *hahn-space-decomp-def*
proof –
define S **where** $S = (\bigsqcup$ *pos-img* $)$
define A **where** $A = (\bigcup$ *i. seq-meas-rep i* $)$
have *pos-meas-set* A **unfolding** *A-def* **using** *assms seq-meas-rep-pos* **by** *simp*
have *neg-meas-set* $((\text{space } M) - A)$
using *seq-meas-rep-compl* *assms* **unfolding** *A-def* **by** *simp*
show $\exists M1 M2. \text{pos-meas-set } M1 \wedge \text{neg-meas-set } M2 \wedge \text{space } M = M1 \cup M2 \wedge$
 $M1 \cap M2 = \{\}$
proof (*intro exI conjI*)
show *pos-meas-set* A **using** \langle *pos-meas-set* $A\rangle$.
show *neg-meas-set* $(\text{space } M - A)$ **using** \langle *neg-meas-set* $(\text{space } M - A)\rangle$.
show $\text{space } M = A \cup (\text{space } M - A)$
by (*metis* *Diff-partition* \langle *pos-meas-set* $A\rangle$ *inf.absorb-iff2* *pos-meas-setD1*
sets.Int-space-eq1)

```

  show  $A \cap (\text{space } M - A) = \{\}$  by auto
qed
qed

theorem hahn-decomposition:
  shows  $\exists M1 M2. \text{hahn-space-decomp } M1 M2$ 
proof (cases  $\forall E \in \text{sets } M. \mu E < \infty$ )
  case True
  thus ?thesis using hahn-decomp-finite by simp
next
  case False
  define  $m$  where  $m = (\lambda A . - \mu A)$ 
  have  $\exists M1 M2. \text{signed-measure-space.hahn-space-decomp } M m M1 M2$ 
  proof (rule signed-measure-space.hahn-decomp-finite)
    show signed-measure-space  $M m$ 
    using signed-measure-minus sgn-meas  $\langle m = (\lambda A . - \mu A) \rangle$ 
    by (unfold-locales, simp)
    show  $\forall E \in \text{sets } M. m E < \infty$ 
  proof
    fix  $E$ 
    assume  $E \in \text{sets } M$ 
    show  $m E < \infty$ 
  proof
    show  $m E \neq \infty$ 
  proof (rule ccontr)
    assume  $\neg m E \neq \infty$ 
    have  $m E = \infty$ 
    using  $\langle \neg m E \neq \infty \rangle$  by auto
    have signed-measure  $M m$ 
    using  $\langle \text{signed-measure-space } M m \rangle$  signed-measure-space-def by auto
    moreover have  $m E = - \mu E$  using  $\langle m = (\lambda A . - \mu A) \rangle$  by auto
    then have  $\infty \notin \text{range } m$  using  $\langle \text{signed-measure } M m \rangle$ 
    by (metis (no-types, lifting) False ereal-less-PIfty
    ereal-uminus-eq-reorder image-iff inf-range m-def rangeI)
    show False using  $\langle m E = \infty \rangle \langle \infty \notin \text{range } m \rangle$ 
    by (metis rangeI)
  qed
  qed
  qed
  qed
  hence  $\exists M1 M2. (\text{neg-meas-set } M1) \wedge (\text{pos-meas-set } M2) \wedge (\text{space } M = M1 \cup M2) \wedge (M1 \cap M2 = \{\})$ 
  using pos-meas-set-opp neg-meas-set-opp unfolding m-def
  by (metis sgn-meas signed-measure-minus signed-measure-space-def
  signed-measure-space.hahn-space-decomp-def)
  thus ?thesis using hahn-space-decomp-def by (metis inf-commute sup-commute)
qed

```


5 The Jordan decomposition theorem

definition *jordan-decomp* where

$$\begin{aligned} \text{jordan-decomp } m1 \ m2 \longleftrightarrow & ((\text{measure-space } (\text{space } M) (\text{sets } M) \ m1) \wedge \\ & (\text{measure-space } (\text{space } M) (\text{sets } M) \ m2) \wedge \\ & (\forall A \in \text{sets } M. \ 0 \leq m1 \ A) \wedge \\ & (\forall A \in \text{sets } M. \ 0 \leq m2 \ A) \wedge \\ & (\forall A \in \text{sets } M. \ \mu \ A = (m1 \ A) - (m2 \ A)) \wedge \\ & (\forall P \ N \ A. \ \text{hahn-space-decomp } P \ N \longrightarrow \\ & \quad (A \in \text{sets } M \longrightarrow A \subseteq P \longrightarrow (m2 \ A) = 0) \wedge \\ & \quad (A \in \text{sets } M \longrightarrow A \subseteq N \longrightarrow (m1 \ A) = 0)) \wedge \\ & ((\forall A \in \text{sets } M. \ m1 \ A < \infty) \vee (\forall A \in \text{sets } M. \ m2 \ A < \infty))) \end{aligned}$$

lemma *jordan-decomp-pos-meas*:

assumes *jordan-decomp* $m1 \ m2$

and *hahn-space-decomp* $P \ N$

and $A \in \text{sets } M$

shows $m1 \ A = \mu \ (A \cap P)$

proof –

have $A \cap P \in \text{sets } M$ **using** *assms unfolding hahn-space-decomp-def*

by (*simp add: pos-meas-setD1 sets.Int*)

have $A \cap N \in \text{sets } M$ **using** *assms unfolding hahn-space-decomp-def*

by (*simp add: neg-meas-setD1 sets.Int*)

have $(A \cap P) \cap (A \cap N) = \{\}$ **using** *assms unfolding hahn-space-decomp-def*

by *auto*

have $A = (A \cap P) \cup (A \cap N)$ **using** *assms unfolding hahn-space-decomp-def*

by (*metis Int-Un-distrib sets.Int-space-eq2*)

hence $m1 \ A = m1 \ ((A \cap P) \cup (A \cap N))$ **by** *simp*

also have $\dots = m1 \ (A \cap P) + m1 \ (A \cap N)$

using *assms pos-e2ennreal-additive[of M m1] ⟨A∩P ∈ sets M⟩ ⟨A∩N ∈ sets M⟩*

$\langle A \cap P \cap (A \cap N) = \{\} \rangle$

unfolding *jordan-decomp-def additive-def* **by** *simp*

also have $\dots = m1 \ (A \cap P)$ **using** *assms unfolding jordan-decomp-def*

by (*metis Int-lower2 ⟨A ∩ N ∈ sets M⟩ add.right-neutral*)

also have $\dots = m1 \ (A \cap P) - m2 \ (A \cap P)$

using *assms unfolding jordan-decomp-def*

by (*metis Int-subset-iff ⟨A ∩ P ∈ sets M⟩ ereal-minus(7)*)

local.pos-wtn-base pos-wtn-subset)

also have $\dots = \mu \ (A \cap P)$ **using** *assms ⟨A ∩ P ∈ sets M⟩*

unfolding *jordan-decomp-def* **by** *simp*

finally show *?thesis* .

qed

lemma *jordan-decomp-neg-meas*:

assumes *jordan-decomp* $m1 \ m2$

and *hahn-space-decomp* $P \ N$

and $A \in \text{sets } M$

shows $m2 \ A = -\mu \ (A \cap N)$

proof –
have $A \cap P \in \text{sets } M$ **using** *assms unfolding hahn-space-decomp-def*
by (*simp add: pos-meas-setD1 sets.Int*)
have $A \cap N \in \text{sets } M$ **using** *assms unfolding hahn-space-decomp-def*
by (*simp add: neg-meas-setD1 sets.Int*)
have $(A \cap P) \cap (A \cap N) = \{\}$
using *assms unfolding hahn-space-decomp-def by auto*
have $A = (A \cap P) \cup (A \cap N)$
using *assms unfolding hahn-space-decomp-def*
by (*metis Int-Un-distrib sets.Int-space-eq2*)
hence $m2 A = m2 ((A \cap P) \cup (A \cap N))$ **by** *simp*
also have $\dots = m2 (A \cap P) + m2 (A \cap N)$
using *pos-e2ennreal-additive[of M m2] assms*
 $\langle A \cap P \in \text{sets } M \rangle \langle A \cap N \in \text{sets } M \rangle \langle A \cap P \cap (A \cap N) = \{\} \rangle$
unfolding *jordan-decomp-def additive-def* **by** *simp*
also have $\dots = m2 (A \cap N)$ **using** *assms unfolding jordan-decomp-def*
by (*metis Int-lower2 \langle A \cap P \in \text{sets } M \rangle add commute add.right-neutral*)
also have $\dots = m2 (A \cap N) - m1 (A \cap N)$
using *assms unfolding jordan-decomp-def*
by (*metis Int-lower2 \langle A \cap N \in \text{sets } M \rangle ereal-minus(7)*)
also have $\dots = -\mu (A \cap N)$ **using** *assms \langle A \cap P \in \text{sets } M \rangle*
unfolding *jordan-decomp-def*
by (*metis Diff-cancel Diff-eq-empty-iff Int-Un-eq(2) \langle A \cap N \in \text{sets } M \rangle*
 $\langle m2 (A \cap N) = m2 (A \cap N) - m1 (A \cap N) \rangle$ *ereal-minus(8)*
ereal-uminus-eq-reorder sup.bounded-iff)
finally show *?thesis* .
qed

lemma *pos-inter-neg-0*:

assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp P N*
and $A \in \text{sets } M$
and $A \subseteq N$
shows $\mu (A \cap M1) = 0$

proof –
have $\mu (A \cap M1) = \mu (A \cap ((M1 \cap P) \cup (M1 \cap (\text{sym-diff } M1 P))))$
by (*metis Diff-subset-conv Int-Un-distrib Un-upper1 inf.orderE*)
also have $\dots = \mu ((A \cap (M1 \cap P)) \cup (A \cap (M1 \cap (\text{sym-diff } M1 P))))$
by (*simp add: Int-Un-distrib*)
also have $\dots = \mu (A \cap (M1 \cap P)) + \mu (A \cap (M1 \cap (\text{sym-diff } M1 P)))$
proof (*rule signed-measure-add*)
show *signed-measure M \mu using sgn-meas* .
show $A \cap (M1 \cap P) \in \text{sets } M$
by (*meson assms(1) assms(2) assms(3) hahn-space-decomp-def sets.Int*
signed-measure-space.pos-meas-setD1 signed-measure-space-axioms)
show $A \cap (M1 \cap \text{sym-diff } M1 P) \in \text{sets } M$
by (*meson Diff-subset assms(1) assms(2) assms(3) hahn-space-decomp-def*
pos-meas-setD1 pos-meas-set-union pos-meas-subset sets.Diff sets.Int)
show $A \cap (M1 \cap P) \cap (A \cap (M1 \cap \text{sym-diff } M1 P)) = \{\}$ **by** *auto*

qed
also have $\dots = \mu (A \cap (M1 \cap (\text{sym-diff } M1 P)))$
proof –
have $A \cap (M1 \cap P) = \{\}$ **using** *assms hahn-space-decomp-def* **by** *auto*
thus *?thesis* **using** *signed-measure-empty[OF sgn-meas]* **by** *simp*
qed
also have $\dots = 0$
proof (*rule hahn-decomp-ess-unique[OF assms(1) assms(2)]*)
show $A \cap (M1 \cap \text{sym-diff } M1 P) \subseteq \text{sym-diff } M1 P \cup \text{sym-diff } M2 N$ **by** *auto*
show $A \cap (M1 \cap \text{sym-diff } M1 P) \in \text{sets } M$
proof –
have $\text{sym-diff } M1 P \in \text{sets } M$ **using** *assms*
by (*meson hahn-space-decomp-def sets.Diff sets.Un signed-measure-space.pos-meas-setD1 signed-measure-space-axioms*)
hence $M1 \cap \text{sym-diff } M1 P \in \text{sets } M$
by (*meson assms(1) hahn-space-decomp-def pos-meas-setD1 sets.Int*)
thus *?thesis* **by** (*simp add: assms sets.Int*)
qed
qed
finally show *?thesis* .
qed

lemma *neg-inter-pos-0*:
assumes *hahn-space-decomp M1 M2*
and *hahn-space-decomp P N*
and $A \in \text{sets } M$
and $A \subseteq P$
shows $\mu (A \cap M2) = 0$
proof –
have $\mu (A \cap M2) = \mu (A \cap ((M2 \cap N) \cup (M2 \cap (\text{sym-diff } M2 N))))$
by (*metis Diff-subset-conv Int-Un-distrib Un-upper1 inf.orderE*)
also have $\dots = \mu ((A \cap (M2 \cap N)) \cup (A \cap (M2 \cap (\text{sym-diff } M2 N))))$
by (*simp add: Int-Un-distrib*)
also have $\dots = \mu (A \cap (M2 \cap N)) + \mu (A \cap (M2 \cap (\text{sym-diff } M2 N)))$
proof (*rule signed-measure-add*)
show *signed-measure M* μ **using** *sgn-meas* .
show $A \cap (M2 \cap N) \in \text{sets } M$
by (*meson assms(1) assms(2) assms(3) hahn-space-decomp-def sets.Int signed-measure-space.neg-meas-setD1 signed-measure-space-axioms*)
show $A \cap (M2 \cap \text{sym-diff } M2 N) \in \text{sets } M$
by (*meson Diff-subset assms(1) assms(2) assms(3) hahn-space-decomp-def neg-meas-setD1 neg-meas-set-union neg-meas-subset sets.Diff sets.Int*)
show $A \cap (M2 \cap N) \cap (A \cap (M2 \cap \text{sym-diff } M2 N)) = \{\}$ **by** *auto*
qed
also have $\dots = \mu (A \cap (M2 \cap (\text{sym-diff } M2 N)))$
proof –
have $A \cap (M2 \cap N) = \{\}$ **using** *assms hahn-space-decomp-def* **by** *auto*
thus *?thesis* **using** *signed-measure-empty[OF sgn-meas]* **by** *simp*
qed

also have ... = 0
proof (rule hahn-decomp-ess-unique[OF assms(1) assms(2)])
show $A \cap (M2 \cap \text{sym-diff } M2 \ N) \subseteq \text{sym-diff } M1 \ P \cup \text{sym-diff } M2 \ N$ **by** auto
show $A \cap (M2 \cap \text{sym-diff } M2 \ N) \in \text{sets } M$
proof –
have $\text{sym-diff } M2 \ N \in \text{sets } M$ **using** assms
by (meson hahn-space-decomp-def sets.Diff sets.Un
signed-measure-space.neg-meas-setD1 signed-measure-space-axioms)
hence $M2 \cap \text{sym-diff } M2 \ N \in \text{sets } M$
by (meson assms(1) hahn-space-decomp-def neg-meas-setD1 sets.Int)
thus ?thesis **by** (simp add: assms sets.Int)
qed
qed
finally show ?thesis .
qed

lemma jordan-decomposition :
shows $\exists m1 \ m2. \text{jordan-decomp } m1 \ m2$
proof –
have $\exists M1 \ M2. \text{hahn-space-decomp } M1 \ M2$ **using** hahn-decomposition
unfolding hahn-space-decomp-def **by** simp
from this **obtain** $M1 \ M2$ **where** hahn-space-decomp $M1 \ M2$ **by** auto
note Mprops = this
define m1 **where** $m1 = (\lambda A. \mu (A \cap M1))$
define m2 **where** $m2 = (\lambda A. -\mu (A \cap M2))$
show ?thesis **unfolding** jordan-decomp-def
proof (intro exI allI impI conjI ballI)
show measure-space (space M) (sets M) ($\lambda x. e2ennreal (m1 \ x)$)
using pos-signed-to-meas-space Mprops m1-def
unfolding hahn-space-decomp-def **by** auto
next
show measure-space (space M) (sets M) ($\lambda x. e2ennreal (m2 \ x)$)
using neg-signed-to-meas-space Mprops m2-def
unfolding hahn-space-decomp-def **by** auto
next
fix A
assume $A \in \text{sets } M$
thus $0 \leq m1 \ A$ **unfolding** m1-def **using** Mprops
unfolding hahn-space-decomp-def
by (meson inf-sup-ord(2) pos-meas-setD1 sets.Int
signed-measure-space.pos-measure-meas signed-measure-space-axioms)
next
fix A
assume $A \in \text{sets } M$
thus $0 \leq m2 \ A$ **unfolding** m2-def **using** Mprops
unfolding hahn-space-decomp-def
by (metis ereal-0-le-uminus-iff inf-sup-ord(2) neg-meas-self
neg-meas-setD1 neg-meas-subset sets.Int)
next

```

fix A
assume A ∈ sets M
have μ A = μ ((A ∩ M1) ∪ (A ∩ M2)) using Mprops
  unfolding hahn-space-decomp-def
  by (metis Int-Un-distrib ⟨A ∈ sets M⟩ sets.Int-space-eq2)
also have ... = μ (A ∩ M1) + μ (A ∩ M2)
proof (rule signed-measure-add)
  show signed-measure M μ using sgn-meas .
  show A ∩ M1 ∈ sets M using Mprops ⟨A ∈ sets M⟩
    unfolding hahn-space-decomp-def
    by (simp add: pos-meas-setD1 sets.Int)
  show A ∩ M2 ∈ sets M using Mprops ⟨A ∈ sets M⟩
    unfolding hahn-space-decomp-def
    by (simp add: neg-meas-setD1 sets.Int)
  show A ∩ M1 ∩ (A ∩ M2) = {} using Mprops
    unfolding hahn-space-decomp-def by auto
qed
also have ... = m1 A - m2 A using m1-def m2-def by simp
finally show μ A = m1 A - m2 A .
next
fix P N A
assume hahn-space-decomp P N and A ∈ sets M and A ⊆ N
note hn = this
have μ (A ∩ M1) = 0
proof (rule pos-inter-neg-0[OF - hn])
  show hahn-space-decomp M1 M2 using Mprops
    unfolding hahn-space-decomp-def by simp
qed
thus m1 A = 0 unfolding m1-def by simp
next
fix P N A
assume hahn-space-decomp P N and A ∈ sets M and A ⊆ P
note hp = this
have μ (A ∩ M2) = 0
proof (rule neg-inter-pos-0[OF - hp])
  show hahn-space-decomp M1 M2 using Mprops
    unfolding hahn-space-decomp-def by simp
qed
thus m2 A = 0 unfolding m2-def by simp
next
show (∀ E ∈ sets M. m1 E < ∞) ∨ (∀ E ∈ sets M. m2 E < ∞)
proof (cases ∀ E ∈ sets M. m1 E < ∞)
  case True
  thus ?thesis by simp
next
  case False
  have ∀ E ∈ sets M. m2 E < ∞
  proof
    fix E

```

```

assume  $E \in \text{sets } M$ 
show  $m2 E < \infty$ 
proof –
  have  $(m2 E) = -\mu (E \cap M2)$  using m2-def by simp
  also have  $\dots \neq \infty$  using False sgn-meas inf-range
    by (metis ereal-less-PIfty ereal-uminus-uminus m1-def rangeI)
  finally have  $m2 E \neq \infty$  .
  thus ?thesis by (simp add: top.not-eq-extremum)
qed
qed
thus ?thesis by simp
qed
qed
qed

```

```

lemma jordan-decomposition-unique :
  assumes jordan-decomp m1 m2
    and jordan-decomp n1 n2
    and  $A \in \text{sets } M$ 
  shows  $m1 A = n1 A$   $m2 A = n2 A$ 
proof –
  have  $\exists M1 M2.$  hahn-space-decomp M1 M2 using hahn-decomposition by simp
  with jordan-decomp-neg-meas jordan-decomp-pos-meas
  show  $m1 A = n1 A$   $m2 A = n2 A$  by (metis assms)+
qed

end

end

```

References

- [1] E. DiBenedetto. *Real Analysis*. Birkhäuser Advanced Texts. Birkhäuser.