Fun With Tilings

Tobias Nipkow and Lawrence Paulson

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Abstract

Tilings are defined inductively. It is shown that one form of mutilated chess board cannot be tiled with dominoes, while another one can be tiled with L-shaped tiles.

Sections 1 and 2 are by Paulson and described elsewhere [1]. Section 3 is by Nipkow and formalizes a well-known argument from the literature [2].

Please add further fun examples of this kind!

theory *Tilings* **imports** *Main* **begin**

1 Inductive Tiling

inductive-set
  tiling :: *'a set set* ⇒ *'a set set*
  for *A* :: *'a set set* **where**
    empty [*simp*, *intro*]: {} ∈ *tiling A* |
    Un [*simp*, *intro*]: ⟦ *a* ∈ *A*; *t* ∈ *tiling A*; *a* ∩ *t* = {} ⟧
      ⟹ *a* ∪ *t* ∈ *tiling A*

lemma *tiling-UnI* [*intro*]:
  ⟦ *t* ∈ *tiling A*; *u* ∈ *tiling A*; *t* ∩ *u* = {} ⟧ ⟹ *t* ∪ *u* ∈ *tiling A*
  ⟨*proof*⟩

lemma *tiling-Diff1E*:
  assumes *t*−*a* ∈ *tiling A* **and** *a* ∈ *A* **and** *a* ⊆ *t*
  shows *t* ∈ *tiling A*
  ⟨*proof*⟩

lemma *tiling-finite*:
  assumes ⋀*a*. *a* ∈ *A* ⟹ *finite a*
  shows *t* ∈ *tiling A* ⟹ *finite t*
  ⟨*proof*⟩

2 The Mutilated Chess Board Cannot be Tiled by Dominoes

The originator of this problem is Max Black, according to J A Robinson. It was popularized as the *Mutilated Checkerboard Problem* by J McCarthy.

inductive-set *domino* :: (*nat* × *nat*) *set set* **where**
horiz [*simp*]: {(*i*, *j*), (*i*, *Suc j*)} ∈ *domino* |
vertl [*simp*]: {(*i*, *j*), (*Suc i*, *j*)} ∈ *domino*

lemma *domino-finite*: *d* ∈ *domino* ⟹ *finite d*
⟨*proof*⟩

declare *tiling-finite*[*OF domino-finite*, *simp*]

Sets of squares of the given colour

definition
coloured :: *nat* ⇒ (*nat* × *nat*) *set* **where**
coloured b = {(*i*, *j*). (*i* + *j*) *mod* 2 = *b*}

abbreviation
whites :: (*nat* × *nat*) *set* **where**
whites ≡ *coloured* 0

abbreviation
blacks :: (*nat* × *nat*) *set* **where**
blacks ≡ *coloured* (*Suc* 0)

Chess boards

lemma *Sigma-Suc1* [*simp*]:
{0..< *Suc n*} × *B* = ({*n*} × *B*) ∪ ({0..<*n*} × *B*)
⟨*proof*⟩

lemma *Sigma-Suc2* [*simp*]:
A × {0..< *Suc n*} = (*A* × {*n*}) ∪ (*A* × {0..<*n*})
⟨*proof*⟩

lemma *dominoes-tile-row* [*intro*!]: {*i*} × {0..< 2∗*n*} ∈ *tiling domino*
⟨*proof*⟩

lemma *dominoes-tile-matrix*: {0..<*m*} × {0..< 2∗*n*} ∈ *tiling domino*
⟨*proof*⟩

coloured and Dominoes

lemma *coloured-insert* [*simp*]:
coloured b ∩ (*insert* (*i*, *j*) *t*) =
(*if* (*i* + *j*) *mod* 2 = *b then insert* (*i*, *j*) (*coloured b* ∩ *t*)
else coloured b ∩ *t*)

⟨*proof*⟩

lemma *domino-singletons*:
  $d \in domino \Longrightarrow$
   $(\exists\, i\ j.\ whites \cap d = \{(i,\ j)\}) \wedge$
   $(\exists\, m\ n.\ blacks \cap d = \{(m,\ n)\})$
  ⟨*proof*⟩

Tilings of dominoes

declare
  Int-Un-distrib [*simp*]
  Diff-Int-distrib [*simp*]

lemma *tiling-domino-0-1*:
  $t \in tiling\ domino \Longrightarrow card(whites \cap t) = card(blacks \cap t)$
⟨*proof*⟩

Final argument is surprisingly complex

theorem *gen-mutil-not-tiling*:
  assumes $t \in tiling\ domino\ (i + j)\ mod\ 2 = 0$
    $(m + n)\ mod\ 2 = 0\ \{(i,\ j),\ (m,\ n)\} \subseteq t$
  shows $t - \{(i,\ j)\} - \{(m,\ n)\} \notin tiling\ domino$
⟨*proof*⟩

Apply the general theorem to the well-known case

theorem *mutil-not-tiling*:
  assumes $t = \{0..< 2 * Suc\ m\} \times \{0..< 2 * Suc\ n\}$
  shows $t - \{(0,0)\} - \{(Suc(2 * m),\ Suc(2 * n))\} \notin tiling\ domino$
⟨*proof*⟩

3 The Mutilated Chess Board Can be Tiled by Ls

Remove a arbitrary square from a chess board of size $2^n \times 2^n$. The result can be tiled by L-shaped tiles: ⊞. The four possible L-shaped tiles are obtained by dropping one of the four squares from $\{(x, y), (x + 1, y), (x, y + 1), (x + 1, y + 1)\}$:

definition *L2* (*x*::*nat*) (*y*::*nat*) = {(*x*,*y*), (*x*+*1*,*y*), (*x*, *y*+*1*)}
definition *L3* (*x*::*nat*) (*y*::*nat*) = {(*x*,*y*), (*x*+*1*,*y*), (*x*+*1*, *y*+*1*)}
definition *L0* (*x*::*nat*) (*y*::*nat*) = {(*x*+*1*,*y*), (*x*,*y*+*1*), (*x*+*1*, *y*+*1*)}
definition *L1* (*x*::*nat*) (*y*::*nat*) = {(*x*,*y*), (*x*,*y*+*1*), (*x*+*1*, *y*+*1*)}

All tiles:

definition *Ls* :: (*nat* ∗ *nat*) *set set* **where**
  Ls ≡ { *L0 x y* | *x y*. *True*} ∪ { *L1 x y* | *x y*. *True*} ∪
     { *L2 x y* | *x y*. *True*} ∪ { *L3 x y* | *x y*. *True*}

lemma *LinLs*: *L0 i j* : *Ls* & *L1 i j* : *Ls* & *L2 i j* : *Ls* & *L3 i j* : *Ls*

⟨*proof*⟩

Square $2^n \times 2^n$ grid, shifted by i and j:

definition *square2* (*n*::*nat*) (*i*::*nat*) (*j*::*nat*) = {*i*..< 2^n+*i*} × {*j*..< 2^n+*j*}

lemma *in-square2*[*simp*]:
(*a*,*b*) : *square2 n i j* ⟷ *i*≤*a* ∧ *a*<2^n+*i* ∧ *j*≤*b* ∧ *b*<2^n+*j*
⟨*proof*⟩

lemma *square2-Suc*: *square2* (*Suc n*) *i j* =
square2 n i j ∪ *square2 n* (2^n + *i*) *j* ∪ *square2 n i* (2^n + *j*) ∪
square2 n (2^n + *i*) (2^n + *j*)
⟨*proof*⟩

lemma *square2-disj*: *square2 n i j* ∩ *square2 n x y* = {} ⟷
(2^n+*i* ≤ *x* ∨ 2^n+*x* ≤ *i*) ∨ (2^n+*j* ≤ *y* ∨ 2^n+*y* ≤ *j*) (**is** *?A* = *?B*)
⟨*proof*⟩

Some specific lemmas:

lemma *pos-pow2*: (*0*::*nat*) < 2^n
⟨*proof*⟩

declare *nat-zero-less-power-iff*[*simp del*] *zero-less-power*[*simp del*]

lemma *Diff-insert-if*: **shows**
B ≠ {} ⟹ *a* ∈ *A* ⟹ *A* − *insert a B* = (*A*−*B* − {*a*}) **and**
B ≠ {} ⟹ *a* ∉ *A* ⟹ *A* − *insert a B* = *A*−*B*
⟨*proof*⟩

lemma *DisjI1*: *A* ∩ *B* = {} ⟹ (*A*−*X*) ∩ *B* = {}
⟨*proof*⟩

lemma *DisjI2*: *A* ∩ *B* = {} ⟹ *A* ∩ (*B*−*X*) = {}
⟨*proof*⟩

The main theorem:

theorem *Ls-can-tile*: *i* ≤ *a* ⟹ *a* < 2^n + *i* ⟹ *j* ≤ *b* ⟹ *b* < 2^n + *j*
⟹ *square2 n i j* − {(*a*,*b*)} ∈ *tiling Ls*
⟨*proof*⟩

corollary *Ls-can-tile00*:
a < 2^n ⟹ *b* < 2^n ⟹ *square2 n 0 0* − {(*a*, *b*)} ∈ *tiling Ls*
⟨*proof*⟩

end

References

[1] Lawrence C. Paulson. A simple formalization and proof for the mutilated chess board. *Logic J. of the IGPL*, 9(3), 2001.

[2] Velleman. *How to Prove it.* Cambridge University Press, 1994.