

Fun With Tilings

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Abstract

Tilings are defined inductively. It is shown that one form of mutilated chess board cannot be tiled with dominoes, while another one can be tiled with L-shaped tiles.

Sections 1 and 2 are by Paulson and described elsewhere [1]. Section 3 is by Nipkow and formalizes a well-known argument from the literature [2].

Please add further fun examples of this kind!

theory *Tilings* **imports** *Main* **begin**

1 Inductive Tiling

inductive-set

```
tiling :: 'a set set  $\Rightarrow$  'a set set
for A :: 'a set set where
empty [simp, intro]: {}  $\in$  tiling A |
Un [simp, intro]:  [| a  $\in$  A; t  $\in$  tiling A; a  $\cap$  t = {} |]
                     $\Longrightarrow$  a  $\cup$  t  $\in$  tiling A
```

lemma *tiling-UnI* [*intro*]:

```
[| t  $\in$  tiling A; u  $\in$  tiling A; t  $\cap$  u = {} |]  $\Longrightarrow$  t  $\cup$  u  $\in$  tiling A
<proof>
```

lemma *tiling-Diff1E*:

```
assumes t-a  $\in$  tiling A and a  $\in$  A and a  $\subseteq$  t
shows t  $\in$  tiling A
<proof>
```

lemma *tiling-finite*:

```
assumes  $\bigwedge a. a \in A \Longrightarrow$  finite a
shows t  $\in$  tiling A  $\Longrightarrow$  finite t
<proof>
```

2 The Mutilated Chess Board Cannot be Tiled by Dominoes

The originator of this problem is Max Black, according to J A Robinson. It was popularized as the *Mutilated Checkerboard Problem* by J McCarthy.

inductive-set *domino* :: (nat × nat) set set **where**
horiz [simp]: {(i, j), (i, Suc j)} ∈ *domino* |
vertl [simp]: {(i, j), (Suc i, j)} ∈ *domino*

lemma *domino-finite*: $d \in \text{domino} \implies \text{finite } d$
 ⟨*proof*⟩

declare *tiling-finite*[OF *domino-finite*, *simp*]

Sets of squares of the given colour

definition
coloured :: nat ⇒ (nat × nat) set **where**
coloured b = {(i, j). (i + j) mod 2 = b}

abbreviation
whites :: (nat × nat) set **where**
whites ≡ *coloured* 0

abbreviation
blacks :: (nat × nat) set **where**
blacks ≡ *coloured* (Suc 0)

Chess boards

lemma *Sigma-Suc1* [simp]:
 $\{0..< \text{Suc } n\} \times B = (\{n\} \times B) \cup (\{0..<n\} \times B)$
 ⟨*proof*⟩

lemma *Sigma-Suc2* [simp]:
 $A \times \{0..< \text{Suc } n\} = (A \times \{n\}) \cup (A \times \{0..<n\})$
 ⟨*proof*⟩

lemma *dominoes-tile-row* [intro!]: $\{i\} \times \{0..< 2*n\} \in \text{tiling } \text{domino}$
 ⟨*proof*⟩

lemma *dominoes-tile-matrix*: $\{0..<m\} \times \{0..< 2*n\} \in \text{tiling } \text{domino}$
 ⟨*proof*⟩

coloured and Dominoes

lemma *coloured-insert* [simp]:
coloured b ∩ (*insert* (i, j) t) =
 (if (i + j) mod 2 = b then *insert* (i, j) (*coloured* b ∩ t)
 else *coloured* b ∩ t)

<proof>

lemma *domino-singletons*:

$d \in \text{domino} \implies$
 $(\exists i j. \text{whites} \cap d = \{(i, j)\}) \wedge$
 $(\exists m n. \text{blacks} \cap d = \{(m, n)\})$

<proof>

Tilings of dominoes

declare

Int-Un-distrib [*simp*]
Diff-Int-distrib [*simp*]

lemma *tiling-domino-0-1*:

$t \in \text{tiling domino} \implies \text{card}(\text{whites} \cap t) = \text{card}(\text{blacks} \cap t)$

<proof>

Final argument is surprisingly complex

theorem *gen-mutil-not-tiling*:

$t \in \text{tiling domino} \implies$
 $(i + j) \bmod 2 = 0 \implies (m + n) \bmod 2 = 0 \implies$
 $\{(i, j), (m, n)\} \subseteq t$
 $\implies (t - \{(i, j)\} - \{(m, n)\}) \notin \text{tiling domino}$

<proof>


Apply the general theorem to the well-known case

theorem *mutil-not-tiling*:

$t = \{0..< 2 * \text{Suc } m\} \times \{0..< 2 * \text{Suc } n\}$
 $\implies t - \{(0, 0)\} - \{(\text{Suc } (2 * m), \text{Suc } (2 * n))\} \notin \text{tiling domino}$

<proof>

3 The Mutilated Chess Board Can be Tiled by Ls

Remove a arbitrary square from a chess board of size $2^n \times 2^n$. The result can be tiled by L-shaped tiles: . The four possible L-shaped tiles are obtained by dropping one of the four squares from $\{(x, y), (x + 1, y), (x, y + 1), (x + 1, y + 1)\}$:

definition *L2* $(x::\text{nat}) (y::\text{nat}) = \{(x, y), (x + 1, y), (x, y + 1)\}$

definition *L3* $(x::\text{nat}) (y::\text{nat}) = \{(x, y), (x + 1, y), (x + 1, y + 1)\}$

definition *L0* $(x::\text{nat}) (y::\text{nat}) = \{(x + 1, y), (x, y + 1), (x + 1, y + 1)\}$

definition *L1* $(x::\text{nat}) (y::\text{nat}) = \{(x, y), (x, y + 1), (x + 1, y + 1)\}$

All tiles:

definition *Ls* :: $(\text{nat} * \text{nat}) \text{ set set where}$

$Ls \equiv \{ L0 \ x \ y \mid x \ y. \ \text{True} \} \cup \{ L1 \ x \ y \mid x \ y. \ \text{True} \} \cup$
 $\{ L2 \ x \ y \mid x \ y. \ \text{True} \} \cup \{ L3 \ x \ y \mid x \ y. \ \text{True} \}$

lemma *LinLs*: $L0\ i\ j : Ls \ \&\ L1\ i\ j : Ls \ \&\ L2\ i\ j : Ls \ \&\ L3\ i\ j : Ls$
 ⟨proof⟩

Square $2^n \times 2^n$ grid, shifted by i and j :

definition *square2* $(n::nat)\ (i::nat)\ (j::nat) = \{i..< 2^{\widehat{n}+i}\} \times \{j..< 2^{\widehat{n}+j}\}$

lemma *in-square2*[*simp*]:

$(a,b) : \text{square2}\ n\ i\ j \iff i \leq a \wedge a < 2^{\widehat{n}+i} \wedge j \leq b \wedge b < 2^{\widehat{n}+j}$
 ⟨proof⟩

lemma *square2-Suc*: $\text{square2}\ (Suc\ n)\ i\ j =$

$\text{square2}\ n\ i\ j \cup \text{square2}\ n\ (2^{\widehat{n}} + i)\ j \cup \text{square2}\ n\ i\ (2^{\widehat{n}} + j) \cup$
 $\text{square2}\ n\ (2^{\widehat{n}} + i)\ (2^{\widehat{n}} + j)$
 ⟨proof⟩

lemma *square2-disj*: $\text{square2}\ n\ i\ j \cap \text{square2}\ n\ x\ y = \{\} \iff$

$(2^{\widehat{n}+i} \leq x \vee 2^{\widehat{n}+x} \leq i) \vee (2^{\widehat{n}+j} \leq y \vee 2^{\widehat{n}+y} \leq j)$ (is ?A = ?B)
 ⟨proof⟩

Some specific lemmas:

lemma *pos-pow2*: $(0::nat) < 2^{\widehat{n}(n::nat)}$
 ⟨proof⟩

declare *nat-zero-less-power-iff*[*simp del*] *zero-less-power*[*simp del*]

lemma *Diff-insert-if*: **shows**

$B \neq \{\} \implies a:A \implies A - \text{insert}\ a\ B = (A - B - \{a\})$ **and**
 $B \neq \{\} \implies a \sim : A \implies A - \text{insert}\ a\ B = A - B$
 ⟨proof⟩

lemma *DisjI1*: $A\ \text{Int}\ B = \{\} \implies (A - X)\ \text{Int}\ B = \{\}$
 ⟨proof⟩

lemma *DisjI2*: $A\ \text{Int}\ B = \{\} \implies A\ \text{Int}\ (B - X) = \{\}$
 ⟨proof⟩

The main theorem:

theorem *Ls-can-tile*: $i \leq a \implies a < 2^{\widehat{n}} + i \implies j \leq b \implies b < 2^{\widehat{n}} + j$
 $\implies \text{square2}\ n\ i\ j - \{(a,b)\} : \text{tiling}\ Ls$
 ⟨proof⟩

corollary *Ls-can-tile00*:

$a < 2^{\widehat{n}} \implies b < 2^{\widehat{n}} \implies \text{square2}\ n\ 0\ 0 - \{(a,b)\} \in \text{tiling}\ Ls$
 ⟨proof⟩

end

References

- [1] Lawrence C. Paulson. A simple formalization and proof for the mutilated chess board. *Logic J. of the IGPL*, 9(3), 2001.
- [2] Velleman. *How to Prove it*. Cambridge University Press, 1994.