

Fun With Functions

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Abstract

This is a collection of cute puzzles of the form “Show that if a function satisfies the following constraints, it must be ...” Please add further examples to this collection!

Apart from the one about factorial, they all come from the delightful booklet by Terence Tao [1] but go back to Math Olympiads and similar events.

Please add further examples of this kind, either directly or by sending them to me. Let us make this a growing body of *fun!*

theory *FunWithFunctions* **imports** *Complex-Main* **begin**

See [1]. Was first brought to our attention by Herbert Ehler who provided a similar proof.

theorem *identity1*: **fixes** $f :: nat \Rightarrow nat$
assumes $fff: \bigwedge n. f(f(n)) < f(Suc(n))$
shows $f(n) = n$
<proof>

See [1]. Possible extension: Should also hold if the range of f is the reals!

lemma *identity2*: **fixes** $f :: nat \Rightarrow nat$
assumes $f(k) = k$ **and** $k \geq 2$
and f -times: $\bigwedge m n. f(m*n) = f(m)*f(n)$
and f -mono: $\bigwedge m n. m < n \implies f m < f n$
shows $f(n) = n$
<proof>

One more from Tao’s booklet. If f is also assumed to be continuous, $f x = x + 1$ holds for all reals, not only rationals. Extend the proof!

theorem *plus1*:
fixes $f :: real \Rightarrow real$
assumes $0: f 0 = 1$ **and** f -add: $\bigwedge x y. f(x+y+1) = f x + f y$
assumes $r : \mathbb{Q}$ **shows** $f(r) = r + 1$
<proof>

The only total model of a naive recursion equation of factorial on integers is 0 for all negative arguments. Probably folklore.

theorem *ifac-neg0*: **fixes** *ifac* :: *int* \Rightarrow *int*
assumes *ifac-rec*: $\bigwedge i. \text{ifac } i = (\text{if } i=0 \text{ then } 1 \text{ else } i * \text{ifac}(i - 1))$
shows $i < 0 \implies \text{ifac } i = 0$
(*proof*)

end

References

- [1] Terence Tao. *Solving Mathematical Problems*. Oxford University Press, 2006.