

Foundation of geometry in planes, and some complements : Excluding the parallel axioms

Fumiya Iwama

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Abstract

“Foundations of Geometry” is a mathematical book written by Hilbert in 1899. This entry is a complete formalization of “Incidence” (excluding cubic axioms), “Order” and “Congruence” (excluding point sequences) of the axioms constructed in this book. In addition, the theorem of the problem about the part that is treated implicitly and is not clearly stated in it is being carried out in parallel.

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1 Incidence

D.Hilbert made a rigorous reconstruction of Euclidean geometry in Chapter 1 of [1]. There, five types of axioms are listed and 32 theorems are proved. In Hilbert’s axiom system, basic concepts such as points and lines are treated as undefined terms, and only their relationships are defined by axioms. In addition, the continuity axiom stipulates that the Euclidean plane is essentially equivalent to the real plane \mathbb{R}^2 , ensuring that the axiom system is categorical.

Implement each axiom and definition and prove the theorem (Coupling axioms related to space geometry axiom 4 to 8 are excluded).

datatype *Point* = *char*

datatype *Segment* = *Se Point Point*

datatype *Line* = *Li Point Point*

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datatype Angle = An Point Point Point
datatype Triangle = Tr Point Point Point
datatype Geo-object =
  Poi Point
  | Seg Segment
  | Lin Line
  | Ang Angle
  | Tri Triangle
datatype sign = add | sub
datatype Geo-objects = Emp | Geos Geo-object sign Geo-objects

locale Eq-relation =
  fixes Eq :: Geo-objects  $\Rightarrow$  Geo-objects  $\Rightarrow$  bool
  and Inv :: bool  $\Rightarrow$  bool
  assumes Eq-refl [simp,intro] : Eq obs obs
  and Eq-rev :  $\llbracket$ Eq obs1 obs2 $\rrbracket \Longrightarrow$  Eq obs2 obs1
  and Eq-trans :  $\llbracket$ Eq obs1 obs3; Eq obs2 obs3 $\rrbracket \Longrightarrow$  Eq obs1 obs2
  and Inv-def : Inv b1  $\longleftrightarrow$   $\neg$  b1

locale Definition-1 = Eq-relation +
  fixes Line-on :: Line  $\Rightarrow$  Point  $\Rightarrow$  bool

locale Axiom-1 = Definition-1 +
  assumes Line-exist :  $\llbracket$  $\neg$  Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp) $\rrbracket$ 
     $\Longrightarrow$   $\exists$  l. Line-on l p1  $\wedge$  Line-on l p2
  and Line-unique :  $\llbracket$ Line-on l1 p1; Line-on l1 p2; Line-on l2 p1; Line-on l2 p2;
     $\neg$  Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp) $\rrbracket \Longrightarrow$  Eq (Geos
(Lin l1) add Emp) (Geos (Lin l2) add Emp)
  and Line-on-exist :  $\exists$  p q. Line-on l1 p  $\wedge$  Line-on l1 q
     $\wedge$   $\neg$  Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp)
  and Line-not-on-exist :  $\exists$  p q r.  $\neg$  Line-on l1 p  $\wedge$   $\neg$  Line-on l1 q  $\wedge$   $\neg$  Line-on
l1 r
     $\wedge$   $\neg$  Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp)
     $\wedge$   $\neg$  Eq (Geos (Poi q) add Emp) (Geos (Poi r) add Emp)
     $\wedge$   $\neg$  Eq (Geos (Poi r) add Emp) (Geos (Poi p) add Emp)

locale Incidence-Rule = Axiom-1 +
  assumes Point-Eq :  $\llbracket$ P1(p1); Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add
Emp) $\rrbracket \Longrightarrow$  P1(p2)
  and Line-on-trans :  $\llbracket$ Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp);
Line-on l1 p1 $\rrbracket$ 
     $\Longrightarrow$  Line-on l2 p1
  and Line-on-rule : Line-on (Li p1 p2) p1  $\wedge$  Line-on (Li p1 p2) p2

lemma(in Incidence-Rule) Eq-not-trans :
  assumes N :
     $\neg$  Eq obs1 obs2
    Eq obs2 obs3

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shows $\neg Eq\ obs1\ obs3$
(proof)

lemma(in *Incidence-Rule*) *Line-rev* :
assumes $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p2)\ add\ Emp)$
shows $Eq\ (Geos\ (Lin\ (Li\ p1\ p2))\ add\ Emp)\ (Geos\ (Lin\ (Li\ p2\ p1))\ add\ Emp)$
(proof)

lemma(in *Incidence-Rule*) *Line-not-on-Point* :
assumes N :
 $\neg Line-on\ (Li\ p1\ p2)\ p3$
shows $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p3)\ add\ Emp)$
(proof)

lemma(in *Incidence-Rule*) *Line-not-on-trans* :
assumes
 $Eq\ (Geos\ (Lin\ l1)\ add\ Emp)\ (Geos\ (Lin\ l2)\ add\ Emp)$
 $\neg Line-on\ l1\ p1$
shows $\neg Line-on\ l2\ p1$
(proof)

lemma(in *Incidence-Rule*) *Line-on-rev* :
assumes
 $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p2)\ add\ Emp)$
 $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p3)\ add\ Emp)$
 $Line-on\ (Li\ p1\ p2)\ p3$
shows $Line-on\ (Li\ p1\ p3)\ p2$
(proof)

lemma(in *Incidence-Rule*) *Line-not-Eq* :
assumes
 $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p2)\ add\ Emp)$
 $\neg Line-on\ (Li\ p1\ p2)\ p3$
shows $\neg Eq\ (Geos\ (Lin\ (Li\ p1\ p2))\ add\ Emp)\ (Geos\ (Lin\ (Li\ p1\ p3))\ add\ Emp)$
(proof)

lemma(in *Incidence-Rule*) *Line-not-Eq-on* :
assumes N :
 $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p2)\ add\ Emp)$
 $\neg Eq\ (Geos\ (Poi\ p1)\ add\ Emp)\ (Geos\ (Poi\ p3)\ add\ Emp)$
 $\neg Eq\ (Geos\ (Lin\ (Li\ p1\ p2))\ add\ Emp)\ (Geos\ (Lin\ (Li\ p1\ p3))\ add\ Emp)$
shows $\neg Line-on\ (Li\ p1\ p2)\ p3$
(proof)

lemma(in *Incidence-Rule*) *Line-unique-Point* :
assumes
 $\neg Eq\ (Geos\ (Lin\ l1)\ add\ Emp)\ (Geos\ (Lin\ l2)\ add\ Emp)$
 $Line-on\ l1\ p1\ Line-on\ l1\ p2$
 $Line-on\ l2\ p1\ Line-on\ l2\ p2$

shows $Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)$
 $\langle proof \rangle$

lemma(in *Incidence-Rule*) *Line-not-on-Eq* :
assumes N :
 $\neg Line-on\ l1\ p1$
 $Line-on\ l2\ p1$
shows $\neg Eq (Geos (Lin\ l1) add Emp) (Geos (Lin\ l2) add Emp)$
 $\langle proof \rangle$

lemma(in *Incidence-Rule*) *Line-cross-not-on* :
assumes
 $\neg Eq (Geos (Poi\ p1) add Emp) (Geos (Poi\ p2) add Emp)$
 $\neg Eq (Geos (Poi\ p2) add Emp) (Geos (Poi\ p4) add Emp)$
 $\neg Line-on (Li\ p1\ p2)\ p3$
 $Line-on (Li\ p2\ p3)\ p4$
shows $\neg Line-on (Li\ p1\ p2)\ p4$
 $\langle proof \rangle$

end

2 Order

locale *Definition-2 = Incidence-Rule +*
fixes $Line-on-Seg :: Line \Rightarrow Segment \Rightarrow bool$
and $Bet-Point :: Segment \Rightarrow Point \Rightarrow bool$
and $Seg-on-Seg :: Segment \Rightarrow Segment \Rightarrow bool$
and $Line-on-Line :: Line \Rightarrow Line \Rightarrow bool$
and $Plane-sameside :: Line \Rightarrow Point \Rightarrow Point \Rightarrow bool$
and $Plane-diffside :: Line \Rightarrow Point \Rightarrow Point \Rightarrow bool$
assumes $Bet-Point-def : \llbracket Bet-Point (Se\ p1\ p2)\ p3 \rrbracket \Longrightarrow \neg Eq (Geos (Poi\ p1) add Emp) (Geos (Poi\ p2) add Emp)$
 $\wedge \neg Eq (Geos (Poi\ p2) add Emp) (Geos (Poi\ p3) add Emp) \wedge \neg Eq (Geos (Poi\ p3) add Emp) (Geos (Poi\ p1) add Emp)$
and $Bet-rev : \llbracket Bet-Point (Se\ p1\ p2)\ p3 \rrbracket \Longrightarrow Bet-Point (Se\ p2\ p1)\ p3$
and $Line-Bet-exist : \llbracket Bet-Point (Se\ p1\ p2)\ p3 \rrbracket \Longrightarrow \exists l. Line-on\ l\ p1 \wedge Line-on\ l\ p2 \wedge Line-on\ l\ p3$
and $Seg-rev : Eq (Geos (Seg (Se\ p1\ p2)) add Emp) (Geos (Seg (Se\ p2\ p1)) add Emp)$
and $Plane-sameside-def : Plane-sameside\ l1\ p1\ p2 \longleftrightarrow \neg Line-on-Seg\ l1\ (Se\ p1\ p2) \wedge \neg Line-on\ l1\ p1 \wedge \neg Line-on\ l1\ p2 \wedge \neg Eq (Geos (Poi\ p1) add Emp) (Geos (Poi\ p2) add Emp)$
and $Plane-diffside-def : Plane-diffside\ l1\ p1\ p2 \longleftrightarrow (\exists p. Bet-Point (Se\ p1\ p2)\ p \wedge Line-on\ l1\ p \wedge \neg Line-on\ l1\ p1 \wedge \neg Line-on\ l1\ p2)$

locale *Axiom-2 = Definition-2 +*
assumes $Bet-extension : \llbracket Line-on\ l1\ p1; Line-on\ l1\ p2; \neg Eq (Geos (Poi\ p1) add Emp) (Geos (Poi\ p2) add Emp) \rrbracket \Longrightarrow \exists p. Bet-Point (Se\ p1\ p)\ p2 \wedge Line-on\ l1\ p$
and $Bet-iff : \llbracket Bet-Point (Se\ p1\ p2)\ p3 \rrbracket \Longrightarrow Inv (Bet-Point (Se\ p2\ p3)\ p1) \wedge$

Inv (*Bet-Point* (*Se p3 p1*) *p2*)
and *Pachets-axiom* : $\llbracket \neg \text{Line-on } (Li\ p1\ p2)\ p3; \text{Bet-Point } (Se\ p1\ p2)\ p4; \text{Line-on } l1\ p4;$
 $\neg \text{Line-on } l1\ p1; \neg \text{Line-on } l1\ p2; \neg \text{Line-on } l1\ p3 \rrbracket \implies$
 $\text{Line-on-Seg } l1\ (Se\ p1\ p3) \wedge \neg \text{Line-on-Seg } l1\ (Se\ p2\ p3)$
 $\vee \text{Line-on-Seg } l1\ (Se\ p2\ p3) \wedge \neg \text{Line-on-Seg } l1\ (Se\ p1\ p3)$
and *Seg-move-sameside* : $\llbracket \text{Line-on } l1\ p1; \text{Line-on } l1\ p2; \neg \text{Eq } (Geos\ (Poi\ p1)$
add Emp) (*Geos* (*Poi p2*) *add Emp*);
 $\neg \text{Eq } (Geos\ (Poi\ p3)\ \text{add Emp})\ (Geos\ (Poi\ p4)\ \text{add Emp}) \rrbracket \implies$
 $\exists p. \text{Eq } (Geos\ (Seg\ (Se\ p3\ p4))\ \text{add Emp})\ (Geos\ (Seg\ (Se\ p1\ p))\ \text{add Emp}) \wedge$
 $\neg \text{Bet-Point } (Se\ p\ p2)\ p1 \wedge \text{Line-on } l1\ p \wedge \neg \text{Eq } (Geos\ (Poi\ p1)\ \text{add Emp})\ (Geos$
(*Poi p*) *add Emp*)
and *Seg-move-diffside* : $\llbracket \text{Line-on } l1\ p1; \text{Line-on } l1\ p2; \neg \text{Eq } (Geos\ (Poi\ p1)\ \text{add}$
Emp) (*Geos* (*Poi p2*) *add Emp*);
 $\neg \text{Eq } (Geos\ (Poi\ p3)\ \text{add Emp})\ (Geos\ (Poi\ p4)\ \text{add Emp}) \rrbracket \implies$
 $\exists p. \text{Eq } (Geos\ (Seg\ (Se\ p3\ p4))\ \text{add Emp})\ (Geos\ (Seg\ (Se\ p1\ p))\ \text{add Emp})$
 $\wedge \text{Bet-Point } (Se\ p\ p2)\ p1 \wedge \text{Line-on } l1\ p \wedge \neg \text{Eq } (Geos\ (Poi\ p1)\ \text{add Emp})\ (Geos$
(*Poi p*) *add Emp*)

locale *Order-Rule* = *Axiom-2* +
assumes *Bet-Point-Eq* : $\llbracket \text{Bet-Point } (Se\ p1\ p2)\ p3; \text{Eq } (Geos\ (Poi\ p1)\ \text{add Emp})$
(*Geos* (*Poi p4*) *add Emp*) $\rrbracket \implies \text{Bet-Point } (Se\ p4\ p2)\ p3$
and *Line-on-Seg-rule* : $\text{Line-on-Seg } l1\ (Se\ p1\ p2) \longleftrightarrow (\exists p. \text{Line-on } l1\ p \wedge$
Bet-Point (*Se p1 p2*) *p*)
and *Seg-on-Seg-rule* : $\text{Seg-on-Seg } (Se\ p1\ p2)\ (Se\ p3\ p4) \longleftrightarrow (\exists p. \text{Bet-Point}$
(*Se p1 p2*) *p* $\wedge \text{Bet-Point } (Se\ p3\ p4)\ p)$
and *Line-on-Line-rule* : $\text{Line-on-Line } l1\ l2 \longleftrightarrow (\exists p. \text{Line-on } l1\ p \wedge \text{Line-on}$
l2 p)
and *Seg-Point-Eq* : $\llbracket \text{Eq } (Geos\ (Poi\ p1)\ \text{add Emp})\ (Geos\ (Poi\ p2)\ \text{add Emp}) \rrbracket$
 $\implies \text{Eq } (Geos\ (Seg\ (Se\ p3\ p1))\ \text{add Emp})\ (Geos\ (Seg\ (Se\ p3\ p2))\ \text{add Emp})$

lemma(*in Order-Rule*) *Line-Bet-on* :
assumes
 $\text{Bet-Point } (Se\ p1\ p2)\ p3$
shows $\text{Line-on } (Li\ p1\ p2)\ p3$ **and** $\text{Line-on } (Li\ p2\ p1)\ p3$
and $\text{Line-on } (Li\ p2\ p3)\ p1$ **and** $\text{Line-on } (Li\ p3\ p2)\ p1$
and $\text{Line-on } (Li\ p1\ p3)\ p2$ **and** $\text{Line-on } (Li\ p3\ p1)\ p2$
<proof>

lemma(*in Order-Rule*) *Line-Bet-not-Eq* :
assumes *N* :
 $\text{Bet-Point } (Se\ p1\ p2)\ p3$
 $\neg \text{Line-on } (Li\ p1\ p2)\ p4$
shows $\neg \text{Eq } (Geos\ (Lin\ (Li\ p4\ p3))\ \text{add Emp})\ (Geos\ (Lin\ (Li\ p4\ p2))\ \text{add Emp})$
<proof>

Theorem3

theorem(*in Order-Rule*) *Seg-density* :
assumes $\neg \text{Eq } (Geos\ (Poi\ A)\ \text{add Emp})\ (Geos\ (Poi\ C)\ \text{add Emp})$

shows $\exists p. \text{Bet-Point } (Se\ A\ C)\ p$
(proof)

lemma(in *Order-Rule*) *Line-Bet-not-on* :

assumes

Line-on (*Li* *p1* *p2*) *p3*

\neg *Line-on* (*Li* *p1* *p2*) *p4*

Bet-Point (*Se* *p3* *p4*) *p5*

shows *Inv* (*Line-on* (*Li* *p1* *p2*) *p5*)

(proof)

lemma(in *Order-Rule*) *Line-not-on-ex* :

assumes *N* :

\neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*) (*Geos* (*Poi* *p2*) *add Emp*)

\neg *Line-on* (*Li* *p1* *p2*) *p3*

Line-on (*Li* *p1* *p4*) *p2*

shows \neg *Line-on* (*Li* *p1* *p4*) *p3*

(proof)

lemma(in *Order-Rule*) *Line-on-dens* :

assumes

\neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*) (*Geos* (*Poi* *p3*) *add Emp*)

\neg *Eq* (*Geos* (*Poi* *p2*) *add Emp*) (*Geos* (*Poi* *p4*) *add Emp*)

Line-on (*Li* *p1* *p2*) *p3*

Line-on (*Li* *p1* *p4*) *p3*

shows *Line-on* (*Li* *p2* *p4*) *p3*

(proof)

lemma(in *Order-Rule*) *Bet-case-lemma1* :

assumes

Line-on *l1* *A*

Line-on *l1* *B*

Line-on *l1* *C*

\neg *Bet-Point* (*Se* *B* *A*) *C*

\neg *Bet-Point* (*Se* *C* *B*) *A*

\neg *Eq* (*Geos* (*Poi* *A*) *add Emp*) (*Geos* (*Poi* *B*) *add Emp*)

\neg *Eq* (*Geos* (*Poi* *B*) *add Emp*) (*Geos* (*Poi* *C*) *add Emp*)

\neg *Eq* (*Geos* (*Poi* *C*) *add Emp*) (*Geos* (*Poi* *A*) *add Emp*)

\neg *Line-on* (*Li* *A* *C*) *D*

Bet-Point (*Se* *B* *G*) *D*

shows $\exists p. \text{Line-on } (Li\ A\ D)\ p \wedge \text{Bet-Point } (Se\ G\ C)\ p$

(proof)

lemma(in *Order-Rule*) *Bet-case-lemma2* :

assumes

Line-on *l1* *A*

Line-on *l1* *B*

Line-on *l1* *C*

\neg *Bet-Point* (*Se* *B* *A*) *C*

$\neg \text{Bet-Point } (Se\ C\ B)\ A$
 $\neg \text{Eq } (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ B)\ add\ Emp)$
 $\neg \text{Eq } (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ C)\ add\ Emp)$
 $\neg \text{Eq } (Geos\ (Poi\ C)\ add\ Emp)\ (Geos\ (Poi\ A)\ add\ Emp)$
shows $\text{Bet-Point } (Se\ A\ C)\ B$
 <proof>

Theorem4

lemma(in *Order-Rule*) *Bet-case* :

assumes
 $\text{Line-on } l1\ A$
 $\text{Line-on } l1\ B$
 $\text{Line-on } l1\ C$
 $\neg \text{Eq } (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ B)\ add\ Emp)$
 $\neg \text{Eq } (Geos\ (Poi\ B)\ add\ Emp)\ (Geos\ (Poi\ C)\ add\ Emp)$
 $\neg \text{Eq } (Geos\ (Poi\ C)\ add\ Emp)\ (Geos\ (Poi\ A)\ add\ Emp)$
shows $\text{Bet-Point } (Se\ A\ C)\ B \vee \text{Bet-Point } (Se\ C\ B)\ A \vee \text{Bet-Point } (Se\ B\ A)\ C$
 <proof>

lemma(in *Order-Rule*) *Bet-case-fact* :

assumes
 $\text{Bet-Point } (Se\ A\ C)\ B \vee \text{Bet-Point } (Se\ C\ B)\ A \vee \text{Bet-Point } (Se\ B\ A)\ C$
shows
 $\text{Bet-Point } (Se\ A\ C)\ B \wedge \neg \text{Bet-Point } (Se\ C\ B)\ A \wedge \neg \text{Bet-Point } (Se\ B\ A)\ C$
 $\vee \neg \text{Bet-Point } (Se\ A\ C)\ B \wedge \text{Bet-Point } (Se\ C\ B)\ A \wedge \neg \text{Bet-Point } (Se\ B\ A)\ C$
 $\vee \neg \text{Bet-Point } (Se\ A\ C)\ B \wedge \neg \text{Bet-Point } (Se\ C\ B)\ A \wedge \text{Bet-Point } (Se\ B\ A)\ C$
 <proof>

lemma(in *Order-Rule*) *Bet-swap-lemma-1* :

assumes
 $\neg \text{Eq } (Geos\ (Poi\ A)\ add\ Emp)\ (Geos\ (Poi\ D)\ add\ Emp)$
 $\text{Bet-Point } (Se\ A\ C)\ B$
 $\text{Bet-Point } (Se\ B\ D)\ C$
shows $\text{Line-on } (Li\ A\ D)\ B \wedge \text{Line-on } (Li\ A\ D)\ C$
 <proof>

lemma(in *Order-Rule*) *Bet-swap-lemma-2* :

assumes
 $\text{Bet-Point } (Se\ p1\ p3)\ p2$
 $\neg \text{Line-on } (Li\ p1\ p3)\ p4$
 $\neg \text{Line-on } (Li\ p2\ p5)\ p3$
 $\neg \text{Line-on } (Li\ p2\ p5)\ p1$
 $\neg \text{Line-on } (Li\ p2\ p5)\ p4$
 $\text{Bet-Point } (Se\ p3\ p5)\ p4$
shows $\exists p. \text{Line-on } (Li\ p2\ p5)\ p \wedge \text{Bet-Point } (Se\ p1\ p4)\ p$
 <proof>

lemma(in *Order-Rule*) *Bet-swap-lemma-3* :

assumes

Bet-Point (Se p1 p3) p2
Bet-Point (Se p3 p5) p4
 \neg *Line-on* (Li p1 p3) p5
shows $\exists p.$ *Bet-Point* (Se p1 p4) p \wedge *Bet-Point* (Se p5 p2) p
 <proof>

lemma(in *Order-Rule*) *Bet-swap-lemma-4* :
assumes
 \neg *Eq* (*Geos* (*Poi* A) *add Emp*) (*Geos* (*Poi* D) *add Emp*)
Bet-Point (Se A E) G
Bet-Point (Se D G) H
 \neg *Line-on* (Li A D) E
shows $\exists p.$ *Line-on* (Li H E) p \wedge *Bet-Point* (Se D A) p
 <proof>

lemma(in *Order-Rule*) *Bet-swap-lemma-5* :
assumes
Bet-Point (Se A C) B
Bet-Point (Se B D) C
Bet-Point (Se C F) E
 \neg *Line-on* (Li A D) F
 \neg *Line-on* (Li A C) F
shows *Bet-Point* (Se A D) C
 <proof>

theorem(in *Order-Rule*) *Bet-swap-234-134* :
assumes
Bet-Point (Se A C) B
Bet-Point (Se B D) C
shows *Bet-Point* (Se A D) C
 <proof>

theorem(in *Order-Rule*) *Bet-swap-234-124* :
assumes
Bet-Point (Se A C) B
Bet-Point (Se B D) C
shows *Bet-Point* (Se A D) B
 <proof>

theorem(in *Order-Rule*) *Bet-swap-134-234* :
assumes
Bet-Point (Se A C) B
Bet-Point (Se A D) C
shows *Bet-Point* (Se B D) C
 <proof>

lemma(in *Order-Rule*) *Bet-swap-134-124* :
assumes
Bet-Point (Se A C) B

$Bet\text{-}Point (Se A D) C$
shows $Bet\text{-}Point (Se A D) B$
 <proof>

theorem(in *Order-Rule*) *Bet-swap-243-124* :
assumes
 $Bet\text{-}Point (Se A D) B$
 $Bet\text{-}Point (Se B D) C$
shows $Bet\text{-}Point (Se A C) B$
 <proof>

theorem(in *Order-Rule*) *Bet-swap-243-143* :
assumes
 $Bet\text{-}Point (Se A D) B$
 $Bet\text{-}Point (Se B D) C$
shows $Bet\text{-}Point (Se A D) C$
 <proof>

Theorem5

lemma(in *Order-Rule*) *Bet-four-Point-case* :
assumes
 $Line\text{-}on\ l1\ P$
 $Line\text{-}on\ l1\ Q$
 $Line\text{-}on\ l1\ R$
 $Line\text{-}on\ l1\ S$
 $Bet\text{-}Point (Se P R) Q$
 $\neg Eq (Geos (Poi P) add Emp) (Geos (Poi S) add Emp)$
 $\neg Eq (Geos (Poi Q) add Emp) (Geos (Poi S) add Emp)$
 $\neg Eq (Geos (Poi R) add Emp) (Geos (Poi S) add Emp)$
shows $Bet\text{-}Point (Se P S) R \vee Bet\text{-}Point (Se R S) P$
 $\vee Bet\text{-}Point (Se P R) S \wedge Bet\text{-}Point (Se P S) Q$
 $\vee Bet\text{-}Point (Se P Q) S \vee Bet\text{-}Point (Se Q S) P$
 <proof>

lemma(in *Order-Rule*) *Plane-diffside-rev* :
assumes
 $Plane\text{-}diffside\ l1\ p1\ p2$
shows $Plane\text{-}diffside\ l1\ p2\ p1$
 <proof>

lemma(in *Order-Rule*) *Plane-sameside-rev* :
assumes
 $Plane\text{-}sameside\ l1\ p1\ p2$
shows $Plane\text{-}sameside\ l1\ p2\ p1$
 <proof>

lemma(in *Order-Rule*) *Plane-sameside-not-diffside* :
assumes N :
 $Plane\text{-}sameside\ l1\ p1\ p2$

shows \neg *Plane-diffside* *l1* *p1* *p2*
(*proof*)

lemma(in *Order-Rule*) *Plane-diffside-not-sameside* :
assumes *N* :
 Plane-diffside *l1* *p1* *p2*
shows \neg *Plane-sameside* *l1* *p1* *p2*
(*proof*)

lemma(in *Order-Rule*) *Plane-not-sameside-diffside* :
assumes \neg *Plane-sameside* *l1* *p1* *p2*
 \neg *Line-on* *l1* *p1* \neg *Line-on* *l1* *p2*
 \neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*) (*Geos* (*Poi* *p2*) *add Emp*)
shows *Plane-diffside* *l1* *p1* *p2*
(*proof*)

lemma(in *Order-Rule*) *Plane-not-diffside-sameside* :
assumes \neg *Plane-diffside* *l1* *p1* *p2*
 \neg *Line-on* *l1* *p1* \neg *Line-on* *l1* *p2*
 \neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*) (*Geos* (*Poi* *p2*) *add Emp*)
shows *Plane-sameside* *l1* *p1* *p2*
(*proof*)

lemma(in *Order-Rule*) *Plane-Line-diff-trans* :
assumes
 Plane-diffside *l1* *p1* *p2*
 Eq (*Geos* (*Lin* *l1*) *add Emp*) (*Geos* (*Lin* *l2*) *add Emp*)
shows *Plane-diffside* *l2* *p1* *p2*
(*proof*)

lemma(in *Order-Rule*) *Plane-Line-trans* :
assumes
 Plane-sameside *l1* *p1* *p2*
 Eq (*Geos* (*Lin* *l1*) *add Emp*) (*Geos* (*Lin* *l2*) *add Emp*)
shows *Plane-sameside* *l2* *p1* *p2*
(*proof*)

lemma(in *Order-Rule*) *Line-other-Point* :
assumes *Line-on* *l1* *p1*
shows $\exists p.$ *Line-on* *l1* *p* \wedge \neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*) (*Geos* (*Poi* *p*) *add Emp*)
(*proof*)

lemma(in *Order-Rule*) *Plane-Bet-sameside* :
assumes
 Bet-Point (*Se* *p1* *p3*) *p2*
 Line-on *l1* *p1*
 \neg *Eq* (*Geos* (*Lin* (*Li* *p1* *p3*)) *add Emp*) (*Geos* (*Lin* *l1*) *add Emp*)
shows *Plane-sameside* *l1* *p2* *p3*

<proof>

lemma(in *Order-Rule*) *Plane-Bet-diffside* :

assumes

Bet-Point (*Se p1 p3*) *p2*

Line-on l1 p2

\neg *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*)

shows *Plane-diffside l1 p1 p3*

<proof>

lemma(in *Order-Rule*) *Plane-trans-inv* :

assumes

Plane-diffside l1 A B

Plane-diffside l1 A C

\neg *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)

shows *Plane-sameside l1 B C*

<proof>

lemma(in *Order-Rule*) *Plane-trans* :

assumes

Plane-sameside l1 A B

Plane-diffside l1 A C

shows *Plane-diffside l1 B C*

<proof>

lemma(in *Order-Rule*) *Plane-sameside-trans* :

assumes

Plane-sameside l1 A B

Plane-sameside l1 B C

\neg *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)

shows *Plane-sameside l1 A C*

<proof>

lemma (in *Order-Rule*) *Seg-Bet-not-on* :

assumes

Bet-Point (*Se p1 p3*) *p2*

shows \neg *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*)

<proof>

end

3 Congruence

Of the equivalence relations for angles, only the transitive law is not included in the axiom, but is mentioned by the theorem. However, in the proofs before that, there are some scenes where it is regarded as congruence by the congruence relation with the same angle. Therefore, we add a weak transitive law that “when two angles are congruent, the same angle as one

is congruent with the other”. Also, the uniqueness of the large and small relationship between the two angles and the transitive relation of three or more those have not been proved. Therefore, each proof regarding these is added to this section. Furthermore, regarding Theorem 23, the proof is omitted because the “large and small relationship of line segments”, which is treated as a premise, is undefined. As a result, the proof process of Theorem 24 is different from the existing one.

locale *Definition-3 = Order-Rule* +
fixes *Def* :: *Geo-object* \Rightarrow *bool*
and *Cong* :: *Geo-objects* \Rightarrow *Geo-objects* \Rightarrow *bool*
and *Gr* :: *Geo-objects* \Rightarrow *Geo-objects* \Rightarrow *bool*
and *Ang-inside* :: *Angle* \Rightarrow *Point* \Rightarrow *bool*
and *Right-angle* :: *Angle* \Rightarrow *bool*
assumes *Tri-def* : *Def* (*Tri* (*Tr* *p1 p2 p3*)) \longleftrightarrow \neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*)
(*Geos* (*Poi* *p2*) *add Emp*)
 \wedge \neg *Eq* (*Geos* (*Poi* *p2*) *add Emp*) (*Geos* (*Poi* *p3*) *add Emp*) \wedge \neg *Eq* (*Geos*
(*Poi* *p3*) *add Emp*) (*Geos* (*Poi* *p1*) *add Emp*)
 \wedge \neg *Bet-Point* (*Se* *p1 p2*) *p3* \wedge \neg *Bet-Point* (*Se* *p2 p3*) *p1* \wedge \neg *Bet-Point*
(*Se* *p3 p1*) *p2*
 \wedge \neg *Seg-on-Seg* (*Se* *p1 p2*) (*Se* *p2 p3*) \wedge \neg *Seg-on-Seg* (*Se* *p2 p3*) (*Se* *p3 p1*)
 \wedge \neg *Seg-on-Seg* (*Se* *p3 p1*) (*Se* *p1 p2*)
and *Cong-refl* [*simp,intro*] : *Cong* *obs obs*
and *Ang-def* : *Def* (*Ang* (*An* *p1 p2 p3*)) \longleftrightarrow \neg *Eq* (*Geos* (*Poi* *p1*) *add Emp*)
(*Geos* (*Poi* *p2*) *add Emp*)
 \wedge \neg *Eq* (*Geos* (*Poi* *p2*) *add Emp*) (*Geos* (*Poi* *p3*) *add Emp*) \wedge \neg *Eq* (*Geos*
(*Poi* *p3*) *add Emp*) (*Geos* (*Poi* *p1*) *add Emp*)
 \wedge \neg *Eq* (*Geos* (*Lin* (*Li* *p2 p1*)) *add Emp*) (*Geos* (*Lin* (*Li* *p2 p3*)) *add Emp*)
and *Ang-rev* : \llbracket *Cong* (*Geos* (*ang1*) *add Emp*) (*Geos* (*ang2*) *add Emp*) \rrbracket \Longrightarrow
Cong (*Geos* (*ang2*) *add Emp*) (*Geos* (*ang1*) *add Emp*)
and *Ang-roll* : *Cong* (*Geos* (*Ang* (*An* *p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An* *p3*
p2 p1)) *add Emp*)
 \wedge *Eq* (*Geos* (*Ang* (*An* *p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An* *p3 p2 p1*)) *add*
Emp)
and *Ang-inside-def* : *Ang-inside* (*An* *p1 p2 p3*) *p* \longleftrightarrow *Def* (*Ang* (*An* *p1 p2*
p3)) \wedge *Plane-sameside* (*Li* *p2 p1*) *p3 p* \wedge *Plane-sameside* (*Li* *p2 p3*) *p1 p*
and *Ang-Point-swap* : \llbracket *Def* (*Ang* (*An* *p1 p2 p3*)); *Line-on* (*Li* *p2 p1*) *p4*; \neg
Bet-Point (*Se* *p1 p4*) *p2*;
Line-on (*Li* *p2 p3*) *p5*; \neg *Bet-Point* (*Se* *p3 p5*) *p2*; \neg *Eq* (*Geos* (*Poi* *p2*) *add*
Emp) (*Geos* (*Poi* *p4*) *add Emp*);
 \neg *Eq* (*Geos* (*Poi* *p2*) *add Emp*) (*Geos* (*Poi* *p5*) *add Emp*) \rrbracket \Longrightarrow
Eq (*Geos* (*Ang* (*An* *p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An* *p4 p2 p5*)) *add*
Emp) \wedge *Def* (*Ang* (*An* *p4 p2 p5*))
and *Ang-Right-angle-def* : *Right-angle* (*An* *p1 p2 p3*) \longleftrightarrow
 $(\exists p. \text{Cong} (\text{Geos} (\text{Ang} (\text{An} p1 p2 p3)) \text{add Emp}) (\text{Geos} (\text{Ang} (\text{An} p1 p2 p))$
add Emp)
 \wedge *Bet-Point* (*Se* *p3 p*) *p2* \wedge *Def* (*Ang* (*An* *p1 p2 p3*)) \wedge *Def* (*Ang* (*An* *p1*
p2 p)))
and *Tri-Cong-def* : *Cong* (*Geos* (*Tri* (*Tr* *p11 p12 p13*)) *add Emp*) (*Geos* (*Tri*

$(Tr\ p21\ p22\ p23))\ add\ Emp)$
 $\longleftrightarrow Eq\ (Geos\ (Seg\ (Se\ p11\ p12))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p21\ p22))\ add\ Emp)$
 $\wedge Eq\ (Geos\ (Seg\ (Se\ p12\ p13))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p22\ p23))\ add\ Emp)$
 $\wedge Eq\ (Geos\ (Seg\ (Se\ p13\ p11))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p23\ p21))\ add\ Emp)$
 $\wedge Cong\ (Geos\ (Ang\ (An\ p12\ p11\ p13))\ add\ Emp)\ (Geos\ (Ang\ (An\ p22\ p21\ p23))\ add\ Emp)$
 $\wedge Cong\ (Geos\ (Ang\ (An\ p13\ p12\ p11))\ add\ Emp)\ (Geos\ (Ang\ (An\ p23\ p22\ p21))\ add\ Emp)$
 $\wedge Cong\ (Geos\ (Ang\ (An\ p11\ p13\ p12))\ add\ Emp)\ (Geos\ (Ang\ (An\ p21\ p23\ p22))\ add\ Emp)$
and *Ang-greater-def* : $\llbracket Cong\ (Geos\ (Ang\ an1)\ add\ Emp)\ (Geos\ (Ang\ (An\ p4\ p2\ p3))\ add\ Emp)$;
 $Plane-sameside\ (Li\ p2\ p3)\ p4\ p1 \rrbracket \implies$
 $Ang-inside\ (An\ p1\ p2\ p3)\ p4 \longleftrightarrow Gr\ (Geos\ (Ang\ (An\ p1\ p2\ p3))\ add\ Emp)$
 $(Geos\ (Ang\ an1)\ add\ Emp)$
and *Ang-less-def* : $\llbracket Cong\ (Geos\ (Ang\ an1)\ add\ Emp)\ (Geos\ (Ang\ (An\ p4\ p2\ p3))\ add\ Emp)$;
 $Plane-sameside\ (Li\ p2\ p3)\ p4\ p1;$ $\neg Ang-inside\ (An\ p1\ p2\ p3)\ p4;$
 $\neg Eq\ (Geos\ (Lin\ (Li\ p2\ p1))\ add\ Emp)\ (Geos\ (Lin\ (Li\ p2\ p4))\ add\ Emp) \rrbracket$
 \implies
 $Gr\ (Geos\ (Ang\ an1)\ add\ Emp)\ (Geos\ (Ang\ (An\ p1\ p2\ p3))\ add\ Emp)$

locale *Axiom-3 = Definition-3 +*

assumes *Seg-add* : $\llbracket Line-on\ l1\ p11;$ $Line-on\ l1\ p12;$ $Line-on\ l1\ p13;$ $\neg Seg-on-Seg\ (Se\ p11\ p12)\ (Se\ p12\ p13);$
 $Line-on\ l2\ p21;$ $Line-on\ l2\ p22;$ $Line-on\ l2\ p23;$ $\neg Seg-on-Seg\ (Se\ p21\ p22)\ (Se\ p22\ p23);$
 $Eq\ (Geos\ (Seg\ (Se\ p11\ p12))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p21\ p22))\ add\ Emp);$
 $Eq\ (Geos\ (Seg\ (Se\ p12\ p13))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p22\ p23))\ add\ Emp) \rrbracket$
 \implies
 $Eq\ (Geos\ (Seg\ (Se\ p11\ p13))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p21\ p23))\ add\ Emp)$
and *Seg-sub* : $\llbracket Line-on\ l1\ p11;$ $Line-on\ l1\ p12;$ $Line-on\ l1\ p13;$ $\neg Seg-on-Seg\ (Se\ p11\ p12)\ (Se\ p12\ p13);$
 $Line-on\ l2\ p21;$ $Line-on\ l2\ p22;$ $Line-on\ l2\ p23;$ $\neg Seg-on-Seg\ (Se\ p21\ p22)\ (Se\ p22\ p23);$
 $Eq\ (Geos\ (Seg\ (Se\ p11\ p13))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p21\ p23))\ add\ Emp) \rrbracket$
 \implies
 $Eq\ (Geos\ (Seg\ (Se\ p11\ p12))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p21\ p22))\ add\ Emp)$
 $\wedge Eq\ (Geos\ (Seg\ (Se\ p12\ p13))\ add\ Emp)\ (Geos\ (Seg\ (Se\ p22\ p23))\ add\ Emp)$
 $Emp)$
and *Ang-move-sameside* : $\llbracket \neg Line-on\ (Li\ p1\ p2)\ p3;$ $Def\ (Ang\ a1) \rrbracket \implies \exists p.\ Cong\ (Geos\ (Ang\ a1)\ add\ Emp)\ (Geos\ (Ang\ (An\ p\ p1\ p2))\ add\ Emp) \wedge Plane-sameside\ (Li\ p1\ p2)\ p\ p3$
and *Ang-move-diffside* : $\llbracket \neg Line-on\ (Li\ p1\ p2)\ p3;$ $Def\ (Ang\ a1) \rrbracket \implies \exists p.\ Cong\ (Geos\ (Ang\ a1)\ add\ Emp)\ (Geos\ (Ang\ (An\ p\ p1\ p2))\ add\ Emp) \wedge Plane-diffside\ (Li\ p1\ p2)\ p\ p3$

and *Ang-move-unique* : $\llbracket \text{Cong} (\text{Geos} (\text{Ang } an1) \text{ add Emp}) (\text{Geos} (\text{Ang} (An\ p1\ p2\ p3)) \text{ add Emp})$
 $\text{Cong} (\text{Geos} (\text{Ang } an1) \text{ add Emp}) (\text{Geos} (\text{Ang} (An\ p4\ p2\ p3)) \text{ add Emp});$
 $\text{Plane-sameside} (Li\ p2\ p3)\ p1\ p4 \rrbracket \implies$
 $\text{Eq} (\text{Geos} (Lin (Li\ p1\ p2)) \text{ add Emp}) (\text{Geos} (Lin (Li\ p4\ p2)) \text{ add Emp}) \wedge \neg$
 $\text{Bet-Point} (Se\ p1\ p4)\ p2$
and *Tri-week-SAS* : $\llbracket \text{Def} (\text{Tri} (Tr\ p11\ p12\ p13)); \text{Def} (\text{Tri} (Tr\ p21\ p22\ p23));$
 $\text{Eq} (\text{Geos} (\text{Seg} (Se\ p11\ p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (Se\ p21\ p22)) \text{ add Emp});$
 $\text{Eq} (\text{Geos} (\text{Seg} (Se\ p11\ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (Se\ p21\ p23)) \text{ add Emp});$
 $\text{Cong} (\text{Geos} (\text{Ang} (An\ p12\ p11\ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (An\ p22\ p21$
 $p23)) \text{ add Emp}) \rrbracket$
 $\implies \text{Cong} (\text{Geos} (\text{Ang} (An\ p13\ p12\ p11)) \text{ add Emp}) (\text{Geos} (\text{Ang} (An\ p23$
 $p22\ p21)) \text{ add Emp})$

locale *Congruence-Rule* = *Axiom-3* +
assumes *Ang-weektrans* : $\llbracket \text{Eq} (\text{Geos} (\text{Ang } an1) \text{ add Emp}) (\text{Geos} (\text{Ang } an2) \text{ add$
 $\text{Emp});$
 $\text{Cong} (\text{Geos} (\text{Ang } an2) \text{ add Emp}) (\text{Geos} (\text{Ang } an3) \text{ add Emp}) \rrbracket \implies \text{Cong}$
 $(\text{Geos} (\text{Ang } an1) \text{ add Emp}) (\text{Geos} (\text{Ang } an3) \text{ add Emp})$

lemma (*in* *Congruence-Rule*) *Seg-Bet-add* :
assumes
 $\text{Bet-Point} (Se\ p11\ p13)\ p12$
 $\text{Bet-Point} (Se\ p21\ p23)\ p22$
 $\text{Eq} (\text{Geos} (\text{Seg} (Se\ p11\ p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (Se\ p21\ p22)) \text{ add Emp})$
 $\text{Eq} (\text{Geos} (\text{Seg} (Se\ p12\ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (Se\ p22\ p23)) \text{ add Emp})$
shows $\text{Eq} (\text{Geos} (\text{Seg} (Se\ p11\ p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (Se\ p21\ p23)) \text{ add$
 $\text{Emp})$
 $\langle \text{proof} \rangle$

lemma (*in* *Congruence-Rule*) *Tri-simple-def* :
assumes
 $\neg \text{Eq} (\text{Geos} (\text{Poi } A) \text{ add Emp}) (\text{Geos} (\text{Poi } B) \text{ add Emp})$
 $\neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C) \text{ add Emp})$
 $\neg \text{Eq} (\text{Geos} (\text{Poi } C) \text{ add Emp}) (\text{Geos} (\text{Poi } A) \text{ add Emp})$
 $\neg \text{Line-on} (Li\ A\ B)\ C$
shows $\text{Def} (\text{Tri} (Tr\ A\ B\ C))$
 $\langle \text{proof} \rangle$

lemma (*in* *Congruence-Rule*) *Tri-def-Line* :
assumes
 $\text{Def} (\text{Tri} (Tr\ A\ B\ C))$
shows $\neg \text{Line-on} (Li\ A\ B)\ C \wedge \neg \text{Line-on} (Li\ B\ C)\ A \wedge \neg \text{Line-on} (Li\ C\ A)\ B$
 $\langle \text{proof} \rangle$

lemma (*in* *Congruence-Rule*) *Tri-def-trans* :
assumes
 $\text{Def} (\text{Tri} (Tr\ A\ B\ C))$
shows $\text{Def} (\text{Tri} (Tr\ B\ C\ A))$

<proof>

lemma (in *Congruence-Rule*) *Tri-def-rev* :

assumes

$Def (Tri (Tr A B C))$

shows $Def (Tri (Tr C B A))$

<proof>

lemma (in *Congruence-Rule*) *Tri-def-extension* :

assumes

$Def (Tri (Tr A B C))$

$\neg Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)$

$Line-on (Li B C) D$

shows $Def (Tri (Tr A B D))$

<proof>

lemma (in *Congruence-Rule*) *Ang-to-Tri* :

assumes

$Def (Ang (An A B C))$

shows $Def (Tri (Tr A B C))$

<proof>

lemma (in *Congruence-Rule*) *Ang-simple-def* :

assumes

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$

$\neg Line-on (Li A B) C$

shows $Def (Ang (An A B C))$

<proof>

lemma (in *Congruence-Rule*) *Tri-to-Ang* :

assumes

$Def (Tri (Tr A B C))$

shows $Def (Ang (An A B C))$

<proof>

lemma (in *Congruence-Rule*) *Ang-def-rev* :

assumes

$Def (Ang (An A B C))$

shows $Def (Ang (An C B A))$

<proof>

lemma (in *Congruence-Rule*) *Ang-def-inv* :

assumes

$Def (Ang (An A B C))$

shows $Def (Ang (An A C B))$

<proof>

lemma (in *Congruence-Rule*) *Ang-def-extension* :

assumes

$Def (Ang (An A B C))$
 $\neg Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)$
 $Line-on (Li B C) D$
shows $Def (Ang (An A B D))$
 $\langle proof \rangle$

lemma (in Congruence-Rule) Bet-end-Point :
shows $\neg Bet-Point (Se p1 p1) p2$
 $\langle proof \rangle$

lemma (in Congruence-Rule) Seg-Plane-sameside :
assumes
 $Line-on l1 A$
 $Line-on l1 B$
 $Line-on l1 C$
 $\neg Line-on l1 D$
 $\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$
 $\neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$
 $\neg Bet-Point (Se B C) A$
shows $Plane-sameside (Li D A) B C \vee Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$
 $\langle proof \rangle$

lemma (in Congruence-Rule) Seg-move-unique :
assumes
 $Line-on l1 A$
 $Line-on l1 B$
 $Line-on l1 C$
 $\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$
 $\neg Eq (Geos (Poi A) add Emp) (Geos (Poi C) add Emp)$
 $Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$
 $\neg Bet-Point (Se B C) A$
shows $Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$
 $\langle proof \rangle$

lemma (in Congruence-Rule) Seg-not-Eq-Point :
assumes
 $\neg Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$
shows $\neg Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)$
 $\langle proof \rangle$

lemma (in Congruence-Rule) Ang-replace :
assumes
 $Def (Ang (An A B C))$
 $Def (Ang (An A1 B1 C1))$
 $Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)$
shows $\exists p. Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An p B1 C1)) add Emp)$

$\wedge \text{Eq} (\text{Geos} (\text{Ang} (\text{An} A1 B1 C1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} p B1 C1)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B A)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 p)) \text{ add Emp}) \wedge$
Line-on (*Li* B1 A1) p $\wedge \neg \text{Bet-Point} (\text{Se} p A1) B1 \wedge \text{Def} (\text{Ang} (\text{An} p B1 C1))$
and $\exists p. \text{Cong} (\text{Geos} (\text{Ang} (\text{An} A B C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} A1 B1 p)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Ang} (\text{An} A1 B1 C1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} A1 B1 p)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 p)) \text{ add Emp}) \wedge$
Line-on (*Li* B1 C1) p $\wedge \neg \text{Bet-Point} (\text{Se} p C1) B1 \wedge \text{Def} (\text{Ang} (\text{An} A1 B1 p))$
and $\exists p q. \text{Cong} (\text{Geos} (\text{Ang} (\text{An} A B C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} p B1 q)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Ang} (\text{An} A1 B1 C1)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} p B1 q)) \text{ add Emp})$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B A)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 p)) \text{ add Emp}) \wedge$
Line-on (*Li* B1 A1) p $\wedge \neg \text{Bet-Point} (\text{Se} p A1) B1$
 $\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 q)) \text{ add Emp}) \wedge$
Line-on (*Li* B1 C1) q $\wedge \neg \text{Bet-Point} (\text{Se} q C1) B1 \wedge \text{Def} (\text{Ang} (\text{An} p B1 q))$
 ⟨proof⟩

Theorem11

theorem (in *Congruence-Rule*) *Tri-isosceles*:

assumes

$\text{Def} (\text{Tri} (\text{Tr} A B C))$

$\text{Eq} (\text{Geos} (\text{Seg} (\text{Se} A B)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} A C)) \text{ add Emp})$

shows $\text{Cong} (\text{Geos} (\text{Ang} (\text{An} A B C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} A C B)) \text{ add Emp})$

⟨proof⟩

lemma (in *Congruence-Rule*) *Tri-week-ASA* :

assumes *N* :

$\text{Def} (\text{Tri} (\text{Tr} A B C))$

$\text{Def} (\text{Tri} (\text{Tr} A1 B1 C1))$

$\text{Eq} (\text{Geos} (\text{Seg} (\text{Se} A B)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} A1 B1)) \text{ add Emp})$

$\text{Cong} (\text{Geos} (\text{Ang} (\text{An} C A B)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} C1 A1 B1)) \text{ add Emp})$

$\text{Cong} (\text{Geos} (\text{Ang} (\text{An} C B A)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} C1 B1 A1)) \text{ add Emp})$

shows $\neg\neg \text{Eq} (\text{Geos} (\text{Seg} (\text{Se} B C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} B1 C1)) \text{ add Emp})$

⟨proof⟩

Theorem12

theorem (in *Congruence-Rule*) *Tri-SAS*:

assumes

$\text{Def} (\text{Tri} (\text{Tr} A B C))$

$\text{Def} (\text{Tri} (\text{Tr} A1 B1 C1))$

$\text{Eq} (\text{Geos} (\text{Seg} (\text{Se} A B)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} A1 B1)) \text{ add Emp})$

$\text{Eq} (\text{Geos} (\text{Seg} (\text{Se} A C)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se} A1 C1)) \text{ add Emp})$

$\text{Cong} (\text{Geos} (\text{Ang} (\text{An} B A C)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An} B1 A1 C1)) \text{ add Emp})$

shows $Cong (Geos (Tri (Tr A B C)) add Emp) (Geos (Tri (Tr A1 B1 C1)) add Emp)$
 ⟨proof⟩

Theorem13

theorem (in *Congruence-Rule*) *Tri-ASA*:

assumes

$Def (Tri (Tr A B C))$

$Def (Tri (Tr A1 B1 C1))$

$Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A1 B1)) add Emp)$

$Cong (Geos (Ang (An C B A)) add Emp) (Geos (Ang (An C1 B1 A1)) add Emp)$

$Cong (Geos (Ang (An C A B)) add Emp) (Geos (Ang (An C1 A1 B1)) add Emp)$

shows $Cong (Geos (Tri (Tr A B C)) add Emp) (Geos (Tri (Tr A1 B1 C1)) add Emp)$

⟨proof⟩

Theorem14

theorem (in *Congruence-Rule*) *Ang-complementary* :

assumes

$Def (Ang (An A B C))$

$Def (Ang (An A1 B1 C1))$

$Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)$

$Bet-Point (Se A D) B$

$Bet-Point (Se A1 D1) B1$

shows

$Cong (Geos (Ang (An C B D)) add Emp) (Geos (Ang (An C1 B1 D1)) add Emp)$

⟨proof⟩

theorem (in *Congruence-Rule*) *Ang-vertical* :

assumes

$Def (Ang (An A B C))$

$Bet-Point (Se A D) B$

$Bet-Point (Se C E) B$

shows $Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An D B E)) add Emp)$

and $Cong (Geos (Ang (An C B D)) add Emp) (Geos (Ang (An A B E)) add Emp)$

⟨proof⟩

lemma (in *Congruence-Rule*) *Ang-inside-Planeside* :

assumes $Ang-inside (An A B C) D$

shows $Plane-diffside (Li B D) A C$

⟨proof⟩

lemma (in *Congruence-Rule*) *Ang-inside-Bet-Point* :

assumes

Bet-Point (*Se* $p1$ $p3$) $p2$
 \neg *Eq* (*Geos* (*Lin* (*Li* $p4$ $p1$)) *add Emp*) (*Geos* (*Lin* (*Li* $p4$ $p3$)) *add Emp*)
 \neg *Eq* (*Geos* (*Poi* $p4$) *add Emp*) (*Geos* (*Poi* $p1$) *add Emp*)
 \neg *Eq* (*Geos* (*Poi* $p4$) *add Emp*) (*Geos* (*Poi* $p3$) *add Emp*)

shows *Ang-inside* (*An* $p1$ $p4$ $p3$) $p2$

\langle *proof* \rangle

lemma (**in** *Congruence-Rule*) *Ang-inside-HalfLine* :

assumes

Ang-inside (*An* A B C) D
 \neg *Eq* (*Geos* (*Poi* B) *add Emp*) (*Geos* (*Poi* E) *add Emp*)
Line-on (*Li* B D) E
 \neg *Bet-Point* (*Se* E D) B

shows

Ang-inside (*An* A B C) E

\langle *proof* \rangle

lemma (**in** *Congruence-Rule*) *Ang-outside-Planeside* :

assumes

Def (*Ang* (*An* A B C))
 \neg *Ang-inside* (*An* A B C) D
shows \neg (*Plane-sameside* (*Li* B A) C D \wedge *Plane-sameside* (*Li* B C) A D)
and \neg *Plane-sameside* (*Li* B A) C D \wedge *Plane-sameside* (*Li* B C) A D
 \vee *Plane-sameside* (*Li* B A) C D \wedge \neg *Plane-sameside* (*Li* B C) A D
 \vee \neg *Plane-sameside* (*Li* B A) C D \wedge \neg *Plane-sameside* (*Li* B C) A D

\langle *proof* \rangle

lemma (**in** *Congruence-Rule*) *Ang-outside-exclusive* :

assumes

Plane-sameside (*Li* B C) A D
 \neg *Eq* (*Geos* (*Poi* B) *add Emp*) (*Geos* (*Poi* C) *add Emp*)
 \neg *Eq* (*Geos* (*Lin* (*Li* B A)) *add Emp*) (*Geos* (*Lin* (*Li* B D)) *add Emp*)

shows

\neg (\neg *Ang-inside* (*An* A B C) D \wedge \neg *Ang-inside* (*An* D B C) A)

\langle *proof* \rangle

lemma (**in** *Congruence-Rule*) *Ang-inside-case* :

assumes

Def (*Ang* (*An* A B C))
Def (*Ang* (*An* D B C))
Plane-sameside (*Li* B C) A D
 \neg *Eq* (*Geos* (*Lin* (*Li* B A)) *add Emp*) (*Geos* (*Lin* (*Li* B D)) *add Emp*)

shows

Ang-inside (*An* A B C) D \wedge \neg *Ang-inside* (*An* D B C) A
 \vee \neg *Ang-inside* (*An* A B C) D \wedge *Ang-inside* (*An* D B C) A

\langle *proof* \rangle

lemma (**in** *Congruence-Rule*) *Plane-sameside-HalfLine* :

assumes

Plane-sameside l1 p1 p2
Line-on l1 p3
Line-on (Li p3 p1) p4
 \neg *Bet-Point (Se p4 p1) p3*
 \neg *Eq (Geos (Poi p1) add Emp) (Geos (Poi p4) add Emp)*
 \neg *Eq (Geos (Poi p3) add Emp) (Geos (Poi p4) add Emp)*

shows *Plane-sameside l1 p1 p4*
{proof}

lemma (in Congruence-Rule) Plane-Bet-sameside-rev :

assumes

Plane-sameside l1 p1 p3
Line-on l1 p2
 \neg *Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)*
 \neg *Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp)*
 \neg *Eq (Geos (Poi p3) add Emp) (Geos (Poi p1) add Emp)*
Line-on l2 p1 Line-on l2 p2 Line-on l2 p3
 \neg *Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp)*

shows *Bet-Point (Se p3 p2) p1 \vee Bet-Point (Se p2 p1) p3*
{proof}

lemma (in Congruence-Rule) Seg-Bet-relation :

assumes *N :*

Bet-Point (Se p1 p2) p3
shows \neg *Eq (Geos (Seg (Se p1 p2)) add Emp) (Geos (Seg (Se p1 p3)) add Emp)*

{proof}

lemma (in Congruence-Rule) Seg-Bet-move-lemma1 :

assumes

Bet-Point (Se p11 p13) p12
Line-on l1 p21 Line-on l1 p22 Line-on l1 p23
 \neg *Eq (Geos (Poi p21) add Emp) (Geos (Poi p22) add Emp)*
 \neg *Eq (Geos (Poi p21) add Emp) (Geos (Poi p23) add Emp)*
Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add Emp)
Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 p23)) add Emp)
 \neg *Bet-Point (Se p22 p23) p21*

shows *Bet-Point (Se p21 p23) p22*
{proof}

lemma (in Congruence-Rule) Seg-Bet-move-sameside :

assumes

Bet-Point (Se p11 p13) p12
Line-on l1 p21 Line-on l1 p4
 \neg *Eq (Geos (Poi p21) add Emp) (Geos (Poi p4) add Emp)*
shows $\exists p q.$ *Bet-Point (Se p21 q) p \wedge Line-on l1 p \wedge Line-on l1 q*
 \wedge *Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p)) add Emp)*
 \wedge *Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 q)) add Emp)*
 \wedge \neg *Bet-Point (Se p p4) p21 \wedge \neg Bet-Point (Se q p4) p21*

<proof>

lemma (in *Congruence-Rule*) *Seg-Bet-move-diffside* :

assumes

Bet-Point (*Se p11 p13*) *p12*

Line-on l1 p21 Line-on l1 p4

$\neg \text{Eq} (\text{Geos} (\text{Poi } p21) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$

shows $\exists p q. \text{Bet-Point} (\text{Se } p21 q) p \wedge \text{Line-on } l1 p \wedge \text{Line-on } l1 q$

$\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 p)) \text{ add Emp})$

$\wedge \text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 q)) \text{ add Emp})$

$\wedge \text{Bet-Point} (\text{Se } p p4) p21 \wedge \text{Bet-Point} (\text{Se } q p4) p21$

<proof>

lemma (in *Congruence-Rule*) *Seg-Bet-wrong-relation* :

assumes

Bet-Point (*Se p11 p13*) *p12*

Bet-Point (*Se p21 p22*) *p23*

$\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 p12)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 p22)) \text{ add Emp})$

$\text{Eq} (\text{Geos} (\text{Seg} (\text{Se } p11 p13)) \text{ add Emp}) (\text{Geos} (\text{Seg} (\text{Se } p21 p23)) \text{ add Emp})$

shows *False*

<proof>

lemma (in *Congruence-Rule*) *Ang-inside-trans* :

assumes

Ang-inside (*An A B C*) *D Def* (*Ang* (*An A B C*))

Line-on (*Li B A1*) *A* $\neg \text{Bet-Point} (\text{Se } A A1) B$

Line-on (*Li B C1*) *C* $\neg \text{Bet-Point} (\text{Se } C C1) B$

$\neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } A1) \text{ add Emp})$

$\neg \text{Eq} (\text{Geos} (\text{Poi } B) \text{ add Emp}) (\text{Geos} (\text{Poi } C1) \text{ add Emp})$

shows *Ang-inside* (*An A1 B C1*) *D*

<proof>

lemma (in *Congruence-Rule*) *Ang-sub-lemma1* :

assumes

Plane-sameside (*Li o1 l1*) *h1 k1*

$\neg \text{Eq} (\text{Geos} (\text{Poi } o1) \text{ add Emp}) (\text{Geos} (\text{Poi } l1) \text{ add Emp})$

Plane-sameside (*Li o2 l2*) *h2 k2*

$\neg \text{Eq} (\text{Geos} (\text{Poi } o2) \text{ add Emp}) (\text{Geos} (\text{Poi } l2) \text{ add Emp})$

Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)

Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)

$\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o1 h1)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o1 k1)) \text{ add Emp})$

$\neg \text{Eq} (\text{Geos} (\text{Lin} (\text{Li } o2 h2)) \text{ add Emp}) (\text{Geos} (\text{Lin} (\text{Li } o2 k2)) \text{ add Emp})$

Ang-inside (*An k1 o1 l1*) *h1*

shows

Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)

Ang-inside (*An k2 o2 l2*) *h2*

<proof>

Theorem15

theorem (in *Congruence-Rule*) *Ang-sub* :

assumes

Plane-sameside (*Li o1 l1*) *h1 k1*

\neg *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)

Plane-sameside (*Li o2 l2*) *h2 k2*

\neg *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)

Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)

Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)

\neg *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)

\neg *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)

shows

Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)

<proof>

theorem (in *Congruence-Rule*) *Ang-add* :

assumes

Plane-diffside (*Li o1 l1*) *h1 k1*

\neg *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)

Plane-diffside (*Li o2 l2*) *h2 k2*

\neg *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)

Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)

Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)

\neg *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)

\neg *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)

shows

Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)

<proof>

lemma (in *Congruence-Rule*) *Ang-split-lemma1* :

assumes *N* :

Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))

Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)

Cong (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*)

Plane-sameside (*Li o1 k1*) *h1 l1*

Plane-sameside (*Li o2 k2*) *h2 l2*

\neg *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*)

shows

\neg *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*)

<proof>

Theorem16

theorem (in *Congruence-Rule*) *Ang-split* :

assumes

$Def (Ang (An h1 o1 k1)) Def (Ang (An h2 o2 k2))$
 $Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 k2)) add Emp)$
 $Ang-inside (An h1 o1 k1) l1$
shows
 $\exists p. Ang-inside (An h2 o2 k2) p$
 $\wedge Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 p)) add Emp)$
 $\wedge Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An k2 o2 p)) add Emp)$
 $\langle proof \rangle$

theorem (in Congruence-Rule) Ang-split-unique :

assumes
 $Def (Ang (An h1 o1 k1)) Def (Ang (An h2 o2 k2))$
 $Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 k2)) add Emp)$
 $Ang-inside (An h1 o1 k1) l1$
 $Ang-inside (An h2 o2 k2) l21$
 $Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 l21)) add Emp)$
 $Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An k2 o2 l21)) add Emp)$
 $Ang-inside (An h2 o2 k2) l22$
 $Cong (Geos (Ang (An h1 o1 l1)) add Emp) (Geos (Ang (An h2 o2 l22)) add Emp)$
 $Cong (Geos (Ang (An k1 o1 l1)) add Emp) (Geos (Ang (An k2 o2 l22)) add Emp)$
shows
 $Eq (Geos (Lin (Li o2 l21)) add Emp) (Geos (Lin (Li o2 l22)) add Emp)$
 $\langle proof \rangle$

lemma (in Congruence-Rule) Tri-week-SSS-lemma1 :

assumes
 $Plane-diffside (Li x y) z1 z2$
 $\neg Eq (Geos (Poi x) add Emp) (Geos (Poi y) add Emp)$
 $Eq (Geos (Seg (Se x z1)) add Emp) (Geos (Seg (Se x z2)) add Emp)$
 $Eq (Geos (Seg (Se y z1)) add Emp) (Geos (Seg (Se y z2)) add Emp)$
 $\exists p. Bet-Point (Se z1 z2) p \wedge Line-on (Li x y) p \wedge Eq (Geos (Poi x) add Emp) (Geos (Poi p) add Emp)$
shows $Cong (Geos (Ang (An x z1 y)) add Emp) (Geos (Ang (An x z2 y)) add Emp)$
 $\langle proof \rangle$

Theorem17

theorem (in Congruence-Rule) Tri-week-SSS :

assumes
 $Plane-diffside (Li x y) z1 z2$
 $\neg Eq (Geos (Poi x) add Emp) (Geos (Poi y) add Emp)$

$Eq (Geos (Seg (Se x z1)) add Emp) (Geos (Seg (Se x z2)) add Emp)$
 $Eq (Geos (Seg (Se y z1)) add Emp) (Geos (Seg (Se y z2)) add Emp)$
shows $Cong (Geos (Ang (An x y z1)) add Emp) (Geos (Ang (An x y z2)) add Emp)$
 Emp)
 <proof>

Theorem18

theorem (in *Congruence-Rule*) *Tri-SSS* :

assumes
 $Def (Tri (Tr A1 B1 C1)) Def (Tri (Tr A2 B2 C2))$
 $Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$
 $Eq (Geos (Seg (Se B1 C1)) add Emp) (Geos (Seg (Se B2 C2)) add Emp)$
 $Eq (Geos (Seg (Se C1 A1)) add Emp) (Geos (Seg (Se C2 A2)) add Emp)$
shows $Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B2 C2)) add Emp)$
 Emp)
 <proof>

Theorem19

theorem (in *Congruence-Rule*) *Ang-trans* :

assumes
 $Def (Ang (An A1 B1 C1)) Def (Ang (An A2 B2 C2)) Def (Ang (An A3 B3 C3))$
 $Cong (Geos (Ang (An A2 B2 C2)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)$
 $Cong (Geos (Ang (An A3 B3 C3)) add Emp) (Geos (Ang (An A1 B1 C1)) add Emp)$
shows $Cong (Geos (Ang (An A2 B2 C2)) add Emp) (Geos (Ang (An A3 B3 C3)) add Emp)$
 Emp)
 <proof>

lemma (in *Congruence-Rule*) *Ang-move-unique-inv* :

assumes
 $Def (Ang (An p1 p2 p3)) Def (Ang (An p4 p2 p3))$
 $Plane-sameside (Li p2 p3) p1 p4$
 $Eq (Geos (Lin (Li p2 p1)) add Emp) (Geos (Lin (Li p2 p4)) add Emp)$
shows
 $Cong (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p4 p2 p3)) add Emp)$
 Emp)
 <proof>

Theorem20

theorem (in *Congruence-Rule*) *Ang-move-Greater* :

assumes
 $Def (Ang (An h1 o1 k1)) Def (Ang (An h2 o2 l2))$
 $Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 k2)) add Emp)$
 $Plane-sameside (Li o2 h2) k2 l2$
 $Cong (Geos (Ang (An h2 o2 l2)) add Emp) (Geos (Ang (An h1 o1 l1)) add Emp)$
 Emp)

Plane-sameside (Li o1 h1) k1 l1
Ang-inside (An h2 o2 l2) k2
shows
 \neg *Ang-inside (An h1 o1 k1) l1*
 \neg *Eq (Geos (Lin (Li o1 k1)) add Emp) (Geos (Lin (Li o1 l1)) add Emp)*
 <proof>

theorem (in Congruence-Rule) Ang-move-Smaller :

assumes
Def (Ang (An h1 o1 k1)) Def (Ang (An h2 o2 l2))
Cong (Geos (Ang (An h1 o1 k1)) add Emp) (Geos (Ang (An h2 o2 k2)) add Emp)
Plane-sameside (Li o2 h2) k2 l2
Cong (Geos (Ang (An h2 o2 l2)) add Emp) (Geos (Ang (An h1 o1 l1)) add Emp)
Plane-sameside (Li o1 h1) k1 l1
 \neg *Ang-inside (An h2 o2 l2) k2*
 \neg *Eq (Geos (Lin (Li o2 k2)) add Emp) (Geos (Lin (Li o2 l2)) add Emp)*
shows *Ang-inside (An h1 o1 k1) l1*
 <proof>

lemma (in Congruence-Rule) Ang-not-Gr-Eq-rev :

assumes
Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))
 \neg *Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)*
shows
Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)
 \vee *Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)*
 <proof>

lemma (in Congruence-Rule) Ang-not-Eq-Gr :

assumes
Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))
 \neg *Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)*
shows
 \neg *Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)*
 \vee *Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)*
 <proof>

lemma (in Congruence-Rule) Ang-relation-case :

assumes
Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))
shows

$Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\vee Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\vee Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
 <proof>

lemma (in Congruence-Rule) Ang-not-Gr-lemma1 :

assumes
 $Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))$
 $Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
shows
 $\neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 <proof>

lemma (in Congruence-Rule) Ang-not-Gr :

assumes
 $Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))$
 $Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
shows
 $\neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\neg Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
 <proof>

lemma (in Congruence-Rule) Ang-Gr-not-Eq-rev :

assumes
 $Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))$
 $Gr (Geos (Ang (An p21 p22 p23)) add Emp) (Geos (Ang (An p11 p12 p13)) add Emp)$
shows
 $\neg Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 <proof>

lemma (in Congruence-Rule) Ang-relation-case-fact :

assumes
 $Def (Ang (An p11 p12 p13)) Def (Ang (An p21 p22 p23))$
shows
 $Cong (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$
 $\wedge \neg Gr (Geos (Ang (An p11 p12 p13)) add Emp) (Geos (Ang (An p21 p22 p23)) add Emp)$

$p23)) \text{ add Emp}$
 $\wedge \neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp})$
 $\vee \neg \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp})$
 $\wedge \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp})$
 $\wedge \neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp})$
 $\vee \neg \text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp})$
 $\wedge \neg \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp})$
 $\wedge \text{Gr} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp})$
 $\langle \text{proof} \rangle$

lemma (in Congruence-Rule) Ang-Gr-trans-Eq-Gr :

assumes

$\text{Def} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{Def} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{Def} (\text{Ang} (\text{An } p31 \ p32 \ p33))$

$\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp})$

$\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p31 \ p32 \ p33)) \text{ add Emp})$

shows

$\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p31 \ p32 \ p33)) \text{ add Emp})$

$\langle \text{proof} \rangle$

lemma (in Congruence-Rule) Ang-Gr-trans-Gr-Eq :

assumes

$\text{Def} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{Def} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{Def} (\text{Ang} (\text{An } p31 \ p32 \ p33))$

$\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp})$

$\text{Cong} (\text{Geos} (\text{Ang} (\text{An } p21 \ p22 \ p23)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p31 \ p32 \ p33)) \text{ add Emp})$

shows

$\text{Gr} (\text{Geos} (\text{Ang} (\text{An } p11 \ p12 \ p13)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p31 \ p32 \ p33)) \text{ add Emp})$

$\langle \text{proof} \rangle$

lemma (in Congruence-Rule) Ang-Eq-Point :

assumes

$\text{Def} (\text{Ang} (\text{An } p1 \ p2 \ p3))$

$\text{Eq} (\text{Geos} (\text{Poi } p1) \text{ add Emp}) (\text{Geos} (\text{Poi } p4) \text{ add Emp})$

shows

$\text{Eq} (\text{Geos} (\text{Ang} (\text{An } p1 \ p2 \ p3)) \text{ add Emp}) (\text{Geos} (\text{Ang} (\text{An } p4 \ p2 \ p3)) \text{ add Emp})$

\wedge *Def* (*Ang* (*An* *p4* *p2* *p3*))
 ⟨*proof*⟩

lemma (in *Congruence-Rule*) *Planeside-wrong-relation* :

assumes

Plane-diffside (*Li* *p1* *p2*) *p3* *p4*
Plane-diffside (*Li* *p1* *p3*) *p2* *p4*
Plane-sameside (*Li* *p1* *p5*) *p3* *p2*
Plane-sameside (*Li* *p1* *p5*) *p4* *p2*

shows *False*

⟨*proof*⟩

lemma (in *Congruence-Rule*) *Ang-Gr-trans-Gr-Gr* :

assumes

Def (*Ang* (*An* *p11* *p12* *p13*)) *Def* (*Ang* (*An* *p21* *p22* *p23*)) *Def* (*Ang* (*An* *p31*
p32 *p33*))
Gr (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p21* *p22* *p23*))
add Emp)
Gr (*Geos* (*Ang* (*An* *p21* *p22* *p23*)) *add Emp*) (*Geos* (*Ang* (*An* *p31* *p32* *p33*))
add Emp)

shows

Gr (*Geos* (*Ang* (*An* *p11* *p12* *p13*)) *add Emp*) (*Geos* (*Ang* (*An* *p31* *p32* *p33*))
add Emp)
 ⟨*proof*⟩

lemma (in *Congruence-Rule*) *Ang-complementary-inside* :

assumes

Def (*Ang* (*An* *p1* *p2* *p3*))
Bet-Point (*Se* *p3* *p4*) *p2*
Ang-inside (*An* *p5* *p2* *p3*) *p1*

shows

Ang-inside (*An* *p1* *p2* *p4*) *p5*
 ⟨*proof*⟩

Theorem21

theorem (in *Congruence-Rule*) *Ang-Right-angle-Cong* :

assumes

Right-angle (*An* *l1* *o1* *h1*) *Right-angle* (*An* *l2* *o2* *h2*)

shows

Cong (*Geos* (*Ang* (*An* *l1* *o1* *h1*)) *add Emp*) (*Geos* (*Ang* (*An* *l2* *o2* *h2*)) *add*
Emp)
 ⟨*proof*⟩

lemma (in *Congruence-Rule*) *Ang-external-Gr-lemma1* :

assumes *N* :

Def (*Tri* (*Tr* *A* *B* *C*))
Bet-Point (*Se* *B* *D*) *A*

shows \neg *Cong* (*Geos* (*Ang* (*An* *C* *A* *D*)) *add Emp*) (*Geos* (*Ang* (*An* *A* *C* *B*))
add Emp)

$\langle proof \rangle$

lemma (in *Congruence-Rule*) *Ang-external-Gr-lemma2* :

assumes N :

$Def (Tri (Tr A B C))$

$Bet-Point (Se B D) A$

shows $\neg Gr (Geos (Ang (An A C B)) add Emp) (Geos (Ang (An C A D)) add Emp)$

$\langle proof \rangle$

Theorem22

theorem (in *Congruence-Rule*) *Ang-external-Gr* :

assumes

$Def (Tri (Tr A B C))$

$Bet-Point (Se B D) A$

shows

$Gr (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An A C B)) add Emp)$

$Gr (Geos (Ang (An C A D)) add Emp) (Geos (Ang (An A B C)) add Emp)$

$\langle proof \rangle$

lemma (in *Congruence-Rule*) *Seg-not-Eq-move* :

assumes

$\neg Eq (Geos (Poi A1) add Emp) (Geos (Poi B1) add Emp)$

$\neg Eq (Geos (Poi A2) add Emp) (Geos (Poi B2) add Emp)$

$\neg Eq (Geos (Poi A2) add Emp) (Geos (Poi B3) add Emp)$

$Line-on l1 A2 Line-on l1 B2 Line-on l1 B3$

$\neg Bet-Point (Se B3 B2) A2$

$Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B3)) add Emp)$

$\neg Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$

shows

$Bet-Point (Se B2 A2) B3 \wedge \neg Bet-Point (Se A2 B3) B2$

$\vee \neg Bet-Point (Se B2 A2) B3 \wedge Bet-Point (Se A2 B3) B2$

$\langle proof \rangle$

lemma (in *Congruence-Rule*) *Tri-Seg-diagonal* :

assumes

$Def (Tri (Tr A B C))$

$Bet-Point (Se B C) D$

$Eq (Geos (Seg (Se A C)) add Emp) (Geos (Seg (Se C D)) add Emp)$

shows

$Gr (Geos (Ang (An B A C)) add Emp) (Geos (Ang (An A B C)) add Emp)$

$\langle proof \rangle$

lemma (in *Congruence-Rule*) *Tri-Bet-Ang-Gr* :

assumes

$Def (Tri (Tr A B C))$

$Bet-Point (Se A C) D$

$Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$

shows

$Gr (Geos (Ang (An A D B)) add Emp) (Geos (Ang (An A B D)) add Emp)$
 ⟨proof⟩

Theorem24

theorem (in *Congruence-Rule*) *Tri-isosceles-inv* :

assumes N :

$Def (Tri (Tr A B C))$

$Cong (Geos (Ang (An A B C)) add Emp) (Geos (Ang (An A C B)) add Emp)$

shows

$\neg\neg Eq (Geos (Seg (Se A B)) add Emp) (Geos (Seg (Se A C)) add Emp)$

⟨proof⟩

lemma (in *Congruence-Rule*) *Tri-AAS-lemma1* :

assumes

$Def (Tri (Tr A1 B1 C1)) Def (Tri (Tr A2 B2 C2))$

$Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$

$Cong (Geos (Ang (An A1 C1 B1)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp)$

$Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos (Ang (An B2 A2 C2)) add Emp)$

shows

$Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B2 C2)) add Emp)$

⟨proof⟩

Theorem25

theorem (in *Congruence-Rule*) *Tri-AAS* :

assumes

$Def (Tri (Tr A1 B1 C1)) Def (Tri (Tr A2 B2 C2))$

$Eq (Geos (Seg (Se A1 B1)) add Emp) (Geos (Seg (Se A2 B2)) add Emp)$

$Cong (Geos (Ang (An A1 C1 B1)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp)$

$Cong (Geos (Ang (An A1 B1 C1)) add Emp) (Geos (Ang (An A2 B2 C2)) add Emp)$

$\vee Cong (Geos (Ang (An B1 A1 C1)) add Emp) (Geos (Ang (An B2 A2 C2)) add Emp)$

shows

$Cong (Geos (Tri (Tr A1 B1 C1)) add Emp) (Geos (Tri (Tr A2 B2 C2)) add Emp)$

⟨proof⟩

Theorem26

theorem (in *Congruence-Rule*) *Seg-bisection* :

assumes

$\neg Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)$

shows

$\exists p. Eq (Geos (Seg (Se A p)) add Emp) (Geos (Seg (Se p B)) add Emp) \wedge$
Bet-Point (Se A B) p

⟨proof⟩

theorem (in *Congruence-Rule*) *Ang-bisection* :

assumes

$Def (Ang (An A B C))$

shows

$\exists p. Cong (Geos (Ang (An A B p)) add Emp) (Geos (Ang (An p B C)) add Emp)$

$\wedge Ang-inside (An A B C) p \wedge Def (Ang (An A B p)) \wedge Def (Ang (An p B C))$
 $\langle proof \rangle$

end

References

- [1] D. Hilbert. *The Foundations of Geometry*. <https://math.berkeley.edu/wodzicki/160/Hilbert.pdf>.