Foundation of geometry in planes, and some complements : Excluding the parallel axioms

Fumiya Iwama

May 14, 2024

Abstract

"Foundations of Geometry" is a mathematical book written by Hilbert in 1899. This entry is a complete formalization of "Incidence" (excluding cubic axioms), "Order" and "Congruence" (excluding point sequences) of the axioms constructed in this book. In addition, the theorem of the problem about the part that is treated implicitly and is not clearly stated in it is being carried out in parallel.

Contents

1 Incidence **1**

2 Order **4**

3 Congruence **11**

1 Incidence

D.Hilbert made a rigorous reconstruction of Euclidean geometry in Chapter 1 of [1]. There, five types of axioms are listed and 32 theorems are proved. In Hilbert's axiom system, basic concepts such as points and lines are treated as undefined terms, and only their relationships are defined by axioms. In addition, the continuity axiom stipulates that the Euclidean plane is essentially equivalent to the real plane R2, ensuring that the axiom system is categorical.

Implement each axiom and definition and prove the theorem (Coupling axioms related to space geometry axiom 4 to 8 are excluded).

datatype *Point* = *char*
datatype *Segment* = *Se Point Point*
datatype *Line* = *Li Point Point*

```
datatype Angle = An Point Point Point
datatype Triangle = Tr Point Point Point
datatype Geo-object =
  Poi Point
  | Seg Segment
  | Lin Line
  | Ang Angle
  | Tri Triangle
datatype sign = add | sub
datatype Geo-objects = Emp | Geos Geo-object sign Geo-objects

locale Eq-relation =
  fixes Eq :: Geo-objects ⇒ Geo-objects ⇒ bool
    and Inv :: bool ⇒ bool
 assumes Eq-refl [simp,intro] : Eq obs obs
    and Eq-rev : ⟦Eq obs1 obs2⟧ ⟹ Eq obs2 obs1
    and Eq-trans : ⟦Eq obs1 obs3; Eq obs2 obs3⟧ ⟹ Eq obs1 obs2
    and Inv-def : Inv b1 ⟷ ¬ b1

locale Definition-1 = Eq-relation +
  fixes Line-on :: Line ⇒ Point ⇒ bool

locale Axiom-1 = Definition-1 +
  assumes Line-exist : ⟦¬ Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)⟧

      ⟹ ∃ l. Line-on l p1 ∧ Line-on l p2
    and Line-unique : ⟦Line-on l1 p1; Line-on l1 p2; Line-on l2 p1; Line-on l2 p2;
       ¬ Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)⟧ ⟹ Eq (Geos
(Lin l1) add Emp) (Geos (Lin l2) add Emp)
    and Line-on-exist : ∃ p q. Line-on l1 p ∧ Line-on l1 q
      ∧ ¬ Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp)
    and Line-not-on-exist : ∃ p q r. ¬ Line-on l1 p ∧ ¬ Line-on l1 q ∧ ¬ Line-on
l1 r
      ∧ ¬ Eq (Geos (Poi p) add Emp) (Geos (Poi q) add Emp)
      ∧ ¬ Eq (Geos (Poi q) add Emp) (Geos (Poi r) add Emp)
      ∧ ¬ Eq (Geos (Poi r) add Emp) (Geos (Poi p) add Emp)

locale Incidence-Rule = Axiom-1 +
  assumes Point-Eq : ⟦P1(p1); Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add
Emp)⟧ ⟹ P1(p2)
    and Line-on-trans : ⟦Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp);
Line-on l1 p1⟧
      ⟹ Line-on l2 p1
    and Line-on-rule : Line-on (Li p1 p2) p1 ∧ Line-on (Li p1 p2) p2

lemma(in Incidence-Rule) Eq-not-trans :
  assumes N :
    ¬ Eq obs1 obs2
    Eq obs2 obs3
```

shows ¬ *Eq obs1 obs3*
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-rev* :
assumes ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
shows *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*)
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-not-on-Point* :
assumes *N* :
¬ *Line-on* (*Li p1 p2*) *p3*
shows ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-not-on-trans* :
assumes
Eq (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
¬ *Line-on l1 p1*
shows ¬ *Line-on l2 p1*
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-on-rev* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
Line-on (*Li p1 p2*) *p3*
shows *Line-on* (*Li p1 p3*) *p2*
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-not-Eq* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Line-on* (*Li p1 p2*) *p3*
shows ¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*)
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-not-Eq-on* :
assumes *N* :
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*)
shows ¬ *Line-on* (*Li p1 p2*) *p3*
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-unique-Point* :
assumes
¬ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
Line-on l1 p1 Line-on l1 p2
Line-on l2 p1 Line-on l2 p2

shows *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-not-on-Eq* :
assumes *N* :
¬ *Line-on l1 p1*
Line-on l2 p1
shows ¬ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
⟨*proof*⟩

lemma(**in** *Incidence-Rule*) *Line-cross-not-on* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
¬ *Line-on* (*Li p1 p2*) *p3*
Line-on (*Li p2 p3*) *p4*
shows ¬ *Line-on* (*Li p1 p2*) *p4*
⟨*proof*⟩

end

2 Order

locale *Definition-2* = *Incidence-Rule* +
fixes *Line-on-Seg* :: *Line* ⇒ *Segment* ⇒ *bool*
and *Bet-Point* :: *Segment* ⇒ *Point* ⇒ *bool*
and *Seg-on-Seg* :: *Segment* ⇒ *Segment* ⇒ *bool*
and *Line-on-Line* :: *Line* ⇒ *Line* ⇒ *bool*
and *Plane-sameside* :: *Line* ⇒ *Point* ⇒ *Point* ⇒ *bool*
and *Plane-diffside* :: *Line* ⇒ *Point* ⇒ *Point* ⇒ *bool*
assumes *Bet-Point-def* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
and *Bet-rev* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ *Bet-Point* (*Se p2 p1*) *p3*
and *Line-Bet-exist* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ ∃ *l*. *Line-on l p1* ∧ *Line-on l p2* ∧ *Line-on l p3*
and *Seg-rev* : *Eq* (*Geos* (*Seg* (*Se p1 p2*)) *add Emp*) (*Geos* (*Seg* (*Se p2 p1*)) *add Emp*)
and *Plane-sameside-def* : *Plane-sameside l1 p1 p2* ⟷ ¬ *Line-on-Seg l1* (*Se p1 p2*) ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
and *Plane-diffside-def* : *Plane-diffside l1 p1 p2* ⟷ (∃ *p*. *Bet-Point* (*Se p1 p2*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2*)

locale *Axiom-2* = *Definition-2* +
assumes *Bet-extension* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧ ⟹ ∃ *p*. *Bet-Point* (*Se p1 p*) *p2* ∧ *Line-on l1 p*
and *Bet-iff* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ *Inv* (*Bet-Point* (*Se p2 p3*) *p1*) ∧

Inv (*Bet-Point* (*Se p3 p1*) *p2*)
and *Pachets-axiom* : ⟦¬ *Line-on* (*Li p1 p2*) *p3*; *Bet-Point* (*Se p1 p2*) *p4*; *Line-on l1 p4*;
¬ *Line-on l1 p1*; ¬ *Line-on l1 p2*; ¬ *Line-on l1 p3*⟧ ⟹
Line-on-Seg l1 (*Se p1 p3*) ∧ ¬ *Line-on-Seg l1* (*Se p2 p3*)
∨ *Line-on-Seg l1* (*Se p2 p3*) ∧ ¬ *Line-on-Seg l1* (*Se p1 p3*)
and *Seg-move-sameside* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*);
¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)⟧ ⟹
∃ *p*. *Eq* (*Geos* (*Seg* (*Se p3 p4*)) *add Emp*) (*Geos* (*Seg* (*Se p1 p*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p p2*) *p1* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)
and *Seg-move-diffside* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*);
¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)⟧ ⟹
∃ *p*. *Eq* (*Geos* (*Seg* (*Se p3 p4*)) *add Emp*) (*Geos* (*Seg* (*Se p1 p*)) *add Emp*) ∧ *Bet-Point* (*Se p p2*) *p1* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)

locale *Order-Rule* = *Axiom-2* +
assumes *Bet-Point-Eq* : ⟦*Bet-Point* (*Se p1 p2*) *p3*; *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)⟧ ⟹ *Bet-Point* (*Se p4 p2*) *p3*
and *Line-on-Seg-rule* : *Line-on-Seg l1* (*Se p1 p2*) ⟷ (∃ *p*. *Line-on l1 p* ∧ *Bet-Point* (*Se p1 p2*) *p*)
and *Seg-on-Seg-rule* : *Seg-on-Seg* (*Se p1 p2*) (*Se p3 p4*) ⟷ (∃ *p*. *Bet-Point* (*Se p1 p2*) *p* ∧ *Bet-Point* (*Se p3 p4*) *p*)
and *Line-on-Line-rule* : *Line-on-Line l1 l2* ⟷ (∃ *p*. *Line-on l1 p* ∧ *Line-on l2 p*)
and *Seg-Point-Eq* : ⟦*Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧ ⟹ *Eq* (*Geos* (*Seg* (*Se p3 p1*)) *add Emp*) (*Geos* (*Seg* (*Se p3 p2*)) *add Emp*)

lemma(**in** *Order-Rule*) *Line-Bet-on* :
assumes
Bet-Point (*Se p1 p2*) *p3*
shows *Line-on* (*Li p1 p2*) *p3* **and** *Line-on* (*Li p2 p1*) *p3*
and *Line-on* (*Li p2 p3*) *p1* **and** *Line-on* (*Li p3 p2*) *p1*
and *Line-on* (*Li p1 p3*) *p2* **and** *Line-on* (*Li p3 p1*) *p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Line-Bet-not-Eq* :
assumes *N* :
Bet-Point (*Se p1 p2*) *p3*
¬ *Line-on* (*Li p1 p2*) *p4*
shows ¬ *Eq* (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*)
⟨*proof*⟩

Theorem3

theorem(**in** *Order-Rule*) *Seg-density* :
assumes ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)

shows $\exists\, p.$ *Bet-Point* (*Se A C*) *p*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Line-Bet-not-on* :
assumes
Line-on (*Li p1 p2*) *p3*
¬ *Line-on* (*Li p1 p2*) *p4*
Bet-Point (*Se p3 p4*) *p5*
shows *Inv* (*Line-on* (*Li p1 p2*) *p5*)
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Line-not-on-ex* :
assumes *N* :
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Line-on* (*Li p1 p2*) *p3*
Line-on (*Li p1 p4*) *p2*
shows ¬ *Line-on* (*Li p1 p4*) *p3*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Line-on-dens* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
Line-on (*Li p1 p2*) *p3*
Line-on (*Li p1 p4*) *p3*
shows *Line-on* (*Li p2 p4*) *p3*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-case-lemma1* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Bet-Point* (*Se B A*) *C*
¬ *Bet-Point* (*Se C B*) *A*
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
¬ *Line-on* (*Li A C*) *D*
Bet-Point (*Se B G*) *D*
shows $\exists\, p.$ *Line-on* (*Li A D*) *p* ∧ *Bet-Point* (*Se G C*) *p*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-case-lemma2* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Bet-Point* (*Se B A*) *C*

¬ *Bet-Point* (*Se C B*) *A*
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
shows *Bet-Point* (*Se A C*) *B*
⟨*proof*⟩

Theorem4

lemma(**in** *Order-Rule*) *Bet-case* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
shows *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-case-fact* :
assumes
Bet-Point (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C*
shows
Bet-Point (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C*
∨ ¬ *Bet-Point* (*Se A C*) *B* ∧ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C*
∨ ¬ *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* ∧ *Bet-Point* (*Se B A*) *C*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-swap-lemma-1* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
shows *Line-on* (*Li A D*) *B* ∧ *Line-on* (*Li A D*) *C*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-swap-lemma-2* :
assumes
Bet-Point (*Se p1 p3*) *p2*
¬ *Line-on* (*Li p1 p3*) *p4*
¬ *Line-on* (*Li p2 p5*) *p3*
¬ *Line-on* (*Li p2 p5*) *p1*
¬ *Line-on* (*Li p2 p5*) *p4*
Bet-Point (*Se p3 p5*) *p4*
shows ∃ *p*. *Line-on* (*Li p2 p5*) *p* ∧ *Bet-Point* (*Se p1 p4*) *p*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-swap-lemma-3* :
assumes

Bet-Point (*Se p1 p3*) *p2*
Bet-Point (*Se p3 p5*) *p4*
¬ *Line-on* (*Li p1 p3*) *p5*
shows ∃ *p*. *Bet-Point* (*Se p1 p4*) *p* ∧ *Bet-Point* (*Se p5 p2*) *p*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-swap-lemma-4* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Bet-Point (*Se A E*) *G*
Bet-Point (*Se D G*) *H*
¬ *Line-on* (*Li A D*) *E*
shows ∃ *p*. *Line-on* (*Li H E*) *p* ∧ *Bet-Point* (*Se D A*) *p*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-swap-lemma-5* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
Bet-Point (*Se C F*) *E*
¬ *Line-on* (*Li A D*) *F*
¬ *Line-on* (*Li A C*) *F*
shows *Bet-Point* (*Se A D*) *C*
⟨*proof*⟩

theorem(**in** *Order-Rule*) *Bet-swap-234-134* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A D*) *C*
⟨*proof*⟩

theorem(**in** *Order-Rule*) *Bet-swap-234-124* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A D*) *B*
⟨*proof*⟩

theorem(**in** *Order-Rule*) *Bet-swap-134-234* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se A D*) *C*
shows *Bet-Point* (*Se B D*) *C*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Bet-swap-134-124* :
assumes
Bet-Point (*Se A C*) *B*

Bet-Point (*Se A D*) *C*
shows *Bet-Point* (*Se A D*) *B*
⟨*proof*⟩

theorem(**in** *Order-Rule*) *Bet-swap-243-124* :
assumes
Bet-Point (*Se A D*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A C*) *B*
⟨*proof*⟩

theorem(**in** *Order-Rule*) *Bet-swap-243-143* :
assumes
Bet-Point (*Se A D*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A D*) *C*
⟨*proof*⟩

Theorem5

lemma(**in** *Order-Rule*) *Bet-four-Point-case* :
assumes
Line-on l1 P
Line-on l1 Q
Line-on l1 R
Line-on l1 S
Bet-Point (*Se P R*) *Q*
¬ *Eq* (*Geos* (*Poi P*) *add Emp*) (*Geos* (*Poi S*) *add Emp*)
¬ *Eq* (*Geos* (*Poi Q*) *add Emp*) (*Geos* (*Poi S*) *add Emp*)
¬ *Eq* (*Geos* (*Poi R*) *add Emp*) (*Geos* (*Poi S*) *add Emp*)
shows *Bet-Point* (*Se P S*) *R* ∨ *Bet-Point* (*Se R S*) *P*
∨ *Bet-Point* (*Se P R*) *S* ∧ *Bet-Point* (*Se P S*) *Q*
∨ *Bet-Point* (*Se P Q*) *S* ∨ *Bet-Point* (*Se Q S*) *P*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-diffside-rev* :
assumes
Plane-diffside l1 p1 p2
shows *Plane-diffside l1 p2 p1*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-sameside-rev* :
assumes
Plane-sameside l1 p1 p2
shows *Plane-sameside l1 p2 p1*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-sameside-not-diffside* :
assumes *N* :
Plane-sameside l1 p1 p2

shows ¬ *Plane-diffside l1 p1 p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-diffside-not-sameside* :
assumes *N* :
Plane-diffside l1 p1 p2
shows ¬ *Plane-sameside l1 p1 p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-not-sameside-diffside* :
assumes ¬ *Plane-sameside l1 p1 p2*
¬ *Line-on l1 p1* ¬ *Line-on l1 p2*
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
shows *Plane-diffside l1 p1 p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-not-diffside-sameside* :
assumes ¬ *Plane-diffside l1 p1 p2*
¬ *Line-on l1 p1* ¬ *Line-on l1 p2*
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
shows *Plane-sameside l1 p1 p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-Line-diff-trans* :
assumes
Plane-diffside l1 p1 p2
Eq (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
shows *Plane-diffside l2 p1 p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-Line-trans* :
assumes
Plane-sameside l1 p1 p2
Eq (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
shows *Plane-sameside l2 p1 p2*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Line-other-Point* :
assumes *Line-on l1 p1*
shows ∃ *p*. *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-Bet-sameside* :
assumes
Bet-Point (*Se p1 p3*) *p2*
Line-on l1 p1
¬ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*)
shows *Plane-sameside l1 p2 p3*

⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-Bet-diffside* :
  assumes
    Bet-Point (*Se p1 p3*) *p2*
    Line-on l1 p2
    ¬ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*)
  shows *Plane-diffside l1 p1 p3*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-trans-inv* :
  assumes
    Plane-diffside l1 A B
    Plane-diffside l1 A C
    ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
  shows *Plane-sameside l1 B C*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-trans* :
  assumes
    Plane-sameside l1 A B
    Plane-diffside l1 A C
  shows *Plane-diffside l1 B C*
⟨*proof*⟩

lemma(**in** *Order-Rule*) *Plane-sameside-trans* :
  assumes
    Plane-sameside l1 A B
    Plane-sameside l1 B C
    ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
  shows *Plane-sameside l1 A C*
⟨*proof*⟩

lemma (**in** *Order-Rule*) *Seg-Bet-not-on* :
  assumes
    Bet-Point (*Se p1 p3*) *p2*
  shows ¬ *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*)
⟨*proof*⟩

end

3 Congruence

Of the equivalence relations for angles, only the transitive law is not included in the axiom, but is mentioned by the theorem. However, in the proofs before that, there are some scenes where it is regarded as congruence by the congruence relation with the same angle. Therefore, we add a weak transitive law that "when two angles are congruent, the same angle as one is congruent with the other". Also, the uniqueness of the large and small relationship between the two angles and the transitive relation of three or more those have not been proved. Therefore, each proof regarding these is added to this section. Furthermore, regarding Theorem 23, the proof is omitted because the "large and small relationship of line segments", which is treated as a premise, is undefined. As a result, the proof process of Theorem 24 is different from the existing one.

locale *Definition-3* = *Order-Rule* +
fixes *Def* :: *Geo-object* ⇒ *bool*
and *Cong* :: *Geo-objects* ⇒ *Geo-objects* ⇒ *bool*
and *Gr* :: *Geo-objects* ⇒ *Geo-objects* ⇒ *bool*
and *Ang-inside* :: *Angle* ⇒ *Point* ⇒ *bool*
and *Right-angle* :: *Angle* ⇒ *bool*
assumes *Tri-def* : *Def* (*Tri* (*Tr p1 p2 p3*)) ⟷ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
∧ ¬ *Bet-Point* (*Se p1 p2*) *p3* ∧ ¬ *Bet-Point* (*Se p2 p3*) *p1* ∧ ¬ *Bet-Point* (*Se p3 p1*) *p2*
∧ ¬ *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ∧ ¬ *Seg-on-Seg* (*Se p2 p3*) (*Se p3 p1*) ∧ ¬ *Seg-on-Seg* (*Se p3 p1*) (*Se p1 p2*)
and *Cong-refl* [*simp*,*intro*] : *Cong obs obs*
and *Ang-def* : *Def* (*Ang* (*An p1 p2 p3*)) ⟷ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*)
and *Ang-rev* : ⟦*Cong* (*Geos* (*ang1*) *add Emp*) (*Geos* (*ang2*) *add Emp*)⟧ ⟹
Cong (*Geos* (*ang2*) *add Emp*) (*Geos* (*ang1*) *add Emp*)
and *Ang-roll* : *Cong* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p3 p2 p1*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p3 p2 p1*)) *add Emp*)
and *Ang-inside-def* : *Ang-inside* (*An p1 p2 p3*) *p* ⟷ *Def* (*Ang* (*An p1 p2 p3*)) ∧ *Plane-sameside* (*Li p2 p1*) *p3 p* ∧ *Plane-sameside* (*Li p2 p3*) *p1 p*
and *Ang-Point-swap* : ⟦*Def* (*Ang* (*An p1 p2 p3*)); *Line-on* (*Li p2 p1*) *p4*; ¬ *Bet-Point* (*Se p1 p4*) *p2*;
Line-on (*Li p2 p3*) *p5*; ¬ *Bet-Point* (*Se p3 p5*) *p2*; ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*);
¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*)⟧ ⟹
Eq (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p5*)) *add Emp*) ∧ *Def* (*Ang* (*An p4 p2 p5*))
and *Ang-Right-angle-def* : *Right-angle* (*An p1 p2 p3*) ⟷
(∃ *p*. *Cong* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p1 p2 p*)) *add Emp*)
∧ *Bet-Point* (*Se p3 p*) *p2* ∧ *Def* (*Ang* (*An p1 p2 p3*)) ∧ *Def* (*Ang* (*An p1 p2 p*)))
and *Tri-Cong-def* : *Cong* (*Geos* (*Tri* (*Tr p11 p12 p13*)) *add Emp*) (*Geos* (*Tri*

(*Tr p21 p22 p23*)) *add Emp*)
⟷ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p13 p11*)) *add Emp*) (*Geos* (*Seg* (*Se p23 p21*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An p12 p11 p13*)) *add Emp*) (*Geos* (*Ang* (*An p22 p21 p23*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An p11 p13 p12*)) *add Emp*) (*Geos* (*Ang* (*An p21 p23 p22*)) *add Emp*)
and *Ang-greater-def* : ⟦*Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*);
Plane-sameside (*Li p2 p3*) *p4 p1*⟧ ⟹
Ang-inside (*An p1 p2 p3*) *p4* ⟷ *Gr* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang an1*) *add Emp*)
and *Ang-less-def* : ⟦*Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*);
Plane-sameside (*Li p2 p3*) *p4 p1*; ¬ *Ang-inside* (*An p1 p2 p3*) *p4*;
¬ *Eq* (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*)⟧ ⟹
Gr (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*)

locale *Axiom-3* = *Definition-3* +
assumes *Seg-add* : ⟦*Line-on l1 p11*; *Line-on l1 p12*; *Line-on l1 p13*; ¬ *Seg-on-Seg* (*Se p11 p12*) (*Se p12 p13*);
Line-on l2 p21; *Line-on l2 p22*; *Line-on l2 p23*; ¬ *Seg-on-Seg* (*Se p21 p22*) (*Se p22 p23*);
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*);
Eq (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)⟧ ⟹
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
and *Seg-sub* : ⟦*Line-on l1 p11*; *Line-on l1 p12*; *Line-on l1 p13*; ¬ *Seg-on-Seg* (*Se p11 p12*) (*Se p12 p13*);
Line-on l2 p21; *Line-on l2 p22*; *Line-on l2 p23*; ¬ *Seg-on-Seg* (*Se p21 p22*) (*Se p22 p23*);
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)⟧ ⟹
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)
and *Ang-move-sameside* : ⟦¬ *Line-on* (*Li p1 p2*) *p3*; *Def* (*Ang a1*)⟧ ⟹ ∃ *p*. *Cong* (*Geos* (*Ang a1*) *add Emp*) (*Geos* (*Ang* (*An p p1 p2*)) *add Emp*) ∧ *Plane-sameside* (*Li p1 p2*) *p p3*
and *Ang-move-diffside* : ⟦¬ *Line-on* (*Li p1 p2*) *p3*; *Def* (*Ang a1*)⟧ ⟹ ∃ *p*. *Cong* (*Geos* (*Ang a1*) *add Emp*) (*Geos* (*Ang* (*An p p1 p2*)) *add Emp*) ∧ *Plane-diffside* (*Li p1 p2*) *p p3*

and *Ang-move-unique* : ⟦*Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*);
Cong (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*);
Plane-sameside (*Li p2 p3*) *p1 p4*⟧ ⟹
Eq (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p1 p4*) *p2*
and *Tri-week-SAS* : ⟦*Def* (*Tri* (*Tr p11 p12 p13*)); *Def* (*Tri* (*Tr p21 p22 p23*));
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*);
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*);
Cong (*Geos* (*Ang* (*An p12 p11 p13*)) *add Emp*) (*Geos* (*Ang* (*An p22 p21 p23*)) *add Emp*)⟧
⟹ *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*)

locale *Congruence-Rule* = *Axiom-3* +
assumes *Ang-weektrans* : ⟦*Eq* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang an2*) *add Emp*);
Cong (*Geos* (*Ang an2*) *add Emp*) (*Geos* (*Ang an3*) *add Emp*)⟧ ⟹ *Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang an3*) *add Emp*)

lemma (**in** *Congruence-Rule*) *Seg-Bet-add* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Bet-Point (*Se p21 p23*) *p22*
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)
shows *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-sinple-def* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
¬ *Line-on* (*Li A B*) *C*
shows *Def* (*Tri* (*Tr A B C*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-def-Line* :
assumes
Def (*Tri* (*Tr A B C*))
shows ¬ *Line-on* (*Li A B*) *C* ∧ ¬ *Line-on* (*Li B C*) *A* ∧ ¬ *Line-on* (*Li C A*) *B*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-def-trans* :
assumes
Def (*Tri* (*Tr A B C*))
shows *Def* (*Tri* (*Tr B C A*))

⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-def-rev* :
assumes
Def (*Tri* (*Tr A B C*))
shows *Def* (*Tri* (*Tr C B A*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-def-extension* :
assumes
Def (*Tri* (*Tr A B C*))
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Line-on (*Li B C*) *D*
shows *Def* (*Tri* (*Tr A B D*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-to-Tri* :
assumes
Def (*Ang* (*An A B C*))
shows *Def* (*Tri* (*Tr A B C*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-sinple-def* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Line-on* (*Li A B*) *C*
shows *Def* (*Ang* (*An A B C*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-to-Ang* :
assumes
Def (*Tri* (*Tr A B C*))
shows *Def* (*Ang* (*An A B C*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-def-rev* :
assumes
Def (*Ang* (*An A B C*))
shows *Def* (*Ang* (*An C B A*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-def-inv* :
assumes
Def (*Ang* (*An A B C*))
shows *Def* (*Ang* (*An A C B*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-def-extension* :
assumes

Def (*Ang* (*An A B C*))
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Line-on (*Li B C*) *D*
shows *Def* (*Ang* (*An A B D*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Bet-end-Point* :
shows ¬ *Bet-Point* (*Se p1 p1*) *p2*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-Plane-sameside* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Line-on l1 D*
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Bet-Point* (*Se B C*) *A*
shows *Plane-sameside* (*Li D A*) *B C* ∨ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-move-unique* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
¬ *Bet-Point* (*Se B C*) *A*
shows *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-not-Eq-Point* :
assumes
¬ *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
shows ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-replace* :
assumes
Def (*Ang* (*An A B C*))
Def (*Ang* (*An A1 B1 C1*))
Cong (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
shows ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)

$\wedge$ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) $\wedge$ *Line-on* (*Li B1 A1*) *p* $\wedge$ $\neg$ *Bet-Point* (*Se p A1*) *B1* $\wedge$ *Def* (*Ang* (*An p B1 C1*))
and $\exists$ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 p*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 p*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) $\wedge$ *Line-on* (*Li B1 C1*) *p* $\wedge$ $\neg$ *Bet-Point* (*Se p C1*) *B1* $\wedge$ *Def* (*Ang* (*An A1 B1 p*))
and $\exists$ *p q*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) $\wedge$ *Line-on* (*Li B1 A1*) *p* $\wedge$ $\neg$ *Bet-Point* (*Se p A1*) *B1*
$\wedge$ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 q*)) *add Emp*) $\wedge$ *Line-on* (*Li B1 C1*) *q* $\wedge$ $\neg$ *Bet-Point* (*Se q C1*) *B1* $\wedge$ *Def* (*Ang* (*An p B1 q*))
⟨*proof*⟩

Theorem11

theorem (**in** *Congruence-Rule*) *Tri-isosceles*:
assumes
Def (*Tri* (*Tr A B C*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-week-ASA* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Def (*Tri* (*Tr A1 B1 C1*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 A1*)) *add Emp*)
shows $\neg\neg$ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*)
⟨*proof*⟩

Theorem12

theorem (**in** *Congruence-Rule*) *Tri-SAS*:
assumes
Def (*Tri* (*Tr A B C*))
Def (*Tri* (*Tr A1 B1 C1*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*)

shows *Cong* (*Geos* (*Tri* (*Tr A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*)
⟨*proof*⟩

Theorem13

theorem (**in** *Congruence-Rule*) *Tri-ASA*:
assumes
Def (*Tri* (*Tr A B C*))
Def (*Tri* (*Tr A1 B1 C1*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 A1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*)
shows *Cong* (*Geos* (*Tri* (*Tr A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*)
⟨*proof*⟩

Theorem14

theorem (**in** *Congruence-Rule*) *Ang-complementary* :
assumes
Def (*Ang* (*An A B C*))
Def (*Ang* (*An A1 B1 C1*))
Cong (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
Bet-Point (*Se A D*) *B*
Bet-Point (*Se A1 D1*) *B1*
shows
Cong (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 D1*)) *add Emp*)
⟨*proof*⟩

theorem (**in** *Congruence-Rule*) *Ang-vertical* :
assumes
Def (*Ang* (*An A B C*))
Bet-Point (*Se A D*) *B*
Bet-Point (*Se C E*) *B*
shows *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D B E*)) *add Emp*)
and *Cong* (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An A B E*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-inside-Planeside* :
assumes *Ang-inside* (*An A B C*) *D*
shows *Plane-diffside* (*Li B D*) *A C*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-inside-Bet-Point* :

assumes
Bet-Point (*Se p1 p3*) *p2*
¬ *Eq* (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*)
¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
shows *Ang-inside* (*An p1 p4 p3*) *p2*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-inside-HalfLine* :
assumes
Ang-inside (*An A B C*) *D*
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi E*) *add Emp*)
Line-on (*Li B D*) *E*
¬ *Bet-Point* (*Se E D*) *B*
shows
Ang-inside (*An A B C*) *E*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-outside-Planeside* :
assumes
Def (*Ang* (*An A B C*))
¬ *Ang-inside* (*An A B C*) *D*
shows ¬ (*Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*)
and ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-outside-exclusive* :
assumes
Plane-sameside (*Li B C*) *A D*
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*)
shows
¬ (¬ *Ang-inside* (*An A B C*) *D* ∧ ¬ *Ang-inside* (*An D B C*) *A*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-inside-case* :
assumes
Def (*Ang* (*An A B C*))
Def (*Ang* (*An D B C*))
Plane-sameside (*Li B C*) *A D*
¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*)
shows
Ang-inside (*An A B C*) *D* ∧ ¬ *Ang-inside* (*An D B C*) *A*
∨ ¬ *Ang-inside* (*An A B C*) *D* ∧ *Ang-inside* (*An D B C*) *A*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Plane-sameside-HalfLine* :

assumes
Plane-sameside l1 p1 p2
Line-on l1 p3
Line-on (Li p3 p1) p4
¬ *Bet-Point (Se p4 p1) p3*
¬ *Eq (Geos (Poi p1) add Emp) (Geos (Poi p4) add Emp)*
¬ *Eq (Geos (Poi p3) add Emp) (Geos (Poi p4) add Emp)*
shows *Plane-sameside l1 p1 p4*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Plane-Bet-sameside-rev* :
assumes
Plane-sameside l1 p1 p3
Line-on l1 p2
¬ *Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)*
¬ *Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp)*
¬ *Eq (Geos (Poi p3) add Emp) (Geos (Poi p1) add Emp)*
Line-on l2 p1 Line-on l2 p2 Line-on l2 p3
¬ *Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp)*
shows *Bet-Point (Se p3 p2) p1* ∨ *Bet-Point (Se p2 p1) p3*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-Bet-relation* :
assumes *N* :
Bet-Point (Se p1 p2) p3
shows ¬ *Eq (Geos (Seg (Se p1 p2)) add Emp) (Geos (Seg (Se p1 p3)) add Emp)*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-Bet-move-lemma1* :
assumes
Bet-Point (Se p11 p13) p12
Line-on l1 p21 Line-on l1 p22 Line-on l1 p23
¬ *Eq (Geos (Poi p21) add Emp) (Geos (Poi p22) add Emp)*
¬ *Eq (Geos (Poi p21) add Emp) (Geos (Poi p23) add Emp)*
Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add Emp)
Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 p23)) add Emp)
¬ *Bet-Point (Se p22 p23) p21*
shows *Bet-Point (Se p21 p23) p22*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-Bet-move-sameside* :
assumes
Bet-Point (Se p11 p13) p12
Line-on l1 p21 Line-on l1 p4
¬ *Eq (Geos (Poi p21) add Emp) (Geos (Poi p4) add Emp)*
shows ∃ *p q. Bet-Point (Se p21 q) p* ∧ *Line-on l1 p* ∧ *Line-on l1 q*
∧ *Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p)) add Emp)*
∧ *Eq (Geos (Seg (Se p11 p13)) add Emp) (Geos (Seg (Se p21 q)) add Emp)*
∧ ¬ *Bet-Point (Se p p4) p21* ∧ ¬ *Bet-Point (Se q p4) p21*

⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-Bet-move-diffside* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Line-on l1 p21 Line-on l1 p4
¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
shows ∃ *p q*. *Bet-Point* (*Se p21 q*) *p* ∧ *Line-on l1 p* ∧ *Line-on l1 q*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 q*)) *add Emp*)
∧ *Bet-Point* (*Se p p4*) *p21* ∧ *Bet-Point* (*Se q p4*) *p21*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-Bet-wrong-relation* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Bet-Point (*Se p21 p22*) *p23*
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
shows *False*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-inside-trans* :
assumes
Ang-inside (*An A B C*) *D Def* (*Ang* (*An A B C*))
Line-on (*Li B A1*) *A* ¬ *Bet-Point* (*Se A A1*) *B*
Line-on (*Li B C1*) *C* ¬ *Bet-Point* (*Se C C1*) *B*
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*)
shows *Ang-inside* (*An A1 B C1*) *D*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-sub-lemma1* :
assumes
Plane-sameside (*Li o1 l1*) *h1 k1*
¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)
Plane-sameside (*Li o2 l2*) *h2 k2*
¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)
Ang-inside (*An k1 o1 l1*) *h1*
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An k2 o2 l2*) *h2*
⟨*proof*⟩

Theorem15

theorem (**in** *Congruence-Rule*) *Ang-sub* :
assumes
Plane-sameside (*Li o1 l1*) *h1 k1*
¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)
Plane-sameside (*Li o2 l2*) *h2 k2*
¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
⟨*proof*⟩

theorem (**in** *Congruence-Rule*) *Ang-add* :
assumes
Plane-diffside (*Li o1 l1*) *h1 k1*
¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)
Plane-diffside (*Li o2 l2*) *h2 k2*
¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-split-lemma1* :
assumes *N* :
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*)
Plane-sameside (*Li o1 k1*) *h1 l1*
Plane-sameside (*Li o2 k2*) *h2 l2*
¬ *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*)
shows
¬ *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*)
⟨*proof*⟩

Theorem16

theorem (**in** *Congruence-Rule*) *Ang-split* :
assumes

Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An h1 o1 k1*) *l1*
shows
∃ *p*. *Ang-inside* (*An h2 o2 k2*) *p*
∧ *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 p*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 p*)) *add Emp*)
⟨*proof*⟩

theorem (**in** *Congruence-Rule*) *Ang-split-unique* :
assumes
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An h1 o1 k1*) *l1*
Ang-inside (*An h2 o2 k2*) *l21*
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l21*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l21*)) *add Emp*)
Ang-inside (*An h2 o2 k2*) *l22*
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l22*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l22*)) *add Emp*)
shows
Eq (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l22*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-week-SSS-lemma1* :
assumes
Plane-diffside (*Li x y*) *z1 z2*
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi y*) *add Emp*)
Eq (*Geos* (*Seg* (*Se x z1*)) *add Emp*) (*Geos* (*Seg* (*Se x z2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se y z1*)) *add Emp*) (*Geos* (*Seg* (*Se y z2*)) *add Emp*)
∃ *p*. *Bet-Point* (*Se z1 z2*) *p* ∧ *Line-on* (*Li x y*) *p* ∧ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An x z1 y*)) *add Emp*) (*Geos* (*Ang* (*An x z2 y*)) *add Emp*)
⟨*proof*⟩

Theorem17

theorem (**in** *Congruence-Rule*) *Tri-week-SSS* :
assumes
Plane-diffside (*Li x y*) *z1 z2*
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi y*) *add Emp*)

Eq (*Geos* (*Seg* (*Se x z1*)) *add Emp*) (*Geos* (*Seg* (*Se x z2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se y z1*)) *add Emp*) (*Geos* (*Seg* (*Se y z2*)) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*)
⟨*proof*⟩

Theorem18

theorem (**in** *Congruence-Rule*) *Tri-SSS* :
assumes
Def (*Tri* (*Tr A1 B1 C1*)) *Def* (*Tri* (*Tr A2 B2 C2*))
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 C2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se C1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se C2 A2*)) *add Emp*)
shows *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*)
⟨*proof*⟩

Theorem19

theorem (**in** *Congruence-Rule*) *Ang-trans* :
assumes
Def (*Ang* (*An A1 B1 C1*)) *Def* (*Ang* (*An A2 B2 C2*)) *Def* (*Ang* (*An A3 B3 C3*))
Cong (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-move-unique-inv* :
assumes
Def (*Ang* (*An p1 p2 p3*)) *Def* (*Ang* (*An p4 p2 p3*))
Plane-sameside (*Li p2 p3*) *p1 p4*
Eq (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*)
⟨*proof*⟩

Theorem20

theorem (**in** *Congruence-Rule*) *Ang-move-Greater* :
assumes
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 l2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Plane-sameside (*Li o2 h2*) *k2 l2*
Cong (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*)

Plane-sameside (*Li o1 h1*) *k1 l1*
Ang-inside (*An h2 o2 l2*) *k2*
shows
¬ *Ang-inside* (*An h1 o1 k1*) *l1*
¬ *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*)
⟨*proof*⟩

theorem (**in** *Congruence-Rule*) *Ang-move-Smaller* :
assumes
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 l2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Plane-sameside (*Li o2 h2*) *k2 l2*
Cong (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*)
Plane-sameside (*Li o1 h1*) *k1 l1*
¬ *Ang-inside* (*An h2 o2 l2*) *k2*
¬ *Eq* (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*)
shows *Ang-inside* (*An h1 o1 k1*) *l1*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-not-Gr-Eq-rev* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-not-Eq-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-relation-case* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
shows

Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-not-Gr-lemma1* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
shows
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-not-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
shows
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-Gr-not-Eq-rev* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
shows
¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-relation-case-fact* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
shows
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22

p23)) *add Emp*)
 ∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
 ∨ ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
 ∧ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
 ∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
 ∨ ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
 ∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
 ∧ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-Gr-trans-Eq-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*)) *Def* (*Ang* (*An p31 p32 p33*))
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-Gr-trans-Gr-Eq* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*)) *Def* (*Ang* (*An p31 p32 p33*))
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
Cong (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-Eq-Point* :
assumes
Def (*Ang* (*An p1 p2 p3*))
Eq (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
shows
Eq (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*)

∧ *Def* (*Ang* (*An p4 p2 p3*))
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Planeside-wrong-relation* :
assumes
Plane-diffside (*Li p1 p2*) *p3 p4*
Plane-diffside (*Li p1 p3*) *p2 p4*
Plane-sameside (*Li p1 p5*) *p3 p2*
Plane-sameside (*Li p1 p5*) *p4 p2*
shows *False*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-Gr-trans-Gr-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*)) *Def* (*Ang* (*An p31 p32 p33*))
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-complementary-inside* :
assumes
Def (*Ang* (*An p1 p2 p3*))
Bet-Point (*Se p3 p4*) *p2*
Ang-inside (*An p5 p2 p3*) *p1*
shows
Ang-inside (*An p1 p2 p4*) *p5*
⟨*proof*⟩

Theorem21

theorem (**in** *Congruence-Rule*) *Ang-Right-angle-Cong* :
assumes
Right-angle (*An l1 o1 h1*) *Right-angle* (*An l2 o2 h2*)
shows
Cong (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-external-Gr-lemma1* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B D*) *A*
shows ¬ *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)

⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Ang-external-Gr-lemma2* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B D*) *A*
shows ¬ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*)
⟨*proof*⟩

Theorem22

theorem (**in** *Congruence-Rule*) *Ang-external-Gr* :
assumes
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B D*) *A*
shows
Gr (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
Gr (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Seg-not-Eq-move* :
assumes
¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*)
¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B3*) *add Emp*)
Line-on l1 A2 Line-on l1 B2 Line-on l1 B3
¬ *Bet-Point* (*Se B3 B2*) *A2*
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B3*)) *add Emp*)
¬ *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
shows
Bet-Point (*Se B2 A2*) *B3* ∧ ¬ *Bet-Point* (*Se A2 B3*) *B2*
∨ ¬ *Bet-Point* (*Se B2 A2*) *B3* ∧ *Bet-Point* (*Se A2 B3*) *B2*
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-Seg-diagonal* :
assumes
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B C*) *D*
Eq (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se C D*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-Bet-Ang-Gr* :
assumes
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se A C*) *D*
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
shows

Gr (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*)
⟨*proof*⟩

Theorem24

theorem (**in** *Congruence-Rule*) *Tri-isosceles-inv* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Cong (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
shows
¬¬ *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
⟨*proof*⟩

lemma (**in** *Congruence-Rule*) *Tri-AAS-lemma1* :
assumes
Def (*Tri* (*Tr A1 B1 C1*)) *Def* (*Tri* (*Tr A2 B2 C2*))
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*)
shows
Cong (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*)
⟨*proof*⟩

Theorem25

theorem (**in** *Congruence-Rule*) *Tri-AAS* :
assumes
Def (*Tri* (*Tr A1 B1 C1*)) *Def* (*Tri* (*Tr A2 B2 C2*))
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*)
∨ *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*)
shows
Cong (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*)
⟨*proof*⟩

Theorem26

theorem (**in** *Congruence-Rule*) *Seg-bisection* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
shows
∃ *p*. *Eq* (*Geos* (*Seg* (*Se A p*)) *add Emp*) (*Geos* (*Seg* (*Se p B*)) *add Emp*) ∧ *Bet-Point* (*Se A B*) *p*
⟨*proof*⟩

theorem (**in** *Congruence-Rule*) *Ang-bisection* :
assumes
Def (*Ang* (*An A B C*))
shows
$\exists\, p.$ *Cong* (*Geos* (*Ang* (*An A B p*)) *add Emp*) (*Geos* (*Ang* (*An p B C*)) *add Emp*)
$\wedge$ *Ang-inside* (*An A B C*) *p* $\wedge$ *Def* (*Ang* (*An A B p*)) $\wedge$ *Def* (*Ang* (*An p B C*))
$\langle$*proof*$\rangle$

end

References

[1] D. Hilbert. *The Foundations of Geometry*. https://math.berkeley.edu/wodzicki/160/Hilbert.pdf.