Foundation of geometry in planes, and some complements : Excluding the parallel axioms

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September 1, 2025

Abstract

"Foundations of Geometry" is a mathematical book written by Hilbert in 1899. This entry is a complete formalization of "Incidence" (excluding cubic axioms), "Order" and "Congruence" (excluding point sequences) of the axioms constructed in this book. In addition, the theorem of the problem about the part that is treated implicitly and is not clearly stated in it is being carried out in parallel.

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1 Incidence

D.Hilbert made a rigorous reconstruction of Euclidean geometry in Chapter 1 of [1]. There, five types of axioms are listed and 32 theorems are proved. In Hilbert's axiom system, basic concepts such as points and lines are treated as undefined terms, and only their relationships are defined by axioms. In addition, the continuity axiom stipulates that the Euclidean plane is essentially equivalent to the real plane R2, ensuring that the axiom system is categorical.

Implement each axiom and definition and prove the theorem (Coupling axioms related to space geometry axiom 4 to 8 are excluded).

datatype *Point* = *char*
datatype *Segment* = *Se Point Point*
datatype *Line* = *Li Point Point*

datatype *Angle* = *An Point Point Point*
datatype *Triangle* = *Tr Point Point Point*
datatype *Geo-object* =
Poi Point
| *Seg Segment*
| *Lin Line*
| *Ang Angle*
| *Tri Triangle*
datatype *sign* = *add* | *sub*
datatype *Geo-objects* = *Emp* | *Geos Geo-object sign Geo-objects*

locale *Eq-relation* =
fixes *Eq* :: *Geo-objects* ⇒ *Geo-objects* ⇒ *bool*
and *Inv* :: *bool* ⇒ *bool*
assumes *Eq-refl* [*simp*,*intro*] : *Eq obs obs*
and *Eq-rev* : ⟦*Eq obs1 obs2*⟧ ⟹ *Eq obs2 obs1*
and *Eq-trans* : ⟦*Eq obs1 obs3*; *Eq obs2 obs3*⟧ ⟹ *Eq obs1 obs2*
and *Inv-def* : *Inv b1* ⟷ ¬ *b1*

locale *Definition-1* = *Eq-relation* +
fixes *Line-on* :: *Line* ⇒ *Point* ⇒ *bool*

locale *Axiom-1* = *Definition-1* +
assumes *Line-exist* : ⟦¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧

⟹ ∃ *l*. *Line-on l p1* ∧ *Line-on l p2*
and *Line-unique* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; *Line-on l2 p1*; *Line-on l2 p2*;
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧ ⟹ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
and *Line-on-exist* : ∃ *p q*. *Line-on l1 p* ∧ *Line-on l1 q*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*)
and *Line-not-on-exist* : ∃ *p q r*. ¬ *Line-on l1 p* ∧ ¬ *Line-on l1 q* ∧ ¬ *Line-on l1 r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)

locale *Incidence-Rule* = *Axiom-1* +
assumes *Point-Eq* : ⟦*P1*(*p1*); *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧ ⟹ *P1*(*p2*)
and *Line-on-trans* : ⟦*Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*); *Line-on l1 p1*⟧
⟹ *Line-on l2 p1*
and *Line-on-rule* : *Line-on* (*Li p1 p2*) *p1* ∧ *Line-on* (*Li p1 p2*) *p2*

lemma(**in** *Incidence-Rule*) *Eq-not-trans* :
assumes *N* :
¬ *Eq obs1 obs2*
Eq obs2 obs3

shows ¬ *Eq obs1 obs3*
proof
assume *W* : *Eq obs1 obs3*
from *assms* **have** *P1* : *Eq obs3 obs2* **by** (*simp add*:*Eq-rev*)
from *W P1* **have** *P2* : *Eq obs1 obs2* **by** (*blast intro*:*Eq-trans*)
from *N P2* **show** *False* **by** *simp*
qed

lemma(**in** *Incidence-Rule*) *Line-rev* :
assumes ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
shows *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*)
proof −
from *assms* **have** *P1* : *Line-on* (*Li p1 p2*) *p1* ∧ *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p2 p1*) *p1* ∧ *Line-on* (*Li p2 p1*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2* **show** *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
qed

lemma(**in** *Incidence-Rule*) *Line-not-on-Point* :
assumes *N* :
¬ *Line-on* (*Li p1 p2*) *p3*
shows ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
proof
assume *W* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
have *P1* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
from *W P1* **have** *P2* : *Line-on* (*Li p1 p2*) *p3* **by** (*simp add*:*Point-Eq*)
from *N P2* **show** *False* **by** *simp*
qed

lemma(**in** *Incidence-Rule*) *Line-not-on-trans* :
assumes
Eq (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
¬ *Line-on l1 p1*
shows ¬ *Line-on l2 p1*
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Lin l2*) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Eq-rev*)
from *P1* **have** *P2* : *Line-on l2 p1* ⟹ *Line-on l1 p1* **by** (*simp add*:*Line-on-trans*)
from *assms P2* **show** ¬ *Line-on l2 p1* **by** *blast*
qed

lemma(**in** *Incidence-Rule*) *Line-on-rev* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
Line-on (*Li p1 p2*) *p3*
shows *Line-on* (*Li p1 p3*) *p2*
proof −

have *P1* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2 P3* **have** *P4* : *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P5* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *P4 P5* **show** *Line-on* (*Li p1 p3*) *p2* **by** (*simp add*:*Line-on-trans*)
qed

lemma(**in** *Incidence-Rule*) *Line-not-Eq* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Line-on* (*Li p1 p2*) *p3*
shows ¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*)
proof −
have *P1* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp add*:*Eq-rev*)
from *P1 P2* **have** *P3* : *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) ⟹
Line-on (*Li p1 p2*) *p3* **by** (*simp add*:*Line-on-trans*)
from *assms P3* **show** ¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** *blast*
qed

lemma(**in** *Incidence-Rule*) *Line-not-Eq-on* :
assumes *N* :
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*)
shows ¬ *Line-on* (*Li p1 p2*) *p3*
proof
assume *W* : *Line-on* (*Li p1 p2*) *p3*
have *P1* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *N W P1 P2 P3* **have** *P4* : *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *N P4* **show** *False* **by** *simp*
qed

lemma(**in** *Incidence-Rule*) *Line-unique-Point* :
assumes
¬ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
Line-on l1 p1 Line-on l1 p2
Line-on l2 p1 Line-on l2 p2
shows *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)

proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
⟹
Eq (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P1* **show** *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** *blast*
qed

lemma(**in** *Incidence-Rule*) *Line-not-on-Eq* :
assumes *N* :
¬ *Line-on l1 p1*
Line-on l2 p1
shows ¬ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
proof
assume *W* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
from *N W* **have** *P1* : *Line-on l1 p1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *N P1* **show** *False* **by** *simp*
qed

lemma(**in** *Incidence-Rule*) *Line-cross-not-on* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
¬ *Line-on* (*Li p1 p2*) *p3*
Line-on (*Li p2 p3*) *p4*
shows ¬ *Line-on* (*Li p1 p2*) *p4*
proof −
have *P1* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p2 p3*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2* **have** *P3* : *Line-on* (*Li p1 p2*) *p4* ⟹ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P4* : *Line-on* (*Li p2 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *P3 P4* **have** *P5* : *Line-on* (*Li p1 p2*) *p4* ⟹ *Line-on* (*Li p1 p2*) *p3* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *assms P5* **show** ¬ *Line-on* (*Li p1 p2*) *p4* **by** *blast*
qed

end

2 Order

locale *Definition-2* = *Incidence-Rule* +
fixes *Line-on-Seg* :: *Line* ⇒ *Segment* ⇒ *bool*
and *Bet-Point* :: *Segment* ⇒ *Point* ⇒ *bool*
and *Seg-on-Seg* :: *Segment* ⇒ *Segment* ⇒ *bool*
and *Line-on-Line* :: *Line* ⇒ *Line* ⇒ *bool*
and *Plane-sameside* :: *Line* ⇒ *Point* ⇒ *Point* ⇒ *bool*
and *Plane-diffside* :: *Line* ⇒ *Point* ⇒ *Point* ⇒ *bool*
assumes *Bet-Point-def* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ ¬ *Eq* (*Geos* (*Poi p1*)

add Emp) (*Geos* (*Poi p2*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
and *Bet-rev* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ *Bet-Point* (*Se p2 p1*) *p3*
and *Line-Bet-exist* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ ∃ *l*. *Line-on l p1* ∧ *Line-on l p2* ∧ *Line-on l p3*
and *Seg-rev* : *Eq* (*Geos* (*Seg* (*Se p1 p2*)) *add Emp*) (*Geos* (*Seg* (*Se p2 p1*)) *add Emp*)
and *Plane-sameside-def* : *Plane-sameside l1 p1 p2* ⟷ ¬ *Line-on-Seg l1* (*Se p1 p2*) ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
and *Plane-diffside-def* : *Plane-diffside l1 p1 p2* ⟷ (∃ *p*. *Bet-Point* (*Se p1 p2*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2*)

locale *Axiom-2* = *Definition-2* +
assumes *Bet-extension* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧ ⟹ ∃ *p*. *Bet-Point* (*Se p1 p*) *p2* ∧ *Line-on l1 p*
and *Bet-iff* : ⟦*Bet-Point* (*Se p1 p2*) *p3*⟧ ⟹ *Inv* (*Bet-Point* (*Se p2 p3*) *p1*) ∧ *Inv* (*Bet-Point* (*Se p3 p1*) *p2*)
and *Pachets-axiom* : ⟦¬ *Line-on* (*Li p1 p2*) *p3*; *Bet-Point* (*Se p1 p2*) *p4*; *Line-on l1 p4*;
¬ *Line-on l1 p1*; ¬ *Line-on l1 p2*; ¬ *Line-on l1 p3*⟧ ⟹
Line-on-Seg l1 (*Se p1 p3*) ∧ ¬ *Line-on-Seg l1* (*Se p2 p3*)
∨ *Line-on-Seg l1* (*Se p2 p3*) ∧ ¬ *Line-on-Seg l1* (*Se p1 p3*)
and *Seg-move-sameside* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*);
¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)⟧ ⟹
∃ *p*. *Eq* (*Geos* (*Seg* (*Se p3 p4*)) *add Emp*) (*Geos* (*Seg* (*Se p1 p*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p p2*) *p1* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)
and *Seg-move-diffside* : ⟦*Line-on l1 p1*; *Line-on l1 p2*; ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*);
¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)⟧ ⟹
∃ *p*. *Eq* (*Geos* (*Seg* (*Se p3 p4*)) *add Emp*) (*Geos* (*Seg* (*Se p1 p*)) *add Emp*) ∧ *Bet-Point* (*Se p p2*) *p1* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)

locale *Order-Rule* = *Axiom-2* +
assumes *Bet-Point-Eq* : ⟦*Bet-Point* (*Se p1 p2*) *p3*; *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)⟧ ⟹ *Bet-Point* (*Se p4 p2*) *p3*
and *Line-on-Seg-rule* : *Line-on-Seg l1* (*Se p1 p2*) ⟷ (∃ *p*. *Line-on l1 p* ∧ *Bet-Point* (*Se p1 p2*) *p*)
and *Seg-on-Seg-rule* : *Seg-on-Seg* (*Se p1 p2*) (*Se p3 p4*) ⟷ (∃ *p*. *Bet-Point* (*Se p1 p2*) *p* ∧ *Bet-Point* (*Se p3 p4*) *p*)
and *Line-on-Line-rule* : *Line-on-Line l1 l2* ⟷ (∃ *p*. *Line-on l1 p* ∧ *Line-on l2 p*)
and *Seg-Point-Eq* : ⟦*Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)⟧ ⟹ *Eq* (*Geos* (*Seg* (*Se p3 p1*)) *add Emp*) (*Geos* (*Seg* (*Se p3 p2*)) *add Emp*)

lemma(**in** *Order-Rule*) *Line-Bet-on* :
assumes
Bet-Point (*Se p1 p2*) *p3*
shows *Line-on* (*Li p1 p2*) *p3* **and** *Line-on* (*Li p2 p1*) *p3*
and *Line-on* (*Li p2 p3*) *p1* **and** *Line-on* (*Li p3 p2*) *p1*
and *Line-on* (*Li p1 p3*) *p2* **and** *Line-on* (*Li p3 p1*) *p2*
proof −
from *assms* **have** ∃ *l*. *Line-on l p1* ∧ *Line-on l p2* ∧ *Line-on l p3* **by** (*blast intro*:*Line-Bet-exist*)
then obtain *l1* :: *Line* **where** *P1* : *Line-on l1 p1* ∧ *Line-on l1 p2* ∧ *Line-on l1 p3* **by** *blast*
from *assms* **have** *P2* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
have *P3* : *Line-on* (*Li p1 p2*) *p1* ∧ *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *P1* **have** *P4* : *Line-on l1 p1* **by** *simp*
from *P1* **have** *P5* : *Line-on l1 p2* **by** *simp*
from *P2 P3 P4 P5* **have** *P6* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1 P6* **show** *P7* : *Line-on* (*Li p1 p2*) *p3* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** *P8* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P2 P7 P8* **show** *Line-on* (*Li p1 p3*) *p2* **by** (*blast intro*:*Line-on-rev Eq-rev*)
from *P2 P7 P8* **show** *Line-on* (*Li p3 p1*) *p2* **by** (*blast intro*:*Line-on-trans Line-on-rev Eq-rev Line-rev*)
from *P2 P7* **show** *Line-on* (*Li p2 p1*) *p3* **by** (*blast intro*:*Line-on-trans Line-rev*)
from *assms* **have** *P9* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P2 P7 P9* **show** *Line-on* (*Li p2 p3*) *p1* **by** (*blast intro*:*Line-on-rev Line-on-trans Line-rev Eq-rev*)
from *P9* **have** *P10* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P2 P7 P8 P10* **show** *Line-on* (*Li p3 p2*) *p1* **by** (*blast intro*:*Line-on-rev Bet-Point-def Line-on-trans Eq-rev Line-rev*)
qed

lemma(**in** *Order-Rule*) *Line-Bet-not-Eq* :
assumes *N* :
Bet-Point (*Se p1 p2*) *p3*
¬ *Line-on* (*Li p1 p2*) *p4*
shows ¬ *Eq* (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*)
proof
assume *W* : *Eq* (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*)
have *P1* : *Line-on* (*Li p4 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *W P1* **have** *P2* : *Line-on* (*Li p4 p2*) *p3* **by** (*simp add*:*Line-on-trans*)
have *P3* : *Line-on* (*Li p4 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P4* : *Line-on* (*Li p1 p2*) *p3* **by** (*simp add*:*Line-Bet-on*)
have *P5* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P6* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)

by (*simp add*:*Bet-Point-def*)
from *P2 P3 P4 P5 P6* **have** *P7* : *Eq* (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P8* : *Line-on* (*Li p4 p2*) *p4* **by** (*simp add*:*Line-on-rule*)
from *P7 P8* **have** *P9* : *Line-on* (*Li p1 p2*) *p4* **by** (*simp add*:*Line-on-trans*)
from *N P9* **show** *False* **by** *simp*
qed

Theorem3

theorem(**in** *Order-Rule*) *Seg-density* :
assumes ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
shows ∃ *p*. *Bet-Point* (*Se A C*) *p*
proof −
have ∃ *p q r*. ¬ *Line-on* (*Li A C*) *p* ∧ ¬ *Line-on* (*Li A C*) *q* ∧ ¬ *Line-on* (*Li A C*) *r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *E* :: *Point* **where** *P1* : ¬ *Line-on* (*Li A C*) *E* **by** *blast*
then have *P2* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Line-not-on-Point*)
have *P3* : *Line-on* (*Li A E*) *A* ∧ *Line-on* (*Li A E*) *E* **by** (*simp add*:*Line-on-rule*)
from *P2 P3* **have** ∃ *p*. *Bet-Point* (*Se A p*) *E* ∧ *Line-on* (*Li A E*) *p* **by** (*simp add*:*Bet-extension*)
then obtain *F* :: *Point* **where** *P4* : *Bet-Point* (*Se A F*) *E* ∧ *Line-on* (*Li A E*) *F* **by** *blast*
then have *P5* : *Line-on* (*Li A F*) *E* **by** (*simp add*:*Line-Bet-on*)
from *P4* **have** *P6* : *Bet-Point* (*Se A F*) *E* **by** *simp*
from *P6* **have** *P7* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P2 P4 P6 P7* **have** *P8* : *Line-on* (*Li A E*) *F* **by** (*simp add*:*Line-on-rev*)
from *assms P1* **have** *P9* : ¬ *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li A E*)) *add Emp*) **by** (*simp add*:*Line-not-Eq*)
have *P10* : *Line-on* (*Li A F*) *A* **by** (*simp add*:*Line-on-rule*)
from *P2 P3 P5 P10* **have** *P11* : *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li A F*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *P9 P11* **have** *P12* : ¬ *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li A F*)) *add Emp*) **by** (*simp add*:*Eq-not-trans*)
from *assms P7 P12* **have** *P13* : ¬ *Line-on* (*Li A C*) *F* **by** (*simp add*:*Line-not-Eq-on*)
from *assms P7 P13* **have** *P14* : ¬ *Line-on* (*Li A F*) *C* **by** (*blast intro*:*Line-on-rev*)
have *Line-on* (*Li A F*) *F* **by** (*simp add*:*Line-on-rule*)
then have *P15* : *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on* (*Li A F*) *C* **by** (*simp add*:*Point-Eq*)
from *P14 P15* **have** *P16* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
have *P17* : *Line-on* (*Li F C*) *F* ∧ *Line-on* (*Li F C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P16 P17* **have** ∃ *p*. *Bet-Point* (*Se F p*) *C* ∧ *Line-on* (*Li F C*) *p* **by** (*simp add*:*Bet-extension*)

then obtain G :: *Point* **where** *P18* : *Bet-Point* (*Se F G*) *C* ∧ *Line-on* (*Li F C*) *G* **by** *blast*
from *P18* **have** *P19* : *Line-on* (*Li F G*) *C* **by** (*simp add*:*Line-Bet-on*)
from *P18* **have** *P20* : *Bet-Point* (*Se F G*) *C* **by** *simp*
then have *P21* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P20* **have** *P22* : *Line-on* (*Li F C*) *G* **by** (*simp add*:*Line-Bet-on*)
from *P7 P14 P21 P22* **have** *P23* : ¬ *Line-on* (*Li A F*) *G* **by** (*simp add*:*Line-cross-not-on*)
from *P6 P23* **have** *P24* : ¬ *Eq* (*Geos* (*Lin* (*Li G E*)) *add Emp*) (*Geos* (*Lin* (*Li G F*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P5* **have** *P25* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) ⟹ *Line-on* (*Li A F*) *G* **by** (*simp add*:*Point-Eq*)
from *P23 P25* **have** *P26* : ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P21* **have** *P27* : ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P24 P26 P27* **have** *P28* : ¬ *Line-on* (*Li G E*) *F* **by** (*simp add*:*Line-not-Eq-on*)
from *P26 P28* **have** *P29* : ¬ *Line-on* (*Li E G*) *F* **by** (*blast intro*:*Line-rev Line-on-trans Eq-rev*)
have *P30* : *Line-on* (*Li E G*) *E* **by** (*simp add*:*Line-on-rule*)
have *P31* : *Line-on* (*Li A E*) *E* **by** (*simp add*:*Line-on-rule*)
have *P32* : *Line-on* (*Li A E*) *A* **by** (*simp add*:*Line-on-rule*)
from *P2 P30 P31 P32* **have** *P33* : *Line-on* (*Li E G*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li E G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P8 P33* **have** *P34* : *Line-on* (*Li E G*) *A* ⟹ *Line-on* (*Li E G*) *F* **by** (*simp add*:*Line-on-trans*)
from *P29 P34* **have** *P35* : ¬ *Line-on* (*Li E G*) *A* **by** *blast*
have *P36* : *Line-on* (*Li E G*) *G* **by** (*simp add*:*Line-on-rule*)
have *P37* : *Line-on* (*Li F G*) *G* **by** (*simp add*:*Line-on-rule*)
from *P20* **have** *P38* : ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P19 P36 P37 P38* **have** *P39* : *Line-on* (*Li E G*) *C* ⟹ *Eq* (*Geos* (*Lin* (*Li F G*)) *add Emp*) (*Geos* (*Lin* (*Li E G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P40* : *Line-on* (*Li F G*) *F* **by** (*simp add*:*Line-on-rule*)
from *P39 P40* **have** *P41* : *Line-on* (*Li E G*) *C* ⟹ *Line-on* (*Li E G*) *F* **by** (*simp add*:*Line-on-trans*)
from *P29 P41* **have** *P42* : ¬ *Line-on* (*Li E G*) *C* **by** *blast*
from *P6 P14 P29 P30 P35 P42* **have** *P43* : *Line-on-Seg* (*Li E G*) (*Se A C*) ∧ ¬ *Line-on-Seg* (*Li E G*) (*Se F C*) ∨ *Line-on-Seg* (*Li E G*) (*Se F C*) ∧ ¬ *Line-on-Seg* (*Li E G*) (*Se A C*) **by** (*simp add*:*Pachets-axiom*)
then have *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ ∃ *p*. *Line-on* (*Li E G*) *p* ∧ *Bet-Point* (*Se F C*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain D :: *Point* **where** *P44* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ *Line-on* (*Li E G*) *D* ∧ *Bet-Point* (*Se F C*) *D* **by** *blast*
from *P44* **have** *P46* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ *Bet-Point* (*Se F C*) *D* **by** *simp*
from *P46* **have** *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P46* **have** *P47* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ *Line-on* (*Li F D*) *C*

by (*simp add*:*Line-Bet-on*)
have *P48* : *Line-on* (*Li F D*) *F* **by** (*simp add*:*Line-on-rule*)
have *P49* : *Line-on* (*Li F G*) *F* **by** (*simp add*:*Line-on-rule*)
from *P16 P19 P47 P48 P49* **have** *P50* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹
Eq (*Geos* (*Lin* (*Li F D*)) *add Emp*) (*Geos* (*Lin* (*Li F G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P51* : *Line-on* (*Li F D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P50 P51* **have** *P52* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ *Line-on* (*Li F G*) *D* **by** (*simp add*:*Line-on-trans*)
have *P53* : *Line-on* (*Li F G*) *G* **by** (*simp add*:*Line-on-rule*)
have *P54* : *Line-on* (*Li E G*) *G* **by** (*simp add*:*Line-on-rule*)
from *P46* **have** *P55* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi G*) *add Emp*)
⟹ *Bet-Point* (*Se F C*) *G* **by** (*simp add*:*Point-Eq*)
from *P20* **have** *Inv* (*Bet-Point* (*Se G C*) *F*) ∧ *Inv* (*Bet-Point* (*Se C F*) *G*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se C F*) *G* **by** (*simp add*:*Inv-def*)
then have *P56* : ¬ *Bet-Point* (*Se F C*) *G* **by** (*blast intro*:*Bet-rev*)
from *P55 P56* **have** *P57* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹ ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** *blast*
from *P44 P52 P53 P54 P57* **have** *P58* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹
Eq (*Geos* (*Lin* (*Li E G*)) *add Emp*) (*Geos* (*Lin* (*Li F G*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *P26* **have** *P59* : *Eq* (*Geos* (*Lin* (*Li E G*)) *add Emp*) (*Geos* (*Lin* (*Li G E*)) *add Emp*) **by** (*simp add*:*Line-rev Eq-rev*)
from *P27* **have** *P60* : *Eq* (*Geos* (*Lin* (*Li F G*)) *add Emp*) (*Geos* (*Lin* (*Li G F*)) *add Emp*) **by** (*simp add*:*Line-rev Eq-rev*)
from *P58 P59 P60* **have** *P61* : *Line-on-Seg* (*Li E G*) (*Se F C*) ⟹
Eq (*Geos* (*Lin* (*Li G E*)) *add Emp*) (*Geos* (*Lin* (*Li G F*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P24 P61* **have** *P62* : ¬ *Line-on-Seg* (*Li E G*) (*Se F C*) **by** *blast*
from *P43 P62* **have** *Line-on-Seg* (*Li E G*) (*Se A C*) ∧ ¬ *Line-on-Seg* (*Li E G*) (*Se F C*) **by** *blast*
then have ∃ *p*. *Line-on* (*Li E G*) *p* ∧ *Bet-Point* (*Se A C*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
thus ∃ *p*. *Bet-Point* (*Se A C*) *p* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Line-Bet-not-on* :
assumes
Line-on (*Li p1 p2*) *p3*
¬ *Line-on* (*Li p1 p2*) *p4*
Bet-Point (*Se p3 p4*) *p5*
shows *Inv* (*Line-on* (*Li p1 p2*) *p5*)
proof −
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P1* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P2* : *Line-on* (*Li p3 p5*) *p4* **by** (*simp add*:*Line-Bet-on*)

have *P3* : *Line-on* (*Li p3 p5*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li p3 p5*) *p5* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P3 P4* **have** *P5* : *Line-on* (*Li p1 p2*) *p5* ⟹ *Eq* (*Geos* (*Lin* (*Li p3 p5*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P2 P5* **have** *P6* : *Line-on* (*Li p1 p2*) *p5* ⟹ *Line-on* (*Li p1 p2*) *p4* **by** (*simp add*:*Line-on-trans*)
from *assms P6* **have** ¬ *Line-on* (*Li p1 p2*) *p5* **by** *blast*
thus *Inv* (*Line-on* (*Li p1 p2*) *p5*) **by** (*simp add*:*Inv-def*)
qed

lemma(**in** *Order-Rule*) *Line-not-on-ex* :
assumes *N* :
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
¬ *Line-on* (*Li p1 p2*) *p3*
Line-on (*Li p1 p4*) *p2*
shows ¬ *Line-on* (*Li p1 p4*) *p3*
proof
assume *W* : *Line-on* (*Li p1 p4*) *p3*
have *P1* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p1 p4*) *p1* **by** (*simp add*:*Line-on-rule*)
from *N P1 P2 P3* **have** *P4* : *Eq* (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *W P4* **have** *P5* : *Line-on* (*Li p1 p2*) *p3* **by** (*simp add*:*Line-on-trans*)
from *N P5* **show** *False* **by** *simp*
qed

lemma(**in** *Order-Rule*) *Line-on-dens* :
assumes
¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
Line-on (*Li p1 p2*) *p3*
Line-on (*Li p1 p4*) *p3*
shows *Line-on* (*Li p2 p4*) *p3*
proof −
have *P1* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p1 p4*) *p1* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2* **have** *P3* : *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P4* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *P3 P4* **have** *P5* : *Line-on* (*Li p1 p4*) *p2* **by** (*simp add*:*Line-on-trans*)
have *P6* : *Line-on* (*Li p1 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P7* : *Line-on* (*Li p2 p4*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P8* : *Line-on* (*Li p2 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
from *assms P5 P6 P7 P8* **have** *P9* : *Eq* (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P9* **show** *Line-on* (*Li p2 p4*) *p3* **by** (*simp add*:*Line-on-trans*)
qed

lemma(**in** *Order-Rule*) *Bet-case-lemma1* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Bet-Point* (*Se B A*) *C*
¬ *Bet-Point* (*Se C B*) *A*
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
¬ *Line-on* (*Li A C*) *D*
Bet-Point (*Se B G*) *D*
shows ∃ *p*. *Line-on* (*Li A D*) *p* ∧ *Bet-Point* (*Se G C*) *p*
proof −
have *P1* : *Line-on* (*Li A C*) *A* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2* **have** *P3* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P3* **have** *P4* : *Line-on* (*Li A C*) *B* **by** (*simp add*:*Line-on-trans*)
have *P11* : *Line-on* (*Li B G*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms P2 P4 P11* **have** *P12* : *Line-on* (*Li B G*) *C* ⟹ *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms* **have** *P13* : *Line-on* (*Li B G*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P12 P13* **have** *P14* : *Line-on* (*Li B G*) *C* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *assms P14* **have** *P15* : ¬ *Line-on* (*Li B G*) *C* **by** *blast*
have *P16* : *Line-on* (*Li A D*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P4 P16* **have** *P17* : *Line-on* (*Li A D*) *B* ⟹ *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P18* : *Line-on* (*Li A D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P17 P18* **have** *P19* : *Line-on* (*Li A D*) *B* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *assms P19* **have** *P20* : ¬ *Line-on* (*Li A D*) *B* **by** *blast*
from *assms P1 P2 P16* **have** *P21* : *Line-on* (*Li A D*) *C* ⟹ *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P18 P21* **have** *P22* : *Line-on* (*Li A D*) *C* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *assms P22* **have** *P23* : ¬ *Line-on* (*Li A D*) *C* **by** *blast*
from *assms P1 P4 P11* **have** *P24* : *Line-on* (*Li B G*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P13 P24* **have** *P25* : *Line-on* (*Li B G*) *A* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *assms P25* **have** *P26* : ¬ *Line-on* (*Li B G*) *A* **by** *blast*
have *P27* : *Line-on* (*Li B G*) *G* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P28* : ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P13 P18 P27 P28* **have** *P29* : *Line-on* (*Li A D*) *G* ⟹ *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li B G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P16 P29* **have** *P30* : *Line-on* (*Li A D*) *G* ⟹ *Line-on* (*Li B G*) *A* **by**

(*simp add*:*Line-on-trans*)
from *P26 P30* **have** *P31* : ¬ *Line-on* (*Li A D*) *G* **by** *blast*
from *assms P15 P18 P20 P23 P31* **have** *P32* : *Line-on-Seg* (*Li A D*) (*Se B C*) ∧ ¬ *Line-on-Seg* (*Li A D*) (*Se G C*)
∨ *Line-on-Seg* (*Li A D*) (*Se G C*) ∧ ¬ *Line-on-Seg* (*Li A D*) (*Se B C*) **by** (*simp add*:*Pachets-axiom*)
have *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ ∃ *p*. *Line-on* (*Li A D*) *p* ∧ *Bet-Point* (*Se B C*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *A2* :: *Point* **where** *P33* : *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ *Line-on* (*Li A D*) *A2* ∧ *Bet-Point* (*Se B C*) *A2* **by** *blast*
from *assms* **have** *P34* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P34* **have** *P35* : ¬ *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Line-not-Eq*)
have *P36* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P37* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P2 P4 P36 P37* **have** *P38* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P33* **have** *P39* : *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ *Bet-Point* (*Se B C*) *A2* **by** *simp*
then have *P40* : *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ *Line-on* (*Li B C*) *A2* **by** (*simp add*:*Line-Bet-on*)
from *P38 P40* **have** *P41* : *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ *Line-on* (*Li A C*) *A2* **by** (*simp add*:*Line-on-trans*)
from *P1 P16 P33 P35 P41* **have** *P42* : *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P39 P42* **have** *P43* : *Line-on-Seg* (*Li A D*) (*Se B C*) ⟹ *Bet-Point* (*Se B C*) *A* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *P44* : ¬ *Bet-Point* (*Se B C*) *A* **by** (*blast intro*:*Bet-rev*)
from *P43 P44* **have** *P45* : ¬ *Line-on-Seg* (*Li A D*) (*Se B C*) **by** *blast*
from *P32 P45* **have** *Line-on-Seg* (*Li A D*) (*Se G C*) ∧ ¬ *Line-on-Seg* (*Li A D*) (*Se B C*) **by** *blast*
thus ∃ *p*. *Line-on* (*Li A D*) *p* ∧ *Bet-Point* (*Se G C*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
qed

lemma(**in** *Order-Rule*) *Bet-case-lemma2* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Bet-Point* (*Se B A*) *C*
¬ *Bet-Point* (*Se C B*) *A*
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
shows *Bet-Point* (*Se A C*) *B*
proof −
have *P1* : *Line-on* (*Li A C*) *A* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)

from *assms P1 P2* **have** *P3* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P3* **have** *P4* : *Line-on* (*Li A C*) *B* **by** (*simp add*:*Line-on-trans*)
have ∃ *p q r*. ¬ *Line-on* (*Li A C*) *p* ∧ ¬ *Line-on* (*Li A C*) *q* ∧ ¬ *Line-on* (*Li A C*) *r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *D* :: *Point* **where** *P5* : ¬ *Line-on* (*Li A C*) *D* **by** *blast*
from *P4* **have** *P6* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Point-Eq*)
from *P5 P6* **have** *P7* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
have *P8* : *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P7 P8 P9* **have** ∃ *p*. *Bet-Point* (*Se B p*) *D* ∧ *Line-on* (*Li B D*) *p* **by** (*simp add*:*Bet-extension*)
then obtain *G* :: *Point* **where** *P10* : *Bet-Point* (*Se B G*) *D* **by** *blast*
from *assms P5 P10* **have** ∃ *p*. *Line-on* (*Li A D*) *p* ∧ *Bet-Point* (*Se G C*) *p* **by** (*simp add*:*Bet-case-lemma1*)
then obtain *E* :: *Point* **where** *P11* : *Line-on* (*Li A D*) *E* ∧ *Bet-Point* (*Se G C*) *E* **by** *blast*
from *assms* **have** *P12* : ¬ *Bet-Point* (*Se B C*) *A* **by** (*blast intro*:*Bet-rev*)
from *assms* **have** *P13* : ¬ *Bet-Point* (*Se A B*) *C* **by** (*blast intro*:*Bet-rev*)
from *assms* **have** *P14* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P15* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P16* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P14* **have** *P17* : *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P5 P17* **have** *P18* : ¬ *Line-on* (*Li C A*) *D* **by** (*simp add*:*Line-not-on-trans*)
from *assms P10 P12 P13 P14 P15 P16 P18* **have** ∃ *p*. *Line-on* (*Li C D*) *p* ∧ *Bet-Point* (*Se G A*) *p* **by** (*simp add*:*Bet-case-lemma1*)
then obtain *F* :: *Point* **where** *P19* : *Line-on* (*Li C D*) *F* ∧ *Bet-Point* (*Se G A*) *F* **by** *blast*
have *P20* : *Line-on* (*Li B G*) *B* **by** (*simp add*:*Line-on-rule*)
have *P21* : *Line-on* (*Li B G*) *G* **by** (*simp add*:*Line-on-rule*)
from *P10* **have** *P22* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P4 P20 P21 P22* **have** *P23* : *Line-on* (*Li A C*) *G* ⟹ *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P10* **have** *P24* : *Line-on* (*Li B G*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P23 P24* **have** *P25* : *Line-on* (*Li A C*) *G* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P5 P25* **have** *P26* : ¬ *Line-on* (*Li A C*) *G* **by** *blast*
from *P11* **have** *P27* : *Bet-Point* (*Se C G*) *E* **by** (*blast intro*:*Bet-rev*)

have *P28* : *Line-on* (*Li C G*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2 P28* **have** *P29* : *Line-on* (*Li C G*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li C G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P30* : *Line-on* (*Li C G*) *G* **by** (*simp add*:*Line-on-rule*)
from *P29 P30* **have** *P31* : *Line-on* (*Li C G*) *A* ⟹ *Line-on* (*Li A C*) *G* **by** (*simp add*:*Line-on-trans*)
from *P26 P31* **have** *P32* : ¬ *Line-on* (*Li C G*) *A* **by** *blast*
from *P27 P32* **have** ¬ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li A G*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
then have *P33* : ¬ *Eq* (*Geos* (*Lin* (*Li A G*)) *add Emp*) (*Geos* (*Lin* (*Li A E*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P19* **have** *P34* : *Bet-Point* (*Se A G*) *F* **by** (*blast intro*:*Bet-rev*)
then have *P35* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P27* **have** *P36* : *Line-on* (*Li C G*) *E* **by** (*simp add*:*Line-Bet-on*)
then have *P37* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Line-on* (*Li C G*) *A* **by** (*simp add*:*Point-Eq*)
from *P32 P37* **have** *P38* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P33 P35 P38* **have** *P39* : ¬ *Line-on* (*Li A G*) *E* **by** (*simp add*:*Line-not-Eq-on*)
from *P14 P26 P35* **have** *P40* : ¬ *Line-on* (*Li A G*) *C* **by** (*blast intro*:*Line-on-rev*)
from *P34 P40* **have** *P41* : ¬ *Eq* (*Geos* (*Lin* (*Li C F*)) *add Emp*) (*Geos* (*Lin* (*Li C G*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P34* **have** *P42* : *Line-on* (*Li A G*) *F* **by** (*simp add*:*Line-Bet-on*)
then have *P43* : *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on* (*Li A G*) *C* **by** (*simp add*:*Point-Eq*)
from *P40 P43* **have** *P44* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P27* **have** *P45* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P41 P44 P45* **have** *P46* : ¬ *Line-on* (*Li C F*) *G* **by** (*simp add*:*Line-not-Eq-on*)
from *P35* **have** *P47* : *Eq* (*Geos* (*Lin* (*Li A G*)) *add Emp*) (*Geos* (*Lin* (*Li G A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P40 P47* **have** *P48* : ¬ *Line-on* (*Li G A*) *C* **by** (*simp add*:*Line-not-on-trans*)
from *P19* **have** *P49* : *Bet-Point* (*Se G A*) *F* **by** *simp*
from *P48 P49* **have** *P50* : ¬ *Eq* (*Geos* (*Lin* (*Li C F*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *assms P44 P50* **have** *P51* : ¬ *Line-on* (*Li C F*) *A* **by** (*simp add*:*Line-not-Eq-on*)
have *P52* : *Line-on* (*Li C F*) *C* **by** (*simp add*:*Line-on-rule*)
from *P27* **have** *P53* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P28 P36 P52 P53* **have** *P54* : *Line-on* (*Li C F*) *E* ⟹ *Eq* (*Geos* (*Lin* (*Li C G*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P30 P54* **have** *P55* : *Line-on* (*Li C F*) *E* ⟹ *Line-on* (*Li C F*) *G* **by** (*simp add*:*Line-on-trans*)
from *P46 P55* **have** *P56* : ¬ *Line-on* (*Li C F*) *E* **by** *blast*
have *P57* : *Line-on* (*Li C F*) *F* **by** (*simp add*:*Line-on-rule*)
from *P34 P39 P46 P51 P56 P57* **have** *P58* : *Line-on-Seg* (*Li C F*) (*Se A E*) ∧ ¬ *Line-on-Seg* (*Li C F*) (*Se G E*)

$\vee$ *Line-on-Seg* (*Li C F*) (*Se G E*) $\wedge$ $\neg$ *Line-on-Seg* (*Li C F*) (*Se A E*) **by** (*simp add*:*Pachets-axiom*)
have *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ $\exists$ *p*. *Line-on* (*Li C F*) *p* $\wedge$ *Bet-Point* (*Se G E*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *D2* :: *Point* **where** *P59* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Line-on* (*Li C F*) *D2* $\wedge$ *Bet-Point* (*Se G E*) *D2* **by** *blast*
then have *P60* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Bet-Point* (*Se G E*) *D2* **by** *simp*
then have *P61* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Line-on* (*Li G E*) *D2* **by** (*simp add*:*Line-Bet-on*)
have *P62* : *Line-on* (*Li G E*) *G* **by** (*simp add*:*Line-on-rule*)
have *P63* : *Line-on* (*Li G E*) *E* **by** (*simp add*:*Line-on-rule*)
from *P27* **have** *P64* : $\neg$ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P27* **have** *P66* : *Line-on* (*Li G E*) *C* **by** (*simp add*:*Line-Bet-on*)
from *P59* **have** *P67* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Line-on* (*Li C F*) *D2* **by** *simp*
from *P60* **have** *P68* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se G E*) *C* **by** (*simp add*:*Point-Eq*)
from *P27* **have** *Inv* (*Bet-Point* (*Se G E*) *C*) $\wedge$ *Inv* (*Bet-Point* (*Se E C*) *G*) **by** (*simp add*:*Bet-iff*)
then have *P69* : $\neg$ *Bet-Point* (*Se G E*) *C* $\wedge$ $\neg$ *Bet-Point* (*Se E C*) *G* **by** (*simp add*:*Inv-def*)
from *P68 P69* **have** *P70* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
from *P52 P61 P66 P67 P70* **have** *P71* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li G E*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P63 P71* **have** *P72* : *Line-on-Seg* (*Li C F*) (*Se G E*) $\Longrightarrow$ *Line-on* (*Li C F*) *E* **by** (*simp add*:*Line-on-trans*)
from *P56 P72* **have** *P73* : $\neg$ *Line-on-Seg* (*Li C F*) (*Se G E*) **by** *blast*
from *P58 P73* **have** *Line-on-Seg* (*Li C F*) (*Se A E*) $\wedge$ $\neg$ *Line-on-Seg* (*Li C F*) (*Se G E*) **by** *blast*
then have $\exists$ *p*. *Line-on* (*Li C F*) *p* $\wedge$ *Bet-Point* (*Se A E*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *D3* :: *Point* **where** *P74* : *Line-on* (*Li C F*) *D3* $\wedge$ *Bet-Point* (*Se A E*) *D3* **by** *blast*
then have *P75* : *Line-on* (*Li C F*) *D3* **by** *simp*
from *P74* **have** *P76* : *Bet-Point* (*Se A E*) *D3* **by** *simp*
then have *P77* : *Line-on* (*Li A E*) *D3* **by** (*simp add*:*Line-Bet-on*)
from *P19* **have** *P78* : *Line-on* (*Li C D*) *F* **by** *simp*
from *P2* **have** *P79* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Point-Eq*)
from *P5 P79* **have** *P80* : $\neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P44 P78 P80* **have** *P81* : *Line-on* (*Li C F*) *D* **by** (*simp add*:*Line-on-rev*)
from *P11* **have** *P82* : *Line-on* (*Li A D*) *E* **by** *simp*
from *P1* **have** *P83* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Point-Eq*)

from *P5 P83* **have** *P84* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P38 P82 P84* **have** *P85* : *Line-on* (*Li A E*) *D* **by** (*simp add*:*Line-on-rev*)
have *P86* : *Line-on* (*Li A E*) *E* **by** (*simp add*:*Line-on-rule*)
then have *P87* : *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) ⟹ *Line-on* (*Li C F*) *E* **by** (*simp add*:*Line-on-trans*)
from *P56 P87* **have** *P88* : ¬ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) **by** *blast*
from *P75 P77 P81 P85 P88* **have** *P89* : *Eq* (*Geos* (*Poi D3*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P76 P89* **have** *P90* : *Bet-Point* (*Se A E*) *D* **by** (*simp add*:*Point-Eq*)
have *P91* : *Line-on* (*Li A E*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms P1 P2 P91* **have** *P92* : *Line-on* (*Li A E*) *C* ⟹ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P85 P92* **have** *P93* : *Line-on* (*Li A E*) *C* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P5 P93* **have** *P94* : ¬ *Line-on* (*Li A E*) *C* **by** *blast*
from *assms P1 P4 P20* **have** *P95* : *Line-on* (*Li B G*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P24 P95* **have** *P96* : *Line-on* (*Li B G*) *A* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P5 P96* **have** *P97* : ¬ *Line-on* (*Li B G*) *A* **by** *blast*
from *assms P2 P4 P20* **have** *P98* : *Line-on* (*Li B G*) *C* ⟹ *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P24 P98* **have** *P99* : *Line-on* (*Li B G*) *C* ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P5 P99* **have** *P100* : ¬ *Line-on* (*Li B G*) *C* **by** *blast*
from *P21 P62 P63 P64* **have** *P101* : *Line-on* (*Li B G*) *E* ⟹ *Eq* (*Geos* (*Lin* (*Li G E*)) *add Emp*) (*Geos* (*Lin* (*Li B G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P66 P101* **have** *P102* : *Line-on* (*Li B G*) *E* ⟹ *Line-on* (*Li B G*) *C* **by** (*simp add*:*Line-on-trans*)
from *P100 P102* **have** *P103* : ¬ *Line-on* (*Li B G*) *E* **by** *blast*
from *P24 P90 P94 P97 P100 P103* **have** *P104* : *Line-on-Seg* (*Li B G*) (*Se A C*) ∧ ¬ *Line-on-Seg* (*Li B G*) (*Se E C*)
∨ *Line-on-Seg* (*Li B G*) (*Se E C*) ∧ ¬ *Line-on-Seg* (*Li B G*) (*Se A C*) **by** (*simp add*:*Pachets-axiom*)
have *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ ∃ *p*. *Line-on* (*Li B G*) *p* ∧ *Bet-Point* (*Se E C*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *B2* :: *Point* **where** *P105* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Line-on* (*Li B G*) *B2* ∧ *Bet-Point* (*Se E C*) *B2* **by** *blast*
then have *P106* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Bet-Point* (*Se E C*) *B2* **by** *simp*
then have *P107* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Line-on* (*Li E C*) *B2* **by** (*simp add*:*Line-Bet-on*)
from *P105* **have** *P108* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Line-on* (*Li B G*) *B2* **by** *simp*
have *P109* : *Line-on* (*Li E C*) *E* **by** (*simp add*:*Line-on-rule*)
have *P110* : *Line-on* (*Li E C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P28 P36 P53 P109 P110* **have** *P111* : *Eq* (*Geos* (*Lin* (*Li C G*)) *add Emp*)

(*Geos* (*Lin* (*Li E C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P30 P111* **have** *P112* : *Line-on* (*Li E C*) *G* **by** (*simp add*:*Line-on-trans*)
from *P106* **have** *P113* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) ⟹
Bet-Point (*Se E C*) *G* **by** (*simp add*:*Point-Eq*)
from *P69 P113* **have** *P114* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ ¬ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** *blast*
from *P21 P107 P108 P112 P114* **have** *P115* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Eq* (*Geos* (*Lin* (*Li E C*)) *add Emp*) (*Geos* (*Lin* (*Li B G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P109 P115* **have** *P116* : *Line-on-Seg* (*Li B G*) (*Se E C*) ⟹ *Line-on* (*Li B G*) *E* **by** (*simp add*:*Line-on-trans*)
from *P103 P116* **have** *P117* : ¬ *Line-on-Seg* (*Li B G*) (*Se E C*) **by** *blast*
from *P104 P117* **have** *Line-on-Seg* (*Li B G*) (*Se A C*) **by** *blast*
then have ∃ *p*. *Line-on* (*Li B G*) *p* ∧ *Bet-Point* (*Se A C*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *B3* :: *Point* **where** *P118* : *Line-on* (*Li B G*) *B3* ∧ *Bet-Point* (*Se A C*) *B3* **by** *blast*
from *P24* **have** *P119* : *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) ⟹ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P5 P119* **have** *P120* : ¬ *Eq* (*Geos* (*Lin* (*Li B G*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** *blast*
from *P118* **have** *P121* : *Line-on* (*Li B G*) *B3* **by** *simp*
from *P118* **have** *P122* : *Bet-Point* (*Se A C*) *B3* **by** *simp*
then have *P123* : *Line-on* (*Li A C*) *B3* **by** (*simp add*:*Line-Bet-on*)
from *P4 P20 P120 P121 P123* **have** *P124* : *Eq* (*Geos* (*Poi B3*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P122 P124* **show** *Bet-Point* (*Se A C*) *B* **by** (*simp add*:*Point-Eq*)
qed

Theorem4

lemma(**in** *Order-Rule*) *Bet-case* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
shows *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C*
proof −
from *assms* **have** *P1* : ¬ *Bet-Point* (*Se B A*) *C* ∧ ¬ *Bet-Point* (*Se C B*) *A* ⟹ *Bet-Point* (*Se A C*) *B* **by** (*simp add*:*Bet-case-lemma2*)
from *assms* **have** *P2* : ¬ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se A C*) *B* ⟹ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Bet-case-lemma2*)
from *assms* **have** *P3* : ¬ *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se B A*) *C* ⟹ *Bet-Point* (*Se C B*) *A* **by** (*simp add*:*Bet-case-lemma2*)
from *P1 P2 P3* **show** *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Bet-case-fact* :
assumes
Bet-Point (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C*
shows
Bet-Point (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C*
∨ ¬ *Bet-Point* (*Se A C*) *B* ∧ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C*
∨ ¬ *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* ∧ *Bet-Point* (*Se B A*) *C*
proof −
have *Bet-Point* (*Se A C*) *B* ⟹ *Inv* (*Bet-Point* (*Se C B*) *A*) ∧ *Inv* (*Bet-Point* (*Se B A*) *C*) **by** (*simp add*:*Bet-iff*)
then have *P1* : *Bet-Point* (*Se A C*) *B* ⟹ ¬ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Inv-def*)
have *Bet-Point* (*Se C B*) *A* ⟹ *Inv* (*Bet-Point* (*Se A C*) *B*) ∧ *Inv* (*Bet-Point* (*Se B A*) *C*) **by** (*simp add*:*Bet-iff*)
then have *P2* : *Bet-Point* (*Se C B*) *A* ⟹ ¬ *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Inv-def*)
have *Bet-Point* (*Se B A*) *C* ⟹ *Inv* (*Bet-Point* (*Se A C*) *B*) ∧ *Inv* (*Bet-Point* (*Se C B*) *A*) **by** (*simp add*:*Bet-iff*)
then have *P3* : *Bet-Point* (*Se B A*) *C* ⟹ ¬ *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* **by** (*simp add*:*Inv-def*)
from *assms P1 P2 P3* **show** *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C*
∨ ¬ *Bet-Point* (*Se A C*) *B* ∧ *Bet-Point* (*Se C B*) *A* ∧ ¬ *Bet-Point* (*Se B A*) *C*
∨ ¬ *Bet-Point* (*Se A C*) *B* ∧ ¬ *Bet-Point* (*Se C B*) *A* ∧ *Bet-Point* (*Se B A*) *C* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Bet-swap-lemma-1* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
shows *Line-on* (*Li A D*) *B* ∧ *Line-on* (*Li A D*) *C*
proof −
from *assms* **have** *P1* : *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-Bet-on*)
have *P2* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P5* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P1 P2 P3 P4 P5* **have** *P6* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms* **have** *P7* : *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P6 P7* **have** *P8* : *Line-on* (*Li A B*) *D* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P9* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)

from *assms P8 P9* **have** *P10* : *Line-on* (*Li A D*) *B* **by** (*simp add*:*Line-on-rev*)
have *P11* : *Line-on* (*Li A D*) *D* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P12* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P4 P7 P10 P11 P12* **have** *P13* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P3 P13* **have** *P14* : *Line-on* (*Li A D*) *C* **by** (*simp add*:*Line-on-trans*)
from *P10 P14* **show** *Line-on* (*Li A D*) *B* ∧ *Line-on* (*Li A D*) *C* **by** *simp*
qed

lemma(**in** *Order-Rule*) *Bet-swap-lemma-2* :
assumes
Bet-Point (*Se p1 p3*) *p2*
¬ *Line-on* (*Li p1 p3*) *p4*
¬ *Line-on* (*Li p2 p5*) *p3*
¬ *Line-on* (*Li p2 p5*) *p1*
¬ *Line-on* (*Li p2 p5*) *p4*
Bet-Point (*Se p3 p5*) *p4*
shows ∃ *p*. *Line-on* (*Li p2 p5*) *p* ∧ *Bet-Point* (*Se p1 p4*) *p*
proof −
have *P1* : *Line-on* (*Li p2 p5*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms P1* **have** *P2* : *Line-on-Seg* (*Li p2 p5*) (*Se p1 p4*) ∧ ¬ *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ∨ *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ∧ ¬ *Line-on-Seg* (*Li p2 p5*) (*Se p1 p4*) **by** (*simp add*:*Pachets-axiom*)
then have *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ ∃ *p*. *Line-on* (*Li p2 p5*) *p* ∧ *Bet-Point* (*Se p3 p4*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *p6* :: *Point* **where** *P3* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ *Line-on* (*Li p2 p5*) *p6* ∧ *Bet-Point* (*Se p3 p4*) *p6* **by** *blast*
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P4* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P3* **have** *P5* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ *Bet-Point* (*Se p3 p4*) *p6* **by** *simp*
from *P3* **have** *P6* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ *Line-on* (*Li p3 p6*) *p4* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *P7* : *Line-on* (*Li p3 p5*) *p4* **by** (*simp add*:*Line-Bet-on*)
have *P8* : *Line-on* (*Li p3 p6*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li p3 p5*) *p3* **by** (*simp add*:*Line-on-rule*)
from *P4 P6 P7 P8 P9* **have** *P10* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹
Eq (*Geos* (*Lin* (*Li p3 p5*)) *add Emp*) (*Geos* (*Lin* (*Li p3 p6*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P11* : *Line-on* (*Li p3 p5*) *p5* **by** (*simp add*:*Line-on-rule*)
from *P10 P11* **have** *P12* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ *Line-on* (*Li p3 p6*) *p5* **by** (*simp add*:*Line-on-trans*)
have *P13* : *Line-on* (*Li p2 p5*) *p5* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li p3 p6*) *p6* **by** (*simp add*:*Line-on-rule*)
from *P5* **have** *P15* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ *Eq* (*Geos* (*Poi p6*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) ⟹

Bet-Point (*Se p3 p4*) *p5* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se p5 p4*) *p3*) ∧ *Inv* (*Bet-Point* (*Se p4 p3*) *p5*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se p4 p3*) *p5* **by** (*simp add*:*Inv-def*)
then have *P16* : ¬ *Bet-Point* (*Se p3 p4*) *p5* **by** (*blast intro*:*Bet-rev*)
from *P15 P16* **have** *P17* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ ¬ *Eq* (*Geos* (*Poi p6*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) **by** *blast*
from *P3 P12 P13 P14 P17* **have** *P18* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹
Eq (*Geos* (*Lin* (*Li p3 p6*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p5*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P8 P18* **have** *P19* : *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) ⟹ *Line-on* (*Li p2 p5*) *p3* **by** (*simp add*:*Line-on-trans*)
from *assms P19* **have** *P20* : ¬ *Line-on-Seg* (*Li p2 p5*) (*Se p3 p4*) **by** *blast*
from *P2 P3 P20* **have** *Line-on-Seg* (*Li p2 p5*) (*Se p1 p4*) **by** *blast*
thus ∃ *p*. *Line-on* (*Li p2 p5*) *p* ∧ *Bet-Point* (*Se p1 p4*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
qed

lemma(**in** *Order-Rule*) *Bet-swap-lemma-3* :
assumes
Bet-Point (*Se p1 p3*) *p2*
Bet-Point (*Se p3 p5*) *p4*
¬ *Line-on* (*Li p1 p3*) *p5*
shows ∃ *p*. *Bet-Point* (*Se p1 p4*) *p* ∧ *Bet-Point* (*Se p5 p2*) *p*
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P2* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *assms P2* **have** *P3* : ¬ *Line-on* (*Li p3 p1*) *p5* **by** (*simp add*:*Line-not-on-trans*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P1* **have** *P5* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P3 P4 P5* **have** *P6* : ¬ *Line-on* (*Li p3 p5*) *p1* **by** (*blast intro*:*Line-on-rev*)
from *assms* **have** *P7* : *Bet-Point* (*Se p5 p3*) *p4* **by** (*simp add*:*Bet-rev*)
from *P4* **have** *P8* : *Eq* (*Geos* (*Lin* (*Li p3 p5*)) *add Emp*) (*Geos* (*Lin* (*Li p5 p3*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P8 P6* **have** *P9* : ¬ *Line-on* (*Li p5 p3*) *p1* **by** (*simp add*:*Line-not-on-trans*)
from *P7 P9* **have** *P10* : ¬ *Eq* (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *assms* **have** *Line-on* (*Li p3 p5*) *p4* **by** (*simp add*:*Line-Bet-on*)
then have *P11* : *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) ⟹ *Line-on* (*Li p3 p5*) *p1* **by** (*simp add*:*Point-Eq*)
from *P6 P11* **have** ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** *blast*
then have *P12* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P10 P12* **have** *P13* : ¬ *Line-on* (*Li p1 p4*) *p3* **by** (*simp add*:*Line-not-Eq-on*)
from *P1 P12 P13* **have** *P14* : ¬ *Line-on* (*Li p1 p3*) *p4* **by** (*blast intro*:*Line-on-rev*)

from *assms* **have** *P15* : ¬ *Eq* (*Geos* (*Lin* (*Li p5 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p5 p3*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *assms* **have** *P16* : *Line-on* (*Li p1 p3*) *p2* **by** (*simp add*:*Line-Bet-on*)
then have *P17* : *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) ⟹ *Line-on* (*Li p1 p3*) *p5* **by** (*simp add*:*Point-Eq*)
from *assms P17* **have** ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) **by** *blast*
then have *P18* : ¬ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P4* **have** *P19* : ¬ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P15 P18 P19* **have** *P20* : ¬ *Line-on* (*Li p5 p2*) *p3* **by** (*simp add*:*Line-not-Eq-on*)
from *P18* **have** *P21* : *Eq* (*Geos* (*Lin* (*Li p5 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p5*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P20 P21* **have** *P22* : ¬ *Line-on* (*Li p2 p5*) *p3* **by** (*simp add*:*Line-not-on-trans*)

from *assms* **have** *P23* :*Bet-Point* (*Se p3 p1*) *p2* **by** (*blast intro*:*Bet-rev*)
from *P3 P23* **have** *P24* : ¬ *Eq* (*Geos* (*Lin* (*Li p5 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p5 p1*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
have *Line-on* (*Li p3 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
then have *P25* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) ⟹ *Line-on* (*Li p3 p1*) *p5* **by** (*simp add*:*Point-Eq*)
from *P3 P25* **have** *P26* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) **by** *blast*
then have *P27* : ¬ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P18 P24 P27* **have** *P28* : ¬ *Line-on* (*Li p5 p2*) *p1* **by** (*simp add*:*Line-not-Eq-on*)
from *P21 P28* **have** *P29* : ¬ *Line-on* (*Li p2 p5*) *p1* **by** (*simp add*:*Line-not-on-trans*)

from *assms* **have** *P31* : *Line-on* (*Li p3 p4*) *p5* **by** (*simp add*:*Line-Bet-on*)
have *P32* : *Line-on* (*Li p3 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P33* : *Line-on* (*Li p2 p5*) *p5* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P34* : ¬ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P31 P32 P33 P34* **have** *P35* : *Line-on* (*Li p2 p5*) *p4* ⟹ *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p5*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P36* : *Line-on* (*Li p3 p4*) *p3* **by** (*simp add*:*Line-on-rule*)
from *P35 P36* **have** *P37* : *Line-on* (*Li p2 p5*) *p4* ⟹ *Line-on* (*Li p2 p5*) *p3* **by** (*simp add*:*Line-on-trans*)
from *P22 P37* **have** *P38* : ¬ *Line-on* (*Li p2 p5*) *p4* **by** *blast*
from *assms P14 P22 P29 P38* **have** ∃ *p*. *Line-on* (*Li p2 p5*) *p* ∧ *Bet-Point* (*Se p1 p4*) *p* **by** (*simp add*:*Bet-swap-lemma-2*)
then obtain *p6* :: *Point* **where** *P39* : *Line-on* (*Li p2 p5*) *p6* ∧ *Bet-Point* (*Se p1 p4*) *p6* **by** *blast*
from *P12* **have** *P40* : *Eq* (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P13 P40* **have** *P41* : ¬ *Line-on* (*Li p4 p1*) *p3* **by** (*simp add*:*Line-not-on-trans*)

from *assms P6* **have** *P42* : ¬ *Eq* (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p5*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P12 P26 P42* **have** *P43* : ¬ *Line-on* (*Li p1 p4*) *p5* **by** (*simp add*:*Line-not-Eq-on*)
from *P40 P43* **have** *P44* : ¬ *Line-on* (*Li p4 p1*) *p5* **by** (*simp add*:*Line-not-on-trans*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P45* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P16 P45* **have** *P47* : *Line-on* (*Li p1 p2*) *p3* **by** (*simp add*:*Line-on-rev*)
from *P47* **have** *P48* : *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) ⟹ *Line-on* (*Li p1 p4*) *p3* **by** (*simp add*:*Line-on-trans*)
from *P13 P48* **have** *P49* : ¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) **by** *blast*
from *P12 P45 P49* **have** *P50* : ¬ *Line-on* (*Li p1 p2*) *p4* **by** (*simp add*:*Line-not-Eq-on*)
from *P12 P45 P50* **have** *P51* : ¬ *Line-on* (*Li p1 p4*) *p2* **by** (*blast intro*:*Line-on-rev*)
from *P40 P51* **have** *P52* : ¬ *Line-on* (*Li p4 p1*) *p2* **by** (*simp add*:*Line-not-on-trans*)

from *P18 P19 P20* **have** *P53* : ¬ *Line-on* (*Li p5 p3*) *p2* **by** (*blast intro*:*Line-on-rev*)
from *P7 P23 P41 P44 P52 P53* **have** ∃ *p*. *Line-on* (*Li p4 p1*) *p* ∧ *Bet-Point* (*Se p5 p2*) *p* **by** (*simp add*:*Bet-swap-lemma-2*)
then obtain *p7* :: *Point* **where** *P54* : *Line-on* (*Li p4 p1*) *p7* ∧ *Bet-Point* (*Se p5 p2*) *p7* **by** *blast*
from *P33 P44* **have** *P55* : ¬ *Eq* (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p5*)) *add Emp*) **by** (*simp add*:*Line-not-on-Eq*)
from *P39* **have** *P56* : *Line-on* (*Li p4 p1*) *p6* **by** (*simp add*:*Line-Bet-on*)
from *P54* **have** *P57* : *Line-on* (*Li p2 p5*) *p7* **by** (*simp add*:*Line-Bet-on*)
from *P39 P54 P55 P56 P57* **have** *P58* : *Eq* (*Geos* (*Poi p7*) *add Emp*) (*Geos* (*Poi p6*) *add Emp*) **by** (*blast intro*:*Line-unique-Point*)
from *P54* **have** *P59* : *Bet-Point* (*Se p5 p2*) *p7* **by** *simp*
from *P58 P59* **have** *P60* : *Bet-Point* (*Se p5 p2*) *p6* **by** (*simp add*:*Point-Eq*)
from *P39 P60* **show** ∃ *p*. *Bet-Point* (*Se p1 p4*) *p* ∧ *Bet-Point* (*Se p5 p2*) *p* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Bet-swap-lemma-4* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Bet-Point (*Se A E*) *G*
Bet-Point (*Se D G*) *H*
¬ *Line-on* (*Li A D*) *E*
shows ∃ *p*. *Line-on* (*Li H E*) *p* ∧ *Bet-Point* (*Se D A*) *p*
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P1* **have** *P2* : ¬ *Line-on* (*Li A E*) *D* **by** (*blast intro*:*Line-on-rev*)
from *P1* **have** *P3* : *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li E A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P2 P3* **have** *P4* : ¬ *Line-on* (*Li E A*) *D* **by** (*simp add*:*Line-not-on-trans*)
from *assms* **have** *P5* : *Bet-Point* (*Se E A*) *G* **by** (*simp add*:*Bet-rev*)

from *P4 P5* **have** *P6* : ¬ *Eq* (*Geos* (*Lin* (*Li D G*)) *add Emp*) (*Geos* (*Lin* (*Li D A*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *assms* **have** *P7* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms* **have** *P8* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P6 P7 P8* **have** *P9* : ¬ *Line-on* (*Li D G*) *A* **by** (*simp add*:*Line-not-Eq-on*)
from *assms P2* **have** *P10* : ¬ *Eq* (*Geos* (*Lin* (*Li D G*)) *add Emp*) (*Geos* (*Lin* (*Li D E*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
have *Line-on* (*Li A D*) *D* **by** (*simp add*:*Line-on-rule*)
then have *P11* : *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li A D*) *E* **by** (*simp add*:*Point-Eq*)
from *assms P11* **have** *P12* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** *blast*
from *P7 P10 P12* **have** *P13* : ¬ *Line-on* (*Li D G*) *E* **by** (*simp add*:*Line-not-Eq-on*)
from *assms P13* **have** *P14* : ¬ *Eq* (*Geos* (*Lin* (*Li E H*)) *add Emp*) (*Geos* (*Lin* (*Li E G*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *assms* **have** *Line-on* (*Li D G*) *H* **by** (*simp add*:*Line-Bet-on*)
then have *P15* : *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li D G*) *E* **by** (*simp add*:*Point-Eq*)
from *P13 P15* **have** *P16* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi H*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *Line-on* (*Li D G*) *G* **by** (*simp add*:*Line-on-rule*)
then have *P17* : *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li D G*) *E* **by** (*simp add*:*Point-Eq*)
from *P13 P17* **have** *P18* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P14 P16 P18* **have** *P19* : ¬ *Line-on* (*Li E H*) *G* **by** (*simp add*:*Line-not-Eq-on*)
from *P7* **have** *P20* : *Eq* (*Geos* (*Lin* (*Li D G*)) *add Emp*) (*Geos* (*Lin* (*Li G D*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P13 P20* **have** *P21* : ¬ *Line-on* (*Li G D*) *E* **by** (*simp add*:*Line-not-on-trans*)
from *assms* **have** *P22* : *Bet-Point* (*Se G D*) *H* **by** (*simp add*:*Bet-rev*)
from *P21 P22* **have** *P23* : ¬ *Eq* (*Geos* (*Lin* (*Li E H*)) *add Emp*) (*Geos* (*Lin* (*Li E D*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P12* **have** *P24* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P16 P23 P24* **have** *P25* : ¬ *Line-on* (*Li E H*) *D* **by** (*simp add*:*Line-not-Eq-on*)
from *P16* **have** *P26* : *Eq* (*Geos* (*Lin* (*Li E H*)) *add Emp*) (*Geos* (*Lin* (*Li H E*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P25 P26* **have** *P27* : ¬ *Line-on* (*Li H E*) *D* **by** (*simp add*:*Line-not-on-trans*)
have *P28* : *Line-on* (*Li A E*) *A* **by** (*simp add*:*Line-on-rule*)
have *P29* : *Line-on* (*Li A E*) *E* **by** (*simp add*:*Line-on-rule*)
have *P30* : *Line-on* (*Li E H*) *E* **by** (*simp add*:*Line-on-rule*)
from *P1 P28 P29 P30* **have** *P31* : *Line-on* (*Li E H*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li E H*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms* **have** *P32* : *Line-on* (*Li A E*) *G* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P33* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by**

(*blast intro*:*Eq-rev*)
from *P31 P32* **have** *P34* : *Line-on* (*Li E H*) *A* ⟹ *Line-on* (*Li E H*) *G* **by** (*simp add*:*Line-on-trans*)
from *P19 P34* **have** *P35* : ¬ *Line-on* (*Li E H*) *A* **by** *blast*
from *P26 P35* **have** *P36* : ¬ *Line-on* (*Li H E*) *A* **by** (*simp add*:*Line-not-on-trans*)
from *P26 P19* **have** *P37* : ¬ *Line-on* (*Li H E*) *G* **by** (*simp add*:*Line-not-on-trans*)
have *P38* : *Line-on* (*Li H E*) *H* **by** (*simp add*:*Line-on-rule*)
from *assms P9 P27 P36 P37 P38* **have** *P39* : *Line-on-Seg* (*Li H E*) (*Se D A*) ∧ ¬ *Line-on-Seg* (*Li H E*) (*Se G A*) ∨ *Line-on-Seg* (*Li H E*) (*Se G A*) ∧ ¬ *Line-on-Seg* (*Li H E*) (*Se D A*) **by** (*simp add*:*Pachets-axiom*)
then have *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ ∃ *p*. *Line-on* (*Li H E*) *p* ∧ *Bet-Point* (*Se G A*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *C2* :: *Point* **where** *P40* : *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ *Line-on* (*Li H E*) *C2* ∧ *Bet-Point* (*Se G A*) *C2* **by** *blast*
from *assms* **have** *P41* : *Line-on* (*Li G A*) *E* **by** (*simp add*:*Line-Bet-on*)
from *P40* **have** *P42* : *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ *Line-on* (*Li G A*) *C2* **by** (*simp add*:*Line-Bet-on*)
have *P43* : *Line-on* (*Li H E*) *E* **by** (*simp add*:*Line-on-rule*)
from *P40* **have** *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ *Bet-Point* (*Se G A*) *C2* **by** *simp*
then have *P44* : *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Bet-Point (*Se G A*) *E* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se E G*) *A*) ∧ *Inv* (*Bet-Point* (*Se G A*) *E*) **by** (*simp add*:*Bet-iff*)
then have *P45* : ¬ *Bet-Point* (*Se G A*) *E* **by** (*simp add*:*Inv-def*)
from *P44 P45* **have** *P46* : *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ ¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** *blast*
from *P40 P41 P42 P43 P46* **have** *P47* : *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹
Eq (*Geos* (*Lin* (*Li G A*)) *add Emp*) (*Geos* (*Lin* (*Li H E*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P48* : *Line-on* (*Li G A*) *G* **by** (*simp add*:*Line-on-rule*)
from *P47 P48* **have** *P49* : *Line-on-Seg* (*Li H E*) (*Se G A*) ⟹ *Line-on* (*Li H E*) *G* **by** (*simp add*:*Line-on-trans*)
from *P37 P49* **have** *P50* : ¬ *Line-on-Seg* (*Li H E*) (*Se G A*) **by** *blast*
from *P39 P40 P50* **have** *Line-on-Seg* (*Li H E*) (*Se D A*) **by** *blast*
thus ∃ *p*. *Line-on* (*Li H E*) *p* ∧ *Bet-Point* (*Se D A*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
qed

lemma(**in** *Order-Rule*) *Bet-swap-lemma-5* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
Bet-Point (*Se C F*) *E*
¬ *Line-on* (*Li A D*) *F*
¬ *Line-on* (*Li A C*) *F*
shows *Bet-Point* (*Se A D*) *C*
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)

$\Longrightarrow$ *Bet-Point* (*Se D C*) *B* **by** (*simp add*:*Bet-Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se D C*) *B*) $\wedge$ *Inv* (*Bet-Point* (*Se C B*) *D*) **by** (*simp add*:*Bet-iff*)
then have *P2* : $\neg$ *Bet-Point* (*Se D C*) *B* **by** (*simp add*:*Inv-def*)
from *P1 P2* **have** *P3* : $\neg$ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *assms P3* **have** *P4* : *Line-on* (*Li A D*) *B* $\wedge$ *Line-on* (*Li A D*) *C* **by** (*simp add*:*Bet-swap-lemma-1*)
then have *P5* : *Line-on* (*Li A D*) *C* **by** *simp*
from *assms* **have** $\exists$ *p*. *Bet-Point* (*Se A E*) *p* $\wedge$ *Bet-Point* (*Se F B*) *p* **by** (*simp add*:*Bet-swap-lemma-3*)
then obtain *G* :: *Point* **where** *P6* : *Bet-Point* (*Se A E*) *G* $\wedge$ *Bet-Point* (*Se F B*) *G* **by** *blast*
from *P3* **have** *P7* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li D A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P4 P7* **have** *P8* : *Line-on* (*Li D A*) *B* **by** (*blast intro*:*Line-on-trans*)
from *assms P7* **have** *P9* : $\neg$ *Line-on* (*Li D A*) *F* **by** (*simp add*:*Line-not-on-trans*)
have *P10* : *Line-on* (*Li D A*) *D* **by** (*simp add*:*Line-on-rule*)
have *P11* : *Line-on* (*Li D B*) *D* **by** (*simp add*:*Line-on-rule*)
have *P12* : *Line-on* (*Li D B*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P13* : $\neg$ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P8 P10 P11 P12 P13* **have** *P14* : *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin* (*Li D B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P9 P14* **have** *P15* : $\neg$ *Line-on* (*Li D B*) *F* **by** (*simp add*:*Line-not-on-trans*)

from *assms* **have** *P16* : *Bet-Point* (*Se D B*) *C* **by** (*simp add*:*Bet-rev*)
from *P6* **have** *P17* : *Bet-Point* (*Se B F*) *G* **by** (*simp add*:*Bet-rev*)
from *P15 P16 P17* **have** $\exists$ *p*. *Bet-Point* (*Se D G*) *p* $\wedge$ *Bet-Point* (*Se F C*) *p* **by** (*simp add*:*Bet-swap-lemma-3*)
then obtain *H* :: *Point* **where** *P18* : *Bet-Point* (*Se D G*) *H* $\wedge$ *Bet-Point* (*Se F C*) *H* **by** *blast*
from *assms* **have** *P19* : $\neg$ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P20* : *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *assms P20* **have** *P21* : $\neg$ *Line-on* (*Li C A*) *F* **by** (*simp add*:*Line-not-on-trans*)
from *P19* **have** *P22* : $\neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P23* : $\neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P21 P22 P23* **have** *P24* : $\neg$ *Line-on* (*Li C F*) *A* **by** (*blast intro*:*Line-on-rev*)
from *assms* **have** *P25* : *Bet-Point* (*Se F C*) *E* **by** (*simp add*:*Bet-rev*)
from *P23* **have** *P26* : *Eq* (*Geos* (*Lin* (*Li C F*)) *add Emp*) (*Geos* (*Lin* (*Li F C*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P24 P26* **have** *P27* : $\neg$ *Line-on* (*Li F C*) *A* **by** (*simp add*:*Line-not-on-trans*)
from *P25 P27* **have** *P28* : $\neg$ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P25* **have** *Line-on* (*Li F C*) *E* **by** (*simp add*:*Line-Bet-on*)

then have *P29* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Line-on* (*Li F C*) *A* **by** (*simp add*:*Point-Eq*)
from *P27 P29* **have** ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** *blast*
then have *P30* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P19 P28 P30* **have** *P31* : ¬ *Line-on* (*Li A E*) *C* **by** (*simp add*:*Line-not-Eq-on*)
from *P19 P30 P31* **have** *P32* : ¬ *Line-on* (*Li A C*) *E* **by** (*blast intro*:*Line-on-rev*)

from *P5 P19 P32* **have** *P33* : ¬ *Line-on* (*Li A D*) *E* **by** (*simp add*:*Line-not-on-ex*)

from *P3 P30 P33* **have** *P34* : ¬ *Line-on* (*Li A E*) *D* **by** (*blast intro*:*Line-on-rev*)
from *P30* **have** *P35* : *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li E A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P18* **have** *P36* : *Bet-Point* (*Se D G*) *H* **by** *simp*
from *P6* **have** *P37* : *Bet-Point* (*Se A E*) *G* **by** *simp*
from *P3 P18 P33 P37* **have** ∃ *p*. *Line-on* (*Li H E*) *p* ∧ *Bet-Point* (*Se D A*) *p* **by** (*simp add*:*Bet-swap-lemma-4*)
then obtain *C2* :: *Point* **where** *P38* : *Line-on* (*Li H E*) *C2* ∧ *Bet-Point* (*Se D A*) *C2* **by** *blast*
have *Line-on* (*Li H E*) *E* **by** (*simp add*:*Line-on-rule*)
then have *P39* : *Eq* (*Geos* (*Lin* (*Li H E*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) ⟹ *Line-on* (*Li A D*) *E* **by** (*simp add*:*Line-on-trans*)
from *P33 P39* **have** *P40* : ¬ *Eq* (*Geos* (*Lin* (*Li H E*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** *blast*
from *P23* **have** *P41* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P25* **have** *P42* : *Line-on* (*Li F E*) *C* **by** (*simp add*:*Line-Bet-on*)
from *P18* **have** *P43* : *Line-on* (*Li F H*) *C* **by** (*simp add*:*Line-Bet-on*)
from *P36* **have** *P44* : *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Bet-Point* (*Se D G*) *E* **by** (*simp add*:*Point-Eq*)
then have *P45* : *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li D G*) *E* **by** (*simp add*:*Line-Bet-on*)
have *P46* : *Line-on* (*Li D G*) *G* **by** (*simp add*:*Line-on-rule*)
have *P47* : *Line-on* (*Li A E*) *E* **by** (*simp add*:*Line-on-rule*)
from *P37* **have** *P48* : *Line-on* (*Li A E*) *G* **by** (*simp add*:*Line-Bet-on*)
from *P44* **have** *P49* : *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P45 P46 P47 P48 P49* **have** *P50* : *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li D G*)) *add Emp*) (*Geos* (*Lin* (*Li A E*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P51* : *Line-on* (*Li D G*) *D* **by** (*simp add*:*Line-on-rule*)
from *P50 P51* **have** *P52* : *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li A E*) *D* **by** (*simp add*:*Line-on-trans*)
from *P34 P52* **have** *P53* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi H*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P41 P42 P43 P53* **have** *P54* : *Line-on* (*Li E H*) *C* **by** (*blast intro*:*Line-on-dens*)
from *P53* **have** *P55* : *Eq* (*Geos* (*Lin* (*Li E H*)) *add Emp*) (*Geos* (*Lin* (*Li H E*))

add Emp) **by** (*simp add*:*Line-rev*)
from *P54 P55* **have** *P56* : *Line-on* (*Li H E*) *C* **by** (*blast intro*:*Line-on-trans*)
from *P38* **have** *P57* : *Line-on* (*Li A D*) *C2* **by** (*simp add*:*Line-Bet-on*)
from *P5 P38 P40 P56 P57* **have** *P58* : *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Line-unique-Point*)
from *P38* **have** *P59* : *Bet-Point* (*Se D A*) *C2* **by** *simp*
from *P58 P59* **have** *Bet-Point* (*Se D A*) *C* **by** (*simp add*:*Point-Eq*)
thus *Bet-Point* (*Se A D*) *C* **by** (*simp add*:*Bet-rev*)
qed

theorem(**in** *Order-Rule*) *Bet-swap-234-134* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A D*) *C*
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Bet-Point* (*Se D C*) *B* **by** (*simp add*:*Bet-Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se D C*) *B*) ∧ *Inv* (*Bet-Point* (*Se C B*) *D*) **by** (*simp add*:*Bet-iff*)
then have *P2* : ¬ *Bet-Point* (*Se D C*) *B* **by** (*simp add*:*Inv-def*)
from *P1 P2* **have** *P3* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *assms P3* **have** *Line-on* (*Li A D*) *B* ∧ *Line-on* (*Li A D*) *C* **by** (*simp add*:*Bet-swap-lemma-1*)
then have *P4* : *Line-on* (*Li A D*) *C* **by** *simp*
have ∃ *p q r*. ¬ *Line-on* (*Li A D*) *p* ∧ ¬ *Line-on* (*Li A D*) *q* ∧ ¬ *Line-on* (*Li A D*) *r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *F* :: *Point* **where** *P5* : ¬ *Line-on* (*Li A D*) *F* **by** *blast*
from *P4* **have** *P6* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P5 P6* **have** ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** *blast*
then have ∃ *p*. *Bet-Point* (*Se C F*) *p* **by** (*simp add*:*Seg-density*)
then obtain *E* :: *Point* **where** *P7* : *Bet-Point* (*Se C F*) *E* **by** *blast*
have *P8* : *Line-on* (*Li A D*) *A* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P10* : *Line-on* (*Li A C*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P11* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P4 P8 P9 P10 P11* **have** *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P12* : *Line-on* (*Li A C*) *F* ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Line-on-trans*)
from *P5 P12* **have** *P13* : ¬ *Line-on* (*Li A C*) *F* **by** *blast*

from *assms P5 P7 P13* **show** *Bet-Point* (*Se A D*) *C* **by** (*blast intro*:*Bet-swap-lemma-5*)
qed

theorem(**in** *Order-Rule*) *Bet-swap-234-124* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A D*) *B*
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Bet-Point* (*Se D C*) *B* **by** (*simp add*:*Bet-Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se D C*) *B*) ∧ *Inv* (*Bet-Point* (*Se C B*) *D*) **by** (*simp add*:*Bet-iff*)
then have *P2* : ¬ *Bet-Point* (*Se D C*) *B* **by** (*simp add*:*Inv-def*)
from *P1 P2* **have** *P3* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *assms P3* **have** *Line-on* (*Li A D*) *B* ∧ *Line-on* (*Li A D*) *C* **by** (*simp add*:*Bet-swap-lemma-1*)
then have *P4* : *Line-on* (*Li A D*) *B* **by** *simp*
have ∃ *p q r*. ¬ *Line-on* (*Li A D*) *p* ∧ ¬ *Line-on* (*Li A D*) *q* ∧ ¬ *Line-on* (*Li A D*) *r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *F* :: *Point* **where** *P5* : ¬ *Line-on* (*Li A D*) *F* **by** *blast*
from *P4* **have** *P6* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P5 P6* **have** ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** *blast*
then have ∃ *p*. *Bet-Point* (*Se B F*) *p* **by** (*simp add*:*Seg-density*)
then obtain *E* :: *Point* **where** *P7* : *Bet-Point* (*Se B F*) *E* **by** *blast*
from *assms* **have** *P8* : *Bet-Point* (*Se D B*) *C* **by** (*simp add*:*Bet-rev*)
from *assms* **have** *P9* : *Bet-Point* (*Se C A*) *B* **by** (*simp add*:*Bet-rev*)
from *P3* **have** *P10* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li D A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P5 P10* **have** *P11* : ¬ *Line-on* (*Li D A*) *F* **by** (*simp add*:*Line-not-on-trans*)
from *P4 P10* **have** *P12* : *Line-on* (*Li D A*) *B* **by** (*simp add*:*Line-on-trans*)
have *P13* : *Line-on* (*Li D A*) *D* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li D B*) *D* **by** (*simp add*:*Line-on-rule*)
have *P15* : *Line-on* (*Li D B*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P16* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P12 P13 P14 P15 P16* **have** *Eq* (*Geos* (*Lin* (*Li D B*)) *add Emp*) (*Geos* (*Lin* (*Li D A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P17* : *Line-on* (*Li D B*) *F* ⟹ *Line-on* (*Li D A*) *F* **by** (*simp add*:*Line-on-trans*)
from *P11 P17* **have** *P18* : ¬ *Line-on* (*Li D B*) *F* **by** *blast*
from *P7 P8 P9 P11 P18* **have** *Bet-Point* (*Se D A*) *B* **by** (*blast intro*:*Bet-swap-lemma-5*)

thus *Bet-Point* (*Se A D*) *B* **by** (*blast intro*:*Bet-rev*)
qed

theorem(**in** *Order-Rule*) *Bet-swap-134-234* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se A D*) *C*
shows *Bet-Point* (*Se B D*) *C*
proof −
from *assms* **have** *P2* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P4* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P5* : *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-Bet-on*)
have *P6* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P7* : *Line-on* (*Li A D*) *C* **by** (*simp add*:*Line-Bet-on*)
have *P8* : *Line-on* (*Li A D*) *A* **by** (*simp add*:*Line-on-rule*)
from *P2 P5 P6 P7 P8* **have** *P9* : *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P10* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *P9 P10* **have** *P11* : *Line-on* (*Li A D*) *B* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** *P12* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P13* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li D A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P11 P13* **have** *P14* : *Line-on* (*Li D A*) *B* **by** (*simp add*:*Line-on-trans*)
from *P12* **have** *P15* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P16* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Bet-Point* (*Se A C*) *D* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se D C*) *A*) ∧ *Inv* (*Bet-Point* (*Se C A*) *D*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se C A*) *D* **by** (*simp add*:*Inv-def*)
then have *P17* : ¬ *Bet-Point* (*Se A C*) *D* **by** (*blast intro*:*Bet-rev*)
from *P16 P17* **have** *P18* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P14 P15 P18* **have** *P19* : *Line-on* (*Li D B*) *A* **by** (*simp add*:*Line-on-rev*)
from *P18* **have** *P20* : *Eq* (*Geos* (*Lin* (*Li D B*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P19 P20* **have** *P21* : *Line-on* (*Li B D*) *A* **by** (*simp add*:*Line-on-trans*)
have *P22* : *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
from *P4 P8 P11 P21 P22* **have** *P23* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P23* **have** *P24* : *Line-on* (*Li B D*) *C* **by** (*simp add*:*Line-on-trans*)
have ∃ *p q r*. ¬ *Line-on* (*Li A D*) *p* ∧ ¬ *Line-on* (*Li A D*) *q* ∧ ¬ *Line-on* (*Li A D*) *r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi*

q) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *F* :: *Point* **where** *P25* : ¬ *Line-on* (*Li A D*) *F* **by** *blast*
from *P11* **have** *P26* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P25 P26* **have** ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** *blast*
then have ∃ *p*. *Bet-Point* (*Se B F*) *p* **by** (*simp add*:*Seg-density*)
then obtain *G* :: *Point* **where** *P27* : *Bet-Point* (*Se B F*) *G* **by** *blast*
from *P11 P25 P27* **have** *Inv* (*Line-on* (*Li A D*) *G*) **by** (*simp add*:*Line-Bet-not-on*)
then have *P28* : ¬ *Line-on* (*Li A D*) *G* **by** (*simp add*:*Inv-def*)
from *assms P25* **have** *P29* : ¬ *Eq* (*Geos* (*Lin* (*Li F C*)) *add Emp*) (*Geos* (*Lin* (*Li F D*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P7* **have** *P30* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P25 P30* **have** *P31* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P32* : *Line-on* (*Li A D*) *D* **by** (*simp add*:*Line-on-rule*)
then have *P33* : *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P25 P33* **have** *P34* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P29 P31 P34* **have** *P35* : ¬ *Line-on* (*Li F C*) *D* **by** (*simp add*:*Line-not-Eq-on*)
from *P31* **have** *P36* : *Eq* (*Geos* (*Lin* (*Li F C*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P35 P36* **have** *P37* : ¬ *Line-on* (*Li C F*) *D* **by** (*simp add*:*Line-not-on-trans*)

from *assms* **have** *P38* : *Bet-Point* (*Se D A*) *C* **by** (*simp add*:*Bet-rev*)
from *P13 P25* **have** *P39* : ¬ *Line-on* (*Li D A*) *F* **by** (*simp add*:*Line-not-on-trans*)
from *P38 P39* **have** *P40* : ¬ *Eq* (*Geos* (*Lin* (*Li F C*)) *add Emp*) (*Geos* (*Lin* (*Li F A*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *P8* **have** *P41* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P25 P41* **have** *P42* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P31 P40 P42* **have** *P43* : ¬ *Line-on* (*Li F C*) *A* **by** (*simp add*:*Line-not-Eq-on*)
from *P36 P43* **have** *P44* : ¬ *Line-on* (*Li C F*) *A* **by** (*simp add*:*Line-not-on-trans*)

from *P2* **have** *P45* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P31* **have** *P46* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P44 P45 P46* **have** *P47* : ¬ *Line-on* (*Li C A*) *F* **by** (*blast intro*:*Line-on-rev*)
from *P45* **have** *P48* : *Eq* (*Geos* (*Lin* (*Li C A*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P47 P48* **have** *P49* : ¬ *Line-on* (*Li A C*) *F* **by** (*simp add*:*Line-not-on-trans*)
from *assms P49* **have** *P50* : ¬ *Eq* (*Geos* (*Lin* (*Li F B*)) *add Emp*) (*Geos* (*Lin* (*Li F C*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)

from *P11* **have** *P51* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Point-Eq*)
from *P25 P51* **have** *P52* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P31 P50 P52* **have** *P53* : ¬ *Line-on* (*Li F B*) *C* **by** (*simp add*:*Line-not-Eq-on*)
from *P27* **have** *P54* : *Line-on* (*Li B F*) *G* **by** (*simp add*:*Line-Bet-on*)
from *P27* **have** *P56* : *Line-on* (*Li F B*) *G* **by** (*simp add*:*Line-Bet-on*)
have *P57* : *Line-on* (*Li F B*) *F* **by** (*simp add*:*Line-on-rule*)
have *P58* : *Line-on* (*Li C F*) *F* **by** (*simp add*:*Line-on-rule*)
from *P27* **have** *P59* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi G*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P56 P57 P58 P59* **have** *P60* : *Line-on* (*Li C F*) *G* ⟹ *Eq* (*Geos* (*Lin* (*Li C F*)) *add Emp*) (*Geos* (*Lin* (*Li F B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P61* : *Line-on* (*Li C F*) *C* **by** (*simp add*:*Line-on-rule*)
from *P60 P61* **have** *P62* : *Line-on* (*Li C F*) *G* ⟹ *Line-on* (*Li F B*) *C* **by** (*simp add*:*Line-on-trans*)
from *P53 P62* **have** *P63* : ¬ *Line-on* (*Li C F*) *G* **by** *blast*
have *P64* : *Line-on* (*Li C F*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P28 P37 P44 P63 P64* **have** *P65* : *Line-on-Seg* (*Li C F*) (*Se A G*) ∧ ¬ *Line-on-Seg* (*Li C F*) (*Se D G*) ∨ *Line-on-Seg* (*Li C F*) (*Se D G*) ∧ ¬ *Line-on-Seg* (*Li C F*) (*Se A G*) **by** (*simp add*:*Pachets-axiom*)
then have *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ ∃ *p*. *Line-on* (*Li C F*) *p* ∧ *Bet-Point* (*Se A G*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *H* :: *Point* **where** *P66* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li C F*) *H* ∧ *Bet-Point* (*Se A G*) *H* **by** *blast*
from *P9* **have** *P67* : *Line-on* (*Li A B*) *F* ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Line-on-trans*)
from *P25 P67* **have** *P68* : ¬ *Line-on* (*Li A B*) *F* **by** *blast*
from *P4* **have** *P69* : *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P68 P69* **have** *P70* : ¬ *Line-on* (*Li B A*) *F* **by** (*simp add*:*Line-not-on-trans*)
from *P3 P27 P66 P70* **have** *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ ∃ *p*. *Line-on* (*Li H F*) *p* ∧ *Bet-Point* (*Se A B*) *p* **by** (*simp add*:*Bet-swap-lemma-4*)
then obtain *E* :: *Point* **where** *P71* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li H F*) *E* ∧ *Bet-Point* (*Se A B*) *E* **by** *blast*
then have *P72* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li A B*) *E* **by** (*simp add*:*Line-Bet-on*)
from *P36* **have** *P73* : *Eq* (*Geos* (*Lin* (*Li C F*)) *add Emp*) (*Geos* (*Lin* (*Li F C*)) *add Emp*) **by** (*simp add*:*Eq-rev*)
from *P66 P73* **have** *P74* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li F C*) *H* **by** (*simp add*:*Line-on-trans*)
from *P66* **have** *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Bet-Point* (*Se A G*) *H* **by** *simp*
then have *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Bet-Point (*Se A G*) *F* **by** (*simp add*:*Point-Eq*)
then have *P75* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Line-on (*Li A G*) *F* **by** (*simp add*:*Line-Bet-on*)

have *P76* : *Line-on* (*Li A G*) *G* **by** (*simp add*:*Line-on-rule*)
have *P77* : *Line-on* (*Li B F*) *F* **by** (*simp add*:*Line-on-rule*)
from *P54 P59 P75 P76 P77* **have** *P78* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li A G*)) *add Emp*) (*Geos* (*Lin* (*Li B F*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P79* : *Line-on* (*Li A G*) *A* **by** (*simp add*:*Line-on-rule*)
from *P78 P79* **have** *P80* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Line-on (*Li B F*) *A* **by** (*simp add*:*Line-on-trans*)
have *P81* : *Line-on* (*Li B F*) *B* **by** (*simp add*:*Line-on-rule*)
from *P4 P6 P10 P80 P81* **have** *P82* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li B F*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P77 P82* **have** *P83* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi H*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Line-on (*Li A B*) *F* **by** (*simp add*:*Line-on-trans*)
from *P68 P83* **have** *P84* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi H*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P46* **have** *P85* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P74 P84 P85* **have** *P86* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li F H*) *C* **by** (*blast intro*:*Line-on-rev*)
from *P84* **have** *P87* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Lin* (*Li F H*)) *add Emp*) (*Geos* (*Lin* (*Li H F*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P86 P87* **have** *P88* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li H F*) *C* **by** (*simp add*:*Line-on-trans*)
from *P71* **have** *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Bet-Point* (*Se A B*) *E* **by** *simp*
then have *P89* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Bet-Point* (*Se A B*) *C* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se C B*) *A*) ∧ *Inv* (*Bet-Point* (*Se B A*) *C*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Inv-def*)
then have *P90* : ¬ *Bet-Point* (*Se A B*) *C* **by** (*blast intro*:*Bet-rev*)
from *P89 P90* **have** *P91* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
from *P5 P71 P72 P88 P91* **have** *P92* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹
Eq (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li H F*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P4 P11 P12* **have** *P93* : *Line-on* (*Li A B*) *D* **by** (*simp add*:*Line-on-rev*)
from *P92 P93* **have** *P94* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li H F*) *D* **by** (*simp add*:*Line-on-trans*)
have *P95* : *Line-on* (*Li C F*) *F* **by** (*simp add*:*Line-on-rule*)
have *P96* : *Line-on* (*Li H F*) *H* **by** (*simp add*:*Line-on-rule*)
have *P97* : *Line-on* (*Li H F*) *F* **by** (*simp add*:*Line-on-rule*)
from *P66 P84 P95 P96 P97* **have** *P98* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹
Eq (*Geos* (*Lin* (*Li H F*)) *add Emp*) (*Geos* (*Lin* (*Li C F*)) *add Emp*) **by** (*simp*

add:*Line-unique*)
from *P94 P98* **have** *P99* : *Line-on-Seg* (*Li C F*) (*Se A G*) ⟹ *Line-on* (*Li C F*) *D* **by** (*simp add*:*Line-on-trans*)
from *P37 P99* **have** *P100* : ¬ *Line-on-Seg* (*Li C F*) (*Se A G*) **by** *blast*
from *P65 P100* **have** *Line-on-Seg* (*Li C F*) (*Se D G*) **by** *blast*
then have ∃ *p*. *Line-on* (*Li C F*) *p* ∧ *Bet-Point* (*Se D G*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *H2* :: *Point* **where** *P101* : *Line-on* (*Li C F*) *H2* ∧ *Bet-Point* (*Se D G*) *H2* **by** *blast*
from *P23* **have** *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Eq-rev*)
then have *P102* : *Line-on* (*Li B D*) *F* ⟹ *Line-on* (*Li A D*) *F* **by** (*simp add*:*Line-on-trans*)
from *P25 P102* **have** *P103* : ¬ *Line-on* (*Li B D*) *F* **by** *blast*
from *P18* **have** *P104* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P27 P101 P103 P104* **have** ∃ *p*. *Line-on* (*Li H2 F*) *p* ∧ *Bet-Point* (*Se D B*) *p* **by** (*simp add*:*Bet-swap-lemma-4*)
then obtain *C2* :: *Point* **where** *P105* : *Line-on* (*Li H2 F*) *C2* ∧ *Bet-Point* (*Se D B*) *C2* **by** *blast*
have *Line-on* (*Li H2 F*) *F* **by** (*simp add*:*Line-on-rule*)
then have *P106* : *Eq* (*Geos* (*Lin* (*Li H2 F*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) ⟹ *Line-on* (*Li B D*) *F* **by** (*simp add*:*Line-on-trans*)
from *P103 P106* **have** *P107* : ¬ *Eq* (*Geos* (*Lin* (*Li H2 F*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** *blast*
from *P73 P101* **have** *P108* : *Line-on* (*Li F C*) *H2* **by** (*simp add*:*Line-on-trans*)
from *P101* **have** *Bet-Point* (*Se D G*) *H2* **by** *simp*
then have *Eq* (*Geos* (*Poi H2*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Bet-Point* (*Se D G*) *F* **by** (*simp add*:*Point-Eq*)
then have *P109* : *Eq* (*Geos* (*Poi H2*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li D G*) *F* **by** (*simp add*:*Line-Bet-on*)
have *P110* : *Line-on* (*Li D G*) *G* **by** (*simp add*:*Line-on-rule*)
from *P54 P59 P77 P109 P110* **have** *P111* : *Eq* (*Geos* (*Poi H2*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li B F*)) *add Emp*) (*Geos* (*Lin* (*Li D G*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P81 P111* **have** *P112* : *Eq* (*Geos* (*Poi H2*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li D G*) *B* **by** (*simp add*:*Line-on-trans*)
have *P113* : *Line-on* (*Li D G*) *D* **by** (*simp add*:*Line-on-rule*)
from *P11 P18 P32 P112 P113* **have** *P114* : *Eq* (*Geos* (*Poi H2*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li D G*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P110 P114* **have** *P115* : *Eq* (*Geos* (*Poi H2*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Line-on* (*Li A D*) *G* **by** (*simp add*:*Line-on-trans*)
from *P28 P115* **have** *P116* : ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi H2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P31 P108 P116* **have** *P117* : *Line-on* (*Li F H2*) *C* **by** (*simp add*:*Line-on-rev*)
from *P116* **have** *P118* : *Eq* (*Geos* (*Lin* (*Li F H2*)) *add Emp*) (*Geos* (*Lin* (*Li H2 F*)) *add Emp*) **by** (*simp add*:*Line-rev*)

from *P117 P118* **have** *P119* : *Line-on* (*Li H2 F*) *C* **by** (*simp add*:*Line-on-trans*)
from *P105* **have** *P121* : *Line-on* (*Li B D*) *C2* **by** (*simp add*:*Line-Bet-on*)
from *P24 P105 P107 P119 P121* **have** *P122* : *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Line-unique-Point*)
from *P105* **have** *P123* : *Bet-Point* (*Se D B*) *C2* **by** *simp*
from *P122 P123* **have** *Bet-Point* (*Se D B*) *C* **by** (*simp add*:*Point-Eq*)
thus *Bet-Point* (*Se B D*) *C* **by** (*blast intro*:*Bet-rev*)
qed

lemma(**in** *Order-Rule*) *Bet-swap-134-124* :
assumes
Bet-Point (*Se A C*) *B*
Bet-Point (*Se A D*) *C*
shows *Bet-Point* (*Se A D*) *B*
proof −
from *assms* **have** *P1* : *Bet-Point* (*Se B D*) *C* **by** (*blast intro*:*Bet-swap-134-234*)
from *assms P1* **show** *Bet-Point* (*Se A D*) *B* **by** (*blast intro*:*Bet-swap-234-124*)
qed

theorem(**in** *Order-Rule*) *Bet-swap-243-124* :
assumes
Bet-Point (*Se A D*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A C*) *B*
proof −
from *assms* **have** *P1* : *Bet-Point* (*Se D B*) *C* **by** (*simp add*:*Bet-rev*)
from *assms* **have** *P2* : *Bet-Point* (*Se D A*) *B* **by** (*simp add*:*Bet-rev*)
from *P1 P2* **have** *Bet-Point* (*Se C A*) *B* **by** (*blast intro*:*Bet-swap-134-234*)
thus *Bet-Point* (*Se A C*) *B* **by** (*simp add*:*Bet-rev*)
qed

theorem(**in** *Order-Rule*) *Bet-swap-243-143* :
assumes
Bet-Point (*Se A D*) *B*
Bet-Point (*Se B D*) *C*
shows *Bet-Point* (*Se A D*) *C*
proof −
from *assms* **have** *P1* : *Bet-Point* (*Se D B*) *C* **by** (*simp add*:*Bet-rev*)
from *assms* **have** *P2* : *Bet-Point* (*Se D A*) *B* **by** (*simp add*:*Bet-rev*)
from *P1 P2* **have** *Bet-Point* (*Se D A*) *C* **by** (*blast intro*:*Bet-swap-134-124*)
thus *Bet-Point* (*Se A D*) *C* **by** (*simp add*:*Bet-rev*)
qed

Theorem5

lemma(**in** *Order-Rule*) *Bet-four-Point-case* :
assumes
Line-on l1 P
Line-on l1 Q
Line-on l1 R

Line-on l1 S
Bet-Point (Se P R) Q
¬ *Eq* (*Geos* (*Poi P*) *add Emp*) (*Geos* (*Poi S*) *add Emp*)
¬ *Eq* (*Geos* (*Poi Q*) *add Emp*) (*Geos* (*Poi S*) *add Emp*)
¬ *Eq* (*Geos* (*Poi R*) *add Emp*) (*Geos* (*Poi S*) *add Emp*)
shows *Bet-Point (Se P S) R* ∨ *Bet-Point (Se R S) P*
∨ *Bet-Point (Se P R) S* ∧ *Bet-Point (Se P S) Q*
∨ *Bet-Point (Se P Q) S* ∨ *Bet-Point (Se Q S) P*
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi P*) *add Emp*) (*Geos* (*Poi R*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms* **have** *P2* : ¬ *Eq* (*Geos* (*Poi S*) *add Emp*) (*Geos* (*Poi P*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P1 P2* **have** *Bet-Point (Se P S) R* ∨ *Bet-Point (Se S R) P* ∨ *Bet-Point (Se R P) S* **by** (*simp add*:*Bet-case*)
then have *P3* : *Bet-Point (Se P S) R* ∨ *Bet-Point (Se R S) P* ∨ *Bet-Point (Se P R) S* **by** (*blast intro*:*Bet-rev*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi S*) *add Emp*) (*Geos* (*Poi Q*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P5* : ¬ *Eq* (*Geos* (*Poi Q*) *add Emp*) (*Geos* (*Poi P*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P4 P5* **have** *Bet-Point (Se P Q) S* ∨ *Bet-Point (Se Q S) P* ∨ *Bet-Point (Se S P) Q* **by** (*simp add*:*Bet-case*)
then have *P6* : *Bet-Point (Se P Q) S* ∨ *Bet-Point (Se Q S) P* ∨ *Bet-Point (Se P S) Q* **by** (*blast intro*:*Bet-rev*)
from *P3 P6* **show** *Bet-Point (Se P S) R* ∨ *Bet-Point (Se R S) P*
∨ *Bet-Point (Se P R) S* ∧ *Bet-Point (Se P S) Q*
∨ *Bet-Point (Se P Q) S* ∨ *Bet-Point (Se Q S) P* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Plane-diffside-rev* :
assumes
Plane-diffside l1 p1 p2
shows *Plane-diffside l1 p2 p1*
proof −
from *assms* **have** ∃ *p*. *Bet-Point* (*Se p1 p2*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2* **by** (*simp add*:*Plane-diffside-def*)
then obtain *p3* :: *Point* **where** *P1* : *Bet-Point* (*Se p1 p2*) *p3* ∧ *Line-on l1 p3* ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2* **by** *blast*
then have *P2* : *Bet-Point* (*Se p2 p1*) *p3* **by** (*simp add*:*Bet-rev*)
from *P1 P2* **have** ∃ *p*. *Bet-Point* (*Se p2 p1*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 p2* ∧ ¬ *Line-on l1 p1* **by** *blast*
thus *Plane-diffside l1 p2 p1* **by** (*simp add*:*Plane-diffside-def*)
qed

lemma(**in** *Order-Rule*) *Plane-sameside-rev* :
assumes
Plane-sameside l1 p1 p2
shows *Plane-sameside l1 p2 p1*

proof −
have *Line-on-Seg l1 (Se p2 p1)* ⟹ ∃ *p. Line-on l1 p* ∧ *Bet-Point (Se p2 p1) p*
by (*simp add*:*Line-on-Seg-rule*)
then obtain *p3* :: *Point* **where** *P1* : *Line-on-Seg l1 (Se p2 p1)* ⟹
Line-on l1 p3 ∧ *Bet-Point (Se p2 p1) p3* **by** *blast*
then have *P2* : *Line-on-Seg l1 (Se p2 p1)* ⟹ *Bet-Point (Se p1 p2) p3* **by** (*simp add*:*Bet-rev*)
from *P1 P2* **have** *Line-on-Seg l1 (Se p2 p1)* ⟹ ∃ *p. Line-on l1 p* ∧ *Bet-Point (Se p1 p2) p* **by** *blast*
then have *Line-on-Seg l1 (Se p2 p1)* ⟹ *Line-on-Seg l1 (Se p1 p2)* **by** (*simp add*:*Line-on-Seg-rule*)
then have *P3* : ¬ *Line-on-Seg l1 (Se p1 p2)* ⟹ ¬ *Line-on-Seg l1 (Se p2 p1)* **by** *blast*
from *assms* **have** *P4* : ¬ *Line-on-Seg l1 (Se p1 p2)* ∧ ¬ *Line-on l1 p1*
∧ ¬ *Line-on l1 p2* ∧ ¬ *Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)*
by (*simp add*:*Plane-sameside-def*)
from *P3 P4* **have** *P5* : ¬ *Line-on-Seg l1 (Se p2 p1)* **by** *blast*
from *P4* **have** *P6* : ¬ *Eq (Geos (Poi p2) add Emp) (Geos (Poi p1) add Emp)*
by (*blast intro*:*Eq-rev*)
from *P4 P5 P6* **show** *Plane-sameside l1 p2 p1* **by** (*simp add*:*Plane-sameside-def*)
qed

lemma(**in** *Order-Rule*) *Plane-sameside-not-diffside* :
assumes *N* :
Plane-sameside l1 p1 p2
shows ¬ *Plane-diffside l1 p1 p2*
proof
assume *W* : *Plane-diffside l1 p1 p2*
then have ∃ *p. Bet-Point (Se p1 p2) p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 p1* ∧ ¬ *Line-on l1 p2* **by** (*simp add*:*Plane-diffside-def*)
then have ∃ *p. Line-on l1 p* ∧ *Bet-Point (Se p1 p2) p* **by** *blast*
then have *P1* : *Line-on-Seg l1 (Se p1 p2)* **by** (*simp add*:*Line-on-Seg-rule*)
from *N* **have** *P2* : ¬ *Line-on-Seg l1 (Se p1 p2)* **by** (*simp add*:*Plane-sameside-def*)
from *P1 P2* **show** *False* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Plane-diffside-not-sameside* :
assumes *N* :
Plane-diffside l1 p1 p2
shows ¬ *Plane-sameside l1 p1 p2*
proof
assume *W* : *Plane-sameside l1 p1 p2*
then have *P1* : ¬ *Plane-diffside l1 p1 p2* **by** (*simp add*:*Plane-sameside-not-diffside*)
from *N P1* **show** *False* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Plane-not-sameside-diffside* :
assumes ¬ *Plane-sameside l1 p1 p2*
¬ *Line-on l1 p1* ¬ *Line-on l1 p2*

```
    ¬ Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)
  shows Plane-diffside l1 p1 p2
proof −
  from assms have P1 : ¬ Line-on-Seg l1 (Se p1 p2) ⟹ Plane-sameside l1 p1 p2
by (simp add:Plane-sameside-def)
  from assms P1 have P2 : Line-on-Seg l1 (Se p1 p2) by blast
   from P2 have P3 : ∃ p. Line-on l1 p ∧ Bet-Point (Se p1 p2) p by (simp
add:Line-on-Seg-rule)
  from assms P3 have ∃ p. Bet-Point (Se p1 p2) p
    ∧ Line-on l1 p ∧ ¬ Line-on l1 p1 ∧ ¬ Line-on l1 p2 by blast
  thus Plane-diffside l1 p1 p2 by (simp add:Plane-diffside-def)
qed

lemma(in Order-Rule) Plane-not-diffside-sameside :
  assumes ¬ Plane-diffside l1 p1 p2
    ¬ Line-on l1 p1 ¬ Line-on l1 p2
    ¬ Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)
  shows Plane-sameside l1 p1 p2
proof −
  from assms have P1 : ¬ Plane-sameside l1 p1 p2 ⟹ Plane-diffside l1 p1 p2 by
(simp add:Plane-not-sameside-diffside)
  from assms P1 show Plane-sameside l1 p1 p2 by blast
qed

lemma(in Order-Rule) Plane-Line-diff-trans :
  assumes
    Plane-diffside l1 p1 p2
    Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp)
  shows Plane-diffside l2 p1 p2
proof −
  from assms have ∃ p. Bet-Point (Se p1 p2) p ∧ Line-on l1 p ∧ ¬ Line-on l1 p1
∧ ¬ Line-on l1 p2 by (simp add:Plane-diffside-def)
  then obtain p3 :: Point where P1 : Bet-Point (Se p1 p2) p3 ∧ Line-on l1 p3
∧ ¬ Line-on l1 p1 ∧ ¬ Line-on l1 p2 by blast
  from assms P1 have P2 : Line-on l2 p3 by (simp add:Line-on-trans)
  from assms P1 have P3 : ¬ Line-on l2 p1 by (simp add:Line-not-on-trans)
  from assms P1 have P4 : ¬ Line-on l2 p2 by (simp add:Line-not-on-trans)
  from P1 P2 P3 P4 have ∃ p. Bet-Point (Se p1 p2) p ∧ Line-on l2 p ∧ ¬ Line-on
l2 p1 ∧ ¬ Line-on l2 p2 by blast
  thus Plane-diffside l2 p1 p2 by (simp add:Plane-diffside-def)
qed

lemma(in Order-Rule) Plane-Line-trans :
  assumes
    Plane-sameside l1 p1 p2
    Eq (Geos (Lin l1) add Emp) (Geos (Lin l2) add Emp)
  shows Plane-sameside l2 p1 p2
proof −
  have Line-on-Seg l2 (Se p1 p2) ⟹ ∃ p. Line-on l2 p ∧ Bet-Point (Se p1 p2) p
```

by (*simp add*:*Line-on-Seg-rule*)
then obtain *p3* :: *Point* **where** *P1* : *Line-on-Seg l2* (*Se p1 p2*) ⟹ *Line-on l2 p3* ∧ *Bet-Point* (*Se p1 p2*) *p3* **by** *blast*
from *assms P1* **have** *P2* : *Line-on-Seg l2* (*Se p1 p2*) ⟹ *Line-on l1 p3* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P2* **have** *Line-on-Seg l2* (*Se p1 p2*) ⟹ ∃ *p*. *Line-on l1 p* ∧ *Bet-Point* (*Se p1 p2*) *p* **by** *blast*
then have *P3* : *Line-on-Seg l2* (*Se p1 p2*) ⟹ *Line-on-Seg l1* (*Se p1 p2*) **by** (*simp add*:*Line-on-Seg-rule*)
from *assms* **have** *P4* : ¬ *Line-on-Seg l1* (*Se p1 p2*) ∧ ¬ *Line-on l1 p1*
∧ ¬ *Line-on l1 p2* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
by (*simp add*:*Plane-sameside-def*)
from *P3 P4* **have** *P5* : ¬ *Line-on-Seg l2* (*Se p1 p2*) **by** *blast*
from *assms P4* **have** *P6* : *Line-on l2 p1* ⟹ *Line-on l1 p1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P4 P6* **have** *P7* : ¬ *Line-on l2 p1* **by** *blast*
from *assms P4* **have** *P8* : *Line-on l2 p2* ⟹ *Line-on l1 p2* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P4 P8* **have** *P9* : ¬ *Line-on l2 p2* **by** *blast*
from *P4 P5 P7 P9* **show** *Plane-sameside l2 p1 p2* **by** (*simp add*:*Plane-sameside-def*)
qed

lemma(**in** *Order-Rule*) *Line-other-Point* :
assumes *Line-on l1 p1*
shows ∃ *p*. *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)
proof −
have ∃ *p q*. *Line-on l1 p* ∧ *Line-on l1 q* ∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) **by** (*blast intro*:*Line-on-exist*)
then obtain *p2 p3* :: *Point* **where** *P1* : *Line-on l1 p2* ∧ *Line-on l1 p3* ∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** *blast*
then have *P2* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ⟹
Eq (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P1 P2* **have** ¬ (*Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)) **by** *blast*
then have *P3* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
∨ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
∨ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** *blast*
from *P1* **have** *P4* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ⟹
∃ *p*. *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** *blast*
from *P1* **have** *P5* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ∧ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) ⟹

$\exists p.$ *Line-on l1 p* $\wedge \neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** *blast*
from *P1* **have** *P6* : $\neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) $\wedge$ $\neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) $\Longrightarrow$
$\exists p.$ *Line-on l1 p* $\wedge \neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** *blast*
from *P3 P4 P5 P6* **show** $\exists p.$ *Line-on l1 p* $\wedge \neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** *blast*
qed

lemma(**in** *Order-Rule*) *Plane-Bet-sameside* :
assumes
Bet-Point (*Se p1 p3*) *p2*
Line-on l1 p1
$\neg$ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*)
shows *Plane-sameside l1 p2 p3*
proof −
from *assms* **have** $\exists p.$ *Line-on l1 p* $\wedge \neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Line-other-Point*)
then obtain *p4* :: *Point* **where** *P1* : *Line-on l1 p4* $\wedge \neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** *blast*
have *P2* : *Line-on* (*Li p4 p1*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p4 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
have *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$
($\exists p.$ *Bet-Point* (*Se p2 p3*) *p* $\wedge$ *Line-on* (*Li p4 p1*) *p* $\wedge \neg$ *Line-on* (*Li p4 p1*) *p2* $\wedge \neg$ *Line-on* (*Li p4 p1*) *p3*) **by** (*simp add*:*Plane-diffside-def*)
then obtain *p5* :: *Point* **where** *P4* : *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$
Bet-Point (*Se p2 p3*) *p5* $\wedge$ *Line-on* (*Li p4 p1*) *p5* $\wedge \neg$ *Line-on* (*Li p4 p1*) *p2* $\wedge \neg$ *Line-on* (*Li p4 p1*) *p3* **by** *blast*
then have *P5* : *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$ *Bet-Point* (*Se p3 p2*) *p5* **by** (*simp add*:*Bet-rev*)
from *assms* **have** *P6* : *Bet-Point* (*Se p3 p1*) *p2* **by** (*simp add*:*Bet-rev*)
from *P5 P6* **have** *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$ *Bet-Point* (*Se p3 p1*) *p5* **by** (*blast intro*:*Bet-swap-134-124*)
then have *P7* : *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$ *Line-on* (*Li p3 p1*) *p5* **by** (*simp add*:*Line-Bet-on*)
have *P8* : *Line-on* (*Li p3 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
from *P4* **have** *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$ *Bet-Point* (*Se p2 p3*) *p5* **by** *simp*
then have *P9* : *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se p2 p3*) *p1* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se p3 p2*) *p1*) $\wedge$ *Inv* (*Bet-Point* (*Se p2 p1*) *p3*) **by** (*simp add*:*Bet-iff*)
then have $\neg$ *Bet-Point* (*Se p3 p2*) *p1* **by** (*simp add*:*Inv-def*)
then have *P10* : $\neg$ *Bet-Point* (*Se p2 p3*) *p1* **by** (*blast intro*:*Bet-rev*)
from *P9 P10* **have** *P11* : *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow \neg$ *Eq* (*Geos* (*Poi p5*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** *blast*
from *P3 P4 P7 P8 P11* **have** *P12* : *Plane-diffside* (*Li p4 p1*) *p2 p3* $\Longrightarrow$

Eq (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P13* : *Line-on* (*Li p3 p1*) *p3* **by** (*simp add*:*Line-on-rule*)
from *P12 P13* **have** *P14* : *Plane-diffside* (*Li p4 p1*) *p2 p3* ⟹ *Line-on* (*Li p4 p1*) *p3* **by** (*simp add*:*Line-on-trans*)
from *P4 P14* **have** *P15* : ¬ *Plane-diffside* (*Li p4 p1*) *p2 p3* **by** *blast*
from *assms P1 P2 P3* **have** *Eq* (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P16* : *Plane-diffside l1 p2 p3* ⟹ *Plane-diffside* (*Li p4 p1*) *p2 p3* **by** (*blast intro*:*Plane-Line-diff-trans Eq-rev*)
from *P15 P16* **have** *P17* : ¬ *Plane-diffside l1 p2 p3* **by** *blast*
from *assms* **have** *P18* : *Line-on* (*Li p1 p3*) *p2* **by** (*simp add*:*Line-Bet-on*)
have *P19* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P20* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P21* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P19 P20 P21* **have** *P22* : *Line-on l1 p3* ⟹ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P22* **have** *P23* : ¬ *Line-on l1 p3* **by** *blast*
from *assms* **have** *P24* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P18 P19 P24* **have** *P25* : *Line-on l1 p2* ⟹ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P25* **have** *P26* : ¬ *Line-on l1 p2* **by** *blast*
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P27* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P17 P23 P26 P27* **show** *Plane-sameside l1 p2 p3* **by** (*simp add*:*Plane-not-diffside-sameside*)
qed

lemma(**in** *Order-Rule*) *Plane-Bet-diffside* :
assumes
Bet-Point (*Se p1 p3*) *p2*
Line-on l1 p2
¬ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*)
shows *Plane-diffside l1 p1 p3*
proof −
from *assms* **have** ∃ *p*. *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Line-other-Point*)
then obtain *p4* :: *Point* **where** *P1* : *Line-on l1 p4* ∧ ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** *blast*
from *assms* **have** *P2* : *Line-on* (*Li p1 p3*) *p2* **by** (*simp add*:*Line-Bet-on*)
from *assms P1 P2* **have** *P3* : *Line-on* (*Li p1 p3*) *p4* ⟹ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P3* **have** *P4* : ¬ *Line-on* (*Li p1 p3*) *p4* **by** *blast*
have *P5* : *Line-on* (*Li p4 p2*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P6* : *Line-on* (*Li p4 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms P4* **have** *P7* : ¬ *Eq* (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*) (*Geos* (*Lin*

(*Li p4 p3*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
from *assms* **have** *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) ⟹ *Bet-Point* (*Se p1 p3*) *p4* **by** (*simp add*:*Point-Eq*)
then have *P8* : *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) ⟹ *Line-on* (*Li p1 p3*) *p4* **by** (*simp add*:*Line-Bet-on*)
from *assms P4 P8* **have** *P9* : ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
then have *P10* : *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) ⟹ *Line-on* (*Li p1 p3*) *p4* **by** (*simp add*:*Point-Eq*)
from *assms P4 P10* **have** *P11* : ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P7 P9 P11* **have** *P12* : ¬ *Line-on* (*Li p4 p2*) *p3* **by** (*simp add*:*Line-not-Eq-on*)
from *assms* **have** *P13* : *Bet-Point* (*Se p3 p1*) *p2* **by** (*simp add*:*Bet-rev*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P14* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *assms P4 P14* **have** *P15* : ¬ *Line-on* (*Li p3 p1*) *p4* **by** (*simp add*:*Line-not-on-trans*)
from *P13 P15* **have** *P16* : ¬ *Eq* (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) **by** (*simp add*:*Line-Bet-not-Eq*)
have *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
then have *P17* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) ⟹ *Line-on* (*Li p1 p3*) *p4* **by** (*simp add*:*Point-Eq*)
from *assms P4 P17* **have** *P18* : ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9 P16 P18* **have** *P19* : ¬ *Line-on* (*Li p4 p2*) *p1* **by** (*simp add*:*Line-not-Eq-on*)
from *assms P6 P12 P19* **have** ∃ *p*. *Bet-Point* (*Se p1 p3*) *p* ∧ *Line-on* (*Li p4 p2*) *p* ∧ ¬ *Line-on* (*Li p4 p2*) *p1* ∧ ¬ *Line-on* (*Li p4 p2*) *p3* **by** *blast*
then have *P20* : *Plane-diffside* (*Li p4 p2*) *p1 p3* **by** (*simp add*:*Plane-diffside-def*)
from *assms P1 P5 P6* **have** *P21* : *Eq* (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P20 P21* **show** *Plane-diffside l1 p1 p3* **by** (*simp add*:*Plane-Line-diff-trans*)
qed

lemma(**in** *Order-Rule*) *Plane-trans-inv* :
assumes
Plane-diffside l1 A B
Plane-diffside l1 A C
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
shows *Plane-sameside l1 B C*
proof −
from *assms* **have** ∃ *p*. *Bet-Point* (*Se A B*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 B* **by** (*simp add*:*Plane-diffside-def*)
then obtain *D* :: *Point* **where** *P1* : *Bet-Point* (*Se A B*) *D* ∧ *Line-on l1 D* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 B* **by** *blast*
then have *P2* : *Bet-Point* (*Se A B*) *D* **by** *simp*
from *assms* **have** ∃ *p*. *Bet-Point* (*Se A C*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 C* **by** (*simp add*:*Plane-diffside-def*)

then obtain *p2* :: *Point* **where** *P3* : *Bet-Point* (*Se A C*) *p2* ∧ *Line-on l1 p2* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 C* **by** *blast*
then have *Bet-Point* (*Se A C*) *p2* **by** *simp*
then have *P4* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P3* **have** *P5* : ¬ *Line-on l1 C* **by** *simp*
from *P1* **have** *P6* : *Line-on l1 D* **by** *simp*
from *P1* **have** *P7* : ¬ *Line-on l1 A* **by** *simp*
from *P1* **have** *P8* : ¬ *Line-on l1 B* **by** *simp*
from *P2 P5 P6 P7 P8* **have** *P9* : ¬ *Line-on* (*Li A B*) *C* ⟹ *Line-on-Seg l1* (*Se A C*) ∧ ¬ *Line-on-Seg l1* (*Se B C*)
∨ *Line-on-Seg l1* (*Se B C*) ∧ ¬ *Line-on-Seg l1* (*Se A C*) **by** (*simp add*:*Pachets-axiom*)
from *P3* **have** *Bet-Point* (*Se A C*) *p2* ∧ *Line-on l1 p2* **by** *simp*
then have ∃ *p*. *Line-on l1 p* ∧ *Bet-Point* (*Se A C*) *p* **by** *blast*
then have *P10* : *Line-on-Seg l1* (*Se A C*) **by** (*simp add*:*Line-on-Seg-rule*)
from *P9 P10* **have** *P11* : ¬ *Line-on* (*Li A B*) *C* ⟹ ¬ *Line-on-Seg l1* (*Se B C*) **by** *blast*
from *assms P5 P8 P11* **have** *P12* : ¬ *Line-on* (*Li A B*) *C* ⟹ *Plane-sameside l1 B C* **by** (*simp add*:*Plane-sameside-def*)
from *P6* **have** *P13* : *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on l1 C* **by** (*simp add*:*Point-Eq*)
from *P5 P13* **have** *P14* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
from *P2* **have** *P15* : *Line-on* (*Li A B*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P2* **have** *P16* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
from *P2* **have** *P17* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms P2 P4 P14 P15 P16 P17* **have** *P18* : *Line-on* (*Li A B*) *C* ⟹ *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se B C*) *A*
∨ *Bet-Point* (*Se A B*) *C* ∧ *Bet-Point* (*Se A C*) *D* ∨ *Bet-Point* (*Se A D*) *C* ∨ *Bet-Point* (*Se D C*) *A* **by** (*simp add*:*Bet-four-Point-case*)
from *P2* **have** *P19* : *Line-on* (*Li A B*) *C* ⟹ *Bet-Point* (*Se A C*) *B* ⟹ *Bet-Point* (*Se D C*) *B* **by** (*blast intro*:*Bet-swap-134-234*)
have *Line-on* (*Li D C*) *C* **by** (*simp add*:*Line-on-rule*)
then have *P20* : *Eq* (*Geos* (*Lin* (*Li D C*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) ⟹ *Line-on l1 C* **by** (*simp add*:*Line-on-trans*)
from *P5 P20* **have** *P21* : ¬ *Eq* (*Geos* (*Lin* (*Li D C*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** *blast*
from *P6 P19 P21* **have** *P22* : *Line-on* (*Li A B*) *C* ⟹ *Bet-Point* (*Se A C*) *B* ⟹ *Plane-sameside l1 B C* **by** (*simp add*:*Plane-Bet-sameside*)
from *P2* **have** *Bet-Point* (*Se B A*) *D* **by** (*simp add*:*Bet-rev*)
then have *P23* : *Bet-Point* (*Se B C*) *A* ⟹ *Bet-Point* (*Se D C*) *A* **by** (*blast intro*:*Bet-swap-134-234*)
from *P6 P21 P23* **have** *P24* : *Bet-Point* (*Se B C*) *A* ⟹ *Plane-sameside l1 A C* **by** (*simp add*:*Plane-Bet-sameside*)
from *assms* **have** *P25* : ¬ *Plane-sameside l1 A C* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P24 P25* **have** *P26* : ¬ *Bet-Point* (*Se B C*) *A* **by** *blast*
have *Bet-Point* (*Se A B*) *C* ∧ *Bet-Point* (*Se A C*) *D* ⟹ *Bet-Point* (*Se B A*) *C* ∧ *Bet-Point* (*Se C A*) *D* **by** (*simp add*:*Bet-rev*)

then have *P27* : *Bet-Point* (*Se A B*) *C* ∧ *Bet-Point* (*Se A C*) *D* ⟹ *Bet-Point* (*Se D B*) *C* **by** (*blast intro*:*Bet-swap-243-124 Bet-rev*)
have *Line-on* (*Li D B*) *B* **by** (*simp add*:*Line-on-rule*)
then have *P28* : *Eq* (*Geos* (*Lin* (*Li D B*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) ⟹ *Line-on l1 B* **by** (*simp add*:*Line-on-trans*)
from *P8 P28* **have** *P29* : ¬ *Eq* (*Geos* (*Lin* (*Li D B*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** *blast*
from *P6 P27 P29* **have** *P30* : *Bet-Point* (*Se A B*) *C* ∧ *Bet-Point* (*Se A C*) *D* ⟹ *Plane-sameside l1 B C* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
have *P31* : *Bet-Point* (*Se A D*) *C* ⟹ *Bet-Point* (*Se D A*) *C* **by** (*simp add*:*Bet-rev*)
have *Line-on* (*Li D A*) *A* **by** (*simp add*:*Line-on-rule*)
then have *P32* : *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) ⟹ *Line-on l1 A* **by** (*simp add*:*Line-on-trans*)
from *P7 P32* **have** *P33* : ¬ *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** *blast*
from *P6 P31 P33* **have** *P34* : *Bet-Point* (*Se A D*) *C* ⟹ *Plane-sameside l1 A C* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
from *P25 P34* **have** *P35* : ¬ *Bet-Point* (*Se A D*) *C* **by** *blast*
from *P6 P21* **have** *P36* : *Bet-Point* (*Se D C*) *A* ⟹ *Plane-sameside l1 A C* **by** (*simp add*:*Plane-Bet-sameside*)
from *P25 P36* **have** *P37* : ¬ *Bet-Point* (*Se D C*) *A* **by** *blast*
from *P18 P22 P26 P30 P35 P37* **have** *P38* : *Line-on* (*Li A B*) *C* ⟹ *Plane-sameside l1 B C* **by** *blast*
from *P12 P38* **show** *Plane-sameside l1 B C* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Plane-trans* :
assumes
Plane-sameside l1 A B
Plane-diffside l1 A C
shows *Plane-diffside l1 B C*
proof −
from *assms* **have** ∃ *p*. *Bet-Point* (*Se A C*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 C* **by** (*simp add*:*Plane-diffside-def*)
then obtain *D* :: *Point* **where** *P1* : *Bet-Point* (*Se A C*) *D* ∧ *Line-on l1 D* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 C* **by** *blast*
from *assms* **have** *P2* : ¬ *Line-on l1 B* **by** (*simp add*:*Plane-sameside-def*)
from *P1* **have** *P3* : *Bet-Point* (*Se A C*) *D* **by** *simp*
from *P1* **have** *P4* : ¬ *Line-on l1 A* **by** *simp*
from *P1* **have** *P5* : ¬ *Line-on l1 C* **by** *simp*
from *P1* **have** *P6* : *Line-on l1 D* **by** *simp*
from *P2 P3 P4 P5 P6* **have** *P7* : ¬ *Line-on* (*Li A C*) *B* ⟹ *Line-on-Seg l1* (*Se A B*) ∧ ¬ *Line-on-Seg l1* (*Se C B*)
∨ *Line-on-Seg l1* (*Se C B*) ∧ ¬ *Line-on-Seg l1* (*Se A B*) **by** (*simp add*:*Pachets-axiom*)
have *P8* : *Line-on-Seg l1* (*Se A B*) ⟹ ∃ *p*. *Line-on l1 p* ∧ *Bet-Point* (*Se A B*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
from *P2 P4 P8* **have** *Line-on-Seg l1* (*Se A B*) ⟹ ∃ *p*. *Bet-Point* (*Se A B*) *p* ∧ *Line-on l1 p* ∧ ¬ *Line-on l1 A* ∧ ¬ *Line-on l1 B* **by** *blast*

then have *Line-on-Seg l1 (Se A B)* $\Longrightarrow$ *Plane-diffside l1 A B* **by** (*simp add*:*Plane-diffside-def*)
then have *P9* : *Line-on-Seg l1 (Se A B)* $\Longrightarrow$ $\neg$ *Plane-sameside l1 A B* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *assms P9* **have** *P10* : $\neg$ *Line-on-Seg l1 (Se A B)* **by** *blast*
from *P7 P10* **have** $\neg$ *Line-on (Li A C) B* $\Longrightarrow$ *Line-on-Seg l1 (Se C B)***by** *blast*
then have *P11* : $\neg$ *Line-on (Li A C) B* $\Longrightarrow$ $\exists$ *p. Line-on l1 p* $\wedge$ *Bet-Point (Se C B) p* **by** (*simp add*:*Line-on-Seg-rule*)
from *P2 P5 P11* **have** $\neg$ *Line-on (Li A C) B* $\Longrightarrow$ $\exists$ *p. Bet-Point (Se C B) p* $\wedge$ *Line-on l1 p* $\wedge$ $\neg$ *Line-on l1 C* $\wedge$ $\neg$ *Line-on l1 B* **by** *blast*
then have $\neg$ *Line-on (Li A C) B* $\Longrightarrow$ *Plane-diffside l1 C B* **by** (*simp add*:*Plane-diffside-def*)
then have *P12* : $\neg$ *Line-on (Li A C) B* $\Longrightarrow$ *Plane-diffside l1 B C* **by** (*simp add*:*Plane-diffside-rev*)
have *P13* : *Line-on (Li A C) A* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on (Li A C) C* **by** (*simp add*:*Line-on-rule*)
from *P3* **have** *P15* : *Line-on (Li A C) D* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *Eq (Geos (Poi C) add Emp) (Geos (Poi B) add Emp)* $\Longrightarrow$ *Plane-sameside l1 A C* **by** (*blast intro*:*Point-Eq Eq-rev*)
then have *P16* : *Eq (Geos (Poi C) add Emp) (Geos (Poi B) add Emp)* $\Longrightarrow$ $\neg$ *Plane-diffside l1 A C* **by** (*simp add*:*Plane-sameside-not-diffside*)
from *assms P16* **have** *P17* : $\neg$ *Eq (Geos (Poi C) add Emp) (Geos (Poi B) add Emp)* **by** *blast*
from *P6* **have** *P18* : *Eq (Geos (Poi D) add Emp) (Geos (Poi B) add Emp)* $\Longrightarrow$ *Line-on l1 B* **by** (*simp add*:*Point-Eq*)
from *P2 P18* **have** *P19* : $\neg$ *Eq (Geos (Poi D) add Emp) (Geos (Poi B) add Emp)* **by** *blast*
from *assms* **have** *P20* : $\neg$ *Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)* **by** (*simp add*:*Plane-sameside-def*)
from *assms P3 P13 P14 P15 P17 P19 P20* **have** *P21* : *Line-on (Li A C) B* $\Longrightarrow$ *Bet-Point (Se A B) C* $\vee$ *Bet-Point (Se C B) A*
$\vee$ *Bet-Point (Se A C) B* $\wedge$ *Bet-Point (Se A B) D* $\vee$ *Bet-Point (Se A D) B* $\vee$ *Bet-Point (Se D B) A* **by** (*simp add*:*Bet-four-Point-case*)
from *P3* **have** *P22* : *Bet-Point (Se A B) C* $\Longrightarrow$ *Bet-Point (Se A B) D* **by** (*blast intro*:*Bet-swap-134-124*)
have *Line-on (Li A B) A* **by** (*simp add*:*Line-on-rule*)
then have *P23* : *Eq (Geos (Lin (Li A B)) add Emp) (Geos (Lin l1) add Emp)* $\Longrightarrow$ *Line-on l1 A* **by** (*simp add*:*Line-on-trans*)
from *P4 P23* **have** *P24* : $\neg$ *Eq (Geos (Lin (Li A B)) add Emp) (Geos (Lin l1) add Emp)* **by** *blast*
from *P6 P22 P24* **have** *Bet-Point (Se A B) C* $\Longrightarrow$ *Plane-diffside l1 A B* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P25* : *Bet-Point (Se A B) C* $\Longrightarrow$ $\neg$ *Plane-sameside l1 A B* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *assms P25* **have** *P26* : $\neg$ *Bet-Point (Se A B) C* **by** *blast*
from *P3* **have** *P27* : *Bet-Point (Se C A) D* **by** (*simp add*:*Bet-rev*)
from *P27* **have** *P28* : *Bet-Point (Se C B) A* $\Longrightarrow$ *Bet-Point (Se C B) D* **by** (*blast intro*:*Bet-swap-134-124*)
have *Line-on (Li C B) B* **by** (*simp add*:*Line-on-rule*)
then have *P29* : *Eq (Geos (Lin (Li C B)) add Emp) (Geos (Lin l1) add Emp)* $\Longrightarrow$ *Line-on l1 B* **by** (*simp add*:*Line-on-trans*)

from *P2 P29* **have** *P30* : ¬ *Eq* (*Geos* (*Lin* (*Li C B*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** *blast*
from *P6 P28 P30* **have** *Bet-Point* (*Se C B*) *A* ⟹ *Plane-diffside l1 C B* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P31* : *Bet-Point* (*Se C B*) *A* ⟹ *Plane-diffside l1 B C* **by** (*blast intro*:*Plane-diffside-rev*)
from *P6 P24* **have** *Bet-Point* (*Se A B*) *D* ⟹ *Plane-diffside l1 A B* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P32* : *Bet-Point* (*Se A B*) *D* ⟹ ¬ *Plane-sameside l1 A B* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *assms P32* **have** ¬ *Bet-Point* (*Se A B*) *D* **by** *blast*
then have *P33* : ¬ (*Bet-Point* (*Se A C*) *B* ∧ *Bet-Point* (*Se A B*) *D*) **by** *blast*
from *P3* **have** *P34* : *Bet-Point* (*Se A D*) *B* ⟹ *Bet-Point* (*Se C B*) *D* **by** (*blast intro*:*Bet-swap-134-234 Bet-rev*)
from *P6 P30 P34* **have** *Bet-Point* (*Se A D*) *B* ⟹ *Plane-diffside l1 C B* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P35* : *Bet-Point* (*Se A D*) *B* ⟹ *Plane-diffside l1 B C* **by** (*simp add*:*Plane-diffside-rev*)
from *P27* **have** *P36* : *Bet-Point* (*Se D B*) *A* ⟹ *Bet-Point* (*Se C B*) *D* **by** (*blast intro*:*Bet-swap-234-124 Bet-rev*)
from *P6 P30 P36* **have** *Bet-Point* (*Se D B*) *A* ⟹ *Plane-diffside l1 C B* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P37* : *Bet-Point* (*Se D B*) *A* ⟹ *Plane-diffside l1 B C* **by** (*simp add*:*Plane-diffside-rev*)
from *P21 P26 P31 P33 P35 P37* **have** *P38* : *Line-on* (*Li A C*) *B* ⟹ *Plane-diffside l1 B C* **by** *blast*
from *P12 P38* **show** *Plane-diffside l1 B C* **by** *blast*
qed

lemma(**in** *Order-Rule*) *Plane-sameside-trans* :
assumes
Plane-sameside l1 A B
Plane-sameside l1 B C
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
shows *Plane-sameside l1 A C*
proof −
from *assms* **have** *P1* : *Plane-diffside l1 A C* ⟹ *Plane-diffside l1 B C* **by** (*blast intro*:*Plane-trans*)
from *assms* **have** *P2* : ¬ *Plane-diffside l1 B C* **by** (*simp add*:*Plane-sameside-not-diffside*)
from *P1 P2* **have** *P3* : ¬ *Plane-diffside l1 A C* **by** *blast*
from *assms* **have** *P4* : ¬ *Line-on l1 A* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P5* : ¬ *Line-on l1 C* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P6* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P3 P4 P5 P6* **show** *Plane-sameside l1 A C* **by** (*simp add*:*Plane-not-diffside-sameside*)
qed

lemma (**in** *Order-Rule*) *Seg-Bet-not-on* :
assumes

Bet-Point (*Se p1 p3*) *p2*
shows ¬ *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*)
proof −
from *assms* **have** ∃ *l*. *Line-on l p1* ∧ *Line-on l p3* ∧ *Line-on l p2* **by** (*simp add*:*Line-Bet-exist*)
then obtain *l1* :: *Line* **where** *P1* : *Line-on l1 p1* ∧ *Line-on l1 p3* ∧ *Line-on l1 p2* **by** *blast*
have *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ ∃ *p*. *Bet-Point* (*Se p1 p2*) *p* ∧ *Bet-Point* (*Se p2 p3*) *p* **by** (*simp add*:*Seg-on-Seg-rule*)
then obtain *p4* :: *Point* **where** *P2* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Bet-Point* (*Se p1 p2*) *p4* ∧ *Bet-Point* (*Se p2 p3*) *p4* **by** *blast*
then have *P3* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Bet-Point* (*Se p2 p1*) *p4* **by** (*blast intro*:*Bet-rev*)
from *assms* **have** *P4* : *Bet-Point* (*Se p3 p1*) *p2* **by** (*simp add*:*Bet-rev*)
from *P3 P4* **have** *P5* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Bet-Point* (*Se p3 p1*) *p4* **by** (*blast intro*:*Bet-swap-243-143*)
have ∃ *p q r*. ¬ *Line-on l1 p* ∧ ¬ *Line-on l1 q* ∧ ¬ *Line-on l1 r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *p5* :: *Point* **where** *P6* : ¬ *Line-on l1 p5* **by** *blast*
have *P7* : *Line-on* (*Li p5 p4*) *p5* **by** (*simp add*:*Line-on-rule*)
have *P8* : *Line-on* (*Li p3 p1*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li p3 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P10* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P1 P8 P9 P10* **have** *Eq* (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P11* : *Line-on* (*Li p3 p1*) *p5* ⟹ *Line-on l1 p5* **by** (*simp add*:*Line-on-trans*)
from *P6 P11* **have** *P12* : ¬ *Line-on* (*Li p3 p1*) *p5* **by** *blast*
from *P7* **have** *P13* : *Eq* (*Geos* (*Lin* (*Li p5 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) ⟹ *Line-on* (*Li p3 p1*) *p5* **by** (*simp add*:*Line-on-trans*)
from *P12 P13* **have** *P14* : ¬ *Eq* (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p5 p4*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P15* : *Line-on* (*Li p5 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
from *P5 P14 P15* **have** *P16* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Plane-diffside* (*Li p5 p4*) *p3 p1* **by** (*simp add*:*Plane-Bet-diffside*)
have *P17* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P18* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P19* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P1 P17 P18 P19* **have** *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P20* : *Line-on* (*Li p1 p2*) *p5* ⟹ *Line-on l1 p5* **by** (*simp add*:*Line-on-trans*)
from *P6 P20* **have** *P21* : ¬ *Line-on* (*Li p1 p2*) *p5* **by** *blast*
from *P7* **have** *P22* : *Eq* (*Geos* (*Lin* (*Li p5 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) ⟹ *Line-on* (*Li p1 p2*) *p5* **by** (*simp add*:*Line-on-trans*)
from *P21 P22* **have** *P23* : ¬ *Eq* (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin*

(*Li p5 p4*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P2* **have** *P24* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Bet-Point* (*Se p1 p2*) *p4* **by** *simp*
from *P15 P23 P24* **have** *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Plane-diffside* (*Li p5 p4*) *p1 p2* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P25* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Plane-diffside* (*Li p5 p4*) *p2 p1* **by** (*simp add*:*Plane-diffside-rev*)
have *P26* : *Line-on* (*Li p2 p3*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P27* : *Line-on* (*Li p2 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P28* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P1 P26 P27 P28* **have** *Eq* (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P29* : *Line-on* (*Li p2 p3*) *p5* ⟹ *Line-on l1 p5* **by** (*simp add*:*Line-on-trans*)
from *P6 P29* **have** *P30* : ¬ *Line-on* (*Li p2 p3*) *p5* **by** *blast*
from *P7* **have** *P31* : *Eq* (*Geos* (*Lin* (*Li p5 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*) ⟹ *Line-on* (*Li p2 p3*) *p5* **by** (*simp add*:*Line-on-trans*)
from *P30 P31* **have** *P32* : ¬ *Eq* (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p5 p4*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P2* **have** *P33* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Bet-Point* (*Se p2 p3*) *p4* **by** *simp*
from *P15 P32 P33* **have** *P34* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Plane-diffside* (*Li p5 p4*) *p2 p3* **by** (*simp add*:*Plane-Bet-diffside*)
from *P10 P25 P28 P34* **have** *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Plane-sameside* (*Li p5 p4*) *p1 p3* **by** (*blast intro*:*Plane-trans-inv*)
then have *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ *Plane-sameside* (*Li p5 p4*) *p3 p1* **by** (*simp add*:*Plane-sameside-rev*)
then have *P35* : *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) ⟹ ¬ *Plane-diffside* (*Li p5 p4*) *p3 p1* **by** (*simp add*:*Plane-sameside-not-diffside*)
from *P16 P35* **show** ¬ *Seg-on-Seg* (*Se p1 p2*) (*Se p2 p3*) **by** *blast*
qed

end

3 Congruence

Of the equivalence relations for angles, only the transitive law is not included in the axiom, but is mentioned by the theorem. However, in the proofs before that, there are some scenes where it is regarded as congruence by the congruence relation with the same angle. Therefore, we add a weak transitive law that "when two angles are congruent, the same angle as one is congruent with the other". Also, the uniqueness of the large and small relationship between the two angles and the transitive relation of three or more those have not been proved. Therefore, each proof regarding these is added to this section. Furthermore, regarding Theorem 23, the proof is omitted because the "large and small relationship of line segments", which is treated as a premise, is undefined. As a result, the proof process of Theorem

24 is different from the existing one.

locale *Definition-3 = Order-Rule +*
fixes *Def :: Geo-object ⇒ bool*
and *Cong :: Geo-objects ⇒ Geo-objects ⇒ bool*
and *Gr :: Geo-objects ⇒ Geo-objects ⇒ bool*
and *Ang-inside :: Angle ⇒ Point ⇒ bool*
and *Right-angle :: Angle ⇒ bool*
assumes *Tri-def : Def (Tri (Tr p1 p2 p3)) ⟷ ¬ Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)*
∧ ¬ Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp) ∧ ¬ Eq (Geos (Poi p3) add Emp) (Geos (Poi p1) add Emp)
∧ ¬ Bet-Point (Se p1 p2) p3 ∧ ¬ Bet-Point (Se p2 p3) p1 ∧ ¬ Bet-Point (Se p3 p1) p2
∧ ¬ Seg-on-Seg (Se p1 p2) (Se p2 p3) ∧ ¬ Seg-on-Seg (Se p2 p3) (Se p3 p1) ∧ ¬ Seg-on-Seg (Se p3 p1) (Se p1 p2)
and *Cong-refl [simp,intro] : Cong obs obs*
and *Ang-def : Def (Ang (An p1 p2 p3)) ⟷ ¬ Eq (Geos (Poi p1) add Emp) (Geos (Poi p2) add Emp)*
∧ ¬ Eq (Geos (Poi p2) add Emp) (Geos (Poi p3) add Emp) ∧ ¬ Eq (Geos (Poi p3) add Emp) (Geos (Poi p1) add Emp)
∧ ¬ Eq (Geos (Lin (Li p2 p1)) add Emp) (Geos (Lin (Li p2 p3)) add Emp)
and *Ang-rev : ⟦Cong (Geos (ang1) add Emp) (Geos (ang2) add Emp)⟧ ⟹*
Cong (Geos (ang2) add Emp) (Geos (ang1) add Emp)
and *Ang-roll : Cong (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p3 p2 p1)) add Emp)*
∧ Eq (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p3 p2 p1)) add Emp)
and *Ang-inside-def : Ang-inside (An p1 p2 p3) p ⟷ Def (Ang (An p1 p2 p3)) ∧ Plane-sameside (Li p2 p1) p3 p ∧ Plane-sameside (Li p2 p3) p1 p*
and *Ang-Point-swap : ⟦Def (Ang (An p1 p2 p3)); Line-on (Li p2 p1) p4; ¬ Bet-Point (Se p1 p4) p2;*
Line-on (Li p2 p3) p5; ¬ Bet-Point (Se p3 p5) p2; ¬ Eq (Geos (Poi p2) add Emp) (Geos (Poi p4) add Emp);
¬ Eq (Geos (Poi p2) add Emp) (Geos (Poi p5) add Emp)⟧ ⟹
Eq (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p4 p2 p5)) add Emp) ∧ Def (Ang (An p4 p2 p5))
and *Ang-Right-angle-def : Right-angle (An p1 p2 p3) ⟷*
(∃ p. Cong (Geos (Ang (An p1 p2 p3)) add Emp) (Geos (Ang (An p1 p2 p)) add Emp)
∧ Bet-Point (Se p3 p) p2 ∧ Def (Ang (An p1 p2 p3)) ∧ Def (Ang (An p1 p2 p)))
and *Tri-Cong-def : Cong (Geos (Tri (Tr p11 p12 p13)) add Emp) (Geos (Tri (Tr p21 p22 p23)) add Emp)*
⟷ Eq (Geos (Seg (Se p11 p12)) add Emp) (Geos (Seg (Se p21 p22)) add Emp)
∧ Eq (Geos (Seg (Se p12 p13)) add Emp) (Geos (Seg (Se p22 p23)) add Emp)
∧ Eq (Geos (Seg (Se p13 p11)) add Emp) (Geos (Seg (Se p23 p21)) add Emp)

∧ *Cong* (*Geos* (*Ang* (*An p12 p11 p13*)) *add Emp*) (*Geos* (*Ang* (*An p22 p21 p23*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An p11 p13 p12*)) *add Emp*) (*Geos* (*Ang* (*An p21 p23 p22*)) *add Emp*)
and *Ang-greater-def* : ⟦*Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*);
Plane-sameside (*Li p2 p3*) *p4 p1*⟧ ⟹
Ang-inside (*An p1 p2 p3*) *p4* ⟷ *Gr* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang an1*) *add Emp*)
and *Ang-less-def* : ⟦*Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*);
Plane-sameside (*Li p2 p3*) *p4 p1*; ¬ *Ang-inside* (*An p1 p2 p3*) *p4*;
¬ *Eq* (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*)⟧ ⟹
Gr (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*)

locale *Axiom-3 = Definition-3* +
assumes *Seg-add* : ⟦*Line-on l1 p11*; *Line-on l1 p12*; *Line-on l1 p13*; ¬ *Seg-on-Seg* (*Se p11 p12*) (*Se p12 p13*);
Line-on l2 p21; *Line-on l2 p22*; *Line-on l2 p23*; ¬ *Seg-on-Seg* (*Se p21 p22*) (*Se p22 p23*);
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*);
Eq (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)⟧ ⟹
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
and *Seg-sub* : ⟦*Line-on l1 p11*; *Line-on l1 p12*; *Line-on l1 p13*; ¬ *Seg-on-Seg* (*Se p11 p12*) (*Se p12 p13*);
Line-on l2 p21; *Line-on l2 p22*; *Line-on l2 p23*; ¬ *Seg-on-Seg* (*Se p21 p22*) (*Se p22 p23*);
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)⟧ ⟹
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)
and *Ang-move-sameside* : ⟦¬ *Line-on* (*Li p1 p2*) *p3*; *Def* (*Ang a1*)⟧ ⟹ ∃ *p*. *Cong* (*Geos* (*Ang a1*) *add Emp*) (*Geos* (*Ang* (*An p p1 p2*)) *add Emp*) ∧ *Plane-sameside* (*Li p1 p2*) *p p3*
and *Ang-move-diffside* : ⟦¬ *Line-on* (*Li p1 p2*) *p3*; *Def* (*Ang a1*)⟧ ⟹ ∃ *p*. *Cong* (*Geos* (*Ang a1*) *add Emp*) (*Geos* (*Ang* (*An p p1 p2*)) *add Emp*) ∧ *Plane-diffside* (*Li p1 p2*) *p p3*
and *Ang-move-unique* : ⟦*Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*);
Cong (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*);
Plane-sameside (*Li p2 p3*) *p1 p4*⟧ ⟹
Eq (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p2*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p1 p4*) *p2*
and *Tri-week-SAS* : ⟦*Def* (*Tri* (*Tr p11 p12 p13*)); *Def* (*Tri* (*Tr p21 p22 p23*));

Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*);
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*);
Cong (*Geos* (*Ang* (*An p12 p11 p13*)) *add Emp*) (*Geos* (*Ang* (*An p22 p21 p23*)) *add Emp*)⟧
⟹ *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*)

locale *Congruence-Rule* = *Axiom-3* +
assumes *Ang-weektrans* : ⟦*Eq* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang an2*) *add Emp*);
Cong (*Geos* (*Ang an2*) *add Emp*) (*Geos* (*Ang an3*) *add Emp*)⟧ ⟹ *Cong* (*Geos* (*Ang an1*) *add Emp*) (*Geos* (*Ang an3*) *add Emp*)

lemma (**in** *Congruence-Rule*) *Seg-Bet-add* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Bet-Point (*Se p21 p23*) *p22*
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p23*)) *add Emp*)
shows *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
proof −
from *assms* **have** ∃ *l*. *Line-on l p11* ∧ *Line-on l p13* ∧ *Line-on l p12* **by** (*simp add*:*Line-Bet-exist*)
then obtain *l1* :: *Line* **where** *P1* : *Line-on l1 p11* ∧ *Line-on l1 p13* ∧ *Line-on l1 p12* **by** *blast*
from *assms* **have** ∃ *l*. *Line-on l p21* ∧ *Line-on l p23* ∧ *Line-on l p22* **by** (*simp add*:*Line-Bet-exist*)
then obtain *l2* :: *Line* **where** *P2* : *Line-on l2 p21* ∧ *Line-on l2 p23* ∧ *Line-on l2 p22* **by** *blast*
from *assms* **have** *P3* : ¬ *Seg-on-Seg* (*Se p11 p12*) (*Se p12 p13*) **by** (*simp add*:*Seg-Bet-not-on*)
from *assms* **have** *P4* : ¬ *Seg-on-Seg* (*Se p21 p22*) (*Se p22 p23*) **by** (*simp add*:*Seg-Bet-not-on*)
from *assms P1 P2 P3 P4* **show** *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*) **by** (*blast intro*:*Seg-add*)
qed

lemma (**in** *Congruence-Rule*) *Tri-sinple-def* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
¬ *Line-on* (*Li A B*) *C*
shows *Def* (*Tri* (*Tr A B C*))
proof −
have *P1* : *Bet-Point* (*Se A B*) *C* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-Bet-on*)
from *assms P1* **have** *P2* : ¬ *Bet-Point* (*Se A B*) *C* **by** *blast*
from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)

by (*blast intro*:*Eq-rev*)
from *P3* **have** *P4* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-rev*)
have *P5* : *Bet-Point* (*Se B C*) *A* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-Bet-on*)
from *assms P5* **have** *P6* : ¬ *Bet-Point* (*Se B C*) *A* **by** *blast*
have *P7* : *Bet-Point* (*Se C A*) *B* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-Bet-on*)
from *assms P7* **have** *P8* : ¬ *Bet-Point* (*Se C A*) *B* **by** *blast*
have *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ ∃ *p*. *Bet-Point* (*Se A B*) *p* ∧ *Bet-Point* (*Se B C*) *p* **by** (*simp add*:*Seg-on-Seg-rule*)
then obtain *D* :: *Point* **where** *P9* : *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ *Bet-Point* (*Se A B*) *D* ∧ *Bet-Point* (*Se B C*) *D* **by** *blast*
have *P10* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *P9* **have** *P11* : *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ *Line-on* (*Li A B*) *D* **by** (*simp add*:*Line-Bet-on*)
have *P12* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
from *P9* **have** *P13* : *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P9* **have** *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ *Bet-Point* (*Se A B*) *D* **by** *simp*
then have *P14* : *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P10 P11 P12 P13 P14* **have** *P15* : *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P16* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P15 P16* **have** *P17* : *Seg-on-Seg* (*Se A B*) (*Se B C*) ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *assms P17* **have** *P18* : ¬ *Seg-on-Seg* (*Se A B*) (*Se B C*) **by** *blast*
have *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ ∃ *p*. *Bet-Point* (*Se B C*) *p* ∧ *Bet-Point* (*Se C A*) *p* **by** (*simp add*:*Seg-on-Seg-rule*)
then obtain *E* :: *Point* **where** *P19* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Bet-Point* (*Se B C*) *E* ∧ *Bet-Point* (*Se C A*) *E* **by** *blast*
then have *P20* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Line-on* (*Li B C*) *E* **by** (*simp add*:*Line-Bet-on*)
have *P21* : *Line-on* (*Li C A*) *C* **by** (*simp add*:*Line-on-rule*)
from *P19* **have** *P22* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Line-on* (*Li C A*) *E* **by** (*simp add*:*Line-Bet-on*)
from *P19* **have** *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Bet-Point* (*Se B C*) *E* **by** *simp*
then have *P23* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P16 P20 P21 P22 P23* **have** *P24* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Eq* (*Geos* (*Lin* (*Li C A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P25* : *Line-on* (*Li C A*) *A* **by** (*simp add*:*Line-on-rule*)
from *P24 P25* **have** *P26* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-on-trans*)
from *assms P3 P26* **have** *P27* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Line-on* (*Li B A*) *C* **by** (*blast intro*:*Line-on-rev*)

from *P4 P27* **have** *P28* : *Seg-on-Seg* (*Se B C*) (*Se C A*) ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *assms P28* **have** *P29* : ¬ *Seg-on-Seg* (*Se B C*) (*Se C A*) **by** *blast*
have *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ ∃ *p*. *Bet-Point* (*Se C A*) *p* ∧ *Bet-Point* (*Se A B*) *p* **by** (*simp add*:*Seg-on-Seg-rule*)
then obtain *F* :: *Point* **where** *P30* : *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ *Bet-Point* (*Se C A*) *F* ∧ *Bet-Point* (*Se A B*) *F* **by** *blast*
then have *P31* : *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ *Line-on* (*Li C A*) *F* **by** (*simp add*:*Line-Bet-on*)
have *P32* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
from *P30* **have** *P33* : *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ *Line-on* (*Li A B*) *F* **by** (*simp add*:*Line-Bet-on*)
from *P30* **have** *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ *Bet-Point* (*Se C A*) *F* **by** *simp*
then have *P34* : *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P25 P31 P32 P33 P34* **have** *P35* : *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ *Eq* (*Geos* (*Lin* (*Li C A*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P21 P35* **have** *P36* : *Seg-on-Seg* (*Se C A*) (*Se A B*) ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *assms P36* **have** *P37* : ¬ *Seg-on-Seg* (*Se C A*) (*Se A B*) **by** *blast*
from *assms P2 P6 P8 P18 P29 P37* **show** *Def* (*Tri* (*Tr A B C*)) **by** (*simp add*:*Tri-def*)
qed

lemma (**in** *Congruence-Rule*) *Tri-def-Line* :
assumes
Def (*Tri* (*Tr A B C*))
shows ¬ *Line-on* (*Li A B*) *C* ∧ ¬ *Line-on* (*Li B C*) *A* ∧ ¬ *Line-on* (*Li C A*) *B*
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
∧ ¬ *Bet-Point* (*Se A B*) *C* ∧ ¬ *Bet-Point* (*Se B C*) *A* ∧ ¬ *Bet-Point* (*Se C A*) *B* **by** (*simp add*:*Tri-def*)
have *P2* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
from *P1 P2 P3* **have** *P4* : *Line-on* (*Li A B*) *C* ⟹ *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Bet-case*)
from *P1* **have** *P5* : ¬ *Bet-Point* (*Se A C*) *B* **by** (*blast intro*:*Bet-rev*)
from *P1* **have** *P6* : ¬ *Bet-Point* (*Se C B*) *A* **by** (*blast intro*:*Bet-rev*)
from *P1* **have** *P7* : ¬ *Bet-Point* (*Se B A*) *C* **by** (*blast intro*:*Bet-rev*)
from *P4 P5 P6 P7* **have** *P8* : ¬ *Line-on* (*Li A B*) *C* **by** *blast*
have *P9* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P10* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P1 P9 P10* **have** *P11* : *Line-on* (*Li B C*) *A* ⟹ *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Bet-case*)
from *P5 P6 P7 P11* **have** *P12* : ¬ *Line-on* (*Li B C*) *A* **by** *blast*

have *P13* : *Line-on* (*Li C A*) *C* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li C A*) *A* **by** (*simp add*:*Line-on-rule*)
from *P1 P13 P14* **have** *P15* : *Line-on* (*Li C A*) *B* ⟹ *Bet-Point* (*Se A C*) *B* ∨ *Bet-Point* (*Se C B*) *A* ∨ *Bet-Point* (*Se B A*) *C* **by** (*simp add*:*Bet-case*)
from *P5 P6 P7 P15* **have** *P16* : ¬ *Line-on* (*Li C A*) *B* **by** *blast*
from *P8 P12 P16* **show** ¬ *Line-on* (*Li A B*) *C* ∧ ¬ *Line-on* (*Li B C*) *A* ∧ ¬ *Line-on* (*Li C A*) *B* **by** *simp*
qed

lemma (**in** *Congruence-Rule*) *Tri-def-trans* :
assumes
Def (*Tri* (*Tr A B C*))
shows *Def* (*Tri* (*Tr B C A*))
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ∧ ¬ *Bet-Point* (*Se A B*) *C* ∧ ¬ *Bet-Point* (*Se B C*) *A* ∧ ¬ *Bet-Point* (*Se C A*) *B* **by** (*simp add*:*Tri-def*)
from *assms* **have** *P2* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Tri-def-Line*)
from *P1 P2* **show** *Def* (*Tri* (*Tr B C A*)) **by** (*simp add*:*Tri-sinple-def*)
qed

lemma (**in** *Congruence-Rule*) *Tri-def-rev* :
assumes
Def (*Tri* (*Tr A B C*))
shows *Def* (*Tri* (*Tr C B A*))
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ∧ ¬ *Bet-Point* (*Se A B*) *C* ∧ ¬ *Bet-Point* (*Se B C*) *A* ∧ ¬ *Bet-Point* (*Se C A*) *B* **by** (*simp add*:*Tri-def*)
from *assms* **have** *P2* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Tri-def-Line*)
from *P1* **have** *P3* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li C B*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P2 P3* **have** *P4* : ¬ *Line-on* (*Li C B*) *A* **by** (*simp add*:*Line-not-on-trans*)
from *P1* **have** *P5* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1* **have** *P6* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1* **have** *P7* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P4 P5 P6 P7* **show** *Def* (*Tri* (*Tr C B A*)) **by** (*simp add*:*Tri-sinple-def*)
qed

lemma (**in** *Congruence-Rule*) *Tri-def-extension* :
assumes
Def (*Tri* (*Tr A B C*))

```
    ¬ Eq (Geos (Poi B) add Emp) (Geos (Poi D) add Emp)
    Line-on (Li B C) D
  shows Def (Tri (Tr A B D))
proof −
  from assms have P1 : ¬ Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)
by (simp add:Tri-def)
  from assms have P2 : ¬ Line-on (Li B C) A by (simp add:Tri-def-Line)
  from assms have P3 : Eq (Geos (Poi D) add Emp) (Geos (Poi A) add Emp)
⟹ Line-on (Li B C) A by (simp add:Point-Eq)
  from P2 P3 have P4 : ¬ Eq (Geos (Poi D) add Emp) (Geos (Poi A) add Emp)
by blast
  from assms have P5 : ¬ Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp)
by (simp add:Tri-def)
  from assms P5 have P6 : Line-on (Li B D) C by (simp add:Line-on-rev)
  have P7 : Line-on (Li B D) B by (simp add:Line-on-rule)
  have P8 : Line-on (Li B C) B by (simp add:Line-on-rule)
  have P9 : Line-on (Li B C) C by (simp add:Line-on-rule)
  from P5 P6 P7 P8 P9 have Eq (Geos (Lin (Li B D)) add Emp) (Geos (Lin (Li
B C)) add Emp) by (simp add:Line-unique)
   then have P10 : Line-on (Li B D) A ⟹ Line-on (Li B C) A by (simp
add:Line-on-trans)
  from P2 P10 have P11 : ¬ Line-on (Li B D) A by blast
  from assms P1 P4 P11 have Def (Tri (Tr B D A)) by (simp add:Tri-sinple-def)
  thus Def (Tri (Tr A B D)) by (simp add:Tri-def-trans)
qed

lemma (in Congruence-Rule) Ang-to-Tri :
  assumes
    Def (Ang (An A B C))
  shows Def (Tri (Tr A B C))
proof −
  from assms have P1 : ¬ Eq (Geos (Poi A) add Emp) (Geos (Poi B) add Emp)
    ∧ ¬ Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) ∧ ¬ Eq (Geos (Poi
C) add Emp) (Geos (Poi A) add Emp)
    ∧ ¬ Eq (Geos (Lin (Li B A)) add Emp) (Geos (Lin (Li B C)) add Emp) by
(simp add:Ang-def)
  have P2 : Line-on (Li B A) B by (simp add:Line-on-rule)
  have P3 : Line-on (Li B C) B by (simp add:Line-on-rule)
  have P4 : Line-on (Li B C) C by (simp add:Line-on-rule)
  from P1 have P5 : ¬ Eq (Geos (Poi B) add Emp) (Geos (Poi C) add Emp) by
simp
  from P1 have Eq (Geos (Lin (Li A B)) add Emp) (Geos (Lin (Li B A)) add
Emp) by (simp add:Line-rev)
   then have P6 : Line-on (Li A B) C ⟹ Line-on (Li B A) C by (simp
add:Line-on-trans)
  from P2 P3 P4 P5 P6 have P7 : Line-on (Li A B) C ⟹ Eq (Geos (Lin (Li
B A)) add Emp) (Geos (Lin (Li B C)) add Emp) by (simp add:Line-unique)
  from P1 P7 have P8 : ¬ Line-on (Li A B) C by blast
  from P1 P8 show Def (Tri (Tr A B C)) by (simp add:Tri-sinple-def)
```

qed

lemma (**in** *Congruence-Rule*) *Ang-sinple-def* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Line-on* (*Li A B*) *C*
shows *Def* (*Ang* (*An A B C*))
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *assms P1* **have** *P2* : ¬ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-not-on-trans*)
have *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
then have *P3* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Point-Eq*)
from *P2 P3* **have** *P4* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
have *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
then have *P5* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Point-Eq*)
from *P2 P5* **have** *P6* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
then have *P7* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-on-trans*)
from *P2 P7* **have** *P8* : ¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P4 P6 P8* **show** *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Ang-def*)
qed

lemma (**in** *Congruence-Rule*) *Tri-to-Ang* :
assumes
Def (*Tri* (*Tr A B C*))
shows *Def* (*Ang* (*An A B C*))
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *assms* **have** *P2* : ¬ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Tri-def-Line*)
from *P1 P2* **show** *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Ang-sinple-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-def-rev* :
assumes
Def (*Ang* (*An A B C*))
shows *Def* (*Ang* (*An C B A*))
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)

$\wedge \neg$ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Ang-def*)
have *P2* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
from *P1* **have** *P5* : $\neg$ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** *simp*
from *P2 P3 P4 P5* **have** *P6* : *Line-on* (*Li B C*) *A* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1 P6* **have** *P7* : $\neg$ *Line-on* (*Li B C*) *A* **by** *blast*
from *P1* **have** *P8* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li C B*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P7 P8* **have** *P9* : $\neg$ *Line-on* (*Li C B*) *A* **by** (*simp add*:*Line-not-on-trans*)
from *P1* **have** *P10* : $\neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9 P10* **show** *Def* (*Ang* (*An C B A*)) **by** (*simp add*:*Ang-sinple-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-def-inv* :
assumes
Def (*Ang* (*An A B C*))
shows *Def* (*Ang* (*An A C B*))
proof −
from *assms* **have** *P1* : $\neg$ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) $\wedge \neg$ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\wedge \neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*)
$\wedge \neg$ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Ang-def*)
have *P2* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
from *P1* **have** *P5* : $\neg$ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** *simp*
from *P2 P3 P4 P5* **have** *P6* : *Line-on* (*Li B C*) *A* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1 P6* **have** *P7* : $\neg$ *Line-on* (*Li B C*) *A* **by** *blast*
have *P8* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P1 P4 P8 P9* **have** *P10* : *Line-on* (*Li A C*) *B* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P11* : *Line-on* (*Li A C*) *A* **by** (*simp add*:*Line-on-rule*)
from *P10 P11* **have** *P12* : *Line-on* (*Li A C*) *B* $\Longrightarrow$ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-on-trans*)
from *P7 P12* **have** *P13* : $\neg$ *Line-on* (*Li A C*) *B* **by** *blast*
from *P1* **have** *P14* : $\neg$ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P13 P14* **show** *Def* (*Ang* (*An A C B*)) **by** (*simp add*:*Ang-sinple-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-def-extension* :
assumes
Def (*Ang* (*An A B C*))
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*)
Line-on (*Li B C*) *D*
shows *Def* (*Ang* (*An A B D*))
proof −
from *assms* **have** *P1* : *Def* (*Tri* (*Tr A B C*)) **by** (*simp add*:*Ang-to-Tri*)
from *assms P1* **have** *Def* (*Tri* (*Tr A B D*)) **by** (*simp add*:*Tri-def-extension*)
thus *Def* (*Ang* (*An A B D*)) **by** (*simp add*:*Tri-to-Ang*)
qed

lemma (**in** *Congruence-Rule*) *Bet-end-Point* :
shows ¬ *Bet-Point* (*Se p1 p1*) *p2*
proof
assume *W* : *Bet-Point* (*Se p1 p1*) *p2*
then have *P1* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def del*:*Eq-refl*)
have *P2* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** *simp*
from *P1 P2* **show** *False* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-Plane-sameside* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Line-on l1 D*
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Bet-Point* (*Se B C*) *A*
shows *Plane-sameside* (*Li D A*) *B C* ∨ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
proof −
have *Line-on* (*Li D A*) *D* **by** (*simp add*:*Line-on-rule*)
then have *P1* : *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) ⟹ *Line-on l1 D* **by** (*simp add*:*Line-on-trans*)
from *assms P1* **have** *P2* : ¬ *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** *blast*
have *Plane-diffside* (*Li D A*) *B C* ⟹ ∃ *p*. *Bet-Point* (*Se B C*) *p* ∧ *Line-on* (*Li D A*) *p* ∧ ¬ *Line-on* (*Li D A*) *B* ∧ ¬ *Line-on* (*Li D A*) *C* **by** (*simp add*:*Plane-diffside-def*)
then obtain *E* :: *Point* **where** *P3* : *Plane-diffside* (*Li D A*) *B C* ⟹ *Bet-Point* (*Se B C*) *E* ∧ *Line-on* (*Li D A*) *E* **by** *blast*
then have *P4* : *Plane-diffside* (*Li D A*) *B C* ⟹ *Line-on* (*Li B C*) *E* **by** (*simp add*:*Line-Bet-on*)
have *P5* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P6* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P5 P6* **have** *P7* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add*

Emp) $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P4 P7* **have** *P8* : *Plane-diffside* (*Li D A*) *B C* $\Longrightarrow$ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\Longrightarrow$
Line-on l1 E **by** (*simp add*:*Line-on-trans*)
have *P9* : *Line-on* (*Li D A*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms P2 P3 P8 P9* **have** *P10* : *Plane-diffside* (*Li D A*) *B C* $\Longrightarrow$ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\Longrightarrow$
Eq (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P3* **have** *P11* : *Plane-diffside* (*Li D A*) *B C* $\Longrightarrow$ *Bet-Point* (*Se B C*) *E* **by** *simp*
from *P10 P11* **have** *P12* : *Plane-diffside* (*Li D A*) *B C* $\Longrightarrow$ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se B C*) *A* **by** (*simp add*:*Point-Eq*)
from *assms P12* **have** ¬ *Plane-diffside* (*Li D A*) *B C* ∨ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
then have *P13* : ¬ *Plane-diffside* (*Li D A*) *B C* ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
∨ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
from *assms P9* **have** *P14* : *Line-on* (*Li D A*) *B* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P2 P14* **have** *P15* : ¬ *Line-on* (*Li D A*) *B* **by** *blast*
from *assms P9* **have** *P16* : *Line-on* (*Li D A*) *C* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li D A*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P2 P16* **have** *P17* : ¬ *Line-on* (*Li D A*) *C* **by** *blast*
from *P15 P17* **have** *P18* : ¬ *Plane-diffside* (*Li D A*) *B C* ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\Longrightarrow$
Plane-sameside (*Li D A*) *B C* **by** (*simp add*:*Plane-not-diffside-sameside*)
from *P13 P18* **show** *Plane-sameside* (*Li D A*) *B C* ∨ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-move-unique* :
assumes
Line-on l1 A
Line-on l1 B
Line-on l1 C
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
¬ *Bet-Point* (*Se B C*) *A*
shows *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
proof −
have ∃ *p q r*. ¬ *Line-on l1 p* ∧ ¬ *Line-on l1 q* ∧ ¬ *Line-on l1 r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *D* :: *Point* **where** *P1* : ¬ *Line-on l1 D* **by** *blast*

have *P2* : *Line-on* (*Li A D*) *D* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms P3 P4* **have** *P5* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P5* **have** *P6* : *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *P1 P5* **have** *P7* : ¬ *Line-on* (*Li A B*) *D* **by** (*simp add*:*Line-not-on-trans*)
from *assms P7* **have** *Def* (*Ang* (*An A B D*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P8* : *Def* (*Ang* (*An D A B*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P9* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P10* : ¬ *Bet-Point* (*Se D D*) *A* **by** (*simp add*:*Bet-end-Point*)
from *assms P2 P6 P8 P9 P10* **have** *Eq* (*Geos* (*Ang* (*An D A B*)) *add Emp*) (*Geos* (*Ang* (*An D A C*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
then have *P11* : *Cong* (*Geos* (*Ang* (*An D A B*)) *add Emp*) (*Geos* (*Ang* (*An D A C*)) *add Emp*) **by** (*simp add*:*Ang-weektrans*)
from *assms P7* **have** *Def* (*Tri* (*Tr A B D*)) **by** (*blast intro*:*Ang-sinple-def Ang-to-Tri*)
then have *Def* (*Tri* (*Tr D B A*)) **by** (*simp add*:*Tri-def-rev*)
then have *P12* : *Def* (*Tri* (*Tr A D B*)) **by** (*simp add*:*Tri-def-trans*)
have *P13* : *Line-on* (*Li A C*) *A* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P13 P14* **have** *P15* : *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1 P15* **have** *P16* : ¬ *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-not-on-trans*)
from *assms P16* **have** *Def* (*Tri* (*Tr A C D*)) **by** (*blast intro*:*Ang-sinple-def Ang-to-Tri*)
then have *Def* (*Tri* (*Tr D C A*)) **by** (*simp add*:*Tri-def-rev*)
then have *P17* : *Def* (*Tri* (*Tr A D C*)) **by** (*simp add*:*Tri-def-trans*)
from *assms P11 P12 P17* **have** *P18* : *Cong* (*Geos* (*Ang* (*An B D A*)) *add Emp*) (*Geos* (*Ang* (*An C D A*)) *add Emp*) **by** (*simp add*:*Tri-week-SAS*)
have *P19* : *Cong* (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An B D A*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P20* : *Eq* (*Geos* (*Ang* (*An B D A*)) *add Emp*) (*Geos* (*Ang* (*An A D B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P18 P20* **have** *P21* : *Cong* (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An C D A*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
from *assms P1* **have** *P22* : *Plane-sameside* (*Li D A*) *B C* ∨ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Seg-Plane-sameside*)
from *P19 P21* **have** *P23* : *Plane-sameside* (*Li D A*) *B C* ⟹ *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li C D*)) *add Emp*) **by** (*simp add*:*Ang-move-unique*)
have *P24* : *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
from *P23 P24* **have** *P25* : *Plane-sameside* (*Li D A*) *B C* ⟹ *Line-on* (*Li C D*) *B* **by** (*simp add*:*Line-on-trans*)
have *P26* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P27* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P28* : *Line-on* (*Li C D*) *C* **by** (*simp add*:*Line-on-rule*)

from *P25 P26 P27 P28* **have** *P29* : *Plane-sameside* (*Li D A*) *B C* ⟹ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li C D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P30* : *Line-on* (*Li C D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P29 P30* **have** *P31* : *Plane-sameside* (*Li D A*) *B C* ⟹ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹
Line-on (*Li B C*) *D* **by** (*simp add*:*Line-on-trans*)
from *assms P26 P27* **have** *P32* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P31 P32* **have** *P33* : *Plane-sameside* (*Li D A*) *B C* ⟹ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹
Line-on l1 D **by** (*simp add*:*Line-on-trans*)
from *P1 P33* **have** *P34* : *Plane-sameside* (*Li D A*) *B C* ⟹ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
from *P22 P34* **show** *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-not-Eq-Point* :
assumes
¬ *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
shows ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
proof −
have *P1* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*) **by** (*simp add*:*Seg-Point-Eq*)
from *assms P1* **show** ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-replace* :
assumes
Def (*Ang* (*An A B C*))
Def (*Ang* (*An A1 B1 C1*))
Cong (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
shows ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ *Line-on* (*Li B1 A1*) *p* ∧ ¬ *Bet-Point* (*Se p A1*) *B1* ∧ *Def* (*Ang* (*An p B1 C1*))
and ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧

Line-on (*Li B1 C1*) *p* ∧ ¬ *Bet-Point* (*Se p C1*) *B1* ∧ *Def* (*Ang* (*An A1 B1 p*))
and ∃ *p q*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ *Line-on* (*Li B1 A1*) *p* ∧ ¬ *Bet-Point* (*Se p A1*) *B1*
∧ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 q*)) *add Emp*) ∧ *Line-on* (*Li B1 C1*) *q* ∧ ¬ *Bet-Point* (*Se q C1*) *B1* ∧ *Def* (*Ang* (*An p B1 q*))
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P2* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P3* : *Line-on* (*Li B1 A1*) *A1* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li B1 A1*) *B1* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P5* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P2 P3 P4 P5* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p A1*) *B1* ∧ *Line-on* (*Li B1 A1*) *p* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *A2* :: *Point* **where** *P6* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 A2*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se A2 A1*) *B1* ∧ *Line-on* (*Li B1 A1*) *A2* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** *blast*
from *assms* **have** *P7* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) **by** (*simp add*:*Ang-def*)
have *P8* : *Line-on* (*Li B1 C1*) *B1* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li B1 C1*) *C1* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P10* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P7 P8 P9 P10* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p C1*) *B1* ∧ *Line-on* (*Li B1 C1*) *p* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *C2* :: *Point* **where** *P11* : *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 C2*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se C2 C1*) *B1* ∧ *Line-on* (*Li B1 C1*) *C2* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** *blast*
have *P12* : ¬ *Bet-Point* (*Se C1 C1*) *B1* **by** (*simp add*:*Bet-end-Point*)
from *P6* **have** *P13* : ¬ *Bet-Point* (*Se A1 A2*) *B1* **by** (*blast intro*:*Bet-rev*)
from *assms P3 P6 P7 P9 P12 P13* **have** *P14* : *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 C1*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *assms P14* **have** *P15* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 C1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms* **have** *P16* : ¬ *Eq* (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P6* **have** *P17* : *Line-on* (*Li B1 A1*) *A2* **by** *simp*
from *P6* **have** *P18* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** *simp*

from *P4 P8 P17 P18* **have** *P19* : *Line-on* (*Li B1 C1*) *A2* ⟹ *Eq* (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P16 P19* **have** *P20* : ¬ *Line-on* (*Li B1 C1*) *A2* **by** *blast*
from *P7 P20* **have** *Def* (*Ang* (*An B1 C1 A2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P21* : *Def* (*Ang* (*An A2 B1 C1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P6 P14 P15 P21* **show** ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ *Line-on* (*Li B1 A1*) *p* ∧ ¬ *Bet-Point* (*Se p A1*) *B1* ∧ *Def* (*Ang* (*An p B1 C1*)) **by** *blast*
have *P22* : ¬ *Bet-Point* (*Se A1 A1*) *B1* **by** (*simp add*:*Bet-end-Point*)
from *P11* **have** *P23* : ¬ *Bet-Point* (*Se C1 C2*) *B1* **by** (*blast intro*:*Bet-rev*)
from *assms P2 P3 P7 P11 P22 P23* **have** *P24* : *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C2*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *assms P24* **have** *P25* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P11* **have** *P26* : *Line-on* (*Li B1 C1*) *C2* **by** *simp*
from *P11* **have** *P27* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** *simp*
from *P4 P8 P26 P27* **have** *P28* : *Line-on* (*Li B1 A1*) *C2* ⟹ *Eq* (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P16 P28* **have** *P29*: ¬ *Line-on* (*Li B1 A1*) *C2* **by** *blast*
from *P2 P29* **have** *Def* (*Ang* (*An B1 A1 C2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P30* : *Def* (*Ang* (*An A1 B1 C2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P11 P24 P25 P30* **show** ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ *Line-on* (*Li B1 C1*) *p* ∧ ¬ *Bet-Point* (*Se p C1*) *B1* ∧ *Def* (*Ang* (*An A1 B1 p*)) **by** *blast*
from *assms P6 P11 P13 P17 P23 P26* **have** *P31* : *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 C2*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *assms P31* **have** *P32* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P33* : *Line-on* (*Li B1 C2*) *B1* **by** (*simp add*:*Line-on-rule*)
have *P34* : *Line-on* (*Li B1 C2*) *C2* **by** (*simp add*:*Line-on-rule*)
from *P8 P26 P27 P33 P34* **have** *Eq* (*Geos* (*Lin* (*Li B1 C2*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P35* : *Line-on* (*Li B1 C2*) *A2* ⟹ *Line-on* (*Li B1 C1*) *A2* **by** (*simp add*:*Line-on-trans*)
from *P20 P35* **have** *P36* : ¬ *Line-on* (*Li B1 C2*) *A2* **by** *blast*
from *P11 P36* **have** *Def* (*Ang* (*An B1 C2 A2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P37* : *Def* (*Ang* (*An A2 B1 C2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P6 P11 P21 P30 P31 P32 P37* **show** ∃ *p q*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 q*)) *add*

Emp)
∧ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*) ∧ *Line-on* (*Li B1 A1*) *p* ∧ ¬ *Bet-Point* (*Se p A1*) *B1*
∧ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 q*)) *add Emp*) ∧ *Line-on* (*Li B1 C1*) *q* ∧ ¬ *Bet-Point* (*Se q C1*) *B1* ∧ *Def* (*Ang* (*An p B1 q*))
by *blast*
qed

Theorem11

theorem (**in** *Congruence-Rule*) *Tri-isosceles*:
assumes
Def (*Tri* (*Tr A B C*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se A B*)) *add Emp*) **by** (*simp add*:*Eq-rev*)
have *P2* : *Cong* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An C A B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms* **have** *Def* (*Tri* (*Tr C B A*)) **by** (*simp add*:*Tri-def-rev*)
then have *P3* : *Def* (*Tri* (*Tr A C B*)) **by** (*simp add*:*Tri-def-trans*)
from *assms P1 P2 P3* **have** *P4* : *Cong* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*simp add*:*Tri-week-SAS*)
have *P5* : *Eq* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P4 P5* **have** *P6* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
have *P7* : *Eq* (*Geos* (*Ang* (*An B C A*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P6 P7* **show** *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev Ang-rev*)
qed

lemma (**in** *Congruence-Rule*) *Tri-week-ASA* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Def (*Tri* (*Tr A1 B1 C1*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 A1*)) *add Emp*)
shows ¬¬ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*)
proof
assume *W* : ¬ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*)
have *P1* : *Line-on* (*Li B1 C1*) *B1* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li B1 C1*) *C1* **by** (*simp add*:*Line-on-rule*)

from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P1 P2 P3 P4* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p C1*) *B1* ∧ *Line-on* (*Li B1 C1*) *p* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *D1* :: *Point* **where** *P5* : *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 D1*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se D1 C1*) *B1* ∧ *Line-on* (*Li B1 C1*) *D1* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** *blast*
from *W* **have** *P6* : ¬ *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B C*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P5 P6* **have** ¬ *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B1 D1*)) *add Emp*) **by** (*simp add*:*Eq-not-trans*)
then have *P7* : ¬ *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** (*simp add*:*Seg-not-Eq-Point*)
from *assms* **have** *P8* : ¬ *Line-on* (*Li B1 C1*) *A1* **by** (*simp add*:*Tri-def-Line*)
from *P5* **have** *P9* : *Line-on* (*Li B1 C1*) *D1* **by** *simp*
then have *P10* : *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Line-on* (*Li B1 C1*) *A1* **by** (*simp add*:*Point-Eq*)
from *P8 P10* **have** *P11* : ¬ *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** *blast*
from *assms* **have** *P12* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*) **by** (*simp add*:*Tri-def*)
have *P13* : *Line-on* (*Li A1 B1*) *B1* **by** (*simp add*:*Line-on-rule*)
from *P5* **have** *P14* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** *simp*
from *assms P7 P9 P14* **have** *Def* (*Tri* (*Tr A1 B1 D1*)) **by** (*blast intro*:*Tri-def-extension*)
then have *Def* (*Tri* (*Tr D1 B1 A1*)) **by** (*simp add*:*Tri-def-rev*)
then have *P15* : *Def* (*Tri* (*Tr B1 A1 D1*)) **by** (*simp add*:*Tri-def-trans*)
from *assms* **have** *P16* : *Def* (*Tri* (*Tr B A C*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms* **have** *P17* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-trans*)
from *P5* **have** *P18* : *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 D1*)) *add Emp*) **by** *simp*
have *P19* : *Line-on* (*Li B1 A1*) *A1* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *Def* (*Tri* (*Tr C1 B1 A1*)) **by** (*simp add*:*Tri-def-rev*)
then have *P20* : *Def* (*Ang* (*An C1 B1 A1*)) **by** (*simp add*:*Tri-to-Ang*)
have *P21* : *Line-on* (*Li B1 A1*) *A1* **by** (*simp add*:*Line-on-rule*)
have *P22* : ¬ *Bet-Point* (*Se A1 A1*) *B1* **by** (*simp add*:*Bet-end-Point*)
from *P5* **have** *P23* : ¬ *Bet-Point* (*Se C1 D1*) *B1* **by** (*blast intro*:*Bet-rev*)
from *P12* **have** *P24* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9 P14 P19 P20 P21 P22 P23 P24* **have** *P25* : *Eq* (*Geos* (*Ang* (*An C1 B1 A1*)) *add Emp*) (*Geos* (*Ang* (*An D1 B1 A1*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *assms P25* **have** *P26* : *Cong* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos*

(*Ang* (*An D1 B1 A1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P27* : *Eq* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P26 P27* **have** *P28* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D1 B1 A1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
have *P29* : *Eq* (*Geos* (*Ang* (*An D1 B1 A1*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 D1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P28 P29* **have** *P30* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 D1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev Ang-rev*)
from *P15 P16 P17 P18 P30* **have** *P31* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An D1 A1 B1*)) *add Emp*) **by** (*simp add*:*Tri-week-SAS*)
from *P1 P2 P3 P5 P8 P23* **have** *P32* : *Plane-sameside* (*Li A1 B1*) *C1 D1* ∨ *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** (*simp add*:*Seg-Plane-sameside*)
from *assms P31* **have** *P33* : *Plane-sameside* (*Li A1 B1*) *C1 D1* ⟹ *Eq* (*Geos* (*Lin* (*Li C1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li D1 A1*)) *add Emp*) **by** (*simp add*:*Ang-move-unique*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P34* : *Eq* (*Geos* (*Lin* (*Li C1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 C1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P11* **have** *P35* : *Eq* (*Geos* (*Lin* (*Li D1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P33 P34 P35* **have** *P36* : *Plane-sameside* (*Li A1 B1*) *C1 D1* ⟹ *Eq* (*Geos* (*Lin* (*Li A1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
have *P37* : *Line-on* (*Li A1 C1*) *C1* **by** (*simp add*:*Line-on-rule*)
have *P38* : *Line-on* (*Li A1 C1*) *A1* **by** (*simp add*:*Line-on-rule*)
have *P39* : *Line-on* (*Li A1 D1*) *D1* **by** (*simp add*:*Line-on-rule*)
have *P40* : *Line-on* (*Li A1 D1*) *A1* **by** (*simp add*:*Line-on-rule*)
from *P37* **have** *P41* : *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) ⟹ *Line-on* (*Li A1 C1*) *D1* **by** (*simp add*:*Point-Eq*)
from *P11 P38 P39 P40 P41* **have** *P42* : *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li A1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P32 P36 P42* **have** *P43* : *Eq* (*Geos* (*Lin* (*Li A1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** *blast*
from *P2 P7 P9 P37* **have** *P44* : *Line-on* (*Li A1 C1*) *D1* ⟹ *Eq* (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 C1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1 P44* **have** *P45* : *Line-on* (*Li A1 C1*) *D1* ⟹ *Line-on* (*Li A1 C1*) *B1* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** *Def* (*Tri* (*Tr C1 B1 A1*)) **by** (*simp add*:*Tri-def-rev*)
then have *P46* : ¬ *Line-on* (*Li A1 C1*) *B1* **by** (*simp add*:*Tri-def-Line*)
from *P45 P46* **have** *P47* : ¬ *Line-on* (*Li A1 C1*) *D1* **by** *blast*
have *P48* : *Line-on* (*Li A1 D1*) *D1* **by** (*simp add*:*Line-on-rule*)
from *P47 P48* **have** *P49* : ¬ *Eq* (*Geos* (*Lin* (*Li A1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** (*simp add*:*Line-not-on-Eq*)
from *P43 P49* **show** *False* **by** *blast*
qed

Theorem12

theorem (**in** *Congruence-Rule*) *Tri-SAS*:
assumes
Def (*Tri* (*Tr A B C*))
Def (*Tri* (*Tr A1 B1 C1*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*)
shows *Cong* (*Geos* (*Tri* (*Tr A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*)
proof −
from *assms* **have** *P1* : *Cong* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 A1*)) *add Emp*) **by** (*simp add*:*Tri-week-SAS*)
have *P2* : *Eq* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An C A B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P3* : *Eq* (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P2* **have** *P4* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
from *P3 P4* **have** *P5* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms P1 P5* **have** *P6* : ¬¬ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) **by** (*simp add*:*Tri-week-ASA*)
from *assms* **have** *P7* : *Eq* (*Geos* (*Seg* (*Se C A*)) *add Emp*) (*Geos* (*Seg* (*Se C1 A1*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-rev Eq-trans*)
from *assms* **have** *P8* : *Def* (*Tri* (*Tr A C B*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms* **have** *P9* : *Def* (*Tri* (*Tr A1 C1 B1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms P5 P8 P9* **have** *P10* : *Cong* (*Geos* (*Ang* (*An B C A*)) *add Emp*) (*Geos* (*Ang* (*An B1 C1 A1*)) *add Emp*) **by** (*simp add*:*Tri-week-SAS*)
have *P11* : *Eq* (*Geos* (*Ang* (*An B C A*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P12* : *Eq* (*Geos* (*Ang* (*An B1 C1 A1*)) *add Emp*) (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P10 P11* **have** *P13* : *Cong* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An B1 C1 A1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
from *P10 P12 P13* **have** *P14* : *Cong* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms P1 P6 P7 P14* **show** *Cong* (*Geos* (*Tri* (*Tr A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
qed

Theorem13

theorem (**in** *Congruence-Rule*) *Tri-ASA*:
assumes
Def (*Tri* (*Tr A B C*))
Def (*Tri* (*Tr A1 B1 C1*))
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*)

Cong (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 A1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*)
shows *Cong* (*Geos* (*Tri* (*Tr A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*)
proof −
from *assms* **have** *P1* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-rev Eq-trans*)
from *assms* **have** *P2* : *Def* (*Tri* (*Tr B A C*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms* **have** *P3* : *Def* (*Tri* (*Tr B1 A1 C1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms P1 P2 P3* **have** *P4* : ¬¬ *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) **by** (*simp add*:*Tri-week-ASA*)
have *P5* : *Eq* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P6* : *Eq* (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P5* **have** *P7* : *Cong* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
from *P6 P7* **have** *P8* : *Cong* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev Ang-rev*)
from *assms P1 P4 P8* **show** *Cong* (*Geos* (*Tri* (*Tr A B C*)) *add Emp*) (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
qed

Theorem14

theorem (**in** *Congruence-Rule*) *Ang-complementary* :
assumes
Def (*Ang* (*An A B C*))
Def (*Ang* (*An A1 B1 C1*))
Cong (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
Bet-Point (*Se A D*) *B*
Bet-Point (*Se A1 D1*) *B1*
shows
Cong (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 D1*)) *add Emp*)
proof −
from *assms* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B1 C1*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*)
∧ *Line-on* (*Li B1 A1*) *p* ∧ ¬ *Bet-Point* (*Se p A1*) *B1* ∧ *Def* (*Ang* (*An p B1 C1*)) **by** (*simp add*:*Ang-replace*)
then obtain *A2* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 C1*)) *add Emp*)

∧ *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B1 A2*)) *add Emp*) ∧ *Line-on* (*Li B1 A1*) *A2*

∧ ¬ *Bet-Point* (*Se A2 A1*) *B1* ∧ *Def* (*Ang* (*An A2 B1 C1*)) **by** *blast*

from *assms P1* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 p*)) *add Emp*)

∧ *Eq* (*Geos* (*Ang* (*An A2 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 p*)) *add Emp*)

∧ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*)

∧ *Line-on* (*Li B1 C1*) *p* ∧ ¬ *Bet-Point* (*Se p C1*) *B1* ∧ *Def* (*Ang* (*An A2 B1 p*)) **by** (*simp add*:*Ang-replace*)

then obtain *C2* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A2 B1 C2*)) *add Emp*)

∧ *Eq* (*Geos* (*Seg* (*Se B C*)) *add Emp*) (*Geos* (*Seg* (*Se B1 C2*)) *add Emp*) ∧ *Line-on* (*Li B1 C1*) *C2*

∧ ¬ *Bet-Point* (*Se C2 C1*) *B1* ∧ *Def* (*Ang* (*An A2 B1 C2*)) **by** *blast*

from *assms* **have** *Def* (*Tri* (*Tr A B C*)) **by** (*simp add*:*Ang-to-Tri*)

then have *P3* : *Def* (*Tri* (*Tr B A C*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)

from *P2* **have** *Def* (*Tri* (*Tr A2 B1 C2*)) **by** (*simp add*:*Ang-to-Tri*)

then have *P4* : *Def* (*Tri* (*Tr B1 A2 C2*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)

from *P1 P2 P3 P4* **have** *P5* : *Cong* (*Geos* (*Tri* (*Tr B A C*)) *add Emp*) (*Geos* (*Tri* (*Tr B1 A2 C2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)

then have *P6* : *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)

from *P5* **have** *P7* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 B1*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)

have *P8* : *Line-on* (*Li B1 D1*) *B1* **by** (*simp add*:*Line-on-rule*)

from *assms* **have** *P9* : *Line-on* (*Li A1 D1*) *B1* **by** (*simp add*:*Line-Bet-on*)

from *assms* **have** *p10* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)

from *assms* **have** *P11* : *Line-on* (*Li B1 A1*) *D1* **by** (*simp add*:*Line-Bet-on*)

have *P12* : *Line-on* (*Li B1 D1*) *D1* **by** (*simp add*:*Line-on-rule*)

have *P13* : *Line-on* (*Li B1 A1*) *B1* **by** (*simp add*:*Line-on-rule*)

from *assms* **have** *P14* : ¬ *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)

from *P8 P11 P12 P13 P14* **have** *P15* : *Eq* (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 D1*)) *add Emp*) **by** (*simp add*:*Line-unique*)

from *P1 P15* **have** *P16* : *Line-on* (*Li B1 D1*) *A2* **by** (*simp add*:*Line-on-trans*)

from *P4* **have** *P17* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Tri-def*)

from *assms* **have** *P18* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)

then have *P19* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)

from *P8 P16 P17 P19* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se B D*)) *add Emp*) (*Geos* (*Seg* (*Se B1 p*)) *add Emp*)

∧ *Bet-Point* (*Se p A2*) *B1* ∧ *Line-on* (*Li B1 D1*) *p* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-diffside*)

then obtain *D2* :: *Point* **where** *P20* : *Eq* (*Geos* (*Seg* (*Se B D*)) *add Emp*) (*Geos* (*Seg* (*Se B1 D2*)) *add Emp*)

∧ *Bet-Point* (*Se D2 A2*) *B1* ∧ *Line-on* (*Li B1 D1*) *D2* ∧ ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi D2*) *add Emp*) **by** *blast*
from *P1* **have** *P21* : *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B1*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-trans*)
from *P20* **have** *P22* : *Bet-Point* (*Se A2 D2*) *B1* **by** (*blast intro*:*Bet-rev*)
from *assms P20 P21 P22* **have** *P23* : *Eq* (*Geos* (*Seg* (*Se A D*)) *add Emp*) (*Geos* (*Seg* (*Se A2 D2*)) *add Emp*) **by** (*blast intro*:*Seg-Bet-add*)
from *P3* **have** *P24* : *Def* (*Tri* (*Tr C A B*)) **by** (*blast intro*:*Tri-def-rev*)
from *assms* **have** *P25* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms* **have** *P26* : *Line-on* (*Li A B*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P24 P25 P26* **have** *P27* : *Def* (*Tri* (*Tr C A D*)) **by** (*simp add*:*Tri-def-extension*)
from *P4* **have** *P28* : *Def* (*Tri* (*Tr C2 A2 B1*)) **by** (*blast intro*:*Tri-def-rev*)
from *P22* **have** *P29* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi D2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P22* **have** *P30* : *Line-on* (*Li A2 B1*) *D2* **by** (*simp add*:*Line-Bet-on*)
from *P28 P29 P30* **have** *P31* : *Def* (*Tri* (*Tr C2 A2 D2*)) **by** (*simp add*:*Tri-def-extension*)
from *P24* **have** *P32* : *Def* (*Ang* (*An C A B*)) **by** (*simp add*:*Tri-to-Ang*)
from *P27* **have** *P33* : *Def* (*Ang* (*An C A D*)) **by** (*simp add*:*Tri-to-Ang*)
have *P34* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P35* : ¬ *Bet-Point* (*Se C C*) *A* **by** (*simp add*:*Bet-end-Point*)
from *assms* **have** *Inv* (*Bet-Point* (*Se D B*) *A*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se D B*) *A* **by** (*simp add*:*Inv-def*)
then have *P36* : ¬ *Bet-Point* (*Se B D*) *A* **by** (*blast intro*:*Bet-rev*)
from *P33* **have** ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P37* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P25 P26 P32 P33 P34 P35 P36 P37* **have** *P38* : *Eq* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *P28* **have** *P39* : *Def* (*Ang* (*An C2 A2 B1*)) **by** (*simp add*:*Tri-to-Ang*)
from *P31* **have** *P40* : *Def* (*Ang* (*An C2 A2 D2*)) **by** (*simp add*:*Tri-to-Ang*)
have *P41* : *Line-on* (*Li A2 C2*) *C2* **by** (*simp add*:*Line-on-rule*)
have *P42* : ¬ *Bet-Point* (*Se C2 C2*) *A2* **by** (*simp add*:*Bet-end-Point*)
from *P20* **have** *Inv* (*Bet-Point* (*Se B1 D2*) *A2*) **by** (*simp add*:*Bet-iff*)
then have *P43* : ¬ *Bet-Point* (*Se B1 D2*) *A2* **by** (*simp add*:*Inv-def*)
from *P40* **have** ¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P44* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P29 P30 P39 P40 P41 P42 P43 P44* **have** *P45* : *Eq* (*Geos* (*Ang* (*An C2 A2 B1*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 D2*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *P7 P38* **have** *P46* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 B1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
from *P45 P46* **have** *P47* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 D2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev Ang-rev*)
from *P27* **have** *P48* : *Def* (*Tri* (*Tr A C D*)) **by** (*blast intro*:*Tri-def-trans Tri-def-rev*)
from *P31* **have** *P49* : *Def* (*Tri* (*Tr A2 C2 D2*)) **by** (*blast intro*:*Tri-def-trans*

Tri-def-rev)
from *P6 P23 P47 P48 P49* **have** *P50* : *Cong* (*Geos* (*Tri* (*Tr A C D*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 C2 D2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P51* : *Cong* (*Geos* (*Ang* (*An A D C*)) *add Emp*) (*Geos* (*Ang* (*An A2 D2 C2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P50* **have** *Eq* (*Geos* (*Seg* (*Se C D*)) *add Emp*) (*Geos* (*Seg* (*Se C2 D2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
then have *P52* : *Eq* (*Geos* (*Seg* (*Se D C*)) *add Emp*) (*Geos* (*Seg* (*Se D2 C2*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-trans*)
from *assms* **have** *P53* : *Line-on* (*Li D A*) *B* **by** (*simp add*:*Bet-rev Line-Bet-on*)
from *assms* **have** *Inv* (*Bet-Point* (*Se B A*) *D*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se B A*) *D* **by** (*simp add*:*Inv-def*)
then have *P54* : ¬ *Bet-Point* (*Se A B*) *D* **by** (*blast intro*:*Bet-rev*)
have *P55* : *Line-on* (*Li D C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P56* : ¬ *Bet-Point* (*Se C C*) *D* **by** (*simp add*:*Bet-end-Point*)
from *P48* **have** *P57* : *Def* (*Ang* (*An A D C*)) **by** (*simp add*:*Tri-to-Ang Ang-def-inv*)
from *P57* **have** *P58* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P18 P53 P54 P55 P56 P57 P58* **have** *P59* : *Eq* (*Geos* (*Ang* (*An A D C*)) *add Emp*) (*Geos* (*Ang* (*An B D C*)) *add Emp*) ∧ *Def* (*Ang* (*An B D C*)) **by** (*simp add*:*Ang-Point-swap*)
from *P22* **have** *P60* : *Line-on* (*Li D2 A2*) *B1* **by** (*simp add*:*Line-Bet-on*)
from *P20* **have** *Inv* (*Bet-Point* (*Se A2 B1*) *D2*) **by** (*simp add*:*Bet-iff*)
then have *P61* : ¬ *Bet-Point* (*Se A2 B1*) *D2* **by** (*simp add*:*Inv-def*)
have *P62* : *Line-on* (*Li D2 C2*) *C2* **by** (*simp add*:*Line-on-rule*)
have *P63* : ¬ *Bet-Point* (*Se C2 C2*) *D2* **by** (*simp add*:*Bet-end-Point*)
from *P49* **have** *P64* : *Def* (*Ang* (*An A2 D2 C2*)) **by** (*simp add*:*Tri-to-Ang Ang-def-inv*)
from *P20* **have** *P65* : ¬ *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P64* **have** *P66* : ¬ *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P60 P61 P62 P63 P64 P65 P66* **have** *P67* : *Eq* (*Geos* (*Ang* (*An A2 D2 C2*)) *add Emp*) (*Geos* (*Ang* (*An B1 D2 C2*)) *add Emp*) ∧ *Def* (*Ang* (*An B1 D2 C2*)) **by** (*simp add*:*Ang-Point-swap*)
from *P51 P59* **have** *P68* : *Cong* (*Geos* (*Ang* (*An B D C*)) *add Emp*) (*Geos* (*Ang* (*An A2 D2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev*)
from *P67 P68* **have** *P69* : *Cong* (*Geos* (*Ang* (*An B D C*)) *add Emp*) (*Geos* (*Ang* (*An B1 D2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Eq-rev Ang-rev*)
from *P59* **have** *Def* (*Tri* (*Tr B D C*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P70* : *Def* (*Tri* (*Tr D B C*)) **by** (*blast intro*:*Tri-def-trans Tri-def-rev*)
from *P67* **have** *Def* (*Tri* (*Tr B1 D2 C2*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P71* : *Def* (*Tri* (*Tr D2 B1 C2*)) **by** (*blast intro*:*Tri-def-trans Tri-def-rev*)
from *P20* **have** *Eq* (*Geos* (*Seg* (*Se B D*)) *add Emp*) (*Geos* (*Seg* (*Se B1 D2*)) *add Emp*) **by** *simp*
then have *P72* : *Eq* (*Geos* (*Seg* (*Se D B*)) *add Emp*) (*Geos* (*Seg* (*Se D2 B1*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-trans*)
from *P52 P69 P70 P71 P72* **have** *Cong* (*Geos* (*Tri* (*Tr D B C*)) *add Emp*)

(*Geos* (*Tri* (*Tr D2 B1 C2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P73* : *Cong* (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An C2 B1 D2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P71* **have** *Def* (*Tri* (*Tr C2 B1 D2*)) **by** (*blast intro*:*Tri-def-rev*)
then have *P74* : *Def* (*Ang* (*An C2 B1 D2*)) **by** (*simp add*:*Tri-to-Ang*)
from *assms* **have** *P75* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P71* **have** *P76* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P2 P75 P76* **have** *P77* : *Line-on* (*Li B1 C2*) *C1* **by** (*simp add*:*Line-on-rev*)
from *P14* **have** *P78* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P74* **have** *P79* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi D2*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P20 P78 P79* **have** *P80* : *Line-on* (*Li B1 D2*) *D1* **by** (*simp add*:*Line-on-rev*)
from *assms* **have** ¬ *Eq* (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P81* : ¬ *Eq* (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P82* : *Line-on* (*Li A1 D1*) *D1* **by** (*simp add*:*Line-on-rule*)
from *P9 P11 P13 P14 P82* **have** *P83* : *Eq* (*Geos* (*Lin* (*Li B1 A1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P81 P83* **have** *P84* : ¬ *Eq* (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) **by** (*simp add*:*Eq-not-trans*)
then have *P85* : ¬ *Eq* (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P86* : *Line-on* (*Li B1 C1*) *B1* **by** (*simp add*:*Line-on-rule*)
from *assms P85 P86* **have** *P87* : *Plane-diffside* (*Li B1 C1*) *A1 D1* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P88* : *Plane-diffside* (*Li B1 C1*) *D1 A1* **by** (*simp add*:*Plane-diffside-rev*)
have *P89* : *Bet-Point* (*Se D2 D1*) *B1* ⟹ *Line-on* (*Li D2 D1*) *B1* **by** (*simp add*:*Line-Bet-on*)
have *P90* : *Line-on* (*Li D2 D1*) *D1* **by** (*simp add*:*Line-on-rule*)
from *P9 P14 P82 P89 P90* **have** *P91* : *Bet-Point* (*Se D2 D1*) *B1* ⟹ *Eq* (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) (*Geos* (*Lin* (*Li D2 D1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P84 P91* **have** *Bet-Point* (*Se D2 D1*) *B1* ⟹ ¬ *Eq* (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li D2 D1*)) *add Emp*) **by** (*simp add*:*Eq-not-trans*)
then have *P92* : *Bet-Point* (*Se D2 D1*) *B1* ⟹ ¬ *Eq* (*Geos* (*Lin* (*Li D2 D1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P86 P92* **have** *Bet-Point* (*Se D2 D1*) *B1* ⟹ *Plane-diffside* (*Li B1 C1*) *D2 D1* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P93* : *Bet-Point* (*Se D2 D1*) *B1* ⟹ *Plane-diffside* (*Li B1 C1*) *D1 D2* **by** (*simp add*:*Plane-diffside-rev*)
from *P20* **have** *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Bet-Point* (*Se A1 A2*) *B1* **by** (*blast intro*:*Bet-Point-Eq*)
then have *P94* : *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Bet-Point* (*Se A2 A1*) *B1* **by** (*simp add*:*Bet-rev*)
from *P1 P94* **have** *P95* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi D2*) *add*

Emp) **by** (*blast intro*:*Eq-rev*)
from *P88 P93 P95* **have** *Bet-Point* (*Se D2 D1*) *B1* ⟹ *Plane-sameside* (*Li B1 C1*) *A1 D2* **by** (*blast intro*:*Plane-trans-inv*)
then have *P96* : *Bet-Point* (*Se D2 D1*) *B1* ⟹ *Plane-sameside* (*Li B1 C1*) *D2 A1* **by** (*simp add*:*Plane-sameside-rev*)
from *P1* **have** *Def* (*Tri* (*Tr A2 B1 C1*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P97* : ¬ *Line-on* (*Li B1 C1*) *A2* **by** (*simp add*:*Tri-def-Line*)
have *Line-on* (*Li D2 A2*) *A2* **by** (*simp add*:*Line-on-rule*)
then have *P98* : *Eq* (*Geos* (*Lin* (*Li D2 A2*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) ⟹ *Line-on* (*Li B1 C1*) *A2* **by** (*simp add*:*Line-on-trans*)
from *P97 P98* **have** *P99* : ¬ *Eq* (*Geos* (*Lin* (*Li D2 A2*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** *blast*
from *P20* **have** *P100* : *Bet-Point* (*Se D2 A2*) *B1* **by** *simp*
from *P86 P99 P100* **have** *P101* : *Plane-diffside* (*Li B1 C1*) *D2 A2* **by** (*simp add*:*Plane-Bet-diffside*)
from *P96 P101* **have** ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) ⟹ *Bet-Point* (*Se D2 D1*) *B1* ⟹
Plane-diffside (*Li B1 C1*) *A1 A2* **by** (*simp add*:*Plane-trans*)
then have *P102* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) ⟹ *Bet-Point* (*Se D2 D1*) *B1* ⟹
¬ *Plane-sameside* (*Li B1 C1*) *A1 A2* **by** (*simp add*:*Plane-diffside-not-sameside*)
have *P103* : *Line-on* (*Li B1 A1*) *A1* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *Def* (*Tri* (*Tr C1 B1 A1*)) **by** (*simp add*:*Ang-to-Tri Tri-def-rev*)
then have *P104* : ¬ *Line-on* (*Li B1 A1*) *C1* **by** (*simp add*:*Tri-def-Line*)
from *P4* **have** *P105* : ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P1 p10 P13 P103 P104 P105* **have** *P106* : *Plane-sameside* (*Li C1 B1*) *A2 A1* ∨ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** (*simp add*:*Seg-Plane-sameside*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P107* : *Eq* (*Geos* (*Lin* (*Li C1 B1*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*simp add*:*Line-rev Eq-rev*)
from *P106 P107* **have** *P108* : *Plane-sameside* (*Li B1 C1*) *A1 A2* ∨ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*blast intro*:*Plane-sameside-rev Plane-Line-trans Eq-rev*)
from *P102 P108* **have** *P109* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) ⟹ ¬ *Bet-Point* (*Se D2 D1*) *B1* **by** *blast*
from *P22* **have** *P110* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) ⟹ *Bet-Point* (*Se A1 D2*) *B1* **by** (*blast intro*:*Bet-Point-Eq Eq-rev*)
then have *P111* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) ⟹ *Line-on* (*Li A1 D2*) *B1* **by** (*simp add*:*Line-Bet-on*)
have *P112* : *Line-on* (*Li A1 D2*) *A1* **by** (*simp add*:*Line-on-rule*)
have *P113* : *Line-on* (*Li A1 D1*) *A1* **by** (*simp add*:*Line-on-rule*)
from *p10* **have** *P114* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9 P111 P112 P113 P114* **have** *P115* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) ⟹ *Eq* (*Geos* (*Lin* (*Li A1 D1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D2*)) *add Emp*) **by** (*simp add*:*Line-unique*)

from *P84 P115* **have** *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) (*Geos* (*Lin* (*Li A1 D2*)) *add Emp*) **by** (*simp add*:*Eq-not-trans*)
then have *P116* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Lin* (*Li A1 D2*)) *add Emp*) (*Geos* (*Lin* (*Li B1 C1*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P86 P110 P116* **have** *P117* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ *Plane-diffside* (*Li B1 C1*) *A1 D2* **by** (*simp add*:*Plane-Bet-diffside*)
have *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ *Bet-Point* (*Se D2 D1*) *B1* $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Poi D2*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P118* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ *Bet-Point* (*Se D2 D1*) *B1* $\Longrightarrow$
$\neg$ *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi D2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P87 P117 P118* **have** *P119* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ *Bet-Point* (*Se D2 D1*) *B1* $\Longrightarrow$
Plane-sameside (*Li B1 C1*) *D1 D2* **by** (*blast intro*:*Plane-trans-inv*)
from *P93* **have** *P120* : *Bet-Point* (*Se D2 D1*) *B1* $\Longrightarrow$ $\neg$ *Plane-sameside* (*Li B1 C1*) *D1 D2* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P119 P120* **have** *P121* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) $\Longrightarrow$ $\neg$ *Bet-Point* (*Se D2 D1*) *B1* **by** *blast*
from *P109 P121* **have** *P122* : $\neg$ *Bet-Point* (*Se D2 D1*) *B1* **by** *blast*
from *P2 P74 P75 P77 P78 P80 P122* **have** *P123* : *Eq* (*Geos* (*Ang* (*An C2 B1 D2*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 D1*)) *add Emp*) **by** (*simp add*:*Ang-Point-swap*)
from *P73 P123* **show** *Cong* (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An C1 B1 D1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
qed

theorem (**in** *Congruence-Rule*) *Ang-vertical* :
assumes
Def (*Ang* (*An A B C*))
Bet-Point (*Se A D*) *B*
Bet-Point (*Se C E*) *B*
shows *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D B E*)) *add Emp*)
and *Cong* (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An A B E*)) *add Emp*)
proof −
have *P1* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An C B A*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms* **have** *P2* : *Def* (*Ang* (*An C B A*)) **by** (*simp add*:*Ang-def-rev*)
from *assms P1 P2* **show** *Cong* (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An A B E*)) *add Emp*) **by** (*simp add*:*Ang-complementary*)
from *assms* **have** *P3* : *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** $\neg$ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P4* : $\neg$ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P2 P3 P4* **have** *P5* : *Def* (*Ang* (*An C B D*)) **by** (*simp add*:*Ang-def-extension*)

then have *P6* : *Def* (*Ang* (*An D B C*)) **by** (*simp add*:*Ang-def-rev*)
have *P7* : *Cong* (*Geos* (*Ang* (*An C B D*)) *add Emp*) (*Geos* (*Ang* (*An D B C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms* **have** *P8* : *Bet-Point* (*Se D A*) *B* **by** (*simp add*:*Bet-rev*)
from *assms P5 P6 P7 P8* **have** *P9* : *Cong* (*Geos* (*Ang* (*An D B E*)) *add Emp*) (*Geos* (*Ang* (*An C B A*)) *add Emp*) **by** (*simp add*:*Ang-complementary*)
from *P1* **have** *P10* : *Eq* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P9 P10* **show** *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D B E*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
qed

lemma (**in** *Congruence-Rule*) *Ang-inside-Planeside* :
assumes *Ang-inside* (*An A B C*) *D*
shows *Plane-diffside* (*Li B D*) *A C*
proof −
from *assms* **have** *P1* : *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Ang-inside-def*)
then have *P2* : ¬ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Plane-sameside-def*)
have *P3* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
then have *P4* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Point-Eq*)
from *P2 P4* **have** *P5* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P6* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P7* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P5 P6 P7* **have** ∃ *p*. *Bet-Point* (*Se C p*) *B* ∧ *Line-on* (*Li B C*) *p* **by** (*simp add*:*Bet-extension*)
then obtain *E* :: *Point* **where** *P8* : *Bet-Point* (*Se C E*) *B* ∧ *Line-on* (*Li B C*) *E* **by** *blast*
then have *P9* : *Line-on* (*Li C E*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P10* : *Line-on* (*Li C E*) *C* **by** (*simp add*:*Line-on-rule*)
from *P5 P6 P7 P9 P10* **have** *P11* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li C E*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1* **have** *P12* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Plane-sameside-def*)
from *P11 P12* **have** *P13* : ¬ *Line-on* (*Li C E*) *A* **by** (*simp add*:*Line-not-on-trans*)
have *P14* : *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
from *P1* **have** *P15* : ¬ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *P3* **have** *P16* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Point-Eq*)
from *P15 P16* **have** *P17* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P6* **have** *P18* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Point-Eq*)
from *P12 P18* **have** *P19* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** *blast*
from *P17 P19* **have** *P20* : *Line-on* (*Li B D*) *A* ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-rev*)
from *P15 P20* **have** *P21* : ¬ *Line-on* (*Li B D*) *A* **by** *blast*

from *P1* **have** *P22* : ¬ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *P5* **have** *P23* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P17 P23* **have** *P24* : *Line-on* (*Li B D*) *C* ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-on-rev*)
from *P22 P24* **have** *P25* : ¬ *Line-on* (*Li B D*) *C* **by** *blast*
from *P8* **have** *P26* : *Bet-Point* (*Se C E*) *B* **by** *simp*
then have *P27* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P8* **have** *P28* : *Line-on* (*Li B C*) *E* **by** *simp*
from *P6 P14 P27 P28* **have** *P29* : *Line-on* (*Li B D*) *E* ⟹ *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P29* **have** *P30* : *Line-on* (*Li B D*) *E* ⟹ *Line-on* (*Li B D*) *C* **by** (*simp add*:*Line-on-trans*)
from *P25 P30* **have** *P31* : ¬ *Line-on* (*Li B D*) *E* **by** *blast*
from *P13 P14 P21 P25 P26 P31* **have** *P32* : *Line-on-Seg* (*Li B D*) (*Se C A*) ∧ ¬ *Line-on-Seg* (*Li B D*) (*Se E A*)
∨ *Line-on-Seg* (*Li B D*) (*Se E A*) ∧ ¬ *Line-on-Seg* (*Li B D*) (*Se C A*) **by** (*simp add*:*Pachets-axiom*)
have *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ ∃ *p*. *Line-on* (*Li B D*) *p* ∧ *Bet-Point* (*Se E A*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *F* :: *Point* **where** *P33* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Line-on* (*Li B D*) *F* ∧ *Bet-Point* (*Se E A*) *F* **by** *blast*
then have *P34* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Bet-Point* (*Se A E*) *F* **by** (*simp add*:*Bet-rev*)
from *P3* **have** *P35* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li A E*)) *add Emp*) ⟹ *Line-on* (*Li A E*) *B* **by** (*simp add*:*Line-on-trans*)
have *P36* : *Line-on* (*Li A E*) *E* **by** (*simp add*:*Line-on-rule*)
from *P6 P27 P28 P35 P36* **have** *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li A E*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P37* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li A E*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
have *P38* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
from *P37 P38* **have** *P39* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-on-trans*)
from *P12 P39* **have** *P40* : ¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** *blast*
from *P37 P40* **have** *P41* : ¬ *Eq* (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P34 P38 P41* **have** *P42* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Plane-sameside* (*Li B A*) *E F* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
from *P11* **have** *P43* : *Eq* (*Geos* (*Lin* (*Li C E*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) ⟹ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P40 P43* **have** *P44* : ¬ *Eq* (*Geos* (*Lin* (*Li C E*)) *add Emp*) (*Geos* (*Lin* (*Li*

B A)) *add Emp*) **by** *blast*
from *P3 P26 P44* **have** *P45* : *Plane-diffside* (*Li B A*) *E C* **by** (*simp add*:*Plane-Bet-diffside Plane-diffside-rev*)
from *P34* **have** *P46* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Line-on* (*Li A E*) *F* **by** (*simp add*:*Line-Bet-on*)
have *P47* : *Line-on* (*Li C E*) *E* **by** (*simp add*:*Line-on-rule*)
have *P48* : *Line-on* (*Li A E*) *A* **by** (*simp add*:*Line-on-rule*)
from *P42 P45* **have** *P49* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Plane-diffside* (*Li B A*) *F C* **by** (*simp add*:*Plane-trans*)
from *P1* **have** *P50* : *Plane-sameside* (*Li B A*) *C D* **by** (*simp add*:*Point-Eq*)
then have *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ *Plane-sameside* (*Li B A*) *C F* **by** (*simp add*:*Point-Eq*)
then have *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ ¬ *Plane-diffside* (*Li B A*) *C F* **by** (*simp add*:*Plane-sameside-not-diffside*)
then have *P51* : *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) ⟹ ¬ *Plane-diffside* (*Li B A*) *F C* **by** (*blast intro*:*Plane-diffside-rev*)
from *P49 P51* **have** *P52* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** *blast*
from *P49 P50* **have** *P53* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Plane-diffside* (*Li B A*) *D F* **by** (*simp add*:*Plane-trans Plane-diffside-rev*)
from *P46* **have** *P54* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹
Line-on (*Li A E*) *B* **by** (*simp add*:*Point-Eq*)
from *P26* **have** *P55* : *Line-on* (*Li C E*) *B* **by** (*simp add*:*Line-Bet-on*)
from *P27 P36 P47 P54 P55* **have** *P56* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li A E*)) *add Emp*) (*Geos* (*Lin* (*Li C E*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P48 P56* **have** *P57* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹
Line-on (*Li C E*) *A* **by** (*simp add*:*Line-on-trans*)
from *P13 P57* **have** *P58* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** *blast*
have *P59* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P14 P17 P33 P52 P58 P59* **have** *P60* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Bet-Point* (*Se B F*) *D* ∨ *Bet-Point* (*Se F D*) *B* ∨ *Bet-Point* (*Se D B*) *F* **by** (*simp add*:*Bet-case*)
have *P61* : *Line-on* (*Li B F*) *B* **by** (*simp add*:*Line-on-rule*)
have *P62* : *Line-on* (*Li B F*) *F* **by** (*simp add*:*Line-on-rule*)
from *P33* **have** *P63* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Line-on* (*Li B D*) *F* **by** *simp*
from *P14 P58 P61 P62 P63* **have** *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Eq* (*Geos* (*Lin* (*Li B F*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P64* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Eq* (*Geos* (*Lin* (*Li B F*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P38* **have** *P65* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B D*))

add Emp) $\Longrightarrow$ *Line-on* (*Li B D*) *A* **by** (*simp add*:*Line-on-trans*)
from *P21 P65* **have** *P66* : $\neg$ *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P64 P66* **have** *P67* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Lin* (*Li B F*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** *blast*
from *P3 P67* **have** *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Bet-Point* (*Se B F*) *D* $\Longrightarrow$ *Plane-sameside* (*Li B A*) *D F* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
then have *P68* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Bet-Point* (*Se B F*) *D* $\Longrightarrow$ $\neg$ *Plane-diffside* (*Li B A*) *D F* **by** (*simp add*:*Plane-sameside-not-diffside*)
from *P53 P68* **have** *P69* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ $\neg$ *Bet-Point* (*Se B F*) *D* **by** *blast*
have *P70* : *Bet-Point* (*Se D B*) *F* $\Longrightarrow$ *Bet-Point* (*Se B D*) *F* **by** (*simp add*:*Bet-rev*)
from *P3 P66 P70* **have** *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Bet-Point* (*Se D B*) *F* $\Longrightarrow$ *Plane-sameside* (*Li B A*) *D F* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
then have *P71* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Bet-Point* (*Se D B*) *F* $\Longrightarrow$ $\neg$ *Plane-diffside* (*Li B A*) *D F* **by** (*simp add*:*Plane-sameside-not-diffside*)
from *P53 P71* **have** *P72* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ $\neg$ *Bet-Point* (*Se D B*) *F* **by** *blast*
from *P60 P69 P72* **have** *P73* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Bet-Point* (*Se F D*) *B* **by** *blast*
have *Line-on* (*Li F D*) *D* **by** (*simp add*:*Line-on-rule*)
then have *P74* : *Eq* (*Geos* (*Lin* (*Li F D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P22 P74* **have** *P75* : $\neg$ *Eq* (*Geos* (*Lin* (*Li F D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** *blast*
from *P6 P73 P75* **have** *P76* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Plane-diffside* (*Li B C*) *F D* **by** (*simp add*:*Plane-Bet-diffside*)
from *P33* **have** *P77* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Bet-Point* (*Se E A*) *F* **by** *simp*
have *Line-on* (*Li E A*) *A* **by** (*simp add*:*Line-on-rule*)
then have *P78* : *Eq* (*Geos* (*Lin* (*Li E A*)) *add Emp*) (*Geos* (*Lin* (*Li C E*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li C E*) *A* **by** (*simp add*:*Line-on-trans*)
from *P13 P78* **have** *P79* : $\neg$ *Eq* (*Geos* (*Lin* (*Li E A*)) *add Emp*) (*Geos* (*Lin* (*Li C E*)) *add Emp*) **by** *blast*
from *P47 P77 P79* **have** *P80* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Plane-sameside* (*Li C E*) *F A* **by** (*simp add*:*Plane-Bet-sameside*)
from *P11 P80* **have** *P81* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Plane-sameside* (*Li B C*) *F A* **by** (*blast intro*:*Eq-rev Plane-Line-trans*)
from *P76 P81* **have** *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ *Plane-diffside* (*Li B C*) *A D* **by** (*simp add*:*Plane-trans*)
then have *P82* : *Line-on-Seg* (*Li B D*) (*Se E A*) $\Longrightarrow$ $\neg$ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P1 P82* **have** *P83* : $\neg$ *Line-on-Seg* (*Li B D*) (*Se E A*) **by** *blast*
from *P32 P83* **have** *Line-on-Seg* (*Li B D*) (*Se C A*) **by** *blast*
then have *P84* : $\exists p$. *Line-on* (*Li B D*) *p* $\wedge$ *Bet-Point* (*Se C A*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
from *P21 P25 P84* **have** $\exists p$. *Bet-Point* (*Se C A*) *p* $\wedge$ *Line-on* (*Li B D*) *p* $\wedge$ $\neg$

Line-on (*Li B D*) *C* ∧ ¬ *Line-on* (*Li B D*) *A* **by** *blast*
then have *Plane-diffside* (*Li B D*) *C A* **by** (*simp add*:*Plane-diffside-def*)
thus *Plane-diffside* (*Li B D*) *A C* **by** (*simp add*:*Plane-diffside-rev*)
qed

lemma (**in** *Congruence-Rule*) *Ang-inside-Bet-Point* :
assumes
Bet-Point (*Se p1 p3*) *p2*
¬ *Eq* (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*)
¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
shows *Ang-inside* (*An p1 p4 p3*) *p2*
proof −
have *P1* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p4 p1*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li p4 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P5* : *Line-on* (*Li p4 p3*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P6* : *Line-on* (*Li p4 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P7* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
by (*simp add*:*Bet-Point-def*)
from *P2* **have** *P8* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p1*))
add Emp) ⟹ *Line-on* (*Li p4 p1*) *p3* **by** (*simp add*:*Line-on-trans*)
from *assms P3 P5 P6 P8* **have** *P9* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos*
(*Lin* (*Li p4 p1*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*) **by** (*simp*
add:*Line-unique*)
from *assms P9* **have** *P10* : ¬ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin*
(*Li p4 p1*)) *add Emp*) **by** *blast*
from *P1* **have** *P11* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p4*
p3)) *add Emp*) ⟹ *Line-on* (*Li p4 p3*) *p1* **by** (*simp add*:*Line-on-trans*)
from *assms P3 P4 P5 P11* **have** *P12* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*)
(*Geos* (*Lin* (*Li p4 p3*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li p4 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p4 p3*)) *add Emp*) **by** (*simp*
add:*Line-unique*)
from *assms P12* **have** *P13* : ¬ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin*
(*Li p4 p3*)) *add Emp*) **by** *blast*
from *assms P4 P10* **have** *Plane-sameside* (*Li p4 p1*) *p2 p3* **by** (*simp add*:*Plane-Bet-sameside*)
then have *P14* : *Plane-sameside* (*Li p4 p1*) *p3 p2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P15* : *Bet-Point* (*Se p3 p1*) *p2* **by** (*simp add*:*Bet-rev*)
from *P7* **have** *P16* : *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p3*
p1)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P13 P16* **have** *P17* : ¬ *Eq* (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin*
(*Li p4 p3*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *assms P6 P15 P17* **have** *Plane-sameside* (*Li p4 p3*) *p2 p1* **by** (*simp*
add:*Plane-Bet-sameside*)
then have *P18* : *Plane-sameside* (*Li p4 p3*) *p1 p2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P19* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
by (*blast intro*:*Eq-rev*)

from *P7* **have** *P20* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P19 P20* **have** *P21* : *Def* (*Ang* (*An p1 p4 p3*)) **by** (*simp add*:*Ang-def*)
from *P14 P18 P21* **show** *Ang-inside* (*An p1 p4 p3*) *p2* **by** (*simp add*:*Ang-inside-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-inside-HalfLine* :
assumes
Ang-inside (*An A B C*) *D*
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi E*) *add Emp*)
Line-on (*Li B D*) *E*
¬ *Bet-Point* (*Se E D*) *B*
shows
Ang-inside (*An A B C*) *E*
proof −
from *assms* **have** *P1* : *Def* (*Ang* (*An A B C*)) ∧ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Ang-inside-def*)
have *Plane-diffside* (*Li B A*) *C E* ⟹ ∃ *p*. *Bet-Point* (*Se C E*) *p* ∧ *Line-on* (*Li B A*) *p*
∧ ¬ *Line-on* (*Li B A*) *C* ∧ ¬ *Line-on* (*Li B A*) *E* **by** (*simp add*:*Plane-diffside-def*)
then obtain *p1* :: *Point* **where** *Plane-diffside* (*Li B A*) *C E* ⟹ *Bet-Point* (*Se C E*) *p1* **by** *blast*
from *assms P1* **have** *Plane-diffside* (*Li B A*) *C E* ⟹ *Plane-diffside* (*Li B A*) *D E* **by** (*blast intro*:*Plane-trans*)
then have *Plane-diffside* (*Li B A*) *C E* ⟹ ∃ *p*. *Bet-Point* (*Se D E*) *p* ∧ *Line-on* (*Li B A*) *p*
∧ ¬ *Line-on* (*Li B A*) *D* ∧ ¬ *Line-on* (*Li B A*) *E* **by** (*simp add*:*Plane-diffside-def*)
then obtain *F* :: *Point* **where** *P2* : *Plane-diffside* (*Li B A*) *C E* ⟹ *Bet-Point* (*Se D E*) *F* ∧ *Line-on* (*Li B A*) *F* **by** *blast*
then have *Plane-diffside* (*Li B A*) *C E* ⟹ *Bet-Point* (*Se E D*) *F* **by** (*simp add*:*Bet-rev*)
then have *P3* : *Plane-diffside* (*Li B A*) *C E* ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
⟹ *Bet-Point* (*Se E D*) *B* **by** (*simp add*:*Point-Eq*)
from *assms P3* **have** *P4* : *Plane-diffside* (*Li B A*) *C E* ⟹ ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** *blast*
have *P5* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
have *P6* : *Line-on* (*Li E D*) *E* **by** (*simp add*:*Line-on-rule*)
have *P7* : *Line-on* (*Li E D*) *D* **by** (*simp add*:*Line-on-rule*)
from *assms P5 P6 P7* **have** *P8* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li E D*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P2* **have** *P9* : *Plane-diffside* (*Li B A*) *C E* ⟹ *Line-on* (*Li E D*) *F* **by** (*simp add*:*Line-Bet-on*)
from *P8 P9* **have** *P10* : *Plane-diffside* (*Li B A*) *C E* ⟹
¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B D*) *F* **by** (*simp add*:*Line-on-trans*)
have *P11* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)

have *P12* : *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
from *P2 P4 P10 P11 P12* **have** *P13* : *Plane-diffside* (*Li B A*) *C E* ⟹
¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *P5 P13* **have** *P14* : *Plane-diffside* (*Li B A*) *C E* ⟹
¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-trans*)
from *P1* **have** *P15* : ¬ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *P14 P15* **have** *P16* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ ¬ *Plane-diffside* (*Li B A*) *C E* **by** *blast*
from *assms P1* **have** *Plane-diffside* (*Li B C*) *A E* ⟹ *Plane-diffside* (*Li B C*) *D E* **by** (*blast intro*:*Plane-trans*)
then have *Plane-diffside* (*Li B C*) *A E* ⟹ ∃ *p*. *Bet-Point* (*Se D E*) *p* ∧ *Line-on* (*Li B C*) *p*
∧ ¬ *Line-on* (*Li B C*) *D* ∧ ¬ *Line-on* (*Li B C*) *E* **by** (*simp add*:*Plane-diffside-def*)
then obtain *G* :: *Point* **where** *P17* : *Plane-diffside* (*Li B C*) *A E* ⟹ *Bet-Point* (*Se D E*) *G* ∧ *Line-on* (*Li B C*) *G* **by** *blast*
then have *Plane-diffside* (*Li B C*) *A E* ⟹ *Bet-Point* (*Se E D*) *G* **by** (*simp add*:*Bet-rev*)
then have *P18* : *Plane-diffside* (*Li B C*) *A E* ⟹ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
⟹ *Bet-Point* (*Se E D*) *B* **by** (*simp add*:*Point-Eq*)
from *assms P18* **have** *P19* : *Plane-diffside* (*Li B C*) *A E* ⟹ ¬ *Eq* (*Geos* (*Poi G*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** *blast*
from *P17* **have** *P20* : *Plane-diffside* (*Li B C*) *A E* ⟹ *Line-on* (*Li E D*) *G* **by** (*simp add*:*Line-Bet-on*)
from *P8 P20* **have** *P21* : *Plane-diffside* (*Li B C*) *A E* ⟹
¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B D*) *G* **by** (*simp add*:*Line-on-trans*)
have *P22* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
from *P12 P17 P19 P21 P22* **have** *P23* : *Plane-diffside* (*Li B C*) *A E* ⟹
¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*)
⟹ *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *P5 P23* **have** *P24* : *Plane-diffside* (*Li B C*) *A E* ⟹
¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P1* **have** *P25* : ¬ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *P24 P25* **have** *P26* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ ¬ *Plane-diffside* (*Li B C*) *A E* **by** *blast*
from *P1* **have** *P27* : ¬ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Plane-sameside-def*)
from *P8 P12* **have** *P28* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li E D*) *B* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *assms P6 P11 P28* **have** *P29* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B A*) *E* ⟹
Eq (*Geos* (*Lin* (*Li E D*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P29* **have** *P30* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹

Line-on (*Li B A*) *E* ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-trans*)
from *P15 P30* **have** *P31* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
¬ *Line-on* (*Li B A*) *E* **by** *blast*
from *P1* **have** *P32* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Ang-def*)
have *P33* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P12 P22 P32 P33* **have** *P34* : *Line-on* (*Li B D*) *C* ⟹
Eq (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P5 P34* **have** *P35* : *Line-on* (*Li B D*) *C* ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P25 P35* **have** *P36* : ¬ *Line-on* (*Li B D*) *C* **by** *blast*
from *assms* **have** *P37* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) ⟹ *Line-on* (*Li B D*) *C* **by** (*simp add*:*Point-Eq*)
from *P36 P37* **have** *P38* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P16 P27 P31 P38* **have** *P39* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Plane-sameside (*Li B A*) *C E* **by** (*simp add*:*Plane-not-diffside-sameside*)
from *P1* **have** *P40* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Plane-sameside-def*)
from *assms P6 P22 P28* **have** *P41* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B C*) *E* ⟹
Eq (*Geos* (*Lin* (*Li E D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P41* **have** *P42* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Line-on (*Li B C*) *E* ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-on-trans*)
from *P25 P42* **have** *P43* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ ¬ *Line-on* (*Li B C*) *E* **by** *blast*
from *P39* **have** *P44* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ ¬ *Line-on* (*Li B A*) *E* **by** (*simp add*:*Plane-sameside-def*)
have *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
then have *P45* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Eq (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B A*) *E* **by** (*simp add*:*Point-Eq*)
from *P44 P45* **have** *P46* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** *blast*
from *P26 P40 P43 P46* **have** *P47* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Plane-sameside (*Li B C*) *A E* **by** (*simp add*:*Plane-not-diffside-sameside*)
from *P1 P39 P47* **have** *P48* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Ang-inside* (*An A B C*) *E* **by** (*simp add*:*Ang-inside-def*)
from *assms* **have** *P49* : *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Ang-inside* (*An A B C*) *E* **by** (*simp add*:*Point-Eq*)
from *P48 P49* **show** *Ang-inside* (*An A B C*) *E* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-outside-Planeside* :
assumes
Def (*Ang* (*An A B C*))
¬ *Ang-inside* (*An A B C*) *D*
shows ¬ (*Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*)
and ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D*
proof −
from *assms* **have** *P1* : *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* ⟹ *Ang-inside* (*An A B C*) *D* **by** (*simp add*:*Ang-inside-def*)
from *assms P1* **show** ¬ (*Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*) **by** *blast*
thus ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-outside-exclusive* :
assumes
Plane-sameside (*Li B C*) *A D*
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*)
shows
¬ (¬ *Ang-inside* (*An A B C*) *D* ∧ ¬ *Ang-inside* (*An D B C*) *A*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Plane-sameside-def*)
from *assms P1* **have** *Def* (*Ang* (*An B C A*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P2* : *Def* (*Ang* (*An A B C*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have *P3* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Ang-outside-Planeside*)
from *assms* **have** *P4* : ¬ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *assms P4* **have** *Def* (*Ang* (*An B C D*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P5* : *Def* (*Ang* (*An D B C*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have *P6* : ¬ *Ang-inside* (*An D B C*) *A* ⟹ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A*
∨ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A* **by** (*simp add*:*Ang-outside-Planeside*)
from *P3 P6* **have** *P7* : ¬ *Ang-inside* (*An A B C*) *D* ∧ ¬ *Ang-inside* (*An D B C*) *A* ⟹
¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* ∧ ¬

Plane-sameside (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A* **by** *blast*
from *assms* **have** *P8* : *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ ¬ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A* ⟹ *False* **by** *blast*
from *assms* **have** *Plane-sameside* (*Li B C*) *D A* **by** (*simp add*:*Plane-sameside-rev*)
then have *P9* : ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A*
∨ ¬ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ ¬ *Plane-sameside* (*Li B C*) *D A* ⟹ *False* **by** *blast*
have *P10* : *Line-on* (*Li C B*) *C* **by** (*simp add*:*Line-on-rule*)
have *P11* : *Line-on* (*Li C B*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms P10 P11* **have** ∃ *p*. *Bet-Point* (*Se C p*) *B* ∧ *Line-on* (*Li C B*) *p* **by** (*blast intro*:*Bet-extension Eq-rev*)
then obtain *E* :: *Point* **where** *P12* : *Bet-Point* (*Se C E*) *B* ∧ *Line-on* (*Li C B*) *E* **by** *blast*
then have *P13* : *Line-on* (*Li E C*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P14* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
have *P15* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
have *P16* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P17* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P16* **have** *P18* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Point-Eq*)
from *P1 P18* **have** *P19* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** *blast*
from *P13 P14 P15 P19* **have** *P20* : *Line-on* (*Li E C*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) **by** (*simp add*:*Line-unique*)

have *P21* : *Line-on* (*Li E C*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms P13 P16 P17 P21* **have** *P22* : *Eq* (*Geos* (*Lin* (*Li E C*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P20 P22* **have** *P23* : *Line-on* (*Li E C*) *A* ⟹ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P15 P23* **have** *P24* : *Line-on* (*Li E C*) *A* ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-on-trans*)
from *P1 P24* **have** *P25* : ¬ *Line-on* (*Li E C*) *A* **by** *blast*
have *P26* : *Line-on* (*Li E C*) *E* **by** (*simp add*:*Line-on-rule*)
from *P21* **have** *P27* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Line-on* (*Li E C*) *A* **by** (*simp add*:*Point-Eq*)
from *P25 P27* **have** *P28* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** *blast*
from *P12* **have** *P29* : *Bet-Point* (*Se C E*) *B* **by** *simp*
then have *P30* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Bet-Point-def Eq-rev*)
have *Plane-diffside* (*Li B D*) *C A* ⟹ ∃ *p*. *Bet-Point* (*Se C A*) *p* ∧ *Line-on* (*Li B D*) *p* ∧ ¬ *Line-on* (*Li B D*) *C* ∧ ¬ *Line-on* (*Li B D*) *A* **by** (*simp add*:*Plane-diffside-def*)
then have *P31* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Line-on* (*Li B D*) *C* ∧ ¬ *Line-on* (*Li B D*) *A* **by** *blast*
have *P32* : *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
from *P29* **have** *P33* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P13 P26 P32 P33* **have** *P34* : *Line-on* (*Li B D*) *E* ⟹ *Eq* (*Geos* (*Lin* (*Li E C*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P21 P34* **have** *P35* : *Line-on* (*Li B D*) *E* ⟹ *Line-on* (*Li B D*) *C* **by** (*simp add*:*Line-on-trans*)
from *P31 P35* **have** *P36* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Line-on* (*Li B D*) *E* **by** *blast*
from *P29* **have** *P37* : *Bet-Point* (*Se E C*) *B* **by** (*simp add*:*Bet-rev*)
from *P25 P31 P32 P36 P37* **have** *P38* : *Plane-diffside* (*Li B D*) *C A* ⟹
Line-on-Seg (*Li B D*) (*Se E A*) ∧ ¬ *Line-on-Seg* (*Li B D*) (*Se C A*)
∨ *Line-on-Seg* (*Li B D*) (*Se C A*) ∧ ¬ *Line-on-Seg* (*Li B D*) (*Se E A*) **by** (*simp add*:*Pachets-axiom*)
have *P39* : *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ ∃ *p*. *Line-on* (*Li B D*) *p* ∧ *Bet-Point* (*Se E A*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
from *P31 P36 P39* **have** *Plane-diffside* (*Li B D*) *C A* ∧ *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹
∃ *p*. *Bet-Point* (*Se E A*) *p* ∧ *Line-on* (*Li B D*) *p* ∧ ¬ *Line-on* (*Li B D*) *E* ∧ ¬ *Line-on* (*Li B D*) *A* **by** *blast*
then have *P40* : *Plane-diffside* (*Li B D*) *C A* ∧ *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Plane-diffside* (*Li B D*) *E A* **by** (*simp add*:*Plane-diffside-def*)
from *P26* **have** *P41* : *Eq* (*Geos* (*Lin* (*Li E C*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) ⟹ *Line-on* (*Li B D*) *E* **by** (*simp add*:*Line-on-trans*)
from *P36 P41* **have** *P42* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Eq* (*Geos* (*Lin* (*Li E C*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** *blast*
from *P32 P37 P42* **have** *P43* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Plane-diffside* (*Li B D*) *E C* **by** (*simp add*:*Plane-Bet-diffside*)

from *P28* **have** *P44* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P30 P40 P43 P44* **have** *Plane-diffside* (*Li B D*) *C A* ∧ *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Plane-sameside* (*Li B D*) *A C* **by** (*blast intro*:*Plane-trans-inv*)
then have *P45* : *Plane-diffside* (*Li B D*) *C A* ∧ *Line-on-Seg* (*Li B D*) (*Se E A*) ⟹ *Plane-sameside* (*Li B D*) *C A* **by** (*blast intro*:*Plane-sameside-rev*)
have *P46* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Plane-sameside* (*Li B D*) *C A* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P45 P46* **have** *P47* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Line-on-Seg* (*Li B D*) (*Se E A*) **by** *blast*
from *P38 P47* **have** *Plane-diffside* (*Li B D*) *C A* ⟹ *Line-on-Seg* (*Li B D*) (*Se C A*) **by** *blast*
then have *Plane-diffside* (*Li B D*) *C A* ⟹ ∃ *p*. *Line-on* (*Li B D*) *p* ∧ *Bet-Point* (*Se C A*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *F* :: *Point* **where** *P48* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Line-on* (*Li B D*) *F* ∧ *Bet-Point* (*Se C A*) *F* **by** *blast*
from *P15* **have** *P49* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-on-trans*)
from *P1 P49* **have** *P50* : ¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** *blast*
from *P48* **have** *P51* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se A C*) *F* **by** (*simp add*:*Bet-rev*)
from *assms P19 P50 P51* **have** *P52* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Ang-inside* (*An A B C*) *F* **by** (*simp add*:*Ang-inside-Bet-Point*)
then have *Plane-diffside* (*Li B D*) *C A* ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Ang-inside* (*An A B C*) *D* **by** (*simp add*:*Point-Eq*)
then have *P53* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi F*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P48* **have** *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se C A*) *F* **by** *simp*
then have *Plane-diffside* (*Li B D*) *C A* ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹ *Bet-Point* (*Se C A*) *B* **by** (*simp add*:*Point-Eq*)
then have *P54* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-Bet-on*)
from *P1 P54* **have** *P55* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Eq* (*Geos* (*Poi F*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** *blast*
have *P56* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P16* **have** *P57* : *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Point-Eq*)
from *P4 P57* **have** *P58* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P32 P48 P53 P55 P56 P58* **have** ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹
Bet-Point (*Se B F*) *D* ∨ *Bet-Point* (*Se F D*) *B* ∨ *Bet-Point* (*Se D B*) *F* **by** (*simp add*:*Bet-case*)
then have *P59* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹
Bet-Point (*Se B F*) *D* ∧ ¬ *Bet-Point* (*Se F D*) *B* ∧ ¬ *Bet-Point* (*Se D B*) *F*
∨ ¬ *Bet-Point* (*Se B F*) *D* ∧ *Bet-Point* (*Se F D*) *B* ∧ ¬ *Bet-Point* (*Se D B*) *F*
∨ ¬ *Bet-Point* (*Se B F*) *D* ∧ ¬ *Bet-Point* (*Se F D*) *B* ∧ *Bet-Point* (*Se D B*)

F **by** (*simp add*:*Bet-case-fact*)
from *P26* **have** *P60* : *Eq* (*Geos* (*Lin* (*Li E C*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) ⟹ *Line-on* (*Li B D*) *E* **by** (*simp add*:*Line-on-trans*)
from *P36 P60* **have** *P61* : *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P62* : *Bet-Point* (*Se F D*) *B* ⟹ *Line-on* (*Li F D*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P63* : *Line-on* (*Li F D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P32 P56 P58 P62 P63* **have** *P64* : *Bet-Point* (*Se F D*) *B* ⟹ *Eq* (*Geos* (*Lin* (*Li F D*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P61 P64* **have** *P65* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se F D*) *B* ⟹
¬ *Eq* (*Geos* (*Lin* (*Li F D*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P13 P65* **have** *P66* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se F D*) *B* ⟹ *Plane-diffside* (*Li E C*) *F D* **by** (*simp add*:*Plane-Bet-diffside*)
have *Line-on* (*Li C A*) *A* **by** (*simp add*:*Line-on-rule*)
then have *P67* : *Eq* (*Geos* (*Lin* (*Li C A*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) ⟹ *Line-on* (*Li E C*) *A* **by** (*simp add*:*Line-on-trans*)
from *P25 P67* **have** *P68* : ¬ *Eq* (*Geos* (*Lin* (*Li C A*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) **by** *blast*
from *P48* **have** *P69* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se C A*) *F* **by** *simp*
from *P21 P68 P69* **have** *P71* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Plane-sameside* (*Li E C*) *F A* **by** (*simp add*:*Plane-Bet-sameside*)
from *P66 P71* **have** *P72* : *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se F D*) *B* ⟹
Plane-diffside (*Li E C*) *A D* **by** (*simp add*:*Plane-trans*)
from *P22 P72* **have** *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se F D*) *B* ⟹

Plane-diffside (*Li B C*) *A D* **by** (*simp add*:*Plane-Line-diff-trans*)
then have *Plane-diffside* (*Li B D*) *C A* ⟹ *Bet-Point* (*Se F D*) *B* ⟹
¬ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Plane-diffside-not-sameside*)
then have *P73* : *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Bet-Point* (*Se F D*) *B* **by** *blast*
have *P74* : *Bet-Point* (*Se B F*) *D* ⟹ *Line-on* (*Li B F*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P59* **have** *P75* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹
Bet-Point (*Se B F*) *D* ⟹ ¬ *Bet-Point* (*Se D F*) *B* **by** (*blast intro*:*Bet-rev*)
from *P52 P58 P74 P75* **have** ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹
Bet-Point (*Se B F*) *D* ⟹ *Ang-inside* (*An A B C*) *D* **by** (*simp add*:*Ang-inside-HalfLine*)
then have *P76* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Bet-Point* (*Se B F*) *D* **by** *blast*
have *P77* : *Bet-Point* (*Se D B*) *F* ⟹ *Line-on* (*Li B F*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P59* **have** *P78* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹
Bet-Point (*Se D B*) *F* ⟹ ¬ *Bet-Point* (*Se D F*) *B* **by** (*blast intro*:*Bet-rev*)
from *P52 P58 P77 P78* **have** ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹

Bet-Point (*Se D B*) *F* ⟹ *Ang-inside* (*An A B C*) *D* **by** (*simp add*:*Ang-inside-HalfLine*)
then have *P79* : ¬ *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *C A* ⟹ ¬ *Bet-Point* (*Se D B*) *F* **by** *blast*
from *P59 P73 P76 P79* **have** ¬ *Ang-inside* (*An A B C*) *D* ∧ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-diffside* (*Li B D*) *C A* ⟹ *False* **by** *blast*
then have *P80* : ¬ *Ang-inside* (*An A B C*) *D* ∧ ¬ *Plane-sameside* (*Li B A*) *C D*
∧ *Plane-sameside* (*Li B C*) *A D* ∧ *Plane-diffside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A* ⟹ *False* **by** *simp*
from *P5* **have** *Def* (*Tri* (*Tr B D C*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P81* : ¬ *Line-on* (*Li B D*) *C* **by** (*simp add*:*Tri-def-Line*)
from *P14 P15 P19 P32* **have** *P82* : *Line-on* (*Li B D*) *A* ⟹
Eq (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P82* **have** *P83* : ¬ *Line-on* (*Li B D*) *A* **by** *blast*
from *P28 P81 P83* **have** *P84* : ¬ *Plane-sameside* (*Li B D*) *C A* ⟹ *Plane-diffside* (*Li B D*) *C A* **by** (*simp add*:*Plane-not-sameside-diffside*)
from *P80 P84* **have** *P85* : ¬ *Ang-inside* (*An A B C*) *D* ∧ ¬ *Plane-sameside* (*Li B A*) *C D*
∧ *Plane-sameside* (*Li B C*) *A D* ∧ ¬ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A* ⟹ *False* **by** *simp*
from *P7 P8 P9 P85* **show** ¬ (¬ *Ang-inside* (*An A B C*) *D* ∧ ¬ *Ang-inside* (*An D B C*) *A*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-inside-case* :
assumes
Def (*Ang* (*An A B C*))
Def (*Ang* (*An D B C*))
Plane-sameside (*Li B C*) *A D*
¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B D*)) *add Emp*)
shows
Ang-inside (*An A B C*) *D* ∧ ¬ *Ang-inside* (*An D B C*) *A*
∨ ¬ *Ang-inside* (*An A B C*) *D* ∧ *Ang-inside* (*An D B C*) *A*
proof −
have *P1* : *Ang-inside* (*An A B C*) *D* ⟹ *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Ang-inside-def*)
have *P2* : *Ang-inside* (*An D B C*) *A* ⟹ *Plane-sameside* (*Li B D*) *C A* ∧ *Plane-sameside* (*Li B C*) *D A* **by** (*simp add*:*Ang-inside-def*)
have *P3* : *Ang-inside* (*An A B C*) *D* ⟹ *Plane-diffside* (*Li B D*) *A C* **by** (*simp add*:*Ang-inside-Planeside*)
have *Plane-diffside* (*Li B D*) *A C* ⟹ *Plane-diffside* (*Li B D*) *C A* **by** (*simp add*:*Plane-diffside-rev*)
then have *P4* : *Plane-diffside* (*Li B D*) *A C* ⟹ ¬ *Plane-sameside* (*Li B D*) *C A* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P3 P4* **have** *P5* : *Ang-inside* (*An A B C*) *D* ⟹ ¬ *Plane-sameside* (*Li B D*) *C A* **by** (*simp add*:*Plane-diffside-rev*)
from *P2 P5* **have** *P6* : *Ang-inside* (*An A B C*) *D* ∧ *Ang-inside* (*An D B C*) *A*

$\Longrightarrow$ *False* **by** *simp*
from *assms* **have** *P7* : $\neg$ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *assms P7* **have** *P8* : $\neg$ *Ang-inside* (*An A B C*) *D* $\wedge$ $\neg$ *Ang-inside* (*An D B C*) *A* $\Longrightarrow$ *False* **by** (*simp add*:*Ang-outside-exclusive*)
from *P6 P8* **show** *Ang-inside* (*An A B C*) *D* $\wedge$ $\neg$ *Ang-inside* (*An D B C*) *A*
$\vee$ $\neg$ *Ang-inside* (*An A B C*) *D* $\wedge$ *Ang-inside* (*An D B C*) *A* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Plane-sameside-HalfLine* :
assumes
Plane-sameside l1 p1 p2
Line-on l1 p3
Line-on (*Li p3 p1*) *p4*
$\neg$ *Bet-Point* (*Se p4 p1*) *p3*
$\neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
$\neg$ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
shows *Plane-sameside l1 p1 p4*
proof $-$
from *assms* **have** *P1* : $\neg$ *Line-on l1 p1* $\wedge$ $\neg$ *Line-on l1 p2*
$\wedge$ $\neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Plane-sameside-def*)
have *Plane-diffside l1 p1 p4* $\Longrightarrow$
$\exists$*p*. *Bet-Point* (*Se p1 p4*) *p* $\wedge$ *Line-on l1 p* $\wedge$ $\neg$ *Line-on l1 p1* $\wedge$ $\neg$ *Line-on l1 p4* **by** (*simp add*:*Plane-diffside-def*)
then obtain *p5* :: *Point* **where** *P2* : *Plane-diffside l1 p1 p4* $\Longrightarrow$
Bet-Point (*Se p1 p4*) *p5* $\wedge$ *Line-on l1 p5* $\wedge$ $\neg$ *Line-on l1 p4* **by** *blast*
from *assms* **have** *P3* : *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) $\Longrightarrow$ *Line-on l1 p1* **by** (*simp add*:*Point-Eq*)
from *P1 P3* **have** *P4* : $\neg$ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** *blast*
then have *P5* : *Eq* (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *assms P5* **have** *P6* : *Line-on* (*Li p1 p3*) *p4* **by** (*simp add*:*Line-rev Line-on-trans*)
from *P4* **have** *P7* : $\neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P6 P7* **have** *P8* : *Line-on* (*Li p1 p4*) *p3* **by** (*simp add*:*Line-on-rev*)
from *P2* **have** *P9* : *Plane-diffside l1 p1 p4* $\Longrightarrow$ *Line-on* (*Li p1 p4*) *p5* **by** (*simp add*:*Line-Bet-on*)
from *assms P2 P8 P9* **have** *P10* : *Plane-diffside l1 p1 p4* $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li p1 p4*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P11* : *Line-on* (*Li p1 p4*) *p1* **by** (*simp add*:*Line-on-rule*)
from *P10 P11* **have** *P12* : *Plane-diffside l1 p1 p4* $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) $\Longrightarrow$
Line-on l1 p1 **by** (*simp add*:*Line-on-trans*)
from *P1 P12* **have** *P13* : *Plane-diffside l1 p1 p4* $\Longrightarrow$ *Eq* (*Geos* (*Poi p5*) *add*

Emp) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P2* **have** *P14* : *Plane-diffside l1 p1 p4* $\Longrightarrow$ *Bet-Point* (*Se p1 p4*) *p5* **by** *simp*
from *P13 P14* **have** *Plane-diffside l1 p1 p4* $\Longrightarrow$ *Bet-Point* (*Se p1 p4*) *p3* **by** (*simp add*:*Point-Eq*)
then have *P15* : *Plane-diffside l1 p1 p4* $\Longrightarrow$ *Bet-Point* (*Se p4 p1*) *p3* **by** (*simp add*:*Bet-rev*)
from *assms P15* **have** *P16* : $\neg$ *Plane-diffside l1 p1 p4* **by** *blast*
have *P17* : *Line-on* (*Li p3 p1*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms P17* **have** *P18* : *Line-on l1 p4* $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li p3 p1*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P19* : *Line-on* (*Li p3 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
from *P18 P19* **have** *P20* : *Line-on l1 p4* $\Longrightarrow$ *Line-on l1 p1* **by** (*simp add*:*Line-on-trans*)
from *P1 P20* **have** *P21* : $\neg$ *Line-on l1 p4* **by** *blast*
from *assms P1 P16 P21* **show** *Plane-sameside l1 p1 p4* **by** (*simp add*:*Plane-not-diffside-sameside*)
qed

lemma (**in** *Congruence-Rule*) *Plane-Bet-sameside-rev* :
assumes
Plane-sameside l1 p1 p3
Line-on l1 p2
$\neg$ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
$\neg$ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*)
$\neg$ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*)
Line-on l2 p1 Line-on l2 p2 Line-on l2 p3
$\neg$ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin l2*) *add Emp*)
shows *Bet-Point* (*Se p3 p2*) *p1* $\vee$ *Bet-Point* (*Se p2 p1*) *p3*
proof $-$
from *assms* **have** *P1* : *Bet-Point* (*Se p1 p3*) *p2* $\vee$ *Bet-Point* (*Se p3 p2*) *p1* $\vee$ *Bet-Point* (*Se p2 p1*) *p3* **by** (*simp add*:*Bet-case*)
have *P2* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms P2 P3* **have** *P4* : *Eq* (*Geos* (*Lin l2*) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P4* **have** $\neg$ *Eq* (*Geos* (*Lin l1*) *add Emp*) (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
then have *P6* : $\neg$ *Eq* (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin l1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P6* **have** *Bet-Point* (*Se p1 p3*) *p2* $\Longrightarrow$ *Plane-diffside l1 p1 p3* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P7* : *Bet-Point* (*Se p1 p3*) *p2* $\Longrightarrow$ $\neg$ *Plane-sameside l1 p1 p3* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *assms P7* **have** *P8* : $\neg$ *Bet-Point* (*Se p1 p3*) *p2* **by** *blast*
from *P1 P8* **show** *Bet-Point* (*Se p3 p2*) *p1* $\vee$ *Bet-Point* (*Se p2 p1*) *p3* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-Bet-relation* :
assumes *N* :

Bet-Point (Se p1 p2) p3
shows ¬ *Eq* (*Geos* (*Seg* (*Se p1 p2*)) *add Emp*) (*Geos* (*Seg* (*Se p1 p3*)) *add Emp*)
proof
assume *W* : *Eq* (*Geos* (*Seg* (*Se p1 p2*)) *add Emp*) (*Geos* (*Seg* (*Se p1 p3*)) *add Emp*)
from *N* **have** *Inv* (*Bet-Point* (*Se p2 p3*) *p1*) ∧ *Inv* (*Bet-Point* (*Se p3 p1*) *p2*) **by** (*simp add*:*Bet-iff*)
then have *P1* : ¬ *Bet-Point* (*Se p2 p3*) *p1* **by** (*simp add*:*Inv-def*)
have *P2* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P4* : *Line-on* (*Li p1 p2*) *p3* **by** (*simp add*:*Line-Bet-on*)
from *N* **have** *P5* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *N* **have** ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P6* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *W P1 P2 P3 P4 P5 P6* **have** *P7* : *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*blast intro*:*Seg-move-unique*)
from *N* **have** *P8* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P7 P8* **show** *False* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-Bet-move-lemma1* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Line-on l1 p21 Line-on l1 p22 Line-on l1 p23
¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p22*) *add Emp*)
¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*)
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
¬ *Bet-Point* (*Se p22 p23*) *p21*
shows *Bet-Point* (*Se p21 p23*) *p22*
proof −
from *assms* **have** *P1* : ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p21*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p11*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P2* : ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P1 P2* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p21*) *p22* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *p211* :: *Point* **where** *P3* : *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p211*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p211 p21*) *p22* ∧ *Line-on l1 p211* ∧ ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p211*) *add Emp*) **by** *blast*

from *assms* **have** ¬ *Eq* (*Geos* (*Poi p13*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P4* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P3 P4* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p*)) *add Emp*)
∧ *Bet-Point* (*Se p p211*) *p22* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-diffside*)
then obtain *p231* :: *Point* **where** *P5* : *Eq* (*Geos* (*Seg* (*Se p12 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p231*)) *add Emp*)
∧ *Bet-Point* (*Se p231 p211*) *p22* ∧ *Line-on l1 p231* ∧ ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p231*) *add Emp*) **by** *blast*
have *P6* : *Eq* (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p21*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *assms* **have** *P7* : *Eq* (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*) (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) **by** (*simp add*:*Eq-rev*)
from *P3 P7* **have** *P8* : *Eq* (*Geos* (*Seg* (*Se p22 p211*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P6 P8* **have** *P9* : *Eq* (*Geos* (*Seg* (*Se p22 p211*)) *add Emp*) (*Geos* (*Seg* (*Se p22 p21*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *assms P1 P3 P9* **have** *P10* : *Eq* (*Geos* (*Poi p211*) *add Emp*) (*Geos* (*Poi p21*) *add Emp*) **by** (*blast intro*:*Seg-move-unique*)
from *P5* **have** *P11* : *Bet-Point* (*Se p211 p231*) *p22* **by** (*simp add*:*Bet-rev*)
from *P10 P11* **have** *P12* : *Bet-Point* (*Se p21 p231*) *p22* **by** (*simp add*:*Bet-Point-Eq*)
have *P13* : *Line-on* (*Li p11 p12*) *p11* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li p11 p12*) *p12* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P15* : *Line-on* (*Li p11 p12*) *p13* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *P16* : ¬ *Seg-on-Seg* (*Se p11 p12*) (*Se p12 p13*) **by** (*simp add*:*Seg-Bet-not-on*)
from *P12* **have** *P17* : ¬ *Seg-on-Seg* (*Se p21 p22*) (*Se p22 p231*) **by** (*simp add*:*Seg-Bet-not-on*)
from *assms P5 P13 P14 P15 P16 P17* **have** *P18* : *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p231*)) *add Emp*) **by** (*simp add*:*Seg-add*)
from *assms P18* **have** *P19* : *Eq* (*Geos* (*Seg* (*Se p21 p231*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P12* **have** *P20* : ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p231*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P12* **have** *P21* : *Bet-Point* (*Se p231 p21*) *p22* **by** (*simp add*:*Bet-rev*)
from *P21* **have** *P22* : *Bet-Point* (*Se p231 p23*) *p21* ⟹ *Bet-Point* (*Se p22 p23*) *p21* **by** (*blast intro*:*Bet-swap-134-234*)
from *assms P22* **have** *P23* : ¬ *Bet-Point* (*Se p231 p23*) *p21* **by** *blast*
from *assms P5 P19 P20 P23* **have** *P24* : *Eq* (*Geos* (*Poi p231*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** (*blast intro*:*Seg-move-unique*)
from *P21 P24* **have** *Bet-Point* (*Se p23 p21*) *p22* **by** (*simp add*:*Bet-Point-Eq*)
thus *Bet-Point* (*Se p21 p23*) *p22* **by** (*simp add*:*Bet-rev*)
qed

lemma (**in** *Congruence-Rule*) *Seg-Bet-move-sameside* :
assumes

Bet-Point (*Se p11 p13*) *p12*
Line-on l1 p21 Line-on l1 p4
¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
shows ∃ *p q*. *Bet-Point* (*Se p21 q*) *p* ∧ *Line-on l1 p* ∧ *Line-on l1 q*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 q*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p4*) *p21* ∧ ¬ *Bet-Point* (*Se q p4*) *p21*
proof −
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p11*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P1* : ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P1* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p4*) *p21* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *p22* :: *Point* **where** *P2* : *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p22 p4*) *p21* ∧ *Line-on l1 p22* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p22*) *add Emp*) **by** *blast*
from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P2 P3* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p22*) *p21* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *p23* :: *Point* **where** *P4* : *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p23 p22*) *p21* ∧ *Line-on l1 p23* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** *blast*
from *P4* **have** ¬ *Bet-Point* (*Se p23 p22*) *p21* **by** *simp*
then have *P5* : ¬ *Bet-Point* (*Se p22 p23*) *p21* **by** (*blast intro*:*Bet-rev*)
from *assms P2 P4 P5* **have** *P6* : *Bet-Point* (*Se p21 p23*) *p22* **by** (*blast intro*:*Seg-Bet-move-lemma1*)
then have *P7* : *Bet-Point* (*Se p23 p21*) *p22* **by** (*simp add*:*Bet-rev*)
from *P7* **have** *P8* : *Bet-Point* (*Se p23 p4*) *p21* ⟹ *Bet-Point* (*Se p22 p4*) *p21* **by** (*blast intro*:*Bet-swap-134-234*)
from *P2 P8* **have** *P9* : ¬ *Bet-Point* (*Se p23 p4*) *p21* **by** *blast*
from *P2 P4 P6 P9* **show** ∃ *p q*. *Bet-Point* (*Se p21 q*) *p* ∧ *Line-on l1 p* ∧ *Line-on l1 q*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 q*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p4*) *p21* ∧ ¬ *Bet-Point* (*Se q p4*) *p21* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-Bet-move-diffside* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Line-on l1 p21 Line-on l1 p4

¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
shows ∃ *p q*. *Bet-Point* (*Se p21 q*) *p* ∧ *Line-on l1 p* ∧ *Line-on l1 q*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 q*)) *add Emp*)
∧ *Bet-Point* (*Se p p4*) *p21* ∧ *Bet-Point* (*Se q p4*) *p21*
proof −
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p11*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P1* : ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P1* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Bet-Point* (*Se p p4*) *p21* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-diffside*)
then obtain *p22* :: *Point* **where** *P2* : *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
∧ *Bet-Point* (*Se p22 p4*) *p21* ∧ *Line-on l1 p22* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p22*) *add Emp*) **by** *blast*
from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P2 P3* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p22*) *p21* ∧ *Line-on l1 p* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *p23* :: *Point* **where** *P4* : *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p23 p22*) *p21* ∧ *Line-on l1 p23* ∧ ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** *blast*
from *P4* **have** ¬ *Bet-Point* (*Se p23 p22*) *p21* **by** *simp*
then have *P5* : ¬ *Bet-Point* (*Se p22 p23*) *p21* **by** (*blast intro*:*Bet-rev*)
from *assms P2 P4 P5* **have** *P6* : *Bet-Point* (*Se p21 p23*) *p22* **by** (*blast intro*:*Seg-Bet-move-lemma1*)
then have *P7* : *Bet-Point* (*Se p23 p21*) *p22* **by** (*simp add*:*Bet-rev*)
from *P2 P7* **have** *P8* : *Bet-Point* (*Se p23 p4*) *p21* **by** (*blast intro*:*Bet-swap-234-134*)
from *P2 P4 P6 P8* **show** ∃ *p q*. *Bet-Point* (*Se p21 q*) *p* ∧ *Line-on l1 p* ∧ *Line-on l1 q*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 q*)) *add Emp*)
∧ *Bet-Point* (*Se p p4*) *p21* ∧ *Bet-Point* (*Se q p4*) *p21* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Seg-Bet-wrong-relation* :
assumes
Bet-Point (*Se p11 p13*) *p12*
Bet-Point (*Se p21 p22*) *p23*
Eq (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*)
shows *False*
proof −

have *P1* : *Line-on* (*Li p21 p22*) *p21* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li p21 p22*) *p22* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p22*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P1 P2 P3* **have** ∃ *p q*. *Bet-Point* (*Se p21 q*) *p* ∧ *Line-on* (*Li p21 p22*) *p* ∧ *Line-on* (*Li p21 p22*) *q*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 q*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p p22*) *p21* ∧ ¬ *Bet-Point* (*Se q p22*) *p21* **by** (*simp add*:*Seg-Bet-move-sameside*)
then obtain *pn2 pn3* :: *Point* **where** *P4* : *Bet-Point* (*Se p21 pn3*) *pn2* ∧ *Line-on* (*Li p21 p22*) *pn2* ∧ *Line-on* (*Li p21 p22*) *pn3*
∧ *Eq* (*Geos* (*Seg* (*Se p11 p12*)) *add Emp*) (*Geos* (*Seg* (*Se p21 pn2*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se p11 p13*)) *add Emp*) (*Geos* (*Seg* (*Se p21 pn3*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se pn2 p22*) *p21* ∧ ¬ *Bet-Point* (*Se pn3 p22*) *p21* **by** *blast*
then have *P5* : *Bet-Point* (*Se p21 pn3*) *pn2* **by** *simp*
then have ¬ *Eq* (*Geos* (*Poi pn2*) *add Emp*) (*Geos* (*Poi p21*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P6* : ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi pn2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P4* **have** *P7* : *Eq* (*Geos* (*Seg* (*Se p21 pn2*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p22*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P1 P2 P3 P4 P6 P7* **have** *P8* : *Eq* (*Geos* (*Poi pn2*) *add Emp*) (*Geos* (*Poi p22*) *add Emp*) **by** (*blast intro*:*Seg-move-unique*)
from *P5 P8* **have** *Bet-Point* (*Se p21 pn3*) *p22* **by** (*simp add*:*Point-Eq*)
then have *P9* : *Bet-Point* (*Se pn3 p21*) *p22* **by** (*simp add*:*Bet-rev*)
from *assms* **have** *P10* : *Line-on* (*Li p21 p22*) *p23* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p23*) *add Emp*) (*Geos* (*Poi p21*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P11* : ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P5* **have** *P12* : ¬ *Eq* (*Geos* (*Poi p21*) *add Emp*) (*Geos* (*Poi pn3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms P4* **have** *P13* : *Eq* (*Geos* (*Seg* (*Se p21 pn3*)) *add Emp*) (*Geos* (*Seg* (*Se p21 p23*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *assms* **have** *P14* : *Bet-Point* (*Se p22 p21*) *p23* **by** (*simp add*:*Bet-rev*)
have *P15* : *Bet-Point* (*Se pn3 p23*) *p21* ⟹ *Bet-Point* (*Se p23 pn3*) *p21* **by** (*simp add*:*Bet-rev*)
from *P14 P15* **have** *Bet-Point* (*Se pn3 p23*) *p21* ⟹ *Bet-Point* (*Se p22 pn3*) *p21* **by** (*blast intro*:*Bet-swap-234-134*)
then have *P16* : *Bet-Point* (*Se pn3 p23*) *p21* ⟹ *Bet-Point* (*Se pn3 p22*) *p21* **by** (*simp add*:*Bet-rev*)
from *P4 P16* **have** *P17* : ¬ *Bet-Point* (*Se pn3 p23*) *p21* **by** *blast*
from *P1 P4 P10 P11 P12 P13 P17* **have** *P18* : *Eq* (*Geos* (*Poi pn3*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** (*blast intro*:*Seg-move-unique*)
from *P9 P18* **have** *P19* : *Bet-Point* (*Se p23 p21*) *p22* **by** (*simp add*:*Bet-Point-Eq*)
from *assms* **have** *Inv* (*Bet-Point* (*Se p22 p23*) *p21*) ∧ *Inv* (*Bet-Point* (*Se p23 p21*) *p22*) **by** (*simp add*:*Bet-iff*)
then have *P20* : ¬ *Bet-Point* (*Se p23 p21*) *p22* **by** (*simp add*:*Inv-def*)

from *P19 P20* **show** *False* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-inside-trans* :
assumes
Ang-inside (*An A B C*) *D Def* (*Ang* (*An A B C*))
Line-on (*Li B A1*) *A* ¬ *Bet-Point* (*Se A A1*) *B*
Line-on (*Li B C1*) *C* ¬ *Bet-Point* (*Se C C1*) *B*
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*)
¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*)
shows *Ang-inside* (*An A1 B C1*) *D*
proof −
from *assms* **have** *P1* : *Plane-sameside* (*Li B A*) *C D* ∧ *Plane-sameside* (*Li B C*) *A D* **by** (*simp add*:*Ang-inside-def*)
have *P2* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
have *P3* : *Line-on* (*Li B A1*) *B* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P5* : *Def* (*Tri* (*Tr A B C*)) **by** (*simp add*:*Ang-to-Tri*)
from *P5* **have** *P6* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *assms P2 P3 P4 P6* **have** *P7* : *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B A1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P8* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P9* : *Line-on* (*Li B C1*) *B* **by** (*simp add*:*Line-on-rule*)
have *P10* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P5* **have** *P11* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *assms P8 P9 P10 P11* **have** *P12* : *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li B C1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P13* : *Plane-diffside* (*Li B A*) *C1 D* ⟹ *Plane-diffside* (*Li B A*) *D C1* **by** (*simp add*:*Plane-diffside-rev*)
from *P1* **have** *P14* : *Plane-sameside* (*Li B A*) *D C* **by** (*simp add*:*Plane-sameside-rev*)
from *P13 P14* **have** *P15* : *Plane-diffside* (*Li B A*) *C1 D* ⟹ *Plane-diffside* (*Li B A*) *C C1* **by** (*simp add*:*Plane-trans*)
then have *Plane-diffside* (*Li B A*) *C1 D* ⟹ ∃ *p*. *Bet-Point* (*Se C C1*) *p* ∧ *Line-on* (*Li B A*) *p* ∧ ¬ *Line-on* (*Li B A*) *C* ∧ ¬ *Line-on* (*Li B A*) *C1* **by** (*simp add*:*Plane-diffside-def*)
then obtain *B1* :: *Point* **where** *P16* : *Plane-diffside* (*Li B A*) *C1 D* ⟹ *Bet-Point* (*Se C C1*) *B1* ∧ *Line-on* (*Li B A*) *B1* ∧ ¬ *Line-on* (*Li B A*) *C* ∧ ¬ *Line-on* (*Li B A*) *C1* **by** *blast*
from *P16* **have** *P17* : *Plane-diffside* (*Li B A*) *C1 D* ⟹ *Bet-Point* (*Se C C1*) *B1* **by** *simp*
then have *P18* : *Plane-diffside* (*Li B A*) *C1 D* ⟹ *Line-on* (*Li C C1*) *B1* **by** (*simp add*:*Line-Bet-on*)
have *P19* : *Line-on* (*Li B C1*) *C1* **by** (*simp add*:*Line-on-rule*)
have *P20* : *Line-on* (*Li C C1*) *C* **by** (*simp add*:*Line-on-rule*)
have *P21* : *Line-on* (*Li C C1*) *C1* **by** (*simp add*:*Line-on-rule*)
from *assms P19 P20 P21* **have** *P22* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi*

C1) *add Emp*) ⟹
Plane-diffside (*Li B A*) *C1 D* ⟹ *Eq* (*Geos* (*Lin* (*Li B C1*)) *add Emp*) (*Geos* (*Lin* (*Li C C1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P9 P22* **have** *P23* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹
Plane-diffside (*Li B A*) *C1 D* ⟹ *Line-on* (*Li C C1*) *B* **by** (*simp add*:*Line-on-trans*)
from *P21* **have** *P24* : *Eq* (*Geos* (*Lin* (*Li C C1*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) ⟹ *Line-on* (*Li B A*) *C1* **by** (*simp add*:*Line-on-trans*)
from *P16 P24* **have** *P25* : *Plane-diffside* (*Li B A*) *C1 D* ⟹
¬ *Eq* (*Geos* (*Lin* (*Li C C1*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** *blast*
from *P2 P16 P18 P23 P25* **have** *P26* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹
Plane-diffside (*Li B A*) *C1 D* ⟹ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P17 P26* **have** *P27* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹
Plane-diffside (*Li B A*) *C1 D* ⟹ *Bet-Point* (*Se C C1*) *B* **by** (*simp add*:*Point-Eq*)
from *assms P27* **have** *P28* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹
¬ *Plane-diffside* (*Li B A*) *C1 D* **by** *blast*
from *assms P2 P9 P19* **have** *P29* : *Line-on* (*Li B A*) *C1* ⟹
Eq (*Geos* (*Lin* (*Li B C1*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P12 P29* **have** *P30* : *Line-on* (*Li B A*) *C1* ⟹
Eq (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P10 P30* **have** *P31* : *Line-on* (*Li B A*) *C1* ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-on-trans*)
from *P1* **have** *P32* : ¬ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Plane-sameside-def*)
from *P31 P32* **have** *P33* : ¬ *Line-on* (*Li B A*) *C1* **by** *blast*
from *P1* **have** *P34* : ¬ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *P12 P19* **have** *Line-on* (*Li B C*) *C1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
then have *P35* : *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Point-Eq*)
from *P1* **have** *P36* : ¬ *Line-on* (*Li B C*) *D* **by** (*simp add*:*Plane-sameside-def*)
from *P35 P36* **have** *P37* : ¬ *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P28 P33 P34 P37* **have** *P38* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹
Plane-sameside (*Li B A*) *C1 D* **by** (*simp add*:*Plane-not-diffside-sameside*)
from *P14* **have** *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹ *Plane-sameside* (*Li B A*) *D C1* **by** (*simp add*:*Point-Eq*)
then have *P39* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) ⟹ *Plane-sameside* (*Li B A*) *C1 D* **by** (*simp add*:*Plane-sameside-rev*)
from *P38 P39* **have** *P40* : *Plane-sameside* (*Li B A*) *C1 D* **by** *blast*
from *P7 P40* **have** *P41* : *Plane-sameside* (*Li B A1*) *C1 D* **by** (*simp add*:*Plane-Line-trans*)
have *P42* : *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Plane-diffside* (*Li B C*) *D A1* **by** (*simp add*:*Plane-diffside-rev*)
from *P1* **have** *P43* : *Plane-sameside* (*Li B C*) *D A* **by** (*simp add*:*Plane-sameside-rev*)

from *P42 P43* **have** *P44* : *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Plane-diffside* (*Li B C*) *A A1* **by** (*simp add*:*Plane-trans*)
then have *Plane-diffside* (*Li B C*) *A1 D* ⟹ ∃ *p*. *Bet-Point* (*Se A A1*) *p* ∧ *Line-on* (*Li B C*) *p* ∧ ¬ *Line-on* (*Li B C*) *A* ∧ ¬ *Line-on* (*Li B C*) *A1* **by** (*simp add*:*Plane-diffside-def*)
then obtain *B2* :: *Point* **where** *P45* : *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Bet-Point* (*Se A A1*) *B2* ∧ *Line-on* (*Li B C*) *B2* ∧ ¬ *Line-on* (*Li B C*) *A* ∧ ¬ *Line-on* (*Li B C*) *A1* **by** *blast*
from *P45* **have** *P46* : *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Bet-Point* (*Se A A1*) *B2* **by** *simp*
then have *P47* : *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Line-on* (*Li A A1*) *B2* **by** (*simp add*:*Line-Bet-on*)
have *P48* : *Line-on* (*Li B A1*) *A1* **by** (*simp add*:*Line-on-rule*)
have *P49* : *Line-on* (*Li A A1*) *A* **by** (*simp add*:*Line-on-rule*)
have *P50* : *Line-on* (*Li A A1*) *A1* **by** (*simp add*:*Line-on-rule*)
from *assms P48 P49 P50* **have** *P51* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Eq* (*Geos* (*Lin* (*Li B A1*)) *add Emp*) (*Geos* (*Lin* (*Li A A1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P3 P51* **have** *P52* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Line-on* (*Li A A1*) *B* **by** (*simp add*:*Line-on-trans*)
from *P50* **have** *P53* : *Eq* (*Geos* (*Lin* (*Li A A1*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) ⟹ *Line-on* (*Li B C*) *A1* **by** (*simp add*:*Line-on-trans*)
from *P45 P53* **have** *P54* : *Plane-diffside* (*Li B C*) *A1 D* ⟹ ¬ *Eq* (*Geos* (*Lin* (*Li A A1*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** *blast*
from *P8 P45 P47 P52 P54* **have** *P55* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P46 P55* **have** *P56* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Plane-diffside* (*Li B C*) *A1 D* ⟹ *Bet-Point* (*Se A A1*) *B* **by** (*simp add*:*Point-Eq*)
from *assms P56* **have** *P57* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ ¬ *Plane-diffside* (*Li B C*) *A1 D* **by** *blast*
from *assms P3 P8 P48* **have** *P58* : *Line-on* (*Li B C*) *A1* ⟹ *Eq* (*Geos* (*Lin* (*Li B A1*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P58* **have** *P59* : *Line-on* (*Li B C*) *A1* ⟹ *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P10 P59* **have** *P60* : *Line-on* (*Li B C*) *A1* ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-on-trans*)
from *P32 P60* **have** *P61* : ¬ *Line-on* (*Li B C*) *A1* **by** *blast*
from *P7 P48* **have** *Line-on* (*Li B A*) *A1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
then have *P62* : *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) ⟹

Line-on (*Li B A*) *D* **by** (*simp add*:*Point-Eq*)
from *P34 P62* **have** *P63* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *P36 P57 P61 P63* **have** *P64* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹
Plane-sameside (*Li B C*) *A1 D* **by** (*simp add*:*Plane-not-diffside-sameside*)
from *P43* **have** *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Plane-sameside* (*Li B C*) *D A1* **by** (*simp add*:*Point-Eq*)
then have *P65* : *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) ⟹ *Plane-sameside* (*Li B C*) *A1 D* **by** (*simp add*:*Plane-sameside-rev*)
from *P64 P65* **have** *P66* : *Plane-sameside* (*Li B C*) *A1 D* **by** *blast*
from *P12 P66* **have** *P67* : *Plane-sameside* (*Li B C1*) *A1 D* **by** (*simp add*:*Plane-Line-trans*)
from *P12* **have** *P68* : *Line-on* (*Li B C1*) *A1* ⟹ *Line-on* (*Li B C*) *A1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P61 P68* **have** *P69* : ¬ *Line-on* (*Li B C1*) *A1* **by** *blast*
from *assms P69* **have** *Def* (*Ang* (*An B C1 A1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P70* : *Def* (*Ang* (*An A1 B C1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P41 P67 P70* **show** *Ang-inside* (*An A1 B C1*) *D* **by** (*simp add*:*Ang-inside-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-sub-lemma1* :
assumes
Plane-sameside (*Li o1 l1*) *h1 k1*
¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)
Plane-sameside (*Li o2 l2*) *h2 k2*
¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)
Ang-inside (*An k1 o1 l1*) *h1*
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An k2 o2 l2*) *h2*
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li o1 l1*) *h1* **by** (*simp add*:*Plane-sameside-def*)
from *assms P1* **have** *Def* (*Ang* (*An o1 l1 h1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P2* : *Def* (*Ang* (*An h1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P3* : ¬ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Plane-sameside-def*)
from *assms P3* **have** *Def* (*Ang* (*An o1 l1 k1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P4* : *Def* (*Ang* (*An k1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P5* : ¬ *Line-on* (*Li o2 l2*) *h2* **by** (*simp add*:*Plane-sameside-def*)
from *assms P5* **have** *Def* (*Ang* (*An o2 l2 h2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P6* : *Def* (*Ang* (*An h2 o2 l2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P7* : ¬ *Line-on* (*Li o2 l2*) *k2* **by** (*simp add*:*Plane-sameside-def*)
from *assms P7* **have** *Def* (*Ang* (*An o2 l2 k2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P8* : *Def* (*Ang* (*An k2 o2 l2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)

from *assms P4 P8* **have** $\exists p\ q.$ *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An p o2 q*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An p o2 q*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Seg* (*Se o1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se o2 p*)) *add Emp*) $\wedge$ *Line-on* (*Li o2 k2*) *p* $\wedge \neg$ *Bet-Point* (*Se p k2*) *o2*
$\wedge$ *Eq* (*Geos* (*Seg* (*Se o1 l1*)) *add Emp*) (*Geos* (*Seg* (*Se o2 q*)) *add Emp*) $\wedge$ *Line-on* (*Li o2 l2*) *q* $\wedge \neg$ *Bet-Point* (*Se q l2*) *o2* $\wedge$ *Def* (*Ang* (*An p o2 q*)) **by** (*simp add*:*Ang-replace*)
then obtain *k21 l21* :: *Point* **where** *P9* : *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 l21*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 l21*)) *add Emp*)
$\wedge$ *Eq* (*Geos* (*Seg* (*Se o1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se o2 k21*)) *add Emp*) $\wedge$ *Line-on* (*Li o2 k2*) *k21* $\wedge \neg$ *Bet-Point* (*Se k21 k2*) *o2*
$\wedge$ *Eq* (*Geos* (*Seg* (*Se o1 l1*)) *add Emp*) (*Geos* (*Seg* (*Se o2 l21*)) *add Emp*) $\wedge$ *Line-on* (*Li o2 l2*) *l21* $\wedge \neg$ *Bet-Point* (*Se l21 l2*) *o2* $\wedge$ *Def* (*Ang* (*An k21 o2 l21*)) **by** *blast*
from *assms* **have** *Plane-diffside* (*Li o1 h1*) *k1 l1* **by** (*simp add*:*Ang-inside-Planeside*)
then have $\exists p.$ *Bet-Point* (*Se k1 l1*) *p* $\wedge$ *Line-on* (*Li o1 h1*) *p* $\wedge \neg$ *Line-on* (*Li o1 h1*) *k1* $\wedge \neg$ *Line-on* (*Li o1 h1*) *l1* **by** (*simp add*:*Plane-diffside-def*)
then obtain *h11* :: *Point* **where** *P10* : *Bet-Point* (*Se k1 l1*) *h11* $\wedge$ *Line-on* (*Li o1 h1*) *h11* $\wedge \neg$ *Line-on* (*Li o1 h1*) *k1* $\wedge \neg$ *Line-on* (*Li o1 h1*) *l1* **by** *blast*
then have *Eq* (*Geos* (*Poi h11*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) $\Longrightarrow$ *Bet-Point* (*Se k1 l1*) *o1* **by** (*blast intro*:*Point-Eq*)
then have *P11* : *Eq* (*Geos* (*Poi h11*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Line-Bet-on*)
from *P3 P11* **have** *P12* : $\neg$ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h11*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P13* : *Line-on* (*Li o2 h2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li o2 h2*) *h2* **by** (*simp add*:*Line-on-rule*)
from *P6* **have** $\neg$ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P15* : $\neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P12 P13 P14 P15* **have** $\exists p.$ *Eq* (*Geos* (*Seg* (*Se o1 h11*)) *add Emp*) (*Geos* (*Seg* (*Se o2 p*)) *add Emp*)
$\wedge \neg$ *Bet-Point* (*Se p h2*) *o2* $\wedge$ *Line-on* (*Li o2 h2*) *p* $\wedge \neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *h21* :: *Point* **where** *P16* : *Eq* (*Geos* (*Seg* (*Se o1 h11*)) *add Emp*) (*Geos* (*Seg* (*Se o2 h21*)) *add Emp*)
$\wedge \neg$ *Bet-Point* (*Se h21 h2*) *o2* $\wedge$ *Line-on* (*Li o2 h2*) *h21* $\wedge \neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) **by** *blast*
have *P17* : *Line-on* (*Li o1 l1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li h1 h11*) *h1* **by** (*simp add*:*Line-on-rule*)
then have *P18* : *Eq* (*Geos* (*Lin* (*Li h1 h11*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o1 l1*) *h1* **by** (*simp add*:*Line-on-trans*)
from *P1 P18* **have** *P19* : $\neg$ *Eq* (*Geos* (*Lin* (*Li h1 h11*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) **by** *blast*
from *P17 P19* **have** *Bet-Point* (*Se h1 h11*) *o1* $\Longrightarrow$ *Plane-diffside* (*Li o1 l1*) *h1*

h11 **by** (*simp add*:*Plane-Bet-diffside*)
then have *P20* : *Bet-Point* (*Se h1 h11*) *o1* ⟹ *Plane-diffside* (*Li o1 l1*) *h11 h1* **by** (*simp add*:*Plane-diffside-rev*)
from *P10* **have** *P21* : *Bet-Point* (*Se l1 k1*) *h11* **by** (*simp add*:*Bet-rev*)
have *P22* : *Line-on* (*Li o1 l1*) *l1* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li l1 k1*) *k1* **by** (*simp add*:*Line-on-rule*)
then have *P23* : *Eq* (*Geos* (*Lin* (*Li l1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Line-on-trans*)
from *P3 P23* **have** *P24* : ¬ *Eq* (*Geos* (*Lin* (*Li l1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) **by** *blast*
from *P21 P22 P24* **have** *P25* : *Plane-sameside* (*Li o1 l1*) *h11 k1* **by** (*simp add*:*Plane-Bet-sameside*)
from *P20 P25* **have** *Bet-Point* (*Se h1 h11*) *o1* ⟹ *Plane-diffside* (*Li o1 l1*) *h1 k1* **by** (*simp add*:*Plane-trans Plane-diffside-rev*)
then have *P26* : *Bet-Point* (*Se h1 h11*) *o1* ⟹ ¬ *Plane-sameside* (*Li o1 l1*) *h1 k1* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *assms P26* **have** *P27* : ¬ *Bet-Point* (*Se h1 h11*) *o1* **by** *blast*
have *P28* : ¬ *Bet-Point* (*Se l1 l1*) *o1* **by** (*simp add*:*Bet-end-Point*)
from *assms P2 P10 P22 P27 P28 P12* **have** *P29* : *Eq* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h11 o1 l1*)) *add Emp*) ∧ *Def* (*Ang* (*An h11 o1 l1*)) **by** (*simp add*:*Ang-Point-swap*)
from *P9* **have** *P30* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P16* **have** *P31* : ¬ *Bet-Point* (*Se h2 h21*) *o2* **by** (*blast intro*:*Bet-rev*)
from *P9* **have** *P32* : ¬ *Bet-Point* (*Se l2 l21*) *o2* **by** (*blast intro*:*Bet-rev*)
from *P6 P9 P16 P30 P31 P32* **have** *P33* : *Eq* (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An h21 o2 l21*)) *add Emp*) ∧ *Def* (*Ang* (*An h21 o2 l21*)) **by** (*simp add*:*Ang-Point-swap*)
from *assms P29* **have** *P34* : *Cong* (*Geos* (*Ang* (*An h11 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P33 P34* **have** *P35* : *Cong* (*Geos* (*Ang* (*An h11 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h21 o2 l21*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P29* **have** *Def* (*Tri* (*Tr h11 o1 l1*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P36* : *Def* (*Tri* (*Tr o1 h11 l1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P33* **have** *Def* (*Tri* (*Tr h21 o2 l21*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P37* : *Def* (*Tri* (*Tr o2 h21 l21*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P9 P16 P35 P36 P37* **have** *P38* : *Cong* (*Geos* (*Tri* (*Tr o1 h11 l1*)) *add Emp*) (*Geos* (*Tri* (*Tr o2 h21 l21*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P39* : *Cong* (*Geos* (*Ang* (*An o1 l1 h11*)) *add Emp*) (*Geos* (*Ang* (*An o2 l21 h21*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P4* **have** *P40* : *Def* (*Tri* (*Tr o1 k1 l1*)) **by** (*simp add*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P9* **have** *P41* : *Def* (*Tri* (*Tr o2 k21 l21*)) **by** (*simp add*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P9 P40 P41* **have** *P42* : *Cong* (*Geos* (*Tri* (*Tr o1 k1 l1*)) *add Emp*) (*Geos* (*Tri* (*Tr o2 k21 l21*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P43* : *Cong* (*Geos* (*Ang* (*An o1 l1 k1*)) *add Emp*) (*Geos* (*Ang* (*An o2 l21 k21*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P4* **have** *P44* : *Def* (*Ang* (*An o1 l1 k1*)) **by** (*blast intro*:*Ang-def-rev*

Ang-def-inv)
have *P45* : *Line-on* (*Li l1 o1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *P46* : ¬ *Bet-Point* (*Se o1 o1*) *l1* **by** (*simp add*:*Bet-end-Point*)
from *P10* **have** *P47* : *Line-on* (*Li l1 k1*) *h11* **by** (*simp add*:*Line-Bet-on*)
from *P10* **have** *Inv*(*Bet-Point* (*Se h11 k1*) *l1*) **by** (*simp add*:*Bet-iff*)
then have ¬ *Bet-Point* (*Se h11 k1*) *l1* **by** (*simp add*:*Inv-def*)
then have *P48* : ¬ *Bet-Point* (*Se k1 h11*) *l1* **by** (*blast intro*:*Bet-rev*)
from *assms* **have** *P49* : ¬ *Eq* (*Geos* (*Poi l1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P10* **have** *Bet-Point* (*Se k1 l1*) *h11* **by** *simp*
then have *P50* : ¬ *Eq* (*Geos* (*Poi l1*) *add Emp*) (*Geos* (*Poi h11*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P44 P45 P46 P47 P48 P49 P50* **have** *P51* : *Eq* (*Geos* (*Ang* (*An o1 l1 k1*)) *add Emp*) (*Geos* (*Ang* (*An o1 l1 h11*)) *add Emp*) ∧ *Def* (*Ang* (*An o1 l1 h11*)) **by** (*simp add*:*Ang-Point-swap*)
have *P52* : *Eq* (*Geos* (*Ang* (*An o2 l21 k21*)) *add Emp*) (*Geos* (*Ang* (*An k21 l21 o2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P43 P52* **have** *P53* : *Cong* (*Geos* (*Ang* (*An o1 l1 k1*)) *add Emp*) (*Geos* (*Ang* (*An k21 l21 o2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P51 P53* **have** *P54* : *Cong* (*Geos* (*Ang* (*An o1 l1 h11*)) *add Emp*) (*Geos* (*Ang* (*An k21 l21 o2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P55* : *Eq* (*Geos* (*Ang* (*An o2 l21 h21*)) *add Emp*) (*Geos* (*Ang* (*An h21 l21 o2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P39 P55* **have** *P56* : *Cong* (*Geos* (*Ang* (*An o1 l1 h11*)) *add Emp*) (*Geos* (*Ang* (*An h21 l21 o2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P57* : *Line-on* (*Li o2 l2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P58* : *Line-on* (*Li l21 o2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P59* : *Line-on* (*Li l21 o2*) *l21* **by** (*simp add*:*Line-on-rule*)
from *P9* **have** *P60* : *Line-on* (*Li o2 l2*) *l21* **by** *simp*
from *P30 P57 P58 P59 P60* **have** *P61* : *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P16 P57* **have** *P62* : ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *h21 h2* **by** (*blast intro*:*Plane-sameside-HalfLine Plane-sameside-rev*)
from *assms* **have** *P63* : *Plane-sameside* (*Li o2 l2*) *k2 h2* **by** (*simp add*:*Plane-sameside-rev*)
from *P41* **have** *P64* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P9 P57 P63 P64* **have** *P65* : ¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k2 k21* **by** (*simp add*:*Plane-sameside-HalfLine*)
then have *P66* : ¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹

Plane-sameside (*Li o2 l2*) *k21 k2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms P62* **have** *P67* : ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *h21 k2* **by** (*blast intro*:*Plane-sameside-trans*)
have *P68* : *Line-on* (*Li o2 k2*) *o2* **by** (*simp add*:*Line-on-rule*)

from *P9* **have** *P69* : *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹ *Line-on* (*Li o2 k2*) *h21* **by** (*blast intro*:*Point-Eq*)
from *P9 P13 P16 P68 P69* **have** *P70* : *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *assms P70* **have** *P71* : ¬ *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) **by** *blast*
from *P65 P67 P71* **have** *P72* : ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k21 h21* **by** (*blast intro*:*Plane-sameside-trans Plane-sameside-rev*)
from *P66* **have** *P73* : ¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Eq (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k21 h21* **by** (*simp add*:*Point-Eq*)
from *P71 P72 P73* **have** *P74* : ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k21 h21* **by** *blast*
from *P63* **have** *P75* : *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k2 h21* **by** (*simp add*:*Point-Eq*)
from *P66 P71 P75* **have** *P76* : *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k21 h21* **by** (*blast intro*:*Plane-sameside-trans Eq-rev*)

from *assms* **have** *P77* : *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *h2 k21* **by** (*simp add*:*Point-Eq*)
from *P62 P71 P77* **have** *P78* : ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Eq (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k21 h21* **by** (*blast intro*:*Plane-sameside-trans Plane-sameside-rev*)

from *P77* **have** *P79* : *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) ⟹
Eq (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) ⟹
Plane-sameside (*Li o2 l2*) *k21 h21* **by** (*blast intro*:*Point-Eq Plane-sameside-rev*)

from *P71 P74 P76 P78 P79* **have** *P80* : *Plane-sameside* (*Li o2 l2*) *k21 h21* **by** *blast*
from *P61 P74 P80* **have** *P81* : *Plane-sameside* (*Li l21 o2*) *k21 h21* **by** (*simp add*:*Plane-Line-trans*)
from *P54 P56 P74 P81* **have** *P82* : *Eq* (*Geos* (*Lin* (*Li k21 l21*)) *add Emp*) (*Geos* (*Lin* (*Li h21 l21*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se k21 h21*) *l21* **by** (*simp*

add:*Ang-move-unique*)
have *P83* : *Line-on* (*Li h21 l21*) *h21* **by** (*simp add*:*Line-on-rule*)
from *P74 P82 P83* **have** *P84* : *Line-on* (*Li k21 l21*) *h21* **by** (*blast intro*:*Line-on-trans Eq-rev*)
have *P85* : *Line-on* (*Li k21 l21*) *k21* **by** (*simp add*:*Line-on-rule*)
have *P86* : *Line-on* (*Li k21 l21*) *l21* **by** (*simp add*:*Line-on-rule*)
from *P9* **have** ¬ *Eq* (*Geos* (*Poi l21*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P87* : ¬ *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P33* **have** *P88* : ¬ *Eq* (*Geos* (*Poi l21*) *add Emp*) (*Geos* (*Poi h21*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P71* **have** *P89* : ¬ *Eq* (*Geos* (*Poi h21*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P81* **have** *P90* : ¬ *Line-on* (*Li l21 o2*) *k21* **by** (*simp add*:*Plane-sameside-def*)
from *P85* **have** *P91* : *Eq* (*Geos* (*Lin* (*Li k21 l21*)) *add Emp*) (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) ⟹
Line-on (*Li l21 o2*) *k21* **by** (*simp add*:*Line-on-trans*)
from *P90 P91* **have** *P92* : ¬ *Eq* (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) (*Geos* (*Lin* (*Li k21 l21*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P59 P81 P84 P85 P86 P87 P88 P89 P92* **have** *P93* : *Bet-Point* (*Se h21 l21*) *k21* ∨ *Bet-Point* (*Se l21 k21*) *h21* **by** (*simp add*:*Plane-Bet-sameside-rev*)
from *P10* **have** *P94* : *Bet-Point* (*Se k1 l1*) *h11* **by** *simp*
then have *P95* : *Bet-Point* (*Se l1 k1*) *h11* **by** (*simp add*:*Bet-rev*)
from *P42* **have** *P96* : *Eq* (*Geos* (*Seg* (*Se k1 l1*)) *add Emp*) (*Geos* (*Seg* (*Se k21 l21*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
then have *P97* : *Eq* (*Geos* (*Seg* (*Se l1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se l21 k21*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans Seg-rev*)
from *P38* **have** *Eq* (*Geos* (*Seg* (*Se h11 l1*)) *add Emp*) (*Geos* (*Seg* (*Se h21 l21*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
then have *P98* : *Eq* (*Geos* (*Seg* (*Se l1 h11*)) *add Emp*) (*Geos* (*Seg* (*Se l21 h21*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans Seg-rev*)
from *P95 P97 P98* **have** *P99* : ¬ *Bet-Point* (*Se l21 h21*) *k21* **by** (*blast intro*:*Seg-Bet-wrong-relation*)
then have *P100* : ¬ *Bet-Point* (*Se h21 l21*) *k21* **by** (*blast intro*:*Bet-rev*)
from *P93 P100* **have** *Bet-Point* (*Se l21 k21*) *h21* **by** *blast*
then have *P101* : ¬ *Seg-on-Seg* (*Se k21 h21*) (*Se h21 l21*) **by** (*simp add*:*Bet-rev Seg-Bet-not-on*)
have *P102* : *Line-on* (*Li k1 l1*) *k1* **by** (*simp add*:*Line-on-rule*)
have *P103* : *Line-on* (*Li k1 l1*) *l1* **by** (*simp add*:*Line-on-rule*)
from *P94* **have** *P104* : *Line-on* (*Li k1 l1*) *h11* **by** (*simp add*:*Line-Bet-on*)
from *P94* **have** *P105* : ¬ *Seg-on-Seg* (*Se k1 h11*) (*Se h11 l1*) **by** (*simp add*:*Seg-Bet-not-on*)
from *assms P84 P85 P86 P96 P101 P102 P103 P104 P105* **have** *P106* : *Eq* (*Geos* (*Seg* (*Se k1 h11*)) *add Emp*) (*Geos* (*Seg* (*Se k21 h21*)) *add Emp*) **by** (*simp add*:*Seg-sub*)
from *P42* **have** *P107* : *Cong* (*Geos* (*Ang* (*An l1 k1 o1*)) *add Emp*) (*Geos* (*Ang* (*An l21 k21 o2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P4* **have** *P108* : *Def* (*Ang* (*An l1 k1 o1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)

from *P94* **have** *Inv* (*Bet-Point* (*Se l1 h11*) *k1*) **by** (*simp add*:*Bet-iff*)
then have *P109* : ¬ *Bet-Point* (*Se l1 h11*) *k1* **by** (*simp add*:*Inv-def*)
have *P110* : *Line-on* (*Li k1 o1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *P111* : ¬ *Bet-Point* (*Se o1 o1*) *k1* **by** (*simp add*:*Bet-end-Point*)
from *P94* **have** ¬ *Eq* (*Geos* (*Poi h11*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P112* : ¬ *Eq* (*Geos* (*Poi k1*) *add Emp*) (*Geos* (*Poi h11*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P108* **have** *P113* : ¬ *Eq* (*Geos* (*Poi k1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P104 P108 P109 P110 P111 P112 P113* **have** *P114* : *Eq* (*Geos* (*Ang* (*An l1 k1 o1*)) *add Emp*) (*Geos* (*Ang* (*An h11 k1 o1*)) *add Emp*) ∧ *Def* (*Ang* (*An h11 k1 o1*)) **by** (*simp add*:*Ang-Point-swap*)
from *P9* **have** *P115* : *Def* (*Ang* (*An l21 k21 o2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
have *P116* : *Line-on* (*Li k21 o2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P117* : ¬ *Bet-Point* (*Se o2 o2*) *k21* **by** (*simp add*:*Bet-end-Point*)
from *P64* **have** *P119* : ¬ *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P71 P84 P99 P115 P116 P117 P119* **have** *P120* : *Eq* (*Geos* (*Ang* (*An l21 k21 o2*)) *add Emp*) (*Geos* (*Ang* (*An h21 k21 o2*)) *add Emp*) ∧ *Def* (*Ang* (*An h21 k21 o2*)) **by** (*simp add*:*Ang-Point-swap*)
from *P107 P114* **have** *P121* : *Cong* (*Geos* (*Ang* (*An h11 k1 o1*)) *add Emp*) (*Geos* (*Ang* (*An l21 k21 o2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P120 P121* **have** *P122* : *Cong* (*Geos* (*Ang* (*An h11 k1 o1*)) *add Emp*) (*Geos* (*Ang* (*An h21 k21 o2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P114* **have** *P123* : *Def* (*Tri* (*Tr k1 h11 o1*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P120* **have** *P124* : *Def* (*Tri* (*Tr k21 h21 o2*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P9* **have** *P125* : *Eq* (*Geos* (*Seg* (*Se k1 o1*)) *add Emp*) (*Geos* (*Seg* (*Se k21 o2*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-trans Eq-rev*)
from *P106 P122 P123 P124 P125* **have** *Cong* (*Geos* (*Tri* (*Tr k1 h11 o1*)) *add Emp*) (*Geos* (*Tri* (*Tr k21 h21 o2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P126* : *Cong* (*Geos* (*Ang* (*An k1 o1 h11*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 h21*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
have *P127* : *Eq* (*Geos* (*Ang* (*An k1 o1 h11*)) *add Emp*) (*Geos* (*Ang* (*An h11 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P128* : *Eq* (*Geos* (*Ang* (*An k21 o2 h21*)) *add Emp*) (*Geos* (*Ang* (*An h21 o2 k21*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P126 P127* **have** *P129* : *Cong* (*Geos* (*Ang* (*An h11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 h21*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P128 P129* **have** *P130* : *Cong* (*Geos* (*Ang* (*An h11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h21 o2 k21*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P114* **have** *P131* : *Def* (*Ang* (*An h11 o1 k1*)) **by** (*simp add*:*Ang-def-inv*)
from *P2* **have** ¬ *Eq* (*Geos* (*Poi h1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P132* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)

from *P10 P12 P132* **have** *P133* : *Line-on* (*Li o1 h11*) *h1* **by** (*blast intro*:*Line-on-rev*)
from *P27* **have** *P134* : ¬ *Bet-Point* (*Se h11 h1*) *o1* **by** (*blast intro*:*Bet-rev*)
have *P135* : *Line-on* (*Li o1 k1*) *k1* **by** (*simp add*:*Line-on-rule*)
have *P136* : ¬ *Bet-Point* (*Se k1 k1*) *o1* **by** (*simp add*:*Bet-end-Point*)
from *P4* **have** ¬ *Eq* (*Geos* (*Poi k1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P137* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P131 P132 P133 P134 P135 P136 P137* **have** *P138* : *Eq* (*Geos* (*Ang* (*An h11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) ∧ *Def* (*Ang* (*An h1 o1 k1*)) **by** (*simp add*:*Ang-Point-swap*)
from *P120* **have** *P139* : *Def* (*Ang* (*An h21 o2 k21*)) **by** (*simp add*:*Ang-def-inv*)
from *P15 P16* **have** *P140* : *Line-on* (*Li o2 h21*) *h2* **by** (*simp add*:*Line-on-rev*)
from *P16* **have** *P141* : ¬ *Bet-Point* (*Se h21 h2*) *o2* **by** (*blast intro*:*Bet-rev*)
from *P8* **have** ¬ *Eq* (*Geos* (*Poi k2*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P142* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9 P64 P142* **have** *P143* : *Line-on* (*Li o2 k21*) *k2* **by** (*simp add*:*Line-on-rev*)
from *P9 P15 P139 P140 P141 P142 P143* **have** *P143* : *Eq* (*Geos* (*Ang* (*An h21 o2 k21*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) ∧ *Def* (*Ang* (*An h2 o2 k2*)) **by** (*simp add*:*Ang-Point-swap*)
from *P130 P138* **have** *P145* : *Cong* (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h21 o2 k21*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P143 P145* **show** *Cong* (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P93 P100* **have** *P146* : *Bet-Point* (*Se k21 l21*) *h21* **by** (*blast intro*:*Bet-rev*)
from *P9* **have** *P147* : ¬ *Eq* (*Geos* (*Lin* (*Li o2 k21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P30 P64 P146 P147* **have** *P148* : *Ang-inside* (*An k21 o2 l21*) *h21* **by** (*simp add*:*Ang-inside-Bet-Point*)
from *P8* **have** *P149* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P9 P142 P148 P149* **have** *P150* : *Ang-inside* (*An k2 o2 l2*) *h21* **by** (*simp add*:*Ang-inside-trans*)
from *P15 P16* **have** *P151* : *Line-on* (*Li o2 h21*) *h2* **by** (*simp add*:*Line-on-rev*)
from *P16* **have** *P152* : ¬ *Bet-Point* (*Se h2 h21*) *o2* **by** (*blast intro*:*Bet-rev*)
from *P15 P150 P151 P152* **show** *Ang-inside* (*An k2 o2 l2*) *h2* **by** (*simp add*:*Ang-inside-HalfLine*)
qed

Theorem15

theorem (**in** *Congruence-Rule*) *Ang-sub* :
assumes
Plane-sameside (*Li o1 l1*) *h1 k1*
¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)
Plane-sameside (*Li o2 l2*) *h2 k2*
¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add*

Emp)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)
¬ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li o1 l1*) *h1* **by** (*simp add*:*Plane-sameside-def*)
from *assms P1* **have** *Def* (*Ang* (*An o1 l1 h1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P2* : *Def* (*Ang* (*An h1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P3* : ¬ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Plane-sameside-def*)
from *assms P3* **have** *Def* (*Ang* (*An o1 l1 k1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P4* : *Def* (*Ang* (*An k1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P2 P4* **have** *P5* : *Ang-inside* (*An h1 o1 l1*) *k1* ∧ ¬ *Ang-inside* (*An k1 o1 l1*) *h1*
∨ ¬ *Ang-inside* (*An h1 o1 l1*) *k1* ∧ *Ang-inside* (*An k1 o1 l1*) *h1* **by** (*simp add*:*Ang-inside-case*)
from *assms* **have** *P6* : *Ang-inside* (*An k1 o1 l1*) *h1* ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) **by** (*simp add*:*Ang-sub-lemma1*)
from *assms* **have** *P7* : *Plane-sameside* (*Li o1 l1*) *k1 h1* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P8* : *Plane-sameside* (*Li o2 l2*) *k2 h2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P9* : ¬ *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P10* : ¬ *Eq* (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P7 P8 P9 P10* **have** *P11* : *Ang-inside* (*An h1 o1 l1*) *k1* ⟹
Cong (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-sub-lemma1*)
have *P12* : *Eq* (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P13* : *Eq* (*Geos* (*Ang* (*An k2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P11 P12* **have** *P14* : *Ang-inside* (*An h1 o1 l1*) *k1* ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P13 P14* **have** *P15* : *Ang-inside* (*An h1 o1 l1*) *k1* ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P5 P6 P15* **show** *Cong* (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) **by** *blast*
qed

theorem (**in** *Congruence-Rule*) *Ang-add* :
assumes
Plane-diffside (*Li o1 l1*) *h1 k1*
¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*)
Plane-diffside (*Li o2 l2*) *h2 k2*

$\neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*)
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*)
$\neg$ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)
$\neg$ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
proof −
from *assms* **have** $\exists$ *p*. *Bet-Point* (*Se h1 k1*) *p* $\wedge$ *Line-on* (*Li o1 l1*) *p*
$\wedge$ $\neg$ *Line-on* (*Li o1 l1*) *h1* $\wedge$ $\neg$ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Plane-diffside-def*)
then obtain *l11* :: *Point* **where** *P1* : *Bet-Point* (*Se h1 k1*) *l11* $\wedge$ *Line-on* (*Li o1 l1*) *l11*
$\wedge$ $\neg$ *Line-on* (*Li o1 l1*) *h1* $\wedge$ $\neg$ *Line-on* (*Li o1 l1*) *k1* **by** *blast*
have *P2* : *Line-on* (*Li o1 l1*) *o1* **by** (*simp add*:*Line-on-rule*)
then have *P3* : *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Point-Eq*)
from *P1 P3* **have** *P4* : $\neg$ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) **by** *blast*
have *P5* : *Line-on* (*Li o1 k1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *P6* : *Line-on* (*Li o1 k1*) *k1* **by** (*simp add*:*Line-on-rule*)
from *P4 P5 P6* **have** $\exists$ *p*. *Eq* (*Geos* (*Seg* (*Se o1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se o1 p*)) *add Emp*)
$\wedge$ *Bet-Point* (*Se p k1*) *o1* $\wedge$ *Line-on* (*Li o1 k1*) *p* $\wedge$ $\neg$ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-diffside*)
then obtain *k11* :: *Point* **where** *P7* : *Eq* (*Geos* (*Seg* (*Se o1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se o1 k11*)) *add Emp*)
$\wedge$ *Bet-Point* (*Se k11 k1*) *o1* $\wedge$ *Line-on* (*Li o1 k1*) *k11* $\wedge$ $\neg$ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k11*) *add Emp*) **by** *blast*
from *P7* **have** *P8* : *Bet-Point* (*Se k11 k1*) *o1* **by** *blast*
have *Line-on* (*Li k11 k1*) *k1* **by** (*simp add*:*Line-on-rule*)
then have *P9* : *Eq* (*Geos* (*Lin* (*Li k11 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o1 l1*) *k1* **by** (*simp add*:*Line-on-trans*)
from *P1 P9* **have** *P10* : $\neg$ *Eq* (*Geos* (*Lin* (*Li k11 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) **by** *blast*
from *P2 P8 P10* **have** *Plane-diffside* (*Li o1 l1*) *k11 k1* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P11* : *Plane-diffside* (*Li o1 l1*) *k1 k11* **by** (*simp add*:*Plane-diffside-rev*)
from *assms* **have** *P12* : *Plane-diffside* (*Li o1 l1*) *k1 h1* **by** (*simp add*:*Plane-diffside-rev*)
from *P2* **have** *P13* : *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o1 l1*) *h1* **by** (*simp add*:*Point-Eq*)
from *P1 P13* **have** *P14* : $\neg$ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) **by** *blast*
have *P15* : *Line-on* (*Li o1 h1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *P16* : *Line-on* (*Li o1 h1*) *h1* **by** (*simp add*:*Line-on-rule*)
from *P5 P14 P15 P16* **have** *P17* : *Line-on* (*Li o1 k1*) *h1*
$\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P17* **have** *P18* : $\neg$ *Line-on* (*Li o1 k1*) *h1* **by** *blast*

from *P7* **have** *P19* : *Eq* (*Geos* (*Poi k11*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o1 k1*) *h1* **by** (*blast intro*:*Point-Eq*)
from *P18 P19* **have** *P20* : $\neg$ *Eq* (*Geos* (*Poi k11*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) **by** *blast*
from *P11 P12 P20* **have** *Plane-sameside* (*Li o1 l1*) *k11 h1* **by** (*blast intro*:*Plane-trans-inv*)
then have *P21* : *Plane-sameside* (*Li o1 l1*) *h1 k11* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** $\exists$ *p*. *Bet-Point* (*Se h2 k2*) *p* $\wedge$ *Line-on* (*Li o2 l2*) *p* $\wedge$ $\neg$ *Line-on* (*Li o2 l2*) *h2* $\wedge$ $\neg$ *Line-on* (*Li o2 l2*) *k2* **by** (*simp add*:*Plane-diffside-def*)
then obtain *l21* :: *Point* **where** *P22* : *Bet-Point* (*Se h2 k2*) *l21* $\wedge$ *Line-on* (*Li o2 l2*) *l21* $\wedge$ $\neg$ *Line-on* (*Li o2 l2*) *h2* $\wedge$ $\neg$ *Line-on* (*Li o2 l2*) *k2* **by** *blast*
have *P23* : *Line-on* (*Li o2 l2*) *o2* **by** (*simp add*:*Line-on-rule*)
then have *P24* : *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k2*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o2 l2*) *k2* **by** (*simp add*:*Point-Eq*)
from *P22 P24* **have** *P25* : $\neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k2*) *add Emp*) **by** *blast*
have *P26* : *Line-on* (*Li o2 k2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P27* : *Line-on* (*Li o2 k2*) *k2* **by** (*simp add*:*Line-on-rule*)
from *P4 P25 P26 P27* **have** $\exists$ *p*. *Eq* (*Geos* (*Seg* (*Se o1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se o2 p*)) *add Emp*) $\wedge$ *Bet-Point* (*Se p k2*) *o2* $\wedge$ *Line-on* (*Li o2 k2*) *p* $\wedge$ $\neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-diffside*)
then obtain *k21* :: *Point* **where** *P28* : *Eq* (*Geos* (*Seg* (*Se o1 k1*)) *add Emp*) (*Geos* (*Seg* (*Se o2 k21*)) *add Emp*) $\wedge$ *Bet-Point* (*Se k21 k2*) *o2* $\wedge$ *Line-on* (*Li o2 k2*) *k21* $\wedge$ $\neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k21*) *add Emp*) **by** *blast*
from *P28* **have** *P29* : *Bet-Point* (*Se k21 k2*) *o2* **by** *blast*
have *Line-on* (*Li k21 k2*) *k2* **by** (*simp add*:*Line-on-rule*)
then have *P30* : *Eq* (*Geos* (*Lin* (*Li k21 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o2 l2*) *k2* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** $\exists$ *p*. *Bet-Point* (*Se h2 k2*) *p* $\wedge$ *Line-on* (*Li o2 l2*) *p* $\wedge$ $\neg$ *Line-on* (*Li o2 l2*) *h2* $\wedge$ $\neg$ *Line-on* (*Li o2 l2*) *k2* **by** (*simp add*:*Plane-diffside-def*)
from *P22 P30* **have** *P31* : $\neg$ *Eq* (*Geos* (*Lin* (*Li k21 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) **by** *blast*
from *P23 P29 P31* **have** *Plane-diffside* (*Li o2 l2*) *k21 k2* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P32* : *Plane-diffside* (*Li o2 l2*) *k2 k21* **by** (*simp add*:*Plane-diffside-rev*)
from *assms* **have** *P33* : *Plane-diffside* (*Li o2 l2*) *k2 h2* **by** (*simp add*:*Plane-diffside-rev*)
from *P23* **have** *P34* : *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li o2 l2*) *h2* **by** (*simp add*:*Point-Eq*)
from *P22 P34* **have** *P35* : $\neg$ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*) **by** *blast*
have *P36* : *Line-on* (*Li o2 h2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P37* : *Line-on* (*Li o2 h2*) *h2* **by** (*simp add*:*Line-on-rule*)
from *P26 P35 P36 P37* **have** *P38* : *Line-on* (*Li o2 k2*) *h2* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms P38* **have** *P39* : $\neg$ *Line-on* (*Li o2 k2*) *h2* **by** *blast*
from *P28* **have** *P40* : *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*)

$\Longrightarrow$ *Line-on* (*Li o2 k2*) *h2* **by** (*blast intro*:*Point-Eq*)
from *P39 P40* **have** *P41* : ¬ *Eq* (*Geos* (*Poi k21*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*) **by** *blast*
from *P32 P33 P41* **have** *Plane-sameside* (*Li o2 l2*) *k21 h2* **by** (*blast intro*:*Plane-trans-inv*)
then have *P42* : *Plane-sameside* (*Li o2 l2*) *h2 k21* **by** (*simp add*:*Plane-sameside-rev*)
from *assms P1* **have** *Def* (*Ang* (*An o1 l1 k1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P43* : *Def* (*Ang* (*An k1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P22* **have** *Def* (*Ang* (*An o2 l2 k2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P44* : *Def* (*Ang* (*An k2 o2 l2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P8* **have** *P45* : *Bet-Point* (*Se k1 k11*) *o1* **by** (*simp add*:*Bet-rev*)
from *P28* **have** *P46* : *Bet-Point* (*Se k2 k21*) *o2* **by** (*simp add*:*Bet-rev*)
from *assms P43 P44 P45 P46* **have** *P47* : *Cong* (*Geos* (*Ang* (*An l1 o1 k11*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k21*)) *add Emp*) **by** (*simp add*:*Ang-complementary*)
have *P48* : *Eq* (*Geos* (*Ang* (*An l1 o1 k11*)) *add Emp*) (*Geos* (*Ang* (*An k11 o1 l1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P49* : *Eq* (*Geos* (*Ang* (*An l2 o2 k21*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 l2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P47 P48* **have** *P50* : *Cong* (*Geos* (*Ang* (*An k11 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k21*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P49 P50* **have** *P51* : *Cong* (*Geos* (*Ang* (*An k11 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P52* : *Line-on* (*Li o1 k11*) *k11* **by** (*simp add*:*Line-on-rule*)
have *P53* : *Line-on* (*Li o1 k11*) *o1* **by** (*simp add*:*Line-on-rule*)
from *P5 P7 P52 P53* **have** *P54* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k11*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *P54* **have** *P55* : *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k11*)) *add Emp*)
$\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *assms P55* **have** *P56* : ¬ *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k11*)) *add Emp*) **by** *blast*
have *P57* : *Line-on* (*Li o2 k21*) *k21* **by** (*simp add*:*Line-on-rule*)
have *P58* : *Line-on* (*Li o2 k21*) *o2* **by** (*simp add*:*Line-on-rule*)
from *P26 P28 P57 P58* **have** *P59* : *Eq* (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k21*)) *add Emp*) **by** (*blast intro*:*Line-unique*)
from *P59* **have** *P60* : *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k21*)) *add Emp*)
$\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *assms P60* **have** *P61* : ¬ *Eq* (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k21*)) *add Emp*) **by** *blast*
from *assms P21 P42 P51 P56 P61* **have** *P62* : *Cong* (*Geos* (*Ang* (*An h1 o1 k11*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k21*)) *add Emp*) **by** (*simp add*:*Ang-sub*)
have *P63* : *Eq* (*Geos* (*Ang* (*An h1 o1 k11*)) *add Emp*) (*Geos* (*Ang* (*An k11 o1 h1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P64* : *Eq* (*Geos* (*Ang* (*An h2 o2 k21*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P62 P63* **have** *P65* : *Cong* (*Geos* (*Ang* (*An k11 o1 h1*)) *add Emp*) (*Geos*

(*Ang* (*An h2 o2 k21*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P64 P65* **have** *P66* : *Cong* (*Geos* (*Ang* (*An k11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An k21 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P54* **have** *P67* : *Line-on* (*Li o1 k11*) *h1* ⟹ *Line-on* (*Li o1 k1*) *h1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P18 P67* **have** *P68* : ¬ *Line-on* (*Li o1 k11*) *h1* **by** *blast*
from *P7 P68* **have** *Def* (*Ang* (*An o1 k11 h1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P69* : *Def* (*Ang* (*An k11 o1 h1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P59* **have** *P70* : *Line-on* (*Li o2 k21*) *h2* ⟹ *Line-on* (*Li o2 k2*) *h2* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P39 P70* **have** *P71* : ¬ *Line-on* (*Li o2 k21*) *h2* **by** *blast*
from *P28 P71* **have** *Def* (*Ang* (*An o2 k21 h2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P72* : *Def* (*Ang* (*An k21 o2 h2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P8 P29 P66 P69 P72* **show** *Cong* (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) **by** (*simp add*:*Ang-complementary*)
qed

lemma (**in** *Congruence-Rule*) *Ang-split-lemma1* :
assumes *N* :
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*)
Plane-sameside (*Li o1 k1*) *h1 l1*
Plane-sameside (*Li o2 k2*) *h2 l2*
¬ *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*)
shows
¬ *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*)
proof
assume *W* : *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*)
have *P1* : *Line-on* (*Li o2 k2*) *o2* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P2* : ¬ *Line-on* (*Li o2 k2*) *h2* ∧ ¬ *Line-on* (*Li o2 k2*) *l2*
∧ ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** (*simp add*:*Plane-sameside-def*)
from *P1 P2* **have** *Bet-Point* (*Se h2 l2*) *o2* ⟹ (∃ *p*. *Bet-Point* (*Se h2 l2*) *p*
∧ *Line-on* (*Li o2 k2*) *p* ∧ ¬ *Line-on* (*Li o2 k2*) *h2* ∧ ¬ *Line-on* (*Li o2 k2*) *l2*) **by** *blast*
then have *Bet-Point* (*Se h2 l2*) *o2* ⟹ *Plane-diffside* (*Li o2 k2*) *h2 l2* **by** (*simp add*:*Plane-diffside-def*)
then have *P3* : *Bet-Point* (*Se h2 l2*) *o2* ⟹ ¬ *Plane-sameside* (*Li o2 k2*) *h2 l2* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *N P3* **have** *P4* : ¬ *Bet-Point* (*Se h2 l2*) *o2* **by** *blast*
have *P5* : *Line-on* (*Li o2 l2*) *l2* **by** (*simp add*:*Line-on-rule*)
from *W P5* **have** *P6* : *Line-on* (*Li o2 h2*) *l2* **by** (*simp add*:*Line-on-trans*)
have *P7* : *Line-on* (*Li o2 k2*) *k2* **by** (*simp add*:*Line-on-rule*)
have *P8* : ¬ *Bet-Point* (*Se k2 k2*) *o2* **by** (*simp add*:*Bet-end-Point*)
from *P1* **have** *P9* : *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) ⟹ *Line-on* (*Li o2 k2*) *l2* **by** (*simp add*:*Point-Eq*)

from *P2 P9* **have** *P10* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** *blast*
from *N* **have** *Def* (*Tri* (*Tr h2 o2 k2*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P11* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k2*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *N P4 P6 P7 P8 P10 P11* **have** *P12* :
Eq (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) ∧ *Def* (*Ang* (*An l2 o2 k2*)) **by** (*simp add*:*Ang-Point-swap*)
from *N P12* **have** *P13* : *Cong* (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *N* **have** *P14* : *Cong* (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** (*blast intro*:*Ang-rev*)
from *N P13 P14* **have** *P15* : *Eq* (*Geos* (*Lin* (*Li h1 o1*)) *add Emp*) (*Geos* (*Lin* (*Li l1 o1*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se h1 l1*) *o1* **by** (*simp add*:*Ang-move-unique*)
from *N* **have** *Def* (*Tri* (*Tr h1 o1 k1*)) **by** (*simp add*:*Ang-to-Tri*)
then have ¬ *Eq* (*Geos* (*Poi h1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P16* : *Eq* (*Geos* (*Lin* (*Li h1 o1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *N* **have** *P17* : ¬ *Line-on* (*Li o1 k1*) *h1* ∧ ¬ *Line-on* (*Li o1 k1*) *l1*
∧ ¬ *Eq* (*Geos* (*Poi h1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*) **by** (*simp add*:*Plane-sameside-def*)
have *P18* : *Line-on* (*Li o1 k1*) *o1* **by** (*simp add*:*Line-on-rule*)
then have *P19* : *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*) ⟹ *Line-on* (*Li o1 k1*) *l1* **by** (*simp add*:*Point-Eq*)
from *P17 P19* **have** ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*) **by** *blast*
then have *P20* : *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li l1 o1*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P15 P16* **have** *P21* : *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li l1 o1*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P20 P21* **have** *P22* : *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *N P22* **show** *False* **by** *blast*
qed

Theorem16

theorem (**in** *Congruence-Rule*) *Ang-split* :
assumes
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An h1 o1 k1*) *l1*
shows
∃ *p*. *Ang-inside* (*An h2 o2 k2*) *p*
∧ *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 p*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 p*)) *add Emp*)
proof −

from *assms* **have** *P1* : *Plane-sameside* (*Li o1 h1*) *k1 l1* ∧ *Plane-sameside* (*Li o1 k1*) *h1 l1* **by** (*simp add*:*Ang-inside-def*)
from *assms* **have** *P2* : ¬ *Line-on* (*Li o2 k2*) *h2* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
from *P1* **have** *P3* : ¬ *Line-on* (*Li o1 k1*) *l1* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P4* : *Def* (*Tri* (*Tr h1 o1 k1*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P5* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P3 P5* **have** *Def* (*Ang* (*An o1 k1 l1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P6* : *Def* (*Ang* (*An k1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P2 P6* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An p o2 k2*)) *add Emp*)
∧ *Plane-sameside* (*Li o2 k2*) *p h2* **by** (*simp add*:*Ang-move-sameside*)
then obtain *l2* :: *Point* **where** *P7* : *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*)
∧ *Plane-sameside* (*Li o2 k2*) *l2 h2* **by** *blast*
from *assms* **have** *Def* (*Tri* (*Tr h2 o2 k2*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P8* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k2*) *add Emp*) **by** (*simp add*:*Tri-def*)
have *P9* : *Eq* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P7 P9* **have** *P10* : *Cong* (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P1* **have** *P11* : ¬ *Line-on-Seg* (*Li o1 h1*) (*Se k1 l1*) ∧ ¬ *Line-on* (*Li o1 h1*) *k1*
∧ ¬ *Line-on* (*Li o1 h1*) *l1* ∧ ¬ *Eq* (*Geos* (*Poi k1*) *add Emp*) (*Geos* (*Poi l1*) *add Emp*) **by** (*simp add*:*Plane-sameside-def*)
have *Line-on* (*Li o1 l1*) *l1* **by** (*simp add*:*Line-on-rule*)
then have *P12* : *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) ⟹ *Line-on* (*Li o1 h1*) *l1* **by** (*simp add*:*Line-on-trans*)
from *P11 P12* **have** *P13* : ¬ *Eq* (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) **by** *blast*
from *P7* **have** *P14* : *Plane-sameside* (*Li o2 k2*) *h2 l2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms P1 P10 P13 P14* **have** *P15* : ¬ *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-split-lemma1*)
from *P1* **have** *P16* : *Plane-sameside* (*Li o1 k1*) *l1 h1* **by** (*simp add*:*Plane-sameside-rev*)
from *P7* **have** *P17* : *Plane-sameside* (*Li o2 k2*) *l2 h2* **by** *simp*
from *assms P5 P8 P10 P13 P15 P16 P17* **have** *P18* :
Cong (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*)
∧ *Ang-inside* (*An h2 o2 k2*) *l2* **by** (*simp add*:*Ang-sub-lemma1*)
have *P19* : *Eq* (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P7 P19* **have** *P20* : *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P21* : *Eq* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P22* : *Eq* (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)

from *P18 P21* **have** *P23* : *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P22 P23* **have** *P24* : *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P18 P20 P24* **show** ∃ *p*. *Ang-inside* (*An h2 o2 k2*) *p*
∧ *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 p*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 p*)) *add Emp*) **by** *blast*
qed

theorem (**in** *Congruence-Rule*) *Ang-split-unique* :
assumes
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 k2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Ang-inside (*An h1 o1 k1*) *l1*
Ang-inside (*An h2 o2 k2*) *l21*
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l21*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l21*)) *add Emp*)
Ang-inside (*An h2 o2 k2*) *l22*
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l22*)) *add Emp*)
Cong (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l22*)) *add Emp*)
shows
Eq (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l22*)) *add Emp*)
proof −
from *assms* **have** *Plane-sameside* (*Li o2 h2*) *k2 l21* ∧ *Plane-sameside* (*Li o2 k2*) *h2 l21* **by** (*simp add*:*Ang-inside-def*)
then have *P1* : *Plane-sameside* (*Li o2 k2*) *l21 h2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P2* : *Plane-sameside* (*Li o2 h2*) *k2 l22* ∧ *Plane-sameside* (*Li o2 k2*) *h2 l22* **by** (*simp add*:*Ang-inside-def*)
from *P1 P2* **have** *P3* : ¬ *Eq* (*Geos* (*Poi l22*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹
Plane-sameside (*Li o2 k2*) *l21 l22* **by** (*simp add*:*Plane-sameside-trans*)
have *P4* : *Eq* (*Geos* (*Ang* (*An k2 o2 l21*)) *add Emp*) (*Geos* (*Ang* (*An l21 o2 k2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P4* **have** *P5* : *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l21 o2 k2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P6* : *Eq* (*Geos* (*Ang* (*An k2 o2 l22*)) *add Emp*) (*Geos* (*Ang* (*An l22 o2 k2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P6* **have** *P7* : *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l22 o2 k2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P3 P5 P7* **have** *P8* : ¬ *Eq* (*Geos* (*Poi l22*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) (*Geos* (*Lin* (*Li l22 o2*)) *add Emp*) **by**

(*simp add*:*Ang-move-unique*)
have *P9* : *Line-on* (*Li o2 k2*) *o2* **by** (*simp add*:*Line-on-rule*)
from *P1* **have** *P10* : ¬ *Line-on* (*Li o2 k2*) *l21* **by** (*simp add*:*Plane-sameside-def*)
from *P9* **have** *P11* : *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹ *Line-on* (*Li o2 k2*) *l21* **by** (*simp add*:*Point-Eq*)
from *P10 P11* **have** *P12* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) **by** *blast*
then have *P13* : *Eq* (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P2* **have** *P14* : ¬ *Line-on* (*Li o2 k2*) *l22* **by** (*simp add*:*Plane-sameside-def*)
from *P9* **have** *P15* : *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l22*) *add Emp*) ⟹ *Line-on* (*Li o2 k2*) *l22* **by** (*simp add*:*Point-Eq*)
from *P14 P15* **have** ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l22*) *add Emp*) **by** *blast*
then have *P16* : *Eq* (*Geos* (*Lin* (*Li o2 l22*)) *add Emp*) (*Geos* (*Lin* (*Li l22 o2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P8 P13* **have** *P17* : ¬ *Eq* (*Geos* (*Poi l22*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li l22 o2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P16 P17* **have** *P18* : ¬ *Eq* (*Geos* (*Poi l22*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l22*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
have *P19* : *Line-on* (*Li o2 l21*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P20* : *Line-on* (*Li o2 l21*) *l21* **by** (*simp add*:*Line-on-rule*)
have *P21* : *Line-on* (*Li o2 l22*) *o2* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li o2 l22*) *l22* **by** (*simp add*:*Line-on-rule*)
then have *P22* : *Eq* (*Geos* (*Poi l22*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹ *Line-on* (*Li o2 l22*) *l21* **by** (*simp add*:*Point-Eq*)
from *P12 P19 P20 P21 P22* **have** *P23* : *Eq* (*Geos* (*Poi l22*) *add Emp*) (*Geos* (*Poi l21*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l22*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P18 P23* **show** *Eq* (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l22*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Tri-week-SSS-lemma1* :
assumes
Plane-diffside (*Li x y*) *z1 z2*
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi y*) *add Emp*)
Eq (*Geos* (*Seg* (*Se x z1*)) *add Emp*) (*Geos* (*Seg* (*Se x z2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se y z1*)) *add Emp*) (*Geos* (*Seg* (*Se y z2*)) *add Emp*)
∃ *p*. *Bet-Point* (*Se z1 z2*) *p* ∧ *Line-on* (*Li x y*) *p* ∧ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi p*) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An x z1 y*)) *add Emp*) (*Geos* (*Ang* (*An x z2 y*)) *add Emp*)
proof −

from *assms* **have** *P1* : $\exists$ *p*. *Bet-Point* (*Se z1 z2*) *p* $\wedge$ *Line-on* (*Li x y*) *p* $\wedge$ $\neg$ *Line-on* (*Li x y*) *z1* $\wedge$ $\neg$ *Line-on* (*Li x y*) *z2* **by** (*simp add*:*Plane-diffside-def*)
from *assms* **obtain** *pn* :: *Point* **where** *P2* : *Bet-Point* (*Se z1 z2*) *pn* $\wedge$ *Line-on* (*Li x y*) *pn* $\wedge$ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) **by** *blast*
from *P2* **have** *P3* : *Bet-Point* (*Se z1 z2*) *pn* **by** *simp*
then have *P4* : *Line-on* (*Li z1 z2*) *pn* **by** (*simp add*:*Line-Bet-on*)
from *P2 P4* **have** *P5* : *Line-on* (*Li z1 z2*) *x* **by** (*blast intro*:*Eq-rev Point-Eq*)
from *assms P3* **have** *P6* : *Bet-Point* (*Se z1 z2*) *x* **by** (*blast intro*:*Eq-rev Point-Eq*)
have *P7* : *Line-on* (*Li x y*) *x* **by** (*simp add*:*Line-on-rule*)
have *P8* : *Line-on* (*Li x y*) *y* **by** (*simp add*:*Line-on-rule*)
from *assms P5 P7 P8* **have** *P9* : *Line-on* (*Li z1 z2*) *y* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li z1 z2*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P10* : *Line-on* (*Li z1 z2*) *z1* **by** (*simp add*:*Line-on-rule*)
from *P9 P10* **have** *P11* : *Line-on* (*Li z1 z2*) *y* $\Longrightarrow$ *Line-on* (*Li x y*) *z1* **by** (*simp add*:*Line-on-trans*)
from *P1 P11* **have** *P12* : $\neg$ *Line-on* (*Li z1 z2*) *y* **by** *blast*
from *P3* **have** *P13* : $\neg$ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi z2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P8* **have** *P14* : *Eq* (*Geos* (*Poi z2*) *add Emp*) (*Geos* (*Poi y*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li x y*) *z2* **by** (*blast intro*:*Eq-rev Point-Eq*)
from *P1 P14* **have** *P15* : $\neg$ *Eq* (*Geos* (*Poi z2*) *add Emp*) (*Geos* (*Poi y*) *add Emp*) **by** *blast*
from *P8* **have** *P16* : *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li x y*) *z1* **by** (*blast intro*:*Eq-rev Point-Eq*)
from *P1 P16* **have** *P17* : $\neg$ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) **by** *blast*
from *P12 P13* **have** *P18* : *Def* (*Tri* (*Tr z1 z2 y*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
then have *P19* : *Def* (*Tri* (*Tr y z1 z2*)) **by** (*blast intro*:*Tri-def-trans*)
from *assms P19* **have** *P20* : *Cong* (*Geos* (*Ang* (*An y z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An y z2 z1*)) *add Emp*) **by** (*simp add*:*Tri-isosceles*)
from *P18* **have** *P21* : *Def* (*Ang* (*An z1 z2 y*)) **by** (*simp add*:*Tri-to-Ang*)
from *P6* **have** *P22* : *Line-on* (*Li z2 z1*) *x* **by** (*simp add*:*Line-Bet-on*)
from *P6* **have** *P23* : *Inv* (*Bet-Point* (*Se z2 x*) *z1*) $\wedge$ *Inv* (*Bet-Point* (*Se x z1*) *z2*) **by** (*simp add*:*Bet-iff*)
then have $\neg$ *Bet-Point* (*Se x z1*) *z2* **by** (*simp add*:*Inv-def*)
then have *P24* : $\neg$ *Bet-Point* (*Se z1 x*) *z2* **by** (*blast intro*:*Bet-rev*)
have *P25* : *Line-on* (*Li z2 y*) *y* **by** (*simp add*:*Line-on-rule*)
have *P26* : $\neg$ *Bet-Point* (*Se y y*) *z2* **by** (*simp add*:*Bet-end-Point*)
from *P6* **have** *P27* : $\neg$ *Eq* (*Geos* (*Poi z2*) *add Emp*) (*Geos* (*Poi x*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P15 P21 P22 P24 P25 P26 P27* **have** *P28* :
Eq (*Geos* (*Ang* (*An z1 z2 y*)) *add Emp*) (*Geos* (*Ang* (*An x z2 y*)) *add Emp*) $\wedge$ *Def* (*Ang* (*An x z2 y*)) **by** (*simp add*:*Ang-Point-swap*)
have *P29* : *Eq* (*Geos* (*Ang* (*An y z2 z1*)) *add Emp*) (*Geos* (*Ang* (*An z1 z2 y*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P20 P29* **have** *P30* : *Cong* (*Geos* (*Ang* (*An y z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An z1 z2 y*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P28 P30* **have** *P31* : *Cong* (*Geos* (*Ang* (*An y z1 z2*)) *add Emp*) (*Geos* (*Ang*

(*An x z2 y*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P19* **have** *P32* : *Def* (*Ang* (*An z2 z1 y*)) **by** (*blast intro*:*Tri-to-Ang Tri-def-rev*)
from *P6* **have** *P33* : *Line-on* (*Li z1 z2*) *x* **by** (*simp add*:*Line-Bet-on*)
from *P23* **have** *P34* : ¬ *Bet-Point* (*Se z2 x*) *z1* **by** (*simp add*:*Inv-def*)
have *P35* : *Line-on* (*Li z1 y*) *y* **by** (*simp add*:*Line-on-rule*)
have *P36* : ¬ *Bet-Point* (*Se y y*) *z1* **by** (*simp add*:*Bet-end-Point*)
from *P6* **have** ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P37* : ¬ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi x*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P17* **have** *P38* : ¬ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi y*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P32 P33 P34 P35 P36 P37 P38* **have** *P39* :
Eq (*Geos* (*Ang* (*An z2 z1 y*)) *add Emp*) (*Geos* (*Ang* (*An x z1 y*)) *add Emp*) ∧ *Def* (*Ang* (*An x z1 y*)) **by** (*simp add*:*Ang-Point-swap*)
have *P40* : *Eq* (*Geos* (*Ang* (*An y z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An z2 z1 y*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P31 P40* **have** *P41* : *Cong* (*Geos* (*Ang* (*An z2 z1 y*)) *add Emp*) (*Geos* (*Ang* (*An x z2 y*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P39 P41* **show** *Cong* (*Geos* (*Ang* (*An x z1 y*)) *add Emp*) (*Geos* (*Ang* (*An x z2 y*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
qed

Theorem17

theorem (**in** *Congruence-Rule*) *Tri-week-SSS* :
assumes
Plane-diffside (*Li x y*) *z1 z2*
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi y*) *add Emp*)
Eq (*Geos* (*Seg* (*Se x z1*)) *add Emp*) (*Geos* (*Seg* (*Se x z2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se y z1*)) *add Emp*) (*Geos* (*Seg* (*Se y z2*)) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*)
proof −
from *assms* **have** ∃ *p*. *Bet-Point* (*Se z1 z2*) *p* ∧ *Line-on* (*Li x y*) *p*
∧ ¬ *Line-on* (*Li x y*) *z1* ∧ ¬ *Line-on* (*Li x y*) *z2* **by** (*simp add*:*Plane-diffside-def*)
then obtain *pn* :: *Point* **where** *P1* : *Bet-Point* (*Se z1 z2*) *pn* ∧ *Line-on* (*Li x y*) *pn*
∧ ¬ *Line-on* (*Li x y*) *z1* ∧ ¬ *Line-on* (*Li x y*) *z2* **by** *blast*
have *P2* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Eq (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi y*) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *assms P2* **have** ¬ (*Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*)) **by** *blast*
then have *P3* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*)
∨ ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*)
∨ ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi

y) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) **by** *blast*
from *P1* **have** *P4* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
∃ *p*. *Bet-Point* (*Se z1 z2*) *p* ∧ *Line-on* (*Li x y*) *p* ∧ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** *blast*
from *assms P4* **have** *P5* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x z1 y*)) *add Emp*) (*Geos* (*Ang* (*An x z2 y*)) *add Emp*) **by** (*simp add*:*Tri-week-SSS-lemma1*)
have *P6* : *Line-on* (*Li x y*) *x* **by** (*simp add*:*Line-on-rule*)
then have *P7* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) ⟹ *Line-on* (*Li x y*) *z1* **by** (*simp add*:*Point-Eq*)
from *P1 P7* **have** *P8* : ¬ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi x*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P9* : *Line-on* (*Li x y*) *y* **by** (*simp add*:*Line-on-rule*)
then have *P10* : *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) ⟹ *Line-on* (*Li x y*) *z1* **by** (*simp add*:*Point-Eq*)
from *P1 P10* **have** *P11* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) **by** *blast*
from *assms P1* **have** *Def* (*Tri* (*Tr x y z1*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
then have *P12* : *Def* (*Tri* (*Tr z1 x y*)) **by** (*simp add*:*Tri-def-trans*)
from *P6* **have** *P13* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi z2*) *add Emp*) ⟹ *Line-on* (*Li x y*) *z2* **by** (*simp add*:*Point-Eq*)
from *P1 P13* **have** *P14* : ¬ *Eq* (*Geos* (*Poi z2*) *add Emp*) (*Geos* (*Poi x*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9* **have** *P15* : *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi z2*) *add Emp*) ⟹ *Line-on* (*Li x y*) *z2* **by** (*simp add*:*Point-Eq*)
from *P1 P15* **have** *P16* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi z2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P1* **have** *Def* (*Tri* (*Tr x y z2*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
then have *P17* : *Def* (*Tri* (*Tr z2 x y*)) **by** (*simp add*:*Tri-def-trans*)
have *P18* : *Eq* (*Geos* (*Seg* (*Se x z1*)) *add Emp*) (*Geos* (*Seg* (*Se z1 x*)) *add Emp*) **by** (*blast intro*:*Seg-rev*)
have *P19* : *Eq* (*Geos* (*Seg* (*Se x z2*)) *add Emp*) (*Geos* (*Seg* (*Se z2 x*)) *add Emp*) **by** (*blast intro*:*Seg-rev*)
from *assms P18* **have** *P20* : *Eq* (*Geos* (*Seg* (*Se z1 x*)) *add Emp*) (*Geos* (*Seg* (*Se x z2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P19 P20* **have** *P21* : *Eq* (*Geos* (*Seg* (*Se z1 x*)) *add Emp*) (*Geos* (*Seg* (*Se z2 x*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
have *P22* : *Eq* (*Geos* (*Seg* (*Se y z1*)) *add Emp*) (*Geos* (*Seg* (*Se z1 y*)) *add Emp*) **by** (*blast intro*:*Seg-rev*)
have *P23* : *Eq* (*Geos* (*Seg* (*Se y z2*)) *add Emp*) (*Geos* (*Seg* (*Se z2 y*)) *add Emp*) **by** (*blast intro*:*Seg-rev*)
from *assms P22* **have** *P24* : *Eq* (*Geos* (*Seg* (*Se z1 y*)) *add Emp*) (*Geos* (*Seg* (*Se y z2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P23 P24* **have** *P25* : *Eq* (*Geos* (*Seg* (*Se z1 y*)) *add Emp*) (*Geos* (*Seg* (*Se z2 y*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P5 P12 P17 P21 P25* **have** *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Tri* (*Tr z1 x y*)) *add Emp*) (*Geos* (*Tri* (*Tr z2 x y*)) *add Emp*) **by**

(*simp add*:*Tri-SAS*)
then have *P26* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An z1 y x*)) *add Emp*) (*Geos* (*Ang* (*An z2 y x*)) *add Emp*)
by (*simp add*:*Tri-Cong-def*)
have *P27* : *Eq* (*Geos* (*Ang* (*An z1 y x*)) *add Emp*) (*Geos* (*Ang* (*An x y z1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P28* : *Eq* (*Geos* (*Ang* (*An z2 y x*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P26 P27* **have** *P29* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An z2 y x*)) *add Emp*)
by (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P28 P29* **have** *P30* : *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*)
by (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms* **have** *P31* : *Eq* (*Geos* (*Lin* (*Li x y*)) *add Emp*) (*Geos* (*Lin* (*Li y x*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *assms P31* **have** *P32* : *Plane-diffside* (*Li y x*) *z1 z2* **by** (*simp add*:*Plane-Line-diff-trans*)
from *assms* **have** *P33* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi x*) *add Emp*)
by (*blast intro*:*Eq-rev*)
from *P1 P31* **have** *P34* : *Line-on* (*Li y x*) *pn* **by** (*blast intro*:*Line-on-trans*)
from *P1 P34* **have** *P35* : *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
∃ *p*. *Bet-Point* (*Se z1 z2*) *p* ∧ *Line-on* (*Li y x*) *p* ∧ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** *blast*
from *assms P32 P33 P35* **have** *P36* : *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An y z1 x*)) *add Emp*) (*Geos* (*Ang* (*An y z2 x*)) *add Emp*)
by (*simp add*:*Tri-week-SSS-lemma1*)
from *P12* **have** *P37* : *Def* (*Tri* (*Tr z1 y x*)) **by** (*blast intro*:*Tri-def-trans Tri-def-rev*)
from *P17* **have** *P38* : *Def* (*Tri* (*Tr z2 y x*)) **by** (*blast intro*:*Tri-def-trans Tri-def-rev*)
from *P21 P25 P36 P37 P38* **have** *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Tri* (*Tr z1 y x*)) *add Emp*) (*Geos* (*Tri* (*Tr z2 y x*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P39* : *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*)
by (*simp add*:*Tri-Cong-def*)
from *P1* **have** *P40* : *Bet-Point* (*Se z1 z2*) *pn* **by** *simp*
then have *P41* : *Line-on* (*Li z1 z2*) *pn* **by** (*simp add*:*Line-Bet-on*)
have *Line-on* (*Li z1 z2*) *z1* **by** (*simp add*:*Line-on-rule*)
then have *P42* : *Eq* (*Geos* (*Lin* (*Li z1 z2*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) ⟹ *Line-on* (*Li x y*) *z1* **by** (*simp add*:*Line-on-trans*)
from *P1 P42* **have** *P43* : ¬ *Eq* (*Geos* (*Lin* (*Li z1 z2*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) **by** *blast*
from *P1 P6 P41* **have** *P44* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add*

Emp) ⟹ *Line-on* (*Li z1 z2*) *x* ⟹
Eq (*Geos* (*Lin* (*Li z1 z2*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P43 P44* **have** *P45* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
¬ *Line-on* (*Li z1 z2*) *x* **by** *blast*
from *P40* **have** *P46* : ¬ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi z2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P45 P46* **have** ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Tri* (*Tr z1 z2 x*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
then have *P47* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Tri* (*Tr x z1 z2*)) **by** (*simp add*:*Tri-def-trans*)
from *assms P47* **have** *P48* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An x z2 z1*)) *add Emp*) **by** (*simp add*:*Tri-isosceles*)
have *P49* : *Line-on* (*Li z1 x*) *x* **by** (*simp add*:*Line-on-rule*)
have *P50* : ¬ *Bet-Point* (*Se x x*) *z1* **by** (*simp add*:*Bet-end-Point*)
from *P40* **have** *P51* : *Inv* (*Bet-Point* (*Se z2 pn*) *z1*) ∧ *Inv* (*Bet-Point* (*Se pn z1*) *z2*) **by** (*simp add*:*Bet-iff*)
then have *P52* : ¬ *Bet-Point* (*Se z2 pn*) *z1* **by** (*simp add*:*Inv-def*)
from *P40* **have** ¬ *Eq* (*Geos* (*Poi pn*) *add Emp*) (*Geos* (*Poi z1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P53* : ¬ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P47* **have** *P54* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Ang* (*An x z1 z2*)) **by** (*simp add*:*Tri-to-Ang*)
from *P8 P41 P49 P50 P52 P53 P54* **have** *P55* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An x z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An x z1 pn*)) *add Emp*) ∧ *Def* (*Ang* (*An x z1 pn*)) **by** (*simp add*:*Ang-Point-swap*)
from *P48 P55* **have** *P56* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x z1 pn*)) *add Emp*) (*Geos* (*Ang* (*An x z2 z1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P47* **have** *P57* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Ang* (*An x z2 z1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans Tri-to-Ang*)
have *P58* : *Line-on* (*Li z2 x*) *x* **by** (*simp add*:*Line-on-rule*)
have *P59* : ¬ *Bet-Point* (*Se x x*) *z2* **by** (*simp add*:*Bet-end-Point*)
from *P40* **have** *P60* : *Line-on* (*Li z2 z1*) *pn* **by** (*simp add*:*Line-Bet-on*)
from *P51* **have** ¬ *Bet-Point* (*Se pn z1*) *z2* **by** (*simp add*:*Inv-def*)
then have *P61* : ¬ *Bet-Point* (*Se z1 pn*) *z2* **by** (*blast intro*:*Bet-rev*)
from *P40* **have** *P62* : ¬ *Eq* (*Geos* (*Poi z2*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P14 P57 P58 P59 P60 P61 P62* **have** *P63* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹

Eq (*Geos* (*Ang* (*An x z2 z1*)) *add Emp*) (*Geos* (*Ang* (*An x z2 pn*)) *add Emp*) ∧ *Def* (*Ang* (*An x z2 pn*)) **by** (*simp add*:*Ang-Point-swap*)
from *P56 P63* **have** *P64* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x z1 pn*)) *add Emp*) (*Geos* (*Ang* (*An x z2 pn*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P1 P9 P41* **have** *P65* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹ *Line-on* (*Li z1 z2*) *y* ⟹
Eq (*Geos* (*Lin* (*Li z1 z2*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P43 P65* **have** *P66* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
¬ *Line-on* (*Li z1 z2*) *y* **by** *blast*
from *P46 P66* **have** ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Tri* (*Tr z1 z2 y*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
then have *P67* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Tri* (*Tr y z1 z2*)) **by** (*simp add*:*Tri-def-trans*)
from *assms P67* **have** *P68* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An y z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An y z2 z1*)) *add Emp*) **by** (*simp add*:*Tri-isosceles*)
from *P67* **have** *P69* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Ang* (*An y z1 z2*)) **by** (*simp add*:*Tri-to-Ang*)
have *P70* : *Line-on* (*Li z1 y*) *y* **by** (*simp add*:*Line-on-rule*)
have *P71* : ¬ *Bet-Point* (*Se y y*) *z1* **by** (*simp add*:*Bet-end-Point*)
from *P11* **have** *P72* : ¬ *Eq* (*Geos* (*Poi z1*) *add Emp*) (*Geos* (*Poi y*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P41 P52 P53 P69 P70 P71 P72* **have** *P73* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An y z1 z2*)) *add Emp*) (*Geos* (*Ang* (*An y z1 pn*)) *add Emp*) ∧ *Def* (*Ang* (*An y z1 pn*)) **by** (*simp add*:*Ang-Point-swap*)
from *P68 P73* **have** *P74* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An y z1 pn*)) *add Emp*) (*Geos* (*Ang* (*An y z2 z1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P69* **have** *P75* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Def (*Ang* (*An y z2 z1*)) **by** (*simp add*:*Ang-def-inv*)
have *P76* : *Line-on* (*Li z2 y*) *y* **by** (*simp add*:*Line-on-rule*)
have *P77* : ¬ *Bet-Point* (*Se y y*) *z2* **by** (*simp add*:*Bet-end-Point*)
from *P16* **have** *P78* : ¬ *Eq* (*Geos* (*Poi z2*) *add Emp*) (*Geos* (*Poi y*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P60 P61 P62 P75 P76 P77 P78* **have** *P79* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An y z2 z1*)) *add Emp*) (*Geos* (*Ang* (*An y z2 pn*)) *add Emp*) ∧ *Def* (*Ang* (*An y z2 pn*)) **by** (*simp add*:*Ang-Point-swap*)
from *P74 P79* **have** *P80* : ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add*

Emp) $\Longrightarrow$
Cong (*Geos* (*Ang* (*An y z1 pn*)) *add Emp*) (*Geos* (*Ang* (*An y z2 pn*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P81* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) $\Longrightarrow$
¬ *Eq* (*Geos* (*Poi pn*) *add Emp*) (*Geos* (*Poi x*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P1 P6 P9 P81* **have**
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se x pn*) *y* ∨ *Bet-Point* (*Se pn y*) *x* ∨ *Bet-Point* (*Se y x*) *pn* **by** (*simp add*:*Bet-case*)
then have *P82* :
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se x pn*) *y* ∧ ¬ *Bet-Point* (*Se pn y*) *x* ∧ ¬ *Bet-Point* (*Se y x*) *pn*
∨ ¬ *Bet-Point* (*Se x pn*) *y* ∧ *Bet-Point* (*Se pn y*) *x* ∧ ¬ *Bet-Point* (*Se y x*) *pn*
∨ ¬ *Bet-Point* (*Se x pn*) *y* ∧ ¬ *Bet-Point* (*Se pn y*) *x* ∧ *Bet-Point* (*Se y x*) *pn* **by** (*simp add*:*Bet-case-fact*)
have *P83* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$ *Bet-Point* (*Se pn x*) *y* **by** (*simp add*:*Bet-rev*)
have *P84* : *Line-on* (*Li z1 pn*) *pn* **by** (*simp add*:*Line-on-rule*)
have *P85* : *Line-on* (*Li pn x*) *x* **by** (*simp add*:*Line-on-rule*)
from *P83* **have** *P86* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$ *Line-on* (*Li pn x*) *y* **by** (*simp add*:*Line-Bet-on*)
from *assms P6 P9 P85 P86* **have** *P87* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li pn x*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P88* : *Line-on* (*Li z1 pn*) *z1* **by** (*simp add*:*Line-on-rule*)
from *P87 P88* **have** *P89* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li pn x*)) *add Emp*) (*Geos* (*Lin* (*Li z1 pn*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li x y*) *z1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P89* **have** *P90* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$
¬ *Eq* (*Geos* (*Lin* (*Li pn x*)) *add Emp*) (*Geos* (*Lin* (*Li z1 pn*)) *add Emp*) **by** *blast*
from *P83 P84 P90* **have** *P91* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$ *Plane-sameside* (*Li z1 pn*) *y x* **by** (*simp add*:*Plane-Bet-sameside*)
have *P92* : *Line-on* (*Li z2 pn*) *pn* **by** (*simp add*:*Line-on-rule*)
have *P93* : *Line-on* (*Li z2 pn*) *z2* **by** (*simp add*:*Line-on-rule*)
from *P87 P93* **have** *P94* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li pn x*)) *add Emp*) (*Geos* (*Lin* (*Li z2 pn*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li x y*) *z2* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P94* **have** *P95* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$
¬ *Eq* (*Geos* (*Lin* (*Li pn x*)) *add Emp*) (*Geos* (*Lin* (*Li z2 pn*)) *add Emp*) **by** *blast*
from *P83 P92 P95* **have** *P96* : *Bet-Point* (*Se x pn*) *y* $\Longrightarrow$ *Plane-sameside* (*Li z2 pn*) *y x* **by** (*simp add*:*Plane-Bet-sameside*)
from *P37* **have** *Def* (*Ang* (*An y z1 x*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-rev Ang-def-inv*)
then have *P97* : ¬ *Eq* (*Geos* (*Lin* (*Li z1 y*)) *add Emp*) (*Geos* (*Lin* (*Li z1 x*)) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P38* **have** *Def* (*Ang* (*An y z2 x*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-rev*

Ang-def-inv)
then have *P98* : ¬ *Eq* (*Geos* (*Lin* (*Li z2 y*)) *add Emp*) (*Geos* (*Lin* (*Li z2 x*)) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P53 P62 P64 P80 P91 P96 P97 P98* **have** *P99* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹ *Bet-Point* (*Se x pn*) *y* ⟹
Cong (*Geos* (*Ang* (*An y z1 x*)) *add Emp*) (*Geos* (*Ang* (*An y z2 x*)) *add Emp*) **by** (*simp add*:*Ang-sub*)
have *P100* : *Line-on* (*Li pn y*) *y* **by** (*simp add*:*Line-on-rule*)
from *P85* **have** *P101* : *Bet-Point* (*Se pn y*) *x* ⟹ *Line-on* (*Li pn y*) *x* **by** (*simp add*:*Line-Bet-on*)
from *assms P6 P9 P100 P101* **have** *P102* : *Bet-Point* (*Se pn y*) *x* ⟹
Eq (*Geos* (*Lin* (*Li pn y*)) *add Emp*) (*Geos* (*Lin* (*Li x y*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P88 P102* **have** *P103* : *Bet-Point* (*Se pn y*) *x* ⟹
Eq (*Geos* (*Lin* (*Li pn y*)) *add Emp*) (*Geos* (*Lin* (*Li z1 pn*)) *add Emp*) ⟹ *Line-on* (*Li x y*) *z1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P103* **have** *P104* : *Bet-Point* (*Se pn y*) *x* ⟹
¬ *Eq* (*Geos* (*Lin* (*Li pn y*)) *add Emp*) (*Geos* (*Lin* (*Li z1 pn*)) *add Emp*) **by** *blast*
from *P84 P104* **have** *Bet-Point* (*Se pn y*) *x* ⟹ *Plane-sameside* (*Li z1 pn*) *x y* **by** (*simp add*:*Plane-Bet-sameside*)
then have *P105* : *Bet-Point* (*Se pn y*) *x* ⟹ *Plane-sameside* (*Li z1 pn*) *y x* **by** (*simp add*:*Plane-sameside-rev*)
from *P93 P102* **have** *P106* : *Bet-Point* (*Se pn y*) *x* ⟹
Eq (*Geos* (*Lin* (*Li pn y*)) *add Emp*) (*Geos* (*Lin* (*Li z2 pn*)) *add Emp*) ⟹ *Line-on* (*Li x y*) *z2* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P106* **have** *P107* : *Bet-Point* (*Se pn y*) *x* ⟹
¬ *Eq* (*Geos* (*Lin* (*Li pn y*)) *add Emp*) (*Geos* (*Lin* (*Li z2 pn*)) *add Emp*) **by** *blast*
from *P92 P107* **have** *Bet-Point* (*Se pn y*) *x* ⟹ *Plane-sameside* (*Li z2 pn*) *x y* **by** (*simp add*:*Plane-Bet-sameside*)
then have *P108* : *Bet-Point* (*Se pn y*) *x* ⟹ *Plane-sameside* (*Li z2 pn*) *y x* **by** (*simp add*:*Plane-sameside-rev*)
from *P53 P62 P64 P80 P97 P98 P105 P108* **have** *P109* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹ *Bet-Point* (*Se pn y*) *x* ⟹
Cong (*Geos* (*Ang* (*An y z1 x*)) *add Emp*) (*Geos* (*Ang* (*An y z2 x*)) *add Emp*) **by** (*simp add*:*Ang-sub*)
have *P110* : *Bet-Point* (*Se y x*) *pn* ⟹ *Bet-Point* (*Se x y*) *pn* **by** (*simp add*:*Bet-rev*)
from *P88* **have** *P111* : *Eq* (*Geos* (*Lin* (*Li x y*)) *add Emp*) (*Geos* (*Lin* (*Li z1 pn*)) *add Emp*)
⟹ *Line-on* (*Li x y*) *z1* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P111* **have** *P112* : ¬ *Eq* (*Geos* (*Lin* (*Li x y*)) *add Emp*) (*Geos* (*Lin* (*Li z1 pn*)) *add Emp*) **by** *blast*
from *P84 P110 P112* **have** *Bet-Point* (*Se y x*) *pn* ⟹ *Plane-diffside* (*Li z1 pn*)

x y **by** (*simp add*:*Plane-Bet-diffside*)
then have *P113* : *Bet-Point* (*Se y x*) *pn* ⟹ *Plane-diffside* (*Li z1 pn*) *y x* **by** (*simp add*:*Plane-diffside-rev*)
from *P93* **have** *P114* : *Eq* (*Geos* (*Lin* (*Li x y*)) *add Emp*) (*Geos* (*Lin* (*Li z2 pn*)) *add Emp*)
⟹ *Line-on* (*Li x y*) *z2* **by** (*blast intro*:*Line-on-trans Eq-rev*)
from *P1 P114* **have** *P115* : ¬ *Eq* (*Geos* (*Lin* (*Li x y*)) *add Emp*) (*Geos* (*Lin* (*Li z2 pn*)) *add Emp*) **by** *blast*
from *P92 P110 P115* **have** *Bet-Point* (*Se y x*) *pn* ⟹ *Plane-diffside* (*Li z2 pn*) *x y* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P116* : *Bet-Point* (*Se y x*) *pn* ⟹ *Plane-diffside* (*Li z2 pn*) *y x* **by** (*simp add*:*Plane-diffside-rev*)
from *P53 P62 P64 P80 P97 P98 P113 P116* **have** *P117* : ¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹ *Bet-Point* (*Se y x*) *pn* ⟹
Cong (*Geos* (*Ang* (*An y z1 x*)) *add Emp*) (*Geos* (*Ang* (*An y z2 x*)) *add Emp*) **by** (*simp add*:*Ang-add*)
from *P82 P99 P109 P117* **have** *P118* :
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An y z1 x*)) *add Emp*) (*Geos* (*Ang* (*An y z2 x*)) *add Emp*) **by** *blast*
from *P21 P25 P37 P38 P118* **have**
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Tri* (*Tr z1 y x*)) *add Emp*) (*Geos* (*Tri* (*Tr z2 y x*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P119* :
¬ *Eq* (*Geos* (*Poi x*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi y*) *add Emp*) (*Geos* (*Poi pn*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P3 P30 P39 P119* **show** *Cong* (*Geos* (*Ang* (*An x y z1*)) *add Emp*) (*Geos* (*Ang* (*An x y z2*)) *add Emp*) **by** *blast*
qed

Theorem18

theorem (**in** *Congruence-Rule*) *Tri-SSS* :
assumes
Def (*Tri* (*Tr A1 B1 C1*)) *Def* (*Tri* (*Tr A2 B2 C2*))
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 C2*)) *add Emp*)
Eq (*Geos* (*Seg* (*Se C1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se C2 A2*)) *add Emp*)
shows *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*)
proof −
from *assms* **have** *Def* (*Tri* (*Tr C2 B2 A2*)) **by** (*simp add*:*Tri-def-rev*)
then have *P1* : ¬ *Line-on* (*Li A2 C2*) *B2* **by** (*simp add*:*Tri-def-Line*)

from *assms* **have** *P2* : *Def* (*Ang* (*An B1 A1 C1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans Tri-to-Ang*)
from *P1 P2* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p A2 C2*)) *add Emp*) ∧ *Plane-sameside* (*Li A2 C2*) *p B2* **by** (*simp add*:*Ang-move-sameside*)
then obtain *B21* :: *Point* **where** *P3* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B21 A2 C2*)) *add Emp*)
∧ *Plane-sameside* (*Li A2 C2*) *B21 B2* **by** *blast*
then have *P4* : ¬ *Line-on* (*Li A2 C2*) *B21* **by** (*simp add*:*Plane-sameside-def*)
from *P2 P4* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p A2 C2*)) *add Emp*) ∧ *Plane-diffside* (*Li A2 C2*) *p B21* **by** (*simp add*:*Ang-move-diffside*)
then obtain *B22* :: *Point* **where** *P5* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B22 A2 C2*)) *add Emp*)
∧ *Plane-diffside* (*Li A2 C2*) *B22 B21* **by** *blast*
have *P6* : *Line-on* (*Li A2 B21*) *A2* **by** (*simp add*:*Line-on-rule*)
have *P7* : *Line-on* (*Li A2 B21*) *B21* **by** (*simp add*:*Line-on-rule*)
have *P8* : *Line-on* (*Li A2 C2*) *A2* **by** (*simp add*:*Line-on-rule*)
then have *P9* : *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B21*) *add Emp*) ⟹ *Line-on* (*Li A2 C2*) *B21* **by** (*simp add*:*Point-Eq*)
from *P4 P9* **have** *P10* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B21*) *add Emp*) **by** *blast*
from *assms* **have** *P11* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P6 P7 P10 P11* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p B21*) *A2* ∧ *Line-on* (*Li A2 B21*) *p* ∧ ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *B211* :: *Point* **where** *P12* : *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B211*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se B211 B21*) *A2* ∧ *Line-on* (*Li A2 B21*) *B211* ∧ ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) **by** *blast*
have *P13* : *Line-on* (*Li A2 B22*) *A2* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li A2 B22*) *B22* **by** (*simp add*:*Line-on-rule*)
from *P8* **have** *P15* : *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B22*) *add Emp*) ⟹ *Line-on* (*Li A2 C2*) *B22* **by** (*simp add*:*Point-Eq*)
from *P5* **have** *P16* : ∃ *p*. *Bet-Point* (*Se B22 B21*) *p* ∧ *Line-on* (*Li A2 C2*) *p*
∧ ¬ *Line-on* (*Li A2 C2*) *B22* ∧ ¬ *Line-on* (*Li A2 C2*) *B21* **by** (*simp add*:*Plane-diffside-def*)
from *P15 P16* **have** *P17* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B22*) *add Emp*) **by** *blast*
from *P11 P13 P14 P17* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p B22*) *A2* ∧ *Line-on* (*Li A2 B22*) *p* ∧ ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *B221* :: *Point* **where** *P18* : *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B221*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se B221 B22*) *A2* ∧ *Line-on* (*Li A2 B22*) *B221* ∧ ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B221*) *add Emp*) **by** *blast*

from *assms* **have** *P19* : ¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P20* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P4 P20* **have** *P21* : *Def* (*Ang* (*An B21 A2 C2*)) **by** (*blast intro*:*Ang-sinple-def Ang-def-rev Ang-def-inv*)
from *P12* **have** ¬ *Bet-Point* (*Se B211 B21*) *A2* **by** *blast*
then have *P22* : ¬ *Bet-Point* (*Se B21 B211*) *A2* **by** (*blast intro*:*Bet-rev*)
have *P23* : *Line-on* (*Li A2 C2*) *C2* **by** (*simp add*:*Line-on-rule*)
have *P24* : ¬ *Bet-Point* (*Se C2 C2*) *A2* **by** (*simp add*:*Bet-end-Point*)
from *P12 P20 P21 P22 P23 P24* **have** *P25* :
Eq (*Geos* (*Ang* (*An B21 A2 C2*)) *add Emp*) (*Geos* (*Ang* (*An B211 A2 C2*)) *add Emp*) ∧ *Def* (*Ang* (*An B211 A2 C2*)) **by** (*simp add*:*Ang-Point-swap*)
from *P3 P25* **have** *P26* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B211 A2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms* **have** *P27* : *Def* (*Tri* (*Tr A1 B1 C1*)) **by** (*simp add*:*Ang-to-Tri*)
from *P25* **have** *P28* : *Def* (*Tri* (*Tr A2 B211 C2*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-trans Tri-def-rev*)
from *assms* **have** *P29* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) **by** (*blast intro*:*Seg-rev Eq-rev Eq-trans*)
from *P12 P26 P27 P28 P29* **have** *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B211 C2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P30* : *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B211 C2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P16 P20* **have** *P31* : *Def* (*Ang* (*An B22 A2 C2*)) **by** (*blast intro*:*Ang-sinple-def Ang-def-rev Ang-def-inv*)
from *P18* **have** ¬ *Bet-Point* (*Se B221 B22*) *A2* **by** *blast*
then have *P32* : ¬ *Bet-Point* (*Se B22 B221*) *A2* **by** (*blast intro*:*Bet-rev*)
from *P18 P20 P23 P24 P31 P32* **have** *P33* :
Eq (*Geos* (*Ang* (*An B22 A2 C2*)) *add Emp*) (*Geos* (*Ang* (*An B221 A2 C2*)) *add Emp*) ∧ *Def* (*Ang* (*An B221 A2 C2*)) **by** (*simp add*:*Ang-Point-swap*)
from *P5 P33* **have** *P34* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B221 A2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P33* **have** *P35* : *Def* (*Tri* (*Tr A2 B221 C2*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-trans Tri-def-rev*)
from *P18 P27 P29 P34 P35* **have** *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B221 C2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P36* : *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B221 C2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P12* **have** *P37* : *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B211*)) *add Emp*) **by** *simp*
from *P18* **have** *P38* : *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B221*)) *add Emp*) **by** *simp*
from *P37 P38* **have** *P39* : *Eq* (*Geos* (*Seg* (*Se A2 B221*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B211*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *assms P38* **have** *P40* : *Eq* (*Geos* (*Seg* (*Se A2 B221*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P30 P36* **have** *P41* : *Eq* (*Geos* (*Seg* (*Se B221 C2*)) *add Emp*) (*Geos* (*Seg* (*Se B211 C2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)

from *assms P36* **have** *P42* : *Eq* (*Geos* (*Seg* (*Se B221 C2*)) *add Emp*) (*Geos* (*Seg* (*Se B2 C2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P5* **have** *P43* : *Plane-diffside* (*Li A2 C2*) *B22 B21* **by** *simp*
then have *P44* : *Eq* (*Geos* (*Poi B21*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹ *Plane-diffside* (*Li A2 C2*) *B22 B211* **by** (*blast intro*:*Point-Eq Eq-rev*)
from *P6 P8 P20 P23* **have** *P45* : *Line-on* (*Li A2 B21*) *C2* ⟹ *Eq* (*Geos* (*Lin* (*Li A2 B21*)) *add Emp*) (*Geos* (*Lin* (*Li A2 C2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P45* **have** *P46* : *Line-on* (*Li A2 B21*) *C2* ⟹ *Line-on* (*Li A2 C2*) *B21* **by** (*simp add*:*Line-on-trans*)
from *P4 P46* **have** *P47* : ¬ *Line-on* (*Li A2 B21*) *C2* **by** *blast*
from *P6 P7 P10 P12 P22 P47* **have** *P48* : *Plane-sameside* (*Li C2 A2*) *B21 B211* ∨ *Eq* (*Geos* (*Poi B21*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) **by** (*simp add*:*Seg-Plane-sameside*)
from *P20* **have** *P49* : *Plane-sameside* (*Li C2 A2*) *B21 B211* ⟹ *Plane-sameside* (*Li A2 C2*) *B21 B211* **by** (*blast intro*:*Line-rev Plane-Line-trans Eq-rev*)
from *P43* **have** *P50* : *Plane-diffside* (*Li A2 C2*) *B21 B22* **by** (*simp add*:*Plane-diffside-rev*)
from *P49 P50* **have** *P51* : *Plane-sameside* (*Li C2 A2*) *B21 B211* ⟹ *Plane-diffside* (*Li A2 C2*) *B22 B211* **by** (*simp add*:*Plane-trans Plane-diffside-rev*)
from *P44 P48 P51* **have** *P52* : *Plane-diffside* (*Li A2 C2*) *B22 B211* **by** *blast*
then have *Plane-diffside* (*Li A2 C2*) *B211 B22* **by** (*blast intro*:*Plane-diffside-rev*)
then have *P53* : *Eq* (*Geos* (*Poi B22*) *add Emp*) (*Geos* (*Poi B221*) *add Emp*) ⟹ *Plane-diffside* (*Li A2 C2*) *B211 B221* **by** (*blast intro*:*Point-Eq Eq-rev*)
from *P8 P13 P20 P23* **have** *P54* : *Line-on* (*Li A2 B22*) *C2* ⟹ *Eq* (*Geos* (*Lin* (*Li A2 B22*)) *add Emp*) (*Geos* (*Lin* (*Li A2 C2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P14 P54* **have** *P55* : *Line-on* (*Li A2 B22*) *C2* ⟹ *Line-on* (*Li A2 C2*) *B22* **by** (*simp add*:*Line-on-trans*)
from *P16 P55* **have** *P56* : ¬ *Line-on* (*Li A2 B22*) *C2* **by** *blast*
from *P31* **have** ¬ *Eq* (*Geos* (*Poi B22*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P57* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B22*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P13 P14 P18 P32 P56 P57* **have** *P58* : *Plane-sameside* (*Li C2 A2*) *B22 B221* ∨ *Eq* (*Geos* (*Poi B22*) *add Emp*) (*Geos* (*Poi B221*) *add Emp*) **by** (*simp add*:*Seg-Plane-sameside*)
from *P20* **have** *P59* : *Plane-sameside* (*Li C2 A2*) *B22 B221* ⟹ *Plane-sameside* (*Li A2 C2*) *B22 B221* **by** (*blast intro*:*Line-rev Plane-Line-trans Eq-rev*)
from *P52 P59* **have** *P60* : *Plane-sameside* (*Li C2 A2*) *B22 B221* ⟹ *Plane-diffside* (*Li A2 C2*) *B211 B221* **by** (*simp add*:*Plane-trans Plane-diffside-rev*)
from *P53 P58 P60* **have** *Plane-diffside* (*Li A2 C2*) *B211 B221* **by** *blast*
then have *P61* : *Plane-diffside* (*Li A2 C2*) *B221 B211* **by** (*simp add*:*Plane-diffside-rev*)
from *P20 P61* **have** *P62* : *Plane-diffside* (*Li C2 A2*) *B221 B211* **by** (*blast intro*:*Line-rev Plane-Line-diff-trans*)
have *P63* : *Eq* (*Geos* (*Seg* (*Se B221 C2*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B221*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P64* : *Eq* (*Geos* (*Seg* (*Se B211 C2*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B211*))

add Emp) **by** (*simp add*:*Seg-rev*)
from *P41 P63 P64* **have** *P65* : *Eq* (*Geos* (*Seg* (*Se C2 B221*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B211*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P19 P39 P62 P65* **have** *P66* : *Cong* (*Geos* (*Ang* (*An C2 A2 B221*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 B211*)) *add Emp*) **by** (*simp add*:*Tri-week-SSS*)
have *P67* : *Eq* (*Geos* (*Ang* (*An C2 A2 B211*)) *add Emp*) (*Geos* (*Ang* (*An B211 A2 C2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P66 P67* **have** *P68* : *Cong* (*Geos* (*Ang* (*An C2 A2 B221*)) *add Emp*) (*Geos* (*Ang* (*An B211 A2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P3* **have** *P69* : *Plane-sameside* (*Li A2 C2*) *B21 B2* **by** *simp*
from *P50 P69* **have** *P70* : *Plane-diffside* (*Li A2 C2*) *B2 B22* **by** (*simp add*:*Plane-trans*)
then have *P71* : *Plane-diffside* (*Li A2 C2*) *B22 B2* **by** (*simp add*:*Plane-diffside-rev*)
from *P70* **have** *P72* : *Eq* (*Geos* (*Poi B22*) *add Emp*) (*Geos* (*Poi B221*) *add Emp*) ⟹
Plane-diffside (*Li A2 C2*) *B221 B2* **by** (*blast intro*:*Point-Eq Plane-diffside-rev*)
from *P59 P71* **have** *P73* : *Plane-sameside* (*Li C2 A2*) *B22 B221* ⟹
Plane-diffside (*Li A2 C2*) *B221 B2* **by** (*simp add*:*Plane-trans*)
from *P58 P72 P73* **have** *P74* : *Plane-diffside* (*Li A2 C2*) *B221 B2* **by** *blast*
from *P20 P74* **have** *P75* : *Plane-diffside* (*Li C2 A2*) *B221 B2* **by** (*blast intro*:*Line-rev Plane-Line-diff-trans*)
have *P76* : *Eq* (*Geos* (*Seg* (*Se B221 C2*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B221*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P77* : *Eq* (*Geos* (*Seg* (*Se B2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B2*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P42 P76 P77* **have** *P78* : *Eq* (*Geos* (*Seg* (*Se C2 B221*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P19 P40 P75 P78* **have** *P79* : *Cong* (*Geos* (*Ang* (*An C2 A2 B221*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 B2*)) *add Emp*) **by** (*simp add*:*Tri-week-SSS*)
have *P80* : *Eq* (*Geos* (*Ang* (*An C2 A2 B2*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P79 P80* **have** *P81* : *Cong* (*Geos* (*Ang* (*An C2 A2 B221*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P52 P71* **have** *P82* : ¬ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹
Plane-sameside (*Li A2 C2*) *B2 B211* **by** (*blast intro*:*Plane-trans-inv*)
from *P68 P81 P82* **have** *P83* : ¬ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li B2 A2*)) *add Emp*) (*Geos* (*Lin* (*Li B211 A2*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se B2 B211*) *A2* **by** (*simp add*:*Ang-move-unique*)
from *assms* **have** *P84* : *Def* (*Ang* (*An B2 A2 C2*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-rev Ang-def-inv*)
then have ¬ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P85* : *Eq* (*Geos* (*Lin* (*Li B2 A2*)) *add Emp*) (*Geos* (*Lin* (*Li A2 B2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
have *P86* : *Line-on* (*Li B211 A2*) *B211* **by** (*simp add*:*Line-on-rule*)
from *P83 P85 P86* **have** *P87* : ¬ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹
Line-on (*Li A2 B2*) *B211* **by** (*blast intro*:*Eq-rev Line-on-trans*)

have *Line-on* (*Li A2 B2*) *B2* **by** (*simp add*:*Line-on-rule*)
then have *P88* : *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹ *Line-on* (*Li A2 B2*) *B211* **by** (*simp add*:*Point-Eq*)
from *P87 P88* **have** *P89* : *Line-on* (*Li A2 B2*) *B211* **by** *blast*
have *P90* : ¬ *Bet-Point* (*Se B2 B2*) *A2* **by** (*simp add*:*Bet-end-Point*)
have *P91* : *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹ *Bet-Point* (*Se B211 B2*) *A2* ⟹ *Bet-Point* (*Se B2 B2*) *A2* **by** (*blast intro*:*Eq-rev Bet-Point-Eq*)
from *P90 P91* **have** *P92* : *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B211*) *add Emp*) ⟹ ¬ *Bet-Point* (*Se B2 B211*) *A2* **by** (*blast intro*:*Bet-rev*)
from *P83 P92* **have** *P93* : ¬ *Bet-Point* (*Se B2 B211*) *A2* **by** *blast*
from *P12 P20 P23 P24 P84 P89 P93* **have** *P94* : *Eq* (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) (*Geos* (*Ang* (*An B211 A2 C2*)) *add Emp*) ∧ *Def* (*Ang* (*An B211 A2 C2*)) **by** (*simp add*:*Ang-Point-swap*)
from *P25 P94* **have** *P95* : *Eq* (*Geos* (*Ang* (*An B21 A2 C2*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P3 P95* **have** *P96* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P97* : *Eq* (*Geos* (*Seg* (*Se C1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P98* : *Eq* (*Geos* (*Seg* (*Se C2 A2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *assms P97 P98* **have** *P99* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *assms* **have** *P100* : *Def* (*Tri* (*Tr A1 B1 C1*)) **by** (*simp add*:*Ang-to-Tri*)
from *assms* **have** *P101* : *Def* (*Tri* (*Tr A2 B2 C2*)) **by** (*simp add*:*Ang-to-Tri*)
from *assms P96 P99 P100 P101* **show** *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
qed

Theorem19

theorem (**in** *Congruence-Rule*) *Ang-trans* :
assumes
Def (*Ang* (*An A1 B1 C1*)) *Def* (*Ang* (*An A2 B2 C2*)) *Def* (*Ang* (*An A3 B3 C3*))
Cong (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*)
shows *Cong* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*)
proof −
from *assms* **have** *P1* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) **by** (*simp add*:*Ang-rev*)
from *assms P1* **have** ∃ *p q*. *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B2 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An p B2 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 p*)) *add Emp*)

∧ *Line-on* (*Li B2 A2*) *p* ∧ ¬ *Bet-Point* (*Se p A2*) *B2*
∧ *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 q*)) *add Emp*)
∧ *Line-on* (*Li B2 C2*) *q* ∧ ¬ *Bet-Point* (*Se q C2*) *B2* ∧ *Def* (*Ang* (*An p B2 q*)) **by** (*simp add*:*Ang-replace*)
then obtain *A21 C21* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A21 B2 C21*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A21 B2 C21*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 A21*)) *add Emp*)
∧ *Line-on* (*Li B2 A2*) *A21* ∧ ¬ *Bet-Point* (*Se A21 A2*) *B2*
∧ *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 C21*)) *add Emp*)
∧ *Line-on* (*Li B2 C2*) *C21* ∧ ¬ *Bet-Point* (*Se C21 C2*) *B2* ∧ *Def* (*Ang* (*An A21 B2 C21*)) **by** *blast*
from *assms* **have** *P3* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) **by** (*simp add*:*Ang-rev*)
from *assms P3* **have** ∃ *p q*. *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An p B3 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) (*Geos* (*Ang* (*An p B3 q*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 p*)) *add Emp*)
∧ *Line-on* (*Li B3 A3*) *p* ∧ ¬ *Bet-Point* (*Se p A3*) *B3*
∧ *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 q*)) *add Emp*)
∧ *Line-on* (*Li B3 C3*) *q* ∧ ¬ *Bet-Point* (*Se q C3*) *B3* ∧ *Def* (*Ang* (*An p B3 q*)) **by** (*simp add*:*Ang-replace*)
then obtain *A31 C31* :: *Point* **where** *P4* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A31 B3 C31*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) (*Geos* (*Ang* (*An A31 B3 C31*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 A31*)) *add Emp*)
∧ *Line-on* (*Li B3 A3*) *A31* ∧ ¬ *Bet-Point* (*Se A31 A3*) *B3*
∧ *Eq* (*Geos* (*Seg* (*Se B1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se B3 C31*)) *add Emp*)
∧ *Line-on* (*Li B3 C3*) *C31* ∧ ¬ *Bet-Point* (*Se C31 C3*) *B3* ∧ *Def* (*Ang* (*An A31 B3 C31*)) **by** *blast*
from *assms* **have** *P5* : *Def* (*Tri* (*Tr B1 A1 C1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans Ang-to-Tri*)
from *P2* **have** *P6* : *Def* (*Tri* (*Tr B2 A21 C21*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans Ang-to-Tri*)
from *P2 P5 P6* **have** *Cong* (*Geos* (*Tri* (*Tr B1 A1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr B2 A21 C21*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P7* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A21 C21*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P4* **have** *P8* : *Def* (*Tri* (*Tr B3 A31 C31*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans Ang-to-Tri*)
from *P4 P5 P8* **have** *Cong* (*Geos* (*Tri* (*Tr B1 A1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr B3 A31 C31*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P9* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A31 C31*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P6* **have** *P10* : *Def* (*Tri* (*Tr A21 C21 B2*)) **by** (*blast intro*:*Tri-def-trans*)
from *P8* **have** *P11* : *Def* (*Tri* (*Tr A31 C31 B3*)) **by** (*blast intro*:*Tri-def-trans*)

from *P7 P9* **have** *P12* : *Eq* (*Geos* (*Seg* (*Se A21 C21*)) *add Emp*) (*Geos* (*Seg* (*Se A31 C31*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P2 P4* **have** *P13* : *Eq* (*Geos* (*Seg* (*Se B2 A21*)) *add Emp*) (*Geos* (*Seg* (*Se B3 A31*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P2 P4* **have** *P14* : *Eq* (*Geos* (*Seg* (*Se B2 C21*)) *add Emp*) (*Geos* (*Seg* (*Se B3 C31*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
have *P15* : *Eq* (*Geos* (*Seg* (*Se B2 C21*)) *add Emp*) (*Geos* (*Seg* (*Se C21 B2*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P16* : *Eq* (*Geos* (*Seg* (*Se B3 C31*)) *add Emp*) (*Geos* (*Seg* (*Se C31 B3*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P14 P15 P16* **have** *P17* : *Eq* (*Geos* (*Seg* (*Se C21 B2*)) *add Emp*) (*Geos* (*Seg* (*Se C31 B3*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P10 P11 P12 P13 P17* **have** *Cong* (*Geos* (*Tri* (*Tr A21 C21 B2*)) *add Emp*) (*Geos* (*Tri* (*Tr A31 C31 B3*)) *add Emp*) **by** (*simp add*:*Tri-SSS*)
then have *P18* : *Cong* (*Geos* (*Ang* (*An A21 B2 C21*)) *add Emp*) (*Geos* (*Ang* (*An A31 B3 C31*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *P2 P18* **have** *P19* : *Cong* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A31 B3 C31*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev*)
from *P4 P19* **show** *Cong* (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) (*Geos* (*Ang* (*An A3 B3 C3*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev*)
qed

lemma (**in** *Congruence-Rule*) *Ang-move-unique-inv* :
assumes
Def (*Ang* (*An p1 p2 p3*)) *Def* (*Ang* (*An p4 p2 p3*))
Plane-sameside (*Li p2 p3*) *p1 p4*
Eq (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*)
proof −
have *P1* : *Line-on* (*Li p2 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
from *assms P1* **have** *P2* : *Line-on* (*Li p2 p1*) *p4* **by** (*blast intro*:*Line-on-trans Eq-rev*)
have *P3* : *Line-on* (*Li p2 p3*) *p2* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P4* : ¬ *Line-on* (*Li p2 p3*) *p1* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P5* : ¬ *Line-on* (*Li p2 p3*) *p4* **by** (*simp add*:*Plane-sameside-def*)
from *P3 P4 P5* **have** *Bet-Point* (*Se p1 p4*) *p2* ⟹ ∃ *p*. *Bet-Point* (*Se p1 p4*) *p* ∧ *Line-on* (*Li p2 p3*) *p* ∧
¬ *Line-on* (*Li p2 p3*) *p1* ∧ ¬ *Line-on* (*Li p2 p3*) *p4* **by** *blast*
then have *Bet-Point* (*Se p1 p4*) *p2* ⟹ *Plane-diffside* (*Li p2 p3*) *p1 p4* **by** (*simp add*:*Plane-diffside-def*)
then have *P6* : *Bet-Point* (*Se p1 p4*) *p2* ⟹ ¬ *Plane-sameside* (*Li p2 p3*) *p1 p4* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *assms P6* **have** *P7* : ¬ *Bet-Point* (*Se p1 p4*) *p2* **by** *blast*
have *P8* : *Line-on* (*Li p2 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P9* : ¬ *Bet-Point* (*Se p3 p3*) *p2* **by** (*simp add*:*Bet-end-Point*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Ang-def*)

then have *P10* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P11* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *assms P2 P7 P8 P9 P10 P11* **have** *Eq* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*)
∧ *Def* (*Ang* (*An p4 p2 p3*)) **by** (*simp add*:*Ang-Point-swap*)
thus *Cong* (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
qed

Theorem20

theorem (**in** *Congruence-Rule*) *Ang-move-Greater* :
assumes
Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 l2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Plane-sameside (*Li o2 h2*) *k2 l2*
Cong (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*)
Plane-sameside (*Li o1 h1*) *k1 l1*
Ang-inside (*An h2 o2 l2*) *k2*
shows
¬ *Ang-inside* (*An h1 o1 k1*) *l1*
¬ *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li o2 h2*) *k2* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P2* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P2* **have** *Def* (*Ang* (*An o2 h2 k2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P3* : *Def* (*Ang* (*An h2 o2 k2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P3* **have** *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ ∃ *p*. *Ang-inside* (*An h2 o2 k2*) *p*
∧ *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 p*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 p*)) *add Emp*) **by** (*simp add*:*Ang-split*)
then obtain *l21* :: *Point* **where** *P4* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Ang-inside* (*An h2 o2 k2*) *l21*
∧ *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l21*)) *add Emp*)
∧ *Cong* (*Geos* (*Ang* (*An k1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 l21*)) *add Emp*) **by** *blast*
then have *P5* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Plane-sameside (*Li o2 k2*) *h2 l21* ∧ *Plane-sameside* (*Li o2 h2*) *k2 l21* **by** (*simp add*:*Ang-inside-def*)
from *assms* **have** *P6* : *Plane-diffside* (*Li o2 k2*) *h2 l2* **by** (*simp add*:*Ang-inside-Planeside*)

from *P5 P6* **have** *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Plane-diffside (*Li o2 k2*) *l21 l2* **by** (*blast intro*:*Plane-trans*)
then have *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ ∃ *p*. *Bet-Point* (*Se l21 l2*) *p* ∧ *Line-on* (*Li o2 k2*) *p*
∧ ¬ *Line-on* (*Li o2 k2*) *l21* ∧ ¬ *Line-on* (*Li o2 k2*) *l2* **by** (*simp add*:*Plane-diffside-def*)
then obtain *pn* :: *Point* **where** *P7* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Bet-Point (*Se l21 l2*) *pn* ∧ *Line-on* (*Li o2 k2*) *pn*
∧ ¬ *Line-on* (*Li o2 k2*) *l21* ∧ ¬ *Line-on* (*Li o2 k2*) *l2* **by** *blast*
then have *P8* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Bet-Point* (*Se l21 l2*) *pn* **by** *simp*
then have *P9* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
¬ *Eq* (*Geos* (*Poi l21*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms* **have** *P10* : *Plane-sameside* (*Li o2 h2*) *l2 k2* **by** (*simp add*:*Plane-sameside-rev*)
from *P5 P9 P10* **have** *P11* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Plane-sameside* (*Li o2 h2*) *l2 l21* **by** (*blast intro*:*Plane-sameside-trans*)
from *P8* **have** *P12* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Poi pn*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) ⟹ *Bet-Point* (*Se l21 l2*) *o2* **by** (*simp add*:*Point-Eq*)
have *P13* : *Line-on* (*Li o2 h2*) *o2* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li l21 l2*) *l2* **by** (*simp add*:*Line-on-rule*)
from *P14* **have** *P15* : *Eq* (*Geos* (*Lin* (*Li l21 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) ⟹
Line-on (*Li o2 h2*) *l2* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** *Def* (*Tri* (*Tr o2 h2 l2*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P16* : ¬ *Line-on* (*Li o2 h2*) *l2* **by** (*simp add*:*Tri-def-Line*)
from *P15 P16* **have** *P17* : ¬ *Eq* (*Geos* (*Lin* (*Li l21 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 h2*)) *add Emp*) **by** *blast*
from *P12 P13 P17* **have** *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Poi pn*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) ⟹
Plane-diffside (*Li o2 h2*) *l2 l21* **by** (*simp add*:*Plane-Bet-diffside Plane-diffside-rev*)
then have *P18* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Poi pn*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) ⟹
¬ *Plane-sameside* (*Li o2 h2*) *l2 l21* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P11 P18* **have** *P19* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
¬ *Eq* (*Geos* (*Poi pn*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** *blast*
have *P20* : *Line-on* (*Li o2 k2*) *o2* **by** (*simp add*:*Line-on-rule*)
from *P8* **have** *P21* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Line-on* (*Li l21 l2*) *pn* **by** (*simp add*:*Line-Bet-on*)
from *P7 P19 P20 P21* **have** *P22* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Line-on* (*Li l21 l2*) *o2* ⟹
Eq (*Geos* (*Lin* (*Li l21 l2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P14 P22* **have** *P23* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Line-on (*Li l21 l2*) *o2* ⟹ *Line-on* (*Li o2 k2*) *l2* **by** (*simp add*:*Line-on-trans*)
from *P7 P23* **have** *P24* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ ¬ *Line-on* (*Li l21 l2*) *o2* **by** *blast*
from *P9 P24* **have** *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Def* (*Ang* (*An l21 l2 o2*)) **by** (*simp add*:*Ang-sinple-def*)

then have *P25* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Def* (*Ang* (*An l21 o2 l2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have *P26* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
¬ *Eq* (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) **by** (*simp add*:*Ang-def*)
have *P27* : *Eq* (*Geos* (*Ang* (*An h2 o2 l21*)) *add Emp*) (*Geos* (*Ang* (*An l21 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P4 P27* **have** *P28* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Cong (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l21 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P29* : *Eq* (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P29* **have** *P30* : *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P11 P28 P30* **have** *P31* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*) (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) **by** (*simp add*:*Ang-move-unique*)
from *P25* **have** *P32* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ *Def* (*Tri* (*Tr l21 o2 l2*)) **by** (*simp add*:*Ang-to-Tri*)
then have *Ang-inside* (*An h1 o1 k1*) *l1* ⟹ ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P33* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P32* **have** *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
¬ *Eq* (*Geos* (*Poi l21*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P34* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P31 P33* **have** *P35* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li l21 o2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P34 P35* **have** *P36* : *Ang-inside* (*An h1 o1 k1*) *l1* ⟹
Eq (*Geos* (*Lin* (*Li o2 l21*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P26 P36* **show** ¬ *Ang-inside* (*An h1 o1 k1*) *l1* **by** *blast*
from *assms* **have** *P37* : *Def* (*Ang* (*An k1 o1 h1*)) **by** (*blast intro*:*Ang-def-rev*)
from *assms* **have** *P38* : ¬ *Line-on* (*Li o1 h1*) *l1* **by** (*simp add*:*Plane-sameside-def*)
from *P37* **have** *P39* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P38 P39* **have** *Def* (*Ang* (*An o1 h1 l1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P40* : *Def* (*Ang* (*An l1 o1 h1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P37 P40* **have** *P41* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** (*simp add*:*Ang-move-unique-inv*)
have *P42* : *Cong* (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P37 P40 P41 P42* **have** *P43* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*)

(*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
have *P44* : *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P40* **have** *P45* : *Def* (*Ang* (*An h1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev*)
from *assms P40 P43 P44 P45* **have** *P46* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *assms P45 P46* **have** *P47* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *assms P3 P47* **have** *P48* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P29 P48* **have** *P49* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P50* : *Cong* (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P49 P50* **have** *P51* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li k2 o2*)) *add Emp*) (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se k2 l2*) *o2* **by** (*simp add*:*Ang-move-unique*)
from *assms* **have** *Def* (*Ang* (*An h2 o2 l2*)) ∧ *Plane-sameside* (*Li o2 h2*) *l2 k2*
∧ *Plane-sameside* (*Li o2 l2*) *h2 k2* **by** (*simp add*:*Ang-inside-def*)
then have *P52* : ¬ *Line-on* (*Li o2 l2*) *k2* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P53* : *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P51 P53* **have** *P54* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li k2 o2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
have *P55* : *Line-on* (*Li k2 o2*) *k2* **by** (*simp add*:*Line-on-rule*)
from *P54 P55* **have** *P56* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Line-on (*Li o2 l2*) *k2* **by** (*simp add*:*Line-on-trans*)
from *P52 P56* **show** ¬ *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) **by** *blast*
qed

theorem (**in** *Congruence-Rule*) *Ang-move-Smaller* :
assumes

Def (*Ang* (*An h1 o1 k1*)) *Def* (*Ang* (*An h2 o2 l2*))
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*)
Plane-sameside (*Li o2 h2*) *k2 l2*
Cong (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*)
Plane-sameside (*Li o1 h1*) *k1 l1*
¬ *Ang-inside* (*An h2 o2 l2*) *k2*
¬ *Eq* (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*)
shows *Ang-inside* (*An h1 o1 k1*) *l1*
proof −
have *P1* : *Ang-inside* (*An l2 o2 h2*) *k2* ⟹ *Ang-inside* (*An h2 o2 l2*) *k2* **by** (*simp add*:*Ang-inside-def Ang-def-rev*)
from *assms P1* **have** *P2* : ¬ *Ang-inside* (*An l2 o2 h2*) *k2* **by** *blast*
from *assms* **have** *P3* : ¬ *Line-on* (*Li o2 h2*) *k2* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi h2*) *add Emp*) (*Geos* (*Poi o2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P4* : ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi h2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P3 P4* **have** *Def* (*Ang* (*An o2 h2 k2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P5* : *Def* (*Ang* (*An k2 o2 h2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P6* : *Def* (*Ang* (*An l2 o2 h2*)) **by** (*blast intro*:*Ang-def-rev*)
from *assms P5 P6* **have** *P7* : *Ang-inside* (*An k2 o2 h2*) *l2* ∧ ¬ *Ang-inside* (*An l2 o2 h2*) *k2*
∨ ¬ *Ang-inside* (*An k2 o2 h2*) *l2* ∧ *Ang-inside* (*An l2 o2 h2*) *k2* **by** (*simp add*:*Ang-inside-case*)
from *P2 P7* **have** *Ang-inside* (*An k2 o2 h2*) *l2* **by** *blast*
then have *P8* : *Ang-inside* (*An h2 o2 k2*) *l2* **by** (*simp add*:*Ang-inside-def Ang-def-rev*)
from *assms* **have** *P9* : ¬ *Line-on* (*Li o1 h1*) *l1* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi h1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P10* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P9 P10* **have** *Def* (*Ang* (*An o1 h1 l1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P11* : *Def* (*Ang* (*An h1 o1 l1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P5* **have** *P12* : *Def* (*Ang* (*An h2 o2 k2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P13* : *Cong* (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-rev*)
from *assms* **have** *P14* : *Plane-sameside* (*Li o2 h2*) *l2 k2* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P15* : *Cong* (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) **by** (*blast intro*:*Ang-rev*)
from *assms* **have** *P16* : *Plane-sameside* (*Li o1 h1*) *l1 k1* **by** (*simp add*:*Plane-sameside-rev*)
from *P8 P11 P12 P13 P14 P15 P16* **have** *P17* : ¬ *Ang-inside* (*An h1 o1 l1*) *k1* **by** (*simp add*:*Ang-move-Greater*)
have *P18* : *Ang-inside* (*An l1 o1 h1*) *k1* ⟹ *Ang-inside* (*An h1 o1 l1*) *k1* **by** (*simp add*:*Ang-inside-def Ang-def-rev*)
from *P17 P18* **have** *P19* : ¬ *Ang-inside* (*An l1 o1 h1*) *k1* **by** *blast*

from *assms* **have** *P20* : *Def* (*Ang* (*An k1 o1 h1*)) **by** (*blast intro*:*Ang-def-rev*)
from *P11* **have** *P21* : *Def* (*Ang* (*An l1 o1 h1*)) **by** (*blast intro*:*Ang-def-rev*)
have *Line-on* (*Li o1 k1*) *k1* **by** (*simp add*:*Line-on-rule*)
then have *P22* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Line-on (*Li o1 l1*) *k1* **by** (*simp add*:*Line-on-trans*)
have *P23* : *Line-on* (*Li o1 h1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li l1 k1*) *l1* **by** (*simp add*:*Line-on-rule*)
then have *P24* : *Eq* (*Geos* (*Lin* (*Li l1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) ⟹ *Line-on* (*Li o1 h1*) *l1* **by** (*simp add*:*Line-on-trans*)
from *P9 P24* **have** *P25* : ¬ *Eq* (*Geos* (*Lin* (*Li l1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) **by** *blast*
from *P23 P25* **have** *Bet-Point* (*Se l1 k1*) *o1* ⟹ *Plane-diffside* (*Li o1 h1*) *l1 k1* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P26* : *Bet-Point* (*Se l1 k1*) *o1* ⟹ ¬ *Plane-sameside* (*Li o1 h1*) *l1 k1* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P16 P26* **have** *P27* : ¬ *Bet-Point* (*Se l1 k1*) *o1* **by** *blast*
have *P28* : *Line-on* (*Li o1 h1*) *h1* **by** (*simp add*:*Line-on-rule*)
have *P29* : ¬ *Bet-Point* (*Se h1 h1*) *o1* **by** (*simp add*:*Bet-end-Point*)
from *assms* **have** *P30* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P10 P21 P22 P27 P28 P29 P30* **have** *P31* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) ∧ *Def* (*Ang* (*An k1 o1 h1*)) **by** (*simp add*:*Ang-Point-swap*)
have *P32* : *Eq* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P31 P32* **have** *P33* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
have *P34* : *Eq* (*Geos* (*Ang* (*An k1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P33 P34* **have** *P35* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An h1 o1 l1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *assms P35* **have** *P36* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P37* : *Eq* (*Geos* (*Ang* (*An h2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P37* **have** *P38* : *Cong* (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An k2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P39* : *Eq* (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P36 P39* **have** *P40* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹

Cong (*Geos* (*Ang* (*An h1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms P38 P40* **have** *P41* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li k2 o2*)) *add Emp*) (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se k2 l2*) *o2* **by** (*simp add*:*Ang-move-unique*)
from *P12* **have** ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi k2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P42* : *Eq* (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li k2 o2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P41 P42* **have** *P43* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi o2*) *add Emp*) (*Geos* (*Poi l2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P44* : *Eq* (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) (*Geos* (*Lin* (*Li l2 o2*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P43 P44* **have** *P45* : *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li o2 k2*)) *add Emp*) (*Geos* (*Lin* (*Li o2 l2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *assms P45* **have** *P46* : ¬ *Eq* (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) **by** *blast*
from *assms P20 P21 P46* **have** *P47* : *Ang-inside* (*An k1 o1 h1*) *l1* ∧ ¬ *Ang-inside* (*An l1 o1 h1*) *k1*
∨ ¬ *Ang-inside* (*An k1 o1 h1*) *l1* ∧ *Ang-inside* (*An l1 o1 h1*) *k1* **by** (*simp add*:*Ang-inside-case*)
from *P19 P47* **have** *Ang-inside* (*An k1 o1 h1*) *l1* **by** *blast*
thus *Ang-inside* (*An h1 o1 k1*) *l1* **by** (*simp add*:*Ang-inside-def Ang-def-rev*)
qed

lemma (**in** *Congruence-Rule*) *Ang-not-Gr-Eq-rev* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
shows
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
proof −
from *assms* **have** ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
then have ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p p11* **using** *assms* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p4* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p4 p12 p13*)) *add Emp*)

∧ *Plane-sameside* (*Li p12 p13*) *p4 p11* **by** *blast*
from *assms P1* **have** *P2* : ¬ *Ang-inside* (*An p11 p12 p13*) *p4* ∧ ¬ *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) ⟹
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-less-def*)
from *assms P2* **have** *P3* : *Ang-inside* (*An p11 p12 p13*) *p4* ∨ *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) **by** *blast*
from *P1* **have** *P4* : *Plane-sameside* (*Li p12 p13*) *p11 p4* **by** (*simp add*:*Plane-sameside-rev*)
then have *P5* : ¬ *Line-on* (*Li p12 p13*) *p4* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P6* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P5 P6* **have** *Def* (*Ang* (*An p12 p13 p4*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P7* : *Def* (*Ang* (*An p4 p12 p13*)) **by** (*blast intro*:*Ang-def-inv Ang-def-rev*)
from *assms P4 P7* **have** *P8* : *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p4 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-move-unique-inv*)
from *assms P1 P7 P8* **have** *P9* : *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P1* **have** *P10* : *Ang-inside* (*An p11 p12 p13*) *p4* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P3 P9 P10* **show** *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-not-Eq-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
proof −
from *assms* **have** ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
then have ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p p11* **using** *assms* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p4* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p4 p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p4 p11* **by** *blast*
from *P1* **have** *P2* : *Plane-sameside* (*Li p12 p13*) *p11 p4* **by** (*simp add*:*Plane-sameside-rev*)

then have *P3* : ¬ *Line-on* (*Li p12 p13*) *p4* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P3 P4* **have** *Def* (*Ang* (*An p12 p13 p4*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P5* : *Def* (*Ang* (*An p4 p12 p13*)) **by** (*blast intro*:*Ang-def-inv Ang-def-rev*)
from *assms P2 P5* **have** *P6* : *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p4 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-move-unique-inv*)
from *assms P1 P5 P6* **have** *P7* : *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *assms P7* **have** *P8* : ¬ *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p4*)) *add Emp*) **by** *blast*
from *assms P2 P5 P8* **have** *P9* : *Ang-inside* (*An p11 p12 p13*) *p4* ∧ ¬ *Ang-inside* (*An p4 p12 p13*) *p11*
∨ ¬ *Ang-inside* (*An p11 p12 p13*) *p4* ∧ *Ang-inside* (*An p4 p12 p13*) *p11* **by** (*simp add*:*Ang-inside-case*)
from *P1* **have** *P10* : *Ang-inside* (*An p11 p12 p13*) *p4* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P1 P8* **have** *P11* : ¬ *Ang-inside* (*An p11 p12 p13*) *p4* ⟹
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-less-def*)
from *P9 P10 P11* **show** *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-relation-case* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
shows
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-not-Eq-Gr*)
then have *P2* : ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang*

(*An p21 p22 p23*)) *add Emp*) ⟹
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** *blast*
from *P1* **have** *P3* : ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) ⟹
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** *blast*
from *assms* **have** ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr-Eq-rev*)
then have *P4* : ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) ⟹
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** *blast*
from *P2 P3 P4* **show** *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-not-Gr-lemma1* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
shows
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
proof −
from *assms* **have** ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
then have ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p p11* **using** *assms* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p14* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p14 p11* **by** *blast*
from *P1* **have** *P2* : *Plane-sameside* (*Li p12 p13*) *p11 p14* **by** (*simp add*:*Plane-sameside-rev*)
then have *P3* : ¬ *Line-on* (*Li p12 p13*) *p14* **by** (*simp add*:*Plane-sameside-def*)

from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P3 P4* **have** *Def* (*Ang* (*An p12 p13 p14*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P5* : *Def* (*Ang* (*An p14 p12 p13*)) **by** (*blast intro*:*Ang-def-inv Ang-def-rev*)
from *assms P1 P5* **have** *P6* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
have *P7* : *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms* **have** *P8* : *Def* (*Ang* (*An p13 p12 p11*)) **by** (*simp add*:*Ang-def-rev*)
from *assms P5 P6 P7 P8* **have** *P9* : *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P2 P7 P9* **have** *P10* : *Eq* (*Geos* (*Lin* (*Li p11 p12*)) *add Emp*) (*Geos* (*Lin* (*Li p14 p12*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p11 p14*) *p12* **by** (*simp add*:*Ang-move-unique*)
from *P1* **have** *Ang-inside* (*An p11 p12 p13*) *p14* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
then have *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Ang-inside (*An p11 p12 p13*) *p14* **by** *blast*
then have *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Plane-sameside (*Li p12 p11*) *p13 p14* **by** (*simp add*:*Ang-inside-def*)
then have *P11* : *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
¬ *Line-on* (*Li p12 p11*) *p14* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P12* : *Eq* (*Geos* (*Lin* (*Li p11 p12*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P10 P12* **have** *P13* : *Eq* (*Geos* (*Lin* (*Li p14 p12*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
have *P14* : *Line-on* (*Li p14 p12*) *p14* **by** (*simp add*:*Line-on-rule*)
from *P13 P14* **have** *P15* : *Line-on* (*Li p12 p11*) *p14* **by** (*simp add*:*Line-on-trans*)
from *P11 P15* **show** ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-not-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
shows
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
proof −

from *assms* **show** *P1* : ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr-lemma1*)
from *assms* **have** *P2* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-rev*)
from *assms P2* **show** ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr-lemma1*)
qed

lemma (**in** *Congruence-Rule*) *Ang-Gr-not-Eq-rev* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
shows
¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
proof −
from *assms* **have** ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
then have ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p p11* **using** *assms* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p14* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p14 p11* **by** *blast*
from *P1* **have** *P2* : *Plane-sameside* (*Li p12 p13*) *p11 p14* **by** (*simp add*:*Plane-sameside-rev*)
then have *P3* : ¬ *Line-on* (*Li p12 p13*) *p14* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P3 P4* **have** *Def* (*Ang* (*An p12 p13 p14*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P5* : *Def* (*Ang* (*An p14 p12 p13*)) **by** (*blast intro*:*Ang-def-inv Ang-def-rev*)
from *assms* **have** ¬ *Line-on* (*Li p22 p23*) *p21* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
then have ∃ *p*. *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p p22 p23*)) *add Emp*)
∧ *Plane-sameside* (*Li p22 p23*) *p p21* **using** *assms* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p24* :: *Point* **where** *P6* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p24 p22 p23*)) *add Emp*)
∧ *Plane-sameside* (*Li p22 p23*) *p24 p21* **by** *blast*
then have *P7* : ¬ *Line-on* (*Li p22 p23*) *p24* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P8* : ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P7 P8* **have** *Def* (*Ang* (*An p22 p23 p24*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P9* : *Def* (*Ang* (*An p24 p22 p23*)) **by** (*blast intro*:*Ang-def-inv Ang-def-rev*)
from *P6* **have** *P10* : *Plane-sameside* (*Li p22 p23*) *p21 p24* **by** (*simp add*:*Plane-sameside-rev*)
from *assms* **have** *P11* : *Def* (*Ang* (*An p13 p12 p11*)) **by** (*simp add*:*Ang-def-rev*)
from *assms* **have** *P12* : *Def* (*Ang* (*An p23 p22 p21*)) **by** (*simp add*:*Ang-def-rev*)

from *P5* **have** *P13* : *Def* (*Ang* (*An p13 p12 p14*)) **by** (*simp add*:*Ang-def-rev*)
from *P9* **have** *P14* : *Def* (*Ang* (*An p23 p22 p24*)) **by** (*simp add*:*Ang-def-rev*)
have *P15* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P16* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P17* : *Cong* (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p13 p12 p14*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P18* : *Cong* (*Geos* (*Ang* (*An p24 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p24*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P6 P9 P11 P15* **have** *P19* :
Cong (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p24 p22 p23*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P9 P11 P14 P18 P19* **have** *P20* :
Cong (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p24*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *assms P1 P5 P12 P16* **have** *P21* :
Cong (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P5 P12 P13 P17 P21* **have** *P22* :
Cong (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*) (*Geos* (*Ang* (*An p13 p12 p14*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P6* **have** *P23* : *Ang-inside* (*An p21 p22 p23*) *p24* ⟷
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *assms P23* **have** *P24* : *Ang-inside* (*An p21 p22 p23*) *p24* **by** *blast*
from *P12 P24* **have** *P25* : *Ang-inside* (*An p23 p22 p21*) *p24* **by** (*simp add*:*Ang-inside-def*)
from *P2 P6 P11 P12 P20 P22 P25* **have** *P26* : ¬ *Ang-inside* (*An p13 p12 p11*) *p14*
∧ ¬ *Eq* (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p14*)) *add Emp*) **by** (*simp add*:*Ang-move-Greater*)
from *P1* **have** *Ang-inside* (*An p11 p12 p13*) *p14* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
then have *P27* : *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Ang-inside (*An p11 p12 p13*) *p14* **by** *blast*
from *P11 P27* **have** *P28* : *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Ang-inside (*An p13 p12 p11*) *p14* **by** (*simp add*:*Ang-inside-def*)
from *P26 P28* **show** ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** *blast*
from *assms P1 P5* **have** *P29* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
have *P30* : *Cong* (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P5 P11 P29 P30* **have** *P31* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*))

add Emp) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p13 p12 p11*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P2 P30 P31* **have** *P32* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li p11 p12*)) *add Emp*) (*Geos* (*Lin* (*Li p14 p12*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se p11 p14*) *p12* **by** (*simp add*:*Ang-move-unique*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p11*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P33* : *Eq* (*Geos* (*Lin* (*Li p11 p12*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P32 P33* **have** *P34* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p14 p12*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P5* **have** ¬ *Eq* (*Geos* (*Poi p14*) *add Emp*) (*Geos* (*Poi p12*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P35* : *Eq* (*Geos* (*Lin* (*Li p14 p12*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p14*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P34 P35* **have** *P36* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li p12 p11*)) *add Emp*) (*Geos* (*Lin* (*Li p12 p14*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P26 P36* **show** ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-relation-case-fact* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*))
shows
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)

proof −
from *assms* **have** *P1* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-relation-case*)
from *assms* **have** *P2* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ==>
¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr*)
from *assms* **have** *P3* : *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) ⟹
¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-Gr-not-Eq-rev*)
from *assms* **have** *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
¬ *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-Gr-not-Eq-rev*)
then have *P4* : *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) ⟹
¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-rev*)
from *P1 P2 P3 P4* **show** *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** *blast*

qed

lemma (**in** *Congruence-Rule*) *Ang-Gr-trans-Eq-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*)) *Def* (*Ang* (*An p31 p32 p33*))
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li p22 p23*) *p21* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
from *assms P1* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p p22 p23*)) *add Emp*)
∧ *Plane-sameside* (*Li p22 p23*) *p p21* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p24* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p24 p22 p23*)) *add Emp*)
∧ *Plane-sameside* (*Li p22 p23*) *p24 p21* **by** *blast*
then have *P3* : *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p24 p22 p23*)) *add Emp*) **by** *simp*
from *P2* **have** *P4* : *Plane-sameside* (*Li p22 p23*) *p24 p21* **by** *simp*
from *assms P3 P4* **have** *Ang-inside* (*An p21 p22 p23*) *p24* **by** (*simp add*:*Ang-greater-def*)
then have *P5* : *Ang-inside* (*An p23 p22 p21*) *p24* **by** (*simp add*:*Ang-inside-def Ang-def-rev*)
from *assms* **have** *P6* : ¬ *Line-on* (*Li p32 p33*) *p31* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
from *assms P6* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p32 p33*)) *add Emp*)
∧ *Plane-sameside* (*Li p32 p33*) *p p31* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p34* :: *Point* **where** *P7* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p34 p32 p33*)) *add Emp*)
∧ *Plane-sameside* (*Li p32 p33*) *p34 p31* **by** *blast*
then have *P8* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p34 p32 p33*)) *add Emp*) **by** *simp*
from *P7* **have** *P9* : *Plane-sameside* (*Li p32 p33*) *p34 p31* **by** *simp*
from *assms* **have** *P10* : *Def* (*Ang* (*An p33 p32 p31*)) **by** (*blast intro*:*Ang-def-rev*)
from *assms* **have** *P11* : *Def* (*Ang* (*An p23 p22 p21*)) **by** (*blast intro*:*Ang-def-rev*)
from *P4* **have** *P12* : ¬ *Line-on* (*Li p22 p23*) *p24* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P13* : ¬ *Eq* (*Geos* (*Poi p22*) *add Emp*) (*Geos* (*Poi p23*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P12 P13* **have** *Def* (*Ang* (*An p22 p23 p24*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P14* : *Def* (*Ang* (*An p24 p22 p23*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have *P15* : *Def* (*Ang* (*An p23 p22 p24*)) **by** (*blast intro*:*Ang-def-rev*)
have *P16* : *Cong* (*Geos* (*Ang* (*An p24 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p23*

p22 p24)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P3 P14 P15 P16* **have** *P17* :
Cong (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p24*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
have *P18* : *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p33 p32 p31*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P10 P15 P17 P18* **have** *P19* :
Cong (*Geos* (*Ang* (*An p33 p32 p31*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p24*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P9* **have** *P20* : ¬ *Line-on* (*Li p32 p33*) *p34* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P21* : ¬ *Eq* (*Geos* (*Poi p32*) *add Emp*) (*Geos* (*Poi p33*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P20 P21* **have** *Def* (*Ang* (*An p32 p33 p34*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P22* : *Def* (*Ang* (*An p34 p32 p33*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have *P23* : *Def* (*Ang* (*An p33 p32 p34*)) **by** (*blast intro*:*Ang-def-rev*)
have *P24* : *Cong* (*Geos* (*Ang* (*An p34 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p33 p32 p34*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P8 P22 P23 P24* **have** *P25* :
Cong (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p33 p32 p34*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
have *P26* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P11 P23 P25 P26* **have** *P27* :
Cong (*Geos* (*Ang* (*An p23 p22 p21*)) *add Emp*) (*Geos* (*Ang* (*An p33 p32 p34*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P9* **have** *P28* : *Plane-sameside* (*Li p32 p33*) *p31 p34* **by** (*simp add*:*Plane-sameside-rev*)
from *P4 P5 P10 P11 P19 P27 P28* **have** *P29* : ¬ *Ang-inside* (*An p33 p32 p31*) *p34*
∧ ¬ *Eq* (*Geos* (*Lin* (*Li p32 p31*)) *add Emp*) (*Geos* (*Lin* (*Li p32 p34*)) *add Emp*) **by** (*simp add*:*Ang-move-Greater*)
have *P30* : *Ang-inside* (*An p31 p32 p33*) *p34* ⟹ *Ang-inside* (*An p33 p32 p31*) *p34* **by** (*simp add*:*Ang-inside-def Ang-def-rev*)
from *P29 P30* **have** *P31* : ¬ *Ang-inside* (*An p31 p32 p33*) *p34* **by** *blast*
from *assms P8 P22* **have** *P32* : *Cong* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p34 p32 p33*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P9 P29 P31 P32* **show** *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** (*simp add*:*Ang-less-def*)
qed

lemma (**in** *Congruence-Rule*) *Ang-Gr-trans-Gr-Eq* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*)) *Def* (*Ang* (*An p31 p32 p33*))
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
Cong (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
shows

Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
from *assms P1* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p p11* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p14* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p14 p11* **by** *blast*
then have *P3* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** *simp*
from *P2* **have** *P4* : *Plane-sameside* (*Li p12 p13*) *p14 p11* **by** *simp*
from *P3 P4* **have** *P5* : *Ang-inside* (*An p11 p12 p13*) *p14* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *assms P5* **have** *P6* : *Ang-inside* (*An p11 p12 p13*) *p14* **by** *simp*
from *P4* **have** *P7* : ¬ *Line-on* (*Li p12 p13*) *p14* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P8* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P7 P8* **have** *Def* (*Ang* (*An p12 p13 p14*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P9* : *Def* (*Ang* (*An p14 p12 p13*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P3 P9* **have** *P10* : *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P4 P10* **have** *P11* : *Ang-inside* (*An p11 p12 p13*) *p14* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P6 P11* **show** *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** *simp*
qed

lemma (**in** *Congruence-Rule*) *Ang-Eq-Point* :
assumes
Def (*Ang* (*An p1 p2 p3*))
Eq (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*)
shows
Eq (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*) ∧ *Def* (*Ang* (*An p4 p2 p3*))
proof −
have *Line-on* (*Li p2 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
then have *P1* : *Line-on* (*Li p2 p1*) *p4* **using** *assms* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *P2* : *Bet-Point* (*Se p1 p4*) *p2* ⟹ *Bet-Point* (*Se p4 p4*) *p2* **by** (*simp add*:*Bet-Point-Eq*)
have *P3* : ¬ *Bet-Point* (*Se p4 p4*) *p2* **by** (*simp add*:*Bet-end-Point*)
from *P2 P3* **have** *P4* : ¬ *Bet-Point* (*Se p1 p4*) *p2* **by** *blast*
have *P5* : *Line-on* (*Li p2 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P6* : ¬ *Bet-Point* (*Se p3 p3*) *p2* **by** (*simp add*:*Bet-end-Point*)

from *assms* **have** *P7* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *assms* **have** *P8* : *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) ⟹
Eq (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P7 P8* **have** *P9* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p4*) *add Emp*) **by** *blast*
from *assms* **have** *P10* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *assms P1 P4 P5 P6 P9 P10* **show**
Eq (*Geos* (*Ang* (*An p1 p2 p3*)) *add Emp*) (*Geos* (*Ang* (*An p4 p2 p3*)) *add Emp*) ∧ *Def* (*Ang* (*An p4 p2 p3*)) **by** (*simp add*:*Ang-Point-swap*)
qed

lemma (**in** *Congruence-Rule*) *Planeside-wrong-relation* :
assumes
Plane-diffside (*Li p1 p2*) *p3 p4*
Plane-diffside (*Li p1 p3*) *p2 p4*
Plane-sameside (*Li p1 p5*) *p3 p2*
Plane-sameside (*Li p1 p5*) *p4 p2*
shows *False*
proof −
from *assms* **have** ∃ *p*. *Bet-Point* (*Se p3 p4*) *p* ∧ *Line-on* (*Li p1 p2*) *p* ∧ ¬ *Line-on* (*Li p1 p2*) *p3* ∧ ¬ *Line-on* (*Li p1 p2*) *p4* **by** (*simp add*:*Plane-diffside-def*)
then obtain *p6* :: *Point* **where** *P1* : *Bet-Point* (*Se p3 p4*) *p6* ∧ *Line-on* (*Li p1 p2*) *p6*
∧ ¬ *Line-on* (*Li p1 p2*) *p3* ∧ ¬ *Line-on* (*Li p1 p2*) *p4* **by** *blast*
from *assms* **have** ∃ *p*. *Bet-Point* (*Se p2 p4*) *p* ∧ *Line-on* (*Li p1 p3*) *p* ∧ ¬ *Line-on* (*Li p1 p3*) *p2* ∧ ¬ *Line-on* (*Li p1 p3*) *p4* **by** (*simp add*:*Plane-diffside-def*)
then obtain *p7* :: *Point* **where** *P2* : *Bet-Point* (*Se p2 p4*) *p7* ∧ *Line-on* (*Li p1 p3*) *p7*
∧ ¬ *Line-on* (*Li p1 p3*) *p2* ∧ ¬ *Line-on* (*Li p1 p3*) *p4* **by** *blast*
then have *P3* : *Bet-Point* (*Se p2 p4*) *p7* **by** *simp*
then have *P4* : *Line-on* (*Li p2 p4*) *p7* **by** (*blast intro*:*Line-Bet-on*)
from *P3* **have** *P5* : ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p7*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
have *P6* : *Line-on* (*Li p2 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
have *P7* : *Line-on* (*Li p3 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
from *P4 P5 P6 P7* **have** *P8* : *Line-on* (*Li p3 p4*) *p7* ⟹
Eq (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P9* : *Line-on* (*Li p2 p4*) *p2* **by** (*simp add*:*Line-on-rule*)
from *P8 P9* **have** *P10* : *Line-on* (*Li p3 p4*) *p7* ⟹ *Line-on* (*Li p3 p4*) *p2* **by** (*simp add*:*Line-on-trans*)
from *P7* **have** *P11* : *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) ⟹
Line-on (*Li p1 p2*) *p4* **by** (*simp add*:*Line-on-trans*)
from *P1 P11* **have** *P12* : ¬ *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li*

p1 p2)) *add Emp*) **by** *blast*
from *P1* **have** *P13* : *Bet-Point* (*Se p3 p4*) *p6* **by** *simp*
then have *P14* : *Line-on* (*Li p3 p4*) *p6* **by** (*blast intro*:*Line-Bet-on*)
have *P15* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *P1 P10 P12 P14 P15* **have** *P16* : *Line-on* (*Li p3 p4*) *p7* ⟹
Eq (*Geos* (*Poi p6*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P13 P16* **have** *P17* : *Line-on* (*Li p3 p4*) *p7* ⟹ *Bet-Point* (*Se p3 p4*) *p2*
by (*simp add*:*Point-Eq*)
from *P7* **have** *P18* : *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1*
p3)) *add Emp*) ⟹
Line-on (*Li p1 p3*) *p4* **by** (*simp add*:*Line-on-trans*)
from *P2 P18* **have** *P19* : ¬ *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li*
p1 p3)) *add Emp*) **by** *blast*
have *P20* : *Line-on* (*Li p1 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *P17 P19 P20* **have** *Line-on* (*Li p3 p4*) *p7* ⟹ *Plane-sameside* (*Li p1 p3*)
p2 p4 **by** (*simp add*:*Plane-Bet-sameside*)
then have *Line-on* (*Li p3 p4*) *p7* ⟹ ¬ *Plane-diffside* (*Li p1 p3*) *p2 p4* **by** (*simp*
add:*Plane-sameside-not-diffside*)
then have *P21* : ¬ *Line-on* (*Li p3 p4*) *p7* **using** *assms* **by** *blast*
from *P3* **have** *P22* : ¬ *Eq* (*Geos* (*Poi p7*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*)
by (*simp add*:*Bet-Point-def*)
from *P4 P9 P15 P22* **have** *P23* : *Line-on* (*Li p1 p2*) *p7* ⟹
Eq (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp*
add:*Line-unique*)
from *P6 P23* **have** *P24* : *Line-on* (*Li p1 p2*) *p7* ⟹ *Line-on* (*Li p1 p2*) *p4* **by**
(*simp add*:*Line-on-trans*)
from *P1 P24* **have** *P25* : ¬ *Line-on* (*Li p1 p2*) *p7* **by** *blast*
from *P1 P13 P21 P25* **have** *P26* : *Line-on-Seg* (*Li p1 p2*) (*Se p3 p7*) ∧ ¬
Line-on-Seg (*Li p1 p2*) (*Se p4 p7*)
∨ *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ∧ ¬ *Line-on-Seg* (*Li p1 p2*) (*Se p3 p7*)
by (*simp add*:*Pachets-axiom*)
have *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹ ∃ *p*. *Line-on* (*Li p1 p2*) *p* ∧ *Bet-Point*
(*Se p4 p7*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *p8* :: *Point* **where** *P27* : *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹
Line-on (*Li p1 p2*) *p8* ∧ *Bet-Point* (*Se p4 p7*) *p8* **by** *blast*
then have *P28* : *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹ *Line-on* (*Li p4 p7*) *p8*
by (*blast intro*:*Line-Bet-on*)
from *P3* **have** *P29* : *Line-on* (*Li p4 p7*) *p2* **by** (*blast intro*:*Line-Bet-on*)
from *P27* **have** *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹ *Bet-Point* (*Se p4 p7*) *p8*
by *simp*
then have *P30* : *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹
Eq (*Geos* (*Poi p8*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ⟹ *Bet-Point* (*Se p4*
p7) *p2* **by** (*simp add*:*Point-Eq*)
from *P3* **have** *Inv* (*Bet-Point* (*Se p4 p7*) *p2*) **by** (*simp add*:*Bet-iff*)
then have *P31* : ¬ *Bet-Point* (*Se p4 p7*) *p2* **by** (*simp add*:*Inv-def*)
from *P30 P31* **have** *P32* : *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹
¬ *Eq* (*Geos* (*Poi p8*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** *blast*
from *P15 P27 P28 P29 P32* **have** *P33* : *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹
Eq (*Geos* (*Lin* (*Li p4 p7*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by**

(*simp add*:*Line-unique*)
have *P34* : *Line-on* (*Li p4 p7*) *p4* **by** (*simp add*:*Line-on-rule*)
from *P33 P34* **have** *P35* : *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) ⟹ *Line-on* (*Li p1 p2*) *p4* **by** (*simp add*:*Line-on-trans*)
from *P1 P35* **have** *P36* : ¬ *Line-on-Seg* (*Li p1 p2*) (*Se p4 p7*) **by** *blast*
from *P26 P36* **have** *Line-on-Seg* (*Li p1 p2*) (*Se p3 p7*) **by** *blast*
then have ∃ *p*. *Line-on* (*Li p1 p2*) *p* ∧ *Bet-Point* (*Se p3 p7*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *p8* :: *Point* **where** *P37* : *Line-on* (*Li p1 p2*) *p8* ∧ *Bet-Point* (*Se p3 p7*) *p8* **by** *blast*
have *Line-on* (*Li p3 p4*) *p3* **by** (*simp add*:*Line-on-rule*)
then have *P38* : *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p7*) *add Emp*) ⟹ *Line-on* (*Li p3 p4*) *p7* **by** (*simp add*:*Point-Eq*)
from *P21 P38* **have** *P39* : ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p7*) *add Emp*) **by** *blast*
have *P40* : *Line-on* (*Li p3 p7*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P41* : *Line-on* (*Li p3 p7*) *p7* **by** (*simp add*:*Line-on-rule*)
from *P2* **have** *P42* : *Line-on* (*Li p1 p3*) *p7* **by** *simp*
from *P20 P39 P40 P41 P42* **have** *P43* :
Eq (*Geos* (*Lin* (*Li p1 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p3 p7*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P44* : *Line-on* (*Li p1 p3*) *p1* **by** (*simp add*:*Line-on-rule*)
from *P43 P44* **have** *P45* : *Line-on* (*Li p3 p7*) *p1* **by** (*simp add*:*Line-on-trans*)
from *P37* **have** *P46* : *Line-on* (*Li p3 p7*) *p8* **by** (*blast intro*:*Line-Bet-on*)
have *P47* : *Line-on* (*Li p1 p2*) *p1* **by** (*simp add*:*Line-on-rule*)
from *P40* **have** *P48* : *Eq* (*Geos* (*Lin* (*Li p3 p7*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) ⟹
Line-on (*Li p1 p2*) *p3* **by** (*simp add*:*Line-on-trans*)
from *P1 P48* **have** *P49* : ¬ *Eq* (*Geos* (*Lin* (*Li p3 p7*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** *blast*
from *P37 P45 P46 P48 P47 P49* **have** *P50* : *Eq* (*Geos* (*Poi p8*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P37* **have** *P51* : *Bet-Point* (*Se p3 p7*) *p8* **by** *simp*
from *P50 P51* **have** *P52* : *Bet-Point* (*Se p3 p7*) *p1* **by** (*simp add*:*Point-Eq*)
from *P44* **have** *P53* : *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) ⟹ *Line-on* (*Li p1 p3*) *p2* **by** (*simp add*:*Point-Eq*)
from *P2 P53* **have** *P54* : ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** *blast*
have *P55* : *Line-on* (*Li p1 p2*) *p2* **by** (*simp add*:*Line-on-rule*)
from *P45 P47 P54 P55* **have** *P56* : *Line-on* (*Li p3 p7*) *p2* ⟹
Eq (*Geos* (*Lin* (*Li p3 p7*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p2*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P49 P56* **have** *P57* : ¬ *Line-on* (*Li p3 p7*) *p2* **by** *blast*
from *assms* **have** *P58* : ¬ *Line-on-Seg* (*Li p1 p5*) (*Se p3 p2*) ∧ ¬ *Line-on* (*Li p1 p5*) *p3*
∧ ¬ *Line-on* (*Li p1 p5*) *p2* ∧ ¬ *Eq* (*Geos* (*Poi p3*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Plane-sameside-def*)
from *P52* **have** *P59* : ¬ *Eq* (*Geos* (*Poi p7*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)

have *P60* : *Line-on* (*Li p1 p5*) *p1* **by** (*simp add*:*Line-on-rule*)
from *P41 P45 P59 P60* **have** *P61* : *Line-on* (*Li p1 p5*) *p7* ⟹
Eq (*Geos* (*Lin* (*Li p3 p7*)) *add Emp*) (*Geos* (*Lin* (*Li p1 p5*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P40 P61* **have** *P62* : *Line-on* (*Li p1 p5*) *p7* ⟹ *Line-on* (*Li p1 p5*) *p3* **by** (*simp add*:*Line-on-trans*)
from *P58 P62* **have** *P63* : ¬ *Line-on* (*Li p1 p5*) *p7* **by** *blast*
from *P58* **have** *P64* : ¬ *Line-on* (*Li p1 p5*) *p3* **by** *simp*
from *P58* **have** *P65* : ¬ *Line-on* (*Li p1 p5*) *p2* **by** *simp*
from *P52 P57 P60 P63 P64 P65* **have** *P66* : *Line-on-Seg* (*Li p1 p5*) (*Se p3 p2*) ∧ ¬ *Line-on-Seg* (*Li p1 p5*) (*Se p7 p2*)
∨ *Line-on-Seg* (*Li p1 p5*) (*Se p7 p2*) ∧ ¬ *Line-on-Seg* (*Li p1 p5*) (*Se p3 p2*) **by** (*simp add*:*Pachets-axiom*)
from *P58 P66* **have** *Line-on-Seg* (*Li p1 p5*) (*Se p7 p2*) **by** *blast*
then have ∃ *p*. *Line-on* (*Li p1 p5*) *p* ∧ *Bet-Point* (*Se p7 p2*) *p* **by** (*simp add*:*Line-on-Seg-rule*)
then obtain *p9* :: *Point* **where** *P67* : *Line-on* (*Li p1 p5*) *p9* ∧ *Bet-Point* (*Se p7 p2*) *p9* **by** *blast*
then have *P68* : *Bet-Point* (*Se p2 p7*) *p9* **by** (*simp add*:*Bet-rev*)
from *P3 P68* **have** *Bet-Point* (*Se p2 p4*) *p9* **by** (*blast intro*:*Bet-swap-134-124*)
then have *P69* : *Bet-Point* (*Se p4 p2*) *p9* **by** (*simp add*:*Bet-rev*)
from *P67 P69* **have** ∃ *p*. *Line-on* (*Li p1 p5*) *p* ∧ *Bet-Point* (*Se p4 p2*) *p* **by** *blast*
then have *P70* : *Line-on-Seg* (*Li p1 p5*) (*Se p4 p2*) **by** (*simp add*:*Line-on-Seg-rule*)
from *assms* **have** *P71* : ¬ *Line-on-Seg* (*Li p1 p5*) (*Se p4 p2*) **by** (*simp add*:*Plane-sameside-def*)
from *P70 P71* **show** *False* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-Gr-trans-Gr-Gr* :
assumes
Def (*Ang* (*An p11 p12 p13*)) *Def* (*Ang* (*An p21 p22 p23*)) *Def* (*Ang* (*An p31 p32 p33*))
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*)
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
from *assms P1* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p p11* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p14* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p14 p11* **by** *blast*
from *assms P1* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p p12 p13*)) *add Emp*)

∧ *Plane-sameside* (*Li p12 p13*) *p p11* **by** (*simp add*:*Ang-move-sameside*)
then obtain *p15* :: *Point* **where** *P3* : *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*)
∧ *Plane-sameside* (*Li p12 p13*) *p15 p11* **by** *blast*
from *P2* **have** *P4* : *Cong* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) **by** *simp*
from *P2* **have** *P5* : *Plane-sameside* (*Li p12 p13*) *p14 p11* **by** *simp*
from *P3* **have** *P6* : *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*) **by** *simp*
from *P3* **have** *P7* : *Plane-sameside* (*Li p12 p13*) *p15 p11* **by** *simp*
from *P4 P5* **have** *Ang-inside* (*An p11 p12 p13*) *p14* ⟷
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
then have *P8* : *Ang-inside* (*An p11 p12 p13*) *p14* **using** *assms* **by** *blast*
from *P5* **have** *P9* : ¬ *Line-on* (*Li p12 p13*) *p14* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P10* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P9 P10* **have** *Def* (*Ang* (*An p12 p13 p14*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P11* : *Def* (*Ang* (*An p14 p12 p13*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P4 P11* **have** *P12* : *Gr* (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
from *P11* **have** *P13* : *Eq* (*Geos* (*Poi p14*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) ⟹
Eq (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*) ∧ *Def* (*Ang* (*An p15 p12 p13*)) **by** (*simp add*:*Ang-Eq-Point*)
then have *P14* : *Eq* (*Geos* (*Poi p14*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
from *assms P4 P11 P13 P14* **have** *P15* : *Eq* (*Geos* (*Poi p14*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *assms P6 P13 P15* **have** *P16* : *Eq* (*Geos* (*Poi p14*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *assms* **have** *P17* : ¬ *Cong* (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) **by** (*simp add*:*Ang-Gr-not-Eq-rev*)
from *P16 P17* **have** *P18* : ¬ *Eq* (*Geos* (*Poi p15*) *add Emp*) (*Geos* (*Poi p14*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P7* **have** *P19* : *Plane-sameside* (*Li p12 p13*) *p11 p15* **by** (*simp add*:*Plane-sameside-rev*)
from *P5 P18 P19* **have** *P20* : *Plane-sameside* (*Li p12 p13*) *p15 p14* **by** (*blast intro*:*Plane-sameside-trans Plane-sameside-rev*)
from *P6 P20* **have** *P21* : *Ang-inside* (*An p14 p12 p13*) *p15* ⟷
Gr (*Geos* (*Ang* (*An p14 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P12 P21* **have** *Ang-inside* (*An p14 p12 p13*) *p15* **by** *simp*
then have *P22* : *Plane-sameside* (*Li p12 p14*) *p13 p15* ∧ *Plane-sameside* (*Li*

p12 p13) *p14 p15* **by** (*simp add*:*Ang-inside-def*)
from *P8* **have** *P23* : *Plane-sameside* (*Li p12 p11*) *p13 p14* ∧ *Plane-sameside* (*Li p12 p13*) *p11 p14* **by** (*simp add*:*Ang-inside-def*)
then have *P24* : *Plane-diffside* (*Li p12 p11*) *p13 p15* ⟹ *Plane-diffside* (*Li p12 p11*) *p14 p15* **by** (*blast intro*:*Plane-trans*)
from *P8* **have** *P25* : *Plane-diffside* (*Li p12 p14*) *p11 p13* **by** (*simp add*:*Ang-inside-Planeside*)
from *P22 P25* **have** *P26* : *Plane-diffside* (*Li p12 p14*) *p11 p15* **by** (*blast intro*:*Plane-trans Plane-diffside-rev*)
from *P5 P7 P24 P26* **have** *Plane-diffside* (*Li p12 p11*) *p13 p15* ⟹ *False* **by** (*blast intro*:*Planeside-wrong-relation*)
then have *P27* : ¬ *Plane-diffside* (*Li p12 p11*) *p13 p15* **by** *blast*
from *P23* **have** *P28* : ¬ *Line-on* (*Li p12 p11*) *p13* **by** (*simp add*:*Plane-sameside-def*)
have *P29* : *Line-on* (*Li p12 p13*) *p12* **by** (*simp add*:*Line-on-rule*)
from *P19* **have** *P30* : ¬ *Line-on* (*Li p12 p13*) *p11* **by** (*simp add*:*Plane-sameside-def*)
from *P19* **have** *P31* : ¬ *Line-on* (*Li p12 p13*) *p15* **by** (*simp add*:*Plane-sameside-def*)
from *P29 P30 P31* **have** *Bet-Point* (*Se p11 p15*) *p12* ⟹ ∃ *p*. *Bet-Point* (*Se p11 p15*) *p* ∧
Line-on (*Li p12 p13*) *p* ∧ ¬ *Line-on* (*Li p12 p13*) *p11* ∧ ¬ *Line-on* (*Li p12 p13*) *p15* **by** *blast*
then have *Bet-Point* (*Se p11 p15*) *p12* ⟹ *Plane-diffside* (*Li p12 p13*) *p11 p15* **by** (*simp add*:*Plane-diffside-def*)
then have *P32* : *Bet-Point* (*Se p11 p15*) *p12* ⟹ ¬ *Plane-sameside* (*Li p12 p13*) *p15 p11* **by** (*simp add*:*Plane-diffside-rev Plane-diffside-not-sameside*)
from *P7 P32* **have** *P33* : ¬ *Bet-Point* (*Se p11 p15*) *p12* **by** *blast*
have *P34* : *Line-on* (*Li p12 p13*) *p13* **by** (*simp add*:*Line-on-rule*)
have *P35* : ¬ *Bet-Point* (*Se p13 p13*) *p12* **by** (*simp add*:*Bet-end-Point*)
from *P29* **have** *P36* : *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) ⟹
Line-on (*Li p12 p13*) *p15* **by** (*simp add*:*Point-Eq*)
from *P31 P36* **have** *P37* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) **by** *blast*
from *assms* **have** *P38* : ¬ *Eq* (*Geos* (*Poi p12*) *add Emp*) (*Geos* (*Poi p13*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *assms P33 P34 P35 P37 P38* **have** *P39* : *Line-on* (*Li p12 p11*) *p15* ⟹
Eq (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*) ∧ *Def* (*Ang* (*An p15 p12 p13*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P40* : *Line-on* (*Li p12 p11*) *p15* ⟹
Cong (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p15 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
from *assms P6 P39 P40* **have** *P41* : *Line-on* (*Li p12 p11*) *p15* ⟹
Cong (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *assms P41* **have** *P42* : *Line-on* (*Li p12 p11*) *p15* ⟹
Gr (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq*)
from *assms* **have** *P43* : ¬ *Gr* (*Geos* (*Ang* (*An p21 p22 p23*)) *add Emp*) (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) **by** (*simp add*:*Ang-Gr-not-Eq-rev*)
from *P42 P43* **have** *P44*: ¬ *Line-on* (*Li p12 p11*) *p15* **by** *blast*
from *P34* **have** *P45* : *Eq* (*Geos* (*Poi p13*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*)

$\Longrightarrow$
Line-on (*Li p12 p13*) *p15* **by** (*simp add*:*Point-Eq*)
from *P31 P45* **have** *P46* : ¬ *Eq* (*Geos* (*Poi p13*) *add Emp*) (*Geos* (*Poi p15*) *add Emp*) **by** *blast*
from *P27 P28 P44 P46* **have** *P47* : *Plane-sameside* (*Li p12 p11*) *p13 p15* **by** (*simp add*:*Plane-not-diffside-sameside*)
from *assms P19 P47* **have** *P48* : *Ang-inside* (*An p11 p12 p13*) *p15* **by** (*simp add*:*Ang-inside-def*)
from *P6 P7* **have** *P49* : *Ang-inside* (*An p11 p12 p13*) *p15* $\longleftrightarrow$
Gr (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P48 P49* **show** *Gr* (*Geos* (*Ang* (*An p11 p12 p13*)) *add Emp*) (*Geos* (*Ang* (*An p31 p32 p33*)) *add Emp*) **by** *simp*
qed

lemma (**in** *Congruence-Rule*) *Ang-complementary-inside* :
assumes
Def (*Ang* (*An p1 p2 p3*))
Bet-Point (*Se p3 p4*) *p2*
Ang-inside (*An p5 p2 p3*) *p1*
shows
Ang-inside (*An p1 p2 p4*) *p5*
proof −
from *assms* **have** *P1* : *Plane-sameside* (*Li p2 p5*) *p3 p1* ∧ *Plane-sameside* (*Li p2 p3*) *p5 p1* **by** (*simp add*:*Ang-inside-def*)
from *assms* **have** *P2* : *Line-on* (*Li p2 p3*) *p4* **by** (*simp add*:*Line-Bet-on*)
have *P3* : *Line-on* (*Li p2 p3*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P4* : *Line-on* (*Li p2 p4*) *p2* **by** (*simp add*:*Line-on-rule*)
have *P5* : *Line-on* (*Li p2 p4*) *p4* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P6* : ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P2 P3 P4 P5 P6* **have** *P7* : *Eq* (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p4*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P1 P7* **have** *P8* : *Plane-sameside* (*Li p2 p4*) *p5 p1* **by** (*blast intro*:*Plane-Line-trans*)
have *P9* : *Line-on* (*Li p2 p1*) *p2* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li p2 p1*) *p1* **by** (*simp add*:*Line-on-rule*)
then have *P10* : *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) $\Longrightarrow$
Line-on (*Li p3 p4*) *p1* **by** (*blast intro*:*Eq-rev Line-on-trans*)
from *P1* **have** *P11* : ¬ *Line-on* (*Li p2 p3*) *p1* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** *P12* : *Line-on* (*Li p3 p4*) *p2* **by** (*simp add*:*Line-Bet-on*)
have *P13* : *Line-on* (*Li p3 p4*) *p3* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li p2 p3*) *p3* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P15* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p3*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P3 P12 P13 P14 P15* **have** *P16* : *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p3*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P10 P16* **have** *P17* : *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) $\Longrightarrow$

Line-on (*Li p2 p3*) *p1* **by** (*simp add*:*Line-on-trans*)
from *P11 P17* **have** *P18* : ¬ *Eq* (*Geos* (*Lin* (*Li p3 p4*)) *add Emp*) (*Geos* (*Lin* (*Li p2 p1*)) *add Emp*) **by** *blast*
from *assms P9 P18* **have** *P19* : *Plane-diffside* (*Li p2 p1*) *p3 p4* **by** (*simp add*:*Plane-Bet-diffside*)
from *assms* **have** *P20* : *Plane-diffside* (*Li p2 p1*) *p3 p5* **by** (*simp add*:*Ang-inside-Planeside Plane-diffside-rev*)
from *P5* **have** *P21* : *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) ⟹ *Line-on* (*Li p2 p4*) *p5* **by** (*simp add*:*Point-Eq*)
from *P8* **have** *P22* : ¬ *Line-on* (*Li p2 p4*) *p5* **by** (*simp add*:*Plane-sameside-def*)
from *P21 P22* **have** *P23* : ¬ *Eq* (*Geos* (*Poi p4*) *add Emp*) (*Geos* (*Poi p5*) *add Emp*) **by** *blast*
from *P19 P20 P23* **have** *P24* : *Plane-sameside* (*Li p2 p1*) *p4 p5* **by** (*simp add*:*Plane-trans-inv*)
then have *P25* : ¬ *Line-on* (*Li p2 p1*) *p4* **by** (*simp add*:*Plane-sameside-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi p1*) *add Emp*) (*Geos* (*Poi p2*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P26* : ¬ *Eq* (*Geos* (*Poi p2*) *add Emp*) (*Geos* (*Poi p1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P25 P26* **have** *Def* (*Ang* (*An p2 p1 p4*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P27* : *Def* (*Ang* (*An p1 p2 p4*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P8* **have** *P28* : *Plane-sameside* (*Li p2 p4*) *p1 p5* **by** (*simp add*:*Plane-sameside-rev*)
from *P24 P27 P28* **show** *Ang-inside* (*An p1 p2 p4*) *p5* **by** (*simp add*:*Ang-inside-def*)
qed

Theorem21

theorem (**in** *Congruence-Rule*) *Ang-Right-angle-Cong* :
assumes
Right-angle (*An l1 o1 h1*) *Right-angle* (*An l2 o2 h2*)
shows
Cong (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*)
proof −
from *assms* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 p*)) *add Emp*)
∧ *Bet-Point* (*Se h1 p*) *o1* ∧ *Def* (*Ang* (*An l1 o1 h1*)) ∧ *Def* (*Ang* (*An l1 o1 p*)) **by** (*simp add*:*Ang-Right-angle-def*)
then obtain *k1* :: *Point* **where** *P1* : *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*)
∧ *Bet-Point* (*Se h1 k1*) *o1* ∧ *Def* (*Ang* (*An l1 o1 h1*)) ∧ *Def* (*Ang* (*An l1 o1 k1*)) **by** *blast*
from *assms* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 p*)) *add Emp*)
∧ *Bet-Point* (*Se h2 p*) *o2* ∧ *Def* (*Ang* (*An l2 o2 h2*)) ∧ *Def* (*Ang* (*An l2 o2 p*)) **by** (*simp add*:*Ang-Right-angle-def*)
then obtain *k2* :: *Point* **where** *P2* : *Cong* (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*)
∧ *Bet-Point* (*Se h2 k2*) *o2* ∧ *Def* (*Ang* (*An l2 o2 h2*)) ∧ *Def* (*Ang* (*An l2 o2 k2*)) **by** *blast*

from *P1* **have** *P3* : ¬ *Line-on* (*Li o1 h1*) *l1* **by** (*simp add*:*Ang-to-Tri Tri-def-Line*)
from *P1 P2 P3* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An p o1 h1*)) *add Emp*)
∧ *Plane-sameside* (*Li o1 h1*) *p l1* **by** (*simp add*:*Ang-move-sameside*)
then obtain *l11* :: *Point* **where** *P4* : *Cong* (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*)
∧ *Plane-sameside* (*Li o1 h1*) *l11 l1* **by** *blast*
then have *P5* : ¬ *Line-on* (*Li o1 h1*) *l11* **by** (*simp add*:*Plane-sameside-def*)
from *P1* **have** *P6* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi h1*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P5 P6* **have** *Def* (*Ang* (*An o1 h1 l11*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P7* : *Def* (*Ang* (*An l11 o1 h1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P2 P4 P7* **have** *P8* : *Cong* (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P2* **have** *P9* : *Def* (*Ang* (*An h2 o2 l2*)) **by** (*simp add*:*Ang-def-rev*)
from *P7* **have** *P10* : *Def* (*Ang* (*An h1 o1 l11*)) **by** (*simp add*:*Ang-def-rev*)
have *P11* : *Cong* (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P12* : *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l11*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P2 P4 P7 P9 P11* **have** *P13* : *Cong* (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P7 P9 P10 P12 P13* **have** *P14* : *Cong* (*Geos* (*Ang* (*An h2 o2 l2*)) *add Emp*) (*Geos* (*Ang* (*An h1 o1 l11*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P1 P2 P9 P10 P14* **have** *P15* : *Cong* (*Geos* (*Ang* (*An l2 o2 k2*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-complementary*)
from *P1 P4 P7* **have** *P16* : *Eq* (*Geos* (*Lin* (*Li o1 l11*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** (*simp add*:*Ang-move-unique-inv*)
from *P1 P2 P4 P7 P16* **have** *Eq* (*Geos* (*Lin* (*Li o1 l11*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) ⟹
Cong (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
then have *P17* : ¬ *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Lin* (*Li o1 l11*)) *add Emp*) (*Geos* (*Lin* (*Li o1 l1*)) *add Emp*) **by** *blast*
from *P1 P4 P7 P17* **have** *P18* : ¬ *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) ⟹
Ang-inside (*An l11 o1 h1*) *l1* ∧ ¬ *Ang-inside* (*An l1 o1 h1*) *l11*
∨ ¬ *Ang-inside* (*An l11 o1 h1*) *l1* ∧ *Ang-inside* (*An l1 o1 h1*) *l11* **by** (*simp add*:*Ang-inside-case*)
from *P4* **have** *P19* : *Plane-sameside* (*Li o1 h1*) *l1 l11* **by** (*simp add*:*Plane-sameside-rev*)
have *P20* : *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** *simp*
from *P19 P20* **have** *P21* : *Ang-inside* (*An l11 o1 h1*) *l1* ⟹
Gr (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)

from *P1* **have** *P22* : *Line-on* (*Li o1 h1*) *k1* **by** (*simp add*:*Line-Bet-on*)
have *P23* : *Line-on* (*Li o1 h1*) *o1* **by** (*simp add*:*Line-on-rule*)
have *P24* : *Line-on* (*Li o1 k1*) *k1* **by** (*simp add*:*Line-on-rule*)
have *P25* : *Line-on* (*Li o1 k1*) *o1* **by** (*simp add*:*Line-on-rule*)
from *P1* **have** *P26* : *Bet-Point* (*Se h1 k1*) *o1* **by** *simp*
then have *P27* : ¬ *Eq* (*Geos* (*Poi k1*) *add Emp*) (*Geos* (*Poi o1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P22 P23 P24 P25 P27* **have** *P28* : *Eq* (*Geos* (*Lin* (*Li o1 h1*)) *add Emp*) (*Geos* (*Lin* (*Li o1 k1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P19 P28* **have** *P29* : *Plane-sameside* (*Li o1 k1*) *l1 l11* **by** (*simp add*:*Plane-Line-trans*)
then have *P30* : *Plane-sameside* (*Li o1 k1*) *l11 l1* **by** (*simp add*:*Plane-sameside-rev*)
then have *P31* : ¬ *Line-on* (*Li o1 k1*) *l11* **by** (*simp add*:*Plane-sameside-def*)
from *P1* **have** *P32* : ¬ *Eq* (*Geos* (*Poi o1*) *add Emp*) (*Geos* (*Poi k1*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P31 P32* **have** *Def* (*Ang* (*An o1 k1 l11*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P33* : *Def* (*Ang* (*An l11 o1 k1*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P2 P7 P8 P15 P33* **have** *P34* : *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** (*blast intro*:*Ang-rev Ang-trans*)
from *P1* **have** *P35* : *Def* (*Ang* (*An l1 o1 h1*)) **by** *simp*
from *P26 P35* **have** *P36* : *Ang-inside* (*An l11 o1 h1*) *l1* ⟹ *Ang-inside* (*An l1 o1 k1*) *l11* **by** (*simp add*:*Ang-complementary-inside*)
have *P37* : *Cong* (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** *simp*
from *P30 P36 P37* **have** *P38* : *Ang-inside* (*An l11 o1 h1*) *l1* ⟹ *Gr* (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P1 P7 P21* **have** *P39* : *Ang-inside* (*An l11 o1 h1*) *l1* ⟹ *Gr* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq*)
from *P1 P7 P33 P38 P39* **have** *P40* : *Ang-inside* (*An l11 o1 h1*) *l1* ⟹ *Gr* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Gr*)
from *P7 P33* **have** *P41* :
Cong (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*)

∧ *Gr* (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*simp add*:*Ang-relation-case-fact*)
from *P40 P41* **have** *P42* : *Ang-inside* (*An l11 o1 h1*) *l1* ⟹
¬ *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** *blast*
from *P34 P42* **have** *P43* : ¬ *Ang-inside* (*An l11 o1 h1*) *l1* **by** *blast*
have *P44* : *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** *simp*
from *P4 P44* **have** *P45* : *Ang-inside* (*An l1 o1 h1*) *l11* ⟹
Gr (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P1 P7* **have** *P46* : *Ang-inside* (*An l1 o1 h1*) *l11* ⟹ *Ang-inside* (*An l11 o1 k1*) *l1* **by** (*simp add*:*Ang-complementary-inside*)
have *P47* : *Cong* (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** *simp*
from *P29 P46 P47* **have** *P48* : *Ang-inside* (*An l1 o1 h1*) *l11* ⟹
Gr (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 k1*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *P1 P33 P48* **have** *P49* : *Ang-inside* (*An l1 o1 h1*) *l11* ⟹
Gr (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P1 P7 P33 P45 P49* **have** *P50* : *Ang-inside* (*An l1 o1 h1*) *l11* ⟹
Gr (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Gr*)
from *P41 P50* **have** *P51* : *Ang-inside* (*An l1 o1 h1*) *l11* ⟹
¬ *Cong* (*Geos* (*Ang* (*An l11 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l11 o1 k1*)) *add Emp*) **by** *blast*
from *P34 P51* **have** *P52* : ¬ *Ang-inside* (*An l1 o1 h1*) *l11* **by** *blast*
from *P18 P43 P52* **show** *Cong* (*Geos* (*Ang* (*An l1 o1 h1*)) *add Emp*) (*Geos* (*Ang* (*An l2 o2 h2*)) *add Emp*) **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-external-Gr-lemma1* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B D*) *A*
shows ¬ *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
proof
assume *W* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
from *N* **have** *P1* : *Line-on* (*Li B D*) *A* **by** (*simp add*:*Line-Bet-on*)
have *P2* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
from *N* **have** ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P3* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *N* **have** *P4* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Tri-def*)

then have *P5* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P2 P3 P5* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se C B*)) *add Emp*) (*Geos* (*Seg* (*Se A p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p D*) *A* ∧ *Line-on* (*Li B D*) *p* ∧ ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *D1* :: *Point* **where** *P6* : *Eq* (*Geos* (*Seg* (*Se C B*)) *add Emp*) (*Geos* (*Seg* (*Se A D1*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se D1 D*) *A* ∧ *Line-on* (*Li B D*) *D1* ∧ ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** *blast*
have *P7* : *Line-on* (*Li A D*) *A* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P8* : *Line-on* (*Li A D*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P9* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
have *P10* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P11* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P7 P8 P9 P10 P11* **have** *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P12* : *Line-on* (*Li A D*) *C* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *N* **have** *P13* : ¬ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Tri-def-Line*)
from *P12 P13* **have** *P14* : ¬ *Line-on* (*Li A D*) *C* **by** *blast*
from *P3 P14* **have** *Def* (*Ang* (*An A D C*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P15* : *Def* (*Ang* (*An C A D*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
have *P16* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P17* : ¬ *Bet-Point* (*Se C C*) *A* **by** (*simp add*:*Bet-end-Point*)
have *P18* : *Line-on* (*Li A D*) *D* **by** (*simp add*:*Line-on-rule*)
from *P1 P2 P3 P7 P18* **have** *P19* : *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li A D*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P6 P19* **have** *P20* : *Line-on* (*Li A D*) *D1* **by** (*simp add*:*Line-on-trans*)
from *P6* **have** *P21* : ¬ *Bet-Point* (*Se D D1*) *A* **by** (*blast intro*:*Bet-rev*)
from *N* **have** ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P22* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P6 P15 P16 P17 P20 P21 P22* **have** *P23* :
Eq (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An C A D1*)) *add Emp*)
∧ *Def* (*Ang* (*An C A D1*)) **by** (*simp add*:*Ang-Point-swap*)
have *P24* : *Cong* (*Geos* (*Ang* (*An C A D1*)) *add Emp*) (*Geos* (*Ang* (*An C A D1*)) *add Emp*) **by** *simp*
from *P23 P24* **have** *P25* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An C A D1*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
from *P23* **have** *P26* : *Def* (*Tri* (*Tr A C D1*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *N* **have** *P27* : *Def* (*Tri* (*Tr C A B*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
have *P28* : *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se C A*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P6* **have** *P29* : *Eq* (*Geos* (*Seg* (*Se A D1*)) *add Emp*) (*Geos* (*Seg* (*Se C B*))

add Emp) **by** (*simp add*:*Eq-rev*)
from *P29 P26 P27 P28* **have**
Cong (*Geos* (*Ang* (*An C A D1*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
$\Longrightarrow$
Cong (*Geos* (*Tri* (*Tr A C D1*)) *add Emp*) (*Geos* (*Tri* (*Tr C A B*)) *add Emp*)
by (*simp add*:*Tri-SAS*)
then have *P29* : *Cong* (*Geos* (*Ang* (*An C A D1*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Ang* (*An D1 C A*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*)
by (*simp add*:*Tri-Cong-def*)
from *N* **have** *P30* : *Def* (*Ang* (*An A C B*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-inv*)
from *W P15 P23 P25 P30* **have** *P31* : *Cong* (*Geos* (*Ang* (*An C A D1*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P29 P31* **have** *P32* : *Cong* (*Geos* (*Ang* (*An D1 C A*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** *simp*
have *P33* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P34* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *P4 P33 P34* **have** $\exists$ *p*. *Bet-Point* (*Se B p*) *C* $\wedge$ *Line-on* (*Li B C*) *p* **by** (*simp add*:*Bet-extension*)
then obtain *E* :: *Point* **where** *P35* : *Bet-Point* (*Se B E*) *C* $\wedge$ *Line-on* (*Li B C*) *E* **by** *blast*
have *P36* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An D A C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P15* **have** *P37* : *Def* (*Ang* (*An D A C*)) **by** (*simp add*:*Ang-def-rev*)
have *P38* : *Cong* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P30* **have** *P39* : *Def* (*Ang* (*An B C A*)) **by** (*simp add*:*Ang-def-rev*)
from *W P15 P30 P36 P37 P38* **have** *P40* : *Cong* (*Geos* (*Ang* (*An D A C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P30 P37 P38 P39 P40* **have** *P41* : *Cong* (*Geos* (*Ang* (*An D A C*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *N* **have** *P42* : *Bet-Point* (*Se D B*) *A* **by** (*simp add*:*Bet-rev*)
from *P35 P37 P39 P41 P42* **have** *P43* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An A C E*)) *add Emp*) **by** (*simp add*:*Ang-complementary*)
have *P44* : *Cong* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An C A B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P39* **have** *P45* : *Def* (*Ang* (*An B A C*)) **by** (*simp add*:*Ang-def-inv*)
then have *P46* : *Def* (*Ang* (*An C A B*)) **by** (*simp add*:*Ang-def-rev*)
from *P23* **have** *P47* : *Def* (*Ang* (*An D1 C A*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P32 P44 P45 P46 P47* **have** *P48* : *Cong* (*Geos* (*Ang* (*An D1 C A*)) *add Emp*) (*Geos* (*Ang* (*An C A B*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P35* **have** *P49* : *Bet-Point* (*Se B E*) *C* **by** *simp*
then have *P50* : $\neg$ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P35* **have** *P51* : *Line-on* (*Li B C*) *E* **by** *simp*
from *P16 P34 P50 P51* **have** *P52* : *Line-on* (*Li A C*) *E* $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*simp add*:*Line-unique*)

from *P33 P52* **have** *P53* : *Line-on* (*Li A C*) *E* $\Longrightarrow$ *Line-on* (*Li A C*) *B* **by** (*simp add*:*Line-on-trans*)
from *N* **have** *Def* (*Tri* (*Tr A C B*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
then have *P54* : $\neg$ *Line-on* (*Li A C*) *B* **by** (*simp add*:*Tri-def-Line*)
from *P53 P54* **have** *P55* : $\neg$ *Line-on* (*Li A C*) *E* **by** *blast*
from *P22 P55* **have** *P56* : *Def* (*Ang* (*An A C E*)) **by** (*simp add*:*Ang-sinple-def*)
from *P43 P46 P47 P48 P56* **have** *P57* : *Cong* (*Geos* (*Ang* (*An D1 C A*)) *add Emp*) (*Geos* (*Ang* (*An A C E*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
then have *P58* : *Cong* (*Geos* (*Ang* (*An A C E*)) *add Emp*) (*Geos* (*Ang* (*An D1 C A*)) *add Emp*) **by** (*simp add*:*Ang-rev*)
have *P59* : *Cong* (*Geos* (*Ang* (*An A C E*)) *add Emp*) (*Geos* (*Ang* (*An E C A*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
have *P60* : *Line-on* (*Li C A*) *A* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li B D*) *B* **by** (*simp add*:*Line-on-rule*)
then have *P61* : *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li C A*) *B* **by** (*simp add*:*Line-on-trans*)
from *N* **have** *P62* : $\neg$ *Line-on* (*Li C A*) *B* **by** (*simp add*:*Tri-def-Line*)
from *P61 P62* **have** *P63* : $\neg$ *Eq* (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** *blast*
from *N P60 P63* **have** *P64* : *Plane-diffside* (*Li C A*) *B D* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P65* : *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) $\Longrightarrow$
Plane-diffside (*Li C A*) *B D1* **by** (*simp add*:*Point-Eq*)
from *P6* **have** *P66* : *Line-on* (*Li B D*) *D1* **by** *simp*
from *P6* **have** *P67* : $\neg$ *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P2 P3 P66 P67* **have** *P68* : $\neg$ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se D A*) *D1* $\vee$ *Bet-Point* (*Se A D1*) *D* $\vee$ *Bet-Point* (*Se D1 D*) *A* **by** (*simp add*:*Bet-case*)
from *P19* **have** *P69* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) $\Longrightarrow$
Eq (*Geos* (*Lin* (*Li B D*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P63 P69* **have** *P70* : $\neg$ *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** *blast*
from *P60 P70* **have** *P71* : *Bet-Point* (*Se A D*) *D1* $\Longrightarrow$ *Plane-sameside* (*Li C A*) *D D1* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
have *Line-on* (*Li A D1*) *D1* **by** (*simp add*:*Line-on-rule*)
then have *P72* : *Eq* (*Geos* (*Lin* (*Li A D1*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) $\Longrightarrow$
Line-on (*Li C A*) *D1* **by** (*simp add*:*Line-on-trans*)
from *P23* **have** *Def* (*Tri* (*Tr C A D1*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P73* : $\neg$ *Line-on* (*Li C A*) *D1* **by** (*simp add*:*Tri-def-Line*)
from *P72 P73* **have** *P74* : $\neg$ *Eq* (*Geos* (*Lin* (*Li A D1*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** *blast*
from *P60 P74* **have** *P75* : *Bet-Point* (*Se A D1*) *D* $\Longrightarrow$ *Plane-sameside* (*Li C A*) *D D1* **by** (*simp add*:*Plane-Bet-sameside*)
from *P6 P64 P68 P71 P75* **have** *P76* : $\neg$ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) $\Longrightarrow$

Plane-diffside (*Li C A*) *B D1* **by** (*blast intro*:*Plane-trans Plane-diffside-rev Bet-rev*)
from *P65 P76* **have** *P77* : *Plane-diffside* (*Li C A*) *B D1* **by** *blast*
have *P78* : *Line-on* (*Li C A*) *C* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li B E*) *B* **by** (*simp add*:*Line-on-rule*)
then have *P79* : *Eq* (*Geos* (*Lin* (*Li B E*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) ⟹ *Line-on* (*Li C A*) *B* **by** (*simp add*:*Line-on-trans*)
from *P62 P79* **have** *P80* : ¬ *Eq* (*Geos* (*Lin* (*Li B E*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** *blast*
from *P49 P78 P80* **have** *P81* : *Plane-diffside* (*Li C A*) *B E* **by** (*simp add*:*Plane-Bet-diffside*)
from *P51* **have** *P82* : *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B C*) *D1* **by** (*blast intro*:*Eq-rev Point-Eq*)
from *P77* **have** ∃ *p*. *Bet-Point* (*Se B D1*) *p* ∧ *Line-on* (*Li C A*) *p* ∧ ¬ *Line-on* (*Li C A*) *B* ∧ ¬ *Line-on* (*Li C A*) *D1* **by** (*simp add*:*Plane-diffside-def*)
then obtain *F* :: *Point* **where** *Bet-Point* (*Se B D1*) *F* **by** *blast*
then have *P83* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P8 P20 P33 P82 P83* **have** *P84* : *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹
Eq (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P7 P84* **have** *P85* : *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Line-on-trans*)
from *N* **have** *P86* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Tri-def-Line*)
from *P85 P86* **have** *P87* : ¬ *Eq* (*Geos* (*Poi D1*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** *blast*
from *P77 P81 P87* **have** *P88* : *Plane-sameside* (*Li C A*) *D1 E* **by** (*simp add*:*Plane-trans-inv*)
from *P58 P59 P88* **have** *P89* : *Eq* (*Geos* (*Lin* (*Li D1 C*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) ∧ ¬ *Bet-Point* (*Se D1 E*) *C* **by** (*simp add*:*Ang-move-unique*)
from *P49* **have** *P90* : *Line-on* (*Li E C*) *B* **by** (*simp add*:*Line-Bet-on*)
from *P89 P90* **have** *P91* : *Line-on* (*Li D1 C*) *B* **by** (*blast intro*:*Eq-rev Line-on-trans*)
have *P92* : *Line-on* (*Li D1 C*) *D1* **by** (*simp add*:*Line-on-rule*)
from *P8 P20 P83 P91 P92* **have** *P93* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li D1 C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P89 P93* **have** *P94* : *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li E C*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P7 P94* **have** *P95* : *Line-on* (*Li E C*) *A* **by** (*simp add*:*Line-on-trans*)
from *P56* **have** *Def* (*Tri* (*Tr E C A*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P96* : ¬ *Line-on* (*Li E C*) *A* **by** (*simp add*:*Tri-def-Line*)
from *P95 P96* **show** *False* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Ang-external-Gr-lemma2* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B D*) *A*
shows ¬ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add*

Emp)
proof
assume *W* : *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*)
from *N* **have** ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P2* : *Line-on* (*Li A D*) *A* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P3* : *Line-on* (*Li A D*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P4* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
have *P5* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P6* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P2 P3 P4 P5 P6* **have** *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P7* : *Line-on* (*Li A D*) *C* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *N* **have** *P8* : ¬ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Tri-def-Line*)
from *P7 P8* **have** *P9* : ¬ *Line-on* (*Li A D*) *C* **by** *blast*
from *P1 P9* **have** *Def* (*Ang* (*An A D C*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P10* : *Def* (*Ang* (*An C A D*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
have *P11* : *Cong* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *N* **have** *P12* : *Def* (*Ang* (*An A C B*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-inv*)
then have *P13* : *Def* (*Ang* (*An B C A*)) **by** (*simp add*:*Ang-def-rev*)
from *W P10 P11 P12 P13* **have** *P14* : *Gr* (*Geos* (*Ang* (*An B C A*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
from *N* **have** *P15* : ¬ *Line-on* (*Li C A*) *B* **by** (*simp add*:*Tri-def-Line*)
from *N* **have** *P16* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P10 P15* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An p C A*)) *add Emp*)
∧ *Plane-sameside* (*Li C A*) *p B* **by** (*simp add*:*Ang-move-sameside*)
then obtain *B1* :: *Point* **where** *P17* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An B1 C A*)) *add Emp*)
∧ *Plane-sameside* (*Li C A*) *B1 B* **by** *blast*
then have *P18* : ¬ *Line-on* (*Li C A*) *B1* **by** (*simp add*:*Plane-sameside-def*)
from *P16 P18* **have** *Def* (*Ang* (*An C A B1*)) **by** (*blast intro*:*Ang-sinple-def Eq-rev*)
then have *P19* : *Def* (*Ang* (*An B1 C A*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *N* **have** *P20* : *Def* (*Ang* (*An A C B*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans Tri-to-Ang*)
from *P17* **have** *P21* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An B1 C A*)) *add Emp*) **by** *simp*
from *P17* **have** *P22* : *Plane-sameside* (*Li C A*) *B1 B* **by** *simp*
from *P10 P14 P19 P20 P21 P22* **have** *Ang-inside* (*An B C A*) *B1* **by** (*simp add*:*Ang-greater-def*)
then have *Plane-diffside* (*Li C B1*) *B A* **by** (*simp add*:*Ang-inside-Planeside*)

then have $\exists p.$ *Bet-Point* (*Se B A*) *p* $\wedge$ *Line-on* (*Li C B1*) *p* $\wedge$ $\neg$ *Line-on* (*Li C B1*) *B* $\wedge$ $\neg$ *Line-on* (*Li C B1*) *A* **by** (*simp add*:*Plane-diffside-def*)
then obtain *B2* :: *Point* **where** *P23* : *Bet-Point* (*Se B A*) *B2* $\wedge$ *Line-on* (*Li C B1*) *B2*
$\wedge$ $\neg$ *Line-on* (*Li C B1*) *B* $\wedge$ $\neg$ *Line-on* (*Li C B1*) *A* **by** *blast*
then have *Line-on* (*Li B A*) *B2* **by** (*simp add*:*Line-Bet-on*)
then have *P24* : *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) $\Longrightarrow$ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Point-Eq*)
from *N* **have** *Def* (*Tri* (*Tr B A C*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
then have *P25* : $\neg$ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Tri-def-Line*)
from *P24 P25* **have** *P26* : $\neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P23* **have** *P27* : *Bet-Point* (*Se A B*) *B2* **by** (*simp add*:*Bet-rev*)
have *P28* : *Line-on* (*Li C A*) *A* **by** (*simp add*:*Line-on-rule*)
from *P5* **have** *P29* : *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) $\Longrightarrow$
Line-on (*Li C A*) *B* **by** (*simp add*: *Line-on-trans*)
from *P15 P29* **have** *P30* : $\neg$ *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** *blast*
from *P27 P28 P30* **have** *P31* : *Plane-sameside* (*Li C A*) *B B2* **by** (*simp add*:*Plane-Bet-sameside Plane-sameside-rev*)
have *P32* : *Line-on* (*Li C A*) *C* **by** (*simp add*:*Line-on-rule*)
have *Line-on* (*Li B1 B2*) *B1* **by** (*simp add*:*Line-on-rule*)
then have *P33* : *Eq* (*Geos* (*Lin* (*Li B1 B2*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) $\Longrightarrow$
Line-on (*Li C A*) *B1* **by** (*simp add*: *Line-on-trans*)
from *P18 P33* **have** *P34* : $\neg$ *Eq* (*Geos* (*Lin* (*Li B1 B2*)) *add Emp*) (*Geos* (*Lin* (*Li C A*)) *add Emp*) **by** *blast*
from *P32 P34* **have** *Bet-Point* (*Se B1 B2*) *C* $\Longrightarrow$ *Plane-diffside* (*Li C A*) *B1 B2* **by** (*simp add*:*Plane-Bet-diffside*)
then have *P35* : *Bet-Point* (*Se B1 B2*) *C* $\Longrightarrow$ $\neg$ *Plane-sameside* (*Li C A*) *B1 B2* **by** (*simp add*:*Plane-diffside-not-sameside*)
from *P22 P31* **have** *P36* : $\neg$ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) $\Longrightarrow$
Plane-sameside (*Li C A*) *B1 B2* **by** (*blast intro*:*Plane-sameside-trans Eq-rev*)
from *P35 P36* **have** *P37* : $\neg$ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) $\Longrightarrow$
$\neg$ *Bet-Point* (*Se B1 B2*) *C* **by** *blast*
have *P38* : *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) $\Longrightarrow$
Bet-Point (*Se B1 B2*) *C* $\Longrightarrow$ *Bet-Point* (*Se B2 B2*) *C* **by** (*simp add*:*Bet-Point-Eq*)
have *P39* : $\neg$ *Bet-Point* (*Se B2 B2*) *C* **by** (*simp add*:*Bet-end-Point*)
from *P38 P39* **have** *P40* : *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) $\Longrightarrow$
$\neg$ *Bet-Point* (*Se B1 B2*) *C* **by** *blast*
from *P37 P40* **have** *P41* : $\neg$ *Bet-Point* (*Se B1 B2*) *C* **by** *blast*
have *P42* : $\neg$ *Bet-Point* (*Se A A*) *C* **by** (*simp add*:*Bet-end-Point*)
from *P16 P19 P23 P26 P28 P41 P42* **have** *P43* :
Eq (*Geos* (*Ang* (*An B1 C A*)) *add Emp*) (*Geos* (*Ang* (*An B2 C A*)) *add Emp*) $\wedge$ *Def* (*Ang* (*An B2 C A*)) **by** (*simp add*:*Ang-Point-swap*)

from *P21 P43* **have** *P44* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An B2 C A*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P43* **have** *P45* : *Def* (*Tri* (*Tr A B2 C*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P27* **have** *P46* : *Bet-Point* (*Se B A*) *B2* **by** (*simp add*:*Bet-rev*)
from *N P46* **have** *P47* : *Bet-Point* (*Se B2 D*) *A* **by** (*blast intro*:*Bet-swap-134-234*)
from *P45 P47* **have** *P48* : ¬ *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B2*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr-lemma1*)
have *P49* : *Cong* (*Geos* (*Ang* (*An B2 C A*)) *add Emp*) (*Geos* (*Ang* (*An A C B2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P43* **have** *P50* : *Def* (*Ang* (*An A C B2*)) **by** (*simp add*:*Ang-def-rev*)
from *P10 P43 P44 P49 P50* **have** *P51* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B2*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P48 P51* **show** *False* **by** *blast*
qed

Theorem22

theorem (**in** *Congruence-Rule*) *Ang-external-Gr* :
assumes
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B D*) *A*
shows
Gr (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
Gr (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
proof −
from *assms* **have** ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P1* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P2* : *Line-on* (*Li A D*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P3* : *Line-on* (*Li A D*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P4* : *Line-on* (*Li A B*) *A* **by** (*simp add*:*Line-on-rule*)
have *P5* : *Line-on* (*Li A B*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P6* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P2 P3 P4 P5 P6* **have** *Eq* (*Geos* (*Lin* (*Li A D*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P7* : *Line-on* (*Li A D*) *C* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *assms* **have** *P8* : ¬ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Tri-def-Line*)
from *P7 P8* **have** *P9* : ¬ *Line-on* (*Li A D*) *C* **by** *blast*
from *P1 P9* **have** *Def* (*Ang* (*An A D C*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P10* : *Def* (*Ang* (*An C A D*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P11* : *Def* (*Ang* (*An A C B*)) **by** (*simp add*:*Tri-to-Ang Ang-def-inv*)
from *P10 P11* **have** *P12* :
Cong (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*)

by (*simp add*:*Ang-relation-case*)
from *assms* **have** *P13* : ¬ *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr-lemma1*)
from *assms* **have** *P14* : ¬ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr-lemma2*)
from *P12 P13 P14* **show** *P15* : *Gr* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** *blast*
have *P16* : *Line-on* (*Li C A*) *C* **by** (*simp add*:*Line-on-rule*)
have *P17* : *Line-on* (*Li C A*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P18* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P16 P17 P18* **have** ∃ *p*. *Bet-Point* (*Se C p*) *A* ∧ *Line-on* (*Li C A*) *p* **by** (*simp add*:*Bet-extension*)
then obtain *E* :: *Point* **where** *P19* : *Bet-Point* (*Se C E*) *A* **by** *blast*
from *assms* **have** *P20* : *Bet-Point* (*Se D B*) *A* **by** (*simp add*:*Bet-rev*)
from *P10 P19 P20* **have** *P21* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An E A B*)) *add Emp*) **by** (*simp add*:*Ang-vertical*)
have *P22* : *Eq* (*Geos* (*Ang* (*An E A B*)) *add Emp*) (*Geos* (*Ang* (*An B A E*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P21 P22* **have** *P23* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An B A E*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P19* **have** *P24* : *Line-on* (*Li C E*) *A* **by** (*simp add*:*Line-Bet-on*)
have *P25* : *Line-on* (*Li C E*) *E* **by** (*simp add*:*Line-on-rule*)
from *P19* **have** *P26* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P4 P24 P25 P26* **have** *P27* : *Line-on* (*Li A B*) *E* ⟹
Eq (*Geos* (*Lin* (*Li C E*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P28* : *Line-on* (*Li C E*) *C* **by** (*simp add*:*Line-on-rule*)
from *P27 P28* **have** *P29* : *Line-on* (*Li A B*) *E* ⟹ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *P8 P29* **have** *P30* : ¬ *Line-on* (*Li A B*) *E* **by** *blast*
from *P6 P30* **have** *Def* (*Ang* (*An A B E*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P31* : *Def* (*Ang* (*An B A E*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P11* **have** *P32* : *Def* (*Tri* (*Tr A C B*)) **by** (*simp add*:*Ang-to-Tri*)
from *P19 P32* **have** *P33* : ¬ *Cong* (*Geos* (*Ang* (*An B A E*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr-lemma1*)
from *P19 P32* **have** *P34* : ¬ *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An B A E*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr-lemma2*)
from *assms* **have** *P35* : *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Tri-to-Ang*)
from *P31 P35* **have** *P36* :
Cong (*Geos* (*Ang* (*An B A E*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An B A E*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
∨ *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An B A E*)) *add Emp*)
by (*simp add*:*Ang-relation-case*)
from *P33 P34 P36* **have** *P37* : *Gr* (*Geos* (*Ang* (*An B A E*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** *blast*
from *P10 P23 P31 P35 P37* **show** *Gr* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr*)

qed

lemma (**in** *Congruence-Rule*) *Seg-not-Eq-move* :
assumes
$\neg$ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi B1*) *add Emp*)
$\neg$ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*)
$\neg$ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B3*) *add Emp*)
Line-on l1 A2 Line-on l1 B2 Line-on l1 B3
$\neg$ *Bet-Point* (*Se B3 B2*) *A2*
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B3*)) *add Emp*)
$\neg$ *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
shows
Bet-Point (*Se B2 A2*) *B3* $\wedge$ $\neg$ *Bet-Point* (*Se A2 B3*) *B2*
$\vee$ $\neg$ *Bet-Point* (*Se B2 A2*) *B3* $\wedge$ *Bet-Point* (*Se A2 B3*) *B2*
proof $-$
from *assms* **have** *P1* : *Eq* (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B3*)) *add Emp*) $\Longrightarrow$
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *assms P1* **have** $\neg$ *Eq* (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B3*)) *add Emp*) **by** *blast*
then have *P2* : $\neg$ *Eq* (*Geos* (*Poi B2*) *add Emp*) (*Geos* (*Poi B3*) *add Emp*) **by** (*simp add*:*Seg-not-Eq-Point*)
from *assms* **have** *P3* : $\neg$ *Eq* (*Geos* (*Poi B3*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P4* : *Line-on l1 A2* **by** *simp*
from *assms* **have** *P5* : *Line-on l1 B2* **by** *simp*
from *assms* **have** *P6* : *Line-on l1 B3* **by** *simp*
from *assms* **have** *P7* : $\neg$ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) **by** *simp*
from *P2 P3 P4 P5 P6 P7* **have** *Bet-Point* (*Se A2 B3*) *B2* $\vee$ *Bet-Point* (*Se B3 B2*) *A2* $\vee$ *Bet-Point* (*Se B2 A2*) *B3* **by** (*simp add*:*Bet-case*)
then have *P8* :
Bet-Point (*Se A2 B3*) *B2* $\wedge$ $\neg$ *Bet-Point* (*Se B3 B2*) *A2* $\wedge$ $\neg$ *Bet-Point* (*Se B2 A2*) *B3*
$\vee$ $\neg$ *Bet-Point* (*Se A2 B3*) *B2* $\wedge$ *Bet-Point* (*Se B3 B2*) *A2* $\wedge$ $\neg$ *Bet-Point* (*Se B2 A2*) *B3*
$\vee$ $\neg$ *Bet-Point* (*Se A2 B3*) *B2* $\wedge$ $\neg$ *Bet-Point* (*Se B3 B2*) *A2* $\wedge$ *Bet-Point* (*Se B2 A2*) *B3* **by** (*simp add*:*Bet-case-fact*)
from *assms P8* **show** *Bet-Point* (*Se B2 A2*) *B3* $\wedge$ $\neg$ *Bet-Point* (*Se A2 B3*) *B2*
$\vee$ $\neg$ *Bet-Point* (*Se B2 A2*) *B3* $\wedge$ *Bet-Point* (*Se A2 B3*) *B2* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Tri-Seg-diagonal* :
assumes
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se B C*) *D*
Eq (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se C D*)) *add Emp*)
shows

Gr (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
proof −
from *assms* **have** *P1* : ¬ *Line-on* (*Li A B*) *C* **by** (*simp add*:*Tri-def-Line*)
have *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
then have *P2* : *Eq* (*Geos* (*Lin* (*Li A C*)) *add Emp*) (*Geos* (*Lin* (*Li A B*)) *add Emp*) ⟹
Line-on (*Li A B*) *C* **by** (*simp add*:*Line-on-trans*)
from *P1 P2* **have** *P3* : ¬ *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li A C*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P5* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P3 P4 P5* **have** *P6* : *Ang-inside* (*An B A C*) *D* **by** (*simp add*:*Ang-inside-Bet-Point*)
then have *Plane-sameside* (*Li A C*) *B D* **by** (*simp add*:*Ang-inside-def*)
then have *P7* : *Plane-sameside* (*Li A C*) *D B* **by** (*simp add*:*Plane-sameside-rev*)
have *P8* : *Cong* (*Geos* (*Ang* (*An D A C*)) *add Emp*) (*Geos* (*Ang* (*An D A C*)) *add Emp*) **by** *simp*
from *P6 P7 P8* **have** *P9* : *Gr* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An D A C*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)
from *assms* **have** *P10* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *assms* **have** *P11* : *Line-on* (*Li C B*) *D* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *P12* : *Def* (*Tri* (*Tr A C B*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P10 P11 P12* **have** *Def* (*Tri* (*Tr A C D*)) **by** (*simp add*:*Tri-def-extension*)
then have *P13* : *Def* (*Tri* (*Tr C A D*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
have *P14* : *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se C A*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *assms P14* **have** *P15* : *Eq* (*Geos* (*Seg* (*Se C A*)) *add Emp*) (*Geos* (*Seg* (*Se C D*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P13 P15* **have** *P16* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An C D A*)) *add Emp*) **by** (*simp add*:*Tri-isosceles*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P17* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P18* : *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-Bet-on*)
from *assms P17 P18* **have** *Def* (*Tri* (*Tr A B D*)) **by** (*simp add*:*Tri-def-extension*)
then have *P19* : *Def* (*Tri* (*Tr D B A*)) **by** (*simp add*:*Tri-def-rev*)
from *assms P19* **have** *P20* : *Gr* (*Geos* (*Ang* (*An A D C*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
from *assms* **have** *P21* : *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Tri-to-Ang*)
have *P22* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
have *P23* : ¬ *Bet-Point* (*Se A A*) *B* **by** (*simp add*:*Bet-end-Point*)
from *assms* **have** *P24* : *Line-on* (*Li B C*) *D* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *Inv* (*Bet-Point* (*Se C D*) *B*) **by** (*simp add*:*Bet-iff*)

then have *P25* : ¬ *Bet-Point* (*Se C D*) *B* **by** (*simp add*:*Inv-def*)
from *P4* **have** *P26* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P17 P21 P22 P23 P24 P25 P26* **have** *P27* :
Eq (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) ∧ *Def* (*Ang* (*An A B D*)) **by** (*simp add*:*Ang-Point-swap*)
have *P28* : *Cong* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P27 P28* **have** *P29* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev*)
have *P30* : *Eq* (*Geos* (*Ang* (*An C D A*)) *add Emp*) (*Geos* (*Ang* (*An A D C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P16 P30* **have** *P31* : *Cong* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A D C*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P13* **have** *P32* : *Def* (*Ang* (*An C A D*)) **by** (*simp add*:*Tri-to-Ang*)
then have *P33* : *Def* (*Ang* (*An A D C*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P27* **have** *P34* : *Def* (*Ang* (*An D B A*)) **by** (*blast intro*:*Ang-def-rev*)
from *P20 P31 P32 P33 P34* **have** *P35* : *Gr* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr*)
from *P21 P29 P32 P34 P35* **have** *P36* : *Gr* (*Geos* (*Ang* (*An C A D*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
have *P37* : *Cong* (*Geos* (*Ang* (*An D A C*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P32* **have** *P38* : *Def* (*Ang* (*An D A C*)) **by** (*simp add*:*Ang-def-rev*)
from *assms* **have** *P39* : *Def* (*Ang* (*An B A C*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-rev Ang-def-inv*)
from *P9 P32 P37 P38 P39* **have** *P40* : *Gr* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An C A D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq*)
from *P21 P32 P36 P39 P40* **show** *Gr* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Gr*)
qed

lemma (**in** *Congruence-Rule*) *Tri-Bet-Ang-Gr* :
assumes
Def (*Tri* (*Tr A B C*))
Bet-Point (*Se A C*) *D*
Eq (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
shows
Gr (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*)
proof −
from *assms* **have** *P1* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*simp add*:*Tri-isosceles*)
from *assms* **have** *P2* : *Def* (*Tri* (*Tr B C A*)) **by** (*simp add*:*Tri-def-trans*)
from *assms* **have** *P3* : *Line-on* (*Li C A*) *D* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P2 P3 P4* **have** *Def* (*Tri* (*Tr B C D*)) **by** (*simp add*:*Tri-def-extension*)
then have *P5* : *Def* (*Tri* (*Tr D C B*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms* **have** *P6* : *Bet-Point* (*Se C A*) *D* **by** (*simp add*:*Bet-rev*)

from *P5 P6* **have** *P7* : *Gr* (*Geos* (*Ang* (*An B D A*)) *add Emp*) (*Geos* (*Ang* (*An D C B*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
from *P5* **have** *P8* : *Def* (*Ang* (*An D C B*)) **by** (*simp add*:*Tri-to-Ang*)
from *assms* **have** *P9* : *Line-on* (*Li C D*) *A* **by** (*simp add*:*Line-Bet-on*)
from *assms* **have** *Inv* (*Bet-Point* (*Se D A*) *C*) **by** (*simp add*:*Bet-iff*)
then have *P10* : ¬ *Bet-Point* (*Se D A*) *C* **by** (*simp add*:*Inv-def*)
have *P11* : *Line-on* (*Li C B*) *B* **by** (*simp add*:*Line-on-rule*)
have *P12* : ¬ *Bet-Point* (*Se B B*) *C* **by** (*simp add*:*Bet-end-Point*)
from *P6* **have** *P13* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P5* **have** *P14* : ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P8 P9 P10 P11 P12 P13 P14* **have** *P15* : *Eq* (*Geos* (*Ang* (*An D C B*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ *Def* (*Ang* (*An A C B*)) **by** (*simp add*:*Ang-Point-swap*)
from *P1 P15* **have** *P16* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D C B*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev*)
have *P17* : *Cong* (*Geos* (*Ang* (*An B D A*)) *add Emp*) (*Geos* (*Ang* (*An A D B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P2* **have** *P18* : *Def* (*Tri* (*Tr B A C*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms* **have** *P19* : *Line-on* (*Li A C*) *D* **by** (*simp add*:*Line-Bet-on*)
from *P6* **have** *P20* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P18 P19 P20* **have** *Def* (*Tri* (*Tr B A D*)) **by** (*simp add*:*Tri-def-extension*)
then have *P21* : *Def* (*Ang* (*An B D A*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-inv*)
then have *P22* : *Def* (*Ang* (*An A D B*)) **by** (*simp add*:*Ang-def-rev*)
from *P7 P8 P17 P21 P22* **have** *P23* : *Gr* (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An D C B*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
from *assms* **have** *P24* : *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Tri-to-Ang*)
from *P8 P16 P22 P23 P24* **have** *P25* : *Gr* (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P24* **have** ¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P26* : ¬ *Eq* (*Geos* (*Lin* (*Li B C*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P27* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P28* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P6 P26 P27 P28* **have** *P29* : *Ang-inside* (*An C B A*) *D* **by** (*simp add*:*Ang-inside-Bet-Point*)
then have *P30* : *Plane-sameside* (*Li B A*) *D C* **by** (*simp add*:*Ang-inside-def Plane-sameside-rev*)
have *P31* : *Cong* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** *simp*
from *P29 P30 P31* **have** *P32* : *Gr* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*simp add*:*Ang-greater-def*)

have *P33* : *Cong* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P24* **have** *P34* : *Def* (*Ang* (*An C B A*)) **by** (*simp add*:*Ang-def-rev*)
from *P22* **have** *P35* : *Def* (*Ang* (*An D B A*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P24 P32 P33 P34 P35* **have** *P36* : *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
have *P37* : *Cong* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P35* **have** *P38* : *Def* (*Ang* (*An A B D*)) **by** (*simp add*:*Ang-def-rev*)
from *P24 P35 P36 P37 P38* **have** *P39* : *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P22 P24 P25 P38 P39* **show** *Gr* (*Geos* (*Ang* (*An A D B*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Gr Ang-rev*)
qed

Theorem24

theorem (**in** *Congruence-Rule*) *Tri-isosceles-inv* :
assumes *N* :
Def (*Tri* (*Tr A B C*))
Cong (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
shows
¬¬ *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
proof
assume *W* : ¬ *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A C*)) *add Emp*)
have *P1* : *Line-on* (*Li A C*) *A* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
from *N* **have** *P3* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *N* **have** ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P4* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P2 P3 P4* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p C*) *A* ∧ *Line-on* (*Li A C*) *p* ∧ ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *D* :: *Point* **where** *P5* : *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se A D*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se D C*) *A* ∧ *Line-on* (*Li A C*) *D* ∧ ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** *blast*
from *W P1 P2 P3 P4 P5* **have** *P6* : *Bet-Point* (*Se C A*) *D* ∧ ¬ *Bet-Point* (*Se A D*) *C*
∨ ¬ *Bet-Point* (*Se C A*) *D* ∧ *Bet-Point* (*Se A D*) *C* **by** (*simp add*:*Seg-not-Eq-move*)
from *N* **have** *P7* : *Def* (*Tri* (*Tr B C A*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
have *P8* : *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se B A*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P5 P8* **have** *P9* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se A*

D)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
from *P7 P9* **have** *P10* : *Bet-Point* (*Se C A*) *D* ⟹
Gr (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*simp add*:*Tri-Seg-diagonal*)
have *P11* : *Cong* (*Geos* (*Ang* (*An C B A*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms* **have** *P12* : *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Tri-to-Ang*)
then have *P13* : *Def* (*Ang* (*An C B A*)) **by** (*simp add*:*Ang-def-rev*)
then have *P14* : *Def* (*Ang* (*An B C A*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P10 P11 P12 P13 P14* **have** *P15* : *Bet-Point* (*Se C A*) *D* ⟹
Gr (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An B C A*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
from *P14* **have** *P16* : *Def* (*Ang* (*An A C B*)) **by** (*simp add*:*Ang-def-rev*)
have *P17* : *Cong* (*Geos* (*Ang* (*An B C A*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P12 P14 P15 P16 P17* **have** *P18* : *Bet-Point* (*Se C A*) *D* ⟹
Gr (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P12 P16* **have** *P19* : *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
∨ ¬ *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*)
∧ *Gr* (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*)
by (*simp add*:*Ang-relation-case-fact*)
from *P18 P19* **have** *P20* : *Bet-Point* (*Se C A*) *D* ⟹
¬ *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** *blast*
from *assms* **have** *P21* : *Def* (*Tri* (*Tr B A C*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P5 P21* **have** *Def* (*Tri* (*Tr B A D*)) **by** (*simp add*:*Tri-def-extension*)
then have *P22* : *Def* (*Tri* (*Tr A B D*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P5 P22* **have** *P23* : *Bet-Point* (*Se A D*) *C* ⟹
Gr (*Geos* (*Ang* (*An A C B*)) *add Emp*) (*Geos* (*Ang* (*An A B C*)) *add Emp*) **by** (*simp add*:*Tri-Bet-Ang-Gr*)
from *P19 P23* **have** *P24* : *Bet-Point* (*Se A D*) *C* ⟹
¬ *Cong* (*Geos* (*Ang* (*An A B C*)) *add Emp*) (*Geos* (*Ang* (*An A C B*)) *add Emp*) **by** *blast*
from *assms P6 P20 P24* **show** *False* **by** *blast*
qed

lemma (**in** *Congruence-Rule*) *Tri-AAS-lemma1* :
assumes

Def (*Tri* (*Tr A1 B1 C1*)) *Def* (*Tri* (*Tr A2 B2 C2*))
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*)
shows
Cong (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*)
proof −
have *P1* : *Line-on* (*Li A2 C2*) *A2* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li A2 C2*) *C2* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi C1*) *add Emp*) (*Geos* (*Poi A1*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P3* : ¬ *Eq* (*Geos* (*Poi A1*) *add Emp*) (*Geos* (*Poi C1*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms* **have** *P4* : ¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*simp add*:*Tri-def*)
then have *P5* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P2 P3 P5* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p C2*) *A2* ∧ *Line-on* (*Li A2 C2*) *p* ∧ ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *C3* :: *Point* **where** *P6* : *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se C3 C2*) *A2* ∧ *Line-on* (*Li A2 C2*) *C3* ∧ ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi C3*) *add Emp*) **by** *blast*
from *P6* **have** *P7* : *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Eq (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *assms P7* **have** *P8* : *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Cong (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
from *P6* **have** *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) ⟹
Eq (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
then have *P9* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Seg* (*Se A1 C1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) **by** *blast*
from *P1 P2 P3 P5 P6 P9* **have** *P10* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Bet-Point (*Se C2 A2*) *C3* ∧ ¬ *Bet-Point* (*Se A2 C3*) *C2*
∨ ¬ *Bet-Point* (*Se C2 A2*) *C3* ∧ *Bet-Point* (*Se A2 C3*) *C2* **by** (*simp add*:*Seg-not-Eq-move*)

from *assms* **have** *P11* : *Def* (*Tri* (*Tr B2 A2 C2*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P6 P11* **have** *Def* (*Tri* (*Tr B2 A2 C3*)) **by** (*simp add*:*Tri-def-extension*)
then have *P12* : *Def* (*Tri* (*Tr A2 B2 C3*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P11* **have** *P13* : *Def* (*Ang* (*An B2 A2 C2*)) **by** (*simp add*:*Tri-to-Ang*)
have *P14* : *Line-on* (*Li A2 B2*) *B2* **by** (*simp add*:*Line-on-rule*)
have *P15* : ¬ *Bet-Point* (*Se B2 B2*) *A2* **by** (*simp add*:*Bet-end-Point*)
from *P6* **have** *P16* : ¬ *Bet-Point* (*Se C2 C3*) *A2* **by** (*blast intro*:*Bet-rev*)
from *assms* **have** *P17* : ¬ *Eq* (*Geos* (*Poi A2*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P6 P13 P14 P15 P16 P17* **have** *P18* :
Eq (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C3*)) *add Emp*) ∧ *Def* (*Ang* (*An B2 A2 C3*)) **by** (*simp add*:*Ang-Point-swap*)
from *assms P18* **have** *P19* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C3*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *assms P6 P12 P19* **have** *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C3*)) *add Emp*) **by** (*simp add*:*Tri-SAS*)
then have *P20* : *Cong* (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
from *assms* **have** *P21* : *Def* (*Ang* (*An A1 C1 B1*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-inv*)
from *assms* **have** *P22* : *Def* (*Ang* (*An A2 C2 B2*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-inv*)
from *P18* **have** *P23* : *Def* (*Ang* (*An A2 C3 B2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms P20 P21 P22 P23* **have** *P24* : *Cong* (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*) **by** (*blast intro*:*Ang-trans Ang-rev*)
from *P22 P23 P24* **have** *P25* : ¬ *Gr* (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*) (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr*)
from *assms* **have** *P26* : *Def* (*Tri* (*Tr B2 C2 A2*)) **by** (*blast intro*:*Tri-def-trans*)
have *P27* : *Bet-Point* (*Se C2 A2*) *C3* ⟹ ¬ *Eq* (*Geos* (*Poi C3*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
have *P28* : *Bet-Point* (*Se A2 C3*) *C2* ⟹ ¬ *Eq* (*Geos* (*Poi C3*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P10 P27 P28* **have** *P29* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi C3*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P5 P6* **have** *P30* : *Line-on* (*Li C2 A2*) *C3* **by** (*blast intro*:*Line-rev Line-on-trans*)
from *P26 P29 P30* **have** ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Def (*Tri* (*Tr B2 C2 C3*)) **by** (*simp add*:*Tri-def-extension*)
then have *P31* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Def (*Tri* (*Tr C3 C2 B2*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
then have *P32* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹

Bet-Point (Se C2 A2) C3 ⟹ Gr (Geos (Ang (An B2 C3 A2)) add Emp) (Geos (Ang (An C3 C2 B2)) add Emp) **by** (*simp add*:*Ang-external-Gr*)
have *P33* : *Cong (Geos (Ang (An B2 C3 A2)) add Emp) (Geos (Ang (An A2 C3 B2)) add Emp)* **by** (*simp add*:*Ang-roll*)
from *P31* **have** *P34* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Def (Ang (An C3 C2 B2)) **by** (*simp add*:*Tri-to-Ang*)
from *P23* **have** *P35* : *Def (Ang (An B2 C3 A2))* **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P23 P32 P33 P34 P35* **have** *P36* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Bet-Point (Se C2 A2) C3 ⟹ Gr (Geos (Ang (An A2 C3 B2)) add Emp) (Geos (Ang (An C3 C2 B2)) add Emp) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
have *P37* : *Bet-Point (Se C2 A2) C3 ⟹ Line-on (Li C2 C3) A2* **by** (*simp add*:*Line-Bet-on*)
from *P10* **have** *P38* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Bet-Point (Se C2 A2) C3 ⟹ ¬ *Bet-Point (Se C3 A2) C2* **by** (*blast intro*:*Bet-rev*)
have *P39* : *Line-on (Li C2 B2) B2* **by** (*simp add*:*Line-on-rule*)
have *P40* : ¬ *Bet-Point (Se B2 B2) C2* **by** (*simp add*:*Bet-end-Point*)
from *P13* **have** *P41* : ¬ *Eq (Geos (Poi C2) add Emp) (Geos (Poi B2) add Emp)* **by** (*simp add*:*Ang-def*)
from *P4 P34 P37 P38 P39 P40 P41* **have** ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Bet-Point (Se C2 A2) C3 ⟹ Eq (Geos (Ang (An C3 C2 B2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp)
∧ Def (Ang (An A2 C2 B2)) **by** (*simp add*:*Ang-Point-swap*)
then have *P42* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Bet-Point (Se C2 A2) C3 ⟹ Cong (Geos (Ang (An C3 C2 B2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp) **by** (*blast intro*:*Ang-weektrans*)
from *P22 P23 P34 P36 P42* **have** *P43* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Bet-Point (Se C2 A2) C3 ⟹ Gr (Geos (Ang (An A2 C3 B2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P31* **have** ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Def (Tri (Tr C2 C3 B2)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
then have *P44* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Bet-Point (Se A2 C3) C2 ⟹ Gr (Geos (Ang (An B2 C2 A2)) add Emp) (Geos (Ang (An C2 C3 B2)) add Emp) **by** (*simp add*:*Ang-external-Gr Bet-rev*)
have *P45* : *Cong (Geos (Ang (An B2 C2 A2)) add Emp) (Geos (Ang (An A2 C2 B2)) add Emp)* **by** (*simp add*:*Ang-roll*)
from *P26* **have** *P46* : *Def (Ang (An B2 C2 A2))* **by** (*simp add*:*Tri-to-Ang*)
from *P34* **have** *P47* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp) (Geos (Seg (Se A2 C3)) add Emp) ⟹*
Def (Ang (An C2 C3 B2)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P22 P44 P45 P46 P47* **have** *P48* : ¬ *Eq (Geos (Seg (Se A2 C2)) add Emp)*

(*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Bet-Point (*Se A2 C3*) *C2* ⟹ *Gr* (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) (*Geos* (*Ang* (*An C2 C3 B2*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Eq-Gr Ang-rev*)
have *P49* : *Bet-Point* (*Se A2 C3*) *C2* ⟹ *Line-on* (*Li C3 C2*) *A2* **by** (*simp add*:*Line-Bet-on*)
from *P10* **have** *P50* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Bet-Point (*Se A2 C3*) *C2* ⟹ ¬ *Bet-Point* (*Se C2 A2*) *C3* **by** (*blast intro*:*Bet-rev*)
have *P51* : *Line-on* (*Li C3 B2*) *B2* **by** (*simp add*:*Line-on-rule*)
have *P52* : ¬ *Bet-Point* (*Se B2 B2*) *C3* **by** (*simp add*:*Bet-end-Point*)
from *P6* **have** *P53* : ¬ *Eq* (*Geos* (*Poi C3*) *add Emp*) (*Geos* (*Poi A2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P47* **have** *P54* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
¬ *Eq* (*Geos* (*Poi C3*) *add Emp*) (*Geos* (*Poi B2*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P47 P49 P50 P51 P52 P53 P54* **have** ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Bet-Point (*Se A2 C3*) *C2* ⟹ *Eq* (*Geos* (*Ang* (*An C2 C3 B2*)) *add Emp*) (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*)
∧ *Def* (*Ang* (*An A2 C3 B2*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P55* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Bet-Point (*Se A2 C3*) *C2* ⟹ *Cong* (*Geos* (*Ang* (*An C2 C3 B2*)) *add Emp*) (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
from *P22 P23 P47 P48 P55* **have** *P56* : ¬ *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) ⟹
Bet-Point (*Se A2 C3*) *C2* ⟹ *Gr* (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) (*Geos* (*Ang* (*An A2 C3 B2*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P10 P25 P43 P56* **have** *P57* : *Eq* (*Geos* (*Seg* (*Se A2 C2*)) *add Emp*) (*Geos* (*Seg* (*Se A2 C3*)) *add Emp*) **by** *blast*
from *P8 P57* **show** *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*) **by** *blast*
qed

Theorem25

theorem (**in** *Congruence-Rule*) *Tri-AAS* :
assumes
Def (*Tri* (*Tr A1 B1 C1*)) *Def* (*Tri* (*Tr A2 B2 C2*))
Eq (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*)
Cong (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*)
∨ *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*)
shows
Cong (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*)
proof −

from *assms* **have** *P1* : *Def* (*Tri* (*Tr B1 A1 C1*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *assms* **have** *P2* : *Def* (*Tri* (*Tr B2 A2 C2*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
have *P3* : *Eq* (*Geos* (*Seg* (*Se A1 B1*)) *add Emp*) (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P4* : *Eq* (*Geos* (*Seg* (*Se A2 B2*)) *add Emp*) (*Geos* (*Seg* (*Se B2 A2*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *assms P3 P4* **have** *P5* : *Eq* (*Geos* (*Seg* (*Se B1 A1*)) *add Emp*) (*Geos* (*Seg* (*Se B2 A2*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
have *P6* : *Eq* (*Geos* (*Ang* (*An A1 C1 B1*)) *add Emp*) (*Geos* (*Ang* (*An B1 C1 A1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *assms P6* **have** *P7* : *Cong* (*Geos* (*Ang* (*An B1 C1 A1*)) *add Emp*) (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P8* : *Eq* (*Geos* (*Ang* (*An A2 C2 B2*)) *add Emp*) (*Geos* (*Ang* (*An B2 C2 A2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P7 P8* **have** *P9* : *Cong* (*Geos* (*Ang* (*An B1 C1 A1*)) *add Emp*) (*Geos* (*Ang* (*An B2 C2 A2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P1 P2 P5 P9* **have** *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Tri* (*Tr B1 A1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr B2 A2 C2*)) *add Emp*) **by** (*simp add*:*Tri-AAS-lemma1*)
then have *P10* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 B2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
have *P11* : *Eq* (*Geos* (*Ang* (*An C1 A1 B1*)) *add Emp*) (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P10 P11* **have** *P12* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An C2 A2 B2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P13* : *Eq* (*Geos* (*Ang* (*An C2 A2 B2*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P12 P13* **have** *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
then have *P14* : *Cong* (*Geos* (*Ang* (*An A1 B1 C1*)) *add Emp*) (*Geos* (*Ang* (*An A2 B2 C2*)) *add Emp*)
$\vee$ *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) **by** *blast*
from *assms* **have** *P15* : *Cong* (*Geos* (*Ang* (*An B1 A1 C1*)) *add Emp*) (*Geos* (*Ang* (*An B2 A2 C2*)) *add Emp*) $\Longrightarrow$
Cong (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos* (*Tri* (*Tr A2 B2 C2*)) *add Emp*) **by** (*simp add*:*Tri-AAS-lemma1*)
from *assms P14 P15* **show** *Cong* (*Geos* (*Tri* (*Tr A1 B1 C1*)) *add Emp*) (*Geos*

(*Tri* (*Tr A2 B2 C2*)) *add Emp*) **by** *blast*
qed

Theorem26

theorem (**in** *Congruence-Rule*) *Seg-bisection* :
assumes
¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*)
shows
∃ *p*. *Eq* (*Geos* (*Seg* (*Se A p*)) *add Emp*) (*Geos* (*Seg* (*Se p B*)) *add Emp*) ∧ *Bet-Point* (*Se A B*) *p*
proof −
have ∃ *p q r*. ¬ *Line-on* (*Li A B*) *p* ∧ ¬ *Line-on* (*Li A B*) *q* ∧ ¬ *Line-on* (*Li A B*) *r*
∧ ¬ *Eq* (*Geos* (*Poi p*) *add Emp*) (*Geos* (*Poi q*) *add Emp*) ∧ ¬ *Eq* (*Geos* (*Poi q*) *add Emp*) (*Geos* (*Poi r*) *add Emp*)
∧ ¬ *Eq* (*Geos* (*Poi r*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*blast intro*:*Line-not-on-exist*)
then obtain *C* :: *Point* **where** *P1* : ¬ *Line-on* (*Li A B*) *C* **by** *blast*
from *assms P1* **have** *P2* : *Def* (*Ang* (*An A B C*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P2* : *Def* (*Ang* (*An C A B*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *assms* **have** *P3* : *Eq* (*Geos* (*Lin* (*Li A B*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-rev*)
from *P1 P3* **have** *P4* : ¬ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-not-on-trans*)
from *P2 P4* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An p B A*)) *add Emp*)
∧ *Plane-diffside* (*Li B A*) *p C* **by** (*simp add*:*Ang-move-diffside*)
then obtain *D1* :: *Point* **where** *P5* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An D1 B A*)) *add Emp*)
∧ *Plane-diffside* (*Li B A*) *D1 C* **by** *blast*
then have ∃ *p*. *Bet-Point* (*Se D1 C*) *p* ∧ *Line-on* (*Li B A*) *p* ∧ ¬ *Line-on* (*Li B A*) *D1* ∧ ¬ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Plane-diffside-def*)
then have *P6* : ¬ *Line-on* (*Li B A*) *D1* **by** *blast*
from *assms* **have** *P7* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P6 P7* **have** *Def* (*Ang* (*An B A D1*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P8* : *Def* (*Ang* (*An D1 B A*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P2 P5 P8* **have** ∃ *p*. *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An p B A*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An D1 B A*)) *add Emp*) (*Geos* (*Ang* (*An p B A*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se B p*)) *add Emp*) ∧ *Line-on* (*Li B D1*) *p*
∧ ¬ *Bet-Point* (*Se p D1*) *B* ∧ *Def* (*Ang* (*An p B A*)) **by** (*simp add*:*Ang-replace*)
then obtain *D* :: *Point* **where** *P9* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*)
∧ *Eq* (*Geos* (*Ang* (*An D1 B A*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*)
∧ *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se B D*)) *add Emp*) ∧ *Line-on* (*Li B D1*) *D*
∧ ¬ *Bet-Point* (*Se D D1*) *B* ∧ *Def* (*Ang* (*An D B A*)) **by** *blast*
have *Plane-diffside* (*Li B A*) *D D1* ⟹

$\exists p.$ *Bet-Point* (*Se D D1*) $p \wedge$ *Line-on* (*Li B A*) $p \wedge \neg$ *Line-on* (*Li B A*) $D \wedge \neg$ *Line-on* (*Li B A*) *D1* **by** (*simp add*:*Plane-diffside-def*)
then obtain *B1* :: *Point* **where** *P10* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Bet-Point* (*Se D D1*) *B1* $\wedge$ *Line-on* (*Li B A*) *B1* $\wedge \neg$ *Line-on* (*Li B A*) *D* $\wedge \neg$ *Line-on* (*Li B A*) *D1* **by** *blast*
then have *P11* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Line-on* (*Li D D1*) *B1* **by** (*simp add*:*Line-Bet-on*)
from *P10* **have** *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Bet-Point* (*Se D D1*) *B1* **by** *simp*
then have *P12* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi D1*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
have *P13* : *Line-on* (*Li B D1*) *D1* **by** (*simp add*:*Line-on-rule*)
have *P14* : *Line-on* (*Li D D1*) *D* **by** (*simp add*:*Line-on-rule*)
have *P15* : *Line-on* (*Li D D1*) *D1* **by** (*simp add*:*Line-on-rule*)
from *P9* **have** *P16* : *Line-on* (*Li B D1*) *D* **by** *simp*
from *P12 P13 P14 P15 P16* **have** *P17* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Eq* (*Geos* (*Lin* (*Li B D1*)) *add Emp*) (*Geos* (*Lin* (*Li D D1*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P18* : *Line-on* (*Li B D1*) *B* **by** (*simp add*:*Line-on-rule*)
from *P17 P18* **have** *P19* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Line-on* (*Li D D1*) *B* **by** (*simp add*:*Line-on-trans*)
from *P14* **have** *P20* : *Eq* (*Geos* (*Lin* (*Li D D1*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) $\Longrightarrow$ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-trans*)
from *P10 P20* **have** *P21* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ $\neg$ *Eq* (*Geos* (*Lin* (*Li D D1*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** *blast*
have *P22* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
from *P10 P11 P19 P21 P22* **have** *P23* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Eq* (*Geos* (*Poi B1*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Line-unique-Point*)
from *P10 P23* **have** *P24* : *Plane-diffside* (*Li B A*) *D D1* $\Longrightarrow$ *Bet-Point* (*Se D D1*) *B* **by** (*blast intro*:*Point-Eq*)
from *P9 P24* **have** *P25* : $\neg$ *Plane-diffside* (*Li B A*) *D D1* **by** *blast*
from *P5* **have** *P26* : *Plane-sameside* (*Li B A*) *C D* $\Longrightarrow$ *Plane-diffside* (*Li B A*) *D D1* **by** (*simp add*:*Plane-diffside-rev Plane-trans*)
from *P25 P26* **have** *P27* : $\neg$ *Plane-sameside* (*Li B A*) *C D* **by** *blast*
from *P5* **have** *P28* : *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) $\Longrightarrow$ *Plane-diffside* (*Li B A*) *D1 D* **by** (*blast intro*:*Point-Eq*)
from *P25 P28* **have** *P29* : $\neg$ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Plane-diffside-rev*)
from *P9* **have** *Def* (*Tri* (*Tr B A D*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P30* : $\neg$ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Tri-def-Line*)
from *P4 P27 P29 P30* **have** *Plane-diffside* (*Li B A*) *C D* **by** (*simp add*:*Plane-not-sameside-diffside*)
then have $\exists p.$ *Bet-Point* (*Se C D*) $p \wedge$ *Line-on* (*Li B A*) $p \wedge \neg$ *Line-on* (*Li B A*) $C \wedge \neg$ *Line-on* (*Li B A*) D **by** (*simp add*:*Plane-diffside-def*)
then obtain *E* :: *Point* **where** *P31* : *Bet-Point* (*Se C D*) *E* $\wedge$ *Line-on* (*Li B A*) *E* $\wedge \neg$ *Line-on* (*Li B A*) *C* $\wedge \neg$ *Line-on* (*Li B A*) *D* **by** *blast*
then have *P32* : *Bet-Point* (*Se C D*) *E* **by** *simp*

then have *P33* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Bet-Point* (*Se D C*) *A* **by** (*blast intro*:*Point-Eq Bet-rev*)
from *P9* **have** *P34* : *Def* (*Tri* (*Tr A D B*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P33 P34* **have** *P35* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹
Gr (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
from *P32* **have** *P36* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹ *Bet-Point* (*Se C D*) *B* **by** (*simp add*:*Point-Eq*)
from *P2* **have** *P37* : *Def* (*Tri* (*Tr B C A*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P36 P37* **have** *P38* : *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) ⟹
Gr (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
have *P39* : *Eq* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P9 P39* **have** *P40* : *Cong* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
have *P41* : *Eq* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P40 P41* **have** *P42* : *Cong* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P2* **have** *P43* : *Def* (*Ang* (*An B A C*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P9* **have** *P44* : *Def* (*Ang* (*An A B D*)) **by** (*simp add*:*Ang-def-rev*)
from *P42 P43 P44* **have** *P45* : ¬ *Gr* (*Geos* (*Ang* (*An B A C*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr*)
from *P35 P45* **have** *P46* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** *blast*
from *P38 P45* **have** *P47* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
have *P48* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
have *P49* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms P31 P46 P47 P48 P49* **have** *P50* : *Bet-Point* (*Se A E*) *B* ∨ *Bet-Point* (*Se E B*) *A*
∨ *Bet-Point* (*Se B A*) *E* **by** (*simp add*:*Bet-case*)
have *P51* : *Bet-Point* (*Se A E*) *B* ⟹ *Line-on* (*Li B E*) *A* **by** (*simp add*:*Line-Bet-on*)
have *P52* : *Line-on* (*Li B E*) *B* **by** (*simp add*:*Line-on-rule*)
from *assms P48 P49 P51 P52* **have** *Bet-Point* (*Se A E*) *B* ⟹
Eq (*Geos* (*Lin* (*Li B E*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P53* : *Bet-Point* (*Se A E*) *B* ⟹ *Line-on* (*Li B E*) *D* ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-trans*)
from *P30 P53* **have** *P54* : *Bet-Point* (*Se A E*) *B* ⟹ ¬ *Line-on* (*Li B E*) *D* **by** *blast*

from *P47 P54* **have** *P55* : *Bet-Point* (*Se A E*) *B* ⟹ *Def* (*Tri* (*Tr B E D*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
have *P56* : *Bet-Point* (*Se A E*) *B* ⟹ *Bet-Point* (*Se E A*) *B* **by** (*simp add*:*Bet-rev*)
from *P55 P56* **have** *P57* : *Bet-Point* (*Se A E*) *B* ⟹
Gr (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An B E D*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
have *P58* : *Bet-Point* (*Se A E*) *B* ⟹ *Line-on* (*Li E A*) *B* **by** (*simp add*:*Line-Bet-on*)
have *P59* : *Line-on* (*Li E A*) *A* **by** (*simp add*:*Line-on-rule*)
from *assms P48 P49 P58 P59* **have** *P60* : *Bet-Point* (*Se A E*) *B* ⟹
Eq (*Geos* (*Lin* (*Li E A*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P61* : *Bet-Point* (*Se A E*) *B* ⟹ *Line-on* (*Li E A*) *C* ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-on-trans*)
from *P31 P61* **have** *P62* : *Bet-Point* (*Se A E*) *B* ⟹ ¬ *Line-on* (*Li E A*) *C* **by** *blast*
from *P46 P62* **have** *Bet-Point* (*Se A E*) *B* ⟹ *Def* (*Tri* (*Tr E A C*)) **by** (*simp add*:*Ang-sinple-def Ang-to-Tri*)
then have *P63* : *Bet-Point* (*Se A E*) *B* ⟹ *Def* (*Tri* (*Tr E C A*)) **by** (*blast intro*:*Tri-def-rev Tri-def-trans*)
from *P31 P63* **have** *P64* : *Bet-Point* (*Se A E*) *B* ⟹
Gr (*Geos* (*Ang* (*An A E D*)) *add Emp*) (*Geos* (*Ang* (*An E A C*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
from *P63* **have** *P65* : *Bet-Point* (*Se A E*) *B* ⟹ *Def* (*Ang* (*An E A C*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-inv*)
have *P66* : *Bet-Point* (*Se A E*) *B* ⟹ *Line-on* (*Li A E*) *B* **by** (*simp add*:*Line-Bet-on*)
have *Bet-Point* (*Se A E*) *B* ⟹ *Inv* (*Bet-Point* (*Se E B*) *A*) **by** (*simp add*:*Bet-iff*)
then have *P67* : *Bet-Point* (*Se A E*) *B* ⟹ ¬ *Bet-Point* (*Se E B*) *A* **by** (*simp add*:*Inv-def*)
have *P68* : *Line-on* (*Li A C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P69* : ¬ *Bet-Point* (*Se C C*) *A* **by** (*simp add*:*Bet-end-Point*)
from *P2* **have** ¬ *Eq* (*Geos* (*Poi C*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P70* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *assms P65 P66 P67 P68 P69 P70* **have** *P71* : *Bet-Point* (*Se A E*) *B* ⟹
Eq (*Geos* (*Ang* (*An E A C*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) ∧ *Def* (*Ang* (*An B A C*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P72* : *Bet-Point* (*Se A E*) *B* ⟹
Cong (*Geos* (*Ang* (*An E A C*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev*)
from *P60* **have** *P73* : *Bet-Point* (*Se A E*) *B* ⟹ *Line-on* (*Li E A*) *D* ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-trans*)
from *P31 P73* **have** *P74* : *Bet-Point* (*Se A E*) *B* ⟹ ¬ *Line-on* (*Li E A*) *D* **by** *blast*
from *P46 P74* **have** *Bet-Point* (*Se A E*) *B* ⟹ *Def* (*Ang* (*An E A D*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P75* : *Bet-Point* (*Se A E*) *B* ⟹ *Def* (*Ang* (*An A E D*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
from *P64 P65 P71 P72 P75* **have** *P76* : *Bet-Point* (*Se A E*) *B* ⟹

Gr (*Geos* (*Ang* (*An A E D*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
have *Bet-Point* (*Se A E*) *B* ⟹ *Inv* (*Bet-Point* (*Se B A*) *E*) **by** (*simp add*:*Bet-iff*)
then have *Bet-Point* (*Se A E*) *B* ⟹ ¬ *Bet-Point* (*Se B A*) *E* **by** (*simp add*:*Inv-def*)
then have *P77* : *Bet-Point* (*Se A E*) *B* ⟹ ¬ *Bet-Point* (*Se A B*) *E* **by** (*blast intro*:*Bet-rev*)
have *P78* : *Line-on* (*Li E D*) *D* **by** (*simp add*:*Line-on-rule*)
have *P79* : ¬ *Bet-Point* (*Se D D*) *E* **by** (*simp add*:*Bet-end-Point*)
from *P47* **have** *P80* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P32* **have** ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
then have *P81* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P58 P75 P77 P78 P79 P80 P81* **have** *P82* : *Bet-Point* (*Se A E*) *B* ⟹
Eq (*Geos* (*Ang* (*An A E D*)) *add Emp*) (*Geos* (*Ang* (*An B E D*)) *add Emp*) ∧ *Def* (*Ang* (*An B E D*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P83* : *Bet-Point* (*Se A E*) *B* ⟹
Cong (*Geos* (*Ang* (*An A E D*)) *add Emp*) (*Geos* (*Ang* (*An B E D*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
from *P9 P57 P75 P82 P83* **have** *P84* : *Bet-Point* (*Se A E*) *B* ⟹
Gr (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An A E D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P9 P71 P75 P76 P84* **have** *P85* : *Bet-Point* (*Se A E*) *B* ⟹
Gr (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Gr*)
from *P9 P40 P71* **have** *P86* : *Bet-Point* (*Se A E*) *B* ⟹
¬ *Gr* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An B A C*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr*)
from *P85 P86* **have** *P87* : ¬ *Bet-Point* (*Se A E*) *B* **by** *blast*
have *P88* : *Bet-Point* (*Se E B*) *A* ⟹ *Line-on* (*Li E A*) *B* **by** (*simp add*:*Line-Bet-on*)
from *assms P48 P49 P59 P88* **have** *P89* : *Bet-Point* (*Se E B*) *A* ⟹
Eq (*Geos* (*Lin* (*Li E A*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P90* : *Bet-Point* (*Se E B*) *A* ⟹ *Line-on* (*Li E A*) *C* ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-on-trans*)
from *P31 P90* **have** *P91* : *Bet-Point* (*Se E B*) *A* ⟹ ¬ *Line-on* (*Li E A*) *C* **by** *blast*
from *P46 P91* **have** *Bet-Point* (*Se E B*) *A* ⟹ *Def* (*Ang* (*An E A C*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P92* : *Bet-Point* (*Se E B*) *A* ⟹ *Def* (*Tri* (*Tr A E C*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P93* : *Bet-Point* (*Se E B*) *A* ⟹ *Gr* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An A E C*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
have *P94* : *Bet-Point* (*Se E B*) *A* ⟹ *Line-on* (*Li B E*) *A* **by** (*simp add*:*Line-Bet-on*)
from *assms P48 P49 P52 P94* **have** *Bet-Point* (*Se E B*) *A* ⟹
Eq (*Geos* (*Lin* (*Li B E*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)

then have *P95* : *Bet-Point* (*Se E B*) *A* ⟹ *Line-on* (*Li B E*) *D* ⟹ *Line-on* (*Li B A*) *D* **by** (*simp add*:*Line-on-trans*)
from *P31 P95* **have** *P96* : *Bet-Point* (*Se E B*) *A* ⟹ ¬ *Line-on* (*Li B E*) *D* **by** *blast*
from *P47 P96* **have** *Bet-Point* (*Se E B*) *A* ⟹ *Def* (*Ang* (*An B E D*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P97* : *Bet-Point* (*Se E B*) *A* ⟹ *Def* (*Tri* (*Tr E D B*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P32* **have** *P98* : *Bet-Point* (*Se D C*) *E* **by** (*simp add*:*Bet-rev*)
from *P97 P98* **have** *P99* : *Bet-Point* (*Se E B*) *A* ⟹
Gr (*Geos* (*Ang* (*An B E C*)) *add Emp*) (*Geos* (*Ang* (*An E B D*)) *add Emp*) **by** (*simp add*:*Ang-external-Gr*)
from *P92* **have** *P100* : *Bet-Point* (*Se E B*) *A* ⟹ *Def* (*Ang* (*An A E C*)) **by** (*simp add*:*Tri-to-Ang*)
have *Bet-Point* (*Se E B*) *A* ⟹ *Inv* (*Bet-Point* (*Se B A*) *E*) **by** (*simp add*:*Bet-iff*)
then have *Bet-Point* (*Se E B*) *A* ⟹ ¬ *Bet-Point* (*Se B A*) *E* **by** (*simp add*:*Inv-def*)
then have *P101* : *Bet-Point* (*Se E B*) *A* ⟹ ¬ *Bet-Point* (*Se A B*) *E* **by** (*blast intro*:*Bet-rev*)
have *P102* : *Line-on* (*Li E C*) *C* **by** (*simp add*:*Line-on-rule*)
have *P103* : ¬ *Bet-Point* (*Se C C*) *E* **by** (*simp add*:*Bet-end-Point*)
from *P32* **have** *P104* : ¬ *Eq* (*Geos* (*Poi E*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P80 P88 P100 P101 P102 P103 P104* **have** *P105* : *Bet-Point* (*Se E B*) *A* ⟹
Eq (*Geos* (*Ang* (*An A E C*)) *add Emp*) (*Geos* (*Ang* (*An B E C*)) *add Emp*) ∧ *Def* (*Ang* (*An B E C*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P106* : *Bet-Point* (*Se E B*) *A* ⟹
Cong (*Geos* (*Ang* (*An A E C*)) *add Emp*) (*Geos* (*Ang* (*An B E C*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
have *P107* : *Bet-Point* (*Se E B*) *A* ⟹ *Line-on* (*Li B A*) *E* **by** (*simp add*:*Line-Bet-on*)
have *Bet-Point* (*Se E B*) *A* ⟹ *Inv* (*Bet-Point* (*Se A E*) *B*) **by** (*simp add*:*Bet-iff*)
then have *P108* : *Bet-Point* (*Se E B*) *A* ⟹ ¬ *Bet-Point* (*Se A E*) *B* **by** (*simp add*:*Inv-def*)
have *P109* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
have *P110* : ¬ *Bet-Point* (*Se D D*) *B* **by** (*simp add*:*Bet-end-Point*)
from *P44* **have** *P111* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P44 P47 P107 P108 P109 P110 P111* **have** *P112* : *Bet-Point* (*Se E B*) *A* ⟹
Eq (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An E B D*)) *add Emp*) ∧ *Def* (*Ang* (*An E B D*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P113* : *Bet-Point* (*Se E B*) *A* ⟹
Cong (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An E B D*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans*)
from *P2 P93 P100 P105 P106* **have** *P114* : *Bet-Point* (*Se E B*) *A* ⟹
Gr (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An B E C*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq*)
from *P44 P99 P105 P112 P113* **have** *P115* : *Bet-Point* (*Se E B*) *A* ⟹

Gr (*Geos* (*Ang* (*An B E C*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Eq Ang-rev*)
from *P2 P44 P105 P114 P115* **have** *P116* : *Bet-Point* (*Se E B*) *A* ⟹
Gr (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*blast intro*:*Ang-Gr-trans-Gr-Gr*)
from *P9 P41* **have** *P117* : *Cong* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P2 P44 P117* **have** *P118* : ¬ *Gr* (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*)
∧ ¬ *Gr* (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An C A B*)) *add Emp*) **by** (*simp add*:*Ang-not-Gr*)
from *P116 P118* **have** *P119* : ¬ *Bet-Point* (*Se E B*) *A* **by** *blast*
from *P50 P87 P119* **have** *P120* : *Bet-Point* (*Se B A*) *E* **by** *blast*
then have *P121* : *Line-on* (*Li B E*) *A* **by** (*simp add*:*Line-Bet-on*)
from *assms P48 P49 P52 P121* **have** *Eq* (*Geos* (*Lin* (*Li B E*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
then have *P122* : *Line-on* (*Li B E*) *C* ⟹ *Line-on* (*Li B A*) *C* **by** (*simp add*:*Line-on-trans*)
from *P31 P122* **have** *P123* : ¬ *Line-on* (*Li B E*) *C* **by** *blast*
from *P47 P123* **have** *P124* : *Def* (*Ang* (*An B E C*)) **by** (*simp add*:*Ang-sinple-def*)
from *P32 P120 P124* **have** *P125* : *Cong* (*Geos* (*Ang* (*An C E A*)) *add Emp*) (*Geos* (*Ang* (*An B E D*)) *add Emp*) **by** (*simp add*:*Ang-vertical*)
have *P126* : *Eq* (*Geos* (*Seg* (*Se A C*)) *add Emp*) (*Geos* (*Seg* (*Se C A*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P127* : *Eq* (*Geos* (*Seg* (*Se B D*)) *add Emp*) (*Geos* (*Seg* (*Se D B*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P9 P126 P127* **have** *P128* : *Eq* (*Geos* (*Seg* (*Se C A*)) *add Emp*) (*Geos* (*Seg* (*Se D B*)) *add Emp*) **by** (*blast intro*:*Eq-trans Eq-rev*)
have *P129* : *Eq* (*Geos* (*Ang* (*An B E D*)) *add Emp*) (*Geos* (*Ang* (*An D E B*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P125 P129* **have** *P130* : *Cong* (*Geos* (*Ang* (*An C E A*)) *add Emp*) (*Geos* (*Ang* (*An D E B*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P120* **have** *P131* : *Line-on* (*Li A B*) *E* **by** (*simp add*:*Line-Bet-on*)
from *P119* **have** *P132* : ¬ *Bet-Point* (*Se B E*) *A* **by** (*blast intro*:*Bet-rev*)
from *P46* **have** *P133* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi E*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P2 P68 P69 P70 P131 P132 P133* **have** *P134* :
Eq (*Geos* (*Ang* (*An C A B*)) *add Emp*) (*Geos* (*Ang* (*An C A E*)) *add Emp*) ∧ *Def* (*Ang* (*An C A E*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P135* : *Def* (*Tri* (*Tr C A E*)) **by** (*simp add*:*Ang-to-Tri*)
from *P9 P134* **have** *P136* : *Cong* (*Geos* (*Ang* (*An C A E*)) *add Emp*) (*Geos* (*Ang* (*An D B A*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P9* **have** *P137* : *Def* (*Ang* (*An D B A*)) **by** *simp*
from *P31 P47 P87 P109 P110 P111 P137* **have** *P138* :
Eq (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An D B E*)) *add Emp*) ∧ *Def* (*Ang* (*An D B E*)) **by** (*simp add*:*Ang-Point-swap*)
then have *P139* : *Def* (*Tri* (*Tr D B E*)) **by** (*simp add*:*Ang-to-Tri*)
from *P136 P138* **have** *P140* : *Cong* (*Geos* (*Ang* (*An C A E*)) *add Emp*) (*Geos* (*Ang* (*An D B E*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)

from *P128 P130 P135 P139 P140* **have** *Cong* (*Geos* (*Tri* (*Tr C A E*)) *add Emp*) (*Geos* (*Tri* (*Tr D B E*)) *add Emp*) **by** (*simp add*:*Tri-AAS*)
then have *P141* : *Eq* (*Geos* (*Seg* (*Se A E*)) *add Emp*) (*Geos* (*Seg* (*Se B E*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
have *P142* : *Eq* (*Geos* (*Seg* (*Se B E*)) *add Emp*) (*Geos* (*Seg* (*Se E B*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P141 P142* **have** *P143* : *Eq* (*Geos* (*Seg* (*Se A E*)) *add Emp*) (*Geos* (*Seg* (*Se E B*)) *add Emp*) **by** (*blast intro*:*Eq-trans*)
from *P120* **have** *P144* : *Bet-Point* (*Se A B*) *E* **by** (*simp add*:*Bet-rev*)
from *P143 P144* **show** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se A p*)) *add Emp*) (*Geos* (*Seg* (*Se p B*)) *add Emp*) ∧ *Bet-Point* (*Se A B*) *p* **by** *blast*
qed

theorem (**in** *Congruence-Rule*) *Ang-bisection* :
assumes
Def (*Ang* (*An A B C*))
shows
∃ *p*. *Cong* (*Geos* (*Ang* (*An A B p*)) *add Emp*) (*Geos* (*Ang* (*An p B C*)) *add Emp*)
∧ *Ang-inside* (*An A B C*) *p* ∧ *Def* (*Ang* (*An A B p*)) ∧ *Def* (*Ang* (*An p B C*))
proof −
have *P1* : *Line-on* (*Li B C*) *B* **by** (*simp add*:*Line-on-rule*)
have *P2* : *Line-on* (*Li B C*) *C* **by** (*simp add*:*Line-on-rule*)
from *assms* **have** *P3* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C*) *add Emp*) **by** (*simp add*:*Ang-def*)
from *assms* **have** ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*simp add*:*Ang-def*)
then have *P4* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P1 P2 P3 P4* **have** ∃ *p*. *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B p*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se p C*) *B* ∧ *Line-on* (*Li B C*) *p* ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi p*) *add Emp*) **by** (*simp add*:*Seg-move-sameside*)
then obtain *C2* :: *Point* **where** *P5* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B C2*)) *add Emp*)
∧ ¬ *Bet-Point* (*Se C2 C*) *B* ∧ *Line-on* (*Li B C*) *C2* ∧ ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** *blast*
then have *P6* : *Line-on* (*Li B C*) *C2* **by** *simp*
then have *P7* : *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) ⟹ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Point-Eq*)
from *assms* **have** *P8* : *Def* (*Tri* (*Tr A B C*)) **by** (*simp add*:*Ang-to-Tri*)
then have *P9* : ¬ *Line-on* (*Li B C*) *A* **by** (*simp add*:*Tri-def-Line*)
from *P7 P9* **have** *P10* : ¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi A*) *add Emp*) **by** *blast*
then have ∃ *p*. *Eq* (*Geos* (*Seg* (*Se C2 p*)) *add Emp*) (*Geos* (*Seg* (*Se p A*)) *add Emp*) ∧ *Bet-Point* (*Se C2 A*) *p* **by** (*simp add*:*Seg-bisection*)
then obtain *D* :: *Point* **where** *P11* : *Eq* (*Geos* (*Seg* (*Se C2 D*)) *add Emp*) (*Geos* (*Seg* (*Se D A*)) *add Emp*)
∧ *Bet-Point* (*Se C2 A*) *D* **by** *blast*

have *P12* : *Line-on* (*Li B A*) *B* **by** (*simp add*:*Line-on-rule*)
from *P1 P5 P6 P12* **have** *P13* : *Line-on* (*Li B A*) *C2* ⟹
Eq (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *assms* **have** *P14* : ¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C*)) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P13 P14* **have** *P15* : ¬ *Line-on* (*Li B A*) *C2* **by** *blast*
from *P11* **have** *P16* : *Bet-Point* (*Se C2 A*) *D* **by** *simp*
then have *P17* : *Line-on* (*Li C2 A*) *D* **by** (*simp add*:*Line-Bet-on*)
have *P18* : *Line-on* (*Li C2 A*) *A* **by** (*simp add*:*Line-on-rule*)
have *P19* : *Line-on* (*Li B A*) *A* **by** (*simp add*:*Line-on-rule*)
from *P16* **have** *P20* : ¬ *Eq* (*Geos* (*Poi A*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P17 P18 P19 P20* **have** *P21* : *Line-on* (*Li B A*) *D* ⟹
Eq (*Geos* (*Lin* (*Li C2 A*)) *add Emp*) (*Geos* (*Lin* (*Li B A*)) *add Emp*) **by** (*simp add*:*Line-unique*)
have *P22* : *Line-on* (*Li C2 A*) *C2* **by** (*simp add*:*Line-on-rule*)
from *P21 P22* **have** *P23* : *Line-on* (*Li B A*) *D* ⟹ *Line-on* (*Li B A*) *C2* **by** (*simp add*:*Line-on-trans*)
from *P15 P23* **have** *P24* : ¬ *Line-on* (*Li B A*) *D* **by** *blast*
from *P4 P24* **have** *Def* (*Ang* (*An B A D*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P25* : *Def* (*Tri* (*Tr A B D*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
from *P4 P15* **have** *P26* : *Def* (*Ang* (*An B A C2*)) **by** (*simp add*:*Ang-sinple-def*)
then have *Def* (*Tri* (*Tr C2 B A*)) **by** (*blast intro*:*Ang-to-Tri Tri-def-rev Tri-def-trans*)
then have *P27* : ¬ *Line-on* (*Li C2 B*) *A* **by** (*simp add*:*Tri-def-Line*)
have *P28* : *Line-on* (*Li C2 B*) *C2* **by** (*simp add*:*Line-on-rule*)
from *P16* **have** *P29* : ¬ *Eq* (*Geos* (*Poi D*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*simp add*:*Bet-Point-def*)
from *P17 P22 P28 P29* **have** *P30* : *Line-on* (*Li C2 B*) *D* ⟹
Eq (*Geos* (*Lin* (*Li C2 A*)) *add Emp*) (*Geos* (*Lin* (*Li C2 B*)) *add Emp*) **by** (*simp add*:*Line-unique*)
from *P18 P30* **have** *P31* : *Line-on* (*Li C2 B*) *D* ⟹ *Line-on* (*Li C2 B*) *A* **by** (*simp add*:*Line-on-trans*)
from *P27 P31* **have** *P32* : ¬ *Line-on* (*Li C2 B*) *D* **by** *blast*
from *P5* **have** *P33* : ¬ *Eq* (*Geos* (*Poi C2*) *add Emp*) (*Geos* (*Poi B*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P32 P33* **have** *Def* (*Ang* (*An C2 B D*)) **by** (*simp add*:*Ang-sinple-def*)
then have *P34* : *Def* (*Tri* (*Tr C2 B D*)) **by** (*simp add*:*Ang-to-Tri*)
have *P35* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se A B*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
have *P36* : *Eq* (*Geos* (*Seg* (*Se B C2*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B*)) *add Emp*) **by** (*simp add*:*Seg-rev*)
from *P5* **have** *P37* : *Eq* (*Geos* (*Seg* (*Se B A*)) *add Emp*) (*Geos* (*Seg* (*Se B C2*)) *add Emp*) **by** *simp*
from *P35 P36 P37* **have** *P38* : *Eq* (*Geos* (*Seg* (*Se A B*)) *add Emp*) (*Geos* (*Seg* (*Se C2 B*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
have *P39* : *Eq* (*Geos* (*Seg* (*Se C2 D*)) *add Emp*) (*Geos* (*Seg* (*Se D C2*)) *add Emp*) **by** (*simp add*:*Seg-rev*)

from *P11 P39* **have** *P40* : *Eq* (*Geos* (*Seg* (*Se D A*)) *add Emp*) (*Geos* (*Seg* (*Se D C2*)) *add Emp*) **by** (*blast intro*:*Eq-rev Eq-trans*)
from *P25 P34 P38 P40* **have** *Cong* (*Geos* (*Tri* (*Tr A B D*)) *add Emp*) (*Geos* (*Tri* (*Tr C2 B D*)) *add Emp*) **by** (*simp add*:*Tri-SSS*)
then have *P41* : *Cong* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An D B C2*)) *add Emp*) **by** (*simp add*:*Tri-Cong-def*)
have *P42* : *Eq* (*Geos* (*Ang* (*An D B A*)) *add Emp*) (*Geos* (*Ang* (*An A B D*)) *add Emp*) **by** (*simp add*:*Ang-roll*)
from *P41 P42* **have** *P43* : *Cong* (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An D B C2*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P34* **have** *P44* : *Def* (*Ang* (*An D B C2*)) **by** (*blast intro*:*Tri-to-Ang Ang-def-rev Ang-def-inv*)
have *P45* : *Line-on* (*Li B D*) *D* **by** (*simp add*:*Line-on-rule*)
have *P46* : ¬ *Bet-Point* (*Se D D*) *B* **by** (*simp add*:*Bet-end-Point*)
from *P33* **have** *P47* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi C2*) *add Emp*) **by** (*blast intro*:*Eq-rev*)
from *P3 P6 P47* **have** *P48* : *Line-on* (*Li B C2*) *C* **by** (*simp add*:*Line-on-rev*)
from *P34* **have** *P49* : ¬ *Eq* (*Geos* (*Poi B*) *add Emp*) (*Geos* (*Poi D*) *add Emp*) **by** (*simp add*:*Tri-def*)
from *P3 P5 P44 P45 P46 P48 P49* **have** *P50* : *Eq* (*Geos* (*Ang* (*An D B C2*)) *add Emp*) (*Geos* (*Ang* (*An D B C*)) *add Emp*)
∧ *Def* (*Ang* (*An D B C*)) **by** (*simp add*:*Ang-Point-swap*)
from *P43 P50* **have** *P51* : *Cong* (*Geos* (*Ang* (*An A B D*)) *add Emp*) (*Geos* (*Ang* (*An D B C*)) *add Emp*) **by** (*blast intro*:*Ang-weektrans Ang-rev Eq-rev*)
from *P25* **have** *P52* : *Def* (*Ang* (*An A B D*)) **by** (*simp add*:*Tri-to-Ang*)
from *P26* **have** *P53* : *Def* (*Ang* (*An A B C2*)) **by** (*blast intro*:*Ang-def-rev Ang-def-inv*)
then have *P54* : ¬ *Eq* (*Geos* (*Lin* (*Li B A*)) *add Emp*) (*Geos* (*Lin* (*Li B C2*)) *add Emp*) **by** (*simp add*:*Ang-def*)
from *P16* **have** *P55* : *Bet-Point* (*Se A C2*) *D* **by** (*simp add*:*Bet-rev*)
from *P4 P5 P54 P55* **have** *P56* : *Ang-inside* (*An A B C2*) *D* **by** (*simp add*:*Ang-inside-Bet-Point*)
have *P57* : ¬ *Bet-Point* (*Se A A*) *B* **by** (*simp add*:*Bet-end-Point*)
from *P3 P4 P5 P6 P19 P53 P56 P57* **have** *P58* : *Ang-inside* (*An A B C*) *D* **by** (*simp add*:*Ang-inside-trans*)
from *P50 P51 P52 P58* **show** ∃*p*. *Cong* (*Geos* (*Ang* (*An A B p*)) *add Emp*) (*Geos* (*Ang* (*An p B C*)) *add Emp*)
∧ *Ang-inside* (*An A B C*) *p* ∧ *Def* (*Ang* (*An A B p*)) ∧ *Def* (*Ang* (*An p B C*)) **by** *blast*
qed

end

References

[1] D. Hilbert. *The Foundations of Geometry*. https://math.berkeley.edu/wodzicki/160/Hilbert.pdf.