

The Földes–Hammer Characterization of Split Graphs

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Abstract

This entry formalizes the classical Földes–Hammer theorem characterizing finite split graphs as exactly the graphs with no induced copy of $2K_2$, C_4 , or C_5 . It builds on the AFP session `Undirected_Graph_Theory`.

1 Overview

A finite graph is split when its vertex set can be partitioned into a clique and an independent set. The main theorem of this entry is `foldes_hammer` in `Foldes_Hammer.thy`, which proves the equivalence between split graphs and the absence of induced $2K_2$, C_4 , and C_5 .

The forward direction is established by analyzing how any split partition excludes each forbidden configuration. The converse direction follows an extremal argument: choose a maximum clique whose complement minimizes the number of induced edges, orient any remaining outside edge according to nested nonneighbor sets in the clique, and swap one clique vertex with an outside vertex to derive a contradiction.

2 Sources

The entry follows the classical theorem due to Földes and Hammer [2] and standard graph-class background as presented by Brandstädt, Le, and Spinrad [1].

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theory *Foldes-Hammer*

imports *Undirected-Graph-Theory.Undirected-Graphs-Root*
begin

context *fin-sgraph*
begin

definition *is-clique* :: 'a set $\Rightarrow$ bool **where**
is-clique $C \longleftrightarrow C \subseteq V \wedge \text{induced-edges } C = \text{all-edges } C$

definition *cliques* :: 'a set set **where**
cliques = $\{C. \text{is-clique } C\}$

definition *maximum-clique* :: 'a set $\Rightarrow$ bool **where**
maximum-clique $K \longleftrightarrow K \in \text{cliques} \wedge (\forall C \in \text{cliques}. \text{card } C \leq \text{card } K)$

definition *outside-edge-count* :: 'a set $\Rightarrow$ nat **where**
outside-edge-count $K = \text{card } (\text{induced-edges } (V - K))$

definition *is-split-partition* :: 'a set $\Rightarrow$ 'a set $\Rightarrow$ bool **where**
is-split-partition $C \ I \longleftrightarrow C \cap I = \{\} \wedge C \cup I = V \wedge \text{is-clique } C \wedge \text{is-independent-set } I$

definition *is-split-graph* :: bool **where**
is-split-graph $\longleftrightarrow (\exists C \ I. \text{is-split-partition } C \ I)$

definition *induces-2K2* :: 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ bool **where**
induces-2K2 $a \ b \ c \ d \longleftrightarrow$
 $\text{distinct } [a, b, c, d] \wedge$
 $\text{vert-adj } a \ b \wedge \text{vert-adj } c \ d \wedge$
 $\neg \text{vert-adj } a \ c \wedge \neg \text{vert-adj } a \ d \wedge \neg \text{vert-adj } b \ c \wedge \neg \text{vert-adj } b \ d$

definition *induces-C4* :: 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ bool **where**
induces-C4 $a \ b \ c \ d \longleftrightarrow$
 $\text{distinct } [a, b, c, d] \wedge$
 $\text{vert-adj } a \ b \wedge \text{vert-adj } b \ c \wedge \text{vert-adj } c \ d \wedge \text{vert-adj } d \ a \wedge$
 $\neg \text{vert-adj } a \ c \wedge \neg \text{vert-adj } b \ d$

definition *induces-C5* :: 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ 'a $\Rightarrow$ bool **where**
induces-C5 $a \ b \ c \ d \ e \longleftrightarrow$
 $\text{distinct } [a, b, c, d, e] \wedge$
 $\text{vert-adj } a \ b \wedge \text{vert-adj } b \ c \wedge \text{vert-adj } c \ d \wedge \text{vert-adj } d \ e \wedge \text{vert-adj } e \ a \wedge$
 $\neg \text{vert-adj } a \ c \wedge \neg \text{vert-adj } a \ d \wedge \neg \text{vert-adj } b \ d \wedge \neg \text{vert-adj } b \ e \wedge \neg \text{vert-adj } c \ e$

definition *has-induced-2K2* :: bool **where**
has-induced-2K2 $\longleftrightarrow (\exists a \ b \ c \ d. \text{induces-2K2 } a \ b \ c \ d)$

definition *has-induced-C4* :: bool **where**

has-induced-C4 $\longleftrightarrow (\exists a\ b\ c\ d. \text{induces-C4}\ a\ b\ c\ d)$

definition *has-induced-C5* :: bool **where**

has-induced-C5 $\longleftrightarrow (\exists a\ b\ c\ d\ e. \text{induces-C5}\ a\ b\ c\ d\ e)$

definition *foldes-hammer-free* :: bool **where**

foldes-hammer-free $\longleftrightarrow \neg \text{has-induced-2K2} \wedge \neg \text{has-induced-C4} \wedge \neg \text{has-induced-C5}$

lemma *is-clique-alt*:

is-clique *C* $\longleftrightarrow C \subseteq V \wedge (\forall u \in C. \forall v \in C. u \neq v \longrightarrow \text{vert-adj}\ u\ v)$
<proof>

lemma *clique-pair-adj*:

assumes *is-clique* *C* *u* $\in$ *C* *v* $\in$ *C* *u* $\neq$ *v*

shows *vert-adj* *u* *v*

<proof>

lemma *clique-mono*:

assumes *is-clique* *C* *D* $\subseteq$ *C*

shows *is-clique* *D*

<proof>

lemma *vert-adj-neg*:

assumes *vert-adj* *u* *v*

shows *u* $\neq$ *v*

<proof>

lemma *empty-clique*: *is-clique* {}

<proof>

lemma *singleton-clique*:

assumes *v* $\in$ *V*

shows *is-clique* {*v*}

<proof>

lemma *finite-cliques*: *finite* *cliques*

<proof>

lemma *cliques-nonempty*: {} $\in$ *cliques*

<proof>

lemma *maximum-clique-exists*:

obtains *K* **where** *maximum-clique* *K*

<proof>

lemma *maximum-clique-finite*:

assumes *maximum-clique* *K*

shows *finite* K
 $\langle \text{proof} \rangle$

lemma *finite-induced-edges-subset-V*:
 assumes $S \subseteq V$
 shows *finite* (*induced-edges* S)
 $\langle \text{proof} \rangle$

lemma *obtain-best-clique*:
 obtains K where *maximum-clique* K
 and $\forall L. \text{maximum-clique } L \longrightarrow \text{outside-edge-count } K \leq \text{outside-edge-count } L$
 $\langle \text{proof} \rangle$

lemma *insert-clique-if-all-adj*:
 assumes C : *is-clique* C
 and vV : $v \in V$
 and $vnot$: $v \notin C$
 and adj : $\forall u \in C. \text{vert-adj } u \ v$
 shows *is-clique* (*insert* $v \ C$)
 $\langle \text{proof} \rangle$

lemma *maximum-clique-outside-has-nonneighbor*:
 assumes $maxK$: *maximum-clique* K
 and vV : $v \in V$
 and $vnot$: $v \notin K$
 shows $\exists u \in K. \neg \text{vert-adj } u \ v$
 $\langle \text{proof} \rangle$

lemma *split-graph-alt*:
is-split-graph $\longleftrightarrow (\exists C. \text{is-clique } C \wedge \text{is-independent-set } (V - C))$
 $\langle \text{proof} \rangle$

lemma *independent-pair-not-adj*:
 assumes *is-independent-set* $I \ u \in I \ v \in I$
 shows $\neg \text{vert-adj } u \ v$
 $\langle \text{proof} \rangle$

lemma *split-partition-edge-hits-clique*:
 assumes sp : *is-split-partition* $C \ I$
 and uv : *vert-adj* $u \ v$
 shows $u \in C \vee v \in C$
 $\langle \text{proof} \rangle$

lemma *split-partition-nonedge-not-both-clique*:
 assumes sp : *is-split-partition* $C \ I$
 and uV : $u \in V$
 and vV : $v \in V$
 and uv : $u \neq v$
 and $not-adj$: $\neg \text{vert-adj } u \ v$

shows $\neg (u \in C \wedge v \in C)$
 $\langle proof \rangle$

lemma *induces-2K2-inV*:
assumes *induces-2K2 a b c d*
shows $a \in V \ b \in V \ c \in V \ d \in V$
 $\langle proof \rangle$

lemma *induces-C4-inV*:
assumes *induces-C4 a b c d*
shows $a \in V \ b \in V \ c \in V \ d \in V$
 $\langle proof \rangle$

lemma *induces-C5-inV*:
assumes *induces-C5 a b c d e*
shows $a \in V \ b \in V \ c \in V \ d \in V \ e \in V$
 $\langle proof \rangle$

lemma *split-partition-forbids-2K2*:
assumes *sp: is-split-partition C I*
and *H: induces-2K2 a b c d*
shows *False*
 $\langle proof \rangle$

lemma *split-partition-forbids-C4*:
assumes *sp: is-split-partition C I*
and *H: induces-C4 a b c d*
shows *False*
 $\langle proof \rangle$

lemma *split-partition-forbids-C5*:
assumes *sp: is-split-partition C I*
and *H: induces-C5 a b c d e*
shows *False*
 $\langle proof \rangle$

lemma *split-graph-imp-folde-hammer-free*:
assumes *is-split-graph*
shows *folde-hammer-free*
 $\langle proof \rangle$

lemma *outside-adjacent-nonneighbor-sets-nested*:
assumes *fh: folde-hammer-free*
and *Kmax: maximum-clique K*
and *uO: u ∈ V − K*
and *vO: v ∈ V − K*
and *uv: vert-adj u v*
shows $\{x \in K. \neg \text{vert-adj } x \ u\} \subseteq \{x \in K. \neg \text{vert-adj } x \ v\} \vee$
 $\{x \in K. \neg \text{vert-adj } x \ v\} \subseteq \{x \in K. \neg \text{vert-adj } x \ u\}$

$\langle proof \rangle$

lemma *outside-adjacent-smaller-nonneighbor-set-singleton*:

assumes *fh*: *foldes-hammer-free*
and *Kmax*: *maximum-clique K*
and *uO*: $u \in V - K$
and *vO*: $v \in V - K$
and *uv*: *vert-adj u v*
and *AB*: $\{x \in K. \neg \text{vert-adj } x \ u\} \subseteq \{x \in K. \neg \text{vert-adj } x \ v\}$
obtains *a* **where** $\{x \in K. \neg \text{vert-adj } x \ u\} = \{a\}$ and $a \in K$ and $\neg \text{vert-adj } a \ u$
and $\neg \text{vert-adj } a \ v$
 $\langle proof \rangle$

lemma *orient-outside-edge-for-swap*:

assumes *fh*: *foldes-hammer-free*
and *Kmax*: *maximum-clique K*
and *uO*: $u \in V - K$
and *vO*: $v \in V - K$
and *uv*: *vert-adj u v*
obtains *u' v' a* **where**
 $\{u', v'\} = \{u, v\}$
and $u' \in V - K$
and $v' \in V - K$
and *vert-adj u' v'*
and $\{x \in K. \neg \text{vert-adj } x \ u'\} = \{a\}$
and $a \in K$
and $\neg \text{vert-adj } a \ u'$
and $\neg \text{vert-adj } a \ v'$
 $\langle proof \rangle$

lemma *replace-unique-nonneighbor-maximum-clique*:

assumes *Kmax*: *maximum-clique K*
and *uO*: $u \in V - K$
and *Au*: $\{x \in K. \neg \text{vert-adj } x \ u\} = \{a\}$
shows *maximum-clique (insert u (K - {a}))*
 $\langle proof \rangle$

lemma *replacement-edge-transfer*:

assumes *fh*: *foldes-hammer-free*
and *Kmax*: *maximum-clique K*
and *uO*: $u \in V - K$
and *vO*: $v \in V - K$
and *uv*: *vert-adj u v*
and *Au*: $\{x \in K. \neg \text{vert-adj } x \ u\} = \{a\}$
and *aNv*: $\neg \text{vert-adj } a \ v$
and *zO*: $z \in V - K$
and *zne*: $z \neq u$
and *az*: *vert-adj a z*
shows *vert-adj u z*

$\langle proof \rangle$

lemma *outside-set-after-swap*:

assumes uO : $u \in V - K$
and aV : $a \in V$
and aK : $a \in K$
shows $V - \text{insert } u (K - \{a\}) = \text{insert } a ((V - K) - \{u\})$
 $\langle proof \rangle$

lemma *swap-outside-edges-inj*:

assumes uO : $u \in V - K$
and aK : $a \in K$
shows $\text{inj-on } (\%e. \text{ if } a \in e \text{ then insert } u (e - \{a\}) \text{ else } e)$
 $(\text{induced-edges } (V - \text{insert } u (K - \{a\})))$
 $\langle proof \rangle$

lemma *swap-outside-edges-image-subset*:

assumes fh : *foldes-hammer-free*
and $Kmax$: *maximum-clique* K
and uO : $u \in V - K$
and vO : $v \in V - K$
and uv : *vert-adj* $u v$
and Au : $\{x \in K. \neg \text{vert-adj } x u\} = \{a\}$
and aNv : $\neg \text{vert-adj } a v$
shows $(\%e. \text{ if } a \in e \text{ then insert } u (e - \{a\}) \text{ else } e) \subseteq$
 $\text{induced-edges } (V - \text{insert } u (K - \{a\}))$
 $\subseteq \text{induced-edges } (V - K)$
 $\langle proof \rangle$

lemma *swap-outside-edges-missing-uv*:

assumes uO : $u \in V - K$
and vO : $v \in V - K$
and uv : *vert-adj* $u v$
and aK : $a \in K$
and aNv : $\neg \text{vert-adj } a v$
shows $\{u, v\} \in \text{induced-edges } (V - K)$
and $\{u, v\} \notin (\%e. \text{ if } a \in e \text{ then insert } u (e - \{a\}) \text{ else } e)$
 $\text{induced-edges } (V - \text{insert } u (K - \{a\}))$
 $\langle proof \rangle$

lemma *outside-edge-count-swap-strict*:

assumes fh : *foldes-hammer-free*
and $Kmax$: *maximum-clique* K
and uO : $u \in V - K$
and vO : $v \in V - K$
and uv : *vert-adj* $u v$
and Au : $\{x \in K. \neg \text{vert-adj } x u\} = \{a\}$
and aNv : $\neg \text{vert-adj } a v$
shows $\text{outside-edge-count } (\text{insert } u (K - \{a\})) < \text{outside-edge-count } K$

$\langle proof \rangle$

theorem *foldes-hammer-converse*:

assumes *fh*: *foldes-hammer-free*

shows *is-split-graph*

$\langle proof \rangle$

theorem *foldes-hammer*:

is-split-graph $\longleftrightarrow$ *foldes-hammer-free*

$\langle proof \rangle$

end

end

References

- [1] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey*. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999. DOI: <https://doi.org/10.1137/1.9780898719796>.
- [2] S. Földes and P. L. Hammer. Split graphs. *Congressus Numerantium*, 19:311–315, 1977.