

# Fisher’s Inequality: Linear Algebraic Proof Techniques for Combinatorics

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## Abstract

Linear algebraic techniques are powerful, yet often underrated tools in combinatorial proofs. This formalisation provides a library including matrix representations of incidence set systems, general formal proof techniques for the rank argument and linear bound argument, and finally a formalisation of a number of variations of the well-known Fisher’s inequality. We build on our prior work formalising combinatorial design theory using a locale-centric approach, including extensions such as constant intersect designs and dual incidence systems. In addition to Fisher’s inequality, we also formalise proofs on other incidence system properties using the incidence matrix representation, such as design existence, dual system relationships and incidence system isomorphisms. This formalisation is presented in the paper “Formalising Fisher’s Inequality: Formal Linear Algebraic Techniques in Combinatorics”, accepted to ITP 2022.

## Contents

<b>1</b>	<b>Micellaneous Multiset/Set Extras</b>	<b>3</b>
1.1	Set extras . . . . .	3
1.2	Multiset Extras . . . . .	3
1.3	Permutation on Sets and Multisets . . . . .	4
1.4	Lists . . . . .	4
1.5	Summation Rules . . . . .	5
<b>2</b>	<b>Matrix and Vector Additions</b>	<b>6</b>
2.1	Vector Extras . . . . .	6
2.2	Matrix Extras . . . . .	10
2.3	Vector and Matrix Homomorphism . . . . .	12
2.4	Zero One injections and homomorphisms . . . . .	13
<b>3</b>	<b>Micellaneous Design Extras</b>	<b>16</b>
3.1	Extensions to existing Locales and Properties . . . . .	16
3.2	New Design Locales . . . . .	20

<b>4</b>	<b>Incidence Vectors and Matrices</b>	<b>22</b>
4.1	Incidence Vectors . . . . .	22
4.2	Incidence Matrices . . . . .	23
4.3	0-1 Matrices . . . . .	27
4.4	Ordered Incidence Systems . . . . .	32
4.5	Incidence Matrices on Design Subtypes . . . . .	41
4.6	Zero One Matrix Incidence System Existence . . . . .	45
4.7	Isomorphisms and Incidence Matrices . . . . .	47
<b>5</b>	<b>Dual Systems</b>	<b>48</b>
5.1	Dual Blocks . . . . .	48
5.2	Basic Dual Properties . . . . .	49
5.3	Incidence System Dual Properties . . . . .	50
5.4	Dual Properties for Design sub types . . . . .	51
5.5	Generalise Dual Concept . . . . .	53
<b>6</b>	<b>Rank Argument - General</b>	<b>54</b>
6.1	Row/Column Operations . . . . .	55
6.2	Rank and Linear Independence . . . . .	59
<b>7</b>	<b>Linear Bound Argument - General</b>	<b>59</b>
7.1	Vec Space Extensions . . . . .	60
7.2	Linear Bound Argument Lemmas . . . . .	61
<b>8</b>	<b>Fisher's Inequality</b>	<b>62</b>
8.1	Uniform Fisher's Inequality . . . . .	62
8.2	Generalised Fisher's Inequality . . . . .	64
<b>9</b>	<b>Matrices/Vectors mod x</b>	<b>65</b>
9.1	Basic Mod Context . . . . .	65
9.2	Mod Type . . . . .	68
9.3	Mat mod type . . . . .	70
<b>10</b>	<b>Variations on Fisher's Inequality</b>	<b>74</b>
10.1	Matrix mod properties . . . . .	74
10.2	The Odd-town Problem . . . . .	75

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# 1 Micellaneous Multiset/Set Extras

**theory** *Set-Multiset-Extras* **imports** *Design-Theory.Multisets-Extras HOL-Combinatorics.Multiset-Permutat*  
**begin**

## 1.1 Set extras

Minor set extras on cardinality and filtering

**lemma** *equal-card-inter-fin-eq-sets*:  $finite\ A \implies finite\ B \implies card\ A = card\ B \implies$

$card\ (A \cap B) = card\ A \implies A = B$   
(*proof*)

**lemma** *insert-filter-set-true*:  $P\ x \implies \{a \in (insert\ x\ A) . P\ a\} = insert\ x\ \{a \in A . P\ a\}$   
(*proof*)

**lemma** *insert-filter-set-false*:  $\neg\ P\ x \implies \{a \in (insert\ x\ A) . P\ a\} = \{a \in A . P\ a\}$   
(*proof*)

## 1.2 Multiset Extras

Minor multiset extras on size and element reasoning

**lemma** *obtain-two-items-mset*:

**assumes**  $size\ A > 1$

**obtains**  $x\ y$  **where**  $x \in\# A$  **and**  $y \in\# A - \{x\}$

(*proof*)

**lemma** *obtain-two-items-mset-filter*:

**assumes**  $size\ \{a \in\# A . P\ a\} > 1$

**obtains**  $x\ y$  **where**  $x \in\# A$  **and**  $y \in\# A - \{x\}$  **and**  $P\ x$  **and**  $P\ y$

(*proof*)

**lemma** *size-count-mset-ss*:

**assumes**  $finite\ B$

**assumes**  $(set\ mset\ A) \subseteq B$

**shows**  $size\ A = (\sum\ x \in B . count\ A\ x)$

(*proof*)

**lemma** *mset-list-by-index*:  $mset\ (xs) = image\ mset\ (\lambda\ i . (xs\ !\ i))\ (mset\ set\ \{..<length\ xs\})$

(*proof*)

**lemma** *count-mset-split-image-filter*:

**assumes**  $\bigwedge\ x . x \in\# A \implies a \neq g\ x$

**shows**  $count\ (image\ mset\ (\lambda\ x . if\ P\ x\ then\ a\ else\ g\ x)\ A)\ a = size\ (filter\ mset\ P\ A)$

(*proof*)

(*proof*)

**lemma** *count-mset-split-image-filter-not:*

**assumes**  $\bigwedge x. x \in \#A \implies b \neq f x$

**shows**  $\text{count } (\text{image-mset } (\lambda x. \text{if } P x \text{ then } f x \text{ else } b) A) b = \text{size } (\text{filter-mset } (\lambda x. \neg P x) A)$

*<proof>*

**lemma** *removeAll-size-lt:*  $\text{size } (\text{removeAll-mset } C M) \leq \text{size } M$

*<proof>*

**lemma** *mset-image-eq-filter-eq:*  $A = \text{image-mset } f B \implies$

$\text{filter-mset } P A = (\text{image-mset } f (\text{filter-mset } (\lambda x. P (f x)) B))$

*<proof>*

### 1.3 Permutation on Sets and Multisets

**lemma** *elem-permutation-of-set-empty-iff:*  $\text{finite } A \implies xs \in \text{permutations-of-set } A \implies$

$xs = [] \iff A = \{\}$

*<proof>*

**lemma** *elem-permutation-of-mset-empty-iff:*  $xs \in \text{permutations-of-multiset } A \implies$

$xs = [] \iff A = \{\#\}$

*<proof>*

### 1.4 Lists

Further lemmas on the relationship between lists and multisets

**lemma** *count-distinct-mset-list-index:*  $i1 < \text{length } xs \implies i2 < \text{length } xs \implies i1 \neq i2 \implies$

$\text{distinct-mset } (\text{mset } xs) \implies xs ! i1 \neq xs ! i2$

*<proof>*

**lemma** *index-remove1-mset-ne:*

**assumes**  $x \in \# (\text{mset } xs)$

**assumes**  $y \in \# \text{remove1-mset } x (\text{mset } xs)$

**assumes**  $xs ! j1 = x$

**assumes**  $j1 < \text{length } xs$

**obtains**  $j2$  **where**  $xs ! j2 = y$  **and**  $j2 < \text{length } xs$  **and**  $j1 \neq j2$

*<proof>*

**lemma** *count-list-mset:*  $\text{count-list } xs x = \text{count } (\text{mset } xs) x$

*<proof>*

**lemma** *count-min-2-indices-lt:*

**assumes**  $i1 < i2$

**assumes**  $xs ! i1 = x$

**assumes**  $xs ! i2 = x$

**assumes**  $i1 < \text{length } xs$   $i2 < \text{length } xs$

**shows**  $\text{count } (\text{mset } xs) x \geq 2$

*<proof>*

**lemma** *count-min-2-indices*:  $i1 \neq i2 \implies xs ! i1 = x \implies xs ! i2 = x \implies i1 < \text{length } xs \implies i2 < \text{length } xs \implies \text{count } (\text{mset } xs) x \geq 2$   
*<proof>*

**lemma** *obtain-set-list-item*:  
**assumes**  $x \in \text{set } xs$   
**obtains**  $i$  **where**  $i < \text{length } xs$  **and**  $xs ! i = x$   
*<proof>*

## 1.5 Summation Rules

Some lemmas to make it simpler to work with double and triple summations

**context** *comm-monoid-add*  
**begin**

**lemma** *sum-reorder-triple*:  $(\sum l \in A . (\sum i \in B . (\sum j \in C . g l i j))) = (\sum i \in B . (\sum j \in C . (\sum l \in A . g l i j)))$   
*<proof>*

**lemma** *double-sum-mult-hom*:  
**fixes**  $k :: 'b :: \{\text{comm-ring-1}\}$   
**shows**  $(\sum i \in A . (\sum j \in g i . k * (f i j))) = k * (\sum i \in A . (\sum j \in g i . f i j))$   
*<proof>*

**lemma** *double-sum-split-case*:  
**assumes** *finite A*  
**shows**  $(\sum i \in A . (\sum j \in A . f i j)) = (\sum i \in A . (f i i)) + (\sum i \in A . (\sum j \in (A - \{i\}) . f i j))$   
*<proof>*

**lemma** *double-sum-split-case2*:  $(\sum i \in A . (\sum j \in A . g i j)) = (\sum i \in A . (g i i)) + (\sum i \in A . (\sum j \in \{a \in A . a \neq i\} . g i j))$   
*<proof>*

**end**

**context** *comm-ring-1*  
**begin**

**lemma** *double-sum-split-case-square*:  
**assumes** *finite A*  
**shows**  $(\sum i \in A . f i)^2 = (\sum i \in A . (f i * f i)) + (\sum i \in A . (\sum j \in (A - \{i\}) . f i * f j))$   
*<proof>*

**lemma** *double-sum-split-square-diff*: *finite*  $\{0..<x\} \implies$

$$\begin{aligned} & (\sum i \in \{0..<x\} . (\sum j \in (\{0..<x\} - \{i\}) . c i * c j)) = \\ & (\sum i \in \{0..<x\} . c i)^2 - (\sum i \in \{0..<x\} . c i * c i) \\ & \langle proof \rangle \end{aligned}$$

**end**  
**end**

## 2 Matrix and Vector Additions

**theory** *Matrix-Vector-Extras* **imports** *Set-Multiset-Extras Jordan-Normal-Form.Matrix*

*Design-Theory.Multisets-Extras Groebner-Bases.Macaulay-Matrix Polynomial-Factorization.Missing-List*  
**begin**

### 2.1 Vector Extras

For ease of use, a number of additions to the existing vector library as initially developed in the JNF AFP Entry, are given below

We define the concept of summing up elements of a vector

**definition** (in *comm-monoid-add*) *sum-vec* :: 'a vec  $\Rightarrow$  'a **where**  
*sum-vec* v  $\equiv$  *sum* ( $\lambda$  i . v \$ i) {0..*dim-vec* v}

**lemma** *sum-vec-vNil[simp]*: *sum-vec* vNil = 0  
*<proof>*

**lemma** *sum-vec-vCons*: *sum-vec* (vCons a v) = a + *sum-vec* v  
*<proof>*

**lemma** *sum-vec-list*: *sum-list* (*list-of-vec* v) = *sum-vec* v  
*<proof>*

**lemma** *sum-vec-mset*: *sum-vec* v = ( $\sum$  x  $\in$  # (*mset* (*list-of-vec* v)) . x)  
*<proof>*

**lemma** *dim-vec-vCons-ne-0*: *dim-vec* (vCons a v) > 0  
*<proof>*

**lemma** *sum-vec-vCons-lt*:

**assumes**  $\bigwedge$  i. i < *dim-vec* (vCons a v)  $\implies$  (vCons a v) \$ i  $\leq$  (n :: int)

**assumes** *sum-vec* v  $\leq$  m

**shows** *sum-vec* (vCons a v)  $\leq$  n + m

*<proof>*

**lemma** *sum-vec-one-zero*:

**assumes**  $\bigwedge$  i. i < *dim-vec* (v :: int vec)  $\implies$  v \$ i  $\leq$  (1 :: int)

**shows** *sum-vec* v  $\leq$  *dim-vec* v

*<proof>*

Definition to convert a vector to a multiset

**definition** *vec-mset*:: 'a vec  $\Rightarrow$  'a multiset **where**  
*vec-mset* v  $\equiv$  *image-mset* (*vec-index* v) (*mset-set* {..*dim-vec* v})

**lemma** *vec-elem-exists-mset*:  $(\exists i \in \{..*dim-vec* v\}. v \$ i = x) \longleftrightarrow x \in \# \textit{vec-mset}  
*v*  
*<proof>*$

**lemma** *mset-vec-same-size*: *dim-vec* v = *size* (*vec-mset* v)  
*<proof>*

**lemma** *mset-vec-eq-mset-list*: *vec-mset* v = *mset* (*list-of-vec* v)  
*<proof>*

**lemma** *vec-mset-img-map*: *image-mset* f (*mset* (xs)) = *vec-mset* (*map-vec* f (*vec-of-list*  
xs))  
*<proof>*

**lemma** *vec-mset-vNil*: *vec-mset* vNil = {#}  
*<proof>*

**lemma** *vec-mset-vCons*: *vec-mset* (vCons x v) = *add-mset* x (*vec-mset* v)  
*<proof>*

**lemma** *vec-mset-set*: *vec-set* v = *set-mset* (*vec-mset* v)  
*<proof>*

**lemma** *vCons-set-contains-in*:  $a \in \textit{set}_v v \implies \textit{set}_v (\textit{vCons} a v) = \textit{set}_v v$   
*<proof>*

**lemma** *vCons-set-contains-add*:  $a \notin \textit{set}_v v \implies \textit{set}_v (\textit{vCons} a v) = \textit{set}_v v \cup \{a\}$   
*<proof>*

**lemma** *setv-vec-mset-not-in-iff*:  $a \notin \textit{set}_v v \longleftrightarrow a \notin \# \textit{vec-mset} v$   
*<proof>*

Abbreviation for counting occurrences of an element in a vector

**abbreviation** *count-vec* v a  $\equiv$  *count* (*vec-mset* v) a

**lemma** *vec-count-lt-dim*: *count-vec* v a  $\leq$  *dim-vec* v  
*<proof>*

**lemma** *count-vec-empty*: *dim-vec* v = 0  $\implies$  *count-vec* v a = 0  
*<proof>*

**lemma** *count-vec-vNil*: *count-vec* vNil a = 0  
*<proof>*

**lemma** *count-vec-vCons*: *count-vec* (vCons aa v) a = (if (aa = a) then *count-vec*

$v\ a + 1$  else  $\text{count-vec } v\ a$ )  
(proof)

**lemma** *elem-exists-count-min*:  $\exists i \in \{.. < \text{dim-vec } v\}. v\ \$\ i = x \implies \text{count-vec } v\ x \geq 1$   
(proof)

**lemma** *count-vec-count-mset*:  $\text{vec-mset } v = \text{image-mset } f\ A \implies \text{count-vec } v\ a = \text{count } (\text{image-mset } f\ A)\ a$   
(proof)

**lemma** *count-vec-alt-list*:  $\text{count-vec } v\ a = \text{length } (\text{filter } (\lambda y. a = y)\ (\text{list-of-vec } v))$   
(proof)

**lemma** *count-vec-alt*:  $\text{count-vec } v\ x = \text{card } \{ i. v\ \$\ i = x \wedge i < \text{dim-vec } v \}$   
(proof)

**lemma** *count-vec-sum-ones*:  
fixes  $v :: 'a :: \{\text{ring-1}\}$  *vec*  
assumes  $\bigwedge i. i < \text{dim-vec } v \implies v\ \$\ i = 1 \vee v\ \$\ i = 0$   
shows  $\text{of-nat } (\text{count-vec } v\ 1) = \text{sum-vec } v$   
(proof)

**lemma** *count-vec-two-elems*:  
fixes  $v :: 'a :: \{\text{zero-neq-one}\}$  *vec*  
assumes  $\bigwedge i. i < \text{dim-vec } v \implies v\ \$\ i = 1 \vee v\ \$\ i = 0$   
shows  $\text{count-vec } v\ 1 + \text{count-vec } v\ 0 = \text{dim-vec } v$   
(proof)

**lemma** *count-vec-sum-zeros*:  
fixes  $v :: 'a :: \{\text{ring-1}\}$  *vec*  
assumes  $\bigwedge i. i < \text{dim-vec } v \implies v\ \$\ i = 1 \vee v\ \$\ i = 0$   
shows  $\text{of-nat } (\text{count-vec } v\ 0) = \text{of-nat } (\text{dim-vec } v) - \text{sum-vec } v$   
(proof)

**lemma** *count-vec-sum-ones-alt*:  
fixes  $v :: 'a :: \{\text{ring-1}\}$  *vec*  
assumes  $\text{vec-set } v \subseteq \{0, 1\}$   
shows  $\text{of-nat } (\text{count-vec } v\ 1) = \text{sum-vec } v$   
(proof)

**lemma** *setv-not-in-count0-iff*:  $a \notin \text{set}_v\ v \iff \text{count-vec } v\ a = 0$   
(proof)

**lemma** *sum-count-vec*:  
assumes  $\text{finite } (\text{set}_v\ v)$   
shows  $(\sum i \in \text{set}_v\ v. \text{count-vec } v\ i) = \text{dim-vec } v$   
(proof)



**lemma** *sum-setv-subset-eq*:  
**assumes** *finite A*  
**assumes**  $set_v v \subseteq A$   
**shows**  $(\sum i \in set_v v. count-vec v i) = (\sum i \in A. count-vec v i)$   
 $\langle proof \rangle$

**lemma** *sum-count-vec-subset*:  $finite A \implies set_v v \subseteq A \implies (\sum i \in A. count-vec v i) = dim-vec v$   
 $\langle proof \rangle$

An abbreviation for checking if an element is in a vector

**abbreviation** *vec-contains* ::  $'a \Rightarrow 'a\ vec \Rightarrow bool$  (**infix**  $\in\ \$ 50$ )**where**  
 $a \in\ \$ v \equiv a \in set_v v$

**lemma** *vec-set-mset-contains-iff*:  $a \in\ \$ v \longleftrightarrow a \in\ \# vec-mset v$   
 $\langle proof \rangle$

**lemma** *vec-contains-count-gt1-iff*:  $a \in\ \$ v \longleftrightarrow count-vec v a \geq 1$   
 $\langle proof \rangle$

**lemma** *vec-contains-obtains-index*:  
**assumes**  $a \in\ \$ v$   
**obtains**  $i$  **where**  $i < dim-vec v$  **and**  $v \$ i = a$   
 $\langle proof \rangle$

**lemma** *vec-count-eq-list-count*:  $count (mset xs) a = count-vec (vec-of-list xs) a$   
 $\langle proof \rangle$

**lemma** *vec-contains-col-elements-mat*:  
**assumes**  $j < dim-col M$   
**assumes**  $a \in\ \$ col M j$   
**shows**  $a \in elements-mat M$   
 $\langle proof \rangle$

**lemma** *vec-contains-row-elements-mat*:  
**assumes**  $i < dim-row M$   
**assumes**  $a \in\ \$ row M i$   
**shows**  $a \in elements-mat M$   
 $\langle proof \rangle$

**lemma** *vec-contains-img*:  $a \in\ \$ v \implies f a \in\ \$ (map-vec f v)$   
 $\langle proof \rangle$

The existing vector library contains the identity and zero vectors, but no definition of a vector where all elements are 1, as defined below

**definition** *all-ones-vec* ::  $nat \Rightarrow 'a :: \{zero, one\} vec (u_v)$  **where**  
 $u_v n \equiv vec n (\lambda i. 1)$

**lemma** *dim-vec-all-ones[simp]*:  $dim-vec (u_v n) = n$

*<proof>*

**lemma** *all-ones-index* [simp]:  $i < n \implies u_v n \$ i = 1$

*<proof>*

**lemma** *dim-vec-mult-vec-mat* [simp]:  $\dim\text{-vec } (v \cdot_v A) = \dim\text{-col } A$

*<proof>*

**lemma** *all-ones-vec-smult*[simp]:  $i < n \implies ((k :: ('a :: \{one, zero, monoid-mult\}))$

$\cdot_v (u_v n)) \$ i = k$

*<proof>*

Extra lemmas on existing vector operations

**lemma** *smult-scalar-prod-sum*:

**fixes**  $x :: 'a :: \{comm-ring-1\}$

**assumes**  $vx \in \text{carrier-vec } n$

**assumes**  $vy \in \text{carrier-vec } n$

**shows**  $(\sum i \in \{0..<n\} . ((x \cdot_v vx) \$ i) * ((y \cdot_v vy) \$ i)) = x * y * (vx \cdot_v vy)$

*<proof>*

**lemma** *scalar-prod-double-sum-fn-vec*:

**fixes**  $c :: nat \Rightarrow ('a :: \{comm-semiring-0\})$

**fixes**  $f :: nat \Rightarrow 'a \text{ vec}$

**assumes**  $\bigwedge j . j < k \implies \dim\text{-vec } (f j) = n$

**shows**  $(\text{vec } n (\lambda i . \sum j = 0..<k . c j * (f j) \$ i)) \cdot (\text{vec } n (\lambda i . \sum j = 0..<k . c j * (f j) \$ i)) =$

$(\sum j1 \in \{0..<k\} . c j1 * c j1 * ((f j1) \cdot (f j1))) +$

$(\sum j1 \in \{0..<k\} . (\sum j2 \in (\{0..<k\} - \{j1\}) . c j1 * c j2 * ((f j1) \cdot (f j2))))$

*<proof>*

**lemma** *vec-prod-zero*:  $(0_v n) \cdot (0_v n) = 0$

*<proof>*

**lemma** *map-vec-compose*:  $\text{map-vec } f (\text{map-vec } g v) = \text{map-vec } (f \circ g) v$

*<proof>*

## 2.2 Matrix Extras

As with vectors, the all ones mat definition defines the concept of a matrix where all elements are 1

**definition** *all-ones-mat* ::  $nat \Rightarrow 'a :: \{zero, one\} \text{ mat } (J_m)$  **where**

$J_m n \equiv \text{mat } n n (\lambda (i,j) . 1)$

**lemma** *all-ones-mat-index*[simp]:  $i < \dim\text{-row } (J_m n) \implies j < \dim\text{-col } (J_m n)$

$\implies J_m n \$ \$ (i, j) = 1$

*<proof>*

**lemma** *all-ones-mat-dim-row*[simp]:  $\dim\text{-row } (J_m n) = n$

*<proof>*

**lemma** *all-ones-mat-dim-col[simp]*:  $\dim\text{-col } (J_m \ n) = n$   
 ⟨proof⟩

Basic lemmas on existing matrix operations

**lemma** *index-mult-vec-mat[simp]*:  $j < \dim\text{-col } A \implies (v \ v * A) \$ j = v \cdot \text{col } A \ j$   
 ⟨proof⟩

**lemma** *transpose-mat-mult-entries*:  $i < \dim\text{-row } A \implies j < \dim\text{-row } A \implies$   
 $(A * A^T) \$\$ (i, j) = (\sum k \in \{0..<(\dim\text{-col } A)\}. (A \$\$ (i, k)) * (A \$\$ (j, k)))$   
 ⟨proof⟩

**lemma** *transpose-mat-elems*:  $\text{elements-mat } A = \text{elements-mat } A^T$   
 ⟨proof⟩

**lemma** *row-elems-subset-mat*:  $i < \dim\text{-row } N \implies \text{vec-set } (\text{row } N \ i) \subseteq \text{elements-mat } N$   
 ⟨proof⟩

**lemma** *col-elems-subset-mat*:  $i < \dim\text{-col } N \implies \text{vec-set } (\text{col } N \ i) \subseteq \text{elements-mat } N$   
 ⟨proof⟩

**lemma** *obtain-row-index*:  
 assumes  $r \in \text{set } (\text{rows } M)$   
 obtains  $i$  where  $\text{row } M \ i = r$  and  $i < \dim\text{-row } M$   
 ⟨proof⟩

**lemma** *row-prop-cond*:  $(\bigwedge i. i < \dim\text{-row } M \implies P (\text{row } M \ i)) \implies r \in \text{set } (\text{rows } M) \implies P \ r$   
 ⟨proof⟩

**lemma** *obtain-col-index*:  
 assumes  $c \in \text{set } (\text{cols } M)$   
 obtains  $j$  where  $\text{col } M \ j = c$  and  $j < \dim\text{-col } M$   
 ⟨proof⟩

**lemma** *col-prop-cond*:  $(\bigwedge j. j < \dim\text{-col } M \implies P (\text{col } M \ j)) \implies c \in \text{set } (\text{cols } M) \implies P \ c$   
 ⟨proof⟩

Lemmas on the *map-mat* definition

**lemma** *row-map-mat[simp]*:  
 assumes  $i < \dim\text{-row } A$  shows  $\text{row } (\text{map-mat } f \ A) \ i = \text{map-vec } f \ (\text{row } A \ i)$   
 ⟨proof⟩

**lemma** *map-vec-mat-rows*:  $\text{map } (\text{map-vec } f) \ (\text{rows } M) = \text{rows } ((\text{map-mat } f) \ M)$   
 ⟨proof⟩

**lemma** *map-vec-mat-cols*:  $\text{map } (\text{map-vec } f) (\text{cols } M) = \text{cols } ((\text{map-mat } f) M)$   
 ⟨proof⟩

**lemma** *map-mat-compose*:  $\text{map-mat } f (\text{map-mat } g A) = \text{map-mat } (f \circ g) A$   
 ⟨proof⟩

**lemmas** *map-mat-elements = elements-mat-map*

Reasoning on sets and multisets of matrix elements

**lemma** *set-cols-carrier*:  $A \in \text{carrier-mat } m \ n \implies v \in \text{set } (\text{cols } A) \implies v \in \text{carrier-vec } m$   
 ⟨proof⟩

**lemma** *mset-cols-index-map*:  $\text{image-mset } (\lambda j. \text{col } M \ j) (\text{mset-set } \{0..< \text{dim-col } M\}) = \text{mset } (\text{cols } M)$   
 ⟨proof⟩

**lemma** *mset-rows-index-map*:  $\text{image-mset } (\lambda i. \text{row } M \ i) (\text{mset-set } \{0..< \text{dim-row } M\}) = \text{mset } (\text{rows } M)$   
 ⟨proof⟩

**lemma** *index-to-col-card-size-prop*:

**assumes**  $i < \text{dim-row } M$

**assumes**  $\bigwedge j. j < \text{dim-col } M \implies P \ j \longleftrightarrow Q \ (\text{col } M \ j)$

**shows**  $\text{card } \{j . j < \text{dim-col } M \wedge P \ j\} = \text{size } \{\#c \in \# (\text{mset } (\text{cols } M)) . Q \ c \ \#\}$   
 ⟨proof⟩

**lemma** *index-to-row-card-size-prop*:

**assumes**  $j < \text{dim-col } M$

**assumes**  $\bigwedge i. i < \text{dim-row } M \implies P \ i \longleftrightarrow Q \ (\text{row } M \ i)$

**shows**  $\text{card } \{i . i < \text{dim-row } M \wedge P \ i\} = \text{size } \{\#r \in \# (\text{mset } (\text{rows } M)) . Q \ r \ \#\}$   
 ⟨proof⟩

**lemma** *setv-row-subset-mat-elems*:

**assumes**  $v \in \text{set } (\text{rows } M)$

**shows**  $\text{set}_v \ v \subseteq \text{elements-mat } M$   
 ⟨proof⟩

**lemma** *setv-col-subset-mat-elems*:

**assumes**  $v \in \text{set } (\text{cols } M)$

**shows**  $\text{set}_v \ v \subseteq \text{elements-mat } M$   
 ⟨proof⟩

## 2.3 Vector and Matrix Homomorphism

We extend on the existing lemmas on homomorphism mappings as applied to vectors and matrices

**context** *semiring-hom*

**begin**

**lemma** *vec-hom-smult2*:

**assumes** *dim-vec*  $v2 \leq \text{dim-vec } v1$

**shows**  $\text{hom } (v1 \cdot v2) = \text{vec}_h v1 \cdot \text{vec}_h v2$

*<proof>*

**end**

**lemma** *map-vec-vCons*:  $\text{vCons } (f a) (\text{map-vec } f v) = \text{map-vec } f (\text{vCons } a v)$

*<proof>*

**context** *inj-zero-hom*

**begin**

**lemma** *vec-hom-zero-iff[simp]*:  $(\text{map-vec } \text{hom } x = 0_v n) = (x = 0_v n)$

*<proof>*

**lemma** *mat-hom-inj*:  $\text{map-mat } \text{hom } A = \text{map-mat } \text{hom } B \implies A = B$

*<proof>*

**lemma** *vec-hom-inj*:  $\text{map-vec } \text{hom } v = \text{map-vec } \text{hom } w \implies v = w$

*<proof>*

**lemma** *vec-hom-set-distinct-iff*:

**fixes**  $xs :: 'a \text{ vec list}$

**shows**  $\text{distinct } xs \longleftrightarrow \text{distinct } (\text{map } (\text{map-vec } \text{hom}) xs)$

*<proof>*

**lemma** *vec-hom-mset*:  $\text{image-mset } \text{hom } (\text{vec-mset } v) = \text{vec-mset } (\text{map-vec } \text{hom } v)$

*<proof>*

**lemma** *vec-hom-set*:  $\text{hom } ' \text{set}_v v = \text{set}_v (\text{map-vec } \text{hom } v)$

*<proof>*

**end**

## 2.4 Zero One injections and homomorphisms

Define a locale to encapsulate when a function is injective on a certain set (i.e. not a universal homomorphism for the type

**locale** *injective-lim* =

**fixes**  $A :: 'a \text{ set}$

**fixes**  $f :: 'a \implies 'b$  **assumes** *injectivity-lim*:  $\bigwedge x y. x \in A \implies y \in A \implies f x = f y \implies x = y$

**begin**

**lemma** *eq-iff[simp]*:  $x \in A \implies y \in A \implies f x = f y \longleftrightarrow x = y$  *<proof>*

**lemma** *inj-on-f*: *inj-on*  $f A$  *<proof>*

**end**

**sublocale** *injective*  $\subseteq$  *injective-lim Univ*  
⟨*proof*⟩

**context** *injective-lim*  
**begin**

**lemma** *mat-hom-inj-lim*:  
  **assumes** *elements-mat*  $M \subseteq A$  **and** *elements-mat*  $N \subseteq A$   
  **shows** *map-mat*  $f M = \text{map-mat } f N \implies M = N$   
  ⟨*proof*⟩

**lemma** *vec-hom-inj-lim*: **assumes** *set<sub>v</sub>*  $v \subseteq A$  *set<sub>v</sub>*  $w \subseteq A$   
  **shows** *map-vec*  $f v = \text{map-vec } f w \implies v = w$   
  ⟨*proof*⟩

**lemma** *lim-inj-hom-count-vec*:  
  **assumes** *set<sub>v</sub>*  $v \subseteq A$   
  **assumes**  $x \in A$   
  **shows** *count-vec*  $v x = \text{count-vec } (\text{map-vec } f v) (f x)$   
  ⟨*proof*⟩

**lemma** *vec-hom-lim-set-distinct-iff*:  
  **fixes**  $xs :: 'a \text{ vec list}$   
  **assumes**  $\bigwedge v . v \in \text{set } (xs) \implies \text{set}_v v \subseteq A$   
  **shows** *distinct*  $xs \longleftrightarrow \text{distinct } (\text{map } (\text{map-vec } f) xs)$   
  ⟨*proof*⟩

**lemma** *mat-rows-hom-lim-distinct-iff*:  
  **assumes** *elements-mat*  $M \subseteq A$   
  **shows** *distinct* (*rows*  $M$ )  $\longleftrightarrow \text{distinct } (\text{map } (\text{map-vec } f) (\text{rows } M))$   
  ⟨*proof*⟩

**lemma** *mat-cols-hom-lim-distinct-iff*:  
  **assumes** *elements-mat*  $M \subseteq A$   
  **shows** *distinct* (*cols*  $M$ )  $\longleftrightarrow \text{distinct } (\text{map } (\text{map-vec } f) (\text{cols } M))$   
  ⟨*proof*⟩

**end**

**locale** *inj-on-01-hom* = *zero-hom* + *one-hom* + *injective-lim*  $\{0, 1\}$  *hom*  
**begin**

**lemma** *inj-0-iff*:  $x \in \{0, 1\} \implies \text{hom } x = 0 \longleftrightarrow x = 0$   
  ⟨*proof*⟩

**lemma** *inj-1-iff*:  $x \in \{0, 1\} \implies \text{hom } x = 1 \longleftrightarrow x = 1$

*<proof>*

**end**

**context** *zero-neq-one*

**begin**

**definition** *of-zero-neq-one* :: 'b :: {zero-neq-one}  $\Rightarrow$  'a **where**  
*of-zero-neq-one* x  $\equiv$  if (x = 0) then 0 else 1

**lemma** *of-zero-neq-one-1 [simp]*: *of-zero-neq-one* 1 = 1  
*<proof>*

**lemma** *of-zero-neq-one-0 [simp]*: *of-zero-neq-one* 0 = 0  
*<proof>*

**lemma** *of-zero-neq-one-0-iff [iff]*: *of-zero-neq-one* x = 0  $\longleftrightarrow$  x = 0  
*<proof>*

**lemma** *of-zero-neq-one-lim-eq*: x  $\in$  {0, 1}  $\Longrightarrow$  y  $\in$  {0, 1}  $\Longrightarrow$  *of-zero-neq-one* x =  
*of-zero-neq-one* y  $\longleftrightarrow$  x = y  
*<proof>*

**end**

**interpretation** *of-zero-hom*: *zero-hom-0 of-zero-neq-one*  
*<proof>*

**interpretation** *of-injective-lim*: *injective-lim* {0, 1} *of-zero-neq-one*  
*<proof>*

**interpretation** *of-inj-on-01-hom*: *inj-on-01-hom of-zero-neq-one*  
*<proof>*

We want the ability to transform any 0-1 vector or matrix to another 'c type

**definition** *lift-01-vec* :: 'b :: {zero-neq-one} vec  $\Rightarrow$  'c :: {zero-neq-one} vec **where**  
*lift-01-vec* v  $\equiv$  *map-vec of-zero-neq-one v*

**lemma** *lift-01-vec-simp [simp]*: *dim-vec* (*lift-01-vec* v) = *dim-vec* v  
i < *dim-vec* v  $\Longrightarrow$  (*lift-01-vec* v) \$ i = *of-zero-neq-one* (v \$ i)  
*<proof>*

**lemma** *lift-01-vec-count*:

**assumes** *set\_v* v  $\subseteq$  {0, 1}

**assumes** x  $\in$  {0, 1}

**shows** *count-vec* v x = *count-vec* (*lift-01-vec* v) (*of-zero-neq-one* x)

*<proof>*

**definition** *lift-01-mat* :: 'b :: {zero-neq-one} mat  $\Rightarrow$  'c :: {zero-neq-one} mat **where**

*lift-01-mat* M  $\equiv$  map-mat of-zero-neq-one M

**lemma** *lift-01-mat-simp*[simp]: *dim-row* (*lift-01-mat* M) = *dim-row* M

*dim-col* (*lift-01-mat* M) = *dim-col* M

$i < \text{dim-row } M \implies j < \text{dim-col } M \implies (\text{lift-01-mat } M) \text{ \$(\$ (i, j)) = of-zero-neq-one (M \$(\$ (i, j))$

*<proof>*

**lemma** *lift-01-mat-carrier*: *lift-01-mat* M  $\in$  carrier-mat (*dim-row* M) (*dim-col* M)

*<proof>*

**end**

### 3 Micellaneous Design Extras

Extension's to the author's previous entry on Design Theory

**theory** *Design-Extras* **imports** *Set-Multiset-Extras Design-Theory.BIBD*

**begin**

#### 3.1 Extensions to existing Locales and Properties

Extend lemmas on intersection number

**lemma** *inter-num-max-bound*:

**assumes** *finite* b1 *finite* b2

**shows**  $b1 \mid\mid b2 \leq \text{card } b1 \ b1 \mid\mid b2 \leq \text{card } b2$

*<proof>*

**lemma** *inter-eq-blocks-eq-card*:  $\text{card } b1 = \text{card } b2 \implies \text{finite } b1 \implies \text{finite } b2 \implies b1 \mid\mid b2 = \text{card } b1$

$\implies b1 = b2$

*<proof>*

**lemma** *inter-num-of-eq-blocks*:  $b1 = b2 \implies b1 \mid\mid b2 = \text{card } b1$

*<proof>*

**lemma** *intersect-num-same-eq-size*[simp]:  $b1 \mid\mid b1 = \text{card } b1$

*<proof>*

**lemma** *index-lt-rep-general*:  $x \in ps \implies B \text{ index } ps \leq B \text{ rep } x$

*<proof>*

**context** *incidence-system*

**begin**

**lemma** *block-size-alt*:



**assumes**  $bl \in \# \mathcal{B}$   
**shows**  $card\ bl = card\ \{x \in \mathcal{V} . x \in bl\}$   
 $\langle proof \rangle$

**lemma** *complement-image*:  $\mathcal{B}^C = image\ mset\ block\ complement\ \mathcal{B}$   
 $\langle proof \rangle$

**lemma** *point-in-block-rep-min-iff*:  
**assumes**  $x \in \mathcal{V}$   
**shows**  $\exists bl . bl \in \# \mathcal{B} \wedge x \in bl \longleftrightarrow (\mathcal{B}\ rep\ x > 0)$   
 $\langle proof \rangle$

**lemma** *points-inter-num-rep*:  
**assumes**  $b1 \in \# \mathcal{B}$  **and**  $b2 \in \# \mathcal{B} - \{\#b1\# \}$   
**shows**  $card\ \{v \in \mathcal{V} . v \in b1 \wedge v \in b2\} = b1\ |\cap|\ b2$   
 $\langle proof \rangle$

Extensions on design operation lemmas

**lemma** *del-block-b*:  
 $bl \in \# \mathcal{B} \implies size\ (del\ block\ bl) = b - 1$   
 $bl \notin \# \mathcal{B} \implies size\ (del\ block\ bl) = b$   
 $\langle proof \rangle$

**lemma** *del-block-points-index*:  
**assumes**  $ps \subseteq \mathcal{V}$   
**assumes**  $card\ ps = 2$   
**assumes**  $bl \in \# \mathcal{B}$   
**shows**  $ps \subseteq bl \implies points\ index\ (del\ block\ bl)\ ps = points\ index\ \mathcal{B}\ ps - 1$   
 $\neg (ps \subseteq bl) \implies points\ index\ (del\ block\ bl)\ ps = points\ index\ \mathcal{B}\ ps$   
 $\langle proof \rangle$

**end**

Extensions to properties of design sub types

**context** *finite-incidence-system*  
**begin**

**lemma** *complete-block-size-eq-points*:  $bl \in \# \mathcal{B} \implies card\ bl = v \implies bl = \mathcal{V}$   
 $\langle proof \rangle$

**lemma** *complete-block-all-subsets*:  $bl \in \# \mathcal{B} \implies card\ bl = v \implies ps \subseteq \mathcal{V} \implies ps \subseteq bl$   
 $\langle proof \rangle$

**lemma** *del-block-complete-points-index*:  $ps \subseteq \mathcal{V} \implies card\ ps = 2 \implies bl \in \# \mathcal{B} \implies card\ bl = v \implies points\ index\ (del\ block\ bl)\ ps = points\ index\ \mathcal{B}\ ps - 1$   
 $\langle proof \rangle$

```

end

context design
begin

lemma block-num-rep-bound:  $b \leq (\sum x \in \mathcal{V}. \mathcal{B} \text{ rep } x)$ 
⟨proof⟩

end

context proper-design
begin

lemma del-block-proper:
  assumes  $b > 1$ 
  shows proper-design  $\mathcal{V}$  (del-block bl)
⟨proof⟩

end

context simple-design
begin

lemma inter-num-lt-block-size-strict:
  assumes  $bl1 \in \# \mathcal{B}$ 
  assumes  $bl2 \in \# \mathcal{B}$ 
  assumes  $bl1 \neq bl2$ 
  assumes  $\text{card } bl1 = \text{card } bl2$ 
  shows  $bl1 \sqcap bl2 < \text{card } bl1$   $bl1 \sqcap bl2 < \text{card } bl2$ 
⟨proof⟩

lemma block-mset-distinct: distinct-mset  $\mathcal{B}$  ⟨proof⟩

end

context constant-rep-design
begin

lemma index-lt-const-rep:
  assumes  $ps \subseteq \mathcal{V}$ 
  assumes  $ps \neq \{\}$ 
  shows  $\mathcal{B} \text{ index } ps \leq r$ 
⟨proof⟩

end

context t-wise-balance
begin

```

**lemma** *obtain-t-subset-with-point*:  
**assumes**  $x \in \mathcal{V}$   
**obtains**  $ps$  where  $ps \subseteq \mathcal{V}$  and  $\text{card } ps = t$  and  $x \in ps$   
 $\langle \text{proof} \rangle$

**lemma** *const-index-lt-rep*:  
**assumes**  $x \in \mathcal{V}$   
**shows**  $\Lambda_t \leq \mathcal{B}$  rep  $x$   
 $\langle \text{proof} \rangle$

**end**

**context** *pairwise-balance*  
**begin**

**lemma** *index-zero-iff*:  $\Lambda = 0 \iff (\forall bl \in \# \mathcal{B} . \text{card } bl = 1)$   
 $\langle \text{proof} \rangle$

**lemma** *count-complete-lt-balance*:  $\text{count } \mathcal{B} \mathcal{V} \leq \Lambda$   
 $\langle \text{proof} \rangle$

**lemma** *eq-index-rep-imp-complete*:  
**assumes**  $\Lambda = \mathcal{B}$  rep  $x$   
**assumes**  $x \in \mathcal{V}$   
**assumes**  $bl \in \# \mathcal{B}$   
**assumes**  $x \in bl$   
**shows**  $\text{card } bl = v$   
 $\langle \text{proof} \rangle$

**lemma** *incomplete-index-strict-lt-rep*:  
**assumes**  $\bigwedge bl. bl \in \# \mathcal{B} \implies \text{incomplete-block } bl$   
**assumes**  $x \in \mathcal{V}$   
**assumes**  $\Lambda > 0$   
**shows**  $\Lambda < \mathcal{B}$  rep  $x$   
 $\langle \text{proof} \rangle$

Construct new PBD's from existing PBD's

**lemma** *remove-complete-block-pbd*:  
**assumes**  $b \geq 2$   
**assumes**  $bl \in \# \mathcal{B}$   
**assumes**  $\text{card } bl = v$   
**shows** *pairwise-balance*  $\mathcal{V} (\text{del-block } bl) (\Lambda - 1)$   
 $\langle \text{proof} \rangle$

**lemma** *remove-complete-block-pbd-alt*:  
**assumes**  $b \geq 2$   
**assumes**  $bl \in \# \mathcal{B}$   
**assumes**  $bl = \mathcal{V}$   
**shows** *pairwise-balance*  $\mathcal{V} (\text{del-block } bl) (\Lambda - 1)$

*<proof>*

**lemma** *b-gt-index*:  $b \geq \Lambda$   
*<proof>*

**lemma** *remove-complete-blocks-set-pbd*:  
  **assumes**  $x < \Lambda$   
  **assumes**  $\text{size } A = x$   
  **assumes**  $A \subset \# \mathcal{B}$   
  **assumes**  $\bigwedge a. a \in \# A \implies a = \mathcal{V}$   
  **shows** *pairwise-balance*  $\mathcal{V} (\mathcal{B} - A) (\Lambda - x)$   
*<proof>*

**lemma** *remove-all-complete-blocks-pbd*:  
  **assumes**  $\text{count } \mathcal{B} \mathcal{V} < \Lambda$   
  **shows** *pairwise-balance*  $\mathcal{V} (\text{removeAll-mset } \mathcal{V} \mathcal{B}) (\Lambda - (\text{count } \mathcal{B} \mathcal{V}))$  (**is pairwise-balance**  $\mathcal{V} ?\mathcal{B} ?\Lambda$ )  
*<proof>*

**end**

**context** *bid*  
**begin**  
**lemma** *symmetric-bibdIII*:  $r = k \implies \text{symmetric-bid } \mathcal{V} \mathcal{B} k \Lambda$   
*<proof>*  
**end**

### 3.2 New Design Locales

We establish a number of new locales and link them to the existing locale hierarchy in order to reason in contexts requiring specific combinations of contexts

Regular t-wise balance

**locale** *regular-t-wise-balance* = *t-wise-balance* + *constant-rep-design*  
**begin**

**lemma** *reg-index-lt-rep*:  
  **shows**  $\Lambda_t \leq r$   
*<proof>*

**end**

**locale** *regular-pairwise-balance* = *regular-t-wise-balance*  $\mathcal{V} \mathcal{B} \geq \Lambda r$  + *pairwise-balance*  $\mathcal{V} \mathcal{B} \Lambda$   
**for**  $\mathcal{V}$  **and**  $\mathcal{B}$  **and**  $\Lambda$  **and**  $r$

Const Intersect Design

This is the dual of a balanced design, and used extensively in the re-

maining formalisation

**locale** *const-intersect-design* = *proper-design* +

**fixes**  $m :: \text{nat}$

**assumes** *const-intersect*:  $b1 \in\# \mathcal{B} \implies b2 \in\# (\mathcal{B} - \{\#b1\}) \implies b1 \cap b2 = m$

**sublocale** *symmetric-bibd*  $\subseteq$  *const-intersect-design*  $\vee \mathcal{B} \wedge$

*<proof>*

**context** *const-intersect-design*

**begin**

**lemma** *inter-num-le-block-size*:

**assumes**  $bl \in\# \mathcal{B}$

**assumes**  $b \geq 2$

**shows**  $m \leq \text{card } bl$

*<proof>*

**lemma** *const-inter-multiplicity-one*:

**assumes**  $bl \in\# \mathcal{B}$

**assumes**  $m < \text{card } bl$

**shows** *multiplicity*  $bl = 1$

*<proof>*

**lemma** *mult-blocks-const-inter*:

**assumes**  $bl \in\# \mathcal{B}$

**assumes** *multiplicity*  $bl > 1$

**assumes**  $b \geq 2$

**shows**  $m = \text{card } bl$

*<proof>*

**lemma** *simple-const-inter-block-size*:  $(\bigwedge bl. bl \in\# \mathcal{B} \implies m < \text{card } bl) \implies \text{simple-design } \vee \mathcal{B}$

*<proof>*

**lemma** *simple-const-inter-iff*:

**assumes**  $b \geq 2$

**shows**  $\text{size } \{\#bl \in\# \mathcal{B} . \text{card } bl = m \ \#\} \leq 1 \iff \text{simple-design } \vee \mathcal{B}$

*<proof>*

**lemma** *empty-inter-implies-rep-one*:

**assumes**  $m = 0$

**assumes**  $x \in \mathcal{V}$

**shows**  $\mathcal{B} \text{ rep } x \leq 1$

*<proof>*

**lemma** *empty-inter-implies-b-lt-v*:

**assumes**  $m = 0$

**shows**  $b \leq v$

*<proof>*

**end**

**locale** *simple-const-intersect-design* = *const-intersect-design* + *simple-design*

**end**

## 4 Incidence Vectors and Matrices

Incidence Matrices are an important representation for any incidence set system. The majority of basic definitions and properties proved in this theory are based on Stinson [8] and Colbourn [3].

**theory** *Incidence-Matrices* **imports** *Design-Extras Matrix-Vector-Extras List-Index.List-Index Design-Theory.Design-Isomorphisms*

**begin**

### 4.1 Incidence Vectors

A function which takes an ordered list of points, and a block, returning a 0-1 vector  $v$  where there is a 1 in the  $i$ th position if point  $i$  is in that block

**definition** *inc-vec-of* :: 'a list  $\Rightarrow$  'a set  $\Rightarrow$  ('b :: {ring-1}) vec **where**  
*inc-vec-of* Vs bl  $\equiv$  vec (length Vs) ( $\lambda$  i . if (Vs ! i)  $\in$  bl then 1 else 0)

**lemma** *inc-vec-one-zero-elems*:  $set_v$  (inc-vec-of Vs bl)  $\subseteq$  {0, 1}  
*<proof>*

**lemma** *finite-inc-vec-elems*: finite (set\_v (inc-vec-of Vs bl))  
*<proof>*

**lemma** *inc-vec-elems-max-two*: card (set\_v (inc-vec-of Vs bl))  $\leq$  2  
*<proof>*

**lemma** *inc-vec-dim*: dim-vec (inc-vec-of Vs bl) = length Vs  
*<proof>*

**lemma** *inc-vec-index*:  $i < \text{length } Vs \implies \text{inc-vec-of } Vs \text{ bl } \$ i = (\text{if } (Vs ! i) \in \text{bl} \text{ then } 1 \text{ else } 0)$   
*<proof>*

**lemma** *inc-vec-index-one-iff*:  $i < \text{length } Vs \implies \text{inc-vec-of } Vs \text{ bl } \$ i = 1 \iff Vs ! i \in \text{bl}$   
*<proof>*

**lemma** *inc-vec-index-zero-iff*:  $i < \text{length } Vs \implies \text{inc-vec-of } Vs \text{ bl } \$ i = 0 \iff Vs ! i \notin \text{bl}$   
*<proof>*

**lemma** *inc-vec-of-bij-betw*:  
**assumes** *inj-on f (set Vs)*  
**assumes** *bl ⊆ (set Vs)*  
**shows** *inc-vec-of Vs bl = inc-vec-of (map f Vs) (f ` bl)*  
*<proof>*

## 4.2 Incidence Matrices

A function which takes a list of points, and list of sets of points, and returns a  $v \times b$  0-1 matrix  $M$ , where  $v$  is the number of points, and  $b$  the number of sets, such that there is a 1 in the  $i, j$  position if and only if point  $i$  is in block  $j$ . The matrix has type *'b mat* to allow for operations commonly used on matrices [8]

**definition** *inc-mat-of* :: *'a list*  $\Rightarrow$  *'a set list*  $\Rightarrow$  (*'b* :: {*ring-1*}) *mat* **where**  
*inc-mat-of Vs Bs*  $\equiv$  *mat (length Vs) (length Bs) ( $\lambda (i,j) . \text{if } (Vs ! i) \in (Bs ! j) \text{ then } 1 \text{ else } 0$ )*

Basic lemmas on the *inc-mat-of* matrix result (elements/dimensions/indexing)

**lemma** *inc-mat-one-zero-elems*: *elements-mat (inc-mat-of Vs Bs) ⊆ {0, 1}*  
*<proof>*

**lemma** *fin-incidence-mat-elems*: *finite (elements-mat (inc-mat-of Vs Bs))*  
*<proof>*

**lemma** *inc-matrix-elems-max-two*: *card (elements-mat (inc-mat-of Vs Bs)) ≤ 2*  
*<proof>*

**lemma** *inc-mat-of-index [simp]*: *i < dim-row (inc-mat-of Vs Bs)  $\implies$  j < dim-col (inc-mat-of Vs Bs)  $\implies$*   
*inc-mat-of Vs Bs \$\$ (i, j) = (if (Vs ! i) ∈ (Bs ! j) then 1 else 0)*  
*<proof>*

**lemma** *inc-mat-dim-row*: *dim-row (inc-mat-of Vs Bs) = length Vs*  
*<proof>*

**lemma** *inc-mat-dim-vec-row*: *dim-vec (row (inc-mat-of Vs Bs) i) = length Bs*  
*<proof>*

**lemma** *inc-mat-dim-col*: *dim-col (inc-mat-of Vs Bs) = length Bs*  
*<proof>*

**lemma** *inc-mat-dim-vec-col*: *dim-vec (col (inc-mat-of Vs Bs) i) = length Vs*  
*<proof>*

**lemma** *inc-matrix-point-in-block-one*: *i < length Vs  $\implies$  j < length Bs  $\implies$  Vs ! i ∈ Bs ! j*  
 $\implies$  *(inc-mat-of Vs Bs) \$\$ (i, j) = 1*

*<proof>*

**lemma** *inc-matrix-point-not-in-block-zero*:  $i < \text{length } Vs \implies j < \text{length } Bs \implies Vs ! i \notin Bs ! j \implies$   
 $(\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 0$   
*<proof>*

**lemma** *inc-matrix-point-in-block*:  $i < \text{length } Vs \implies j < \text{length } Bs \implies (\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 1$   
 $\implies Vs ! i \in Bs ! j$   
*<proof>*

**lemma** *inc-matrix-point-not-in-block*:  $i < \text{length } Vs \implies j < \text{length } Bs \implies$   
 $(\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 0 \implies Vs ! i \notin Bs ! j$   
*<proof>*

**lemma** *inc-matrix-point-not-in-block-iff*:  $i < \text{length } Vs \implies j < \text{length } Bs \implies$   
 $(\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 0 \iff Vs ! i \notin Bs ! j$   
*<proof>*

**lemma** *inc-matrix-point-in-block-iff*:  $i < \text{length } Vs \implies j < \text{length } Bs \implies$   
 $(\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 1 \iff Vs ! i \in Bs ! j$   
*<proof>*

**lemma** *inc-matrix-subset-implies-one*:  
**assumes**  $I \subseteq \{.. < \text{length } Vs\}$   
**assumes**  $j < \text{length } Bs$   
**assumes**  $(!) Vs ' I \subseteq Bs ! j$   
**assumes**  $i \in I$   
**shows**  $(\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 1$   
*<proof>*

**lemma** *inc-matrix-one-implies-membership*:  $I \subseteq \{.. < \text{length } Vs\} \implies j < \text{length } Bs \implies$   
 $(\bigwedge i. i \in I \implies (\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 1) \implies i \in I \implies Vs ! i \in Bs ! j$   
*<proof>*

**lemma** *inc-matrix-elems-one-zero*:  $i < \text{length } Vs \implies j < \text{length } Bs \implies$   
 $(\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 0 \vee (\text{inc-mat-of } Vs \ Bs) \$\$ (i, j) = 1$   
*<proof>*

Reasoning on Rows/Columns of the incidence matrix

**lemma** *inc-mat-col-def*:  $j < \text{length } Bs \implies i < \text{length } Vs \implies$   
 $(\text{col } (\text{inc-mat-of } Vs \ Bs) \ j) \$ i = (\text{if } (Vs ! i \in Bs ! j) \text{ then } 1 \text{ else } 0)$   
*<proof>*

**lemma** *inc-mat-col-list-map-elem*:  $j < \text{length } Bs \implies i < \text{length } Vs \implies$   
 $\text{col } (\text{inc-mat-of } Vs \ Bs) \ j \$ i = \text{map-vec } (\lambda x. \text{if } (x \in (Bs ! j)) \text{ then } 1 \text{ else } 0)$   
 $(\text{vec-of-list } Vs) \$ i$



*<proof>*

**lemma** *inc-mat-col-list-map*:  $j < \text{length } Bs \implies$   
 $\text{col } (\text{inc-mat-of } Vs \ Bs) \ j = \text{map-vec } (\lambda \ x . \text{if } (x \in (Bs \ ! \ j)) \text{ then } 1 \text{ else } 0)$   
*(vec-of-list Vs)*  
*<proof>*

**lemma** *inc-mat-row-def*:  $j < \text{length } Bs \implies i < \text{length } Vs \implies$   
 $(\text{row } (\text{inc-mat-of } Vs \ Bs) \ i) \ \$ \ j = (\text{if } (Vs \ ! \ i \in Bs \ ! \ j) \text{ then } 1 \text{ else } 0)$   
*<proof>*

**lemma** *inc-mat-row-list-map-elem*:  $j < \text{length } Bs \implies i < \text{length } Vs \implies$   
 $\text{row } (\text{inc-mat-of } Vs \ Bs) \ i \ \$ \ j = \text{map-vec } (\lambda \ bl . \text{if } ((Vs \ ! \ i) \in bl) \text{ then } 1 \text{ else } 0)$   
*(vec-of-list Bs) \\$ j*  
*<proof>*

**lemma** *inc-mat-row-list-map*:  $i < \text{length } Vs \implies$   
 $\text{row } (\text{inc-mat-of } Vs \ Bs) \ i = \text{map-vec } (\lambda \ bl . \text{if } ((Vs \ ! \ i) \in bl) \text{ then } 1 \text{ else } 0)$   
*(vec-of-list Bs)*  
*<proof>*

Connecting *inc-vec-of* and *inc-mat-of*

**lemma** *inc-mat-col-inc-vec*:  $j < \text{length } Bs \implies \text{col } (\text{inc-mat-of } Vs \ Bs) \ j = \text{inc-vec-of}$   
 $Vs \ (Bs \ ! \ j)$   
*<proof>*

**lemma** *inc-mat-of-cols-inc-vecs*:  $\text{cols } (\text{inc-mat-of } Vs \ Bs) = \text{map } (\lambda \ j . \text{inc-vec-of}$   
 $Vs \ j) \ Bs$   
*<proof>*

**lemma** *inc-mat-of-bij-betw*:  
**assumes** *inj-on f (set Vs)*  
**assumes**  $\bigwedge \ bl . bl \in (\text{set } Bs) \implies bl \subseteq (\text{set } Vs)$   
**shows**  $\text{inc-mat-of } Vs \ Bs = \text{inc-mat-of } (\text{map } f \ Vs) \ (\text{map } ((\cdot) \ f) \ Bs)$   
*<proof>*

Definitions for the incidence matrix representation of common incidence system properties

**definition** *non-empty-col* ::  $('a :: \{\text{zero-neq-one}\}) \text{ mat} \Rightarrow \text{nat} \Rightarrow \text{bool}$  **where**  
 $\text{non-empty-col } M \ j \equiv \exists \ k . k \neq 0 \wedge k \in \$ \text{col } M \ j$

**definition** *proper-inc-mat* ::  $('a :: \{\text{zero-neq-one}\}) \text{ mat} \Rightarrow \text{bool}$  **where**  
 $\text{proper-inc-mat } M \equiv (\text{dim-row } M > 0 \wedge \text{dim-col } M > 0)$

Matrix version of the representation number property (*rep*)

**definition** *mat-rep-num* ::  $('a :: \{\text{zero-neq-one}\}) \text{ mat} \Rightarrow \text{nat} \Rightarrow \text{nat}$  **where**  
 $\text{mat-rep-num } M \ i \equiv \text{count-vec } (\text{row } M \ i) \ 1$

Matrix version of the points index property (*index*)

**definition** *mat-point-index* :: ('a :: {zero-neq-one}) mat  $\Rightarrow$  nat set  $\Rightarrow$  nat **where**  
*mat-point-index* M I  $\equiv$  card {j . j < dim-col M  $\wedge$  ( $\forall$  i  $\in$  I. M \$\$ (i, j) = 1)}

**definition** *mat-inter-num* :: ('a :: {zero-neq-one}) mat  $\Rightarrow$  nat  $\Rightarrow$  nat  $\Rightarrow$  nat **where**  
*mat-inter-num* M j1 j2  $\equiv$  card {i . i < dim-row M  $\wedge$  M \$\$ (i, j1) = 1  $\wedge$  M \$\$ (i, j2) = 1}

Matrix version of the block size property

**definition** *mat-block-size* :: ('a :: {zero-neq-one}) mat  $\Rightarrow$  nat  $\Rightarrow$  nat **where**  
*mat-block-size* M j  $\equiv$  count-vec (col M j) 1

**lemma** *non-empty-col-obtains*:  
**assumes** *non-empty-col* M j  
**obtains** i **where** i < dim-row M **and** (col M j) \$ i  $\neq$  0  
 <proof>

**lemma** *non-empty-col-alt-def*:  
**assumes** j < dim-col M  
**shows** *non-empty-col* M j  $\longleftrightarrow$  ( $\exists$  i. i < dim-row M  $\wedge$  M \$\$ (i, j)  $\neq$  0)  
 <proof>

**lemma** *proper-inc-mat-map*: *proper-inc-mat* M  $\implies$  *proper-inc-mat* (map-mat f M)  
 <proof>

**lemma** *mat-point-index-alt*: *mat-point-index* M I = card {j  $\in$  {0.. $\dim$ -col M} .  
 ( $\forall$  i  $\in$  I . M \$\$ (i, j) = 1)}  
 <proof>

**lemma** *mat-block-size-sum-alt*:  
**fixes** M :: 'a :: {ring-1} mat  
**shows** *elements-mat* M  $\subseteq$  {0, 1}  $\implies$  j < dim-col M  $\implies$  of-nat (*mat-block-size*  
 M j) = sum-vec (col M j)  
 <proof>

**lemma** *mat-rep-num-sum-alt*:  
**fixes** M :: 'a :: {ring-1} mat  
**shows** *elements-mat* M  $\subseteq$  {0, 1}  $\implies$  i < dim-row M  $\implies$  of-nat (*mat-rep-num*  
 M i) = sum-vec (row M i)  
 <proof>

**lemma** *mat-point-index-two-alt*:  
**assumes** i1 < dim-row M  
**assumes** i2 < dim-row M  
**shows** *mat-point-index* M {i1, i2} = card {j . j < dim-col M  $\wedge$  M \$\$ (i1, j) = 1  
 $\wedge$  M \$\$ (i2, j) = 1}  
 <proof>

Transpose symmetries

**lemma** *trans-mat-rep-block-size-sym*: j < dim-col M  $\implies$  *mat-block-size* M j =  
*mat-rep-num* M<sup>T</sup> j

$i < \dim\text{-row } M \implies \text{mat-rep-num } M \ i = \text{mat-block-size } M^T \ i$   
 ⟨proof⟩

**lemma** *trans-mat-point-index-inter-sym*:

$i1 < \dim\text{-row } M \implies i2 < \dim\text{-row } M \implies \text{mat-point-index } M \ \{i1, i2\} =$   
 $\text{mat-inter-num } M^T \ i1 \ i2$   
 $j1 < \dim\text{-col } M \implies j2 < \dim\text{-col } M \implies \text{mat-inter-num } M \ j1 \ j2 = \text{mat-point-index}$   
 $M^T \ \{j1, j2\}$   
 ⟨proof⟩

### 4.3 0-1 Matrices

Incidence matrices contain only two elements: 0 and 1. We define a locale which provides a context to work in for matrices satisfying this condition for any 'b type.

**locale** *zero-one-matrix* =  
**fixes** *matrix* :: 'b :: {zero-neq-one} *mat* (M)  
**assumes** *elems01*: *elements-mat* M  $\subseteq \{0, 1\}$   
**begin**

Row and Column Properties of the Matrix

**lemma** *row-elems-ss01*:  $i < \dim\text{-row } M \implies \text{vec-set } (\text{row } M \ i) \subseteq \{0, 1\}$   
 ⟨proof⟩

**lemma** *col-elems-ss01*:  
**assumes**  $j < \dim\text{-col } M$   
**shows**  $\text{vec-set } (\text{col } M \ j) \subseteq \{0, 1\}$   
 ⟨proof⟩

**lemma** *col-nth-0-or-1-iff*:  
**assumes**  $j < \dim\text{-col } M$   
**assumes**  $i < \dim\text{-row } M$   
**shows**  $\text{col } M \ j \ \$ \ i = 0 \longleftrightarrow \text{col } M \ j \ \$ \ i \neq 1$   
 ⟨proof⟩

**lemma** *row-nth-0-or-1-iff*:  
**assumes**  $j < \dim\text{-col } M$   
**assumes**  $i < \dim\text{-row } M$   
**shows**  $\text{row } M \ i \ \$ \ j = 0 \longleftrightarrow \text{row } M \ i \ \$ \ j \neq 1$   
 ⟨proof⟩

**lemma** *transpose-entries*:  $\text{elements-mat } (M^T) \subseteq \{0, 1\}$   
 ⟨proof⟩

**lemma** *M-not-zero-simp*:  $j < \dim\text{-col } M \implies i < \dim\text{-row } M \implies M \ \$ \$ \ (i, j) \neq 0$   
 $\implies M \ \$ \$ \ (i, j) = 1$   
 ⟨proof⟩

**lemma** *M-not-one-simp*:  $j < \dim\text{-col } M \implies i < \dim\text{-row } M \implies M \$\$ (i, j) \neq 1$   
 $\implies M \$\$ (i, j) = 0$

*<proof>*

Definition for mapping a column to a block

**definition** *map-col-to-block* :: 'a :: {zero-neq-one} vec  $\Rightarrow$  nat set **where**  
*map-col-to-block*  $c \equiv \{ i \in \{..<\dim\text{-vec } c\} . c \$ i = 1 \}$

**lemma** *map-col-to-block-alt*:  $\text{map-col-to-block } c = \{ i . i < \dim\text{-vec } c \wedge c \$ i = 1 \}$

*<proof>*

**lemma** *map-col-to-block-elem*:  $i < \dim\text{-vec } c \implies i \in \text{map-col-to-block } c \longleftrightarrow c \$ i = 1$

*<proof>*

**lemma** *in-map-col-valid-index*:  $i \in \text{map-col-to-block } c \implies i < \dim\text{-vec } c$

*<proof>*

**lemma** *map-col-to-block-size*:  $j < \dim\text{-col } M \implies \text{card } (\text{map-col-to-block } (\text{col } M j)) = \text{mat-block-size } M j$

*<proof>*

**lemma** *in-map-col-valid-index-M*:  $j < \dim\text{-col } M \implies i \in \text{map-col-to-block } (\text{col } M j) \implies i < \dim\text{-row } M$

*<proof>*

**lemma** *map-col-to-block-elem-not*:  $c \in \text{set } (\text{cols } M) \implies i < \dim\text{-vec } c \implies i \notin \text{map-col-to-block } c \longleftrightarrow c \$ i = 0$

*<proof>*

**lemma** *obtain-block-index-map-block-set*:

**assumes**  $bl \in \# \{ \# \text{map-col-to-block } c . c \in \# \text{mset } (\text{cols } M) \# \}$

**obtains**  $j$  **where**  $j < \dim\text{-col } M$  **and**  $bl = \text{map-col-to-block } (\text{col } M j)$

*<proof>*

**lemma** *mat-ord-inc-sys-point[simp]*:  $x < \dim\text{-row } M \implies [0..<(\dim\text{-row } M)] ! x = x$

*<proof>*

**lemma** *mat-ord-inc-sys-block[simp]*:  $j < \dim\text{-col } M \implies (\text{map } (\text{map-col-to-block}) (\text{cols } M)) ! j = \text{map-col-to-block } (\text{col } M j)$

*<proof>*

**lemma** *ordered-to-mset-col-blocks*:

$\{ \# \text{map-col-to-block } c . c \in \# \text{mset } (\text{cols } M) \# \} = \text{mset } (\text{map } (\text{map-col-to-block}) (\text{cols } M))$

*<proof>*

Lemmas on incidence matrix properties

**lemma** *non-empty-col-01*:

**assumes**  $j < \dim\text{-col } M$   
**shows**  $\text{non-empty-col } M j \longleftrightarrow 1 \in \$ \text{ col } M j$   
 <proof>

**lemma** *mat-rep-num-alt*:  
**assumes**  $i < \dim\text{-row } M$   
**shows**  $\text{mat-rep-num } M i = \text{card } \{j . j < \dim\text{-col } M \wedge M \$\$ (i, j) = 1\}$   
 <proof>

**lemma** *mat-rep-num-alt-col*:  $i < \dim\text{-row } M \implies \text{mat-rep-num } M i = \text{size } \{\#c \in \# (\text{mset } (\text{cols } M)) . c \$ i = 1\#\}$   
 <proof>

A zero one matrix is an incidence system

**lemma** *map-col-to-block-wf*:  $\bigwedge c. c \in \text{set } (\text{cols } M) \implies \text{map-col-to-block } c \subseteq \{0..<\dim\text{-row } M\}$   
 <proof>

**lemma** *one-implies-block-nempty*:  $j < \dim\text{-col } M \implies 1 \in \$ (\text{col } M j) \implies \text{map-col-to-block } (\text{col } M j) \neq \{\}$   
 <proof>

**interpretation** *incidence-sys*:  $\text{incidence-system } \{0..<\dim\text{-row } M\}$   
 $\{\# \text{ map-col-to-block } c . c \in \# \text{ mset } (\text{cols } M)\#\}$   
 <proof>

**interpretation** *fin-incident-sys*:  $\text{finite-incident-system } \{0..<\dim\text{-row } M\}$   
 $\{\# \text{ map-col-to-block } c . c \in \# \text{ mset } (\text{cols } M)\#\}$   
 <proof>

**lemma** *block-nempty-implies-all-zeros*:  $j < \dim\text{-col } M \implies \text{map-col-to-block } (\text{col } M j) = \{\} \implies$   
 $i < \dim\text{-row } M \implies \text{col } M j \$ i = 0$   
 <proof>

**lemma** *block-nempty-implies-no-one*:  $j < \dim\text{-col } M \implies \text{map-col-to-block } (\text{col } M j) = \{\} \implies \neg (1 \in \$ (\text{col } M j))$   
 <proof>

**lemma** *mat-is-design*:  
**assumes**  $\bigwedge j. j < \dim\text{-col } M \implies 1 \in \$ (\text{col } M j)$   
**shows**  $\text{design } \{0..<\dim\text{-row } M\} \{\# \text{ map-col-to-block } c . c \in \# \text{ mset } (\text{cols } M)\#\}$   
 <proof>

**lemma** *mat-is-proper-design*:  
**assumes**  $\bigwedge j. j < \dim\text{-col } M \implies 1 \in \$ (\text{col } M j)$   
**assumes**  $\dim\text{-col } M > 0$   
**shows**  $\text{proper-design } \{0..<\dim\text{-row } M\} \{\# \text{ map-col-to-block } c . c \in \# \text{ mset } (\text{cols } M)\#\}$

*<proof>*

Show the 01 injective function preserves system properties

**lemma** *inj-on-01-hom-index*:

**assumes** *inj-on-01-hom*  $f$

**assumes**  $i < \text{dim-row } M$   $j < \text{dim-col } M$

**shows**  $M \text{ \#\# } (i, j) = 1 \iff (\text{map-mat } f \ M) \text{ \#\# } (i, j) = 1$

**and**  $M \text{ \#\# } (i, j) = 0 \iff (\text{map-mat } f \ M) \text{ \#\# } (i, j) = 0$

*<proof>*

**lemma** *preserve-non-empty*:

**assumes** *inj-on-01-hom*  $f$

**assumes**  $j < \text{dim-col } M$

**shows**  $\text{non-empty-col } M \ j \iff \text{non-empty-col } (\text{map-mat } f \ M) \ j$

*<proof>*

**lemma** *preserve-mat-rep-num*:

**assumes** *inj-on-01-hom*  $f$

**assumes**  $i < \text{dim-row } M$

**shows**  $\text{mat-rep-num } M \ i = \text{mat-rep-num } (\text{map-mat } f \ M) \ i$

*<proof>*

**lemma** *preserve-mat-block-size*:

**assumes** *inj-on-01-hom*  $f$

**assumes**  $j < \text{dim-col } M$

**shows**  $\text{mat-block-size } M \ j = \text{mat-block-size } (\text{map-mat } f \ M) \ j$

*<proof>*

**lemma** *preserve-mat-point-index*:

**assumes** *inj-on-01-hom*  $f$

**assumes**  $\bigwedge i. i \in I \implies i < \text{dim-row } M$

**shows**  $\text{mat-point-index } M \ I = \text{mat-point-index } (\text{map-mat } f \ M) \ I$

*<proof>*

**lemma** *preserve-mat-inter-num*:

**assumes** *inj-on-01-hom*  $f$

**assumes**  $j1 < \text{dim-col } M$   $j2 < \text{dim-col } M$

**shows**  $\text{mat-inter-num } M \ j1 \ j2 = \text{mat-inter-num } (\text{map-mat } f \ M) \ j1 \ j2$

*<proof>*

**lemma** *lift-mat-01-index-iff*:

$i < \text{dim-row } M \implies j < \text{dim-col } M \implies (\text{lift-01-mat } M) \text{ \#\# } (i, j) = 0 \iff M \text{ \#\# } (i, j) = 0$

$i < \text{dim-row } M \implies j < \text{dim-col } M \implies (\text{lift-01-mat } M) \text{ \#\# } (i, j) = 1 \iff M \text{ \#\# } (i, j) = 1$

*<proof>*

**lemma** *lift-mat-elems*:  $\text{elements-mat } (\text{lift-01-mat } M) \subseteq \{0, 1\}$

*<proof>*

**lemma** *lift-mat-is-0-1: zero-one-matrix (lift-01-mat M)*  
*<proof>*

**lemma** *lift-01-mat-distinct-cols: distinct (cols M)  $\implies$  distinct (cols (lift-01-mat M))*  
*<proof>*

**end**

Some properties must be further restricted to matrices having a 'a type

**locale** *zero-one-matrix-ring-1 = zero-one-matrix M for M :: 'b :: {ring-1} mat*  
**begin**

**lemma** *map-col-block-eq:*  
**assumes**  $c \in \text{set}(\text{cols } M)$   
**shows**  $\text{inc-vec-of } [0..<\text{dim-vec } c] (\text{map-col-to-block } c) = c$   
*<proof>*

**lemma** *inc-mat-of-map-rev: inc-mat-of [0..<dim-row M] (map map-col-to-block (cols M)) = M*  
*<proof>*

**lemma** *M-index-square-itself:  $j < \text{dim-col } M \implies i < \text{dim-row } M \implies (M \$\$ (i, j))^{\wedge 2} = M \$\$ (i, j)$*   
*<proof>*

**lemma** *M-col-index-square-itself:  $j < \text{dim-col } M \implies i < \text{dim-row } M \implies ((\text{col } M j) \$ i)^{\wedge 2} = (\text{col } M j) \$ i$*   
*<proof>*

Scalar Prod Alternative definitions for matrix properties

**lemma** *scalar-prod-inc-vec-block-size-mat:*  
**assumes**  $j < \text{dim-col } M$   
**shows**  $(\text{col } M j) \cdot (\text{col } M j) = \text{of-nat } (\text{mat-block-size } M j)$   
*<proof>*

**lemma** *scalar-prod-inc-vec-mat-inter-num:*  
**assumes**  $j1 < \text{dim-col } M$   $j2 < \text{dim-col } M$   
**shows**  $(\text{col } M j1) \cdot (\text{col } M j2) = \text{of-nat } (\text{mat-inter-num } M j1 j2)$   
*<proof>*

**end**

Any matrix generated by *inc-mat-of* is a 0-1 matrix.

**lemma** *inc-mat-of-01-mat: zero-one-matrix-ring-1 (inc-mat-of Vs Bs)*  
*<proof>*

## 4.4 Ordered Incidence Systems

We impose an arbitrary ordering on the point set and block collection to enable matrix reasoning. Note that this is also common in computer algebra representations of designs

**locale** *ordered-incidence-system* =  
**fixes**  $\mathcal{V}s :: 'a \text{ list}$  **and**  $\mathcal{B}s :: 'a \text{ set list}$   
**assumes** *wf-list*:  $b \in \# (\text{mset } \mathcal{B}s) \implies b \subseteq \text{set } \mathcal{V}s$   
**assumes** *distinct*: *distinct*  $\mathcal{V}s$

An ordered incidence system, as it is defined on lists, can only represent finite incidence systems

**sublocale** *ordered-incidence-system*  $\subseteq$  *finite-incidence-system* *set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   
 $\langle \text{proof} \rangle$

**lemma** *ordered-incidence-sysI*:  
**assumes** *finite-incidence-system*  $\mathcal{V}$   $\mathcal{B}$   
**assumes**  $\mathcal{V}s \in \text{permutations-of-set } \mathcal{V}$  **and**  $\mathcal{B}s \in \text{permutations-of-multiset } \mathcal{B}$   
**shows** *ordered-incidence-system*  $\mathcal{V}s$   $\mathcal{B}s$   
 $\langle \text{proof} \rangle$

**lemma** *ordered-incidence-sysII*:  
**assumes** *finite-incidence-system*  $\mathcal{V}$   $\mathcal{B}$  **and** *set*  $\mathcal{V}s = \mathcal{V}$  **and** *distinct*  $\mathcal{V}s$  **and** *mset*  $\mathcal{B}s = \mathcal{B}$   
**shows** *ordered-incidence-system*  $\mathcal{V}s$   $\mathcal{B}s$   
 $\langle \text{proof} \rangle$

**context** *ordered-incidence-system*  
**begin**

For ease of notation, establish the same notation as for incidence systems

**abbreviation**  $\mathcal{V} \equiv \text{set } \mathcal{V}s$   
**abbreviation**  $\mathcal{B} \equiv \text{mset } \mathcal{B}s$

Basic properties on ordered lists

**lemma** *points-indexing*:  $\mathcal{V}s \in \text{permutations-of-set } \mathcal{V}$   
 $\langle \text{proof} \rangle$

**lemma** *blocks-indexing*:  $\mathcal{B}s \in \text{permutations-of-multiset } \mathcal{B}$   
 $\langle \text{proof} \rangle$

**lemma** *points-list-empty-iff*:  $\mathcal{V}s = [] \iff \mathcal{V} = \{\}$   
 $\langle \text{proof} \rangle$

**lemma** *points-indexing-inj*:  $\forall i \in I . i < \text{length } \mathcal{V}s \implies \text{inj-on } (!) \mathcal{V}s \ I$   
 $\langle \text{proof} \rangle$

**lemma** *blocks-list-empty-iff*:  $\mathcal{B}s = [] \iff \mathcal{B} = \{\#\}$   
 $\langle \text{proof} \rangle$



**lemma** *blocks-list-empty: proper-design*  $\mathcal{V} \mathcal{B} \implies \mathcal{B}s \neq []$   
 ⟨proof⟩

**lemma** *points-list-empty: proper-design*  $\mathcal{V} \mathcal{B} \implies \mathcal{V}s \neq []$   
 ⟨proof⟩

**lemma** *points-list-length: length*  $\mathcal{V}s = v$   
 ⟨proof⟩

**lemma** *blocks-list-length: length*  $\mathcal{B}s = b$   
 ⟨proof⟩

**lemma** *valid-points-index:  $i < v \implies \mathcal{V}s ! i \in \mathcal{V}$*   
 ⟨proof⟩

**lemma** *valid-points-index-cons:  $x \in \mathcal{V} \implies \exists i. \mathcal{V}s ! i = x \wedge i < v$*   
 ⟨proof⟩

**lemma** *valid-points-index-obtains:*  
**assumes**  $x \in \mathcal{V}$   
**obtains**  $i$  **where**  $\mathcal{V}s ! i = x \wedge i < v$   
 ⟨proof⟩

**lemma** *valid-blocks-index:  $j < b \implies \mathcal{B}s ! j \in \# \mathcal{B}$*   
 ⟨proof⟩

**lemma** *valid-blocks-index-cons:  $bl \in \# \mathcal{B} \implies \exists j. \mathcal{B}s ! j = bl \wedge j < b$*   
 ⟨proof⟩

**lemma** *valid-blocks-index-obtains:*  
**assumes**  $bl \in \# \mathcal{B}$   
**obtains**  $j$  **where**  $\mathcal{B}s ! j = bl \wedge j < b$   
 ⟨proof⟩

**lemma** *block-points-valid-point-index:*  
**assumes**  $bl \in \# \mathcal{B} \ x \in bl$   
**obtains**  $i$  **where**  $i < \text{length } \mathcal{V}s \wedge \mathcal{V}s ! i = x$   
 ⟨proof⟩

**lemma** *points-set-index-img:  $\mathcal{V} = \text{image}(\lambda i. (\mathcal{V}s ! i)) (\{..<v\})$*   
 ⟨proof⟩

**lemma** *blocks-mset-image:  $\mathcal{B} = \text{image-mset} (\lambda i. (\mathcal{B}s ! i)) (\text{mset-set } \{..<b\})$*   
 ⟨proof⟩

**lemma** *incidence-cond-indexed[simp]:  $i < v \implies j < b \implies \text{incident } (\mathcal{V}s ! i) (\mathcal{B}s ! j)$*   
 $j \longleftrightarrow (\mathcal{V}s ! i) \in (\mathcal{B}s ! j)$   
 ⟨proof⟩

**lemma** *bij-betw-points-index*: *bij-betw* ( $\lambda i. \mathcal{V}s ! i$ )  $\{0..<v\}$   $\mathcal{V}$   
 $\langle proof \rangle$

Some lemmas on cardinality due to different set descriptor filters

**lemma** *card-filter-point-indices*:  $card \{i \in \{0..<v\}. P (\mathcal{V}s ! i)\} = card \{v \in \mathcal{V} . P v\}$   
 $\langle proof \rangle$

**lemma** *card-block-points-filter*:  
**assumes**  $j < b$   
**shows**  $card (\mathcal{B}s ! j) = card \{i \in \{0..<v\} . (\mathcal{V}s ! i) \in (\mathcal{B}s ! j)\}$   
 $\langle proof \rangle$

**lemma** *obtains-two-diff-block-indices*:  
**assumes**  $j1 < b$   
**assumes**  $j2 < b$   
**assumes**  $j1 \neq j2$   
**assumes**  $b \geq 2$   
**obtains**  $bl1 bl2$  **where**  $bl1 \in \# \mathcal{B}$  **and**  $\mathcal{B}s ! j1 = bl1$  **and**  $bl2 \in \# \mathcal{B} - \{\#bl1\#}$   
**and**  $\mathcal{B}s ! j2 = bl2$   
 $\langle proof \rangle$

**lemma** *filter-size-blocks-eq-card-indices*:  $size \{\# b \in \# \mathcal{B} . P b \#\} = card \{j \in \{..<(b)\}. P (\mathcal{B}s ! j)\}$   
 $\langle proof \rangle$

**lemma** *blocks-index-ne-belong*:  
**assumes**  $i1 < length \mathcal{B}s$   
**assumes**  $i2 < length \mathcal{B}s$   
**assumes**  $i1 \neq i2$   
**shows**  $\mathcal{B}s ! i2 \in \# \mathcal{B} - \{\#(\mathcal{B}s ! i1)\#}$   
 $\langle proof \rangle$

**lemma** *inter-num-points-filter-def*:  
**assumes**  $j1 < b$   $j2 < b$   $j1 \neq j2$   
**shows**  $card \{x \in \{0..<v\} . ((\mathcal{V}s ! x) \in (\mathcal{B}s ! j1) \wedge (\mathcal{V}s ! x) \in (\mathcal{B}s ! j2))\} = (\mathcal{B}s ! j1) \cap (\mathcal{B}s ! j2)$   
 $\langle proof \rangle$

Define an incidence matrix for this ordering of an incidence system

**abbreviation**  $N :: int\ mat$  **where**  
 $N \equiv inc\ mat\ of \ \mathcal{V}s \ \mathcal{B}s$

**sublocale** *zero-one-matrix-ring-1*  $N$   
 $\langle proof \rangle$

**lemma** *N-alt-def-dim*:  $N = mat\ v\ b \ (\lambda (i,j) . if (incident (\mathcal{V}s ! i) (\mathcal{B}s ! j)) then 1 else 0)$

*<proof>*

Matrix Dimension related lemmas

**lemma** *N-carrier-mat*:  $N \in \text{carrier-mat } v \text{ b}$   
*<proof>*

**lemma** *dim-row-is-v[simp]*:  $\text{dim-row } N = v$   
*<proof>*

**lemma** *dim-col-is-b[simp]*:  $\text{dim-col } N = b$   
*<proof>*

**lemma** *dim-vec-row-N*:  $\text{dim-vec } (\text{row } N \ i) = b$   
*<proof>*

**lemma** *dim-vec-col-N*:  $\text{dim-vec } (\text{col } N \ i) = v$  *<proof>*

**lemma** *dim-vec-N-col*:

**assumes**  $j < b$

**shows**  $\text{dim-vec } (\text{cols } N \ ! \ j) = v$

*<proof>*

**lemma** *N-carrier-mat-01-lift*:  $\text{lift-01-mat } N \in \text{carrier-mat } v \text{ b}$   
*<proof>*

Transpose properties

**lemma** *transpose-N-mult-dim*:  $\text{dim-row } (N * N^T) = v$   $\text{dim-col } (N * N^T) = v$   
*<proof>*

**lemma** *N-trans-index-val*:  $i < \text{dim-col } N \implies j < \text{dim-row } N \implies$   
 $N^T \ \ \$\$ \ (i, j) = (\text{if } (\mathcal{V}s \ ! \ j) \in (\mathcal{B}s \ ! \ i) \ \text{then } 1 \ \text{else } 0)$   
*<proof>*

Matrix element and index related lemmas

**lemma** *mat-row-elems*:  $i < v \implies \text{vec-set } (\text{row } N \ i) \subseteq \{0, 1\}$   
*<proof>*

**lemma** *mat-col-elems*:  $j < b \implies \text{vec-set } (\text{col } N \ j) \subseteq \{0, 1\}$   
*<proof>*

**lemma** *matrix-elems-one-zero*:  $i < v \implies j < b \implies N \ \ \$\$ \ (i, j) = 0 \vee N \ \ \$\$ \ (i, j) = 1$   
*<proof>*

**lemma** *matrix-point-in-block-one*:  $i < v \implies j < b \implies (\mathcal{V}s \ ! \ i) \in (\mathcal{B}s \ ! \ j) \implies N \ \ \$\$ \ (i, j) = 1$   
*<proof>*

**lemma** *matrix-point-not-in-block-zero*:  $i < v \implies j < b \implies \mathcal{V}s \ ! \ i \notin \mathcal{B}s \ ! \ j \implies N \ \ \$\$ \ (i, j) = 0$

*<proof>*

**lemma** *matrix-point-in-block*:  $i < v \implies j < b \implies N \$\$ (i, j) = 1 \implies \mathcal{V}s ! i \in \mathcal{B}s ! j$   
*<proof>*

**lemma** *matrix-point-not-in-block*:  $i < v \implies j < b \implies N \$\$ (i, j) = 0 \implies \mathcal{V}s ! i \notin \mathcal{B}s ! j$   
*<proof>*

**lemma** *matrix-point-not-in-block-iff*:  $i < v \implies j < b \implies N \$\$ (i, j) = 0 \iff \mathcal{V}s ! i \notin \mathcal{B}s ! j$   
*<proof>*

**lemma** *matrix-point-in-block-iff*:  $i < v \implies j < b \implies N \$\$ (i, j) = 1 \iff \mathcal{V}s ! i \in \mathcal{B}s ! j$   
*<proof>*

**lemma** *matrix-subset-implies-one*:  $I \subseteq \{..< v\} \implies j < b \implies (!) \mathcal{V}s ' I \subseteq \mathcal{B}s ! j \implies i \in I \implies N \$\$ (i, j) = 1$   
*<proof>*

**lemma** *matrix-one-implies-membership*:  $I \subseteq \{..< v\} \implies j < \text{size } \mathcal{B} \implies \forall i \in I. N \$\$ (i, j) = 1 \implies i \in I \implies \mathcal{V}s ! i \in \mathcal{B}s ! j$   
*<proof>*

Incidence Vector's of Incidence Matrix columns

**lemma** *col-inc-vec-of*:  $j < \text{length } \mathcal{B}s \implies \text{inc-vec-of } \mathcal{V}s (\mathcal{B}s ! j) = \text{col } N j$   
*<proof>*

**lemma** *inc-vec-eq-iff-blocks*:  
assumes  $bl \in \# \mathcal{B}$   
assumes  $bl' \in \# \mathcal{B}$   
shows  $\text{inc-vec-of } \mathcal{V}s bl = \text{inc-vec-of } \mathcal{V}s bl' \iff bl = bl'$   
*<proof>*

Incidence matrix column properties

**lemma** *N-col-def*:  $j < b \implies i < v \implies (\text{col } N j) \$ i = (\text{if } (\mathcal{V}s ! i \in \mathcal{B}s ! j) \text{ then } 1 \text{ else } 0)$   
*<proof>*

**lemma** *N-col-def-indiv*:  $j < b \implies i < v \implies \mathcal{V}s ! i \in \mathcal{B}s ! j \implies (\text{col } N j) \$ i = 1$   
 $j < b \implies i < v \implies \mathcal{V}s ! i \notin \mathcal{B}s ! j \implies (\text{col } N j) \$ i = 0$   
*<proof>*

**lemma** *N-col-list-map-elem*:  $j < b \implies i < v \implies \text{col } N j \$ i = \text{map-vec } (\lambda x . \text{if } (x \in (\mathcal{B}s ! j)) \text{ then } 1 \text{ else } 0) (\text{vec-of-list } \mathcal{V}s) \$ i$

*<proof>*

**lemma** *N-col-list-map*:  $j < b \implies \text{col } N j = \text{map-vec } (\lambda x . \text{if } (x \in (\mathcal{B}s ! j)) \text{ then } 1 \text{ else } 0) \text{ (vec-of-list } \mathcal{V}s)$   
*<proof>*

**lemma** *N-col-mset-point-set-img*:  $j < b \implies \text{vec-mset } (\text{col } N j) = \text{image-mset } (\lambda x . \text{if } (x \in (\mathcal{B}s ! j)) \text{ then } 1 \text{ else } 0) \text{ (mset-set } \mathcal{V})$   
*<proof>*

**lemma** *matrix-col-to-block*:  
**assumes**  $j < b$   
**shows**  $\mathcal{B}s ! j = (\lambda k . \mathcal{V}s ! k) \text{ ' } \{i \in \{..< v\} . (\text{col } N j) \$ i = 1\}$   
*<proof>*

**lemma** *matrix-col-to-block-v2*:  $j < b \implies \mathcal{B}s ! j = (\lambda k . \mathcal{V}s ! k) \text{ ' map-col-to-block } (\text{col } N j)$   
*<proof>*

**lemma** *matrix-col-in-blocks*:  $j < b \implies (!) \mathcal{V}s \text{ ' map-col-to-block } (\text{col } N j) \in \# \mathcal{B}$   
*<proof>*

**lemma** *inc-matrix-col-block*:  
**assumes**  $c \in \text{set } (\text{cols } N)$   
**shows**  $(\lambda x . \mathcal{V}s ! x) \text{ ' (map-col-to-block } c) \in \# \mathcal{B}$   
*<proof>*

#### Incidence Matrix Row Definitions

**lemma** *N-row-def*:  $j < b \implies i < v \implies (\text{row } N i) \$ j = (\text{if } (\mathcal{V}s ! i \in \mathcal{B}s ! j) \text{ then } 1 \text{ else } 0)$   
*<proof>*

**lemma** *N-row-list-map-elem*:  $j < b \implies i < v \implies \text{row } N i \$ j = \text{map-vec } (\lambda bl . \text{if } ((\mathcal{V}s ! i) \in bl) \text{ then } 1 \text{ else } 0) \text{ (vec-of-list } \mathcal{B}s) \$ j$   
*<proof>*

**lemma** *N-row-list-map*:  $i < v \implies \text{row } N i = \text{map-vec } (\lambda bl . \text{if } ((\mathcal{V}s ! i) \in bl) \text{ then } 1 \text{ else } 0) \text{ (vec-of-list } \mathcal{B}s)$   
*<proof>*

**lemma** *N-row-mset-blocks-img*:  $i < v \implies \text{vec-mset } (\text{row } N i) = \text{image-mset } (\lambda x . \text{if } ((\mathcal{V}s ! i) \in x) \text{ then } 1 \text{ else } 0) \mathcal{B}$   
*<proof>*

#### Alternate Block representations

**lemma** *block-mat-cond-rep*:  
**assumes**  $j < \text{length } \mathcal{B}s$   
**shows**  $(\mathcal{B}s ! j) = \{\mathcal{V}s ! i \mid i . i < \text{length } \mathcal{V}s \wedge N \$\$ (i, j) = 1\}$

*<proof>*

**lemma** *block-mat-cond-rep'*:  $j < \text{length } \mathcal{B}s \implies (\mathcal{B}s ! j) = ((!) \mathcal{V}s) \text{ ' } \{i . i < \text{length } \mathcal{V}s \wedge N \$\$ (i, j) = 1\}$   
*<proof>*

**lemma** *block-mat-cond-rev*:  
**assumes**  $j < \text{length } \mathcal{B}s$   
**shows**  $\{i . i < \text{length } \mathcal{V}s \wedge N \$\$ (i, j) = 1\} = ((\text{List-Index.index}) \mathcal{V}s) \text{ ' } (\mathcal{B}s ! j)$   
*<proof>*

Incidence Matrix incidence system properties

**lemma** *incomplete-block-col*:  
**assumes**  $j < b$   
**assumes** *incomplete-block*  $(\mathcal{B}s ! j)$   
**shows**  $0 \in \$ (\text{col } N j)$   
*<proof>*

**lemma** *mat-rep-num-N-row*:  
**assumes**  $i < v$   
**shows** *mat-rep-num*  $N i = \mathcal{B} \text{ rep } (\mathcal{V}s ! i)$   
*<proof>*

**lemma** *point-rep-mat-row-sum*:  $i < v \implies \text{sum-vec } (\text{row } N i) = \mathcal{B} \text{ rep } (\mathcal{V}s ! i)$   
*<proof>*

**lemma** *mat-block-size-N-col*:  
**assumes**  $j < b$   
**shows** *mat-block-size*  $N j = \text{card } (\mathcal{B}s ! j)$   
*<proof>*

**lemma** *block-size-mat-rep-sum*:  $j < b \implies \text{sum-vec } (\text{col } N j) = \text{mat-block-size } N j$   
*<proof>*

**lemma** *mat-point-index-rep*:  
**assumes**  $I \subseteq \{..<v\}$   
**shows** *mat-point-index*  $N I = \mathcal{B} \text{ index } ((\lambda i. \mathcal{V}s ! i) \text{ ' } I)$   
*<proof>*

**lemma** *incidence-mat-two-index*:  $i1 < v \implies i2 < v \implies$   
 $\text{mat-point-index } N \{i1, i2\} = \mathcal{B} \text{ index } \{\mathcal{V}s ! i1, \mathcal{V}s ! i2\}$   
*<proof>*

**lemma** *ones-incidence-mat-block-size*:  
**assumes**  $j < b$   
**shows**  $((u_v v) v^* N) \$ j = \text{mat-block-size } N j$   
*<proof>*

**lemma** *mat-block-size-conv*:  $j < \text{dim-col } N \implies \text{card } (\mathcal{B}s ! j) = \text{mat-block-size } N j$

*<proof>*

**lemma** *mat-inter-num-conv:*

**assumes**  $j1 < \text{dim-col } N$   $j2 < \text{dim-col } N$

**shows**  $(\mathcal{B}s ! j1) \mid \cap \mid (\mathcal{B}s ! j2) = \text{mat-inter-num } N \ j1 \ j2$

*<proof>*

**lemma** *non-empty-col-map-conv:*

**assumes**  $j < \text{dim-col } N$

**shows**  $\text{non-empty-col } N \ j \longleftrightarrow \mathcal{B}s ! j \neq \{\}$

*<proof>*

**lemma** *scalar-prod-inc-vec-inter-num:*

**assumes**  $j1 < b$   $j2 < b$

**shows**  $(\text{col } N \ j1) \cdot (\text{col } N \ j2) = (\mathcal{B}s ! j1) \mid \cap \mid (\mathcal{B}s ! j2)$

*<proof>*

**lemma** *scalar-prod-block-size-lift-01:*

**assumes**  $i < b$

**shows**  $((\text{col } (\text{lift-01-mat } N) \ i) \cdot (\text{col } (\text{lift-01-mat } N) \ i)) = (\text{of-nat } (\text{card } (\mathcal{B}s ! i)))$   
 $:: ('b :: \{\text{ring-1}\})$

*<proof>*

**lemma** *scalar-prod-inter-num-lift-01:*

**assumes**  $j1 < b$   $j2 < b$

**shows**  $((\text{col } (\text{lift-01-mat } N) \ j1) \cdot (\text{col } (\text{lift-01-mat } N) \ j2)) = (\text{of-nat } ((\mathcal{B}s ! j1)$   
 $\mid \cap \mid (\mathcal{B}s ! j2))) :: ('b :: \{\text{ring-1}\})$

*<proof>*

The System complement's incidence matrix flips 0's and 1's

**lemma** *map-block-complement-entry:*  $j < b \implies (\text{map block-complement } \mathcal{B}s) ! j =$   
 $\text{block-complement } (\mathcal{B}s ! j)$

*<proof>*

**lemma** *complement-mat-entries:*

**assumes**  $i < v$  **and**  $j < b$

**shows**  $(\mathcal{V}s ! i \notin \mathcal{B}s ! j) \longleftrightarrow (\mathcal{V}s ! i \in (\text{map block-complement } \mathcal{B}s) ! j)$

*<proof>*

**lemma** *length-blocks-complement:*  $\text{length } (\text{map block-complement } \mathcal{B}s) = b$

*<proof>*

**lemma** *ordered-complement:*  $\text{ordered-incidence-system } \mathcal{V}s \ (\text{map block-complement } \mathcal{B}s)$

*<proof>*

**interpretation** *ordered-comp:*  $\text{ordered-incidence-system } \mathcal{V}s \ (\text{map block-complement } \mathcal{B}s)$

*<proof>*

**lemma** *complement-mat-entries-val*:

**assumes**  $i < v$  **and**  $j < b$

**shows**  $\text{ordered-comp}.N \ \$\$ (i, j) = (\text{if } \mathcal{V}s ! i \in \mathcal{B}s ! j \text{ then } 0 \text{ else } 1)$

*<proof>*

**lemma** *ordered-complement-mat*:  $\text{ordered-comp}.N = \text{mat } v \ b \ (\lambda (i,j) . \text{if } (\mathcal{V}s ! i) \in (\mathcal{B}s ! j) \text{ then } 0 \text{ else } 1)$

*<proof>*

**lemma** *ordered-complement-mat-map*:  $\text{ordered-comp}.N = \text{map-mat } (\lambda x. \text{if } x = 1 \text{ then } 0 \text{ else } 1) \ N$

*<proof>*

**end**

Establishing connection between incidence system and ordered incidence system locale

**lemma** (**in** *incidence-system*) *alt-ordering-sysI*:  $Vs \in \text{permutations-of-set } \mathcal{V} \implies Bs \in \text{permutations-of-multiset } \mathcal{B} \implies$

*ordered-incidence-system*  $Vs \ Bs$

*<proof>*

**lemma** (**in** *finite-incidence-system*) *exists-ordering-sysI*:  $\exists Vs \ Bs . Vs \in \text{permutations-of-set } \mathcal{V} \wedge$

$Bs \in \text{permutations-of-multiset } \mathcal{B} \wedge \text{ordered-incidence-system } Vs \ Bs$

*<proof>*

**lemma** *inc-sys-orderedI*:

**assumes** *incidence-system*  $V \ B$  **and** *distinct*  $Vs$  **and** *set*  $Vs = V$  **and** *mset*  $Bs = B$

**shows** *ordered-incidence-system*  $Vs \ Bs$

*<proof>*

Generalise the idea of an incidence matrix to an unordered context

**definition** *is-incidence-matrix* ::  $'c :: \{\text{ring-1}\} \text{ mat} \Rightarrow 'a \text{ set} \Rightarrow 'a \text{ set multiset} \Rightarrow \text{bool}$  **where**

*is-incidence-matrix*  $N \ V \ B \longleftrightarrow$

$(\exists Vs \ Bs . (Vs \in \text{permutations-of-set } V \wedge Bs \in \text{permutations-of-multiset } B \wedge N = (\text{inc-mat-of } Vs \ Bs)))$

**lemma** (**in** *incidence-system*) *is-incidence-mat-alt*: *is-incidence-matrix*  $N \ \mathcal{V} \ \mathcal{B} \longleftrightarrow$

$(\exists Vs \ Bs . (\text{set } Vs = \mathcal{V} \wedge \text{mset } Bs = \mathcal{B} \wedge \text{ordered-incidence-system } Vs \ Bs \wedge N = (\text{inc-mat-of } Vs \ Bs)))$

*<proof>*

**lemma** (**in** *ordered-incidence-system*) *is-incidence-mat-true*: *is-incidence-matrix*  $N \ \mathcal{V} \ \mathcal{B} = \text{True}$



*<proof>*

## 4.5 Incidence Matrices on Design Subtypes

**locale** *ordered-design* = *ordered-incidence-system*  $\mathcal{V}s$   $\mathcal{B}s$  + *design set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   
  **for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$   
**begin**

**lemma** *incidence-mat-non-empty-blocks*:

**assumes**  $j < b$

**shows**  $1 \in \$ (col\ N\ j)$

*<proof>*

**lemma** *all-cols-non-empty*:  $j < dim-col\ N \implies non-empty-col\ N\ j$

*<proof>*

**end**

**locale** *ordered-simple-design* = *ordered-design*  $\mathcal{V}s$   $\mathcal{B}s$  + *simple-design* (*set*  $\mathcal{V}s$ ) *mset*  $\mathcal{B}s$   
**for**  $\mathcal{V}s$   $\mathcal{B}s$

**begin**

**lemma** *block-list-distinct*: *distinct*  $\mathcal{B}s$

*<proof>*

**lemma** *distinct-cols-N*: *distinct* (*cols*  $N$ )

*<proof>*

**lemma** *simp-blocks-length-card*: *length*  $\mathcal{B}s = card\ (set\ \mathcal{B}s)$

*<proof>*

**lemma** *blocks-index-inj-on*: *inj-on*  $(\lambda\ i.\ \mathcal{B}s\ !\ i)\ \{0..<length\ \mathcal{B}s\}$

*<proof>*

**lemma** *x-in-block-set-img*: **assumes**  $x \in set\ \mathcal{B}s$  **shows**  $x \in (!)\ \mathcal{B}s\ ' \{0..<length\ \mathcal{B}s\}$

*<proof>*

**lemma** *blocks-index-simp-bij-betw*: *bij-betw*  $(\lambda\ i.\ \mathcal{B}s\ !\ i)\ \{0..<length\ \mathcal{B}s\}\ (set\ \mathcal{B}s)$

*<proof>*

**lemma** *blocks-index-simp-unique*:  $i1 < length\ \mathcal{B}s \implies i2 < length\ \mathcal{B}s \implies i1 \neq i2 \implies \mathcal{B}s\ !\ i1 \neq \mathcal{B}s\ !\ i2$

*<proof>*

**lemma** *lift-01-distinct-cols-N*: *distinct* (*cols* (*lift-01-mat*  $N$ ))

*<proof>*

**end**

**locale** *ordered-proper-design* = *ordered-design*  $\mathcal{V}s$   $\mathcal{B}s$  + *proper-design set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   
**for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$   
**begin**

**lemma** *mat-is-proper: proper-inc-mat*  $N$   
*<proof>*

**end**

**locale** *ordered-constant-rep* = *ordered-proper-design*  $\mathcal{V}s$   $\mathcal{B}s$  + *constant-rep-design set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   $r$   
**for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$  **and**  $r$

**begin**

**lemma** *incidence-mat-rep-num:  $i < v \implies \text{mat-rep-num } N\ i = r$*   
*<proof>*

**lemma** *incidence-mat-rep-num-sum:  $i < v \implies \text{sum-vec (row } N\ i) = r$*   
*<proof>*

**lemma** *transpose-N-mult-diag:*  
**assumes**  $i = j$  **and**  $i < v$  **and**  $j < v$   
**shows**  $(N * N^T)\ \$(i, j) = r$   
*<proof>*

**end**

**locale** *ordered-block-design* = *ordered-proper-design*  $\mathcal{V}s$   $\mathcal{B}s$  + *block-design set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   $k$   
**for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$  **and**  $k$

**begin**

**lemma** *incidence-mat-block-size:  $j < b \implies \text{mat-block-size } N\ j = k$*   
*<proof>*

**lemma** *incidence-mat-block-size-sum:  $j < b \implies \text{sum-vec (col } N\ j) = k$*   
*<proof>*

**lemma** *ones-mult-incidence-mat-k-index:  $j < b \implies ((u_v\ v)\ v * N)\ \$\ j = k$*   
*<proof>*

**lemma** *ones-mult-incidence-mat-k:  $((u_v\ v)\ v * N) = k \cdot_v (u_v\ b)$*   
*<proof>*

**end**

**locale** *ordered-incomplete-design* = *ordered-block-design*  $\mathcal{V} s \mathcal{B} s k$  + *incomplete-design*  
 $\mathcal{V} \mathcal{B} k$   
**for**  $\mathcal{V} s$  **and**  $\mathcal{B} s$  **and**  $k$

**begin**

**lemma** *incidence-mat-incomplete*:  $j < b \implies 0 \in \$ (col N j)$   
 $\langle proof \rangle$

**end**

**locale** *ordered-t-wise-balance* = *ordered-proper-design*  $\mathcal{V} s \mathcal{B} s$  + *t-wise-balance set*  
 $\mathcal{V} s mset \mathcal{B} s t \Lambda_t$   
**for**  $\mathcal{V} s$  **and**  $\mathcal{B} s$  **and**  $t$  **and**  $\Lambda_t$

**begin**

**lemma** *incidence-mat-des-index*:  
**assumes**  $I \subseteq \{0..<v\}$   
**assumes**  $card I = t$   
**shows** *mat-point-index*  $N I = \Lambda_t$   
 $\langle proof \rangle$

**end**

**locale** *ordered-pairwise-balance* = *ordered-t-wise-balance*  $\mathcal{V} s \mathcal{B} s 2 \Lambda$  + *pairwise-balance set*  
 $\mathcal{V} s mset \mathcal{B} s \Lambda$   
**for**  $\mathcal{V} s$  **and**  $\mathcal{B} s$  **and**  $\Lambda$

**begin**

**lemma** *incidence-mat-des-two-index*:  
**assumes**  $i1 < v$   
**assumes**  $i2 < v$   
**assumes**  $i1 \neq i2$   
**shows** *mat-point-index*  $N \{i1, i2\} = \Lambda$   
 $\langle proof \rangle$

**lemma** *transpose-N-mult-off-diag*:  
**assumes**  $i \neq j$  **and**  $i < v$  **and**  $j < v$   
**shows**  $(N * N^T) \$$ (i, j) = \Lambda$   
 $\langle proof \rangle$

**end**

**context** *pairwise-balance*

**begin**

**lemma** *ordered-pbdI*:

**assumes**  $\mathcal{B} = \text{mset } \mathcal{B}s$  **and**  $\mathcal{V} = \text{set } \mathcal{V}s$  **and** *distinct*  $\mathcal{V}s$   
**shows** *ordered-pairwise-balance*  $\mathcal{V}s$   $\mathcal{B}s$   $\Lambda$   
 ⟨*proof*⟩  
**end**

**locale** *ordered-regular-pairwise-balance* = *ordered-pairwise-balance*  $\mathcal{V}s$   $\mathcal{B}s$   $\Lambda$  +  
*regular-pairwise-balance set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   $\Lambda$  **r** **for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$  **and**  $\Lambda$  **and** **r**

**sublocale** *ordered-regular-pairwise-balance*  $\subseteq$  *ordered-constant-rep*  
 ⟨*proof*⟩

**context** *ordered-regular-pairwise-balance*  
**begin**

Stinson's Theorem 1.15. Stinson [8] gives an iff condition for incidence matrices of regular pairwise balanced designs. The other direction is proven in the *zero-one-matrix* context

**lemma** *rpbd-incidence-matrix-cond*:  $N * (N^T) = \Lambda \cdot_m (J_m \mathbf{v}) + (\mathbf{r} - \Lambda) \cdot_m (1_m \mathbf{v})$   
 ⟨*proof*⟩  
**end**

**locale** *ordered-bibd* = *ordered-proper-design*  $\mathcal{V}s$   $\mathcal{B}s$  + *bidb set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   $k$   $\Lambda$   
**for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$  **and**  $k$  **and**  $\Lambda$

**sublocale** *ordered-bibd*  $\subseteq$  *ordered-incomplete-design*  
 ⟨*proof*⟩

**sublocale** *ordered-bibd*  $\subseteq$  *ordered-constant-rep*  $\mathcal{V}s$   $\mathcal{B}s$  **r**  
 ⟨*proof*⟩

**sublocale** *ordered-bibd*  $\subseteq$  *ordered-pairwise-balance*  
 ⟨*proof*⟩

**locale** *ordered-sym-bibd* = *ordered-bibd*  $\mathcal{V}s$   $\mathcal{B}s$   $k$   $\Lambda$  + *symmetric-bibd set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   $k$   $\Lambda$   
**for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$  **and**  $k$  **and**  $\Lambda$

**sublocale** *ordered-sym-bibd*  $\subseteq$  *ordered-simple-design*  
 ⟨*proof*⟩

**locale** *ordered-const-intersect-design* = *ordered-proper-design*  $\mathcal{V}s$   $\mathcal{B}s$  + *const-intersect-design set*  $\mathcal{V}s$  *mset*  $\mathcal{B}s$   $m$   
**for**  $\mathcal{V}s$   $\mathcal{B}s$   $m$

**locale** *simp-ordered-const-intersect-design* = *ordered-const-intersect-design* + *ordered-simple-design*

**begin**

**lemma** *max-one-block-size-inter:*

**assumes**  $b \geq 2$   
**assumes**  $bl \in \# \mathcal{B}$   
**assumes**  $\text{card } bl = m$   
**assumes**  $bl2 \in \# \mathcal{B} - \{\#bl\# \}$   
**shows**  $m < \text{card } bl2$

*<proof>*

**lemma** *block-size-inter-num-cases:*

**assumes**  $bl \in \# \mathcal{B}$   
**assumes**  $b \geq 2$   
**shows**  $m < \text{card } bl \vee (\text{card } bl = m \wedge (\forall bl' \in \# (\mathcal{B} - \{\#bl\# \}) . m < \text{card } bl'))$

*<proof>*

**lemma** *indexed-const-intersect:*

**assumes**  $j1 < b$   
**assumes**  $j2 < b$   
**assumes**  $j1 \neq j2$   
**shows**  $(\mathcal{B}s ! j1) \cap (\mathcal{B}s ! j2) = m$

*<proof>*

**lemma** *const-intersect-block-size-diff:*

**assumes**  $j' < b$  **and**  $j < b$  **and**  $j \neq j'$  **and**  $\text{card } (\mathcal{B}s ! j') = m$  **and**  $b \geq 2$   
**shows**  $\text{card } (\mathcal{B}s ! j) - m > 0$

*<proof>*

**lemma** *scalar-prod-inc-vec-const-inter:*

**assumes**  $j1 < b$   $j2 < b$   $j1 \neq j2$   
**shows**  $(\text{col } N j1) \cdot (\text{col } N j2) = m$

*<proof>*

**end**

## 4.6 Zero One Matrix Incidence System Existence

We prove 0-1 matrices with certain properties imply the existence of an incidence system with particular properties. This leads to Stinson's theorem in the other direction [8]

**context** *zero-one-matrix*

**begin**

**lemma** *mat-is-ordered-incidence-sys:* *ordered-incidence-system*  $[0..<(\text{dim-row } M)]$   
 $(\text{map } (\text{map-col-to-block}) (\text{cols } M))$

*<proof>*

**interpretation** *mat-ord-inc-sys:* *ordered-incidence-system*  $[0..<(\text{dim-row } M)]$   $(\text{map } (\text{map-col-to-block}) (\text{cols } M))$

*<proof>*

**lemma** *mat-ord-inc-sys-N*: *mat-ord-inc-sys.N = lift-01-mat M*

*<proof>*

**lemma** *map-col-to-block-mat-rep-num*:

**assumes**  $x < \text{dim-row } M$

**shows**  $(\{\# \text{ map-col-to-block } c . c \in \# \text{ mset } (\text{cols } M)\# \} \text{ rep } x) = \text{mat-rep-num } M$   
 $x$

*<proof>*

**end**

**context** *zero-one-matrix-ring-1*

**begin**

**lemma** *transpose-cond-index-vals*:

**assumes**  $M * (M^T) = \Lambda \cdot_m (J_m (\text{dim-row } M)) + (r - \Lambda) \cdot_m (1_m (\text{dim-row } M))$

**assumes**  $i < \text{dim-row } (M * (M^T))$

**assumes**  $j < \text{dim-col } (M * (M^T))$

**shows**  $i = j \implies (M * (M^T)) \text{ $$ } (i, j) = r$   $i \neq j \implies (M * (M^T)) \text{ $$ } (i, j) = \Lambda$

*<proof>*

**end**

**locale** *zero-one-matrix-int = zero-one-matrix-ring-1 M for M :: int mat*

**begin**

Some useful conditions on the transpose product for matrix system properties

**lemma** *transpose-cond-diag-r*:

**assumes**  $i < \text{dim-row } (M * (M^T))$

**assumes**  $\bigwedge j . i = j \implies (M * (M^T)) \text{ $$ } (i, j) = r$

**shows**  $\text{mat-rep-num } M \ i = r$

*<proof>*

**lemma** *transpose-cond-non-diag*:

**assumes**  $i1 < \text{dim-row } (M * (M^T))$

**assumes**  $i2 < \text{dim-row } (M * (M^T))$

**assumes**  $i1 \neq i2$

**assumes**  $\bigwedge j . j \neq i \implies i < \text{dim-row } (M * (M^T)) \implies j < \text{dim-row } (M * (M^T)) \implies (M * (M^T)) \text{ $$ } (i, j) = \Lambda$

**shows**  $\Lambda = \text{mat-point-index } M \ \{i1, i2\}$

*<proof>*

**lemma** *trans-cond-implies-map-rep-num*:

**assumes**  $M * (M^T) = \Lambda \cdot_m (J_m (\text{dim-row } M)) + (r - \Lambda) \cdot_m (1_m (\text{dim-row } M))$

**assumes**  $x < \text{dim-row } M$

**shows** (*image-mset map-col-to-block* (*mset* (*cols* *M*))) *rep* *x* = *r*  
 ⟨*proof*⟩

**lemma** *trans-cond-implies-map-index*:

**assumes**  $M * (M^T) = \Lambda \cdot_m (J_m (\dim\text{-row } M)) + (r - \Lambda) \cdot_m (1_m (\dim\text{-row } M))$

**assumes**  $ps \subseteq \{0..<\dim\text{-row } M\}$

**assumes**  $\text{card } ps = 2$

**shows** (*image-mset map-col-to-block* (*mset* (*cols* *M*))) *index* *ps* =  $\Lambda$

⟨*proof*⟩

Stinson Theorem 1.15 existence direction

**lemma** *rpbid-exists*:

**assumes**  $\dim\text{-row } M \geq 2$  — Min two points

**assumes**  $\dim\text{-col } M \geq 1$  — Min one block

**assumes**  $\bigwedge j. j < \dim\text{-col } M \implies 1 \in \$ \text{col } M j$  — no empty blocks

**assumes**  $M * (M^T) = \Lambda \cdot_m (J_m (\dim\text{-row } M)) + (r - \Lambda) \cdot_m (1_m (\dim\text{-row } M))$

**shows** *ordered-regular-pairwise-balance* [ $0..<\dim\text{-row } M$ ] (*map map-col-to-block* (*cols* *M*))  $\Lambda$  *r*

⟨*proof*⟩

**lemma** *vec-k-uniform-mat-block-size*:

**assumes**  $((u_v (\dim\text{-row } M)) \cdot_v * M) = k \cdot_v (u_v (\dim\text{-col } M))$

**assumes**  $j < \dim\text{-col } M$

**shows** *mat-block-size* *M* *j* = *k*

⟨*proof*⟩

**lemma** *vec-k-impl-uniform-block-size*:

**assumes**  $((u_v (\dim\text{-row } M)) \cdot_v * M) = k \cdot_v (u_v (\dim\text{-col } M))$

**assumes**  $bl \in \# (\text{image-mset map-col-to-block } (\text{mset } (\text{cols } M)))$

**shows**  $\text{card } bl = k$

⟨*proof*⟩

**lemma** *bibd-exists*:

**assumes**  $\dim\text{-col } M \geq 1$  — Min one block

**assumes**  $\bigwedge j. j < \dim\text{-col } M \implies 1 \in \$ \text{col } M j$  — no empty blocks

**assumes**  $M * (M^T) = \Lambda \cdot_m (J_m (\dim\text{-row } M)) + (r - \Lambda) \cdot_m (1_m (\dim\text{-row } M))$

**assumes**  $((u_v (\dim\text{-row } M)) \cdot_v * M) = k \cdot_v (u_v (\dim\text{-col } M))$

**assumes**  $(r :: \text{nat}) \geq 0$

**assumes**  $k \geq 2$   $k < \dim\text{-row } M$

**shows** *ordered-bibd* [ $0..<\dim\text{-row } M$ ] (*map map-col-to-block* (*cols* *M*)) *k*  $\Lambda$

⟨*proof*⟩

**end**

## 4.7 Isomorphisms and Incidence Matrices

If two incidence systems have the same incidence matrix, they are isomorphic. Similarly if two incidence systems are isomorphic there exists an ordering such that they have the same incidence matrix

**locale** *two-ordered-sys* = *D1: ordered-incidence-system*  $\mathcal{V}s$   $\mathcal{B}s$  + *D2: ordered-incidence-system*  $\mathcal{V}s'$   $\mathcal{B}s'$

**for**  $\mathcal{V}s$  **and**  $\mathcal{B}s$  **and**  $\mathcal{V}s'$  **and**  $\mathcal{B}s'$

**begin**

**lemma** *equal-inc-mat-isomorphism*:

**assumes**  $D1.N = D2.N$

**shows** *incidence-system-isomorphism*  $D1.\mathcal{V}$   $D1.\mathcal{B}$   $D2.\mathcal{V}$   $D2.\mathcal{B}$   $(\lambda x . \mathcal{V}s'!(List-Index.index$   
 $\mathcal{V}s x))$

*<proof>*

**lemma** *equal-inc-mat-isomorphism-ex*:  $D1.N = D2.N \implies \exists \pi .$  *incidence-system-isomorphism*  
 $D1.\mathcal{V}$   $D1.\mathcal{B}$   $D2.\mathcal{V}$   $D2.\mathcal{B}$   $\pi$

*<proof>*

**lemma** *equal-inc-mat-isomorphism-obtain*:

**assumes**  $D1.N = D2.N$

**obtains**  $\pi$  **where** *incidence-system-isomorphism*  $D1.\mathcal{V}$   $D1.\mathcal{B}$   $D2.\mathcal{V}$   $D2.\mathcal{B}$   $\pi$

*<proof>*

**end**

**context** *incidence-system-isomorphism*

**begin**

**lemma** *exists-eq-inc-mats*:

**assumes** *finite*  $\mathcal{V}$  *finite*  $\mathcal{V}'$

**obtains**  $N$  **where** *is-incidence-matrix*  $N$   $\mathcal{V}$   $\mathcal{B}$  **and** *is-incidence-matrix*  $N$   $\mathcal{V}'$   $\mathcal{B}'$   
*<proof>*

**end**

**end**

## 5 Dual Systems

The concept of a dual incidence system [3] is an important property in design theory. It enables us to reason on the existence of several different types of design constructs through dual properties [8]

**theory** *Dual-Systems* **imports** *Incidence-Matrices*

**begin**

### 5.1 Dual Blocks

A dual design of  $(\mathcal{V}, \mathcal{B})$ , is the design where each block in  $\mathcal{B}$  represents a point  $x$ , and a block in a dual design is a set of blocks which  $x$  is in from the original design. It is important to note that if a block repeats in  $\mathcal{B}$ , each



instance of the block is a distinct point. As such the definition below uses each block's list index as its identifier. The list of points would simply be the indices  $0..<\text{length } \mathcal{B}s$

**definition** *dual-blocks* :: 'a set  $\Rightarrow$  'a set list  $\Rightarrow$  nat set multiset **where**  
*dual-blocks*  $\mathcal{V} \mathcal{B}s \equiv \{\# \{y . y < \text{length } \mathcal{B}s \wedge x \in \mathcal{B}s ! y\} . x \in \# (\text{mset-set } \mathcal{V})\# \}$

**lemma** *dual-blocks-wf*:  $b \in \# \text{ dual-blocks } \mathcal{V} \mathcal{B}s \Longrightarrow b \subseteq \{0..<\text{length } \mathcal{B}s\}$   
 <proof>

**context** *ordered-incidence-system*  
**begin**

**definition** *dual-blocks-ordered* :: nat set list ( $\mathcal{B}s^*$ ) **where**  
*dual-blocks-ordered*  $\equiv \text{map } (\lambda x . \{y . y < \text{length } \mathcal{B}s \wedge x \in \mathcal{B}s ! y\}) \mathcal{V}s$

**lemma** *dual-blocks-ordered-eq*: *dual-blocks*  $\mathcal{V} \mathcal{B}s = \text{mset } (\mathcal{B}s^*)$   
 <proof>

**lemma** *dual-blocks-len*:  $\text{length } \mathcal{B}s^* = \text{length } \mathcal{V}s$   
 <proof>

A dual system is an incidence system

**sublocale** *dual-sys*: *finite-incidence-system*  $\{0..<\text{length } \mathcal{B}s\}$  *dual-blocks*  $\mathcal{V} \mathcal{B}s$   
 <proof>

**lemma** *dual-is-ordered-inc-sys*: *ordered-incidence-system*  $[0..<\text{length } \mathcal{B}s]$   $\mathcal{B}s^*$   
 <proof>

**interpretation** *ordered-dual-sys*: *ordered-incidence-system*  $[0..<\text{length } \mathcal{B}s]$   $\mathcal{B}s^*$   
 <proof>

## 5.2 Basic Dual Properties

**lemma** *ord-dual-blocks-b*: *ordered-dual-sys*.b = v  
 <proof>

**lemma** *dual-blocks-b*: *dual-sys*.b = v  
 <proof>

**lemma** *dual-blocks-v*: *dual-sys*.v = b  
 <proof>

**lemma** *ord-dual-blocks-v*: *ordered-dual-sys*.v = b  
 <proof>

**lemma** *dual-point-block*:  $i < v \Longrightarrow \mathcal{B}s^* ! i = \{y . y < \text{length } \mathcal{B}s \wedge (\mathcal{V}s ! i) \in \mathcal{B}s ! y\}$   
 <proof>

**lemma** *dual-incidence-iff*:  $i < v \implies j < b \implies \mathcal{B}s ! j = bl \implies \mathcal{V}s ! i = x \implies (x \in bl \iff j \in \mathcal{B}s* ! i)$

*<proof>*

**lemma** *dual-incidence-iff2*:  $i < v \implies j < b \implies (\mathcal{V}s ! i \in \mathcal{B}s ! j \iff j \in \mathcal{B}s* ! i)$

*<proof>*

**lemma** *dual-blocks-point-exists*:  $bl \in \# \text{ dual-blocks } \mathcal{V} \mathcal{B}s \implies$

$\exists x. x \in \mathcal{V} \wedge bl = \{y . y < \text{length } \mathcal{B}s \wedge x \in \mathcal{B}s ! y\}$

*<proof>*

**lemma** *dual-blocks-ne-index-ne*:  $j1 < \text{length } \mathcal{B}s* \implies j2 < \text{length } \mathcal{B}s* \implies \mathcal{B}s* ! j1 \neq \mathcal{B}s* ! j2 \implies j1 \neq j2$

*<proof>*

**lemma** *dual-blocks-list-index-img*:  $\text{image-mset } (\lambda x . \mathcal{B}s* ! x) (\text{mset-set } \{0..<\text{length } \mathcal{B}s*\}) = \text{mset } \mathcal{B}s*$

*<proof>*

**lemma** *dual-blocks-elem-iff*:

**assumes**  $j < v$

**shows**  $x \in (\mathcal{B}s* ! j) \iff \mathcal{V}s ! j \in \mathcal{B}s ! x \wedge x < b$

*<proof>*

The incidence matrix of the dual of a design is just the transpose

**lemma** *dual-incidence-mat-eq-trans*:  $\text{ordered-dual-sys}.N = N^T$

*<proof>*

**lemma** *dual-incidence-mat-eq-trans-rev*:  $(\text{ordered-dual-sys}.N)^T = N$

*<proof>*

### 5.3 Incidence System Dual Properties

Many common design properties have a dual in the dual design which enables extensive reasoning Using incidence matrices and the transpose property these are easy to prove. We leave examples of counting proofs (commented out), to demonstrate how incidence matrices can significantly simplify reasoning

**lemma** *dual-blocks-nempty*:

**assumes**  $(\bigwedge x . x \in \mathcal{V} \implies \mathcal{B} \text{ rep } x > 0)$

**assumes**  $bl \in \# \text{ dual-blocks } \mathcal{V} \mathcal{B}s$

**shows**  $bl \neq \{\}$

*<proof>*

**lemma** *dual-blocks-size-is-rep*:  $j < \text{length } \mathcal{B}s* \implies \text{card } (\mathcal{B}s* ! j) = \mathcal{B} \text{ rep } (\mathcal{V}s ! j)$

*<proof>*

**lemma** *dual-blocks-size-is-rep-obtain:*  
**assumes**  $bl \in \# \text{ dual-blocks } \mathcal{V} \mathcal{B}s$   
**obtains**  $x$  **where**  $x \in \mathcal{V}$  **and**  $\text{card } bl = \mathcal{B} \text{ rep } x$   
 $\langle \text{proof} \rangle$

**lemma** *dual-blocks-rep-is-size:*  
**assumes**  $i < \text{length } \mathcal{B}s$   
**shows**  $(\text{mset } \mathcal{B}s^*) \text{ rep } i = \text{card } (\mathcal{B}s ! i)$   
 $\langle \text{proof} \rangle$

**lemma** *dual-blocks-inter-index:*  
**assumes**  $j1 < \text{length } \mathcal{B}s^* \ j2 < \text{length } \mathcal{B}s^*$   
**shows**  $(\mathcal{B}s^* ! j1) \mid \cap \mid (\mathcal{B}s^* ! j2) = \text{points-index } \mathcal{B} \{ \mathcal{V}s ! j1, \mathcal{V}s ! j2 \}$   
 $\langle \text{proof} \rangle$

**lemma** *dual-blocks-points-index-inter:*  
**assumes**  $i1 < b \ i2 < b$   
**shows**  $(\text{mset } \mathcal{B}s^*) \text{ index } \{i1, i2\} = (\mathcal{B}s ! i1) \mid \cap \mid (\mathcal{B}s ! i2)$   
 $\langle \text{proof} \rangle$

**end**

## 5.4 Dual Properties for Design sub types

**context** *ordered-design*  
**begin**

**lemma** *dual-is-design:*  
**assumes**  $(\bigwedge x . x \in \mathcal{V} \implies \mathcal{B} \text{ rep } x > 0)$  — Required to ensure no blocks are empty  
**shows** *design*  $\{0..<\text{length } \mathcal{B}s\}$  (*dual-blocks*  $\mathcal{V} \mathcal{B}s$ )  
 $\langle \text{proof} \rangle$   
**end**

**context** *ordered-proper-design*  
**begin**

**lemma** *dual-sys-b-non-zero:*  $\text{dual-sys.b} \neq 0$   
 $\langle \text{proof} \rangle$

**lemma** *dual-is-proper-design:*  
**assumes**  $(\bigwedge x . x \in \mathcal{V} \implies \mathcal{B} \text{ rep } x > 0)$  — Required to ensure no blocks are empty  
**shows** *proper-design*  $\{0..<\text{length } \mathcal{B}s\}$  (*dual-blocks*  $\mathcal{V} \mathcal{B}s$ )

$\langle proof \rangle$   
**end**  
**context** *ordered-block-design*  
**begin**  
**lemma** *dual-blocks-const-rep*:  $i \in \{0..<length \mathcal{B}s\} \implies (mset \mathcal{B}s^*) \text{ rep } i = k$   
 $\langle proof \rangle$   
**lemma** *dual-blocks-constant-rep-design*:  
**assumes**  $(\bigwedge x . x \in \mathcal{V} \implies \mathcal{B} \text{ rep } x > 0)$   
**shows** *constant-rep-design*  $\{0..<length \mathcal{B}s\} (dual-blocks \vee \mathcal{B}s) k$   
 $\langle proof \rangle$   
**end**  
**context** *ordered-constant-rep*  
**begin**  
**lemma** *dual-blocks-const-size*:  $j < length \mathcal{B}s^* \implies card (\mathcal{B}s^* ! j) = r$   
 $\langle proof \rangle$   
**lemma** *dual-is-block-design*: *block-design*  $\{0..<length \mathcal{B}s\} (dual-blocks \vee \mathcal{B}s) r$   
 $\langle proof \rangle$   
**end**  
**context** *ordered-pairwise-balance*  
**begin**  
**lemma** *dual-blocks-const-intersect*:  
**assumes**  $j1 < length \mathcal{B}s^* \ j2 < length \mathcal{B}s^*$   
**assumes**  $j1 \neq j2$   
**shows**  $(\mathcal{B}s^* ! j1) \cap (\mathcal{B}s^* ! j2) = \Lambda$   
 $\langle proof \rangle$   
**lemma** *dual-is-const-intersect-des*:  
**assumes**  $\Lambda > 0$   
**shows** *const-intersect-design*  $\{0..<(length \mathcal{B}s)\} (dual-blocks \vee \mathcal{B}s) \Lambda$   
 $\langle proof \rangle$   
**lemma** *dual-is-simp-const-inter-des*:  
**assumes**  $\Lambda > 0$   
**assumes**  $\bigwedge bl . bl \in \# \mathcal{B} \implies incomplete-block \ bl$   
**shows** *simple-const-intersect-design*  $\{0..<(length \mathcal{B}s)\} (dual-blocks \vee \mathcal{B}s) \Lambda$   
 $\langle proof \rangle$

**end**

**context** *ordered-const-intersect-design*  
**begin**

**lemma** *dual-is-balanced:*  
  **assumes**  $ps \subseteq \{0..<length\ \mathcal{B}s\}$   
  **assumes**  $card\ ps = 2$   
  **shows**  $(dual-blocks\ \mathcal{V}\ \mathcal{B}s)\ index\ ps = m$   
*<proof>*

**lemma** *dual-is-pbd:*  
  **assumes**  $(\bigwedge x . x \in \mathcal{V} \implies \mathcal{B}\ rep\ x > 0)$   
  **assumes**  $b \geq 2$   
  **shows**  $pairwise-balance\ \{0..<(length\ \mathcal{B}s)\}\ (dual-blocks\ \mathcal{V}\ \mathcal{B}s)\ m$   
*<proof>*

**end**

**context** *ordered-sym-bibd*  
**begin**

**lemma** *dual-is-balanced:*  
  **assumes**  $ps \subseteq \{0..<length\ \mathcal{B}s\}$   
  **assumes**  $card\ ps = 2$   
  **shows**  $(dual-blocks\ \mathcal{V}\ \mathcal{B}s)\ index\ ps = \Lambda$   
*<proof>*

**lemma** *dual-bibd: bibd*  $\{0..<(length\ \mathcal{B}s)\}\ (dual-blocks\ \mathcal{V}\ \mathcal{B}s)\ r\ \Lambda$   
*<proof>*

The dual of a BIBD must be symmetric

**lemma** *dual-bibd-symmetric: symmetric-bibd*  $\{0..<(length\ \mathcal{B}s)\}\ (dual-blocks\ \mathcal{V}\ \mathcal{B}s)$   
 $r\ \Lambda$   
*<proof>*

**end**

## 5.5 Generalise Dual Concept

The above formalisation relies on one translation of a dual design. However, any design with an ordering of points and blocks such that the matrix is the transpose of the original is a dual. The definition below encapsulates this concept. Additionally, we prove an isomorphism exists between the generated dual from *dual-blocks* and any design satisfying the is dual definition

**context** *ordered-incidence-system*  
**begin**

**definition** *is-dual*: 'b list  $\Rightarrow$  'b set list  $\Rightarrow$  bool **where**  
*is-dual*  $Vs' Bs' \equiv$  ordered-incidence-system  $Vs' Bs' \wedge$  (inc-mat-of  $Vs' Bs' = N^T$ )

**lemma** *is-dualI*:  
**assumes** ordered-incidence-system  $Vs' Bs'$   
**assumes** (inc-mat-of  $Vs' Bs' = N^T$ )  
**shows** *is-dual*  $Vs' Bs'$   
 <proof>

**lemma** *is-dualD1*:  
**assumes** *is-dual*  $Vs' Bs'$   
**shows** (inc-mat-of  $Vs' Bs' = N^T$ )  
 <proof>

**lemma** *is-dualD2*:  
**assumes** *is-dual*  $Vs' Bs'$   
**shows** ordered-incidence-system  $Vs' Bs'$   
 <proof>

**lemma** *generated-is-dual*: *is-dual*  $[0..<(\text{length } Bs)] Bs^*$   
 <proof>

**lemma** *is-dual-isomorphism-generated*:  
**assumes** *is-dual*  $Vs' Bs'$   
**shows**  $\exists \pi$ . incidence-system-isomorphism (set  $Vs'$ ) (mset  $Bs'$ ) ( $\{0..<(\text{length } Bs)\}$ ) (dual-blocks  $\mathcal{V} Bs$ )  $\pi$   
 <proof>

**interpretation** *ordered-dual-sys*: ordered-incidence-system  $[0..<\text{length } Bs] Bs^*$   
 <proof>

Original system is dual of the dual

**lemma** *is-dual-rev*: ordered-dual-sys.*is-dual*  $\mathcal{V} Bs$   
 <proof>

**end**

**end**

## 6 Rank Argument - General

General lemmas to enable reasoning using the rank argument. This is described by Godsil [5] and Bukh [2], both of whom present it as a foundational technique

**theory** *Rank-Argument-General imports* *Dual-Systems Jordan-Normal-Form.Determinant Jordan-Normal-Form.DL-Rank Jordan-Normal-Form.Ring-Hom-Matrix BenOr-Kozen-Reif.More-Matrix*  
**begin**

## 6.1 Row/Column Operations

Extensions to the existing elementary operations are made to enable reasoning on multiple operations at once, similar to mathematical literature

**lemma** *index-mat-addrow-basic* [simp]:

$$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies \text{addrow } a \ k \ l \ A \ \$\$ (i,j) = (\text{if } k = i \text{ then } (a * (A \ \$\$ (l,j)) + (A \ \$\$ (i,j))) \text{ else } A \ \$\$ (i,j))$$

$$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies \text{addrow } a \ i \ l \ A \ \$\$ (i,j) = (a * (A \ \$\$ (l,j)) + (A \ \$\$ (i,j)))$$

$$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies k \neq i \implies \text{addrow } a \ k \ l \ A \ \$\$ (i,j) = A \ \$\$ (i,j)$$

$$\dim\text{-row } (\text{addrow } a \ k \ l \ A) = \dim\text{-row } A \quad \dim\text{-col } (\text{addrow } a \ k \ l \ A) = \dim\text{-col } A$$

*<proof>*

Function to add a column to multiple other columns

**fun** *add-col-to-multiple* :: 'a :: semiring-1  $\Rightarrow$  nat list  $\Rightarrow$  nat  $\Rightarrow$  'a mat  $\Rightarrow$  'a mat  
**where**  
*add-col-to-multiple* a [] l A = A |  
*add-col-to-multiple* a (k # ks) l A = (addcol a k l (add-col-to-multiple a ks l A))

Function to add a row to multiple other rows

**fun** *add-row-to-multiple* :: 'a :: semiring-1  $\Rightarrow$  nat list  $\Rightarrow$  nat  $\Rightarrow$  'a mat  $\Rightarrow$  'a mat  
**where**  
*add-row-to-multiple* a [] l A = A |  
*add-row-to-multiple* a (k # ks) l A = (addrow a k l (add-row-to-multiple a ks l A))

Function to add multiple rows to a single row

**fun** *add-multiple-rows* :: 'a :: semiring-1  $\Rightarrow$  nat  $\Rightarrow$  nat list  $\Rightarrow$  'a mat  $\Rightarrow$  'a mat  
**where**  
*add-multiple-rows* a k [] A = A |  
*add-multiple-rows* a k (l # ls) A = (addrow a k l (add-multiple-rows a k ls A))

Function to add multiple columns to a single col

**fun** *add-multiple-cols* :: 'a :: semiring-1  $\Rightarrow$  nat  $\Rightarrow$  nat list  $\Rightarrow$  'a mat  $\Rightarrow$  'a mat  
**where**  
*add-multiple-cols* a k [] A = A |  
*add-multiple-cols* a k (l # ls) A = (addcol a k l (add-multiple-cols a k ls A))

Basic lemmas on dimension and indexing of resulting matrix from above functions

**lemma** *add-multiple-rows-dim* [simp]:

$$\dim\text{-row } (\text{add-multiple-rows } a \ k \ ls \ A) = \dim\text{-row } A$$

$$\dim\text{-col } (\text{add-multiple-rows } a \ k \ ls \ A) = \dim\text{-col } A$$

*<proof>*

**lemma** *add-multiple-rows-index-unchanged* [simp]:

$$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies k \neq i \implies \text{add-multiple-rows } a \ k \ ls \ A \ \$\$ (i,j) = A \ \$\$ (i,j)$$

*<proof>*

**lemma** *add-multiple-rows-index-eq*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $i \notin \text{set } ls$  **and**  $\bigwedge l. l \in \text{set } ls \implies l < \dim\text{-row } A$

**shows**  $\text{add-multiple-rows } a \ i \ ls \ A \ \$\$ \ (i,j) = (\sum l \leftarrow ls. a * A \ \$\$ \ (l,j)) + A \ \$\$ \ (i,j)$   
*<proof>*

**lemma** *add-multiple-rows-index-eq-bounds*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $i < low \vee i \geq up$  **and**  $up \leq \dim\text{-row } A$

**shows**  $\text{add-multiple-rows } a \ i \ [low..<up] \ A \ \$\$ \ (i,j) = (\sum l=low..<up. a * A \ \$\$ \ (l,j)) + A \ \$\$ \ (i,j)$   
*<proof>*

**lemma** *add-multiple-cols-dim [simp]*:

$\dim\text{-row } (\text{add-multiple-cols } a \ k \ ls \ A) = \dim\text{-row } A$

$\dim\text{-col } (\text{add-multiple-cols } a \ k \ ls \ A) = \dim\text{-col } A$

*<proof>*

**lemma** *add-multiple-cols-index-unchanged [simp]*:

$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies k \neq j \implies \text{add-multiple-cols } a \ k \ ls \ A \ \$\$ \ (i,j) = A \ \$\$ \ (i,j)$

*<proof>*

**lemma** *add-multiple-cols-index-eq*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $j \notin \text{set } ls$  **and**  $\bigwedge l. l \in \text{set } ls \implies l < \dim\text{-col } A$

**shows**  $\text{add-multiple-cols } a \ j \ ls \ A \ \$\$ \ (i,j) = (\sum l \leftarrow ls. a * A \ \$\$ \ (i,l)) + A \ \$\$ \ (i,j)$   
*<proof>*

**lemma** *add-multiple-cols-index-eq-bounds*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $j < low \vee j \geq up$  **and**  $up \leq \dim\text{-col } A$

**shows**  $\text{add-multiple-cols } a \ j \ [low..<up] \ A \ \$\$ \ (i,j) = (\sum l=low..<up. a * A \ \$\$ \ (i,l)) + A \ \$\$ \ (i,j)$   
*<proof>*

**lemma** *add-row-to-multiple-dim [simp]*:

$\dim\text{-row } (\text{add-row-to-multiple } a \ ks \ l \ A) = \dim\text{-row } A$

$\dim\text{-col } (\text{add-row-to-multiple } a \ ks \ l \ A) = \dim\text{-col } A$

*<proof>*

**lemma** *add-row-to-multiple-index-unchanged [simp]*:

$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies i \notin \text{set } ks \implies \text{add-row-to-multiple } a \ ks \ l \ A \ \$\$ \ (i,j) = A \ \$\$ \ (i,j)$

*<proof>*

**lemma** *add-row-to-multiple-index-unchanged-bound*:

$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies i < low \implies i \geq up \implies$



*add-row-to-multiple*  $a$  [ $low..<up$ ]  $l$   $A$   $\$ \$ (i,j) = A \$ \$ (i,j)$   
 ⟨*proof*⟩

**lemma** *add-row-to-multiple-index-change*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $i \in \text{set } ks$  **and** *distinct*  $ks$  **and**  
 $l \notin \text{set } ks$   
**and**  $l < \dim\text{-row } A$   
**shows** *add-row-to-multiple*  $a$   $ks$   $l$   $A$   $\$ \$ (i,j) = (a * A \$ \$ (l, j)) + A \$ \$ (i,j)$   
 ⟨*proof*⟩

**lemma** *add-row-to-multiple-index-change-bounds*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $i \geq low$  **and**  $i < up$  **and**  $l < low \vee l \geq up$   
**and**  $l < \dim\text{-row } A$   
**shows** *add-row-to-multiple*  $a$  [ $low..<up$ ]  $l$   $A$   $\$ \$ (i,j) = (a * A \$ \$ (l, j)) + A \$ \$ (i,j)$   
 ⟨*proof*⟩

**lemma** *add-col-to-multiple-dim* [*simp*]:

$\dim\text{-row } (\text{add-col-to-multiple } a \text{ } ks \text{ } l \text{ } A) = \dim\text{-row } A$   
 $\dim\text{-col } (\text{add-col-to-multiple } a \text{ } ks \text{ } l \text{ } A) = \dim\text{-col } A$   
 ⟨*proof*⟩

**lemma** *add-col-to-multiple-index-unchanged* [*simp*]:

$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies j \notin \text{set } ks \implies \text{add-col-to-multiple } a \text{ } ks \text{ } l \text{ } A$   
 $\$ \$ (i,j) = A \$ \$ (i,j)$   
 ⟨*proof*⟩

**lemma** *add-col-to-multiple-index-unchanged-bound*:

$i < \dim\text{-row } A \implies j < \dim\text{-col } A \implies j < low \implies j \geq up \implies$   
 $\text{add-col-to-multiple } a$  [ $low..<up$ ]  $l$   $A$   $\$ \$ (i,j) = A \$ \$ (i,j)$   
 ⟨*proof*⟩

**lemma** *add-col-to-multiple-index-change*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $j \in \text{set } ks$  **and** *distinct*  $ks$  **and**  
 $l \notin \text{set } ks$   
**and**  $l < \dim\text{-col } A$   
**shows** *add-col-to-multiple*  $a$   $ks$   $l$   $A$   $\$ \$ (i,j) = (a * A \$ \$ (i, l)) + A \$ \$ (i,j)$   
 ⟨*proof*⟩

**lemma** *add-col-to-multiple-index-change-bounds*:

**assumes**  $i < \dim\text{-row } A$  **and**  $j < \dim\text{-col } A$  **and**  $j \geq low$  **and**  $j < up$  **and**  $l < low \vee l \geq up$   
**and**  $l < \dim\text{-col } A$   
**shows** *add-col-to-multiple*  $a$  [ $low..<up$ ]  $l$   $A$   $\$ \$ (i,j) = (a * A \$ \$ (i, l)) + A \$ \$ (i,j)$   
 ⟨*proof*⟩

Operations specifically on 1st row/column

**lemma** *add-first-row-to-multiple-index*:

**assumes**  $i < \dim\text{-row } M$  **and**  $j < \dim\text{-col } M$   
**shows**  $i = 0 \implies (\text{add-row-to-multiple } a \ [1..<\dim\text{-row } M] \ 0 \ M) \ \$$ \$(i, j) = M \ \$$  
 $(i, j)$   
**and**  $i \neq 0 \implies (\text{add-row-to-multiple } a \ [1..<\dim\text{-row } M] \ 0 \ M) \ \$$ \$(i, j) = (a *  
 $M \ \$(0, j)) + M \ \$(i, j)$   
 $\langle \text{proof} \rangle$$$

**lemma** *add-all-cols-to-first:*

**assumes**  $i < \dim\text{-row } (M)$   
**assumes**  $j < \dim\text{-col } (M)$   
**shows**  $j \neq 0 \implies \text{add-multiple-cols } 1 \ 0 \ [1..<\dim\text{-col } M] \ M \ \$$ \$(i, j) = M \ \$$ (i,  
 $j)$   
**and**  $j = 0 \implies \text{add-multiple-cols } 1 \ 0 \ [1..<\dim\text{-col } M] \ M \ \$$ \$(i, j) = (\sum l =  
 $1..<\dim\text{-col } M. \ M \ \$(i, l)) + M \ \$(i, 0)$   
 $\langle \text{proof} \rangle$$$

Lemmas on the determinant of the matrix under extended row/column operations

**lemma** *add-row-to-multiple-carrier:*

$A \in \text{carrier-mat } n \ n \implies \text{add-row-to-multiple } a \ ks \ l \ A \in \text{carrier-mat } n \ n$   
 $\langle \text{proof} \rangle$

**lemma** *add-col-to-multiple-carrier:*

$A \in \text{carrier-mat } n \ n \implies \text{add-col-to-multiple } a \ ks \ l \ A \in \text{carrier-mat } n \ n$   
 $\langle \text{proof} \rangle$

**lemma** *add-multiple-rows-carrier:*

$A \in \text{carrier-mat } n \ n \implies \text{add-multiple-rows } a \ k \ ls \ A \in \text{carrier-mat } n \ n$   
 $\langle \text{proof} \rangle$

**lemma** *add-multiple-cols-carrier:*

$A \in \text{carrier-mat } n \ n \implies \text{add-multiple-cols } a \ k \ ls \ A \in \text{carrier-mat } n \ n$   
 $\langle \text{proof} \rangle$

**lemma** *add-row-to-multiple-det:*

**assumes**  $l \notin \text{set } ks$  **and**  $l < n$  **and**  $A \in \text{carrier-mat } n \ n$   
**shows**  $\det (\text{add-row-to-multiple } a \ ks \ l \ A) = \det A$   
 $\langle \text{proof} \rangle$

**lemma** *add-col-to-multiple-det:*

**assumes**  $l \notin \text{set } ks$  **and**  $l < n$  **and**  $A \in \text{carrier-mat } n \ n$   
**shows**  $\det (\text{add-col-to-multiple } a \ ks \ l \ A) = \det A$   
 $\langle \text{proof} \rangle$

**lemma** *add-multiple-cols-det:*

**assumes**  $k \notin \text{set } ls$  **and**  $\bigwedge l. l \in \text{set } ls \implies l < n$  **and**  $A \in \text{carrier-mat } n \ n$   
**shows**  $\det (\text{add-multiple-cols } a \ k \ ls \ A) = \det A$   
 $\langle \text{proof} \rangle$

**lemma** *add-multiple-rows-det:*

**assumes**  $k \notin \text{set } ls$  **and**  $\bigwedge l. l \in \text{set } ls \implies l < n$  **and**  $A \in \text{carrier-mat } n \ n$

**shows**  $\text{det } (\text{add-multiple-rows } a \ k \ ls \ A) = \text{det } A$

*<proof>*

## 6.2 Rank and Linear Independence

**abbreviation**  $\text{rank } v \ M \equiv \text{vec-space.rank } v \ M$

Basic lemma: the rank of the multiplication of two matrices will be less than the minimum of the individual ranks of those matrices. This directly follows from an existing lemmas in the linear algebra library which show independently that the resulting matrices rank is less than either the right or left matrix rank in the product

**lemma** *rank-mat-mult-lt-min-rank-factor:*

**fixes**  $A :: 'a :: \{\text{conjugatable-ordered-field}\} \text{ mat}$

**assumes**  $A \in \text{carrier-mat } n \ m$

**assumes**  $B \in \text{carrier-mat } m \ nc$

**shows**  $\text{rank } n \ (A * B) \leq \min (\text{rank } n \ A) (\text{rank } m \ B)$

*<proof>*

Rank Argument 1: Given two a  $x \times y$  matrix  $M$  where  $MM^T$  has rank  $x$ ,  $x \leq y$

**lemma** *rank-argument:*

**fixes**  $M :: ('c :: \{\text{conjugatable-ordered-field}\}) \text{ mat}$

**assumes**  $M \in \text{carrier-mat } x \ y$

**assumes**  $\text{vec-space.rank } x \ (M * M^T) = x$

**shows**  $x \leq y$

*<proof>*

Generalise the rank argument to use the determinant. If the determinant of the matrix is non-zero, than it's rank must be equal to  $x$ . This removes the need for someone to use facts on rank in their proofs.

**lemma** *rank-argument-det:*

**fixes**  $M :: ('c :: \{\text{conjugatable-ordered-field}\}) \text{ mat}$

**assumes**  $M \in \text{carrier-mat } x \ y$

**assumes**  $\text{det } (M * M^T) \neq 0$

**shows**  $x \leq y$

*<proof>*

**end**

## 7 Linear Bound Argument - General

Lemmas to enable general reasoning using the linear bound argument for combinatorial proofs. Jukna [7] presents a good overview of the mathematical background this theory is based on and applications

**theory** *Linear-Bound-Argument* **imports** *Incidence-Matrices Jordan-Normal-Form.DL-Rank*

*Jordan-Normal-Form.Ring-Hom-Matrix*

**begin**

## 7.1 Vec Space Extensions

Simple extensions to the existing vector space locale on linear independence

**context** *vec-space*

**begin**

**lemma** *lin-indpt-set-card-lt-dim*:

**fixes**  $A :: 'a \text{ vec set}$

**assumes**  $A \subseteq \text{carrier-vec } n$

**assumes** *lin-indpt*  $A$

**shows**  $\text{card } A \leq \text{dim}$

*<proof>*

**lemma** *lin-indpt-dim-col-lt-dim*:

**fixes**  $A :: 'a \text{ mat}$

**assumes**  $A \in \text{carrier-mat } n \text{ nc}$

**assumes** *distinct* (*cols*  $A$ )

**assumes** *lin-indpt* (*set* (*cols*  $A$ ))

**shows**  $\text{nc} \leq \text{dim}$

*<proof>*

**lemma** *lin-comb-imp-lin-dep-fin*:

**fixes**  $A :: 'a \text{ vec set}$

**assumes** *finite*  $A$

**assumes**  $A \subseteq \text{carrier-vec } n$

**assumes** *lincomb*  $c \ A = 0_v \ n$

**assumes**  $\exists a. a \in A \wedge c \ a \neq 0$

**shows** *lin-dep*  $A$

*<proof>*

While a trivial definition, this enables us to directly reference the definition outside of a locale context, as *lin-indpt* is an inherited definition

**definition** *lin-indpt-vs*::  $'a \text{ vec set} \Rightarrow \text{bool}$  **where**

*lin-indpt-vs*  $A \iff \text{lin-indpt } A$

**lemma** *lin-comb-sum-lin-indpt*:

**fixes**  $A :: 'a \text{ vec list}$

**assumes**  $\text{set } (A) \subseteq \text{carrier-vec } n$

**assumes** *distinct*  $A$

**assumes**  $\bigwedge f. \text{lincomb-list } (\lambda i. f \ (A \ ! \ i)) \ A = 0_v \ n \implies \forall v \in (\text{set } A). f \ v = 0$

**shows** *lin-indpt* (*set*  $A$ )

*<proof>*

**lemma** *lin-comb-mat-lin-indpt*:

**fixes**  $A :: 'a \text{ vec list}$

**assumes**  $set (A) \subseteq carrier-vec\ n$   
**assumes**  $distinct\ A$   
**assumes**  $\bigwedge f. mat-of-cols\ n\ A\ *_v\ vec\ (length\ A)\ (\lambda i. f\ (A\ !\ i)) = 0_v\ n \implies \forall v \in$   
 $(set\ A). f\ v = 0$   
**shows**  $lin-indpt\ (set\ A)$   
 $\langle proof \rangle$

**lemma**  $lin-comb-mat-lin-indpt-vs$ :  
**fixes**  $A :: 'a\ vec\ list$   
**assumes**  $set (A) \subseteq carrier-vec\ n$   
**assumes**  $distinct\ A$   
**assumes**  $\bigwedge f. mat-of-cols\ n\ A\ *_v\ vec\ (length\ A)\ (\lambda i. f\ (A\ !\ i)) = 0_v\ n \implies \forall v \in$   
 $(set\ A). f\ v = 0$   
**shows**  $lin-indpt-vs\ (set\ A)$   
 $\langle proof \rangle$

**end**

## 7.2 Linear Bound Argument Lemmas

Three general representations of the linear bound argument, requiring a direct fact of linear independence on the rows of the vector space over either a set  $A$  of vectors, list  $xs$  of vectors or a Matrix' columns

**lemma**  $lin-bound-arg-general-set$ :  
**fixes**  $A :: ('a :: \{field\})\ vec\ set$   
**assumes**  $A \subseteq carrier-vec\ nr$   
**assumes**  $vec-space.lin-indpt-vs\ nr\ A$   
**shows**  $card\ A \leq nr$   
 $\langle proof \rangle$

**lemma**  $lin-bound-arg-general-list$ :  
**fixes**  $xs :: ('a :: \{field\})\ vec\ list$   
**assumes**  $distinct\ xs$   
**assumes**  $(set\ xs) \subseteq carrier-vec\ nr$   
**assumes**  $vec-space.lin-indpt-vs\ nr\ (set\ xs)$   
**shows**  $length\ (xs) \leq nr$   
 $\langle proof \rangle$

**lemma**  $lin-bound-arg-general$ :  
**fixes**  $A :: ('a :: \{field\})\ mat$   
**assumes**  $distinct\ (cols\ A)$   
**assumes**  $A \in carrier-mat\ nr\ nc$   
**assumes**  $vec-space.lin-indpt-vs\ nr\ (set\ (cols\ A))$   
**shows**  $nc \leq nr$   
 $\langle proof \rangle$

The linear bound argument lemma on a matrix requiring the lower level assumption on a linear combination. This removes the need to directly refer

to any aspect of the linear algebra libraries

**lemma** *lin-bound-argument*:

**fixes**  $A :: ('a :: \{field\})\ mat$   
**assumes**  $distinct\ (cols\ A)$   
**assumes**  $A \in carrier\text{-}mat\ nr\ nc$   
**assumes**  $\bigwedge f. A *_{vec}\ nc\ (\lambda i. f\ (col\ A\ i)) = 0_v\ nr \implies \forall v \in (set\ (cols\ A)). f\ v = 0$   
**shows**  $nc \leq nr$   
 $\langle proof \rangle$

A further extension to present the linear combination directly as a sum. This manipulation from vector product to a summation was found to commonly be repeated in proofs applying this rule

**lemma** *lin-bound-argument2*:

**fixes**  $A :: ('a :: \{field\})\ mat$   
**assumes**  $distinct\ (cols\ A)$   
**assumes**  $A \in carrier\text{-}mat\ nr\ nc$   
**assumes**  $\bigwedge f. vec\ nr\ (\lambda i. \sum j \in \{0..<nc\}. f\ (col\ A\ j) * (col\ A\ j)\ \$\ i) = 0_v\ nr \implies \forall v \in (set\ (cols\ A)). f\ v = 0$   
**shows**  $nc \leq nr$   
 $\langle proof \rangle$

**end**

## 8 Fisher's Inequality

This theory presents the proof of Fisher's Inequality [4] on BIBD's (i.e. uniform Fisher's) and the generalised nonuniform Fisher's Inequality

**theory** *Fishers-Inequality* **imports** *Rank-Argument-General-Linear-Bound-Argument*  
**begin**

### 8.1 Uniform Fisher's Inequality

**context** *ordered-bibd*  
**begin**

Row/Column transformation steps

Following design theory lecture notes from MATH3301 at the University of Queensland [6], a simple transformation to produce an upper triangular matrix using elementary row operations is to (1) Subtract the first row from each other row, and (2) add all columns to the first column

**lemma** *transform-N-step1-vals*:

**defines**  $mdef: M \equiv (N * N^T)$   
**assumes**  $i < dim\text{-}row\ M$   
**assumes**  $j < dim\text{-}col\ M$

**shows**  $i = 0 \implies j = 0 \implies (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } M] 0 M) \text{ \textasciitilde\textasciitilde}$   
 $(i, j) = (\text{int } r) - \text{top left elem}$   
**and**  $i \neq 0 \implies j = 0 \implies (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } M] 0 M) \text{ \textasciitilde\textasciitilde}$   
 $(i, j) = (\text{int } \Lambda) - (\text{int } r) - \text{first column ex. 1}$   
**and**  $i = 0 \implies j \neq 0 \implies (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } M] 0 M) \text{ \textasciitilde\textasciitilde}$   
 $(i, j) = (\text{int } \Lambda) - \text{first row ex. 1}$   
**and**  $i \neq 0 \implies j \neq 0 \implies i = j \implies (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } M]$   
 $0 M) \text{ \textasciitilde\textasciitilde}$   $(i, j) = (\text{int } r) - (\text{int } \Lambda) - \text{diagonal ex. 1}$   
**and**  $i \neq 0 \implies j \neq 0 \implies i \neq j \implies (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } M]$   
 $0 M) \text{ \textasciitilde\textasciitilde}$   $(i, j) = 0 - \text{everything else}$   
 $\langle \text{proof} \rangle$

**lemma** *transform-N-step2-vals:*

**defines**  $m\text{def}: M \equiv (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } (N * N^T)] 0 (N * N^T))$   
**assumes**  $i < \text{dim-row } (M)$   
**assumes**  $j < \text{dim-col } (M)$   
**shows**  $i = 0 \implies j = 0 \implies \text{add-multiple-cols } 1 0 [1..<\text{dim-col } M] M \text{ \textasciitilde\textasciitilde}$   $(i, j) =$   
 $(\text{int } r) + (\text{int } \Lambda) * (v - 1) - \text{top left element}$   
**and**  $i = 0 \implies j \neq 0 \implies \text{add-multiple-cols } 1 0 [1..<\text{dim-col } M] M \text{ \textasciitilde\textasciitilde}$   $(i, j) =$   
 $(\text{int } \Lambda) - \text{top row}$   
**and**  $i \neq 0 \implies i = j \implies \text{add-multiple-cols } 1 0 [1..<\text{dim-col } M] M \text{ \textasciitilde\textasciitilde}$   $(i, j) =$   
 $(\text{int } r) - (\text{int } \Lambda) - \text{Diagonal}$   
**and**  $i \neq 0 \implies i \neq j \implies \text{add-multiple-cols } 1 0 [1..<\text{dim-col } M] M \text{ \textasciitilde\textasciitilde}$   $(i, j) = 0$   
 $- \text{Everything else}$   
 $\langle \text{proof} \rangle$

Transformed matrix is upper triangular

**lemma** *transform-upper-triangular:*

**defines**  $m\text{def}: M \equiv (\text{add-row-to-multiple } (-1) [1..<\text{dim-row } (N * N^T)] 0 (N * N^T))$   
**shows** *upper-triangular*  $(\text{add-multiple-cols } 1 0 [1..<\text{dim-col } M] M)$   
 $\langle \text{proof} \rangle$

Find the determinant of the  $NN^T$  matrix using transformed matrix values

**lemma** *determinant-inc-mat-square:*  $\det (N * N^T) = (r + \Lambda * (v - 1)) * (r - \Lambda) \wedge (v - 1)$   
 $\langle \text{proof} \rangle$

Fisher's Inequality using the rank argument. Note that to use the rank argument we must first map  $N$  to a real matrix. It is useful to explicitly include the parameters which should be used in the application of the *rank-argument-det* lemma

**theorem** *Fishers-Inequality-BIBD:*  $v \leq b$   
 $\langle \text{proof} \rangle$

**end**

## 8.2 Generalised Fisher's Inequality

**context** *simp-ordered-const-intersect-design*  
**begin**

Lemma to reason on sum coefficients

**lemma** *sum-split-coeffs-0*:

**fixes**  $c :: \text{nat} \Rightarrow \text{real}$

**assumes**  $b \geq 2$

**assumes**  $m > 0$

**assumes**  $j' < b$

**assumes**  $0 = (\sum j \in \{0..<b\} . (c\ j)^2 * ((\text{card } (\mathcal{B}s\ !\ j)) - (\text{int } m))) +$   
 $m * ((\sum j \in \{0..<b\} . c\ j)^2)$

**shows**  $c\ j' = 0$

*<proof>*

The general non-uniform version of fisher's inequality is also known as the "Block town problem". In this case we are working in a constant intersect design, hence the inequality is the opposite way around compared to the BIBD version. The theorem below is the more traditional set theoretic approach. This proof is based off one by Jukna [7]

**theorem** *general-fishers-inequality*:  $b \leq v$

*<proof>*

**end**

Using the dual design concept, it is easy to translate the set theoretic general definition of Fisher's inequality to a more traditional design theoretic version on pairwise balanced designs. Two versions of this are given using different trivial (but crucial) conditions on design properties

**context** *ordered-pairwise-balance*

**begin**

**corollary** *general-nonuniform-fishers*: — only valid on incomplete designs

**assumes**  $\Lambda > 0$

**assumes**  $\bigwedge bl. bl \in \# \mathcal{B} \implies \text{incomplete-block } bl$

— i.e. not a super trivial design with only complete blocks

**shows**  $v \leq b$

*<proof>*

**corollary** *general-nonuniform-fishers-comp*:

**assumes**  $\Lambda > 0$

**assumes**  $\text{count } \mathcal{B} \ \mathcal{V} < \Lambda$  — i.e. not a super trivial design with only complete blocks and single blocks

**shows**  $v \leq b$

*<proof>*

**end**

**end**



## 9 Matrices/Vectors mod x

This section formalises operations and properties mod some integer x on integer matrices and vectors. Much of this file was no longer needed for fishers once the generic idea of lifting a 0-1 matrix was introduced, however it is left as an example for future work on matrices mod n, beyond 0-1 matrices

**theory** *Vector-Matrix-Mod* **imports** *Matrix-Vector-Extras*  
*Berlekamp-Zassenhaus.Finite-Field* *Berlekamp-Zassenhaus.More-Missing-Multiset*  
**begin**

Simple abbreviations for main mapping functions

**abbreviation** *to-int-mat* :: 'a :: finite mod-ring mat  $\Rightarrow$  int mat **where**  
*to-int-mat*  $\equiv$  *map-mat to-int-mod-ring*

**abbreviation** *to-int-vec* :: 'a :: finite mod-ring vec  $\Rightarrow$  int vec **where**  
*to-int-vec*  $\equiv$  *map-vec to-int-mod-ring*

**interpretation** *of-int-mod-ring-hom-sr*: *semiring-hom of-int-mod-ring*  
(*proof*)

NOTE: The context directly below is copied from Matrix Vector Extras, as for some reason they can't be used locally if not? Ideally remove in future if possible

**context** *inj-zero-hom*  
**begin**

**lemma** *vec-hom-zero-iff[simp]*: (*map-vec hom x = 0<sub>v</sub> n*) = (*x = 0<sub>v</sub> n*)  
(*proof*)

**lemma** *mat-hom-inj*: *map-mat hom A = map-mat hom B*  $\implies$  *A = B*  
(*proof*)

**lemma** *vec-hom-inj*: *map-vec hom v = map-vec hom w*  $\implies$  *v = w*  
(*proof*)

**lemma** *vec-hom-set-distinct-iff*:  
**fixes** *xs* :: 'a vec list  
**shows** *distinct xs*  $\longleftrightarrow$  *distinct (map (map-vec hom) xs)*  
(*proof*)  
**end**

### 9.1 Basic Mod Context

**locale** *mat-mod* = **fixes** *m* :: int  
**assumes** *non-triv-m*: *m > 1*  
**begin**

First define the mod operations on vectors

**definition** *vec-mod* :: *int vec*  $\Rightarrow$  *int vec* **where**  
*vec-mod* *v*  $\equiv$  *map-vec* ( $\lambda x . x \bmod m$ ) *v*

**lemma** *vec-mod-dim[simp]*: *dim-vec* (*vec-mod* *v*) = *dim-vec* *v*  
 ⟨*proof*⟩

**lemma** *vec-mod-index[simp]*:  $i < \text{dim-vec } v \implies (\text{vec-mod } v) \$ i = (v \$ i) \bmod m$   
 ⟨*proof*⟩

**lemma** *vec-mod-index-same[simp]*:  $i < \text{dim-vec } v \implies v \$ i < m \implies v \$ i \geq 0$   
 $\implies (\text{vec-mod } v) \$ i = v \$ i$   
 ⟨*proof*⟩

**lemma** *vec-setI2*:  $i < \text{dim-vec } v \implies v \$ i \in \text{set}_v v$   
 ⟨*proof*⟩

**lemma** *vec-mod-eq*:  $\text{set}_v v \subseteq \{0..<m\} \implies \text{vec-mod } v = v$   
 ⟨*proof*⟩

**lemma** *vec-mod-set-vec-same*:  $\text{set}_v v \subseteq \{0..<m\} \implies \text{set}_v (\text{vec-mod } v) = \text{set}_v v$   
 ⟨*proof*⟩

Define the mod operation on matrices

**definition** *mat-mod* :: *int mat*  $\Rightarrow$  *int mat* **where**  
*mat-mod* *M*  $\equiv$  *map-mat* ( $\lambda x . x \bmod m$ ) *M*

**lemma** *mat-mod-dim[simp]*: *dim-row* (*mat-mod* *M*) = *dim-row* *M* *dim-col* (*mat-mod* *M*) = *dim-col* *M*  
 ⟨*proof*⟩

**lemma** *mat-mod-index [simp]*:  $i < \text{dim-row } M \implies j < \text{dim-col } M \implies (\text{mat-mod } M) \$\$ (i, j) = (M \$\$ (i, j)) \bmod m$   
 ⟨*proof*⟩

**lemma** *mat-mod-index-same[simp]*:  $i < \text{dim-row } M \implies j < \text{dim-col } M \implies M \$\$ (i, j) < m \implies$   
 $M \$\$ (i, j) \geq 0 \implies \text{mat-mod } M \$\$ (i, j) = M \$\$ (i, j)$   
 ⟨*proof*⟩

**lemma** *elements-matI2*:  $i < \text{dim-row } A \implies j < \text{dim-col } A \implies A \$\$ (i, j) \in \text{elements-mat } A$   
 ⟨*proof*⟩

**lemma** *mat-mod-elements-in*:  
**assumes**  $x \in \text{elements-mat } M$   
**shows**  $x \bmod m \in \text{elements-mat } (\text{mat-mod } M)$   
 ⟨*proof*⟩

**lemma** *mat-mod-elements-map*:

**assumes**  $x \in \text{elements-mat } M$

**shows**  $\text{elements-mat } (\text{mat-mod } M) = (\lambda x. x \text{ mod } m) ` (\text{elements-mat } M)$

*<proof>*

**lemma** *mat-mod-eq-cond*:

**assumes**  $\text{elements-mat } M \subseteq \{0..<m\}$

**shows**  $\text{mat-mod } M = M$

*<proof>*

**lemma** *mat-mod-eq-elements-cond*:  $\text{elements-mat } M \subseteq \{0..<m\} \implies \text{elements-mat } (\text{mat-mod } M) = \text{elements-mat } M$

*<proof>*

**lemma** *mat-mod-vec-mod-row*:  $i < \text{dim-row } A \implies \text{row } (\text{mat-mod } A) i = \text{vec-mod } (\text{row } A i)$

*<proof>*

**lemma** *mat-mod-vec-mod-col*:  $j < \text{dim-col } A \implies \text{col } (\text{mat-mod } A) j = \text{vec-mod } (\text{col } A j)$

*<proof>*

**lemma** *count-vec-mod-eq*:  $\text{set}_v v \subseteq \{0..<m\} \implies \text{count-vec } v x = \text{count-vec } (\text{vec-mod } v) x$

*<proof>*

**lemma** *elems-mat-setv-row-0m*:  $i < \text{dim-row } M \implies \text{elements-mat } M \subseteq \{0..<m\} \implies \text{set}_v (\text{row } M i) \subseteq \{0..<m\}$

*<proof>*

**lemma** *elems-mat-setv-col-0m*:  $j < \text{dim-col } M \implies \text{elements-mat } M \subseteq \{0..<m\} \implies \text{set}_v (\text{col } M j) \subseteq \{0..<m\}$

*<proof>*

**lemma** *mat-mod-count-row-eq*:  $i < \text{dim-row } M \implies \text{elements-mat } M \subseteq \{0..<m\} \implies$

$\text{count-vec } (\text{row } (\text{mat-mod } M) i) x = \text{count-vec } (\text{row } M i) x$

*<proof>*

**lemma** *mat-mod-count-col-eq*:  $j < \text{dim-col } M \implies \text{elements-mat } M \subseteq \{0..<m\} \implies$

$\text{count-vec } (\text{col } (\text{mat-mod } M) j) x = \text{count-vec } (\text{col } M j) x$

*<proof>*

**lemma** *mod-mat-one*:  $\text{mat-mod } (1_m \ n) = (1_m \ n)$

*<proof>*

**lemma** *mod-mat-zero*:  $\text{mat-mod } (0_m \ nr \ nc) = (0_m \ nr \ nc)$

*<proof>*

**lemma** *vec-mod-unit*:  $vec-mod (unit-vec\ n\ i) = (unit-vec\ n\ i)$   
*<proof>*

**lemma** *vec-mod-zero*:  $vec-mod (0_v\ n) = (0_v\ n)$   
*<proof>*

**lemma** *mat-mod-cond-iff*:  $elements-mat\ M \subseteq \{0..<m\} \implies P\ M \longleftrightarrow P\ (mat-mod\ M)$   
*<proof>*

**end**

## 9.2 Mod Type

The below locale takes lemmas from the Poly Mod Finite Field theory in the Berlekamp Zassenhaus AFP entry, however has removed any excess material on polynomials mod, and only included the general factors. Ideally, this could be used as the base locale for both in the future

**locale** *mod-type* =  
  **fixes**  $m :: int$  **and**  $ty :: 'a :: nontriv\ itself$   
  **assumes**  $m: m = CARD('a)$   
**begin**

**lemma** *m1*:  $m > 1$  *<proof>*

**definition**  $M :: int \Rightarrow int$  **where**  $M\ x = x\ mod\ m$

**lemma** *M-0[simp]*:  $M\ 0 = 0$   
*<proof>*

**lemma** *M-M[simp]*:  $M\ (M\ x) = M\ x$   
*<proof>*

**lemma** *M-plus[simp]*:  $M\ (M\ x + y) = M\ (x + y)$   $M\ (x + M\ y) = M\ (x + y)$   
*<proof>*

**lemma** *M-minus[simp]*:  $M\ (M\ x - y) = M\ (x - y)$   $M\ (x - M\ y) = M\ (x - y)$   
*<proof>*

**lemma** *M-times[simp]*:  $M\ (M\ x * y) = M\ (x * y)$   $M\ (x * M\ y) = M\ (x * y)$   
*<proof>*

**lemma** *M-1[simp]*:  $M\ 1 = 1$  *<proof>*

**lemma** *M-sum*:  $M\ (sum\ (\lambda\ x.\ M\ (f\ x))\ A) = M\ (sum\ f\ A)$   
*<proof>*

**definition**  $inv-M :: int \Rightarrow int$  **where**

$inv-M\ x = (if\ x + x \leq m\ then\ x\ else\ x - m)$

**lemma**  $M-inv-M-id[simp]$ :  $M\ (inv-M\ x) = M\ x$

$\langle proof \rangle$

**definition**  $M-Rel :: int \Rightarrow 'a\ mod-ring \Rightarrow bool$

**where**  $M-Rel\ x\ x' \equiv (M\ x = to-int-mod-ring\ x')$

**lemma**  $to-int-mod-ring-plus$ :  $to-int-mod-ring\ ((x :: 'a\ mod-ring) + y) = M\ (to-int-mod-ring\ x + to-int-mod-ring\ y)$

$\langle proof \rangle$

**lemma**  $to-int-mod-ring-times$ :  $to-int-mod-ring\ ((x :: 'a\ mod-ring) * y) = M\ (to-int-mod-ring\ x * to-int-mod-ring\ y)$

$\langle proof \rangle$

**lemma**  $eq-M-Rel[transfer-rule]$ :  $(M-Rel\ ==>\ M-Rel\ ==>\ (=))\ (\lambda\ x\ y.\ M\ x = M\ y)\ (=)$

$\langle proof \rangle$

**lemma**  $one-M-Rel[transfer-rule]$ :  $M-Rel\ 1\ 1$

$\langle proof \rangle$

**lemma**  $zero-M-Rel[transfer-rule]$ :  $M-Rel\ 0\ 0$

$\langle proof \rangle$

**lemma**  $M-to-int-mod-ring$ :  $M\ (to-int-mod-ring\ (x :: 'a\ mod-ring)) = to-int-mod-ring\ x$

$\langle proof \rangle$

**lemma**  $right-total-M-Rel[transfer-rule]$ :  $right-total\ M-Rel$

$\langle proof \rangle$

**lemma**  $left-total-M-Rel[transfer-rule]$ :  $left-total\ M-Rel$

$\langle proof \rangle$

**lemma**  $bi-total-M-Rel[transfer-rule]$ :  $bi-total\ M-Rel$

$\langle proof \rangle$

**lemma**  $to-int-mod-ring-of-int-M$ :  $to-int-mod-ring\ (of-int\ x :: 'a\ mod-ring) = M\ x$

$\langle proof \rangle$

**lemma**  $UNIV-M-Rel[transfer-rule]$ :  $rel-set\ M-Rel\ \{0..<m\}\ UNIV$

$\langle proof \rangle$

**end**

### 9.3 Mat mod type

Define a context to work on matrices and vectors of type  $'a \text{ mod-ring}$

**locale**  $\text{mat-mod-type} = \text{mat-mod} + \text{mod-type}$   
**begin**

**lemma**  $\text{to-int-mod-ring-plus}$ :  $\text{to-int-mod-ring} ((x :: 'a \text{ mod-ring}) + y) = (\text{to-int-mod-ring } x + \text{to-int-mod-ring } y) \text{ mod } m$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{to-int-mod-ring-times}$ :  $\text{to-int-mod-ring} ((x :: 'a \text{ mod-ring}) * y) = (\text{to-int-mod-ring } x * \text{to-int-mod-ring } y) \text{ mod } m$   
 $\langle \text{proof} \rangle$

Set up transfer relation for matrices and vectors

**definition**  $\text{MM-Rel} :: \text{int mat} \Rightarrow 'a \text{ mod-ring mat} \Rightarrow \text{bool}$   
**where**  $\text{MM-Rel } f f' \equiv (\text{mat-mod } f = \text{to-int-mat } f')$

**definition**  $\text{MV-Rel} :: \text{int vec} \Rightarrow 'a \text{ mod-ring vec} \Rightarrow \text{bool}$   
**where**  $\text{MV-Rel } v v' \equiv (\text{vec-mod } v = \text{to-int-vec } v')$

**lemma**  $\text{to-int-mat-index[simp]}$ :  $i < \text{dim-row } N \Longrightarrow j < \text{dim-col } N \Longrightarrow (\text{to-int-mat } N \ \$\$ (i, j)) = \text{to-int-mod-ring } (N \ \$\$ (i, j))$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{to-int-vec-index[simp]}$ :  $i < \text{dim-vec } v \Longrightarrow (\text{to-int-vec } v \ \$i) = \text{to-int-mod-ring } (v \ \$i)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{eq-dim-row-MM-Rel[transfer-rule]}$ :  $(\text{MM-Rel} \Longrightarrow (=)) \text{ dim-row dim-row}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lt-dim-row-MM-Rel[transfer-rule]}$ :  $(\text{MM-Rel} \Longrightarrow (=) \Longrightarrow (=)) (\lambda M i. i < \text{dim-row } M) (\lambda M i. i < \text{dim-row } M)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{eq-dim-col-MM-Rel[transfer-rule]}$ :  $(\text{MM-Rel} \Longrightarrow (=)) \text{ dim-col dim-col}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lt-dim-col-MM-Rel[transfer-rule]}$ :  $(\text{MM-Rel} \Longrightarrow (=) \Longrightarrow (=)) (\lambda M j. j < \text{dim-col } M) (\lambda M j. j < \text{dim-col } M)$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{eq-dim-vec-MV-Rel[transfer-rule]}$ :  $(\text{MV-Rel} \Longrightarrow (=)) \text{ dim-vec dim-vec}$   
 $\langle \text{proof} \rangle$

**lemma**  $\text{lt-dim-vec-MV-Rel[transfer-rule]}$ :  $(\text{MV-Rel} \Longrightarrow (=) \Longrightarrow (=)) (\lambda v j. j < \text{dim-vec } v) (\lambda v j. j < \text{dim-vec } v)$

*<proof>*

**lemma** *eq-MM-Rel[transfer-rule]*: ( $MM\text{-Rel} \implies MM\text{-Rel} \implies (=)$ ) ( $\lambda f f' . mat\text{-mod } f = mat\text{-mod } f'$ ) (=)  
*<proof>*

**lemma** *eq-MV-Rel[transfer-rule]*: ( $MV\text{-Rel} \implies MV\text{-Rel} \implies (=)$ ) ( $\lambda v v' . vec\text{-mod } v = vec\text{-mod } v'$ ) (=)  
*<proof>*

**lemma** *index-MV-Rel[transfer-rule]*: ( $MV\text{-Rel} \implies (=) \implies M\text{-Rel}$ )  
( $\lambda v i . \text{if } i < dim\text{-vec } v \text{ then } v \$ i \text{ else } 0$ ) ( $\lambda v i . \text{if } i < dim\text{-vec } v \text{ then } v \$ i \text{ else } 0$ )  
*<proof>*

**lemma** *index-MM-Rel[transfer-rule]*: ( $MM\text{-Rel} \implies (=) \implies (=) \implies M\text{-Rel}$ )  
( $\lambda M i j . \text{if } (i < dim\text{-row } M \wedge j < dim\text{-col } M) \text{ then } M \$\$ (i, j) \text{ else } 0$ )  
( $\lambda M i j . \text{if } (i < dim\text{-row } M \wedge j < dim\text{-col } M) \text{ then } M \$\$ (i, j) \text{ else } 0$ )  
*<proof>*

**lemma** *index-MM-Rel-explicit*:  
**assumes**  $MM\text{-Rel } N N'$   
**assumes**  $i < dim\text{-row } N \wedge j < dim\text{-col } N$   
**shows** ( $N \$\$ (i, j) \text{ mod } m = to\text{-int-mod-ring } (N' \$\$ (i, j))$ )  
*<proof>*

**lemma** *one-MV-Rel[transfer-rule]*:  $MV\text{-Rel } (unit\text{-vec } n \ i) (unit\text{-vec } n \ i)$   
*<proof>*

**lemma** *one-MM-Rel[transfer-rule]*:  $MM\text{-Rel } (1_m \ n) (1_m \ n)$   
*<proof>*

**lemma** *zero-MM-Rel[transfer-rule]*:  $MM\text{-Rel } (0_m \ nr \ nc) (0_m \ nr \ nc)$   
*<proof>*

**lemma** *zero-MV-Rel[transfer-rule]*:  $MV\text{-Rel } (0_v \ n) (0_v \ n)$   
*<proof>*

**lemma** *right-unique-MV-Rel[transfer-rule]*: *right-unique*  $MV\text{-Rel}$   
*<proof>*

**lemma** *right-unique-MM-Rel[transfer-rule]*: *right-unique*  $MM\text{-Rel}$   
*<proof>*

**lemma** *mod-to-int-mod-ring*: ( $to\text{-int-mod-ring } (x :: 'a \text{ mod-ring}) \text{ mod } m = to\text{-int-mod-ring } x$ )  
*<proof>*

**lemma** *mat-mod-to-int-mat*: *mat-mod* (*to-int-mat* ( $N :: 'a \text{ mod-ring mat}$ )) = *to-int-mat*  $N$   
 ⟨*proof*⟩

**lemma** *vec-mod-to-int-vec*: *vec-mod* (*to-int-vec* ( $v :: 'a \text{ mod-ring vec}$ )) = *to-int-vec*  $v$   
 ⟨*proof*⟩

**lemma** *right-total-MM-Rel[transfer-rule]*: *right-total MM-Rel*  
 ⟨*proof*⟩

**lemma** *right-total-MV-Rel[transfer-rule]*: *right-total MV-Rel*  
 ⟨*proof*⟩

**lemma** *to-int-mod-ring-of-int-mod*: *to-int-mod-ring* (*of-int*  $x :: 'a \text{ mod-ring}$ ) =  $x \text{ mod } m$   
 ⟨*proof*⟩

**lemma** *vec-mod-v-representative*: *vec-mod*  $v = \text{to-int-vec}$  (*map-vec of-int*  $v :: 'a \text{ mod-ring vec}$ )  
 ⟨*proof*⟩

**lemma** *mat-mod-N-representative*: *mat-mod*  $N = \text{to-int-mat}$  (*map-mat of-int*  $N :: 'a \text{ mod-ring mat}$ )  
 ⟨*proof*⟩

**lemma** *left-total-MV-Rel[transfer-rule]*: *left-total MV-Rel*  
 ⟨*proof*⟩

**lemma** *left-total-MM-Rel[transfer-rule]*: *left-total MM-Rel*  
 ⟨*proof*⟩

**lemma** *bi-total-MV-Rel[transfer-rule]*: *bi-total MV-Rel*  
 ⟨*proof*⟩

**lemma** *bi-total-MM-Rel[transfer-rule]*: *bi-total MM-Rel*  
 ⟨*proof*⟩

**lemma** *domain-MV-rel[transfer-domain-rule]*: *Domainp MV-Rel* = ( $\lambda f. \text{True}$ )  
 ⟨*proof*⟩

**lemma** *domain-MM-rel[transfer-domain-rule]*: *Domainp MM-Rel* = ( $\lambda f. \text{True}$ )  
 ⟨*proof*⟩

**lemma** *mem-MV-Rel[transfer-rule]*:  
 (*MV-Rel*  $\implies$  *rel-set MV-Rel*  $\implies$   $(=)$ ) ( $\lambda x Y. \exists y \in Y. \text{vec-mod } x = \text{vec-mod } y$ ) ( $\in$ )  
 ⟨*proof*⟩



**lemma** *mem-MM-Rel[transfer-rule]*:

( $MM-Rel \implies rel-set\ MM-Rel \implies (=)$ ) ( $\lambda x\ Y. \exists y \in Y. mat-mod\ x = mat-mod\ y$ ) ( $\in$ )  
*<proof>*

**lemma** *conversep-MM-Rel-OO-MM-Rel [simp]*:  $MM-Rel^{-1-1}\ OO\ MM-Rel = (=)$   
*<proof>*

**lemma** *MM-Rel-OO-conversep-MM-Rel [simp]*:  $MM-Rel\ OO\ MM-Rel^{-1-1} = (\lambda\ M\ M'.\ mat-mod\ M = mat-mod\ M')$   
*<proof>*

**lemma** *conversep-MM-Rel-OO-eq-m [simp]*:  $MM-Rel^{-1-1}\ OO\ (\lambda\ M\ M'.\ mat-mod\ M = mat-mod\ M') = MM-Rel^{-1-1}$   
*<proof>*

**lemma** *eq-m-OO-MM-Rel [simp]*:  $(\lambda\ M\ M'.\ mat-mod\ M = mat-mod\ M')\ OO\ MM-Rel = MM-Rel$   
*<proof>*

**lemma** *eq-mset-MM-Rel [transfer-rule]*:

( $rel-mset\ MM-Rel \implies rel-mset\ MM-Rel \implies (=)$ ) ( $rel-mset\ (\lambda\ M\ M'.\ mat-mod\ M = mat-mod\ M')$ ) ( $=$ )  
*<proof>*

**lemma** *vec-mset-MV-Rel[transfer-rule]*:

( $MV-Rel \implies (=)$ ) ( $\lambda\ v.\ vec-mset\ (vec-mod\ v)$ ) ( $\lambda\ v.\ image-mset\ (to-int-mod-ring\ (vec-mset\ v))$ )  
*<proof>*

**lemma** *vec-count-MV-Rel-direct*:

**assumes**  $MV-Rel\ v1\ v2$

**shows**  $count-vec\ v2\ i = count-vec\ (vec-mod\ v1)\ (to-int-mod-ring\ i)$

*<proof>*

**lemma** *MM-Rel-MV-Rel-row*:  $MM-Rel\ A\ B \implies i < dim-row\ A \implies MV-Rel\ (row\ A\ i)\ (row\ B\ i)$

*<proof>*

**lemma** *MM-Rel-MV-Rel-col*:  $MM-Rel\ A\ B \implies j < dim-col\ A \implies MV-Rel\ (col\ A\ j)\ (col\ B\ j)$

*<proof>*

**end**

**end**

## 10 Variations on Fisher's Inequality

**theory** *Fishers-Inequality-Variations* **imports** *Dual-Systems Rank-Argument-General*  
*Vector-Matrix-Mod Linear-Bound-Argument*  
**begin**

### 10.1 Matrix mod properties

This is reasoning on properties specific to incidence matrices under *mat-mod*.  
Ultimately, this definition was not used in the final proof but it is left as is  
in case of future use

**context** *mat-mod*  
**begin**

**lemma** *mat-mod-proper-iff*:  $\text{proper-inc-mat } (\text{mat-mod } N) \longleftrightarrow \text{proper-inc-mat } N$   
*<proof>*

**lemma** *mat-mod-rep-num-eq*:  $i < \text{dim-row } N \implies \text{elements-mat } N \subseteq \{0..<m\}$   
 $\implies$   
 $\text{mat-rep-num } (\text{mat-mod } N) \ i = \text{mat-rep-num } N \ i$   
*<proof>*

**lemma** *mat-point-index-eq*:  $\text{elements-mat } N \subseteq \{0..<m\} \implies$   
 $\text{mat-point-index } (\text{mat-mod } N) \ I = \text{mat-point-index } N \ I$   
*<proof>*

**lemma** *mod-mat-inter-num-eq*:  $\text{elements-mat } N \subseteq \{0..<m\} \implies$   
 $\text{mat-inter-num } (\text{mat-mod } N) \ j1 \ j2 = \text{mat-inter-num } N \ j1 \ j2$   
*<proof>*

**lemma** *mod-mat-block-size*:  $\text{elements-mat } N \subseteq \{0..<m\} \implies \text{mat-block-size } (\text{mat-mod}$   
 $N) \ j = \text{mat-block-size } N \ j$   
*<proof>*

**lemma** *mat-mod-non-empty-col-iff*:  $\text{elements-mat } M \subseteq \{0..<m\} \implies$   
 $\text{non-empty-col } (\text{mat-mod } M) \ j \longleftrightarrow \text{non-empty-col } M \ j$   
*<proof>*

**end**

**context** *mat-mod-type*  
**begin**

**lemma** *mat-rep-num-MM-Rel*:  
**assumes** *MM-Rel*  $A \ B$   
**assumes**  $i < \text{dim-row } A$   
**shows**  $\text{mat-rep-num } (\text{mat-mod } A) \ i = \text{mat-rep-num } B \ i$   
*<proof>*

**lemma** *mat-block-size-MM-Rel*:  
**assumes** *MM-Rel A B*  
**assumes**  $j < \text{dim-col } A$   
**shows**  $\text{mat-block-size } (\text{mat-mod } A) j = \text{mat-block-size } B j$   
 $\langle \text{proof} \rangle$

**lemma** *mat-inter-num-MM-Rel*:  
**assumes** *MM-Rel A B*  
**assumes**  $j1 < \text{dim-col } A$   $j2 < \text{dim-col } B$   
**shows**  $\text{mat-inter-num } (\text{mat-mod } A) j1 j2 = \text{mat-inter-num } B j1 j2$   
 $\langle \text{proof} \rangle$

Lift 01 and mat mod equivalence on 0-1 matrices

**lemma** *of-int-mod-ring-lift-01-eq*:  
**assumes** *zero-one-matrix N*  
**shows**  $\text{map-mat } (\text{of-int-mod-ring}) N = (\text{lift-01-mat}) N$   
 $\langle \text{proof} \rangle$

**lemma** *to-int-mod-ring-lift-01-eq*:  
**assumes** *zero-one-matrix N*  
**shows**  $\text{to-int-mat } N = (\text{lift-01-mat}) N$   
 $\langle \text{proof} \rangle$

**end**

## 10.2 The Odd-town Problem

The odd-town problem [1] is perhaps one of the most common introductory problems for applying the linear algebra bound method to a combinatorial problem. In mathematical literature, it is considered simpler than Fisher's Inequality, however presents some interesting challenges to formalisation. Most significantly, it considers the incidence matrix to have elements of types integers mod 2.

Initially, define a locale to represent the odd town context (a town with  $v$  people, and  $b$  groups) which must each be of odd size, but have an even intersection number with any other group

**locale** *odd-town* = *ordered-design* +  
**assumes** *odd-groups*:  $bl \in \# \mathcal{B} \implies \text{odd } (\text{card } bl)$   
**and** *even-inters*:  $bl1 \in \# \mathcal{B} \implies bl2 \in \# (\mathcal{B} - \{\#bl1\}) \implies \text{even } (bl1 \cap bl2)$   
**begin**

**lemma** *odd-town-no-repeat-clubs*: *distinct-mset*  $\mathcal{B}$   
 $\langle \text{proof} \rangle$

**lemma** *odd-blocks-mat-block-size*:  $j < \text{dim-col } N \implies \text{odd } (\text{mat-block-size } N j)$   
 $\langle \text{proof} \rangle$

**lemma** *odd-block-size-mod-2*:  
**assumes**  $CARD('b::prime-card) = 2$   
**assumes**  $j < b$   
**shows**  $of\_nat (card (\mathcal{B}s ! j)) = (1 :: 'b \text{ mod-ring})$   
 $\langle proof \rangle$

**lemma** *valid-indices-block-min*:  $j1 < dim\_col N \implies j2 < dim\_col N \implies j1 \neq j2$   
 $\implies b \geq 2$   
 $\langle proof \rangle$

**lemma** *even-inter-mat-intersections*:  $j1 < dim\_col N \implies j2 < dim\_col N \implies j1 \neq j2$   
 $\implies even (mat\_inter\_num N j1 j2)$   
 $\langle proof \rangle$

**lemma** *even-inter-mod-2*:  
**assumes**  $CARD('b::prime-card) = 2$   
**assumes**  $i < b$  **and**  $jlt: j < b$  **and**  $ne: i \neq j$   
**shows**  $of\_nat ((\mathcal{B}s ! i) |\cap| (\mathcal{B}s ! j)) = (0 :: 'b \text{ mod-ring})$   
 $\langle proof \rangle$

**end**

The odd town locale must be simple by definition

**sublocale** *odd-town*  $\subseteq$  *ordered-simple-design*  
 $\langle proof \rangle$

**context** *odd-town*  
**begin**

The upper bound lemma (i.e. variation on Fisher's) for the odd town property using the linear bound argument. Notice it follows exactly the same pattern as the generalised version, however it's sum manipulation argument is significantly simpler (in line with the mathematical proofs)

**lemma** *upper-bound-clubs*:  
**assumes**  $CARD('b::prime-card) = 2$   
**shows**  $b \leq v$   
 $\langle proof \rangle$

**end**

**end**

**theory** *Fishers-Inequality-Root*  
**imports**  
*Set-Multiset-Extras*  
*Matrix-Vector-Extras*  
*Design-Extras*

*Incidence-Matrices*  
*Dual-Systems*  
*Rank-Argument-General*  
*Linear-Bound-Argument*

*Fishers-Inequality*  
*Vector-Matrix-Mod*  
*Fishers-Inequality-Variations*

**begin**

**end**

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