First Welfare Theorem *

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Abstract

Contents

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1 Introducing Syntax

Syntax, abbreviations and type-synonyms

theory *Syntax*
  imports *Main*
begin

type-synonym $'a$ *relation* $= ('a \times 'a)$ *set*

abbreviation *gen-weak-stx* :: $'a \Rightarrow 'a$ *relation* $\Rightarrow 'a \Rightarrow$ *bool*
  (- $\succeq$[-] - [*51*,*100*,*51*] *60*)
  where
    $x \succeq[P]\ y \equiv (x,\ y) \in P$

abbreviation *gen-indif-stx* :: $'a \Rightarrow 'a$ *relation* $\Rightarrow 'a \Rightarrow$ *bool*
  (- $\approx$[-] - [*51*,*100*,*51*] *60*)
  where
    $x \approx[P]\ y \equiv x \succeq[P]\ y \wedge y \succeq[P]\ x$

abbreviation *gen-strc-stx* :: $'a \Rightarrow 'a$ *relation* $\Rightarrow 'a \Rightarrow$ *bool*
(- $\succ$[-] - [*51*,*100*,*51*] *60*)
where
$x \succ[P]\ y \equiv\ x \succeq[P]\ y \wedge \neg y \succeq[P]\ x$

end

2 Arg Min and Arg Max sets

theory *Argmax*
imports
Complex-Main
begin

2.1 Definitions and Lemmas by Julian Parsert

definition of argmax and argmin returing a set.

definition *arg-min-set* :: $('a \Rightarrow 'b::ord) \Rightarrow 'a\ set \Rightarrow 'a\ set$
where
arg-min-set f S = $\{x.$ *is-arg-min f* $(\lambda x.\ x{\in}S)\ x\}$

definition *arg-max-set* :: $('a \Rightarrow 'b::ord) \Rightarrow 'a\ set \Rightarrow 'a\ set$
where
arg-max-set f S = $\{x.$ *is-arg-max f* $(\lambda x.\ x{\in}S)\ x\}$

Useful lemmas for *arg-max-set* and *arg-min-set*.

lemma *no-better-in-s*:
assumes $x \in$ *arg-max-set f S*
shows $\nexists y.\ y \in S \wedge (f\ y) > (f\ x)$
⟨*proof*⟩

lemma *argmax-sol-in-s*:
assumes $x \in$ *arg-max-set f S*
shows $x \in S$
⟨*proof*⟩

lemma *leq-all-in-sol*:
fixes f :: $'a \Rightarrow ('b$ :: *preorder*)
assumes $x \in$ *arg-max-set f S*
shows $\forall y \in S.\ f\ y \geq f\ x \longrightarrow y \in$ *arg-max-set f S*
⟨*proof*⟩

lemma *all-leq*:
fixes f :: $'a \Rightarrow ('b$ :: *linorder*)
assumes $x \in$ *arg-max-set f S*
shows $\forall y \in S.\ f\ x \geq f\ y$
⟨*proof*⟩

lemma *all-in-argmax-equal*:
  fixes f :: $'a \Rightarrow ('b$ :: *linorder*)
  assumes $x \in$ *arg-max-set* f S
  shows $\forall y \in$ *arg-max-set* f S. $f\ x = f\ y$
    ⟨*proof*⟩

end

3 Preference Relations

Preferences modeled as a set of pairs

theory *Preferences*
  imports
    HOL−Analysis.Multivariate-Analysis
    Syntax
begin

3.1 Basic Preference Relation

Basic preference relation locale with carrier and relation modeled as a set of pairs.

locale *preference* =
  fixes *carrier* :: *'a set*
  fixes *relation* :: *'a relation*
  assumes *not-outside*: $(x,y) \in$ *relation* $\Longrightarrow x \in$ *carrier*
    and $(x,y) \in$ *relation* $\Longrightarrow y \in$ *carrier*
  assumes *trans-refl*: *preorder-on carrier relation*

context *preference*
begin

abbreviation *geq* (- $\succeq$ - [*51*,*51*] *60*)
  where
    $x \succeq y \equiv x \succeq$[*relation*] y

abbreviation *str-gr* (- $\succ$ - [*51*,*51*] *60*)
  where
    $x \succ y \equiv x \succeq y \wedge \neg y \succeq x$

abbreviation *indiff* (- $\approx$ - [*51*,*51*] *60*)
  where
    $x \approx y \equiv x \succeq y \wedge y \succeq x$

lemma *reflexivity*: *refl-on carrier relation*
  ⟨*proof*⟩

lemma *transitivity*: *trans relation*
⟨*proof*⟩

lemma *indiff-trans* [*simp*]: $x \approx y \Longrightarrow y \approx z \Longrightarrow x \approx z$
⟨*proof*⟩

end

3.1.1 Contour sets

definition *at-least-as-good* :: $'a \Rightarrow 'a\ set \Rightarrow 'a\ relation \Rightarrow 'a\ set$
where
at-least-as-good $x\ B\ P = \{y \in B.\ y \succeq[P]\ x\ \}$

definition *no-better-than* :: $'a \Rightarrow 'a\ set \Rightarrow 'a\ relation \Rightarrow 'a\ set$
where
no-better-than $x\ B\ P = \{y \in B.\ x \succeq[P]\ y\}$

definition *as-good-as* :: $'a \Rightarrow 'a\ set \Rightarrow 'a\ relation \Rightarrow 'a\ set$
where
as-good-as $x\ B\ P = \{y \in B.\ x \approx[P]\ y\}$

lemma *at-lst-asgd-ge*:
assumes $x \in$ *at-least-as-good* $y\ B\ Pr$
shows $x \succeq[Pr]\ y$
⟨*proof*⟩

lemma *strict-contour-is-diff*:
$\{a \in B.\ a \succ[Pr]\ y\} =$ *at-least-as-good* $y\ B\ Pr\ -$ *as-good-as* $y\ B\ Pr$
⟨*proof*⟩

lemma *strict-countour-def* [*simp*]:
(*at-least-as-good* $y\ B\ Pr$) $-$ *as-good-as* $y\ B\ Pr = \{x \in B.\ x \succ[Pr]\ y\}$
⟨*proof*⟩

lemma *at-least-as-goodD* [*dest*]:
assumes $z \in$ *at-least-as-good* $y\ B\ Pr$
shows $z \succeq[Pr]\ y$
⟨*proof*⟩

3.2 Rational Preference Relation

Rational preferences add totality to the basic preferences.

locale *rational-preference* = *preference* +
assumes *total*: *total-on carrier relation*
begin

lemma *compl*: $\forall\, x \in$ *carrier* . $\forall\, y \in$ *carrier* . $x \succeq y \lor y \succeq x$
⟨*proof*⟩

lemma *strict-not-refl-weak* [*iff*]: $x \in$ *carrier* $\wedge$ $y \in$ *carrier* $\Longrightarrow \neg\ (y \succeq x) \longleftrightarrow x$
$\succ y$
⟨*proof*⟩

lemma *strict-trans* [*simp*]: $x \succ y \Longrightarrow y \succ z \Longrightarrow x \succ z$
⟨*proof*⟩

lemma *completeD* [*dest*]: $x \in$ *carrier* $\Longrightarrow y \in$ *carrier* $\Longrightarrow x \neq y \Longrightarrow x \succeq y \vee y$
$\succeq x$
⟨*proof*⟩

lemma *pref-in-at-least-as*:
assumes $x \succeq y$
shows $x \in$ *at-least-as-good y carrier relation*
⟨*proof*⟩

lemma *worse-in-no-better*:
assumes $x \succeq y$
shows $y \in$ *no-better-than y carrier relation*
⟨*proof*⟩

lemma *strict-is-neg-transitive* :
assumes $x \in$ *carrier* $\wedge$ $y \in$ *carrier* $\wedge$ $z \in$ *carrier*
shows $x \succ y \Longrightarrow x \succ z \vee z \succ y$
⟨*proof*⟩

lemma *weak-is-transitive*:
assumes $x \in$ *carrier* $\wedge$ $y \in$ *carrier* $\wedge$ $z \in$ *carrier*
shows $x \succeq y \Longrightarrow y \succeq z \Longrightarrow x \succeq z$
⟨*proof*⟩

lemma *no-better-than-nonepty*:
assumes *carrier* $\neq \{\}$
shows $\bigwedge x.\ x \in$ *carrier* $\Longrightarrow$ (*no-better-than x carrier relation*) $\neq \{\}$
⟨*proof*⟩

lemma *no-better-subset-pref* :
assumes $x \succeq y$
shows *no-better-than y carrier relation* $\subseteq$ *no-better-than x carrier relation*
⟨*proof*⟩

lemma *no-better-thansubset-rel* :
assumes $x \in$ *carrier* **and** $y \in$ *carrier*
assumes *no-better-than y carrier relation* $\subseteq$ *no-better-than x carrier relation*
shows $x \succeq y$
⟨*proof*⟩

lemma *nbt-nest* :

shows $(\textit{no-better-than } y \textit{ carrier relation} \subseteq \textit{no-better-than } x \textit{ carrier relation}) \vee$
$(\textit{no-better-than } x \textit{ carrier relation} \subseteq \textit{no-better-than } y \textit{ carrier relation})$
⟨*proof*⟩

lemma *at-lst-asgd-not-ge*:
assumes $\textit{carrier} \neq \{\}$
assumes $x \in \textit{carrier}$ **and** $y \in \textit{carrier}$
assumes $x \notin \textit{at-least-as-good } y \textit{ carrier relation}$
shows $\neg\ x \succeq y$
⟨*proof*⟩

lemma *as-good-as-sameIff* [*iff*]:
$z \in \textit{as-good-as } y \textit{ carrier relation} \longleftrightarrow z \succeq y \wedge y \succeq z$
⟨*proof*⟩

lemma *same-at-least-as-equal*:
assumes $z \approx y$
shows $\textit{at-least-as-good } z \textit{ carrier relation} =$
$\textit{at-least-as-good } y \textit{ carrier relation}$ (**is** $?az = ?ay$)
⟨*proof*⟩

lemma *as-good-asIff* [*iff*]:
$x \in \textit{as-good-as } y \textit{ carrier relation} \longleftrightarrow x \approx[\textit{relation}]\ y$
⟨*proof*⟩

lemma *nbt-subset*:
assumes *finite carrier*
assumes $x \in \textit{carrier}$ **and** $y \in \textit{carrier}$
shows $\textit{no-better-than } x \textit{ carrier relation} \subseteq \textit{no-better-than } x \textit{ carrier relation} \vee$
$\textit{no-better-than } x \textit{ carrier relation} \subseteq \textit{no-better-than } x \textit{ carrier relation}$
⟨*proof*⟩

lemma *fnt-carrier-fnt-rel*: *finite carrier* $\Longrightarrow$ *finite relation*
⟨*proof*⟩

lemma *nbt-subset-carrier*:
assumes $x \in \textit{carrier}$
shows $\textit{no-better-than } x \textit{ carrier relation} \subseteq \textit{carrier}$
⟨*proof*⟩

lemma *xy-in-eachothers-nbt*:
assumes $x \in \textit{carrier}\ y \in \textit{carrier}$
shows $x \in \textit{no-better-than } y \textit{ carrier relation} \vee$
$y \in \textit{no-better-than } x \textit{ carrier relation}$
⟨*proof*⟩

lemma *same-nbt-same-pref*:
assumes $x \in \textit{carrier}\ y \in \textit{carrier}$
shows $x \in \textit{no-better-than } y \textit{ carrier relation} \wedge$

$y \in$ *no-better-than x carrier relation* $\longleftrightarrow x \approx y$
⟨*proof*⟩

lemma *indifferent-imp-weak-pref*:
assumes $x \approx y$
shows $x \succeq y$ $y \succeq x$
⟨*proof*⟩

3.3 Finite carrier

context
assumes *finite carrier*
begin

lemma *fnt-carrier-fnt-nbt*:
shows $\forall x{\in}carrier.$ *finite* (*no-better-than x carrier relation*)
⟨*proof*⟩

lemma *nbt-subset-imp-card-leq*:
assumes $x \in$ *carrier* **and** $y \in$ *carrier*
shows *no-better-than x carrier relation* $\subseteq$ *no-better-than y carrier relation* $\longleftrightarrow$
card (*no-better-than x carrier relation*) $\leq$ *card* (*no-better-than y carrier relation*)
(**is** *?nbt* $\longleftrightarrow$ *?card*)
⟨*proof*⟩

lemma *card-leq-pref*:
assumes $x \in$ *carrier* **and** $y \in$ *carrier*
shows *card* (*no-better-than x carrier relation*) $\leq$ *card* (*no-better-than y carrier relation*)
$\longleftrightarrow y \succeq x$
⟨*proof*⟩

lemma *finite-ne-remove-induct*:
assumes *finite* B $B \neq \{\}$
$\bigwedge A.$ *finite* $A \implies A \subseteq B \implies A \neq \{\} \implies$
$(\bigwedge x.\ x \in A \implies A - \{x\} \neq \{\} \implies P\ (A - \{x\})) \implies P\ A$
shows $P\ B$
⟨*proof*⟩

lemma *finite-nempty-preorder-has-max*:
assumes *finite* B $B \neq \{\}$ *refl-on* B R *trans* R *total-on* B R
shows $\exists x \in B.\ \forall y \in B.\ (x, y) \in R$
⟨*proof*⟩

lemma *finite-nempty-preorder-has-min*:
assumes *finite* B $B \neq \{\}$ *refl-on* B R *trans* R *total-on* B R
shows $\exists x \in B.\ \forall y \in B.\ (y, x) \in R$
⟨*proof*⟩

lemma *finite-nonempty-carrier-has-maximum*:
  assumes *carrier* $\neq$ {}
  shows $\exists\, e \in$ *carrier*. $\forall\, m \in$ *carrier*. $e \succeq$[*relation*] m
  ⟨*proof*⟩

lemma *finite-nonempty-carrier-has-minimum*:
  assumes *carrier* $\neq$ {}
  shows $\exists\, e \in$ *carrier*. $\forall\, m \in$ *carrier*. $m \succeq$[*relation*] e
  ⟨*proof*⟩

end

lemma *all-carrier-ex-sub-rel*:
  $\forall\, c \subseteq$ *carrier*. $\exists\, r \subseteq$ *relation*. *rational-preference* c r
⟨*proof*⟩

end

3.4 Local Non-Satiation

Defining local non-satiation.

definition *local-nonsatiation*
  where
  local-nonsatiation B $P \longleftrightarrow$
   ($\forall\, x{\in}B$. $\forall\, e{>}0$. $\exists\, y{\in}B$. *norm* $(y - x) \leq e \wedge y \succ$[$P$] x)

Alternate definitions and intro/dest rules with them

lemma *lns-alt-def1* [*iff*] :
  shows *local-nonsatiation* B $P \longleftrightarrow$ ($\forall\, x \in B$. $\forall\, e{>}0$. ($\exists\, y \in B$. *dist* y $x \leq e \wedge y$ $\succ$[P] x))
  ⟨*proof*⟩

lemma *lns-normI* [*intro*]:
  assumes $\bigwedge x\ e.\ x \in B \Longrightarrow e > 0 \Longrightarrow$ ($\exists\, y{\in}B$. *norm* $(y - x) \leq e \wedge y \succ$[$P$] x)
  shows *local-nonsatiation* B P
  ⟨*proof*⟩

lemma *lns-distI* [*intro*]:
  assumes $\bigwedge x\ e.\ x \in B \Longrightarrow e > 0 \Longrightarrow$ ($\exists\, y{\in}B$. (*dist* y x) $\leq e \wedge y \succ$[P] x)
  shows *local-nonsatiation* B P
  ⟨*proof*⟩

lemma *lns-alt-def2* [*iff*]:
  local-nonsatiation B $P \longleftrightarrow$ ($\forall\, x{\in}B$. $\forall\, e{>}0$. ($\exists\, y$. $y \in$ (*ball* x e) $\wedge y \in B \wedge y$ $\succ$[P] x))
⟨*proof*⟩

lemma *lns-normD* [*dest*]:
assumes *local-nonsatiation B P*
shows $\forall x \in B.\ \forall e{>}0.\ \exists y \in B.\ (norm\ (y - x) \leq e \wedge y \succ[P]\ x)$
⟨*proof*⟩

3.5 Convex preferences

definition *weak-convex-pref* :: ('*a*::*real-vector*) *relation* ⇒ *bool*
where
$weak\text{-}convex\text{-}pref\ Pr \longleftrightarrow (\forall x\ y.\ x \succeq[Pr]\ y \longrightarrow$
$(\forall \alpha\ \beta.\ \alpha + \beta = 1 \wedge \alpha > 0 \wedge \beta > 0 \longrightarrow \alpha *_R x + \beta *_R y \succeq[Pr]\ y))$

definition *convex-pref* :: ('*a*::*real-vector*) *relation* ⇒ *bool*
where
$convex\text{-}pref\ Pr \longleftrightarrow (\forall x\ y.\ x \succ[Pr]\ y \longrightarrow$
$(\forall \alpha.\ 1 > \alpha \wedge \alpha > 0 \longrightarrow \alpha *_R x + (1{-}\alpha) *_R y \succ[Pr]\ y))$

definition *strict-convex-pref* :: ('*a*::*real-vector*) *relation* ⇒ *bool*
where
$strict\text{-}convex\text{-}pref\ Pr \longleftrightarrow (\forall x\ y.\ x \succeq[Pr]\ y \wedge x \neq y \longrightarrow$
$(\forall \alpha.\ 1 > \alpha \wedge \alpha > 0 \longrightarrow \alpha *_R x + (1{-}\alpha) *_R y \succ[Pr]\ y))$

lemma *convex-ge-imp-conved*:
assumes $\forall x\ y.\ x \succeq[Pr]\ y \longrightarrow (\forall \alpha\ \beta.\ \alpha + \beta = 1 \wedge \alpha \geq 0 \wedge \beta \geq 0 \longrightarrow \alpha *_R x + \beta *_R y \succeq[Pr]\ y)$
shows *weak-convex-pref Pr*
⟨*proof*⟩

lemma *weak-convexI* [*intro*]:
assumes $\bigwedge x\ y\ \alpha\ \beta.\ x \succeq[Pr]\ y \Longrightarrow \alpha + \beta = 1 \Longrightarrow 0 < \alpha \Longrightarrow 0 < \beta \Longrightarrow \alpha *_R x + \beta *_R y \succeq[Pr]\ y$
shows *weak-convex-pref Pr*
⟨*proof*⟩

lemma *weak-convexD* [*dest*]:
assumes *weak-convex-pref Pr* **and** $x \succeq[Pr]\ y$ **and** $0 < u$ **and** $0 < v$ **and** $u + v = 1$
shows $u *_R x + v *_R y \succeq[Pr]\ y$
⟨*proof*⟩

3.6 Real Vector Preferences

Preference relations on real vector type class.

locale *real-vector-rpr* = *rational-preference carrier relation*
for *carrier* :: '*a*::*real-vector set*
and *relation* :: '*a relation*

sublocale *real-vector-rpr* ⊆ *rational-preference carrier relation*
⟨*proof*⟩

context *real-vector-rpr*
begin

lemma *have-rpr*: *rational-preference carrier relation*
⟨*proof*⟩

Multiple convexity alternate definitions intro/dest rules.

lemma *weak-convex1D* [*dest*]:
assumes *weak-convex-pref relation* **and** $x \succeq[relation]\ y$ **and** $0 \leq u$ **and** $0 \leq v$ **and** $u + v = 1$
shows $u *_R x + v *_R y \succeq[relation]\ y$
⟨*proof*⟩

lemma *weak-convex1I* [*intro*] :
assumes $\forall x.$ *convex* (*at-least-as-good x carrier relation*)
shows *weak-convex-pref relation*
⟨*proof*⟩

Definition of convexity in "Handbook of Social Choice and Welfare"[1].

lemma *convex-def-alt*:
assumes *rational-preference carrier relation*
assumes *weak-convex-pref relation*
shows ($\forall x \in$ *carrier*. *convex* (*at-least-as-good x carrier relation*))
⟨*proof*⟩

lemma *convex-imp-convex-str-upper-cnt*:
assumes $\forall x \in$ *carrier*. *convex* (*at-least-as-good x carrier relation*)
shows *convex* (*at-least-as-good x carrier relation* − *as-good-as x carrier relation*)
(**is** *convex* (*?a* − *?b*))
⟨*proof*⟩

end

3.6.1 Monotone preferences

definition *weak-monotone-prefs* :: *'a set* ⇒ (*'a*::*ord*) *relation* ⇒ *bool*
where
weak-monotone-prefs B P ⟷ ($\forall x \in B.\ \forall y \in B.\ x \geq y \longrightarrow x \succeq[P]y$)

definition *monotone-preference* :: *'a set* ⇒ (*'a*::*ord*) *relation* ⇒ *bool*
where
monotone-preference B P ⟷ ($\forall x \in B.\ \forall y \in B.\ x > y \longrightarrow x \succ[P]\ y$)

Given a carrier set that is unbounded above (not the "standard" mathematical definition), monotonicity implies local non-satiation.

lemma *unbounded-above-mono-imp-lns*:
assumes $\forall M \in$ *carrier*. ($\forall x > M.\ x \in$ *carrier*)

assumes *mono*: *monotone-preference* (*carrier*:: $'a$::*ordered-euclidean-space set*) *relation*
shows *local-nonsatiation carrier relation*
⟨*proof*⟩

end

4 Utility Functions

Utility functions and results involving them.

theory *Utility-Functions*
imports
Preferences
begin

4.1 Ordinal utility functions

Ordinal utility function locale

locale *ordinal-utility* =
fixes *carrier* :: $'a$ *set*
fixes *relation* :: $'a$ *relation*
fixes *u* :: $'a \Rightarrow$ *real*
assumes *util-def*[*iff*]: $x \in carrier \Longrightarrow y \in carrier \Longrightarrow x \succeq[relation]\ y \longleftrightarrow u\ x \geq u\ y$
assumes *not-outside*: $x \succeq[relation]\ y \Longrightarrow x \in carrier$
and $x \succeq[relation]\ y \Longrightarrow y \in carrier$
begin

lemma *util-def-conf*: $x \in carrier \Longrightarrow y \in carrier \Longrightarrow u\ x \geq u\ y \longleftrightarrow x \succeq[relation]\ y$
⟨*proof*⟩

lemma *relation-subset-crossp*:
$relation \subseteq carrier \times carrier$
⟨*proof*⟩

Utility function implies totality of relation

lemma *util-imp-total*: *total-on carrier relation*
⟨*proof*⟩

lemma *x-y-in-carrier*: $x \succeq[relation]\ y \Longrightarrow x \in carrier \wedge y \in carrier$
⟨*proof*⟩

Utility function implies transitivity of relation.

lemma *util-imp-trans*: *trans relation*
⟨*proof*⟩

lemma *util-imp-refl*: *refl-on carrier relation*
⟨*proof*⟩

lemma *affine-trans-is-u*:
shows $\forall\,\alpha>0.\ (\forall\,\beta.\ \textit{ordinal-utility carrier relation}\ (\lambda x.\ u(x)*\alpha + \beta))$
⟨*proof*⟩

This utility function definition is ordinal. Hence they are only unique up to a monotone transformation.

lemma *ordinality-of-utility-function* :
fixes f :: *real* ⇒ *real*
assumes *monot*: *monotone* (>) (>) f
shows $(f \circ u)\ x > (f \circ u)\ y \longleftrightarrow u\ x > u\ y$
⟨*proof*⟩

corollary *utility-prefs-corresp* :
fixes f :: *real* ⇒ *real*
assumes *monotonicity* : *monotone* (>) (>) f
shows $\forall\, x{\in}\textit{carrier}.\ \forall\, y{\in}\textit{carrier}.\ (x,y) \in \textit{relation} \longleftrightarrow (f \circ u)\ x \geq (f \circ u)\ y$
⟨*proof*⟩

corollary *monotone-comp-is-utility*:
fixes f :: *real* ⇒ *real*
assumes *monot*: *monotone* (>) (>) f
shows *ordinal-utility carrier relation* $(f \circ u)$
⟨*proof*⟩

lemma *ordinal-utility-left*:
assumes $x \succeq[\textit{relation}]\ y$
shows $u\ x \geq u\ y$
⟨*proof*⟩

lemma *add-right*:
assumes $\bigwedge x\ y.\ x \succeq[\textit{relation}]\ y \Longrightarrow f\,x \geq f\,y$
shows *ordinal-utility carrier relation* $(\lambda x.\ u\ x + f\,x)$
⟨*proof*⟩

lemma *add-left*:
assumes $\bigwedge x\ y.\ x \succeq[\textit{relation}]\ y \Longrightarrow f\,x \geq f\,y$
shows *ordinal-utility carrier relation* $(\lambda x.\ f\,x + u\ x)$
⟨*proof*⟩

lemma *ordinal-utility-scale-transl*:
assumes $(c{::}\textit{real}) > 0$
shows *ordinal-utility carrier relation* $(\lambda x.\ c * (u\ x) + d)$
⟨*proof*⟩

lemma *strict-prefernce-iff-strict-utility*:

assumes $x \in carrier$
assumes $y \in carrier$
shows $x \succ[relation]\ y \longleftrightarrow u\ x > u\ y$
⟨*proof*⟩

end

A utility function implies a rational preference relation. Hence a utility function contains exactly the same amount of information as a RPR

sublocale *ordinal-utility* ⊆ *rational-preference carrier relation*
⟨*proof*⟩

Given a finite carrier set. We can guarantee that given a rational preference relation, there must also exist a utility function representing this relation. Construction of witness roughly follows from.

theorem *fnt-carrier-exists-util-fun*:
assumes *finite carrier*
assumes *rational-preference carrier relation*
shows $\exists\, u.$ *ordinal-utility carrier relation u*
⟨*proof*⟩

corollary *obt-u-fnt-carrier*:
assumes *finite carrier*
assumes *rational-preference carrier relation*
obtains *u* **where** *ordinal-utility carrier relation u*
⟨*proof*⟩

theorem *ordinal-util-imp-rat-prefs*:
assumes *ordinal-utility carrier relation u*
shows *rational-preference carrier relation*
⟨*proof*⟩

4.2 Utility function on Euclidean Space

locale *eucl-ordinal-utility* = *ordinal-utility carrier relation u*
for *carrier* :: (*'a*::*euclidean-space*) *set*
and *relation* :: *'a relation*
and *u* :: *'a* ⇒ *real*

sublocale *eucl-ordinal-utility* ⊆ *rational-preference carrier relation*
⟨*proof*⟩

lemma *ord-eucl-utility-imp-rpr*: *eucl-ordinal-utility s rel u* ⟶ *real-vector-rpr s rel*
⟨*proof*⟩

context *eucl-ordinal-utility*
begin

Local non-satiation on utility functions

lemma *lns-pref-lns-util* [*iff*]:
  local-nonsatiation carrier relation $\longleftrightarrow$
  $(\forall\, x\in carrier.\ \forall\, e > 0.\ \exists\, y\in carrier.$
  $norm\ (y - x) \leq e \land u\ y > u\ x)$ (**is** - $\longleftrightarrow$ *?alt*)
⟨*proof*⟩

end

lemma *finite-carrier-rpr-iff-u*:
  assumes *finite carrier*
    and $(relation{::}'a\ relation) \subseteq carrier \times carrier$
  shows *rational-preference carrier relation* $\longleftrightarrow$ $(\exists\, u.\ ordinal\text{-}utility\ carrier\ relation\ u)$
⟨*proof*⟩

end

5 Consumers

Consumption sets

theory *Consumers*
  imports
    HOL−Analysis.Multivariate-Analysis
    ../Syntax
begin

5.1 Pre Arrow-Debreu consumption set

It turns out that the First Welfare Theorem does not require any particular limitations on the consumption set

locale *pre-arrow-debreu-consumption-set* =
  fixes *consumption-set* :: $('a{::}euclidean\text{-}space)\ set$
  assumes $x \in (UNIV{::}\ 'a\ set) \Longrightarrow x \in consumption\text{-}set$
begin
end

5.2 Arrow-Debreu model consumption set

The Arrow-Debreu model consumption set includes more and stricter assumptions which are necessary for further results.

locale *gen-pre-arrow-debreu-consum-set* =
  fixes *consumption-set* :: $('a{::}ordered\text{-}euclidean\text{-}space)\ set$
begin

end

locale *arrow-debreu-consum-set* =
  fixes *consumption-set* :: ('*a*::*ordered-euclidean-space*) *set*
  assumes *r-plus*: *consumption-set* $\subseteq$ {(*x*::'*a*). $x \geq 0$}
  assumes *closed*: *closed consumption-set*
  assumes *convex*: *convex consumption-set*
  assumes *non-empty*: *consumption-set* $\neq$ {}
  assumes $\forall$ *M* $\in$ *consumption-set*. ($\forall$ *x* > *M*. *x* $\in$ *consumption-set*)
begin

lemma *x-larger-0*: *x* $\in$ *consumption-set* $\Longrightarrow$ $x \geq 0$
  ⟨*proof*⟩

lemma *larger-in-consump-set*:
  x $\in$ *consumption-set* $\wedge$ $y \geq x$ $\Longrightarrow$ *y* $\in$ *consumption-set*
  ⟨*proof*⟩

end

end

theory *Common*
  imports
    ../*Preferences*
    ../*Utility-Functions*
    ../*Argmax*
begin

6 Pareto Ordering

Allows us to define a Pareto Ordering.

locale *pareto-ordering* =
  fixes *agents* :: '*i set*
  fixes *U* :: '*i* $\Rightarrow$ '*a* $\Rightarrow$ *real*
begin
notation *U* (*U*[-])

definition *pareto-dominating* (**infix** $\succ$*Pareto* 60)
  where
    X $\succ$*Pareto* *Y* $\longleftrightarrow$
      ($\forall$ *i* $\in$ *agents*. *U*[*i*] (*X i*) $\geq$ *U*[*i*] (*Y i*)) $\wedge$
      ($\exists$ *i* $\in$ *agents*. *U*[*i*] (*X i*) > *U*[*i*] (*Y i*))

lemma *trans-strict-pareto*: *X* $\succ$*Pareto* *Y* $\Longrightarrow$ *Y* $\succ$*Pareto* *Z* $\Longrightarrow$ *X* $\succ$*Pareto* *Z*

$\langle proof \rangle$

lemma *anti-sym-strict-pareto*: $X \succ Pareto\ Y \Longrightarrow \neg Y \succ Pareto\ X$
$\langle proof \rangle$

end

6.1 Budget constraint

Definition returns all afforedable bundles given wealth W

f is a function that computes the value given a bundle

definition *budget-constraint*
where
$budget\text{-}constraint\ f\ S\ W = \{x \in S.\ f\ x \leq W\}$

6.2 Feasiblity

definition *feasible-private-ownership*
where
$feasible\text{-}private\text{-}ownership\ A\ F\ \mathcal{E}\ Cs\ Ps\ X\ Y \longleftrightarrow$
$(\sum i{\in}A.\ X\ i) \leq (\sum i{\in}A.\ \mathcal{E}\ i) + (\sum j{\in}F.\ Y\ j) \wedge$
$(\forall i{\in}A.\ X\ i \in Cs) \wedge (\forall j{\in}F.\ Y\ j \in Ps\ j)$

lemma *feasible-private-ownershipD*:
assumes $feasible\text{-}private\text{-}ownership\ A\ F\ \mathcal{E}\ Cs\ Ps\ X\ Y$
shows $(\sum i{\in}A.\ X\ i) \leq (\sum i{\in}A.\ \mathcal{E}\ i) + (\sum j{\in}F.\ Y\ j)$
and $(\forall i{\in}A.\ X\ i \in Cs)$ **and** $(\forall j{\in}F.\ Y\ j \in Ps\ j)$
$\langle proof \rangle$

end

theory *Exchange-Economy*
imports
../Preferences
../Utility-Functions
../Argmax
Consumers
Common
begin

7 Exchange Economy

Define the exchange economy model

locale *exchange-economy* =

fixes *consumption-set* :: (*'a*::*ordered-euclidean-space*) *set*
fixes *agents* :: *'i set*
fixes $\mathcal{E}$ :: *'i* ⇒ *'a*
fixes *Pref* :: *'i* ⇒ *'a relation*
fixes *U* :: *'i* ⇒ *'a* ⇒ *real*
assumes *cons-set-props*: *pre-arrow-debreu-consumption-set consumption-set*
assumes *agent-props*: *i* ∈ *agents* ⟹ *eucl-ordinal-utility consumption-set* (*Pref* *i*) (*U i*)
assumes *finite-agents*: *finite agents* **and** *agents* ≠ {}

sublocale *exchange-economy* ⊆ *pareto-ordering agents U*
⟨*proof*⟩

context *exchange-economy*
begin

context
begin

notation *U* (*U*[-])
notation *Pref* (*Pr*[-])
notation $\mathcal{E}$ ($\mathcal{E}$[-])

lemma *base-pref-is-ord-eucl-rpr*: *i* ∈ *agents* ⟹ *rational-preference consumption-set* *Pr*[*i*]
⟨*proof*⟩ **abbreviation** *calculate-value*
where
calculate-value P x ≡ *P* · *x*

7.1 Feasibility

definition *feasible-allocation*
where
feasible-allocation A E ⟷
(∑ *i*∈*agents*. *A i*) ≤ (∑ *i*∈*agents*. *E i*)

7.2 Pareto optimality

definition *pareto-optimal-endow*
where
pareto-optimal-endow X E ⟷
(*feasible-allocation X E* ∧
(∄ *X'*. *feasible-allocation X' E* ∧ *X'* ≻*Pareto X*))

7.3 Competitive Equilibrium in Exchange Economy

Competitive Equilibirum or Walrasian Equilibrium definition.

definition *comp-equilib-endow*
where

comp-equilib-endow P X E $\equiv$
feasible-allocation X E $\wedge$
$(\forall\, i \in$ *agents*. $X\ i \in$ *arg-max-set* $U[i]$
(*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot E\ i)))$

7.4 Lemmas for final result

lemma *utility-function-def* [*iff*]:
assumes $i \in$ *agents*
shows $U[i]\ x \geq U[i]\ y \longleftrightarrow x \succeq[Pr[i]]\ y$
⟨*proof*⟩

lemma *budget-constraint-is-feasible*:
assumes $i \in$ *agents*
assumes $X \in$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i]))$
shows $P \cdot X \leq P \cdot \mathcal{E}[i]$
⟨*proof*⟩

lemma *arg-max-set-therefore-no-better* :
assumes $i \in$ *agents*
assumes $x \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i]))$
shows $U[i]\ y > U[i]\ x \longrightarrow y \notin$ *budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i])$
⟨*proof*⟩

Since we need no restriction on the consumption set for the First Welfare Theorem

lemma *consumption-set-member*: $\forall\, x.\ x \in$ *consumption-set*
⟨*proof*⟩

Under the assumption of Local non-satiation, agents will utilise their entire budget.

lemma *argmax-entire-budget* :
assumes $i \in$ *agents*
assumes *local-nonsatiation consumption-set* $Pr[i]$
assumes $X \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i]))$
shows $P \cdot X = P \cdot \mathcal{E}[i]$
⟨*proof*⟩

All bundles that would be strictly preferred to any argmax result, are more expensive.

lemma *pref-more-expensive*:
assumes $i \in$ *agents*
assumes $x \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i]))$
assumes $U[i]\ y > U[i]\ x$

shows $y \cdot P > P \cdot \mathcal{E}[i]$
⟨*proof*⟩

Greater or equal utility implies greater or equal price.

lemma *same-util-is-equal-or-more-expensive*:
assumes $i \in$ *agents*
assumes *local-nonsatiation consumption-set Pr*$[i]$
assumes $x \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i])$)
assumes $U[i]\ y \geq U[i]\ x$
shows $y \cdot P \geq P \cdot \mathcal{E}[i]$
⟨*proof*⟩

lemma *all-in-argmax-same-price*:
assumes $i \in$ *agents*
assumes *local-nonsatiation consumption-set Pr*$[i]$
assumes $x \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i])$)
and $y \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* $(P \cdot \mathcal{E}[i])$)
shows $P \cdot x = P \cdot y$
⟨*proof*⟩

All rationally acting agents (which is every agent by assumption) will not decrease his utility

lemma *individual-rationalism* :
assumes *comp-equilib-endow* $P\ X\ \mathcal{E}$
shows $\forall i \in$ *agents*. $X\ i \succeq$[*Pref* i] $\mathcal{E}[i]$
⟨*proof*⟩

lemma *walras-law-per-agent* :
assumes $\bigwedge i.\ i \in$ *agents* $\Longrightarrow$ *local-nonsatiation consumption-set Pr*$[i]$
assumes *comp-equilib-endow* $P\ X\ \mathcal{E}$
shows $\forall i \in$ *agents*. $P \cdot X\ i = P \cdot \mathcal{E}[i]$
⟨*proof*⟩

Walras Law holds in our Exchange Economy model. It states that in an equilibrium, demand equals supply

lemma *walras-law*:
assumes $\bigwedge i.\ i \in$ *agents* $\Longrightarrow$ *local-nonsatiation consumption-set Pr*$[i]$
assumes *comp-equilib-endow* $P\ X\ \mathcal{E}$
shows $(\sum i{\in}\textit{agents}.\ P \cdot (X\ i)) - (\sum i{\in}\textit{agents}.\ P \cdot \mathcal{E}[i]) = 0$
⟨*proof*⟩

lemma *inner-with-ge-0*: $(P::(\textit{real}, \textit{'n::finite})\ \textit{vec}) > 0 \Longrightarrow A \geq B \Longrightarrow P \cdot A \geq P \cdot B$
⟨*proof*⟩

7.5 First Welfare Theorem in Exchange Economy

We prove the first welfare theorem in our Exchange Economy model.

theorem *first-welfare-theorem-exchange*:
  assumes *lns* : $\bigwedge i.\ i \in agents \Longrightarrow$ *local-nonsatiation consumption-set Pr*$[i]$
    and *price-cond*: *Price* $>$ *0*
  assumes *equilibrium* : *comp-equilib-endow Price* $\mathcal{X}$ $\mathcal{E}$
  shows *pareto-optimal-endow* $\mathcal{X}$ $\mathcal{E}$
$\langle proof \rangle$

Monotone preferences can be used instead of local non-satiation. Many textbooks etc. do not introduce the concept of local non-satiation and use monotonicity instead.

corollary *first-welfare-exch-thm-monot*:
  assumes $\forall M \in carrier.\ (\forall x > M.\ x \in carrier)$
  assumes $\bigwedge i.\ i \in agents \Longrightarrow$ *monotone-preference consumption-set Pr*$[i]$
  and *price-cond*: *Price* $>$ *0*
  assumes *comp-equilib-endow Price* $\mathcal{X}$ $\mathcal{E}$
  shows *pareto-optimal-endow* $\mathcal{X}$ $\mathcal{E}$
  $\langle proof \rangle$

end

end

end

8 Pre Arrow-Debreu model

Model similar to Arrow-Debreu model but with fewer assumptions, since we only need assumptions strong enough to proof the First Welfare Theorem.

theory *Private-Ownership-Economy*
  imports
    ../Preferences
    ../Preferences
    ../Utility-Functions
    ../Argmax
    Consumers
    Common
begin

locale *pre-arrow-debreu-model* =
  fixes *production-sets* :: *'f* $\Rightarrow$ (*'a*::*ordered-euclidean-space*) *set*
  fixes *consumption-set* :: *'a set*
  fixes *agents* :: *'i set*
  fixes *firms* :: *'f set*

fixes $\mathcal{E}$ :: $'i \Rightarrow 'a$ ($\mathcal{E}$[-])
fixes *Pref* :: $'i \Rightarrow 'a$ *relation* (*Pr*[-])
fixes *U* :: $'i \Rightarrow 'a \Rightarrow$ *real* (*U*[-])
fixes Θ :: $'i \Rightarrow 'f \Rightarrow$ *real* (Θ[-,-])
assumes *cons-set-props*: *pre-arrow-debreu-consumption-set consumption-set*
assumes *agent-props*: $i \in$ *agents* $\Longrightarrow$ *eucl-ordinal-utility consumption-set* (*Pr*[*i*]) (*U*[*i*])
assumes *firms-comp-owned*: $j \in$ *firms* $\Longrightarrow$ ($\sum$ *i*$\in$*agents*. Θ[*i*,*j*]) = *1*
assumes *finite-nonepty-agents*: *finite agents* **and** *agents* $\neq$ {}

sublocale *pre-arrow-debreu-model* $\subseteq$ *pareto-ordering agents U*
⟨*proof*⟩

context *pre-arrow-debreu-model*
begin

No restrictions on consumption set needed

lemma *all-larger-zero-in-csset*: $\forall x.\ x \in$ *consumption-set*
⟨*proof*⟩

context
begin

Calculate wealth of individual i in context of Private Ownership economy.

private abbreviation *poe-wealth*
where
poe-wealth P i Y $\equiv P \cdot \mathcal{E}[i] + (\sum j{\in}\mathit{firms}.\ \Theta[i,j] *_R (P \cdot Y\ j))$

8.1 Feasiblity

private abbreviation *feasible*
where
feasible X Y $\equiv$ *feasible-private-ownership agents firms* $\mathcal{E}$ *consumption-set production-sets X Y*

private abbreviation *calculate-value*
where
calculate-value P x $\equiv P \cdot x$

8.2 Profit maximisation

In a production economy we need to specify profit maximisation.

definition *profit-maximisation*
where
profit-maximisation P S = *arg-max-set* ($\lambda x.\ P \cdot x$) *S*

8.3 Competitive Equilibrium

Competitive equilibrium in context of production economy with private ownership. This includes the profit maximisation condition.

definition *competitive-equilibrium*
  where
    competitive-equilibrium P X Y $\longleftrightarrow$ *feasible* X Y $\wedge$
    $(\forall j \in$ *firms*. $(Y\, j) \in$ *profit-maximisation* P (*production-sets* j)) $\wedge$
    $(\forall i \in$ *agents*. $(X\ i) \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* P i Y)))

lemma *competitive-equilibriumD* [*dest*]:
  assumes *competitive-equilibrium* P X Y
  shows *feasible* X Y $\wedge$
    $(\forall j \in$ *firms*. $(Y\, j) \in$ *profit-maximisation* P (*production-sets* j)) $\wedge$
    $(\forall i \in$ *agents*. $(X\ i) \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* P i Y)))
  ⟨*proof*⟩

lemma *compet-max-profit*:
  assumes $j \in$ *firms*
  assumes *competitive-equilibrium* P X Y
  shows $Y\, j \in$ *profit-maximisation* P (*production-sets* j)
  ⟨*proof*⟩

8.4 Pareto Optimality

definition *pareto-optimal*
  where
    pareto-optimal X Y $\longleftrightarrow$
      (*feasible* X Y $\wedge$
      $(\nexists X'\ Y'.$ *feasible* $X'\ Y' \wedge X' \succ$*Pareto* X))

lemma *pareto-optimalI* [*intro*]:
  assumes *feasible* X Y
    and $\nexists X'\ Y'.$ *feasible* $X'\ Y' \wedge X' \succ$*Pareto* X
  shows *pareto-optimal* X Y
  ⟨*proof*⟩

lemma *pareto-optimalD* [*dest*]:
  assumes *pareto-optimal* X Y
  shows *feasible* X Y **and** $\nexists X'\ Y'.$ *feasible* $X'\ Y' \wedge X' \succ$*Pareto* X
  ⟨*proof*⟩

lemma *util-fun-def-holds*: $i \in$ *agents* $\Longrightarrow x \succeq[Pr[i]]\ y \longleftrightarrow U[i]\ x \geq U[i]\ y$
  ⟨*proof*⟩

lemma *base-pref-is-ord-eucl-rpr*: $i \in$ *agents* $\Longrightarrow$ *rational-preference consumption-set* $Pr[i]$

⟨*proof*⟩

lemma *prof-max-ge-all-in-pset*:
assumes $j \in$ *firms*
assumes $Y\ j \in$ *profit-maximisation* P (*production-sets* j)
shows $\forall\, y \in$ *production-sets* j. $P \cdot Y\ j \geq P \cdot y$
⟨*proof*⟩

8.5 Lemmas for final result

Strictly preferred bundles are strictly more expensive.

lemma *all-prefered-are-more-expensive*:
assumes *i-agt*: $i \in$ *agents*
assumes *equil*: *competitive-equilibrium* $P\ \mathcal{X}\ \mathcal{Y}$
assumes $z \in$ *consumption-set*
assumes $(U\ i)\ z > (U\ i)\ (\mathcal{X}\ i)$
shows $z \cdot P > P \cdot (\mathcal{X}\ i)$
⟨*proof*⟩

Given local non-satiation, argmax will use the entire budget.

lemma *am-utilises-entire-bgt*:
assumes *i-agts*: $i \in$ *agents*
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes *argmax-sol* : $X \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$))
shows $P \cdot X = P \cdot \mathcal{E}[i] + (\sum j{\in}\mathit{firms}.\ \Theta[i,j] *_R (P \cdot Y\ j))$
⟨*proof*⟩

corollary *x-equil-x-ext-budget*:
assumes *i-agt*: $i \in$ *agents*
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes *equilibrium* : *competitive-equilibrium* $P\ X\ Y$
shows $P \cdot X\ i = P \cdot \mathcal{E}[i] + (\sum j{\in}\mathit{firms}.\ \Theta[i,j] *_R (P \cdot Y\ j))$
⟨*proof*⟩

lemma *same-price-in-argmax* :
assumes *i-agt*: $i \in$ *agents*
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes $x \in$ *arg-max-set* $(U[i])$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$))
assumes $y \in$ *arg-max-set* $(U[i])$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$))
shows $(P \cdot x) = (P \cdot y)$
⟨*proof*⟩

Greater or equal utility implies greater or equal value.

lemma *utility-ge-price-ge* :
assumes *agts*: $i \in$ *agents*

assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes *equil*: *competitive-equilibrium* $P\ X\ Y$
assumes *geq*: $U[i]\ z \geq U[i]\ (X\ i)$
and $z \in$ *consumption-set*
shows $P \cdot z \geq P \cdot (X\ i)$
⟨*proof*⟩

lemma *commutativity-sums-over-funs*:
fixes $X :: 'x\ set$
fixes $Y :: 'y\ set$
shows $(\sum i{\in}X.\ \sum j{\in}Y.\ (f\ i\ j *_R C \cdot g\ j)) = (\sum j{\in}Y. \sum i{\in}X.\ (f\ i\ j *_R C \cdot g\ j))$
⟨*proof*⟩

lemma *assoc-fun-over-sum*:
fixes $X :: 'x\ set$
fixes $Y :: 'y\ set$
shows $(\sum j{\in}Y.\ \sum i{\in}X.\ f\ i\ j *_R C \cdot g\ j) = (\sum j{\in}Y.\ (\sum i{\in}X.\ f\ i\ j) *_R C \cdot g\ j)$
⟨*proof*⟩

Walras' law in context of production economy with private ownership. That is, in an equilibrium demand equals supply.

lemma *walras-law*:
assumes $\bigwedge i.\ i{\in}agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
assumes $(\forall i \in agents.\ (X\ i) \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$)))
shows $P \cdot (\sum i{\in}agents.\ (X\ i)) = P \cdot ((\sum i{\in}agents.\ \mathcal{E}[i]) + (\sum j{\in}firms.\ Y\ j))$
⟨*proof*⟩

lemma *walras-law-in-compeq*:
assumes $\bigwedge i.\ i{\in}agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
assumes *competitive-equilibrium* $P\ X\ Y$
shows $P \cdot ((\sum i{\in}agents.\ (X\ i)) - (\sum i{\in}agents.\ \mathcal{E}[i]) - (\sum j{\in}firms.\ Y\ j)) = 0$
⟨*proof*⟩

8.6 First Welfare Theorem

Proof of First Welfare Theorem in context of production economy with private ownership.

theorem *first-welfare-theorem-priv-own*:
assumes $\bigwedge i.\ i \in agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
and $Price > 0$
assumes *competitive-equilibrium* $Price\ \mathcal{X}\ \mathcal{Y}$
shows *pareto-optimal* $\mathcal{X}\ \mathcal{Y}$
⟨*proof*⟩

Equilibrium cannot be Pareto dominated.

lemma *equilibria-dom-eachother*:
  assumes $\bigwedge i.\ i \in agents \Longrightarrow local\text{-}nonsatiation\ consumption\text{-}set\ Pr[i]$
    and $Price > 0$
  assumes *equil*: $competitive\text{-}equilibrium\ Price\ \mathcal{X}\ \mathcal{Y}$
  shows $\nexists X'\ Y'.\ competitive\text{-}equilibrium\ P\ X'\ Y' \wedge X' \succ Pareto\ \mathcal{X}$
$\langle proof \rangle$

Using monotonicity instead of local non-satiation proves the First Welfare Theorem.

corollary *first-welfare-thm-monotone*:
  assumes $\forall M \in carrier.\ (\forall x > M.\ x \in carrier)$
  assumes $\bigwedge i.\ i \in agents \Longrightarrow monotone\text{-}preference\ consumption\text{-}set\ Pr[i]$
    and $Price > 0$
  assumes $competitive\text{-}equilibrium\ Price\ \mathcal{X}\ \mathcal{Y}$
  shows $pareto\text{-}optimal\ \mathcal{X}\ \mathcal{Y}$
  $\langle proof \rangle$

end

end

end

9 Arrow-Debreu model

theory *Arrow-Debreu-Model*
  imports
    ../Preferences
    ../Preferences
    ../Utility-Functions
    ../Argmax
    Consumers
    Common
begin

locale *pre-arrow-debreu-model* =
  fixes $production\text{-}sets :: 'f \Rightarrow ('a::ordered\text{-}euclidean\text{-}space)\ set$
  fixes $consumption\text{-}set :: 'a\ set$
  fixes $agents :: 'i\ set$
  fixes $firms :: 'f\ set$
  fixes $\mathcal{E} :: 'i \Rightarrow 'a\ \ (\mathcal{E}[\text{-}])$
  fixes $Pref :: 'i \Rightarrow 'a\ relation\ (Pr[\text{-}])$
  fixes $U :: 'i \Rightarrow 'a \Rightarrow real\ (U[\text{-}])$
  fixes $\Theta :: 'i \Rightarrow 'f \Rightarrow real\ \ (\Theta[\text{-},\text{-}])$
  assumes *cons-set-props*: $arrow\text{-}debreu\text{-}consum\text{-}set\ consumption\text{-}set$
  assumes *agent-props*: $i \in agents \Longrightarrow eucl\text{-}ordinal\text{-}utility\ consumption\text{-}set\ (Pr[i])$
$(U[i])$
  assumes *firms-comp-owned*: $j \in firms \Longrightarrow (\sum i{\in}agents.\ \Theta[i,j]) = 1$
  assumes *finite-nonepty-agents*: $finite\ agents$ **and** $agents \neq \{\}$

sublocale *pre-arrow-debreu-model* $\subseteq$ *pareto-ordering agents U*
⟨*proof*⟩

context *pre-arrow-debreu-model*
begin

Calculate wealth of individual i in context of Private Ownership economy.

context
begin

private abbreviation *poe-wealth*
where
poe-wealth $P\ i\ Y \equiv P \cdot \mathcal{E}[i] + (\sum j{\in}firms.\ \Theta[i,j] *_R (P \cdot Y\,j))$

9.1 Feasiblity

private abbreviation *feasible*
where
feasible X Y $\equiv$ *feasible-private-ownership agents firms* $\mathcal{E}$ *consumption-set production-sets X Y*

private abbreviation *calculate-value*
where
calculate-value $P\ x \equiv P \cdot x$

9.2 Profit maximisation

In a production economy (which this is) we need to specify profit maximisation.

definition *profit-maximisation*
where
profit-maximisation $P\ S =$ *arg-max-set* $(\lambda x.\ P \cdot x)\ S$

9.3 Competitive Equilibirium

Competitive equilibrium in context of production economy with private ownership. This includes the profit maximisation condition.

definition *competitive-equilibrium*
where
competitive-equilibrium $P\ X\ Y \longleftrightarrow$ *feasible* $X\ Y \wedge$
$(\forall j \in firms.\ (Y\,j) \in$ *profit-maximisation* P $(production\text{-}sets\ j)) \wedge$
$(\forall i \in agents.\ (X\ i) \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$)))

lemma *competitive-equilibriumD* [*dest*]:

assumes *competitive-equilibrium* $P\ X\ Y$
shows *feasible* $X\ Y \wedge$
$(\forall j \in \textit{firms}.\ (Y\ j) \in \textit{profit-maximisation}\ P\ (\textit{production-sets}\ j)) \wedge$
$(\forall i \in \textit{agents}.\ (X\ i) \in \textit{arg-max-set}\ U[i]\ (\textit{budget-constraint}\ (\textit{calculate-value}\ P)\ \textit{consumption-set}\ (\textit{poe-wealth}\ P\ i\ Y)))$
⟨*proof*⟩

lemma *compet-max-profit*:
assumes $j \in \textit{firms}$
assumes *competitive-equilibrium* $P\ X\ Y$
shows $Y\ j \in \textit{profit-maximisation}\ P\ (\textit{production-sets}\ j)$
⟨*proof*⟩

9.4 Pareto Optimality

definition *pareto-optimal*
where
pareto-optimal $X\ Y \longleftrightarrow$
$(\textit{feasible}\ X\ Y \wedge$
$(\nexists X'\ Y'.\ \textit{feasible}\ X'\ Y' \wedge X' \succ\textit{Pareto}\ X))$

lemma *pareto-optimalI*[*intro*]:
assumes *feasible* $X\ Y$
and $\nexists X'\ Y'.\ \textit{feasible}\ X'\ Y' \wedge X' \succ\textit{Pareto}\ X$
shows *pareto-optimal* $X\ Y$
⟨*proof*⟩

lemma *pareto-optimalD*[*dest*]:
assumes *pareto-optimal* $X\ Y$
shows *feasible* $X\ Y$ **and** $\nexists X'\ Y'.\ \textit{feasible}\ X'\ Y' \wedge X' \succ\textit{Pareto}\ X$
⟨*proof*⟩

lemma *util-fun-def-holds*:
assumes $i \in \textit{agents}$
and $x \in \textit{consumption-set}$
and $y \in \textit{consumption-set}$
shows $x \succeq[\textit{Pr}[i]]\ y \longleftrightarrow U[i]\ x \geq U[i]\ y$
⟨*proof*⟩

lemma *base-pref-is-ord-eucl-rpr*: $i \in \textit{agents} \Longrightarrow \textit{rational-preference consumption-set}\ \textit{Pr}[i]$
⟨*proof*⟩

lemma *prof-max-ge-all-in-pset*:
assumes $j \in \textit{firms}$
assumes $Y\ j \in \textit{profit-maximisation}\ P\ (\textit{production-sets}\ j)$
shows $\forall y \in \textit{production-sets}\ j.\ P \cdot Y\ j \geq P \cdot y$
⟨*proof*⟩

9.5 Lemmas for final result

Strictly preferred bundles are strictly more expensive.

lemma *all-prefered-are-more-expensive*:
assumes *i-agt*: $i \in agents$
assumes *equil*: *competitive-equilibrium* $P\ \mathcal{X}\ \mathcal{Y}$
assumes $z \in consumption\text{-}set$
assumes $(U\ i)\ z > (U\ i)\ (\mathcal{X}\ i)$
shows $z \cdot P > P \cdot (\mathcal{X}\ i)$
⟨*proof*⟩

Given local non-satiation, argmax will use the entire budget.

lemma *am-utilises-entire-bgt*:
assumes *i-agts*: $i \in agents$
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes *argmax-sol* : $X \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$))
shows $P \cdot X = P \cdot \mathcal{E}[i] + (\sum j{\in}firms.\ \Theta[i,j] *_R (P \cdot Y\ j))$
⟨*proof*⟩

corollary *x-equil-x-ext-budget*:
assumes *i-agt*: $i \in agents$
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes *equilibrium* : *competitive-equilibrium* $P\ X\ Y$
shows $P \cdot X\ i = P \cdot \mathcal{E}[i] + (\sum j{\in}firms.\ \Theta[i,j] *_R (P \cdot Y\ j))$
⟨*proof*⟩

lemma *same-price-in-argmax* :
assumes *i-agt*: $i \in agents$
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes $x \in$ *arg-max-set* $(U[i])$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$))
assumes $y \in$ *arg-max-set* $(U[i])$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$))
shows $(P \cdot x) = (P \cdot y)$
⟨*proof*⟩

Greater or equal utility implies greater or equal value.

lemma *utility-ge-price-ge* :
assumes *agts*: $i \in agents$
assumes *lns* : *local-nonsatiation consumption-set* $Pr[i]$
assumes *equil*: *competitive-equilibrium* $P\ X\ Y$
assumes *geq*: $U[i]\ z \geq U[i]\ (X\ i)$
and $z \in consumption\text{-}set$
shows $P \cdot z \geq P \cdot (X\ i)$
⟨*proof*⟩

lemma *commutativity-sums-over-funs*:
fixes $X :: {'x}\ set$

fixes Y :: $'y$ *set*
shows $(\sum i{\in}X.\ \sum j{\in}Y.\ (f\ i\ j *_R C \cdot g\ j)) = (\sum j{\in}Y.\sum i{\in}X.\ (f\ i\ j *_R C \cdot g\ j))$
⟨*proof*⟩

lemma *assoc-fun-over-sum*:
fixes X :: $'x$ *set*
fixes Y :: $'y$ *set*
shows $(\sum j{\in}Y.\ \sum i{\in}X.\ f\ i\ j *_R C \cdot g\ j) = (\sum j{\in}Y.\ (\sum i{\in}X.\ f\ i\ j) *_R C \cdot g\ j)$
⟨*proof*⟩

Walras' law in context of production economy with private ownership. That is, in an equilibrium demand equals supply.

lemma *walras-law*:
assumes $\bigwedge i.\ i{\in}agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
assumes $(\forall\, i \in agents.\ (X\ i) \in$ *arg-max-set* $U[i]$ (*budget-constraint* (*calculate-value* P) *consumption-set* (*poe-wealth* $P\ i\ Y$)))
shows $P \cdot (\sum i{\in}agents.\ (X\ i)) = P \cdot ((\sum i{\in}agents.\ \mathcal{E}[i]) + (\sum j{\in}firms.\ Y\ j))$
⟨*proof*⟩

lemma *walras-law-in-compeq*:
assumes $\bigwedge i.\ i{\in}agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
assumes *competitive-equilibrium* $P\ X\ Y$
shows $P \cdot ((\sum i{\in}agents.\ (X\ i)) - (\sum i{\in}agents.\ \mathcal{E}[i]) - (\sum j{\in}firms.\ Y\ j)) = 0$
⟨*proof*⟩

9.6 First Welfare Theorem

Proof of First Welfare Theorem in context of production economy with private ownership.

theorem *first-welfare-theorem-priv-own*:
assumes $\bigwedge i.\ i \in agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
and $Price > 0$
assumes *competitive-equilibrium* $Price\ \mathcal{X}\ \mathcal{Y}$
shows *pareto-optimal* $\mathcal{X}\ \mathcal{Y}$
⟨*proof*⟩

Equilibrium cannot be Pareto dominated.

lemma *equilibria-dom-eachother*:
assumes $\bigwedge i.\ i \in agents \Longrightarrow$ *local-nonsatiation consumption-set* $Pr[i]$
and $Price > 0$
assumes *equil*: *competitive-equilibrium* $Price\ \mathcal{X}\ \mathcal{Y}$
shows $\nexists X'\ Y'.$ *competitive-equilibrium* $P\ X'\ Y' \wedge X' \succ Pareto\ \mathcal{X}$
⟨*proof*⟩

Using monotonicity instead of local non-satiation proves the First Welfare Theorem.

corollary *first-welfare-thm-monotone*:
assumes $\forall M \in carrier. (\forall x > M. x \in carrier)$
assumes $\bigwedge i.\ i \in agents \Longrightarrow monotone\text{-}preference\ consumption\text{-}set\ Pr[i]$
and $Price > 0$
assumes $competitive\text{-}equilibrium\ Price\ \mathcal{X}\ \mathcal{Y}$
shows $pareto\text{-}optimal\ \mathcal{X}\ \mathcal{Y}$
$\langle proof \rangle$

end

end

end

10 Related work

[2]

References

[1] K. J. Arrow, A. Sen, and K. Suzumura. *Handbook of Social Choice and Welfare*, volume 2. Elsevier, 2010.

[2] S. Tadelis. *Game Theory: An Introduction*. Princeton University Press, 2013.