

# Data refinement of representation of a file

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## Abstract

This document illustrates the verification of basic file operations (file creation, file read and file write) in Isabelle theorem prover [4]. We describe a file at two levels of abstraction: an abstract file represented as a resizable array, and a concrete file represented using data blocks.

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## 1 Introduction

This document is based on [1], which explores the challenges of verifying the core operations of a Unix-like file system [5, 3]. The paper [1] formalizes

the specification of the file system as a map from file names to sequences of bytes, then formalizes an implementation that uses such standard file system data structures as i-nodes and fixed-sized disk blocks. The correctness of the implementation is verified by proving the existence of a simulation relation [2] between the specification and the implementation. The original effort of [1] started in Isabelle. The process of developing the proof in Isabelle helped to remove the initial bugs in the concrete and abstract models (though the proof has not been completed so far).

Here we present a completed proof for a simplified problem: data refinement of a single file. We present operations on both abstract and concrete files, define a function mapping concrete files to abstract files, and prove that this function is a simulation relation.

We use two libraries of arrays: arrays without bounds checks, which can be thought of as an array with an unbounded number of elements, and resizable arrays, which have a notion of the current size, but expand in response to array writes that are outside the current bounds.

## 2 Arrays without bounds

**theory** *CArrays* **imports** *Main* **begin**

For these arrays there is no built-in protection against reading or writing out-of-bounds.

**type-synonym** *'a cArray* = *nat => 'a*

**definition** *makeCArray* :: *nat => 'a => 'a cArray* **where**  
*makeCArray* *arraySize* *fillValue* *index* ==  
 if *index* < *arraySize* then *fillValue* else *undefined*

**definition** *readCArray* :: *'a cArray => nat => 'a* **where**  
*readCArray* *array* *index* == *array* *index*

**definition** *writeCArray* :: *'a cArray => nat => 'a => 'a cArray* **where**  
*writeCArray* *array* *index* *value* == *array*(*index* := *value*)

**lemma** *makeCArrayCorrect*:  
*index* < *arraySize* ==>  
*readCArray* (*makeCArray* *arraySize* *fillValue*) *index* = *fillValue*  
 <proof>

**lemma** *writeCArrayCorrect1*:  
*readCArray* (*writeCArray* *array* *index* *value*) *index* = *value*  
 <proof>

```

lemma writeCArrayCorrect2:
  index1 ~ = index2 ==>
    readCArray (writeCArray array index1 value) index2 =
      readCArray array index2
⟨proof⟩

end

```

### 3 Resizable arrays

```

theory ResizableArrays imports Main begin

```

These arrays resize themselves, padding with fillValue.

```

type-synonym 'a rArray = nat * (nat => 'a)

```

```

definition fillAndUpdate :: nat => (nat => 'a) => nat => 'a => 'a => (nat
=> 'a) where
  fillAndUpdate len f i value fillValue j ==
    if j=i then value
    else if (len <= j & j < i) then fillValue
    else f j

```

```

definition raWrite :: 'a rArray => nat => 'a => 'a => 'a rArray where
  raWrite arr i value fillValue ==
    (let len = fst arr;
      f = snd arr
    in
      if i < len then (len,f(i:=value))
      else (i+1, fillAndUpdate len f i value fillValue)
    )

```

```

definition cutoff :: 'a => 'a rArray => 'a rArray where
  cutoff fill arr ==
    (fst arr,
     % i. if i < fst arr then snd arr i else fill)

```

```

lemma raWriteSizeSame [simp]: i < fst arr ==> fst (raWrite arr i value fillValue)
= fst arr
⟨proof⟩

```

```

lemma raWriteSizeGrows [simp]: (fst arr <= i) ==> fst (raWrite arr i value
fillValue) = i+1
⟨proof⟩

```

```

lemma raWriteContentChanged [simp]: snd (raWrite arr i value fillValue) i =
value
⟨proof⟩

```

```

lemma raWriteContentOld [simp]: [ j < fst arr; i ~ = j ] ==>

```

$snd (raWrite\ arr\ i\ value\ fillValue)\ j = snd\ arr\ j$

*<proof>*

**lemma** *raWriteContentFill* [simp]:  $[[\ fst\ arr < j; j < i ]]$   $\implies$   
 $snd (raWrite\ arr\ i\ value\ fillValue)\ j = fillValue$

*<proof>*

**end**

## 4 Data refinement of representation of a file

**theory** *FileRefinement* **imports** *Complex-Main CArrays ResizableArrays* **begin**

We describe a file at two levels of abstraction: an abstract file, represented as a resizable array, and a concrete file, represented using data blocks. We consider the following operations:

```
makeAFS      :: AFile
afsRead      :: "AFile => nat \<rightarrow> byte"
afsWrite     :: "AFile => nat => byte \<rightarrow> AFile"
afsFileSize  :: "AFile => nat"
```

**typedecl**

— unit of file content

*byte*

**consts**

— value used for padding

*fillByte* :: *byte*

**axiomatization**

*blockSize* :: *nat* — in bytes **and**

*numBlocks* :: *nat* — total number of blocks in the file system

**where**

*nonZeroBlockSize*:  $blockSize > 0$  **and**

*nonZeroNumBlocks*:  $numBlocks > 0$

### 4.1 Abstract File

**type-synonym** *AFile* = *byte rArray* — abstract file is a resizable array of bytes

**definition** *makeAF* :: *AFile*

**where** — initial file has size 0

$makeAF == (0, \% index.\ fillByte)$

**definition** *afSize* :: *AFile* => *nat* **where**

— file size is the length of the resizable array

$afSize\ afile == fst\ afile$

**definition** *afRead* :: *AFile* => *nat* → *byte* **where**  
 — reading from a file looks up the byte, reporting *None* if the index is out of file bounds  
*afRead* *afile* *byteIndex* ==  
 if *byteIndex* < *fst* *afile* then *Some* ((*snd* *afile*) *byteIndex*) else *None*

**definition** *afWrite* :: *AFile* => *nat* => *byte* → *AFile* **where**  
 — writing to a file updates the file content and extends the file if there is enough space  
*afWrite* *afile* *byteIndex* *value* ==  
 if *byteIndex* *div* *blockSize* < *numBlocks* then  
   *Some* (*raWrite* *afile* *byteIndex* *value* *fillByte*)  
 else *None*

## 4.2 Concrete File

**type-synonym** *Block* = *byte* *cArray* — array of *blockSize* bytes

**record** *CFile* =  
*fileSize* :: *nat* — in bytes  
*nextFreeBlock* :: *nat* — next block available for allocation  
*data* :: *Block* *cArray* — array of up to *numBlocks* blocks

**definition** *makeCF* :: *CFile*  
**where** — initial file has no allocated blocks  
*makeCF* ==  
 (| *fileSize* = 0,  
   *nextFreeBlock* = 0,  
   *data* = *makeCArray* *numBlocks* (*makeCArray* *blockSize* *fillByte*)  
 |)

**definition** *cfSize* :: *CFile* => *nat* **where**  
*cfSize* *cfile* == *fileSize* *cfile*

**definition** *cfRead* :: *CFile* => *nat* → *byte* **where**  
 — Looks up correct data block and reads its content, if *byteIndex* is within bounds, else returns *None*.  
*cfRead* *cfile* *byteIndex* ==  
 if *byteIndex* < *fileSize* *cfile* then  
   (let *i* = *byteIndex* *div* *blockSize* in  
     (let *j* = *byteIndex* *mod* *blockSize* in  
       *Some* (*readCArray* (*readCArray* (*data* *cfile*) *i*) *j*)))  
 else *None*

### 4.2.1 Writing File

We first present some auxiliary operations.

**definition** *cfWriteNoExtend* :: *CFile* => *nat* => *byte* => *CFile* **where**

— Writing to a file when *byteIndex* is within bounds.

```

cfWriteNoExtend cfile byteIndex value ==
  let i = byteIndex div blockSize in
    let j = byteIndex mod blockSize in
      let block = readCArray (data cfile) i in
        cfile(| data :=
          writeCArray (data cfile) i (writeCArray block j value) |)

```

**definition** *cfExtendFile* :: *CFile* => *nat* => *CFile* **where**

— Writing to a file when *byteIndex* is out of bounds. Involves allocating a new block.

```

cfExtendFile cfile byteIndex ==
  cfile(| fileSize := Suc byteIndex,
    nextFreeBlock := Suc (byteIndex div blockSize) |)

```

The main file write operation.

**definition** *cfWrite* :: *CFile* => *nat* => *byte* → *CFile* **where**

— Writes the file at byte location *byteIndex*, automatically extending the file to that byte location if *byteIndex* is not within bounds.

```

cfWrite cfile byteIndex value ==
  if byteIndex div blockSize < numBlocks then
    if byteIndex < fileSize cfile then
      Some (cfWriteNoExtend cfile byteIndex value)
    else
      Some (cfWriteNoExtend (cfExtendFile cfile byteIndex) byteIndex value)
  else None

```

### 4.3 Reachability Invariants for Concrete File

**definition** *nextFreeBlockInvariant* :: *CFile* => *bool* **where**

```

nextFreeBlockInvariant state ==
  (fileSize state + blockSize - 1) div blockSize = nextFreeBlock state

```

**definition** *unallocatedBlocksInvariant* :: *CFile* => *bool* **where**

— This invariant of the implementation is needed to prove *writeExtendCorrect*. It says that any unallocated block contains *fillByte*'s.

```

unallocatedBlocksInvariant state ==
  ∀ blockNum i .
    ~ blockNum < nextFreeBlock state & blockNum < numBlocks & i < blockSize
    --> data state blockNum i = fillByte

```

**definition** *lastBlockInvariant* :: *CFile* => *bool* **where**

```

lastBlockInvariant state ==
  ∀ index .
    ~ index < fileSize state & nextFreeBlock state = index div blockSize + 1
    --> data state (index div blockSize) (index mod blockSize) = fillByte

```

**definition** *reachabilityInvariant* :: *CFile* => *bool* **where**

```

reachabilityInvariant cfile ==

```

*nextFreeBlockInvariant cfile &*  
*unallocatedBlocksInvariant cfile &*  
*lastBlockInvariant cfile*

#### 4.4 Initial File Satisfies Invariants

We prove each invariant individually and then summarize.

**lemma** *nextFreeBlockInvariant1:*  
*nextFreeBlockInvariant makeCF*  
{proof}

**lemma** *unallocatedBlocksInvariant1:*  
*unallocatedBlocksInvariant makeCF*  
{proof}

**lemma** *lastBlockInvariant1:*  
*lastBlockInvariant makeCF*  
{proof}

**lemma** *makeCFpreserves: reachabilityInvariant makeCF*  
{proof}

#### 4.5 Properties of Concrete File Operations

**lemma** *cfWriteNoExtendPreservesFileSize:*  
[[ *index < fileSize cfile1;*  
*cfWrite cfile1 index value = Some cfile2*  
]] ==>  
*fileSize cfile2 = fileSize cfile1*  
{proof}

**lemma** *cfWriteExtendFileSize:*  
[[  $\sim$  *index < fileSize cfile1;*  
*cfWrite cfile1 index value = Some cfile2*  
]] ==> *fileSize cfile2 = Suc index*  
{proof}

**lemma** *fileSizeIncreases:*  
*cfWrite cfile1 index value = Some cfile2*  
==> *fileSize cfile1 <= fileSize cfile2*  
{proof}

**lemma** *nextFreeBlockIncreases:*  
[[ *nextFreeBlockInvariant cfile1;*  
*cfWrite cfile1 index value = Some cfile2*  
]] ==> *nextFreeBlock cfile1 <= nextFreeBlock cfile2*  
{proof}

## 4.6 Concrete File Operations Preserve Invariants

There is only one top-level concrete operation: write, and we show that it preserves all reachability invariants.

**lemma** *cfWritePreservesNextFreeBlockInvariant:*

```

[[ reachabilityInvariant cfile1;
   cfWrite cfile1 byteIndex value = Some cfile2
  ]] ==> nextFreeBlockInvariant cfile2
<proof>

```

**lemma** *modInequalityLemma:*

```

(a::nat) ~ = b & a mod c = b mod c ==> a div c ~ = b div c
<proof>

```

**lemma** *mod-round-lt:*

```

[[ 0 < (c::nat);
   a < b
  ]] ==> a div c < (b + c - 1) div c
<proof>

```

**lemma** *blockNumNELemma:*

```

!!blockNum i.
[[ nextFreeBlockInvariant cfile1;
   cfile1
   (| data :=
      writeCArray (data cfile1) (byteIndex div blockSize)
      (writeCArray
       (readCArray (data cfile1) (byteIndex div blockSize))
       (byteIndex mod blockSize) value) |) =
   cfile2;
   ~ blockNum < nextFreeBlock cfile2; blockNum < numBlocks;
   i < blockSize; byteIndex div blockSize < numBlocks;
   byteIndex < fileSize cfile1 ]]
==> blockNum ~ = byteIndex div blockSize
<proof>

```

**lemma** *cfWritePreservesUnallocatedBlocksInvariant:*

```

[[ reachabilityInvariant cfile1;
   cfWrite cfile1 byteIndex value = Some cfile2
  ]] ==> unallocatedBlocksInvariant cfile2
<proof>

```

**lemma** *cfWritePreservesLastBlockInvariant:*

```

[[ reachabilityInvariant cfile1;
   cfWrite cfile1 byteIndex value = Some cfile2 ]] ==>
lastBlockInvariant cfile2
<proof>

```

Final statement that all invariants are preserved.



**lemma** *cfWritePreserves*:  
 $\llbracket \text{reachabilityInvariant } cfile1; \text{cfWrite } cfile1 \text{ byteIndex } value = \text{Some } cfile2 \rrbracket \implies$   
 $\text{reachabilityInvariant } cfile2$   
 $\langle \text{proof} \rangle$

## 4.7 Commuting Diagrams for Simulation Relation

Here we show correctness of file system operations.

### 4.7.1 Abstraction Function

**definition** *abstFn* :: *CFile* => *AFile* **where**  
 $\text{abstFn } cfile == (\text{fileSize } cfile,$   
 $\quad \% \text{ index } . \text{case } cfRead \text{ } cfile \text{ index of}$   
 $\quad \quad \text{None} \quad \quad \implies \text{fillByte}$   
 $\quad \quad | \text{Some } value \implies value)$

**primrec** *oAbstFn* :: *CFile option* => *AFile option* **where**  
 $\text{oAbstFn } \text{None} = \text{None}$   
 $| \text{oAbstFn } (\text{Some } s) = \text{Some } (\text{abstFn } s)$

### 4.7.2 Creating a File

**lemma** *makeCFCorrect*:  $\text{abstFn } \text{makeCF} = \text{makeAF}$   
 $\langle \text{proof} \rangle$

### 4.7.3 File Size

**lemma** *fileSizeCorrect*:  
 $\text{cfSize } cfile = \text{afSize } (\text{abstFn } cfile)$   
 $\langle \text{proof} \rangle$

### 4.7.4 Read Operation

**lemma** *readCorrect*:  
 $\text{cfRead } cfile = \text{afRead } (\text{abstFn } cfile)$   
 $\langle \text{proof} \rangle$

### 4.7.5 Write Operation

**lemma** *writeNoExtendCorrect*:  
 $\llbracket \text{index} < \text{fileSize } cfile1; \text{Some } cfile2 = \text{cfWrite } cfile1 \text{ index } value \rrbracket \implies$   
 $\text{Some } (\text{abstFn } cfile2) = \text{afWrite } (\text{abstFn } cfile1) \text{ index } value$   
 $\langle \text{proof} \rangle$

**lemma** *writeExtendCorrect*:  
 $\llbracket \text{nextFreeBlockInvariant } cfile1; \text{unallocatedBlocksInvariant } cfile1; \rrbracket$

$lastBlockInvariant\ cfile1;$   
 $\sim\ index < fileSize\ cfile1;$   
 $Some\ cfile2 = cfWrite\ cfile1\ index\ value$   
 $\|\ ==> Some\ (abstFn\ cfile2) = afWrite\ (abstFn\ cfile1)\ index\ value$   
 $\langle proof \rangle$

**lemma** *writeSucceedCorrect:*

$\|\ nextFreeBlockInvariant\ cfile1;$   
 $unallocatedBlocksInvariant\ cfile1;$   
 $lastBlockInvariant\ cfile1;$   
 $Some\ cfile2 = cfWrite\ cfile1\ index\ value$   
 $\|\ ==> Some\ (abstFn\ cfile2) = afWrite\ (abstFn\ cfile1)\ index\ value$   
 $\langle proof \rangle$

**lemma** *writeFailCorrect:*

$cfWrite\ cfile1\ index\ value = None ==>$   
 $afWrite\ (abstFn\ cfile1)\ index\ value = None$   
 $\langle proof \rangle$

**lemma** *writeCorrect:*

$reachabilityInvariant\ cfile1 ==>$   
 $\circ AbstFn\ (cfWrite\ cfile1\ index\ value) = afWrite\ (abstFn\ cfile1)\ index\ value$   
 $\langle proof \rangle$

**end**

## References

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