Feuerbach's Theorem

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Abstract

Feuerbach's theorem is a classical result in Euclidean geometry about certain circles associated with a triangle. It states that the nine-point circle is tangent internally to the triangle's incircle and tangent externally to each of the three excircles, where the *incircle* is the circle tangent to all three sides of the triangle; the *excircles* are circles tangent to one side and the extensions of the other two; the *nine-point circle* is the circle passing through nine special points of the triangle.

This development is an experiment in automatic translation from one proof assistant to another using AI. The HOL Light file `feuerbach.ml` was given, one proof at a time, to Claude 4.6 using the Isabelle Assistant plug-in.[1] Despite coaxing and hints, the autoformalisation process transformed a 214-line HOL Light file to nearly 1500 lines of Isabelle/HOL, though one that could be processed in five seconds. Part of the excess length comes from spelling out two transformations (translation of a point to zero, followed by rotation of a side to the vector $(1, 0)$) that in HOL Light is done silently by WLOG tactics. The current proof has been heavily simplified, in part using Claude but mostly manually, and is now under 750 lines. It is notable that Claude often relied on the **algebra** proof method, despite its relative obscurity.

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[1]Part of AutoCorrode: `https://github.com/awslabs/AutoCorrode`

theory *Feuerbach*
imports *HOL−Analysis.Analysis*
begin

1 Algebraic condition for two circles to be tangent to each other

lemma *inner-real2*: (*x*::*real*^*2*) · *y* = *x*$*1* ∗ *y*$*1* + *x*$*2* ∗ *y*$*2*
⟨*proof*⟩

lemma *norm-real2-sq*: (*norm* (*x*::*real*^*2*))^*2* = (*x*$*1*)^*2* + (*x*$*2*)^*2*
⟨*proof*⟩

lemma *dist-real2-sq*: (*dist* (*a*::*real*^*2*) *b*)^*2* = (*a*$*1* − *b*$*1*)^*2* + (*a*$*2* − *b*$*2*)^*2*
⟨*proof*⟩

lemma *collinear-3-2d*:
collinear {(*a*::*real*^*2*), *b*, *c*} ⟷
(*b*$*1* − *a*$*1*) ∗ (*c*$*2* − *a*$*2*) = (*c*$*1* − *a*$*1*) ∗ (*b*$*2* − *a*$*2*)
⟨*proof*⟩

2 Geometric invariance lemmas

lemma *collinear-linear-image*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ ′*b*::*euclidean-space*
assumes *linear f inj f*
shows *collinear* (*f* ‘ *S*) = *collinear S*
⟨*proof*⟩

lemma *collinear-orthogonal-transformation*:
fixes *f* :: ′*a*::*euclidean-space* ⇒ ′*b*::*euclidean-space*
assumes *orthogonal-transformation f*
shows *collinear* {*f a*, *f b*, *f c*} = *collinear* {*a*, *b*, *c*}
⟨*proof*⟩

lemma *rotation-to-x-axis*:
fixes *v* :: *real*^*2*
assumes *v* ≠ *0*
obtains *R* :: *real*^*2* ⇒ *real*^*2* **where** *orthogonal-transformation R* (*R v*)$*1* = *norm v* (*R v*)$*2* = *0*
⟨*proof*⟩

lemma *collinear-translate*:
fixes *x* :: ′*a*::*euclidean-space*
shows *collinear* {*x*, *y*, *z*} = *collinear* {*x* − *a*, *y* − *a*, *z* − *a*}
⟨*proof*⟩

lemma *int-case-lemma*:

fixes *c1 c2* :: (*real*,*2*) *vec*
assumes *0* ≤ *r1 r1* < *r2* **and** *d*: *dist c1 c2* = |*r1* − *r2*| **and** *int*: *dist c1 c2* = |*r1* − *r2*|
shows *c1* = *c2* ∧ *r1* = *r2* ∨ (∃!*x*. *dist c1 x* = *r1* ∧ *dist c2 x* = *r2*)
⟨*proof*⟩

theorem *circles-tangent*:
fixes *r1 r2* :: *real* **and** *c1 c2* :: *real*^*2*
assumes *0* ≤ *r1 0* ≤ *r2*
and *dist c1 c2* = *r1* + *r2* ∨ *dist c1 c2* = |*r1* − *r2*|
shows *c1* = *c2* ∧ *r1* = *r2* ∨ (∃!*x*::*real*^*2*. *dist c1 x* = *r1* ∧ *dist c2 x* = *r2*)
⟨*proof*⟩

3 Feuerbach's theorem

Given a non-degenerate triangle *abc*, let the circle passing through the midpoints of its sides (the "9 point circle") have center *ncenter* and radius *nradius*. Now suppose the circle with center *icenter* and radius *iradius* is tangent to three sides (either internally or externally to one side and two produced sides), meaning that it's either the inscribed circle or one of the three escribed circles. Then the two circles are tangent to each other, i.e. either they are identical or they touch at exactly one point.

locale *Feuerbach* =
fixes *a b c* :: *real*^*2*
and *mbc mac mab* :: *real*^*2*
and *pbc pac pab* :: *real*^*2*
and *ncenter icenter* :: *real*^*2*
and *nradius iradius* :: *real*
assumes *mab-def*: *mab* = *midpoint a b*
and *mbc-def*: *mbc* = *midpoint b c*
and *mac-def*: *mac* = *midpoint c a*
assumes *nr1*: *dist ncenter mbc* = *nradius*
and *nr2*: *dist ncenter mac* = *nradius*
and *nr3*: *dist ncenter mab* = *nradius*
and *ir1*: *dist icenter pbc* = *iradius*
and *ir2*: *dist icenter pac* = *iradius*
and *ir3*: *dist icenter pab* = *iradius*
and *col-ab*: *collinear* {*a*, *b*, *pab*} **and** *orth-ab*: *orthogonal* (*a* − *b*) (*icenter* − *pab*)
and *col-bc*: *collinear* {*b*, *c*, *pbc*} **and** *orth-bc*: *orthogonal* (*b* − *c*) (*icenter* − *pbc*)
and *col-ac*: *collinear* {*a*, *c*, *pac*} **and** *orth-ac*: *orthogonal* (*a* − *c*) (*icenter* − *pac*)

begin

lemma *special*:

assumes *bx*: *b$2 = 0* **and** *bne*: *b$1 ≠ 0* **and** *cne*: *c$2 ≠ 0* **and** [*simp*]: *a=0*
shows *ncenter = icenter ∧ nradius = iradius ∨*
(∃!*x*::*real^2*. *dist ncenter x = nradius ∧ dist icenter x = iradius*)
⟨*proof*⟩

theorem *main*:
assumes ¬ *collinear* {*a*, *b*, *c*}
shows *ncenter = icenter ∧ nradius = iradius ∨*
(∃!*x*::*real^2*. *dist ncenter x = nradius ∧ dist icenter x = iradius*)
⟨*proof*⟩

end

4 Nine-point circle

As a little bonus, verify that the circle passing through the midpoints of the sides is indeed a 9-point circle, i.e. it passes through the feet of the altitudes and the midpoints of the lines joining the vertices to the orthocenter (where the altitudes intersect).

lemma *nine-point-circle-1-scalar*:
fixes *bv cx cy n1 n2 f1 f2 g1 g2 h1 h2 nr* :: *real*
assumes *cy-ne*: *cy ≠ 0* **and** *bv-ne*: *bv ≠ 0*
— Nine-point center equidistant to midpoints of sides
— midpoint(0,b) = (bv/2, 0), midpoint(b,c) = ((bv+cx)/2, cy/2), midpoint(c,0) = (cx/2, cy/2)
and *nr1*: *(n1 − (bv+cx)/2)^2 + (n2 − cy/2)^2 = nr^2*
and *nr2*: *(n1 − cx/2)^2 + (n2 − cy/2)^2 = nr^2*
and *nr3*: *(n1 − bv/2)^2 + n2^2 = nr^2*
— fab = (f1, f2): foot of altitude from c to ab (the x-axis)
— collinear 0, (bv,0), (f1,f2): bv * f2 = f1 * 0, so f2 = 0
and *fab-col*: *bv ∗ f2 = 0*
— orthogonal (0-(bv,0)) ((cx,cy)-(f1,f2)): -bv*(cx-f1) + 0*(cy-f2) = 0
and *fab-orth*: *bv ∗ (cx − f1) = 0*
— fbc = (g1, g2): foot of altitude from 0 to bc
— collinear (bv,0), (cx,cy), (g1,g2): (cx-bv)*(g2-0) = (g1-bv)*(cy-0)
and *fbc-col*: *(cx − bv) ∗ g2 = (g1 − bv) ∗ cy*
— orthogonal ((bv,0)-(cx,cy)) ((0,0)-(g1,g2)): (bv-cx)*(-g1) + (-cy)*(-g2) = 0
and *fbc-orth*: *(bv − cx) ∗ g1 = cy ∗ g2*
— fac = (h1, h2): foot of altitude from b to ca
— collinear (cx,cy), 0, (h1,h2): (-cx)*(h2-cy) = (h1-cx)*(-cy)
and *fac-col*: *cx ∗ h2 = h1 ∗ cy*
— orthogonal ((cx,cy)-0) ((bv,0)-(h1,h2)): cx*(bv-h1) + cy*(0-h2) = 0
and *fac-orth*: *cx ∗ (bv − h1) = cy ∗ h2*
shows *(n1 − f1)^2 + (n2 − f2)^2 = nr^2 ∧ (n1 − g1)^2 + (n2 − g2)^2 = nr^2 ∧ (n1 − h1)^2 + (n2 − h2)^2 = nr^2*
⟨*proof*⟩

theorem *nine-point-circle-1*:

fixes *a b c* :: *real^2*
and *mbc mac mab* :: *real^2*
and *fbc fac fab* :: *real^2*
and *ncenter* :: *real^2*
and *nradius* :: *real*
assumes ¬ *collinear* {*a*, *b*, *c*}
and *midpoint a b* = *mab*
and *midpoint b c* = *mbc*
and *midpoint c a* = *mac*
and *dist ncenter mbc* = *nradius*
and *dist ncenter mac* = *nradius*
and *dist ncenter mab* = *nradius*
and *collinear* {*a*, *b*, *fab*} *orthogonal* (*a* − *b*) (*c* − *fab*)
and *collinear* {*b*, *c*, *fbc*} *orthogonal* (*b* − *c*) (*a* − *fbc*)
and *collinear* {*c*, *a*, *fac*} *orthogonal* (*c* − *a*) (*b* − *fac*)
shows *dist ncenter fab* = *nradius* ∧ *dist ncenter fbc* = *nradius* ∧ *dist ncenter fac* = *nradius*
⟨*proof*⟩

theorem *nine-point-circle-2*:
fixes *a b c* :: *real^2*
and *mbc mac mab* :: *real^2*
and *fbc fac fab* :: *real^2*
and *ncenter oc* :: *real^2*
and *nradius* :: *real*
assumes ¬ *collinear* {*a*, *b*, *c*}
and *midpoint a b* = *mab*
and *midpoint b c* = *mbc*
and *midpoint c a* = *mac*
and *dist ncenter mbc* = *nradius*
and *dist ncenter mac* = *nradius*
and *dist ncenter mab* = *nradius*
and *collinear* {*a*, *b*, *fab*} *orthogonal* (*a* − *b*) (*c* − *fab*)
and *collinear* {*b*, *c*, *fbc*} *orthogonal* (*b* − *c*) (*a* − *fbc*)
and *collinear* {*c*, *a*, *fac*} *orthogonal* (*c* − *a*) (*b* − *fac*)
and *collinear* {*oc*, *a*, *fbc*} *collinear* {*oc*, *b*, *fac*} *collinear* {*oc*, *c*, *fab*}
shows *dist ncenter* (*midpoint oc a*) = *nradius* ∧
dist ncenter (*midpoint oc b*) = *nradius* ∧
dist ncenter (*midpoint oc c*) = *nradius*
⟨*proof*⟩

end