Farey Sequences and Ford Circles

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Abstract

The sequence F_n of *Farey fractions* of order n has the form

$$\frac{0}{1}, \frac{1}{n}, \frac{1}{n-1}, \ldots, \frac{n-1}{n}, \frac{1}{1}$$

where the fractions appear in numerical order and have denominators at most n. The transformation from F_n to F_{n+1} can be effected by combining adjacent elements of the sequence F_n, using an operation called the *mediant*. Adjacent (reduced) fractions $(a/b) < (c/d)$ satisfy the *unimodular* relation $bc - ad = 1$ and their mediant is $\frac{a+c}{b+d}$. A *Ford circle* is specified by a rational number, and interesting consequences follow in the case of Ford circles obtained from some Farey sequence F_n. The formalised material is drawn from Apostol's *Modular Functions and Dirichlet Series in Number Theory* [1].

1 Farey Sequences and Ford Circles

theory *Farey-Ford*
 imports *HOL−Analysis.Analysis HOL−Number-Theory.Totient HOL−Library.Sublist*

begin

1.1 Farey sequences

lemma *sorted-two-sublist*:
 fixes x:: $'a$::*order*
 assumes *sorted*: *sorted-wrt* (<) l
 shows *sublist* $[x,\ y]$ $l \longleftrightarrow x < y \land x \in set\ l \land y \in set\ l \land (\forall z \in set\ l.\ z \leq x$
$\lor\ z \geq y)$
⟨*proof*⟩

lemma *transp-add1-int*:
 assumes $\bigwedge n$::*int*. $R\ (f\ n)\ (f\ (1 + n))$
  and $n < n'$
  and *transp* R
 shows $R\ (f\ n)\ (f\ n')$
⟨*proof*⟩

lemma *refl-transp-add1-int*:
 assumes $\bigwedge n$::*int*. $R\ (f\ n)\ (f\ (1 + n))$
   and $n \leq n'$
   and *reflp* R *transp* R
  shows $R\ (f\ n)\ (f\ n')$
 ⟨*proof*⟩

lemma *transp-Suc*:
 assumes $\bigwedge n.\ R\ (f\ n)\ (f\ (Suc\ n))$
   and $n < n'$
   and *transp* R
  shows $R\ (f\ n)\ (f\ n')$
⟨*proof*⟩

lemma *refl-transp-Suc*:
 assumes $\bigwedge n.\ R\ (f\ n)\ (f\ (Suc\ n))$
   and $n \leq n'$
   and *reflp* R *transp* R
  shows $R\ (f\ n)\ (f\ n')$
 ⟨*proof*⟩

lemma *coprime-unimodular-int*:
 fixes $a\ b$::*int*
 assumes *coprime* $a\ b$ $a>1$ $b>1$
 obtains $x\ y$ **where** $a*x - b*y = 1$ $0 < x$ $x < b$ $0 < y$ $y < a$
⟨*proof*⟩

1.2 Farey Fractions

type-synonym *farey* = *rat*

definition *num-farey* :: *farey* ⇒ *int*
where *num-farey* ≡ λ*x*. *fst* (*quotient-of x*)

definition *denom-farey* :: *farey* ⇒ *int*
where *denom-farey* ≡ λ*x*. *snd* (*quotient-of x*)

definition *farey* :: [*int*,*int*] ⇒ *farey*
where *farey* ≡ λ*a b*. *max 0* (*min 1* (*Fract a b*))

lemma *farey01* [*simp*]: *0* ≤ *farey a b farey a b* ≤ *1*
⟨*proof*⟩

lemma *farey-0* [*simp*]: *farey 0 n* = *0*
⟨*proof*⟩

lemma *farey-1* [*simp*]: *farey 1 1* = *1*
⟨*proof*⟩

lemma *num-farey-nonneg*: *x* ∈ {*0..1*} ⟹ *num-farey x* ≥ *0*
⟨*proof*⟩

lemma *num-farey-le-denom*: *x* ∈ {*0..1*} ⟹ *num-farey x* ≤ *denom-farey x*
⟨*proof*⟩

lemma *denom-farey-pos*: *denom-farey x* > *0*
⟨*proof*⟩

lemma *coprime-num-denom-farey* [*intro*]: *coprime* (*num-farey x*) (*denom-farey x*)
⟨*proof*⟩

lemma *rat-of-farey-conv-num-denom*:
x = *rat-of-int* (*num-farey x*) / *rat-of-int* (*denom-farey x*)
⟨*proof*⟩

lemma *num-denom-farey-eqI*:
assumes *x* = *of-int a* / *of-int b b* > *0 coprime a b*
shows *num-farey x* = *a denom-farey x* = *b*
⟨*proof*⟩

lemma *farey-cases* [*cases type*, *case-names farey*]:
assumes *x* ∈ {*0..1*}
obtains *a b* **where** *0*≤*a a*≤*b coprime a b x* = *Fract a b*
⟨*proof*⟩

lemma *rat-of-farey*: ⟦*x* = *of-int a* / *of-int b*; *x* ∈ {*0..1*}⟧ ⟹ *x* = *farey a b*
⟨*proof*⟩

lemma *farey-num-denom-eq* [*simp*]: $x \in \{0..1\} \Longrightarrow$ *farey* (*num-farey* x) (*denom-farey* x) $= x$
⟨*proof*⟩

lemma *farey-eqI*:
assumes *num-farey* $x =$ *num-farey* y *denom-farey* $x =$ *denom-farey* y
shows $x=y$
⟨*proof*⟩

lemma
assumes *coprime* a b $0 \le a$ $a<b$
shows *num-farey-eq* [*simp*]: *num-farey* (*farey* a b) $= a$
and *denom-farey-eq* [*simp*]: *denom-farey* (*farey* a b) $= b$
⟨*proof*⟩

lemma
assumes $0 \le a$ $a \le b$ $0<b$
shows *num-farey*: *num-farey* (*farey* a b) $= a$ *div* (*gcd* a b)
and *denom-farey*: *denom-farey* (*farey* a b) $= b$ *div* (*gcd* a b)
⟨*proof*⟩

lemma
assumes *coprime* a b $0<b$
shows *num-farey-Fract* [*simp*]: *num-farey* (*Fract* a b) $= a$
and *denom-farey-Fract* [*simp*]: *denom-farey* (*Fract* a b) $= b$
⟨*proof*⟩

lemma *num-farey-0* [*simp*]: *num-farey* $0 = 0$
and *denom-farey-0* [*simp*]: *denom-farey* $0 = 1$
and *num-farey-1* [*simp*]: *num-farey* $1 = 1$
and *denom-farey-1* [*simp*]: *denom-farey* $1 = 1$
⟨*proof*⟩

lemma *num-farey-neq-denom*: *denom-farey* $x \neq 1 \Longrightarrow$ *num-farey* $x \neq$ *denom-farey* x
⟨*proof*⟩

lemma *num-farey-0-iff* [*simp*]: *num-farey* $x = 0 \longleftrightarrow x = 0$
⟨*proof*⟩

lemma *denom-farey-le1-cases*:
assumes *denom-farey* $x \le 1$ $x \in \{0..1\}$
shows $x = 0 \lor x = 1$
⟨*proof*⟩

definition *mediant* :: *farey* ⇒ *farey* ⇒ *farey* **where**
mediant ≡ λx y. *Fract* (*fst* (*quotient-of* x) + *fst* (*quotient-of* y))
(*snd* (*quotient-of* x) + *snd* (*quotient-of* y))

lemma *mediant-eq-Fract*:
$\mathit{mediant}\ x\ y = \mathit{Fract}\ (\mathit{num\text{-}farey}\ x + \mathit{num\text{-}farey}\ y)\ (\mathit{denom\text{-}farey}\ x + \mathit{denom\text{-}farey}\ y)$
⟨*proof*⟩

lemma *mediant-eq-farey*:
assumes $x \in \{0..1\}$ $y \in \{0..1\}$
shows $\mathit{mediant}\ x\ y = \mathit{farey}\ (\mathit{num\text{-}farey}\ x + \mathit{num\text{-}farey}\ y)\ (\mathit{denom\text{-}farey}\ x + \mathit{denom\text{-}farey}\ y)$
⟨*proof*⟩

definition *farey-unimodular* :: *farey* ⇒ *farey* ⇒ *bool* **where**
$\mathit{farey\text{-}unimodular}\ x\ y \longleftrightarrow$
$\mathit{denom\text{-}farey}\ x * \mathit{num\text{-}farey}\ y - \mathit{num\text{-}farey}\ x * \mathit{denom\text{-}farey}\ y = 1$

lemma *farey-unimodular-imp-less*:
assumes *farey-unimodular x y*
shows $x < y$
⟨*proof*⟩

lemma *denom-mediant*: $\mathit{denom\text{-}farey}\ (\mathit{mediant}\ x\ y) \le \mathit{denom\text{-}farey}\ x + \mathit{denom\text{-}farey}\ y$
⟨*proof*⟩

lemma *unimodular-imp-both-coprime*:
fixes a:: $'a$::{*algebraic-semidom*,*comm-ring-1*}
assumes $b*c - a*d = 1$
shows *coprime a b coprime c d*
⟨*proof*⟩

lemma *unimodular-imp-coprime*:
fixes a:: $'a$::{*algebraic-semidom*,*comm-ring-1*}
assumes $b*c - a*d = 1$
shows *coprime* $(a+c)$ $(b+d)$
⟨*proof*⟩

definition *fareys* :: *int* ⇒ *rat list*
where $\mathit{fareys}\ n \equiv \mathit{sorted\text{-}list\text{-}of\text{-}set}\ \{x \in \{0..1\}.\ \mathit{denom\text{-}farey}\ x \le n\}$

lemma *strict-sorted-fareys*: *sorted-wrt* $(<)$ (*fareys n*)
⟨*proof*⟩

lemma *farey-set-UN-farey*: $\{x \in \{0..1\}.\ \mathit{denom\text{-}farey}\ x \le n\} = (\bigcup b \in \{1..n\}.\ \bigcup a \in \{0..b\}.\ \{\mathit{farey}\ a\ b\})$
⟨*proof*⟩

lemma *farey-set-UN-farey'*: $\{x \in \{0..1\}.\ \mathit{denom\text{-}farey}\ x \le n\} = (\bigcup b \in \{1..n\}.$

$\bigcup a \in \{0..b\}.$ *if coprime a b then* $\{$*farey a b*$\}$ *else* $\{\})$
⟨*proof*⟩

lemma *farey-set-UN-Fract*: $\{x \in \{0..1\}.$ *denom-farey* $x \le n\} = (\bigcup b \in \{1..n\}.$ $\bigcup a \in \{0..b\}.$ $\{$*Fract a b*$\})$
⟨*proof*⟩

lemma *farey-set-UN-Fract'*: $\{x \in \{0..1\}.$ *denom-farey* $x \le n\} = (\bigcup b \in \{1..n\}.$ $\bigcup a \in \{0..b\}.$ *if coprime a b then* $\{$*Fract a b*$\}$ *else* $\{\})$
⟨*proof*⟩

lemma *finite-farey-set*: *finite* $\{x \in \{0..1\}.$ *denom-farey* $x \le n\}$
⟨*proof*⟩

lemma *denom-in-fareys-iff*: $x \in$ *set* (*fareys n*) $\longleftrightarrow$ *denom-farey* $x \le$ *int* $n \wedge x \in \{0..1\}$
⟨*proof*⟩

lemma *denom-fareys-leI*: $x \in$ *set* (*fareys n*) $\Longrightarrow$ *denom-farey* $x \le n$
⟨*proof*⟩

lemma *denom-fareys-leD*: ⟦*denom-farey* $x \le$ *int n*; $x \in \{0..1\}$⟧ $\Longrightarrow x \in$ *set* (*fareys n*)
⟨*proof*⟩

lemma *fareys-increasing-1*: *set* (*fareys n*) $\subseteq$ *set* (*fareys* $(1 + n)$)
⟨*proof*⟩

definition *fareys-new* :: *int* $\Rightarrow$ *rat set* **where**
fareys-new $n \equiv \{$*Fract a n*$|$ *a. coprime a n* $\wedge\ a \in \{0..n\}\}$

lemma *fareys-new-0* [*simp*]: *fareys-new* $0 = \{\}$
⟨*proof*⟩

lemma *fareys-new-1* [*simp*]: *fareys-new* $1 = \{0,1\}$
⟨*proof*⟩

lemma *fareys-new-not01*:
assumes $n>1$
shows $0 \notin$ (*fareys-new n*) $1 \notin$ (*fareys-new n*)
⟨*proof*⟩

lemma *inj-num-farey*: *inj-on num-farey* (*fareys-new n*)
⟨*proof*⟩

lemma *finite-fareys-new* [*simp*]: *finite* (*fareys-new n*)
⟨*proof*⟩

lemma *card-fareys-new*:

assumes $n > 1$
shows *card* (*fareys-new* (*int* n)) = *totient* n
⟨*proof*⟩

lemma *disjoint-fareys-plus1*:
assumes $n > 0$
shows *disjnt* (*set* (*fareys* n)) (*fareys-new* ($1 + n$))
⟨*proof*⟩

lemma *set-fareys-plus1*: *set* (*fareys* ($1 + n$)) = *set* (*fareys* n) ∪ *fareys-new* ($1 + n$)
⟨*proof*⟩

lemma *length-fareys-Suc*:
assumes $n>0$
shows *length* (*fareys* (1 + *int* n)) = *length* (*fareys* n) + *totient* (*Suc* n)
⟨*proof*⟩

lemma *fareys-0* [*simp*]: *fareys* 0 = []
⟨*proof*⟩

lemma *fareys-1* [*simp*]: *fareys* 1 = [0, 1]
⟨*proof*⟩

lemma *fareys-2* [*simp*]: *fareys* 2 = [0, *farey* 1 2, 1]
⟨*proof*⟩

lemma *length-fareys-1*:
shows *length* (*fareys* 1) = 1 + *totient* 1
⟨*proof*⟩

lemma *length-fareys*: $n>0 \Longrightarrow$ *length* (*fareys* n) = 1 + ($\sum k=1..n.$ *totient* k)
⟨*proof*⟩

lemma *subseq-fareys-1*: *subseq* (*fareys* n) (*fareys* ($1 + n$))
⟨*proof*⟩

lemma *monotone-fareys*: *monotone* (≤) *subseq fareys*
⟨*proof*⟩

lemma *farey-unimodular-0-1* [*simp*, *intro*]: *farey-unimodular* 0 1
⟨*proof*⟩

Apostol's Theorem 5.2 for integers

lemma *mediant-lies-betw-int*:
fixes a b c d::*int*
assumes *rat-of-int* a / *of-int* b < *of-int* c / *of-int* d $b>0$ $d>0$

shows *rat-of-int a* / *of-int b* < (*of-int a* + *of-int c*) / (*of-int b* + *of-int d*)
(*rat-of-int a* + *of-int c*) / (*of-int b* + *of-int d*) < *of-int c* / *of-int d*
⟨*proof*⟩

Apostol's Theorem 5.2

theorem *mediant-inbetween*:
fixes *x y*::*farey*
assumes *x* < *y*
shows *x* < *mediant x y* *mediant x y* < *y*
⟨*proof*⟩

lemma *sublist-fareysD*:
assumes *sublist* [*x*,*y*] (*fareys n*)
obtains *x* ∈ *set* (*fareys n*) *y* ∈ *set* (*fareys n*)
⟨*proof*⟩

Adding the denominators of two consecutive Farey fractions

lemma *sublist-fareys-add-denoms*:
fixes *a b c d*::*int*
defines *x* ≡ *Fract a b*
defines *y* ≡ *Fract c d*
assumes *sub*: *sublist* [*x*,*y*] (*fareys* (*int n*)) **and** *b*>*0* *d*>*0* *coprime a b* *coprime c d*
shows *b* + *d* > *n*
⟨*proof*⟩

1.3 Apostol's Theorems 5.3–5.5

theorem *consec-subset-fareys*:
fixes *a b c d*::*int*
assumes *abcd*: *0* ≤ *Fract a b* *Fract a b* < *Fract c d* *Fract c d* ≤ *1*
and *consec*: *b*∗*c* − *a*∗*d* = *1*
and *max*: *max b d* ≤ *n* *n* < *b*+*d*
and *b*>*0*
shows *sublist* [*Fract a b*, *Fract c d*] (*fareys n*)
⟨*proof*⟩

lemma *farey-unimodular-mediant*:
assumes *farey-unimodular x y*
shows *farey-unimodular x* (*mediant x y*) *farey-unimodular* (*mediant x y*) *y*
⟨*proof*⟩

Apostol's Theorem 5.4

theorem *mediant-unimodular*:
fixes *a b c d*::*int*
assumes *abcd*: *0* ≤ *Fract a b* *Fract a b* < *Fract c d* *Fract c d* ≤ *1*
and *consec*: *b*∗*c* − *a*∗*d* = *1*
and *0*: *b*>*0* *d*>*0*
defines *h* ≡ *a*+*c*

defines $k \equiv b+d$
obtains *Fract a b* $<$ *Fract h k Fract h k* $<$ *Fract c d coprime h k*
$b{*}h - a{*}k = 1$ $c{*}k - d{*}h = 1$
⟨*proof*⟩

Apostol's Theorem 5.5, first part: "Each fraction in $F\ (n + 1)$ which is not in $F\ n$ is the mediant of a pair of consecutive fractions in $F\ n$

lemma *get-consecutive-parents*:
fixes *m n*::*int*
assumes *coprime m n* $0<m$ $m<n$
obtains *a b c d* **where** $m = a+c$ $n = b+d$ $b{*}c - a{*}d = 1$ $a\geq 0$ $b>0$ $c>0$ $d>0$ $a<b$ $c\leq d$
⟨*proof*⟩

theorem *fareys-new-eq-mediant*:
assumes $x \in$ *fareys-new n* $n>1$
obtains *a b c d* **where**
sublist [*Fract a b*, *Fract c d*] (*fareys* $(n-1)$)
$x =$ *mediant* (*Fract a b*) (*Fract c d*) *coprime a b coprime c d* $a\geq 0$ $b>0$ $c>0$ $d>0$
⟨*proof*⟩

Apostol's Theorem 5.5, second part: "Moreover, if $a\ /\ b < c\ /\ d$ are consecutive in any $F\ n$, then they satisfy the unimodular relation $bc - ad = 1$.

theorem *consec-imp-unimodular*:
assumes *sublist* [*Fract a b*, *Fract c d*] (*fareys* (*int n*)) $b>0$ $d>0$ *coprime a b coprime c d*
shows $b{*}c - a{*}d = 1$
⟨*proof*⟩

1.4 Ford circles

definition *Ford-center* :: *rat* ⇒ *complex* **where**
Ford-center r ≡ (λ(*h*,*k*). *Complex* (h/k) $(1/(2 * k\hat{~}2))$) (*quotient-of r*)

definition *Ford-radius* :: *rat* ⇒ *real* **where**
Ford-radius r ≡ (λ(*h*,*k*). $1/(2 * k\hat{~}2)$) (*quotient-of r*)

definition *Ford-tan* :: [*rat*,*rat*] ⇒ *bool* **where**
Ford-tan r s ≡ *dist* (*Ford-center r*) (*Ford-center s*) = *Ford-radius r* + *Ford-radius s*

definition *Ford-circle* :: *rat* ⇒ *complex set* **where**
Ford-circle r ≡ *sphere* (*Ford-center r*) (*Ford-radius r*)

lemma *Im-Ford-center* [*simp*]: *Im* (*Ford-center r*) = *Ford-radius r*
⟨*proof*⟩

lemma *Ford-radius-nonneg*: *Ford-radius r* $\geq$ *0*
⟨*proof*⟩

lemma *two-Ford-tangent*:
assumes *r*: (*a*,*b*) = *quotient-of r* **and** *s*: (*c*,*d*) = *quotient-of s*
shows (*dist* (*Ford-center r*) (*Ford-center s*))ˆ*2* − (*Ford-radius r* + *Ford-radius s*)ˆ*2*
= ((*a*∗*d* − *b*∗*c*)ˆ*2* − *1*) / (*b*∗*d*)ˆ*2*
⟨*proof*⟩

Apostol's Theorem 5.6

lemma *two-Ford-tangent-iff*:
assumes *r*: (*a*,*b*) = *quotient-of r* **and** *s*: (*c*,*d*) = *quotient-of s*
shows *Ford-tan r s* $\longleftrightarrow$ $|b * c - a * d| = 1$
⟨*proof*⟩

Also Apostol's Theorem 5.6: Distinct Ford circles do not overlap

lemma *Ford-no-overlap*:
assumes *r* $\neq$ *s*
shows *dist* (*Ford-center r*) (*Ford-center s*) $\geq$ *Ford-radius r* + *Ford-radius s*
⟨*proof*⟩

lemma *Ford-aux1*:
assumes *a*$\neq$*0*
shows *cmod* (*Complex* (b / (a ∗ (a^2 + b^2))) (*1* / (*2* ∗ a^2) − *inverse* (a^2 + b^2))) = *1* / (*2* ∗ a^2)
(**is** *cmod ?z* = *?r*)
⟨*proof*⟩

lemma *Ford-aux2*:
assumes *a*$\neq$*0*
shows *cmod* (*Complex* (a / (b ∗ (b^2 + a^2)) − *1* / (a ∗ b)) (*1* / (*2* ∗ a^2) − *inverse* (b^2 + a^2))) = *1* / (*2* ∗ a^2)
(**is** *cmod ?z* = *?r*)
⟨*proof*⟩

definition *Radem-trans* :: *rat* ⇒ *complex* ⇒ *complex* **where**
Radem-trans ≡ λ*r* τ. *let* (*h*,*k*) = *quotient-of r in* − i ∗ *of-int k* ˆ *2* ∗ (τ − *of-rat r*)

Theorem 5.8 first part

lemma *Radem-trans-image*: *Radem-trans r* ‘ *Ford-circle r* = *sphere* (*of-rat* (*1*/*2*)) (*1*/*2*)
⟨*proof*⟩

locale *three-Ford* =
fixes *N*::*nat*
fixes *h1 k1 h k h2 k2*::*int*
assumes *sub1*: *sublist* [*Fract h1 k1*, *Fract h k*] (*fareys* (*int N*))

assumes *sub2*: *sublist* [*Fract h k*, *Fract h2 k2*] (*fareys* (*int N*))
assumes *coprime*: *coprime h1 k1 coprime h k coprime h2 k2*
assumes *k-pos*: $k1 > 0$ $k > 0$ $k2 > 0$

begin

definition $r1 \equiv$ *Fract h1 k1*
definition $r \equiv$ *Fract h k*
definition $r2 \equiv$ *Fract h2 k2*

lemma *N-pos*: $N>0$
⟨*proof*⟩

lemma *r-eq-quotient*:
$(h1,k1)$ = *quotient-of r1* (h,k) = *quotient-of r* $(h2,k2)$ = *quotient-of r2*
⟨*proof*⟩

lemma *r-eq-divide*:
r1 = *of-int h1* / *of-int k1* *r* = *of-int h* / *of-int k* *r2* = *of-int h2*/ *of-int k2*
⟨*proof*⟩

lemma *collapse-r*:
real-of-int h1 / *of-int k1* = *of-rat r1*
real-of-int h / *of-int k* = *of-rat r* *real-of-int h2*/ *of-int k2* = *of-rat r2*
⟨*proof*⟩

lemma *unimod1*: $k1*h - h1*k = 1$
and *unimod2*: $k*h2 - h*k2 = 1$
⟨*proof*⟩

lemma *r-less*: $r1 < r$ $r < r2$
⟨*proof*⟩

lemma *r01*:
obtains $r1 \in \{0..1\}$ $r \in \{0..1\}$ $r2 \in \{0..1\}$
⟨*proof*⟩

lemma *atMost-N*:
obtains $k1 \leq N$ $k \leq N$ $k2 \leq N$
⟨*proof*⟩

lemma *greaterN1*: $k1 + k > N$
⟨*proof*⟩

lemma *greaterN2*: $k + k2 > N$
⟨*proof*⟩

definition *alpha1* $\equiv$ *Complex* ($h/k - k1$ / *of-int*$(k * (k^2 + k1^2))$) (*inverse* (*of-int* $(k^2 + k1^2)$))

definition *alpha2* $\equiv$ *Complex* $(h/k + k2 \ / \ \textit{of-int}(k * (k^2 + k2^2)))$ $(\textit{inverse} \ (\textit{of-int} \ (k^2 + k2^2)))$

definition *zed1* $\equiv$ *Complex* (k^2) $(k{*}k1) \ / \ ((k^2 + k1^2))$
definition *zed2* $\equiv$ *Complex* (k^2) $(- \ k{*}k2) \ / \ ((k^2 + k2^2))$

Apostol's Theorem 5.7

lemma *three-Ford-tangent*:
obtains *alpha1* $\in$ *Ford-circle r alpha1* $\in$ *Ford-circle r1*
alpha2 $\in$ *Ford-circle r alpha2* $\in$ *Ford-circle r2*
⟨*proof*⟩

Theorem 5.8 second part, for alpha1

lemma *Radem-trans-alpha1*: *Radem-trans r alpha1* = *zed1*
⟨*proof*⟩

Theorem 5.8 second part, for alpha2

lemma *Radem-trans-alpha2*: *Radem-trans r alpha2* = *zed2*
⟨*proof*⟩

Theorem 5.9, for zed1

lemma *cmod-zed1*: *cmod zed1* $= k \ / \ \textit{sqrt} \ (k^2 + k1^2)$
⟨*proof*⟩

Theorem 5.9, for zed2

lemma *cmod-zed2*: *cmod zed2* $= k \ / \ \textit{sqrt} \ (k^2 + k2^2)$
⟨*proof*⟩

The last part of theorem 5.9

lemma *RMS-calc*:
assumes $k' > 0$ $k' + k > \textit{int} \ N$
shows $k \ / \ \textit{sqrt} \ (k^2 + k'^2) < \textit{sqrt} \ 2 * k/N$
⟨*proof*⟩

lemma *on-chord-bounded-cmod*:
assumes $z \in$ *closed-segment zed1 zed2*
shows *cmod* $z < \textit{sqrt} \ 2 * k/N$
⟨*proof*⟩

lemma *chord-length-less*: *dist zed1 zed2* $< 2 * \textit{sqrt} \ 2 * k \ / \ N$
⟨*proof*⟩

end

end

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References

[1] T. M. Apostol. *Modular Functions and Dirichlet Series in Number Theory.* Springer, 1990.