The Fair Games Theorem

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Abstract

This is the optional stopping theorem (or fair games theorem) for real-valued discrete-time stochastic processes on a σ-finite filtered measure space. A theory of piecewise-constant stopping times — the function taking i on an $\mathcal{F}_i$-measurable set S and j on its complement — is developed, together with the corresponding decomposition of integrals over the two pieces. The central results then prove both directions of the theorem: if X is a submartingale, then for any bounded stopping times $\tau \leq \pi$ the expected stopped values satisfy $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_\pi]$ (via the telescoping identity and the submartingale set-integral inequality); conversely, monotonicity of expected stopped values over all bounded stopping times characterises submartingales. It is further shown that the stopped process X^τ of a submartingale is again a submartingale. Throughout, the theorems are cross-referenced to their counterparts in Mathlib's **OptionalStopping**, from which it was translated by Claude 4.6. It was polished manually afterwards.

The fair games theorem is #62 on the *Top 100 Mathematical Theorems*.

Contents

1 Integrability of Stopped Values and Processes

theory *Stopped-Value-Integration*
  imports *Doob-Convergence.Stopping-Time*
begin

Integrability of stopped values and stopped processes, and the telescoping identity for differences of stopped values. These results bridge the gap between the existing AFP theories (Martingales, `Doob_Convergence`) and the optional stopping theorem.

context *nat-sigma-finite-filtered-measure*
begin

1.1 Stopped value as a sum of indicators

A stopped value with a bounded stopping time can be written as a finite sum of indicators.

lemma *stopped-value-eq-sum*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow 'b :: real\text{-}vector$
  assumes τ*-st*: *stopping-time* τ **and** τ*-bnd*: $\bigwedge\omega.\ \omega \in space\ M \Longrightarrow \tau\ \omega \leq N$
  assumes ω*-in*: $\omega \in space\ M$
  shows $stopped\text{-}value\ X\ \tau\ \omega = (\sum i{\leq}N.\ indicator\ \{\omega \in space\ M.\ \tau\ \omega = i\}\ \omega *_R X\ i\ \omega)$
⟨*proof*⟩

1.2 Telescoping identity for stopped values

The difference of stopped values can be expressed as a sum of indicator-weighted increments. This corresponds to `stoppedValue_sub_eq_sum'` in Mathlib.

lemma *stopped-value-sub-eq-sum*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow real$
  assumes τ*-st*: *stopping-time* τ **and** π*-st*: *stopping-time* π
    and *le*: $\bigwedge\omega.\ \omega \in space\ M \Longrightarrow \tau\ \omega \leq \pi\ \omega$
    and *bnd*: $\bigwedge\omega.\ \omega \in space\ M \Longrightarrow \pi\ \omega \leq N$
    and ω*-in*: $\omega \in space\ M$
  shows $stopped\text{-}value\ X\ \pi\ \omega - stopped\text{-}value\ X\ \tau\ \omega =$
    $(\sum i{\leq}N.\ indicator\ \{\omega \in space\ M.\ \tau\ \omega \leq i \wedge i < \pi\ \omega\}\ \omega * (X\ (Suc\ i)\ \omega - X\ i\ \omega))$
⟨*proof*⟩

If each $X\ i$ is integrable and the stopping time is bounded, then the stopped value is integrable. This corresponds to `integrable_stoppedValue` in Mathlib.

lemma *integrable-stopped-value*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow real$
  assumes τ*-st*: *stopping-time* τ **and** *int-X*: $\bigwedge i.\ integrable\ M\ (X\ i)$

and τ-*bnd*: $\bigwedge\omega.\ \omega \in$ *space* $M \Longrightarrow \tau\ \omega \leq N$
shows *integrable* M (*stopped-value* $X\ \tau$)
⟨*proof*⟩

1.3 Stopped process

The stopped process X^τ is defined as X (*min* $i\ \tau$).

definition *stopped-process* :: $(nat \Rightarrow 'a \Rightarrow 'b) \Rightarrow ('a \Rightarrow nat) \Rightarrow nat \Rightarrow 'a \Rightarrow 'b$
where
stopped-process $X\ \tau\ i\ \omega \equiv X$ (*min* i $(\tau\ \omega)$) ω

lemma *stopped-process-eq-stopped-value*:
stopped-process $X\ \tau\ i =$ *stopped-value* X ($\lambda\omega.$ *min* i $(\tau\ \omega)$)
⟨*proof*⟩

lemma *integrable-stopped-process*:
fixes X :: $nat \Rightarrow 'a \Rightarrow real$
assumes *stopping-time* τ $\bigwedge i.$ *integrable* M $(X\ i)$
shows *integrable* M (*stopped-process* $X\ \tau\ n$)
⟨*proof*⟩

end

end

2 Piecewise Stopping Times

theory *Piecewise-Stopping-Time*
imports *Stopped-Value-Integration*

begin

Piecewise constant stopping times and their interaction with stopped values and integration. These are needed for the converse direction of the optional stopping theorem.

context *nat-sigma-finite-filtered-measure*
begin

2.1 Piecewise constant stopping times

Given $i \leq j$ and an F i-measurable set S, the function that returns i on S and j on its complement is a stopping time. This corresponds to `isStoppingTime_piecewise_const` in Mathlib.

lemma *stopping-time-piecewise-const*:
assumes $i \leq j$ **and** S: $S \in$ *sets* $(F\ i)$
shows *stopping-time* ($\lambda\omega.$ *if* $\omega \in S$ *then* i *else* j)
⟨*proof*⟩

The stopped value at a piecewise constant stopping time decomposes into a piecewise function. This corresponds to `stoppedValue_piecewise_const` in Mathlib.

lemma *stopped-value-piecewise-const*:
  assumes $S \subseteq$ *space M*
  shows *stopped-value X* $(\lambda\omega.$ *if* $\omega \in S$ *then i else j*$) = (\lambda\omega.$ *if* $\omega \in S$ *then X i* ω *else X j* $\omega)$
  ⟨*proof*⟩

2.2 Integration over piecewise functions

The integral of a piecewise function splits into integrals over the two pieces. This corresponds to `integral_piecewise` in Mathlib.

lemma *piecewise-eq-indicator-sum*:
  fixes *f g* :: $'a \Rightarrow$ *real*
  assumes $S \in$ *sets M* $\omega \in$ *space M*
  shows (*if* $\omega \in S$ *then f* ω *else g* ω) = *indicat-real S* $\omega * f\ \omega$ + *indicat-real* (*space M* $-$ *S*) $\omega * g\ \omega$
  ⟨*proof*⟩

lemma *integral-piecewise*:
  fixes *f g* :: $'a \Rightarrow$ *real*
  assumes *S-meas*: $S \in$ *sets M* **and** *int-f*: *integrable M f* **and** *int-g*: *integrable M g*
  shows $(\int \omega.$ (*if* $\omega \in S$ *then f* ω *else g* ω) $\partial M) =$
    set-lebesgue-integral M S f + *set-lebesgue-integral M* (*space M* $-$ *S*) *g*
⟨*proof*⟩

end

end

3 Fair Games Theorem

theory *Optional-Stopping*
  imports
    Piecewise-Stopping-Time
    Martingales.Martingale
begin

The optional stopping theorem (fair games theorem): an adapted integrable process is a submartingale if and only if for all bounded stopping times τ and π with $\tau \leq \pi$, the expected stopped value at τ is at most that at π. We also prove that the stopped process of a submartingale is a submartingale.

context *nat-sigma-finite-filtered-measure*
begin

3.1 Helper lemmas

lemma *indicator-set-in-F*:
assumes τ*-st*: *stopping-time* τ **and** π*-st*: *stopping-time* π
shows $\{\omega \in \mathit{space}\ M.\ \tau\ \omega \leq i \wedge i < \pi\ \omega\} \in \mathit{sets}\ (F\ i)$
$\langle\mathit{proof}\rangle$

3.2 Forward direction

If X is a submartingale and $\tau \leq \pi$ are bounded stopping times, then $\mathit{integral}^L$ M $(\mathit{stopped\text{-}value}\ X\ \tau) \leq \mathit{integral}^L\ M\ (\mathit{stopped\text{-}value}\ X\ \pi)$. This corresponds to `Submartingale.expected_stoppedValue_mono` in Mathlib.

theorem *expected-stopped-value-mono*:
fixes $X :: \mathit{nat} \Rightarrow {}'a \Rightarrow \mathit{real}$
assumes *sub*: *submartingale-linorder* $M\ F\ 0\ X$
and τ*-st*: *stopping-time* τ **and** π*-st*: *stopping-time* π
and *le*: $\bigwedge\omega.\ \omega \in \mathit{space}\ M \Longrightarrow \tau\ \omega \leq \pi\ \omega$
and *bnd*: $\bigwedge\omega.\ \omega \in \mathit{space}\ M \Longrightarrow \pi\ \omega \leq N$
shows $(\int \omega.\ \mathit{stopped\text{-}value}\ X\ \tau\ \omega\ \partial M) \leq (\int \omega.\ \mathit{stopped\text{-}value}\ X\ \pi\ \omega\ \partial M)$
$\langle\mathit{proof}\rangle$

3.3 Converse direction

If an adapted integrable process satisfies the monotonicity of expected stopped values for all bounded stopping times, then it is a submartingale. This corresponds to `submartingale_of_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-of-expected-stopped-value-mono*:
fixes $X :: \mathit{nat} \Rightarrow {}'a \Rightarrow \mathit{real}$
assumes *adapted*: *adapted-process* $M\ F\ 0\ X$
and *integrable*: $\bigwedge i.\ \mathit{integrable}\ M\ (X\ i)$
and *mono*: $\bigwedge\tau\ \pi\ N.\ \mathit{stopping\text{-}time}\ \tau \Longrightarrow \mathit{stopping\text{-}time}\ \pi \Longrightarrow$
$(\bigwedge\omega.\ \omega \in \mathit{space}\ M \Longrightarrow \tau\ \omega \leq \pi\ \omega) \Longrightarrow (\bigwedge\omega.\ \omega \in \mathit{space}\ M \Longrightarrow \pi\ \omega \leq N) \Longrightarrow$
$(\int \omega.\ \mathit{stopped\text{-}value}\ X\ \tau\ \omega\ \partial M) \leq (\int \omega.\ \mathit{stopped\text{-}value}\ X\ \pi\ \omega\ \partial M)$
shows *submartingale* $M\ F\ 0\ X$
$\langle\mathit{proof}\rangle$

3.4 The optional stopping theorem (iff)

The full characterization: an adapted integrable process is a submartingale iff expected stopped values are monotone for all bounded stopping times. This corresponds to `submartingale_iff_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-iff-expected-stopped-value-mono*:
fixes $X :: \mathit{nat} \Rightarrow {}'a \Rightarrow \mathit{real}$
assumes *adapted-process* $M\ F\ 0\ X$ $\bigwedge i.\ \mathit{integrable}\ M\ (X\ i)$
shows *submartingale* $M\ F\ 0\ X \longleftrightarrow$
$(\forall\tau\ \pi.\ \mathit{stopping\text{-}time}\ \tau \longrightarrow \mathit{stopping\text{-}time}\ \pi \longrightarrow$
$(\forall\omega{\in}\mathit{space}\ M.\ \tau\ \omega \leq \pi\ \omega) \longrightarrow (\exists N.\ \forall\omega{\in}\mathit{space}\ M.\ \pi\ \omega \leq N) \longrightarrow$

$(\int \omega.\ \textit{stopped-value}\ X\ \tau\ \omega\ \partial M) \leq (\int \omega.\ \textit{stopped-value}\ X\ \pi\ \omega\ \partial M))$ (**is** $?L = ?R$)
⟨*proof*⟩

3.5 Stopped process of a submartingale

The stopped process of a submartingale with respect to a stopping time is a submartingale. This corresponds to `Submartingale.stoppedProcess` in Mathlib.

We first show the stopped process is adapted. The proof proceeds by induction: $X^\tau\ 0 = X\ 0$ is trivially $F\ 0$-measurable, and $X^\tau\ (Suc\ n) = \textit{if}\ \tau \leq n\ \textit{then}\ X^\tau\ n\ \textit{else}\ X\ (Suc\ n)$ is $F\ (Suc\ n)$-measurable by the induction hypothesis, adaptedness of X, and the stopping time property.

lemma *adapted-stopped-process*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow real$
  assumes *adapted*: *adapted-process M F 0 X* **and** *τ-st*: *stopping-time τ*
  shows *adapted-process M F 0 (stopped-process X τ)*
⟨*proof*⟩

theorem *stopped-process-submartingale*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow real$
  assumes *sub*: *submartingale-linorder M F 0 X*
    and *τ-st*: *stopping-time τ*
  shows *submartingale M F 0 (stopped-process X τ)*
⟨*proof*⟩

end

end