The Fair Games Theorem

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25 June 2026

Abstract

This is the optional stopping theorem (or fair games theorem) for real-valued discrete-time stochastic processes on a σ-finite filtered measure space. A theory of piecewise-constant stopping times — the function taking i on an $\mathcal{F}_i$-measurable set S and j on its complement — is developed, together with the corresponding decomposition of integrals over the two pieces. The central results then prove both directions of the theorem: if X is a submartingale, then for any bounded stopping times $\tau \leq \pi$ the expected stopped values satisfy $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_\pi]$ (via the telescoping identity and the submartingale set-integral inequality); conversely, monotonicity of expected stopped values over all bounded stopping times characterises submartingales. It is further shown that the stopped process X^τ of a submartingale is again a submartingale. Throughout, the theorems are cross-referenced to their counterparts in Mathlib's **OptionalStopping**, from which it was translated by Claude 4.6. It was polished manually afterwards.

The fair games theorem is #62 on the *Top 100 Mathematical Theorems*.

Contents

1 Integrability of Stopped Values and Processes

theory *Stopped-Value-Integration*
  imports *Doob-Convergence.Stopping-Time*
begin

Integrability of stopped values and stopped processes, and the telescoping identity for differences of stopped values. These results bridge the gap between the existing AFP theories (Martingales, `Doob_Convergence`) and the optional stopping theorem.

context *nat-sigma-finite-filtered-measure*
begin

1.1 Stopped value as a sum of indicators

A stopped value with a bounded stopping time can be written as a finite sum of indicators.

lemma *stopped-value-eq-sum*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow 'b :: real\text{-}vector$
  assumes τ-*st*: *stopping-time* τ **and** τ-*bnd*: $\bigwedge\omega.\ \omega \in space\ M \Longrightarrow \tau\ \omega \le N$
  assumes ω-*in*: $\omega \in space\ M$
  shows $stopped\text{-}value\ X\ \tau\ \omega = (\sum i{\le}N.\ indicator\ \{\omega \in space\ M.\ \tau\ \omega = i\}\ \omega *_R X\ i\ \omega)$
proof −
  have $stopped\text{-}value\ X\ \tau\ \omega = X\ (\tau\ \omega)\ \omega$
    by (*simp add*: *stopped-value-def*)
  also have $\ldots = 1 *_R X\ (\tau\ \omega)\ \omega$ **by** (*metis* (*full-types*) *scale-one*)
  also have $\ldots = indicator\ \{\omega \in space\ M.\ \tau\ \omega = \tau\ \omega\}\ \omega *_R X\ (\tau\ \omega)\ \omega +$
    $(\sum i \in \{..N\} - \{\tau\ \omega\}.\ indicator\ \{\omega \in space\ M.\ \tau\ \omega = i\}\ \omega *_R X\ i\ \omega)$
    using ω-*in* **by** (*simp add*: *indicator-def*)
  also have $\ldots = (\sum i{\le}N.\ indicator\ \{\omega \in space\ M.\ \tau\ \omega = i\}\ \omega *_R X\ i\ \omega)$
    using ω-*in* τ-*bnd*[*OF* ω-*in*]
    by (*subst sum.remove* [**where** $x = \tau\ \omega$]) *simp-all*
  finally show *?thesis* **.**
qed

1.2 Telescoping identity for stopped values

The difference of stopped values can be expressed as a sum of indicator-weighted increments. This corresponds to `stoppedValue_sub_eq_sum’` in Mathlib.

lemma *stopped-value-sub-eq-sum*:
  fixes $X :: nat \Rightarrow 'a \Rightarrow real$
  assumes τ-*st*: *stopping-time* τ **and** π-*st*: *stopping-time* π
    and *le*: $\bigwedge\omega.\ \omega \in space\ M \Longrightarrow \tau\ \omega \le \pi\ \omega$
    and *bnd*: $\bigwedge\omega.\ \omega \in space\ M \Longrightarrow \pi\ \omega \le N$
    and ω-*in*: $\omega \in space\ M$
  shows $stopped\text{-}value\ X\ \pi\ \omega - stopped\text{-}value\ X\ \tau\ \omega =$

$(\sum i{\le}N.\ \mathit{indicator}\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega \le i \wedge i < \pi\ \omega\}\ \omega * (X\ (\mathit{Suc}\ i)\ \omega - X\ i\ \omega))$
proof −
have $(\sum i{\le}N.\ \mathit{indicator}\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega \le i \wedge i < \pi\ \omega\}\ \omega * (X\ (\mathit{Suc}\ i)\ \omega - X\ i\ \omega)) =$
$(\sum i{\in}\{\tau\ \omega..{<}\pi\ \omega\}.\ X\ (\mathit{Suc}\ i)\ \omega - X\ i\ \omega)$
using *ω-in bnd*[*OF ω-in*]
by (*intro sum.mono-neutral-cong-right*) (*auto simp*: *indicator-def*)
also have $\ldots = X\ (\pi\ \omega)\ \omega - X\ (\tau\ \omega)\ \omega$
using *ω-in le sum-Suc-diff′* **by** *fastforce*
finally show *?thesis* **by** (*simp add*: *stopped-value-def*)
qed

If each $X\ i$ is integrable and the stopping time is bounded, then the stopped value is integrable. This corresponds to `integrable_stoppedValue` in Mathlib.

lemma *integrable-stopped-value*:
fixes $X :: \mathit{nat} \Rightarrow {}'a \Rightarrow \mathit{real}$
assumes *τ-st*: *stopping-time τ* **and** *int-X*: $\bigwedge i.\ \mathit{integrable}\ M\ (X\ i)$
and *τ-bnd*: $\bigwedge \omega.\ \omega \in \mathit{space}\ M \Longrightarrow \tau\ \omega \le N$
shows *integrable M* (*stopped-value X τ*)
proof −
— Each indicator set is measurable
have *meas-eq*: $\bigwedge i.\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\} \in \mathit{sets}\ M$
proof −
fix $i :: \mathit{nat}$
have *Measurable.pred* (*F i*) ($\lambda\omega.\ \tau\ \omega = i$)
by (*rule stopping-time-measurable-eq*[*OF τ-st*]) *simp-all*
then have $\{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\} \in \mathit{sets}\ (F\ i)$
by (*metis predE subalg subalgebra-def*)
then show $\{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\} \in \mathit{sets}\ M$
using *sets-F-subset*[*of i*] **by** *blast*
qed
— Each summand is integrable
have *int-summand*: $\bigwedge i.\ \mathit{integrable}\ M\ (\lambda\omega.\ \mathit{indicat\text{-}real}\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\}\ \omega * X\ i\ \omega)$
by (*simp add*: *int-X integrable-real-mult-indicator meas-eq mult.commute*)
— The sum is integrable
have *int-sum*: $\mathit{integrable}\ M\ (\lambda\omega.\ \sum i{\le}N.\ \mathit{indicat\text{-}real}\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\}\ \omega * X\ i\ \omega)$
by (*intro Bochner-Integration.integrable-sum*) (*auto intro*: *int-summand*)
— The stopped value agrees with the sum AE
have *eq*: $\mathit{AE}\ \omega\ \mathit{in}\ M.\ (\sum i{\le}N.\ \mathit{indicat\text{-}real}\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\}\ \omega * X\ i\ \omega) =$
stopped-value X τ ω
by (*intro AE-I2*) (*simp add*: *stopped-value-eq-sum*[*OF τ-st τ-bnd*])
have *eq-space*: $\bigwedge \omega.\ \omega \in \mathit{space}\ M \Longrightarrow$
$\mathit{stopped\text{-}value}\ X\ \tau\ \omega = (\sum i{\le}N.\ \mathit{indicat\text{-}real}\ \{\omega \in \mathit{space}\ M.\ \tau\ \omega = i\}\ \omega * X\ i\ \omega)$

by (*simp add*: *stopped-value-eq-sum*[*OF* τ-*st* τ-*bnd*])
— Measurability via ($\bigwedge w.\ w \in space\ ?M \Longrightarrow\ ?f\ w =\ ?g\ w) \Longrightarrow (?f \in\ ?M \rightarrow_M\ ?M') = (?g \in\ ?M \rightarrow_M\ ?M'$) with the sum
have *meas-sum*: ($\lambda\omega.\ \sum i{\le}N.\ indicat\text{-}real\ \{\omega \in space\ M.\ \tau\ \omega = i\}\ \omega * X\ i\ \omega) \in borel\text{-}measurable\ M$
using *int-sum* **by** (*rule borel-measurable-integrable*)
have *meas-sv*: *stopped-value* $X\ \tau \in$ *borel-measurable* M
using *measurable-cong*[*of M stopped-value X* τ
$\lambda\omega.\ \sum i{\le}N.\ indicat\text{-}real\ \{\omega \in space\ M.\ \tau\ \omega = i\}\ \omega * X\ i\ \omega$ *borel*]
eq-space meas-sum **by** *auto*
— Transfer integrability via AE equality
show *?thesis*
by (*rule Bochner-Integration.integrable-cong-AE-imp*[*OF int-sum meas-sv eq*])
qed

1.3 Stopped process

The stopped process X^τ is defined as $X\ (min\ i\ \tau)$.

definition *stopped-process* :: ($nat \Rightarrow 'a \Rightarrow 'b) \Rightarrow ('a \Rightarrow nat) \Rightarrow nat \Rightarrow 'a \Rightarrow 'b$
where
stopped-process $X\ \tau\ i\ \omega \equiv X\ (min\ i\ (\tau\ \omega))\ \omega$

lemma *stopped-process-eq-stopped-value*:
stopped-process $X\ \tau\ i =$ *stopped-value* $X\ (\lambda\omega.\ min\ i\ (\tau\ \omega))$
unfolding *stopped-process-def stopped-value-def* **by** *simp*

lemma *integrable-stopped-process*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *stopping-time* $\tau\ \bigwedge i.$ *integrable* $M\ (X\ i)$
shows *integrable* M (*stopped-process* $X\ \tau\ n$)
proof −
have *stopping-time* ($\lambda\omega.\ min\ n\ (\tau\ \omega)$)
by (*intro stopping-time-min stopping-time-const assms*) *simp*
with *assms* **show** *?thesis*
unfolding *stopped-process-eq-stopped-value*
using *integrable-stopped-value min.cobounded1* **by** *blast*
qed

end

end

2 Piecewise Stopping Times

theory *Piecewise-Stopping-Time*
imports *Stopped-Value-Integration*

begin

Piecewise constant stopping times and their interaction with stopped values and integration. These are needed for the converse direction of the optional stopping theorem.

```
context nat-sigma-finite-filtered-measure
begin
```

2.1 Piecewise constant stopping times

Given $i \le j$ and an F i-measurable set S, the function that returns i on S and j on its complement is a stopping time. This corresponds to `isStoppingTime_piecewise_const` in Mathlib.

```
lemma stopping-time-piecewise-const:
  assumes i ≤ j and S: S ∈ sets (F i)
  shows stopping-time (λω. if ω ∈ S then i else j)
proof (rule stopping-timeI)
  fix ω assume ω ∈ space M
  show 0 ≤ (if ω ∈ S then i else j) by simp
next
  fix t :: nat assume 0 ≤ t
  have space-eq: space (F t) = space M by simp
  have top: space M ∈ sets (F t) using sets.top[of F t] by simp
  show Measurable.pred (F t) (λω. (if ω ∈ S then i else j) ≤ t)
    unfolding Measurable.pred-def
  proof −
    consider j ≤ t | i ≤ t ¬ j ≤ t | ¬ i ≤ t
      using ‹i ≤ j› by linarith
    then show {ω ∈ space (F t). (if ω ∈ S then i else j) ≤ t} ∈ sets (F t)
    proof cases
      case 1
      then have {ω ∈ space (F t). (if ω ∈ S then i else j) ≤ t} = space (F t)
        using ‹i ≤ j› by (auto simp: space-eq)
      then show ?thesis using top space-eq by simp
    next
      case 2
      then have {ω ∈ space (F t). (if ω ∈ S then i else j) ≤ t} = S
        using S sets.sets-into-space subset-antisym by fastforce
      then show ?thesis
        using S 2 sets-F-mono by force
    next
      case 3
      then show ?thesis
        using ‹i ≤ j› by (auto simp: Measurable.pred-def space-eq split: if-splits)
    qed
  qed
qed
```

The stopped value at a piecewise constant stopping time decomposes into a piecewise function. This corresponds to `stoppedValue_piecewise_const` in Mathlib.

lemma *stopped-value-piecewise-const*:
  assumes $S \subseteq$ *space M*
  shows *stopped-value X* ($\lambda\omega$. *if* $\omega \in S$ *then i else j*) = ($\lambda\omega$. *if* $\omega \in S$ *then X i* ω *else X j* ω)
  unfolding *stopped-value-def* **by** (*simp add*: *if-distrib if-distribR*)

2.2 Integration over piecewise functions

The integral of a piecewise function splits into integrals over the two pieces. This corresponds to `integral_piecewise` in Mathlib.

lemma *piecewise-eq-indicator-sum*:
  fixes *f g* :: $'a \Rightarrow$ *real*
  assumes $S \in$ *sets M* $\omega \in$ *space M*
  shows (*if* $\omega \in S$ *then f* ω *else g* ω) = *indicat-real S* $\omega * f\ \omega$ + *indicat-real* (*space M* $-$ *S*) $\omega * g\ \omega$
  using ‹$\omega \in$ *space M*› **by** (*auto simp*: *indicator-def*)

lemma *integral-piecewise*:
  fixes *f g* :: $'a \Rightarrow$ *real*
  assumes *S-meas*: $S \in$ *sets M* **and** *int-f*: *integrable M f* **and** *int-g*: *integrable M g*
  shows ($\int\omega$. (*if* $\omega \in S$ *then f* ω *else g* ω) ∂M) =
    set-lebesgue-integral M S f + *set-lebesgue-integral M* (*space M* $-$ *S*) *g*
proof $-$
  let *?h* = $\lambda\omega$. *indicat-real S* $\omega * f\ \omega$ + *indicat-real* (*space M* $-$ *S*) $\omega * g\ \omega$
  have *eq-ae*: *AE* ω *in M*. (*if* $\omega \in S$ *then f* ω *else g* ω) = *?h* ω
    by (*rule AE-I2*) (*auto simp*: *indicator-def*)
  have *int-Sf*: *integrable M* ($\lambda\omega$. *indicat-real S* $\omega * f\ \omega$)
    using *integrable-mult-indicator*[*OF S-meas int-f*]
    by (*simp add*: *scaleR-conv-of-real*)
  have *int-Sg*: *integrable M* ($\lambda\omega$. *indicat-real* (*space M* $-$ *S*) $\omega * g\ \omega$)
    using *integrable-mult-indicator*[*OF - int-g*] **by** (*simp add*: *S-meas sets.Diff*)
  have *meas-if*: ($\lambda\omega$. *if* $\omega \in S$ *then f* ω *else g* ω) $\in$ *borel-measurable M*
    by (*intro measurable-If-set*) (*use int-f int-g S-meas* **in** *auto*)
  have *meas-h*: *?h* $\in$ *borel-measurable M*
    using *int-Sf int-Sg* **by** (*intro borel-measurable-add*) *auto*
  have ($\int\omega$. (*if* $\omega \in S$ *then f* ω *else g* ω) ∂M) = ($\int\omega$. *?h* ω ∂M)
    by (*rule integral-cong-AE*[*OF meas-if meas-h eq-ae*])
  also have ... = ($\int\omega$. *indicat-real S* $\omega * f\ \omega$ ∂M) + ($\int\omega$. *indicat-real* (*space M* $-$ *S*) $\omega * g\ \omega$ ∂M)
    by (*rule Bochner-Integration.integral-add*[*OF int-Sf int-Sg*])
  also have ... = *set-lebesgue-integral M S f* + *set-lebesgue-integral M* (*space M* $-$ *S*) *g*
    unfolding *set-lebesgue-integral-def* **by** (*simp add*: *scaleR-conv-of-real*)
  finally show *?thesis* **.**
qed

end

end

3 Fair Games Theorem

theory *Optional-Stopping*
  imports
    Piecewise-Stopping-Time
    Martingales.Martingale
begin

The optional stopping theorem (fair games theorem): an adapted integrable process is a submartingale if and only if for all bounded stopping times τ and π with $\tau \leq \pi$, the expected stopped value at τ is at most that at π. We also prove that the stopped process of a submartingale is a submartingale.

context *nat-sigma-finite-filtered-measure*
begin

3.1 Helper lemmas

lemma *indicator-set-in-F*:
  assumes τ*-st*: *stopping-time* τ **and** π*-st*: *stopping-time* π
  shows $\{\omega \in$ *space* $M.\ \tau\ \omega \leq i \wedge i < \pi\ \omega\} \in$ *sets* $(F\ i)$
proof $-$
  have $*$: *Measurable.pred* $(F\ i)$ $(\lambda\omega.\ \tau\ \omega \leq i \wedge \neg\ \pi\ \omega \leq i)$
    by (*simp add*: π*-st* τ*-st pred-intros-logic stopping-timeD*)
  have *eq*: $\{\omega \in$ *space* $M.\ \tau\ \omega \leq i \wedge i < \pi\ \omega\} = \{\omega \in$ *space* $(F\ i).\ \tau\ \omega \leq i \wedge \neg\ \pi\ \omega \leq i\}$
    by *auto*
  show *?thesis* **using** *Measurable.predE*[*OF* $*$] *sets-F-subset*[*of i*]
    by (*auto simp*: *eq*)
qed

3.2 Forward direction

If X is a submartingale and $\tau \leq \pi$ are bounded stopping times, then *integral*^{L} M (*stopped-value* $X\ \tau$) $\leq$ *integral*^{L} M (*stopped-value* $X\ \pi$). This corresponds to `Submartingale.expected_stoppedValue_mono` in Mathlib.

theorem *expected-stopped-value-mono*:
  fixes $X :: nat \Rightarrow {'a} \Rightarrow real$
  assumes *sub*: *submartingale-linorder* $M\ F\ 0\ X$
    and τ*-st*: *stopping-time* τ **and** π*-st*: *stopping-time* π
    and *le*: $\bigwedge\omega.\ \omega \in$ *space* $M \Longrightarrow \tau\ \omega \leq \pi\ \omega$
    and *bnd*: $\bigwedge\omega.\ \omega \in$ *space* $M \Longrightarrow \pi\ \omega \leq N$
  shows $(\int \omega.$ *stopped-value* $X\ \tau\ \omega\ \partial M) \leq (\int \omega.$ *stopped-value* $X\ \pi\ \omega\ \partial M)$
proof $-$

from *sub* **interpret** *S*: *submartingale-linorder M F 0 X* **.**
have *τ-bnd*: $\bigwedge\omega.\ \omega \in$ *space M* $\Longrightarrow \tau\ \omega \leq N$
using *le bnd* **by** (*meson order-trans*)
— Integrability of stopped values
obtain *int-τ*: *integrable M* (*stopped-value X τ*) **and** *int-π*: *integrable M* (*stopped-value X π*)
by (*meson S.integrable τ-bnd τ-st π-st bnd integrable-stopped-value zero-le*)
— Suffices to show the difference is non-negative
have $0 \leq (\int \omega.$ *stopped-value X π ω ∂M*) − ($\int \omega.$ *stopped-value X τ ω ∂M*)
proof −
have ($\int \omega.$ *stopped-value X π ω ∂M*) − ($\int \omega.$ *stopped-value X τ ω ∂M*) =
($\int \omega.$ *stopped-value X π ω* − *stopped-value X τ ω ∂M*)
by (*rule Bochner-Integration.integral-diff*[*OF int-π int-τ*, *symmetric*])
— Apply the telescoping identity AE
also have ... = ($\int \omega.$ ($\sum i{\leq}N.$ *indicator* {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} *ω* ∗
(*X* (*Suc i*) *ω* − *X i ω*)) *∂M*)
proof (*rule Bochner-Integration.integral-cong-AE*)
show (*λω. stopped-value X π ω* − *stopped-value X τ ω*) ∈ *borel-measurable M*
using *int-π int-τ* **by** (*intro borel-measurable-diff*) *auto*
show (*λω.* $\sum i{\leq}N.$ *indicator* {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} *ω* ∗
(*X* (*Suc i*) *ω* − *X i ω*)) ∈ *borel-measurable M*
proof (*intro borel-measurable-sum borel-measurable-times borel-measurable-indicator borel-measurable-diff borel-measurable-integrable*)
fix *i* **assume** *i* ∈ {..*N*}
show {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} ∈ *sets M*
using *indicator-set-in-F*[*OF τ-st π-st*] *sets-F-subset* **by** *blast*
qed (*auto simp*: *S.integrable*)
show *AE ω in M. stopped-value X π ω* − *stopped-value X τ ω* =
($\sum i{\leq}N.$ *indicator* {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} *ω* ∗ (*X* (*Suc i*) *ω* − *X i ω*))
by (*rule AE-I2*) (*simp add*: *stopped-value-sub-eq-sum*[*OF τ-st π-st le bnd*])
qed
— Exchange integral and sum
also have ... = ($\sum i{\leq}N.$ $\int \omega.$ *indicator* {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} *ω* ∗
(*X* (*Suc i*) *ω* − *X i ω*) *∂M*)
proof (*rule Bochner-Integration.integral-sum*)
fix *i* **assume** *i* ∈ {..*N*}
have {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} ∈ *sets M*
using *indicator-set-in-F*[*OF τ-st π-st*] *sets-F-subset* **by** *blast*
then show *integrable M* (*λω. indicator* {*ω ∈ space M. τ ω ≤ i ∧ i < π ω*} *ω* ∗ (*X* (*Suc i*) *ω* − *X i ω*))
by (*simp add*: *S.integrable integrable-real-mult-indicator mult.commute*)
qed
— Each summand is non-negative via *set-integral-le*
also have ... ≥ *0*
proof (*rule sum-nonneg*)
fix *i* **assume** *i* ∈ {..*N*}

let $?A = \{\omega \in space\ M.\ \tau\ \omega \leq i \wedge i < \pi\ \omega\}$
have *A-Fi*: $?A \in sets\ (F\ i)$
by (*rule indicator-set-in-F*[*OF* τ-*st* π-*st*])
— The summand equals $\int_A X(Suc\ i) - \int_A X\ i$
have *eq*: $(\int \omega.\ indicat\text{-}real\ ?A\ \omega * (X\ (Suc\ i)\ \omega - X\ i\ \omega)\ \partial M) =$
set-lebesgue-integral $M\ ?A\ (X\ (Suc\ i)) -$ *set-lebesgue-integral* $M\ ?A\ (X\ i)$
proof −
have $?A \in sets\ M$
using *A-Fi sets-F-subset* by *blast*
then have *int*: *integrable* $M\ (\lambda x.\ indicat\text{-}real\ ?A\ x * X\ j\ x)$ for j
by (*simp add*: *S.integrable integrable-real-mult-indicator mult.commute*)
have $(\int \omega.\ indicat\text{-}real\ ?A\ \omega * (X\ (Suc\ i)\ \omega - X\ i\ \omega)\ \partial M) =$
$(\int \omega.\ indicat\text{-}real\ ?A\ \omega * X\ (Suc\ i)\ \omega - indicat\text{-}real\ ?A\ \omega * X\ i\ \omega\ \partial M)$
by (*simp add*: *right-diff-distrib*)
also have $\ldots = (\int \omega.\ indicat\text{-}real\ ?A\ \omega * X\ (Suc\ i)\ \omega\ \partial M) - (\int \omega.\ indicat\text{-}real\ ?A\ \omega * X\ i\ \omega\ \partial M)$
using *Bochner-Integration.integral-diff int* by *blast*
also have $\ldots =$ *set-lebesgue-integral* $M\ ?A\ (X\ (Suc\ i)) -$ *set-lebesgue-integral* $M\ ?A\ (X\ i)$
unfolding *set-lebesgue-integral-def* by (*simp add*: *scaleR-conv-of-real*)
finally show *?thesis* .
qed
— Apply the submartingale *set-integral-le* property
have *set-lebesgue-integral* $M\ ?A\ (X\ i) \leq$ *set-lebesgue-integral* $M\ ?A\ (X\ (Suc\ i))$
by (*rule S.set-integral-le*[*OF A-Fi*]) *simp-all*
then show $0 \leq (\int \omega.\ indicat\text{-}real\ ?A\ \omega * (X\ (Suc\ i)\ \omega - X\ i\ \omega)\ \partial M)$
unfolding *eq* by *simp*
qed
finally show *?thesis* by *simp*
qed
then show *?thesis* by *simp*
qed

3.3 Converse direction

If an adapted integrable process satisfies the monotonicity of expected stopped values for all bounded stopping times, then it is a submartingale. This corresponds to `submartingale_of_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-of-expected-stopped-value-mono*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *adapted*: *adapted-process* $M\ F\ 0\ X$
and *integrable*: $\bigwedge i.\ integrable\ M\ (X\ i)$
and *mono*: $\bigwedge \tau\ \pi\ N.\ stopping\text{-}time\ \tau \Longrightarrow stopping\text{-}time\ \pi \Longrightarrow$
$(\bigwedge \omega.\ \omega \in space\ M \Longrightarrow \tau\ \omega \leq \pi\ \omega) \Longrightarrow (\bigwedge \omega.\ \omega \in space\ M \Longrightarrow \pi\ \omega \leq N) \Longrightarrow$
$(\int \omega.\ stopped\text{-}value\ X\ \tau\ \omega\ \partial M) \leq (\int \omega.\ stopped\text{-}value\ X\ \pi\ \omega\ \partial M)$
shows *submartingale* $M\ F\ 0\ X$
proof (*rule submartingale-of-set-integral-le-Suc*[*OF adapted integrable*])
fix $A :: 'a\ set$ and $i :: nat$

```
  assume A-Fi: A ∈ sets (F i)
  — Construct piecewise stopping times
  define τ :: 'a ⇒ nat where τ ω = (if ω ∈ A then i else Suc i) for ω
  define π :: 'a ⇒ nat where π ω = Suc i for ω
  have τ-st: stopping-time τ
    unfolding τ-def by (rule stopping-time-piecewise-const) (simp-all add: A-Fi)
  have π-st: stopping-time π
    unfolding π-def by (rule stopping-time-const) simp
  have τ-le-π: ⋀ω. ω ∈ space M ⟹ τ ω ≤ π ω
    unfolding τ-def π-def by simp
  have π-bnd: ⋀ω. ω ∈ space M ⟹ π ω ≤ Suc i
    unfolding π-def by simp
  — Apply the monotonicity hypothesis
  have ineq: (∫ ω. stopped-value X τ ω ∂M) ≤ (∫ ω. stopped-value X π ω ∂M)
    by (rule mono[OF τ-st π-st τ-le-π π-bnd])
  — Decompose stopped values
  have A-sub: A ⊆ space M
    using A-Fi sets-F-subset sets.sets-into-space by blast
  have sv-τ: stopped-value X τ = (λω. if ω ∈ A then X i ω else X (Suc i) ω)
    unfolding τ-def by (rule stopped-value-piecewise-const[OF A-sub])
  have sv-π: stopped-value X π = X (Suc i)
    unfolding π-def stopped-value-def by simp
  — Decompose the integral of the piecewise function
  have A-meas: A ∈ sets M
    using A-Fi sets-F-subset by blast
  have lhs: (∫ ω. stopped-value X τ ω ∂M) =
     set-lebesgue-integral M A (X i) + set-lebesgue-integral M (space M − A) (X
(Suc i))
    unfolding sv-τ by (rule integral-piecewise[OF A-meas integrable integrable])
  have rhs: (∫ ω. stopped-value X π ω ∂M) =
              set-lebesgue-integral M A (X (Suc i)) + set-lebesgue-integral M (space
M − A) (X (Suc i))
  proof −
    have (∫ ω. stopped-value X π ω ∂M) = (∫ ω. X (Suc i) ω ∂M)
      unfolding sv-π ..
    also have ... = (∫ ω. (if ω ∈ A then X (Suc i) ω else X (Suc i) ω) ∂M)
      by simp
    also have ... = set-lebesgue-integral M A (X (Suc i)) +
      set-lebesgue-integral M (space M − A) (X (Suc i))
      by (rule integral-piecewise[OF A-meas integrable integrable])
    finally show ?thesis .
  qed
  show set-lebesgue-integral M A (X i) ≤ set-lebesgue-integral M A (X (Suc i))
    using ineq lhs rhs by simp
qed
```

3.4 The optional stopping theorem (iff)

The full characterization: an adapted integrable process is a submartingale iff expected stopped values are monotone for all bounded stopping times. This corresponds to `submartingale_iff_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-iff-expected-stopped-value-mono*:
  fixes X :: *nat* ⇒ *'a* ⇒ *real*
  assumes *adapted-process M F 0 X* $\bigwedge i$. *integrable M* $(X\ i)$
  shows *submartingale M F 0 X* ⟷
    (∀ τ π. *stopping-time* τ ⟶ *stopping-time* π ⟶
      (∀ ω∈*space M*. τ ω ≤ π ω) ⟶ (∃ N. ∀ ω∈*space M*. π ω ≤ N) ⟶
      ($\int$ ω. *stopped-value X* τ ω ∂M) ≤ ($\int$ ω. *stopped-value X* π ω ∂M)) (**is** *?L* = *?R*)
proof
  show *?L* ⟹ *?R*
    by (*metis expected-stopped-value-mono submartingale-linorder-def*)
  show *?R* ⟹ *?L*
    by (*intro submartingale-of-expected-stopped-value-mono*[*OF assms*]) *blast*
qed

3.5 Stopped process of a submartingale

The stopped process of a submartingale with respect to a stopping time is a submartingale. This corresponds to `Submartingale.stoppedProcess` in Mathlib.

We first show the stopped process is adapted. The proof proceeds by induction: $X^\tau\ 0 = X\ 0$ is trivially $F\ 0$-measurable, and $X^\tau\ (Suc\ n) = if\ \tau \leq n\ then\ X^\tau\ n\ else\ X\ (Suc\ n)$ is $F\ (Suc\ n)$-measurable by the induction hypothesis, adaptedness of X, and the stopping time property.

lemma *adapted-stopped-process*:
  fixes X :: *nat* ⇒ *'a* ⇒ *real*
  assumes *adapted*: *adapted-process M F 0 X* **and** τ*-st*: *stopping-time* τ
  shows *adapted-process M F 0* (*stopped-process X* τ)
proof (*rule adapted-process.intro*[*OF filtered-measure-axioms*])
  show *adapted-process-axioms F 0* (*stopped-process X* τ)
  proof (*rule adapted-process-axioms.intro*)
    fix *i* :: *nat* **assume** *0* ≤ *i*
    show *stopped-process X* τ *i* ∈ *borel-measurable* (*F i*)
    proof (*induction i*)
      case *0*
      have *stopped-process X* τ *0* = *X 0*
        unfolding *stopped-process-def* **by** *simp*
      then show *?case*
        using *adapted-process.adapted*[*OF adapted*, *of 0*] **by** *simp*
    next
      case (*Suc n*)

— The stopped process at *Suc n* equals a piecewise function on *space M*
have *eq*: $\bigwedge\omega$. *stopped-process X* τ (*Suc n*) ω =
(*if* $\tau\ \omega \leq n$ *then stopped-process X* τ *n* ω *else X* (*Suc n*) ω)
unfolding *stopped-process-def* **by** (*auto simp*: *min-def*)
— *F n* is a sub-sigma-algebra of *F* (*Suc n*)
have *subalg*: *subalgebra* (*F* (*Suc n*)) (*F n*)
unfolding *subalgebra-def* **using** *space-F sets-F-mono*[*of n Suc n*] **by** *auto*
— The induction hypothesis gives *F n*-measurability, lift to *F* (*Suc n*)
have *meas-n*: *stopped-process X* τ *n* $\in$ *borel-measurable* (*F* (*Suc n*))
by (*rule measurable-from-subalg*[*OF subalg Suc.IH*])
— *X* (*Suc n*) is *F* (*Suc n*)-measurable by adaptedness
have *meas-Sn*: *X* (*Suc n*) $\in$ *borel-measurable* (*F* (*Suc n*))
using *adapted-process.adapted*[*OF adapted*] **by** *simp*
— The set $\{\omega.\ \tau\ \omega \leq n\}$ is in *sets* (*F* (*Suc n*))
have *set-Sn*: $\{\omega \in$ *space* (*F* (*Suc n*)). $\tau\ \omega \leq n\} \in$ *sets* (*F* (*Suc n*))
using τ*-st order-less-imp-le predE stopping-time-measurable-le* **by** *blast*
— The piecewise function is *F* (*Suc n*)-measurable
let *?A* = $\{\omega \in$ *space* (*F* (*Suc n*)). $\tau\ \omega \leq n\}$
have *A-sets*: *?A* $\in$ *sets* (*F* (*Suc n*)) **by** (*rule set-Sn*)
have *A-sub*: *?A* $\subseteq$ *space* (*F* (*Suc n*)) **using** *sets.sets-into-space*[*OF A-sets*] **.**
have *meas-pw*: ($\lambda\omega$. *if* $\omega \in$ *?A then stopped-process X* τ *n* ω *else X* (*Suc n*) ω)
$\in$ *borel-measurable* (*F* (*Suc n*))
using *measurable-If-set meas-n meas-Sn set-Sn* **by** *blast*
— Transfer: the stopped process agrees with the piecewise function on *space* (*F* (*Suc n*))
show *?case*
using *measurable-cong*[*of F* (*Suc n*) *stopped-process X* τ (*Suc n*)
$\lambda\omega$. *if* $\omega \in$ *?A then stopped-process X* τ *n* ω *else* - ω]
eq meas-pw **by** *fastforce*
qed
qed
qed

theorem *stopped-process-submartingale*:
fixes *X* :: *nat* $\Rightarrow$ *'a* $\Rightarrow$ *real*
assumes *sub*: *submartingale-linorder M F 0 X*
and τ*-st*: *stopping-time* τ
shows *submartingale M F 0* (*stopped-process X* τ)
proof −
let *?sv* = *stopped-value* (*stopped-process X* τ)
from *sub* **interpret** *S*: *submartingale-linorder M F 0 X* **.**
— Use the converse direction: suffices to show monotonicity of expected stopped values
show *?thesis*
proof (*rule submartingale-of-expected-stopped-value-mono*)
show *adapted-process M F 0* (*stopped-process X* τ)
by (*simp add*: *S.adapted-process-axioms* τ*-st adapted-stopped-process*)
show $\bigwedge i$. *integrable M* (*stopped-process X* τ *i*)

by (*simp add*: *S.integrable* τ-*st* *integrable-stopped-process*)
next
fix σ ϱ :: $'a \Rightarrow$ *nat* **and** N :: *nat*
assume σ-*st*: *stopping-time* σ **and** ϱ-*st*: *stopping-time* ϱ
and $\bigwedge\omega.\ \omega \in$ *space* $M \Longrightarrow \sigma\ \omega \leq \varrho\ \omega$
and *bnd*: $\bigwedge\omega.\ \omega \in$ *space* $M \Longrightarrow \varrho\ \omega \leq N$
then have *le'*: $\bigwedge\omega.\ \omega \in$ *space* $M \Longrightarrow$ *min* $(\sigma\ \omega)\ (\tau\ \omega) \leq$ *min* $(\varrho\ \omega)\ (\tau\ \omega)$
by (*simp add*: *min-le-iff-disj*)
— stopped value of the stopped process equals that of X with *min*
have *sv*: *?sv* σ = *stopped-value* X ($\lambda\omega$. *min* $(\sigma\ \omega)\ (\tau\ \omega)$) *?sv* ϱ = *stopped-value* X ($\lambda\omega$. *min* $(\varrho\ \omega)\ (\tau\ \omega)$)
unfolding *stopped-value-def stopped-process-def* **by** *auto*
— $\lambda\omega$. *min* $(\sigma\ \omega)\ (\tau\ \omega)$ and $\lambda\omega$. *min* $(\varrho\ \omega)\ (\tau\ \omega)$ are stopping times
have *st*-σ': *stopping-time* ($\lambda\omega$. *min* $(\sigma\ \omega)\ (\tau\ \omega)$)
by (*intro stopping-time-min* σ-*st* τ-*st*)
have *st*-ϱ': *stopping-time* ($\lambda\omega$. *min* $(\varrho\ \omega)\ (\tau\ \omega)$)
by (*intro stopping-time-min* ϱ-*st* τ-*st*)
— Apply the forward direction
show $(\int \omega.$ *?sv* $\sigma\ \omega\ \partial M) \leq (\int \omega.$ *?sv* $\varrho\ \omega\ \partial M)$
using *expected-stopped-value-mono*[*OF sub st*-σ' *st*-ϱ' *le'*] *sv bnd min-le-iff-disj*
by *metis*
qed
qed

end

end