Euler's Polyhedron Formula

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Abstract

Euler stated in 1752 that every convex polyhedron satisfied the formula $V - E + F = 2$ where V, E and F are the numbers of its vertices, edges, and faces. For three dimensions, the well-known proof involves removing one face and then flattening the remainder to form a planar graph, which then is iteratively transformed to leave a single triangle. The history of that proof is extensively discussed and elaborated by Imre Lakatos [1], leaving one finally wondering whether the theorem even holds. The formal proof provided here has been ported from HOL Light, where it is credited to Lawrence [2]. The proof generalises Euler's observation from solid polyhedra to convex polytopes of arbitrary dimension.

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1 Euler's Polyhedron Formula

One of the Famous 100 Theorems, ported from HOL Light

Cited source: Lawrence, J. (1997). A Short Proof of Euler's Relation for Convex Polytopes. *Canadian Mathematical Bulletin*, **40**(4), 471–474.

theory *Euler-Formula*
imports
HOL−Analysis.Analysis
begin

Interpret which "side" of a hyperplane a point is on.

definition *hyperplane-side*
where *hyperplane-side* ≡ λ(*a*,*b*). λ*x*. *sgn* (*a* · *x* − *b*)

Equivalence relation imposed by a hyperplane arrangement.

definition *hyperplane-equiv*
where *hyperplane-equiv* ≡ λ*A x y*. ∀ *h* ∈ *A*. *hyperplane-side h x* = *hyperplane-side h y*

lemma *hyperplane-equiv-refl* [*iff*]: *hyperplane-equiv A x x*
⟨*proof*⟩

lemma *hyperplane-equiv-sym*:
hyperplane-equiv A x y ⟷ *hyperplane-equiv A y x*
⟨*proof*⟩

lemma *hyperplane-equiv-trans*:
⟦*hyperplane-equiv A x y*; *hyperplane-equiv A y z*⟧ ⟹ *hyperplane-equiv A x z*
⟨*proof*⟩

lemma *hyperplane-equiv-Un*:
hyperplane-equiv (*A* ∪ *B*) *x y* ⟷ *hyperplane-equiv A x y* ∧ *hyperplane-equiv B x y*
⟨*proof*⟩

1.1 Cells of a hyperplane arrangement

definition *hyperplane-cell* :: ('*a*::*real-inner* × *real*) *set* ⇒ '*a set* ⇒ *bool*
where *hyperplane-cell* ≡ λ*A C*. ∃ *x*. *C* = *Collect* (*hyperplane-equiv A x*)

lemma *hyperplane-cell*: *hyperplane-cell A C* ⟷ (∃ *x*. *C* = {*y*. *hyperplane-equiv A x y*})
⟨*proof*⟩

lemma *not-hyperplane-cell-empty* [*simp*]: ¬ *hyperplane-cell A* {}
⟨*proof*⟩

lemma *nonempty-hyperplane-cell*: *hyperplane-cell A C* ⟹ (*C* ≠ {})

⟨*proof*⟩

lemma *Union-hyperplane-cells*: $\bigcup \{C.\ \textit{hyperplane-cell}\ A\ C\} = \textit{UNIV}$
⟨*proof*⟩

lemma *disjoint-hyperplane-cells*:
$[\![\textit{hyperplane-cell}\ A\ C1;\ \textit{hyperplane-cell}\ A\ C2;\ C1 \neq C2]\!] \Longrightarrow \textit{disjnt}\ C1\ C2$
⟨*proof*⟩

lemma *disjoint-hyperplane-cells-eq*:
$[\![\textit{hyperplane-cell}\ A\ C1;\ \textit{hyperplane-cell}\ A\ C2]\!] \Longrightarrow (\textit{disjnt}\ C1\ C2 \longleftrightarrow (C1 \neq C2))$
⟨*proof*⟩

lemma *hyperplane-cell-empty* [*iff*]: $\textit{hyperplane-cell}\ \{\}\ C \longleftrightarrow C = \textit{UNIV}$
⟨*proof*⟩

lemma *hyperplane-cell-singleton-cases*:
assumes $\textit{hyperplane-cell}\ \{(a,b)\}\ C$
shows $C = \{x.\ a \cdot x = b\} \vee C = \{x.\ a \cdot x < b\} \vee C = \{x.\ a \cdot x > b\}$
⟨*proof*⟩

lemma *hyperplane-cell-singleton*:
$\textit{hyperplane-cell}\ \{(a,b)\}\ C \longleftrightarrow$
$(\textit{if}\ a = 0\ \textit{then}\ C = \textit{UNIV}\ \textit{else}\ C = \{x.\ a \cdot x = b\} \vee C = \{x.\ a \cdot x < b\} \vee C = \{x.\ a \cdot x > b\})$
⟨*proof*⟩

lemma *hyperplane-cell-Un*:
$\textit{hyperplane-cell}\ (A \cup B)\ C \longleftrightarrow$
$C \neq \{\} \wedge$
$(\exists\, C1\ C2.\ \textit{hyperplane-cell}\ A\ C1 \wedge \textit{hyperplane-cell}\ B\ C2 \wedge C = C1 \cap C2)$
⟨*proof*⟩

lemma *finite-hyperplane-cells*:
$\textit{finite}\ A \Longrightarrow \textit{finite}\ \{C.\ \textit{hyperplane-cell}\ A\ C\}$
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cells*:
$\textit{finite}\ A \Longrightarrow \textit{finite}\ \{C.\ \textit{hyperplane-cell}\ A\ C \wedge P\ C\}$
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cells*:
$[\![\textit{finite}\ A;\ \bigwedge C.\ C \in \mathcal{C} \Longrightarrow \textit{hyperplane-cell}\ A\ C]\!] \Longrightarrow \textit{finite}\ \mathcal{C}$
⟨*proof*⟩

lemma *pairwise-disjoint-hyperplane-cells*:
$(\bigwedge C.\ C \in \mathcal{C} \Longrightarrow \textit{hyperplane-cell}\ A\ C) \Longrightarrow \textit{pairwise}\ \textit{disjnt}\ \mathcal{C}$
⟨*proof*⟩

lemma *hyperplane-cell-Int-open-affine*:
assumes *finite A hyperplane-cell A C*
obtains *S T* **where** *open S affine T C* = *S* ∩ *T*
⟨*proof*⟩

lemma *hyperplane-cell-relatively-open*:
assumes *finite A hyperplane-cell A C*
shows *openin* (*subtopology euclidean* (*affine hull C*)) *C*
⟨*proof*⟩

lemma *hyperplane-cell-relative-interior*:
⟦*finite A*; *hyperplane-cell A C*⟧ ⟹ *rel-interior C* = *C*
⟨*proof*⟩

lemma *hyperplane-cell-convex*:
assumes *hyperplane-cell A C*
shows *convex C*
⟨*proof*⟩

lemma *hyperplane-cell-Inter*:
assumes ⋀*C*. *C* ∈ *𝒞* ⟹ *hyperplane-cell A C*
and *𝒞* ≠ {} **and** *INT*: ⋂*𝒞* ≠ {}
shows *hyperplane-cell A* (⋂*𝒞*)
⟨*proof*⟩

lemma *hyperplane-cell-Int*:
⟦*hyperplane-cell A S*; *hyperplane-cell A T*; *S* ∩ *T* ≠ {}⟧ ⟹ *hyperplane-cell A* (*S* ∩ *T*)
⟨*proof*⟩

1.2 A cell complex is considered to be a union of such cells

definition *hyperplane-cellcomplex*
where *hyperplane-cellcomplex A S* ≡
∃*𝒯*. (∀ *C* ∈ *𝒯*. *hyperplane-cell A C*) ∧ *S* = ⋃*𝒯*

lemma *hyperplane-cellcomplex-empty* [*simp*]: *hyperplane-cellcomplex A* {}
⟨*proof*⟩

lemma *hyperplane-cell-cellcomplex*:
hyperplane-cell A C ⟹ *hyperplane-cellcomplex A C*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Union*:
assumes ⋀*S*. *S* ∈ *𝒞* ⟹ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* (⋃ *𝒞*)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Un*:
⟦*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*⟧
⟹ *hyperplane-cellcomplex A* (*S* ∪ *T*)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-UNIV* [*simp*]: *hyperplane-cellcomplex A UNIV*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Inter*:
assumes ⋀*S*. *S* ∈ *C* ⟹ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* (⋂*C*)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Int*:
⟦*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*⟧
⟹ *hyperplane-cellcomplex A* (*S* ∩ *T*)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Compl*:
assumes *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* (− *S*)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-diff*:
⟦*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*⟧
⟹ *hyperplane-cellcomplex A* (*S* − *T*)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-mono*:
assumes *hyperplane-cellcomplex A S A* ⊆ *B*
shows *hyperplane-cellcomplex B S*
⟨*proof*⟩

lemma *finite-hyperplane-cellcomplexes*:
assumes *finite A*
shows *finite* {*C*. *hyperplane-cellcomplex A C*}
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cellcomplexes*:
finite A ⟹ *finite* {*C*. *hyperplane-cellcomplex A C* ∧ *P C*}
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cellcomplex*:
assumes *finite A* ⋀*C*. *C* ∈ *C* ⟹ *hyperplane-cellcomplex A C*
shows *finite C*
⟨*proof*⟩

lemma *cell-subset-cellcomplex*:

⟦*hyperplane-cell A C*; *hyperplane-cellcomplex A S*⟧ ⟹ *C* ⊆ *S* ⟷ ~ *disjnt C S*
⟨*proof*⟩

1.3 Euler characteristic

definition *Euler-characteristic* :: ('*a*::*euclidean-space* × *real*) *set* ⇒ '*a set* ⇒ *int*
where *Euler-characteristic A S* ≡
(∑ *C* | *hyperplane-cell A C* ∧ *C* ⊆ *S*. (−1) ^ *nat* (*aff-dim C*))

lemma *Euler-characteristic-empty* [*simp*]: *Euler-characteristic A* {} = *0*
⟨*proof*⟩

lemma *Euler-characteristic-cell-Union*:
assumes ⋀*C*. *C* ∈ *C* ⟹ *hyperplane-cell A C*
shows *Euler-characteristic A* (⋃ *C*) = (∑ *C*∈*C*. (− *1*) ^ *nat* (*aff-dim C*))
⟨*proof*⟩

lemma *Euler-characteristic-cell*:
hyperplane-cell A C ⟹ *Euler-characteristic A C* = (−*1*) ^ (*nat*(*aff-dim C*))
⟨*proof*⟩

lemma *Euler-characteristic-cellcomplex-Un*:
assumes *finite A hyperplane-cellcomplex A S*
and *AT*: *hyperplane-cellcomplex A T* **and** *disjnt S T*
shows *Euler-characteristic A* (*S* ∪ *T*) =
Euler-characteristic A S + *Euler-characteristic A T*
⟨*proof*⟩

lemma *Euler-characteristic-cellcomplex-Union*:
assumes *finite A*
and *C*: ⋀*C*. *C* ∈ *C* ⟹ *hyperplane-cellcomplex A C pairwise disjnt C*
shows *Euler-characteristic A* (⋃ *C*) = *sum* (*Euler-characteristic A*) *C*
⟨*proof*⟩

lemma *Euler-characteristic*:
fixes *A* :: ('*n*::*euclidean-space* ∗ *real*) *set*
assumes *finite A*
shows *Euler-characteristic A S* =
(∑ *d* = *0*..*DIM*('*n*). (−*1*) ^ *d* ∗ *int* (*card* {*C*. *hyperplane-cell A C* ∧ *C* ⊆ *S* ∧ *aff-dim C* = *int d*}))
(**is** - = *?rhs*)
⟨*proof*⟩

1.4 Show that the characteristic is invariant w.r.t. hyperplane arrangement.

lemma *hyperplane-cells-distinct-lemma*:
{*x*. *a* ∙ *x* = *b*} ∩ {*x*. *a* ∙ *x* < *b*} = {} ∧
{*x*. *a* ∙ *x* = *b*} ∩ {*x*. *a* ∙ *x* > *b*} = {} ∧

$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x > b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x < b\} = \{\}$
⟨*proof*⟩

proposition *Euler-characterstic-lemma*:
assumes *finite A* **and** *hyperplane-cellcomplex A S*
shows *Euler-characteristic* (*insert h A*) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant-aux*:
assumes *finite B finite A hyperplane-cellcomplex A S*
shows *Euler-characteristic* (*A* ∪ *B*) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant*:
assumes *finite A finite B hyperplane-cellcomplex A S hyperplane-cellcomplex B S*
shows *Euler-characteristic A S* = *Euler-characteristic B S*
⟨*proof*⟩

lemma *Euler-characteristic-inclusion-exclusion*:
assumes *finite A finite* $\mathcal{S}$ $\bigwedge K.\ K \in \mathcal{S} \Longrightarrow$ *hyperplane-cellcomplex A K*
shows *Euler-characteristic A* ($\bigcup\ \mathcal{S}$) = ($\sum \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S} \wedge \mathcal{T} \neq \{\}.\ (-\ 1)\ \hat{}\ (card\ \mathcal{T} + 1) * Euler\text{-}characteristic\ A\ (\bigcap \mathcal{T})$)
⟨*proof*⟩

1.5 Euler-type relation for full-dimensional proper polyhedral cones

lemma *Euler-polyhedral-cone*:
fixes *S* :: *'n*::*euclidean-space set*
assumes *polyhedron S conic S* **and** *intS*: *interior S* ≠ {} **and** *S* ≠ *UNIV*
shows ($\sum d = 0..DIM('n).\ (-\ 1)\ \hat{}\ d * int\ (card\ \{f.\ f\ face\text{-}of\ S \wedge aff\text{-}dim\ f = int\ d\})) = 0$ (**is** *?lhs* = *0*)
⟨*proof*⟩

1.6 Euler-Poincare relation for special $(n-1)$-dimensional polytope

lemma *Euler-Poincare-lemma*:
fixes *p* :: *'n*::*euclidean-space set*
assumes *DIM*(*'n*) ≥ *2 polytope p i* ∈ *Basis* **and** *affp*: *affine hull p* = {*x*. *x* · *i* = *1*}
shows ($\sum d = 0..DIM('n) - 1.\ (-1)\ \hat{}\ d * int\ (card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = int\ d\})) = 1$
⟨*proof*⟩

corollary *Euler-poincare-special*:
fixes $p :: 'n::euclidean\text{-}space\ set$
assumes $2 \leq DIM('n)$ *polytope* p $i \in Basis$ **and** *affp*: *affine hull* $p = \{x.\ x \cdot i = 0\}$
shows $(\sum d = 0..DIM('n) - 1.\ (-1) \hat{} d * card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = d\}) = 1$
⟨*proof*⟩

1.7 Now Euler-Poincare for a general full-dimensional polytope

theorem *Euler-Poincare-full*:
fixes $p :: 'n::euclidean\text{-}space\ set$
assumes *polytope* p *aff-dim* $p = DIM('n)$
shows $(\sum d = 0..DIM('n).\ (-1) \hat{} d * (card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = d\})) = 1$
⟨*proof*⟩

In particular, the Euler relation in 3 dimensions

corollary *Euler-relation*:
fixes $p :: 'n::euclidean\text{-}space\ set$
assumes *polytope* p *aff-dim* $p = 3$ $DIM('n) = 3$
shows $(card\ \{v.\ v\ face\text{-}of\ p \wedge aff\text{-}dim\ v = 0\} + card\ \{f.\ f\ face\text{-}of\ p \wedge aff\text{-}dim\ f = 2\}) - card\ \{e.\ e\ face\text{-}of\ p \wedge aff\text{-}dim\ e = 1\} = 2$
⟨*proof*⟩

end

References

[1] I. Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery*. 1976.

[2] J. Lawrence. A short proof of Euler's relation for convex polytopes. *Canadian Mathematical Bulletin*, 40(4):471–474, 1997.