Euler's Polyhedron Formula

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Abstract

Euler stated in 1752 that every convex polyhedron satisfied the formula $V - E + F = 2$ where V, E and F are the numbers of its vertices, edges, and faces. For three dimensions, the well-known proof involves removing one face and then flattening the remainder to form a planar graph, which then is iteratively transformed to leave a single triangle. The history of that proof is extensively discussed and elaborated by Imre Lakatos [1], leaving one finally wondering whether the theorem even holds. The formal proof provided here has been ported from HOL Light, where it is credited to Lawrence [2]. The proof generalises Euler's observation from solid polyhedra to convex polytopes of arbitrary dimension.

Contents

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1 Euler's Polyhedron Formula

One of the Famous 100 Theorems, ported from HOL Light

Cited source: Lawrence, J. (1997). A Short Proof of Euler's Relation for Convex Polytopes. *Canadian Mathematical Bulletin*, **40**(4), 471–474.

theory *Euler-Formula*
  imports
    HOL−Analysis.Analysis
begin

Interpret which "side" of a hyperplane a point is on.

definition *hyperplane-side*
  where *hyperplane-side* ≡ λ(*a*,*b*). λ*x*. *sgn* (*a* · *x* − *b*)

Equivalence relation imposed by a hyperplane arrangement.

definition *hyperplane-equiv*
 where *hyperplane-equiv* ≡ λ*A x y*. ∀ *h* ∈ *A*. *hyperplane-side h x* = *hyperplane-side h y*

lemma *hyperplane-equiv-refl* [*iff*]: *hyperplane-equiv A x x*
  ⟨*proof*⟩

lemma *hyperplane-equiv-sym*:
   hyperplane-equiv A x y ⟷ *hyperplane-equiv A y x*
  ⟨*proof*⟩

lemma *hyperplane-equiv-trans*:
  ⟦*hyperplane-equiv A x y*; *hyperplane-equiv A y z*⟧ ⟹ *hyperplane-equiv A x z*
  ⟨*proof*⟩

lemma *hyperplane-equiv-Un*:
   hyperplane-equiv (*A* ∪ *B*) *x y* ⟷ *hyperplane-equiv A x y* ∧ *hyperplane-equiv B x y*
  ⟨*proof*⟩

1.1 Cells of a hyperplane arrangement

definition *hyperplane-cell* :: (*'a*::*real-inner* × *real*) *set* ⇒ *'a set* ⇒ *bool*
  where *hyperplane-cell* ≡ λ*A C*. ∃ *x*. *C* = *Collect* (*hyperplane-equiv A x*)

lemma *hyperplane-cell*: *hyperplane-cell A C* ⟷ (∃ *x*. *C* = {*y*. *hyperplane-equiv A x y*})
  ⟨*proof*⟩

lemma *not-hyperplane-cell-empty* [*simp*]: ¬ *hyperplane-cell A* {}
  ⟨*proof*⟩

lemma *nonempty-hyperplane-cell*: *hyperplane-cell A C* ⟹ (*C* ≠ {})

⟨*proof*⟩

lemma *Union-hyperplane-cells*: $\bigcup$ {*C*. *hyperplane-cell A C*} = *UNIV*
⟨*proof*⟩

lemma *disjoint-hyperplane-cells*:
⟦*hyperplane-cell A C1*; *hyperplane-cell A C2*; *C1* ≠ *C2*⟧ ⟹ *disjnt C1 C2*
⟨*proof*⟩

lemma *disjoint-hyperplane-cells-eq*:
⟦*hyperplane-cell A C1*; *hyperplane-cell A C2*⟧ ⟹ (*disjnt C1 C2* ⟷ (*C1* ≠ *C2*))
⟨*proof*⟩

lemma *hyperplane-cell-empty* [*iff*]: *hyperplane-cell* {} *C* ⟷ *C* = *UNIV*
⟨*proof*⟩

lemma *hyperplane-cell-singleton-cases*:
assumes *hyperplane-cell* {(*a*,*b*)} *C*
shows *C* = {*x*. *a* · *x* = *b*} ∨ *C* = {*x*. *a* · *x* < *b*} ∨ *C* = {*x*. *a* · *x* > *b*}
⟨*proof*⟩

lemma *hyperplane-cell-singleton*:
hyperplane-cell {(*a*,*b*)} *C* ⟷
(*if a* = *0 then C* = *UNIV else C* = {*x*. *a* · *x* = *b*} ∨ *C* = {*x*. *a* · *x* < *b*} ∨ *C* = {*x*. *a* · *x* > *b*})
⟨*proof*⟩

lemma *hyperplane-cell-Un*:
hyperplane-cell (*A* ∪ *B*) *C* ⟷
C ≠ {} ∧
(∃ *C1 C2*. *hyperplane-cell A C1* ∧ *hyperplane-cell B C2* ∧ *C* = *C1* ∩ *C2*)
⟨*proof*⟩

lemma *finite-hyperplane-cells*:
finite A ⟹ *finite* {*C*. *hyperplane-cell A C*}
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cells*:
finite A ⟹ *finite* {*C*. *hyperplane-cell A C* ∧ *P C*}
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cells*:
⟦*finite A*; ⋀*C*. *C* ∈ $\mathcal{C}$ ⟹ *hyperplane-cell A C*⟧ ⟹ *finite* $\mathcal{C}$
⟨*proof*⟩

lemma *pairwise-disjoint-hyperplane-cells*:
(⋀*C*. *C* ∈ $\mathcal{C}$ ⟹ *hyperplane-cell A C*) ⟹ *pairwise disjnt* $\mathcal{C}$
⟨*proof*⟩

lemma *hyperplane-cell-Int-open-affine*:
assumes *finite A hyperplane-cell A C*
obtains *S T* **where** *open S affine T C* = *S* ∩ *T*
⟨*proof*⟩

lemma *hyperplane-cell-relatively-open*:
assumes *finite A hyperplane-cell A C*
shows *openin* (*subtopology euclidean* (*affine hull C*)) *C*
⟨*proof*⟩

lemma *hyperplane-cell-relative-interior*:
⟦*finite A*; *hyperplane-cell A C*⟧ ⟹ *rel-interior C* = *C*
⟨*proof*⟩

lemma *hyperplane-cell-convex*:
assumes *hyperplane-cell A C*
shows *convex C*
⟨*proof*⟩

lemma *hyperplane-cell-Inter*:
assumes ⋀*C*. *C* ∈ 𝒞 ⟹ *hyperplane-cell A C*
and 𝒞 ≠ {} **and** *INT*: ⋂𝒞 ≠ {}
shows *hyperplane-cell A* (⋂𝒞)
⟨*proof*⟩

lemma *hyperplane-cell-Int*:
⟦*hyperplane-cell A S*; *hyperplane-cell A T*; *S* ∩ *T* ≠ {}⟧ ⟹ *hyperplane-cell A* (*S* ∩ *T*)
⟨*proof*⟩

1.2 A cell complex is considered to be a union of such cells

definition *hyperplane-cellcomplex*
where *hyperplane-cellcomplex A S* ≡
∃𝒯. (∀ *C* ∈ 𝒯. *hyperplane-cell A C*) ∧ *S* = ⋃𝒯

lemma *hyperplane-cellcomplex-empty* [*simp*]: *hyperplane-cellcomplex A* {}
⟨*proof*⟩

lemma *hyperplane-cell-cellcomplex*:
hyperplane-cell A C ⟹ *hyperplane-cellcomplex A C*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Union*:
assumes ⋀*S*. *S* ∈ 𝒞 ⟹ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* (⋃ 𝒞)
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Un*:
$[\![$*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*$]\!]$
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S \cup T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-UNIV* [*simp*]: *hyperplane-cellcomplex A UNIV*
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Inter*:
assumes $\bigwedge S.\ S \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(\bigcap \mathcal{C})$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Int*:
$[\![$*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*$]\!]$
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S \cap T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-Compl*:
assumes *hyperplane-cellcomplex A S*
shows *hyperplane-cellcomplex A* $(- S)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-diff*:
$[\![$*hyperplane-cellcomplex A S*; *hyperplane-cellcomplex A T*$]\!]$
$\Longrightarrow$ *hyperplane-cellcomplex A* $(S - T)$
⟨*proof*⟩

lemma *hyperplane-cellcomplex-mono*:
assumes *hyperplane-cellcomplex A S* $A \subseteq B$
shows *hyperplane-cellcomplex B S*
⟨*proof*⟩

lemma *finite-hyperplane-cellcomplexes*:
assumes *finite A*
shows *finite* $\{C.$ *hyperplane-cellcomplex A C*$\}$
⟨*proof*⟩

lemma *finite-restrict-hyperplane-cellcomplexes*:
finite A $\Longrightarrow$ *finite* $\{C.$ *hyperplane-cellcomplex A C* $\wedge$ *P C*$\}$
⟨*proof*⟩

lemma *finite-set-of-hyperplane-cellcomplex*:
assumes *finite A* $\bigwedge C.\ C \in \mathcal{C} \Longrightarrow$ *hyperplane-cellcomplex A C*
shows *finite* $\mathcal{C}$
⟨*proof*⟩

lemma *cell-subset-cellcomplex*:

$[\![$*hyperplane-cell A C*; *hyperplane-cellcomplex A S*$]\!] \Longrightarrow C \subseteq S \longleftrightarrow$ ~ *disjnt C S*
⟨*proof*⟩

1.3 Euler characteristic

definition *Euler-characteristic* :: (*'a*::*euclidean-space* × *real*) *set* ⇒ *'a set* ⇒ *int*
where *Euler-characteristic A S* ≡
($\sum$ *C* | *hyperplane-cell A C* ∧ *C* ⊆ *S*. (−*1*) ^ *nat* (*aff-dim C*))

lemma *Euler-characteristic-empty* [*simp*]: *Euler-characteristic A* {} = *0*
⟨*proof*⟩

lemma *Euler-characteristic-cell-Union*:
assumes ⋀*C*. *C* ∈ $\mathcal{C}$ ⟹ *hyperplane-cell A C*
shows *Euler-characteristic A* (⋃ $\mathcal{C}$) = ($\sum$ *C*∈$\mathcal{C}$. (− *1*) ^ *nat* (*aff-dim C*))
⟨*proof*⟩

lemma *Euler-characteristic-cell*:
hyperplane-cell A C ⟹ *Euler-characteristic A C* = (−*1*) ^(*nat*(*aff-dim C*))
⟨*proof*⟩

lemma *Euler-characteristic-cellcomplex-Un*:
assumes *finite A hyperplane-cellcomplex A S*
and *AT*: *hyperplane-cellcomplex A T* **and** *disjnt S T*
shows *Euler-characteristic A* (*S* ∪ *T*) =
Euler-characteristic A S + *Euler-characteristic A T*
⟨*proof*⟩

lemma *Euler-characteristic-cellcomplex-Union*:
assumes *finite A*
and $\mathcal{C}$: ⋀*C*. *C* ∈ $\mathcal{C}$ ⟹ *hyperplane-cellcomplex A C pairwise disjnt* $\mathcal{C}$
shows *Euler-characteristic A* (⋃ $\mathcal{C}$) = *sum* (*Euler-characteristic A*) $\mathcal{C}$
⟨*proof*⟩

lemma *Euler-characteristic*:
fixes *A* :: (*'n*::*euclidean-space* ∗ *real*) *set*
assumes *finite A*
shows *Euler-characteristic A S* =
($\sum$ *d* = *0*..*DIM*(*'n*). (−*1*) ^ *d* ∗ *int* (*card* {*C*. *hyperplane-cell A C* ∧ *C* ⊆ *S* ∧ *aff-dim C* = *int d*}))
(**is** - = *?rhs*)
⟨*proof*⟩

1.4 Show that the characteristic is invariant w.r.t. hyperplane arrangement.

lemma *hyperplane-cells-distinct-lemma*:
{*x*. *a* · *x* = *b*} ∩ {*x*. *a* · *x* < *b*} = {} ∧
{*x*. *a* · *x* = *b*} ∩ {*x*. *a* · *x* > *b*} = {} ∧

$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x < b\} \cap \{x.\ a \cdot x > b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x = b\} = \{\} \wedge$
$\{x.\ a \cdot x > b\} \cap \{x.\ a \cdot x < b\} = \{\}$
⟨*proof*⟩

proposition *Euler-characterstic-lemma*:
assumes *finite A* **and** *hyperplane-cellcomplex A S*
shows *Euler-characteristic* (*insert h A*) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant-aux*:
assumes *finite B finite A hyperplane-cellcomplex A S*
shows *Euler-characteristic* ($A \cup B$) *S* = *Euler-characteristic A S*
⟨*proof*⟩

lemma *Euler-characterstic-invariant*:
assumes *finite A finite B hyperplane-cellcomplex A S hyperplane-cellcomplex B S*
shows *Euler-characteristic A S* = *Euler-characteristic B S*
⟨*proof*⟩

lemma *Euler-characteristic-inclusion-exclusion*:
assumes *finite A finite* $\mathcal{S}$ $\bigwedge K.\ K \in \mathcal{S} \Longrightarrow$ *hyperplane-cellcomplex A K*
shows *Euler-characteristic A* ($\bigcup \mathcal{S}$) = ($\sum \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S} \wedge \mathcal{T} \neq \{\}.\ (-\ 1)$ ^ (*card* $\mathcal{T}$ + *1*) $*$ *Euler-characteristic A* ($\bigcap \mathcal{T}$))
⟨*proof*⟩

1.5 Euler-type relation for full-dimensional proper polyhedral cones

lemma *Euler-polyhedral-cone*:
fixes *S* :: *'n::euclidean-space set*
assumes *polyhedron S conic S* **and** *intS*: *interior S* $\neq \{\}$ **and** *S* $\neq$ *UNIV*
shows ($\sum d = 0..DIM('n).\ (-\ 1)$ ^ $d\ *$ *int* (*card* $\{f.\ f$ *face-of S* $\wedge$ *aff-dim f* = *int d*$\}$)) = *0* (**is** *?lhs* = *0*)
⟨*proof*⟩

1.6 Euler-Poincare relation for special $(n-1)$-dimensional polytope

lemma *Euler-Poincare-lemma*:
fixes *p* :: *'n::euclidean-space set*
assumes *DIM('n)* $\geq$ *2 polytope p i* $\in$ *Basis* **and** *affp*: *affine hull p* = $\{x.\ x \cdot i = 1\}$
shows ($\sum d = 0..DIM('n) - 1.\ (-1)$ ^ $d\ *$ *int* (*card* $\{f.\ f$ *face-of p* $\wedge$ *aff-dim f* = *int d*$\}$)) = *1*
⟨*proof*⟩

corollary *Euler-poincare-special*:
fixes p :: $'n$::*euclidean-space set*
assumes $2 \leq DIM('n)$ *polytope* p $i \in$ *Basis* **and** *affp*: *affine hull* $p = \{x.\ x \cdot i = 0\}$
shows $(\sum d = 0..DIM('n) - 1.\ (-1)\ \hat{}\ d * card\ \{f.\ f\ \textit{face-of}\ p \wedge \textit{aff-dim}\ f = d\}) = 1$
⟨*proof*⟩

1.7 Now Euler-Poincare for a general full-dimensional polytope

theorem *Euler-Poincare-full*:
fixes p :: $'n$::*euclidean-space set*
assumes *polytope* p *aff-dim* $p = DIM('n)$
shows $(\sum d = 0..DIM('n).\ (-1)\ \hat{}\ d * (card\ \{f.\ f\ \textit{face-of}\ p \wedge \textit{aff-dim}\ f = d\})) = 1$
⟨*proof*⟩

In particular, the Euler relation in 3 dimensions

corollary *Euler-relation*:
fixes p :: $'n$::*euclidean-space set*
assumes *polytope* p *aff-dim* $p = 3$ $DIM('n) = 3$
shows $(card\ \{v.\ v\ \textit{face-of}\ p \wedge \textit{aff-dim}\ v = 0\} + card\ \{f.\ f\ \textit{face-of}\ p \wedge \textit{aff-dim}\ f = 2\}) - card\ \{e.\ e\ \textit{face-of}\ p \wedge \textit{aff-dim}\ e = 1\} = 2$
⟨*proof*⟩

end

References

[1] I. Lakatos. *Proofs and Refutations: The Logic of Mathematical Discovery.* 1976.

[2] J. Lawrence. A short proof of Euler's relation for convex polytopes. *Canadian Mathematical Bulletin*, 40(4):471–474, 1997.