Epistemic Logic: Completeness of Modal Logics

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Abstract

This work is a formalization of epistemic logic with countably many agents. It includes proofs of soundness and completeness for the axiom system K. The completeness proof is based on the textbook "Reasoning About Knowledge" by Fagin, Halpern, Moses and Vardi (MIT Press 1995) [2]. The extensions of system K (T, KB, K4, S4, S5) and their completeness proofs are based on the textbook "Modal Logic" by Blackburn, de Rijke and Venema (Cambridge University Press 2001) [1].

Contents

theory *Maximal-Consistent-Sets* **imports** *HOL−Cardinals.Cardinal-Order-Relation*
begin

context *wo-rel* **begin**

lemma *underS-bound*: ‹$a \in underS\ n \Longrightarrow b \in underS\ n \Longrightarrow a \in under\ b \vee b \in under\ a$›
⟨*proof*⟩

lemma *finite-underS-bound*:
assumes ‹*finite X*› ‹$X \subseteq underS\ n$› ‹$X \neq \{\}$›
shows ‹$\exists\, a \in X.\ \forall\, b \in X.\ b \in under\ a$›
⟨*proof*⟩

lemma *finite-bound-under*:
assumes ‹*finite p*› ‹$p \subseteq (\bigcup n \in Field\ r.\ f\ n)$›
shows ‹$\exists\, m.\ p \subseteq (\bigcup n \in under\ m.\ f\ n)$›
⟨*proof*⟩

end

locale *MCS-Lim-Ord* =
fixes r :: ‹$'a\ rel$›
assumes *WELL*: ‹*Well-order r*›
and *isLimOrd-r*: ‹*isLimOrd r*›
fixes *consistent* :: ‹$'a\ set \Rightarrow bool$›
assumes *consistent-hereditary*: ‹$consistent\ S \Longrightarrow S' \subseteq S \Longrightarrow consistent\ S'$›
and *inconsistent-finite*: ‹$\bigwedge S.\ \neg\ consistent\ S \Longrightarrow \exists\, S' \subseteq S.\ finite\ S' \wedge \neg\ consistent\ S'$›
begin

definition *extendS* :: ‹$'a\ set \Rightarrow\ 'a \Rightarrow\ 'a\ set \Rightarrow\ 'a\ set$› **where**
‹$extendS\ S\ n\ prev \equiv if\ consistent\ (\{n\} \cup prev)\ then\ \{n\} \cup prev\ else\ prev$›

definition *extendL* :: ‹$('a \Rightarrow\ 'a\ set) \Rightarrow\ 'a \Rightarrow\ 'a\ set$› **where**
‹$extendL\ rec\ n \equiv \bigcup m \in underS\ r\ n.\ rec\ m$›

definition *extend* :: ‹$'a\ set \Rightarrow\ 'a \Rightarrow\ 'a\ set$› **where**
‹$extend\ S\ n \equiv worecZSL\ r\ S\ (extendS\ S)\ extendL\ n$›

lemma *wo-rel-r*: ‹*wo-rel r*›
⟨*proof*⟩

lemma *adm-woL-extendL*: ‹*adm-woL r extendL*›
⟨*proof*⟩

definition *Extend* :: ‹$'a\ set \Rightarrow\ 'a\ set$› **where**

‹$Extend\ S \equiv \bigcup n \in Field\ r.\ extend\ S\ n$›

lemma *extend-subset*: ‹$n \in Field\ r \Longrightarrow S \subseteq extend\ S\ n$›
⟨*proof*⟩

lemma *Extend-subset′*: ‹$Field\ r \neq \{\} \Longrightarrow S \subseteq Extend\ S$›
⟨*proof*⟩

lemma *extend-underS*: ‹$m \in underS\ r\ n \Longrightarrow extend\ S\ m \subseteq extend\ S\ n$›
⟨*proof*⟩

lemma *extend-under*: ‹$m \in under\ r\ n \Longrightarrow extend\ S\ m \subseteq extend\ S\ n$›
⟨*proof*⟩

lemma *consistent-extend*:
assumes ‹*consistent S*›
shows ‹*consistent* (*extend S n*)›
⟨*proof*⟩

lemma *consistent-Extend*:
assumes ‹*consistent S*›
shows ‹*consistent* (*Extend S*)›
⟨*proof*⟩

definition *maximal′* :: ‹′*a set* ⇒ *bool*› **where**
‹$maximal'\ S \equiv \forall p \in Field\ r.\ consistent\ (\{p\} \cup S) \longrightarrow p \in S$›

lemma *Extend-bound*: ‹$n \in Field\ r \Longrightarrow extend\ S\ n \subseteq Extend\ S$›
⟨*proof*⟩

lemma *maximal′-Extend*: ‹*maximal′* (*Extend S*)›
⟨*proof*⟩

end

locale *MCS* =
fixes *consistent* :: ‹′*a set* ⇒ *bool*›
assumes *infinite-UNIV*: ‹*infinite* (*UNIV* :: ′*a set*)›
and ‹$consistent\ S \Longrightarrow S' \subseteq S \Longrightarrow consistent\ S'$›
and ‹$\bigwedge S.\ \neg\ consistent\ S \Longrightarrow \exists S' \subseteq S.\ finite\ S' \wedge \neg\ consistent\ S'$›

sublocale $MCS \subseteq$ *MCS-Lim-Ord* ‹|*UNIV*|›
⟨*proof*⟩

context *MCS* **begin**

lemma *Extend-subset*: ‹$S \subseteq Extend\ S$›
⟨*proof*⟩

definition *maximal* :: ‹*'a set* ⇒ *bool*› **where**
‹*maximal S* ≡ ∀ *p*. *consistent* ({*p*} ∪ *S*) ⟶ *p* ∈ *S*›

lemma *maximal-maximal'*: ‹*maximal S* ⟷ *maximal' S*›
⟨*proof*⟩

lemma *maximal-Extend*: ‹*maximal* (*Extend S*)›
⟨*proof*⟩

end

end

theory *Epistemic-Logic* **imports** *Maximal-Consistent-Sets* **begin**

1 Syntax

type-synonym *id* = *string*

datatype *'i fm*
= *FF* (‹⊥›)
| *Pro id*
| *Dis* ‹*'i fm*› ‹*'i fm*› (**infixr** ‹∨› *60*)
| *Con* ‹*'i fm*› ‹*'i fm*› (**infixr** ‹∧› *65*)
| *Imp* ‹*'i fm*› ‹*'i fm*› (**infixr** ‹⟶› *55*)
| *K 'i* ‹*'i fm*›

abbreviation *TT* (‹⊤›) **where**
‹*TT* ≡ ⊥ ⟶ ⊥›

abbreviation *Neg* (‹¬ -› [*70*] *70*) **where**
‹*Neg p* ≡ *p* ⟶ ⊥›

abbreviation ‹*L i p* ≡ ¬ *K i* (¬ *p*)›

2 Semantics

record (*'i*, *'w*) *frame* =
𝒲 :: ‹*'w set*›
𝒦 :: ‹*'i* ⇒ *'w* ⇒ *'w set*›

record (*'i*, *'w*) *kripke* =
‹(*'i*, *'w*) *frame*› +
π :: ‹*'w* ⇒ *id* ⇒ *bool*›

primrec *semantics* :: ‹(*'i*, *'w*) *kripke* ⇒ *'w* ⇒ *'i fm* ⇒ *bool*› (‹-, - ⊨ -› [*50*, *50*,

50] *50*) **where**
  ‹$M,\ w \models \bot \longleftrightarrow False$›
| ‹$M,\ w \models Pro\ x \longleftrightarrow \pi\ M\ w\ x$›
| ‹$M,\ w \models p \lor q \longleftrightarrow M,\ w \models p \lor M,\ w \models q$›
| ‹$M,\ w \models p \land q \longleftrightarrow M,\ w \models p \land M,\ w \models q$›
| ‹$M,\ w \models p \longrightarrow q \longleftrightarrow M,\ w \models p \longrightarrow M,\ w \models q$›
| ‹$M,\ w \models K\ i\ p \longleftrightarrow (\forall\, v \in \mathcal{W}\ M \cap \mathcal{K}\ M\ i\ w.\ M,\ v \models p)$›

abbreviation *validStar* :: ‹$(('i,\ 'w)\ kripke \Rightarrow bool) \Rightarrow\ 'i\ fm\ set \Rightarrow\ 'i\ fm \Rightarrow bool$›
  (‹-; - $\models\star$ -› [*50*, *50*, *50*] *50*) **where**
  ‹$P;\ G \models\star\ p \equiv \forall\, M.\ P\ M \longrightarrow$
    $(\forall\, w \in \mathcal{W}\ M.\ (\forall\, q \in G.\ M,\ w \models q) \longrightarrow M,\ w \models p)$›

3 S5 Axioms

definition *reflexive* :: ‹$('i,\ 'w,\ 'c)\ frame\text{-}scheme \Rightarrow bool$› **where**
  ‹$reflexive\ M \equiv \forall\, i.\ \forall\, w \in \mathcal{W}\ M.\ w \in \mathcal{K}\ M\ i\ w$›

definition *symmetric* :: ‹$('i,\ 'w,\ 'c)\ frame\text{-}scheme \Rightarrow bool$› **where**
  ‹$symmetric\ M \equiv \forall\, i.\ \forall\, v \in \mathcal{W}\ M.\ \forall\, w \in \mathcal{W}\ M.\ v \in \mathcal{K}\ M\ i\ w \longleftrightarrow w \in \mathcal{K}\ M\ i\ v$›

definition *transitive* :: ‹$('i,\ 'w,\ 'c)\ frame\text{-}scheme \Rightarrow bool$› **where**
  ‹$transitive\ M \equiv \forall\, i.\ \forall\, u \in \mathcal{W}\ M.\ \forall\, v \in \mathcal{W}\ M.\ \forall\, w \in \mathcal{W}\ M.$
    $w \in \mathcal{K}\ M\ i\ v \land u \in \mathcal{K}\ M\ i\ w \longrightarrow u \in \mathcal{K}\ M\ i\ v$›

abbreviation *refltrans* :: ‹$('i,\ 'w,\ 'c)\ frame\text{-}scheme \Rightarrow bool$› **where**
  ‹$refltrans\ M \equiv reflexive\ M \land transitive\ M$›

abbreviation *equivalence* :: ‹$('i,\ 'w,\ 'c)\ frame\text{-}scheme \Rightarrow bool$› **where**
  ‹$equivalence\ M \equiv reflexive\ M \land symmetric\ M \land transitive\ M$›

definition *Euclidean* :: ‹$('i,\ 'w,\ 'c)\ frame\text{-}scheme \Rightarrow bool$› **where**
  ‹$Euclidean\ M \equiv \forall\, i.\ \forall\, u \in \mathcal{W}\ M.\ \forall\, v \in \mathcal{W}\ M.\ \forall\, w \in \mathcal{W}\ M.$
    $v \in \mathcal{K}\ M\ i\ u \longrightarrow w \in \mathcal{K}\ M\ i\ u \longrightarrow w \in \mathcal{K}\ M\ i\ v$›

lemma *Imp-intro* [*intro*]: ‹$(M,\ w \models p \Longrightarrow M,\ w \models q) \Longrightarrow M,\ w \models p \longrightarrow q$›
  ⟨*proof*⟩

theorem *distribution*: ‹$M,\ w \models K\ i\ p \land K\ i\ (p \longrightarrow q) \longrightarrow K\ i\ q$›
⟨*proof*⟩

theorem *generalization*:
  fixes M :: ‹$('i,\ 'w)\ kripke$›
  assumes ‹$\forall\, (M :: ('i,\ 'w)\ kripke).\ \forall\, w \in \mathcal{W}\ M.\ M,\ w \models p$› ‹$w \in \mathcal{W}\ M$›
  shows ‹$M,\ w \models K\ i\ p$›
⟨*proof*⟩

theorem *truth*:
  assumes ‹$reflexive\ M$› ‹$w \in \mathcal{W}\ M$›

shows ‹$M,\ w \models K\ i\ p \longrightarrow p$›
⟨*proof*⟩

theorem *pos-introspection*:
assumes ‹*transitive* M› ‹$w \in \mathcal{W}\ M$›
shows ‹$M,\ w \models K\ i\ p \longrightarrow K\ i\ (K\ i\ p)$›
⟨*proof*⟩

theorem *neg-introspection*:
assumes ‹*symmetric* M› ‹*transitive* M› ‹$w \in \mathcal{W}\ M$›
shows ‹$M,\ w \models \neg\ K\ i\ p \longrightarrow K\ i\ (\neg\ K\ i\ p)$›
⟨*proof*⟩

4 Normal Modal Logic

primrec *eval* :: ‹$(id \Rightarrow bool) \Rightarrow ('i\ fm \Rightarrow bool) \Rightarrow 'i\ fm \Rightarrow bool$› **where**
‹$eval$ - - $\bot = False$›
| ‹$eval\ g$ - $(Pro\ x) = g\ x$›
| ‹$eval\ g\ h\ (p \lor q) = (eval\ g\ h\ p \lor eval\ g\ h\ q)$›
| ‹$eval\ g\ h\ (p \land q) = (eval\ g\ h\ p \land eval\ g\ h\ q)$›
| ‹$eval\ g\ h\ (p \longrightarrow q) = (eval\ g\ h\ p \longrightarrow eval\ g\ h\ q)$›
| ‹$eval$ - $h\ (K\ i\ p) = h\ (K\ i\ p)$›

abbreviation ‹$tautology\ p \equiv \forall\, g\ h.\ eval\ g\ h\ p$›

inductive *AK* :: ‹$('i\ fm \Rightarrow bool) \Rightarrow 'i\ fm \Rightarrow bool$› (‹- ⊢ -› [*50*, *50*] *50*)
for A :: ‹$'i\ fm \Rightarrow bool$› **where**
A1: ‹$tautology\ p \Longrightarrow A \vdash p$›
| *A2*: ‹$A \vdash K\ i\ p \land K\ i\ (p \longrightarrow q) \longrightarrow K\ i\ q$›
| *Ax*: ‹$A\ p \Longrightarrow A \vdash p$›
| *R1*: ‹$A \vdash p \Longrightarrow A \vdash p \longrightarrow q \Longrightarrow A \vdash q$›
| *R2*: ‹$A \vdash p \Longrightarrow A \vdash K\ i\ p$›

primrec *imply* :: ‹$'i\ fm\ list \Rightarrow 'i\ fm \Rightarrow 'i\ fm$› (**infixr** ‹⇝› *56*) **where**
‹$([] \leadsto q) = q$›
| ‹$(p\ \#\ ps \leadsto q) = (p \longrightarrow ps \leadsto q)$›

abbreviation *AK-assms* (‹-; - ⊢ -› [*50*, *50*, *50*] *50*) **where**
‹$A;\ G \vdash p \equiv \exists\, qs.\ set\ qs \subseteq G \land (A \vdash qs \leadsto p)$›

5 Soundness

lemma *eval-semantics*:
‹$eval\ (pi\ w)\ (\lambda q.\ (\!|\mathcal{W} = W,\ \mathcal{K} = r,\ \pi = pi|\!),\ w \models q)\ p = ((\!|\mathcal{W} = W,\ \mathcal{K} = r,\ \pi = pi|\!),\ w \models p)$›
⟨*proof*⟩

lemma *tautology*:

assumes ‹*tautology* p›
shows ‹M, $w \models p$›
⟨*proof*⟩

theorem *soundness*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
shows ‹$A \vdash p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p$›
⟨*proof*⟩

6 Derived rules

lemma *K-A2′*: ‹$A \vdash K\ i\ (p \longrightarrow q) \longrightarrow K\ i\ p \longrightarrow K\ i\ q$›
⟨*proof*⟩

lemma *K-map*:
assumes ‹$A \vdash p \longrightarrow q$›
shows ‹$A \vdash K\ i\ p \longrightarrow K\ i\ q$›
⟨*proof*⟩

lemma *K-LK*: ‹$A \vdash (L\ i\ (\neg\ p) \longrightarrow \neg\ K\ i\ p)$›
⟨*proof*⟩

lemma *K-imply-head*: ‹$A \vdash (p\ \#\ ps \leadsto p)$›
⟨*proof*⟩

lemma *K-imply-Cons*:
assumes ‹$A \vdash ps \leadsto q$›
shows ‹$A \vdash p\ \#\ ps \leadsto q$›
⟨*proof*⟩

lemma *K-right-mp*:
assumes ‹$A \vdash ps \leadsto p$› ‹$A \vdash ps \leadsto (p \longrightarrow q)$›
shows ‹$A \vdash ps \leadsto q$›
⟨*proof*⟩

lemma *tautology-imply-superset*:
assumes ‹*set* $ps \subseteq$ *set* qs›
shows ‹*tautology* $(ps \leadsto r \longrightarrow qs \leadsto r)$›
⟨*proof*⟩

lemma *K-imply-weaken*:
assumes ‹$A \vdash ps \leadsto q$› ‹*set* $ps \subseteq$ *set* ps'›
shows ‹$A \vdash ps' \leadsto q$›
⟨*proof*⟩

lemma *imply-append*: ‹$(ps\ @\ ps' \leadsto q) = (ps \leadsto ps' \leadsto q)$›
⟨*proof*⟩

lemma *K-ImpI*:

assumes ‹$A \vdash p \mathbin{\#} G \leadsto q$›
shows ‹$A \vdash G \leadsto (p \longrightarrow q)$›
⟨*proof*⟩

lemma *K-Boole*:
assumes ‹$A \vdash (\neg\ p) \mathbin{\#} G \leadsto \bot$›
shows ‹$A \vdash G \leadsto p$›
⟨*proof*⟩

lemma *K-DisE*:
assumes ‹$A \vdash p \mathbin{\#} G \leadsto r$› ‹$A \vdash q \mathbin{\#} G \leadsto r$› ‹$A \vdash G \leadsto p \vee q$›
shows ‹$A \vdash G \leadsto r$›
⟨*proof*⟩

lemma *K-mp*: ‹$A \vdash p \mathbin{\#} (p \longrightarrow q) \mathbin{\#} G \leadsto q$›
⟨*proof*⟩

lemma *K-swap*:
assumes ‹$A \vdash p \mathbin{\#} q \mathbin{\#} G \leadsto r$›
shows ‹$A \vdash q \mathbin{\#} p \mathbin{\#} G \leadsto r$›
⟨*proof*⟩

lemma *K-DisL*:
assumes ‹$A \vdash p \mathbin{\#} ps \leadsto q$› ‹$A \vdash p' \mathbin{\#} ps \leadsto q$›
shows ‹$A \vdash (p \vee p') \mathbin{\#} ps \leadsto q$›
⟨*proof*⟩

lemma *K-distrib-K-imp*:
assumes ‹$A \vdash K\ i\ (G \leadsto q)$›
shows ‹$A \vdash map\ (K\ i)\ G \leadsto K\ i\ q$›
⟨*proof*⟩

lemma *K-trans*: ‹$A \vdash (p \longrightarrow q) \longrightarrow (q \longrightarrow r) \longrightarrow p \longrightarrow r$›
⟨*proof*⟩

lemma *K-L-dual*: ‹$A \vdash \neg\ L\ i\ (\neg\ p) \longrightarrow K\ i\ p$›
⟨*proof*⟩

7 Strong Soundness

corollary *soundness-imply*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
shows ‹$A \vdash ps \leadsto p \Longrightarrow P;\ set\ ps \models\!\!\!\models\star\ p$›
⟨*proof*⟩

theorem *strong-soundness*:
assumes ‹$\bigwedge M\ w\ p.\ A\ p \Longrightarrow P\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
shows ‹$A;\ G \vdash p \Longrightarrow P;\ G \models\!\!\!\models\star\ p$›
⟨*proof*⟩

8 Completeness

8.1 Consistent sets

definition *consistent* :: ‹$('i\ fm \Rightarrow bool) \Rightarrow 'i\ fm\ set \Rightarrow bool$› **where**
‹$consistent\ A\ S \equiv \neg\ (A;\ S \vdash \bot)$›

lemma *inconsistent-subset*:
assumes ‹$consistent\ A\ V$› ‹$\neg\ consistent\ A\ (\{p\} \cup V)$›
obtains V' **where** ‹$set\ V' \subseteq V$› ‹$A \vdash p\ \#\ V' \leadsto \bot$›
⟨*proof*⟩

lemma *consistent-consequent*:
assumes ‹$consistent\ A\ V$› ‹$p \in V$› ‹$A \vdash p \longrightarrow q$›
shows ‹$consistent\ A\ (\{q\} \cup V)$›
⟨*proof*⟩

lemma *consistent-consequent′*:
assumes ‹$consistent\ A\ V$› ‹$p \in V$› ‹$tautology\ (p \longrightarrow q)$›
shows ‹$consistent\ A\ (\{q\} \cup V)$›
⟨*proof*⟩

lemma *consistent-disjuncts*:
assumes ‹$consistent\ A\ V$› ‹$(p \vee q) \in V$›
shows ‹$consistent\ A\ (\{p\} \cup V) \vee consistent\ A\ (\{q\} \cup V)$›
⟨*proof*⟩

lemma *exists-finite-inconsistent*:
assumes ‹$\neg\ consistent\ A\ (\{\neg\ p\} \cup V)$›
obtains W **where** ‹$\{\neg\ p\} \cup W \subseteq \{\neg\ p\} \cup V$› ‹$(\neg\ p) \notin W$› ‹$finite\ W$› ‹$\neg$
$consistent\ A\ (\{\neg\ p\} \cup W)$›
⟨*proof*⟩

lemma *inconsistent-imply*:
assumes ‹$\neg\ consistent\ A\ (\{\neg\ p\} \cup set\ G)$›
shows ‹$A \vdash G \leadsto p$›
⟨*proof*⟩

8.2 Maximal consistent sets

lemma *fm-any-size*: ‹$\exists\, p :: 'i\ fm.\ size\ p = n$›
⟨*proof*⟩

lemma *infinite-UNIV-fm*: ‹$infinite\ (UNIV :: 'i\ fm\ set)$›
⟨*proof*⟩

interpretation *MCS* ‹$consistent\ A$› **for** A :: ‹$'i\ fm \Rightarrow bool$›
⟨*proof*⟩

theorem *deriv-in-maximal*:

assumes ‹*consistent* A V› ‹*maximal* A V› ‹$A \vdash p$›
shows ‹$p \in V$›
⟨*proof*⟩

theorem *exactly-one-in-maximal*:
assumes ‹*consistent* A V› ‹*maximal* A V›
shows ‹$p \in V \longleftrightarrow (\neg\ p) \notin V$›
⟨*proof*⟩

theorem *consequent-in-maximal*:
assumes ‹*consistent* A V› ‹*maximal* A V› ‹$p \in V$› ‹$(p \longrightarrow q) \in V$›
shows ‹$q \in V$›
⟨*proof*⟩

theorem *ax-in-maximal*:
assumes ‹*consistent* A V› ‹*maximal* A V› ‹A p›
shows ‹$p \in V$›
⟨*proof*⟩

theorem *mcs-properties*:
assumes ‹*consistent* A V› **and** ‹*maximal* A V›
shows ‹$A \vdash p \Longrightarrow p \in V$›
and ‹$p \in V \longleftrightarrow (\neg\ p) \notin V$›
and ‹$p \in V \Longrightarrow (p \longrightarrow q) \in V \Longrightarrow q \in V$›
⟨*proof*⟩

lemma *maximal-extension*:
fixes V :: ‹*'i fm set*›
assumes ‹*consistent* A V›
obtains W **where** ‹$V \subseteq W$› ‹*consistent* A W› ‹*maximal* A W›
⟨*proof*⟩

8.3 Canonical model

abbreviation *pi* :: ‹*'i fm set* $\Rightarrow$ *id* $\Rightarrow$ *bool*› **where**
‹*pi* V $x \equiv$ *Pro* $x \in V$›

abbreviation *known* :: ‹*'i fm set* $\Rightarrow$ *'i* $\Rightarrow$ *'i fm set*› **where**
‹*known* V $i \equiv \{p.\ K\ i\ p \in V\}$›

abbreviation *reach* :: ‹(*'i fm* $\Rightarrow$ *bool*) $\Rightarrow$ *'i* $\Rightarrow$ *'i fm set* $\Rightarrow$ *'i fm set set*› **where**
‹*reach* A i $V \equiv \{W.$ *known* V $i \subseteq W\}$›

abbreviation *mcss* :: ‹(*'i fm* $\Rightarrow$ *bool*) $\Rightarrow$ *'i fm set set*› **where**
‹*mcss* $A \equiv \{W.$ *consistent* A $W \wedge$ *maximal* A $W\}$›

abbreviation *canonical* :: ‹(*'i fm* $\Rightarrow$ *bool*) $\Rightarrow$ (*'i*, *'i fm set*) *kripke*› **where**
‹*canonical* $A \equiv (\!|\mathcal{W} =$ *mcss* A, $\mathcal{K} =$ *reach* A, $\pi =$ *pi*$|\!)$›

lemma *truth-lemma*:
fixes p :: ‹$'i$ *fm*›
assumes ‹*consistent* A V› **and** ‹*maximal* A V›
shows ‹$p \in V \longleftrightarrow$ *canonical* A, $V \models p$›
⟨*proof*⟩

lemma *canonical-model*:
assumes ‹*consistent* A S› **and** ‹$p \in S$›
defines ‹$V \equiv$ *Extend* A S› **and** ‹$M \equiv$ *canonical* A›
shows ‹M, $V \models p$› **and** ‹*consistent* A V› **and** ‹*maximal* A V›
⟨*proof*⟩

8.4 Completeness

abbreviation *valid* :: ‹(($'i$, $'i$ *fm set*) *kripke* ⇒ *bool*) ⇒ $'i$ *fm set* ⇒ $'i$ *fm* ⇒ *bool*›
(‹-; - ⊫ -› [*50*, *50*, *50*] *50*)
where ‹P; $G \Vdash p \equiv P$; $G \Vdash\star\ p$›

theorem *strong-completeness*:
assumes ‹P; $G \Vdash p$› **and** ‹P (*canonical* A)›
shows ‹A; $G \vdash p$›
⟨*proof*⟩

corollary *completeness*:
assumes ‹P; {} ⊫ p› **and** ‹P (*canonical* A)›
shows ‹$A \vdash p$›
⟨*proof*⟩

corollary *completeness*$_A$:
assumes ‹(λ-. *True*); {} ⊫ p›
shows ‹$A \vdash p$›
⟨*proof*⟩

9 System K

abbreviation *SystemK* (‹- $\vdash_K$ -› [*50*] *50*) **where**
‹$G \vdash_K p \equiv$ (λ-. *False*); $G \vdash p$›

lemma *strong-soundness*$_K$: ‹$G \vdash_K p \Longrightarrow P$; $G \Vdash\star\ p$›
⟨*proof*⟩

abbreviation *validK* (‹- $\Vdash_K$ -› [*50*, *50*] *50*) **where**
‹$G \Vdash_K p \equiv$ (λ-. *True*); $G \Vdash p$›

lemma *strong-completeness*$_K$: ‹$G \Vdash_K p \Longrightarrow G \vdash_K p$›
⟨*proof*⟩

theorem *main*$_K$: ‹$G \Vdash_K p \longleftrightarrow G \vdash_K p$›
⟨*proof*⟩

corollary ‹$G \models_K p \Longrightarrow (\lambda$-. *True*); $G \models\star p$›
  ⟨*proof*⟩

10 System T

Also known as System M

inductive AxT :: ‹$'i\ fm \Rightarrow bool$› **where**
  ‹$AxT\ (K\ i\ p \longrightarrow p)$›

abbreviation *SystemT* (‹- $\vdash_T$ -› [*50*, *50*] *50*) **where**
  ‹$G \vdash_T p \equiv AxT$; $G \vdash p$›

lemma *soundness-AxT*: ‹$AxT\ p \Longrightarrow$ *reflexive* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
  ⟨*proof*⟩

lemma *strong-soundness*$_T$: ‹$G \vdash_T p \Longrightarrow$ *reflexive*; $G \models\star p$›
  ⟨*proof*⟩

lemma *AxT-reflexive*:
  assumes ‹$AxT \le A$› **and** ‹*consistent* $A\ V$› **and** ‹*maximal* $A\ V$›
  shows ‹$V \in$ *reach* $A\ i\ V$›
⟨*proof*⟩

lemma *reflexive*$_T$:
  assumes ‹$AxT \le A$›
  shows ‹*reflexive* (*canonical* A)›
  ⟨*proof*⟩

abbreviation *validT* (‹- $\models_T$ -› [*50*, *50*] *50*) **where**
  ‹$G \models_T p \equiv$ *reflexive*; $G \models p$›

lemma *strong-completeness*$_T$: ‹$G \models_T p \Longrightarrow G \vdash_T p$›
  ⟨*proof*⟩

theorem *main*$_T$: ‹$G \models_T p \longleftrightarrow G \vdash_T p$›
  ⟨*proof*⟩

corollary ‹$G \models_T p \longrightarrow$ *reflexive*; $G \models\star p$›
  ⟨*proof*⟩

11 System KB

inductive AxB :: ‹$'i\ fm \Rightarrow bool$› **where**
  ‹$AxB\ (p \longrightarrow K\ i\ (L\ i\ p))$›

abbreviation *SystemKB* (‹- $\vdash_{KB}$ -› [*50*, *50*] *50*) **where**

$\langle G \vdash_{KB} p \equiv AxB;\ G \vdash p \rangle$

lemma *soundness-AxB*: $\langle AxB\ p \Longrightarrow symmetric\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p \rangle$
⟨*proof*⟩

lemma *strong-soundness*$_{KB}$: $\langle G \vdash_{KB} p \Longrightarrow symmetric;\ G \models\star\ p \rangle$
⟨*proof*⟩

lemma *AxB-symmetric'*:
assumes $\langle AxB \leq A \rangle$ $\langle consistent\ A\ V \rangle$ $\langle maximal\ A\ V \rangle$ $\langle consistent\ A\ W \rangle$ $\langle maximal\ A\ W \rangle$
and $\langle W \in reach\ A\ i\ V \rangle$
shows $\langle V \in reach\ A\ i\ W \rangle$
⟨*proof*⟩

lemma *symmetric*$_{KB}$:
assumes $\langle AxB \leq A \rangle$
shows $\langle symmetric\ (canonical\ A) \rangle$
⟨*proof*⟩

abbreviation *validKB* (‹- $\models_{KB}$ -› [*50*, *50*] *50*) **where**
$\langle G \models_{KB} p \equiv symmetric;\ G \models p \rangle$

lemma *strong-completeness*$_{KB}$: $\langle G \models_{KB} p \Longrightarrow G \vdash_{KB} p \rangle$
⟨*proof*⟩

theorem *main*$_{KB}$: $\langle G \models_{KB} p \longleftrightarrow G \vdash_{KB} p \rangle$
⟨*proof*⟩

corollary $\langle G \models_{KB} p \longrightarrow symmetric;\ G \models\star\ p \rangle$
⟨*proof*⟩

12 System K4

inductive *Ax4* :: ‹*'i fm* ⇒ *bool*› **where**
$\langle Ax4\ (K\ i\ p \longrightarrow K\ i\ (K\ i\ p)) \rangle$

abbreviation *SystemK4* (‹- $\vdash_{K4}$ -› [*50*, *50*] *50*) **where**
$\langle G \vdash_{K4} p \equiv Ax4;\ G \vdash p \rangle$

lemma *soundness-Ax4*: $\langle Ax4\ p \Longrightarrow transitive\ M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M,\ w \models p \rangle$
⟨*proof*⟩

lemma *strong-soundness*$_{K4}$: $\langle G \vdash_{K4} p \Longrightarrow transitive;\ G \models\star\ p \rangle$
⟨*proof*⟩

lemma *Ax4-transitive*:
assumes $\langle Ax4 \leq A \rangle$ $\langle consistent\ A\ V \rangle$ $\langle maximal\ A\ V \rangle$
and $\langle W \in reach\ A\ i\ V \rangle$ $\langle U \in reach\ A\ i\ W \rangle$

shows ‹$U \in$ *reach* A i V›
⟨*proof*⟩

lemma *transitive*$_{K4}$:
assumes ‹*Ax4* $\leq A$›
shows ‹*transitive* (*canonical* A)›
⟨*proof*⟩

abbreviation *validK4* (‹- $\models_{K4}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{K4} p \equiv$ *transitive*; $G \models p$›

lemma *strong-completeness*$_{K4}$: ‹$G \models_{K4} p \Longrightarrow G \vdash_{K4} p$›
⟨*proof*⟩

theorem *main*$_{K4}$: ‹$G \models_{K4} p \longleftrightarrow G \vdash_{K4} p$›
⟨*proof*⟩

corollary ‹$G \models_{K4} p \longrightarrow$ *transitive*; $G \models\star p$›
⟨*proof*⟩

13 System K5

inductive *Ax5* :: ‹*'i fm* $\Rightarrow$ *bool*› **where**
‹*Ax5* (L i $p \longrightarrow K$ i (L i p))›

abbreviation *SystemK5* (‹- $\vdash_{K5}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{K5} p \equiv$ *Ax5*; $G \vdash p$›

lemma *soundness-Ax5*: ‹*Ax5* $p \Longrightarrow$ *Euclidean* $M \Longrightarrow w \in \mathcal{W}$ $M \Longrightarrow M, w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{K5}$: ‹$G \vdash_{K5} p \Longrightarrow$ *Euclidean*; $G \models\star p$›
⟨*proof*⟩

lemma *Ax5-Euclidean*:
assumes ‹*Ax5* $\leq A$›
‹*consistent* A U› ‹*maximal* A U›
‹*consistent* A V› ‹*maximal* A V›
‹*consistent* A W› ‹*maximal* A W›
and ‹$V \in$ *reach* A i U› ‹$W \in$ *reach* A i U›
shows ‹$W \in$ *reach* A i V›
⟨*proof*⟩

lemma *Euclidean*$_{K5}$:
assumes ‹*Ax5* $\leq A$›
shows ‹*Euclidean* (*canonical* A)›
⟨*proof*⟩

abbreviation *validK5* (‹- $\models_{K5}$ -› [*50*, *50*] *50*) **where**

‹$G \models_{K5} p \equiv$ *Euclidean*; $G \models p$›

lemma *strong-completeness*$_{K5}$: ‹$G \models_{K5} p \Longrightarrow G \vdash_{K5} p$›
⟨*proof*⟩

theorem *main*$_{K5}$: ‹$G \models_{K5} p \longleftrightarrow G \vdash_{K5} p$›
⟨*proof*⟩

corollary ‹$G \models_{K5} p \longrightarrow$ *Euclidean*; $G \models_\star p$›
⟨*proof*⟩

14 System S4

abbreviation *Or* :: ‹$('a \Rightarrow bool) \Rightarrow ('a \Rightarrow bool) \Rightarrow 'a \Rightarrow bool$› (**infixl** ‹$\oplus$› *65*) **where**
‹$(A \oplus A')\ p \equiv A\ p \lor A'\ p$›

abbreviation *SystemS4* (‹- $\vdash_{S4}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{S4} p \equiv AxT \oplus Ax4$; $G \vdash p$›

lemma *soundness-AxT4*: ‹$(AxT \oplus Ax4)\ p \Longrightarrow$ *reflexive* $M \land$ *transitive* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M, w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{S4}$: ‹$G \vdash_{S4} p \Longrightarrow$ *refltrans*; $G \models_\star p$›
⟨*proof*⟩

abbreviation *validS4* (‹- $\models_{S4}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{S4} p \equiv$ *refltrans*; $G \models p$›

lemma *strong-completeness*$_{S4}$: ‹$G \models_{S4} p \Longrightarrow G \vdash_{S4} p$›
⟨*proof*⟩

theorem *main*$_{S4}$: ‹$G \models_{S4} p \longleftrightarrow G \vdash_{S4} p$›
⟨*proof*⟩

corollary ‹$G \models_{S4} p \longrightarrow$ *refltrans*; $G \models_\star p$›
⟨*proof*⟩

15 System S5

15.1 T + B + 4

abbreviation *SystemS5* (‹- $\vdash_{S5}$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{S5} p \equiv AxT \oplus AxB \oplus Ax4$; $G \vdash p$›

abbreviation *AxTB4* :: ‹$'i\ fm \Rightarrow bool$› **where**
‹$AxTB4 \equiv AxT \oplus AxB \oplus Ax4$›

lemma *soundness-AxTB4*: ‹*AxTB4* $p \Longrightarrow$ *equivalence* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M$, $w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{S5}$: ‹$G \vdash_{S5} p \Longrightarrow$ *equivalence*; $G \models\star\ p$›
⟨*proof*⟩

abbreviation *validS5* (‹- $\models_{S5}$ -› [*50*, *50*] *50*) **where**
‹$G \models_{S5} p \equiv$ *equivalence*; $G \models p$›

lemma *strong-completeness*$_{S5}$: ‹$G \models_{S5} p \Longrightarrow G \vdash_{S5} p$›
⟨*proof*⟩

theorem *main*$_{S5}$: ‹$G \models_{S5} p \longleftrightarrow G \vdash_{S5} p$›
⟨*proof*⟩

corollary ‹$G \models_{S5} p \longrightarrow$ *equivalence*; $G \models\star\ p$›
⟨*proof*⟩

15.2 T + 5

abbreviation *SystemS5'* (‹- $\vdash_{S5}''$ -› [*50*, *50*] *50*) **where**
‹$G \vdash_{S5}' p \equiv$ *AxT* $\oplus$ *Ax5*; $G \vdash p$›

abbreviation *AxT5* :: ‹*'i fm* $\Rightarrow$ *bool*› **where**
‹*AxT5* $\equiv$ *AxT* $\oplus$ *Ax5*›

lemma *symm-trans-Euclid*: ‹*symmetric* $M \Longrightarrow$ *transitive* $M \Longrightarrow$ *Euclidean* M›
⟨*proof*⟩

lemma *soundness-AxT5*: ‹*AxT5* $p \Longrightarrow$ *equivalence* $M \Longrightarrow w \in \mathcal{W}\ M \Longrightarrow M$, $w \models p$›
⟨*proof*⟩

lemma *strong-soundness*$_{S5}$*'*: ‹$G \vdash_{S5}' p \Longrightarrow$ *equivalence*; $G \models\star\ p$›
⟨*proof*⟩

lemma *refl-Euclid-equiv*: ‹*reflexive* $M \Longrightarrow$ *Euclidean* $M \Longrightarrow$ *equivalence* M›
⟨*proof*⟩

lemma *strong-completeness*$_{S5}$*'*: ‹$G \models_{S5} p \Longrightarrow G \vdash_{S5}' p$›
⟨*proof*⟩

theorem *main*$_{S5}$*'*: ‹$G \models_{S5} p \longleftrightarrow G \vdash_{S5}' p$›
⟨*proof*⟩

15.3 Equivalence between systems

15.3.1 Axiom 5 from B and 4

lemma *K4-L*:
  assumes ‹$Ax4 \leq A$›
  shows ‹$A \vdash L\ i\ (L\ i\ p) \longrightarrow L\ i\ p$›
⟨*proof*⟩

lemma *KB4-5*:
  assumes ‹$AxB \leq A$› ‹$Ax4 \leq A$›
  shows ‹$A \vdash L\ i\ p \longrightarrow K\ i\ (L\ i\ p)$›
⟨*proof*⟩

15.3.2 Axioms B and 4 from T and 5

lemma *T-L*:
  assumes ‹$AxT \leq A$›
  shows ‹$A \vdash p \longrightarrow L\ i\ p$›
⟨*proof*⟩

lemma *S5′-B*:
  assumes ‹$AxT \leq A$› ‹$Ax5 \leq A$›
  shows ‹$A \vdash p \longrightarrow K\ i\ (L\ i\ p)$›
⟨*proof*⟩

lemma *K5-L*:
  assumes ‹$Ax5 \leq A$›
  shows ‹$A \vdash L\ i\ (K\ i\ p) \longrightarrow K\ i\ p$›
⟨*proof*⟩

lemma *S5′-4*:
  assumes ‹$AxT \leq A$› ‹$Ax5 \leq A$›
  shows ‹$A \vdash K\ i\ p \longrightarrow K\ i\ (K\ i\ p)$›
⟨*proof*⟩

lemma *S5-S5′*: ‹$AxTB4 \vdash p \Longrightarrow AxT5 \vdash p$›
⟨*proof*⟩

lemma *S5′-S5*: ‹$AxT5 \vdash p \Longrightarrow AxTB4 \vdash p$›
⟨*proof*⟩

corollary *S5-S5′-assms*: ‹$G \vdash_{S5}\ p \longleftrightarrow G \vdash_{S5}{}'\ p$›
  ⟨*proof*⟩

16 Acknowledgements

The formalization is inspired by Berghofer's formalization of Henkin-style completeness.

- Stefan Berghofer: First-Order Logic According to Fitting. https://www.isa-afp.org/entries/FOL-Fitting.shtml

end

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