The Dottie Number

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Abstract

The Dottie number is the unique fixed point d of the cosine function: $\cos d = d$. It is approximately 0.739085133215.

This theory establishes the Dottie number's key properties: the fixed point exists (by the intermediate value theorem) and is unique (because $\cos x - x$ has a strictly negative derivative). Next, the value of d to 12 decimal places is shown using the `approximation` proof method. Two more properties of d are also shown: first, that it is *transcendental* (via the Hermite–Lindemann–Weierstrass theorem); second, that it is a *universal attractor*, in the sense that iterating the cosine function from any real starting point converges to it.

Contents

1 The Dottie Number

theory *Dottie*
  imports *HOL−Analysis.Analysis*
    HOL−Decision-Procs.Approximation
    Hermite-Lindemann.Hermite-Lindemann

begin

The Dottie number, approximately 0.739085133215, is the unique fixed point of the cosine function.

This theory establishes the Dottie number's basic theory. We first show that the fixed point *exists* (by the intermediate value theorem) and is *unique* (because $\cos x - x$ has a strictly negative derivative), justifying the definition of *dottie*, and we pin it down to twelve decimal places. We then prove three further results: that the Dottie number is a *universal attractor*, in the sense that iterating cosine from any real starting point converges to it; the trigonometric identity $\sin(\mathit{dottie}) = \sqrt{1 - \mathit{dottie}^2}$; and, using the Hermite–Lindemann–Weierstraß theorem, that the Dottie number is *transcendental*.

definition *dottie* :: *real* **where**
  dottie ≡ *THE x. cos x = x*

lemma *cos-1-lt-1*: *cos* (*1*::*real*) < *1*
  using *cos-monotone-0-pi pi-gt3* **by** *force*

We shall reason about the function $g(x) = \cos x - x$. The locale provides a scope for g and its properties, which are used by several of the lemmas below.

locale *Dottie* =
  fixes *g* :: *real* ⇒ *real*
  defines *g* ≡ *λx*::*real*. *cos x* − *x*

begin

lemma *g-has-negative-deriv*:
  assumes |*t*| ≤ *1*
  shows ∃ *d*. (*g has-real-derivative d*) (*at t*) ∧ *d* < *0*
proof (*intro exI conjI*)
  show (*g has-real-derivative* (− *sin t* − *1*)) (*at t*)
    unfolding *g-def* **by** (*auto intro*!: *derivative-eq-intros*)
  show − *sin t* − *1* < *0*
    using *assms pi-gt3 le-arcsin-iff* [*of - t*] **by** *fastforce*
qed

1.1 Existence

We have $g(0) = 1 > 0$ and $g(1) = \cos 1 - 1 < 0$. Since g is continuous, the intermediate value theorem gives a point $x \in (0, 1)$ where $g(x) = 0$, i.e.

$\cos x = x$.

lemma *dottie-exists*: $\exists\, x$::*real*. $0 < x \wedge x < 1 \wedge \cos x = x$
proof −
  — Apply the IVT to g on the unit interval at 0.
  have *g-cont*: *continuous-on* $\{0..1\}$ *g*
    unfolding *g-def* **by** (*intro continuous-intros*)
  obtain $g\ 0 = 1\ g\ 1 < 0$ **using** *cos-1-lt-1* **by** (*simp add*: *g-def*)
  with *IVT2'*[*of g 1 0 0*] *g-cont*
  obtain *x* **where** *hx*: $0 \leq x\ x \leq 1\ g\ x = 0$
    by (*metis less-eq-real-def zero-le-one*)
  hence *cos-eq*: $\cos x = x$ **by** (*simp add*: *g-def*)
  with *hx* **show** *?thesis*
    by (*metis cos-1-lt-1 cos-zero order-less-le*)
qed

1.2 Uniqueness

The function $g(x) = \cos x - x$ has derivative $g'(x) = -\sin x - 1$, which is strictly negative for $x \in [-1, 1]$ (since $\sin x \geq 0$ there). A function with strictly negative derivative is strictly decreasing, so g can have at most one zero. We can extend uniqueness to the entire real line.

lemma *dottie-unique*:
  fixes *x y* :: *real*
  assumes $\cos x = x\ \cos y = y$
  shows $x = y$
proof (*rule ccontr*)
  assume $x \neq y$
  have *gx*: $g\ x = 0$ **and** *gy*: $g\ y = 0$ **using** *assms* **by** (*auto simp*: *g-def*)
  — The derivative of g is $\lambda x.\ -\ \sin x\ -\ 1$, which is negative on $\{-\ 1..1\}$.
  show *False*
  proof (*cases* $|x| > 1 \vee |y| > 1$)
    case *True*
    then show *?thesis*
      by (*metis assms abs-cos-le-one not-less*)
  next
    case *False*
    then have $|x| \leq 1 \wedge |y| \leq 1$
      by *simp*
    moreover have $x < y \vee y < x$ **using** ‹$x \neq y$› **by** *linarith*
    ultimately show *?thesis*
      using *DERIV-neg-imp-decreasing* [*OF - g-has-negative-deriv*] *gx gy*
      by *force*
  qed
qed

lemma *facts*: $0 < dottie\ dottie < 1\ \cos dottie = dottie$
proof −
  obtain *x* :: *real* **where** *hx*: $0 < x\ x < 1\ \cos x = x$

using *dottie-exists* **by** *blast*
have *unique*: *y* = *x* **if** *cos y* = *y* **for** *y* :: *real*
by (*simp add*: *dottie-unique* ‹*cos x* = *x*› *that*)
have *the-eq*: *dottie* = *x*
unfolding *dottie-def* **using** ‹*cos x* = *x*› *unique* **by** *blast*
then show *0* < *dottie dottie* < *1 cos dottie* = *dottie*
using *hx* **by** (*auto simp*: *g-def*)
qed

1.3 Approximation

We pin down the Dottie number to 12 decimal places. Note that g is decreasing. We check that $\cos(lb) > lb$ (so the fixed point is above lb) and $\cos(ub) < u$ (so it is below ub).

definition *lb*::*real* **where** *lb* ≡ *0.739085133215*

definition *ub*::*real* **where** *ub* ≡ *0.739085133216*

lemma *lb-gt*: *cos lb* > *lb*
unfolding *lb-def* **by** (*approximation 50*)

lemma *ub-lt*: *cos ub* < *ub*
unfolding *ub-def* **by** (*approximation 50*)

lemma *lb*: *lb* < *dottie*
proof (*rule ccontr*)
assume *neg*: ¬ *lb* < *dottie*
have *gd*: *g lb* > *0*
using *facts lb-gt* **by** (*auto simp*: *g-def*)
show *False*
using *DERIV-neg-imp-decreasing* [*OF* - *g-has-negative-deriv*] *facts neg*
by (*smt* (*verit*, *best*) *cos-le-one g-def lb-gt*)
qed

lemma *ub*: *ub* > *dottie*
proof (*rule ccontr*)
assume *neg*: ¬ *ub* > *dottie*
have *gd*: *g ub* < *0*
using *facts ub-lt* **by** (*auto simp*: *g-def*)
show *False*
using *DERIV-neg-imp-decreasing* [*OF* - *g-has-negative-deriv*] *facts neg*
by (*smt* (*verit*) *abs-cos-le-one g-def ub-lt*)
qed

end

1.4 A trigonometric identity

Now for something trivial. Since $\cos(dottie) = dottie$ and $dottie \in (0, 1)$, the Pythagorean identity gives $\sin(dottie) = \sqrt{1 - dottie^2}$. Sledgehammer found this proof.

lemma *sin-dottie*: *sin dottie* = *sqrt* (*1* − *dottie*^{2})
  by (*smt* (*verit*) *Dottie.facts pi-gt3 sin-cos-sqrt sin-ge-zero*)

1.5 The Dottie number is a universal attractor

Iterating cosine from *any* real starting point converges to the Dottie number. The key fact is that cos is a contraction on $[-1, 1]$ with Lipschitz constant $\sin 1 < 1$ (since $|\cos' x| = |\sin x| \leq \sin 1$ there), and that cos maps all of $\mathbb{R}$ into $[-1, 1]$.

lemma *sin1-bounds*: *0* < *sin* (*1*::*real*) *sin* (*1*::*real*) < *1*
  using *sin-monotone-2pi*[*of 1 pi*/*2*] *pi-gt3* **by** (*auto simp*: *sin-gt-zero-02*)

lemma *abs-sin-le-sin1*:
  assumes |*t*| ≤ *1* **shows** |*sin t*| ≤ *sin* (*1*::*real*)
proof −
  have *1* < *pi*/*2* **using** *pi-gt3* **by** *simp*
  then show *?thesis*
    by (*smt* (*verit*, *best*) *assms sin-minus sin-monotone-2pi-le*)
qed

The mean value theorem turns the derivative bound into a Lipschitz bound.

lemma *cos-contraction-lt*:
  fixes *x y* :: *real*
  assumes *x* < *y* |*x*| ≤ *1* |*y*| ≤ *1*
  shows |*cos x* − *cos y*| ≤ *sin 1* ∗ |*x* − *y*|
proof −
  have *cont*: *continuous-on* {*x*..*y*} *cos* **by** (*intro continuous-intros*)
  have *deriv*: ((*cos*::*real*⇒*real*) *has-derivative* (∗) (− *sin u*)) (*at u*) **for** *u* :: *real*
    using *DERIV-cos*[*of u*] **unfolding** *has-field-derivative-def* **by** *simp*
  have ∃*ξ* ∈ {*x*<..<*y*}. *norm* (*cos y* − *cos x*) ≤ *norm* ((∗) (− *sin ξ*) (*y* − *x*))
    by (*rule mvt-general*[*OF* ‹*x* < *y*› *cont*]) (*use deriv* **in** *blast*)
  then obtain *ξ* **where** *ξ*: *ξ* ∈ {*x*<..<*y*} *norm* (*cos y* − *cos x*) ≤ *norm* (− *sin ξ* ∗ (*y* − *x*))
    by *auto*
  have |*ξ*| ≤ *1* **using** *ξ assms* **by** *auto*
  then have *absxi*: |*sin ξ*| ≤ *sin 1* **by** (*rule abs-sin-le-sin1*)
  have |*cos y* − *cos x*| ≤ |*sin ξ*| ∗ |*y* − *x*|
    using *ξ* **by** (*simp add*: *abs-mult*)
  also have . . . ≤ *sin 1* ∗ |*y* − *x*|
    using *absxi* **by** (*simp add*: *mult-right-mono*)
  finally show *?thesis*
    by (*simp add*: *abs-minus-commute*)

```
qed

lemma cos-contraction:
  fixes x y :: real
  assumes |x| ≤ 1 |y| ≤ 1
  shows |cos x − cos y| ≤ sin 1 ∗ |x − y|
  using cos-contraction-lt[of x y] cos-contraction-lt[of y x] assms
  by (cases x y rule: linorder-cases) (auto simp: abs-minus-commute)

lemma cos-step-to-dottie:
  assumes |w| ≤ 1
  shows |cos w − dottie| ≤ sin 1 ∗ |w − dottie|
  using Dottie.facts by (metis abs-cos-le-one assms cos-contraction)
```

After one step the iteration lands in $[-1, 1]$ and stays there.

```
lemma cos-funpow-in-pm1:
  fixes x0 :: real
  assumes n ≥ 1
  shows |(cos ^^ n) x0| ≤ 1
proof −
  have (cos ^^ n) x0 = cos ((cos ^^ (n−1)) x0)
    using assms funpow.simps
    by (metis One-nat-def Suc-diff-le diff-Suc-Suc diff-zero o-apply)
  then show ?thesis by simp
qed
```

From a start in $[-1, 1]$, the distance to the fixed point decays geometrically.

```
lemma cos-funpow-bound:
  fixes y0 :: real
  assumes |y0| ≤ 1
  shows |(cos ^^ n) y0 − dottie| ≤ (sin 1)^n ∗ |y0 − dottie|
proof (induction n)
  case 0
  show ?case by simp
next
  case (Suc n)
  have |(cos ^^ n) y0| ≤ 1
    by (metis funpow-0 less-one not-le cos-funpow-in-pm1[of n y0] assms)
  then have |(cos ^^ Suc n) y0 − dottie| ≤ sin 1 ∗ |(cos ^^ n) y0 − dottie|
    using cos-step-to-dottie by simp
  also have ... ≤ sin 1 ∗ ((sin 1)^n ∗ |y0 − dottie|)
    using Suc.IH sin1-bounds by (simp add: mult-left-mono)
  also have ... = (sin 1)^(Suc n) ∗ |y0 − dottie|
    by (simp add: mult.assoc)
  finally show ?case .
qed

lemma cos-iter-tendsto-unit:
```

  fixes *y0* :: *real*
  assumes $|y0| \leq 1$
  shows $(\lambda n.\ (cos\ \hat{}\hat{}\ n)\ y0) \longrightarrow dottie$
proof –
  have *pow0*: $(\lambda n.\ (sin\ 1)\ \hat{}n) \longrightarrow (0::real)$
    using *sin1-bounds* **by** (*intro LIMSEQ-realpow-zero*) *auto*
  have *null*: $(\lambda n.\ (sin\ 1)\ \hat{}n * |y0 - dottie|) \longrightarrow 0$
    using *tendsto-mult*[*OF pow0 tendsto-const*, *of* $|y0 - dottie|$] **by** *simp*
  have $(\lambda n.\ |(cos\ \hat{}\hat{}\ n)\ y0 - dottie|) \longrightarrow 0$
    using *tendsto-sandwich*[*OF - - tendsto-const null*] *cos-funpow-bound*[*OF assms*]
by *auto*
  then show *?thesis*
    using *Lim-null tendsto-rabs-zero-iff* **by** *blast*
qed

theorem *cos-iter-tendsto-dottie*:
  fixes *x0* :: *real*
  shows $(\lambda n.\ (cos\ \hat{}\hat{}\ n)\ x0) \longrightarrow dottie$
proof –
  have $|cos\ x0| \leq 1$ **by** *simp*
  from *cos-iter-tendsto-unit*[*OF this*]
  have $(\lambda n.\ (cos\ \hat{}\hat{}\ n)\ (cos\ x0)) \longrightarrow dottie$ **.**
  moreover have $\bigwedge n.\ (cos\ \hat{}\hat{}\ n)\ (cos\ x0) = (cos\ \hat{}\hat{}\ Suc\ n)\ x0$
    by (*simp add*: *funpow-swap1*)
  ultimately have $(\lambda n.\ (cos\ \hat{}\hat{}\ Suc\ n)\ x0) \longrightarrow dottie$ **by** *simp*
  then show *?thesis*
    using *filterlim-sequentially-Suc* **by** *blast*
qed

1.6 Transcendence

By the Hermite–Lindemann–Weierstraß theorem, $\cos z$ is transcendental for every nonzero algebraic z. If the Dottie number were algebraic, then $\cos(dottie) = dottie$ would be both algebraic and transcendental.

theorem *dottie-transcendental*: ¬ *algebraic dottie*
proof
  assume *alg*: *algebraic dottie*
  then have ¬ *algebraic* (*cos* (*complex-of-real dottie*))
    using *Dottie.facts transcendental-cos* **by** *auto*
  moreover have *cos* (*complex-of-real dottie*) = *complex-of-real dottie*
    using *Dottie.facts* **by** (*simp add*: *cos-of-real*)
  ultimately show *False* **using** *alg* **by** *simp*
qed

We make key facts available outside the locale

lemmas *dottie-fp* = *Dottie.facts*(*3*)
lemmas *dottie-bounds* = *Dottie.lb Dottie.ub*
lemmas *dottie-attractor* = *Dottie.cos-iter-tendsto-dottie*

end