

Disintegration Theorem

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Abstract

We formalize mixture and disintegration of measures. This entry is a formalization of Chapter 14.D of the book by Baccelli et.al. [1]. The main result is the disintegration theorem: let (X, Σ_X) be a measurable space, (Y, Σ_Y) be a standard Borel space, ν be a σ -finite measure on $X \times Y$, and ν_X be the marginal measure on X defined by $\nu_X(A) = \nu(A \times Y)$. Assume that ν_X is σ -finite, then there exists a probability kernel κ from X to Y such that

$$\nu(A \times B) = \int_A \kappa_x(B) \nu_X(dx), \quad A \in \Sigma_X, B \in \Sigma_Y.$$

Such a probability kernel is unique ν_X -almost everywhere.

Contents

1	Lemmas	1
1.1	Lemmas	2
1.2	Equivalence of Measures	3
2	Disintegration Theorem	3
2.1	Definition 14.D.2. (Mixture and Disintegration)	3
2.2	Lemma 14.D.3.	5
2.3	Theorem 14.D.4. (Measure Mixture Theorem)	5
2.4	Marginal Measure	7
2.5	Lemma 14.D.6.	8
2.6	Theorem 14.D.10. (Measure Disintegration Theorem)	12
2.7	Lemma 14.D.12.	12
2.8	Lemma 14.D.13.	12
2.9	Theorem 14.D.14.	13

1 Lemmas

```
theory Lemmas-Disintegration
  imports Standard-Borel-Spaces.StandardBorel
begin
```

1.1 Lemmas

lemma *semiring-of-sets-binary-product-sets[simp]*:

semiring-of-sets (*space* $X \times$ *space* Y) $\{a \times b \mid a \in \text{sets } X \wedge b \in \text{sets } Y\}$
<proof>

lemma *sets-pair-restrict-space*:

sets (*restrict-space* X $A \otimes_M$ *restrict-space* Y B) = *sets* (*restrict-space* $(X \otimes_M$
 $Y)$ $(A \times B)$)
(**is** ?lhs = ?rhs)
<proof>

lemma *restrict-space-space[simp]*: *restrict-space* M (*space* M) = M

<proof>

lemma *atMostq-Int-stable*:

Int-stable $\{\{..r\} \mid r::\text{real. } r \in \mathbb{Q}\}$
<proof>

lemma *rborel-eq-atMostq*:

borel = *sigma UNIV* $\{\{..r\} \mid r::\text{real. } r \in \mathbb{Q}\}$
<proof>

corollary *rborel-eq-atMostq-sets*:

sets borel = *sigma-sets UNIV* $\{\{..r\} \mid r::\text{real. } r \in \mathbb{Q}\}$
<proof>

lemma *mono-absolutely-continuous*:

assumes *sets* $\mu = \text{sets } \nu \wedge A. A \in \text{sets } \mu \implies \mu A \leq \nu A$
shows *absolutely-continuous* $\nu \mu$
<proof>

lemma *ex-measure-countable-space*:

assumes *countable* (*space* X)
and *sets* $X = \text{Pow}$ (*space* X)
shows $\exists \mu. \text{sets } \mu = \text{sets } X \wedge (\forall x \in \text{space } X. \mu \{x\} = f x)$
<proof>

lemma *ex-prob-space-countable*:

assumes *space* $X \neq \{\}$ *countable* (*space* X)
and *sets* $X = \text{Pow}$ (*space* X)
shows $\exists \mu. \text{sets } \mu = \text{sets } X \wedge \text{prob-space } \mu$
<proof>

lemma *AE-I''*:

assumes $N \in \text{null-sets } M$
and $\bigwedge x. x \in \text{space } M \implies x \notin N \implies P x$
shows *AE* x *in* $M. P x$
<proof>

lemma *absolutely-continuous-trans*:
assumes *absolutely-continuous L M absolutely-continuous M N*
shows *absolutely-continuous L N*
<proof>

1.2 Equivalence of Measures

abbreviation *equivalence-measure* :: *'a measure* \Rightarrow *'a measure* \Rightarrow *bool* (**infix** \sim_M 60)
where *equivalence-measure M N* \equiv *absolutely-continuous M N* \wedge *absolutely-continuous N M*

lemma *equivalence-measure-refl*: $M \sim_M M$
<proof>

lemma *equivalence-measure-sym*:
assumes $M \sim_M N$
shows $N \sim_M M$
<proof>

lemma *equivalence-measure-trans*:
assumes $M \sim_M N$ $N \sim_M L$
shows $M \sim_M L$
<proof>

lemma *equivalence-measureI*:
assumes *absolutely-continuous M N absolutely-continuous N M*
shows $M \sim_M N$
<proof>

end

2 Disintegration Theorem

theory *Disintegration*
imports *S-Finite-Measure-Monad.Kernels*
Lemmas-Disintegration
begin

2.1 Definition 14.D.2. (Mixture and Disintegration)

context *measure-kernel*
begin

definition *mixture-of* :: [*'a* \times *'b*] *measure*, *'a measure*] \Rightarrow *bool* **where**
mixture-of $\nu \mu \longleftrightarrow$ *sets* $\nu =$ *sets* $(X \otimes_M Y) \wedge$ *sets* $\mu =$ *sets* $X \wedge (\forall C \in$ *sets* $(X \otimes_M Y). \nu C = (\int^+ x. \int^+ y. \text{indicator } C (x,y) \partial(\kappa x) \partial\mu))$

definition *disintegration* :: [*'a* \times *'b*] *measure*, *'a measure*] \Rightarrow *bool* **where**

disintegration $\nu \mu \longleftrightarrow \text{sets } \nu = \text{sets } (X \otimes_M Y) \wedge \text{sets } \mu = \text{sets } X \wedge (\forall A \in \text{sets } X. \forall B \in \text{sets } Y. \nu (A \times B) = (\int^{+x \in A. (\kappa x B) \partial \mu}))$

lemma *disintegrationI*:

assumes *sets* $\nu = \text{sets } (X \otimes_M Y)$ *sets* $\mu = \text{sets } X$
and $\bigwedge A B. A \in \text{sets } X \implies B \in \text{sets } Y \implies \nu (A \times B) = (\int^{+x \in A. (\kappa x B) \partial \mu})$
proof
shows *disintegration* $\nu \mu$
proof

lemma *mixture-of-disintegration*:

assumes *mixture-of* $\nu \mu$
shows *disintegration* $\nu \mu$
proof

lemma

shows *mixture-of-sets-eq*: *mixture-of* $\nu \mu \implies \text{sets } \nu = \text{sets } (X \otimes_M Y)$ *mixture-of* $\nu \mu \implies \text{sets } \mu = \text{sets } X$
and *mixture-of-space-eq*: *mixture-of* $\nu \mu \implies \text{space } \nu = \text{space } (X \otimes_M Y)$
mixture-of $\nu \mu \implies \text{space } \mu = \text{space } X$
and *disintegration-sets-eq*: *disintegration* $\nu \mu \implies \text{sets } \nu = \text{sets } (X \otimes_M Y)$
disintegration $\nu \mu \implies \text{sets } \mu = \text{sets } X$
and *disintegration-space-eq*: *disintegration* $\nu \mu \implies \text{space } \nu = \text{space } (X \otimes_M Y)$
disintegration $\nu \mu \implies \text{space } \mu = \text{space } X$
proof

lemma

shows *mixture-ofD*: *mixture-of* $\nu \mu \implies C \in \text{sets } (X \otimes_M Y) \implies \nu C = (\int^{+x. \int^{+y. \text{indicator } C (x,y) \partial (\kappa x) \partial \mu})}$
and *disintegrationD*: *disintegration* $\nu \mu \implies A \in \text{sets } X \implies B \in \text{sets } Y \implies \nu (A \times B) = (\int^{+x \in A. (\kappa x B) \partial \mu})$
proof

lemma *disintegration-restrict-space*:

assumes *disintegration* $\nu \mu$ $A \cap \text{space } X \in \text{sets } X$
shows *measure-kernel.disintegration* (*restrict-space* $X A$) $Y \kappa$ (*restrict-space* $\nu (A \times \text{space } Y)$) (*restrict-space* μA)
proof

end

context *subprob-kernel*

begin

lemma *countable-disintegration-AE-unique*:

assumes *countable* (*space* Y) **and** [*measurable-cong*]: *sets* $Y = \text{Pow } (\text{space } Y)$
and *subprob-kernel* $X Y \kappa'$ *sigma-finite-measure* μ
and *disintegration* $\nu \mu$ *measure-kernel.disintegration* $X Y \kappa' \nu \mu$
shows *AE* x *in* $\mu. \kappa x = \kappa' x$
proof

end

lemma(in *subprob-kernel*) *nu-mu-space Y-le*:
 assumes *disintegration* ν μ $A \in \text{sets } X$
 shows $\nu (A \times \text{space } Y) \leq \mu A$
<proof>

context *prob-kernel*
begin

lemma *countable-disintegration-AE-unique-prob*:
 assumes *countable* (*space Y*) **and** [*measurable-cong*]:*sets Y = Pow (space Y)*
 and *prob-kernel X Y* κ' *sigma-finite-measure* μ
 and *disintegration* ν μ *measure-kernel.disintegration X Y* κ' ν μ
 shows *AE x in* μ . $\kappa x = \kappa' x$
<proof>

end

2.2 Lemma 14.D.3.

lemma(in *prob-kernel*) *nu-mu-space Y*:
 assumes *disintegration* ν μ $A \in \text{sets } X$
 shows $\nu (A \times \text{space } Y) = \mu A$
<proof>

corollary(in *subprob-kernel*) *nu-finite*:
 assumes *disintegration* ν μ *finite-measure* μ
 shows *finite-measure* ν
<proof>

corollary(in *subprob-kernel*) *nu-subprob-space*:
 assumes *disintegration* ν μ *subprob-space* μ
 shows *subprob-space* ν
<proof>

corollary(in *prob-kernel*) *nu-prob-space*:
 assumes *disintegration* ν μ *prob-space* μ
 shows *prob-space* ν
<proof>

lemma(in *subprob-kernel*) *nu-sigma-finite*:
 assumes *disintegration* ν μ *sigma-finite-measure* μ
 shows *sigma-finite-measure* ν
<proof>

2.3 Theorem 14.D.4. (Measure Mixture Theorem)

lemma(in *measure-kernel*) *exist-nu*:

assumes sets $\mu = \text{sets } X$
shows $\exists \nu$. disintegration $\nu \ \mu$
 ⟨proof⟩

lemma(in subprob-kernel) exist-unique-nu-sigma-finite':
assumes sets $\mu = \text{sets } X$ sigma-finite-measure μ
shows $\exists ! \nu$. disintegration $\nu \ \mu$
 ⟨proof⟩

lemma(in subprob-kernel) exist-unique-nu-sigma-finite:
assumes sets $\mu = \text{sets } X$ sigma-finite-measure μ
shows $\exists ! \nu$. disintegration $\nu \ \mu \wedge$ sigma-finite-measure ν
 ⟨proof⟩

lemma(in subprob-kernel) exist-unique-nu-finite:
assumes sets $\mu = \text{sets } X$ finite-measure μ
shows $\exists ! \nu$. disintegration $\nu \ \mu \wedge$ finite-measure ν
 ⟨proof⟩

lemma(in subprob-kernel) exist-unique-nu-sub-prob-space:
assumes sets $\mu = \text{sets } X$ subprob-space μ
shows $\exists ! \nu$. disintegration $\nu \ \mu \wedge$ subprob-space ν
 ⟨proof⟩

lemma(in prob-kernel) exist-unique-nu-prob-space:
assumes sets $\mu = \text{sets } X$ prob-space μ
shows $\exists ! \nu$. disintegration $\nu \ \mu \wedge$ prob-space ν
 ⟨proof⟩

lemma(in subprob-kernel) nn-integral-fst-finite':
assumes $f \in \text{borel-measurable } (X \otimes_M Y)$ disintegration $\nu \ \mu$ finite-measure μ
shows $(\int^{+z}. f \ z \ \partial \nu) = (\int^{+x}. \int^{+y}. f \ (x,y) \ \partial(\kappa \ x) \ \partial \mu)$
 ⟨proof⟩

lemma(in prob-kernel) nn-integral-fst:
assumes $f \in \text{borel-measurable } (X \otimes_M Y)$ disintegration $\nu \ \mu$ sigma-finite-measure μ
shows $(\int^{+z}. f \ z \ \partial \nu) = (\int^{+x}. \int^{+y}. f \ (x,y) \ \partial(\kappa \ x) \ \partial \mu)$
 ⟨proof⟩

lemma(in prob-kernel) integrable-eq1:
fixes $f :: - \Rightarrow - :: \{\text{banach, second-countable-topology}\}$
assumes [measurable]: $f \in \text{borel-measurable } (X \otimes_M Y)$
and disintegration $\nu \ \mu$ sigma-finite-measure μ
shows $(\int^{+z}. \text{ennreal } (\text{norm } (f \ z)) \ \partial \nu) < \infty \iff (\int^{+x}. \int^{+y}. \text{ennreal } (\text{norm } (f \ (x,y)))) \ \partial(\kappa \ x) \ \partial \mu < \infty$
 ⟨proof⟩

lemma(in *prob-kernel*) *integrable-kernel-integrable*:
fixes $f :: - \Rightarrow - :: \{\text{banach, second-countable-topology}\}$
assumes *integrable* ν *f disintegration* ν μ *sigma-finite-measure* μ
shows *AE* x in μ . *integrable* (κx) $(\lambda y. f(x,y))$
<proof>

lemma(in *prob-kernel*) *integrable-lebesgue-integral-integrable'*:
fixes $f :: - \Rightarrow - :: \{\text{banach, second-countable-topology}\}$
assumes *integrable* ν *f disintegration* ν μ *sigma-finite-measure* μ
shows *integrable* μ $(\lambda x. \int y. f(x,y) \partial(\kappa x))$
<proof>

lemma(in *prob-kernel*) *integrable-lebesgue-integral-integrable*:
fixes $f :: - \Rightarrow - \Rightarrow - :: \{\text{banach, second-countable-topology}\}$
assumes *integrable* ν $(\lambda(x,y). f x y)$ *disintegration* ν μ *sigma-finite-measure* μ
shows *integrable* μ $(\lambda x. \int y. f x y \partial(\kappa x))$
<proof>

lemma(in *prob-kernel*) *integral-fst*:
fixes $f :: - \Rightarrow - :: \{\text{banach, second-countable-topology}\}$
assumes *integrable* ν *f disintegration* ν μ *sigma-finite-measure* μ
shows $(\int z. f z \partial\nu) = (\int x. \int y. f(x,y) \partial(\kappa x) \partial\mu)$
<proof>

2.4 Marginal Measure

definition *marginal-measure-on* :: [*'a measure, 'b measure, ('a × 'b) measure, 'b set*] \Rightarrow *'a measure* **where**
marginal-measure-on $X Y \nu B = \text{measure-of}(\text{space } X) (\text{sets } X) (\lambda A. \nu(A \times B))$

abbreviation *marginal-measure* :: [*'a measure, 'b measure, ('a × 'b) measure*] \Rightarrow *'a measure* **where**
marginal-measure $X Y \nu \equiv \text{marginal-measure-on } X Y \nu (\text{space } Y)$

lemma *space-marginal-measure*: *space* (*marginal-measure-on* $X Y \nu B$) = *space* X
and *sets-marginal-measure*: *sets* (*marginal-measure-on* $X Y \nu B$) = *sets* X
<proof>

lemma *emeasure-marginal-measure-on*:
assumes *sets* $\nu = \text{sets}(X \otimes_M Y)$ $B \in \text{sets } Y$ $A \in \text{sets } X$
shows *marginal-measure-on* $X Y \nu B A = \nu(A \times B)$
<proof>

lemma *emeasure-marginal-measure*:
assumes *sets* $\nu = \text{sets}(X \otimes_M Y)$ $A \in \text{sets } X$
shows *marginal-measure* $X Y \nu A = \nu(A \times \text{space } Y)$
<proof>

lemma *finite-measure-marginal-measure-on-finite*:

assumes *finite-measure* ν *sets* $\nu = \text{sets } (X \otimes_M Y) B \in \text{sets } Y$
shows *finite-measure (marginal-measure-on* $X Y \nu B$)
 ⟨*proof*⟩

lemma *finite-measure-marginal-measure-finite:*

assumes *finite-measure* ν *sets* $\nu = \text{sets } (X \otimes_M Y)$
shows *finite-measure (marginal-measure* $X Y \nu$)
 ⟨*proof*⟩

lemma *marginal-measure-restrict-space:*

assumes *sets* $\nu = \text{sets } (X \otimes_M Y) B \in \text{sets } Y$
shows *marginal-measure* X (*restrict-space* $Y B$) (*restrict-space* ν (*space* $X \times B$))
 = *marginal-measure-on* $X Y \nu B$
 ⟨*proof*⟩

lemma *restrict-space-marginal-measure-on:*

assumes *sets* $\nu = \text{sets } (X \otimes_M Y) B \in \text{sets } Y A \in \text{sets } X$
shows *restrict-space (marginal-measure-on* $X Y \nu B$) A = *marginal-measure-on*
 (*restrict-space* $X A$) Y (*restrict-space* ν ($A \times \text{space } Y$)) B
 ⟨*proof*⟩

lemma *restrict-space-marginal-measure:*

assumes *sets* $\nu = \text{sets } (X \otimes_M Y) A \in \text{sets } X$
shows *restrict-space (marginal-measure* $X Y \nu$) A = *marginal-measure (restrict-space*
 $X A$) Y (*restrict-space* ν ($A \times \text{space } Y$))
 ⟨*proof*⟩

lemma *marginal-measure-mono:*

assumes *sets* $\nu = \text{sets } (X \otimes_M Y) A \in \text{sets } Y B \in \text{sets } Y A \subseteq B$
shows *emeasure (marginal-measure-on* $X Y \nu A$) \leq *emeasure (marginal-measure-on*
 $X Y \nu B$)
 ⟨*proof*⟩

lemma *marginal-measure-absolutely-countinuous:*

assumes *sets* $\nu = \text{sets } (X \otimes_M Y) A \in \text{sets } Y B \in \text{sets } Y A \subseteq B$
shows *absolutely-continuous (marginal-measure-on* $X Y \nu B$) (*marginal-measure-on*
 $X Y \nu A$)
 ⟨*proof*⟩

lemma *marginal-measure-absolutely-continuous':*

assumes *sets* $\nu = \text{sets } (X \otimes_M Y) A \in \text{sets } Y$
shows *absolutely-continuous (marginal-measure* $X Y \nu$) (*marginal-measure-on* X
 $Y \nu A$)
 ⟨*proof*⟩

2.5 Lemma 14.D.6.

locale *sigma-finite-measure-on-pair* =

fixes $X :: 'a \text{ measure}$ **and** $Y :: 'b \text{ measure}$ **and** $\nu :: ('a \times 'b) \text{ measure}$

assumes *nu-sets*[*measurable-cong*]: *sets* $\nu = \text{sets } (X \otimes_M Y)$
and *sigma-finite*: *sigma-finite-measure* ν
begin

abbreviation $\nu x \equiv \text{marginal-measure } X Y \nu$

end

locale *projection-sigma-finite* =
fixes $X :: 'a \text{ measure}$ **and** $Y :: 'b \text{ measure}$ **and** $\nu :: ('a \times 'b) \text{ measure}$
assumes *nu-sets*[*measurable-cong*]: *sets* $\nu = \text{sets } (X \otimes_M Y)$
and *marginal-sigma-finite*: *sigma-finite-measure* (*marginal-measure* $X Y \nu$)
begin

sublocale νx : *sigma-finite-measure* *marginal-measure* $X Y \nu$
<proof>

lemma *nu-sigma-finite*: *sigma-finite-measure* ν
<proof>

sublocale *sigma-finite-measure-on-pair*
<proof>

definition $\kappa' :: 'a \Rightarrow 'b \text{ set} \Rightarrow \text{ennreal}$ **where**
 $\kappa' x B \equiv \text{RN-deriv } \nu x (\text{marginal-measure-on } X Y \nu B) x$

lemma *kernel-measurable*[*measurable*]:
 $(\lambda x. \text{RN-deriv } (\text{marginal-measure } X Y \nu) (\text{marginal-measure-on } X Y \nu B) x) \in$
borel-measurable νx
<proof>

corollary *κ'-measurable*[*measurable*]:
 $(\lambda x. \kappa' x B) \in \text{borel-measurable } X$
<proof>

lemma *kernel-RN-deriv*:
assumes $A \in \text{sets } X B \in \text{sets } Y$
shows $\nu (A \times B) = (\int^{+x \in A. \kappa' x B} \partial \nu x)$
<proof>

lemma *empty-Y-bot*:
assumes *space* $Y = \{\}$
shows $\nu = \perp$
<proof>

lemma *empty-Y-nux*:
assumes *space* $Y = \{\}$
shows $\nu x A = 0$

<proof>

lemma *kernel-empty0-AE:*

AE x in νx. κ' x {} = 0

<proof>

lemma *kernel-Y1-AE:*

AE x in νx. κ' x (space Y) = 1

<proof>

lemma *kernel-suminf-AE:*

assumes *disjoint-family F*

and $\bigwedge i. F i \in \text{sets } Y$

shows *AE x in νx. $(\sum i. \kappa' x (F i)) = \kappa' x (\bigcup (\text{range } F))$*

<proof>

lemma *kernel-finite-sum-AE:*

assumes *disjoint-family-on F S finite S*

and $\bigwedge i. i \in S \implies F i \in \text{sets } Y$

shows *AE x in νx. $(\sum_{i \in S} \kappa' x (F i)) = \kappa' x (\bigcup_{i \in S} F i)$*

<proof>

lemma *kernel-disjoint-sum-AE:*

assumes *B ∈ sets Y C ∈ sets Y*

and $B \cap C = \{\}$

shows *AE x in νx. $\kappa' x (B \cup C) = \kappa' x B + \kappa' x C$*

<proof>

lemma *kernel-mono-AE:*

assumes *B ∈ sets Y C ∈ sets Y*

and $B \subseteq C$

shows *AE x in νx. $\kappa' x B \leq \kappa' x C$*

<proof>

lemma *kernel-incseq-AE:*

assumes *range B ⊆ sets Y incseq B*

shows *AE x in νx. $\text{incseq } (\lambda n. \kappa' x (B n))$*

<proof>

lemma *kernel-decseq-AE:*

assumes *range B ⊆ sets Y decseq B*

shows *AE x in νx. $\text{decseq } (\lambda n. \kappa' x (B n))$*

<proof>

corollary *kernel-01-AE:*

assumes *B ∈ sets Y*

shows *AE x in νx. $0 \leq \kappa' x B \wedge \kappa' x B \leq 1$*

<proof>

lemma *kernel-get-0*: $0 \leq \kappa' x B$
<proof>

lemma *kernel-le1-AE*:
assumes $B \in \text{sets } Y$
shows $AE x \text{ in } \nu x. \kappa' x B \leq 1$
<proof>

corollary *kernel-n-infty*:
assumes $B \in \text{sets } Y$
shows $AE x \text{ in } \nu x. \kappa' x B \neq \top$
<proof>

corollary *kernel-le-infty*:
assumes $B \in \text{sets } Y$
shows $AE x \text{ in } \nu x. \kappa' x B < \top$
<proof>

lemma *kernel-SUP-incseq*:
assumes $\text{range } B \subseteq \text{sets } Y \text{ incseq } B$
shows $AE x \text{ in } \nu x. \kappa' x (\bigcup (\text{range } B)) = (\bigsqcup n. \kappa' x (B n))$
<proof>

lemma *kernel-lim-incseq*:
assumes $\text{range } B \subseteq \text{sets } Y \text{ incseq } B$
shows $AE x \text{ in } \nu x. (\lambda n. \kappa' x (B n)) \longrightarrow \kappa' x (\bigcup (\text{range } B))$
<proof>

lemma *kernel-INF-decseq*:
assumes $\text{range } B \subseteq \text{sets } Y \text{ decseq } B$
shows $AE x \text{ in } \nu x. \kappa' x (\bigcap (\text{range } B)) = (\bigcap n. \kappa' x (B n))$
<proof>

lemma *kernel-lim-decseq*:
assumes $\text{range } B \subseteq \text{sets } Y \text{ decseq } B$
shows $AE x \text{ in } \nu x. (\lambda n. \kappa' x (B n)) \longrightarrow \kappa' x (\bigcap (\text{range } B))$
<proof>

end

lemma *qlim-eq-lim-mono-at-bot*:
fixes $g :: \text{rat} \Rightarrow 'a :: \text{linorder-topology}$
assumes $\text{mono } f (g \longrightarrow a) \text{ at-bot } \wedge r :: \text{rat}. f (\text{real-of-rat } r) = g r$
shows $(f \longrightarrow a) \text{ at-bot}$
<proof>

lemma *qlim-eq-lim-mono-at-top*:
fixes $g :: \text{rat} \Rightarrow 'a :: \text{linorder-topology}$
assumes $\text{mono } f (g \longrightarrow a) \text{ at-top } \wedge r :: \text{rat}. f (\text{real-of-rat } r) = g r$

shows $(f \longrightarrow a)$ *at-top*
 ⟨*proof*⟩

2.6 Theorem 14.D.10. (Measure Disintegration Theorem)

locale *projection-sigma-finite-standard* = *projection-sigma-finite* + *standard-borel-ne*
 Y
begin

theorem *measure-disintegration*:

$\exists \kappa. \text{prob-kernel } X \ Y \ \kappa \wedge \text{measure-kernel.disintegration } X \ Y \ \kappa \ \nu \ \nu x \wedge$
 $(\forall \kappa''. \text{prob-kernel } X \ Y \ \kappa'' \longrightarrow \text{measure-kernel.disintegration } X \ Y \ \kappa'' \ \nu \ \nu x$
 $\longrightarrow (AE \ x \ \text{in } \nu x. \ \kappa \ x = \kappa'' \ x))$
 ⟨*proof*⟩

end

2.7 Lemma 14.D.12.

lemma *ex-finite-density-measure*:

fixes $A :: \text{nat} \Rightarrow -$
assumes $A: \text{range } A \subseteq \text{sets } M \cup (\text{range } A) = \text{space } M \wedge i. \text{emeasure } M \ (A \ i) \neq \infty$
disjoint-family A
defines $h \equiv (\lambda x. (\sum n. (1/2) \wedge (\text{Suc } n) * (1 / (1 + M \ (A \ n)))) * \text{indicator } (A \ n) \ x)$
shows $h \in \text{borel-measurable } M$
 $\bigwedge x. x \in \text{space } M \implies 0 < h \ x$
 $\bigwedge x. x \in \text{space } M \implies h \ x < 1$
finite-measure (*density* $M \ h$)
 ⟨*proof*⟩

lemma(*in sigma-finite-measure*) *finite-density-measure*:

obtains h **where** $h \in \text{borel-measurable } M$
 $\bigwedge x. x \in \text{space } M \implies 0 < h \ x$
 $\bigwedge x. x \in \text{space } M \implies h \ x < 1$
finite-measure (*density* $M \ h$)
 ⟨*proof*⟩

2.8 Lemma 14.D.13.

lemma (*in measure-kernel*)

assumes *disintegration* $\nu \ \mu$
defines $\nu x \equiv \text{marginal-measure } X \ Y \ \nu$
shows *disintegration-absolutely-continuous*: *absolutely-continuous* $\mu \ \nu x$
and *disintegration-density*: $\nu x = \text{density } \mu \ (\lambda x. \kappa \ x \ (\text{space } Y))$
and *disintegration-absolutely-continuous-iff*:
 $\text{absolutely-continuous } \nu x \ \mu \longleftrightarrow (AE \ x \ \text{in } \mu. \ \kappa \ x \ (\text{space } Y) > 0)$
 ⟨*proof*⟩

2.9 Theorem 14.D.14.

locale *sigma-finite-measure-on-pair-standard* = *sigma-finite-measure-on-pair* + *standard-borel-ne Y*

sublocale *projection-sigma-finite-standard* \subseteq *sigma-finite-measure-on-pair-standard*
<proof>

context *sigma-finite-measure-on-pair-standard*
begin

lemma *measure-disintegration-extension*:

$\exists \mu \kappa$. *finite-measure* $\mu \wedge$ *measure-kernel* $X Y \kappa \wedge$ *measure-kernel.disintegration*
 $X Y \kappa \nu \mu \wedge$

$(\forall x \in \text{space } X. \text{sigma-finite-measure } (\kappa x)) \wedge$

$(\forall x \in \text{space } X. \kappa x (\text{space } Y) > 0) \wedge$

$\mu \sim_M \nu x$ (**is ?goal**)

<proof>

end

lemma(**in** *sigma-finite-measure-on-pair*) *measure-disintegration-extension-AE-unique*:

assumes *sigma-finite-measure* μ *sigma-finite-measure* μ'

measure-kernel $X Y \kappa$ *measure-kernel* $X Y \kappa'$

measure-kernel.disintegration $X Y \kappa \nu \mu$ *measure-kernel.disintegration* X

$Y \kappa' \nu \mu'$

and *absolutely-continuous* $\mu \mu' B \in \text{sets } Y$

shows *AE* x *in* $\mu. \kappa' x B * \text{RN-deriv } \mu \mu' x = \kappa x B$

<proof>

end

References

- [1] F. Baccelli, B. Blaszczyzyn, and M. Karray. *Random Measures, Point Processes, and Stochastic Geometry*. Inria, Jan. 2020.