Dijkstra's Algorithm

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Abstract

We implement and prove correct Dijkstra's algorithm for the single source shortest path problem, conceived in 1956 by E. Dijkstra. The algorithm is implemented using the data refinement framework for monadic, nondeterministic programs. An efficient implementation is derived using data structures from the Isabelle Collection Framework.

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1 Introduction and Overview

Dijkstra's algorithm [1] is an algorithm used to find shortest paths from one given vertex to all other vertices in a non-negatively weighted graph.

The implementation of the algorithm is meant to be an application of our extensions to the Isabelle Collections Framework (ICF) [4, 6, 7]. Moreover, it serves as a test case for our data refinement framework [5]. We use ICF-Maps to efficiently represent the graph and result and the newly introduced unique priority queues for the work list.

For a documentation of the refinement framework see [5], that also contains a userguide and some simpler examples.

The development utilizes a stepwise refinement approach. Starting from an abstract algorithm that has a nice correctness proof, we stepwise refine the algorithm until we end up with an efficient implementation, for that we generate code using Isabelle/HOL's code generator[2, 3].

Structure of the Submission. The abstract version of the algorithm with the correctness proof, as well as the main refinement steps are contained in the theory `Dijkstra`. The refinement steps involving the ICF and code generation are contained in `Dijkstra-Impl`. The theory `Infty` contains an extension of numbers with an infinity element. The theory `Graph` contains a formalization of graphs, paths, and related concepts. The theories `GraphSpec,GraphGA,GraphByMap,HashGraphImpl` contain an ICF-style specification of graphs. The theory `Test` contains a small performance test on random graphs. It uses the ML-code generated by the code generator.

2 Miscellaneous Lemmas

theory *Dijkstra-Misc*
imports *Main*
begin
  inductive-set *least-map* **for** f S **where**
    $[\![x{\in}S; \forall x'{\in}S.\ f\ x \leq f\ x']\!] \Longrightarrow x \in$ *least-map* f S

  lemma *least-map-subset*: *least-map* f $S \subseteq S$
    by (*auto elim*: *least-map.cases*)

  lemmas *least-map-elemD* = *subsetD*[*OF least-map-subset*]

  lemma *least-map-leD*:
    assumes $x \in$ *least-map* f S
    assumes $y{\in}S$
    shows $f\ x \leq f\ y$
    using *assms*

by (*auto elim*: *least-map.cases*)

lemma *least-map-empty*[*simp*]: *least-map f* {} = {}
by (*auto elim*: *least-map.cases*)

lemma *least-map-singleton*[*simp*]: *least-map* (*f*::'*a*⇒'*b*::*order*) {*x*} = {*x*}
by (*auto elim*: *least-map.cases intro*!: *least-map.intros simp*: *refl*)

lemma *least-map-insert-min*:
fixes *f*::'*a*⇒'*b*::*order*
assumes ∀ *y*∈*S*. *f x* ≤ *f y*
shows *x* ∈ *least-map f* (*insert x S*)
using *assms* **by** (*auto intro*: *least-map.intros*)

lemma *least-map-insert-nmin*:
⟦ *x*∈*least-map f S*; *f x* ≤ *f a* ⟧ ⟹ *x*∈*least-map f* (*insert a S*)
by (*auto elim*: *least-map.cases intro*: *least-map.intros*)

context *semilattice-inf*
begin
lemmas [*simp*] = *inf-absorb1 inf-absorb2*

lemma *inf-absorb-less*[*simp*]:
a < *b* ⟹ *inf a b* = *a*
a < *b* ⟹ *inf b a* = *a*
apply (*metis le-iff-inf less-imp-le*)
by (*metis inf-commute le-iff-inf less-imp-le*)
end

end

3 Graphs

theory *Graph*
imports *Main*
begin

This theory defines a notion of graphs. A graph is a record that contains a set of nodes V and a set of labeled edges $E \subseteq V \times W \times V$, where W are the edge labels.

3.1 Definitions

A graph is represented by a record.

record $('v,'w)$ *graph* =
  nodes :: $'v$ *set*
  edges :: $('v \times 'w \times 'v)$ *set*

In a valid graph, edges only go from nodes to nodes.

locale *valid-graph* =
  fixes G :: $('v,'w)$ *graph*
  assumes *E-valid*: *fst*‘*edges* $G \subseteq$ *nodes* G
    snd‘*snd*‘*edges* $G \subseteq$ *nodes* G
begin
  abbreviation $V \equiv$ *nodes* G
  abbreviation $E \equiv$ *edges* G

  lemma *E-validD*: **assumes** $(v,e,v') \in E$
    shows $v \in V$ $v' \in V$
    apply −
    apply (*rule subsetD*[*OF E-valid*(*1*)])
    using *assms* **apply** *force*
    apply (*rule subsetD*[*OF E-valid*(*2*)])
    using *assms* **apply** *force*
    done

end

3.2 Basic operations on Graphs

The empty graph.

definition *empty* **where**
  empty ≡ ⦇ *nodes* = {}, *edges* = {} ⦈

Adds a node to a graph.

definition *add-node* **where**
  add-node v g ≡ ⦇ *nodes* = *insert v* (*nodes g*), *edges*=*edges g*⦈

Deletes a node from a graph. Also deletes all adjacent edges.

definition *delete-node* **where** *delete-node v g* ≡ ⦇
  nodes = *nodes g* − {*v*},
  edges = *edges g* ∩ (−{*v*})×*UNIV*×(−{*v*})
  ⦈

Adds an edge to a graph.

definition *add-edge* **where** *add-edge v e v' g* ≡ ⦇
  nodes = {*v*,*v'*} ∪ *nodes g*,
  edges = *insert* (*v*,*e*,*v'*) (*edges g*)

⦈

Deletes an edge from a graph.

definition *delete-edge* **where** *delete-edge v e v′ g* ≡ ⦇
nodes = *nodes g*, *edges* = *edges g* − {(*v*,*e*,*v′*)} ⦈

Successors of a node.

definition *succ* :: (′*v*,′*w*) *graph* ⇒ ′*v* ⇒ (′*w*×′*v*) *set*
where *succ G v* ≡ {(*w*,*v′*). (*v*,*w*,*v′*)∈*edges G*}

Now follow some simplification lemmas.

lemma *empty-valid*[*simp*]: *valid-graph empty*
unfolding *empty-def* **by** *unfold-locales auto*
lemma *add-node-valid*[*simp*]: **assumes** *valid-graph g*
shows *valid-graph* (*add-node v g*)
proof −
interpret *valid-graph g* **by** *fact*
show *?thesis*
unfolding *add-node-def*
by *unfold-locales* (*auto dest*: *E-validD*)
qed
lemma *delete-node-valid*[*simp*]: **assumes** *valid-graph g*
shows *valid-graph* (*delete-node v g*)
proof −
interpret *valid-graph g* **by** *fact*
show *?thesis*
unfolding *delete-node-def*
by *unfold-locales* (*auto dest*: *E-validD*)
qed
lemma *add-edge-valid*[*simp*]: **assumes** *valid-graph g*
shows *valid-graph* (*add-edge v e v′ g*)
proof −
interpret *valid-graph g* **by** *fact*
show *?thesis*
unfolding *add-edge-def*
by *unfold-locales* (*auto dest*: *E-validD*)
qed
lemma *delete-edge-valid*[*simp*]: **assumes** *valid-graph g*
shows *valid-graph* (*delete-edge v e v′ g*)
proof −
interpret *valid-graph g* **by** *fact*
show *?thesis*
unfolding *delete-edge-def*
by *unfold-locales* (*auto dest*: *E-validD*)
qed

lemma *succ-finite*[*simp*, *intro*]: *finite* (*edges G*) ⟹ *finite* (*succ G v*)
unfolding *succ-def*
by (*rule finite-subset*[**where** *B*=*snd*‘*edges G*]) *force*+

lemma *nodes-empty*[*simp*]: *nodes empty* = {} **unfolding** *empty-def* **by** *simp*
lemma *edges-empty*[*simp*]: *edges empty* = {} **unfolding** *empty-def* **by** *simp*
lemma *succ-empty*[*simp*]: *succ empty v* = {} **unfolding** *empty-def succ-def* **by** *auto*

lemma *nodes-add-node*[*simp*]: *nodes* (*add-node v g*) = *insert v* (*nodes g*)
by (*simp add*: *add-node-def*)
lemma *nodes-add-edge*[*simp*]:
nodes (*add-edge v e v′ g*) = *insert v* (*insert v′* (*nodes g*))
by (*simp add*: *add-edge-def*)
lemma *edges-add-edge*[*simp*]:
edges (*add-edge v e v′ g*) = *insert* (*v*,*e*,*v′*) (*edges g*)
by (*simp add*: *add-edge-def*)
lemma *edges-add-node*[*simp*]:
edges (*add-node v g*) = *edges g*
by (*simp add*: *add-node-def*)

lemma (**in** *valid-graph*) *succ-subset*: *succ G v* ⊆ *UNIV*×*V*
unfolding *succ-def* **using** *E-valid*
by (*force*)

3.3 Paths

A path is represented by a list of adjacent edges.

type-synonym (*′v*,*′w*) *path* = (*′v*×*′w*×*′v*) *list*

context *valid-graph*
begin

The following predicate describes a valid path:

fun *is-path* :: *′v* ⇒ (*′v*,*′w*) *path* ⇒ *′v* ⇒ *bool* **where**
is-path v [] *v′* ⟷ *v*=*v′* ∧ *v′*∈*V* |
is-path v ((*v1*,*w*,*v2*)#*p*) *v′* ⟷ *v*=*v1* ∧ (*v1*,*w*,*v2*)∈*E* ∧ *is-path v2 p v′*

lemma *is-path-simps*[*simp*, *intro*!]:
is-path v [] *v* ⟷ *v*∈*V*
is-path v [(*v*,*w*,*v′*)] *v′* ⟷ (*v*,*w*,*v′*)∈*E*
by (*auto dest*: *E-validD*)

lemma *is-path-memb*[*simp*]:
is-path v p v′ ⟹ *v*∈*V* ∧ *v′*∈*V*
apply (*induct p arbitrary*: *v*)
apply (*auto dest*: *E-validD*)
done

lemma *is-path-split*:
is-path v (*p1*@*p2*) *v′* ⟷ (∃ *u*. *is-path v p1 u* ∧ *is-path u p2 v′*)
by (*induct p1 arbitrary*: *v*) *auto*

```
  lemma is-path-split'[simp]:
    is-path v (p1@(u,w,u')#p2) v'
      ⟷ is-path v p1 u ∧ (u,w,u')∈E ∧ is-path u' p2 v'
    by (auto simp add: is-path-split)
end
```

Set of intermediate vertices of a path. These are all vertices but the last one. Note that, if the last vertex also occurs earlier on the path, it is contained in *int-vertices*.

```
  definition int-vertices :: ('v,'w) path ⇒ 'v set where
    int-vertices p ≡ set (map fst p)

  lemma int-vertices-simps[simp]:
    int-vertices [] = {}
    int-vertices (vv#p) = insert (fst vv) (int-vertices p)
    int-vertices (p1@p2) = int-vertices p1 ∪ int-vertices p2
    by (auto simp add: int-vertices-def)

  lemma (in valid-graph) int-vertices-subset:
    is-path v p v' ⟹ int-vertices p ⊆ V
    apply (induct p arbitrary: v)
    apply (simp)
    apply (force dest: E-validD)
    done

  lemma int-vertices-empty[simp]: int-vertices p = {} ⟷ p=[]
    by (cases p) auto
```

3.3.1 Splitting Paths

Split a path at the point where it first leaves the set W:

```
  lemma (in valid-graph) path-split-set:
    assumes is-path v p v' and v∈W and v'∉W
    obtains p1 p2 u w u' where
    p=p1@(u,w,u')#p2 and
    int-vertices p1 ⊆ W and u∈W and u'∉W
    using assms
  proof (induct p arbitrary: v thesis)
    case Nil thus ?case by auto
  next
    case (Cons vv p)
    note [simp, intro!] = ‹v∈W› ‹v'∉W›
    from Cons.prems obtain w u' where
      [simp]: vv=(v,w,u') and
        REST: is-path u' p v'
      by (cases vv) auto
```

Distinguish wether the second node u' of the path is in W. If yes, the proposition

follows by the induction hypothesis, otherwise it is straightforward, as the split takes place at the first edge of the path.

```
  {
    assume A [simp, intro!]: u'∈W
    from Cons.hyps[OF - REST] obtain p1 uu ww uu' p2 where
      p=p1@(uu,ww,uu')#p2 int-vertices p1 ⊆ W uu ∈ W uu' ∉ W
      by blast
    with Cons.prems(1)[of vv#p1 uu ww uu' p2] have thesis by auto
  } moreover {
    assume u'∉W
    with Cons.prems(1)[of [] v w u' p] have thesis by auto
  } ultimately show thesis by blast
qed
```

Split a path at the point where it first enters the set W:

```
lemma (in valid-graph) path-split-set':
  assumes is-path v p v' and v'∈W
  obtains p1 p2 u where
  p=p1@p2 and
  is-path v p1 u and
  is-path u p2 v' and
  int-vertices p1 ⊆ -W and u∈W
  using assms
proof (cases v∈W)
  case True with that[of [] p] assms show ?thesis
    by auto
next
  case False with assms that show ?thesis
  proof (induct p arbitrary: v thesis)
    case Nil thus ?case by auto
  next
    case (Cons vv p)
    note [simp, intro!] = ‹v'∈W› ‹v∉W›
    from Cons.prems obtain w u' where
      [simp]: vv=(v,w,u') and [simp]: (v,w,u')∈E and
        REST: is-path u' p v'
      by (cases vv) auto
```

Distinguish wether the second node u' of the path is in W. If yes, the proposition is straightforward, otherwise, it follows by the induction hypothesis.

```
    {
      assume A [simp, intro!]: u'∈W
      from Cons.prems(3)[of [vv] p u'] REST have ?case by auto
    } moreover {
      assume [simp, intro!]: u'∉W
      from Cons.hyps[OF REST] obtain p1 p2 u'' where
        [simp]: p=p1@p2 and
          is-path u' p1 u'' and
          is-path u'' p2 v' and
```

```
      int-vertices p1 ⊆ − W and
      u''∈W by blast
    with Cons.prems(3)[of vv#p1] have ?case by auto
  } ultimately show ?case by blast
 qed
qed
```

Split a path at the point where a given vertex is first visited:

```
lemma (in valid-graph) path-split-vertex:
 assumes is-path v p v' and u∈int-vertices p
 obtains p1 p2 where
 p=p1@p2 and
 is-path v p1 u and
 u ∉ int-vertices p1
 using assms
proof (induct p arbitrary: v thesis)
 case Nil thus ?case by auto
next
 case (Cons vv p)
 from Cons.prems obtain w u' where
  [simp]: vv=(v,w,u') v∈V (v,w,u')∈E and
   REST: is-path u' p v'
  by (cases vv) auto

 {
  assume u=v
  with Cons.prems(1)[of [] vv#p] have thesis by auto
 } moreover {
  assume [simp]: u≠v
  with Cons.hyps(1)[OF - REST] Cons.prems(3) obtain p1 p2 where
   p=p1@p2 is-path u' p1 u u∉int-vertices p1
   by auto
  with Cons.prems(1)[of vv#p1 p2] have thesis
   by auto
 } ultimately show ?case by blast
qed
```

3.4 Weighted Graphs

```
locale valid-mgraph = valid-graph G for G::('v,'w::monoid-add) graph

definition path-weight :: ('v,'w::monoid-add) path ⇒ 'w
 where path-weight p ≡ sum-list (map (fst ∘ snd) p)



lemma path-weight-split[simp]:
 (path-weight (p1@p2)::'w::monoid-add) = path-weight p1 + path-weight p2
 unfolding path-weight-def
```

by (*auto*)

lemma *path-weight-empty*[*simp*]: *path-weight* [] = *0*
unfolding *path-weight-def*
by *auto*

lemma *path-weight-cons*[*simp*]:
(*path-weight* (*e*#*p*)::'*w*::*monoid-add*) = *fst* (*snd e*) + *path-weight p*
unfolding *path-weight-def*
by (*auto*)

end

4 Weights for Dijkstra's Algorithm

theory *Weight*
imports *Complex-Main*
begin

In this theory, we set up a type class for weights, and a typeclass for weights with an infinity element. The latter one is used internally in Dijkstra's algorithm.

Moreover, we provide a datatype that adds an infinity element to a given base type.

4.1 Type Classes Setup

class *weight* = *ordered-ab-semigroup-add* + *comm-monoid-add* + *linorder*
begin

lemma *add-nonneg-nonneg* [*simp*]:
assumes *0* ≤ *a* **and** *0* ≤ *b* **shows** *0* ≤ *a* + *b*
proof −
have *0* + *0* ≤ *a* + *b*
using *assms* **by** (*rule add-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-nonpos-nonpos*[*simp*]:
assumes *a* ≤ *0* **and** *b* ≤ *0* **shows** *a* + *b* ≤ *0*
proof −
have *a* + *b* ≤ *0* + *0*
using *assms* **by** (*rule add-mono*)
then show *?thesis* **by** *simp*
qed

lemma *add-nonneg-eq-0-iff*:
assumes *x*: *0* ≤ *x* **and** *y*: *0* ≤ *y*

shows $x + y = 0 \longleftrightarrow x = 0 \wedge y = 0$
by (*metis local.add-0-left local.add-0-right local.add-left-mono local.antisym-conv x y*)

lemma *add-incr*: $0 \leq b \Longrightarrow a \leq a+b$
by (*metis add.comm-neutral add-left-mono*)

lemma *add-incr-left*[*simp*, *intro*!]: $0 \leq b \Longrightarrow a \leq b + a$
by (*metis add-incr add.commute*)

lemma *sum-not-less*[*simp*, *intro*!]:
$0 \leq b \Longrightarrow \neg (a+b < a)$
$0 \leq a \Longrightarrow \neg (a+b < b)$
apply (*metis add-incr less-le-not-le*)
apply (*metis add-incr-left less-le-not-le*)
done

end

instance *nat* :: *weight* **..**
instance *int* :: *weight* **..**
instance *rat* :: *weight* **..**
instance *real* :: *weight* **..**

term *top*

class *top-weight* = *order-top* + *weight* +
assumes *inf-add-right*[*simp*]: $a + top = top$
begin

lemma *inf-add-left*[*simp*]: $top + a = top$
by (*metis add.commute inf-add-right*)

lemmas [*simp*] = *top-unique less-top*[*symmetric*]

lemma *not-less-inf*[*simp*]:
$\neg (a < top) \longleftrightarrow a{=}top$
by *simp*

end

4.2 Adding Infinity

We provide a standard way to add an infinity element to any type.

datatype *'a infty* = *Infty* | *Num 'a*

primrec *val* **where** *val* (*Num d*) = *d*

lemma *num-val-iff*[*simp*]: *e≠Infty* ⟹ *Num* (*val e*) = *e* **by** (*cases e*) *auto*

type-synonym *NatB* = *nat infty*

instantiation *infty* :: (*weight*) *top-weight*
begin
definition (*0*::*'a infty*) == *Num 0*
definition *top* ≡ *Infty*

fun *less-eq-infty* **where**
less-eq Infty (*Num* -) ⟷ *False* |
less-eq - *Infty* ⟷ *True* |
less-eq (*Num a*) (*Num b*) ⟷ *a*≤*b*

lemma [*simp*]: *Infty*≤*a* ⟷ *a*=*Infty*
by (*cases a*) *auto*

fun *less-infty* **where**
less Infty - ⟷ *False* |
less (*Num* -) *Infty* ⟷ *True* |
less (*Num a*) (*Num b*) ⟷ *a*<*b*

lemma [*simp*]: *less a Infty* ⟷ *a* ≠ *Infty*
by (*cases a*) *auto*

fun *plus-infty* **where**
plus - *Infty* = *Infty* |
plus Infty - = *Infty* |
plus (*Num a*) (*Num b*) = *Num* (*a*+*b*)

lemma [*simp*]: *plus Infty a* = *Infty* **by** (*cases a*) *simp-all*

instance
apply (*intro-classes*)
apply (*case-tac* [!] *x*) [*4*]
apply *simp-all*
apply (*case-tac* [!] *y*) [*3*]
apply (*simp-all add*: *less-le-not-le*)
apply (*case-tac z*)
apply (*simp-all add*: *top-infty-def zero-infty-def*)
apply (*case-tac* [!] *a*) [*4*]
apply *simp-all*
apply (*case-tac* [!] *b*) [*3*]
apply (*simp-all add*: *ac-simps*)
apply (*case-tac* [!] *c*) [*2*]
apply (*simp-all add*: *ac-simps add-right-mono*)
apply (*case-tac* (*x*,*y*) *rule*: *less-eq-infty.cases*)
apply (*simp-all add*: *linear*)

done
end

4.2.1 Unboxing

Conversion between the constants defined by the typeclass, and the concrete functions on the *'a infty* type.

lemma *infty-inf-unbox*:
Num a ≠ *top*
top ≠ *Num a*
Infty = *top*
by (*auto simp add*: *top-infty-def*)

lemma *infty-ord-unbox*:
Num a ≤ *Num b* ⟷ *a* ≤ *b*
Num a < *Num b* ⟷ *a* < *b*
by *auto*

lemma *infty-plus-unbox*:
Num a + *Num b* = *Num* (*a*+*b*)
by (*auto*)

lemma *infty-zero-unbox*:
Num a = *0* ⟷ *a* = *0*
Num 0 = *0*
by (*auto simp*: *zero-infty-def*)

lemmas *infty-unbox* =
infty-inf-unbox infty-zero-unbox infty-ord-unbox infty-plus-unbox

lemma *inf-not-zero*[*simp*]:
top≠(*0*::- *infty*) (*0*::- *infty*)≠*top*
apply (*unfold zero-infty-def top-infty-def*)
apply *auto*
done

lemma *num-val-iff'*[*simp*]: *e*≠*top* ⟹ *Num* (*val e*) = *e*
by (*cases e*) (*auto simp add*: *infty-unbox*)

lemma *infty-neE*:
⟦*a*≠*Infty*; ⋀*d*. *a*=*Num d* ⟹ *P*⟧ ⟹ *P*
⟦*a*≠*top*; ⋀*d*. *a*=*Num d* ⟹ *P*⟧ ⟹ *P*
by (*case-tac* [!] *a*) (*auto simp add*: *infty-unbox*)

end

5 Dijkstra's Algorithm

theory *Dijkstra*
imports
Graph
Dijkstra-Misc
Collections.Refine-Dflt-ICF
Weight
begin

This theory defines Dijkstra's algorithm. First, a correct result of Dijkstra's algorithm w.r.t. a graph and a start vertex is specified. Then, the refinement framework is used to specify Dijkstra's Algorithm, prove it correct, and finally refine it to datatypes that are closer to an implementation than the original specification.

5.1 Graph's for Dijkstra's Algorithm

A graph annotated with weights.

locale *weighted-graph* = *valid-graph G*
for $G :: ('V,'W::weight)$ *graph*

5.2 Specification of Correct Result

context *weighted-graph*
begin

A result of Dijkstra's algorithm is correct, if it is a map from nodes v to the shortest path from the start node *v0* to v. Iff there is no such path, the node is not in the map.

definition *is-shortest-path-map* :: $'V \Rightarrow ('V \rightharpoonup ('V,'W)\ path) \Rightarrow bool$
where
is-shortest-path-map v0 res $\equiv \forall\, v{\in}V.$ (*case res v of*
None $\Rightarrow \neg(\exists\, p.$ *is-path v0 p v*) |
Some p $\Rightarrow$ *is-path v0 p v*
$\wedge\ (\forall\, p'.$ *is-path v0 p' v* $\longrightarrow$ *path-weight p* $\leq$ *path-weight p'*)
)
end

The following function returns the weight of an optional path, where *None* is interpreted as infinity.

fun *path-weight'* **where**
path-weight' None = *top* |
path-weight' (Some p) = *Num* (*path-weight p*)

5.3 Dijkstra's Algorithm

The state in the main loop of the algorithm consists of a workset *wl* of vertexes that still need to be explored, and a map *res* that contains the current shortest path for each vertex.

type-synonym $('V,'W)$ *state* $= ('V$ *set*$) \times ('V \rightharpoonup ('V,'W)$ *path*$)$

The preconditions of Dijkstra's algorithm, i.e., that it operates on a valid and finite graph, and that the start node is a node of the graph, are summarized in a locale.

locale *Dijkstra* = *weighted-graph G*
  for *G* :: $('V,'W::$*weight*$)$ *graph*+
  fixes *v0* :: $'V$
  assumes *finite*[*simp*,*intro*!]: *finite V finite E*
  assumes *v0-in-V*[*simp*, *intro*!]: *v0*$\in$*V*
  assumes *nonneg-weights*[*simp*, *intro*]: (*v*,*w*,*v'*)$\in$*edges G* $\Longrightarrow$ *0*$\leq$*w*
begin

Paths have non-negative weights.

lemma *path-nonneg-weight*: *is-path v p v'* $\Longrightarrow$ *0* $\leq$ *path-weight p*
  by (*induct rule*: *is-path.induct*) *auto*

Invariant of the main loop:

- The workset only contains nodes of the graph.
- If the result set contains a path for a node, it is actually a path, and uses only intermediate vertices outside the workset.
- For all vertices outside the workset, the result map contains the shortest path.
- For all vertices in the workset, the result map contains the shortest path among all paths that only use intermediate vertices outside the workset.

definition *dinvar* $\sigma \equiv$ *let* (*wl*,*res*)=σ *in*
  wl $\subseteq$ *V* $\wedge$
  $(\forall\, v{\in}V.\ \forall\, p.$ *res v* = *Some p* $\longrightarrow$ *is-path v0 p v* $\wedge$ *int-vertices p* $\subseteq$ *V*$-$*wl*$)$ $\wedge$
  $(\forall\, v{\in}V{-}wl.\ \forall\, p.$ *is-path v0 p v*
    $\longrightarrow$ *path-weight'* (*res v*) $\leq$ *path-weight'* (*Some p*)$)$ $\wedge$
  $(\forall\, v{\in}wl.\ \forall\, p.$ *is-path v0 p v* $\wedge$ *int-vertices p* $\subseteq$ *V*$-$*wl*
    $\longrightarrow$ *path-weight'* (*res v*) $\leq$ *path-weight'* (*Some p*)
)

Sanity check: The invariant is strong enough to imply correctness of result.

lemma *invar-imp-correct*: *dinvar* ({},*res*) $\Longrightarrow$ *is-shortest-path-map v0 res*

unfolding *dinvar-def is-shortest-path-map-def*
by (*auto simp*: *infty-unbox split*: *option.split*)

The initial workset contains all vertices. The initial result maps *v0* to the empty path, and all other vertices to *None*.

definition *dinit* :: ('*V*,'*W*) *state nres* **where**
dinit ≡ *SPEC* (λ(*wl*,*res*) .
wl=*V* ∧ *res v0* = *Some* [] ∧ (∀ *v*∈*V*−{*v0*}. *res v* = *None*))

The initial state satisfies the invariant.

lemma *dinit-invar*: *dinit* ≤ *SPEC dinvar*
unfolding *dinit-def*
apply (*intro refine-vcg*)
apply (*force simp*: *dinvar-def split*: *option.split*)
done

In each iteration, the main loop of the algorithm pops a minimal node from the workset, and then updates the result map accordingly.

Pop a minimal node from the workset. The node is minimal in the sense that the length of the current path for that node is minimal.

definition *pop-min* :: ('*V*,'*W*) *state* ⇒ ('*V* × ('*V*,'*W*) *state*) *nres* **where**
pop-min σ ≡ *do* {
let (*wl*,*res*)=σ;
ASSERT (*wl*≠{});
v ← *RES* (*least-map* (*path-weight'* ∘ *res*) *wl*);
RETURN (*v*,(*wl*−{*v*},*res*))
}

Updating the result according to a node *v* is done by checking, for each successor node, whether the path over *v* is shorter than the path currently stored into the result map.

inductive *update-spec* :: '*V* ⇒ ('*V*,'*W*) *state* ⇒ ('*V*,'*W*) *state* ⇒ *bool*
where
⟦ ∀ *v'*∈*V*.
res' *v'* ∈ *least-map path-weight'* (
{ *res v'* } ∪ { *Some* (*p*@[(*v*,*w*,*v'*)]) | *p w*. *res v* = *Some p* ∧ (*v*,*w*,*v'*)∈*E* }
)
⟧ ⟹ *update-spec v* (*wl*,*res*) (*wl*,*res'*)

In order to ease the refinement proof, we will assert the following precondition for updating.

definition *update-pre* :: '*V* ⇒ ('*V*,'*W*) *state* ⇒ *bool* **where**
update-pre v σ ≡ *let* (*wl*,*res*)=σ *in v*∈*V*
∧ (∀ *v'*∈*V*−*wl*. *v'*≠*v* ⟶ (∀ *p*. *is-path v0 p v'*
⟶ *path-weight'* (*res v'*) ≤ *path-weight'* (*Some p*)))
∧ (∀ *v'*∈*V*. ∀ *p*. *res v'* = *Some p* ⟶ *is-path v0 p v'*)

```
definition update :: 'V ⇒ ('V,'W) state ⇒ ('V,'W) state nres where
  update v σ ≡ do {ASSERT (update-pre v σ); SPEC (update-spec v σ)}
```

Finally, we define Dijkstra's algorithm:

```
definition dijkstra where
  dijkstra ≡ do {
    σ0←dinit;
    (-,res) ← WHILE_T^dinvar (λ(wl,-). wl≠{})
        (λσ.
          do { (v,σ') ← pop-min σ; update v σ' }
        )
        σ0;
    RETURN res }
```

The following theorem states (total) correctness of Dijkstra's algorithm.

```
theorem dijkstra-correct: dijkstra ≤ SPEC (is-shortest-path-map v0)
  unfolding dijkstra-def
  unfolding dinit-def
  unfolding pop-min-def update-def [abs-def]
  thm refine-vcg

  apply (refine-rcg
    WHILEIT-rule[where R=inv-image {(x,y). x<y} (card ∘ fst)]
    refine-vcg
  )


  supply [[simproc del: defined-all]]
  apply (simp-all split: prod.split-asm)
  apply (tactic ‹
   ALLGOALS ((REPEAT-DETERM o Hypsubst.bound-hyp-subst-tac @{context})
    THEN' asm-full-simp-tac @{context}
    )›)

  proof −
    fix wl res v
    assume INV: dinvar (wl,res)
    and LM: v∈least-map (path-weight' ∘ res) wl
    hence v∈V unfolding dinvar-def by (auto dest: least-map-elemD)
    moreover
    from INV have ∀ v'∈V − (wl−{v}). v' ≠ v ⟶
      (∀ p. is-path v0 p v' ⟶ path-weight' (res v') ≤ Num (path-weight p))
      by (auto simp: dinvar-def)
    moreover from INV have ∀ v'∈V. ∀ p. res v'=Some p ⟶ is-path v0 p v'
      by (auto simp: dinvar-def)
    ultimately show update-pre v (wl−{v},res) by (auto simp: update-pre-def)
  next
```

fix *res*
assume *dinvar* ({}, *res*)
thus *is-shortest-path-map v0 res*
by (*rule invar-imp-correct*)
next
show *wf* (*inv-image* {(*x*, *y*). *x* < *y*} (*card* ∘ *fst*))
by (*blast intro*: *wf-less*)
next
fix *wl res v σ''*
assume
LM: *v*∈*least-map* (*path-weight'* ∘ *res*) *wl* **and**
UD: *update-spec v* (*wl*−{*v*},*res*) *σ''* **and**
INV: *dinvar* (*wl*,*res*)

from *LM* **have** *v*∈*wl* **by** (*auto dest*: *least-map-elemD*)
moreover from *UD* **have** *fst σ'' = wl*−{*v*} **by** (*auto elim*: *update-spec.cases*)
moreover from *INV* **have** *finite wl*
unfolding *dinvar-def* **by** (*auto dest*: *finite-subset*)
ultimately show *card* (*fst σ''*) < *card wl*
apply *simp*
by (*metis card-gt-0-iff diff-Suc-less empty-iff*)
next
fix *a* **and** *res* :: *'V* ⇀ (*'V*,*'W*) *path*
assume *a* = *V* ∧ *res v0* = *Some* [] ∧ (∀ *v*∈*V*−{*v0*}. *res v* = *None*)
thus *dinvar* (*V*,*res*)
by (*force simp*: *dinvar-def split*: *option.split*)
next
fix *wl res*
assume *INV*: *dinvar* (*wl*,*res*)
hence
WL-SUBSET: *wl* ⊆ *V* **and**
PATH-VALID: ∀ *v*∈*V*. ∀ *p*. *res v* = *Some p*
⟶ *is-path v0 p v* ∧ *int-vertices p* ⊆ *V* − *wl* **and**
NWL-MIN: ∀ *v*∈*V* − *wl*. ∀ *p*. *is-path v0 p v*
⟶ *path-weight'* (*res v*) ≤ *Num* (*path-weight p*) **and**
WL-MIN: ∀ *v*∈*wl*. ∀ *p*. *is-path v0 p v* ∧ *int-vertices p* ⊆ *V* − *wl*
⟶ *path-weight'* (*res v*) ≤ *Num* (*path-weight p*)
unfolding *dinvar-def* **by** *auto*

fix *v σ''*
assume *V-LEAST*: *v*∈*least-map* (*path-weight' o res*) *wl*
and *update-spec v* (*wl*−{*v*},*res*) *σ''*
then obtain *res'* **where**
[*simp*]: *σ''*=(*wl*−{*v*},*res'*)
and *CONSIDERED-NEW-PATHS*: ∀ *v'*∈*V*. *res' v'* ∈ *least-map path-weight'*
(*insert* (*res v'*)
({ *Some* (*p*@[(*v*,*w*,*v'*)]) | *p w*. *res v* = *Some p* ∧ (*v*,*w*,*v'*)∈*E* }))
by (*auto elim*!: *update-spec.cases*)

from *V-LEAST* **have** *V-MEM*: $v \in wl$ **by** (*blast intro*: *least-map-elemD*)

show *dinvar* σ''
apply (*unfold dinvar-def*, *simp*)
apply (*intro conjI*)
proof −
from *WL-SUBSET* **show** $wl - \{v\} \subseteq V$ **by** *auto*

show $\forall va \in V.\ \forall p.\ res'\ va = Some\ p$
$\longrightarrow$ *is-path v0 p va* $\wedge$ *int-vertices p* $\subseteq V - (wl - \{v\})$
proof (*intro ballI conjI impI allI*)
fix v' *p*
assume *V'-MEM*: $v' \in V$ **and** [*simp*]: $res'\ v' = Some\ p$

The new paths that we have added are valid and only use intermediate vertices outside the workset.

This proof works as follows: A path $res'\ v'$ is either the old path, or has been assembled as a path over node v. In the former case the proposition follows straightforwardly from the invariant for the old state. In the latter case we get, by the invariant for the old state, that the path over node v is valid. Then, we observe that appending an edge to a valid path yields a valid path again. Also, adding v as intermediate node is legal, as we just removed v from the workset.

with *CONSIDERED-NEW-PATHS* **have** $res'\ v' \in (insert\ (res\ v')$
$(\{\ Some\ (p@[(v,w,v')])\ |\ p\ w.\ res\ v = Some\ p \wedge (v,w,v') \in E\ \}))$
by (*rule-tac least-map-elemD*) *blast*
moreover {
assume [*symmetric,simp*]: $res'\ v' = res\ v'$
from *V'-MEM PATH-VALID* **have**
is-path v0 p v'
int-vertices p $\subseteq V - (wl - \{v\})$
by *force+*
} moreover {
fix *pv w*
assume $res'\ v' = Some\ (pv@[(v,w,v')])$
and [*simp*]: $res\ v = Some\ pv$
and *EDGE*: $(v,w,v') \in E$
hence [*simp*]: $p = pv@[(v,w,v')]$ **by** *simp*

from *bspec*[*OF PATH-VALID rev-subsetD*[*OF V-MEM WL-SUBSET*]]
have
PATHV: *is-path v0 pv v* **and** *IVV*: *int-vertices pv* $\subseteq V - wl$ **by** *auto*
hence
is-path v0 p v'
int-vertices p $\subseteq V - (wl - \{v\})$
by (*auto simp*: *EDGE V'-MEM*)
}
ultimately show
is-path v0 p v'
int-vertices p $\subseteq V - (wl - \{v\})$

```
    by blast+
qed
```

We show that already the *original* result stores the minimal path for all vertices not in the *new* workset. For vertices also not in the original workset, this follows straightforwardly from the invariant.

For the vertex v, that has been removed from the workset, we split a path p' to v at the point u where it first enters the original workset.

As we chose v to be the vertex in the workset with the minimal weight, its weight is less than the current weight of u. As the vertices of the prefix of p' up to u are not in the workset, the current weight of u is less than the weight of the prefix of p', and thus less than the weight of p'. Together, the current weight of v is less than the weight of p'.

```
have RES-MIN: ∀ v∈V − (wl − {v}). ∀ p. is-path v0 p v
  ⟶ path-weight' (res v) ≤ Num (path-weight p)
proof (intro ballI allI impI)
  fix v' p'
  assume NOT-IN-WL: v' ∈ V − (wl − {v})
    and PATH: is-path v0 p' v'
  hence [simp, intro!]: v'∈V by auto

  show path-weight' (res v') ≤ Num (path-weight p')
  proof (cases v' = v)
    assume NE[simp]: v'≠v
    from bspec[OF NWL-MIN, of v'] NOT-IN-WL PATH show
      path-weight' (res v') ≤ Num (path-weight p') by auto
  next
    assume EQ[simp]: v'=v

    from path-split-set'[OF PATH, of wl] V-MEM obtain p1 p2 u where
      [simp]: p'=p1@p2
        and P1: is-path v0 p1 u
        and P2: is-path u p2 v'
        and P1V: int-vertices p1 ⊆ −wl
        and [simp]: u∈wl
      by auto

    from least-map-leD[OF V-LEAST]
    have path-weight' (res v') ≤ path-weight' (res u)by auto
    also from bspec[OF WL-MIN, of u] P1 P1V int-vertices-subset[OF P1]
    have path-weight' (res u) ≤ Num (path-weight p1) by auto
    also have ... ≤ Num (path-weight p')
      using path-nonneg-weight[OF P2]
      apply (auto simp: infty-unbox )
      by (metis add-0-right add-left-mono)
    finally show ?thesis .
  qed
qed
```

With the previous statement, we easily show the third part of the invariant, as the new paths are not longer than the old ones.

show $\forall\, v{\in}V - (wl - \{v\}).\ \forall\, p.$ *is-path v0 p v*
$\longrightarrow$ *path-weight′* (*res′ v*) $\leq$ *Num* (*path-weight p*)
proof (*intro allI ballI impI*)
fix v' *p*
assume *NOT-IN-WL*: $v' \in V - (wl - \{v\})$
and *PATH*: *is-path v0 p* v'
hence [*simp*, *intro*!]: $v'{\in}V$ **by** *auto*
from *bspec*[*OF CONSIDERED-NEW-PATHS*, *of* v']
have *path-weight′* (*res′* v') $\leq$ *path-weight′* (*res* v')
by (*auto dest*: *least-map-leD*)
also from *bspec*[*OF RES-MIN NOT-IN-WL*] *PATH*
have *path-weight′* (*res* v') $\leq$ *Num* (*path-weight p*) **by** *blast*
finally show *path-weight′* (*res′* v') $\leq$ *Num* (*path-weight p*) **.**
qed

Finally, we have to show that for nodes on the worklist, the stored paths are not longer than any path using only nodes not on the worklist. Compared to the situation before the step, those path may also use the node v.

show $\forall\, va{\in}wl - \{v\}.\ \forall\, p.$
is-path v0 p va $\wedge$ *int-vertices p* $\subseteq V - (wl - \{v\})$
$\longrightarrow$ *path-weight′* (*res′ va*) $\leq$ *Num* (*path-weight p*)
proof (*intro allI impI ballI*, *elim conjE*)
fix v' *p*
assume *IWS*: $v'{\in}wl - \{v\}$
and *PATH*: *is-path v0 p* v'
and *VERTICES*: *int-vertices p* $\subseteq V - (wl - \{v\})$
from *IWS WL-SUBSET* **have** [*simp*, *intro*!]: $v'{\in}V$ **by** *auto*

{

If the path is empty, the proposition follows easily from the invariant for the original states, as no intermediate nodes are used at all.

assume [*simp*]: *p*=[]
from *bspec*[*OF CONSIDERED-NEW-PATHS*, *of* v'] **have**
path-weight′ (*res′* v') $\leq$ *path-weight′* (*res* v')
using *IWS WL-SUBSET* **by** (*auto dest*: *least-map-leD*)
also have *int-vertices p* $\subseteq V{-}wl$ **by** *auto*
with *WL-MIN IWS PATH*
have *path-weight′* (*res* v') $\leq$ *Num* (*path-weight p*)
by (*auto simp del*: *path-weight-empty*)
finally have *path-weight′* (*res′* v') $\leq$ *Num* (*path-weight p*) **.**
} moreover {
fix *p1 u w*
assume [*simp*]: *p* = *p1*@[(*u*,*w*,v')]

If the path is not empty, we pick the last but one vertex, and call it u.

from *PATH* **have** *PATH1*: *is-path v0 p1 u* **and** *EDGE*: $(u,w,v')\in E$ **by** *auto*

from *VERTICES* **have** *NIV*: $u\in V - (wl-\{v\})$ **by** *simp*

hence *U-MEM*[*simp*]: $u\in V$ **by** *auto*

From *RES-MIN*, we know that *res u* holds the shortest path to u. Thus p is longer than the path that is constructed by replacing the prefix of p by term "res u"

from *NIV RES-MIN PATH1*
have *G*: *Num* (*path-weight p1*) $\geq$ *path-weight'* (*res u*) **by** *simp*
then obtain *pu* **where** [*simp*]: *res u* = *Some pu*
by (*cases res u*) (*auto simp*: *infty-unbox*)
from *G* **have** *Num* (*path-weight p*) $\geq$ *path-weight'* (*res u*) + *Num w*
by (*auto simp*: *infty-unbox add-right-mono*)
also
have *path-weight'* (*res u*) + *Num w* $\geq$ *path-weight'* (*res' v'*)

The remaining argument depends on wether u equals v. In the case $u \neq v$, all vertices of *res u* are outside the original workset. Thus, appending the edge (u, w, v') to *res u* yields a path to v over intermediate nodes only outside the workset. By the invariant for the original state, *res* v' is shorter than this path. As a step does not replace paths by longer ones, also *res'* v' is shorter.

In the case $u = v$, the step has considered the path to v' over v, and thus the result path is not longer.

proof (*cases u=v*)
assume $u\neq v$
with *NIV* **have** *NIV'*: $u\in V-wl$ **by** *auto*
from *bspec*[*OF PATH-VALID U-MEM*] *NIV'*
have *is-path v0 pu u* **and** *VU*: *int-vertices* (*pu*@[(u,w,v')]) $\subseteq V-wl$
by *auto*
with *EDGE* **have** *PV'*: *is-path v0* (*pu*@[(u,w,v')]) v' **by** *auto*
with *bspec*[*OF WL-MIN*, *of v'*] *IWS VU* **have**
path-weight' (*res v'*) $\leq$ *Num* (*path-weight* (*pu*@[(u,w,v')]))
by *blast*
hence *path-weight'* (*res u*) + *Num w* $\geq$ *path-weight'* (*res v'*)
by (*auto simp*: *infty-unbox*)
also from *CONSIDERED-NEW-PATHS* **have**
path-weight' (*res v'*) $\geq$ *path-weight'* (*res' v'*)
by (*auto dest*: *least-map-leD*)
finally (*order-trans*[*rotated*]) **show** *?thesis* **.**
next
assume [*symmetric*,*simp*]: $u=v$
from *CONSIDERED-NEW-PATHS EDGE* **have**
path-weight' (*res' v'*) $\leq$ *path-weight'* (*Some* (*pu*@[(v,w,v')]))
by (*rule-tac least-map-leD*) *auto*
thus *?thesis* **by** (*auto simp*: *infty-unbox*)
qed
finally (*order-trans*[*rotated*]) **have**
path-weight' (*res' v'*) $\leq$ *Num* (*path-weight p*) **.**
} ultimately show *path-weight'* (*res' v'*) $\leq$ *Num* (*path-weight p*)

```
      using PATH apply (cases p rule: rev-cases) by auto
    qed
  qed
qed
```

5.4 Structural Refinement of Update

Now that we have proved correct the initial version of the algorithm, we start refinement towards an efficient implementation.

First, the update function is refined to iterate over each successor of the selected node, and update the result on demand.

```
definition uinvar
  :: 'V ⇒ 'V set ⇒ - ⇒ ('W×'V) set ⇒ ('V,'W) state ⇒ bool where
  uinvar v wl res it σ ≡ let (wl',res')=σ in wl'=wl
  ∧ (∀ v'∈V.
    res' v' ∈ least-map path-weight' (
      { res v' } ∪ { Some (p@[(v,w,v')]) | p w. res v = Some p
        ∧ (w,v') ∈ succ G v − it }
    ))
  ∧ (∀ v'∈V. ∀ p. res' v' = Some p ⟶ is-path v0 p v')
  ∧ res' v = res v


definition update' :: 'V ⇒ ('V,'W) state ⇒ ('V,'W) state nres where
  update' v σ ≡ do {
    ASSERT (update-pre v σ);
    let (wl,res) = σ;
    let wv = path-weight' (res v);
    let pv = res v;
    FOREACH^uinvar v wl res (succ G v) (λ(w',v') (wl,res).
      if (wv + Num w' < path-weight' (res v')) then do {
          ASSERT (v'∈wl ∧ pv≠None);
          RETURN (wl,res(v' ↦ the pv@[(v,w',v')]))
      } else RETURN (wl,res)
    ) (wl,res)}

lemma update'-refines:
  assumes v'=v and σ'=σ
  shows update' v' σ' ≤ ⇓Id (update v σ)
  apply (simp only: assms)
  unfolding update'-def update-def
  apply (refine-rcg refine-vcg)


  apply (simp-all only: singleton-iff)
proof −
  fix wl res
  assume update-pre v (wl,res)
```

```
  thus uinvar v wl res (succ G v) (wl,res)
    by (simp add: uinvar-def update-pre-def)
next

  fix wl res it wl' res' v' w'
  assume PRE: update-pre v (wl,res)
  assume INV: uinvar v wl res it (wl',res')
  assume MEM: (w',v')∈it
  assume IT-SS: it⊆ succ G v
  assume LESS: path-weight' (res v) + Num w' < path-weight' (res' v')

  from PRE have [simp, intro!]: v∈V by (simp add: update-pre-def)

  from MEM IT-SS have [simp,intro!]: v'∈V using succ-subset
    by auto

  from LESS obtain pv where [simp]: res v = Some pv
    by (cases res v) auto

  thus res v ≠ None by simp

  have [simp]: wl'=wl and [simp]: res' v = res v
    using INV unfolding uinvar-def by auto

  from MEM IT-SS have EDGE[simp]: (v,w',v')∈E
    unfolding succ-def by auto
  with INV have [simp]: is-path v0 pv v
    unfolding uinvar-def by auto

  have 0≤w' by (rule nonneg-weights[OF EDGE])
  hence [simp]: v'≠v using LESS
    by auto
  hence [simp]: v≠v' by blast

  show [simp]: v'∈wl' proof (rule ccontr)
    assume [simp]: v'∉wl'
    hence [simp]: v'∈V−wl and [simp]: v'∉wl by auto
    note LESS
    also
    from INV have path-weight' (res' v') ≤ path-weight' (res v')
      unfolding uinvar-def by (auto dest: least-map-leD)
    also
    from PRE have PW: ⋀p. is-path v0 p v' ⟹
      path-weight' (res v') ≤ path-weight' (Some p)
      unfolding update-pre-def
      by auto
    have P: is-path v0 (pv@[(v,w',v')]) v' by simp
    from PW[OF P] have
      path-weight' (res v') ≤ Num (path-weight (pv@[(v,w',v')]))
```

```
    by auto
  finally show False by (simp add: infty-unbox)
qed

show uinvar v wl res (it−{(w′,v′)}) (wl′,res′(v′↦the (res v)@[(v,w′,v′)]))
proof −
  have (res′(v′↦the (res v)@[(v,w′,v′)])) v = res′ v by simp
  moreover {
    fix v′′ assume VMEM: v′′∈V
    have (res′(v′↦the (res v)@[(v,w′,v′)])) v′′ ∈ least-map path-weight′ (
      { res v′′ } ∪ { Some (p@[(v,w,v′′)]) | p w. res v = Some p
      ∧ (w,v′′) ∈ succ G v − (it − {(w′,v′)}) }
      ) ∧ (∀ p. (res′(v′↦the (res v)@[(v,w′,v′)])) v′′ = Some p
            ⟶ is-path v0 p v′′)
    proof (cases v′′=v′)
      case [simp]: False
      have { Some (p@[(v,w,v′′)]) | p w. res v = Some p
      ∧ (w,v′′) ∈ succ G v − (it − {(w′,v′)}) } =
        { Some (p@[(v,w,v′′)]) | p w. res v = Some p
      ∧ (w,v′′) ∈ succ G v − it }
        by auto
      with INV VMEM show ?thesis unfolding uinvar-def
        by simp
    next
      case [simp]: True
      have EQ: { res v′′ } ∪ { Some (p@[(v,w,v′′)]) | p w. res v = Some p
      ∧ (w,v′′) ∈ succ G v − (it − {(w′,v′)}) } =
      insert (Some (pv@[(v,w′,v′)])) (
        { res v′′ } ∪ { Some (p@[(v,w,v′′)]) | p w. res v = Some p
      ∧ (w,v′′) ∈ succ G v − it })
        using MEM IT-SS
        by auto
      show ?thesis
        apply (subst EQ)
        apply simp
        apply (rule least-map-insert-min)
        apply (rule ballI)
      proof −
        fix r′
        assume A:
          r′ ∈ insert (res v′)
           {Some (pv @ [(v, w, v′)]) |w. (w, v′) ∈ succ G v ∧ (w, v′) ∉ it}

        from LESS have
          path-weight′ (Some (pv @ [(v, w′, v′)])) < path-weight′ (res′ v′)
          by (auto simp: infty-unbox)
        also from INV[unfolded uinvar-def] have
          res′ v′ ∈ least-map path-weight′ (
            insert (res v′)
```

```
        {Some (pv @ [(v, w, v')]) |w. (w, v') ∈ succ G v ∧ (w, v') ∉ it}
      )
      by auto
    with A have path-weight' (res' v') ≤ path-weight' r'
      by (auto dest: least-map-leD)
    finally show
      path-weight' (Some (pv @ [(v, w', v')])) ≤ path-weight' r'
      by simp
  qed
 qed
}
ultimately show ?thesis
 unfolding uinvar-def Let-def
 by auto
qed
next
fix wl res it w' v' wl' res'
assume INV: uinvar v wl res it (wl',res')
and NLESS: ¬ path-weight' (res v) + Num w' < path-weight' (res' v')
and IN-IT: (w',v')∈it
and IT-SS: it ⊆ succ G v

from IN-IT IT-SS have [simp, intro!]: (w',v')∈succ G v by auto
hence [simp,intro!]: v'∈V using succ-subset
 by auto

show uinvar v wl res (it − {(w',v')}) (wl',res')
proof (cases res v)
 case [simp]: None
 from INV show ?thesis
  unfolding uinvar-def by auto
next
 case [simp]: (Some p)
 {
  fix v''
  assume [simp, intro!]: v''∈V
  have res' v'' ∈ least-map path-weight' (
   { res v'' } ∪ { Some (p@[(v,w,v'')]) | p w. res v = Some p
   ∧ (w,v'') ∈ succ G v − (it − {(w',v')}) }
   ) (is - ∈ least-map path-weight' ?S)
  proof (cases v''=v')
   case False with INV show ?thesis
    unfolding uinvar-def by auto
  next
   case [simp]: True

   have EQ: ?S = insert (Some (p@[(v,w',v')])) (
    { res v' } ∪ { Some (p@[(v,w,v'')]) | p w. res v = Some p
                ∧ (w,v'') ∈ succ G v − it }
```

```
        )
        by auto
      from NLESS have
        path-weight' (res' v') ≤ path-weight' (Some (p@[(v,w',v')]))
        by (auto simp: infty-unbox)
      thus ?thesis
        apply (subst EQ)
        apply (rule least-map-insert-nmin)
        using INV unfolding uinvar-def apply auto []
        apply simp
        done
    qed
  } with INV
  show ?thesis
    unfolding uinvar-def by auto
 qed
next
 fix wl res σ'

 assume uinvar v wl res {} σ'
 thus update-spec v (wl,res) σ'
  unfolding uinvar-def
  apply (cases σ')
  apply (auto intro: update-spec.intros simp: succ-def)
  done
next
 show finite (succ G v) by simp
qed
```

We integrate the new update function into the main algorithm:

```
definition dijkstra' where
 dijkstra' ≡ do {
  σ0 ← dinit;
  (-,res) ← WHILE_T^dinvar (λ(wl,-). wl≠{})
       (λσ. do {(v,σ') ← pop-min σ; update' v σ'})
       σ0;
  RETURN res
 }


lemma dijkstra'-refines: dijkstra' ≤ ⇓Id dijkstra
proof −
 note [refine] = update'-refines
 have [refine]: ⋀σ σ'. σ=σ' ⟹ pop-min σ ≤ ⇓Id (pop-min σ') by simp
 show ?thesis
  unfolding dijkstra-def dijkstra'-def
  apply (refine-rcg)
  apply simp-all
  done
```

qed
end

5.5 Refinement to Cached Weights

Next, we refine the data types of the workset and the result map. The workset becomes a map from nodes to their current weights. The result map stores, in addition to the shortest path, also the weight of the shortest path. Moreover, we store the shortest paths in reversed order, which makes appending new edges more effcient.

These refinements allow to implement the workset as a priority queue, and save recomputation of the path weights in the inner loop of the algorithm.

type-synonym $('V,'W)$ *mwl* = $('V \rightharpoonup 'W\ infty)$
type-synonym $('V,'W)$ *mres* = $('V \rightharpoonup (('V,'W)\ path \times 'W))$
type-synonym $('V,'W)$ *mstate* = $('V,'W)\ mwl \times ('V,'W)\ mres$

Map a path with cached weight to one without cached weight.

fun *mpath'* :: $((('V,'W)\ path \times 'W)\ option \rightharpoonup ('V,'W)\ path$ **where**
  mpath' None = *None* |
  mpath' (Some (p,w)) = *Some p*

fun *mpath-weight'* :: $((('V,'W)\ path \times 'W)\ option \Rightarrow ('W::weight)\ infty$ **where**
  mpath-weight' None = *top* |
  mpath-weight' (Some (p,w)) = *Num w*

context *Dijkstra*
begin
  definition $\alpha w::('V,'W)\ mwl \Rightarrow 'V\ set$ **where** $\alpha w \equiv dom$
  definition $\alpha r::('V,'W)\ mres \Rightarrow 'V \rightharpoonup ('V,'W)\ path$ **where**
    $\alpha r \equiv \lambda res\ v.\ case\ res\ v\ of\ None \Rightarrow None \mid Some\ (p,w) \Rightarrow Some\ (rev\ p)$
  definition $\alpha s::('V,'W)\ mstate \Rightarrow ('V,'W)\ state$ **where**
    $\alpha s \equiv map\text{-}prod\ \alpha w\ \alpha r$

Additional invariants for the new state. They guarantee that the cached weights are consistent.

  definition *res-invarm* :: $('V \rightharpoonup (('V,'W)\ path \times 'W)) \Rightarrow bool$ **where**
    res-invarm res $\equiv (\forall v.$ *case res v of*
        None $\Rightarrow$ *True* |
        Some (p,w) $\Rightarrow$ *w* = *path-weight (rev p))*
  definition *dinvarm* :: $('V,'W)\ mstate \Rightarrow bool$ **where**
    dinvarm $\sigma \equiv$ *let (wl,res)* = σ *in*
      $(\forall v \in dom\ wl.$ *the (wl v)* = *mpath-weight' (res v))* $\wedge$ *res-invarm res*

  lemma *mpath-weight'-correct*: $[\![dinvarm\ (wl,res)]\!] \Longrightarrow$
    mpath-weight' (res v) = *path-weight'* $(\alpha r\ res\ v)$

    unfolding *dinvarm-def res-invarm-def* αr*-def*

```
  by (auto split: option.split option.split-asm)

lemma mpath'-correct: ⟦dinvarm (wl,res)⟧ ⟹
  mpath' (res v) = map-option rev (αr res v)
  unfolding dinvarm-def αr-def
  by (auto split: option.split option.split-asm)

lemma wl-weight-correct:
  assumes INV: dinvarm (wl,res)
  assumes WLV: wl v = Some w
  shows path-weight' (αr res v) = w
proof −
  from INV WLV have w = mpath-weight' (res v)
    unfolding dinvarm-def by force
  also from mpath-weight'-correct[OF INV] have
    ... = path-weight' (αr res v) .
  finally show ?thesis by simp
qed
```

The initial state is constructed using an iterator:

```
definition mdinit :: ('V,'W) mstate nres where
  mdinit ≡ do {
    wl ← FOREACH V (λv wl. RETURN (wl(v↦Infty))) Map.empty;
    RETURN (wl(v0↦Num 0),[v0 ↦ ([],0)])
  }

lemma mdinit-refines: mdinit ≤ ⇓(build-rel αs dinvarm) dinit
  unfolding mdinit-def dinit-def
  apply (rule build-rel-SPEC)
  apply (intro FOREACH-rule[where I=λit wl. (∀ v∈V−it. wl v = Some Infty)
∧
    dom wl = V−it]
      refine-vcg)
  apply (auto
    simp: αs-def αw-def αr-def dinvarm-def res-invarm-def infty-unbox
    split: if-split-asm
  )
  done
```

The new pop function:

```
definition
  mpop-min :: ('V,'W) mstate ⇒ ('V × 'W infty × ('V,'W) mstate) nres
  where
  mpop-min σ ≡ do {
    let (wl,res) = σ;
    (v,w,wl')←prio-pop-min wl;
    RETURN (v,w,(wl',res))
  }
```

lemma *mpop-min-refines*:
⟦ (σ,σ′) ∈ *build-rel* αs *dinvarm* ⟧ ⟹
mpop-min σ ≤
⇓(*build-rel*
(λ(v,w,σ). (v,αs σ))
(λ(v,w,σ). *dinvarm* σ ∧ w = *mpath-weight′* (*snd* σ v)))
(*pop-min* σ′)
— The two algorithms are structurally different, so we use the nofail/inres method to prove refinement.
unfolding *mpop-min-def pop-min-def prio-pop-min-def*

apply (*rule pw-ref-I*)
apply *rule*
apply (*auto simp add*: *refine-pw-simps* αs-def αw-def *refine-rel-defs*
split: *prod.split prod.split-asm*)

apply (*auto simp*: *dinvarm-def*) []

apply (*auto simp*: *mpath-weight′-correct wl-weight-correct*) []

apply (*auto*
simp: *wl-weight-correct*
intro!: *least-map.intros*
) []
apply (*metis restrict-map-eq*(*2*))

done

The new update function:

definition *uinvarm v wl res it* σ ≡
uinvar v wl res it (αs σ) ∧ *dinvarm* σ

definition *mupdate* :: ′*V* ⇒ ′*W infty* ⇒ (′*V*,′*W*) *mstate* ⇒ (′*V*,′*W*) *mstate nres*
where
mupdate v wv σ ≡ *do* {
ASSERT (*update-pre v* (αs σ) ∧ *wv*=*mpath-weight′* (*snd* σ v));
let (*wl*,*res*) = σ;
let pv = *mpath′* (*res v*);
FOREACH$^{uinvarm\ v\ (\alpha w\ wl)\ (\alpha r\ res)}$ (*succ G v*) (λ(w′,v′) (*wl*,*res*).
if (*wv* + *Num w*′ < *mpath-weight′* (*res v*′)) *then do* {
ASSERT (*v*′∈*dom wl* ∧ *pv* ≠ *None*);
ASSERT (*wv* ≠ *Infty*);
RETURN (*wl*(*v*′↦*wv* + *Num w*′),
res(*v*′ ↦ ((*v*,*w*′,*v*′)#*the pv*,*val wv* + *w*′)))
} *else RETURN* (*wl*,*res*)
) (*wl*,*res*)
}

lemma *mupdate-refines*:

assumes *SREF*: $(\sigma,\sigma')\in$*build-rel* αs *dinvarm*
assumes *WV*: wv = *mpath-weight'* (*snd* σ v)
assumes *VV'*: $v'=v$
shows *mupdate* v wv $\sigma \leq \Downarrow$(*build-rel* αs *dinvarm*) (*update'* v' σ')
proof (*simp only*: *VV'*)
{

Show that IF-condition is a refinement:

fix *wl res wl' res' it w' v'*
assume *uinvarm v* (αw *wl*) (αr *res*) *it* (*wl'*,*res'*)
and *dinvarm* (*wl*,*res*)
hence *mpath-weight'* (*res v*) + *Num w'* < *mpath-weight'* (*res' v'*) $\longleftrightarrow$
path-weight' (αr *res v*) + *Num w'* < *path-weight'* (αr *res' v'*)
unfolding *uinvarm-def*
by (*auto simp add*: *mpath-weight'-correct*)
} note *COND-refine=this*

{

THEN-case:

fix *wl res wl' res' it w' v'*
assume *UINV*: *uinvarm v* (αw *wl*) (αr *res*) *it* (*wl'*,*res'*)
and *DINV*: *dinvarm* (*wl*,*res*)
and *mpath-weight'* (*res v*) + *Num w'* < *mpath-weight'* (*res' v'*)
and *path-weight'* (αr *res v*) + *Num w'* < *path-weight'* (αr *res' v'*)
and *V'MEM*: $v'\in\alpha w$ *wl'*
and *NN*: αr *res v* $\neq$ *None*

from *NN* **obtain** *pv wv* **where**
ARV: αr *res v* = *Some* (*rev pv*) **and**
RV: *res v* = *Some* (*pv*,*wv*)
unfolding αr*-def* **by** (*auto split*: *option.split-asm*)

with *DINV* **have** [*simp*]: *wv* = *path-weight* (*rev pv*)
unfolding *dinvarm-def res-invarm-def* **by** (*auto split*: *option.split-asm*)

note [*simp*] = *ARV RV*

from *V'MEM NN* **have** $v'\in$*dom wl'* (**is** *?G1*)
and *mpath'* (*res v*) $\neq$ *None* (**is** *?G2*)
unfolding αw*-def* αr*-def* **by** (*auto split*: *option.split-asm*)

hence $\bigwedge x$. αw *wl'* = αw (*wl'*($v'\mapsto x$)) **by** (*auto simp*: αw*-def*)
moreover have *mpath'* (*res v*) = *map-option rev* (αr *res v*) **using** *DINV*
by (*simp add*: *mpath'-correct*)
ultimately have
αw *wl'* = αw (*wl'*(v' $\mapsto$ *mpath-weight'* (*res v*) + *Num w'*))
$\wedge$ (αr *res'*)(v' $\mapsto$ *the* (αr *res v*)@[(v, w', v')])
= αr (*res'*(v' $\mapsto$ ((v, w', v')#*the* (*mpath'* (*res v*)),

```
            val (mpath-weight' (res v)) + w'))) (is ?G3)
      by (auto simp add: αr-def intro!: ext)
    have
      (dinvarm (wl'(v'↦mpath-weight' (res v) + Num w'),
                    res'(v' ↦ ((v,w',v') # the (mpath' (res v)),
                                 val (mpath-weight' (res v)) + w'
                               )))) (is ?G4)
      using UINV unfolding uinvarm-def dinvarm-def res-invarm-def
      by (auto simp: infty-unbox split: option.split option.split-asm)
    note ‹?G1› ‹?G2› ‹?G3› ‹?G4›
  } note THEN-refine=this


  note [refine2] = inj-on-id

  note [simp] = refine-rel-defs

  show mupdate v wv σ ≤ ⇓(build-rel αs dinvarm) (update' v σ')
    using SREF WV
    unfolding mupdate-def update'-def
    apply −

    apply (refine-rcg)

    apply simp-all [3]
    apply (simp add: αs-def uinvarm-def)
    apply (simp-all add: αs-def COND-refine THEN-refine(1−2)) [3]
    apply (rule ccontr,simp)
    using THEN-refine(3,4)
    apply (auto simp: αs-def) []
```

The ELSE-case is trivial:

```
    apply simp
    done
qed
```

Finally, we assemble the refined algorithm:

```
definition mdijkstra where
  mdijkstra ≡ do {
    σ0 ← mdinit;
    (-,res) ← WHILE_T^dinvarm (λ(wl,-). dom wl≠{})
        (λσ. do { (v,wv,σ') ← mpop-min σ; mupdate v wv σ' } )
        σ0;
    RETURN res
  }

lemma mdijkstra-refines: mdijkstra ≤ ⇓(build-rel αr res-invarm) dijkstra'
proof −
  note [refine] = mdinit-refines mpop-min-refines mupdate-refines
```

```
    show ?thesis
      unfolding mdijkstra-def dijkstra'-def
      apply (refine-rcg)
      apply (simp-all split: prod.split
        add: αs-def αw-def dinvarm-def refine-rel-defs)
      done
  qed

end

end
```

6 Graph Interface

```
theory GraphSpec
imports Main Graph
  Collections.Collections


begin
```

This theory defines an ICF-style interface for graphs.

```
  type-synonym ('V,'W,'G) graph-α = 'G ⇒ ('V,'W) graph

  locale graph =
    fixes α :: 'G ⇒ ('V,'W) graph
    fixes invar :: 'G ⇒ bool
    assumes finite[simp, intro!]:
      invar g ⟹ finite (nodes (α g))
      invar g ⟹ finite (edges (α g))
    assumes valid: invar g ⟹ valid-graph (α g)

  type-synonym ('V,'W,'G) graph-empty = unit ⇒ 'G
  locale graph-empty = graph +
    constrains α :: 'G ⇒ ('V,'W) graph
    fixes empty :: unit ⇒ 'G
    assumes empty-correct:
      α (empty ()) = Graph.empty
      invar (empty ())

  type-synonym ('V,'W,'G) graph-add-node = 'V ⇒ 'G ⇒ 'G
  locale graph-add-node = graph +
    constrains α :: 'G ⇒ ('V,'W) graph
    fixes add-node :: 'V ⇒ 'G ⇒ 'G
    assumes add-node-correct:
      invar g ⟹ invar (add-node v g)
      invar g ⟹ α (add-node v g) = Graph.add-node v (α g)

  type-synonym ('V,'W,'G) graph-delete-node = 'V ⇒ 'G ⇒ 'G
```

locale *graph-delete-node* = *graph* +
  constrains α :: $'G \Rightarrow ('V,'W)$ *graph*
  fixes *delete-node* :: $'V \Rightarrow 'G \Rightarrow 'G$
  assumes *delete-node-correct*:
    invar g $\Longrightarrow$ *invar* (*delete-node v g*)
    invar g $\Longrightarrow$ α (*delete-node v g*) = *Graph.delete-node v* (α *g*)

type-synonym $('V,'W,'G)$ *graph-add-edge* = $'V \Rightarrow 'W \Rightarrow 'V \Rightarrow 'G \Rightarrow 'G$
locale *graph-add-edge* = *graph* +
  constrains α :: $'G \Rightarrow ('V,'W)$ *graph*
  fixes *add-edge* :: $'V \Rightarrow 'W \Rightarrow 'V \Rightarrow 'G \Rightarrow 'G$
  assumes *add-edge-correct*:
    invar g $\Longrightarrow$ *invar* (*add-edge v e v' g*)
    invar g $\Longrightarrow$ α (*add-edge v e v' g*) = *Graph.add-edge v e v'* (α *g*)

type-synonym $('V,'W,'G)$ *graph-delete-edge* = $'V \Rightarrow 'W \Rightarrow 'V \Rightarrow 'G \Rightarrow 'G$
locale *graph-delete-edge* = *graph* +
  constrains α :: $'G \Rightarrow ('V,'W)$ *graph*
  fixes *delete-edge* :: $'V \Rightarrow 'W \Rightarrow 'V \Rightarrow 'G \Rightarrow 'G$
  assumes *delete-edge-correct*:
    invar g $\Longrightarrow$ *invar* (*delete-edge v e v' g*)
    invar g $\Longrightarrow$ α (*delete-edge v e v' g*) = *Graph.delete-edge v e v'* (α *g*)

type-synonym $('V,'W,'\sigma,'G)$ *graph-nodes-it* = $'G \Rightarrow ('V,'\sigma)$ *set-iterator*

locale *graph-nodes-it-defs* =
  fixes *nodes-list-it* :: $'G \Rightarrow ('V,'V\ list)$ *set-iterator*
begin
  definition *nodes-it g* $\equiv$ *it-to-it* (*nodes-list-it g*)
end

locale *graph-nodes-it* = *graph* α *invar* + *graph-nodes-it-defs nodes-list-it*
  for α :: $'G \Rightarrow ('V,'W)$ *graph* **and** *invar* **and**
  nodes-list-it :: $'G \Rightarrow ('V,'V\ list)$ *set-iterator*
  +
  assumes *nodes-list-it-correct*:
    invar g $\Longrightarrow$ *set-iterator* (*nodes-list-it g*) (*Graph.nodes* (α *g*))
begin
  lemma *nodes-it-correct*:
    invar g $\Longrightarrow$ *set-iterator* (*nodes-it g*) (*Graph.nodes* (α *g*))
    unfolding *nodes-it-def*
    apply (*rule it-to-it-correct*)
    by (*rule nodes-list-it-correct*)

  lemma *pi-nodes-it*[*icf-proper-iteratorI*]:
    proper-it (*nodes-it S*) (*nodes-it S*)
    unfolding *nodes-it-def*
    by (*intro icf-proper-iteratorI*)

lemma *nodes-it-proper*[*proper-it*]:
proper-it' *nodes-it* *nodes-it*
apply (*rule* *proper-it'I*)
by (*rule* *pi-nodes-it*)

end

type-synonym ('*V*,'*W*,'*σ*,'*G*) *graph-edges-it*
= '*G* ⇒ (('*V*×'*W*×'*V*),'*σ*) *set-iterator*

locale *graph-edges-it-defs* =
fixes *edges-list-it* :: ('*V*,'*W*,('*V*×'*W*×'*V*) *list*,'*G*) *graph-edges-it*
begin
definition *edges-it g* ≡ *it-to-it* (*edges-list-it g*)
end

locale *graph-edges-it* = *graph α invar* + *graph-edges-it-defs edges-list-it*
for *α* :: '*G* ⇒ ('*V*,'*W*) *graph* **and** *invar* **and**
edges-list-it :: ('*V*,'*W*,('*V*×'*W*×'*V*) *list*,'*G*) *graph-edges-it*
+
assumes *edges-list-it-correct*:
invar g ⟹ *set-iterator* (*edges-list-it g*) (*Graph.edges* (*α g*))
begin
lemma *edges-it-correct*:
invar g ⟹ *set-iterator* (*edges-it g*) (*Graph.edges* (*α g*))
unfolding *edges-it-def*
apply (*rule* *it-to-it-correct*)
by (*rule* *edges-list-it-correct*)

lemma *pi-edges-it*[*icf-proper-iteratorI*]:
proper-it (*edges-it S*) (*edges-it S*)
unfolding *edges-it-def*
by (*intro* *icf-proper-iteratorI*)

lemma *edges-it-proper*[*proper-it*]:
proper-it' *edges-it* *edges-it*
apply (*rule* *proper-it'I*)
by (*rule* *pi-edges-it*)

end

type-synonym ('*V*,'*W*,'*σ*,'*G*) *graph-succ-it* =
'*G* ⇒ '*V* ⇒ ('*W*×'*V*,'*σ*) *set-iterator*

locale *graph-succ-it-defs* =
fixes *succ-list-it* :: '*G* ⇒ '*V* ⇒ ('*W*×'*V*,('*W*×'*V*) *list*) *set-iterator*
begin
definition *succ-it g v* ≡ *it-to-it* (*succ-list-it g v*)
end

locale *graph-succ-it* = *graph* α *invar* + *graph-succ-it-defs succ-list-it*
for α :: $'G \Rightarrow ('V,'W)$ *graph* **and** *invar* **and**
succ-list-it :: $'G \Rightarrow 'V \Rightarrow ('W\times'V,('W\times'V)$ *list*) *set-iterator* +
assumes *succ-list-it-correct*:
invar g $\Longrightarrow$ *set-iterator* (*succ-list-it g v*) (*Graph.succ* (α *g*) *v*)
begin
lemma *succ-it-correct*:
invar g $\Longrightarrow$ *set-iterator* (*succ-it g v*) (*Graph.succ* (α *g*) *v*)
unfolding *succ-it-def*
apply (*rule it-to-it-correct*)
by (*rule succ-list-it-correct*)

lemma *pi-succ-it*[*icf-proper-iteratorI*]:
proper-it (*succ-it S v*) (*succ-it S v*)
unfolding *succ-it-def*
by (*intro icf-proper-iteratorI*)

lemma *succ-it-proper*[*proper-it*]:
proper-it' ($\lambda S.$ *succ-it S v*) ($\lambda S.$ *succ-it S v*)
apply (*rule proper-it'I*)
by (*rule pi-succ-it*)

end

6.1 Adjacency Lists

type-synonym $('V,'W)$ *adj-list* = $'V$ *list* $\times$ $('V\times'W\times'V)$ *list*

definition *adjl-*α :: $('V,'W)$ *adj-list* $\Rightarrow$ $('V,'W)$ *graph* **where**
adjl-α *l* $\equiv$ *let* (*nl*,*el*) = *l in* ⦇
nodes = *set nl* $\cup$ *fst*‘*set el* $\cup$ *snd*‘*snd*‘*set el*,
edges = *set el*
⦈

lemma *adjl-is-graph*: *graph adjl-*α (λ-. *True*)
apply (*unfold-locales*)
unfolding *adjl-*α*-def*
by *force*+

type-synonym $('V,'W,'G)$ *graph-from-list* = $('V,'W)$ *adj-list* $\Rightarrow$ $'G$
locale *graph-from-list* = *graph* +
constrains α :: $'G \Rightarrow ('V,'W)$ *graph*
fixes *from-list* :: $('V,'W)$ *adj-list* $\Rightarrow$ $'G$
assumes *from-list-correct*:
invar (*from-list l*)
α (*from-list l*) = *adjl-*α *l*

type-synonym $('V,'W,'G)$ *graph-to-list* $= 'G \Rightarrow ('V,'W)$ *adj-list*
locale *graph-to-list* $=$ *graph* $+$
  constrains $\alpha :: 'G \Rightarrow ('V,'W)$ *graph*
  fixes *to-list* $:: 'G \Rightarrow ('V,'W)$ *adj-list*
  assumes *to-list-correct*:
    invar g $\Longrightarrow$ *adjl-*α (*to-list g*) $= \alpha$ *g*

6.2 Record Based Interface

record $('V,'W,'G)$ *graph-ops* $=$
  gop-$\alpha :: ('V,'W,'G)$ *graph-*α
  gop-invar $:: 'G \Rightarrow$ *bool*
  gop-empty $:: ('V,'W,'G)$ *graph-empty*
  gop-add-node $:: ('V,'W,'G)$ *graph-add-node*
  gop-delete-node $:: ('V,'W,'G)$ *graph-delete-node*
  gop-add-edge $:: ('V,'W,'G)$ *graph-add-edge*
  gop-delete-edge $:: ('V,'W,'G)$ *graph-delete-edge*
  gop-from-list $:: ('V,'W,'G)$ *graph-from-list*
  gop-to-list $:: ('V,'W,'G)$ *graph-to-list*
  gop-nodes-list-it $:: 'G \Rightarrow ('V,'V$ *list*$)$ *set-iterator*
  gop-edges-list-it $:: ('V,'W,('V \times 'W \times 'V)$ *list*$,'G)$ *graph-edges-it*
  gop-succ-list-it $:: 'G \Rightarrow 'V \Rightarrow ('W \times 'V,('W \times 'V)$ *list*$)$ *set-iterator*

locale *StdGraphDefs* $=$
  graph-nodes-it-defs gop-nodes-list-it ops
  $+$ *graph-edges-it-defs gop-edges-list-it ops*
  $+$ *graph-succ-it-defs gop-succ-list-it ops*
  for *ops* $:: ('V,'W,'G,'m)$ *graph-ops-scheme*
begin
  abbreviation α **where** $\alpha \equiv$ *gop-*α *ops*
  abbreviation *invar* **where** *invar* $\equiv$ *gop-invar ops*
  abbreviation *empty* **where** *empty* $\equiv$ *gop-empty ops*
  abbreviation *add-node* **where** *add-node* $\equiv$ *gop-add-node ops*
  abbreviation *delete-node* **where** *delete-node* $\equiv$ *gop-delete-node ops*
  abbreviation *add-edge* **where** *add-edge* $\equiv$ *gop-add-edge ops*
  abbreviation *delete-edge* **where** *delete-edge* $\equiv$ *gop-delete-edge ops*
  abbreviation *from-list* **where** *from-list* $\equiv$ *gop-from-list ops*
  abbreviation *to-list* **where** *to-list* $\equiv$ *gop-to-list ops*
  abbreviation *nodes-list-it* **where** *nodes-list-it* $\equiv$ *gop-nodes-list-it ops*
  abbreviation *edges-list-it* **where** *edges-list-it* $\equiv$ *gop-edges-list-it ops*
  abbreviation *succ-list-it* **where** *succ-list-it* $\equiv$ *gop-succ-list-it ops*
end

locale *StdGraph* $=$ *StdGraphDefs* $+$
  graph α *invar* $+$
  graph-empty α *invar empty* $+$
  graph-add-node α *invar add-node* $+$
  graph-delete-node α *invar delete-node* $+$
  graph-add-edge α *invar add-edge* $+$

graph-delete-edge α *invar delete-edge* +
graph-from-list α *invar from-list* +
graph-to-list α *invar to-list* +
graph-nodes-it α *invar nodes-list-it* +
graph-edges-it α *invar edges-list-it* +
graph-succ-it α *invar succ-list-it*
begin
lemmas *correct* = *empty-correct add-node-correct delete-node-correct*
add-edge-correct delete-edge-correct
from-list-correct to-list-correct

end

6.3 Refinement Framework Bindings

lemma (**in** *graph-nodes-it*) *nodes-it-is-iterator*[*refine-transfer*]:
invar g $\Longrightarrow$ *set-iterator* (*nodes-it g*) (*nodes* (α *g*))
by (*rule nodes-it-correct*)

lemma (**in** *graph-edges-it*) *edges-it-is-iterator*[*refine-transfer*]:
invar g $\Longrightarrow$ *set-iterator* (*edges-it g*) (*edges* (α *g*))
by (*rule edges-it-correct*)

lemma (**in** *graph-succ-it*) *succ-it-is-iterator*[*refine-transfer*]:
invar g $\Longrightarrow$ *set-iterator* (*succ-it g v*) (*Graph.succ* (α *g*) *v*)
by (*rule succ-it-correct*)

lemma (**in** *graph*) *drh*[*refine-dref-RELATES*]: *RELATES* (*build-rel* α *invar*)
by (*simp add*: *RELATES-def*)

end

7 Generic Algorithms for Graphs

theory *GraphGA*
imports
GraphSpec
begin

definition *gga-from-list* ::
('*V*,'*W*,'*G*) *graph-empty* $\Rightarrow$ ('*V*,'*W*,'*G*) *graph-add-node*
$\Rightarrow$ ('*V*,'*W*,'*G*) *graph-add-edge*
$\Rightarrow$ ('*V*,'*W*,'*G*) *graph-from-list*
where
gga-from-list e a u l $\equiv$
let (*nl*,*el*) = *l*;
g1 = *foldl* (λ*g v*. *a v g*) (*e* ()) *nl*

```
    in foldl (λg (v,e,v'). u v e v' g) g1 el

lemma gga-from-list-correct:
  fixes α :: 'G ⇒ ('V,'W) graph
  assumes graph-empty α invar e
  assumes graph-add-node α invar a
  assumes graph-add-edge α invar u
  shows graph-from-list α invar (gga-from-list e a u)
proof −
  interpret
    graph-empty α invar e +
    graph-add-node α invar a +
    graph-add-edge α invar u
    by fact+

  {
    fix nl el
    define g1 where g1 = foldl (λg v. a v g) (e ()) nl
    define g2 where g2 = foldl (λg (v,e,v'). u v e v' g) g1 el
    have invar g1 ∧ α g1 = (| nodes = set nl, edges = {} |)
      unfolding g1-def
      by (induct nl rule: rev-induct)
         (auto simp: empty-correct add-node-correct empty-def add-node-def)
    hence invar g2
      ∧ α g2 = (| nodes = set nl ∪ fst`set el ∪ snd`snd`set el,
                  edges = set el |)
      unfolding g2-def
      by (induct el rule: rev-induct) (auto simp: add-edge-correct add-edge-def)
    hence invar g2 ∧ adjl-α (nl,el) = α g2
      unfolding adjl-α-def by auto
  }
  thus ?thesis
    unfolding gga-from-list-def [abs-def]
    apply unfold-locales
    apply auto
    done
qed

term map-iterator-product


locale gga-edges-it-defs =
  graph-nodes-it-defs nodes-list-it +
  graph-succ-it-defs succ-list-it
  for nodes-list-it :: ('V,'W,'V list,'G) graph-nodes-it
  and succ-list-it :: ('V,'W,('W×'V) list,'G) graph-succ-it
begin
  definition gga-edges-list-it ::
    ('V,'W,('V×'W×'V) list,'G) graph-edges-it
```

```
    where gga-edges-list-it G ≡ set-iterator-product
      (nodes-it G) (succ-it G)
  local-setup ‹Locale-Code.lc-decl-del @{term gga-edges-list-it}›
end
setup ‹
  (Record-Intf.add-unf-thms-global @{thms
    gga-edges-it-defs.gga-edges-list-it-def[abs-def]
  })
›

locale gga-edges-it = gga-edges-it-defs nodes-list-it succ-list-it
  + graph α invar
  + graph-nodes-it α invar nodes-list-it
  + graph-succ-it α invar succ-list-it
  for α :: 'G ⇒ ('V,'W) graph
  and invar
  and nodes-list-it :: ('V,'W,'V list,'G) graph-nodes-it
  and succ-list-it :: ('V,'W,('W×'V) list,'G) graph-succ-it
begin
  lemma gga-edges-list-it-impl:
    shows graph-edges-it α invar gga-edges-list-it
  proof
    fix g
    assume INV: invar g

    from set-iterator-product-correct[OF
      nodes-it-correct[OF INV] succ-it-correct[OF INV]]
    have set-iterator (set-iterator-product (nodes-it g) (λv. succ-it g v))
      (SIGMA v:nodes (α g). succ (α g) v)
      .
    also have (SIGMA v:nodes (α g). succ (α g) v) = edges (α g)
      unfolding succ-def
      by (auto dest: valid-graph.E-validD[OF valid[OF INV]])

    finally show set-iterator (gga-edges-list-it g) (edges (α g))
      unfolding gga-edges-list-it-def .
  qed
end

locale gga-to-list-defs-loc =
  graph-nodes-it-defs nodes-list-it
  + graph-edges-it-defs edges-list-it
  for nodes-list-it :: ('V,'W,'V list,'G) graph-nodes-it
  and edges-list-it :: ('V,'W,('V×'W×'V) list,'G) graph-edges-it
begin
  definition gga-to-list ::
    ('V,'W,'G) graph-to-list
    where
    gga-to-list g ≡
```

(*nodes-it g* (λ-. *True*) (#) [], *edges-it g* (λ-. *True*) (#) [])

end

locale *gga-to-list-loc* = *gga-to-list-defs-loc nodes-list-it edges-list-it* +
graph α *invar*
+ *graph-nodes-it* α *invar nodes-list-it*
+ *graph-edges-it* α *invar edges-list-it*
for α :: *'G* ⇒ (*'V*,*'W*) *graph* **and** *invar*
and *nodes-list-it* :: (*'V*,*'W*,*'V list*,*'G*) *graph-nodes-it*
and *edges-list-it* :: (*'V*,*'W*,(*'V*×*'W*×*'V*) *list*,*'G*) *graph-edges-it*
begin

lemma *gga-to-list-correct*:
shows *graph-to-list* α *invar gga-to-list*
proof
fix *g*
assume [*simp*, *intro*!]: *invar g*
then interpret *valid-graph* α *g* **by** (*rule valid*)

have *set* (*nodes-it g* (λ-. *True*) (#) []) = *V*
apply (*rule-tac I*=λ*it* σ. *set* σ = *V* − *it*
in *set-iterator-rule-P*[*OF nodes-it-correct*])
by *auto*
moreover have *set* (*edges-it g* (λ-. *True*) (#) []) = *E*
apply (*rule-tac I*=λ*it* σ. *set* σ = *E* − *it*
in *set-iterator-rule-P*[*OF edges-it-correct*])
by *auto*
ultimately show *adjl-α* (*gga-to-list g*) = α *g*
unfolding *adjl-α-def gga-to-list-def*
apply *simp*
apply (*rule graph.equality*)
apply (*auto intro*: *E-validD*)
done
qed

end

end

8 Implementing Graphs by Maps

theory *GraphByMap*
imports
GraphSpec
GraphGA
begin

definition *map-Sigma M1 F2* ≡ {

```
  (x,y). ∃ v. M1 x = Some v ∧ y∈F2 v
}

lemma map-Sigma-alt: map-Sigma M1 F2 = Sigma (dom M1) (λx.
  F2 (the (M1 x)))
  unfolding map-Sigma-def
  by auto


lemma ranE:
  assumes v∈ran m
  obtains k where m k = Some v
  using assms
by (metis ran-restrictD restrict-map-self)
lemma option-bind-alt:
  Option.bind x f = (case x of None ⇒ None | Some v ⇒ f v)
  by (auto split: option.split)

locale GraphByMapDefs =
  m1: StdMapDefs m1-ops +
  m2: StdMapDefs m2-ops +
  s3: StdSetDefs s3-ops
  for m1-ops::('V,'m2,'m1,-) map-ops-scheme
  and m2-ops::('V,'s3,'m2,-) map-ops-scheme
  and s3-ops::('W,'s3,-) set-ops-scheme
  and m1-mvif :: ('V ⇒ 'm2 ⇀ 'm2) ⇒ 'm1 ⇒ 'm1
begin
  definition gbm-α :: ('V,'W,'m1) graph-α where
    gbm-α m1 ≡
    (| nodes = dom (m1.α m1),
      edges = {(v,w,v').
        ∃ m2 s3. m1.α m1 v = Some m2
          ∧ m2.α m2 v' = Some s3
          ∧ w∈s3.α s3
      }
    |)

  definition gbm-invar m1 ≡
    m1.invar m1 ∧
    (∀ m2∈ran (m1.α m1). m2.invar m2 ∧
      (∀ s3∈ran (m2.α m2). s3.invar s3)
    ) ∧ valid-graph (gbm-α m1)

  definition gbm-empty :: ('V,'W,'m1) graph-empty where
    gbm-empty ≡ m1.empty

  definition gbm-add-node :: ('V,'W,'m1) graph-add-node where
    gbm-add-node v g ≡ case m1.lookup v g of
    None ⇒ m1.update v (m2.empty ()) g |
```

```
  Some - ⇒ g

definition gbm-delete-node :: ('V,'W,'m1) graph-delete-node where
  gbm-delete-node v g ≡ let g=m1.delete v g in
  m1-mvif (λ- m2. Some (m2.delete v m2)) g

definition gbm-add-edge :: ('V,'W,'m1) graph-add-edge where
  gbm-add-edge v e v' g ≡
  let g = (case m1.lookup v' g of
    None ⇒ m1.update v' (m2.empty ()) g | Some - ⇒ g
  ) in
  case m1.lookup v g of
    None ⇒ (m1.update v (m2.sng v' (s3.sng e)) g) |
    Some m2 ⇒ (case m2.lookup v' m2 of
      None ⇒ m1.update v (m2.update v' (s3.sng e) m2) g |
      Some s3 ⇒ m1.update v (m2.update v' (s3.ins e s3) m2) g)


definition gbm-delete-edge :: ('V,'W,'m1) graph-delete-edge where
  gbm-delete-edge v e v' g ≡
  case m1.lookup v g of
    None ⇒ g |
    Some m2 ⇒ (
      case m2.lookup v' m2 of
        None ⇒ g |
        Some s3 ⇒ m1.update v (m2.update v' (s3.delete e s3) m2) g
    )


definition gbm-nodes-list-it
  :: ('V,'W,'V list,'m1) graph-nodes-it
  where
  gbm-nodes-list-it g ≡ map-iterator-dom (m1.iteratei g)
local-setup ‹Locale-Code.lc-decl-del @{term gbm-nodes-list-it}›

definition gbm-edges-list-it
  :: ('V,'W,('V×'W×'V) list,'m1) graph-edges-it
  where
  gbm-edges-list-it g ≡ set-iterator-image
    (λ((v1,m1),(v2,m2),w). (v1,w,v2))
    (set-iterator-product (m1.iteratei g)
      (λ(v,m2). set-iterator-product
        (m2.iteratei m2) (λ(w,s3). s3.iteratei s3)))

local-setup ‹Locale-Code.lc-decl-del @{term gbm-edges-list-it}›

definition gbm-succ-list-it ::
  ('V,'W,('W×'V) list,'m1) graph-succ-it
  where
```

```
gbm-succ-list-it g v ≡ case m1.lookup v g of
  None ⇒ set-iterator-emp |
  Some m2 ⇒
    set-iterator-image (λ((v',m2),w). (w,v'))
      (set-iterator-product (m2.iteratei m2) (λ(v',s). s3.iteratei s))

local-setup ‹Locale-Code.lc-decl-del @{term gbm-succ-list-it}›

definition
  gbm-from-list ≡ gga-from-list gbm-empty gbm-add-node gbm-add-edge


lemma gbm-nodes-list-it-unf:
  it-to-it (gbm-nodes-list-it g)
  ≡ map-iterator-dom (it-to-it (m1.list-it g))
  apply (rule eq-reflection)
  apply (rule it-to-it-fold)
  unfolding gbm-nodes-list-it-def m1.iteratei-def
  by (intro icf-proper-iteratorI)

lemma gbm-edges-list-it-unf:
  it-to-it (gbm-edges-list-it g)
  ≡ set-iterator-image
    (λ((v1,m1),(v2,m2),w). (v1,w,v2))
    (set-iterator-product (it-to-it (m1.list-it g))
      (λ(v,m2). set-iterator-product
        (it-to-it (m2.list-it m2)) (λ(w,s3). (it-to-it (s3.list-it s3)))))

  apply (rule eq-reflection)
  apply (rule it-to-it-fold)
  unfolding gbm-edges-list-it-def
    m1.iteratei-def m2.iteratei-def s3.iteratei-def
  apply (intro icf-proper-iteratorI allI impI, (simp split: prod.split)?)+
  done

lemma gbm-succ-list-it-unf:
it-to-it (gbm-succ-list-it g v) ≡
  case m1.lookup v g of
    None ⇒ set-iterator-emp |
    Some m2 ⇒
      set-iterator-image (λ((v',m2),w). (w,v'))
        (set-iterator-product (it-to-it (m2.list-it m2))
          (λ(v',s). (it-to-it (s3.list-it s))))

  apply (rule eq-reflection)
  apply (rule it-to-it-fold)
  unfolding gbm-succ-list-it-def
    m2.iteratei-def s3.iteratei-def
  apply (simp split: prod.split option.split)
```

apply (*intro icf-proper-iteratorI allI impI conjI*,
(*simp split*: *prod.split option.split*)*?*)+
done

end

sublocale *GraphByMapDefs* < *graph-nodes-it-defs gbm-nodes-list-it* .
sublocale *GraphByMapDefs* < *graph-edges-it-defs gbm-edges-list-it* .
sublocale *GraphByMapDefs* < *graph-succ-it-defs gbm-succ-list-it* .
sublocale *GraphByMapDefs*
< *gga-to-list-defs-loc gbm-nodes-list-it gbm-edges-list-it* .

context *GraphByMapDefs*
begin

definition [*icf-rec-def*]: *gbm-ops* ≡ ⦇
gop-α = *gbm-α*,
gop-invar = *gbm-invar*,
gop-empty = *gbm-empty*,
gop-add-node = *gbm-add-node*,
gop-delete-node = *gbm-delete-node*,
gop-add-edge = *gbm-add-edge*,
gop-delete-edge = *gbm-delete-edge*,
gop-from-list = *gbm-from-list*,
gop-to-list = *gga-to-list*,
gop-nodes-list-it = *gbm-nodes-list-it*,
gop-edges-list-it = *gbm-edges-list-it*,
gop-succ-list-it = *gbm-succ-list-it*
⦈
local-setup ‹*Locale-Code.lc-decl-del* @{*term gbm-ops*}›
end

locale *GraphByMap* = *GraphByMapDefs m1-ops m2-ops s3-ops m1-mvif* +
m1: *StdMap m1-ops* +
m2: *StdMap m2-ops* +
s3: *StdSet s3-ops* +
m1: *map-value-image-filter m1.α m1.invar m1.α m1.invar m1-mvif*
for *m1-ops*::('*V*,'*m2*,'*m1*,-) *map-ops-scheme*
and *m2-ops*::('*V*,'*s3*,'*m2*,-) *map-ops-scheme*
and *s3-ops*::('*W*,'*s3*,-) *set-ops-scheme*
and *m1-mvif* :: ('*V* ⇒ '*m2* ⇀ '*m2*) ⇒ '*m1* ⇒ '*m1*
begin
lemma *gbm-invar-split*:
assumes *gbm-invar g*
shows
m1.invar g
⋀*v m2*. *m1.α g v* = *Some m2* ⟹ *m2.invar m2*
⋀*v m2 v' s3*. *m1.α g v* = *Some m2* ⟹ *m2.α m2 v'* = *Some s3* ⟹ *s3.invar*

```
s3
    valid-graph (gbm-α g)
    using assms unfolding gbm-invar-def
    by (auto intro: ranI)

end

sublocale GraphByMap < graph gbm-α gbm-invar
proof
  fix g
  assume INV: gbm-invar g
  then interpret vg: valid-graph (gbm-α g) by (simp add: gbm-invar-def)

  from vg.E-valid
  show fst ` edges (gbm-α g) ⊆ nodes (gbm-α g) and
    snd ` snd ` edges (gbm-α g) ⊆ nodes (gbm-α g) .

  from INV show finite (nodes (gbm-α g))
    unfolding gbm-invar-def gbm-α-def by auto

  note [simp] = gbm-invar-split[OF INV]

  show finite (edges (gbm-α g))
    apply (rule finite-imageD[where f=λ(v,e,v'). (v,v',e)])
    apply (rule finite-subset[where B=
      map-Sigma (m1.α g) (λm2. map-Sigma (m2.α m2) (s3.α))])
    apply (auto simp add: map-Sigma-def gbm-α-def) []
    apply (unfold map-Sigma-alt)
    apply (auto intro!: finite-SigmaI inj-onI)
    done
qed

context GraphByMap
begin

  lemma gbm-empty-impl:
    graph-empty gbm-α gbm-invar gbm-empty
    apply (unfold-locales)
    unfolding gbm-α-def gbm-invar-def gbm-empty-def
    apply (auto simp: m1.correct Graph.empty-def)
    apply (unfold-locales)
    apply auto
    done

  lemma gbm-add-node-impl:
    graph-add-node gbm-α gbm-invar gbm-add-node
  proof
    fix g v
    assume INV: gbm-invar g
```

```
  note [simp]= gbm-invar-split[OF INV]
  show gbm-α (gbm-add-node v g) = add-node v (gbm-α g)
    unfolding gbm-α-def gbm-add-node-def
    by (auto simp: m1.correct m2.correct s3.correct add-node-def
      split: option.split if-split-asm)

  thus gbm-invar (gbm-add-node v g)
    unfolding gbm-invar-def
    apply (simp)
    unfolding gbm-α-def gbm-add-node-def add-node-def
    apply (auto simp: m1.correct m2.correct s3.correct add-node-def
      split: option.split if-split-asm elim!: ranE)
    done
qed

lemma gbm-delete-node-impl:
  graph-delete-node gbm-α gbm-invar gbm-delete-node
proof
  fix g v
  assume INV: gbm-invar g
  note [simp]= gbm-invar-split[OF INV]
  show gbm-α (gbm-delete-node v g) = delete-node v (gbm-α g)
    unfolding gbm-α-def gbm-delete-node-def
    by (auto simp: restrict-map-def option-bind-alt
      m1.correct m2.correct s3.correct m1.map-value-image-filter-correct
      delete-node-def
      split: option.split if-split-asm option.split-asm)

  thus gbm-invar (gbm-delete-node v g)
    unfolding gbm-invar-def
    apply (simp)
    unfolding gbm-α-def gbm-delete-node-def delete-node-def
    apply (auto simp: restrict-map-def option-bind-alt
      m1.correct m2.correct s3.correct m1.map-value-image-filter-correct
      split: option.split if-split-asm option.split-asm elim!: ranE)
    done
qed

lemma gbm-add-edge-impl:
  graph-add-edge gbm-α gbm-invar gbm-add-edge
proof
  fix g v e v'
  assume INV: gbm-invar g
  note [simp]= gbm-invar-split[OF INV]
  show gbm-α (gbm-add-edge v e v' g) = add-edge v e v' (gbm-α g)
    unfolding gbm-α-def gbm-add-edge-def
    apply (auto simp: m1.correct m2.correct s3.correct
      Let-def
      split: option.split if-split-asm)
```

```
    unfolding add-edge-def

    apply (fastforce split: if-split-asm
      simp: m1.correct m2.correct s3.correct
    )+
    done

  thus gbm-invar (gbm-add-edge v e v' g)
    unfolding gbm-invar-def
    apply (simp)
    unfolding gbm-α-def gbm-add-edge-def
    apply (force simp: m1.correct m2.correct s3.correct
      Let-def
      split: option.split if-split-asm elim!: ranE)
    done
qed

lemma gbm-delete-edge-impl:
  graph-delete-edge gbm-α gbm-invar gbm-delete-edge
proof
  fix g v e v'
  assume INV: gbm-invar g
  note [simp]= gbm-invar-split[OF INV]
  show gbm-α (gbm-delete-edge v e v' g) = delete-edge v e v' (gbm-α g)
    unfolding gbm-α-def gbm-delete-edge-def delete-edge-def
    apply (auto simp: m1.correct m2.correct s3.correct
      Let-def
      split: option.split if-split-asm)
    done

  thus gbm-invar (gbm-delete-edge v e v' g)
    unfolding gbm-invar-def
    apply (simp)
    unfolding gbm-α-def gbm-delete-edge-def
    apply (auto simp: m1.correct m2.correct s3.correct
      Let-def
      split: option.split if-split-asm elim!: ranE)
    done
qed

lemma gbm-nodes-list-it-impl:
  shows graph-nodes-it gbm-α gbm-invar gbm-nodes-list-it
proof
  fix g
  assume gbm-invar g
  hence MINV: map-op-invar m1-ops g unfolding gbm-invar-def by auto
  from map-iterator-dom-correct[OF m1.iteratei-correct[OF MINV]]
  show set-iterator (gbm-nodes-list-it g) (nodes (gbm-α g))
    unfolding gbm-nodes-list-it-def gbm-α-def by simp
```

qed

lemma *gbm-edges-list-it-impl*:
shows *graph-edges-it gbm-α gbm-invar gbm-edges-list-it*
proof
fix *g*
assume *INV*: *gbm-invar g*

from *INV* **have** *I1*: *m1.invar g* **unfolding** *gbm-invar-def* **by** *auto*
from *INV* **have** *I2*: ⋀*v m2*. (*v*,*m2*)∈*map-to-set* (*m1.α g*) ⟹ *m2.invar m2*
unfolding *gbm-invar-def map-to-set-def*
by (*auto simp*: *ran-def*)
from *INV* **have** *I3*: ⋀*v m2 v′ s*. ⟦
(*v*,*m2*)∈*map-to-set* (*m1.α g*);
(*v′*,*s*)∈*map-to-set* (*m2.α m2*)⟧
⟹ *s3.invar s*
unfolding *gbm-invar-def map-to-set-def*
by (*auto simp*: *ran-def*)

show *set-iterator* (*gbm-edges-list-it g*) (*edges* (*gbm-α g*))
unfolding *gbm-edges-list-it-def*
apply (*rule set-iterator-image-correct*)
apply (*rule set-iterator-product-correct*)
apply (*rule m1.iteratei-correct*)
apply (*rule I1*)
apply (*case-tac a*)
apply *clarsimp*
apply (*rule set-iterator-product-correct*)
apply (*drule I2*)
apply (*subgoal-tac map-iterator* (*m2.iteratei ba*)
(*map-op-α m2-ops* (*snd* (*aa*,*ba*))))
apply *assumption*
apply (*simp add*: *m2.iteratei-correct*)

apply (*case-tac a*)
apply *clarsimp*
apply (*subgoal-tac set-iterator* (*s3.iteratei bb*)
(*s3.α* (*snd* (*ab*,*bb*))))
apply *assumption*
apply (*simp add*: *s3.iteratei-correct I3*)

apply (*auto simp*: *inj-on-def map-to-set-def*) []

apply (*force simp*: *gbm-α-def map-to-set-def*) []
done
qed

lemma *gbm-succ-list-it-impl*:
shows *graph-succ-it gbm-α gbm-invar gbm-succ-list-it*

```
  proof
    fix g v
    assume INV: gbm-invar g
    hence I1[simp]: m1.invar g unfolding gbm-invar-def by auto

    show set-iterator (gbm-succ-list-it g v) (succ (gbm-α g) v)
    proof (cases m1.lookup v g)
      case None hence (succ (gbm-α g) v) = {}
        unfolding succ-def gbm-α-def by (auto simp: m1.lookup-correct)
      with None show ?thesis unfolding gbm-succ-list-it-def
        by (auto simp: set-iterator-emp-correct)
    next
      case (Some m2)
      hence [simp]: m2.invar m2 using gbm-invar-split[OF INV]
        by (simp add: m1.lookup-correct)

      from INV Some have
        I2: ⋀v' s. (v', s) ∈ map-to-set (map-op-α m2-ops m2) ⟹ s3.invar s
        unfolding gbm-invar-def
        by (auto simp: map-to-set-def ran-def m1.lookup-correct)

      from Some show ?thesis
        unfolding gbm-succ-list-it-def apply simp
        apply (rule set-iterator-image-correct)
        apply (rule set-iterator-product-correct)
        apply (rule m2.iteratei-correct)
        apply simp
        apply (case-tac a, simp)
        apply (subgoal-tac set-iterator (s3.iteratei b) (s3.α (snd (aa, b))))
        apply assumption
        apply simp
        apply (rule s3.iteratei-correct)
        apply (simp add: I2)

        apply (auto simp: inj-on-def map-to-set-def) []

        apply (force simp: succ-def gbm-α-def map-to-set-def m1.lookup-correct)
        done
    qed
  qed

  lemma gbm-from-list-impl:
    shows graph-from-list gbm-α gbm-invar gbm-from-list
    unfolding gbm-from-list-def
    apply (rule gga-from-list-correct)
    apply (rule gbm-empty-impl gbm-add-node-impl gbm-add-edge-impl)+
    done

end
```

sublocale *GraphByMap* < *graph-nodes-it gbm-α gbm-invar gbm-nodes-list-it*
  by (*rule gbm-nodes-list-it-impl*)
sublocale *GraphByMap* < *graph-edges-it gbm-α gbm-invar gbm-edges-list-it*
  by (*rule gbm-edges-list-it-impl*)
sublocale *GraphByMap* < *graph-succ-it gbm-α gbm-invar gbm-succ-list-it*
  by (*rule gbm-succ-list-it-impl*)

sublocale *GraphByMap*
  < *gga-to-list-loc gbm-α gbm-invar gbm-nodes-list-it gbm-edges-list-it*
  by *unfold-locales*

context *GraphByMap*
begin
  lemma *gbm-to-list-impl*: *graph-to-list gbm-α gbm-invar gga-to-list*
    by (*rule gga-to-list-correct*)

  lemma *gbm-ops-impl*: *StdGraph gbm-ops*
    apply (*rule StdGraph.intro*)
    apply (*simp-all add*: *icf-rec-unf*)
    apply *icf-locales*
    apply (*rule gbm-empty-impl gbm-add-node-impl gbm-delete-node-impl*
      gbm-add-edge-impl gbm-delete-edge-impl gbm-from-list-impl
      gbm-to-list-impl)+
    done
end

setup ‹
  (*Record-Intf.add-unf-thms-global* @{*thms*
    GraphByMapDefs.gbm-nodes-list-it-unf
    GraphByMapDefs.gbm-edges-list-it-unf
    GraphByMapDefs.gbm-succ-list-it-unf
  })
›

end

9 Graphs by Hashmaps

theory *HashGraphImpl*
imports
  GraphByMap
begin

Abbreviation: hlg

type-synonym ('*V*,'*E*) *hlg* =
  ('*V*,('*V*,'*E ls*) *HashMap.hashmap*) *HashMap.hashmap*

setup *Locale-Code.open-block*
interpretation *hh-mvif*: *g-value-image-filter-loc hm-ops hm-ops*
  by *unfold-locales*
interpretation *hlg-gbm*: *GraphByMap hm-ops hm-ops ls-ops*
  hh-mvif.g-value-image-filter
  by *unfold-locales*
setup *Locale-Code.close-block*

definition [*icf-rec-def*]: *hlg-ops* ≡ *hlg-gbm.gbm-ops*

setup *Locale-Code.open-block*
interpretation *hlg*: *StdGraph hlg-ops*
  unfolding *hlg-ops-def*
  by (*rule hlg-gbm.gbm-ops-impl*)
setup *Locale-Code.close-block*
setup ‹*ICF-Tools.revert-abbrevs HashGraphImpl.hlg*›

thm *map-iterator-dom-def set-iterator-image-def*
  set-iterator-image-filter-def

definition *test-codegen* **where** *test-codegen* ≡ (
  hlg.empty,
  hlg.add-node,
  hlg.delete-node,
  hlg.add-edge,
  hlg.delete-edge,
  hlg.from-list,
  hlg.to-list,
  hlg.nodes-it,
  hlg.edges-it,
  hlg.succ-it
)

export-code *test-codegen* **in** *SML*

end

10 Implementation of Dijkstra's-Algorithm using the ICF

theory *Dijkstra-Impl*
imports
  Dijkstra
  GraphSpec
  HashGraphImpl
  HOL−Library.Code-Target-Numeral
begin

In this second refinement step, we use interfaces from the Isabelle Collec-

tion Framework (ICF) to implement the priority queue and the result map. Moreover, we use a graph interface (that is not contained in the ICF, but in this development) to represent the graph.

The data types of the first refinement step were designed to fit the abstract data types of the used ICF-interfaces, which makes this refinement quite straightforward.

Finally, we instantiate the ICF-interfaces by concrete implementations, obtaining an executable algorithm, for that we generate code using Isabelle/HOL's code generator.

```
locale dijkstraC =
  g: StdGraph g-ops +
  mr: StdMap mr-ops +
  qw: StdUprio qw-ops
  for g-ops :: ('V,'W::weight,'G,'moreg) graph-ops-scheme
  and mr-ops :: ('V, (('V,'W) path × 'W), 'mr,'more-mr) map-ops-scheme
  and qw-ops :: ('V ,'W infty,'qw,'more-qw) uprio-ops-scheme
begin
  definition αsc == map-prod qw.α mr.α
  definition dinvarC-add == λ(wl,res). qw.invar wl ∧ mr.invar res

  definition cdinit :: 'G ⇒ 'V ⇒ ('qw×'mr) nres where
    cdinit g v0 ≡ do {
      wl ← FOREACH (nodes (g.α g))
        (λv wl. RETURN (qw.insert wl v Weight.Infty)) (qw.empty ());
      RETURN (qw.insert wl v0 (Num 0),mr.sng v0 ([],0))
    }

  definition cpop-min :: ('qw×'mr) ⇒ ('V×'W infty×('qw×'mr)) nres where
    cpop-min σ ≡ do {
      let (wl,res) = σ;
      let (v,w,wl')=qw.pop wl;
      RETURN (v,w,(wl',res))
    }

  definition cupdate :: 'G ⇒ 'V ⇒ 'W infty ⇒ ('qw×'mr) ⇒ ('qw×'mr) nres
    where
    cupdate g v wv σ = do {
      ASSERT (dinvarC-add σ);
      let (wl,res)=σ;
      let pv=mpath' (mr.lookup v res);
      FOREACH (succ (g.α g) v) (λ(w',v') (wl,res).
        if (wv + Num w' < mpath-weight' (mr.lookup v' res)) then do {
          RETURN (qw.insert wl v' (wv+Num w'),
                  mr.update v' ((v,w',v')#the pv,val wv + w') res)
        } else RETURN (wl,res)
      ) (wl,res)
    }
```

```
  definition cdijkstra where
    cdijkstra g v0 ≡ do {
      σ0 ← cdinit g v0;
      (-,res) ← WHILE_T (λ(wl,-). ¬ qw.isEmpty wl)
            (λσ. do { (v,wv,σ') ← cpop-min σ; cupdate g v wv σ' } )
            σ0;
      RETURN res
    }

end

locale dijkstraC-fixg = dijkstraC g-ops mr-ops qw-ops +
  Dijkstra ga v0
  for g-ops :: ('V,'W::weight,'G,'moreg) graph-ops-scheme
  and mr-ops :: ('V, (('V,'W) path × 'W), 'mr,'more-mr) map-ops-scheme
  and qw-ops :: ('V ,'W infty,'qw,'more-qw) uprio-ops-scheme
  and ga :: ('V,'W) graph
  and v0 :: 'V +
  fixes g :: 'G
  assumes g-rel: (g,ga)∈br g.α g.invar
begin
  schematic-goal cdinit-refines:
    notes [refine] = inj-on-id
    shows cdinit g v0 ≤⇓?R mdinit
    using g-rel
    unfolding cdinit-def mdinit-def
    apply (refine-rcg)
    apply (refine-dref-type)
    apply (simp-all add: αsc-def dinvarC-add-def refine-rel-defs
      qw.correct mr.correct refine-hsimp)
    done

  schematic-goal cpop-min-refines:
    (σ,σ') ∈ build-rel αsc dinvarC-add
      ⟹ cpop-min σ ≤ ⇓?R (mpop-min σ')
    unfolding cpop-min-def mpop-min-def
    apply (refine-rcg)
    apply (refine-dref-type)
    apply (simp add: αsc-def dinvarC-add-def refine-hsimp refine-rel-defs)
    apply (simp add: αsc-def dinvarC-add-def refine-hsimp refine-rel-defs)
    done

  schematic-goal cupdate-refines:
    notes [refine] = inj-on-id
    shows (σ,σ')∈build-rel αsc dinvarC-add ⟹ v=v' ⟹ wv=wv' ⟹
    cupdate g v wv σ ≤ ⇓?R (mupdate v' wv' σ')
    unfolding cupdate-def mupdate-def
    using g-rel
```

```
    apply (refine-rcg)
    apply (refine-dref-type)
    apply (simp-all add: αsc-def dinvarC-add-def refine-rel-defs
      qw.correct mr.correct refine-hsimp)
    done

  lemma cdijkstra-refines:
    cdijkstra g v0 ≤ ⇓(build-rel mr.α mr.invar) mdijkstra
  proof −
    note [refine] = cdinit-refines cpop-min-refines cupdate-refines
    show ?thesis
      unfolding cdijkstra-def mdijkstra-def
      using g-rel
      apply (refine-rcg)

      apply (auto
        split: prod.split prod.split-asm
        simp add: qw.correct mr.correct dinvarC-add-def αsc-def refine-hsimp
          refine-rel-defs)
      done
  qed
end

context dijkstraC
begin

  thm g.nodes-it-is-iterator

  schematic-goal idijkstra-refines-aux:
    assumes g.invar g
    shows RETURN ?f ≤ cdijkstra g v0
    using assms
    unfolding cdijkstra-def cdinit-def cpop-min-def cupdate-def
    apply (refine-transfer)
    done

  concrete-definition idijkstra for g ?v0.0 uses idijkstra-refines-aux

  lemma idijkstra-refines:
    assumes g.invar g
    shows RETURN (idijkstra g v0) ≤ cdijkstra g v0
    using assms
    by (rule idijkstra.refine)

end
```

The following theorem states correctness of the algorithm independent from the refinement framework.

Intuitively, the first goal states that the abstraction of the returned result

is correct, the second goal states that the result datastructure satisfies its invariant, and the third goal states that the cached weights in the returned result are correct.

Note that this is the main theorem for a user of Dijkstra's algorithm in some bigger context. It may also be specialized for concrete instances of the implementation, as exemplarily done below.

```
theorem (in dijkstraC-fixg) idijkstra-correct:
  shows
  weighted-graph.is-shortest-path-map ga v0 (αr (mr.α (idijkstra g v0)))
    (is ?G1)
  and mr.invar (idijkstra g v0) (is ?G2)
  and Dijkstra.res-invarm (mr.α (idijkstra g v0)) (is ?G3)
proof −
  from g-rel have I: g.invar g by (simp add: refine-rel-defs)

  note idijkstra-refines[OF I]
  also note cdijkstra-refines
  also note mdijkstra-refines
  finally have Z: RETURN (idijkstra g v0) ≤
    ⇓(build-rel (αr ∘ mr.α) (λm. mr.invar m ∧ res-invarm (mr.α m)))
      dijkstra'
    apply (subst (asm) conc-fun-chain)
    apply (simp only: br-chain)
    done
  also note dijkstra'-refines[simplified]
  also note dijkstra-correct
  finally show ?G1 ?G2 ?G3
    by (auto elim: RETURN-ref-SPECD simp: refine-rel-defs)
qed


theorem (in dijkstraC) idijkstra-correct:
  assumes INV: g.invar g
  assumes V0: v0 ∈ nodes (g.α g)
  assumes nonneg-weights: ⋀v w v'. (v,w,v')∈edges (g.α g) ⟹ 0≤w
  shows
  weighted-graph.is-shortest-path-map (g.α g) v0
      (Dijkstra.αr (mr.α (idijkstra g v0))) (is ?G1)
  and Dijkstra.res-invarm (mr.α (idijkstra g v0)) (is ?G2)
proof −
  interpret gv: valid-graph g.α g using g.valid INV .

  interpret dcg: dijkstraC-fixg g-ops mr-ops qw-ops g.α g v0 g
    apply (rule dijkstraC-fixg.intro)
    apply unfold-locales
    apply (simp-all add: hlg.finite INV V0 hlg-ops-def
      nonneg-weights refine-rel-defs)
    done
```

from *dcg.idijkstra-correct* **show** *?G1 ?G2* **by** *simp-all*
qed

Example instantiation with HashSet.based graph, red-black-tree based result map, and finger-tree based priority queue.

setup *Locale-Code.open-block*
interpretation *hrf*: *dijkstraC hlg-ops rm-ops aluprioi-ops*
by *unfold-locales*
setup *Locale-Code.close-block*

definition *hrf-dijkstra* ≡ *hrf.idijkstra*
lemmas *hrf-dijkstra-correct* = *hrf.idijkstra-correct*[*folded hrf-dijkstra-def*]

export-code *hrf-dijkstra* **checking** *SML*
export-code *hrf-dijkstra* **in** *OCaml*
export-code *hrf-dijkstra* **in** *Haskell*
export-code *hrf-dijkstra* **checking** *Scala*

definition *hrfn-dijkstra* :: (*nat*,*nat*) *hlg* ⇒ -
where *hrfn-dijkstra* ≡ *hrf-dijkstra*

export-code *hrfn-dijkstra* **in** *SML*

lemmas *hrfn-dijkstra-correct* =
hrf-dijkstra-correct[**where** *?'a* = *nat* **and** *?'b* = *nat*, *folded hrfn-dijkstra-def*]

term *hrfn-dijkstra*
term *hlg.from-list*

definition *test-hrfn-dijkstra*
≡ *rm.to-list*
(*hrfn-dijkstra* (*hlg.from-list* ([*0*..<*4*],[(*0*,*3*,*1*),(*0*,*4*,*2*),(*2*,*1*,*3*),(*1*,*4*,*3*)])) *0*)

ML-val ‹
@{*code test-hrfn-dijkstra*}

›

end

11 Implementation of Dijkstra's-Algorithm using Automatic Determinization

theory *Dijkstra-Impl-Adet*
imports

Dijkstra
GraphSpec
HashGraphImpl
Collections.Refine-Dflt-ICF
HOL−Library.Code-Target-Numeral
begin

11.1 Setup

11.1.1 Infinity

definition *infty-rel-internal-def*:
infty-rel R ≡ {(*Num a*, *Num a′*)| *a a′*. (*a*,*a′*)∈*R*} ∪ {(*Infty*,*Infty*)}
lemma *infty-rel-def*[*refine-rel-defs*]:
⟨*R*⟩*infty-rel* = {(*Num a*, *Num a′*)| *a a′*. (*a*,*a′*)∈*R*} ∪ {(*Infty*,*Infty*)}
unfolding *infty-rel-internal-def relAPP-def* **by** *simp*

lemma *infty-relI*:
(*Infty*,*Infty*)∈⟨*R*⟩*infty-rel*
(*a*,*a′*)∈*R* ⟹ (*Num a*, *Num a′*)∈⟨*R*⟩*infty-rel*
unfolding *infty-rel-def* **by** *auto*

lemma *infty-relE*:
assumes (*x*,*x′*)∈⟨*R*⟩*infty-rel*
obtains *x*=*Infty* **and** *x′*=*Infty*
| *a a′* **where** *x*=*Num a* **and** *x′*=*Num a′* **and** (*a*,*a′*)∈*R*
using *assms*
unfolding *infty-rel-def*
by *auto*

lemma *infty-rel-simps*[*simp*]:
(*Infty*,*x′*)∈⟨*R*⟩*infty-rel* ⟷ *x′*=*Infty*
(*x*,*Infty*)∈⟨*R*⟩*infty-rel* ⟷ *x*=*Infty*
(*Num a*, *Num a′*)∈⟨*R*⟩*infty-rel* ⟷ (*a*,*a′*)∈*R*
unfolding *infty-rel-def* **by** *auto*

lemma *infty-rel-sv*[*relator-props*]:
single-valued R ⟹ *single-valued* (⟨*R*⟩*infty-rel*)
unfolding *infty-rel-def*
by (*auto intro*: *single-valuedI dest*: *single-valuedD*)

lemma *infty-rel-id*[*simp*, *relator-props*]: ⟨*Id*⟩*infty-rel* = *Id*
apply *rule*
apply (*auto elim*: *infty-relE*) []
apply *safe*
apply (*case-tac b*) **by** *auto*

consts *i-infty* :: *interface* ⇒ *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of infty-rel i-infty*]

lemma *autoref-infty*[*param*,*autoref-rules*]:
(*Infty*,*Infty*)∈⟨*R*⟩*infty-rel*
(*Num*,*Num*)∈*R*→⟨*R*⟩*infty-rel*
(*case-infty*,*case-infty*)∈*Rr*→(*R*→*Rr*)→⟨*R*⟩*infty-rel*→*Rr*
(*rec-infty*,*rec-infty*)∈*Rr*→(*R*→*Rr*)→⟨*R*⟩*infty-rel*→*Rr*
unfolding *infty-rel-def*
by (*auto dest*: *fun-relD*)

definition [*simp*]: *is-Infty x* ≡ *case x of Infty* ⇒ *True* | - ⇒ *False*

context begin interpretation *autoref-syn* .
lemma *pat-is-Infty*[*autoref-op-pat*]:
x=*Infty* ≡ (*OP is-Infty* :::$_i$ ⟨*I*⟩$_i$*i-infty* →$_i$ *i-bool*)$*x*
Infty=*x* ≡ (*OP is-Infty* :::$_i$ ⟨*I*⟩$_i$*i-infty* →$_i$ *i-bool*)$*x*
by (*auto intro*!: *eq-reflection split*: *infty.splits*)
end

lemma *autoref-is-Infty*[*autoref-rules*]:
(*is-Infty*, *is-Infty*)∈⟨*R*⟩*infty-rel* → *bool-rel*
by (*auto split*: *infty.splits*)

definition *infty-eq eq v1 v2* ≡
case (*v1*,*v2*) *of*
(*Infty*,*Infty*) ⇒ *True*
| (*Num a1*, *Num a2*) ⇒ *eq a1 a2*
| - ⇒ *False*

lemma *infty-eq-autoref*[*autoref-rules* (**overloaded**)]:
⟦ *GEN-OP eq* (=) (*R*→*R*→*bool-rel*) ⟧
⟹ (*infty-eq eq*,(=))∈⟨*R*⟩*infty-rel*→⟨*R*⟩*infty-rel*→*bool-rel*
unfolding *infty-eq-def*[*abs-def*]
by (*auto split*: *infty.splits dest*: *fun-relD elim*!: *infty-relE*)

lemma *infty-eq-expand*[*autoref-struct-expand*]: (=) = *infty-eq* (=)
by (*auto intro*!: *ext simp*: *infty-eq-def split*: *infty.splits*)

context begin interpretation *autoref-syn* .
lemma *infty-val-autoref*[*autoref-rules*]:
⟦*SIDE-PRECOND* (*x*≠*Infty*); (*xi*,*x*)∈⟨*R*⟩*infty-rel*⟧
⟹ (*val xi*,(*OP val* ::: ⟨*R*⟩*infty-rel* → *R*) $ *x*)∈*R*
apply (*cases x*)
apply (*auto elim*: *infty-relE*)
done
end

definition *infty-plus* **where**
infty-plus pl a b ≡ *case* (*a*,*b*) *of* (*Num a*, *Num b*) ⇒ *Num* (*pl a b*) | - ⇒ *Infty*

lemma *infty-plus-param*[*param*]:

(*infty-plus*,*infty-plus*) ∈ (*R*→*R*→*R*) → ⟨*R*⟩*infty-rel* → ⟨*R*⟩*infty-rel* → ⟨*R*⟩*infty-rel*
unfolding *infty-plus-def*[*abs-def*]
by *parametricity*

lemma *infty-plus-eq-plus*: *infty-plus* (+) = (+)
unfolding *infty-plus-def*[*abs-def*]
by (*auto intro*!: *ext split*: *infty.split*)

lemma *infty-plus-autoref*[*autoref-rules*]:
GEN-OP pl (+) (*R*→*R*→*R*)
⟹ (*infty-plus pl*,(+)) ∈ ⟨*R*⟩*infty-rel* → ⟨*R*⟩*infty-rel* → ⟨*R*⟩*infty-rel*
apply (*fold infty-plus-eq-plus*)
apply *simp*
apply *parametricity*
done

11.1.2 Graph

consts *i-graph* :: *interface* ⇒ *interface* ⇒ *interface*

definition *graph-more-rel-internal-def*:
graph-more-rel Rm Rv Rw ≡ { (*g*,*g*′).
(*graph.nodes g*, *graph.nodes g*′)∈⟨*Rv*⟩*set-rel*
∧ (*graph.edges g*, *graph.edges g*′)∈⟨⟨*Rv*,⟨*Rw*,*Rv*⟩*prod-rel*⟩*prod-rel*⟩*set-rel*
∧ (*graph.more g*, *graph.more g*′)∈*Rm*}

lemma *graph-more-rel-def*[*refine-rel-defs*]:
⟨*Rm*,*Rv*,*Rw*⟩*graph-more-rel* ≡ { (*g*,*g*′).
(*graph.nodes g*, *graph.nodes g*′)∈⟨*Rv*⟩*set-rel*
∧ (*graph.edges g*, *graph.edges g*′)∈⟨⟨*Rv*,⟨*Rw*,*Rv*⟩*prod-rel*⟩*prod-rel*⟩*set-rel*
∧ (*graph.more g*, *graph.more g*′)∈*Rm*}
unfolding *relAPP-def graph-more-rel-internal-def* **by** *simp*

abbreviation *graph-rel* ≡ ⟨*unit-rel*⟩*graph-more-rel*
lemmas *graph-rel-def* = *graph-more-rel-def*[**where** *Rm*=*unit-rel*, *simplified*]

lemma *graph-rel-id*[*simp*]: ⟨*Id*,*Id*⟩*graph-rel* = *Id*
unfolding *graph-rel-def* **by** *auto*

lemma *graph-more-rel-sv*[*relator-props*]:
⟦*single-valued Rm*; *single-valued Rv*; *single-valued Rw*⟧
⟹ *single-valued* (⟨*Rm*,*Rv*,*Rw*⟩*graph-more-rel*)
unfolding *graph-more-rel-def*
apply (*rule single-valuedI*, *clarsimp*)
apply (*rule graph.equality*)
apply (*erule* (*1*) *single-valuedD*[*rotated*], *tagged-solver*)+
done

lemma [*autoref-itype*]:
graph.nodes ::$_i$ ⟨*Iv*,*Iw*⟩$_i$*i-graph* →$_i$ ⟨*Iv*⟩$_i$*i-set*
by *simp-all*

thm *is-map-to-sorted-list-def*

definition *nodes-to-list g* ≡ *it-to-sorted-list* (λ- -. *True*) (*graph.nodes g*)
lemma *nodes-to-list-itype*[*autoref-itype*]: *nodes-to-list* ::$_i$ ⟨*Iv*,*Iw*⟩$_i$*i-graph* →$_i$ ⟨⟨*Iv*⟩$_i$*i-list*⟩$_i$*i-nres*
by *simp*
lemma *nodes-to-list-pat*[*autoref-op-pat*]: *it-to-sorted-list* (λ- -. *True*) (*graph.nodes g*) ≡ *nodes-to-list g*
unfolding *nodes-to-list-def* **by** *simp*

definition *succ-to-list g v* ≡ *it-to-sorted-list* (λ- -. *True*) (*Graph.succ g v*)
lemma *succ-to-list-itype*[*autoref-itype*]:
succ-to-list ::$_i$ ⟨*Iv*,*Iw*⟩$_i$*i-graph* →$_i$ *Iv* →$_i$ ⟨⟨⟨*Iw*,*Iv*⟩$_i$*i-prod*⟩$_i$*i-list*⟩$_i$*i-nres* **by** *simp*
lemma *succ-to-list-pat*[*autoref-op-pat*]: *it-to-sorted-list* (λ- -. *True*) (*Graph.succ g v*) ≡ *succ-to-list g v*
unfolding *succ-to-list-def* **by** *simp*

context *graph* **begin**
definition *rel-def-internal*: *rel Rv Rw* ≡ *br* α *invar O* ⟨*Rv*,*Rw*⟩*graph-rel*
lemma *rel-def*: ⟨*Rv*,*Rw*⟩*rel* ≡ *br* α *invar O* ⟨*Rv*,*Rw*⟩*graph-rel*
unfolding *relAPP-def rel-def-internal* **by** *simp*

lemma *rel-id*[*simp*]: ⟨*Id*,*Id*⟩*rel* = *br* α *invar* **by** (*simp add*: *rel-def*)

lemma *rel-sv*[*relator-props*]:
⟦*single-valued Rv*; *single-valued Rw*⟧ ⟹ *single-valued* (⟨*Rv*,*Rw*⟩*rel*)
unfolding *rel-def*
by *tagged-solver*

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of rel i-graph*]
end

lemma (**in** *graph-nodes-it*) *autoref-nodes-it*[*autoref-rules*]:
assumes *ID*: *PREFER-id Rv*
shows (λ*s*. *RETURN* (*it-to-list nodes-it s*),*nodes-to-list*) ∈ ⟨*Rv*,*Rw*⟩*rel* → ⟨⟨*Rv*⟩*list-rel*⟩*nres-rel*
unfolding *nodes-to-list-def*[*abs-def*]
proof (*intro fun-relI nres-relI*)
fix *s s*′
from *ID* **have** [*simp*]: *Rv* = *Id* **by** *simp*

assume (*s*,*s*′)∈⟨*Rv*,*Rw*⟩*rel*
hence *INV*: *invar s* **and** [*simp*]: *nodes s*′ = *nodes* (α *s*) **unfolding** *rel-def*
by (*auto simp add*: *br-def graph-rel-def*)

obtain *l* **where**
[*simp*]: *distinct l nodes* (α *s*) = *set l it-to-list nodes-it s* = *l*

unfolding *it-to-list-def*
by (*metis nodes-it-correct*[*OF INV*, *unfolded set-iterator-def set-iterator-genord-def*]

foldli-snoc-id self-append-conv2)

show *RETURN* (*it-to-list nodes-it s*)
≤ ⇓ (⟨*Rv*⟩*list-rel*) (*it-to-sorted-list* (λ- -. *True*) (*nodes s'*))
by (*simp add*: *it-to-sorted-list-def*)
qed

lemma (**in** *graph-succ-it*) *autoref-succ-it*[*autoref-rules*]:
assumes *ID*: *PREFER-id Rv PREFER-id Rw*
shows (λ*s v*. *RETURN* (*it-to-list* (λ*s*. *succ-it s v*) *s*),*succ-to-list*)
∈ ⟨*Rv*,*Rw*⟩*rel* → *Rv* → ⟨⟨⟨*Rw*,*Rv*⟩*prod-rel*⟩*list-rel*⟩*nres-rel*
unfolding *succ-to-list-def*[*abs-def*]
proof (*intro fun-relI nres-relI*)
fix *s s' v v'*
from *ID* **have** [*simp*]: *Rv* = *Id Rw*=*Id* **by** *simp-all*

assume (*v*,*v'*)∈*Rv* **hence** [*simp*]: *v'*=*v* **by** *simp*

assume (*s*,*s'*)∈⟨*Rv*,*Rw*⟩*rel*
hence *INV*: *invar s* **and** [*simp*]: *Graph.succ s'* = *Graph.succ* (α *s*) **unfolding** *rel-def*
by (*auto simp add*: *br-def graph-rel-def succ-def*)

obtain *l* **where**
[*simp*]: *distinct l succ* (α *s*) *v* = *set l it-to-list* (λ*s*. *succ-it s v*) *s* = *l*
unfolding *it-to-list-def*
by (*metis succ-it-correct*[*OF INV*, *unfolded set-iterator-def set-iterator-genord-def*]

foldli-snoc-id self-append-conv2)

show *RETURN* (*it-to-list* (λ*s*. *succ-it s v*) *s*)
≤ ⇓ (⟨⟨*Rw*,*Rv*⟩*prod-rel*⟩*list-rel*) (*it-to-sorted-list* (λ- -. *True*) (*succ s' v'*))
by (*simp add*: *it-to-sorted-list-def*)
qed

11.2 Refinement

locale *dijkstraC* =
g: *StdGraph g-ops* +
mr: *StdMap mr-ops* +
qw: *StdUprio qw-ops*
for *g-ops* :: ('*V*,'*W*::*weight*,'*G*,'*moreg*) *graph-ops-scheme*
and *mr-ops* :: ('*V*, (('*V*,'*W*) *path* × '*W*), '*mr*,'*more-mr*) *map-ops-scheme*
and *qw-ops* :: ('*V* ,'*W infty*,'*qw*,'*more-qw*) *uprio-ops-scheme*
begin

end

locale *dijkstraC-fixg* = *dijkstraC g-ops mr-ops qw-ops* +
  Dijkstra ga v0
  for *g-ops* :: ('*V*,'*W*::*weight*,'*G*,'*moreg*) *graph-ops-scheme*
  and *mr-ops* :: ('*V*, (('*V*,'*W*) *path* × '*W*), '*mr*,'*more-mr*) *map-ops-scheme*
  and *qw-ops* :: ('*V* ,'*W infty*,'*qw*,'*more-qw*) *uprio-ops-scheme*
  and *ga*::('*V*,'*W*) *graph* **and** *v0*::'*V* **and** *g* :: '*G*+
  assumes *ga-trans*: (*g*,*ga*)∈*br g*.α *g.invar*
begin
  abbreviation *v-rel* ≡ *Id* :: ('*V*×'*V*) *set*
  abbreviation *w-rel* ≡ *Id* :: ('*W*×'*W*) *set*

  definition *i-node* :: *interface* **where** *i-node* ≡ *undefined*
  definition *i-weight* :: *interface* **where** *i-weight* ≡ *undefined*

  lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of v-rel i-node*]
  lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of w-rel i-weight*]

  lemma *weight-plus-autoref*[*autoref-rules*]:
    (*0*,*0*) ∈ *w-rel*
    ((+),(+)) ∈ *w-rel* → *w-rel* → *w-rel*
    ((+),(+)) ∈ ⟨*w-rel*⟩*infty-rel* → ⟨*w-rel*⟩*infty-rel* → ⟨*w-rel*⟩*infty-rel*
    ((<),(<)) ∈ ⟨*w-rel*⟩*infty-rel* → ⟨*w-rel*⟩*infty-rel* → *bool-rel*
    by *simp-all*

  lemma [*autoref-rules*]: (*g*,*ga*)∈⟨*v-rel*,*w-rel*⟩*g.rel* **using** *ga-trans*
    by (*simp add*: *g.rel-def*)

  lemma [*autoref-rules*]: (*v0*,*v0*)∈*v-rel* **by** *simp*

  term *mpath-weight'*
  lemma [*autoref-rules*]:
    (*mpath-weight'*,*mpath-weight'*)
      ∈ ⟨⟨*v-rel*×$_r$*w-rel*×$_r$*v-rel*⟩*list-rel*×$_r$*w-rel*⟩*option-rel* → ⟨*w-rel*⟩*infty-rel*
    (*mpath'*, *mpath'*)
      ∈ ⟨⟨*v-rel*×$_r$*w-rel*×$_r$*v-rel*⟩*list-rel*×$_r$*w-rel*⟩*option-rel*
        → ⟨⟨*v-rel*×$_r$*w-rel*×$_r$*v-rel*⟩*list-rel*⟩*option-rel*
    by *auto*

  term *mdinit*

  lemmas [*autoref-tyrel*] =
    ty-REL[**where** *R*=*v-rel*]
    ty-REL[**where** *R*=*w-rel*]
    ty-REL[**where** *R*=⟨*w-rel*⟩*infty-rel*]
    ty-REL[**where** *R*=⟨*v-rel*,⟨*w-rel*⟩*infty-rel*⟩*qw.rel*]
    ty-REL[**where** *R*=⟨*v-rel*,⟨*v-rel*×$_r$*w-rel*×$_r$*v-rel*⟩*list-rel*×$_r$*w-rel*⟩*mr.rel*]
    ty-REL[**where** *R*=⟨*v-rel*×$_r$*w-rel*×$_r$*v-rel*⟩*list-rel*]

lemmas [*autoref-op-pat*] = *uprio-pats*[**where** *'e* = *'V* **and** *'a* = *'W infty*]

schematic-goal *cdijkstra-refines-aux*:
shows (*?c*::*?'c*,
mdijkstra
) ∈ *?R*
apply (*simp only*: *mdijkstra-def mdinit-def mpop-min-def*[*abs-def*] *mupdate-def*)

using [[*goals-limit* = *1*]]

apply (*fold op-map-empty-def*[**where** *'a*=*'V* **and** *'b* = (*'V*×*'W*×*'V*) *list* × *'W*])
apply (*fold op-uprio-empty-def*[**where** *'a*=*'V* **and** *'b* = *'W infty*])

using [[*autoref-trace-failed-id*]]

apply (*autoref-monadic* (*plain*,*trace*))
done

end

context *dijkstraC*
begin

concrete-definition *cdijkstra* **for** *g ?v0.0*
uses *dijkstraC-fixg.cdijkstra-refines-aux*
[*of g-ops mr-ops qw-ops*]

term *cdijkstra*
end

context *dijkstraC-fixg*
begin

term *cdijkstra*
term *mdijkstra*

lemma *cdijkstra-refines*:
RETURN (*cdijkstra g v0*) ≤ ⇓(*build-rel mr.α mr.invar*) *mdijkstra*
apply (*rule cdijkstra.refine*[*THEN nres-relD*, *simplified*])
apply *unfold-locales*
done

theorem *cdijkstra-correct*:
shows
weighted-graph.is-shortest-path-map ga v0 (*αr* (*mr.α* (*cdijkstra g v0*)))

```
    (is ?G1)
    and mr.invar (cdijkstra g v0) (is ?G2)
    and res-invarm (mr.α (cdijkstra g v0)) (is ?G3)
  proof −
    note cdijkstra-refines
    also note mdijkstra-refines
    finally have Z: RETURN (cdijkstra g v0) ≤
      ⇓(build-rel (αr ∘ mr.α) (λm. mr.invar m ∧ res-invarm (mr.α m)))
        dijkstra′
      apply (subst (asm) conc-fun-chain)
      apply (simp only: br-chain)
      done
    also note dijkstra′-refines[simplified]
    also note dijkstra-correct
    finally show ?G1 ?G2 ?G3
      by (auto elim: RETURN-ref-SPECD simp: refine-rel-defs)
  qed

end

theorem (in dijkstraC) cdijkstra-correct:
  assumes INV: g.invar g
  assumes V0: v0 ∈ nodes (g.α g)
  assumes nonneg-weights: ⋀v w v′. (v,w,v′)∈edges (g.α g) ⟹ 0≤w
  shows
  weighted-graph.is-shortest-path-map (g.α g) v0
      (Dijkstra.αr (mr.α (cdijkstra g v0))) (is ?G1)
  and Dijkstra.res-invarm (mr.α (cdijkstra g v0)) (is ?G2)
proof −
  interpret hlgv: valid-graph g.α g using g.valid INV .

  interpret dc: dijkstraC-fixg g-ops mr-ops qw-ops g.α g v0
    apply unfold-locales
    apply (simp-all
      add: hlg.finite INV V0 hlg-ops-def nonneg-weights refine-rel-defs)
    done

  from dc.cdijkstra-correct show ?G1 ?G2 by auto
qed
```

Example instantiation with HashSet.based graph, red-black-tree based result map, and finger-tree based priority queue.

```
setup Locale-Code.open-block
interpretation hrf: dijkstraC hlg-ops rm-ops aluprioi-ops
  by unfold-locales
setup Locale-Code.close-block

definition hrf-dijkstra ≡ hrf.cdijkstra
lemmas hrf-dijkstra-correct = hrf.cdijkstra-correct[folded hrf-dijkstra-def]
```

export-code *hrf-dijkstra* **checking** *SML*
export-code *hrf-dijkstra* **in** *OCaml*
export-code *hrf-dijkstra* **in** *Haskell*
export-code *hrf-dijkstra* **checking** *Scala*

definition *hrfn-dijkstra* :: (*nat*,*nat*) *hlg* ⇒ -
  where *hrfn-dijkstra* ≡ *hrf-dijkstra*

export-code *hrfn-dijkstra* **checking** *SML*

lemmas *hrfn-dijkstra-correct* =
  hrf-dijkstra-correct[**where** *?'a* = *nat* **and** *?'b* = *nat*, *folded hrfn-dijkstra-def*]

end

12 Performance Test

theory *Test*
  imports *Dijkstra-Impl-Adet*
begin

In this theory, we test our implementation of Dijkstra's algorithm for larger, randomly generated graphs.

Simple linear congruence generator for (low-quality) random numbers:

definition *lcg-next s* = ((*81*::*nat*)∗*s* + *173*) *mod 268435456*

Generate a complete graph over the given number of vertices, with random weights:

definition *ran-graph* :: *nat* ⇒ *nat* ⇒ (*nat list*×(*nat* × *nat* × *nat*) *list*) **where**
  ran-graph vertices seed ==
    ([*0*::*nat*..<*vertices*],*fst*
     (*while* (λ (*g*,*v*,*s*). *v* < *vertices*)
     (λ (*g*,*v*,*s*).
     let (*g''*,*v''*,*s''*) = (*while* (λ (*g'*,*v'*,*s'*). *v'* < *vertices*)
      (λ (*g'*,*v'*,*s'*). ((*v*,*s'*,*v'*)#*g'*,*v'*+*1*,*lcg-next s'*))
      (*g*,*0*,*s*))
     in (*g''*,*v*+*1*,*s''*))
     ([],*0*,*lcg-next seed*)))

To experiment with the exported code, we fix the node type to natural numbers, and add a from-list conversion:

type-synonym *nat-res* = (*nat*,((*nat*,*nat*) *path* × *nat*)) *rm*
type-synonym *nat-list-res* = (*nat* × (*nat*,*nat*) *path* × *nat*) *list*

definition *nat-dijkstra* :: (*nat*,*nat*) *hlg* ⇒ *nat* ⇒ *nat-res* **where**
  nat-dijkstra ≡ *hrfn-dijkstra*

```
definition hlg-from-list-nat :: (nat,nat) adj-list ⇒(nat,nat) hlg where
  hlg-from-list-nat ≡ hlg.from-list

definition
  nat-res-to-list :: nat-res ⇒ nat-list-res
  where nat-res-to-list ≡ rm.to-list

value nat-res-to-list (nat-dijkstra (hlg-from-list-nat (ran-graph 4 8912)) 0)

ML-val ‹
let
  (* Configuration of test: *)
  val vertices = @{code nat-of-integer} 1000; (* Number of vertices *)
  val seed = @{code nat-of-integer} 123454; (* Seed for random number generator
*)
  val cfg-print-paths = true; (* Whether to output complete paths *)
  val cfg-print-res = true; (* Whether to output result at all *)

  (* Internals *)
  fun string-of-edge (u,(w,v)) = let
    val u = @{code integer-of-nat} u;
    val w = @{code integer-of-nat} w;
    val v = @{code integer-of-nat} v;
  in
    ( ^ string-of-int u ^ , ^ string-of-int w ^ , ^ string-of-int v ^ )
  end

  fun print-entry (dest,(path,weight)) = let
    val dest = @{code integer-of-nat} dest;
    val weight = @{code integer-of-nat} weight;
  in
    writeln (string-of-int dest ^ :  ^ string-of-int weight ^
      ( if cfg-print-paths then
          via [ ^ commas (map string-of-edge (rev path)) ^ ]
        else
      )
    )
  end

  fun print-res [] = ()
    | print-res (a::l) = let val - = print-entry a in print-res l end;

  val start = Time.now();
  val graph = @{code hlg-from-list-nat} (@{code ran-graph} vertices seed);
  val rt1 = Time.toMilliseconds (Time.now() - start);

  val start = Time.now();
  val res = @{code nat-dijkstra} graph (@{code nat-of-integer} 0);
```

```
  val rt2 = Time.toMilliseconds (Time.now() - start);
in
  writeln (string-of-int (@{code integer-of-nat} vertices) ^ vertices:
  ^ string-of-int rt2 ^ ms +
  ^ string-of-int rt1 ^ ms to create graph =
  ^ string-of-int (rt1+rt2) ^ ms);

  if cfg-print-res then
    print-res (@{code nat-res-to-list} res)
  else ()
end;
›
```

end

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