Differential-Game-Logic

André Platzer

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Abstract

This formalization provides differential game logic (dGL), a logic for proving properties of hybrid game. In addition to the syntax and semantics, it formalizes a uniform substitution calculus for dGL. Church's uniform substitutions substitute a term or formula for a function or predicate symbol everywhere. The uniform substitutions for dGL also substitute hybrid games for a game symbol everywhere. We prove soundness of one-pass uniform substitutions and the axioms of differential game logic with respect to their denotational semantics. One-pass uniform substitutions are faster by postponing soundness-critical admissibility checks with a linear pass homomorphic application and regain soundness by a variable condition at the replacements.

The formalization is based on prior non-mechanized soundness proofs for dGL [1, 2, 4, 1, 3]. This AFP entry formalizes the mathematical proofs [4, 5] till Theorem 19.

Contents

This formalization provides *Differential Game Logic* dGL [5, 4] till Theorem 19, including the corresponding results from [2] till Lemma 13. Differential Game Logic originates from [1].

theory *Lib*
imports
Complex-Main
begin

1 Generic Mathematical Background Lemmas

lemma *finite-subset* [*simp*]: *finite M* $\Longrightarrow$ *finite* $\{x{\in}M.\ P\ x\}$
⟨*proof*⟩

lemma *finite-powerset* [*simp*]: *finite M* $\Longrightarrow$ *finite* $\{S.\ S{\subseteq}M\}$
⟨*proof*⟩

definition *fst-proj*:: $('a{*}'b)\ set \Rightarrow 'a\ set$
where *fst-proj M* $\equiv \{A.\ \exists\, B.\ (A,B){\in}M\}$

definition *snd-proj*:: $('a{*}'b)\ set \Rightarrow 'b\ set$
where *snd-proj M* $\equiv \{B.\ \exists\, A.\ (A,B){\in}M\}$

lemma *fst-proj-mem* [*simp*]: $(A \in$ *fst-proj M*$) = (\exists\, B.\ (A,B){\in}M)$
⟨*proof*⟩

lemma *snd-proj-mem* [*simp*]: $(B \in$ *snd-proj M*$) = (\exists\, A.\ (A,B){\in}M)$
⟨*proof*⟩

lemma *fst-proj-prop*: $\forall\, x{\in}$*fst-proj* $\{(A,B)|\ A\ B.\ P\ A \wedge R\ A\ B\}.\ P(x)$
⟨*proof*⟩

lemma *snd-proj-prop*: $\forall\, x{\in}$*snd-proj* $\{(A,B)\mid A\ B.\ P\ B \wedge R\ A\ B\}.\ P(x)$
⟨*proof*⟩

lemma *map-cons*: *map f* (*Cons x xs*) = *Cons* (*f x*) (*map f xs*)
⟨*proof*⟩
lemma *map-append*: *map f* (*append xs ys*) = *append* (*map f xs*) (*map f ys*)
⟨*proof*⟩

Lockstep induction schema for two simultaneous least fixpoints. If the successor step and supremum step of two least fixpoint inflations preserve a relation, then that relation holds of the two respective least fixpoints.

lemma *lfp-lockstep-induct* [*case-names monof monog step union*]:
fixes *f* :: *'a*::*complete-lattice* $\Rightarrow$ *'a*
and *g* :: *'b*::*complete-lattice* $\Rightarrow$ *'b*
assumes *monof*: *mono f*

and *monog*: *mono g*
and *R-step*: $\bigwedge A\ B.\ A \leq lfp(f) \Longrightarrow B \leq lfp(g) \Longrightarrow R\ A\ B \Longrightarrow R\ (f(A))\ (g(B))$
and *R-Union*: $\bigwedge M{::}('a*'b)\ set.\ (\forall\,(A,B){\in}M.\ R\ A\ B) \Longrightarrow R\ (Sup\ (fst\text{-}proj\ M))\ (Sup\ (snd\text{-}proj\ M))$
shows *R* (*lfp f*) (*lfp g*)
⟨*proof*⟩

lemma *sup-eq-all*: $(\bigwedge A.\ (A{\in}M \Longrightarrow f(A){=}g(A)))$
$\Longrightarrow Sup\ \{f(A) \mid A.\ A{\in}M\} = Sup\ \{g(A) \mid A.\ A{\in}M\}$
⟨*proof*⟩

lemma *sup-corr-eq-chain*: $\bigwedge M{::}('a{::}complete\text{-}lattice*'a)\ set.\ (\forall\,(A,B){\in}M.\ f(A){=}g(B))$
$\Longrightarrow (Sup\ \{f(A) \mid A.\ A{\in}fst\text{-}proj\ M\} = Sup\ \{g(B) \mid B.\ B{\in}snd\text{-}proj\ M\})$
⟨*proof*⟩

end
theory *Identifiers*
imports *Complex-Main*
begin

1.1 Identifier Namespace Configuration

Different configurations are possible for the namespace of identifiers. Finite support is the only important aspect of it.

type-synonym *ident* = *char*

The identifier used for the replacement marker in uniform substitutions

abbreviation *dotid*:: *ident*
where *dotid* ≡ *CHR* ''.''

end
theory *Syntax*
imports
Complex-Main
Identifiers
begin

2 Syntax

Defines the syntax of Differential Game Logic as inductively defined data types. https://doi.org/10.1145/2817824 https://doi.org/10.1007/978-3-319-94205-6_15

2.1 Terms

Numeric literals

type-synonym *lit* = *real*

the set of all real variables

abbreviation *allidents*:: *ident set*
  where *allidents* ≡ {*x* | *x*. *True*}

Variables and differential variables

datatype *variable* =
  RVar ident
| *DVar ident*

datatype *trm* =
  Var variable
| *Number lit*
| *Const ident*
| *Func ident trm*
| *Plus trm trm*
| *Times trm trm*
| *Differential trm*

2.2 Formulas and Hybrid Games

datatype *fml* =
  Pred ident trm
| *Geq trm trm*
| *Not fml* (!)
| *And fml fml* (**infixr** &&& *8*)
| *Exists variable fml*
| *Diamond game fml* ((⟨ - ⟩ -) *20*)
and *game* =
  Game ident
| *Assign variable trm* (**infixr** := *20*)
| *Test fml* (*?*)
| *Choice game game* (**infixr** ∪∪ *10*)
| *Compose game game* (**infixr** ;; *8*)
| *Loop game* (-⁎⁎)
| *Dual game* (-^*d*)
| *ODE ident trm*

Derived operators **definition** *Neg* ::*trm* ⇒ *trm*
where *Neg* ϑ = *Times* (*Number* (−*1*)) ϑ

definition *Minus* ::*trm* ⇒ *trm* ⇒ *trm*
where *Minus* ϑ η = *Plus* ϑ (*Neg* η)

definition *Or* :: *fml* ⇒ *fml* ⇒ *fml* (**infixr** || *7*)
where *Or P Q* = *Not* (*And* (*Not P*) (*Not Q*))

definition *Implies* :: *fml* ⇒ *fml* ⇒ *fml* (**infixr** → *10*)

where *Implies P Q = Or Q (Not P)*

definition *Equiv :: fml ⇒ fml ⇒ fml* (**infixr** ↔ *10*)
where *Equiv P Q = Or (And P Q) (And (Not P) (Not Q))*

definition *Forall :: variable ⇒ fml ⇒ fml*
where *Forall x P = Not (Exists x (Not P))*

definition *Equals :: trm ⇒ trm ⇒ fml*
where *Equals ϑ ϑ′ = ((Geq ϑ ϑ′) && (Geq ϑ′ ϑ))*

definition *Greater :: trm ⇒ trm ⇒ fml*
where *Greater ϑ ϑ′ = ((Geq ϑ ϑ′) && (Not (Geq ϑ′ ϑ)))*

Justification: determinacy theorem justifies this equivalent syntactic abbreviation for box modalities from diamond modalities Theorem 3.1 https://doi.org/10.1145/2817824

definition *Box :: game ⇒ fml ⇒ fml* (([[-]]-) *20*)
where *Box α P = Not (Diamond α (Not P))*

definition *TT ::fml*
where *TT = Geq (Number 0) (Number 0)*

definition *FF ::fml*
where *FF = Geq (Number 0) (Number 1)*

definition *Skip ::game*
where *Skip = Test TT*

Inference: premises, then conclusion

type-synonym *inference = fml list ∗ fml*

type-synonym *sequent = fml list ∗ fml list*

Rule: premises, then conclusion

type-synonym *rule = sequent list ∗ sequent*

2.3 Structural Induction

Induction principles for hybrid games owing to their mutually recursive definition with formulas

lemma *game-induct* [*case-names Game Assign ODE Test Choice Compose Loop Dual*]:
(⋀*a*. *P* (*Game a*))
⟹ (⋀*x ϑ*. *P* (*Assign x ϑ*))
⟹ (⋀*x ϑ*. *P* (*ODE x ϑ*))
⟹ (⋀*φ*. *P* (*? φ*))
⟹ (⋀*α β*. *P α* ⟹ *P β* ⟹ *P* (*α* ∪∪ *β*))

$\Longrightarrow (\bigwedge\alpha\ \beta.\ P\ \alpha \Longrightarrow P\ \beta \Longrightarrow P\ (\alpha\ ;;\ \beta))$
$\Longrightarrow (\bigwedge\alpha.\ P\ \alpha \Longrightarrow P\ (\alpha{*}{*}))$
$\Longrightarrow (\bigwedge\alpha.\ P\ \alpha \Longrightarrow P\ (\alpha\hat{}d))$
$\Longrightarrow P\ \alpha$
⟨*proof*⟩

lemma *fml-induct* [*case-names Pred Geq Not And Exists Diamond*]:
$(\bigwedge x\ \vartheta.\ P\ (Pred\ x\ \vartheta))$
$\Longrightarrow (\bigwedge\vartheta\ \eta.\ P\ (Geq\ \vartheta\ \eta))$
$\Longrightarrow (\bigwedge\varphi.\ P\ \varphi \Longrightarrow P\ (Not\ \varphi))$
$\Longrightarrow (\bigwedge\varphi\ \psi.\ P\ \varphi \Longrightarrow P\ \psi \Longrightarrow P\ (And\ \varphi\ \psi))$
$\Longrightarrow (\bigwedge x\ \varphi.\ P\ \varphi \Longrightarrow P\ (Exists\ x\ \varphi))$
$\Longrightarrow (\bigwedge\alpha\ \varphi.\ P\ \varphi \Longrightarrow P\ (Diamond\ \alpha\ \varphi))$
$\Longrightarrow P\ \varphi$
⟨*proof*⟩

the set of all variables

abbreviation *allvars*:: *variable set*
where $allvars \equiv \{x{::}variable.\ True\}$

end
theory *Denotational-Semantics*
imports
HOL−Analysis.Derivative
Syntax
begin

3 Denotational Semantics

Defines the denotational semantics of Differential Game Logic. https://doi.org/10.1145/2817824 https://doi.org/10.1007/978-3-319-94205-6_15

3.1 States

Vector of reals over ident

type-synonym *Rvec* = *variable* ⇒ *real*
type-synonym *state* = *Rvec*

the set of all worlds

definition *worlds*:: *state set*
where $worlds = \{\nu.\ True\}$

the set of all variables

abbreviation *allvars*:: *variable set*
where $allvars \equiv \{x{::}variable.\ True\}$

the set of all real variables

abbreviation *allrvars*:: *variable set*
  where $allrvars \equiv \{RVar\ x \mid x.\ True\}$

the set of all differential variables

abbreviation *alldvars*:: *variable set*
  where $alldvars \equiv \{DVar\ x \mid x.\ True\}$

lemma *ident-finite*: *finite*($\{x$::*ident*. *True*$\}$)
  ⟨*proof*⟩

lemma *allvar-cases*: $allvars = allrvars \cup alldvars$
  ⟨*proof*⟩

lemma *rvar-finite*: *finite allrvars*
  ⟨*proof*⟩

lemma *dvar-finite*: *finite alldvars*
  ⟨*proof*⟩

lemma *allvars-finite* [*simp*]: *finite*(*allvars*)
  ⟨*proof*⟩

definition *Vagree* :: *state* ⇒ *state* ⇒ *variable set* ⇒ *bool*
  where $Vagree\ \nu\ \nu'\ V \equiv (\forall i.\ i{\in}V \longrightarrow \nu(i) = \nu'(i))$

definition *Uvariation* :: *state* ⇒ *state* ⇒ *variable set* ⇒ *bool*
  where $Uvariation\ \nu\ \nu'\ U \equiv (\forall i.\ {}^{\sim}(i{\in}U) \longrightarrow \nu(i) = \nu'(i))$

lemma *Uvariation-Vagree* [*simp*]: $Uvariation\ \nu\ \nu'\ (-V) = Vagree\ \nu\ \nu'\ V$
  ⟨*proof*⟩

lemma *Vagree-refl* [*simp*]: $Vagree\ \nu\ \nu\ V$
  ⟨*proof*⟩

lemma *Vagree-sym*: $Vagree\ \nu\ \nu'\ V = Vagree\ \nu'\ \nu\ V$
  ⟨*proof*⟩

lemma *Vagree-sym-rel* [*sym*]: $Vagree\ \nu\ \nu'\ V \Longrightarrow Vagree\ \nu'\ \nu\ V$
  ⟨*proof*⟩

lemma *Vagree-union* [*trans*]: $Vagree\ \nu\ \nu'\ V \Longrightarrow Vagree\ \nu\ \nu'\ W \Longrightarrow Vagree\ \nu\ \nu'\ (V \cup W)$
  ⟨*proof*⟩

lemma *Vagree-trans* [*trans*]: $Vagree\ \nu\ \nu'\ V \Longrightarrow Vagree\ \nu'\ \nu''\ W \Longrightarrow Vagree\ \nu\ \nu''\ (V \cap W)$

⟨*proof*⟩

lemma *Vagree-antimon* [*simp*]: *Vagree* $\nu\ \nu'\ V \wedge W \subseteq V \longrightarrow$ *Vagree* $\nu\ \nu'\ W$
⟨*proof*⟩

lemma *Vagree-empty* [*simp*]: *Vagree* $\nu\ \nu'\ \{\}$
⟨*proof*⟩

lemma *Uvariation-empty* [*simp*]: *Uvariation* $\nu\ \nu'\ \{\} = (\nu{=}\nu')$
⟨*proof*⟩

lemma *Vagree-univ* [*simp*]: *Vagree* $\nu\ \nu'$ *allvars* $= (\nu{=}\nu')$
⟨*proof*⟩

lemma *Uvariation-univ* [*simp*]: *Uvariation* $\nu\ \nu'$ *allvars*
⟨*proof*⟩

lemma *Vagree-and* [*simp*]: *Vagree* $\nu\ \nu'\ V \wedge$ *Vagree* $\nu\ \nu'\ W \longleftrightarrow$ *Vagree* $\nu\ \nu'\ (V \cup W)$
⟨*proof*⟩

lemma *Vagree-or*: *Vagree* $\nu\ \nu'\ V \vee$ *Vagree* $\nu\ \nu'\ W \longrightarrow$ *Vagree* $\nu\ \nu'\ (V \cap W)$
⟨*proof*⟩

lemma *Uvariation-refl* [*simp*]: *Uvariation* $\nu\ \nu\ V$
⟨*proof*⟩

lemma *Uvariation-sym*: *Uvariation* $\omega\ \nu\ U =$ *Uvariation* $\nu\ \omega\ U$
⟨*proof*⟩

lemma *Uvariation-sym-rel* [*sym*]: *Uvariation* $\omega\ \nu\ U \Longrightarrow$ *Uvariation* $\nu\ \omega\ U$
⟨*proof*⟩

lemma *Uvariation-trans* [*trans*]: *Uvariation* $\omega\ \nu\ U \Longrightarrow$ *Uvariation* $\nu\ \mu\ V \Longrightarrow$ *Uvariation* $\omega\ \mu\ (U \cup V)$
⟨*proof*⟩

lemma *Uvariation-mon* [*simp*]: $V \supseteq U \Longrightarrow$ *Uvariation* $\omega\ \nu\ U \Longrightarrow$ *Uvariation* $\omega\ \nu\ V$
⟨*proof*⟩

3.2 Interpretations

lemma *mon-mono*: *mono* $r = ((\forall X\ Y.\ (X \subseteq Y \longrightarrow r(X) \subseteq r(Y))))$
⟨*proof*⟩

interpretations of symbols in ident

type-synonym *interp-rep* =
$(ident \Rightarrow real) \times (ident \Rightarrow (real \Rightarrow real)) \times (ident \Rightarrow (real \Rightarrow bool)) \times (ident \Rightarrow$

(*state set* ⇒ *state set*))

definition *is-interp* :: *interp-rep* ⇒ *bool*
where *is-interp I* ≡ *case I of* (-, -, -, *G*) ⇒ (∀ *a*. *mono* (*G a*))

typedef *interp* = {*I*:: *interp-rep*. *is-interp I*}
morphisms *raw-interp well-interp*
⟨*proof*⟩

setup-lifting *type-definition-interp*

lift-definition *Consts*::*interp* ⇒ *ident* ⇒ (*real*) **is** λ(*F0*, -, -, -). *F0* ⟨*proof*⟩
lift-definition *Funcs*:: *interp* ⇒ *ident* ⇒ (*real* ⇒ *real*) **is** λ(-, *F*, -, -). *F* ⟨*proof*⟩
lift-definition *Preds*:: *interp* ⇒ *ident* ⇒ (*real* ⇒ *bool*) **is** λ(-, -, *P*, -). *P* ⟨*proof*⟩
lift-definition *Games*:: *interp* ⇒ *ident* ⇒ (*state set* ⇒ *state set*) **is** λ(-, -, -, *G*). *G* ⟨*proof*⟩

make interpretations

lift-definition *mkinterp*:: (*ident* ⇒ *real*) × (*ident* ⇒ (*real* ⇒ *real*)) × (*ident* ⇒ (*real* ⇒ *bool*)) × (*ident* ⇒ (*state set* ⇒ *state set*))
⇒ *interp*
is λ(*C*, *F*, *P*, *G*). *if* ∀ *a*. *mono* (*G a*) *then* (*C*, *F*, *P*, *G*) *else* (*C*, *F*, *P*, λ- -. {})
⟨*proof*⟩

lemma *Consts-mkinterp* [*simp*]: *Consts* (*mkinterp*(*C*,*F*,*P*,*G*)) = *C*
⟨*proof*⟩

lemma *Funcs-mkinterp* [*simp*]: *Funcs* (*mkinterp*(*C*,*F*,*P*,*G*)) = *F*
⟨*proof*⟩

lemma *Preds-mkinterp* [*simp*]: *Preds* (*mkinterp*(*C*,*F*,*P*,*G*)) = *P*
⟨*proof*⟩

lemma *Games-mkinterp* [*simp*]: (⋀*a*. *mono* (*G a*)) ⟹ *Games* (*mkinterp*(*C*,*F*,*P*,*G*)) = *G*
⟨*proof*⟩

lemma *mkinterp-eq* [*iff*]: (*Consts I* = *Consts J* ∧ *Funcs I* = *Funcs J* ∧ *Preds I* = *Preds J* ∧ *Games I* = *Games J*) = (*I*=*J*)
⟨*proof*⟩

lemma [*simp*]: *X*⊆*Y* ⟹ (*Games I a*)(*X*)⊆(*Games I a*)(*Y*)
⟨*proof*⟩

lifting-update *interp.lifting*
lifting-forget *interp.lifting*

3.3 Semantics

Semantic modification *repv* ω x r replaces the value of variable x in the state ω with r

definition *repv* :: *state* ⇒ *variable* ⇒ *real* ⇒ *state*
where *repv* ω x r = *fun-upd* ω x r

lemma *repv-def-correct*: *repv* ω x r = (λy. *if* $x = y$ *then* r *else* $\omega(y)$)
⟨*proof*⟩

lemma *repv-access* [*simp*]: (*repv* ω x r)(y) = (*if* (x=y) *then* r *else* $\omega(y)$)
⟨*proof*⟩

lemma *repv-self* [*simp*]: *repv* ω x ($\omega(x)$) = ω
⟨*proof*⟩

lemma *Vagree-repv*: *Vagree* ω (*repv* ω x d) ($-\{x\}$)
⟨*proof*⟩

lemma *Vagree-repv-self*: *Vagree* ω (*repv* ω x d) $\{x\}$ = (d=$\omega(x)$)
⟨*proof*⟩

lemma *Uvariation-repv*: *Uvariation* ω (*repv* ω x d) $\{x\}$
⟨*proof*⟩

Semantics of Terms **fun** *term-sem* :: *interp* ⇒ *trm* ⇒ (*state* ⇒ *real*)
where
term-sem I (*Var x*) = ($\lambda\omega$. $\omega(x)$)
| *term-sem I* (*Number r*) = ($\lambda\omega$. *r*)
| *term-sem I* (*Const f*) = ($\lambda\omega$. (*Consts I f*))
| *term-sem I* (*Func f* ϑ) = ($\lambda\omega$. (*Funcs I f*)(*term-sem I* ϑ ω))
| *term-sem I* (*Plus* ϑ η) = ($\lambda\omega$. *term-sem I* ϑ ω + *term-sem I* η ω)
| *term-sem I* (*Times* ϑ η) = ($\lambda\omega$. *term-sem I* ϑ ω ∗ *term-sem I* η ω)
| *term-sem I* (*Differential* ϑ) = ($\lambda\omega$. *sum*(λx. ω(*DVar x*)∗*deriv*(λX. *term-sem I* ϑ (*repv* ω (*RVar x*) *X*))(ω(*RVar x*)))(*allidents*))

Solutions of Differential Equations **type-synonym** *solution* = *real* ⇒ *state*

definition *solves-ODE* :: *interp* ⇒ *solution* ⇒ *ident* ⇒ *trm* ⇒ *bool*
where *solves-ODE I F x* ϑ ≡ ($\forall\zeta$::*real*.
Vagree (*F*(0)) (*F*(ζ)) ($-\{$*RVar x*, *DVar x*$\}$)
∧ *F*(ζ)(*DVar x*) = *deriv*(λt. *F*(*t*)(*RVar x*))(ζ)
∧ *F*(ζ)(*DVar x*) = *term-sem I* ϑ (*F*(ζ)))

Semantics of Formulas and Games **fun** *fml-sem* :: *interp* ⇒ *fml* ⇒ (*state set*) **and**
game-sem :: *interp* ⇒ *game* ⇒ (*state set* ⇒ *state set*)

where
fml-sem I (*Pred p* ϑ) = {ω. (*Preds I p*)(*term-sem I* ϑ ω)}
| *fml-sem I* (*Geq* ϑ η) = {ω. *term-sem I* ϑ $\omega \geq$ *term-sem I* η ω}
| *fml-sem I* (*Not* φ) = {ω. $\omega \notin$ *fml-sem I* φ}
| *fml-sem I* (*And* φ ψ) = *fml-sem I* $\varphi \cap$ *fml-sem I* ψ
| *fml-sem I* (*Exists x* φ) = {ω. $\exists$ *r*. (*repv* ω *x r*) $\in$ *fml-sem I* φ}
| *fml-sem I* (*Diamond* α φ) = *game-sem I* α (*fml-sem I* φ)

| *game-sem I* (*Game a*) = (λX. (*Games I a*)(*X*))
| *game-sem I* (*Assign x* ϑ) = (λX. {ω. (*repv* ω *x* (*term-sem I* ϑ ω)) $\in$ *X*})
| *game-sem I* (*Test* φ) = (λX. *fml-sem I* $\varphi \cap$ *X*)
| *game-sem I* (*Choice* α β) = (λX. *game-sem I* α *X* $\cup$ *game-sem I* β *X*)
| *game-sem I* (*Compose* α β) = (λX. *game-sem I* α (*game-sem I* β *X*))
| *game-sem I* (*Loop* α) = (λX. $\bigcap${*Z*. *X* $\cup$ *game-sem I* α *Z* $\subseteq$ *Z*})
| *game-sem I* (*Dual* α) = (λX. $-$(*game-sem I* α ($-$*X*)))
| *game-sem I* (*ODE x* ϑ) = (λX. {ω. $\exists$ *F T*. *Vagree* ω (*F*(*0*)) ($-${*DVar x*}) $\wedge$ *F*(*T*) $\in$ *X* $\wedge$ *solves-ODE I F x* ϑ})

Validity

definition *valid-in* :: *interp* $\Rightarrow$ *fml* $\Rightarrow$ *bool*
where *valid-in I* $\varphi \equiv$ ($\forall\, \omega$. $\omega \in$ *fml-sem I* φ)

definition *valid* :: *fml* $\Rightarrow$ *bool*
where *valid* $\varphi \equiv$ ($\forall\, I . \forall\, \omega$. $\omega \in$ *fml-sem I* φ)

lemma *valid-is-valid-in-all*: *valid* φ = ($\forall\, I$. *valid-in I* φ)
⟨*proof*⟩

definition *locally-sound* :: *inference* $\Rightarrow$ *bool*
where *locally-sound R* $\equiv$
($\forall\, I$. ($\forall\, k$. *0*$\leq$*k* $\longrightarrow$ *k*<*length* (*fst R*) $\longrightarrow$ *valid-in I* (*nth* (*fst R*) *k*)) $\longrightarrow$ *valid-in I* (*snd R*))

definition *sound* :: *inference* $\Rightarrow$ *bool*
where *sound R* $\equiv$
($\forall\, k$. *0*$\leq$*k* $\longrightarrow$ *k*<*length* (*fst R*) $\longrightarrow$ *valid* (*nth* (*fst R*) *k*)) $\longrightarrow$ *valid* (*snd R*)

lemma *locally-sound-is-sound*: *locally-sound R* $\Longrightarrow$ *sound R*
⟨*proof*⟩

3.4 Monotone Semantics

lemma *monotone-Test* [*simp*]: *X*$\subseteq$*Y* $\Longrightarrow$ *game-sem I* (*Test* φ) *X* $\subseteq$ *game-sem I* (*Test* φ) *Y*
⟨*proof*⟩

lemma *monotone* [*simp*]: *X*$\subseteq$*Y* $\Longrightarrow$ *game-sem I* α *X* $\subseteq$ *game-sem I* α *Y*
⟨*proof*⟩

corollary *game-sem-mono* [*simp*]: *mono* ($\lambda X.$ *game-sem* I α X)
⟨*proof*⟩

corollary *game-union*: *game-sem* I α $(X \cup Y) \supseteq$ *game-sem* I α $X \cup$ *game-sem* I α Y
⟨*proof*⟩

lemmas *game-sem-union* = *game-union*

3.5 Fixpoint Semantics Alternative for Loops

lemma *game-sem-loop-fixpoint-mono*: *mono* ($\lambda Z.$ $X \cup$ *game-sem* I α Z)
⟨*proof*⟩

Consequence of Knaster-Tarski Theorem 3.5 of https://doi.org/10.1145/2817824

lemma *game-sem-loop*: *game-sem* I (*Loop* α) = ($\lambda X.$ *lfp*($\lambda Z.$ $X \cup$ *game-sem* I α Z))
⟨*proof*⟩

corollary *game-sem-loop-back*: ($\lambda X.$ *lfp*($\lambda Z.$ $X \cup$ *game-sem* I α Z)) = *game-sem* I (*Loop* α)
⟨*proof*⟩

corollary *game-sem-loop-iterate*: *game-sem* I (*Loop* α) = ($\lambda X.$ $X \cup$ *game-sem* I α (*game-sem* I (*Loop* α) X))
⟨*proof*⟩

corollary *game-sem-loop-unwind*: *game-sem* I (*Loop* α) = ($\lambda X.$ $X \cup$ *game-sem* I (*Compose* α (*Loop* α)) X)
⟨*proof*⟩

corollary *game-sem-loop-unwind-reduce*: ($\lambda X.$ $X \cup$ *game-sem* I (*Compose* α (*Loop* α)) X) = *game-sem* I (*Loop* α)
⟨*proof*⟩

lemmas *lfp-ordinal-induct-set-cases* = *lfp-ordinal-induct-set* [*case-names mono step union*]

lemma *game-loop-induct* [*case-names step union*]:
($\bigwedge Z.$ $Z \subseteq$ *game-sem* I (*Loop* α) $X \Longrightarrow P(Z) \Longrightarrow P(X \cup$ *game-sem* I α Z))
$\Longrightarrow$ ($\bigwedge M.$ ($\forall Z{\in}M.$ $P(Z)$) $\Longrightarrow P($*Sup* M))
$\Longrightarrow P($*game-sem* I (*Loop* α) X)
⟨*proof*⟩

3.6 Some Simple Obvious Observations

lemma *fml-sem-not* [*simp*]: *fml-sem* I (*Not* φ) = $-$*fml-sem* I φ

⟨*proof*⟩

lemma *fml-sem-not-not* [*simp*]: *fml-sem I* (*Not* (*Not* φ)) = *fml-sem I* φ
⟨*proof*⟩

lemma *fml-sem-or* [*simp*]: *fml-sem I* (*Or* φ ψ) = *fml-sem I* φ ∪ *fml-sem I* ψ
⟨*proof*⟩

lemma *fml-sem-implies* [*simp*]: *fml-sem I* (*Implies* φ ψ) = (−*fml-sem I* φ) ∪ *fml-sem I* ψ
⟨*proof*⟩

lemma *TT-valid* [*simp*]: *valid TT*
⟨*proof*⟩

Semantic equivalence of formulas **definition** *fml-equiv*:: *fml* => *fml* => *bool*
where *fml-equiv* φ ψ ≡ (∀ *I*. *fml-sem I* φ = *fml-sem I* ψ)

Substitutionality for Equivalent Formulas

lemma *fml-equiv-subst*: *fml-equiv* φ ψ ⟹ *P* (*fml-sem I* φ) ⟹ *P* (*fml-sem I* ψ)
⟨*proof*⟩

lemma *valid-fml-equiv*: *valid* (φ ↔ ψ) = *fml-equiv* φ ψ
⟨*proof*⟩

lemma *valid-in-equiv*: *valid-in I* (φ ↔ ψ) = ((*fml-sem I* φ) = (*fml-sem I* ψ))
⟨*proof*⟩

lemma *valid-in-impl*: *valid-in I* (φ → ψ) = ((*fml-sem I* φ) ⊆ (*fml-sem I* ψ))
⟨*proof*⟩

lemma *valid-equiv*: *valid* (φ ↔ ψ) = (∀ *I*. *fml-sem I* φ = *fml-sem I* ψ)
⟨*proof*⟩

lemma *valid-impl*: *valid* (φ → ψ) = (∀ *I*. (*fml-sem I* φ) ⊆ (*fml-sem I* ψ))
⟨*proof*⟩

lemma *fml-sem-equals* [*simp*]: (ω ∈ *fml-sem I* (*Equals* ϑ η)) = (*term-sem I* ϑ ω = *term-sem I* η ω)
⟨*proof*⟩

lemma *equiv-refl-valid* [*simp*]: *valid* (φ ↔ φ)
⟨*proof*⟩

lemma *equal-refl-valid* [*simp*]: *valid* (*Equals* ϑ ϑ)
⟨*proof*⟩

lemma *solves-ODE-alt* : *solves-ODE I F x* ϑ ≡ (∀ ζ::*real*.

Vagree $(F(0))$ $(F(\zeta))$ $(-\{RVar\ x,\ DVar\ x\})$
$\wedge$ $F(\zeta)(DVar\ x) = deriv(\lambda t.\ F(t)(RVar\ x))(\zeta)$
$\wedge$ $F(\zeta) \in$ *fml-sem I* (*Equals* (*Var* (*DVar x*)) ϑ))
⟨*proof*⟩

Semantic equivalence of games **definition** *game-equiv*:: *game* => *game* => *bool*
where *game-equiv* α $\beta \equiv (\forall I\ X.$ *game-sem I* α *X* = *game-sem I* β *X*)

Substitutionality for Equivalent Games

lemma *game-equiv-subst*: *game-equiv* α $\beta \Longrightarrow P$ (*game-sem I* α *X*) $\Longrightarrow P$ (*game-sem I* β *X*)
⟨*proof*⟩

lemma *game-equiv-subst-eq*: *game-equiv* α $\beta \Longrightarrow P$ (*game-sem I* α *X*) == *P* (*game-sem I* β *X*)
⟨*proof*⟩

lemma *skip-id* [*simp*]: *game-sem I Skip X* = *X*
⟨*proof*⟩

lemma *loop-iterate-equiv*: *game-equiv* (*Loop* α) (*Choice Skip* (*Compose* α (*Loop* α)))
⟨*proof*⟩

lemma *fml-equiv-valid*: *fml-equiv* φ $\psi \Longrightarrow$ *valid* φ = *valid* ψ
⟨*proof*⟩

lemma *solves-Vagree*: *solves-ODE I F x* $\vartheta \Longrightarrow (\bigwedge\zeta.$ *Vagree* $(F(\zeta))$ $(F(0))$ $(-\{RVar\ x,DVar\ x\}))$
⟨*proof*⟩

lemma *solves-Vagree-trans*: *Uvariation* $(F(0))$ ω *U* $\Longrightarrow$ *solves-ODE I F x* $\vartheta \Longrightarrow$ *Uvariation* $(F(\zeta))$ ω $(U \cup \{RVar\ x,DVar\ x\})$
⟨*proof*⟩

end
theory *Static-Semantics*
imports
Syntax
Denotational-Semantics
begin

4 Static Semantics

4.1 Semantically-defined Static Semantics

Auxiliary notions of projection of winning conditions upward projection: *restrictto X V* is extends X to the states that agree on V with some state in X, so variables outside V can assume arbitrary values.

definition *restrictto* :: *state set* ⇒ *variable set* ⇒ *state set*
where
restrictto X V = {ν. ∃ω. ω∈*X* ∧ *Vagree* ω ν *V*}

downward projection: *selectlike X* ν *V* selects state ν on V in X, so all variables of V are required to remain constant

definition *selectlike* :: *state set* ⇒ *state* ⇒ *variable set* ⇒ *state set*
where
selectlike X ν *V* = {ω∈*X*. *Vagree* ω ν *V*}

Free variables, semantically characterized. Free variables of a term

definition *FVT* :: *trm* ⇒ *variable set*
where
FVT t = {*x*. ∃*I*.∃ν.∃ω. *Vagree* ν ω (−{*x*}) ∧ ¬(*term-sem I t* ν = *term-sem I t* ω)}

Free variables of a formula

definition *FVF* :: *fml* ⇒ *variable set*
where
FVF φ = {*x*. ∃*I*.∃ν.∃ω. *Vagree* ν ω (−{*x*}) ∧ ν ∈ *fml-sem I* φ ∧ ω ∉ *fml-sem I* φ}

Free variables of a hybrid game

definition *FVG* :: *game* ⇒ *variable set*
where
FVG α = {*x*. ∃*I*.∃ν.∃ω.∃*X*. *Vagree* ν ω (−{*x*}) ∧ ν ∈ *game-sem I* α (*restrictto X* (−{*x*})) ∧ ω ∉ *game-sem I* α (*restrictto X* (−{*x*}))}

Bound variables, semantically characterized. Bound variables of a hybrid game

definition *BVG* :: *game* ⇒ *variable set*
where
BVG α = {*x*. ∃*I*.∃ω.∃*X*. ω ∈ *game-sem I* α *X* ∧ ω ∉ *game-sem I* α (*selectlike X* ω {*x*})}

4.2 Simple Observations

lemma *BVG-elem* [*simp*] :(*x*∈*BVG* α) = (∃*I* ω *X*. ω ∈ *game-sem I* α *X* ∧ ω ∉ *game-sem I* α (*selectlike X* ω {*x*}))

⟨*proof*⟩

lemma *nonBVG-rule*: $(\bigwedge I\ \omega\ X.\ (\omega \in \textit{game-sem}\ I\ \alpha\ X) = (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ \{x\})))$
$\Longrightarrow x \notin BVG\ \alpha$
⟨*proof*⟩

lemma *nonBVG-inc-rule*: $(\bigwedge I\ \omega\ X.\ (\omega \in \textit{game-sem}\ I\ \alpha\ X) \Longrightarrow (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ \{x\})))$
$\Longrightarrow x \notin BVG\ \alpha$
⟨*proof*⟩

lemma *FVT-finite*: *finite*$(FVT\ t)$
⟨*proof*⟩
lemma *FVF-finite*: *finite*$(FVF\ e)$
⟨*proof*⟩
lemma *FVG-finite*: *finite*$(FVG\ a)$
⟨*proof*⟩

end
theory *Coincidence*
imports
Lib
Syntax
Denotational-Semantics
Static-Semantics
HOL.Finite-Set
begin

5 Static Semantics Properties

5.1 Auxiliaries

The state interpolating *stateinterpol* $\nu\ \omega\ S$ between ν and ω that is ν on S and ω elsewhere

definition *stateinterpol*:: *state* ⇒ *state* ⇒ *variable set* ⇒ *state*
where
$\textit{stateinterpol}\ \nu\ \omega\ S = (\lambda x.\ \textit{if}\ (x \in S)\ \textit{then}\ \nu(x)\ \textit{else}\ \omega(x))$

definition *statediff*:: *state* ⇒ *state* ⇒ *variable set*
where $\textit{statediff}\ \nu\ \omega = \{x.\ \nu(x) \neq \omega(x)\}$

lemma *nostatediff*: $x \notin \textit{statediff}\ \nu\ \omega \Longrightarrow \nu(x) = \omega(x)$
⟨*proof*⟩

lemma *stateinterpol-empty*: $\textit{stateinterpol}\ \nu\ \omega\ \{\} = \omega$
⟨*proof*⟩

lemma *stateinterpol-left* [*simp*]: $x{\in}S \Longrightarrow (\mathit{stateinterpol}\ \nu\ \omega\ S)(x){=}\nu(x)$
⟨*proof*⟩

lemma *stateinterpol-right* [*simp*]: $x{\notin}S \Longrightarrow (\mathit{stateinterpol}\ \nu\ \omega\ S)(x){=}\omega(x)$
⟨*proof*⟩

lemma *Vagree-stateinterpol* [*simp*]: $\mathit{Vagree}\ (\mathit{stateinterpol}\ \nu\ \omega\ S)\ \nu\ S$
and $\mathit{Vagree}\ (\mathit{stateinterpol}\ \nu\ \omega\ S)\ \omega\ (-S)$
⟨*proof*⟩

lemma *Vagree-ror*: $\mathit{Vagree}\ \nu\ \nu'\ (V{\cap}W) \Longrightarrow (\exists\omega.\ (\mathit{Vagree}\ \nu\ \omega\ V \wedge \mathit{Vagree}\ \omega\ \nu'\ W))$
⟨*proof*⟩

Remark 8 https://doi.org/10.1007/978-3-319-94205-6_15 about simple properties of projections

lemma *restrictto-extends* [*simp*]: $\mathit{restrictto}\ X\ V \supseteq X$
⟨*proof*⟩

lemma *restrictto-compose* [*simp*]: $\mathit{restrictto}\ (\mathit{restrictto}\ X\ V)\ W = \mathit{restrictto}\ X\ (V{\cap}W)$
⟨*proof*⟩

lemma *restrictto-antimon* [*simp*]: $W{\supseteq}V \Longrightarrow \mathit{restrictto}\ X\ W \subseteq \mathit{restrictto}\ X\ V$
⟨*proof*⟩

lemma *restrictto-empty* [*simp*]: $X{\neq}\{\} \Longrightarrow \mathit{restrictto}\ X\ \{\} = \mathit{worlds}$
⟨*proof*⟩

lemma *selectlike-shrinks* [*simp*]: $\mathit{selectlike}\ X\ \nu\ V \subseteq X$
⟨*proof*⟩

lemma *selectlike-compose* [*simp*]: $\mathit{selectlike}\ (\mathit{selectlike}\ X\ \nu\ V)\ \nu\ W = \mathit{selectlike}\ X\ \nu\ (V{\cup}W)$
⟨*proof*⟩

lemma *selectlike-antimon* [*simp*]: $W{\supseteq}V \Longrightarrow \mathit{selectlike}\ X\ \nu\ W \subseteq \mathit{selectlike}\ X\ \nu\ V$
⟨*proof*⟩

lemma *selectlike-empty* [*simp*]: $\mathit{selectlike}\ X\ \nu\ \{\} = X$
⟨*proof*⟩

lemma *selectlike-self* [*simp*]: $(\nu \in \mathit{selectlike}\ X\ \nu\ V) = (\nu{\in}X)$
⟨*proof*⟩

lemma *selectlike-complement* [*simp*]: $\mathit{selectlike}\ (-X)\ \nu\ V \subseteq -\mathit{selectlike}\ X\ \nu\ V$
⟨*proof*⟩

lemma *selectlike-union*: $\mathit{selectlike}\ (X{\cup}Y)\ \nu\ V = \mathit{selectlike}\ X\ \nu\ V \cup \mathit{selectlike}\ Y$

$\nu\ V$
⟨*proof*⟩

lemma *selectlike-Sup*: *selectlike* (*Sup M*) $\nu\ V$ = *Sup* {*selectlike X* $\nu\ V \mid X.\ X \in M$}
⟨*proof*⟩

lemma *selectlike-equal-cond*: (*selectlike X* $\nu\ V$ = *selectlike Y* $\nu\ V$) = ($\forall \mu.$ *Uvariation* $\mu\ \nu\ (-V) \longrightarrow (\mu \in X) = (\mu \in Y)$)
⟨*proof*⟩

lemma *selectlike-equal-cocond*: (*selectlike X* $\nu\ (-V)$ = *selectlike Y* $\nu\ (-V)$) = ($\forall \mu.$ *Uvariation* $\mu\ \nu\ V \longrightarrow (\mu \in X) = (\mu \in Y)$)
⟨*proof*⟩

lemma *selectlike-equal-cocond-rule*: ($\bigwedge \mu.$ *Uvariation* $\mu\ \nu\ (-V) \Longrightarrow (\mu \in X) = (\mu \in Y)$)
$\Longrightarrow$ (*selectlike X* $\nu\ V$ = *selectlike Y* $\nu\ V$)
⟨*proof*⟩

lemma *selectlike-equal-cocond-corule*: ($\bigwedge \mu.$ *Uvariation* $\mu\ \nu\ V \Longrightarrow (\mu \in X) = (\mu \in Y)$)
$\Longrightarrow$ (*selectlike X* $\nu\ (-V)$ = *selectlike Y* $\nu\ (-V)$)
⟨*proof*⟩

lemma *co-selectlike*: $-$(*selectlike X* $\nu\ V$) = $(-X) \cup \{\omega.\ \neg$ *Vagree* $\omega\ \nu\ V\}$
⟨*proof*⟩

lemma *selectlike-co-selectlike*: *selectlike* ($-$(*selectlike X* $\nu\ V$)) $\nu\ V$ = *selectlike* $(-X)\ \nu\ V$
⟨*proof*⟩

lemma *selectlike-Vagree*: *Vagree* $\nu\ \omega\ V \Longrightarrow$ *selectlike X* $\nu\ V$ = *selectlike X* $\omega\ V$
⟨*proof*⟩

lemma *similar-selectlike-mem*: *Vagree* $\nu\ \omega\ V \Longrightarrow (\nu \in$ *selectlike X* $\omega\ V) = (\nu \in X)$
⟨*proof*⟩

lemma *BVG-nonelem* [*simp*] :$(x \notin BVG\ \alpha) = (\forall I\ \omega\ X.\ (\omega \in$ *game-sem I* $\alpha\ X) =$ $(\omega \in$ *game-sem I* α (*selectlike X* ω $\{x\})))$
⟨*proof*⟩

statediff interoperability

lemma *Vagree-statediff* [*simp*]: *Vagree* $\omega\ \omega'\ S \Longrightarrow$ *statediff* $\omega\ \omega' \subseteq -S$
⟨*proof*⟩

lemma *stateinterpol-diff* [*simp*]: *stateinterpol* $\nu\ \omega$ (*statediff* $\nu\ \omega$) = ν
⟨*proof*⟩

lemma *stateinterpol-insert*: *Vagree* (*stateinterpol v w S*) (*stateinterpol v w* (*insert z S*)) $(-\{z\})$

⟨*proof*⟩

lemma *stateinterpol-FVT* [*simp*]: $z \notin FVT(t) \Longrightarrow$ *term-sem I t* (*stateinterpol* ω ω' *S*) = *term-sem I t* (*stateinterpol* ω ω' (*insert z S*))
⟨*proof*⟩

5.2 Coincidence Lemmas

Coincidence for Terms Lemma 10 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *coincidence-term*: *Vagree* ω ω' (*FVT* ϑ) $\Longrightarrow$ *term-sem I* ϑ ω = *term-sem I* ϑ ω'
⟨*proof*⟩

corollary *coincidence-term-cor*: *Uvariation* ω ω' *U* $\Longrightarrow$ (*FVT* ϑ)$\cap U$={} $\Longrightarrow$ *term-sem I* ϑ ω = *term-sem I* ϑ ω'
⟨*proof*⟩

lemma *stateinterpol-FVF* [*simp*]: $z \notin FVF(e) \Longrightarrow$
((*stateinterpol* ω ω' *S*) $\in$ *fml-sem I e* $\longleftrightarrow$ (*stateinterpol* ω ω' (*insert z S*)) $\in$ *fml-sem I e*)
⟨*proof*⟩

Coincidence for Formulas Lemma 11 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *coincidence-formula*: *Vagree* ω ω' (*FVF* φ) $\Longrightarrow$ ($\omega \in$ *fml-sem I* φ $\longleftrightarrow$ $\omega' \in$ *fml-sem I* φ)
⟨*proof*⟩

corollary *coincidence-formula-cor*: *Uvariation* ω ω' *U* $\Longrightarrow$ (*FVF* φ)$\cap U$={} $\Longrightarrow$ ($\omega \in$ *fml-sem I* φ $\longleftrightarrow$ $\omega' \in$ *fml-sem I* φ)
⟨*proof*⟩

Coincidence for Games *Cignorabimus* α *V* is the set of all sets of variables that can be ignored for the coincidence game lemma

definition *Cignorabimus*:: *game* ⇒ *variable set* ⇒ *variable set set*
where
Cignorabimus α *V* = {*M*. $\forall$ *I*.$\forall \omega$.$\forall \omega'$.$\forall$ *X*. (*Vagree* ω ω' (−*M*) ⟶ ($\omega \in$*game-sem I* α (*restrictto X V*)) ⟶ ($\omega' \in$*game-sem I* α (*restrictto X V*)))}

lemma *Cignorabimus-finite* [*simp*]: *finite* (*Cignorabimus* α *V*)
⟨*proof*⟩

lemma *Cignorabimus-equiv* [*simp*]: *Cignorabimus* α V = {M. $\forall I.\forall\omega.\forall\omega'.\forall X.$ (*Vagree* ω ω' ($-M$) $\longrightarrow$ ($\omega\in$*game-sem* I α (*restrictto* X V)) = ($\omega'\in$*game-sem* I α (*restrictto* X V)))}
⟨*proof*⟩

lemma *Cignorabimus-antimon* [*simp*]: $M \in$ *Cignorabimus* α $V \wedge N\subseteq M \Longrightarrow N \in$ *Cignorabimus* α V
⟨*proof*⟩

lemma *coempty*: $-\{\}$=*allvars*
⟨*proof*⟩

lemma *Cignorabimus-empty* [*simp*]: $\{\} \in$ *Cignorabimus* α V
⟨*proof*⟩

Cignorabimus contains nonfree variables

lemma *Cignorabimus-init*: $V\supseteq FVG(\alpha) \Longrightarrow x\notin V \Longrightarrow \{x\}\in$*Cignorabimus* α V
⟨*proof*⟩

Cignorabimus is closed under union

lemma *Cignorabimus-union*: $M\in$*Cignorabimus* α $V \Longrightarrow N\in$*Cignorabimus* α V $\Longrightarrow (M\cup N)\in$*Cignorabimus* α V
⟨*proof*⟩

lemma *powersetup-induct* [*case-names Base Cup*]:
$\bigwedge C.$ ($\bigwedge M.$ $M\in C \Longrightarrow P\ M$) $\Longrightarrow$
($\bigwedge S.$ ($\bigwedge M.$ $M\in S \Longrightarrow P\ M$) $\Longrightarrow P\ (\bigcup S)$) $\Longrightarrow$
$P\ (\bigcup C)$
⟨*proof*⟩

lemma *Union-insert*: $\bigcup$(*insert* x S) = $x\cup\bigcup S$
⟨*proof*⟩

lemma *powerset2up-induct* [*case-names Finite Nonempty Base Cup*]:
(*finite* C) $\Longrightarrow$ ($C\neq\{\}$) $\Longrightarrow$ ($\bigwedge M.$ $M\in C \Longrightarrow P\ M$) $\Longrightarrow$
($\bigwedge M\ N.$ $P\ M \Longrightarrow P\ N \Longrightarrow P\ (M\cup N)$) $\Longrightarrow$
$P\ (\bigcup C)$
⟨*proof*⟩

lemma *Cignorabimus-step*: ($\bigwedge M.$ $M\in S \Longrightarrow M\in$*Cignorabimus* α V) $\Longrightarrow$ ($\bigcup S$)$\in$*Cignorabimus* α V
⟨*proof*⟩

Lemma 12 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *coincidence-game*: *Vagree* ω ω' $V \Longrightarrow V\supseteq FVG(\alpha) \Longrightarrow$ ($\omega \in$ *game-sem* I α (*restrictto* X V)) = ($\omega' \in$ *game-sem* I α (*restrictto* X V))
⟨*proof*⟩

corollary *coincidence-game-cor*: *Uvariation* $\omega\ \omega'\ U \Longrightarrow U \cap FVG(\alpha) = \{\} \Longrightarrow (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{restrictto}\ X\ (-U))) = (\omega' \in \textit{game-sem}\ I\ \alpha\ (\textit{restrictto}\ X\ (-U)))$
⟨*proof*⟩

5.3 Bound Effect Lemmas

Bignorabimus $\alpha\ V$ is the set of all sets of variables that can be ignored for boundeffect

definition *Bignorabimus*:: *game* ⇒ *variable set set*
where
Bignorabimus $\alpha = \{M.\ \forall I.\forall \omega.\forall X.\ \omega \in \textit{game-sem}\ I\ \alpha\ X \longleftrightarrow \omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ M)\}$

lemma *Bignorabimus-finite* [*simp*]: *finite* (*Bignorabimus* α)
⟨*proof*⟩

lemma *Bignorabimus-single* [*simp*]: *game-sem* $I\ \alpha\ (\textit{selectlike}\ X\ \omega\ M) \subseteq \textit{game-sem}\ I\ \alpha\ X$
⟨*proof*⟩

lemma *Bignorabimus-equiv* [*simp*]: *Bignorabimus* $\alpha = \{M.\ \forall I.\forall \omega.\forall X.\ (\omega \in \textit{game-sem}\ I\ \alpha\ X \longrightarrow \omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ M))\}$

⟨*proof*⟩

lemma *Bignorabimus-empty* [*simp*]: $\{\} \in$ *Bignorabimus* α
⟨*proof*⟩

lemma *Bignorabimus-init*: $x \notin BVG(\alpha) \Longrightarrow \{x\} \in$ *Bignorabimus* α
⟨*proof*⟩

Bignorabimus is closed under union

lemma *Bignorabimus-union*: $M \in$ *Bignorabimus* $\alpha \Longrightarrow N \in$ *Bignorabimus* $\alpha \Longrightarrow (M \cup N) \in$ *Bignorabimus* α
⟨*proof*⟩

lemma *Bignorabimus-step*: $(\bigwedge M.\ M \in S \Longrightarrow M \in \textit{Bignorabimus}\ \alpha) \Longrightarrow (\bigcup S) \in \textit{Bignorabimus}\ \alpha$
⟨*proof*⟩

Lemma 13 https://doi.org/10.1007/978-3-319-94205-6_15

theorem *boundeffect*: $(\omega \in \textit{game-sem}\ I\ \alpha\ X) = (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ (-BVG(\alpha))))$
⟨*proof*⟩

corollary *boundeffect-cor*: $V \cap BVG(\alpha) = \{\} \Longrightarrow (\omega \in \textit{game-sem}\ I\ \alpha\ X) = (\omega \in \textit{game-sem}\ I\ \alpha\ (\textit{selectlike}\ X\ \omega\ V))$
⟨*proof*⟩

5.4 Static Analysis Observations

lemma *BVG-equiv*: *game-equiv* α $\beta \Longrightarrow BVG(\alpha) = BVG(\beta)$
⟨*proof*⟩

lemmas *union-or* = *Set.Un-iff*

lemma *not-union-or*: $(x \notin A \cup B) = (x \notin A \wedge x \notin B)$
⟨*proof*⟩

lemma *repv-selectlike-self*: (*repv* ω x $d \in$ *selectlike* X ω $\{x\}$) = ($d{=}\omega(x) \wedge \omega \in X$)
⟨*proof*⟩

lemma *repv-selectlike-other*: $x \neq y \Longrightarrow$ (*repv* ω x $d \in$ *selectlike* X ω $\{y\}$) = (*repv* ω x $d \in X$)
⟨*proof*⟩

lemma *repv-selectlike-other-converse*: $x \neq y \Longrightarrow$ (*repv* ω x $d \in X$) = (*repv* ω x $d \in$ *selectlike* X ω $\{y\}$)
⟨*proof*⟩

lemma *BVG-assign-other*: $x \neq y \Longrightarrow y \notin BVG(Assign\ x\ \vartheta)$
⟨*proof*⟩

lemma *BVG-assign-meta*: ($\bigwedge I\ \omega$. *term-sem* I ϑ $\omega = \omega(x)$) $\Longrightarrow BVG(Assign\ x\ \vartheta) = \{\}$
and *term-sem* I ϑ $\omega \neq \omega(x) \Longrightarrow BVG(Assign\ x\ \vartheta) = \{x\}$

⟨*proof*⟩

lemma *BVG-assign*: $BVG(Assign\ x\ \vartheta) =$ (*if* ($\forall I\ \omega$. *term-sem* I ϑ $\omega = \omega(x)$) *then* $\{\}$ *else* $\{x\}$)
⟨*proof*⟩

lemma *BVG-ODE-other*: $y \neq RVar\ x \Longrightarrow y \neq DVar\ x \Longrightarrow y \notin BVG(ODE\ x\ \vartheta)$

⟨*proof*⟩

This result could be strengthened to a conditional equality based on the RHS values

lemma *BVG-ODE*: $BVG(ODE\ x\ \vartheta) \subseteq \{RVar\ x, DVar\ x\}$
⟨*proof*⟩

lemma *BVG-test*: $BVG(Test\ \varphi) = \{\}$
⟨*proof*⟩

lemma *BVG-choice*: $BVG(Choice\ \alpha\ \beta) \subseteq BVG(\alpha) \cup BVG(\beta)$
⟨*proof*⟩

lemma *select-nonBV*: $x \notin BVG(\alpha) \Longrightarrow selectlike\ (game\text{-}sem\ I\ \alpha\ (selectlike\ X\ \omega\ \{x\}))\ \omega\ \{x\} = selectlike\ (game\text{-}sem\ I\ \alpha\ X)\ \omega\ \{x\}$
⟨*proof*⟩

lemma *BVG-compose*: $BVG(Compose\ \alpha\ \beta) \subseteq BVG(\alpha) \cup BVG(\beta)$

⟨*proof*⟩

The converse inclusion does not hold generally, because $BVG(x := x{+}1; x := x{-}1) = \{\} \neq BVG(x := x{+}1) \cup BVG(x := x{-}1) = \{x\}$

lemma $BVG(Compose\ (Assign\ x\ (Plus\ (Var\ x)\ (Number\ 1)))\ (Assign\ x\ (Plus\ (Var\ x)\ (Number\ (-1)))))$
$\neq BVG(Assign\ x\ (Plus\ (Var\ x)\ (Number\ 1))) \cup BVG(Assign\ x\ (Plus\ (Var\ x)\ (Number\ (-1))))$
⟨*proof*⟩

lemma *BVG-loop*: $BVG(Loop\ \alpha) \subseteq BVG(\alpha)$
⟨*proof*⟩

lemma *BVG-dual*: $BVG(Dual\ \alpha) \subseteq BVG(\alpha)$

⟨*proof*⟩

end
theory *USubst*
imports
Complex-Main
Syntax
Static-Semantics
Coincidence
Denotational-Semantics
begin

6 Uniform Substitution

uniform substitution representation as tuple of partial maps from identifiers to type-compatible replacements.

type-synonym *usubst* =
$(ident \rightharpoonup trm) \times (ident \rightharpoonup trm) \times (ident \rightharpoonup fml) \times (ident \rightharpoonup game)$

abbreviation *SConst*:: *usubst* ⇒ $(ident \rightharpoonup trm)$
where $SConst \equiv (\lambda(F0, \text{-}, \text{-}, \text{-}).\ F0)$

abbreviation *SFuncs*:: *usubst* ⇒ (*ident* ⇀ *trm*)
where *SFuncs* ≡ (λ(-, *F*, -, -). *F*)
abbreviation *SPreds*:: *usubst* ⇒ (*ident* ⇀ *fml*)
where *SPreds* ≡ (λ(-, -, *P*, -). *P*)
abbreviation *SGames*:: *usubst* ⇒ (*ident* ⇀ *game*)
where *SGames* ≡ (λ(-, -, -, *G*). *G*)

crude approximation of size which is enough for termination arguments

definition *usubstsize*:: *usubst* ⇒ *nat*
where *usubstsize* σ = (*if* (*dom* (*SFuncs* σ) = {} ∧ *dom* (*SPreds* σ) = {}) *then 1 else 2*)

dot is some fixed constant function symbol that is reserved for the purposes of the substitution

definition *dot*:: *trm*
  where *dot* = *Const* (*dotid*)

6.1 Strict Mechanism for Handling Substitution Partiality in Isabelle

Optional terms that result from a substitution, either actually a term or just none to indicate that the substitution clashed

type-synonym *trmo* = *trm option*

abbreviation *undeft*:: *trmo* **where** *undeft* ≡ *None*
abbreviation *Aterm*:: *trm* ⇒ *trmo* **where** *Aterm* ≡ *Some*

lemma *undeft-None*: *undeft*=*None* ⟨*proof*⟩
lemma *Aterm-Some*: *Aterm* ϑ=*Some* ϑ ⟨*proof*⟩

lemma *undeft-equiv*: (ϑ≠*undeft*) = (∃ *t*. ϑ=*Aterm t*)
  ⟨*proof*⟩

Plus on defined terms, strict undeft otherwise

fun *Pluso* :: *trmo* ⇒ *trmo* ⇒ *trmo*
where
  Pluso (*Aterm* ϑ) (*Aterm* η) = *Aterm*(*Plus* ϑ η)
| *Pluso undeft* η = *undeft*
| *Pluso* ϑ *undeft* = *undeft*

Times on defined terms, strict undeft otherwise

fun *Timeso* :: *trmo* ⇒ *trmo* ⇒ *trmo*
where
  Timeso (*Aterm* ϑ) (*Aterm* η) = *Aterm*(*Times* ϑ η)
| *Timeso undeft* η = *undeft*
| *Timeso* ϑ *undeft* = *undeft*

fun *Differentialo* :: *trmo* ⇒ *trmo*
where
Differentialo (*Aterm* ϑ) = *Aterm*(*Differential* ϑ)
| *Differentialo undeft* = *undeft*

lemma *Pluso-undef*: (*Pluso* ϑ η = *undeft*) = (ϑ=*undeft* ∨ η=*undeft*) ⟨*proof*⟩
lemma *Timeso-undef*: (*Timeso* ϑ η = *undeft*) = (ϑ=*undeft* ∨ η=*undeft*) ⟨*proof*⟩

lemma *Differentialo-undef*: (*Differentialo* ϑ = *undeft*) = (ϑ=*undeft*) ⟨*proof*⟩

type-synonym *fmlo* = *fml option*

abbreviation *undeff*:: *fmlo* **where** *undeff* ≡ *None*
abbreviation *Afml*:: *fml* ⇒ *fmlo* **where** *Afml* ≡ *Some*

type-synonym *gameo* = *game option*

abbreviation *undefg*:: *gameo* **where** *undefg* ≡ *None*
abbreviation *Agame*:: *game* ⇒ *gameo* **where** *Agame* ≡ *Some*

lemma *undeff-equiv*: (φ≠*undeff*) = (∃ *f*. φ=*Afml f*)
⟨*proof*⟩

lemma *undefg-equiv*: (α≠*undefg*) = (∃ *g*. α=*Agame g*)
⟨*proof*⟩

Geq on defined terms, strict undeft otherwise

fun *Geqo* :: *trmo* ⇒ *trmo* ⇒ *fmlo*
where
Geqo (*Aterm* ϑ) (*Aterm* η) = *Afml*(*Geq* ϑ η)
| *Geqo undeft* η = *undeff*
| *Geqo* ϑ *undeft* = *undeff*

Not on defined formulas, strict undeft otherwise

fun *Noto* :: *fmlo* ⇒ *fmlo*
where
Noto (*Afml* φ) = *Afml*(*Not* φ)
| *Noto undeff* = *undeff*

And on defined formulas, strict undeft otherwise

fun *Ando* :: *fmlo* ⇒ *fmlo* ⇒ *fmlo*
where
Ando (*Afml* φ) (*Afml* ψ) = *Afml*(*And* φ ψ)
| *Ando undeff* ψ = *undeff*
| *Ando* φ *undeff* = *undeff*

Exists on defined formulas, strict undeft otherwise

fun *Existso* :: *variable* ⇒ *fmlo* ⇒ *fmlo*

where
  Existso x (*Afml* φ) = *Afml*(*Exists x* φ)
| *Existso x undeff* = *undeff*

Diamond on defined games/formulas, strict undeft otherwise

fun *Diamondo* :: *gameo* ⇒ *fmlo* ⇒ *fmlo*
where
  Diamondo (*Agame* α) (*Afml* φ) = *Afml*(*Diamond* α φ)
| *Diamondo undefg* φ = *undeff*
| *Diamondo* α *undeff* = *undeff*

lemma *Geqo-undef*: (*Geqo* ϑ η = *undeff*) = (ϑ=*undeft* ∨ η=*undeft*)
  ⟨*proof*⟩
lemma *Noto-undef*: (*Noto* φ = *undeff*) = (φ=*undeff*)
  ⟨*proof*⟩
lemma *Ando-undef*: (*Ando* φ ψ = *undeff*) = (φ=*undeff* ∨ ψ=*undeff*)
  ⟨*proof*⟩
lemma *Existso-undef*: (*Existso x* φ = *undeff*) = (φ=*undeff*)
  ⟨*proof*⟩
lemma *Diamondo-undef*: (*Diamondo* α φ = *undeff*) = (α=*undefg* ∨ φ=*undeff*)
  ⟨*proof*⟩

Assign on defined terms, strict undefg otherwise

fun *Assigno* :: *variable* ⇒ *trmo* ⇒ *gameo*
where
  Assigno x (*Aterm* ϑ) = *Agame*(*Assign x* ϑ)
| *Assigno x undeft* = *undefg*

fun *ODEo* :: *ident* ⇒ *trmo* ⇒ *gameo*
where
  ODEo x (*Aterm* ϑ) = *Agame*(*ODE x* ϑ)
| *ODEo x undeft* = *undefg*

Test on defined formulas, strict undefg otherwise

fun *Testo* :: *fmlo* ⇒ *gameo*
where
  Testo (*Afml* φ) = *Agame*(*Test* φ)
| *Testo undeff* = *undefg*

Choice on defined games, strict undefg otherwise

fun *Choiceo* :: *gameo* ⇒ *gameo* ⇒ *gameo*
where
  Choiceo (*Agame* α) (*Agame* β) = *Agame*(*Choice* α β)
| *Choiceo* α *undefg* = *undefg*
| *Choiceo undefg* β = *undefg*

Compose on defined games, strict undefg otherwise

fun *Composeo* :: *gameo* ⇒ *gameo* ⇒ *gameo*

where
  Composeo (*Agame* α) (*Agame* β) = *Agame*(*Compose* α β)
| *Composeo* α *undefg* = *undefg*
| *Composeo undefg* β = *undefg*

Loop on defined games, strict undefg otherwise

fun *Loopo* :: *gameo* ⇒ *gameo*
where
  Loopo (*Agame* α) = *Agame*(*Loop* α)
| *Loopo undefg* = *undefg*

Dual on defined games, strict undefg otherwise

fun *Dualo* :: *gameo* ⇒ *gameo*
where
  Dualo (*Agame* α) = *Agame*(*Dual* α)
| *Dualo undefg* = *undefg*

lemma *Assigno-undef*: (*Assigno x* ϑ = *undefg*) = (ϑ=*undeft*) ⟨*proof*⟩
lemma *ODEo-undef*: (*ODEo x* ϑ = *undefg*) = (ϑ=*undeft*) ⟨*proof*⟩
lemma *Testo-undef*: (*Testo* φ = *undefg*) = (φ=*undeff*) ⟨*proof*⟩
lemma *Choiceo-undef*: (*Choiceo* α β = *undefg*) = (α=*undefg* ∨ β=*undefg*) ⟨*proof*⟩
lemma *Composeo-undef*: (*Composeo* α β = *undefg*) = (α=*undefg* ∨ β=*undefg*) ⟨*proof*⟩
lemma *Loopo-undef*: (*Loopo* α = *undefg*) = (α=*undefg*) ⟨*proof*⟩
lemma *Dualo-undef*: (*Dualo* α = *undefg*) = (α=*undefg*) ⟨*proof*⟩

6.2 Recursive Application of One-Pass Uniform Substitution

dotsubstt ϑ is the dot substitution {. ~> ϑ} substituting a term for the . function symbol

definition *dotsubstt*:: *trm* ⇒ *usubst*
  where *dotsubstt* ϑ = (
      (λ*f*. (*if f*=*dotid then* (*Some*(ϑ)) *else None*)),
      (λ-. *None*),
      (λ-. *None*),
      (λ-. *None*)
)

definition *usappconst*:: *usubst* ⇒ *variable set* ⇒ *ident* ⇒ (*trmo*)
where *usappconst* σ *U f* ≡ (*case SConst* σ *f of Some r* ⇒ *if FVT*(*r*)∩*U*={} *then Aterm*(*r*) *else undeft* | *None* ⇒ *Aterm*(*Const f*))

function *usubstappt*:: *usubst* ⇒ *variable set* ⇒ (*trm* ⇒ *trmo*)
where
  usubstappt σ *U* (*Var x*) = *Aterm* (*Var x*)
| *usubstappt* σ *U* (*Number r*) = *Aterm* (*Number r*)
| *usubstappt* σ *U* (*Const f*) = *usappconst* σ *U f*

| *usubstappt* σ U (*Func f* ϑ) =
(*case usubstappt* σ U ϑ *of undeft* ⇒ *undeft*
| *Aterm* $\sigma\vartheta$ ⇒ (*case SFuncs* σ *f of Some r* ⇒ *if FVT*(*r*)∩*U*={}
then usubstappt(*dotsubstt* $\sigma\vartheta$) {} *r else undeft* | *None* ⇒ *Aterm*(*Func f* $\sigma\vartheta$)))
| *usubstappt* σ U (*Plus* ϑ η) = *Pluso* (*usubstappt* σ U ϑ) (*usubstappt* σ U η)
| *usubstappt* σ U (*Times* ϑ η) = *Timeso* (*usubstappt* σ U ϑ) (*usubstappt* σ U η)
| *usubstappt* σ U (*Differential* ϑ) = *Differentialo* (*usubstappt* σ *allvars* ϑ)
⟨*proof*⟩
termination
⟨*proof*⟩

declare *Let-def* [*simp*]

function *usubstappf*:: *usubst* ⇒ *variable set* ⇒ (*fml* ⇒ *fmlo*)
and *usubstappp*:: *usubst* ⇒ *variable set* ⇒ (*game* ⇒ *variable set* × *gameo*)
where
usubstappf σ U (*Pred p* ϑ) =
(*case usubstappt* σ U ϑ *of undeft* ⇒ *undeff*
| *Aterm* $\sigma\vartheta$ ⇒ (*case SPreds* σ *p of Some r* ⇒ *if FVF*(*r*)∩*U*={}
then usubstappf(*dotsubstt* $\sigma\vartheta$) {} *r else undeff* | *None* ⇒ *Afml*(*Pred p* $\sigma\vartheta$)))
| *usubstappf* σ U (*Geq* ϑ η) = *Geqo* (*usubstappt* σ U ϑ) (*usubstappt* σ U η)
| *usubstappf* σ U (*Not* φ) = *Noto* (*usubstappf* σ U φ)
| *usubstappf* σ U (*And* φ ψ) = *Ando* (*usubstappf* σ U φ) (*usubstappf* σ U ψ)
| *usubstappf* σ U (*Exists x* φ) = *Existso x* (*usubstappf* σ (U∪{x}) φ)
| *usubstappf* σ U (*Diamond* α φ) = (*let* $V\alpha$ = *usubstappp* σ U α *in Diamondo*
(*snd* $V\alpha$) (*usubstappf* σ (*fst* $V\alpha$) φ))

| *usubstappp* σ U (*Game a*) =
(*case SGames* σ *a of Some r* ⇒ (U∪*BVG*(*r*),*Agame r*)
| *None* ⇒ (*allvars*,*Agame*(*Game a*)))
| *usubstappp* σ U (*Assign x* ϑ) = (U∪{x}, *Assigno x* (*usubstappt* σ U ϑ))
| *usubstappp* σ U (*Test* φ) = (U, *Testo* (*usubstappf* σ U φ))
| *usubstappp* σ U (*Choice* α β) =
(*let* $V\alpha$ = *usubstappp* σ U α *in*
let $W\beta$ = *usubstappp* σ U β *in*
(*fst* $V\alpha$∪*fst* $W\beta$, *Choiceo* (*snd* $V\alpha$) (*snd* $W\beta$)))
| *usubstappp* σ U (*Compose* α β) =
(*let* $V\alpha$ = *usubstappp* σ U α *in*
let $W\beta$ = *usubstappp* σ (*fst* $V\alpha$) β *in*
(*fst* $W\beta$, *Composeo* (*snd* $V\alpha$) (*snd* $W\beta$)))
| *usubstappp* σ U (*Loop* α) =
(*let* V = *fst*(*usubstappp* σ U α) *in*
(V, *Loopo* (*snd*(*usubstappp* σ V α))))
| *usubstappp* σ U (*Dual* α) =
(*let* $V\alpha$ = *usubstappp* σ U α *in* (*fst* $V\alpha$, *Dualo* (*snd* $V\alpha$)))
| *usubstappp* σ U (*ODE x* ϑ) = (U∪{*RVar x*,*DVar x*}, *ODEo x* (*usubstappt* σ
(U∪{*RVar x*,*DVar x*}) ϑ))
⟨*proof*⟩

termination
⟨*proof*⟩

Induction Principles for Uniform Substitutions

lemmas *usubstappt-induct* = *usubstappt.induct* [*case-names Var Number Const FuncMatch Plus Times Differential*]
lemmas *usubstappfp-induct* = *usubstappf-usubstappp.induct* [*case-names Pred Geq Not And Exists Diamond Game Assign Test Choice Compose Loop Dual ODE*]

Simple Observations for Automation More automation for Case

lemma *usappconst-simp* [*simp*]: *SConst* σ f = *Some* $r \Longrightarrow$ $FVT(r) \cap U = \{\} \Longrightarrow$ *usappconst* σ U f = *Aterm*(r)
and *SConst* σ f = *None* $\Longrightarrow$ *usappconst* σ U f = *Aterm*(*Const* f)
and *SConst* σ f = *Some* $r \Longrightarrow$ $FVT(r) \cap U \neq \{\} \Longrightarrow$ *usappconst* σ U f = *undeft*
⟨*proof*⟩

lemma *usappconst-conv*: *usappconst* σ U $f \neq$ *undeft* $\Longrightarrow$
SConst σ f = *None* $\vee$ $(\exists r.$ *SConst* σ f = *Some* $r \wedge FVT(r) \cap U = \{\})$

⟨*proof*⟩

lemma *usubstappt-const* [*simp*]: *SConst* σ f = *Some* $r \Longrightarrow$ $FVT(r) \cap U = \{\} \Longrightarrow$ *usubstappt* σ U (*Const* f) = *Aterm*(r)
and *SConst* σ f = *None* $\Longrightarrow$ *usubstappt* σ U (*Const* f) = *Aterm*(*Const* f)
and *SConst* σ f = *Some* $r \Longrightarrow$ $FVT(r) \cap U \neq \{\} \Longrightarrow$ *usubstappt* σ U (*Const* f) = *undeft*
⟨*proof*⟩

lemma *usubstappt-const-conv*: *usubstappt* σ U (*Const* f) $\neq$ *undeft* $\Longrightarrow$
SConst σ f = *None* $\vee$ $(\exists r.$ *SConst* σ f = *Some* $r \wedge FVT(r) \cap U = \{\})$
⟨*proof*⟩

lemma *usubstappt-func* [*simp*]: *SFuncs* σ f = *Some* $r \Longrightarrow$ $FVT(r) \cap U = \{\} \Longrightarrow$ *usubstappt* σ U ϑ = *Aterm* $\sigma\vartheta \Longrightarrow$
usubstappt σ U (*Func* f ϑ) = *usubstappt* (*dotsubstt* $\sigma\vartheta$) $\{\}$ r
and *SFuncs* σ f=*None* $\Longrightarrow$ *usubstappt* σ U ϑ = *Aterm* $\sigma\vartheta \Longrightarrow$ *usubstappt* σ U (*Func* f ϑ) = *Aterm*(*Func* f $\sigma\vartheta$)
and *usubstappt* σ U ϑ = *undeft* $\Longrightarrow$ *usubstappt* σ U (*Func* f ϑ) = *undeft*
⟨*proof*⟩

lemma *usubstappt-func2* [*simp*]: *SFuncs* σ f = *Some* $r \Longrightarrow$ $FVT(r) \cap U \neq \{\} \Longrightarrow$ *usubstappt* σ U (*Func* f ϑ) = *undeft*
⟨*proof*⟩

lemma *usubstappt-func-conv*: *usubstappt* σ U (*Func* f ϑ) $\neq$ *undeft* $\Longrightarrow$
usubstappt σ U $\vartheta \neq$ *undeft* $\wedge$
(*SFuncs* σ f = *None* $\vee$ $(\exists r.$ *SFuncs* σ f = *Some* $r \wedge FVT(r) \cap U = \{\})$)
⟨*proof*⟩

lemma *usubstappt-plus-conv*: *usubstappt σ U (Plus ϑ η) ≠ undeft ⟹*
usubstappt σ U ϑ ≠ undeft ∧ usubstappt σ U η ≠ undeft
⟨*proof*⟩

lemma *usubstappt-times-conv*: *usubstappt σ U (Times ϑ η) ≠ undeft ⟹*
usubstappt σ U ϑ ≠ undeft ∧ usubstappt σ U η ≠ undeft
⟨*proof*⟩

lemma *usubstappt-differential-conv*: *usubstappt σ U (Differential ϑ) ≠ undeft ⟹*
usubstappt σ allvars ϑ ≠ undeft
⟨*proof*⟩

lemma *usubstappf-pred* [*simp*]: *SPreds σ p = Some r ⟹ FVF(r)∩U={} ⟹ usubstappt σ U ϑ = Aterm σϑ ⟹*
usubstappf σ U (Pred p ϑ) = usubstappf (dotsubstt σϑ) {} r
and *SPreds σ p = None ⟹ usubstappt σ U ϑ = Aterm σϑ ⟹ usubstappf σ U (Pred p ϑ) = Afml(Pred p σϑ)*
and *usubstappt σ U ϑ = undeft ⟹ usubstappf σ U (Pred p ϑ) = undeff*
⟨*proof*⟩

lemma *usubstappf-pred2* [*simp*]: *SPreds σ p = Some r ⟹ FVF(r)∩U≠{} ⟹ usubstappf σ U (Pred p ϑ) = undeff*
⟨*proof*⟩

lemma *usubstappf-pred-conv*: *usubstappf σ U (Pred p ϑ) ≠ undeff ⟹*
usubstappt σ U ϑ ≠ undeft ∧
(SPreds σ p = None ∨ (∃ r. SPreds σ p = Some r ∧ FVF(r)∩U={}))
⟨*proof*⟩

lemma *usubstappf-geq*: *usubstappt σ U ϑ ≠ undeft ⟹ usubstappt σ U η ≠ undeft ⟹*
usubstappf σ U (Geq ϑ η) = Afml(Geq (the (usubstappt σ U ϑ)) (the (usubstappt σ U η)))
⟨*proof*⟩

lemma *usubstappf-geq-conv*: *usubstappf σ U (Geq ϑ η) ≠ undeff ⟹*
usubstappt σ U ϑ ≠ undeft ∧ usubstappt σ U η ≠ undeft
⟨*proof*⟩

lemma *usubstappf-geqr*: *usubstappf σ U (Geq ϑ η) ≠ undeff ⟹*
usubstappf σ U (Geq ϑ η) = Afml(Geq (the (usubstappt σ U ϑ)) (the (usubstappt σ U η)))
⟨*proof*⟩

lemma *usubstappf-exists*: *usubstappf σ U (Exists x φ) ≠ undeff ⟹*
usubstappf σ U (Exists x φ) = Afml(Exists x (the (usubstappf σ (U∪{x}) φ)))

⟨*proof*⟩

lemma *usubstappp-game* [*simp*]: *SGames* σ *a* = *Some r* ⟹ *usubstappp* σ *U* (*Game a*) = (*U*∪*BVG*(*r*),*Agame*(*r*))
and *SGames* σ *a* = *None* ⟹ *usubstappp* σ *U* (*Game a*) = (*allvars*,*Agame*(*Game a*))
⟨*proof*⟩

lemma *usubstappp-choice* [*simp*]: *usubstappp* σ *U* (*Choice* α β) =
(*fst*(*usubstappp* σ *U* α)∪*fst*(*usubstappp* σ *U* β), *Choiceo* (*snd*(*usubstappp* σ *U* α)) (*snd*(*usubstappp* σ *U* β)))
⟨*proof*⟩

lemma *usubstappp-choice-conv* : *snd*(*usubstappp* σ *U* (*Choice* α β)) ≠ *undefg* ⟹
snd(*usubstappp* σ *U* α) ≠ *undefg* ∧ *snd*(*usubstappp* σ *U* β) ≠ *undefg*
⟨*proof*⟩

lemma *usubstappp-compose* [*simp*]: *usubstappp* σ *U* (*Compose* α β) =
(*fst*(*usubstappp* σ (*fst*(*usubstappp* σ *U* α)) β), *Composeo* (*snd*(*usubstappp* σ *U* α)) (*snd*(*usubstappp* σ (*fst*(*usubstappp* σ *U* α)) β)))
⟨*proof*⟩

lemma *usubstappp-loop*: *usubstappp* σ *U* (*Loop* α) =
(*fst*(*usubstappp* σ *U* α), *Loopo* (*snd*(*usubstappp* σ (*fst*(*usubstappp* σ *U* α)) α)))
⟨*proof*⟩

lemma *usubstappp-dual* [*simp*]: *usubstappp* σ *U* (*Dual* α) =
(*fst*(*usubstappp* σ *U* α), *Dualo* (*snd* (*usubstappp* σ *U* α)))
⟨*proof*⟩

7 Soundness of Uniform Substitution

7.1 USubst Application is a Function of Deterministic Result

lemma *usubstappt-det*: *usubstappt* σ *U* ϑ ≠ *undeft* ⟹ *usubstappt* σ *V* ϑ ≠ *undeft* ⟹
usubstappt σ *U* ϑ = *usubstappt* σ *V* ϑ
⟨*proof*⟩

lemma *usubstappf-and-usubstappp-det*:
shows *usubstappf* σ *U* φ ≠ *undeff* ⟹ *usubstappf* σ *V* φ ≠ *undeff* ⟹ *usubstappf* σ *U* φ = *usubstappf* σ *V* φ
and *snd*(*usubstappp* σ *U* α) ≠ *undefg* ⟹ *snd*(*usubstappp* σ *V* α) ≠ *undefg* ⟹ *snd*(*usubstappp* σ *U* α) = *snd*(*usubstappp* σ *V* α)
⟨*proof*⟩

lemma *usubstappf-det*: *usubstappf* σ *U* φ ≠ *undeff* ⟹ *usubstappf* σ *V* φ ≠ *undeff* ⟹ *usubstappf* σ *U* φ = *usubstappf* σ *V* φ

⟨*proof*⟩

lemma *usubstappp-det*: $snd(usubstappp\ \sigma\ U\ \alpha) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ V\ \alpha) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) = snd(usubstappp\ \sigma\ V\ \alpha)$
⟨*proof*⟩

7.2 Uniform Substitutions are Antimonotone in Taboos

lemma *usubst-taboos-mon*: $fst(usubstappp\ \sigma\ U\ \alpha) \supseteq U$
⟨*proof*⟩

lemma *fst-pair* [*simp*]: $fst\ (a,b) = a$
⟨*proof*⟩

lemma *snd-pair* [*simp*]: $snd\ (a,b) = b$
⟨*proof*⟩

lemma *usubstappt-antimon*: $V \subseteq U \Longrightarrow usubstappt\ \sigma\ U\ \vartheta \neq undeft \Longrightarrow usubstappt\ \sigma\ U\ \vartheta = usubstappt\ \sigma\ V\ \vartheta$
⟨*proof*⟩

Uniform Substitutions of Games have monotone taboo output

lemma *usubstappp-fst-mon*: $U \subseteq V \Longrightarrow fst(usubstappp\ \sigma\ U\ \alpha) \subseteq fst(usubstappp\ \sigma\ V\ \alpha)$
⟨*proof*⟩

lemma *usubstappf-and-usubstappp-antimon*:
shows $V \subseteq U \Longrightarrow usubstappf\ \sigma\ U\ \varphi \neq undeff \Longrightarrow usubstappf\ \sigma\ U\ \varphi = usubstappf\ \sigma\ V\ \varphi$
and $V \subseteq U \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) = snd(usubstappp\ \sigma\ V\ \alpha)$
⟨*proof*⟩

lemma *usubstappf-antimon*: $V \subseteq U \Longrightarrow usubstappf\ \sigma\ U\ \varphi \neq undeff \Longrightarrow usubstappf\ \sigma\ U\ \varphi = usubstappf\ \sigma\ V\ \varphi$
⟨*proof*⟩

lemma *usubstappp-antimon*: $V \subseteq U \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) = snd(usubstappp\ \sigma\ V\ \alpha)$
⟨*proof*⟩

7.3 Taboo Lemmas

lemma *usubstappp-loop-conv*: $snd\ (usubstappp\ \sigma\ U\ (Loop\ \alpha)) \neq undefg \Longrightarrow snd(usubstappp\ \sigma\ U\ \alpha) \neq undefg \wedge snd(usubstappp\ \sigma\ (fst(usubstappp\ \sigma\ U\ \alpha))\ \alpha) \neq undefg$

⟨*proof*⟩

Lemma 13 of http://arxiv.org/abs/1902.07230

lemma *usubst-taboos*: *snd*(*usubstappp* σ U α)≠*undefg* ⟹ *fst*(*usubstappp* σ U α) ⊇ U ∪ *BVG*(*the* (*snd*(*usubstappp* σ U α)))
⟨*proof*⟩

7.4 Substitution Adjoints

Modified interpretation *repI* I f d replaces the interpretation of constant function f in the interpretation I with d

definition *repc* :: *interp* ⇒ *ident* ⇒ *real* ⇒ *interp*
where *repc* I f d ≡ *mkinterp*((λ*c*. *if* $c = f$ *then* d *else Consts* I c), *Funcs* I, *Preds* I, *Games* I)

lemma *repc-consts* [*simp*]: *Consts* (*repc* I f d) c = (*if* (c=f) *then* d *else Consts* I c)
⟨*proof*⟩
lemma *repc-funcs* [*simp*]: *Funcs* (*repc* I f d) = *Funcs* I
⟨*proof*⟩
lemma *repc-preds* [*simp*]: *Preds* (*repc* I f d) = *Preds* I
⟨*proof*⟩
lemma *repc-games* [*simp*]: *Games* (*repc* I f d) = *Games* I
⟨*proof*⟩

lemma *adjoint-stays-mon*: *mono* (*case SGames* σ a *of None* ⇒ *Games* I a | *Some* r ⇒ λX. *game-sem* I r X)
⟨*proof*⟩

adjoint interpretation *adjoint* σ I ω to σ of interpretation I in state ω

definition *adjoint*:: *usubst* ⇒ (*interp* ⇒ *state* ⇒ *interp*)
where *adjoint* σ I ω = *mkinterp*(
(λf. (*case SConst* σ f *of None* ⇒ *Consts* I f| *Some* r ⇒ *term-sem* I r ω)),
(λf. (*case SFuncs* σ f *of None* ⇒ *Funcs* I f | *Some* r ⇒ λd. *term-sem* (*repc* I *dotid* d) r ω)),
(λp. (*case SPreds* σ p *of None* ⇒ *Preds* I p | *Some* r ⇒ λd. ω∈*fml-sem* (*repc* I *dotid* d) r)),
(λa. (*case SGames* σ a *of None* ⇒ *Games* I a | *Some* r ⇒ λX. *game-sem* I r X))
)

Simple Observations about Adjoints **lemma** *adjoint-consts*: *Consts* (*adjoint* σ I ω) f = *term-sem* I (*case SConst* σ f *of Some* r ⇒ r | *None* ⇒ *Const* f) ω
⟨*proof*⟩

lemma *adjoint-funcs*: *Funcs* (*adjoint* σ I ω) f = (*case SFuncs* σ f *of None* ⇒ *Funcs* I f | *Some* r ⇒ λd. *term-sem* (*repc* I *dotid* d) r ω)
⟨*proof*⟩

lemma *adjoint-funcs-match*: *SFuncs* σ *f=Some r* $\Longrightarrow$ *Funcs* (*adjoint* σ *I* ω) *f* = (λ*d*. *term-sem* (*repc I dotid d*) *r* ω)
⟨*proof*⟩

lemma *adjoint-funcs-skip*: *SFuncs* σ *f=None* $\Longrightarrow$ *Funcs* (*adjoint* σ *I* ω) *f* = *Funcs I f*
⟨*proof*⟩

lemma *adjoint-preds*: *Preds* (*adjoint* σ *I* ω) *p* = (*case SPreds* σ *p of None* ⇒ *Preds I p* | *Some r* ⇒ λ*d*. ω∈*fml-sem* (*repc I dotid d*) *r*)
⟨*proof*⟩

lemma *adjoint-preds-skip*: *SPreds* σ *p=None* $\Longrightarrow$ *Preds* (*adjoint* σ *I* ω) *p* = *Preds I p*
⟨*proof*⟩

lemma *adjoint-preds-match*: *SPreds* σ *p=Some r* $\Longrightarrow$ *Preds* (*adjoint* σ *I* ω) *p* = (λ*d*. ω∈*fml-sem* (*repc I dotid d*) *r*)
⟨*proof*⟩

lemma *adjoint-games* [*simp*]: *Games* (*adjoint* σ *I* ω) *a* = (*case SGames* σ *a of None* ⇒ *Games I a* | *Some r* ⇒ λ*X*. *game-sem I r X*)
⟨*proof*⟩

lemma *adjoint-dotsubstt*: *adjoint* (*dotsubstt* ϑ) *I* ω = *repc I dotid* (*term-sem I* ϑ ω)

⟨*proof*⟩

7.5 Uniform Substitution for Terms

Lemma 15 of http://arxiv.org/abs/1902.07230

theorem *usubst-term*: *Uvariation* ν ω *U* $\Longrightarrow$ *usubstappt* σ *U* ϑ≠*undeft* $\Longrightarrow$
term-sem I (*the* (*usubstappt* σ *U* ϑ)) ν = *term-sem* (*adjoint* σ *I* ω) ϑ ν
⟨*proof*⟩

7.6 Uniform Substitution for Formulas and Games

Separately Prove Crucial Ingredient for the ODE Case of *usubst-fml-game*
lemma *same-ODE-same-sol*:
(⋀ν. *Uvariation* ν (*F*(0)) {*RVar x*,*DVar x*} $\Longrightarrow$ *term-sem I* ϑ ν = *term-sem J* η ν)
$\Longrightarrow$ *solves-ODE I F x* ϑ = *solves-ODE J F x* η
⟨*proof*⟩

lemma *usubst-ode*:
assumes *subdef*: *usubstappt* σ {*RVar x*,*DVar x*} ϑ ≠ *undeft*

shows *solves-ODE I F x (the (usubstappt σ {RVar x,DVar x} ϑ)) = solves-ODE (adjoint σ I (F(0))) F x ϑ*
⟨*proof*⟩

lemma *usubst-ode-ext*:
assumes *uv*: *Uvariation (F(0)) ω (U∪{RVar x,DVar x})*
assumes *subdef*: *usubstappt σ (U∪{RVar x,DVar x}) ϑ ≠ undeft*
shows *solves-ODE I F x (the (usubstappt σ (U∪{RVar x,DVar x}) ϑ)) = solves-ODE (adjoint σ I ω) F x ϑ*

⟨*proof*⟩

lemma *usubst-ode-ext2*:
assumes *subdef*: *usubstappt σ (U∪{RVar x,DVar x}) ϑ ≠ undeft*
assumes *uv*: *Uvariation (F(0)) ω (U∪{RVar x,DVar x})*
shows *solves-ODE I F x (the (usubstappt σ (U∪{RVar x,DVar x}) ϑ)) = solves-ODE (adjoint σ I ω) F x ϑ*
⟨*proof*⟩

Separately Prove the Loop Case of *usubst-fml-game* **lemma** *union-comm*: *A∪B=B∪A*
⟨*proof*⟩

definition *loopfpτ*:: *game ⇒ interp ⇒ (state set ⇒ state set)*
where *loopfpτ α I X = lfp(λZ. X ∪ game-sem I α Z)*

lemma *usubst-game-loop*:
assumes *uv*: *Uvariation ν ω U*
and *IHαrec*: *⋀ν ω X. Uvariation ν ω (fst(usubstappp σ U α)) ⟹ snd (usubstappp σ (fst(usubstappp σ U α)) α)≠undefg ⟹ (ν ∈ game-sem I (the (snd (usubstappp σ (fst(usubstappp σ U α)) α))) X) = (ν ∈ game-sem (adjoint σ I ω) α X)*
shows *snd (usubstappp σ U (Loop α))≠undefg ⟹ (ν ∈ game-sem I (the (snd (usubstappp σ U (Loop α))))) X) = (ν ∈ game-sem (adjoint σ I ω) (Loop α) X)*
⟨*proof*⟩

lemma *usubst-fml-game*:
assumes *vaouter*: *Uvariation ν ω U*
shows *usubstappf σ U φ≠undeff ⟹ (ν ∈ fml-sem I (the (usubstappf σ U φ))) = (ν ∈ fml-sem (adjoint σ I ω) φ)*
and *snd (usubstappp σ U α)≠undefg ⟹ (ν ∈ game-sem I (the (snd (usubstappp σ U α))) X) = (ν ∈ game-sem (adjoint σ I ω) α X)*
⟨*proof*⟩

Lemma 16 of http://arxiv.org/abs/1902.07230

theorem *usubst-fml*: *Uvariation ν ω U ⟹ usubstappf σ U φ ≠ undeff ⟹*

$(\nu \in \mathit{fml\text{-}sem}\ I\ (\mathit{the}\ (\mathit{usubstappf}\ \sigma\ U\ \varphi))) = (\nu \in \mathit{fml\text{-}sem}\ (\mathit{adjoint}\ \sigma\ I\ \omega)\ \varphi)$
⟨*proof*⟩

Lemma 17 of http://arxiv.org/abs/1902.07230

theorem *usubst-game*: $\mathit{Uvariation}\ \nu\ \omega\ U \Longrightarrow \mathit{snd}\ (\mathit{usubstappp}\ \sigma\ U\ \alpha) \neq \mathit{undefg} \Longrightarrow$
$(\nu \in \mathit{game\text{-}sem}\ I\ (\mathit{the}\ (\mathit{snd}\ (\mathit{usubstappp}\ \sigma\ U\ \alpha)))\ X) = (\nu \in \mathit{game\text{-}sem}\ (\mathit{adjoint}\ \sigma\ I\ \omega)\ \alpha\ X)$
⟨*proof*⟩

7.7 Soundness of Uniform Substitution of Formulas

abbreviation *usubsta*:: *usubst* ⇒ *fml* ⇒ *fmlo*
where $\mathit{usubsta}\ \sigma\ \varphi \equiv \mathit{usubstappf}\ \sigma\ \{\}\ \varphi$

Theorem 18 of http://arxiv.org/abs/1902.07230

theorem *usubst-sound*: $\mathit{usubsta}\ \sigma\ \varphi \neq \mathit{undeff} \Longrightarrow \mathit{valid}\ \varphi \Longrightarrow \mathit{valid}\ (\mathit{the}\ (\mathit{usubsta}\ \sigma\ \varphi))$
⟨*proof*⟩

7.8 Soundness of Uniform Substitution of Rules

Uniform Substitution applied to a rule or inference

definition *usubstr*:: *usubst* ⇒ *inference* ⇒ *inference option*
where $\mathit{usubstr}\ \sigma\ R \equiv \mathit{if}\ (\mathit{usubstappf}\ \sigma\ \mathit{allvars}\ (\mathit{snd}\ R) \neq \mathit{undeff} \wedge (\forall \varphi{\in}\mathit{set}\ (\mathit{fst}\ R).\ \mathit{usubstappf}\ \sigma\ \mathit{allvars}\ \varphi \neq \mathit{undeff}))\ \mathit{then}$
$\mathit{Some}(\mathit{map}(\mathit{the}\ o\ (\mathit{usubstappf}\ \sigma\ \mathit{allvars}))(\mathit{fst}\ R), \mathit{the}\ (\mathit{usubstappf}\ \sigma\ \mathit{allvars}\ (\mathit{snd}\ R)))$
else
None

Simple observations about applying uniform substitutions to a rule

lemma *usubstr-conv*: $\mathit{usubstr}\ \sigma\ R \neq \mathit{None} \Longrightarrow$
$\mathit{usubstappf}\ \sigma\ \mathit{allvars}\ (\mathit{snd}\ R) \neq \mathit{undeff} \wedge$
$(\forall \varphi{\in}\mathit{set}\ (\mathit{fst}\ R).\ \mathit{usubstappf}\ \sigma\ \mathit{allvars}\ \varphi \neq \mathit{undeff})$
⟨*proof*⟩

lemma *usubstr-union-undef*: $(\mathit{usubstr}\ \sigma\ ((\mathit{append}\ A\ B),\ C) \neq \mathit{None}) = (\mathit{usubstr}\ \sigma\ (A,\ C) \neq \mathit{None} \wedge \mathit{usubstr}\ \sigma\ (B,\ C) \neq \mathit{None})$
⟨*proof*⟩
lemma *usubstr-union-undef2*: $(\mathit{usubstr}\ \sigma\ ((\mathit{append}\ A\ B),\ C) \neq \mathit{None}) \Longrightarrow (\mathit{usubstr}\ \sigma\ (A,\ C) \neq \mathit{None} \wedge \mathit{usubstr}\ \sigma\ (B,\ C) \neq \mathit{None})$
⟨*proof*⟩

lemma *usubstr-cons-undef*: $(\mathit{usubstr}\ \sigma\ ((\mathit{Cons}\ A\ B),\ C) \neq \mathit{None}) = (\mathit{usubstr}\ \sigma\ ([A],\ C) \neq \mathit{None} \wedge \mathit{usubstr}\ \sigma\ (B,\ C) \neq \mathit{None})$
⟨*proof*⟩

lemma *usubstr-cons-undef2*: (*usubstr* σ ((*Cons* *A* *B*), *C*) $\neq$ *None*) $\Longrightarrow$ (*usubstr* σ ([*A*], *C*) $\neq$ *None* $\wedge$ *usubstr* σ (*B*, *C*) $\neq$ *None*)
⟨*proof*⟩

lemma *usubstr-cons*: (*usubstr* σ ((*Cons* *A* *B*), *C*) $\neq$ *None*) $\Longrightarrow$
the (*usubstr* σ ((*Cons* *A* *B*), *C*)) = (*Cons* (*the* (*usubstappf* σ *allvars* *A*)) (*fst* (*the* (*usubstr* σ (*B*, *C*)))), *snd* (*the* (*usubstr* σ ([*A*], *C*))))
⟨*proof*⟩

lemma *usubstr-union*: (*usubstr* σ ((*append* *A* *B*), *C*) $\neq$ *None*) $\Longrightarrow$
the (*usubstr* σ ((*append* *A* *B*), *C*)) = (*append* (*fst* (*the* (*usubstr* σ (*A*, *C*)))) (*fst* (*the* (*usubstr* σ (*B*, *C*)))), *snd* (*the* (*usubstr* σ (*A*, *C*))))
⟨*proof*⟩

lemma *usubstr-length*: *usubstr* σ *R* $\neq$ *None* $\Longrightarrow$ *length* (*fst* (*the* (*usubstr* σ *R*))) = *length* (*fst* *R*)
⟨*proof*⟩

lemma *usubstr-nth*: *usubstr* σ *R* $\neq$ *None* $\Longrightarrow$ 0$\leq$*k* $\Longrightarrow$ *k*<*length* (*fst* *R*) $\Longrightarrow$
nth (*fst* (*the* (*usubstr* σ *R*))) *k* = *the* (*usubstappf* σ *allvars* (*nth* (*fst* *R*) *k*))

⟨*proof*⟩

Theorem 19 of http://arxiv.org/abs/1902.07230

theorem *usubst-rule-sound*: *usubstr* σ *R* $\neq$ *None* $\Longrightarrow$ *locally-sound* *R* $\Longrightarrow$ *locally-sound* (*the* (*usubstr* σ *R*))
⟨*proof*⟩

end
theory *Ids*
imports *Complex-Main*
Syntax
begin

Some specific identifiers used in Axioms

abbreviation *hgid1*::*ident* **where** *hgid1* $\equiv$ *CHR* ''*a*''
abbreviation *hgid2*::*ident* **where** *hgid2* $\equiv$ *CHR* ''*b*''
abbreviation *hgidc*::*ident* **where** *hgidc* $\equiv$ *CHR* ''*c*''
abbreviation *hgidd*::*ident* **where** *hgidd* $\equiv$ *CHR* ''*d*''
abbreviation *pid1*::*ident* **where** *pid1* $\equiv$ *CHR* ''*p*''
abbreviation *pid2*::*ident* **where** *pid2* $\equiv$ *CHR* ''*q*''
abbreviation *fid1*::*ident* **where** *fid1* $\equiv$ *CHR* ''*f*''
abbreviation *xid1*::*variable* **where** *xid1* $\equiv$ *RVar* (*CHR* ''*x*'')
end
theory *Axioms*
imports
Syntax
Denotational-Semantics
Ids

begin

8 Axioms and Axiomatic Proof Rules of Differential Game Logic

8.1 Axioms

abbreviation *pusall*:: *fml*
  where $pusall \equiv \langle Game\ hgidc \rangle\, TT$

abbreviation *nothing*:: *trm*
  where $nothing \equiv Number\ 0$

named-theorems *axiom-defs Axiom definitions*

definition *box-axiom* :: *fml*
  where [*axiom-defs*]:
$box\text{-}axiom \equiv (Box\ (Game\ hgid1)\ pusall) \leftrightarrow Not(Diamond\ (Game\ hgid1)\ (Not(pusall)))$

definition *assigneq-axiom* :: *fml*
  where [*axiom-defs*]:
$assigneq\text{-}axiom \equiv (Diamond\ (Assign\ xid1\ (Const\ fid1))\ pusall) \leftrightarrow Exists\ xid1\ (Equals\ (Var\ xid1)\ (Const\ fid1)\ \&\&\ pusall)$

definition *stutterd-axiom* :: *fml*
  where [*axiom-defs*]:
$stutterd\text{-}axiom \equiv (Diamond\ (Assign\ xid1\ (Var\ xid1))\ pusall) \leftrightarrow pusall$

definition *test-axiom* :: *fml*
  where [*axiom-defs*]:
$test\text{-}axiom \equiv Diamond\ (Test\ (Pred\ pid2\ nothing))\ (Pred\ pid1\ nothing) \leftrightarrow (Pred\ pid2\ nothing\ \&\&\ Pred\ pid1\ nothing)$

definition *choice-axiom* :: *fml*
  where [*axiom-defs*]:
$choice\text{-}axiom \equiv Diamond\ (Choice\ (Game\ hgid1)\ (Game\ hgid2))\ pusall \leftrightarrow (Diamond\ (Game\ hgid1)\ pusall\ ||\ Diamond\ (Game\ hgid2)\ pusall)$

definition *compose-axiom* :: *fml*
  where [*axiom-defs*]:
$compose\text{-}axiom \equiv Diamond\ (Compose\ (Game\ hgid1)\ (Game\ hgid2))\ pusall \leftrightarrow Diamond\ (Game\ hgid1)\ (Diamond\ (Game\ hgid2)\ pusall)$

definition *iterate-axiom* :: *fml*
  where [*axiom-defs*]:
$iterate\text{-}axiom \equiv Diamond\ (Loop\ (Game\ hgid1))\ pusall \leftrightarrow (pusall\ ||\ Diamond\ (Game\ hgid1)\ (Diamond\ (Loop\ (Game\ hgid1))\ pusall))$

definition *dual-axiom* :: *fml*
  where [*axiom-defs*]:
dual-axiom ≡ *Diamond* (*Dual* (*Game hgid1*)) *pusall* ↔ !(*Diamond* (*Game hgid1*) (!*pusall*))

8.2 Axiomatic Rules

named-theorems *rule-defs Rule definitions*

definition *mon-rule* :: *inference*
  where [*rule-defs*]:
mon-rule ≡ ([(⟨*Game hgidc*⟩ *TT*) → (⟨*Game hgidd*⟩ *TT*)], (⟨*Game hgid1*⟩(⟨*Game hgidc*⟩ *TT*)) → (⟨*Game hgid1*⟩(⟨*Game hgidd*⟩ *TT*)))

definition *FP-rule* :: *inference*
  where [*rule-defs*]:
FP-rule ≡ ([((⟨*Game hgidc*⟩ *TT*) || ⟨*Game hgid1*⟩⟨*Game hgidd*⟩ *TT*) → ⟨*Game hgidd*⟩ *TT*], (⟨*Loop* (*Game hgid1*)⟩⟨*Game hgidc*⟩ *TT*) → (⟨*Game hgidd*⟩ *TT*))

definition *MP-rule* :: *inference*
  where [*rule-defs*]:
MP-rule ≡ ([*Pred pid1 nothing* , *Pred pid1 nothing* → *Pred pid2 nothing*], *Pred pid2 nothing*)

definition *gena-rule* :: *inference*
  where [*rule-defs*]:
gena-rule ≡ ([*pusall*], *Exists xid1 pusall*)

8.3 Soundness / Validity Proofs for Axioms

Because an axiom in a uniform substitution calculus is an individual formula, proving the validity of that formula suffices to prove soundness

lemma *box-valid*: *valid box-axiom*
  ⟨*proof*⟩

lemma *assigneq-valid*: *valid assigneq-axiom*
  ⟨*proof*⟩

lemma *stutterd-valid*: *valid stutterd-axiom*
  ⟨*proof*⟩

lemma *test-valid*: *valid test-axiom*
  ⟨*proof*⟩

lemma *choice-valid*: *valid choice-axiom*

⟨*proof*⟩

lemma *compose-valid*: *valid compose-axiom*
⟨*proof*⟩

lemma *dual-valid*: *valid dual-axiom*
⟨*proof*⟩

lemma *iterate-valid*: *valid iterate-axiom*

⟨*proof*⟩

8.4 Local Soundness Proofs for Axiomatic Rules

lemma *mon-locsound*: *locally-sound mon-rule*
⟨*proof*⟩

lemma *FP-locsound*: *locally-sound FP-rule*
⟨*proof*⟩

lemma *MP-locsound*: *locally-sound MP-rule*
⟨*proof*⟩

lemma *gena-locsound*: *locally-sound gena-rule*
⟨*proof*⟩

end

9 dGL Formalization

theory *Differential-Game-Logic*
imports
Complex-Main
Lib
Identifiers
Syntax
Denotational-Semantics
Static-Semantics
Coincidence
USubst
Axioms
begin

This formalization of Differential Game Logic http://arxiv.org/abs/1902.07230 [4] consists of the syntax, denotational semantics, static semantics, uniform substitution lemmas, uniform substitution soundness proofs, and soundness proofs for axioms.

end

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