Derangements

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Abstract

The Derangements Formula describes the number of fixpoint-free permutations as closed-form formula. This theorem is the 88th theorem of the Top 100 Theorems list.

Contents

1 Derangements **1**
1.1 Preliminaries . . . 1
1.1.1 Additions to *HOL.Finite-Set* Theory . . . 1
1.1.2 Additions to *HOL−Combinatorics.Permutations* Theory . . . 2
1.2 Fixpoint-Free Permutations . . . 2
1.3 Properties of Derangements . . . 3
1.4 Construction of Derangements . . . 3
1.5 Cardinality of Derangements . . . 4
1.5.1 Recursive Characterization . . . 4
1.5.2 Closed-Form Characterization . . . 4
1.5.3 Approximation of Cardinality . . . 4

1 Derangements

theory *Derangements*
imports
Complex-Main
HOL−Combinatorics.Permutations
begin

1.1 Preliminaries

1.1.1 Additions to *HOL.Finite-Set* Theory

lemma *card-product-dependent*:
assumes *finite* S $\forall x \in S.$ *finite* $(T\ x)$
shows *card* $\{(x, y).\ x \in S \wedge y \in T\ x\} = (\sum x \in S.\ card\ (T\ x))$
⟨*proof*⟩

1.1.2 Additions to *HOL−Combinatorics.Permutations* Theory

lemma *permutes-imp-bij′*:
  assumes p *permutes* S
  shows *bij* p
⟨*proof*⟩

lemma *permutesE*:
  assumes p *permutes* S
  obtains *bij* p $\forall x.\ x \notin S \longrightarrow p\ x = x$
⟨*proof*⟩

lemma *bij-imp-permutes′*:
  assumes *bij* p $\forall x.\ x \notin A \longrightarrow p\ x = x$
  shows p *permutes* A
⟨*proof*⟩

lemma *permutes-swap*:
  assumes p *permutes* S
  shows *Fun.swap* $x\ y\ p$ *permutes* (*insert* x (*insert* $y\ S$))
  ⟨*proof*⟩

lemma *bij-extends*:
  bij $p \Longrightarrow p\ x = x \Longrightarrow$ *bij* $(p(x := y,\ inv\ p\ y := x))$
⟨*proof*⟩

lemma *permutes-add-one*:
  assumes p *permutes* S $x \notin S$ $y \in S$
  shows $p(x := y,\ inv\ p\ y := x)$ *permutes* (*insert* $x\ S$)
⟨*proof*⟩

lemma *permutations-skip-one*:
  assumes p *permutes* S $x : S$
  shows $p(x := x,\ inv\ p\ x := p\ x)$ *permutes* $(S - \{x\})$
⟨*proof*⟩

lemma *permutes-drop-cycle-size-two*:
  ‹$p \circ$ *Transposition.transpose* x $(p\ x)$ *permutes* $(S - \{x,\ p\ x\})$›
  if ‹p *permutes* S› ‹$p\ (p\ x) = x$›
⟨*proof*⟩

1.2 Fixpoint-Free Permutations

definition *derangements* :: *nat set* ⇒ (*nat* ⇒ *nat*) *set*
where
  derangements $S = \{p.\ p$ *permutes* $S \wedge (\forall x \in S.\ p\ x \neq x)\}$

lemma *derangementsI*:
  assumes p *permutes* S $\bigwedge x.\ x \in S \Longrightarrow p\ x \neq x$
  shows $p \in$ *derangements* S

⟨*proof*⟩

lemma *derangementsE*:
assumes d : *derangements* S
obtains d *permutes* S $\forall x{\in}S.\ d\ x \neq x$
⟨*proof*⟩

1.3 Properties of Derangements

lemma *derangements-inv*:
assumes d: $d \in$ *derangements* S
shows *inv* $d \in$ *derangements* S
⟨*proof*⟩

lemma *derangements-in-image*:
assumes $d \in$ *derangements* A $x \in A$
shows $d\ x \in A$
⟨*proof*⟩

lemma *derangements-in-image-strong*:
assumes $d \in$ *derangements* A $x \in A$
shows $d\ x \in A - \{x\}$
⟨*proof*⟩

lemma *derangements-inverse-in-image*:
assumes $d \in$ *derangements* A $x \in A$
shows *inv* $d\ x \in A$
⟨*proof*⟩

lemma *derangements-fixpoint*:
assumes $d \in$ *derangements* A $x \notin A$
shows $d\ x = x$
⟨*proof*⟩

lemma *derangements-no-fixpoint*:
assumes $d \in$ *derangements* A $x \in A$
shows $d\ x \neq x$
⟨*proof*⟩

lemma *finite-derangements*:
assumes *finite* A
shows *finite* (*derangements* A)
⟨*proof*⟩

1.4 Construction of Derangements

lemma *derangements-empty*[*simp*]:
derangements $\{\} = \{id\}$
⟨*proof*⟩

lemma *derangements-singleton*[*simp*]:
  $derangements\ \{x\} = \{\}$
⟨*proof*⟩

lemma *derangements-swap*:
  assumes $d \in derangements\ S\ x \notin S\ y \notin S\ x \neq y$
  shows $Fun.swap\ x\ y\ d \in derangements\ (insert\ x\ (insert\ y\ S))$
⟨*proof*⟩

lemma *derangements-skip-one*:
  assumes d: $d \in derangements\ S$ **and** $x \in S\ d\ (d\ x) \neq x$
  shows $d(x := x,\ inv\ d\ x := d\ x) \in derangements\ (S - \{x\})$
⟨*proof*⟩

lemma *derangements-add-one*:
  assumes $d \in derangements\ S\ x \notin S\ y \in S$
  shows $d(x := y,\ inv\ d\ y := x) \in derangements\ (insert\ x\ S)$
⟨*proof*⟩

lemma *derangements-drop-minimal-cycle*:
  assumes $d \in derangements\ S\ d\ (d\ x) = x$
  shows $Fun.swap\ x\ (d\ x)\ d \in derangements\ (S - \{x,\ d\ x\})$
⟨*proof*⟩

1.5 Cardinality of Derangements

1.5.1 Recursive Characterization

fun *count-derangements* :: $nat \Rightarrow nat$
where
  $count\text{-}derangements\ 0 = 1$
| $count\text{-}derangements\ (Suc\ 0) = 0$
| $count\text{-}derangements\ (Suc\ (Suc\ n)) = (n + 1) * (count\text{-}derangements\ (Suc\ n) + count\text{-}derangements\ n)$

lemma *card-derangements*:
  assumes $finite\ S\ card\ S = n$
  shows $card\ (derangements\ S) = count\text{-}derangements\ n$
⟨*proof*⟩

1.5.2 Closed-Form Characterization

lemma *count-derangements*:
  $real\ (count\text{-}derangements\ n) = fact\ n * (\sum k \in \{0..n\}.\ (-1)\ \hat{}\ k\ /\ fact\ k)$
⟨*proof*⟩

1.5.3 Approximation of Cardinality

lemma *two-power-fact-le-fact*:
  assumes $n \geq 1$

shows $2\hat{}k * fact\ n \leq (fact\ (n + k) :: 'a :: \{semiring\text{-}char\text{-}0, linordered\text{-}semidom\})$
⟨*proof*⟩

lemma *exp1-approx*:
assumes $n > 0$
shows $exp\ (1{::}real) - (\sum k{<}n.\ 1\ /\ fact\ k) \in \{0..2\ /\ fact\ n\}$
⟨*proof*⟩

lemma *exp1-bounds*: $exp\ 1 \in \{8\ /\ 3..11\ /\ 4 :: real\}$
⟨*proof*⟩

lemma *count-derangements-approximation*:
assumes $n \neq 0$
shows $abs(real\ (count\text{-}derangements\ n) - fact\ n\ /\ exp\ 1) < 1\ /\ 2$
⟨*proof*⟩

theorem *derangements-formula*:
assumes $n \neq 0$ *finite S card S* $= n$
shows $int\ (card\ (derangements\ S)) = round\ (fact\ n\ /\ exp\ 1 :: real)$
⟨*proof*⟩

theorem *derangements-formula'*:
assumes $n \neq 0$ *finite S card S* $= n$
shows $card\ (derangements\ S) = nat\ (round\ (fact\ n\ /\ exp\ 1 :: real))$
⟨*proof*⟩

end

References

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