

Constructing the Reals as Dedekind Cuts of Rationals

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March 24, 2023

Abstract

The type of real numbers is constructed from the positive rationals using the method of Dedekind cuts. This development, briefly described in papers by the authors [1, 2], follows the textbook presentation by Gleason [3]. It's notable that the first formalisation of a significant piece of mathematics, by Jutting [4] in 1977, involved a similar construction.

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Remark. This development was part of the Isabelle distribution from about 1999 to 2022. It has been transferred to the AFP, where it may be more useful.

1 The Reals as Dedekind Sections of Positive Rationals

Fundamentals of Abstract Analysis [Gleason, p. 121] provides some of the definitions.

```
theory Dedekind-Real
imports Complex-Main
begin
```

```
lemma add-eq-exists:  $\exists x. a+x = (b::'a::ab-group-add)$ 
  <proof>
```

1.1 Dedekind cuts or sections

definition

```
cut :: rat set  $\Rightarrow$  bool where
cut A  $\equiv$   $\{ \} \subset A \wedge A \subset \{0<..\} \wedge$ 
   $(\forall y \in A. ((\forall z. 0 < z \wedge z < y \longrightarrow z \in A) \wedge (\exists u \in A. y < u)))$ 
```

lemma *cut-of-rat*:

```
assumes q:  $0 < q$  shows cut  $\{r::rat. 0 < r \wedge r < q\}$  (is cut ?A)
<proof>
```

```
typedef preal = Collect cut
  <proof>
```

lemma *Abs-preal-induct* [*induct type: preal*]:

```
 $(\bigwedge x. \textit{cut } x \Longrightarrow P (\textit{Abs-preal } x)) \Longrightarrow P x$ 
<proof>
```

lemma *cut-Rep-preal* [*simp*]: *cut* (*Rep-preal* *x*)

<proof>

definition

```
psup :: preal set  $\Rightarrow$  preal where
psup P = Abs-preal  $(\bigcup X \in P. \textit{Rep-preal } X)$ 
```

definition

```
add-set :: [rat set, rat set]  $\Rightarrow$  rat set where
add-set A B =  $\{w. \exists x \in A. \exists y \in B. w = x + y\}$ 
```

definition

```
diff-set :: [rat set, rat set]  $\Rightarrow$  rat set where
diff-set A B =  $\{w. \exists x. 0 < w \wedge 0 < x \wedge x \notin B \wedge x + w \in A\}$ 
```

definition

```
mult-set :: [rat set, rat set]  $\Rightarrow$  rat set where
mult-set A B =  $\{w. \exists x \in A. \exists y \in B. w = x * y\}$ 
```

definition

inverse-set :: *rat set* \Rightarrow *rat set* **where**
inverse-set $A \equiv \{x. \exists y. 0 < x \wedge x < y \wedge \text{inverse } y \notin A\}$

instantiation *preal* :: {*ord*, *plus*, *minus*, *times*, *inverse*, *one*}
begin

definition

preal-less-def:
 $r < s \equiv \text{Rep-preal } r < \text{Rep-preal } s$

definition

preal-le-def:
 $r \leq s \equiv \text{Rep-preal } r \subseteq \text{Rep-preal } s$

definition

preal-add-def:
 $r + s \equiv \text{Abs-preal } (\text{add-set } (\text{Rep-preal } r) (\text{Rep-preal } s))$

definition

preal-diff-def:
 $r - s \equiv \text{Abs-preal } (\text{diff-set } (\text{Rep-preal } r) (\text{Rep-preal } s))$

definition

preal-mult-def:
 $r * s \equiv \text{Abs-preal } (\text{mult-set } (\text{Rep-preal } r) (\text{Rep-preal } s))$

definition

preal-inverse-def:
 $\text{inverse } r \equiv \text{Abs-preal } (\text{inverse-set } (\text{Rep-preal } r))$

definition $r \text{ div } s = r * \text{inverse } (s::\text{preal})$

definition

preal-one-def:
 $1 \equiv \text{Abs-preal } \{x. 0 < x \wedge x < 1\}$

instance $\langle \text{proof} \rangle$

end

Reduces equality on abstractions to equality on representatives

declare *Abs-preal-inject* [*simp*]

declare *Abs-preal-inverse* [*simp*]

lemma *rat-mem-preal*: $0 < q \implies \text{cut } \{r::\text{rat}. 0 < r \wedge r < q\}$
 $\langle \text{proof} \rangle$

lemma *preal-nonempty*: $\text{cut } A \implies \exists x \in A. 0 < x$
(proof)

lemma *preal-Ex-mem*: $\text{cut } A \implies \exists x. x \in A$
(proof)

lemma *preal-exists-bound*: $\text{cut } A \implies \exists x. 0 < x \wedge x \notin A$
(proof)

lemma *preal-exists-greater*: $\llbracket \text{cut } A; y \in A \rrbracket \implies \exists u \in A. y < u$
(proof)

lemma *preal-downwards-closed*: $\llbracket \text{cut } A; y \in A; 0 < z; z < y \rrbracket \implies z \in A$
(proof)

Relaxing the final premise

lemma *preal-downwards-closed'*: $\llbracket \text{cut } A; y \in A; 0 < z; z \leq y \rrbracket \implies z \in A$
(proof)

A positive fraction not in a positive real is an upper bound. Gleason p. 122 - Remark (1)

lemma *not-in-preal-ub*:

assumes *A*: $\text{cut } A$
and *notx*: $x \notin A$
and *y*: $y \in A$
and *pos*: $0 < x$
shows $y < x$

(proof)

preal lemmas instantiated to *Rep-preal X*

lemma *mem-Rep-preal-Ex*: $\exists x. x \in \text{Rep-preal } X$

thm *preal-Ex-mem*

(proof)

lemma *Rep-preal-exists-bound*: $\exists x > 0. x \notin \text{Rep-preal } X$

(proof)

lemmas *not-in-Rep-preal-ub* = *not-in-preal-ub* [OF *cut-Rep-preal*]

1.2 Properties of Ordering

instance *preal* :: *order*

(proof)

lemma *preal-imp-pos*: $\llbracket \text{cut } A; r \in A \rrbracket \implies 0 < r$

(proof)

instance *preal* :: *linorder*

(proof)

instantiation *preal* :: *distrib-lattice*
begin

definition
 $(inf :: preal \Rightarrow preal \Rightarrow preal) = min$

definition
 $(sup :: preal \Rightarrow preal \Rightarrow preal) = max$

instance
 $\langle proof \rangle$

end

1.3 Properties of Addition

lemma *preal-add-commute*: $(x::preal) + y = y + x$
 $\langle proof \rangle$

Lemmas for proving that addition of two positive reals gives a positive real

lemma *mem-add-set*:
assumes *cut A cut B*
shows *cut (add-set A B)*
 $\langle proof \rangle$

lemma *preal-add-assoc*: $((x::preal) + y) + z = x + (y + z)$
 $\langle proof \rangle$

instance *preal* :: *ab-semigroup-add*
 $\langle proof \rangle$

1.4 Properties of Multiplication

Proofs essentially same as for addition

lemma *preal-mult-commute*: $(x::preal) * y = y * x$
 $\langle proof \rangle$

Multiplication of two positive reals gives a positive real.

lemma *mem-mult-set*:
assumes *cut A cut B*
shows *cut (mult-set A B)*
 $\langle proof \rangle$

lemma *preal-mult-assoc*: $((x::preal) * y) * z = x * (y * z)$
 $\langle proof \rangle$

instance *preal* :: *ab-semigroup-mult*

<proof>

Positive real 1 is the multiplicative identity element

lemma *preal-mult-1*: $(1::\text{preal}) * z = z$

<proof>

instance *preal* :: *comm-monoid-mult*

<proof>

1.5 Distribution of Multiplication across Addition

lemma *mem-Rep-preal-add-iff*:

$(z \in \text{Rep-preal}(r+s)) = (\exists x \in \text{Rep-preal } r. \exists y \in \text{Rep-preal } s. z = x + y)$

<proof>

lemma *mem-Rep-preal-mult-iff*:

$(z \in \text{Rep-preal}(r*s)) = (\exists x \in \text{Rep-preal } r. \exists y \in \text{Rep-preal } s. z = x * y)$

<proof>

lemma *distrib-subset1*:

$\text{Rep-preal } (w * (x + y)) \subseteq \text{Rep-preal } (w * x + w * y)$

<proof>

lemma *preal-add-mult-distrib-mean*:

assumes *a*: $a \in \text{Rep-preal } w$

and *b*: $b \in \text{Rep-preal } w$

and *d*: $d \in \text{Rep-preal } x$

and *e*: $e \in \text{Rep-preal } y$

shows $\exists c \in \text{Rep-preal } w. a * d + b * e = c * (d + e)$

<proof>

lemma *distrib-subset2*:

$\text{Rep-preal } (w * x + w * y) \subseteq \text{Rep-preal } (w * (x + y))$

<proof>

lemma *preal-add-mult-distrib2*: $(w * ((x::\text{preal}) + y)) = (w * x) + (w * y)$

<proof>

lemma *preal-add-mult-distrib*: $((x::\text{preal}) + y) * w = (x * w) + (y * w)$

<proof>

instance *preal* :: *comm-semiring*

<proof>

1.6 Existence of Inverse, a Positive Real

lemma *mem-inverse-set*:

assumes *cut A* **shows** *cut* (*inverse-set A*)

<proof>

1.7 Gleason's Lemma 9-3.4, page 122

lemma *Gleason9-34-exists:*

assumes A : *cut* A

and $\forall x \in A. x + u \in A$

and $0 \leq z$

shows $\exists b \in A. b + (\text{of-int } z) * u \in A$

<proof>

lemma *Gleason9-34-contr:*

assumes A : *cut* A

shows $\llbracket \forall x \in A. x + u \in A; 0 < u; 0 < y; y \notin A \rrbracket \implies \text{False}$

<proof>

lemma *Gleason9-34:*

assumes *cut* A $0 < u$

shows $\exists r \in A. r + u \notin A$

<proof>

1.8 Gleason's Lemma 9-3.6

lemma *lemma-gleason9-36:*

assumes A : *cut* A

and $x: 1 < x$

shows $\exists r \in A. r * x \notin A$

<proof>

1.9 Existence of Inverse: Part 2

lemma *mem-Rep-preal-inverse-iff:*

$(z \in \text{Rep-preal}(\text{inverse } r)) \longleftrightarrow (0 < z \wedge (\exists y. z < y \wedge \text{inverse } y \notin \text{Rep-preal } r))$

<proof>

lemma *Rep-preal-one:*

$\text{Rep-preal } 1 = \{x. 0 < x \wedge x < 1\}$

<proof>

lemma *subset-inverse-mult-lemma:*

assumes $x\text{pos}: 0 < x$ **and** $x\text{less}: x < 1$

shows $\exists v \ u \ y. 0 < v \wedge v < y \wedge \text{inverse } y \notin \text{Rep-preal } R \wedge$

$u \in \text{Rep-preal } R \wedge x = v * u$

<proof>

lemma *subset-inverse-mult:*

$\text{Rep-preal } 1 \subseteq \text{Rep-preal}(\text{inverse } r * r)$

<proof>

lemma *inverse-mult-subset:* $\text{Rep-preal}(\text{inverse } r * r) \subseteq \text{Rep-preal } 1$

<proof>

lemma *preal-mult-inverse*: $\text{inverse } r * r = (1::\text{preal})$
<proof>

lemma *preal-mult-inverse-right*: $r * \text{inverse } r = (1::\text{preal})$
<proof>

Theorems needing *Gleason9-34*

lemma *Rep-preal-self-subset*: $\text{Rep-preal } (r) \subseteq \text{Rep-preal}(r + s)$
<proof>

lemma *Rep-preal-sum-not-subset*: $\sim \text{Rep-preal } (r + s) \subseteq \text{Rep-preal}(r)$
<proof>

at last, Gleason prop. 9-3.5(iii) page 123

proposition *preal-self-less-add-left*: $(r::\text{preal}) < r + s$
<proof>

1.10 Subtraction for Positive Reals

gleason prop. 9-3.5(iv), page 123: proving $a < b \implies \exists d. a + d = b$. We define the claimed D and show that it is a positive real

lemma *mem-diff-set*:

assumes $r < s$

shows $\text{cut } (\text{diff-set } (\text{Rep-preal } s) (\text{Rep-preal } r))$

<proof>

lemma *mem-Rep-preal-diff-iff*:

$r < s \implies$

$(z \in \text{Rep-preal } (s - r)) \longleftrightarrow$

$(\exists x. 0 < x \wedge 0 < z \wedge x \notin \text{Rep-preal } r \wedge x + z \in \text{Rep-preal } s)$

<proof>

proposition *less-add-left*:

fixes $r::\text{preal}$

assumes $r < s$

shows $r + (s - r) = s$

<proof>

lemma *preal-add-less2-mono1*: $r < (s::\text{preal}) \implies r + t < s + t$
<proof>

lemma *preal-add-less2-mono2*: $r < (s::\text{preal}) \implies t + r < t + s$
<proof>

lemma *preal-add-right-less-cancel*: $r + t < s + t \implies r < (s::\text{preal})$
<proof>

lemma *preal-add-left-less-cancel*: $t + r < t + s \implies r < (s::\text{preal})$

<proof>

lemma *preal-add-less-cancel-left* [*simp*]: $(t + (r::preal) < t + s) \longleftrightarrow (r < s)$
<proof>

lemma *preal-add-less-cancel-right* [*simp*]: $((r::preal) + t < s + t) = (r < s)$
<proof>

lemma *preal-add-le-cancel-left* [*simp*]: $(t + (r::preal) \leq t + s) = (r \leq s)$
<proof>

lemma *preal-add-le-cancel-right* [*simp*]: $((r::preal) + t \leq s + t) = (r \leq s)$
<proof>

lemma *preal-add-right-cancel*: $(r::preal) + t = s + t \implies r = s$
<proof>

lemma *preal-add-left-cancel*: $c + a = c + b \implies a = (b::preal)$
<proof>

instance *preal* :: *linordered-ab-semigroup-add*
<proof>

1.11 Completeness of type *preal*

Prove that supremum is a cut

Part 1 of Dedekind sections definition

lemma *preal-sup*:
assumes *le*: $\bigwedge X. X \in P \implies X \leq Y$ **and** $P \neq \{\}$
shows *cut* $(\bigcup X \in P. \text{Rep-}preal(X))$
<proof>

lemma *preal-psup-le*:
 $\llbracket \bigwedge X. X \in P \implies X \leq Y; x \in P \rrbracket \implies x \leq psup P$
<proof>

lemma *psup-le-ub*: $\llbracket \bigwedge X. X \in P \implies X \leq Y; P \neq \{\} \rrbracket \implies psup P \leq Y$
<proof>

Supremum property

proposition *preal-complete*:
assumes *le*: $\bigwedge X. X \in P \implies X \leq Y$ **and** $P \neq \{\}$
shows $(\exists X \in P. Z < X) \longleftrightarrow (Z < psup P)$ (**is** *?lhs = ?rhs*)
<proof>

1.12 Defining the Reals from the Positive Reals

Here we do quotients the old-fashioned way

definition

$realrel :: ((preal * preal) * (preal * preal)) \text{ set } \mathbf{where}$
 $realrel = \{p. \exists x1\ y1\ x2\ y2. p = ((x1,y1),(x2,y2)) \wedge x1+y2 = x2+y1\}$

definition $Real = UNIV // realrel$

typedef $real = Real$

morphisms $Rep-Real\ Abs-Real$
 $\langle proof \rangle$

This doesn't involve the overloaded "real" function: users don't see it

definition

$real-of-preal :: preal \Rightarrow real \mathbf{where}$
 $real-of-preal\ m = Abs-Real\ (realrel\ \{\{m + 1, 1\}\})$

instantiation $real :: \{zero, one, plus, minus, uminus, times, inverse, ord, abs, sgn\}$

begin

definition

$real-zero-def: 0 = Abs-Real(realrel\ \{\{1, 1\}\})$

definition

$real-one-def: 1 = Abs-Real(realrel\ \{\{1 + 1, 1\}\})$

definition

$real-add-def: z + w =$
 $the\ elem\ (\bigcup (x,y) \in Rep-Real\ z. \bigcup (u,v) \in Rep-Real\ w.$
 $\{ Abs-Real(realrel\ \{\{x+u, y+v\}\}) \})$

definition

$real-minus-def: - r = the\ elem\ (\bigcup (x,y) \in Rep-Real\ r. \{ Abs-Real(realrel\ \{\{y,x\}\}) \})$

definition

$real-diff-def: r - (s::real) = r + - s$

definition

$real-mult-def:$
 $z * w =$
 $the\ elem\ (\bigcup (x,y) \in Rep-Real\ z. \bigcup (u,v) \in Rep-Real\ w.$
 $\{ Abs-Real(realrel\ \{\{x*u + y*v, x*v + y*u\}\}) \})$

definition

$real-inverse-def: inverse\ (r::real) \equiv (THE\ s. (r = 0 \wedge s = 0) \vee s * r = 1)$

definition

$real-divide-def: r\ div\ (s::real) \equiv r * inverse\ s$

definition

real-le-def: $z \leq (w::real) \equiv$
 $(\exists x y u v. x+v \leq u+y \wedge (x,y) \in Rep-Real z \wedge (u,v) \in Rep-Real w)$

definition

real-less-def: $x < (y::real) \equiv x \leq y \wedge x \neq y$

definition

real-abs-def: $|r::real| = (if\ r < 0\ then\ -\ r\ else\ r)$

definition

real-sgn-def: $sgn\ (x::real) = (if\ x=0\ then\ 0\ else\ if\ 0 < x\ then\ 1\ else\ -\ 1)$

instance $\langle proof \rangle$

end

1.13 Equivalence relation over positive reals

lemma *realrel-iff* [*simp*]: $((x1,y1),(x2,y2)) \in realrel = (x1 + y2 = x2 + y1)$
 $\langle proof \rangle$

lemma *preal-trans-lemma*:

assumes $x + y1 = x1 + y$ **and** $x + y2 = x2 + y$

shows $x1 + y2 = x2 + (y1::preal)$

$\langle proof \rangle$

lemma *equiv-realrel*: *equiv UNIV realrel*

$\langle proof \rangle$

Reduces equality of equivalence classes to the *Dedekind-Real.realrel* relation: $(Dedekind-Real.realrel\ \{\{x\} = Dedekind-Real.realrel\ \{\{y\}) = ((x, y) \in Dedekind-Real.realrel)$

lemmas *equiv-realrel-iff* [*simp*] =

eq-equiv-class-iff [*OF equiv-realrel UNIV-I UNIV-I*]

lemma *realrel-in-real* [*simp*]: $realrel\ \{(x,y)\} \in Real$

$\langle proof \rangle$

declare *Abs-Real-inject* [*simp*] *Abs-Real-inverse* [*simp*]

Case analysis on the representation of a real number as an equivalence class of pairs of positive reals.

lemma *eq-Abs-Real* [*case-names Abs-Real, cases type: real*]:

$(\bigwedge x y. z = Abs-Real(realrel\ \{(x,y)\}) \implies P) \implies P$

$\langle proof \rangle$

1.14 Addition and Subtraction

lemma *real-add*:

$Abs-Real (realrel\{\{x,y\}\}) + Abs-Real (realrel\{\{u,v\}\}) =$
 $Abs-Real (realrel\{\{x+u, y+v\}\})$
 <proof>

lemma *real-minus*: $- Abs-Real(realrel\{\{x,y\}\}) = Abs-Real(realrel\{\{y,x\}\})$
 <proof>

instance *real* :: *ab-group-add*
 <proof>

1.15 Multiplication

lemma *real-mult-congruent2-lemma*:
 $!!(x1::preal). \llbracket x1 + y2 = x2 + y1 \rrbracket \implies$
 $x * x1 + y * y1 + (x * y2 + y * x2) =$
 $x * x2 + y * y2 + (x * y1 + y * x1)$
 <proof>

lemma *real-mult-congruent2*:
 $(\lambda p1 p2.$
 $(\lambda(x1,y1). (\lambda(x2,y2).$
 $\{ Abs-Real (realrel\{\{x1*x2 + y1*y2, x1*y2+y1*x2\}\}) \}) p2) p1)$
respects2 realrel
 <proof>

lemma *real-mult*:
 $Abs-Real((realrel\{\{x1,y1\}\})) * Abs-Real((realrel\{\{x2,y2\}\})) =$
 $Abs-Real(realrel\{\{x1*x2+y1*y2,x1*y2+y1*x2\}\})$
 <proof>

lemma *real-mult-commute*: $(z::real) * w = w * z$
 <proof>

lemma *real-mult-assoc*: $((z1::real) * z2) * z3 = z1 * (z2 * z3)$
 <proof>

lemma *real-mult-1*: $(1::real) * z = z$
 <proof>

lemma *real-add-mult-distrib*: $((z1::real) + z2) * w = (z1 * w) + (z2 * w)$
 <proof>

one and zero are distinct

lemma *real-zero-not-eq-one*: $0 \neq (1::real)$
 <proof>

instance *real* :: *comm-ring-1*
 <proof>

1.16 Inverse and Division

lemma *real-zero-iff*: *Abs-Real* (*realrel* “{(x, x)}”) = 0
⟨*proof*⟩

lemma *real-mult-inverse-left-ex*:
assumes $x \neq 0$ obtains $y::real$ where $y*x = 1$
⟨*proof*⟩

lemma *real-mult-inverse-left*:
fixes $x :: real$
assumes $x \neq 0$ shows *inverse* $x * x = 1$
⟨*proof*⟩

1.17 The Real Numbers form a Field

instance *real* :: *field*
⟨*proof*⟩

1.18 The \leq Ordering

lemma *real-le-refl*: $w \leq (w::real)$
⟨*proof*⟩

The arithmetic decision procedure is not set up for type *preal*. This lemma is currently unused, but it could simplify the proofs of the following two lemmas.

lemma *preal-eq-le-imp-le*:
assumes *eq*: $a+b = c+d$ and *le*: $c \leq a$
shows $b \leq (d::preal)$
⟨*proof*⟩

lemma *real-le-lemma*:
assumes *l*: $u1 + v2 \leq u2 + v1$
and $x1 + v1 = u1 + y1$
and $x2 + v2 = u2 + y2$
shows $x1 + y2 \leq x2 + (y1::preal)$
⟨*proof*⟩

lemma *real-le*:
 $Abs-Real(realrel\{\{x1,y1\}\}) \leq Abs-Real(realrel\{\{x2,y2\}\}) \iff x1 + y2 \leq x2 + y1$
⟨*proof*⟩

lemma *real-le-antisym*: $\llbracket z \leq w; w \leq z \rrbracket \implies z = (w::real)$
⟨*proof*⟩

lemma *real-trans-lemma*:
assumes $x + v \leq u + y$

and $u + v' \leq u' + v$
and $x^2 + v^2 = u^2 + y^2$
shows $x + v' \leq u' + (y::\text{preal})$
 $\langle \text{proof} \rangle$

lemma *real-le-trans*: $\llbracket i \leq j; j \leq k \rrbracket \implies i \leq (k::\text{real})$
 $\langle \text{proof} \rangle$

instance *real :: order*
 $\langle \text{proof} \rangle$

instance *real :: linorder*
 $\langle \text{proof} \rangle$

instantiation *real :: distrib-lattice*
begin

definition
 $(\text{inf} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real}) = \text{min}$

definition
 $(\text{sup} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real}) = \text{max}$

instance
 $\langle \text{proof} \rangle$

end

1.19 The Reals Form an Ordered Field

lemma *real-le-eq-diff*: $(x \leq y) \longleftrightarrow (x - y \leq (0::\text{real}))$
 $\langle \text{proof} \rangle$

lemma *real-add-left-mono*:
assumes $le: x \leq y$ **shows** $z + x \leq z + (y::\text{real})$
 $\langle \text{proof} \rangle$

lemma *real-sum-gt-zero-less*: $(0 < s + (-w::\text{real})) \implies (w < s)$
 $\langle \text{proof} \rangle$

lemma *real-less-sum-gt-zero*: $(w < s) \implies (0 < s + (-w::\text{real}))$
 $\langle \text{proof} \rangle$

lemma *real-mult-order*:
fixes $x y::\text{real}$
assumes $0 < x$ $0 < y$
shows $0 < x * y$
 $\langle \text{proof} \rangle$

lemma *real-mult-less-mono2*: $\llbracket (0::\text{real}) < z; x < y \rrbracket \implies z * x < z * y$
 <proof>

instance *real :: linordered-field*
 <proof>

1.20 Completeness of the reals

The function *real-of-preal* requires many proofs, but it seems to be essential for proving completeness of the reals from that of the positive reals.

lemma *real-of-preal-add*:
 $\text{real-of-preal } ((x::\text{preal}) + y) = \text{real-of-preal } x + \text{real-of-preal } y$
 <proof>

lemma *real-of-preal-mult*:
 $\text{real-of-preal } ((x::\text{preal}) * y) = \text{real-of-preal } x * \text{real-of-preal } y$
 <proof>

Gleason prop 9-4.4 p 127

lemma *real-of-preal-trichotomy*:
 $\exists m. (x::\text{real}) = \text{real-of-preal } m \vee x = 0 \vee x = -(\text{real-of-preal } m)$
 <proof>

lemma *real-of-preal-less-iff* [simp]:
 $(\text{real-of-preal } m1 < \text{real-of-preal } m2) = (m1 < m2)$
 <proof>

lemma *real-of-preal-le-iff* [simp]:
 $(\text{real-of-preal } m1 \leq \text{real-of-preal } m2) = (m1 \leq m2)$
 <proof>

lemma *real-of-preal-zero-less* [simp]: $0 < \text{real-of-preal } m$
 <proof>

1.21 Theorems About the Ordering

lemma *real-gt-zero-preal-Ex*: $(0 < x) \longleftrightarrow (\exists y. x = \text{real-of-preal } y)$
 <proof>

1.22 Completeness of Positive Reals

Supremum property for the set of positive reals

Let P be a non-empty set of positive reals, with an upper bound y . Then P has a least upper bound (written S).

FIXME: Can the premise be weakened to $\forall x \in P. x \leq y$?

lemma *posreal-complete*:
assumes *positive-P*: $\forall x \in P. (0::\text{real}) < x$

and *not-empty-P*: $\exists x. x \in P$
and *upper-bound-Ex*: $\exists y. \forall x \in P. x < y$
shows $\exists s. \forall y. (\exists x \in P. y < x) = (y < s)$
 <proof>

1.23 Completeness

lemma *reals-complete*:
fixes $S :: \text{real set}$
assumes *notempty-S*: $\exists X. X \in S$
and *exists-Ub*: *bdd-above S*
shows $\exists x. (\forall s \in S. s \leq x) \wedge (\forall y. (\forall s \in S. s \leq y) \longrightarrow x \leq y)$
 <proof>

1.24 The Archimedean Property of the Reals

theorem *reals-Archimedean*:
fixes $x :: \text{real}$
assumes *x-pos*: $0 < x$
shows $\exists n. \text{inverse (of-nat (Suc n))} < x$
 <proof>

There must be other proofs, e.g. *Suc* of the largest integer in the cut representing x .

lemma *reals-Archimedean2*: $\exists n. (x :: \text{real}) < \text{of-nat (n :: nat)}$
 <proof>

instance *real* :: *archimedean-field*
 <proof>

end

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