

A Proof from THE BOOK: The Partial Fraction Expansion of the Cotangent

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Abstract

In this article, I formalise a proof from THE BOOK [1, Chapter 23]; namely a formula that was called ‘one of the most beautiful formulas involving elementary functions’:

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$

The proof uses Herglotz’s trick to show the real case and analytic continuation for the complex case.

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1 The Partial-Fraction Formula for the Cotangent Function

theory *Cotangent-PFD-Formula*
imports *HOL-Complex-Analysis.Complex-Analysis HOL-Real-Asymp.Real-Asymp*

begin

1.1 Auxiliary lemmas

lemma *uniformly-on-image:*

uniformly-on ($f \text{ ' } A$) $g = \text{filtercomap } (\lambda h. h \circ f)$ (*uniformly-on* A ($g \circ f$))
 $\langle \text{proof} \rangle$

lemma *uniform-limit-image:*

uniform-limit ($f \text{ ' } A$) $g \ h \ F \longleftrightarrow \text{uniform-limit } A$ ($\lambda x \ y. g \ x \ (f \ y)$) ($\lambda x. h \ (f \ x)$) F
 $\langle \text{proof} \rangle$

lemma *Ints-add-iff1* [*simp*]: $x \in \mathbb{Z} \implies x + y \in \mathbb{Z} \longleftrightarrow y \in \mathbb{Z}$

$\langle \text{proof} \rangle$

lemma *Ints-add-iff2* [*simp*]: $y \in \mathbb{Z} \implies x + y \in \mathbb{Z} \longleftrightarrow x \in \mathbb{Z}$

$\langle \text{proof} \rangle$

If a set is discrete (i.e. the difference between any two points is bounded from below), it has no limit points:

lemma *discrete-imp-not-islimgt:*

assumes $e: 0 < e$

and $d: \forall x \in S. \forall y \in S. \text{dist } y \ x < e \longrightarrow y = x$

shows $\neg x \text{ islimgt } S$

$\langle \text{proof} \rangle$

In particular, the integers have no limit point:

lemma *Ints-not-limgt:* $\neg((x :: 'a :: \text{real-normed-algebra-1}) \text{ islimgt } \mathbb{Z})$

$\langle \text{proof} \rangle$

The following lemma allows evaluating telescoping sums of the form

$$\sum_{n=0}^{\infty} (f(n) - f(n+k))$$

where $f(n) \longrightarrow 0$, i.e. where all terms except for the first k are cancelled by later summands.

lemma *sums-long-telescope:*

fixes $f :: \text{nat} \Rightarrow 'a :: \{\text{topological-group-add, topological-comm-monoid-add, ab-group-add}\}$

assumes $\text{lim}: f \longrightarrow 0$

shows $(\lambda n. f \ n - f \ (n + c)) \text{ sums } (\sum k < c. f \ k)$ (**is - sums ?S**)

$\langle \text{proof} \rangle$

1.2 Definition of auxiliary function

The following function is the infinite sum appearing on the right-hand side of the cotangent formula. It can be written either as

$$\sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right)$$

or as

$$2x \sum_{n=1}^{\infty} \frac{1}{x^2 - n^2} .$$

definition *cot-pfd* :: 'a :: {real-normed-field, banach} \Rightarrow 'a **where**
cot-pfd x = $(\sum n. 2 * x / (x^2 - \text{of-nat } (\text{Suc } n)^2))$

The sum in the definition of *cot-pfd* converges uniformly on compact sets. This implies, in particular, that *cot-pfd* is holomorphic (and thus also continuous).

lemma *uniform-limit-cot-pfd-complex*:

assumes $R \geq 0$

shows *uniform-limit* (cball 0 R :: complex set)

$(\lambda N x. \sum n < N. 2 * x / (x^2 - \text{of-nat } (\text{Suc } n)^2))$ *cot-pfd* sequentially

<proof>

lemma *sums-cot-pfd-complex*:

fixes x :: complex

shows $(\lambda n. 2 * x / (x^2 - \text{of-nat } (\text{Suc } n)^2))$ *sums cot-pfd* x

<proof>

lemma *sums-cot-pfd-complex'-aux*:

fixes x :: 'a :: {banach, real-normed-field, field-char-0}

assumes $x \notin \mathbb{Z} - \{0\}$

shows $2 * x / (x^2 - \text{of-nat } (\text{Suc } n)^2) =$

$1 / (x + \text{of-nat } (\text{Suc } n)) + 1 / (x - \text{of-nat } (\text{Suc } n))$

<proof>

lemma *sums-cot-pfd-complex'*:

fixes x :: complex

assumes $x \notin \mathbb{Z} - \{0\}$

shows $(\lambda n. 1 / (x + \text{of-nat } (\text{Suc } n)) + 1 / (x - \text{of-nat } (\text{Suc } n)))$ *sums cot-pfd*

x

<proof>

lemma *summable-cot-pfd-complex*:

fixes x :: complex

shows *summable* $(\lambda n. 2 * x / (x^2 - \text{of-nat } (\text{Suc } n)^2))$

<proof>

lemma *summable-cot-pfd-real*:

fixes $x :: \text{real}$

shows *summable* $(\lambda n. 2 * x / (x \wedge 2 - \text{of-nat } (\text{Suc } n) \wedge 2))$

<proof>

lemma *sums-cot-pfd-real*:

fixes $x :: \text{real}$

shows $(\lambda n. 2 * x / (x \wedge 2 - \text{of-nat } (\text{Suc } n) \wedge 2))$ *sums cot-pfd* x

<proof>

lemma *cot-pfd-complex-of-real [simp]*: *cot-pfd* (*complex-of-real* x) = *of-real* (*cot-pfd* x)

<proof>

lemma *uniform-limit-cot-pfd-real*:

assumes $R \geq 0$

shows *uniform-limit* (*cball* $0 R :: \text{real set}$)

$(\lambda N x. \sum n < N. 2 * x / (x \wedge 2 - \text{of-nat } (\text{Suc } n) \wedge 2))$ *cot-pfd sequentially*

<proof>

1.3 Holomorphicity and continuity

lemma *has-field-derivative-cot-pfd-complex*:

fixes $z :: \text{complex}$

assumes $z: z \in -(\mathbb{Z} - \{0\})$

shows (*cot-pfd has-field-derivative* $(-\text{Polygamma } 1 (1 + z) - \text{Polygamma } 1 (1 - z))$) (*at* z)

<proof>

lemma *has-field-derivative-cot-pfd-complex'* [*derivative-intros*]:

assumes (*g has-field-derivative* g') (*at* x *within* A) **and** $g x \notin \mathbb{Z} - \{0\}$

shows $((\lambda x. \text{cot-pfd } (g x :: \text{complex}))$ *has-field-derivative*

$(-\text{Polygamma } 1 (1 + g x) - \text{Polygamma } 1 (1 - g x)) * g'$) (*at* x *within*

A)

<proof>

lemma *Polygamma-real-conv-complex*: $x \neq 0 \implies \text{Polygamma } n x = \text{Re } (\text{Polygamma } n (\text{of-real } x))$

<proof>

lemma *has-field-derivative-cot-pfd-real* [*derivative-intros*]:

assumes (*g has-field-derivative* g') (*at* x *within* A) **and** $g x \notin \mathbb{Z} - \{0\}$

shows $((\lambda x. \text{cot-pfd } (g x :: \text{real}))$ *has-field-derivative*

$(-\text{Polygamma } 1 (1 + g x) - \text{Polygamma } 1 (1 - g x)) * g'$) (*at* x *within*

A)

<proof>

lemma *holomorphic-on-cot-pfd* [*holomorphic-intros*]:

assumes $A \subseteq -(\mathbb{Z} - \{0\})$

shows *cot-pfd holomorphic-on A*
 ⟨*proof*⟩

lemma *holomorphic-on-cot-pfd'* [*holomorphic-intros*]:
assumes *f holomorphic-on A* $\wedge x. x \in A \implies f x \notin \mathbb{Z} - \{0\}$
shows $(\lambda x. \text{cot-pfd } (f x))$ *holomorphic-on A*
 ⟨*proof*⟩

lemma *continuous-on-cot-pfd-complex* [*continuous-intros*]:
assumes *continuous-on A f* $\wedge z. z \in A \implies f z \notin \mathbb{Z} - \{0\}$
shows *continuous-on A* $(\lambda x. \text{cot-pfd } (f x :: \text{complex}))$
 ⟨*proof*⟩

lemma *continuous-on-cot-pfd-real* [*continuous-intros*]:
assumes *continuous-on A f* $\wedge z. z \in A \implies f z \notin \mathbb{Z} - \{0\}$
shows *continuous-on A* $(\lambda x. \text{cot-pfd } (f x :: \text{real}))$
 ⟨*proof*⟩

1.4 Functional equations

In this section, we will show three few functional equations for the function *cot-pfd*. The first one is trivial; the other two are a bit tedious and not very insightful, so I will not comment on them.

cot-pfd is an odd function:

lemma *cot-pfd-complex-minus* [*simp*]: *cot-pfd* $(-x :: \text{complex}) = -\text{cot-pfd } x$
 ⟨*proof*⟩

lemma *cot-pfd-real-minus* [*simp*]: *cot-pfd* $(-x :: \text{real}) = -\text{cot-pfd } x$
 ⟨*proof*⟩

$(1 :: 'a) / x + \text{cot-pfd } x$ is periodic with period 1:

lemma *cot-pfd-plus-1-complex*:
assumes $x \notin \mathbb{Z}$
shows *cot-pfd* $(x + 1 :: \text{complex}) = \text{cot-pfd } x - 1 / (x + 1) + 1 / x$
 ⟨*proof*⟩

lemma *cot-pfd-plus-1-real*:
assumes $x \notin \mathbb{Z}$
shows *cot-pfd* $(x + 1 :: \text{real}) = \text{cot-pfd } x - 1 / (x + 1) + 1 / x$
 ⟨*proof*⟩

cot-pfd satisfies the following functional equation:

$$2f(x) = f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) + \frac{2}{x+1}$$

lemma *cot-pfd-funeq-complex*:

fixes $x :: \text{complex}$
assumes $x \notin \mathbb{Z}$
shows $2 * \text{cot-pfd } x = \text{cot-pfd } (x / 2) + \text{cot-pfd } ((x + 1) / 2) + 2 / (x + 1)$
 $\langle \text{proof} \rangle$

lemma *cot-pfd-funeq-real*:
fixes $x :: \text{real}$
assumes $x \notin \mathbb{Z}$
shows $2 * \text{cot-pfd } x = \text{cot-pfd } (x / 2) + \text{cot-pfd } ((x + 1) / 2) + 2 / (x + 1)$
 $\langle \text{proof} \rangle$

1.5 The limit at 0

lemma *cot-pfd-real-tendsto-0*: $\text{cot-pfd } -0 \rightarrow (0 :: \text{real})$
 $\langle \text{proof} \rangle$

1.6 Final result

To show the final result, we first prove the real case using Herglotz's trick, following the presentation in 'Proofs from THE BOOK'. [1, Chapter 23].

lemma *cot-pfd-formula-real*:
assumes $x \notin \mathbb{Z}$
shows $\pi * \cot (\pi * x) = 1 / x + \text{cot-pfd } x$
 $\langle \text{proof} \rangle$

We now lift the result from the domain $\mathbb{R} \setminus \mathbb{Z}$ to $\mathbb{C} \setminus \mathbb{Z}$. We do this by noting that $\mathbb{C} \setminus \mathbb{Z}$ is connected and the point $\frac{1}{2}$ is both in $\mathbb{C} \setminus \mathbb{Z}$ and a limit point of $\mathbb{R} \setminus \mathbb{Z}$.

lemma *one-half-limit-point-Reals-minus-Ints*: $(1 / 2 :: \text{complex}) \text{ islimpt } \mathbb{R} - \mathbb{Z}$
 $\langle \text{proof} \rangle$

theorem *cot-pfd-formula-complex*:
fixes $z :: \text{complex}$
assumes $z \notin \mathbb{Z}$
shows $\pi * \cot (\pi * z) = 1 / z + \text{cot-pfd } z$
 $\langle \text{proof} \rangle$

end

References

- [1] M. Aigner and G. M. Ziegler. *Proofs from THE BOOK*. Springer, 4th edition, 2009.