

Light-Weight Containers

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Abstract

This development provides a framework for container types like sets and maps such that generated code implements these containers with different (efficient) data structures. Thanks to type classes and refinement during code generation, this light-weight approach can seamlessly replace Isabelle's default setup for code generation. Heuristics automatically pick one of the available data structures depending on the type of elements to be stored, but users can also choose on their own. The extensible design permits to add more implementations at any time.

To support arbitrary nesting of sets, we define a linear order on sets based on a linear order of the elements and provide efficient implementations. It even allows to compare complements with non-complements.

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Chapter 1

Introduction

This development focuses on generating efficient code for container types like sets and maps. It falls into two parts: First, we define linear order on sets (Ch. 2) that is efficiently executable given a linear order on the elements. Second, we define an extensible framework LC (for light-weight containers) that supports multiple (efficient) implementations of container types (Ch. 3) in generated code. Both parts heavily exploit type classes and the refinement features of the code generator [2]. This way, we are able to implement the HOL types for sets and maps directly, as the name light-weight containers (LC) emphasises.

In comparison with the Isabelle Collections Framework (ICF) [4, 3], the style of refinement is the major difference. In the ICF, the container types are replaced with the types of the data structures inside the logic. Typically, the user has to define his operations that involve maps and sets a second time such that they work on the concrete data structures; then, she has to prove that both definitions agree. With LC, the refinement happens inside the code generator. Hence, the formalisation can stick with the types *'a set* and *('a, 'b) mapping* and there is no need to duplicate definitions or prove refinement. The drawback is that with LC, we can only implement operations that can be fully specified on the abstract container type. In particular, the internal representation of the implementations may not affect the result of the operations. For example, it is not possible to pick non-deterministically an element from a set or fold a set with a non-commutative operation, i.e., the result depends on the order of visiting the elements.

For more documentation and introductory material refer to the userguide (Chapter 4) and the ITP-2013 paper [5].

```
theory Containers-Auxiliary imports  
  HOL-Library.Monad-Syntax  
begin
```


Chapter 2

An executable linear order on sets

2.1 Auxiliary definitions

lemma *insert-bind-set*: $\text{insert } a \ A \ggg f = f \ a \cup (A \ggg f)$
<proof>

lemma *set-bind-iff*:
 $\text{set } (\text{List.bind } xs \ f) = \text{Set.bind } (\text{set } xs) (\text{set } \circ f)$
<proof>

lemma *set-bind-conv-fold*: $\text{set } xs \ggg f = \text{fold } ((\cup) \circ f) \ xs \ \{\}$
<proof>

lemma *card-gt-1D*:
assumes $\text{card } A > 1$
shows $\exists x \ y. \ x \in A \wedge y \in A \wedge x \neq y$
<proof>

lemma *card-eq-1-iff*: $\text{card } A = 1 \longleftrightarrow (\exists x. \ A = \{x\})$
<proof>

lemma *card-eq-Suc-0-ex1*: $\text{card } A = \text{Suc } 0 \longleftrightarrow (\exists! x. \ x \in A)$
<proof>

context *linorder* **begin**

lemma *sorted-last*: $\llbracket \text{sorted } xs; x \in \text{set } xs \rrbracket \implies x \leq \text{last } xs$
<proof>

end

lemma *empty-filter-conv*: $\llbracket \rrbracket = \text{filter } P \ xs \longleftrightarrow (\forall x \in \text{set } xs. \ \neg P \ x)$
<proof>

definition $ID :: 'a \Rightarrow 'a$ **where** $ID = id$

lemma $ID\text{-code}$ $[code, code\text{-unfold}]$: $ID = (\lambda x. x)$
 $\langle proof \rangle$

lemma $ID\text{-Some}$: $ID (Some\ x) = Some\ x$
 $\langle proof \rangle$

lemma $ID\text{-None}$: $ID\ None = None$
 $\langle proof \rangle$

lexicographic order on pairs

context

fixes $leq\text{-}a :: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\langle \sqsubseteq_a \rangle$ 50)
and $less\text{-}a :: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\langle \sqsubset_a \rangle$ 50)
and $leq\text{-}b :: 'b \Rightarrow 'b \Rightarrow bool$ (**infix** $\langle \sqsubseteq_b \rangle$ 50)
and $less\text{-}b :: 'b \Rightarrow 'b \Rightarrow bool$ (**infix** $\langle \sqsubset_b \rangle$ 50)

begin

definition $less\text{-}eq\text{-}prod :: ('a \times 'b) \Rightarrow ('a \times 'b) \Rightarrow bool$ (**infix** $\langle \sqsubseteq \rangle$ 50)
where $less\text{-}eq\text{-}prod = (\lambda(x1, x2)\ (y1, y2). x1 \sqsubset_a y1 \vee x1 \sqsubseteq_a y1 \wedge x2 \sqsubseteq_b y2)$

definition $less\text{-}prod :: ('a \times 'b) \Rightarrow ('a \times 'b) \Rightarrow bool$ (**infix** $\langle \sqsubset \rangle$ 50)
where $less\text{-}prod = (\lambda(x1, x2)\ (y1, y2). x1 \sqsubset_a y1 \vee x1 \sqsubseteq_a y1 \wedge x2 \sqsubset_b y2)$

lemma $less\text{-}eq\text{-}prod\text{-}simps$ $[simp]$:
 $(x1, x2) \sqsubseteq (y1, y2) \longleftrightarrow x1 \sqsubset_a y1 \vee x1 \sqsubseteq_a y1 \wedge x2 \sqsubseteq_b y2$
 $\langle proof \rangle$

lemma $less\text{-}prod\text{-}simps$ $[simp]$:
 $(x1, x2) \sqsubset (y1, y2) \longleftrightarrow x1 \sqsubset_a y1 \vee x1 \sqsubseteq_a y1 \wedge x2 \sqsubset_b y2$
 $\langle proof \rangle$

end

context

fixes $leq\text{-}a :: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\langle \sqsubseteq_a \rangle$ 50)
and $less\text{-}a :: 'a \Rightarrow 'a \Rightarrow bool$ (**infix** $\langle \sqsubset_a \rangle$ 50)
and $leq\text{-}b :: 'b \Rightarrow 'b \Rightarrow bool$ (**infix** $\langle \sqsubseteq_b \rangle$ 50)
and $less\text{-}b :: 'b \Rightarrow 'b \Rightarrow bool$ (**infix** $\langle \sqsubset_b \rangle$ 50)
assumes $lin\text{-}a$: $class.linorder\ leq\text{-}a\ less\text{-}a$
and $lin\text{-}b$: $class.linorder\ leq\text{-}b\ less\text{-}b$

begin

abbreviation $(input)\ less\text{-}eq\text{-}prod' :: ('a \times 'b) \Rightarrow ('a \times 'b) \Rightarrow bool$ (**infix** $\langle \sqsubseteq \rangle$ 50)
where $less\text{-}eq\text{-}prod' \equiv less\text{-}eq\text{-}prod\ leq\text{-}a\ less\text{-}a\ leq\text{-}b$

abbreviation *(input)* $less\text{-}prod' :: ('a \times 'b) \Rightarrow ('a \times 'b) \Rightarrow bool$ (**infix** $\langle \sqsubseteq \rangle$ 50)
where $less\text{-}prod' \equiv less\text{-}prod \text{ leq-}a \text{ less-}a \text{ less-}b$

lemma *linorder-prod*:
 $class.linorder \ (\sqsubseteq) \ (\sqsubset)$
 $\langle proof \rangle$

end

hide-const $less\text{-}eq\text{-}prod' \ less\text{-}prod'$

end

theory *Card-Datatype*
imports *HOL-Library.Cardinality*
begin

2.2 Definitions to prove equations about the cardinality of data types

2.2.1 Specialised *range* constants

definition $rangeIt :: 'a \Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a \ set$
where $rangeIt \ x \ f = range \ (\lambda n. (f \ \frown \ n) \ x)$

definition $rangeC :: ('a \Rightarrow 'b) \ set \Rightarrow 'b \ set$
where $rangeC \ F = (\bigcup f \in F. range \ f)$

lemma *infinite-rangeIt*:
assumes $inj: inj \ f$
and $x: \forall y. x \neq f \ y$
shows $\neg finite \ (rangeIt \ x \ f)$
 $\langle proof \rangle$

lemma *in-rangeC*: $f \in A \implies f \ x \in rangeC \ A$
 $\langle proof \rangle$

lemma *in-rangeCE*: **assumes** $y \in rangeC \ A$
obtains $f \ x$ **where** $f \in A \quad y = f \ x$
 $\langle proof \rangle$

lemma *in-rangeC-singleton*: $f \ x \in rangeC \ \{f\}$
 $\langle proof \rangle$

lemma *in-rangeC-singleton-const*: $x \in rangeC \ \{\lambda-. x\}$
 $\langle proof \rangle$

lemma *rangeC-rangeC*: $f \in rangeC \ A \implies f \ x \in rangeC \ (rangeC \ A)$

$\langle proof \rangle$

lemma *rangeC-eq-empty*: $rangeC\ A = \{\} \longleftrightarrow A = \{\}$
 $\langle proof \rangle$

lemma *Ball-rangeC-iff*:
 $(\forall x \in rangeC\ A. P\ x) \longleftrightarrow (\forall f \in A. \forall x. P\ (f\ x))$
 $\langle proof \rangle$

lemma *Ball-rangeC-singleton*:
 $(\forall x \in rangeC\ \{f\}. P\ x) \longleftrightarrow (\forall x. P\ (f\ x))$
 $\langle proof \rangle$

lemma *Ball-rangeC-rangeC*:
 $(\forall x \in rangeC\ (rangeC\ A). P\ x) \longleftrightarrow (\forall f \in rangeC\ A. \forall x. P\ (f\ x))$
 $\langle proof \rangle$

lemma *finite-rangeC*:
assumes *inj*: $\forall f \in A. inj\ f$
and *disjoint*: $\forall f \in A. \forall g \in A. f \neq g \longrightarrow (\forall x\ y. f\ x \neq g\ y)$
shows $finite\ (rangeC\ (A :: ('a \Rightarrow 'b)\ set)) \longleftrightarrow finite\ A \wedge (A \neq \{\} \longrightarrow finite\ (UNIV :: 'a\ set))$
(is $?lhs \longleftrightarrow ?rhs$
 $\langle proof \rangle$

lemma *finite-rangeC-singleton-const*:
 $finite\ (rangeC\ \{\lambda\cdot. x\})$
 $\langle proof \rangle$

lemma *card-Un*:
 $\llbracket finite\ A; finite\ B \rrbracket \Longrightarrow card\ (A \cup B) = card\ (A) + card\ (B) - card\ (A \cap B)$
 $\langle proof \rangle$

lemma *card-rangeC-singleton-const*:
 $card\ (rangeC\ \{\lambda\cdot. f\}) = 1$
 $\langle proof \rangle$

lemma *card-rangeC*:
assumes *inj*: $\forall f \in A. inj\ f$
and *disjoint*: $\forall f \in A. \forall g \in A. f \neq g \longrightarrow (\forall x\ y. f\ x \neq g\ y)$
shows $card\ (rangeC\ (A :: ('a \Rightarrow 'b)\ set)) = CARD('a) * card\ A$
(is $?lhs = ?rhs$
 $\langle proof \rangle$

lemma *rangeC-Int-rangeC*:
 $\llbracket \forall f \in A. \forall g \in B. \forall x\ y. f\ x \neq g\ y \rrbracket \Longrightarrow rangeC\ A \cap rangeC\ B = \{\}$
 $\langle proof \rangle$

lemmas *rangeC-simps* =

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in-rangeC-singleton
in-rangeC-singleton-const
rangeC-rangeC
rangeC-eq-empty
Ball-rangeC-singleton
Ball-rangeC-rangeC
finite-rangeC
finite-rangeC-singleton-const
card-rangeC-singleton-const
card-rangeC
rangeC-Int-rangeC

bundle *card-datatype* =
rangeC-simps [*simp*]
card-Un [*simp*]
fun-eq-iff [*simp*]
Int-Un-distrib [*simp*]
Int-Un-distrib2 [*simp*]
card-eq-0-iff [*simp*]
imageI [*simp*] *image-eqI* [*simp del*]
conj-cong [*cong*]
infinite-rangeIt [*simp*]

2.2.2 Cardinality primitives for polymorphic HOL types

$\langle ML \rangle$

definition *card-fun* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*
where *card-fun* *a b* = (if *a* $\neq$ 0 $\wedge$ *b* $\neq$ 0 $\vee$ *b* = 1 then *b* $\wedge$ *a* else 0)

lemma *CARD-fun* [*card-simps*]:
 $CARD('a \Rightarrow 'b) = \text{card-fun } CARD('a) \text{ } CARD('b)$
 $\langle \text{proof} \rangle$

definition *card-sum* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*
where *card-sum* *a b* = (if *a* = 0 $\vee$ *b* = 0 then 0 else *a* + *b*)

lemma *CARD-sum* [*card-simps*]:
 $CARD('a + 'b) = \text{card-sum } CARD('a) \text{ } CARD('b)$
 $\langle \text{proof} \rangle$

definition *card-option* :: *nat* $\Rightarrow$ *nat*
where *card-option* *n* = (if *n* = 0 then 0 else *Suc n*)

lemma *CARD-option* [*card-simps*]:
 $CARD('a \text{ option}) = \text{card-option } CARD('a)$
 $\langle \text{proof} \rangle$

definition *card-prod* :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*

where $\text{card-prod } a \ b = a * b$

lemma CARD-prod [*card-simps*]:

$\text{CARD}('a * 'b) = \text{card-prod } \text{CARD}('a) \ \text{CARD}('b)$
 $\langle \text{proof} \rangle$

definition $\text{card-list} :: \text{nat} \Rightarrow \text{nat}$

where $\text{card-list } - = 0$

lemma CARD-list [*card-simps*]: $\text{CARD}('a \ \text{list}) = \text{card-list } \text{CARD}('a)$

$\langle \text{proof} \rangle$

end

theory *List-Fusion*

imports

Main

begin

2.3 Shortcut fusion for lists

lemma Option-map-mono [*partial-function-mono*]:

$\text{mono-option } f \Longrightarrow \text{mono-option } (\lambda x. \text{map-option } g \ (f \ x))$
 $\langle \text{proof} \rangle$

lemma $\text{list-all2-coinduct}$ [*consumes 1, case-names Nil Cons, case-conclusion Cons*
 $\text{hd } tl, \text{coinduct pred: list-all2}$]:

assumes $X: X \ xs \ ys$

and $\text{Nil}' : \bigwedge xs \ ys. X \ xs \ ys \Longrightarrow xs = [] \longleftrightarrow ys = []$

and $\text{Cons}' : \bigwedge xs \ ys. \llbracket X \ xs \ ys; xs \neq []; ys \neq [] \rrbracket \Longrightarrow A \ (\text{hd } xs) \ (\text{hd } ys) \wedge (X \ (tl \ xs) \ (tl \ ys) \vee \text{list-all2 } A \ (tl \ xs) \ (tl \ ys))$

shows $\text{list-all2 } A \ xs \ ys$

$\langle \text{proof} \rangle$

2.3.1 The type of generators for finite lists

type-synonym $('a, 's) \text{raw-generator} = ('s \Rightarrow \text{bool}) \times ('s \Rightarrow 'a \times 's)$

inductive-set $\text{terminates-on} :: ('a, 's) \text{raw-generator} \Rightarrow 's \ \text{set}$

for $g :: ('a, 's) \text{raw-generator}$

where

$\text{stop}: \neg \text{fst } g \ s \Longrightarrow s \in \text{terminates-on } g$

$\mid \text{unfold}: \llbracket \text{fst } g \ s; \text{snd } (\text{snd } g \ s) \in \text{terminates-on } g \rrbracket \Longrightarrow s \in \text{terminates-on } g$

definition $\text{terminates} :: ('a, 's) \text{raw-generator} \Rightarrow \text{bool}$

where $\text{terminates } g \longleftrightarrow (\text{terminates-on } g = \text{UNIV})$

lemma terminatesI [*intro?*]:

$(\bigwedge s. s \in \text{terminates-on } g) \implies \text{terminates } g$
 $\langle \text{proof} \rangle$

lemma *terminatesD*:
 $\text{terminates } g \implies s \in \text{terminates-on } g$
 $\langle \text{proof} \rangle$

lemma *terminates-on-stop*:
 $\text{terminates-on } (\lambda-. \text{False}, \text{next}) = \text{UNIV}$
 $\langle \text{proof} \rangle$

lemma *wf-terminates*:
assumes $\text{wf } R$
and step: $\bigwedge s. \text{fst } g \ s \implies (\text{snd } (\text{snd } g \ s), s) \in R$
shows $\text{terminates } g$
 $\langle \text{proof} \rangle$

lemma *terminates-wfD*:
assumes $\text{terminates } g$
shows $\text{wf } \{(\text{snd } (\text{snd } g \ s), s) \mid s. \text{fst } g \ s\}$
 $\langle \text{proof} \rangle$

lemma *terminates-wfE*:
assumes $\text{terminates } g$
obtains R **where** $\text{wf } R \quad \bigwedge s. \text{fst } g \ s \implies (\text{snd } (\text{snd } g \ s), s) \in R$
 $\langle \text{proof} \rangle$

context **fixes** $g :: ('a, 's) \text{ raw-generator}$ **begin**

partial-function $(\text{option}) \text{ terminates-within} :: 's \Rightarrow \text{nat option}$
where

$\text{terminates-within } s =$
 $(\text{let } (\text{has-next}, \text{next}) = g$
 $\text{in if has-next } s \text{ then}$
 $\quad \text{map-option } (\lambda n. n + 1) (\text{terminates-within } (\text{snd } (\text{next } s)))$
 $\quad \text{else Some } 0)$

lemma *terminates-on-conv-dom-terminates-within*:
 $\text{terminates-on } g = \text{dom } \text{terminates-within}$
 $\langle \text{proof} \rangle$

end

lemma *terminates-within-unfold*:
 $\text{has-next } s \implies$
 $\text{terminates-within } (\text{has-next}, \text{next}) \ s = \text{map-option } (\lambda n. n + 1) (\text{terminates-within}$
 $\text{has-next}, \text{next}) (\text{snd } (\text{next } s)))$
 $\langle \text{proof} \rangle$

```

typedef ('a, 's) generator = {g :: ('a, 's) raw-generator. terminates g}
morphisms generator Generator
⟨proof⟩

```

```

setup-lifting type-definition-generator

```

```

lemma terminates-on-generator-eq-UNIV:
  terminates-on (generator g) = UNIV
⟨proof⟩

```

```

lemma terminates-within-stop:
  terminates-within (λ-. False, next) s = Some 0
⟨proof⟩

```

```

lemma terminates-within-generator-neq-None:
  terminates-within (generator g) s ≠ None
⟨proof⟩

```

```

locale list =
  fixes g :: ('a, 's) generator begin

```

```

definition has-next :: 's ⇒ bool
where has-next = fst (generator g)

```

```

definition next :: 's ⇒ 'a × 's
where next = snd (generator g)

```

```

function unfoldr :: 's ⇒ 'a list
where unfoldr s = (if has-next s then let (a, s') = next s in a # unfoldr s' else [])
⟨proof⟩

```

```

termination
⟨proof⟩

```

```

declare unfoldr.simps [simp del]

```

```

lemma unfoldr-simps:
  has-next s ⇒ unfoldr s = fst (next s) # unfoldr (snd (next s))
  ¬ has-next s ⇒ unfoldr s = []
⟨proof⟩

```

```

end

```

```

declare
  list.has-next-def[code]
  list.next-def[code]
  list.unfoldr.simps[code]

```

```

context includes lifting-syntax
begin

```

lemma *generator-has-next-transfer* [*transfer-rule*]:
 (pcr-generator (=) (=) ==> (=)) fst list.has-next
 <proof>

lemma *generator-next-transfer* [*transfer-rule*]:
 (pcr-generator (=) (=) ==> (=)) snd list.next
 <proof>

end

lemma *unfoldr-eq-Nil-iff* [*iff*]:
 list.unfoldr g s = [] $\longleftrightarrow$ $\neg$ list.has-next g s
 <proof>

lemma *Nil-eq-unfoldr-iff* [*simp*]:
 [] = list.unfoldr g s $\longleftrightarrow$ $\neg$ list.has-next g s
 <proof>

2.3.2 Generators for 'a list

primrec *list-has-next* :: 'a list $\Rightarrow$ bool

where

list-has-next [] $\longleftrightarrow$ False
 | list-has-next (x # xs) $\longleftrightarrow$ True

primrec *list-next* :: 'a list $\Rightarrow$ 'a $\times$ 'a list

where

list-next (x # xs) = (x, xs)

lemma *terminates-list-generator*: terminates (list-has-next, list-next)
 <proof>

lift-definition *list-generator* :: ('a, 'a list) generator
 is (list-has-next, list-next)
 <proof>

lemma *has-next-list-generator* [*simp*]:
 list.has-next list-generator = list-has-next
 <proof>

lemma *next-list-generator* [*simp*]:
 list.next list-generator = list-next
 <proof>

lemma *unfoldr-list-generator*:
 list.unfoldr list-generator xs = xs
 <proof>

lemma *terminates-replicate-generator*:

terminates ($\lambda n :: \text{nat}. 0 < n, \lambda n. (a, n - 1)$)

$\langle \text{proof} \rangle$

lift-definition *replicate-generator* $:: 'a \Rightarrow ('a, \text{nat}) \text{ generator}$

is $\lambda a. (\lambda n. 0 < n, \lambda n. (a, n - 1))$

$\langle \text{proof} \rangle$

lemma *has-next-replicate-generator* [simp]:

list.has-next (*replicate-generator* *a*) *n* $\longleftrightarrow 0 < n$

$\langle \text{proof} \rangle$

lemma *next-replicate-generator* [simp]:

list.next (*replicate-generator* *a*) *n* = (*a*, *n* - 1)

$\langle \text{proof} \rangle$

lemma *unfoldr-replicate-generator*:

list.unfoldr (*replicate-generator* *a*) *n* = *replicate* *n* *a*

$\langle \text{proof} \rangle$

context *fixes* *f* $:: 'a \Rightarrow 'b$ **begin**

lift-definition *map-generator* $:: ('a, 's) \text{ generator} \Rightarrow ('b, 's) \text{ generator}$

is $\lambda (has_next, next). (has_next, \lambda s. \text{let } (a, s') = next\ s \text{ in } (f\ a, s'))$

$\langle \text{proof} \rangle$

lemma *has-next-map-generator* [simp]:

list.has-next (*map-generator* *g*) = *list.has-next* *g*

$\langle \text{proof} \rangle$

lemma *next-map-generator* [simp]:

list.next (*map-generator* *g*) = *apfst* *f* $\circ$ *list.next* *g*

$\langle \text{proof} \rangle$

lemma *unfoldr-map-generator*:

list.unfoldr (*map-generator* *g*) = *map* *f* $\circ$ *list.unfoldr* *g*

(**is** ?lhs = ?rhs)

$\langle \text{proof} \rangle$

end

context *fixes* *g1* $:: ('a, 's1) \text{ raw-generator}$

and *g2* $:: ('a, 's2) \text{ raw-generator}$

begin

fun *append-has-next* $:: 's1 \times 's2 + 's2 \Rightarrow \text{bool}$

where

append-has-next (*Inl* (*s1*, *s2*)) $\longleftrightarrow \text{fst } g1\ s1 \vee \text{fst } g2\ s2$

| *append-has-next* (*Inr* *s2*) $\longleftrightarrow \text{fst } g2\ s2$

```

fun append-next :: 's1 × 's2 + 's2 ⇒ 'a × ('s1 × 's2 + 's2)
where
  append-next (Inl (s1, s2)) =
    (if fst g1 s1 then
      let (x, s1') = snd g1 s1 in (x, Inl (s1', s2))
    else append-next (Inr s2))
  | append-next (Inr s2) = (let (x, s2') = snd g2 s2 in (x, Inr s2'))

end

```

lift-definition *append-generator* :: ('a, 's1) generator ⇒ ('a, 's2) generator ⇒
 ('a, 's1 × 's2 + 's2) generator
is λg1 g2. (append-has-next g1 g2, append-next g1 g2)
 ⟨proof⟩

definition *append-init* :: 's1 ⇒ 's2 ⇒ 's1 × 's2 + 's2
where *append-init* s1 s2 = Inl (s1, s2)

lemma *has-next-append-generator* [simp]:
 list.has-next (append-generator g1 g2) (Inl (s1, s2)) ⟷
 list.has-next g1 s1 ∨ list.has-next g2 s2
 list.has-next (append-generator g1 g2) (Inr s2) ⟷ list.has-next g2 s2
 ⟨proof⟩

lemma *next-append-generator* [simp]:
 list.next (append-generator g1 g2) (Inl (s1, s2)) =
 (if list.has-next g1 s1 then
 let (x, s1') = list.next g1 s1 in (x, Inl (s1', s2))
 else list.next (append-generator g1 g2) (Inr s2))
 list.next (append-generator g1 g2) (Inr s2) = apsnd Inr (list.next g2 s2)
 ⟨proof⟩

lemma *unfoldr-append-generator-Inr*:
 list.unfoldr (append-generator g1 g2) (Inr s2) = list.unfoldr g2 s2
 ⟨proof⟩

lemma *unfoldr-append-generator-Inl*:
 list.unfoldr (append-generator g1 g2) (Inl (s1, s2)) =
 list.unfoldr g1 s1 @ list.unfoldr g2 s2
 ⟨proof⟩

lemma *unfoldr-append-generator*:
 list.unfoldr (append-generator g1 g2) (append-init s1 s2) =
 list.unfoldr g1 s1 @ list.unfoldr g2 s2
 ⟨proof⟩

lift-definition *zip-generator* :: ('a, 's1) generator ⇒ ('b, 's2) generator ⇒ ('a ×

'b, 's1 × 's2) generator
 is $\lambda(\text{has-next1}, \text{next1}) (\text{has-next2}, \text{next2}).$
 $(\lambda(s1, s2). \text{has-next1 } s1 \wedge \text{has-next2 } s2,$
 $\lambda(s1, s2). \text{let } (x, s1') = \text{next1 } s1; (y, s2') = \text{next2 } s2$
 in $((x, y), (s1', s2'))$)
 ⟨proof⟩

abbreviation (input) zip-init :: 's1 ⇒ 's2 ⇒ 's1 × 's2
 where zip-init ≡ Pair

lemma has-next-zip-generator [simp]:
 list.has-next (zip-generator g1 g2) (s1, s2) ⟷
 list.has-next g1 s1 ∧ list.has-next g2 s2
 ⟨proof⟩

lemma next-zip-generator [simp]:
 list.next (zip-generator g1 g2) (s1, s2) =
 ((fst (list.next g1 s1), fst (list.next g2 s2)),
 (snd (list.next g1 s1), snd (list.next g2 s2)))
 ⟨proof⟩

lemma unfoldr-zip-generator:
 list.unfoldr (zip-generator g1 g2) (zip-init s1 s2) =
 zip (list.unfoldr g1 s1) (list.unfoldr g2 s2)
 ⟨proof⟩

context fixes bound :: nat **begin**

lift-definition upt-generator :: (nat, nat) generator
 is $(\lambda n. n < \text{bound}, \lambda n. (n, \text{Suc } n))$
 ⟨proof⟩

lemma has-next-upt-generator [simp]:
 list.has-next upt-generator n ⟷ n < bound
 ⟨proof⟩

lemma next-upt-generator [simp]:
 list.next upt-generator n = (n, Suc n)
 ⟨proof⟩

lemma unfoldr-upt-generator:
 list.unfoldr upt-generator n = [n.. bound]
 ⟨proof⟩

end

context fixes bound :: int **begin**

lift-definition upto-generator :: (int, int) generator

is ($\lambda n. n \leq bound, \lambda n. (n, n + 1)$)
 $\langle proof \rangle$

lemma *has-next-upto-generator* [simp]:
 $list.has_next\ upto_generator\ n \longleftrightarrow n \leq bound$
 $\langle proof \rangle$

lemma *next-upto-generator* [simp]:
 $list.next\ upto_generator\ n = (n, n + 1)$
 $\langle proof \rangle$

lemma *unfoldr-upto-generator*:
 $list.unfoldr\ upto_generator\ n = [n..bound]$
 $\langle proof \rangle$

end

context
fixes $P :: 'a \Rightarrow bool$
begin

context
fixes $g :: ('a, 's)\ raw_generator$
begin

inductive *filter-has-next* :: $'s \Rightarrow bool$

where

$\llbracket fst\ g\ s; P\ (fst\ (snd\ g\ s)) \rrbracket \Longrightarrow filter_has_next\ s$
 $\mid \llbracket fst\ g\ s; \neg P\ (fst\ (snd\ g\ s)); filter_has_next\ (snd\ (snd\ g\ s)) \rrbracket \Longrightarrow filter_has_next\ s$

partial-function (*tailrec*) *filter-next* :: $'s \Rightarrow 'a \times 's$

where

$filter_next\ s = (let\ (x, s') = snd\ g\ s\ in\ if\ P\ x\ then\ (x, s')\ else\ filter_next\ s')$

end

lift-definition *filter-generator* :: $('a, 's)\ generator \Rightarrow ('a, 's)\ generator$

is $\lambda g. (filter_has_next\ g, filter_next\ g)$

$\langle proof \rangle$

lemma *has-next-filter-generator*:

$list.has_next\ (filter_generator\ g)\ s \longleftrightarrow$
 $list.has_next\ g\ s \wedge (let\ (x, s') = list.next\ g\ s\ in\ if\ P\ x\ then\ True\ else\ list.has_next$
 $(filter_generator\ g)\ s')$
 $\langle proof \rangle$

lemma *next-filter-generator*:

$list.next\ (filter_generator\ g)\ s =$
 $(let\ (x, s') = list.next\ g\ s$

in if $P\ x$ then (x, s') else $list.next\ (filter-generator\ g)\ s'$
<proof>

lemma *has-next-filter-generator-induct* [*consumes 1, case-names find step*]:
assumes *list.has-next (filter-generator g) s*
and find: $\bigwedge s. \llbracket list.has-next\ g\ s; P\ (fst\ (list.next\ g\ s)) \rrbracket \implies Q\ s$
and step: $\bigwedge s. \llbracket list.has-next\ g\ s; \neg P\ (fst\ (list.next\ g\ s)); Q\ (snd\ (list.next\ g\ s)) \rrbracket \implies Q\ s$
shows $Q\ s$
<proof>

lemma *filter-generator-empty-conv:*
 $list.has-next\ (filter-generator\ g)\ s \longleftrightarrow (\exists x \in set\ (list.unfoldr\ g\ s). P\ x) \text{ (is ?lhs} \longleftrightarrow ?rhs)$
<proof>

lemma *unfoldr-filter-generator:*
 $list.unfoldr\ (filter-generator\ g)\ s = filter\ P\ (list.unfoldr\ g\ s)$
<proof>

end

2.3.3 Destroying lists

definition *hd-fusion* :: $('a, 's)\ generator \Rightarrow 's \Rightarrow 'a$
where $hd-fusion\ g\ s = hd\ (list.unfoldr\ g\ s)$

lemma *hd-fusion-code* [*code*]:
 $hd-fusion\ g\ s = (if\ list.has-next\ g\ s\ then\ fst\ (list.next\ g\ s)\ else\ undefined)$
<proof>

declare *hd-fusion-def* [*symmetric, code-unfold*]

definition *fold-fusion* :: $('a, 's)\ generator \Rightarrow ('a \Rightarrow 'b \Rightarrow 'b) \Rightarrow 's \Rightarrow 'b \Rightarrow 'b$
where $fold-fusion\ g\ f\ s = fold\ f\ (list.unfoldr\ g\ s)$

lemma *fold-fusion-code* [*code*]:
 $fold-fusion\ g\ f\ s\ b =$
(if *list.has-next g s then*
 $\quad let\ (x, s') = list.next\ g\ s$
 $\quad in\ fold-fusion\ g\ f\ s'\ (f\ x\ b)$
else b)
<proof>

declare *fold-fusion-def* [*symmetric, code-unfold*]

definition *gen-length-fusion* :: $('a, 's)\ generator \Rightarrow nat \Rightarrow 's \Rightarrow nat$
where $gen-length-fusion\ g\ n\ s = n + length\ (list.unfoldr\ g\ s)$

lemma *gen-length-fusion-code* [code]:
 $\text{gen-length-fusion } g \ n \ s =$
 $(\text{if } \text{list.has-next } g \ s \text{ then } \text{gen-length-fusion } g \ (\text{Suc } n) \ (\text{snd } (\text{list.next } g \ s)) \text{ else } n)$
 ⟨proof⟩

definition *length-fusion* :: ('a, 's) generator $\Rightarrow$'s $\Rightarrow$ nat
where *length-fusion* $g \ s = \text{length } (\text{list.unfoldr } g \ s)$

lemma *length-fusion-code* [code]:
 $\text{length-fusion } g = \text{gen-length-fusion } g \ 0$
 ⟨proof⟩

declare *length-fusion-def*[symmetric, code-unfold]

definition *map-fusion* :: ('a $\Rightarrow$ 'b) $\Rightarrow$ ('a, 's) generator $\Rightarrow$'s $\Rightarrow$ 'b list
where *map-fusion* $f \ g \ s = \text{map } f \ (\text{list.unfoldr } g \ s)$

lemma *map-fusion-code* [code]:
 $\text{map-fusion } f \ g \ s =$
 $(\text{if } \text{list.has-next } g \ s \text{ then}$
 $\quad \text{let } (x, s') = \text{list.next } g \ s$
 $\quad \text{in } f \ x \ \# \ \text{map-fusion } f \ g \ s'$
 $\text{else } [])$
 ⟨proof⟩

declare *map-fusion-def*[symmetric, code-unfold]

definition *append-fusion* :: ('a, 's1) generator $\Rightarrow$ ('a, 's2) generator $\Rightarrow$ 's1 $\Rightarrow$ 's2
 $\Rightarrow$ 'a list
where *append-fusion* $g1 \ g2 \ s1 \ s2 = \text{list.unfoldr } g1 \ s1 \ @ \ \text{list.unfoldr } g2 \ s2$

lemma *append-fusion* [code]:
 $\text{append-fusion } g1 \ g2 \ s1 \ s2 =$
 $(\text{if } \text{list.has-next } g1 \ s1 \text{ then}$
 $\quad \text{let } (x, s1') = \text{list.next } g1 \ s1$
 $\quad \text{in } x \ \# \ \text{append-fusion } g1 \ g2 \ s1' \ s2$
 $\text{else } \text{list.unfoldr } g2 \ s2)$
 ⟨proof⟩

declare *append-fusion-def*[symmetric, code-unfold]

definition *zip-fusion* :: ('a, 's1) generator $\Rightarrow$ ('b, 's2) generator $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$
 ('a $\times$ 'b) list
where *zip-fusion* $g1 \ g2 \ s1 \ s2 = \text{zip } (\text{list.unfoldr } g1 \ s1) \ (\text{list.unfoldr } g2 \ s2)$

lemma *zip-fusion-code* [code]:
 $\text{zip-fusion } g1 \ g2 \ s1 \ s2 =$
 $(\text{if } \text{list.has-next } g1 \ s1 \ \wedge \ \text{list.has-next } g2 \ s2 \text{ then}$
 $\quad \text{let } (x, s1') = \text{list.next } g1 \ s1;$

```

      (y, s2') = list.next g2 s2
    in (x, y) # zip-fusion g1 g2 s1' s2'
  else []
<proof>

```

declare *zip-fusion-def*[*symmetric*, *code-unfold*]

definition *list-all-fusion* :: ('a, 's) generator $\Rightarrow$ ('a $\Rightarrow$ bool) $\Rightarrow$'s $\Rightarrow$ bool
where *list-all-fusion* g P s = List.list-all P (list.unfoldr g s)

lemma *list-all-fusion-code* [code]:
list-all-fusion g P s $\longleftrightarrow$
 (list.has-next g s $\longrightarrow$
 (let (x, s') = list.next g s
 in P x $\wedge$ list-all-fusion g P s'))
 <proof>

declare *list-all-fusion-def*[*symmetric*, *code-unfold*]

definition *list-all2-fusion* :: ('a $\Rightarrow$ 'b $\Rightarrow$ bool) $\Rightarrow$ ('a, 's1) generator $\Rightarrow$ ('b, 's2)
 generator $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool
where

list-all2-fusion P g1 g2 s1 s2 =
 list-all2 P (list.unfoldr g1 s1) (list.unfoldr g2 s2)

lemma *list-all2-fusion-code* [code]:
list-all2-fusion P g1 g2 s1 s2 =
 (if list.has-next g1 s1 then
 list.has-next g2 s2 $\wedge$
 (let (x, s1') = list.next g1 s1;
 (y, s2') = list.next g2 s2
 in P x y $\wedge$ list-all2-fusion P g1 g2 s1' s2')
 else $\neg$ list.has-next g2 s2)
 <proof>

declare *list-all2-fusion-def*[*symmetric*, *code-unfold*]

definition *singleton-list-fusion* :: ('a, 'state) generator $\Rightarrow$ 'state $\Rightarrow$ bool
where *singleton-list-fusion* gen state = (case list.unfoldr gen state of [-] $\Rightarrow$ True |
 - $\Rightarrow$ False)

lemma *singleton-list-fusion-code* [code]:
singleton-list-fusion g s $\longleftrightarrow$
 list.has-next g s $\wedge$ $\neg$ list.has-next g (snd (list.next g s))
 <proof>

end

```

theory Lexicographic-Order imports
  List-Fusion
  HOL-Library.Char-ord
begin

hide-const (open) List.lexordp

```

2.4 List fusion for lexicographic order

```

context linorder begin

```

lemma *lexordp-take-index-conv*:

```

  lexordp xs ys  $\longleftrightarrow$ 
    (length xs < length ys  $\wedge$  take (length xs) ys = xs)  $\vee$ 
    ( $\exists i < \min (\text{length } xs) (\text{length } ys).$  take i xs = take i ys  $\wedge$  xs ! i < ys ! i)
    (is ?lhs = ?rhs)
  <proof>

```

lemma *lexordp-lex*: $(xs, ys) \in \text{lex } \{(xs, ys). xs < ys\} \longleftrightarrow \text{lexordp } xs \text{ } ys \wedge \text{length } xs = \text{length } ys$

<proof>

end

2.4.1 Setup for list fusion

```

context ord begin

```

definition *lexord-fusion* :: $('a, 's1) \text{ generator} \Rightarrow ('a, 's2) \text{ generator} \Rightarrow 's1 \Rightarrow 's2 \Rightarrow \text{bool}$

where [code del]: *lexord-fusion g1 g2 s1 s2* = *lexordp (list.unfoldr g1 s1) (list.unfoldr g2 s2)*

definition *lexord-eq-fusion* :: $('a, 's1) \text{ generator} \Rightarrow ('a, 's2) \text{ generator} \Rightarrow 's1 \Rightarrow 's2 \Rightarrow \text{bool}$

where [code del]: *lexord-eq-fusion g1 g2 s1 s2* = *lexordp-eq (list.unfoldr g1 s1) (list.unfoldr g2 s2)*

lemma *lexord-fusion-code*:

```

  lexord-fusion g1 g2 s1 s2  $\longleftrightarrow$ 
    (if list.has-next g1 s1 then
      if list.has-next g2 s2 then
        let (x, s1') = list.next g1 s1;
          (y, s2') = list.next g2 s2
          in x < y  $\vee \neg y < x \wedge \text{lexord-fusion } g1 \text{ } g2 \text{ } s1' \text{ } s2'$ 
        else False
      else list.has-next g2 s2)
  <proof>

```

lemma *lexord-eq-fusion-code*:

```

lexord-eq-fusion g1 g2 s1 s2  $\longleftrightarrow$ 
(list.has-next g1 s1  $\longrightarrow$ 
 list.has-next g2 s2  $\wedge$ 
 (let (x, s1') = list.next g1 s1;
      (y, s2') = list.next g2 s2
      in x < y  $\vee$   $\neg$  y < x  $\wedge$  lexord-eq-fusion g1 g2 s1' s2'))
<proof>

```

end

```

lemmas [code] =
  lexord-fusion-code ord.lexord-fusion-code
  lexord-eq-fusion-code ord.lexord-eq-fusion-code

```

```

lemmas [symmetric, code-unfold] =
  lexord-fusion-def ord.lexord-fusion-def
  lexord-eq-fusion-def ord.lexord-eq-fusion-def

```

end

```

theory Extend-Partial-Order
imports Main
begin

```

2.5 Every partial order can be extended to a total order

```

lemma ChainsD:  $\llbracket x \in C; C \in \text{Chains } r; y \in C \rrbracket \implies (x, y) \in r \vee (y, x) \in r$ 
<proof>

```

```

lemma Chains-Field:  $\llbracket C \in \text{Chains } r; x \in C \rrbracket \implies x \in \text{Field } r$ 
<proof>

```

```

lemma total-onD:
 $\llbracket \text{total-on } A \ r; x \in A; y \in A \rrbracket \implies (x, y) \in r \vee x = y \vee (y, x) \in r$ 
<proof>

```

```

lemma linear-order-imp-linorder: linear-order  $\{(A, B). \text{leq } A \ B\} \implies \text{class.linorder}$ 
leq  $(\lambda x \ y. \text{leq } x \ y \wedge \neg \text{leq } y \ x)$ 
<proof>

```

```

lemma (in linorder) linear-order: linear-order  $\{(A, B). A \leq B\}$ 
<proof>

```

```

definition order-consistent :: ('a  $\times$  'a) set  $\Rightarrow$  ('a  $\times$  'a) set  $\Rightarrow$  bool
where order-consistent r s  $\longleftrightarrow (\forall a \ a'. (a, a') \in r \longrightarrow (a', a) \in s \longrightarrow a = a')$ 

```

2.5. EVERY PARTIAL ORDER CAN BE EXTENDED TO A TOTAL ORDER 27

lemma *order-consistent-sym*:

order-consistent $r\ s \implies \text{order-consistent } s\ r$
 $\langle \text{proof} \rangle$

lemma *antisym-order-consistent-self*:

antisym $r \implies \text{order-consistent } r\ r$
 $\langle \text{proof} \rangle$

lemma *refl-on-trancl*:

assumes *refl-on* $A\ r$
shows *refl-on* $A\ (r^+)$
 $\langle \text{proof} \rangle$

lemma *total-on-refl-on-consistent-into*:

assumes r : *total-on* $A\ r$ *refl-on* $A\ r$
and *consist*: *order-consistent* $r\ s$
and x : $x \in A$ **and** y : $y \in A$ **and** s : $(x, y) \in s$
shows $(x, y) \in r$
 $\langle \text{proof} \rangle$

lemma *porder-linorder-tranclpE* [*consumes 5, case-names base step*]:

assumes r : *partial-order-on* $A\ r$
and s : *linear-order-on* $B\ s$
and *consist*: *order-consistent* $r\ s$
and *B-subset-A*: $B \subseteq A$
and *trancl*: $(x, y) \in (r \cup s)^+$
obtains $(x, y) \in r$
 $| u\ v$ **where** $(x, u) \in r$ $(u, v) \in s$ $(v, y) \in r$
 $\langle \text{proof} \rangle$

lemma *porder-on-consistent-linorder-on-trancl-antisym*:

assumes r : *partial-order-on* $A\ r$
and s : *linear-order-on* $B\ s$
and *consist*: *order-consistent* $r\ s$
and *B-subset-A*: $B \subseteq A$
shows *antisym* $((r \cup s)^+)$
 $\langle \text{proof} \rangle$

lemma *porder-on-linorder-on-tranclp-porder-onI*:

assumes r : *partial-order-on* $A\ r$
and s : *linear-order-on* $B\ s$
and *consist*: *order-consistent* $r\ s$
and *subset*: $B \subseteq A$
shows *partial-order-on* $A\ ((r \cup s)^+)$
 $\langle \text{proof} \rangle$

lemma *porder-extend-to-linorder*:

```

    assumes r: partial-order-on A r
    obtains s where linear-order-on A s    order-consistent r s
  <proof>

end

```

```

theory Set-Linorder
imports
  Containers-Auxiliary
  Lexicographic-Order
  Extend-Partial-Order
  HOL-Library.Cardinality
begin

```

2.6 An executable linear order on sets

2.6.1 Definition of the linear order

Extending finite and cofinite sets

Partition sets into finite and cofinite sets and distribute the rest arbitrarily such that complement switches between the two.

```

consts infinite-complement-partition :: 'a set set

```

```

specification (infinite-complement-partition)
  finite-complement-partition: finite (A :: 'a set) ⟹ A ∈ infinite-complement-partition
  complement-partition:  $\neg \text{finite } (UNIV :: 'a \text{ set})$ 
     $\implies (A :: 'a \text{ set}) \in \text{infinite-complement-partition} \longleftrightarrow - A \notin \text{infinite-complement-partition}$ 
  <proof>

```

```

lemma not-in-complement-partition:
   $\neg \text{finite } (UNIV :: 'a \text{ set})$ 
   $\implies (A :: 'a \text{ set}) \notin \text{infinite-complement-partition} \longleftrightarrow - A \in \text{infinite-complement-partition}$ 
  <proof>

```

```

lemma not-in-complement-partition-False:
   $\llbracket (A :: 'a \text{ set}) \in \text{infinite-complement-partition}; \neg \text{finite } (UNIV :: 'a \text{ set}) \rrbracket$ 
   $\implies - A \in \text{infinite-complement-partition} = \text{False}$ 
  <proof>

```

```

lemma infinite-complement-partition-finite [simp]:
  finite (UNIV :: 'a set) ⟹ infinite-complement-partition = (UNIV :: 'a set set)
  <proof>

```

```

lemma Compl-eq-empty-iff:  $- A = \{\} \longleftrightarrow A = UNIV$ 
  <proof>

```

A lexicographic-style order on finite subsets**context** *ord* **begin****definition** *set-less-aux* :: 'a set $\Rightarrow$ 'a set $\Rightarrow$ bool (**infix** $\sqsubset'$ 50)**where** $A \sqsubset' B \longleftrightarrow \text{finite } A \wedge \text{finite } B \wedge (\exists y \in B - A. \forall z \in (A - B) \cup (B - A). y \leq z \wedge (z \leq y \longrightarrow y = z))$ **definition** *set-less-eq-aux* :: 'a set $\Rightarrow$ 'a set $\Rightarrow$ bool (**infix** $\sqsubseteq'$ 50)**where** $A \sqsubseteq' B \longleftrightarrow A \in \text{infinite-complement-partition} \wedge A = B \vee A \sqsubset' B$ **lemma** *set-less-aux-irrefl* [iff]: $\neg A \sqsubset' A$
(proof)**lemma** *set-less-eq-aux-refl* [iff]: $A \sqsubseteq' A \longleftrightarrow A \in \text{infinite-complement-partition}$
(proof)**lemma** *set-less-aux-empty* [simp]: $\neg A \sqsubset' \{\}$
(proof)**lemma** *set-less-eq-aux-empty* [simp]: $A \sqsubseteq' \{\} \longleftrightarrow A = \{\}$
(proof)**lemma** *set-less-aux-antisym*: $\llbracket A \sqsubset' B; B \sqsubset' A \rrbracket \Longrightarrow \text{False}$
(proof)**lemma** *set-less-aux-conv-set-less-eq-aux*:
 $A \sqsubset' B \longleftrightarrow A \sqsubseteq' B \wedge \neg B \sqsubseteq' A$
(proof)**lemma** *set-less-eq-aux-antisym*: $\llbracket A \sqsubseteq' B; B \sqsubseteq' A \rrbracket \Longrightarrow A = B$
(proof)**lemma** *set-less-aux-finiteD*: $A \sqsubset' B \Longrightarrow \text{finite } A \wedge B \in \text{infinite-complement-partition}$
(proof)**lemma** *set-less-eq-aux-infinite-complement-partitionD*:
 $A \sqsubseteq' B \Longrightarrow A \in \text{infinite-complement-partition} \wedge B \in \text{infinite-complement-partition}$
(proof)**lemma** *Compl-set-less-aux-Compl*:
 $\text{finite } (\text{UNIV} :: 'a \text{ set}) \Longrightarrow \neg A \sqsubset' - B \longleftrightarrow B \sqsubset' A$
(proof)**lemma** *Compl-set-less-eq-aux-Compl*:
 $\text{finite } (\text{UNIV} :: 'a \text{ set}) \Longrightarrow \neg A \sqsubseteq' - B \longleftrightarrow B \sqsubseteq' A$
(proof)**lemma** *set-less-aux-insert-same*:
 $x \in A \longleftrightarrow x \in B \Longrightarrow \text{insert } x A \sqsubset' \text{insert } x B \longleftrightarrow A \sqsubset' B$

$\langle proof \rangle$

lemma *set-less-eq-aux-insert-same*:

$\llbracket A \in \text{infinite-complement-partition}; \text{insert } x \ B \in \text{infinite-complement-partition};$
 $x \in A \longleftrightarrow x \in B \rrbracket$
 $\implies \text{insert } x \ A \sqsubseteq' \text{insert } x \ B \longleftrightarrow A \sqsubseteq' B$

$\langle proof \rangle$

end

context *order* **begin**

lemma *set-less-aux-singleton-iff*: $A \sqsubset' \{x\} \longleftrightarrow \text{finite } A \wedge (\forall a \in A. x < a)$

$\langle proof \rangle$

end

context *linorder* **begin**

lemma *wlog-le* [*case-names sym le*]:

assumes $\bigwedge a \ b. P \ a \ b \implies P \ b \ a$
and $\bigwedge a \ b. a \leq b \implies P \ a \ b$
shows $P \ b \ a$

$\langle proof \rangle$

lemma *empty-set-less-aux* [*simp*]: $\{\} \sqsubset' A \longleftrightarrow A \neq \{\} \wedge \text{finite } A$

$\langle proof \rangle$

lemma *empty-set-less-eq-aux* [*simp*]: $\{\} \sqsubseteq' A \longleftrightarrow \text{finite } A$

$\langle proof \rangle$

lemma *set-less-aux-trans*:

assumes $AB: A \sqsubset' B$ **and** $BC: B \sqsubset' C$
shows $A \sqsubset' C$

$\langle proof \rangle$

lemma *set-less-eq-aux-trans* [*trans*]:

$\llbracket A \sqsubseteq' B; B \sqsubseteq' C \rrbracket \implies A \sqsubseteq' C$

$\langle proof \rangle$

lemma *set-less-trans-set-less-eq* [*trans*]:

$\llbracket A \sqsubset' B; B \sqsubseteq' C \rrbracket \implies A \sqsubset' C$

$\langle proof \rangle$

lemma *set-less-eq-aux-porder*: *partial-order-on infinite-complement-partition* $\{(A, B). A \sqsubseteq' B\}$

$\langle proof \rangle$

lemma *psubset-finite-imp-set-less-aux*:

assumes AsB : $A \subset B$ **and** B : *finite* B
shows $A \sqsubset' B$
 $\langle proof \rangle$

lemma *subset-finite-imp-set-less-eq-aux*:
 $\llbracket A \subseteq B; \text{finite } B \rrbracket \implies A \sqsubseteq' B$
 $\langle proof \rangle$

lemma *empty-set-less-aux-finite-iff*:
 $\text{finite } A \implies \{\} \sqsubset' A \longleftrightarrow A \neq \{\}$
 $\langle proof \rangle$

lemma *set-less-aux-finite-total*:
assumes A : *finite* A **and** B : *finite* B
shows $A \sqsubset' B \vee A = B \vee B \sqsubset' A$
 $\langle proof \rangle$

lemma *set-less-eq-aux-finite-total*:
 $\llbracket \text{finite } A; \text{finite } B \rrbracket \implies A \sqsubseteq' B \vee A = B \vee B \sqsubseteq' A$
 $\langle proof \rangle$

lemma *set-less-eq-aux-finite-total2*:
 $\llbracket \text{finite } A; \text{finite } B \rrbracket \implies A \sqsubseteq' B \vee B \sqsubseteq' A$
 $\langle proof \rangle$

lemma *set-less-aux-rec*:
assumes A : *finite* A **and** B : *finite* B
and A' : $A \neq \{\}$ **and** B' : $B \neq \{\}$
shows $A \sqsubset' B \longleftrightarrow \text{Min } B < \text{Min } A \vee \text{Min } A = \text{Min } B \wedge A - \{\text{Min } A\} \sqsubset' B - \{\text{Min } A\}$
 $\langle proof \rangle$

lemma *set-less-eq-aux-rec*:
assumes *finite* A *finite* B $A \neq \{\}$ $B \neq \{\}$
shows $A \sqsubseteq' B \longleftrightarrow \text{Min } B < \text{Min } A \vee \text{Min } A = \text{Min } B \wedge A - \{\text{Min } A\} \sqsubseteq' B - \{\text{Min } A\}$
 $\langle proof \rangle$

lemma *set-less-aux-Min-antimono*:
 $\llbracket \text{Min } A < \text{Min } B; \text{finite } A; \text{finite } B; A \neq \{\} \rrbracket \implies B \sqsubset' A$
 $\langle proof \rangle$

lemma *sorted-Cons-Min*: $\text{sorted } (x \# xs) \implies \text{Min } (\text{insert } x (\text{set } xs)) = x$
 $\langle proof \rangle$

lemma *set-less-aux-code*:
 $\llbracket \text{sorted } xs; \text{distinct } xs; \text{sorted } ys; \text{distinct } ys \rrbracket$
 $\implies \text{set } xs \sqsubset' \text{set } ys \longleftrightarrow \text{ord.lexordp } (>) \text{ } xs \text{ } ys$
 $\langle proof \rangle$

lemma *set-less-eq-aux-code*:
 assumes *sorted xs distinct xs sorted ys distinct ys*
 shows $\text{set } xs \sqsubseteq' \text{set } ys \longleftrightarrow \text{ord.lexordp-eq } (>) \text{ } xs \text{ } ys$
 $\langle \text{proof} \rangle$

end

Extending $(\sqsubseteq')$ to have $\{\}$ as least element

context *ord* **begin**

definition *set-less-eq-aux'* :: '*a set* $\Rightarrow$ '*a set* $\Rightarrow$ bool (infix $\sqsubseteq''$ 50)
 where $A \sqsubseteq'' B \longleftrightarrow A \sqsubseteq' B \vee A = \{\} \wedge B \in \text{infinite-complement-partition}$

lemma *set-less-eq-aux'-refl*:
 $A \sqsubseteq'' A \longleftrightarrow A \in \text{infinite-complement-partition}$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux'-antisym*: $\llbracket A \sqsubseteq'' B; B \sqsubseteq'' A \rrbracket \Longrightarrow A = B$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux'-infinite-complement-partitionD*:
 $A \sqsubseteq'' B \Longrightarrow A \in \text{infinite-complement-partition} \wedge B \in \text{infinite-complement-partition}$
 $\langle \text{proof} \rangle$

lemma *empty-set-less-eq-def [simp]*: $\{\} \sqsubseteq'' B \longleftrightarrow B \in \text{infinite-complement-partition}$
 $\langle \text{proof} \rangle$

end

context *linorder* **begin**

lemma *set-less-eq-aux'-trans*: $\llbracket A \sqsubseteq'' B; B \sqsubseteq'' C \rrbracket \Longrightarrow A \sqsubseteq'' C$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux'-porder*: *partial-order-on infinite-complement-partition* $\{(A, B). A \sqsubseteq'' B\}$
 $\langle \text{proof} \rangle$

end

Extend $(\sqsubseteq'')$ to a total order on *infinite-complement-partition*

context *ord* **begin**

definition *set-less-eq-aux''* :: '*a set* $\Rightarrow$ '*a set* $\Rightarrow$ bool (infix $\sqsubseteq''''$ 50)
 where *set-less-eq-aux''* =
 (SOME *sleq*).

$(\text{linear-order-on UNIV } \{(a, b). a \leq b\} \longrightarrow \text{linear-order-on infinite-complement-partition } \{(A, B). \text{sleq } A \ B\}) \wedge \text{order-consistent } \{(A, B). A \sqsubseteq'' B\} \{(A, B). \text{sleq } A \ B\})$

lemma *set-less-eq-aux''-spec:*

shows *linear-order* $\{(a, b). a \leq b\} \implies \text{linear-order-on infinite-complement-partition } \{(A, B). A \sqsubseteq''' B\}$
(is PROP ?thesis1)
and *order-consistent* $\{(A, B). A \sqsubseteq'' B\} \{(A, B). A \sqsubseteq''' B\}$ **(is ?thesis2)**
 $\langle \text{proof} \rangle$

end

context *linorder* **begin**

lemma *set-less-eq-aux''-linear-order:*

linear-order-on infinite-complement-partition $\{(A, B). A \sqsubseteq''' B\}$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux''-refl [iff]:* $A \sqsubseteq''' A \longleftrightarrow A \in \text{infinite-complement-partition}$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux'-into-set-less-eq-aux'':*

assumes $A \sqsubseteq'' B$
shows $A \sqsubseteq''' B$
 $\langle \text{proof} \rangle$

lemma *finite-set-less-eq-aux''-finite:*

assumes *finite* A **and** *finite* B
shows $A \sqsubseteq''' B \longleftrightarrow A \sqsubseteq'' B$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux''-finite:*

finite $(\text{UNIV} :: 'a \text{ set}) \implies \text{set-less-eq-aux}'' = \text{set-less-eq-aux}$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux''-antisym:*

$\llbracket A \sqsubseteq''' B; B \sqsubseteq''' A; \\ A \in \text{infinite-complement-partition}; B \in \text{infinite-complement-partition} \rrbracket \\ \implies A = B$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-aux''-trans:* $\llbracket A \sqsubseteq''' B; B \sqsubseteq''' C \rrbracket \implies A \sqsubseteq''' C$

$\langle \text{proof} \rangle$

lemma *set-less-eq-aux''-total:*

$\llbracket A \in \text{infinite-complement-partition}; B \in \text{infinite-complement-partition} \rrbracket \\ \implies A \sqsubseteq''' B \vee B \sqsubseteq''' A$
 $\langle \text{proof} \rangle$

end

Extend ($\sqsubseteq'''$) to cofinite sets

context *ord* **begin**

definition *set-less-eq* :: 'a set $\Rightarrow$ 'a set $\Rightarrow$ bool (**infix** $\langle \sqsubseteq \rangle$ 50)

where

$A \sqsubseteq B \longleftrightarrow$

(if $A \in \text{infinite-complement-partition}$ then $A \sqsubseteq''' B \vee B \notin \text{infinite-complement-partition}$
else $B \notin \text{infinite-complement-partition} \wedge -B \sqsubseteq''' -A$)

definition *set-less* :: 'a set $\Rightarrow$ 'a set $\Rightarrow$ bool (**infix** $\langle \sqsubset \rangle$ 50)

where $A \sqsubset B \longleftrightarrow A \sqsubseteq B \wedge \neg B \sqsubseteq A$

lemma *set-less-eq-def2*:

$A \sqsubseteq B \longleftrightarrow$

(if *finite* (*UNIV* :: 'a set) then $A \sqsubseteq''' B$

else if $A \in \text{infinite-complement-partition}$ then $A \sqsubseteq''' B \vee B \notin \text{infinite-complement-partition}$
else $B \notin \text{infinite-complement-partition} \wedge -B \sqsubseteq''' -A$)

$\langle \text{proof} \rangle$

end

context *linorder* **begin**

lemma *set-less-eq-refl* [*iff*]: $A \sqsubseteq A$

$\langle \text{proof} \rangle$

lemma *set-less-eq-antisym*: $\llbracket A \sqsubseteq B; B \sqsubseteq A \rrbracket \Longrightarrow A = B$

$\langle \text{proof} \rangle$

lemma *set-less-eq-trans*: $\llbracket A \sqsubseteq B; B \sqsubseteq C \rrbracket \Longrightarrow A \sqsubseteq C$

$\langle \text{proof} \rangle$

lemma *set-less-eq-total*: $A \sqsubseteq B \vee B \sqsubseteq A$

$\langle \text{proof} \rangle$

lemma *set-less-eq-linorder*: *class.linorder* ($\sqsubseteq$) ($\sqsubset$)

$\langle \text{proof} \rangle$

lemma *set-less-eq-conv-set-less*: $\text{set-less-eq } A B \longleftrightarrow A = B \vee \text{set-less } A B$

$\langle \text{proof} \rangle$

lemma *Compl-set-less-eq-Compl*: $-A \sqsubseteq -B \longleftrightarrow B \sqsubseteq A$

$\langle \text{proof} \rangle$

lemma *Compl-set-less-Compl*: $-A \sqsubset -B \longleftrightarrow B \sqsubset A$

$\langle \text{proof} \rangle$

lemma *set-less-eq-finite-iff*: $\llbracket \text{finite } A; \text{finite } B \rrbracket \implies A \sqsubseteq B \longleftrightarrow A \sqsubseteq' B$
 $\langle \text{proof} \rangle$

lemma *set-less-finite-iff*: $\llbracket \text{finite } A; \text{finite } B \rrbracket \implies A \sqsubset B \longleftrightarrow A \sqsubset' B$
 $\langle \text{proof} \rangle$

lemma *infinite-set-less-eq-Complement*:
 $\llbracket \text{finite } A; \text{finite } B; \neg \text{finite } (UNIV :: 'a \text{ set}) \rrbracket \implies A \sqsubseteq - B$
 $\langle \text{proof} \rangle$

lemma *infinite-set-less-Complement*:
 $\llbracket \text{finite } A; \text{finite } B; \neg \text{finite } (UNIV :: 'a \text{ set}) \rrbracket \implies A \sqsubset - B$
 $\langle \text{proof} \rangle$

lemma *infinite-Complement-set-less-eq*:
 $\llbracket \text{finite } A; \text{finite } B; \neg \text{finite } (UNIV :: 'a \text{ set}) \rrbracket \implies \neg - A \sqsubseteq B$
 $\langle \text{proof} \rangle$

lemma *infinite-Complement-set-less*:
 $\llbracket \text{finite } A; \text{finite } B; \neg \text{finite } (UNIV :: 'a \text{ set}) \rrbracket \implies \neg - A \sqsubset B$
 $\langle \text{proof} \rangle$

lemma *empty-set-less-eq [iff]*: $\{\} \sqsubseteq A$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-empty [iff]*: $A \sqsubseteq \{\} \longleftrightarrow A = \{\}$
 $\langle \text{proof} \rangle$

lemma *empty-set-less-iff [iff]*: $\{\} \sqsubset A \longleftrightarrow A \neq \{\}$
 $\langle \text{proof} \rangle$

lemma *not-set-less-empty [simp]*: $\neg A \sqsubset \{\}$
 $\langle \text{proof} \rangle$

lemma *set-less-eq-UNIV [iff]*: $A \sqsubseteq UNIV$
 $\langle \text{proof} \rangle$

lemma *UNIV-set-less-eq [iff]*: $UNIV \sqsubseteq A \longleftrightarrow A = UNIV$
 $\langle \text{proof} \rangle$

lemma *set-less-UNIV-iff [iff]*: $A \sqsubset UNIV \longleftrightarrow A \neq UNIV$
 $\langle \text{proof} \rangle$

lemma *not-UNIV-set-less [simp]*: $\neg UNIV \sqsubset A$
 $\langle \text{proof} \rangle$

end

2.6.2 Implementation based on sorted lists

type-synonym 'a proper-interval = 'a option $\Rightarrow$ 'a option $\Rightarrow$ bool

class proper-intvl = ord +
fixes proper-interval :: 'a proper-interval

class proper-interval = proper-intvl +
assumes proper-interval-simps:
 proper-interval None None = True
 proper-interval None (Some y) = ($\exists z. z < y$)
 proper-interval (Some x) None = ($\exists z. x < z$)
 proper-interval (Some x) (Some y) = ($\exists z. x < z \wedge z < y$)

context proper-intvl **begin**

function set-less-eq-aux-Compl :: 'a option $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ bool

where

set-less-eq-aux-Compl ao [] ys $\longleftrightarrow$ True
 | set-less-eq-aux-Compl ao xs [] $\longleftrightarrow$ True
 | set-less-eq-aux-Compl ao (x # xs) (y # ys) $\longleftrightarrow$
 (if x < y then proper-interval ao (Some x) $\vee$ set-less-eq-aux-Compl (Some x) xs
 (y # ys)
 else if y < x then proper-interval ao (Some y) $\vee$ set-less-eq-aux-Compl (Some y)
 (x # xs) ys
 else proper-interval ao (Some y))

$\langle$ proof $\rangle$

termination $\langle$ proof $\rangle$

fun Compl-set-less-eq-aux :: 'a option $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ bool

where

Compl-set-less-eq-aux ao [] [] $\longleftrightarrow \neg$ proper-interval ao None
 | Compl-set-less-eq-aux ao [] (y # ys) $\longleftrightarrow \neg$ proper-interval ao (Some y) $\wedge$ Compl-set-less-eq-aux
 (Some y) [] ys
 | Compl-set-less-eq-aux ao (x # xs) [] $\longleftrightarrow \neg$ proper-interval ao (Some x) $\wedge$ Compl-set-less-eq-aux
 (Some x) xs []
 | Compl-set-less-eq-aux ao (x # xs) (y # ys) $\longleftrightarrow$
 (if x < y then $\neg$ proper-interval ao (Some x) $\wedge$ Compl-set-less-eq-aux (Some x)
 xs (y # ys)
 else if y < x then $\neg$ proper-interval ao (Some y) $\wedge$ Compl-set-less-eq-aux (Some
 y) (x # xs) ys
 else $\neg$ proper-interval ao (Some y))

fun set-less-aux-Compl :: 'a option $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ bool **where**

set-less-aux-Compl ao [] [] $\longleftrightarrow$ proper-interval ao None
 | set-less-aux-Compl ao [] (y # ys) $\longleftrightarrow$ proper-interval ao (Some y) $\vee$ set-less-aux-Compl
 (Some y) [] ys
 | set-less-aux-Compl ao (x # xs) [] $\longleftrightarrow$ proper-interval ao (Some x) $\vee$ set-less-aux-Compl
 (Some x) xs []
 | set-less-aux-Compl ao (x # xs) (y # ys) $\longleftrightarrow$

```

    (if  $x < y$  then proper-interval ao (Some  $x$ )  $\vee$  set-less-aux-Compl (Some  $x$ )  $xs$  ( $y$ 
#  $ys$ )
    else if  $y < x$  then proper-interval ao (Some  $y$ )  $\vee$  set-less-aux-Compl (Some  $y$ ) ( $x$ 
#  $xs$ )  $ys$ 
    else proper-interval ao (Some  $y$ ))

```

```

function Compl-set-less-aux :: 'a option  $\Rightarrow$  'a list  $\Rightarrow$  'a list  $\Rightarrow$  bool where
  Compl-set-less-aux ao []  $\longleftrightarrow$  False
| Compl-set-less-aux ao  $xs$  []  $\longleftrightarrow$  False
| Compl-set-less-aux ao ( $x$  #  $xs$ ) ( $y$  #  $ys$ )  $\longleftrightarrow$ 
  (if  $x < y$  then  $\neg$  proper-interval ao (Some  $x$ )  $\wedge$  Compl-set-less-aux (Some  $x$ )  $xs$ 
( $y$  #  $ys$ )
  else if  $y < x$  then  $\neg$  proper-interval ao (Some  $y$ )  $\wedge$  Compl-set-less-aux (Some  $y$ )
( $x$  #  $xs$ )  $ys$ 
  else  $\neg$  proper-interval ao (Some  $y$ ))
<proof>
termination <proof>

```

end

```

lemmas [code] =
  proper-intrvl.set-less-eq-aux-Compl.simps
  proper-intrvl.set-less-aux-Compl.simps
  proper-intrvl.Compl-set-less-eq-aux.simps
  proper-intrvl.Compl-set-less-aux.simps

```

```

class linorder-proper-interval = linorder + proper-interval
begin

```

```

theorem assumes fin: finite (UNIV :: 'a set)
  and  $xs$ : sorted  $xs$  distinct  $xs$ 
  and  $ys$ : sorted  $ys$  distinct  $ys$ 
  shows set-less-eq-aux-Compl2-conv-set-less-eq-aux-Compl:
     $set\ xs \sqsubseteq' - set\ ys \longleftrightarrow set-less-eq-aux-Compl\ None\ xs\ ys$  (is ?Compl2)
  and Compl1-set-less-eq-aux-conv-Compl-set-less-eq-aux:
     $- set\ xs \sqsubseteq' set\ ys \longleftrightarrow Compl-set-less-eq-aux\ None\ xs\ ys$  (is ?Compl1)
  <proof>

```

```

lemma set-less-aux-Compl-iff:
   $set-less-aux-Compl\ ao\ xs\ ys \longleftrightarrow set-less-eq-aux-Compl\ ao\ xs\ ys \wedge \neg Compl-set-less-eq-aux$ 
 $ao\ ys\ xs$ 
  <proof>

```

```

lemma Compl-set-less-aux-Compl-iff:
   $Compl-set-less-aux\ ao\ xs\ ys \longleftrightarrow Compl-set-less-eq-aux\ ao\ xs\ ys \wedge \neg set-less-eq-aux-Compl$ 
 $ao\ ys\ xs$ 
  <proof>

```

```

theorem assumes fin: finite (UNIV :: 'a set)

```

```

and xs: sorted xs    distinct xs
and ys: sorted ys    distinct ys
shows set-less-aux-Compl2-conv-set-less-aux-Compl:
  set xs  $\sqsubset'$  set ys  $\longleftrightarrow$  set-less-aux-Compl None xs ys (is ?Compl2)
and Compl1-set-less-aux-conv-Compl-set-less-aux:
   $\neg$  set xs  $\sqsubset'$  set ys  $\longleftrightarrow$  Compl-set-less-aux None xs ys (is ?Compl1)
 $\langle$ proof $\rangle$ 

end

```

2.6.3 Implementation of proper intervals for sets

definition *length-last* :: 'a list $\Rightarrow$ nat $\times$ 'a
where *length-last xs* = (*length xs*, *last xs*)

lemma *length-last-Nil* [*code*]:
length-last [] = (0, *undefined*)
 $\langle$ *proof* $\rangle$

lemma *length-last-Cons-code* [*symmetric*, *code*]:
fold ($\lambda x (n, -) . (n + 1, x)$) *xs* (1, *x*) = *length-last* (*x* # *xs*)
 $\langle$ *proof* $\rangle$

context *proper-interval* **begin**

fun *exhaustive-above* :: 'a $\Rightarrow$ 'a list $\Rightarrow$ bool **where**
exhaustive-above *x []* $\longleftrightarrow$ $\neg$ *proper-interval* (*Some x*) *None*
 $\mid$ *exhaustive-above* *x* (*y* # *ys*) $\longleftrightarrow$ $\neg$ *proper-interval* (*Some x*) (*Some y*) $\wedge$ *exhaustive-above* *y* *ys*

fun *exhaustive* :: 'a list $\Rightarrow$ bool **where**
exhaustive [] = *False*
 $\mid$ *exhaustive* (*x* # *xs*) $\longleftrightarrow$ $\neg$ *proper-interval* *None* (*Some x*) $\wedge$ *exhaustive-above* *x* *xs*

fun *proper-interval-set-aux* :: 'a list $\Rightarrow$ 'a list $\Rightarrow$ bool
where
proper-interval-set-aux *xs []* $\longleftrightarrow$ *False*
 $\mid$ *proper-interval-set-aux* [] (*y* # *ys*) $\longleftrightarrow$ *ys* $\neq$ [] $\vee$ *proper-interval* (*Some y*) *None*
 $\mid$ *proper-interval-set-aux* (*x* # *xs*) (*y* # *ys*) $\longleftrightarrow$
 (*if* *x* < *y* *then False*
 else if *y* < *x* *then proper-interval* (*Some y*) (*Some x*) $\vee$ *ys* $\neq$ [] $\vee$ $\neg$ *exhaustive-above* *x* *xs*
 else proper-interval-set-aux *xs* *ys*)

fun *proper-interval-set-Compl-aux* :: 'a option $\Rightarrow$ nat $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ bool
where
proper-interval-set-Compl-aux *ao n [] []* $\longleftrightarrow$
CARD('a) > *n* + 1

```

| proper-interval-set-Compl-aux ao n [] (y # ys) <=>
  (let m = CARD('a) - n; (len-y, y') = length-last (y # ys)
   in m ≠ len-y ∧ (m = len-y + 1 → ¬ proper-interval (Some y') None))
| proper-interval-set-Compl-aux ao n (x # xs) [] <=>
  (let m = CARD('a) - n; (len-x, x') = length-last (x # xs)
   in m ≠ len-x ∧ (m = len-x + 1 → ¬ proper-interval (Some x') None))
| proper-interval-set-Compl-aux ao n (x # xs) (y # ys) <=>
  (if x < y then
    proper-interval ao (Some x) ∨
    proper-interval-set-Compl-aux (Some x) (n + 1) xs (y # ys)
  else if y < x then
    proper-interval ao (Some y) ∨
    proper-interval-set-Compl-aux (Some y) (n + 1) (x # xs) ys
  else proper-interval ao (Some x) ∧
    (let m = card (UNIV :: 'a set) - n in m - length ys ≠ 2 ∨ m - length xs ≠
    2))

```

fun proper-interval-Compl-set-aux :: 'a option ⇒ 'a list ⇒ 'a list ⇒ bool
where

```

proper-interval-Compl-set-aux ao (x # xs) (y # ys) <=>
  (if x < y then
    ¬ proper-interval ao (Some x) ∧
    proper-interval-Compl-set-aux (Some x) xs (y # ys)
  else if y < x then
    ¬ proper-interval ao (Some y) ∧
    proper-interval-Compl-set-aux (Some y) (x # xs) ys
  else ¬ proper-interval ao (Some x) ∧ (ys = [] → xs ≠ []))
| proper-interval-Compl-set-aux ao - - <=> False

```

end

lemmas [code] =
 proper-intrvl.exhaustive-above.simps
 proper-intrvl.exhaustive.simps
 proper-intrvl.proper-interval-set-aux.simps
 proper-intrvl.proper-interval-set-Compl-aux.simps
 proper-intrvl.proper-interval-Compl-set-aux.simps

context linorder-proper-interval **begin**

lemma exhaustive-above-iff:

[[sorted xs; distinct xs; ∀ x' ∈ set xs. x < x']] ⇒ exhaustive-above x xs <=> set
 xs = {z. z > x}
 <proof>

lemma exhaustive-correct:

assumes sorted xs distinct xs
shows exhaustive xs <=> set xs = UNIV
 <proof>

```

theorem proper-interval-set-aux:
  assumes fin: finite (UNIV :: 'a set)
  and xs: sorted xs    distinct xs
  and ys: sorted ys    distinct ys
  shows proper-interval-set-aux xs ys  $\longleftrightarrow$  ( $\exists A. \text{set } xs \sqsubset' A \wedge A \sqsubset' \text{set } ys$ )
   $\langle \text{proof} \rangle$ 

lemma proper-interval-set-Compl-aux:
  assumes fin: finite (UNIV :: 'a set)
  and xs: sorted xs    distinct xs
  and ys: sorted ys    distinct ys
  shows proper-interval-set-Compl-aux None 0 xs ys  $\longleftrightarrow$  ( $\exists A. \text{set } xs \sqsubset' A \wedge A \sqsubset' - \text{set } ys$ )
   $\langle \text{proof} \rangle$ 

lemma proper-interval-Compl-set-aux:
  assumes fin: finite (UNIV :: 'a set)
  and xs: sorted xs    distinct xs
  and ys: sorted ys    distinct ys
  shows proper-interval-Compl-set-aux None xs ys  $\longleftrightarrow$  ( $\exists A. - \text{set } xs \sqsubset' A \wedge A \sqsubset' \text{set } ys$ )
   $\langle \text{proof} \rangle$ 

end

```

2.6.4 Proper intervals for HOL types

```

instantiation unit :: proper-interval begin
fun proper-interval-unit :: unit proper-interval where
  proper-interval-unit None None = True
| proper-interval-unit - - = False
instance  $\langle \text{proof} \rangle$ 
end

```

```

instantiation bool :: proper-interval begin
fun proper-interval-bool :: bool proper-interval where
  proper-interval-bool (Some x) (Some y)  $\longleftrightarrow$  False
| proper-interval-bool (Some x) None  $\longleftrightarrow$   $\neg x$ 
| proper-interval-bool None (Some y)  $\longleftrightarrow$  y
| proper-interval-bool None None = True
instance  $\langle \text{proof} \rangle$ 
end

```

```

instantiation nat :: proper-interval begin
fun proper-interval-nat :: nat proper-interval where
  proper-interval-nat no None = True
| proper-interval-nat None (Some x)  $\longleftrightarrow$   $x > 0$ 
| proper-interval-nat (Some x) (Some y)  $\longleftrightarrow$   $y - x > 1$ 

```

```

instance ⟨proof⟩
end

instantiation int :: proper-interval begin
fun proper-interval-int :: int proper-interval where
  proper-interval-int (Some x) (Some y)  $\longleftrightarrow y - x > 1$ 
| proper-interval-int - - = True
instance ⟨proof⟩
end

instantiation integer :: proper-interval begin
context includes integer.lifting begin
lift-definition proper-interval-integer :: integer proper-interval is proper-interval
  ⟨proof⟩
instance ⟨proof⟩
end
end
lemma proper-interval-integer-simps [code]:
  includes integer.lifting fixes x y :: integer and xo yo :: integer option shows
    proper-interval (Some x) (Some y) = (1 < y - x)
    proper-interval None yo = True
    proper-interval xo None = True
  ⟨proof⟩

instantiation natural :: proper-interval begin
context includes natural.lifting begin
lift-definition proper-interval-natural :: natural proper-interval is proper-interval
  ⟨proof⟩
instance ⟨proof⟩
end
end
lemma proper-interval-natural-simps [code]:
  includes natural.lifting fixes x y :: natural and xo :: natural option shows
    proper-interval xo None = True
    proper-interval None (Some y)  $\longleftrightarrow y > 0$ 
    proper-interval (Some x) (Some y)  $\longleftrightarrow y - x > 1$ 
  ⟨proof⟩

lemma char-less-iff-nat-of-char:  $x < y \longleftrightarrow \text{of-char } x < (\text{of-char } y :: \text{nat})$ 
  ⟨proof⟩

lemma nat-of-char-inject [simp]:  $\text{of-char } x = (\text{of-char } y :: \text{nat}) \longleftrightarrow x = y$ 
  ⟨proof⟩

lemma char-le-iff-nat-of-char:  $x \leq y \longleftrightarrow \text{of-char } x \leq (\text{of-char } y :: \text{nat})$ 
  ⟨proof⟩

instantiation char :: proper-interval
begin

```

```

fun proper-interval-char :: char proper-interval where
  proper-interval-char None None  $\longleftrightarrow$  True
| proper-interval-char None (Some x)  $\longleftrightarrow$  x  $\neq$  CHR 0x00
| proper-interval-char (Some x) None  $\longleftrightarrow$  x  $\neq$  CHR 0xFF
| proper-interval-char (Some x) (Some y)  $\longleftrightarrow$  of-char y - of-char x > (1 :: nat)

instance <proof>

end

instantiation Enum.finite-1 :: proper-interval begin
definition proper-interval-finite-1 :: Enum.finite-1 proper-interval
where proper-interval-finite-1 x y  $\longleftrightarrow$  x = None  $\wedge$  y = None
instance <proof>
end

instantiation Enum.finite-2 :: proper-interval begin
fun proper-interval-finite-2 :: Enum.finite-2 proper-interval where
  proper-interval-finite-2 None None  $\longleftrightarrow$  True
| proper-interval-finite-2 None (Some x)  $\longleftrightarrow$  x = finite-2.a2
| proper-interval-finite-2 (Some x) None  $\longleftrightarrow$  x = finite-2.a1
| proper-interval-finite-2 (Some x) (Some y)  $\longleftrightarrow$  False
instance <proof>
end

instantiation Enum.finite-3 :: proper-interval begin
fun proper-interval-finite-3 :: Enum.finite-3 proper-interval where
  proper-interval-finite-3 None None  $\longleftrightarrow$  True
| proper-interval-finite-3 None (Some x)  $\longleftrightarrow$  x  $\neq$  finite-3.a1
| proper-interval-finite-3 (Some x) None  $\longleftrightarrow$  x  $\neq$  finite-3.a3
| proper-interval-finite-3 (Some x) (Some y)  $\longleftrightarrow$  x = finite-3.a1  $\wedge$  y = finite-3.a3
instance
  <proof>
end

```

2.6.5 List fusion for the order and proper intervals on 'a set

```

definition length-last-fusion :: ('a, 's) generator  $\Rightarrow$  's  $\Rightarrow$  nat  $\times$  'a
where length-last-fusion g s = length-last (list.unfoldr g s)

```

```

lemma length-last-fusion-code [code]:
  length-last-fusion g s =
    (if list.has-next g s then
      let (x, s') = list.next g s
      in fold-fusion g ( $\lambda$ x (n, -). (n + 1, x)) s' (1, x)
    else (0, undefined))
  <proof>

```

declare *length-last-fusion-def* [*symmetric*, *code-unfold*]

context *proper-intvl* **begin**

definition *set-less-eq-aux-Compl-fusion* :: ('a, 's1) generator $\Rightarrow$ ('a, 's2) generator
 $\Rightarrow$ 'a option $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

set-less-eq-aux-Compl-fusion g1 g2 ao s1 s2 =
set-less-eq-aux-Compl ao (list.unfoldr g1 s1) (list.unfoldr g2 s2)

definition *Compl-set-less-eq-aux-fusion* :: ('a, 's1) generator $\Rightarrow$ ('a, 's2) generator
 $\Rightarrow$ 'a option $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

Compl-set-less-eq-aux-fusion g1 g2 ao s1 s2 =
Compl-set-less-eq-aux ao (list.unfoldr g1 s1) (list.unfoldr g2 s2)

definition *set-less-aux-Compl-fusion* :: ('a, 's1) generator $\Rightarrow$ ('a, 's2) generator
 $\Rightarrow$ 'a option $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

set-less-aux-Compl-fusion g1 g2 ao s1 s2 =
set-less-aux-Compl ao (list.unfoldr g1 s1) (list.unfoldr g2 s2)

definition *Compl-set-less-aux-fusion* :: ('a, 's1) generator $\Rightarrow$ ('a, 's2) generator
 $\Rightarrow$ 'a option $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

Compl-set-less-aux-fusion g1 g2 ao s1 s2 =
Compl-set-less-aux ao (list.unfoldr g1 s1) (list.unfoldr g2 s2)

definition *exhaustive-above-fusion* :: ('a, 's) generator $\Rightarrow$ 'a $\Rightarrow$'s $\Rightarrow$ bool

where *exhaustive-above-fusion* g a s = *exhaustive-above* a (list.unfoldr g s)

definition *exhaustive-fusion* :: ('a, 's) generator $\Rightarrow$'s $\Rightarrow$ bool

where *exhaustive-fusion* g s = *exhaustive* (list.unfoldr g s)

definition *proper-interval-set-aux-fusion* :: ('a, 's1) generator $\Rightarrow$ ('a, 's2) generator
 $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

proper-interval-set-aux-fusion g1 g2 s1 s2 =
proper-interval-set-aux (list.unfoldr g1 s1) (list.unfoldr g2 s2)

definition *proper-interval-set-Compl-aux-fusion* ::

('a, 's1) generator $\Rightarrow$ ('a, 's2) generator $\Rightarrow$ 'a option $\Rightarrow$ nat $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

proper-interval-set-Compl-aux-fusion g1 g2 ao n s1 s2 =
proper-interval-set-Compl-aux ao n (list.unfoldr g1 s1) (list.unfoldr g2 s2)

definition *proper-interval-Compl-set-aux-fusion* ::

('a, 's1) generator $\Rightarrow$ ('a, 's2) generator $\Rightarrow$ 'a option $\Rightarrow$ 's1 $\Rightarrow$ 's2 $\Rightarrow$ bool

where

proper-interval-Compl-set-aux-fusion $g1\ g2\ ao\ s1\ s2 =$
proper-interval-Compl-set-aux $ao\ (list.unfoldr\ g1\ s1)\ (list.unfoldr\ g2\ s2)$

lemma *set-less-eq-aux-Compl-fusion-code:*

set-less-eq-aux-Compl-fusion $g1\ g2\ ao\ s1\ s2 \longleftrightarrow$
 $(list.has-next\ g1\ s1 \longrightarrow list.has-next\ g2\ s2 \longrightarrow$
 $(let\ (x,\ s1') = list.next\ g1\ s1;$
 $(y,\ s2') = list.next\ g2\ s2$
 $in\ if\ x < y\ then\ proper-interval\ ao\ (Some\ x) \vee set-less-eq-aux-Compl-fusion\ g1$
 $g2\ (Some\ x)\ s1'\ s2$
 $else\ if\ y < x\ then\ proper-interval\ ao\ (Some\ y) \vee set-less-eq-aux-Compl-fusion$
 $g1\ g2\ (Some\ y)\ s1\ s2'$
 $else\ proper-interval\ ao\ (Some\ y)))$
 $\langle proof \rangle$

lemma *Compl-set-less-eq-aux-fusion-code:*

Compl-set-less-eq-aux-fusion $g1\ g2\ ao\ s1\ s2 \longleftrightarrow$
 $(if\ list.has-next\ g1\ s1\ then$
 $let\ (x,\ s1') = list.next\ g1\ s1$
 $in\ if\ list.has-next\ g2\ s2\ then$
 $let\ (y,\ s2') = list.next\ g2\ s2$
 $in\ if\ x < y\ then\ \neg\ proper-interval\ ao\ (Some\ x) \wedge Compl-set-less-eq-aux-fusion$
 $g1\ g2\ (Some\ x)\ s1'\ s2$
 $else\ if\ y < x\ then\ \neg\ proper-interval\ ao\ (Some\ y) \wedge Compl-set-less-eq-aux-fusion$
 $g1\ g2\ (Some\ y)\ s1\ s2'$
 $else\ \neg\ proper-interval\ ao\ (Some\ y)$
 $else\ \neg\ proper-interval\ ao\ (Some\ x) \wedge Compl-set-less-eq-aux-fusion\ g1\ g2$
 $(Some\ x)\ s1'\ s2$
 $else\ if\ list.has-next\ g2\ s2\ then$
 $let\ (y,\ s2') = list.next\ g2\ s2$
 $in\ \neg\ proper-interval\ ao\ (Some\ y) \wedge Compl-set-less-eq-aux-fusion\ g1\ g2\ (Some$
 $y)\ s1\ s2'$
 $else\ \neg\ proper-interval\ ao\ None)$
 $\langle proof \rangle$

lemma *set-less-aux-Compl-fusion-code:*

set-less-aux-Compl-fusion $g1\ g2\ ao\ s1\ s2 \longleftrightarrow$
 $(if\ list.has-next\ g1\ s1\ then$
 $let\ (x,\ s1') = list.next\ g1\ s1$
 $in\ if\ list.has-next\ g2\ s2\ then$
 $let\ (y,\ s2') = list.next\ g2\ s2$
 $in\ if\ x < y\ then\ proper-interval\ ao\ (Some\ x) \vee set-less-aux-Compl-fusion$
 $g1\ g2\ (Some\ x)\ s1'\ s2$
 $else\ if\ y < x\ then\ proper-interval\ ao\ (Some\ y) \vee set-less-aux-Compl-fusion$
 $g1\ g2\ (Some\ y)\ s1\ s2'$
 $else\ proper-interval\ ao\ (Some\ y)$
 $else\ proper-interval\ ao\ (Some\ x) \vee set-less-aux-Compl-fusion\ g1\ g2\ (Some$
 $x)\ s1'\ s2$

else if $\text{list.has-next } g2 \ s2$ then
 let $(y, s2') = \text{list.next } g2 \ s2$
 in $\text{proper-interval } ao \ (\text{Some } y) \vee \text{set-less-aux-Compl-fusion } g1 \ g2 \ (\text{Some } y) \ s1$
 $s2'$
 else $\text{proper-interval } ao \ \text{None}$
 $\langle \text{proof} \rangle$

lemma *Compl-set-less-aux-fusion-code:*

$\text{Compl-set-less-aux-fusion } g1 \ g2 \ ao \ s1 \ s2 \longleftrightarrow$
 $\text{list.has-next } g1 \ s1 \wedge \text{list.has-next } g2 \ s2 \wedge$
 $(\text{let } (x, s1') = \text{list.next } g1 \ s1;$
 $(y, s2') = \text{list.next } g2 \ s2$
 in if $x < y$ then $\neg \text{proper-interval } ao \ (\text{Some } x) \wedge \text{Compl-set-less-aux-fusion } g1$
 $g2 \ (\text{Some } x) \ s1' \ s2$
 else if $y < x$ then $\neg \text{proper-interval } ao \ (\text{Some } y) \wedge \text{Compl-set-less-aux-fusion}$
 $g1 \ g2 \ (\text{Some } y) \ s1 \ s2'$
 else $\neg \text{proper-interval } ao \ (\text{Some } y)$
 $\langle \text{proof} \rangle$

lemma *exhaustive-above-fusion-code:*

$\text{exhaustive-above-fusion } g \ y \ s \longleftrightarrow$
 (if $\text{list.has-next } g \ s$ then
 let $(x, s') = \text{list.next } g \ s$
 in $\neg \text{proper-interval } (\text{Some } y) \ (\text{Some } x) \wedge \text{exhaustive-above-fusion } g \ x \ s'$
 else $\neg \text{proper-interval } (\text{Some } y) \ \text{None}$)
 $\langle \text{proof} \rangle$

lemma *exhaustive-fusion-code:*

$\text{exhaustive-fusion } g \ s =$
 $(\text{list.has-next } g \ s \wedge$
 $(\text{let } (x, s') = \text{list.next } g \ s$
 in $\neg \text{proper-interval } \text{None} \ (\text{Some } x) \wedge \text{exhaustive-above-fusion } g \ x \ s'))$
 $\langle \text{proof} \rangle$

lemma *proper-interval-set-aux-fusion-code:*

$\text{proper-interval-set-aux-fusion } g1 \ g2 \ s1 \ s2 \longleftrightarrow$
 $\text{list.has-next } g2 \ s2 \wedge$
 $(\text{let } (y, s2') = \text{list.next } g2 \ s2$
 in if $\text{list.has-next } g1 \ s1$ then
 let $(x, s1') = \text{list.next } g1 \ s1$
 in if $x < y$ then False
 else if $y < x$ then $\text{proper-interval } (\text{Some } y) \ (\text{Some } x) \vee \text{list.has-next } g2$
 $s2' \vee \neg \text{exhaustive-above-fusion } g1 \ x \ s1'$
 else $\text{proper-interval-set-aux-fusion } g1 \ g2 \ s1' \ s2'$
 else $\text{list.has-next } g2 \ s2' \vee \text{proper-interval } (\text{Some } y) \ \text{None}$)
 $\langle \text{proof} \rangle$

lemma *proper-interval-set-Compl-aux-fusion-code:*

$\text{proper-interval-set-Compl-aux-fusion } g1 \ g2 \ ao \ n \ s1 \ s2 \longleftrightarrow$

```

(if list.has-next g1 s1 then
  let (x, s1') = list.next g1 s1
  in if list.has-next g2 s2 then
    let (y, s2') = list.next g2 s2
    in if x < y then
      proper-interval ao (Some x) ∨
      proper-interval-set-Compl-aux-fusion g1 g2 (Some x) (n + 1) s1' s2
    else if y < x then
      proper-interval ao (Some y) ∨
      proper-interval-set-Compl-aux-fusion g1 g2 (Some y) (n + 1) s1 s2'
    else
      proper-interval ao (Some x) ∧
      (let m = CARD('a) - n
       in m - length-fusion g2 s2' ≠ 2 ∨ m - length-fusion g1 s1' ≠ 2)
  else
    let m = CARD('a) - n; (len-x, x') = length-last-fusion g1 s1
    in m ≠ len-x ∧ (m = len-x + 1 → ¬ proper-interval (Some x') None)

else if list.has-next g2 s2 then
  let (y, s2') = list.next g2 s2;
  m = CARD('a) - n;
  (len-y, y') = length-last-fusion g2 s2
  in m ≠ len-y ∧ (m = len-y + 1 → ¬ proper-interval (Some y') None)
else CARD('a) > n + 1)
⟨proof⟩

```

lemma *proper-interval-Compl-set-aux-fusion-code*:

```

proper-interval-Compl-set-aux-fusion g1 g2 ao s1 s2 ↔
list.has-next g1 s1 ∧ list.has-next g2 s2 ∧
(let (x, s1') = list.next g1 s1;
 (y, s2') = list.next g2 s2
 in if x < y then
   ¬ proper-interval ao (Some x) ∧ proper-interval-Compl-set-aux-fusion g1 g2
   (Some x) s1' s2
   else if y < x then
     ¬ proper-interval ao (Some y) ∧ proper-interval-Compl-set-aux-fusion g1 g2
     (Some y) s1 s2'
   else ¬ proper-interval ao (Some x) ∧ (list.has-next g2 s2' ∨ list.has-next g1
   s1'))
⟨proof⟩

```

end

lemmas [code] =

```

set-less-eq-aux-Compl-fusion-code proper-intrvl.set-less-eq-aux-Compl-fusion-code
Compl-set-less-eq-aux-fusion-code proper-intrvl.Compl-set-less-eq-aux-fusion-code
set-less-aux-Compl-fusion-code proper-intrvl.set-less-aux-Compl-fusion-code
Compl-set-less-aux-fusion-code proper-intrvl.Compl-set-less-aux-fusion-code
exhaustive-above-fusion-code proper-intrvl.exhaustive-above-fusion-code

```

```

    exhaustive-fusion-code proper-intrvl.exhaustive-fusion-code
    proper-interval-set-aux-fusion-code proper-intrvl.proper-interval-set-aux-fusion-code
    proper-interval-set-Compl-aux-fusion-code proper-intrvl.proper-interval-set-Compl-aux-fusion-code
    proper-interval-Compl-set-aux-fusion-code proper-intrvl.proper-interval-Compl-set-aux-fusion-code

```

```

lemmas [symmetric, code-unfold] =
    set-less-eq-aux-Compl-fusion-def proper-intrvl.set-less-eq-aux-Compl-fusion-def
    Compl-set-less-eq-aux-fusion-def proper-intrvl.Compl-set-less-eq-aux-fusion-def
    set-less-aux-Compl-fusion-def proper-intrvl.set-less-aux-Compl-fusion-def
    Compl-set-less-aux-fusion-def proper-intrvl.Compl-set-less-aux-fusion-def
    exhaustive-above-fusion-def proper-intrvl.exhaustive-above-fusion-def
    exhaustive-fusion-def proper-intrvl.exhaustive-fusion-def
    proper-interval-set-aux-fusion-def proper-intrvl.proper-interval-set-aux-fusion-def
    proper-interval-set-Compl-aux-fusion-def proper-intrvl.proper-interval-set-Compl-aux-fusion-def
    proper-interval-Compl-set-aux-fusion-def proper-intrvl.proper-interval-Compl-set-aux-fusion-def

```

2.6.6 Drop notation

```
context ord begin
```

```

no-notation set-less-aux (infix  $\langle \sqsubset'' \rangle$  50)
  and set-less-eq-aux (infix  $\langle \sqsubseteq'' \rangle$  50)
  and set-less-eq-aux' (infix  $\langle \sqsubseteq''' \rangle$  50)
  and set-less-eq-aux'' (infix  $\langle \sqsubseteq'''' \rangle$  50)
  and set-less-eq (infix  $\langle \sqsubseteq \rangle$  50)
  and set-less (infix  $\langle \sqsubset \rangle$  50)

```

```
end
```

```
end
```

```
theory Containers-Generator
```

```
imports
```

```

    Deriving-Generator-Aux
    Deriving-Derive-Manager
    HOL-Library-Phantom-Type
    Containers-Auxiliary

```

```
begin
```

2.6.7 Introduction

In the following, we provide generators for the major classes of the container framework: `ceq`, `corder`, `cenum`, `set-impl`, and `mapping-impl`.

In this file we provide some common infrastructure on the ML-level which will be used by the individual generators.

```
 $\langle ML \rangle$ 
```

```
end
```

```
theory Collection-Order
imports
  Set-Linorder
  Containers-Generator
  Deriving.Compare-Instances
begin
```

Chapter 3

Light-weight containers

3.1 A linear order for code generation

3.1.1 Optional comparators

```
class ccompare =  
  fixes ccompare :: 'a comparator option  
  assumes ccompare:  $\bigwedge$  comp. ccompare = Some comp  $\implies$  comparator comp  
begin  
  abbreviation ccomp :: 'a comparator where ccomp  $\equiv$  the (ID ccompare)  
  abbreviation cless :: 'a  $\Rightarrow$  'a  $\Rightarrow$  bool where cless  $\equiv$  lt-of-comp (the (ID ccompare))  
  abbreviation cless-eq :: 'a  $\Rightarrow$  'a  $\Rightarrow$  bool where cless-eq  $\equiv$  le-of-comp (the (ID ccompare))  
end  
  
lemma (in ccompare) ID-ccompare':  
   $\bigwedge c. ID\ ccompare = Some\ c \implies$  comparator  $c$   
   $\langle proof \rangle$   
  
lemma (in ccompare) ID-ccompare:  
   $\bigwedge c. ID\ ccompare = Some\ c \implies$  class.linorder (le-of-comp  $c$ ) (lt-of-comp  $c$ )  
   $\langle proof \rangle$   
  
syntax -CCOMPARE :: type  $\Rightarrow$  logic  $\langle \langle 1CCOMPARE / (1'(-')) \rangle \rangle$   
  
syntax-consts -CCOMPARE == ccompare  
  
 $\langle ML \rangle$   
  
definition is-ccompare :: 'a :: ccompare itself  $\Rightarrow$  bool  
where is-ccompare -  $\longleftrightarrow ID\ CCOMPARE('a) \neq None$   
  
context ccompare  
begin
```

```

lemma cless-eq-conv-cless:
  fixes  $a\ b :: 'a$ 
  assumes  $ID\ CCOMPARE('a) \neq None$ 
  shows  $cless\text{-}eq\ a\ b \longleftrightarrow cless\ a\ b \vee a = b$ 
   $\langle proof \rangle$ 
end

```

3.1.2 Generator for the *ccompare*-class

This generator registers itself at the derive-manager for the class *ccompare*. To be more precise, one can choose whether one does not want to support a comparator by passing parameter "no", one wants to register an arbitrary type which is already in class *compare* using parameter "compare", or one wants to generate a new comparator by passing no parameter. In the last case, one demands that the type is a datatype and that all non-recursive types of that datatype already provide a comparator, which can usually be achieved via "derive comparator type" or "derive compare type".

- **instantiation type** :: (type,...,type) (no) corder
- **instantiation datatype** :: (type,...,type) corder
- **instantiation datatype** :: (compare,...,compare) (compare) corder

If the parameter "no" is not used, then the corresponding *is-ccompare*-theorem is automatically generated and attributed with [simp, code-post].

To create a new comparator, we just invoke the functionality provided by the generator. The only difference is the boilerplate-code, which for the generator has to perform the class instantiation for a comparator, whereas here we have to invoke the methods to satisfy the corresponding locale for comparators.

This generator can be used for arbitrary types, not just datatypes. When passing no parameters, we get same limitation as for the order generator.

```

lemma corder-intro:  $class.linorder\ le\ lt \Longrightarrow a = Some\ (le,\ lt) \Longrightarrow a = Some\ (le',lt')$ 
 $\Longrightarrow$ 
 $class.linorder\ le'\ lt' \langle proof \rangle$ 

```

```

lemma comparator-subst:  $c1 = c2 \Longrightarrow comparator\ c1 \Longrightarrow comparator\ c2 \langle proof \rangle$ 

```

```

lemma (in compare) compare-subst:  $\bigwedge\ comp.\ compare = comp \Longrightarrow comparator\ comp$ 
 $\langle proof \rangle$ 

```

$\langle ML \rangle$

3.1.3 Instantiations for HOL types

```

derive (linorder) compare-order
  Enum.finite-1 Enum.finite-2 Enum.finite-3 natural String.literal
derive (compare) ccompare
  unit bool nat int Enum.finite-1 Enum.finite-2 Enum.finite-3 integer natural char
  String.literal
derive (no) ccompare Enum.finite-4 Enum.finite-5

derive ccompare sum list option prod

derive (no) ccompare fun

lemma is-ccompare-fun [simp]:  $\neg$  is-ccompare TYPE('a  $\Rightarrow$  'b)
  <proof>

instantiation set :: (ccompare) ccompare begin
definition CCOMPARE('a set) =
  map-option ( $\lambda$  c. comp-of-ords (ord.set-less-eq (le-of-comp c)) (ord.set-less (le-of-comp
  c))) (ID CCOMPARE('a))
instance <proof>
end

lemma is-ccompare-set [simp, code-post]:
  is-ccompare TYPE('a set)  $\longleftrightarrow$  is-ccompare TYPE('a :: ccompare)
  <proof>

definition cless-eq-set :: 'a :: ccompare set  $\Rightarrow$  'a set  $\Rightarrow$  bool
where [simp, code del]: cless-eq-set = le-of-comp (the (ID CCOMPARE('a set)))

definition cless-set :: 'a :: ccompare set  $\Rightarrow$  'a set  $\Rightarrow$  bool
where [simp, code del]: cless-set = lt-of-comp (the (ID CCOMPARE('a set)))

lemma ccompare-set-code [code]:
  CCOMPARE('a :: ccompare set) =
    (case ID CCOMPARE('a) of None  $\Rightarrow$  None | Some -  $\Rightarrow$  Some (comp-of-ords
    cless-eq-set cless-set))
  <proof>

derive (no) ccompare Predicate.pred

```

3.1.4 Proper intervals

```

class cproper-interval = ccompare +
  fixes cproper-interval :: 'a option  $\Rightarrow$  'a option  $\Rightarrow$  bool
  assumes cproper-interval:
    [ ID CCOMPARE('a)  $\neq$  None; finite (UNIV :: 'a set) ]
     $\Rightarrow$  class.proper-interval cless cproper-interval
begin

```

lemma *ID-ccompare-interval*:

$\llbracket ID\ CCOMPARE('a) = Some\ c; finite\ (UNIV :: 'a\ set) \rrbracket$
 $\impl class.linorder-proper-interval\ (le-of-comp\ c)\ (lt-of-comp\ c)\ cproper-interval$
 $\langle proof \rangle$

end

instantiation *unit* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *unit proper-interval*)

instance $\langle proof \rangle$

end

instantiation *bool* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *bool proper-interval*)

instance $\langle proof \rangle$

end

instantiation *nat* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *nat proper-interval*)

instance $\langle proof \rangle$

end

instantiation *int* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *int proper-interval*)

instance $\langle proof \rangle$

end

instantiation *integer* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *integer proper-interval*)

instance $\langle proof \rangle$

end

instantiation *natural* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *natural proper-interval*)

instance $\langle proof \rangle$

end

instantiation *char* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *char proper-interval*)

instance $\langle proof \rangle$

end

instantiation *Enum.finite-1* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *Enum.finite-1 proper-interval*)

instance $\langle proof \rangle$

end

instantiation *Enum.finite-2* :: *cproper-interval* **begin**

definition *cproper-interval* = (*proper-interval* :: *Enum.finite-2 proper-interval*)
instance $\langle \text{proof} \rangle$
end

instantiation *Enum.finite-3* :: *cproper-interval* **begin**
definition *cproper-interval* = (*proper-interval* :: *Enum.finite-3 proper-interval*)
instance $\langle \text{proof} \rangle$
end

instantiation *Enum.finite-4* :: *cproper-interval* **begin**
definition (*cproper-interval* :: *Enum.finite-4 proper-interval*) - - = *undefined*
instance $\langle \text{proof} \rangle$
end

instantiation *Enum.finite-5* :: *cproper-interval* **begin**
definition (*cproper-interval* :: *Enum.finite-5 proper-interval*) - - = *undefined*
instance $\langle \text{proof} \rangle$
end

lemma *lt-of-comp-sum*: *lt-of-comp* (*comparator-sum ca cb*) *sx sy* = (
case sx of Inl x $\Rightarrow$ (*case sy of Inl y* $\Rightarrow$ *lt-of-comp ca x y* | *Inr y* $\Rightarrow$ *True*)
| *Inr x* $\Rightarrow$ (*case sy of Inl y* $\Rightarrow$ *False* | *Inr y* $\Rightarrow$ *lt-of-comp cb x y*)
 $\langle \text{proof} \rangle$)

instantiation *sum* :: (*cproper-interval*, *cproper-interval*) *cproper-interval* **begin**
fun *cproper-interval-sum* :: ('a + 'b) *proper-interval* **where**
cproper-interval-sum *None None* $\longleftrightarrow$ *True*
| *cproper-interval-sum* *None* (*Some* (*Inl x*)) $\longleftrightarrow$ *cproper-interval* *None* (*Some x*)
| *cproper-interval-sum* *None* (*Some* (*Inr y*)) $\longleftrightarrow$ *True*
| *cproper-interval-sum* (*Some* (*Inl x*)) *None* $\longleftrightarrow$ *True*
| *cproper-interval-sum* (*Some* (*Inl x*)) (*Some* (*Inl y*)) $\longleftrightarrow$ *cproper-interval* (*Some x*) (*Some y*)
| *cproper-interval-sum* (*Some* (*Inl x*)) (*Some* (*Inr y*)) $\longleftrightarrow$ *cproper-interval* (*Some x*) *None* $\vee$ *cproper-interval* *None* (*Some y*)
| *cproper-interval-sum* (*Some* (*Inr y*)) *None* $\longleftrightarrow$ *cproper-interval* (*Some y*) *None*
| *cproper-interval-sum* (*Some* (*Inr y*)) (*Some* (*Inl x*)) $\longleftrightarrow$ *False*
| *cproper-interval-sum* (*Some* (*Inr x*)) (*Some* (*Inr y*)) $\longleftrightarrow$ *cproper-interval* (*Some x*) (*Some y*)
instance
 $\langle \text{proof} \rangle$
end

lemma *lt-of-comp-less-prod*: *lt-of-comp* (*comparator-prod c-a c-b*) =
less-prod (*le-of-comp c-a*) (*lt-of-comp c-a*) (*lt-of-comp c-b*)
 $\langle \text{proof} \rangle$

lemma *lt-of-comp-prod*: *lt-of-comp* (*comparator-prod c-a c-b*) (*x1,x2*) (*y1,y2*) =
(*lt-of-comp c-a x1 y1* $\vee$ *le-of-comp c-a x1 y1* $\wedge$ *lt-of-comp c-b x2 y2*)

<proof>

```

instantiation prod :: (cproper-interval, cproper-interval) cproper-interval begin
fun cproper-interval-prod :: ('a × 'b) proper-interval where
  cproper-interval-prod None None  $\longleftrightarrow$  True
| cproper-interval-prod None (Some (y1, y2))  $\longleftrightarrow$  cproper-interval None (Some
y1)  $\vee$  cproper-interval None (Some y2)
| cproper-interval-prod (Some (x1, x2)) None  $\longleftrightarrow$  cproper-interval (Some x1)
None  $\vee$  cproper-interval (Some x2) None
| cproper-interval-prod (Some (x1, x2)) (Some (y1, y2))  $\longleftrightarrow$ 
  cproper-interval (Some x1) (Some y1)  $\vee$ 
  cless x1 y1  $\wedge$  (cpproper-interval (Some x2) None  $\vee$  cproper-interval None (Some
y2))  $\vee$ 
   $\neg$  cless y1 x1  $\wedge$  cproper-interval (Some x2) (Some y2)
instance
<proof>
end

```

```

instantiation list :: (compare) cproper-interval begin
definition cproper-interval-list :: 'a list proper-interval
where cproper-interval-list xso yso = undefined
instance <proof>
end

```

```

lemma infinite-UNIV-literal:
  infinite (UNIV :: String.literal set)
<proof>

```

```

instantiation String.literal :: cproper-interval begin
definition cproper-interval-literal :: String.literal proper-interval
where cproper-interval-literal xso yso = undefined
instance <proof>
end

```

```

lemma lt-of-comp-option: lt-of-comp (comparator-option c) sx sy = (
  case sx of None  $\Rightarrow$  (case sy of None  $\Rightarrow$  False | Some y  $\Rightarrow$  True)
  | Some x  $\Rightarrow$  (case sy of None  $\Rightarrow$  False | Some y  $\Rightarrow$  lt-of-comp c x y))
<proof>

```

```

instantiation option :: (cpproper-interval) cproper-interval begin
fun cproper-interval-option :: 'a option proper-interval where
  cproper-interval-option None None  $\longleftrightarrow$  True
| cproper-interval-option None (Some x)  $\longleftrightarrow$  x  $\neq$  None
| cproper-interval-option (Some x) None  $\longleftrightarrow$  cproper-interval x None
| cproper-interval-option (Some x) (Some None)  $\longleftrightarrow$  False
| cproper-interval-option (Some x) (Some (Some y))  $\longleftrightarrow$  cproper-interval x (Some
y)

```

```

instance
  <proof>
end

```

```

instantiation set :: (cproper-interval) cproper-interval begin
fun cproper-interval-set :: 'a set proper-interval where
  [code]: cproper-interval-set None None  $\longleftrightarrow$  True
| [code]: cproper-interval-set None (Some B)  $\longleftrightarrow$  (B  $\neq$  {})
| [code]: cproper-interval-set (Some A) None  $\longleftrightarrow$  (A  $\neq$  UNIV)
| cproper-interval-set-Some-Some: — Refine for concrete implementations
  cproper-interval-set (Some A) (Some B)  $\longleftrightarrow$  finite (UNIV :: 'a set)  $\wedge$  ( $\exists$  C. cless
  A C  $\wedge$  cless C B)
instance
  <proof>

```

```

lemma Complement-cproper-interval-set-Complement:
  fixes A B :: 'a set
  assumes corder: ID CCOMPARE('a)  $\neq$  None
  shows cproper-interval (Some (– A)) (Some (– B)) = cproper-interval (Some
  B) (Some A)
  <proof>

end

```

```

instantiation fun :: (type, type) cproper-interval begin

```

No interval checks on functions needed because we have not defined an order on them.

```

definition cproper-interval = (undefined :: ('a  $\Rightarrow$  'b) proper-interval)
instance <proof>
end

end

```

```

theory List-Propor-Interval imports
  HOL-Library.List-Lexorder
  Collection-Order
begin

```

3.2 Instantiate proper-interval of for 'a list

```

lemma Nil-less-conv-neq-Nil: [] < xs  $\longleftrightarrow$  xs  $\neq$  []
  <proof>

```

```

lemma less-append-same-iff:

```

```

fixes  $xs :: 'a :: preorder\ list$ 
shows  $xs < xs @ ys \longleftrightarrow [] < ys$ 
 $\langle proof \rangle$ 

```

```

lemma less-append-same2-iff:
fixes  $xs :: 'a :: preorder\ list$ 
shows  $xs @ ys < xs @ zs \longleftrightarrow ys < zs$ 
 $\langle proof \rangle$ 

```

```

lemma Cons-less-iff:
fixes  $x :: 'a :: preorder$  shows
 $x \# xs < ys \longleftrightarrow (\exists y\ ys'.\ ys = y \# ys' \wedge (x < y \vee x = y \wedge xs < ys'))$ 
 $\langle proof \rangle$ 

```

```

instantiation list :: ({proper-interval, order}) proper-interval begin

```

```

definition proper-interval-list-aux :: 'a list  $\Rightarrow$  'a list  $\Rightarrow$  bool
where proper-interval-list-aux-correct:
 $proper-interval-list-aux\ xs\ ys \longleftrightarrow (\exists zs.\ xs < zs \wedge zs < ys)$ 

```

```

lemma proper-interval-list-aux-simps [code]:
 $proper-interval-list-aux\ xs\ [] \longleftrightarrow False$ 
 $proper-interval-list-aux\ []\ (y \# ys) \longleftrightarrow ys \neq [] \vee proper-interval\ None\ (Some\ y)$ 
 $proper-interval-list-aux\ (x \# xs)\ (y \# ys) \longleftrightarrow x < y \vee x = y \wedge proper-interval-list-aux\ xs\ ys$ 
 $\langle proof \rangle$ 

```

```

fun proper-interval-list :: 'a list option  $\Rightarrow$  'a list option  $\Rightarrow$  bool where
 $proper-interval-list\ None\ None \longleftrightarrow True$ 
|  $proper-interval-list\ None\ (Some\ xs) \longleftrightarrow (xs \neq [])$ 
|  $proper-interval-list\ (Some\ xs)\ None \longleftrightarrow True$ 
|  $proper-interval-list\ (Some\ xs)\ (Some\ ys) \longleftrightarrow proper-interval-list-aux\ xs\ ys$ 
instance
 $\langle proof \rangle$ 
end

```

```

end

```

```

theory Collection-Eq imports
  Containers-Auxiliary
  Containers-Generator
  Deriving.Equality-Instances
begin

```

3.3 A type class for optional equality testing

```

class ceq =
fixes  $ceq :: ('a \Rightarrow 'a \Rightarrow bool)\ option$ 
assumes  $ceq: ceq = Some\ eq \Longrightarrow eq = (=)$ 

```

begin

lemma *ceq-equality*: $ceq = Some\ eq \implies equality\ eq$
 $\langle proof \rangle$

lemma *ID-ceq*: $ID\ ceq = Some\ eq \implies eq = (=)$
 $\langle proof \rangle$

abbreviation $ceq' :: 'a \Rightarrow 'a \Rightarrow bool$ **where** $ceq' \equiv the\ (ID\ ceq)$

end

syntax *-CEQ* :: $type \Rightarrow logic\ (\langle 1CEQ/(1'(-)) \rangle)$

syntax-consts *-CEQ* == *ceq*

$\langle ML \rangle$

definition *is-ceq* :: $'a :: ceq\ itself \Rightarrow bool$
where $is-ceq\ - \longleftrightarrow ID\ CEQ('a) \neq None$

3.3.1 Generator for the *ceq*-class

This generator registers itself at the derive-manager for the class *ceq*. To be more precise, one can choose whether one wants to take $(=)$ as function for $CEQ('a)$ by passing "eq" as parameter, whether equality should not be supported by passing "no" as parameter, or whether an own definition for equality should be derived by not passing any parameters. The last possibility only works for datatypes.

- **instantiation** *type* :: $(type, \dots, type)\ (eq)\ ceq$
- **instantiation** *type* :: $(type, \dots, type)\ (no)\ ceq$
- **instantiation** *datatype* :: $(ceq, \dots, ceq)\ ceq$

If the parameter "no" is not used, then the corresponding *is-ceq*-theorem is also automatically generated and attributed with `[simp, code-post]`.

This generator can be used for arbitrary types, not just datatypes.

lemma *equality-subst*: $c1 = c2 \implies equality\ c1 \implies equality\ c2$ $\langle proof \rangle$

$\langle ML \rangle$

3.3.2 Type class instances for HOL types

derive $(eq)\ ceq\ unit$

declare $[[code\ drop:\ \langle CEQ(unit) \rangle]]$

```

lemma [code]: CEQ(unit) = Some (λ- -. True)
  ⟨proof⟩
derive (eq) ceq
  bool
  nat
  int
  Enum.finite-1
  Enum.finite-2
  Enum.finite-3
  Enum.finite-4
  Enum.finite-5
  integer
  natural
  char
  String.literal
derive ceq sum prod list option
derive (no) ceq fun

lemma is-ceq-fun [simp]: ¬ is-ceq TYPE('a ⇒ 'b)
  ⟨proof⟩

definition set-eq :: 'a set ⇒ 'a set ⇒ bool
  where [simp, code-abbrev]: ⟨set-eq = (=)⟩

lemma set-eq-code [code]:
  ⟨set-eq A B ⟷ A ⊆ B ∧ B ⊆ A⟩
  ⟨proof⟩

instantiation set :: (ceq) ceq begin
definition CEQ('a set) = (case ID CEQ('a) of None ⇒ None | Some - ⇒ Some
  set-eq)
instance ⟨proof⟩
end

lemma is-ceq-set [simp, code-post]: is-ceq TYPE('a set) ⟷ is-ceq TYPE('a ::
  ceq)
  ⟨proof⟩

lemma ID-ceq-set-not-None-iff [simp]: ID CEQ('a set) ≠ None ⟷ ID CEQ('a
  :: ceq) ≠ None
  ⟨proof⟩

Instantiation for 'a Predicate.pred

context fixes eq :: 'a ⇒ 'a ⇒ bool begin

definition member-pred :: 'a Predicate.pred ⇒ 'a ⇒ bool
where member-pred P x ⟷ (∃ y. eq x y ∧ Predicate.eval P y)

definition member-seq :: 'a Predicate.seq ⇒ 'a ⇒ bool

```

where $\text{member-seq } xp = \text{member-pred } (\text{Predicate.pred-of-seq } xp)$

lemma $\text{member-seq-code } [code]:$

$\text{member-seq } \text{seq.Empty } x \longleftrightarrow \text{False}$

$\text{member-seq } (\text{seq.Insert } y P) x \longleftrightarrow \text{eq } x y \vee \text{member-pred } P x$

$\text{member-seq } (\text{seq.Join } Q xq) x \longleftrightarrow \text{member-pred } Q x \vee \text{member-seq } xq x$

$\langle \text{proof} \rangle$

lemma $\text{member-pred-code } [code]:$

$\text{member-pred } (\text{Predicate.Seq } f) = \text{member-seq } (f \ ())$

$\langle \text{proof} \rangle$

definition $\text{leq-pred} :: 'a \text{ Predicate.pred} \Rightarrow 'a \text{ Predicate.pred} \Rightarrow \text{bool}$

where $\text{leq-pred } P Q \longleftrightarrow (\forall x. \text{Predicate.eval } P x \longrightarrow (\exists y. \text{eq } x y \wedge \text{Predicate.eval } Q y))$

definition $\text{leq-seq} :: 'a \text{ Predicate.seq} \Rightarrow 'a \text{ Predicate.pred} \Rightarrow \text{bool}$

where $\text{leq-seq } xp Q \longleftrightarrow \text{leq-pred } (\text{Predicate.pred-of-seq } xp) Q$

lemma $\text{leq-seq-code } [code]:$

$\text{leq-seq } \text{seq.Empty } Q \longleftrightarrow \text{True}$

$\text{leq-seq } (\text{seq.Insert } x P) Q \longleftrightarrow \text{member-pred } Q x \wedge \text{leq-pred } P Q$

$\text{leq-seq } (\text{seq.Join } P xp) Q \longleftrightarrow \text{leq-pred } P Q \wedge \text{leq-seq } xp Q$

$\langle \text{proof} \rangle$

lemma $\text{leq-pred-code } [code]:$

$\text{leq-pred } (\text{Predicate.Seq } f) Q \longleftrightarrow \text{leq-seq } (f \ ()) Q$

$\langle \text{proof} \rangle$

definition $\text{predicate-eq} :: 'a \text{ Predicate.pred} \Rightarrow 'a \text{ Predicate.pred} \Rightarrow \text{bool}$

where $\text{predicate-eq } P Q \longleftrightarrow \text{leq-pred } P Q \wedge \text{leq-pred } Q P$

context **assumes** $\text{eq}: \text{eq} = (=)$ **begin**

lemma $\text{member-pred-eq}: \text{member-pred} = \text{Predicate.eval}$

$\langle \text{proof} \rangle$

lemma $\text{member-seq-eq}: \text{member-seq} = \text{Predicate.member}$

$\langle \text{proof} \rangle$

lemma $\text{leq-pred-eq}: \text{leq-pred} = (\leq)$

$\langle \text{proof} \rangle$

lemma $\text{predicate-eq-eq}: \text{predicate-eq} = (=)$

$\langle \text{proof} \rangle$

end

end

```

instantiation Predicate.pred :: (ceq) ceq begin
definition CEQ('a Predicate.pred) = map-option predicate-eq (ID CEQ('a))
instance <proof>
end

end

```

```

theory Collection-Enum imports
  Containers-Auxiliary
  Containers-Generator
begin

```

3.4 A type class for optional enumerations

3.4.1 Definition

```

class cenum =
  fixes cEnum :: ('a list × (('a ⇒ bool) ⇒ bool) × (('a ⇒ bool) ⇒ bool)) option
  assumes UNIV-cenum: cEnum = Some (enum, enum-all, enum-ex) ⇒ UNIV
  = set enum
  and cenum-all-UNIV: cEnum = Some (enum, enum-all, enum-ex) ⇒ enum-all
  P = Ball UNIV P
  and cenum-ex-UNIV: cEnum = Some (enum, enum-all, enum-ex) ⇒ enum-ex
  P = Bex UNIV P
begin

```

```

lemma ID-cEnum:
  ID cEnum = Some (enum, enum-all, enum-ex)
  ⇒ UNIV = set enum ∧ enum-all = Ball UNIV ∧ enum-ex = Bex UNIV
<proof>

```

```

lemma in-cenum: ID cEnum = Some (enum, rest) ⇒ f ∈ set enum
<proof>

```

```

abbreviation cenum :: 'a list
where cenum ≡ fst (the (ID cEnum))

```

```

abbreviation cenum-all :: ('a ⇒ bool) ⇒ bool
where cenum-all ≡ fst (snd (the (ID cEnum)))

```

```

abbreviation cenum-ex :: ('a ⇒ bool) ⇒ bool
where cenum-ex ≡ snd (snd (the (ID cEnum)))

```

```

end

```

```

syntax -CENUM :: type ⇒ logic (⋀(1CENUM/(1'(-))))

```

```

syntax-consts -CENUM == cEnum

```

$\langle ML \rangle$

3.4.2 Generator for the *cenum*-class

This generator registers itself at the derive-manager for the class *cenum*. To be more precise, one can currently only choose to not support enumeration by passing "no" as parameter.

- `instantiation type :: (type,...,type) (no) cenum`

This generator can be used for arbitrary types, not just datatypes.

$\langle ML \rangle$

3.4.3 Instantiations

```
context fixes cenum-all :: ('a  $\Rightarrow$  bool)  $\Rightarrow$  bool begin
fun all-n-lists :: ('a list  $\Rightarrow$  bool)  $\Rightarrow$  nat  $\Rightarrow$  bool
where [simp del]:
  all-n-lists P n = (if n = 0 then P [] else cenum-all ( $\lambda$ x. all-n-lists ( $\lambda$ xs. P (x #
xs)) (n - 1)))
end
```

```
context fixes cenum-ex :: ('a  $\Rightarrow$  bool)  $\Rightarrow$  bool begin
fun ex-n-lists :: ('a list  $\Rightarrow$  bool)  $\Rightarrow$  nat  $\Rightarrow$  bool
where [simp del]:
  ex-n-lists P n  $\longleftrightarrow$  (if n = 0 then P [] else cenum-ex (%x. ex-n-lists (%xs. P (x
# xs)) (n - 1)))
end
```

```
lemma all-n-lists-iff: fixes cenum shows
  all-n-lists (Ball (set cenum)) P n  $\longleftrightarrow$  ( $\forall$  xs  $\in$  set (List.n-lists n cenum). P xs)
 $\langle$ proof $\rangle$ 
```

```
lemma ex-n-lists-iff: fixes cenum shows
  ex-n-lists (Bex (set cenum)) P n  $\longleftrightarrow$  ( $\exists$  xs  $\in$  set (List.n-lists n cenum). P xs)
 $\langle$ proof $\rangle$ 
```

```
instantiation fun :: (cenum, cenum) cenum begin
```

definition

```
CENUM('a  $\Rightarrow$  'b) =
  (case ID CENUM('a) of None  $\Rightarrow$  None | Some (enum-a, enum-all-a, enum-ex-a)
 $\Rightarrow$ 
    case ID CENUM('b) of None  $\Rightarrow$  None | Some (enum-b, enum-all-b, enum-ex-b)
 $\Rightarrow$  Some
```

```

      (map ( $\lambda$ ys. the o map-of (zip enum-a ys)) (List.n-lists (length enum-a)
enum-b)),
       $\lambda$ P. all-n-lists enum-all-b ( $\lambda$ bs. P (the o map-of (zip enum-a bs))) (length
enum-a),
       $\lambda$ P. ex-n-lists enum-ex-b ( $\lambda$ bs. P (the o map-of (zip enum-a bs))) (length
enum-a)))
instance  $\langle$ proof $\rangle$ 
end

```

instantiation set :: (cenum) cenum **begin**

definition

```

  CENUM('a set) =
  (case ID CENUM('a) of None  $\Rightarrow$  None | Some (enum-a, enum-all-a, enum-ex-a)
 $\Rightarrow$  Some
    (map set (subseqs enum-a),
      $\lambda$ P. list-all P (map set (subseqs enum-a)),
      $\lambda$ P. list-ex P (map set (subseqs enum-a)))))

```

instance

$\langle$ proof $\rangle$

end

instantiation unit :: cenum **begin**

definition CENUM(unit) = Some (enum-class.enum, enum-class.enum-all, enum-class.enum-ex)

instance $\langle$ proof $\rangle$

end

instantiation bool :: cenum **begin**

definition CENUM(bool) = Some (enum-class.enum, enum-class.enum-all, enum-class.enum-ex)

instance $\langle$ proof $\rangle$

end

instantiation prod :: (cenum, cenum) cenum **begin**

definition

```

  CENUM('a  $\times$  'b) =
  (case ID CENUM('a) of None  $\Rightarrow$  None | Some (enum-a, enum-all-a, enum-ex-a)
 $\Rightarrow$ 
    case ID CENUM('b) of None  $\Rightarrow$  None | Some (enum-b, enum-all-b, enum-ex-b)
 $\Rightarrow$  Some
      (List.product enum-a enum-b,
        $\lambda$ P. enum-all-a (%x. enum-all-b (%y. P (x, y))),
        $\lambda$ P. enum-ex-a (%x. enum-ex-b (%y. P (x, y)))))

```

instance

$\langle$ proof $\rangle$

end

instantiation sum :: (cenum, cenum) cenum **begin**

definition

```

  CENUM('a + 'b) =
  (case ID CENUM('a) of None  $\Rightarrow$  None | Some (enum-a, enum-all-a, enum-ex-a)

```

```

⇒
  case ID CENUM('b) of None ⇒ None | Some (enum-b, enum-all-b, enum-ex-b)
⇒ Some
  (map Inl enum-a @ map Inr enum-b,
   λP. enum-all-a (λx. P (Inl x)) ∧ enum-all-b (λx. P (Inr x)),
   λP. enum-ex-a (λx. P (Inl x)) ∨ enum-ex-b (λx. P (Inr x)))
instance
  ⟨proof⟩
end

```

instantiation *option* :: (*cenum*) *cenum* **begin**

definition

```

  CENUM('a option) =
  (case ID CENUM('a) of None ⇒ None | Some (enum-a, enum-all-a, enum-ex-a)
⇒ Some
  (None # map Some enum-a,
   λP. P None ∧ enum-all-a (λx. P (Some x)),
   λP. P None ∨ enum-ex-a (λx. P (Some x))))

```

instance

⟨proof⟩

end

instantiation *Enum.finite-1* :: *cenum* **begin**

definition *CENUM*(*Enum.finite-1*) = *Some* (*enum-class.enum*, *enum-class.enum-all*,
enum-class.enum-ex)

instance ⟨proof⟩

end

instantiation *Enum.finite-2* :: *cenum* **begin**

definition *CENUM*(*Enum.finite-2*) = *Some* (*enum-class.enum*, *enum-class.enum-all*,
enum-class.enum-ex)

instance ⟨proof⟩

end

instantiation *Enum.finite-3* :: *cenum* **begin**

definition *CENUM*(*Enum.finite-3*) = *Some* (*enum-class.enum*, *enum-class.enum-all*,
enum-class.enum-ex)

instance ⟨proof⟩

end

instantiation *Enum.finite-4* :: *cenum* **begin**

definition *CENUM*(*Enum.finite-4*) = *Some* (*enum-class.enum*, *enum-class.enum-all*,
enum-class.enum-ex)

instance ⟨proof⟩

end

instantiation *Enum.finite-5* :: *cenum* **begin**

definition *CENUM*(*Enum.finite-5*) = *Some* (*enum-class.enum*, *enum-class.enum-all*,
enum-class.enum-ex)

```
instance ⟨proof⟩
end
```

```
instantiation char :: cenum begin
definition CENUM(char) = Some (enum-class.enum, enum-class.enum-all, enum-class.enum-ex)
instance ⟨proof⟩
end
```

```
derive (no) cenum list nat int integer natural String.literal
```

```
end
```

```
theory Equal imports Main begin
```

3.5 Locales to abstract over HOL equality

```
locale equal-base = fixes equal :: 'a ⇒ 'a ⇒ bool
```

```
locale equal = equal-base +
  assumes equal-eq: equal = (=)
begin
```

```
lemma equal-conv-eq: equal x y ⟷ x = y
⟨proof⟩
```

```
end
```

```
end
```

```
theory RBT-ext
imports
  HOL-Library.RBT-Impl
  Containers-Auxiliary
  List-Fusion
begin
```

3.6 More on red-black trees

3.6.1 More lemmas

```
context linorder begin
```

```
lemma is-rbt-fold-rbt-insert-impl:
  is-rbt t ⟹ is-rbt (RBT-Impl.fold rbt-insert t' t)
⟨proof⟩
```

lemma *rbt-sorted-fold-insert*: $rbt\text{-}sorted\ t \implies rbt\text{-}sorted\ (RBT\text{-}Impl.fold\ rbt\text{-}insert\ t'\ t)$
 $\langle proof \rangle$

lemma *rbt-lookup-rbt-insert'*: $rbt\text{-}sorted\ t \implies rbt\text{-}lookup\ (rbt\text{-}insert\ k\ v\ t) = (rbt\text{-}lookup\ t)(k \mapsto v)$
 $\langle proof \rangle$

lemma *rbt-lookup-fold-rbt-insert-impl*:
 $rbt\text{-}sorted\ t2 \implies$
 $rbt\text{-}lookup\ (RBT\text{-}Impl.fold\ rbt\text{-}insert\ t1\ t2) = rbt\text{-}lookup\ t2\ ++\ map\text{-}of\ (rev\ (RBT\text{-}Impl.entries\ t1))$
 $\langle proof \rangle$

end

3.6.2 Build the cross product of two RBTs

context *fixes* $f :: 'a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'd \Rightarrow 'e$ **begin**

definition *alist-product* :: $('a \times 'b)\ list \Rightarrow ('c \times 'd)\ list \Rightarrow (('a \times 'c) \times 'e)\ list$
where *alist-product* $xs\ ys = concat\ (map\ (\lambda(a, b). map\ (\lambda(c, d). ((a, c), f\ a\ b\ c\ d))\ ys)\ xs)$

lemma *alist-product-simps* [*simp*]:
 $alist\text{-}product\ []\ ys = []$
 $alist\text{-}product\ xs\ [] = []$
 $alist\text{-}product\ ((a, b) \# xs)\ ys = map\ (\lambda(c, d). ((a, c), f\ a\ b\ c\ d))\ ys\ @\ alist\text{-}product\ xs\ ys$
 $\langle proof \rangle$

lemma *append-alist-product-conv-fold*:
 $zs\ @\ alist\text{-}product\ xs\ ys = rev\ (fold\ (\lambda(a, b). fold\ (\lambda(c, d)\ rest. ((a, c), f\ a\ b\ c\ d)\ #\ rest)\ ys)\ xs\ (rev\ zs))$
 $\langle proof \rangle$

lemma *alist-product-code* [*code*]:
 $alist\text{-}product\ xs\ ys =$
 $rev\ (fold\ (\lambda(a, b). fold\ (\lambda(c, d)\ rest. ((a, c), f\ a\ b\ c\ d)\ #\ rest)\ ys)\ xs\ [])$
 $\langle proof \rangle$

lemma *set-alist-product*:
 $set\ (alist\text{-}product\ xs\ ys) =$
 $(\lambda((a, b), (c, d)). ((a, c), f\ a\ b\ c\ d))\ ` (set\ xs \times set\ ys)$
 $\langle proof \rangle$

lemma *distinct-alist-product*:
 $[distinct\ (map\ fst\ xs); distinct\ (map\ fst\ ys)]$
 $\implies distinct\ (map\ fst\ (alist\text{-}product\ xs\ ys))$

$\langle \text{proof} \rangle$

lemma *map-of-alist-product*:

map-of (*alist-product* *xs* *ys*) (*a*, *c*) =
 (case *map-of* *xs* *a* of *None* $\Rightarrow$ *None*
 | *Some* *b* $\Rightarrow$ *map-option* (*f* *a* *b* *c*) (*map-of* *ys* *c*))

$\langle \text{proof} \rangle$

definition *rbt-product* :: (*'a*, *'b*) *rbt* $\Rightarrow$ (*'c*, *'d*) *rbt* $\Rightarrow$ (*'a* $\times$ *'c*, *'e*) *rbt*

where

rbt-product *rbt1* *rbt2* = *rbtreeify* (*alist-product* (*RBT-Impl.entries* *rbt1*) (*RBT-Impl.entries* *rbt2*))

lemma *rbt-product-code* [*code*]:

rbt-product *rbt1* *rbt2* =
rbtreeify (*rev* (*RBT-Impl.fold* ($\lambda a\ b.$ *RBT-Impl.fold* ($\lambda c\ d\ \text{rest.}$ ((*a*, *c*), *f* *a* *b* *c*
d) # *rest*) *rbt2*) *rbt1* []))

$\langle \text{proof} \rangle$

end

context

fixes *leq-a* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool* (**infix** $\langle \sqsubseteq_a \rangle$ 50)
and *less-a* :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *bool* (**infix** $\langle \sqsubset_a \rangle$ 50)
and *leq-b* :: *'b* $\Rightarrow$ *'b* $\Rightarrow$ *bool* (**infix** $\langle \sqsubseteq_b \rangle$ 50)
and *less-b* :: *'b* $\Rightarrow$ *'b* $\Rightarrow$ *bool* (**infix** $\langle \sqsubset_b \rangle$ 50)
assumes *lin-a*: *class.linorder* *leq-a* *less-a*
and *lin-b*: *class.linorder* *leq-b* *less-b*

begin

abbreviation (*input*) *less-eq-prod'* :: (*'a* $\times$ *'b*) $\Rightarrow$ (*'a* $\times$ *'b*) $\Rightarrow$ *bool* (**infix** $\langle \sqsubseteq \rangle$ 50)

where *less-eq-prod'* $\equiv$ *less-eq-prod* *leq-a* *less-a* *leq-b*

abbreviation (*input*) *less-prod'* :: (*'a* $\times$ *'b*) $\Rightarrow$ (*'a* $\times$ *'b*) $\Rightarrow$ *bool* (**infix** $\langle \sqsubset \rangle$ 50)

where *less-prod'* $\equiv$ *less-prod* *leq-a* *less-a* *less-b*

lemmas *linorder-prod* = *linorder-prod*[*OF* *lin-a* *lin-b*]

lemma *sorted-alist-product*:

assumes *xs*: *linorder.sorted* *leq-a* (*map* *fst* *xs*) *distinct* (*map* *fst* *xs*)
and *ys*: *linorder.sorted* ($\sqsubseteq_b$) (*map* *fst* *ys*)
shows *linorder.sorted* ($\sqsubseteq$) (*map* *fst* (*alist-product* *f* *xs* *ys*))

$\langle \text{proof} \rangle$

lemma *is-rbt-rbt-product*:

$\llbracket$ *ord.is-rbt* ($\sqsubset_a$) *rbt1*; *ord.is-rbt* ($\sqsubset_b$) *rbt2* $\rrbracket$
 $\Rightarrow$ *ord.is-rbt* ($\sqsubset$) (*rbt-product* *f* *rbt1* *rbt2*)

$\langle \text{proof} \rangle$

lemma *rbt-lookup-rbt-product*:

$\llbracket \text{ord.is-rbt } (\sqsubseteq_a) \text{ rbt1}; \text{ord.is-rbt } (\sqsubseteq_b) \text{ rbt2} \rrbracket$
 $\implies \text{ord.rbt-lookup } (\sqsubseteq) (\text{rbt-product } f \text{ rbt1 rbt2}) (a, c) =$
 $(\text{case ord.rbt-lookup } (\sqsubseteq_a) \text{ rbt1 } a \text{ of } \text{None} \Rightarrow \text{None}$
 $\quad | \text{Some } b \Rightarrow \text{map-option } (f \ a \ b \ c) (\text{ord.rbt-lookup } (\sqsubseteq_b) \text{ rbt2 } c))$
 $\langle \text{proof} \rangle$

end

hide-const *less-eq-prod' less-prod'*

3.6.3 Build an RBT where keys are paired with themselves

primrec *RBT-Impl-diag* :: $('a, 'b) \text{ rbt} \Rightarrow ('a \times 'a, 'b) \text{ rbt}$

where

$\text{RBT-Impl-diag } \text{rbt.Empty} = \text{rbt.Empty}$
 $| \text{RBT-Impl-diag } (\text{rbt.Branch } c \ l \ k \ v \ r) = \text{rbt.Branch } c \ (\text{RBT-Impl-diag } l) \ (k, k) \ v$
 $(\text{RBT-Impl-diag } r)$

lemma *entries-RBT-Impl-diag*:

$\text{RBT-Impl.entries } (\text{RBT-Impl-diag } t) = \text{map } (\lambda(k, v). ((k, k), v)) (\text{RBT-Impl.entries } t)$
 $\langle \text{proof} \rangle$

lemma *keys-RBT-Impl-diag*:

$\text{RBT-Impl.keys } (\text{RBT-Impl-diag } t) = \text{map } (\lambda k. (k, k)) (\text{RBT-Impl.keys } t)$
 $\langle \text{proof} \rangle$

lemma *rbt-sorted-RBT-Impl-diag*:

$\text{ord.rbt-sorted } lt \ t \implies \text{ord.rbt-sorted } (\text{less-prod leq } lt \ lt) (\text{RBT-Impl-diag } t)$
 $\langle \text{proof} \rangle$

lemma *bheight-RBT-Impl-diag*:

$\text{bheight } (\text{RBT-Impl-diag } t) = \text{bheight } t$
 $\langle \text{proof} \rangle$

lemma *inv-RBT-Impl-diag*:

assumes $\text{inv1 } t \quad \text{inv2 } t$
shows $\text{inv1 } (\text{RBT-Impl-diag } t) \quad \text{inv2 } (\text{RBT-Impl-diag } t)$
and $\text{color-of } t = \text{color.B} \implies \text{color-of } (\text{RBT-Impl-diag } t) = \text{color.B}$
 $\langle \text{proof} \rangle$

lemma *is-rbt-RBT-Impl-diag*:

$\text{ord.is-rbt } lt \ t \implies \text{ord.is-rbt } (\text{less-prod leq } lt \ lt) (\text{RBT-Impl-diag } t)$
 $\langle \text{proof} \rangle$

lemma **(in** *linorder***)** *rbt-lookup-RBT-Impl-diag*:

$\text{ord.rbt-lookup } (\text{less-prod } (\leq) (<) (<)) (\text{RBT-Impl-diag } t) =$
 $(\lambda(k, k'). \text{ if } k = k' \text{ then } \text{ord.rbt-lookup } (<) \ t \ k \text{ else } \text{None})$

<proof>

3.6.4 Folding and quantifiers over RBTs

definition $RBT-Impl-fold1 :: ('a \Rightarrow 'a \Rightarrow 'a) \Rightarrow ('a, unit) RBT-Impl.rbt \Rightarrow 'a$
where $RBT-Impl-fold1 f rbt = fold f (tl (RBT-Impl.keys rbt)) (hd (RBT-Impl.keys rbt))$

lemma $RBT-Impl-fold1-simps [simp, code]:$

$RBT-Impl-fold1 f rbt.Empty = undefined$
 $RBT-Impl-fold1 f (Branch c rbt.Empty k v r) = RBT-Impl.fold (\lambda k v. f k) r k$
 $RBT-Impl-fold1 f (Branch c (Branch c' l' k' v' r') k v r) =$
 $RBT-Impl.fold (\lambda k v. f k) r (f k (RBT-Impl-fold1 f (Branch c' l' k' v' r')))$

<proof>

definition $RBT-Impl-rbt-all :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a, 'b) rbt \Rightarrow bool$
where $[code del]: RBT-Impl-rbt-all P rbt = (\forall (k, v) \in set (RBT-Impl.entries rbt). P k v)$

lemma $RBT-Impl-rbt-all-simps [simp, code]:$

$RBT-Impl-rbt-all P rbt.Empty \longleftrightarrow True$
 $RBT-Impl-rbt-all P (Branch c l k v r) \longleftrightarrow P k v \wedge RBT-Impl-rbt-all P l \wedge$
 $RBT-Impl-rbt-all P r$
<proof>

definition $RBT-Impl-rbt-ex :: ('a \Rightarrow 'b \Rightarrow bool) \Rightarrow ('a, 'b) rbt \Rightarrow bool$
where $[code del]: RBT-Impl-rbt-ex P rbt = (\exists (k, v) \in set (RBT-Impl.entries rbt). P k v)$

lemma $RBT-Impl-rbt-ex-simps [simp, code]:$

$RBT-Impl-rbt-ex P rbt.Empty \longleftrightarrow False$
 $RBT-Impl-rbt-ex P (Branch c l k v r) \longleftrightarrow P k v \vee RBT-Impl-rbt-ex P l \vee$
 $RBT-Impl-rbt-ex P r$
<proof>

3.6.5 List fusion for RBTs

type-synonym $('a, 'b, 'c) rbt-generator-state = ('c \times ('a, 'b) RBT-Impl.rbt) list$
 $\times ('a, 'b) RBT-Impl.rbt$

fun $rbt-has-next :: ('a, 'b, 'c) rbt-generator-state \Rightarrow bool$

where

$rbt-has-next ([], rbt.Empty) = False$
 $| rbt-has-next - = True$

fun $rbt-keys-next :: ('a, 'b, 'a) rbt-generator-state \Rightarrow 'a \times ('a, 'b, 'a) rbt-generator-state$

where

$rbt-keys-next ((k, t) \# kts, rbt.Empty) = (k, kts, t)$
 $| rbt-keys-next (kts, rbt.Branch c l k v r) = rbt-keys-next ((k, r) \# kts, l)$

lemma *rbt-generator-induct* [case-names empty split shuffle]:
 assumes $P([], \text{rbt.Empty})$
 and $\bigwedge k t \text{ kts}. P(\text{kts}, t) \implies P((k, t) \# \text{kts}, \text{rbt.Empty})$
 and $\bigwedge \text{kts } c \ l \ k \ v \ r. P((f \ k \ v, r) \# \text{kts}, l) \implies P(\text{kts}, \text{Branch } c \ l \ k \ v \ r)$
 shows $P \text{ ktst}$
 <proof>

lemma *terminates-rbt-keys-generator*:
 terminates (*rbt-has-next*, *rbt-keys-next*)
 <proof>

lift-definition *rbt-keys-generator* :: (*'a*, (*'a*, *'b*, *'a*) rbt-generator-state) generator
 is (*rbt-has-next*, *rbt-keys-next*)
 <proof>

definition *rbt-init* :: (*'a*, *'b*) rbt $\Rightarrow$ (*'a*, *'b*, *'c*) rbt-generator-state
 where *rbt-init* = *Pair []*

lemma *has-next-rbt-keys-generator* [simp]:
 list.has-next rbt-keys-generator = *rbt-has-next*
 <proof>

lemma *next-rbt-keys-generator* [simp]:
 list.next rbt-keys-generator = *rbt-keys-next*
 <proof>

lemma *unfoldr-rbt-keys-generator-aux*:
 list.unfoldr rbt-keys-generator (*kts*, *t*) =
RBT-Impl.keys t @ concat (map ($\lambda(k, t). k \# \text{RBT-Impl.keys } t$) kts)
 <proof>

corollary *unfoldr-rbt-keys-generator*:
 list.unfoldr rbt-keys-generator (*rbt-init t*) = *RBT-Impl.keys t*
 <proof>

fun *rbt-entries-next* ::
 (*'a*, *'b*, *'a* $\times$ *'b*) rbt-generator-state $\Rightarrow$ (*'a* $\times$ *'b*) $\times$ (*'a*, *'b*, *'a* $\times$ *'b*) rbt-generator-state
 where
rbt-entries-next ((*kv*, *t*) # *kts*, *rbt.Empty*) = (*kv*, *kts*, *t*)
 | *rbt-entries-next* (*kts*, *rbt.Branch c l k v r*) = *rbt-entries-next* (((*k*, *v*), *r*) # *kts*,
l)

lemma *terminates-rbt-entries-generator*:
 terminates (*rbt-has-next*, *rbt-entries-next*)
 <proof>

lift-definition *rbt-entries-generator* :: (*'a* $\times$ *'b*, (*'a*, *'b*, *'a* $\times$ *'b*) rbt-generator-state)
 generator
 is (*rbt-has-next*, *rbt-entries-next*)

<proof>

lemma *has-next-rbt-entries-generator* [simp]:

list.has-next rbt-entries-generator = rbt-has-next

<proof>

lemma *next-rbt-entries-generator* [simp]:

list.next rbt-entries-generator = rbt-entries-next

<proof>

lemma *unfoldr-rbt-entries-generator-aux*:

list.unfoldr rbt-entries-generator (kts, t) =

RBT-Impl.entries t @ concat (map ($\lambda(k, t). k \# \text{RBT-Impl.entries } t$) kts)

<proof>

corollary *unfoldr-rbt-entries-generator*:

list.unfoldr rbt-entries-generator (rbt-init t) = RBT-Impl.entries t

<proof>

end

theory *RBT-Mapping2*

imports

Collection-Order

RBT-ext

Deriving.RBT-Comparator-Impl

begin

3.7 Mappings implemented by red-black trees

lemma *distinct-map-filterI*: *distinct (map f xs) $\implies$ distinct (map f (filter P xs))*

<proof>

lemma *map-of-filter-apply*:

distinct (map fst xs)

$\implies$ *map-of (filter P xs) k =*

(case map-of xs k of None $\implies$ None | Some v $\implies$ if P (k, v) then Some v else None)

<proof>

3.7.1 Type definition

typedef (overloaded) (*'a, 'b*) *mapping-rbt*

$= \{t :: ('a :: \text{ccompare}, 'b) \text{RBT-Impl.rbt. ord.is-rbt cless } t \vee \text{ID CCOMPARE}('a) = \text{None}\}$

morphisms *impl-of Mapping-RBT'*

<proof>

definition *Mapping-RBT* :: ('a :: ccompare, 'b) rbt $\Rightarrow$ ('a, 'b) mapping-rbt
where

Mapping-RBT t = *Mapping-RBT'*
 (if ord.is-rbt cless t $\vee$ ID CCOMPARE('a) = None then t
 else RBT-Impl.fold (ord.rbt-insert cless) t rbt.Empty)

lemma *Mapping-RBT-inverse*:

fixes y :: ('a :: ccompare, 'b) rbt
assumes y $\in$ {t. ord.is-rbt cless t $\vee$ ID CCOMPARE('a) = None}
shows impl-of (*Mapping-RBT* y) = y
 <proof>

lemma *impl-of-inverse*: *Mapping-RBT* (impl-of t) = t
 <proof>

lemma *type-definition-mapping-rbt'*:

type-definition impl-of Mapping-RBT
 {t :: ('a, 'b) rbt. ord.is-rbt cless t $\vee$ ID CCOMPARE('a :: ccompare) = None}
 <proof>

lemmas *Mapping-RBT-cases*[cases type: mapping-rbt] =
 type-definition.Abs-cases[OF type-definition-mapping-rbt']
and *Mapping-RBT-induct*[induct type: mapping-rbt] =
 type-definition.Abs-induct[OF type-definition-mapping-rbt'] **and**
Mapping-RBT-inject = type-definition.Abs-inject[OF type-definition-mapping-rbt']

lemma *rbt-eq-iff*:

t1 = t2 $\iff$ impl-of t1 = impl-of t2
 <proof>

lemma *rbt-eqI*:

impl-of t1 = impl-of t2 $\implies$ t1 = t2
 <proof>

lemma *Mapping-RBT-impl-of [simp]*:

Mapping-RBT (impl-of t) = t
 <proof>

3.7.2 Operations

setup-lifting *type-definition-mapping-rbt'*

context fixes dummy :: 'a :: ccompare **begin**

lift-definition *lookup* :: ('a, 'b) mapping-rbt $\Rightarrow$ 'a $\rightarrow$ 'b **is** rbt-comp-lookup ccomp
 <proof>

lift-definition *empty* :: ('a, 'b) mapping-rbt **is** RBT-Impl.Empty
 <proof>

lift-definition $insert :: 'a \Rightarrow 'b \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt}$ **is**
 $\text{rbt-comp-insert ccomp}$
 $\langle \text{proof} \rangle$

lift-definition $delete :: 'a \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt}$ **is**
 $\text{rbt-comp-delete ccomp}$
 $\langle \text{proof} \rangle$

lift-definition $bulkload :: ('a \times 'b) \text{ list} \Rightarrow ('a, 'b) \text{ mapping-rbt}$ **is**
 $\text{rbt-comp-bulkload ccomp}$
 $\langle \text{proof} \rangle$

lift-definition $map\text{-entry} :: 'a \Rightarrow ('b \Rightarrow 'b) \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt}$ **is**
 $\text{rbt-comp-map-entry ccomp}$
 $\langle \text{proof} \rangle$

lift-definition $map :: ('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'c) \text{ mapping-rbt}$ **is** RBT-Impl.map
 $\langle \text{proof} \rangle$

lift-definition $entries :: ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a \times 'b) \text{ list}$ **is** RBT-Impl.entries
 $\langle \text{proof} \rangle$

lift-definition $keys :: ('a, 'b) \text{ mapping-rbt} \Rightarrow 'a \text{ set}$ **is** $\text{set} \circ \text{RBT-Impl.keys}$ $\langle \text{proof} \rangle$

lift-definition $fold :: ('a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'c) \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow 'c \Rightarrow 'c$ **is**
 RBT-Impl.fold $\langle \text{proof} \rangle$

lift-definition $is\text{-empty} :: ('a, 'b) \text{ mapping-rbt} \Rightarrow \text{bool}$ **is** $\text{case-rbt True } (\lambda - - - - \text{False})$ $\langle \text{proof} \rangle$

lift-definition $filter :: ('a \times 'b \Rightarrow \text{bool}) \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt}$ **is**
 $\lambda P t. \text{rbtreeify } (\text{List.filter } P (\text{RBT-Impl.entries } t))$
 $\langle \text{proof} \rangle$

lift-definition $join ::$
 $('a \Rightarrow 'b \Rightarrow 'b \Rightarrow 'b) \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt}$
is $\text{rbt-comp-union-with-key ccomp}$
 $\langle \text{proof} \rangle$

lift-definition $meet ::$
 $('a \Rightarrow 'b \Rightarrow 'b \Rightarrow 'b) \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt} \Rightarrow ('a, 'b) \text{ mapping-rbt}$
is $\text{rbt-comp-inter-with-key ccomp}$
 $\langle \text{proof} \rangle$

lift-definition *all* :: ('a $\Rightarrow$ 'b $\Rightarrow$ bool) $\Rightarrow$ ('a, 'b) mapping-rbt $\Rightarrow$ bool
is RBT-Impl-rbt-all <proof>

lift-definition *ex* :: ('a $\Rightarrow$ 'b $\Rightarrow$ bool) $\Rightarrow$ ('a, 'b) mapping-rbt $\Rightarrow$ bool
is RBT-Impl-rbt-ex <proof>

lift-definition *product* ::
 ('a $\Rightarrow$ 'b $\Rightarrow$ 'c $\Rightarrow$ 'd $\Rightarrow$ 'e) $\Rightarrow$ ('a, 'b) mapping-rbt
 $\Rightarrow$ ('c :: ccompare, 'd) mapping-rbt $\Rightarrow$ ('a $\times$ 'c, 'e) mapping-rbt
is rbt-product
 <proof>

lift-definition *diag* ::
 ('a, 'b) mapping-rbt $\Rightarrow$ ('a $\times$ 'a, 'b) mapping-rbt
is RBT-Impl-diag
 <proof>

lift-definition *init* :: ('a, 'b) mapping-rbt $\Rightarrow$ ('a, 'b, 'c) rbt-generator-state
is rbt-init <proof>

end

3.7.3 Properties

lemma *unfoldr-rbt-entries-generator*:
 list.unfoldr rbt-entries-generator (init t) = entries t
 <proof>

lemma *lookup-RBT*:
 ord.is-rbt cless t $\Rightarrow$
 lookup (Mapping-RBT t) = rbt-comp-lookup ccomp t
 <proof>

lemma *lookup-impl-of*:
 rbt-comp-lookup ccomp (impl-of t) = lookup t
 <proof>

lemma *entries-impl-of*:
 RBT-Impl.entries (impl-of t) = entries t
 <proof>

lemma *keys-impl-of*:
 set (RBT-Impl.keys (impl-of t)) = keys t
 <proof>

lemma *lookup-empty [simp]*:
 lookup empty = Map.empty
 <proof>

lemma *fold-conv-fold*:

$\text{fold } f \ t = \text{List.fold } (\text{case-prod } f) \ (\text{entries } t)$
 $\langle \text{proof} \rangle$

lemma *is-empty-empty* [*simp*]:

$\text{is-empty } t \longleftrightarrow t = \text{empty}$
 $\langle \text{proof} \rangle$

context assumes *ID-ccompare-neq-None*: $\text{ID } \text{CCOMPARE}('a :: \text{ccompare}) \neq \text{None}$
begin

lemma *mapping-linorder*: $\text{class.linorder } (\text{cless-eq} :: 'a \Rightarrow 'a \Rightarrow \text{bool}) \ \text{cless}$
 $\langle \text{proof} \rangle$

lemma *mapping-comparator*: $\text{comparator } (\text{ccomp} :: 'a \ \text{comparator})$
 $\langle \text{proof} \rangle$

lemmas *rbt-comp*[*simp*] = *rbt-comp-simps*[*OF mapping-comparator*]

lemma *is-rbt-impl-of* [*simp*, *intro*]:

fixes $t :: ('a, 'b) \ \text{mapping-rbt}$
shows $\text{ord.is-rbt } \text{cless} \ (\text{impl-of } t)$
 $\langle \text{proof} \rangle$

lemma *lookup-insert* [*simp*]:

$\text{lookup } (\text{insert } (k :: 'a) \ v \ t) = (\text{lookup } t)(k \mapsto v)$
 $\langle \text{proof} \rangle$

lemma *lookup-delete* [*simp*]:

$\text{lookup } (\text{delete } (k :: 'a) \ t) = (\text{lookup } t)(k := \text{None})$
 $\langle \text{proof} \rangle$

lemma *map-of-entries* [*simp*]:

$\text{map-of } (\text{entries } (t :: ('a, 'b) \ \text{mapping-rbt})) = \text{lookup } t$
 $\langle \text{proof} \rangle$

lemma *entries-lookup*:

$\text{entries } (t1 :: ('a, 'b) \ \text{mapping-rbt}) = \text{entries } t2 \longleftrightarrow \text{lookup } t1 = \text{lookup } t2$
 $\langle \text{proof} \rangle$

lemma *lookup-bulkload* [*simp*]:

$\text{lookup } (\text{bulkload } xs) = \text{map-of } (xs :: ('a \times 'b) \ \text{list})$
 $\langle \text{proof} \rangle$

lemma *lookup-map-entry* [*simp*]:

$\text{lookup } (\text{map-entry } (k :: 'a) \ f \ t) = (\text{lookup } t)(k := \text{map-option } f \ (\text{lookup } t \ k))$
 $\langle \text{proof} \rangle$

lemma *lookup-map* [simp]:

$\text{lookup } (\text{map } f \ t) \ (k :: 'a) = \text{map-option } (f \ k) \ (\text{lookup } t \ k)$
 $\langle \text{proof} \rangle$

lemma *RBT-lookup-empty* [simp]:

$\text{ord.rbt-lookup } \text{cless } (t :: ('a, 'b) \text{ RBT-Impl.rbt}) = \text{Map.empty} \longleftrightarrow t = \text{RBT-Impl.Empty}$
 $\langle \text{proof} \rangle$

lemma *lookup-empty-empty* [simp]:

$\text{lookup } t = \text{Map.empty} \longleftrightarrow (t :: ('a, 'b) \text{ mapping-rbt}) = \text{empty}$
 $\langle \text{proof} \rangle$

lemma *finite-dom-lookup* [simp]: $\text{finite } (\text{dom } (\text{lookup } (t :: ('a, 'b) \text{ mapping-rbt})))$

$\langle \text{proof} \rangle$

lemma *card-com-lookup* [unfolded length-map, simp]:

$\text{card } (\text{dom } (\text{lookup } (t :: ('a, 'b) \text{ mapping-rbt}))) = \text{length } (\text{List.map } \text{fst } (\text{entries } t))$
 $\langle \text{proof} \rangle$

lemma *lookup-join*:

$\text{lookup } (\text{join } f \ (t1 :: ('a, 'b) \text{ mapping-rbt}) \ t2) =$
 $(\lambda k. \text{case lookup } t1 \ k \text{ of } \text{None} \Rightarrow \text{lookup } t2 \ k \mid \text{Some } v1 \Rightarrow \text{Some } (\text{case lookup } t2$
 $k \text{ of } \text{None} \Rightarrow v1 \mid \text{Some } v2 \Rightarrow f \ k \ v1 \ v2))$
 $\langle \text{proof} \rangle$

lemma *lookup-meet*:

$\text{lookup } (\text{meet } f \ (t1 :: ('a, 'b) \text{ mapping-rbt}) \ t2) =$
 $(\lambda k. \text{case lookup } t1 \ k \text{ of } \text{None} \Rightarrow \text{None} \mid \text{Some } v1 \Rightarrow \text{case lookup } t2 \ k \text{ of } \text{None} \Rightarrow$
 $\text{None} \mid \text{Some } v2 \Rightarrow \text{Some } (f \ k \ v1 \ v2))$
 $\langle \text{proof} \rangle$

lemma *lookup-filter* [simp]:

$\text{lookup } (\text{filter } P \ (t :: ('a, 'b) \text{ mapping-rbt})) \ k =$
 $(\text{case lookup } t \ k \text{ of } \text{None} \Rightarrow \text{None} \mid \text{Some } v \Rightarrow \text{if } P \ (k, v) \text{ then } \text{Some } v \text{ else } \text{None})$
 $\langle \text{proof} \rangle$

lemma *all-conv-all-lookup*:

$\text{all } P \ t \longleftrightarrow (\forall (k :: 'a) \ v. \text{lookup } t \ k = \text{Some } v \longrightarrow P \ k \ v)$
 $\langle \text{proof} \rangle$

lemma *ex-conv-ex-lookup*:

$\text{ex } P \ t \longleftrightarrow (\exists (k :: 'a) \ v. \text{lookup } t \ k = \text{Some } v \wedge P \ k \ v)$
 $\langle \text{proof} \rangle$

lemma *diag-lookup*:

$\text{lookup } (\text{diag } t) = (\lambda (k :: 'a, k'). \text{if } k = k' \text{ then } \text{lookup } t \ k \text{ else } \text{None})$
 $\langle \text{proof} \rangle$

context assumes *ID-ccompare-neq-None'*: $\text{ID } \text{CCOMPARE}('b :: \text{ccompare}) \neq$

None
begin

lemma *mapping-linorder'*: *class.linorder* (*cless-eq* :: '*b* $\Rightarrow$ '*b* $\Rightarrow$ *bool*) *cless*
 <*proof*>

lemma *mapping-comparator'*: *comparator* (*ccomp* :: '*b* *comparator*)
 <*proof*>

lemmas *rbt-comp'*[*simp*] = *rbt-comp-simps*[*OF mapping-comparator'*]

lemma *ccomp-comparator-prod*:
ccomp = (*comparator-prod ccomp ccomp* :: ('*a* $\times$ '*b*) *comparator*)
 <*proof*>

lemma *lookup-product*:
lookup (*product f rbt1 rbt2*) (*a* :: '*a*, *b* :: '*b*) =
 (*case lookup rbt1 a of None* $\Rightarrow$ *None*
 | *Some c* $\Rightarrow$ *map-option* (*f a c b*) (*lookup rbt2 b*)
 <*proof*>
end

end

hide-const (**open**) *impl-of lookup empty insert delete*
entries keys bulkload map-entry map fold join meet filter all ex product diag init

end

theory *AssocList* **imports**
HOL-Library.DAList
begin

3.8 Additional operations for associative lists

3.8.1 Operations on the raw type

primrec *update-with-aux* :: '*val* $\Rightarrow$ '*key* $\Rightarrow$ ('*val* $\Rightarrow$ '*val*) $\Rightarrow$ ('*key* $\times$ '*val*) *list* $\Rightarrow$
 ('*key* $\times$ '*val*) *list*

where

update-with-aux v k f [] = [(*k*, *f v*)]
 | *update-with-aux v k f (p # ps)* = (*if* (*fst p* = *k*) *then* (*k*, *f (snd p)*) # *ps* *else p*
 # *update-with-aux v k f ps*)

Do not use *AList.delete* because this traverses all the list even if it has found the key. We do not have to keep going because we use the invariant that keys are distinct.

fun *delete-aux* :: '*key* $\Rightarrow$ ('*key* $\times$ '*val*) *list* $\Rightarrow$ ('*key* $\times$ '*val*) *list*

where

$delete_aux\ k\ [] = []$
 $| delete_aux\ k\ ((k', v) \# xs) = (if\ k = k'\ then\ xs\ else\ (k', v) \# delete_aux\ k\ xs)$

definition $zip_with_index_from :: nat \Rightarrow 'a\ list \Rightarrow (nat \times 'a)\ list$ **where**

$zip_with_index_from\ n\ xs = zip\ [n..<n + length\ xs]\ xs$

abbreviation $zip_with_index :: 'a\ list \Rightarrow (nat \times 'a)\ list$ **where**

$zip_with_index \equiv zip_with_index_from\ 0$

lemma $update_conv_update_with_aux$:

$AList.update\ k\ v\ xs = update_with_aux\ v\ k\ (\lambda-. v)\ xs$
 $\langle proof \rangle$

lemma $map_of_update_with_aux'$:

$map_of\ (update_with_aux\ v\ k\ f\ ps)\ k' = ((map_of\ ps)(k \mapsto (case\ map_of\ ps\ k\ of\ None \Rightarrow f\ v\ | Some\ v \Rightarrow f\ v)))\ k'$
 $\langle proof \rangle$

lemma $map_of_update_with_aux$:

$map_of\ (update_with_aux\ v\ k\ f\ ps) = (map_of\ ps)(k \mapsto (case\ map_of\ ps\ k\ of\ None \Rightarrow f\ v\ | Some\ v \Rightarrow f\ v))$
 $\langle proof \rangle$

lemma $dom_update_with_aux$: $fst\ 'set\ (update_with_aux\ v\ k\ f\ ps) = \{k\} \cup fst\ 'set\ ps$

$\langle proof \rangle$

lemma $distinct_update_with_aux\ [simp]$:

$distinct\ (map\ fst\ (update_with_aux\ v\ k\ f\ ps)) = distinct\ (map\ fst\ ps)$
 $\langle proof \rangle$

lemma $set_update_with_aux$:

$distinct\ (map\ fst\ xs)$
 $\implies set\ (update_with_aux\ v\ k\ f\ xs) = (set\ xs - \{k\} \times UNIV \cup \{(k, f\ (case\ map_of\ xs\ k\ of\ None \Rightarrow v\ | Some\ v \Rightarrow v))\})$
 $\langle proof \rangle$

lemma set_delete_aux : $distinct\ (map\ fst\ xs) \implies set\ (delete_aux\ k\ xs) = set\ xs - \{k\} \times UNIV$

$\langle proof \rangle$

lemma dom_delete_aux : $distinct\ (map\ fst\ ps) \implies fst\ 'set\ (delete_aux\ k\ ps) = fst\ 'set\ ps - \{k\}$

$\langle proof \rangle$

lemma $distinct_delete_aux\ [simp]$:

$distinct\ (map\ fst\ ps) \implies distinct\ (map\ fst\ (delete_aux\ k\ ps))$
 $\langle proof \rangle$

lemma *map-of-delete-aux'*:

distinct (map fst xs) $\implies$ map-of (delete-aux k xs) = (map-of xs)(k := None)
<proof>

lemma *map-of-delete-aux*:

distinct (map fst xs) $\implies$ map-of (delete-aux k xs) k' = ((map-of xs)(k := None))
k'
<proof>

lemma *delete-aux-eq-Nil-conv*: *delete-aux k ts = [] $\longleftrightarrow$ ts = [] $\vee$ ($\exists v$. ts = [(k, v)])*
<proof>

lemma *zip-with-index-from-simps* [*simp*, *code*]:

zip-with-index-from n [] = []
zip-with-index-from n (x # xs) = (n, x) # zip-with-index-from (Suc n) xs
<proof>

lemma *zip-with-index-from-append* [*simp*]:

zip-with-index-from n (xs @ ys) = zip-with-index-from n xs @ zip-with-index-from
(n + length xs) ys
<proof>

lemma *zip-with-index-from-conv-nth*:

*zip-with-index-from n xs = map (λi . (n + i, xs ! i)) [0..*length xs*]*
<proof>

lemma *map-of-zip-with-index-from* [*simp*]:

map-of (zip-with-index-from n xs) i = (if $i \geq n \wedge i < n + \text{length } xs$ then Some
(xs ! (i - n)) else None)
<proof>

lemma *map-of-map'*: *map-of (map ($\lambda(k, v)$. (k, f k v)) xs) x = map-option (f x)*
(map-of xs x)
<proof>

3.8.2 Operations on the abstract type ('a, 'b) alist

lift-definition *update-with* :: '*v* $\Rightarrow$ '*k* $\Rightarrow$ ('*v* $\Rightarrow$ '*v*) $\Rightarrow$ ('*k*, '*v*) alist $\Rightarrow$ ('*k*, '*v*) alist
is update-with-aux *<proof>*

lift-definition *delete* :: ('*k*, '*v*) alist $\Rightarrow$ ('*k*, '*v*) alist **is** *delete-aux*
<proof>

lift-definition *keys* :: ('*k*, '*v*) alist $\Rightarrow$ '*k* set **is** set $\circ$ map fst *<proof>*

lift-definition *set* :: ('key, 'val) alist $\Rightarrow$ ('key $\times$ 'val) set
is List.set *<proof>*

lift-definition *map-values* :: ('key $\Rightarrow$ 'val $\Rightarrow$ 'val') $\Rightarrow$ ('key, 'val) alist $\Rightarrow$ ('key, 'val') alist **is**
 $\lambda f. \text{map } (\lambda(x,y). (x, f x y))$
 $\langle \text{proof} \rangle$

lemma *lookup-update-with* [simp]:
 $\text{DAList.lookup } (\text{update-with } v \ k \ f \ al) = (\text{DAList.lookup } al)(k \mapsto \text{case } \text{DAList.lookup } al \ k \text{ of } \text{None} \Rightarrow f \ v \mid \text{Some } v \Rightarrow f \ v)$
 $\langle \text{proof} \rangle$

lemma *lookup-delete* [simp]: $\text{DAList.lookup } (\text{delete } k \ al) = (\text{DAList.lookup } al)(k := \text{None})$
 $\langle \text{proof} \rangle$

lemma *finite-dom-lookup* [simp, intro!]: $\text{finite } (\text{dom } (\text{DAList.lookup } m))$
 $\langle \text{proof} \rangle$

lemma *update-conv-update-with*: $\text{DAList.update } k \ v = \text{update-with } v \ k \ (\lambda-. v)$
 $\langle \text{proof} \rangle$

lemma *lookup-update* [simp]: $\text{DAList.lookup } (\text{DAList.update } k \ v \ al) = (\text{DAList.lookup } al)(k \mapsto v)$
 $\langle \text{proof} \rangle$

lemma *dom-lookup-keys*: $\text{dom } (\text{DAList.lookup } al) = \text{keys } al$
 $\langle \text{proof} \rangle$

lemma *keys-empty* [simp]: $\text{keys } \text{DAList.empty} = \{\}$
 $\langle \text{proof} \rangle$

lemma *keys-update-with* [simp]: $\text{keys } (\text{update-with } v \ k \ f \ al) = \text{insert } k \ (\text{keys } al)$
 $\langle \text{proof} \rangle$

lemma *keys-update* [simp]: $\text{keys } (\text{DAList.update } k \ v \ al) = \text{insert } k \ (\text{keys } al)$
 $\langle \text{proof} \rangle$

lemma *keys-delete* [simp]: $\text{keys } (\text{delete } k \ al) = \text{keys } al - \{k\}$
 $\langle \text{proof} \rangle$

lemma *set-empty* [simp]: $\text{set } \text{DAList.empty} = \{\}$
 $\langle \text{proof} \rangle$

lemma *set-update-with*:
 $\text{set } (\text{update-with } v \ k \ f \ al) =$
 $(\text{set } al - \{k\} \times \text{UNIV} \cup \{(k, f (\text{case } \text{DAList.lookup } al \ k \text{ of } \text{None} \Rightarrow v \mid \text{Some } v \Rightarrow v))\})$
 $\langle \text{proof} \rangle$

lemma *set-update*: $\text{set } (DAList.update\ k\ v\ al) = (\text{set } al - \{k\} \times UNIV \cup \{(k, v)\})$
 $\langle proof \rangle$

lemma *set-delete*: $\text{set } (delete\ k\ al) = \text{set } al - \{k\} \times UNIV$
 $\langle proof \rangle$

lemma *size-dalist-transfer* [*transfer-rule*]:
includes *lifting-syntax*
shows $(\text{pcr-alist } (=) (=) ==> (=))\ \text{length}\ \text{size}$
 $\langle proof \rangle$

lemma *size-eq-card-dom-lookup*: $\text{size } al = \text{card } (\text{dom } (DAList.lookup\ al))$
 $\langle proof \rangle$

hide-const (**open**) *update-with keys set delete*

end

theory *DList-Set* **imports**
Collection-Eq
Equal
begin

3.9 Sets implemented by distinct lists

3.9.1 Operations on the raw type with parametrised equality

context *equal-base* **begin**

primrec *list-member* :: $'a\ list \Rightarrow 'a \Rightarrow bool$

where

$list_member\ []\ y \longleftrightarrow False$
 $| list_member\ (x \# xs)\ y \longleftrightarrow equal\ x\ y \vee list_member\ xs\ y$

primrec *list-distinct* :: $'a\ list \Rightarrow bool$

where

$list_distinct\ [] \longleftrightarrow True$
 $| list_distinct\ (x \# xs) \longleftrightarrow \neg list_member\ xs\ x \wedge list_distinct\ xs$

definition *list-insert* :: $'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**

$list_insert\ x\ xs = (\text{if } list_member\ xs\ x \text{ then } xs \text{ else } x \# xs)$

primrec *list-remove1* :: $'a \Rightarrow 'a\ list \Rightarrow 'a\ list$ **where**

$list_remove1\ x\ [] = []$
 $| list_remove1\ x\ (y \# xs) = (\text{if } equal\ x\ y \text{ then } xs \text{ else } y \# list_remove1\ x\ xs)$

primrec *list-remdups* :: $'a\ list \Rightarrow 'a\ list$ **where**

list-remdups [] = []
 | *list-remdups* (x # xs) = (if *list-member* xs x then *list-remdups* xs else x #
list-remdups xs)

lemma *list-member-filterD*: *list-member* (filter *P* xs) x $\implies$ *list-member* xs x
 <proof>

lemma *list-distinct-filter* [*simp*]: *list-distinct* xs $\implies$ *list-distinct* (filter *P* xs)
 <proof>

lemma *list-distinct-tl* [*simp*]: *list-distinct* xs $\implies$ *list-distinct* (tl xs)
 <proof>

end

lemmas [*code*] =
equal-base.list-member.simps
equal-base.list-distinct.simps
equal-base.list-insert-def
equal-base.list-remove1.simps
equal-base.list-remdups.simps

lemmas [*simp*] =
equal-base.list-member.simps
equal-base.list-distinct.simps
equal-base.list-remove1.simps
equal-base.list-remdups.simps

lemma *list-member-conv-member* [*simp*]:
equal-base.list-member (=) = *List.member*
 <proof>

lemma *list-distinct-conv-distinct* [*simp*]:
equal-base.list-distinct (=) = *List.distinct*
 <proof>

lemma *list-insert-conv-insert* [*simp*]:
equal-base.list-insert (=) = *List.insert*
 <proof>

lemma *list-remove1-conv-remove1* [*simp*]:
equal-base.list-remove1 (=) = *List.remove1*
 <proof>

lemma *list-remdups-conv-remdups* [*simp*]:
equal-base.list-remdups (=) = *List.remdups*
 <proof>

context *equal* **begin**

lemma *member-insert* [simp]: *list-member* (*list-insert* *x xs*) *y* $\longleftrightarrow$ *equal* *x y* $\vee$ *list-member* *xs y*
 <proof>

lemma *member-remove1* [simp]:
 $\neg$ *equal* *x y* $\implies$ *list-member* (*list-remove1* *x xs*) *y* = *list-member* *xs y*
 <proof>

lemma *distinct-remove1*:
list-distinct *xs* $\implies$ *list-distinct* (*list-remove1* *x xs*)
 <proof>

lemma *distinct-member-remove1* [simp]:
list-distinct *xs* $\implies$ *list-member* (*list-remove1* *x xs*) = (*list-member* *xs*)(*x* := *False*)
 <proof>

end

lemma *ID-ceq*:
ID *CEQ*('a :: *ceq*) = *Some* *eq* $\implies$ *equal* *eq*
 <proof>

3.9.2 The type of distinct lists

typedef (overloaded) 'a :: *ceq* *set-dlist* =
 {*xs*::'a *list*. *equal-base.list-distinct* *ceq'* *xs* $\vee$ *ID* *CEQ*('a) = *None*}
morphisms *list-of-dlist* *Abs-dlist'*
 <proof>

definition *Abs-dlist* :: 'a :: *ceq* *list* $\Rightarrow$ 'a *set-dlist*
where

Abs-dlist *xs* = *Abs-dlist'*
 (if *equal-base.list-distinct* *ceq'* *xs* $\vee$ *ID* *CEQ*('a) = *None* then *xs*
 else *equal-base.list-remdups* *ceq'* *xs*)

lemma *Abs-dlist-inverse*:
fixes *y* :: 'a :: *ceq* *list*
assumes *y* $\in$ {*xs*. *equal-base.list-distinct* *ceq'* *xs* $\vee$ *ID* *CEQ*('a) = *None*}
shows *list-of-dlist* (*Abs-dlist* *y*) = *y*
 <proof>

lemma *list-of-dlist-inverse*: *Abs-dlist* (*list-of-dlist* *dxs*) = *dxs*
 <proof>

lemma *type-definition-set-dlist'*:
type-definition *list-of-dlist* *Abs-dlist*
 {*xs* :: 'a :: *ceq* *list*. *equal-base.list-distinct* *ceq'* *xs* $\vee$ *ID* *CEQ*('a) = *None*}

<proof>

lemmas *Abs-dlist-cases*[*cases type: set-dlist*] =
type-definition.Abs-cases[*OF type-definition-set-dlist*']
and *Abs-dlist-induct*[*induct type: set-dlist*] =
type-definition.Abs-induct[*OF type-definition-set-dlist*'] **and**
Abs-dlist-inject = *type-definition.Abs-inject*[*OF type-definition-set-dlist*']

setup-lifting *type-definition-set-dlist'*

3.9.3 Operations

lift-definition *empty* :: '*a* :: *ceq set-dlist* **is** []
<proof>

lift-definition *insert* :: '*a* :: *ceq* ⇒ '*a set-dlist* ⇒ '*a set-dlist* **is**
equal-base.list-insert ceq'
<proof>

lift-definition *remove* :: '*a* :: *ceq* ⇒ '*a set-dlist* ⇒ '*a set-dlist* **is**
equal-base.list-remove1 ceq'
<proof>

lift-definition *filter* :: ('*a* :: *ceq* ⇒ *bool*) ⇒ '*a set-dlist* ⇒ '*a set-dlist* **is** *List.filter*
<proof>

Derived operations:

lift-definition *null* :: '*a* :: *ceq set-dlist* ⇒ *bool* **is** *List.null* *<proof>*

lift-definition *member* :: '*a* :: *ceq set-dlist* ⇒ '*a* ⇒ *bool* **is** *equal-base.list-member ceq'* *<proof>*

lift-definition *length* :: '*a* :: *ceq set-dlist* ⇒ *nat* **is** *List.length* *<proof>*

lift-definition *fold* :: ('*a* :: *ceq* ⇒ '*b* ⇒ '*b*) ⇒ '*a set-dlist* ⇒ '*b* ⇒ '*b* **is** *List.fold*
<proof>

lift-definition *foldr* :: ('*a* :: *ceq* ⇒ '*b* ⇒ '*b*) ⇒ '*a set-dlist* ⇒ '*b* ⇒ '*b* **is** *List.foldr*
<proof>

lift-definition *hd* :: '*a* :: *ceq set-dlist* ⇒ '*a* **is** *List.hd* *<proof>*

lift-definition *tl* :: '*a* :: *ceq set-dlist* ⇒ '*a set-dlist* **is** *List.tl*
<proof>

lift-definition *dlist-all* :: ('*a* ⇒ *bool*) ⇒ '*a* :: *ceq set-dlist* ⇒ *bool* **is** *list-all* *<proof>*

lift-definition *dlist-ex* :: ('*a* ⇒ *bool*) ⇒ '*a* :: *ceq set-dlist* ⇒ *bool* **is** *list-ex* *<proof>*

definition $union :: 'a :: ceq\ set-dlist \Rightarrow 'a\ set-dlist \Rightarrow 'a\ set-dlist$ **where**
 $union = fold\ insert$

lift-definition $product :: 'a :: ceq\ set-dlist \Rightarrow 'b :: ceq\ set-dlist \Rightarrow ('a \times 'b)\ set-dlist$
is $\lambda xs\ ys. rev\ (concat\ (map\ (\lambda x. map\ (Pair\ x)\ ys)\ xs))$
 $\langle proof \rangle$

lift-definition $Id-on :: 'a :: ceq\ set-dlist \Rightarrow ('a \times 'a)\ set-dlist$
is $map\ (\lambda x. (x, x))$
 $\langle proof \rangle$

3.9.4 Properties

lemma $member-empty\ [simp]: member\ empty = (\lambda -. False)$
 $\langle proof \rangle$

lemma $null-iff\ [simp]: null\ xs \longleftrightarrow xs = empty$
 $\langle proof \rangle$

lemma $list-of-dlist-empty\ [simp]: list-of-dlist\ DList-Set.empty = []$
 $\langle proof \rangle$

lemma $list-of-dlist-insert\ [simp]: \neg member\ dxs\ x \Longrightarrow list-of-dlist\ (insert\ x\ dxs) =$
 $x \# list-of-dlist\ dxs$
 $\langle proof \rangle$

lemma $list-of-dlist-eq-Nil-iff\ [simp]: list-of-dlist\ dxs = [] \longleftrightarrow dxs = empty$
 $\langle proof \rangle$

lemma $fold-empty\ [simp]: DList-Set.fold\ f\ empty\ b = b$
 $\langle proof \rangle$

lemma $fold-insert\ [simp]: \neg member\ dxs\ x \Longrightarrow DList-Set.fold\ f\ (insert\ x\ dxs)\ b =$
 $DList-Set.fold\ f\ dxs\ (f\ x\ b)$
 $\langle proof \rangle$

lemma $no-memb-fold-insert:$
 $\neg member\ dxs\ x \Longrightarrow fold\ f\ (insert\ x\ dxs)\ b = fold\ f\ dxs\ (f\ x\ b)$
 $\langle proof \rangle$

lemma $set-fold-insert: set\ (List.fold\ List.insert\ xs1\ xs2) = set\ xs1 \cup set\ xs2$
 $\langle proof \rangle$

lemma $list-of-dlist-eq-singleton-conv:$
 $list-of-dlist\ dxs = [x] \longleftrightarrow dxs = DList-Set.insert\ x\ DList-Set.empty$
 $\langle proof \rangle$

lemma $product-code\ [code\ abstract]:$
 $list-of-dlist\ (product\ dxs1\ dxs2) = fold\ (\lambda a. fold\ (\lambda c\ rest. (a, c) \# rest)\ dxs2)$

dxs1 []
 <proof>

lemma *set-list-of-dlist-Abs-dlist*:
 set (list-of-dlist (Abs-dlist xs)) = set xs
 <proof>

context assumes *ID-ceq-neq-None*: ID CEQ('a :: ceq) ≠ None
begin

lemma *equal-ceq*: equal (ceq' :: 'a ⇒ 'a ⇒ bool)
 <proof>

declare *Domainp-forall-transfer*[**where** *A* = *pcr-set-dlist* (=), *simplified set-dlist.domain-eq*,
transfer-rule]

lemma *set-dlist-induct* [case-names Nil insert, induct type: set-dlist]:
 fixes *dxs* :: 'a :: ceq set-dlist
 assumes Nil: *P empty* and Cons: $\bigwedge a \, dxs. \llbracket \neg \text{member } dxs \, a; P \, dxs \rrbracket \implies P$
 (insert *a dxs*)
 shows *P dxs*
 <proof>

context includes *lifting-syntax*
begin

lemma *fold-transfer2* [transfer-rule]:
 assumes *is-equality A*
 shows ((*A* ==> *pcr-set-dlist* (=) ==> *pcr-set-dlist* (=)) ==>
 (*pcr-set-dlist* (=) :: 'a list ⇒ 'a set-dlist ⇒ bool) ==> *pcr-set-dlist* (=) ==>
pcr-set-dlist (=))
 List.fold DList-Set.fold
 <proof>

end

lemma *distinct-list-of-dlist*:
 distinct (list-of-dlist (dxs :: 'a set-dlist))
 <proof>

lemma *member-empty-empty*: (∀ *x* :: 'a. ¬ member *dxs x*) ⇔ *dxs* = empty
 <proof>

lemma *Collect-member*: Collect (member (dxs :: 'a set-dlist)) = set (list-of-dlist
dxs)
 <proof>

lemma *member-insert*: member (insert (*x* :: 'a) *xs*) = (member *xs*)(*x* := True)

<proof>

lemma *member-remove*:

member (remove (x :: 'a) xs) = (member xs)(x := False)

<proof>

lemma *member-union*: *member (union (xs1 :: 'a set-dlist) xs2) x $\longleftrightarrow$ member xs1 x $\vee$ member xs2 x*

<proof>

lemma *member-fold-insert*: *member (List.fold insert xs dxs) (x :: 'a) $\longleftrightarrow$ member dxs x $\vee$ x $\in$ set xs*

<proof>

lemma *card-eq-length* [simp]:

card (Collect (member (dxs :: 'a set-dlist))) = length dxs

<proof>

lemma *finite-member* [simp]:

finite (Collect (member (dxs :: 'a set-dlist)))

<proof>

lemma *member-filter* [simp]: *member (filter P xs) = ($\lambda x :: 'a. \text{member xs } x \wedge P x$)*

<proof>

lemma *dlist-all-conv-member*: *dlist-all P dxs $\longleftrightarrow$ ($\forall x :: 'a. \text{member dxs } x \longrightarrow P x$)*

<proof>

lemma *dlist-ex-conv-member*: *dlist-ex P dxs $\longleftrightarrow$ ($\exists x :: 'a. \text{member dxs } x \wedge P x$)*

<proof>

lemma *member-Id-on*: *member (Id-on dxs) = ($\lambda(x :: 'a, y). x = y \wedge \text{member dxs } x$)*

<proof>

end

lemma *product-member*:

assumes *ID CEQ('a :: ceq) $\neq$ None* *ID CEQ('b :: ceq) $\neq$ None*

shows *member (product dxs1 dxs2) = ($\lambda(a :: 'a, b :: 'b). \text{member dxs1 } a \wedge \text{member dxs2 } b$)*

<proof>

hide-const (**open**) *empty insert remove null member length fold foldr union filter hd tl dlist-all product Id-on*

end

```

theory RBT-Set2
imports
  RBT-Mapping2
begin

```

3.10 Sets implemented by red-black trees

lemma *map-of-map-Pair-const*:

map-of (*map* ($\lambda x. (x, v)$) *xs*) = ($\lambda x. \text{if } x \in \text{set } xs \text{ then Some } v \text{ else None}$)
 <proof>

lemma *map-of-rev-unit [simp]*:

fixes *xs* :: ('a * unit) list
shows *map-of* (*rev xs*) = *map-of xs*
 <proof>

lemma *fold-split-conv-map-fst*: *fold* ($\lambda(x, y). f\ x$) *xs* = *fold* *f* (*map fst xs*)

<proof>

lemma *foldr-split-conv-map-fst*: *foldr* ($\lambda(x, y). f\ x$) *xs* = *foldr* *f* (*map fst xs*)

<proof>

lemma *set-foldr-Cons*:

set (*foldr* ($\lambda x\ xs. \text{if } P\ x\ xs \text{ then } x \# xs \text{ else } xs$) *as* []) $\subseteq$ *set as*
 <proof>

lemma *distinct-fst-foldr-Cons*:

distinct (*map f as*) $\implies$ *distinct* (*map f* (*foldr* ($\lambda x\ xs. \text{if } P\ x\ xs \text{ then } x \# xs \text{ else } xs$) *as* []))
 <proof>

lemma *filter-conv-foldr*:

filter *P xs* = *foldr* ($\lambda x\ xs. \text{if } P\ x \text{ then } x \# xs \text{ else } xs$) *xs* []
 <proof>

lemma *map-of-map-Pair-key*: *map-of* (*map* ($\lambda k. (k, f\ k)$) *xs*) *x* = (*if* *x* $\in$ *set xs* *then* *Some* (*f x*) *else* *None*)

<proof>

lemma *neq-Empty-conv*: *t* $\neq$ *rbt.Empty* $\longleftrightarrow$ ($\exists c\ l\ k\ v\ r. t = \text{Branch } c\ l\ k\ v\ r$)

<proof>

context *linorder* **begin**

lemma *is-rbt-RBT-fold-rbt-insert [simp]*:

is-rbt t $\implies$ *is-rbt* (*fold* ($\lambda(k, v). \text{rbt-insert } k\ v$) *xs t*)
 <proof>

lemma *rbt-lookup-RBT-fold-rbt-insert* [simp]:

$is_rbt\ t \implies rbt_lookup\ (fold\ (\lambda(k, v). rbt_insert\ k\ v)\ xs\ t) = rbt_lookup\ t\ ++$
 $map_of\ (rev\ xs)$
 <proof>

lemma *is-rbt-fold-rbt-delete* [simp]:

$is_rbt\ t \implies is_rbt\ (fold\ rbt_delete\ xs\ t)$
 <proof>

lemma *rbt-lookup-fold-rbt-delete* [simp]:

$is_rbt\ t \implies rbt_lookup\ (fold\ rbt_delete\ xs\ t) = rbt_lookup\ t\ |'\ (-\ set\ xs)$
 <proof>

lemma *is-rbt-fold-rbt-insert*: $is_rbt\ t \implies is_rbt\ (fold\ (\lambda k. rbt_insert\ k\ (f\ k))\ xs\ t)$

<proof>

lemma *rbt-lookup-fold-rbt-insert*:

$is_rbt\ t \implies$
 $rbt_lookup\ (fold\ (\lambda k. rbt_insert\ k\ (f\ k))\ xs\ t) =$
 $rbt_lookup\ t\ ++\ map_of\ (map\ (\lambda k. (k, f\ k))\ xs)$
 <proof>

end

definition *fold-rev* :: $('a \Rightarrow 'b \Rightarrow 'c \Rightarrow 'c) \Rightarrow ('a, 'b)\ rbt \Rightarrow 'c \Rightarrow 'c$

where $fold_rev\ f\ t = List.foldr\ (\lambda(k, v). f\ k\ v)\ (RBT-Impl.entries\ t)$

lemma *fold-rev-simps* [simp, code]:

$fold_rev\ f\ RBT-Impl.Empty = id$
 $fold_rev\ f\ (Branch\ c\ l\ k\ v\ r) = fold_rev\ f\ l\ o\ f\ k\ v\ o\ fold_rev\ f\ r$
 <proof>

context *linorder* **begin**

lemma *sorted-fst-foldr-Cons*:

$sorted\ (map\ f\ as) \implies sorted\ (map\ f\ (foldr\ (\lambda x\ xs. if\ P\ x\ xs\ then\ x\ \# \ xs\ else\ xs)$
 $as\ []))$
 <proof>

end

3.10.1 Type and operations

type-synonym $'a\ set_rbt = ('a, unit)\ mapping_rbt$

translations

$(type)\ 'a\ set_rbt\ <= (type)\ ('a, unit)\ mapping_rbt$

abbreviation *(input)* $\text{Set-RBT} :: ('a :: \text{compare}, \text{unit}) \text{RBT-Impl.rbt} \Rightarrow 'a \text{ set-rbt}$
where $\text{Set-RBT} \equiv \text{Mapping-RBT}$

3.10.2 Primitive operations

lift-definition $\text{member} :: 'a :: \text{compare set-rbt} \Rightarrow 'a \Rightarrow \text{bool}$ **is**
 $\lambda t x. x \in \text{dom} (\text{rbt-comp-lookup ccomp } t) \langle \text{proof} \rangle$

abbreviation $\text{empty} :: 'a :: \text{compare set-rbt}$
where $\text{empty} \equiv \text{RBT-Mapping2.empty}$

abbreviation $\text{insert} :: 'a :: \text{compare} \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a \text{ set-rbt}$
where $\text{insert } k \equiv \text{RBT-Mapping2.insert } k \ ()$

abbreviation $\text{remove} :: 'a :: \text{compare} \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a \text{ set-rbt}$
where $\text{remove} \equiv \text{RBT-Mapping2.delete}$

lift-definition $\text{bulkload} :: 'a :: \text{compare list} \Rightarrow 'a \text{ set-rbt}$ **is**
 $\text{rbt-comp-bulkload ccomp} \circ \text{map } (\lambda x. (x, ()))$
 $\langle \text{proof} \rangle$

abbreviation $\text{is-empty} :: 'a :: \text{compare set-rbt} \Rightarrow \text{bool}$
where $\text{is-empty} \equiv \text{RBT-Mapping2.is-empty}$

abbreviation $\text{union} :: 'a :: \text{compare set-rbt} \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a \text{ set-rbt}$
where $\text{union} \equiv \text{RBT-Mapping2.join } (\lambda - . \text{id})$

abbreviation $\text{inter} :: 'a :: \text{compare set-rbt} \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a \text{ set-rbt}$
where $\text{inter} \equiv \text{RBT-Mapping2.meet } (\lambda - . \text{id})$

lift-definition $\text{inter-list} :: 'a :: \text{compare set-rbt} \Rightarrow 'a \text{ list} \Rightarrow 'a \text{ set-rbt}$ **is**
 $\lambda t xs. \text{fold } (\lambda k. \text{rbt-comp-insert ccomp } k \ ()) [x \leftarrow xs. \text{rbt-comp-lookup ccomp } t \ x$
 $\neq \text{None}] \text{RBT-Impl.Empty}$
 $\langle \text{proof} \rangle$

lift-definition $\text{minus} :: 'a :: \text{compare set-rbt} \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a \text{ set-rbt}$ **is**
 $\text{rbt-comp-minus ccomp}$
 $\langle \text{proof} \rangle$

abbreviation $\text{filter} :: ('a :: \text{compare} \Rightarrow \text{bool}) \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a \text{ set-rbt}$
where $\text{filter } P \equiv \text{RBT-Mapping2.filter } (P \circ \text{fst})$

lift-definition $\text{fold} :: ('a :: \text{compare} \Rightarrow 'b \Rightarrow 'b) \Rightarrow 'a \text{ set-rbt} \Rightarrow 'b \Rightarrow 'b$ **is** $\lambda f.$
 $\text{RBT-Impl.fold } (\lambda a - . f \ a) \langle \text{proof} \rangle$

lift-definition $\text{fold1} :: ('a :: \text{compare} \Rightarrow 'a \Rightarrow 'a) \Rightarrow 'a \text{ set-rbt} \Rightarrow 'a$ **is** RBT-Impl.fold1
 $\langle \text{proof} \rangle$

lift-definition $\text{keys} :: 'a :: \text{compare set-rbt} \Rightarrow 'a \text{ list}$ **is** $\text{RBT-Impl.keys} \langle \text{proof} \rangle$

abbreviation $all :: ('a :: ccompare \Rightarrow bool) \Rightarrow 'a \text{ set-rbt} \Rightarrow bool$
where $all\ P \equiv RBT\text{-Mapping2}.all\ (\lambda k\ -. \ P\ k)$

abbreviation $ex :: ('a :: ccompare \Rightarrow bool) \Rightarrow 'a \text{ set-rbt} \Rightarrow bool$
where $ex\ P \equiv RBT\text{-Mapping2}.ex\ (\lambda k\ -. \ P\ k)$

definition $product :: 'a :: ccompare \text{ set-rbt} \Rightarrow 'b :: ccompare \text{ set-rbt} \Rightarrow ('a \times 'b) \text{ set-rbt}$
where $product\ rbt1\ rbt2 = RBT\text{-Mapping2}.product\ (\lambda\ -\ -\ -. \ ())\ rbt1\ rbt2$

abbreviation $Id\text{-on} :: 'a :: ccompare \text{ set-rbt} \Rightarrow ('a \times 'a) \text{ set-rbt}$
where $Id\text{-on} \equiv RBT\text{-Mapping2}.diag$

abbreviation $init :: 'a :: ccompare \text{ set-rbt} \Rightarrow ('a, unit, 'a) \text{ rbt-generator-state}$
where $init \equiv RBT\text{-Mapping2}.init$

3.10.3 Properties

lemma $member\text{-empty}$ [simp]:
 $member\ empty = (\lambda\ -. \ False)$
 $\langle proof \rangle$

lemma $fold\text{-conv-fold-keys}$: $RBT\text{-Set2}.fold\ f\ rbt\ b = List.fold\ f\ (RBT\text{-Set2}.keys\ rbt)\ b$
 $\langle proof \rangle$

lemma $fold\text{-conv-fold-keys}'$:
 $fold\ f\ t = List.fold\ f\ (RBT\text{-Impl}.keys\ (RBT\text{-Mapping2}.impl\text{-of}\ t))$
 $\langle proof \rangle$

lemma $member\text{-lookup}$ [code]: $member\ t\ x \longleftrightarrow RBT\text{-Mapping2}.lookup\ t\ x = Some\ ()$
 $\langle proof \rangle$

lemma $unfoldr\text{-rbt-keys-generator}$:
 $list.unfoldr\ rbt\text{-keys-generator}\ (init\ t) = keys\ t$
 $\langle proof \rangle$

lemma $keys\text{-eq-Nil-iff}$ [simp]: $keys\ rbt = [] \longleftrightarrow rbt = empty$
 $\langle proof \rangle$

lemma $fold1\text{-conv-fold}$: $fold1\ f\ rbt = List.fold\ f\ (tl\ (keys\ rbt))\ (hd\ (keys\ rbt))$
 $\langle proof \rangle$

context assumes $ID\text{-ccompare-neq-None}$: $ID\ CCOMPARE('a :: ccompare) \neq None$
begin

lemma $set\text{-linorder}$: $class.linorder\ (cless\text{-eq} :: 'a \Rightarrow 'a \Rightarrow bool)\ cless$

$\langle \text{proof} \rangle$

lemma *ccomp-comparator*: *comparator* (*ccomp* :: 'a *comparator*)
 $\langle \text{proof} \rangle$

lemmas *rbt-comps* = *rbt-comp-simps*[*OF ccomp-comparator*] *rbt-comp-minus*[*OF ccomp-comparator*]

lemma *is-rbt-impl-of* [*simp, intro*]:
fixes *t* :: 'a *set-rbt*
shows *ord.is-rbt cless* (*RBT-Mapping2.impl-of t*)
 $\langle \text{proof} \rangle$

lemma *member-RBT*:
ord.is-rbt cless t $\implies$ *member* (*Set-RBT t*) (*x* :: 'a) $\longleftrightarrow$ *ord.rbt-lookup cless t x*
 = *Some* ()
 $\langle \text{proof} \rangle$

lemma *member-impl-of*:
ord.rbt-lookup cless (*RBT-Mapping2.impl-of t*) (*x* :: 'a) = *Some* () $\longleftrightarrow$ *member t x*
 $\langle \text{proof} \rangle$

lemma *member-insert* [*simp*]:
member (*insert x* (*t* :: 'a *set-rbt*)) = (*member t*)(*x* := *True*)
 $\langle \text{proof} \rangle$

lemma *member-fold-insert* [*simp*]:
member (*List.fold insert xs* (*t* :: 'a *set-rbt*)) = ($\lambda x. \text{member } t \ x \ \vee \ x \in \text{set } xs$)
 $\langle \text{proof} \rangle$

lemma *member-remove* [*simp*]:
member (*remove* (*x* :: 'a) *t*) = (*member t*)(*x* := *False*)
 $\langle \text{proof} \rangle$

lemma *member-bulkload* [*simp*]:
member (*bulkload xs*) (*x* :: 'a) $\longleftrightarrow$ *x* $\in$ *set xs*
 $\langle \text{proof} \rangle$

lemma *member-conv-keys*: *member t* = ($\lambda x :: 'a. x \in \text{set } (\text{keys } t)$)
 $\langle \text{proof} \rangle$

lemma *is-empty-empty* [*simp*]:
is-empty t $\longleftrightarrow$ *t* = *empty*
 $\langle \text{proof} \rangle$

lemma *RBT-lookup-empty* [*simp*]:
ord.rbt-lookup cless (*t* :: ('a, *unit*) *rbt*) = *Map.empty* $\longleftrightarrow$ *t* = *RBT-Impl.Empty*
 $\langle \text{proof} \rangle$

lemma *member-empty-empty* [simp]:

$member\ t = (\lambda -. False) \longleftrightarrow (t :: 'a\ set-rbt) = empty$
 <proof>

lemma *member-union* [simp]:

$member\ (union\ (t1 :: 'a\ set-rbt)\ t2) = (\lambda x. member\ t1\ x \vee member\ t2\ x)$
 <proof>

lemma *member-minus* [simp]:

$member\ (minus\ (t1 :: 'a\ set-rbt)\ t2) = (\lambda x. member\ t1\ x \wedge \neg member\ t2\ x)$
 <proof>

lemma *member-inter* [simp]:

$member\ (inter\ (t1 :: 'a\ set-rbt)\ t2) = (\lambda x. member\ t1\ x \wedge member\ t2\ x)$
 <proof>

lemma *member-inter-list* [simp]:

$member\ (inter-list\ (t :: 'a\ set-rbt)\ xs) = (\lambda x. member\ t\ x \wedge x \in set\ xs)$
 <proof>

lemma *member-filter* [simp]:

$member\ (filter\ P\ (t :: 'a\ set-rbt)) = (\lambda x. member\ t\ x \wedge P\ x)$
 <proof>

lemma *distinct-keys* [simp]:

$distinct\ (keys\ (rbt :: 'a\ set-rbt))$
 <proof>

lemma *all-conv-all-member*:

$all\ P\ t \longleftrightarrow (\forall x :: 'a. member\ t\ x \longrightarrow P\ x)$
 <proof>

lemma *ex-conv-ex-member*:

$ex\ P\ t \longleftrightarrow (\exists x :: 'a. member\ t\ x \wedge P\ x)$
 <proof>

lemma *finite-member*: *finite* (*Collect* (*RBT-Set2.member* (*t* :: 'a *set-rbt*)))

<proof>

lemma *member-Id-on*: $member\ (Id-on\ t) = (\lambda(k :: 'a, k'). k = k' \wedge member\ t\ k)$

<proof>

context *assumes* *ID-ccompare-neq-None'*: *ID CCOMPARE*('b :: *ccompare*) $\neq$ *None*

begin

lemma *set-linorder'*: *class.linorder* (*cless-eq* :: 'b $\Rightarrow$ 'b $\Rightarrow$ *bool*) *cless*

<proof>

lemma *member-product*:

member (*product* *rbt1* *rbt2*) = ($\lambda ab :: 'a \times 'b. ab \in \text{Collect } (\text{member } rbt1) \times \text{Collect } (\text{member } rbt2)$)
 $\langle \text{proof} \rangle$

end

end

lemma *sorted-RBT-Set-keys*:

ID *CCOMPARE*('a :: *ccompare*) = *Some* *c*
 $\implies \text{linorder.sorted } (\text{le-of-comp } c) (\text{RBT-Set2.keys } rbt)$
 $\langle \text{proof} \rangle$

context *assumes* *ID-ccompare-neq-None*: *ID* *CCOMPARE*('a :: {*ccompare*, *lattice*}) $\neq \text{None}$
begin

lemma *set-linorder2*: *class.linorder* (*cless-eq* :: 'a $\Rightarrow$ 'a $\Rightarrow$ bool) *cless*
 $\langle \text{proof} \rangle$

end

lemma *set-keys-Mapping-RBT*: *set* (*keys* (*Mapping-RBT* *t*)) = *set* (*RBT-Impl.keys* *t*)
 $\langle \text{proof} \rangle$

hide-const (**open**) *member* *empty* *insert* *remove* *bulkload* *union* *minus*
keys *fold* *fold-rev* *filter* *all* *ex* *product* *Id-on* *init*

end

theory *Closure-Set* **imports** *Equal* **begin**

3.11 Sets implemented as Closures

context *equal-base* **begin**

definition *fun-upd* :: ('a $\Rightarrow$ 'b) $\Rightarrow$ 'a $\Rightarrow$ 'b $\Rightarrow$ 'a $\Rightarrow$ 'b
where *fun-upd-apply*: *fun-upd* *f* *a* *b* *a'* = (if *equal* *a* *a'* then *b* else *f* *a'*)

end

lemmas [*code*] = *equal-base.fun-upd-apply*

lemmas [*simp*] = *equal-base.fun-upd-apply*

lemma *fun-upd-conv-fun-upd*: *equal-base.fun-upd* (=) = *fun-upd*

<proof>

end

theory *Set-Impl* **imports**

Collection-Enum

DList-Set

RBT-Set2

Closure-Set

Containers-Generator

Complex-Main

begin

3.12 Different implementations of sets

3.12.1 Auxiliary functions

A simple quicksort implementation

context *ord* **begin**

function (*sequential*) *quicksort-acc* :: 'a list $\Rightarrow$ 'a list $\Rightarrow$ 'a list

and *quicksort-part* :: 'a list $\Rightarrow$ 'a $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ 'a list

where

quicksort-acc ac [] = ac

| *quicksort-acc* ac [x] = x # ac

| *quicksort-acc* ac (x # xs) = *quicksort-part* ac x [] [] xs

| *quicksort-part* ac x lts eqs gts [] = *quicksort-acc* (eqs @ x # *quicksort-acc* ac gts) lts

| *quicksort-part* ac x lts eqs gts (z # zs) =

(if z > x then *quicksort-part* ac x lts eqs (z # gts) zs

else if z < x then *quicksort-part* ac x (z # lts) eqs gts zs

else *quicksort-part* ac x lts (z # eqs) gts zs)

<proof>

lemma *length-quicksort-accp*:

quicksort-acc-quicksort-part-dom (Inl (ac, xs)) $\implies$ *length* (*quicksort-acc* ac xs)
= *length* ac + *length* xs

and *length-quicksort-partp*:

quicksort-acc-quicksort-part-dom (Inr (ac, x, lts, eqs, gts, zs))

$\implies$ *length* (*quicksort-part* ac x lts eqs gts zs) = *length* ac + 1 + *length* lts +
length eqs + *length* gts + *length* zs

<proof>

termination

<proof>

definition *quicksort* :: 'a list $\Rightarrow$ 'a list

where $quicksort = quicksort\text{-}acc \ []$

lemma $set\text{-}quicksort\text{-}acc$ $[simp]$: $set (quicksort\text{-}acc\ ac\ xs) = set\ ac \cup set\ xs$
and $set\text{-}quicksort\text{-}part$ $[simp]$:
 $set (quicksort\text{-}part\ ac\ x\ lts\ eqs\ gts\ zs) =$
 $set\ ac \cup \{x\} \cup set\ lts \cup set\ eqs \cup set\ gts \cup set\ zs$
 $\langle proof \rangle$

lemma $set\text{-}quicksort$ $[simp]$: $set (quicksort\ xs) = set\ xs$
 $\langle proof \rangle$

lemma $distinct\text{-}quicksort\text{-}acc$:
 $distinct (quicksort\text{-}acc\ ac\ xs) = distinct (ac @ xs)$
and $distinct\text{-}quicksort\text{-}part$:
 $distinct (quicksort\text{-}part\ ac\ x\ lts\ eqs\ gts\ zs) = distinct (ac @ [x] @ lts @ eqs @ gts$
 $@ zs)$
 $\langle proof \rangle$

lemma $distinct\text{-}quicksort$ $[simp]$: $distinct (quicksort\ xs) = distinct\ xs$
 $\langle proof \rangle$

end

lemmas $[code] =$
 $ord.quicksort\text{-}acc.simps\ quicksort\text{-}acc.simps$
 $ord.quicksort\text{-}part.simps\ quicksort\text{-}part.simps$
 $ord.quicksort\text{-}def\ quicksort\text{-}def$

context $linorder$ **begin**

lemma $sorted\text{-}quicksort\text{-}acc$:
 $\llbracket sorted\ ac; \forall x \in set\ xs. \forall a \in set\ ac. x < a \rrbracket$
 $\implies sorted (quicksort\text{-}acc\ ac\ xs)$
and $sorted\text{-}quicksort\text{-}part$:
 $\llbracket sorted\ ac; \forall y \in set\ lts \cup \{x\} \cup set\ eqs \cup set\ gts \cup set\ zs. \forall a \in set\ ac. y < a;$
 $\forall y \in set\ lts. y < x; \forall y \in set\ eqs. y = x; \forall y \in set\ gts. y > x \rrbracket$
 $\implies sorted (quicksort\text{-}part\ ac\ x\ lts\ eqs\ gts\ zs)$
 $\langle proof \rangle$

lemma $sorted\text{-}quicksort$ $[simp]$: $sorted (quicksort\ xs)$
 $\langle proof \rangle$

lemma $insort\text{-}key\text{-}append1$:
 $\forall y \in set\ ys. f\ x < f\ y \implies insort\text{-}key\ f\ x\ (xs @ ys) = insort\text{-}key\ f\ x\ xs @ ys$
 $\langle proof \rangle$

lemma $insort\text{-}key\text{-}append2$:
 $\forall y \in set\ xs. f\ x > f\ y \implies insort\text{-}key\ f\ x\ (xs @ ys) = xs @ insort\text{-}key\ f\ x\ ys$
 $\langle proof \rangle$

lemma *sort-key-append*:

$\forall x \in \text{set } xs. \forall y \in \text{set } ys. f\ x < f\ y \implies \text{sort-key } f\ (xs @ ys) = \text{sort-key } f\ xs @ \text{sort-key } f\ ys$
 $\langle \text{proof} \rangle$

definition *single-list* :: 'a $\Rightarrow$ 'a list **where** *single-list* a = [a]

lemma *to-single-list*: $x \# xs = \text{single-list } x @ xs$

$\langle \text{proof} \rangle$

lemma *sort-snoc*: $\text{sort } (xs @ [x]) = \text{insort } x\ (\text{sort } xs)$

$\langle \text{proof} \rangle$

lemma *sort-append-swap*: $\text{sort } (xs @ ys) = \text{sort } (ys @ xs)$

$\langle \text{proof} \rangle$

lemma *sort-append-swap2*: $\text{sort } (xs @ ys @ zs) = \text{sort } (ys @ xs @ zs)$

$\langle \text{proof} \rangle$

lemma *sort-Cons-append-swap*: $\text{sort } (x \# xs) = \text{sort } (xs @ [x])$

$\langle \text{proof} \rangle$

lemma *sort-append-Cons-swap*: $\text{sort } (ys @ x \# xs) = \text{sort } (ys @ xs @ [x])$

$\langle \text{proof} \rangle$

lemma *quicksort-acc-conv-sort*:

$\text{quicksort-acc } ac\ xs = \text{sort } xs @ ac$

and *quicksort-part-conv-sort*:

$\llbracket \forall y \in \text{set } lts. y < x; \forall y \in \text{set } eqs. y = x; \forall y \in \text{set } gts. y > x \rrbracket$

$\implies \text{quicksort-part } ac\ x\ lts\ eqs\ gts\ zs = \text{sort } (lts @ eqs @ gts @ x \# zs) @ ac$

$\langle \text{proof} \rangle$

lemma *quicksort-conv-sort*: $\text{quicksort } xs = \text{sort } xs$

$\langle \text{proof} \rangle$

lemma *sort-remdups*: $\text{sort } (\text{remdups } xs) = \text{remdups } (\text{sort } xs)$

$\langle \text{proof} \rangle$

end

Removing duplicates from a sorted list

context *ord* **begin**

fun *remdups-sorted* :: 'a list $\Rightarrow$ 'a list

where

$\text{remdups-sorted } [] = []$

| $\text{remdups-sorted } [x] = [x]$

```
| remdups-sorted (x#y#xs) = (if x < y then x # remdups-sorted (y#xs) else
```

```
remdups-sorted (y#xs))
```

```
end
```

```
lemmas [code] = ord.remdups-sorted.simps
```

```
context linorder begin
```

```
lemma [simp]:
```

```
  assumes sorted xs
```

```
  shows sorted-remdups-sorted: sorted (remdups-sorted xs)
```

```
  and set-remdups-sorted: set (remdups-sorted xs) = set xs
```

```
  ⟨proof⟩
```

```
lemma distinct-remdups-sorted [simp]: sorted xs ⟹ distinct (remdups-sorted xs)
```

```
  ⟨proof⟩
```

```
lemma remdups-sorted-conv-remdups: sorted xs ⟹ remdups-sorted xs = remdups
```

```
xs
```

```
  ⟨proof⟩
```

```
end
```

An specialised operation to convert a finite set into a sorted list

```
definition csorted-list-of-set :: 'a :: ccompare set ⇒ 'a list
```

```
where
```

```
  csorted-list-of-set A =
```

```
  (if ID CCOMPARE('a) = None ∨ ¬ finite A then undefined else linorder.sorted-list-of-set
```

```
  cless-eq A)
```

```
lemma csorted-list-of-set-set [simp]:
```

```
  ⟦ ID CCOMPARE('a :: ccompare) = Some c; linorder.sorted (le-of-comp c) xs;
```

```
  distinct xs ⟧
```

```
  ⟹ linorder.sorted-list-of-set (le-of-comp c) (set xs) = xs
```

```
  ⟨proof⟩
```

```
lemma csorted-list-of-set-split:
```

```
  fixes A :: 'a :: ccompare set shows
```

```
  P (csorted-list-of-set A) ⟷
```

```
  (∀ xs. ID CCOMPARE('a) ≠ None ⟶ finite A ⟶ A = set xs ⟶ distinct xs
```

```
  ⟶ linorder.sorted cless-eq xs ⟶ P xs) ∧
```

```
  (ID CCOMPARE('a) = None ∨ ¬ finite A ⟶ P undefined)
```

```
  ⟨proof⟩
```

```
code-identifier code-module Set ↦ (SML) Set-Impl
```

```
  | code-module Set-Impl ↦ (SML) Set-Impl
```

3.12.2 Delete code equation with set as constructor

lemma *is-empty-unfold* [code-unfold]:
 $set\text{-}eq\ A\ \{\} = Set.is\text{-}empty\ A$
 $set\text{-}eq\ \{\}\ A = Set.is\text{-}empty\ A$
 $\langle proof \rangle$

definition *is-UNIV* :: 'a set $\Rightarrow$ bool
where *is-UNIV* $A \longleftrightarrow A = UNIV$

lemma *is-UNIV-unfold* [code-unfold]:
 $A = UNIV \longleftrightarrow is\text{-}UNIV\ A$
 $UNIV = A \longleftrightarrow is\text{-}UNIV\ A$
 $set\text{-}eq\ A\ UNIV \longleftrightarrow is\text{-}UNIV\ A$
 $set\text{-}eq\ UNIV\ A \longleftrightarrow is\text{-}UNIV\ A$
 $\langle proof \rangle$

3.12.3 Set implementations

definition *Collect-set* :: ('a $\Rightarrow$ bool) $\Rightarrow$ 'a set
where [simp]: *Collect-set* = *Collect*

definition *DList-set* :: 'a :: ceq set-dlist $\Rightarrow$ 'a set
where *DList-set* = *Collect* o *DList-Set.member*

definition *RBT-set* :: 'a :: ccompare set-rbt $\Rightarrow$ 'a set
where *RBT-set* = *Collect* o *RBT-Set2.member*

definition *Complement* :: 'a set $\Rightarrow$ 'a set
where [simp]: *Complement* $A = -\ A$

definition *Set-Monad* :: 'a list $\Rightarrow$ 'a set
where [simp]: *Set-Monad* = *set*

code-datatype *Collect-set DList-set RBT-set Set-Monad Complement*

lemma *DList-set-empty* [simp]: *DList-set DList-Set.empty* = {}
 $\langle proof \rangle$

lemma *RBT-set-empty* [simp]: *RBT-set RBT-Set2.empty* = {}
 $\langle proof \rangle$

lemma *RBT-set-conv-keys*:
 $ID\ CCOMPARE('a :: ccompare) \neq None$
 $\implies RBT\text{-}set\ (t :: 'a\ set\text{-}rbt) = set\ (RBT\text{-}Set2.keys\ t)$
 $\langle proof \rangle$

3.12.4 Set operations

named-theorems *set-base-code* $\langle$ Code equations not involving set complement $\rangle$

Various fold operations over sets

```
typedef ('a, 'b) comp-fun-commute = {f :: 'a ⇒ 'b ⇒ 'b. comp-fun-commute f}
morphisms comp-fun-commute-apply Abs-comp-fun-commute
⟨proof⟩
```

setup-lifting *type-definition-comp-fun-commute*

```
lemma comp-fun-commute-apply' [simp]:
  comp-fun-commute-on UNIV (comp-fun-commute-apply f)
⟨proof⟩
```

lift-definition *set-fold-cfc* :: ('a, 'b) *comp-fun-commute* ⇒ 'b ⇒ 'a set ⇒ 'b is *Finite-Set.fold* ⟨proof⟩

```
lemma set-fold-cfc-code [code]:
  fixes xs :: 'a :: ceq list
  and dxs :: 'a :: ceq set-dlist
  and rbt :: 'b :: ccompare set-rbt
  shows [set-base-code]:
    set-fold-cfc f'' b (RBT-set rbt) =
      (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "set-fold-cfc RBT-set:
ccompare = None") (λ-. set-fold-cfc f'' b (RBT-set rbt))
       | Some - ⇒ RBT-Set2.fold (comp-fun-commute-apply f'') rbt
b)
    (is ?RBT-set)
    set-fold-cfc f' b (DList-set dxs) =
      (case ID CEQ('a) of None ⇒ Code.abort (STR "set-fold-cfc DList-set: ceq =
None") (λ-. set-fold-cfc f' b (DList-set dxs))
       | Some - ⇒ DList-Set.fold (comp-fun-commute-apply f') dxs b)
    (is ?DList-set)
    set-fold-cfc f b (Set-Monad xs) =
      (case ID CEQ('a) of None ⇒ Code.abort (STR "set-fold-cfc Set-Monad: ceq
= None") (λ-. set-fold-cfc f b (Set-Monad xs))
       | Some eq ⇒ List.fold (comp-fun-commute-apply f) (equal-base.list-remdups
eq xs) b)
    (is ?Set-Monad)
    set-fold-cfc f''' b (Collect-set P) = Code.abort (STR "set-fold-cfc not supported
on Collect-set") (λ-. set-fold-cfc f''' b (Collect-set P))
  and set-fold-cfc-Complement:
    set-fold-cfc f''' b (Complement A) = Code.abort (STR "set-fold-cfc not supported
on Complement") (λ-. set-fold-cfc f''' b (Complement A))
⟨proof⟩
```

```
typedef ('a, 'b) comp-fun-idem = {f :: 'a ⇒ 'b ⇒ 'b. comp-fun-idem f}
morphisms comp-fun-idem-apply Abs-comp-fun-idem
⟨proof⟩
```

setup-lifting *type-definition-comp-fun-idem*

lemma *comp-fun-idem-apply'* [simp]:
comp-fun-idem-on UNIV (comp-fun-idem-apply f)
 ⟨proof⟩

lift-definition *set-fold-cfi* :: ('a, 'b) *comp-fun-idem* ⇒ 'b ⇒ 'a *set* ⇒ 'b **is** *Finite-Set.fold* ⟨proof⟩

lemma *set-fold-cfi-code* [code]:
fixes *xs* :: 'a *list*
and *dxs* :: 'b :: *ceq set-dlist*
and *rbt* :: 'c :: *ccompare set-rbt*
shows
set-fold-cfi f'' b (RBT-set rbt) =
(case ID CCOMPARE('c) of None ⇒ Code.abort (STR "set-fold-cfi RBT-set:
ccompare = None")) (λ-. set-fold-cfi f'' b (RBT-set rbt))
| Some - ⇒ RBT-Set2.fold (comp-fun-idem-apply f'') rbt b)
(is ?RBT-set)
set-fold-cfi f' b (DList-set dxs) =
(case ID CEQ('b) of None ⇒ Code.abort (STR "set-fold-cfi DList-set: ceq =
None")) (λ-. set-fold-cfi f' b (DList-set dxs))
| Some - ⇒ DList-Set.fold (comp-fun-idem-apply f') dxs b)
(is ?DList-set)
set-fold-cfi f b (Set-Monad xs) = List.fold (comp-fun-idem-apply f) xs b
(is ?Set-Monad)
set-fold-cfi f b (Collect-set P) = Code.abort (STR "set-fold-cfi not supported on
Collect-set")) (λ-. set-fold-cfi f b (Collect-set P))
set-fold-cfi f b (Complement A) = Code.abort (STR "set-fold-cfi not supported
on Complement")) (λ-. set-fold-cfi f b (Complement A))
 ⟨proof⟩

typedef 'a *semilattice-set* = {f :: 'a ⇒ 'a ⇒ 'a. *semilattice-set f*}
morphisms *semilattice-set-apply* *Abs-semilattice-set*
 ⟨proof⟩

setup-lifting *type-definition-semilattice-set*

lemma *semilattice-set-apply'* [simp]:
semilattice-set (semilattice-set-apply f)
 ⟨proof⟩

lemma *comp-fun-idem-semilattice-set-apply* [simp]:
comp-fun-idem-on UNIV (semilattice-set-apply f)
 ⟨proof⟩

lift-definition *set-fold1* :: 'a *semilattice-set* ⇒ 'a *set* ⇒ 'a **is** *semilattice-set.F*
 ⟨proof⟩

lemma (in *semilattice-set*) *F-set-conv-fold*:
xs ≠ [] ⇒ F (set xs) = Finite-Set.fold f (hd xs) (set (tl xs))

<proof>

lemma *set-fold1-code* [code]:

fixes *rbt* :: 'a :: {ccompare, lattice} *set-rbt*
and *dxs* :: 'b :: {ceq, lattice} *set-dlist*
shows [set-base-code]:

$$\text{set-fold1 } f'' \text{ (RBT-set } rbt) =$$

$$\begin{aligned} & (\text{case ID CCOMPARE('a) of None} \Rightarrow \text{Code.abort (STR "set-fold1 RBT-set:} \\ & \text{ccompare = None") } (\lambda\cdot. \text{set-fold1 } f'' \text{ (RBT-set } rbt)) \\ & \quad | \text{Some } - \Rightarrow \text{if RBT-Set2.is-empty } rbt \text{ then Code.abort (STR "set-fold1 RBT-set: empty set") } (\lambda\cdot. \text{set-fold1 } f'' \text{ (RBT-set } rbt)) \\ & \quad \text{else RBT-Set2.fold1 (semilattice-set-apply } f'') \text{ } rbt) \\ & (\text{is ?RBT-set}) \\ & \text{set-fold1 } f' \text{ (DList-set } dxs) = \\ & (\text{case ID CEQ('b) of None} \Rightarrow \text{Code.abort (STR "set-fold1 DList-set: ceq =} \\ & \text{None") } (\lambda\cdot. \text{set-fold1 } f' \text{ (DList-set } dxs)) \\ & \quad | \text{Some } - \Rightarrow \text{if DList-Set.null } dxs \text{ then Code.abort (STR "set-fold1} \\ & \text{DList-set: empty set") } (\lambda\cdot. \text{set-fold1 } f' \text{ (DList-set } dxs)) \\ & \quad \text{else DList-Set.fold (semilattice-set-apply } f') \text{ (DList-Set.tl} \\ & \text{ } dxs) \text{ (DList-Set.hd } dxs)) \\ & (\text{is ?DList-set}) \\ & \text{set-fold1 } f \text{ (Set-Monad (x \# xs)) = fold (semilattice-set-apply } f) \text{ } xs \text{ } x \\ & (\text{is ?Set-Monad}) \\ & \text{set-fold1 } f \text{ (Collect-set } P) = \text{Code.abort (STR "set-fold1: Collect-set") } (\lambda\cdot. \\ & \text{set-fold1 } f \text{ (Collect-set } P)) \\ & \text{and set-fold1-Complement:} \\ & \text{set-fold1 } f \text{ (Complement } A) = \text{Code.abort (STR "set-fold1: Complement") } (\lambda\cdot. \\ & \text{set-fold1 } f \text{ (Complement } A)) \\ & \text{<proof>} \end{aligned}$$

Implementation of set operations

lemma *Collect-code* [code, set-base-code]:

fixes *P* :: 'a :: cenum $\Rightarrow$ bool **shows**
 $\text{Collect } P =$

$$\begin{aligned} & (\text{case ID CENUM('a) of None} \Rightarrow \text{Collect-set } P \\ & \quad | \text{Some (enum, -)} \Rightarrow \text{Set-Monad (filter } P \text{ enum)}) \end{aligned}$$
<proof>

lemma *finite-code* [code]:

fixes *dxs* :: 'a :: ceq *set-dlist*
and *rbt* :: 'b :: ccompare *set-rbt*
and *A* :: 'c :: finite-UNIV *set* **and** *P* :: 'c $\Rightarrow$ bool
shows [set-base-code]:

$$\text{finite (Collect-set } P) \longleftrightarrow$$

$$\text{of-phantom (finite-UNIV :: 'c finite-UNIV) } \vee \text{ Code.abort (STR "finite Col-} \\ \text{lect-set") } (\lambda\cdot. \text{finite (Collect-set } P))$$

$$\text{finite (Set-Monad } xs) = \text{True}$$
and *finite-Complement*:

$$\text{finite (Complement } A) \longleftrightarrow$$

```

    (if of-phantom (finite-UNIV :: 'c finite-UNIV) then True
     else if finite A then False
     else Code.abort (STR "finite Complement: infinite set") (λ-. finite (Complement
A))))
and [set-base-code]:
  finite (RBT-set rbt) =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "finite RBT-set:
ccompare = None") (λ-. finite (RBT-set rbt))
     | Some - ⇒ True)
  finite (DList-set dxs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "finite DList-set: ceq = None")
(λ-. finite (DList-set dxs))
     | Some - ⇒ True)
  ⟨proof⟩

```

lemma *CARD-code* [code-unfold]:
CARD('a :: card-UNIV) = of-phantom (card-UNIV :: 'a card-UNIV)
 ⟨proof⟩

lemma *card-code* [code]:
fixes *dxs :: 'a :: ceq set-dlist and xs :: 'a list*
and *rbt :: 'b :: ccompare set-rbt*
and *A :: 'c :: card-UNIV set*
shows *card-Complement*:
card (Complement A) =
(let a = card A; s = CARD('c)
in if s > 0 then s - a
else if finite A then 0
else Code.abort (STR "card Complement: infinite") (λ-. card (Complement
A)))

```

and [set-base-code]:
  card (Set-Monad xs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "card Set-Monad: ceq =
None") (λ-. card (Set-Monad xs))
     | Some eq ⇒ length (equal-base.list-remdups eq xs))
  card (RBT-set rbt) =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "card RBT-set:
ccompare = None") (λ-. card (RBT-set rbt))
     | Some - ⇒ length (RBT-Set2.keys rbt))
  card (DList-set dxs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "card DList-set: ceq = None")
(λ-. card (DList-set dxs))
     | Some - ⇒ DList-Set.length dxs)
  ⟨proof⟩

```

lemma *is-UNIV-code* [code, set-base-code]:
fixes *rbt :: 'a :: {cproper-interval, finite-UNIV} set-rbt*
and *A :: 'b :: card-UNIV set*

shows
is-UNIV (*RBT-set* *rbt*) =
 (case *ID CCOMPARE*('a) of *None* $\Rightarrow$ *Code.abort* (*STR* "is-UNIV *RBT-set*:
ccompare = *None*") (λ -. *is-UNIV* (*RBT-set* *rbt*))
 | *Some* - $\Rightarrow$ *of-phantom* (*finite-UNIV* :: 'a *finite-UNIV*) $\wedge$
proper-intrvl.exhaustive-fusion cproper-interval rbt-keys-generator (*RBT-Set2.init*
rbt))
 (**is** ?*rbt*)
is-UNIV *A* $\longleftrightarrow$
 (let *a* = *CARD*('b);
 b = *card* *A*
 in if *a* > 0 then *a* = *b*
 else if *b* > 0 then *False*
 else *Code.abort* (*STR* "is-UNIV called on infinite type and set") (λ -.
is-UNIV *A*))
 (**is** ?*generic*)
 <proof>

lemma *is-empty-code* [*code*]:
fixes *dxs* :: 'a :: *ceq set-dlist*
and *rbt* :: 'b :: *ccompare set-rbt*
and *A* :: 'c *set*
shows *is-empty-Complement*:
Set.is-empty (*Complement* *A*) $\longleftrightarrow$ *is-UNIV* *A*
 (**is** ?*Complement*)
and [*set-base-code*]:
Set.is-empty (*RBT-set* *rbt*) $\longleftrightarrow$
 (case *ID CCOMPARE*('b) of *None* $\Rightarrow$ *Code.abort* (*STR* "is-empty *RBT-set*:
ccompare = *None*") (λ -. *Set.is-empty* (*RBT-set* *rbt*))
 | *Some* - $\Rightarrow$ *RBT-Set2.is-empty* *rbt*)
 (**is** ?*RBT-set*)
Set.is-empty (*DList-set* *dxs*) $\longleftrightarrow$
 (case *ID CEQ*('a) of *None* $\Rightarrow$ *Code.abort* (*STR* "is-empty *DList-set*: *ceq* =
None") (λ -. *Set.is-empty* (*DList-set* *dxs*))
 | *Some* - $\Rightarrow$ *DList-Set.null* *dxs*)
 (**is** ?*DList-set*)
Set.is-empty (*Set-Monad* *xs*) $\longleftrightarrow$ *xs* = []
 <proof>

lemma *Set-insert-code* [*code*]:
fixes *dxs* :: 'a :: *ceq set-dlist*
and *rbt* :: 'b :: *ccompare set-rbt*
shows *insert-Complement*:
 $\bigwedge x. \text{Set.insert } x \text{ (Complement } X) = \text{Complement (Set.remove } x \text{ } X)$
and [*set-base-code*]:
 $\bigwedge x. \text{Set.insert } x \text{ (RBT-set } rbt) =$
 (case *ID CCOMPARE*('b) of *None* $\Rightarrow$ *Code.abort* (*STR* "insert *RBT-set*:
ccompare = *None*") (λ -. *Set.insert* *x* (*RBT-set* *rbt*))
 | *Some* - $\Rightarrow$ *RBT-set* (*RBT-Set2.insert* *x* *rbt*))

$\bigwedge x. \text{Set.insert } x \text{ (DList-set } dxs) =$
 $(\text{case ID CEQ('a) of None} \Rightarrow \text{Code.abort (STR "insert DList-set: ceq =$
 $\text{None") } (\lambda-. \text{Set.insert } x \text{ (DList-set } dxs))$
 $\quad | \text{Some } - \Rightarrow \text{DList-set (DList-Set.insert } x \text{ } dxs))$
 $\bigwedge x. \text{Set.insert } x \text{ (Set-Monad } xs) = \text{Set-Monad } (x \# xs)$
 $\bigwedge x. \text{Set.insert } x \text{ (Collect-set } A) =$
 $(\text{case ID CEQ('a) of None} \Rightarrow \text{Code.abort (STR "insert Collect-set: ceq =$
 $\text{None") } (\lambda-. \text{Set.insert } x \text{ (Collect-set } A))$
 $\quad | \text{Some } eq \Rightarrow \text{Collect-set (equal-base.fun-upd eq } A \text{ } x \text{ True))}$
 $\langle \text{proof} \rangle$

lemma *Set-member-code* [code]:

fixes $xs :: 'a :: \text{ceq list}$
shows [set-base-code]:
 $\bigwedge x. x \in \text{Set-Monad } xs \longleftrightarrow$
 $(\text{case ID CEQ('a) of None} \Rightarrow \text{Code.abort (STR "member Set-Monad: ceq =$
 $\text{None") } (\lambda-. x \in \text{Set-Monad } xs)$
 $\quad | \text{Some } eq \Rightarrow \text{equal-base.list-member eq } xs \text{ } x)$
and *mem-Complement*:
 $\bigwedge x. x \in \text{Complement } X \longleftrightarrow x \notin X$
and [set-base-code]:
 $\bigwedge x. x \in \text{RBT-set } rbt \longleftrightarrow \text{RBT-Set2.member } rbt \text{ } x$
 $\bigwedge x. x \in \text{DList-set } dxs \longleftrightarrow \text{DList-Set.member } dxs \text{ } x$
 $\bigwedge x. x \in \text{Collect-set } A \longleftrightarrow A \text{ } x$
 $\langle \text{proof} \rangle$

lemma *Set-remove-code* [code]:

fixes $rbt :: 'a :: \text{compare set-rbt}$
and $dxs :: 'b :: \text{ceq set-dlist}$
shows *remove-Complement*:
 $\bigwedge x A. \text{Set.remove } x \text{ (Complement } A) = \text{Complement (Set.insert } x \text{ } A)$
and [set-base-code]:
 $\bigwedge x. \text{Set.remove } x \text{ (RBT-set } rbt) =$
 $(\text{case ID CCOMPARE('a) of None} \Rightarrow \text{Code.abort (STR "remove RBT-set:$
 $\text{ccompare = None") } (\lambda-. \text{Set.remove } x \text{ (RBT-set } rbt))$
 $\quad | \text{Some } - \Rightarrow \text{RBT-set (RBT-Set2.remove } x \text{ } rbt))$
 $\bigwedge x. \text{Set.remove } x \text{ (DList-set } dxs) =$
 $(\text{case ID CEQ('b) of None} \Rightarrow \text{Code.abort (STR "remove DList-set: ceq =$
 $\text{None") } (\lambda-. \text{Set.remove } x \text{ (DList-set } dxs))$
 $\quad | \text{Some } - \Rightarrow \text{DList-set (DList-Set.remove } x \text{ } dxs))$
 $\bigwedge x. \text{Set.remove } x \text{ (Collect-set } A) =$
 $(\text{case ID CEQ('b) of None} \Rightarrow \text{Code.abort (STR "remove Collect: ceq = None")}$
 $(\lambda-. \text{Set.remove } x \text{ (Collect-set } A))$
 $\quad | \text{Some } eq \Rightarrow \text{Collect-set (equal-base.fun-upd eq } A \text{ } x \text{ False))}$
 $\langle \text{proof} \rangle$

lemma *Set-uminus-code* [code]:

$- (\text{Complement } B) = B$
 $- (\text{Collect-set } P) = \text{Collect-set } (\lambda x. \neg P \text{ } x)$

– $A = \text{Complement } A$
 $\langle \text{proof} \rangle$

These equations represent complements as true complements. If you want that the complement operations returns an explicit enumeration of the elements, use the following set of equations which use *cenum*.

lemma *Set-uminus-cenum*:

fixes $A :: 'a :: \text{cenum set}$
shows $- A =$
 $(\text{case ID CENUM}('a) \text{ of } \text{None} \Rightarrow \text{Complement } A$
 $\quad | \text{Some } (\text{enum}, -) \Rightarrow \text{Set-Monad } (\text{filter } (\lambda x. x \notin A) \text{ enum}))$
and $- (\text{Complement } B) = B$
 $\langle \text{proof} \rangle$

lemma *Set-minus-code* [*code*, *set-base-code*]:

fixes $\text{rbt1 rbt2} :: 'a :: \text{compare set-rbt}$
shows
 $\text{RBT-set rbt1} - \text{RBT-set rbt2} =$
 $(\text{case ID CCOMPARE}('a) \text{ of } \text{None} \Rightarrow \text{Code.abort } (\text{STR } "minus \text{RBT-set}$
 $\text{RBT-set: ccompare} = \text{None}")) (\lambda -. \text{RBT-set rbt1} - \text{RBT-set rbt2})$
 $\quad | \text{Some } - \Rightarrow \text{RBT-set } (\text{RBT-Set2.minus rbt1 rbt2}))$
 $A - B = A \cap (- B)$
 $\langle \text{proof} \rangle$

lemma *Set-union-code* [*code*]:

fixes $\text{rbt1 rbt2} :: 'a :: \text{compare set-rbt}$
and $\text{rbt} :: 'b :: \{\text{compare}, \text{ceq}\} \text{ set-rbt}$
and $\text{dxs} :: 'b \text{ set-dlist}$
and $\text{dxs1 dxs2} :: 'c :: \text{ceq set-dlist}$
shows *Set-union-Complement*:
 $B' \cup \text{Complement } B = \text{Complement } (- B' \cap B)$
 $\text{Complement } B \cup B' = \text{Complement } (B \cap - B')$
and [*set-base-code*]:
 $\text{RBT-set rbt1} \cup \text{RBT-set rbt2} =$
 $(\text{case ID CCOMPARE}('a) \text{ of } \text{None} \Rightarrow \text{Code.abort } (\text{STR } "union \text{RBT-set}$
 $\text{RBT-set: ccompare} = \text{None}")) (\lambda -. \text{RBT-set rbt1} \cup \text{RBT-set rbt2})$
 $\quad | \text{Some } - \Rightarrow \text{RBT-set } (\text{RBT-Set2.union rbt1 rbt2}))$ (**is**
 $? \text{RBT-set-RBT-set}$)
 $\text{RBT-set rbt} \cup \text{DList-set dxs} =$
 $(\text{case ID CCOMPARE}('b) \text{ of } \text{None} \Rightarrow \text{Code.abort } (\text{STR } "union \text{RBT-set}$
 $\text{DList-set: ccompare} = \text{None}")) (\lambda -. \text{RBT-set rbt} \cup \text{DList-set dxs})$
 $\quad | \text{Some } - \Rightarrow$
 $\text{case ID CEQ}('b) \text{ of } \text{None} \Rightarrow \text{Code.abort } (\text{STR } "union \text{RBT-set DList-set:}$
 $\text{ceq} = \text{None}")) (\lambda -. \text{RBT-set rbt} \cup \text{DList-set dxs})$
 $\quad | \text{Some } - \Rightarrow \text{RBT-set } (\text{DList-Set.fold RBT-Set2.insert dxs rbt}))$
(is $? \text{RBT-set-DList-set}$)
 $\text{DList-set dxs} \cup \text{RBT-set rbt} =$
 $(\text{case ID CCOMPARE}('b) \text{ of } \text{None} \Rightarrow \text{Code.abort } (\text{STR } "union \text{DList-set}$
 $\text{RBT-set: ccompare} = \text{None}")) (\lambda -. \text{RBT-set rbt} \cup \text{DList-set dxs})$

```

      | Some - =>
        case ID CEQ('b) of None => Code.abort (STR "union DList-set RBT-set:
ceq = None") (λ-. RBT-set rbt ∪ DList-set dxs)
      | Some - => RBT-set (DList-Set.fold RBT-Set2.insert dxs rbt))
(is ?DList-set-RBT-set)
  DList-set dxs1 ∪ DList-set dxs2 =
    (case ID CEQ('c) of None => Code.abort (STR "union DList-set DList-set:
ceq = None") (λ-. DList-set dxs1 ∪ DList-set dxs2)
    | Some - => DList-set (DList-Set.union dxs1 dxs2)) (is
?DList-set-DList-set)
  Set-Monad zs ∪ RBT-set rbt2 =
    (case ID CCOMPARE('a) of None => Code.abort (STR "union Set-Monad
RBT-set: ccompare = None") (λ-. Set-Monad zs ∪ RBT-set rbt2)
    | Some - => RBT-set (fold RBT-Set2.insert zs rbt2)) (is
?Set-Monad-RBT-set)
  RBT-set rbt1 ∪ Set-Monad zs =
    (case ID CCOMPARE('a) of None => Code.abort (STR "union RBT-set
Set-Monad: ccompare = None") (λ-. RBT-set rbt1 ∪ Set-Monad zs)
    | Some - => RBT-set (fold RBT-Set2.insert zs rbt1)) (is
?RBT-set-Set-Monad)
  Set-Monad ws ∪ DList-set dxs2 =
    (case ID CEQ('c) of None => Code.abort (STR "union Set-Monad DList-set:
ceq = None") (λ-. Set-Monad ws ∪ DList-set dxs2)
    | Some - => DList-set (fold DList-Set.insert ws dxs2)) (is
?Set-Monad-DList-set)
  DList-set dxs1 ∪ Set-Monad ws =
    (case ID CEQ('c) of None => Code.abort (STR "union DList-set Set-Monad:
ceq = None") (λ-. DList-set dxs1 ∪ Set-Monad ws)
    | Some - => DList-set (fold DList-Set.insert ws dxs1)) (is
?DList-set-Set-Monad)
  Set-Monad xs ∪ Set-Monad ys = Set-Monad (xs @ ys)
  B ∪ Collect-set A = Collect-set (λx. A x ∨ x ∈ B)
  Collect-set A ∪ B = Collect-set (λx. A x ∨ x ∈ B)
<proof>

```

lemma *Set-inter-code* [code]:

```

fixes rbt1 rbt2 :: 'a :: ccompare set-rbt
and rbt :: 'b :: {ccompare, ceq} set-rbt
and dxs :: 'b set-dlist
and dxs1 dxs2 :: 'c :: ceq set-dlist
and xs1 xs2 :: 'c list

```

shows [set-base-code]:

```

  RBT-set rbt1 ∩ Set-Monad xs =
    (case ID CCOMPARE('a) of None => Code.abort (STR "inter RBT-set
Set-Monad: ccompare = None") (λ-. RBT-set rbt1 ∩ Set-Monad xs)
    | Some - => RBT-set (RBT-Set2.inter-list rbt1 xs)) (is
?rbt-monad)

```

```

  Set-Monad xs ∩ RBT-set rbt1 =

```

```

    (case ID CCOMPARE('a) of None ⇒ Code.abort (STR "inter Set-Monad
RBT-set: ccompare = None'') (λ-. RBT-set rbt1 ∩ Set-Monad xs)
      | Some - ⇒ RBT-set (RBT-Set2.inter-list rbt1 xs)) (is
?monad-rbt)
    Set-Monad xs' ∩ DList-set dxs2 =
    (case ID CEQ('c) of None ⇒ Code.abort (STR "inter Set-Monad DList-set:
ceq = None'') (λ-. Set-Monad xs' ∩ DList-set dxs2)
      | Some eq ⇒ DList-set (DList-Set.filter (equal-base.list-member
eq xs') dxs2)) (is ?monad-dlist)
    Set-Monad xs1 ∩ Set-Monad xs2 =
    (case ID CEQ('c) of None ⇒ Code.abort (STR "inter Set-Monad Set-Monad:
ceq = None'') (λ-. Set-Monad xs1 ∩ Set-Monad xs2)
      | Some eq ⇒ Set-Monad (filter (equal-base.list-member eq xs2)
xs1)) (is ?monad)
    RBT-set rbt ∩ DList-set dxs =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "inter RBT-set
DList-set: ccompare = None'') (λ-. RBT-set rbt ∩ DList-set dxs)
      | Some - ⇒
      case ID CEQ('b) of None ⇒ Code.abort (STR "inter RBT-set DList-set:
ceq = None'') (λ-. RBT-set rbt ∩ DList-set dxs)
        | Some - ⇒ RBT-set (RBT-Set2.inter-list rbt (list-of-dlist dxs)))
    (is ?rbt-dlist)
    RBT-set rbt1 ∩ RBT-set rbt2 =
    (case ID CCOMPARE('a) of None ⇒ Code.abort (STR "inter RBT-set
RBT-set: ccompare = None'') (λ-. RBT-set rbt1 ∩ RBT-set rbt2)
      | Some - ⇒ RBT-set (RBT-Set2.inter rbt1 rbt2)) (is ?rbt-rbt)
    DList-set dxs1 ∩ Set-Monad xs' =
    (case ID CEQ('c) of None ⇒ Code.abort (STR "inter DList-set Set-Monad:
ceq = None'') (λ-. DList-set dxs1 ∩ Set-Monad xs')
      | Some eq ⇒ DList-set (DList-Set.filter (equal-base.list-member
eq xs') dxs1)) (is ?dlist-monad)
    DList-set dxs1 ∩ DList-set dxs2 =
    (case ID CEQ('c) of None ⇒ Code.abort (STR "inter DList-set DList-set: ceq
= None'') (λ-. DList-set dxs1 ∩ DList-set dxs2)
      | Some - ⇒ DList-set (DList-Set.filter (DList-Set.member dxs2)
dxs1)) (is ?dlist)
    DList-set dxs ∩ RBT-set rbt =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "inter DList-set
RBT-set: ccompare = None'') (λ-. DList-set dxs ∩ RBT-set rbt)
      | Some - ⇒
      case ID CEQ('b) of None ⇒ Code.abort (STR "inter DList-set RBT-set: ceq
= None'') (λ-. DList-set dxs ∩ RBT-set rbt)
        | Some - ⇒ RBT-set (RBT-Set2.inter-list rbt (list-of-dlist dxs)))
    (is ?dlist-rbt)
and Set-inter-Complement:
    Complement B'' ∩ Complement B''' = Complement (B'' ∪ B''') (is ?complement)
and [set-base-code]:
    G ∩ RBT-set rbt2 =
    (case ID CCOMPARE('a) of None ⇒ Code.abort (STR "inter RBT-set2:

```

```

ccompare = None'' (λ-. G ∩ RBT-set rbt2)
              | Some - ⇒ RBT-set (RBT-Set2.filter (λx. x ∈ G) rbt2)) (is
?rbt2)
  RBT-set rbt1 ∩ G =
    (case ID CCOMPARE('a) of None ⇒ Code.abort (STR "inter RBT-set1:
ccompare = None'' (λ-. RBT-set rbt1 ∩ G)
              | Some - ⇒ RBT-set (RBT-Set2.filter (λx. x ∈ G) rbt1)) (is
?rbt1)
  H ∩ DList-set dxs2 =
    (case ID CEQ('c) of None ⇒ Code.abort (STR "inter DList-set2: ceq =
None'' (λ-. H ∩ DList-set dxs2)
              | Some eq ⇒ DList-set (DList-Set.filter (λx. x ∈ H) dxs2)) (is
?dlist2)
  DList-set dxs1 ∩ H =
    (case ID CEQ('c) of None ⇒ Code.abort (STR "inter DList-set1: ceq =
None'' (λ-. DList-set dxs1 ∩ H)
              | Some eq ⇒ DList-set (DList-Set.filter (λx. x ∈ H) dxs1)) (is
?dlist1)
  I ∩ Set-Monad xs'' = Set-Monad (filter (λx. x ∈ I) xs'') (is ?monad2)
  Set-Monad xs'' ∩ I = Set-Monad (filter (λx. x ∈ I) xs'') (is ?monad1)
  J ∩ Collect-set A'' = Collect-set (λx. A'' x ∧ x ∈ J) (is ?collect2)
  Collect-set A'' ∩ J = Collect-set (λx. A'' x ∧ x ∈ J) (is ?collect1)
⟨proof⟩

```

lemma *Set-bind-code* [code, set-base-code]:

```

fixes dxs :: 'a :: ceq set-dlist
and rbt :: 'b :: ccompare set-rbt
shows
  Set.bind (RBT-set rbt) f'' =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "bind RBT-set:
ccompare = None'' (λ-. Set.bind (RBT-set rbt) f'')
              | Some - ⇒ RBT-Set2.fold (union ∘ f'') rbt { }) (is ?RBT)
  Set.bind (DList-set dxs) f' =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "bind DList-set: ceq = None''
(λ-. Set.bind (DList-set dxs) f')
              | Some - ⇒ DList-Set.fold (union ∘ f') dxs { }) (is ?DList)
  Set.bind (Set-Monad xs) f = fold ((∪) ∘ f) xs (Set-Monad []) (is ?Set-Monad)
⟨proof⟩

```

lemma *UNIV-code* [code, set-base-code]:

```

UNIV = - { }
⟨proof⟩

```

lift-definition *inf-sls* :: 'a :: lattice semilattice-set

```

is inf
⟨proof⟩

```

lemma *Inf-fin-code* [code, set-base-code]:

Inf-fin $A = \text{set-fold1 } \text{inf-sls } A$
 $\langle \text{proof} \rangle$

lift-definition *sup-sls* :: 'a :: lattice semilattice-set
is *sup*
 $\langle \text{proof} \rangle$

lemma *Sup-fin-code* [code, set-base-code]:
 $\text{Sup-fin } A = \text{set-fold1 } \text{sup-sls } A$
 $\langle \text{proof} \rangle$

lift-definition *inf-cfi* :: ('a :: lattice, 'a) comp-fun-idem
is *inf*
 $\langle \text{proof} \rangle$

lemma *Inf-code*:
fixes $A :: 'a :: \text{complete-lattice set}$
shows $\text{Inf } A = (\text{if finite } A \text{ then set-fold-cfi } \text{inf-cfi } \text{top } A \text{ else Code.abort (STR$
 $\text{"Inf: infinite"}) } (\lambda-. \text{Inf } A))$
 $\langle \text{proof} \rangle$

lift-definition *sup-cfi* :: ('a :: lattice, 'a) comp-fun-idem
is *sup*
 $\langle \text{proof} \rangle$

lemma *Sup-code*:
fixes $A :: 'a :: \text{complete-lattice set}$
shows $\text{Sup } A = (\text{if finite } A \text{ then set-fold-cfi } \text{sup-cfi } \text{bot } A \text{ else Code.abort (STR$
 $\text{"Sup: infinite"}) } (\lambda-. \text{Sup } A))$
 $\langle \text{proof} \rangle$

lemmas *Inter-code* [code, set-base-code] = *Inf-code*[**where** ?'a = - :: type set]
lemmas *Union-code* [code, set-base-code] = *Sup-code*[**where** ?'a = - :: type set]
lemmas *Predicate-Inf-code* [code, set-base-code] = *Inf-code*[**where** ?'a = - :: type
Predicate.pred]
lemmas *Predicate-Sup-code* [code, set-base-code] = *Sup-code*[**where** ?'a = - :: type
Predicate.pred]
lemmas *Inf-fun-code* [code, set-base-code] = *Inf-code*[**where** ?'a = - :: type $\Rightarrow$ - ::
complete-lattice]
lemmas *Sup-fun-code* [code, set-base-code] = *Sup-code*[**where** ?'a = - :: type $\Rightarrow$ -
:: complete-lattice]

lift-definition *min-sls* :: 'a :: linorder semilattice-set
is *min*
 $\langle \text{proof} \rangle$

lemma *Min-code* [code, set-base-code]:
 $\text{Min } A = \text{set-fold1 } \text{min-sls } A$
 $\langle \text{proof} \rangle$

lift-definition *max-sls* :: 'a :: *linorder semilattice-set*
 is *max*
 <proof>

lemma *Max-code* [*code*, *set-base-code*]:
Max A = set-fold1 max-sls A
 <proof>

We do not implement *Ball*, *Bex*, and *sorted-list-of-set* for *Collect-set* using *CENUM*('a), because it should already have been converted to an explicit list of elements if that is possible.

lemma *Ball-code* [*code*, *set-base-code*]:
 fixes *rbt* :: 'a :: *ccompare set-rbt*
 and *dxs* :: 'b :: *ceq set-dlist*
 shows
Ball (RBT-set rbt) P'' =
 (case ID CCOMPARE('a) of None $\Rightarrow$ Code.abort (STR "Ball RBT-set: ccompare = None") (λ-. *Ball (RBT-set rbt) P''*)
 | Some - $\Rightarrow$ RBT-Set2.all P'' rbt)
Ball (DList-set dxs) P' =
 (case ID CEQ('b) of None $\Rightarrow$ Code.abort (STR "Ball DList-set: ceq = None")
 (λ-. *Ball (DList-set dxs) P'*)
 | Some - $\Rightarrow$ DList-Set.dlist-all P' dxs)
Ball (Set-Monad xs) P = list-all P xs
 <proof>

lemma *Bex-code* [*code*, *set-base-code*]:
 fixes *rbt* :: 'a :: *ccompare set-rbt*
 and *dxs* :: 'b :: *ceq set-dlist*
 shows
Bex (RBT-set rbt) P'' =
 (case ID CCOMPARE('a) of None $\Rightarrow$ Code.abort (STR "Bex RBT-set: ccompare = None") (λ-. *Bex (RBT-set rbt) P''*)
 | Some - $\Rightarrow$ RBT-Set2.ex P'' rbt)
Bex (DList-set dxs) P' =
 (case ID CEQ('b) of None $\Rightarrow$ Code.abort (STR "Bex DList-set: ceq = None")
 (λ-. *Bex (DList-set dxs) P'*)
 | Some - $\Rightarrow$ DList-Set.dlist-ex P' dxs)
Bex (Set-Monad xs) P = list-ex P xs
 <proof>

lemma *csorted-list-of-set-code* [*code*, *set-base-code*]:
 fixes *rbt* :: 'a :: *ccompare set-rbt*
 and *dxs* :: 'b :: {*ccompare*, *ceq*} *set-dlist*
 and *xs* :: 'a :: *ccompare list*
 shows
csorted-list-of-set (Set-Monad xs) =
 (case ID CCOMPARE('a) of None $\Rightarrow$ Code.abort (STR "csorted-list-of-set

```

Set-Monad: ccompare = None'' (λ-. csorted-list-of-set (Set-Monad xs))
          | Some c ⇒ ord.remdups-sorted (lt-of-comp c) (ord.quicksort
(lt-of-comp c) xs))
      csorted-list-of-set (DList-set dxs) =
        (case ID CEQ('b) of None ⇒ Code.abort (STR "csorted-list-of-set DList-set:
ceq = None'') (λ-. csorted-list-of-set (DList-set dxs))
          | Some - ⇒
            case ID CCOMPARE('b) of None ⇒ Code.abort (STR "csorted-list-of-set
DList-set: ccompare = None'') (λ-. csorted-list-of-set (DList-set dxs))
          | Some c ⇒ ord.quicksort (lt-of-comp c) (list-of-dlist dxs))
      csorted-list-of-set (RBT-set rbt) =
        (case ID CCOMPARE('a) of None ⇒ Code.abort (STR "csorted-list-of-set
RBT-set: ccompare = None'') (λ-. csorted-list-of-set (RBT-set rbt))
          | Some - ⇒ RBT-Set2.keys rbt)
    ⟨proof⟩

```

lemma *cless-set-code* [code]:

```

fixes rbt rbt' :: 'a :: ccompare set-rbt
and rbt1 rbt2 :: 'b :: cproper-interval set-rbt
and A B :: 'a set
and A' B' :: 'b set
shows cless-set-rbt-Complement1:
  cless-set (Complement (RBT-set rbt1)) (RBT-set rbt2) ⟷
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "cless-set (Complement
RBT-set) RBT-set: ccompare = None'') (λ-. cless-set (Complement (RBT-set rbt1))
(RBT-set rbt2))
      | Some c ⇒
        finite (UNIV :: 'b set) ∧
          proper-intrvl.Compl-set-less-aux-fusion (lt-of-comp c) cproper-interval
rbt-keys-generator rbt-keys-generator None (RBT-Set2.init rbt1) (RBT-Set2.init
rbt2))
    (is ?Compl-rbt)
and cless-set-rbt-Complement2:
  cless-set (RBT-set rbt1) (Complement (RBT-set rbt2)) ⟷
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "cless-set RBT-set
(Complement RBT-set): ccompare = None'') (λ-. cless-set (RBT-set rbt1) (Complement
(RBT-set rbt2)))
      | Some c ⇒
        finite (UNIV :: 'b set) ⟶
          proper-intrvl.set-less-aux-Compl-fusion (lt-of-comp c) cproper-interval
rbt-keys-generator rbt-keys-generator None (RBT-Set2.init rbt1) (RBT-Set2.init
rbt2))
    (is ?rbt-Compl)
and [set-base-code]:
  cless-set (RBT-set rbt) (RBT-set rbt') ⟷
    (case ID CCOMPARE('a) of None ⇒ Code.abort (STR "cless-set RBT-set
RBT-set: ccompare = None'') (λ-. cless-set (RBT-set rbt) (RBT-set rbt'))
      | Some c ⇒ ord.lexord-fusion (λx y. lt-of-comp c y x) rbt-keys-generator
rbt-keys-generator (RBT-Set2.init rbt) (RBT-Set2.init rbt'))

```

```

    (is ?rbt-rbt)
  and cless-set-Complement12:
    cless-set (Complement A) (Complement B)  $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cless-set Complement
    Complement: ccompare = None") ( $\lambda$ -. cless-set (Complement A) (Complement B))
      | Some -  $\Rightarrow$  cless B A)
    (is ?Compl-Compl)
  and cless-set-Complement1:
    cless-set (Complement A') B'  $\longleftrightarrow$ 
    (case ID CCOMPARE('b) of None  $\Rightarrow$  Code.abort (STR "cless-set Complement1: ccompare = None") ( $\lambda$ -. cless-set (Complement A') B')
      | Some c  $\Rightarrow$ 
      if finite A'  $\wedge$  finite B' then
        finite (UNIV :: 'b set)  $\wedge$ 
        proper-intvl.Compl-set-less-aux (lt-of-comp c) cproper-interval None
        (csorted-list-of-set A') (csorted-list-of-set B')
      else Code.abort (STR "cless-set Complement1: infinite set") ( $\lambda$ -. cless-set
        (Complement A') B'))
    (is ?Compl-fin-fin)
  and cless-set-Complement2:
    cless-set A' (Complement B')  $\longleftrightarrow$ 
    (case ID CCOMPARE('b) of None  $\Rightarrow$  Code.abort (STR "cless-set Complement2: ccompare = None") ( $\lambda$ -. cless-set A' (Complement B'))
      | Some c  $\Rightarrow$ 
      if finite A'  $\wedge$  finite B' then
        finite (UNIV :: 'b set)  $\longrightarrow$ 
        proper-intvl.set-less-aux-Compl (lt-of-comp c) cproper-interval None
        (csorted-list-of-set A') (csorted-list-of-set B')
      else Code.abort (STR "cless-set Complement2: infinite set") ( $\lambda$ -. cless-set
        A' (Complement B')))
    (is ?fin-Compl-fin)
  and [set-base-code]:
    cless-set A B  $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cless-set: ccompare = None") ( $\lambda$ -. cless-set A B)
      | Some c  $\Rightarrow$ 
      if finite A  $\wedge$  finite B then ord.lexordp ( $\lambda$ x y. lt-of-comp c y x) (csorted-list-of-set
        A) (csorted-list-of-set B)
      else Code.abort (STR "cless-set: infinite set") ( $\lambda$ -. cless-set A B))
    (is ?fin-fin)
  <proof>

```

lemma *le-of-comp-set-less-eq*:

le-of-comp (comp-of-ords (ord.set-less-eq le) (ord.set-less le)) = ord.set-less-eq le
 <proof>

lemma *cless-eq-set-code* [code]:

fixes *rbt rbt' :: 'a :: ccompare set-rbt*
and *rbt1 rbt2 :: 'b :: cproper-interval set-rbt*

```

and A B :: 'a set
and A' B' :: 'b set
shows cless-eq-set-rbt-Complement1:
  cless-eq-set (Complement (RBT-set rbt1)) (RBT-set rbt2)  $\longleftrightarrow$ 
    (case ID CCOMPARE('b) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set (Complement
RBT-set) RBT-set: ccompare = None") ( $\lambda$ -. cless-eq-set (Complement (RBT-set
rbt1)) (RBT-set rbt2))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'b set)  $\wedge$ 
          proper-intrvl.Compl-set-less-eq-aux-fusion (lt-of-comp c) cproper-interval
rbt-keys-generator rbt-keys-generator None (RBT-Set2.init rbt1) (RBT-Set2.init
rbt2))
    (is ?Compl-rbt)
and cless-eq-set-rbt-Complement2:
  cless-eq-set (RBT-set rbt1) (Complement (RBT-set rbt2))  $\longleftrightarrow$ 
    (case ID CCOMPARE('b) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set RBT-set
(Complement RBT-set): ccompare = None") ( $\lambda$ -. cless-eq-set (RBT-set rbt1) (Complement
(RBT-set rbt2)))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'b set)  $\longrightarrow$ 
          proper-intrvl.set-less-eq-aux-Compl-fusion (lt-of-comp c) cproper-interval
rbt-keys-generator rbt-keys-generator None (RBT-Set2.init rbt1) (RBT-Set2.init
rbt2))
    (is ?rbt-Compl)
and [set-base-code]:
  cless-eq-set (RBT-set rbt) (RBT-set rbt')  $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set RBT-set
RBT-set: ccompare = None") ( $\lambda$ -. cless-eq-set (RBT-set rbt) (RBT-set rbt'))
      | Some c  $\Rightarrow$  ord.lexord-eq-fusion ( $\lambda$ x y. lt-of-comp c y x)
rbt-keys-generator rbt-keys-generator (RBT-Set2.init rbt) (RBT-Set2.init rbt'))
    (is ?rbt-rbt)
and cless-eq-set-Complement12:
  cless-eq-set (Complement A) (Complement B)  $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set Complement
Complement: ccompare = None") ( $\lambda$ -. cless-eq (Complement A) (Complement
B))
      | Some c  $\Rightarrow$  cless-eq-set B A)
    (is ?Compl-Compl)
and cless-eq-set-Complement1:
  cless-eq-set (Complement A') B'  $\longleftrightarrow$ 
    (case ID CCOMPARE('b) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set Complement1:
ccompare = None") ( $\lambda$ -. cless-eq-set (Complement A') B'))
      | Some c  $\Rightarrow$ 
        if finite A'  $\wedge$  finite B'
        then finite (UNIV :: 'b set)  $\wedge$ 
          proper-intrvl.Compl-set-less-eq-aux (lt-of-comp c) cproper-interval None
(csoted-list-of-set A') (csoted-list-of-set B')
        else Code.abort (STR "cless-eq-set Complement1: infinite set") ( $\lambda$ -. cless-eq-set
(Complement A') B'))

```

```

    (is ?Compl-fin-fin)
  and cless-eq-set-Complement2:
    cless-eq-set A' (Complement B')  $\longleftrightarrow$ 
    (case ID CCOMPARE('b) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set Complement2: ccompare = None") ( $\lambda$ -. cless-eq-set A' (Complement B'))
      | Some c  $\Rightarrow$ 
        if finite A'  $\wedge$  finite B'
        then finite (UNIV :: 'b set)  $\longrightarrow$ 
          proper-intrvl.set-less-eq-aux-Compl (lt-of-comp c) cproper-interval None
          (csorted-list-of-set A') (csorted-list-of-set B')
        else Code.abort (STR "cless-eq-set Complement2: infinite set") ( $\lambda$ -. cless-eq-set A' (Complement B')))
    (is ?fin-Compl-fin)
  and [set-base-code]:
    cless-eq-set A B  $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cless-eq-set: ccompare = None") ( $\lambda$ -. cless-eq-set A B)
      | Some c  $\Rightarrow$ 
        if finite A  $\wedge$  finite B
        then ord.lexordp-eq ( $\lambda$ x y. lt-of-comp c y x) (csorted-list-of-set A) (csorted-list-of-set B)
        else Code.abort (STR "cless-eq-set: infinite set") ( $\lambda$ -. cless-eq-set A B))
    (is ?fin-fin)
  <proof>

```

lemma cproper-interval-set-Some-Some-code [code]:

```

  fixes rbt1 rbt2 :: 'a :: cproper-interval set-rbt
  and A B :: 'a set
  shows cproper-interval-set-Some-Complement-Some-rbt:
    cproper-interval (Some (Complement (RBT-set rbt1))) (Some (RBT-set rbt2))
   $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cpproper-interval
      (Complement RBT-set) RBT-set: ccompare = None") ( $\lambda$ -. cproper-interval (Some
      (Complement (RBT-set rbt1))) (Some (RBT-set rbt2)))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'a set)  $\wedge$  proper-intrvl.proper-interval-Compl-set-aux-fusion
        (lt-of-comp c) cproper-interval rbt-keys-generator rbt-keys-generator None (RBT-Set2.init
        rbt1) (RBT-Set2.init rbt2))
    (is ?Compl-rbt-rbt)
  and cproper-interval-set-Some-rbt-Some-Complement:
    cproper-interval (Some (RBT-set rbt1)) (Some (Complement (RBT-set rbt2)))
   $\longleftrightarrow$ 
    (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cpproper-interval
      RBT-set (Complement RBT-set): ccompare = None") ( $\lambda$ -. cproper-interval (Some
      (RBT-set rbt1)) (Some (Complement (RBT-set rbt2))))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'a set)  $\wedge$  proper-intrvl.proper-interval-set-Compl-aux-fusion
        (lt-of-comp c) cproper-interval rbt-keys-generator rbt-keys-generator None 0 (RBT-Set2.init
        rbt1) (RBT-Set2.init rbt2))

```

```

    (is ?rbt-Compl-rbt)
  and [set-base-code]:
    cproper-interval (Some (RBT-set rbt1)) (Some (RBT-set rbt2))  $\longleftrightarrow$ 
      (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cproper-interval
RBT-set RBT-set: ccompare = None") ( $\lambda$ -. cproper-interval (Some (RBT-set rbt1))
(Some (RBT-set rbt2))))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'a set)  $\wedge$  proper-intrvl.proper-interval-set-aux-fusion (lt-of-comp
c) cproper-interval rbt-keys-generator rbt-keys-generator (RBT-Set2.init rbt1) (RBT-Set2.init
rbt2))
    (is ?rbt-rbt)
  and cproper-interval-set-Some-Complement-Some-Complement:
    cproper-interval (Some (Complement A)) (Some (Complement B))  $\longleftrightarrow$ 
      (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cproper-interval Com-
plement Complement: ccompare = None") ( $\lambda$ -. cproper-interval (Some (Complement
A)) (Some (Complement B))))
      | Some -  $\Rightarrow$  cproper-interval (Some B) (Some A))
    (is ?Compl-Compl)
  and cproper-interval-set-Some-Complement-Some:
    cproper-interval (Some (Complement A)) (Some B)  $\longleftrightarrow$ 
      (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cproper-interval
Complement1: ccompare = None") ( $\lambda$ -. cproper-interval (Some (Complement A))
(Some B)))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'a set)  $\wedge$  proper-intrvl.proper-interval-Compl-set-aux (lt-of-comp
c) cproper-interval None (csorted-list-of-set A) (csorted-list-of-set B))
    (is ?Compl-fin-fin)
  and cproper-interval-set-Some-Some-Complement:
    cproper-interval (Some A) (Some (Complement B))  $\longleftrightarrow$ 
      (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cproper-interval Com-
plement2: ccompare = None") ( $\lambda$ -. cproper-interval (Some A) (Some (Complement
B))))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'a set)  $\wedge$  proper-intrvl.proper-interval-set-Compl-aux (lt-of-comp
c) cproper-interval None 0 (csorted-list-of-set A) (csorted-list-of-set B))
    (is ?fin-Compl-fin)
  and [set-base-code]:
    cproper-interval (Some A) (Some B)  $\longleftrightarrow$ 
      (case ID CCOMPARE('a) of None  $\Rightarrow$  Code.abort (STR "cproper-interval:
ccompare = None") ( $\lambda$ -. cproper-interval (Some A) (Some B)))
      | Some c  $\Rightarrow$ 
        finite (UNIV :: 'a set)  $\wedge$  proper-intrvl.proper-interval-set-aux (lt-of-comp c)
cproper-interval (csorted-list-of-set A) (csorted-list-of-set B))
    (is ?fin-fin)
  <proof>

```

context ord begin

fun sorted-list-subset :: ('a $\Rightarrow$ 'a $\Rightarrow$ bool) $\Rightarrow$ 'a list $\Rightarrow$ 'a list $\Rightarrow$ bool

where

```
sorted-list-subset eq [] ys = True
| sorted-list-subset eq (x # xs) [] = False
| sorted-list-subset eq (x # xs) (y # ys) <=>
  (if eq x y then sorted-list-subset eq xs ys
   else x > y & sorted-list-subset eq (x # xs) ys)
```

end

context *linorder* begin

lemma *sorted-list-subset-correct*:

```
[[ sorted xs; distinct xs; sorted ys; distinct ys ]]
=> sorted-list-subset (=) xs ys <=> set xs <= set ys
<proof>
```

end

context *ord* begin

definition *sorted-list-subset-fusion* :: ('a => 'a => bool) => ('a, 's1) generator => ('a, 's2) generator => 's1 => 's2 => bool

where *sorted-list-subset-fusion* eq g1 g2 s1 s2 = *sorted-list-subset* eq (list.unfoldr g1 s1) (list.unfoldr g2 s2)

lemma *sorted-list-subset-fusion-code*:

```
sorted-list-subset-fusion eq g1 g2 s1 s2 =
  (if list.has-next g1 s1 then
    let (x, s1') = list.next g1 s1
    in list.has-next g2 s2 & (
      let (y, s2') = list.next g2 s2
      in if eq x y then sorted-list-subset-fusion eq g1 g2 s1' s2'
        else y < x & sorted-list-subset-fusion eq g1 g2 s1 s2')
    else True)
<proof>
```

end

lemmas [code, set-base-code] = ord.sorted-list-subset-fusion-code

lemma *subset-eq-code* [code]:

```
fixes A1 A2 :: 'a set
and rbt :: 'b :: ccompare set-rbt
and rbt1 rbt2 :: 'd :: {ccompare, ceq} set-rbt
and dxs :: 'c :: ceq set-dlist
and xs :: 'c list
```

shows [set-base-code]:

```
RBT-set rbt1 <= RBT-set rbt2 <=>
  (case ID CCOMPARE('d) of None => Code.abort (STR "subset RBT-set
```

RBT-set: $ccompare = None''$) ($\lambda\cdot$. *RBT-set* *rbt1* $\subseteq$ *RBT-set* *rbt2*)
 $\quad \quad \quad | \text{Some } c \Rightarrow$
 $\quad \quad \quad (\text{case ID CEQ}('d) \text{ of } None \Rightarrow \text{ord.sorted-list-subset-fusion } (lt\text{-of-comp } c)$
 $\quad \quad \quad (\lambda \ x \ y. \ c \ x \ y = Eq) \ \text{rbt-keys-generator} \ \text{rbt-keys-generator} \ (\text{RBT-Set2.init } \text{rbt1})$
 $\quad \quad \quad (\text{RBT-Set2.init } \text{rbt2}))$
 $\quad \quad \quad | \text{Some } eq \Rightarrow \text{ord.sorted-list-subset-fusion } (lt\text{-of-comp } c) \ eq$
 $\quad \quad \quad \text{rbt-keys-generator} \ \text{rbt-keys-generator} \ (\text{RBT-Set2.init } \text{rbt1}) \ (\text{RBT-Set2.init } \text{rbt2}))$

$(\text{is } ?\text{rbt-rbt})$
and *Complement-subset-eq-Complement*:
 $\text{Complement } A1 \subseteq \text{Complement } A2 \longleftrightarrow A2 \subseteq A1 \ (\text{is } ?\text{Compl})$
and *Collect-subset-eq-Complement*:
 $\text{Collect-set } P \subseteq \text{Complement } A \longleftrightarrow A \subseteq \{x. \neg P \ x\} \ (\text{is } ?\text{Collect-set-Compl})$
and [*set-base-code*]:
 $\text{RBT-set } \text{rbt} \subseteq B \longleftrightarrow$
 $(\text{case ID CCOMPARE}('b) \text{ of } None \Rightarrow \text{Code.abort } (\text{STR } "subset \ \text{RBT-set1}:$
 $\text{ccompare} = None'') (\lambda\cdot$. $\text{RBT-set } \text{rbt} \subseteq B)$
 $\quad \quad \quad | \text{Some } - \Rightarrow \text{list-all-fusion } \text{rbt-keys-generator} \ (\lambda x. \ x \in B)$
 $(\text{RBT-Set2.init } \text{rbt})) \ (\text{is } ?\text{rbt})$
 $\text{DList-set } \text{dxs} \subseteq C \longleftrightarrow$
 $(\text{case ID CEQ}('c) \text{ of } None \Rightarrow \text{Code.abort } (\text{STR } "subset \ \text{DList-set1: } \text{ceq} =$
 $\text{None''}) (\lambda\cdot$. $\text{DList-set } \text{dxs} \subseteq C)$
 $\quad \quad \quad | \text{Some } - \Rightarrow \text{DList-Set.dlist-all } (\lambda x. \ x \in C) \ \text{dxs}) \ (\text{is } ?\text{dlist})$
 $\text{Set-Monad } \text{xs} \subseteq C \longleftrightarrow \text{list-all } (\lambda x. \ x \in C) \ \text{xs} \ (\text{is } ?\text{Set-Monad})$
 $\langle \text{proof} \rangle$

lemma *set-eq-code* [*code*]:
fixes *rbt1* *rbt2* :: 'b :: {*ccompare*, *ceq*} *set-rbt*
shows [*set-base-code*]:
 $\text{set-eq } (\text{RBT-set } \text{rbt1}) \ (\text{RBT-set } \text{rbt2}) =$
 $(\text{case ID CCOMPARE}('b) \text{ of } None \Rightarrow \text{Code.abort } (\text{STR } "set-eq \ \text{RBT-set}$
 $\text{RBT-set: } \text{ccompare} = None'') (\lambda\cdot$. $\text{set-eq } (\text{RBT-set } \text{rbt1}) \ (\text{RBT-set } \text{rbt2}))$
 $\quad \quad \quad | \text{Some } c \Rightarrow$
 $(\text{case ID CEQ}('b) \text{ of } None \Rightarrow \text{list-all2-fusion } (\lambda \ x \ y. \ c \ x \ y = Eq) \ \text{rbt-keys-generator}$
 $\text{rbt-keys-generator} \ (\text{RBT-Set2.init } \text{rbt1}) \ (\text{RBT-Set2.init } \text{rbt2}))$
 $\quad \quad \quad | \text{Some } eq \Rightarrow \text{list-all2-fusion } eq \ \text{rbt-keys-generator} \ \text{rbt-keys-generator}$
 $(\text{RBT-Set2.init } \text{rbt1}) \ (\text{RBT-Set2.init } \text{rbt2})))$
 $(\text{is } ?\text{rbt-rbt})$
and *set-eq-Complement-Complement*:
 $\text{set-eq } (\text{Complement } A) \ (\text{Complement } B) = \text{set-eq } A \ B$
and [*set-base-code*]: $\text{set-eq } A \ B \longleftrightarrow A \subseteq B \wedge B \subseteq A$
 $\langle \text{proof} \rangle$

lemma *Set-project-code* [*code*, *set-base-code*]:
 $\text{Set.filter } P \ A = A \cap \text{Collect-set } P$
 $\langle \text{proof} \rangle$

lemma *Set-image-code* [*code*]:
fixes *dxs* :: 'a :: *ceq* *set-dlist*

```

and rbt :: 'b :: ccompare set-rbt
shows [set-base-code]:
  image h (RBT-set rbt) =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "image RBT-set:
ccompare = None") (λ-. image h (RBT-set rbt))
      | Some - ⇒ RBT-Set2.fold (insert ∘ h) rbt {})

  (is ?rbt)
  image g (DList-set dxs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "image DList-set: ceq =
None") (λ-. image g (DList-set dxs))
      | Some - ⇒ DList-Set.fold (insert ∘ g) dxs {})

  (is ?dlist)
and image-Complement-Complement:
  image f (Complement (Complement B)) = image f B
and [set-base-code]:
  image f (Collect-set A) = Code.abort (STR "image Collect-set") (λ-. image f
(Collect-set A))
  image f (Set-Monad xs) = Set-Monad (map f xs)
⟨proof⟩

```

```

lemma the-elem-code [code]:
  fixes dxs :: 'a :: ceq set-dlist
  and rbt :: 'b :: ccompare set-rbt
  shows
    the-elem (RBT-set rbt) =
      (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "the-elem RBT-set:
ccompare = None") (λ-. the-elem (RBT-set rbt))
        | Some - ⇒
          case RBT-Mapping2.impl-of rbt of RBT-Impl.Branch - RBT-Impl.Empty x
            - RBT-Impl.Empty ⇒ x
            | - ⇒ Code.abort (STR "the-elem RBT-set: not unique") (λ-. the-elem
(RBT-set rbt)))
    the-elem (DList-set dxs) =
      (case ID CEQ('a) of None ⇒ Code.abort (STR "the-elem DList-set: ceq =
None") (λ-. the-elem (DList-set dxs))
        | Some - ⇒
          case list-of-dlist dxs of [x] ⇒ x
          | - ⇒ Code.abort (STR "the-elem DList-set: not unique") (λ-. the-elem
(DList-set dxs)))
    the-elem (Set-Monad [x]) = x
  ⟨proof⟩

```

```

lemma Pow-set-conv-fold:
  Pow (set xs ∪ A) = fold (λx A. A ∪ insert x ' A) xs (Pow A)
  ⟨proof⟩

```

```

lemma Pow-code [code, set-base-code]:
  fixes dxs :: 'a :: ceq set-dlist
  and rbt :: 'b :: ccompare set-rbt

```

shows

```

Pow (RBT-set rbt) =
  (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "Pow RBT-set:
ccompare = None") (λ-. Pow (RBT-set rbt))
   | Some - ⇒ RBT-Set2.fold (λx A. A ∪ insert x 'A) rbt {{{}}})
Pow (DList-set dxs) =
  (case ID CEQ('a) of None ⇒ Code.abort (STR "Pow DList-set: ceq = None")
(λ-. Pow (DList-set dxs))
   | Some - ⇒ DList-Set.fold (λx A. A ∪ insert x 'A) dxs {{{}}})
Pow (Set-Monad xs) = fold (λx A. A ∪ insert x 'A) xs {{{}}})
Pow A = Collect-set (λB. B ⊆ A)
⟨proof⟩

```

lemma *fold-singleton*: $Finite\text{-}Set.fold\ f\ x\ \{y\} = f\ y\ x$
 ⟨proof⟩

lift-definition *sum-cfc* :: $('a \Rightarrow 'b :: comm\text{-}monoid\text{-}add) \Rightarrow ('a, 'b)\ comp\text{-}fun\text{-}commute$
is $\lambda f :: 'a \Rightarrow 'b. plus \circ f$
 ⟨proof⟩

lemma *sum-code* [code]:
 $sum\ f\ A = (if\ finite\ A\ then\ set\text{-}fold\text{-}cfc\ (sum\text{-}cfc\ f)\ 0\ A\ else\ 0)$
 ⟨proof⟩

lift-definition *prod-cfc* :: $('a \Rightarrow 'b :: comm\text{-}monoid\text{-}mult) \Rightarrow ('a, 'b)\ comp\text{-}fun\text{-}commute$
is $\lambda f :: 'a \Rightarrow 'b. times \circ f$
 ⟨proof⟩

lemma *prod-code* [code]:
 $prod\ f\ A = (if\ finite\ A\ then\ set\text{-}fold\text{-}cfc\ (prod\text{-}cfc\ f)\ 1\ A\ else\ 1)$
 ⟨proof⟩

lemma *product-code* [code]:
fixes $dxs :: 'a :: ceq\ set\text{-}dlist$
and $dys :: 'b :: ceq\ set\text{-}dlist$
and $rbt1 :: 'c :: ccompare\ set\text{-}rbt$
and $rbt2 :: 'd :: ccompare\ set\text{-}rbt$
shows

```

Product-Type.product (RBT-set rbt1) (RBT-set rbt2) =
  (case ID CCOMPARE('c) of None ⇒ Code.abort (STR "product RBT-set
RBT-set: ccompare1 = None") (λ-. Product-Type.product (RBT-set rbt1) (RBT-set
rbt2))
   | Some - ⇒
    case ID CCOMPARE('d) of None ⇒ Code.abort (STR "product RBT-set
RBT-set: ccompare2 = None") (λ-. Product-Type.product (RBT-set rbt1) (RBT-set
rbt2))
    | Some - ⇒ RBT-set (RBT-Set2.product rbt1 rbt2))

```

$Product\text{-}Type.product\ A2\ (RBT\text{-}set\ rbt2) =$

```

      (case ID CCOMPARE('d) of None ⇒ Code.abort (STR "product RBT-set:
ccompare2 = None") (λ-. Product-Type.product A2 (RBT-set rbt2))
      | Some - ⇒ RBT-Set2.fold (λy rest. (λx. (x, y)) ' A2 ∪ rest)
rbt2 {})
      (is ?rbt2)

```

```

Product-Type.product (RBT-set rbt1) B2 =
  (case ID CCOMPARE('c) of None ⇒ Code.abort (STR "product RBT-set:
ccompare1 = None") (λ-. Product-Type.product (RBT-set rbt1) B2)
  | Some - ⇒ RBT-Set2.fold (λx rest. Pair x ' B2 ∪ rest) rbt1
{})
(is ?rbt1)

```

```

Product-Type.product (DList-set dxs) (DList-set dys) =
  (case ID CEQ('a) of None ⇒ Code.abort (STR "product DList-set DList-set:
ceq1 = None") (λ-. Product-Type.product (DList-set dxs) (DList-set dys))
  | Some - ⇒
    case ID CEQ('b) of None ⇒ Code.abort (STR "product DList-set DList-set:
ceq2 = None") (λ-. Product-Type.product (DList-set dxs) (DList-set dys))
    | Some - ⇒ DList-Set (DList-Set.product dxs dys)

```

```

Product-Type.product A1 (DList-set dys) =
  (case ID CEQ('b) of None ⇒ Code.abort (STR "product DList-set2: ceq =
None") (λ-. Product-Type.product A1 (DList-set dys))
  | Some - ⇒ DList-Set.fold (λy rest. (λx. (x, y)) ' A1 ∪ rest) dys
{})
(is ?dlist2)

```

```

Product-Type.product (DList-set dxs) B1 =
  (case ID CEQ('a) of None ⇒ Code.abort (STR "product DList-set1: ceq =
None") (λ-. Product-Type.product (DList-set dxs) B1)
  | Some - ⇒ DList-Set.fold (λx rest. Pair x ' B1 ∪ rest) dxs {})
(is ?dlist1)

```

```

Product-Type.product (Set-Monad xs) (Set-Monad ys) =
  Set-Monad (fold (λx. fold (λy rest. (x, y) # rest) ys) xs [])
(is ?Set-Monad)

```

$Product-Type.product\ A\ B = Collect-set\ (\lambda(x, y). x \in A \wedge y \in B)$

<proof>

```

lemma Id-on-code [code]:
  fixes A :: 'a :: ceq set
    and dxs :: 'a set-dlist
    and P :: 'a ⇒ bool
    and rbt :: 'b :: ccompare set-rbt
shows [set-base-code]:
  Id-on (RBT-set rbt) =

```

```

    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "Id-on RBT-set:
ccompare = None") (λ-. Id-on (RBT-set rbt))
    | Some - ⇒ RBT-set (RBT-Set2.Id-on rbt))
  Id-on (DList-set dxs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Id-on DList-set: ceq = None")
    (λ-. Id-on (DList-set dxs))
    | Some - ⇒ DList-set (DList-Set.Id-on dxs))
  Id-on (Collect-set P) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Id-on Collect-set: ceq =
None") (λ-. Id-on (Collect-set P))
    | Some eq ⇒ Collect-set (λ(x, y). eq x y ∧ P x))
and Id-on-Complement:
  Id-on (Complement A) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Id-on Complement: ceq =
None") (λ-. Id-on (Complement A))
    | Some eq ⇒ Collect-set (λ(x, y). eq x y ∧ x ∉ A))
and [set-base-code]:
  Id-on B = (λx. (x, x)) ‘ B
  ⟨proof⟩

```

lemma *Image-code* [code, set-base-code]:

```

fixes dxs :: ('a :: ceq × 'b :: ceq) set-dlist
and rbt :: ('c :: ccompare × 'd :: ccompare) set-rbt
shows
  RBT-set rbt “ C =
    (case ID CCOMPARE('c) of None ⇒ Code.abort (STR "Image RBT-set:
ccompare1 = None") (λ-. RBT-set rbt “ C)
    | Some - ⇒
      case ID CCOMPARE('d) of None ⇒ Code.abort (STR "Image RBT-set:
ccompare2 = None") (λ-. RBT-set rbt “ C)
      | Some - ⇒
        RBT-Set2.fold (λ(x, y) acc. if x ∈ C then insert y acc else acc) rbt {})
    (is ?RBT-set)
  DList-set dxs “ B =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Image DList-set: ceq1 =
None") (λ-. DList-set dxs “ B)
    | Some - ⇒
      case ID CEQ('b) of None ⇒ Code.abort (STR "Image DList-set: ceq2 =
None") (λ-. DList-set dxs “ B)
      | Some - ⇒
        DList-Set.fold (λ(x, y) acc. if x ∈ B then insert y acc else acc) dxs {})
    (is ?DList-set)
  Set-Monad rxs “ A = Set-Monad (fold (λ(x, y) rest. if x ∈ A then y # rest
else rest) rxs [])
    (is ?Set-Monad)
  X “ Y = snd ‘ Set.filter (λ(x, y). x ∈ Y) X
    (is ?generic)
  ⟨proof⟩

```

lemma *insert-relcomp*: $\text{insert } (a, b) A \ O \ B = A \ O \ B \cup \{a\} \times \{c. (b, c) \in B\}$
 $\langle \text{proof} \rangle$

lemma *trancl-code* [code, set-base-code]:
 $\text{trancl } A =$
 $(\text{if finite } A \text{ then ntrancl } (\text{card } A - 1) \ A \text{ else Code.abort } (\text{STR } "trancl: \text{infinite set}")) (\lambda-. \text{trancl } A))$
 $\langle \text{proof} \rangle$

lemma *set-relcomp-set*:
 $\text{set } xs \ O \ \text{set } ys = \text{fold } (\lambda(x, y). \text{fold } (\lambda(y', z) \ A. \text{if } y = y' \text{ then insert } (x, z) \ A \text{ else } A) \ ys) \ xs \ \{\}$
 $\langle \text{proof} \rangle$

lemma *If-not*: $(\text{if } \neg a \text{ then } b \text{ else } c) = (\text{if } a \text{ then } c \text{ else } b)$
 $\langle \text{proof} \rangle$

lemma *relcomp-code* [code, set-base-code]:
fixes $\text{rbt1} :: ('a :: \text{compare} \times 'b :: \text{compare}) \text{ set-rbt}$
and $\text{rbt2} :: ('b \times 'c :: \text{compare}) \text{ set-rbt}$
and $\text{rbt3} :: ('a \times 'd :: \{\text{compare}, \text{ceq}\}) \text{ set-rbt}$
and $\text{rbt4} :: ('d \times 'a) \text{ set-rbt}$
and $\text{rbt5} :: ('b \times 'a) \text{ set-rbt}$
and $\text{dxs1} :: ('d \times 'e :: \text{ceq}) \text{ set-dlist}$
and $\text{dxs2} :: ('e \times 'd) \text{ set-dlist}$
and $\text{dxs3} :: ('e \times 'f :: \text{ceq}) \text{ set-dlist}$
and $\text{dxs4} :: ('f \times 'g :: \text{ceq}) \text{ set-dlist}$
and $\text{xs1} :: ('h \times 'i :: \text{ceq}) \text{ list}$
and $\text{xs2} :: ('i \times 'j) \text{ list}$
and $\text{xs3} :: ('b \times 'h) \text{ list}$
and $\text{xs4} :: ('h \times 'b) \text{ list}$
and $\text{xs5} :: ('f \times 'h) \text{ list}$
and $\text{xs6} :: ('h \times 'f) \text{ list}$
shows
 $\text{RBT-set } \text{rbt1} \ O \ \text{RBT-set } \text{rbt2} =$
 $(\text{case ID CCOMPARE('a) of None} \Rightarrow \text{Code.abort } (\text{STR } "relcomp \text{RBT-set RBT-set: ccompare1} = \text{None}")) (\lambda-. \text{RBT-set } \text{rbt1} \ O \ \text{RBT-set } \text{rbt2})$
 $\quad | \text{Some } - \Rightarrow$
 $\text{case ID CCOMPARE('b) of None} \Rightarrow \text{Code.abort } (\text{STR } "relcomp \text{RBT-set RBT-set: ccompare2} = \text{None}")) (\lambda-. \text{RBT-set } \text{rbt1} \ O \ \text{RBT-set } \text{rbt2})$
 $\quad | \text{Some } c-b \Rightarrow$
 $\text{case ID CCOMPARE('c) of None} \Rightarrow \text{Code.abort } (\text{STR } "relcomp \text{RBT-set RBT-set: ccompare3} = \text{None}")) (\lambda-. \text{RBT-set } \text{rbt1} \ O \ \text{RBT-set } \text{rbt2})$
 $\quad | \text{Some } - \Rightarrow \text{RBT-Set2.fold } (\lambda(x, y). \text{RBT-Set2.fold } (\lambda(y', z) \ A. \text{if } c-b \ y \ y' \neq \text{Eq then } A \text{ else insert } (x, z) \ A) \ \text{rbt2}) \ \text{rbt1} \ \{\}$
 (is ?rbt-rbt)

$\text{RBT-set } \text{rbt3} \ O \ \text{DList-set } \text{dxs1} =$
 $(\text{case ID CCOMPARE('a) of None} \Rightarrow \text{Code.abort } (\text{STR } "relcomp \text{RBT-set$

$DList\text{-}set: ccompare1 = None''$) ($\lambda\text{-}.$ $RBT\text{-}set\ rbt3\ O\ DList\text{-}set\ dxs1$)
 $\quad\quad\quad | \text{Some } - \Rightarrow$
 $\quad\quad\quad case\ ID\ CCOMPARE('d)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ RBT\text{-}set$
 $DList\text{-}set: ccompare2 = None''$) ($\lambda\text{-}.$ $RBT\text{-}set\ rbt3\ O\ DList\text{-}set\ dxs1$)
 $\quad\quad\quad | \text{Some } - \Rightarrow$
 $\quad\quad\quad case\ ID\ CEQ('d)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ RBT\text{-}set\ DList\text{-}set:$
 $ceq2 = None''$) ($\lambda\text{-}.$ $RBT\text{-}set\ rbt3\ O\ DList\text{-}set\ dxs1$)
 $\quad\quad\quad | \text{Some } eq \Rightarrow$
 $\quad\quad\quad case\ ID\ CEQ('e)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ RBT\text{-}set$
 $DList\text{-}set: ceq3 = None''$) ($\lambda\text{-}.$ $RBT\text{-}set\ rbt3\ O\ DList\text{-}set\ dxs1$)
 $\quad\quad\quad | \text{Some } - \Rightarrow RBT\text{-}Set2.fold\ (\lambda(x, y). DList\text{-}Set.fold\ (\lambda(y', z)$
 $A.\ if\ eq\ y\ y'\ then\ insert\ (x, z)\ A\ else\ A)\ dxs1)\ rbt3\ \{\})$
 $(is\ ?rbt\text{-}dlist)$

$DList\text{-}set\ dxs2\ O\ RBT\text{-}set\ rbt4 =$
 $(case\ ID\ CEQ('e)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ RBT\text{-}set:$
 $ceq1 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs2\ O\ RBT\text{-}set\ rbt4$)
 $\quad\quad\quad | \text{Some } - \Rightarrow$
 $\quad\quad\quad case\ ID\ CCOMPARE('d)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set$
 $RBT\text{-}set: ceq2 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs2\ O\ RBT\text{-}set\ rbt4$)
 $\quad\quad\quad | \text{Some } - \Rightarrow$
 $\quad\quad\quad case\ ID\ CEQ('d)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ RBT\text{-}set:$
 $ccompare2 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs2\ O\ RBT\text{-}set\ rbt4$)
 $\quad\quad\quad | \text{Some } eq \Rightarrow$
 $\quad\quad\quad case\ ID\ CCOMPARE('a)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set$
 $RBT\text{-}set: ccompare3 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs2\ O\ RBT\text{-}set\ rbt4$)
 $\quad\quad\quad | \text{Some } - \Rightarrow DList\text{-}Set.fold\ (\lambda(x, y). RBT\text{-}Set2.fold\ (\lambda(y',$
 $z)\ A.\ if\ eq\ y\ y'\ then\ insert\ (x, z)\ A\ else\ A)\ rbt4)\ dxs2\ \{\})$
 $(is\ ?dlist\text{-}rbt)$

$DList\text{-}set\ dxs3\ O\ DList\text{-}set\ dxs4 =$
 $(case\ ID\ CEQ('e)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ DList\text{-}set:$
 $ceq1 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs3\ O\ DList\text{-}set\ dxs4$)
 $\quad\quad\quad | \text{Some } - \Rightarrow$
 $\quad\quad\quad case\ ID\ CEQ('f)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ DList\text{-}set:$
 $ceq2 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs3\ O\ DList\text{-}set\ dxs4$)
 $\quad\quad\quad | \text{Some } eq \Rightarrow$
 $\quad\quad\quad case\ ID\ CEQ('g)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ DList\text{-}set:$
 $ceq3 = None''$) ($\lambda\text{-}.$ $DList\text{-}set\ dxs3\ O\ DList\text{-}set\ dxs4$)
 $\quad\quad\quad | \text{Some } - \Rightarrow DList\text{-}Set.fold\ (\lambda(x, y). DList\text{-}Set.fold\ (\lambda(y', z)$
 $A.\ if\ eq\ y\ y'\ then\ insert\ (x, z)\ A\ else\ A)\ dxs4)\ dxs3\ \{\})$
 $(is\ ?dlist\text{-}dlist)$

$Set\text{-}Monad\ xs1\ O\ Set\text{-}Monad\ xs2 =$
 $(case\ ID\ CEQ('i)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ Set\text{-}Monad\ Set\text{-}Monad:$
 $ceq = None''$) ($\lambda\text{-}.$ $Set\text{-}Monad\ xs1\ O\ Set\text{-}Monad\ xs2$)
 $\quad\quad\quad | \text{Some } eq \Rightarrow fold\ (\lambda(x, y). fold\ (\lambda(y', z)\ A.\ if\ eq\ y\ y'\ then\ insert$
 $(x, z)\ A\ else\ A)\ xs2)\ xs1\ \{\})$
 $(is\ ?monad\text{-}monad)$

$RBT\text{-}set\ rbt1\ O\ Set\text{-}Monad\ xs3 =$
 $(case\ ID\ CCOMPARE('a)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ RBT\text{-}set$
 $Set\text{-}Monad:\ ccompare1 = None'')\ (\lambda\cdot.\ RBT\text{-}set\ rbt1\ O\ Set\text{-}Monad\ xs3)$
 $\quad | Some\ - \Rightarrow$
 $case\ ID\ CCOMPARE('b)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ RBT\text{-}set$
 $Set\text{-}Monad:\ ccompare2 = None'')\ (\lambda\cdot.\ RBT\text{-}set\ rbt1\ O\ Set\text{-}Monad\ xs3)$
 $\quad | Some\ c\text{-}b \Rightarrow RBT\text{-}Set2.fold\ (\lambda(x,\ y).\ fold\ (\lambda(y',\ z)\ A.\ if\ c\text{-}b\ y\ y' \neq$
 $Eq\ then\ A\ else\ insert\ (x,\ z)\ A)\ xs3)\ rbt1\ \{\})$
 $(is\ ?rbt\text{-}monad)$

$Set\text{-}Monad\ xs4\ O\ RBT\text{-}set\ rbt5 =$
 $(case\ ID\ CCOMPARE('a)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ Set\text{-}Monad$
 $RBT\text{-}set:\ ccompare1 = None'')\ (\lambda\cdot.\ Set\text{-}Monad\ xs4\ O\ RBT\text{-}set\ rbt5)$
 $\quad | Some\ - \Rightarrow$
 $case\ ID\ CCOMPARE('b)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ Set\text{-}Monad$
 $RBT\text{-}set:\ ccompare2 = None'')\ (\lambda\cdot.\ Set\text{-}Monad\ xs4\ O\ RBT\text{-}set\ rbt5)$
 $\quad | Some\ c\text{-}b \Rightarrow fold\ (\lambda(x,\ y).\ RBT\text{-}Set2.fold\ (\lambda(y',\ z)\ A.\ if\ c\text{-}b\ y\ y' \neq$
 $Eq\ then\ A\ else\ insert\ (x,\ z)\ A)\ rbt5)\ xs4\ \{\})$
 $(is\ ?monad\text{-}rbt)$

$DList\text{-}set\ dxs3\ O\ Set\text{-}Monad\ xs5 =$
 $(case\ ID\ CEQ('e)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ Set\text{-}Monad:$
 $ceq1 = None'')\ (\lambda\cdot.\ DList\text{-}set\ dxs3\ O\ Set\text{-}Monad\ xs5)$
 $\quad | Some\ - \Rightarrow$
 $case\ ID\ CEQ('f)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ DList\text{-}set\ Set\text{-}Monad:$
 $ceq2 = None'')\ (\lambda\cdot.\ DList\text{-}set\ dxs3\ O\ Set\text{-}Monad\ xs5)$
 $\quad | Some\ eq \Rightarrow DList\text{-}Set.fold\ (\lambda(x,\ y).\ fold\ (\lambda(y',\ z)\ A.\ if\ eq\ y\ y'$
 $then\ insert\ (x,\ z)\ A\ else\ A)\ xs5)\ dxs3\ \{\})$
 $(is\ ?dlist\text{-}monad)$

$Set\text{-}Monad\ xs6\ O\ DList\text{-}set\ dxs4 =$
 $(case\ ID\ CEQ('f)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ Set\text{-}Monad\ DList\text{-}set:$
 $ceq1 = None'')\ (\lambda\cdot.\ Set\text{-}Monad\ xs6\ O\ DList\text{-}set\ dxs4)$
 $\quad | Some\ eq \Rightarrow$
 $case\ ID\ CEQ('g)\ of\ None \Rightarrow Code.abort\ (STR\ "relcomp\ Set\text{-}Monad\ DList\text{-}set:$
 $ceq2 = None'')\ (\lambda\cdot.\ Set\text{-}Monad\ xs6\ O\ DList\text{-}set\ dxs4)$
 $\quad | Some\ - \Rightarrow fold\ (\lambda(x,\ y).\ DList\text{-}Set.fold\ (\lambda(y',\ z)\ A.\ if\ eq\ y\ y'$
 $then\ insert\ (x,\ z)\ A\ else\ A)\ dxs4)\ xs6\ \{\})$
 $(is\ ?monad\text{-}dlist)$
 $\langle proof \rangle$

lemma *irrefl-on-code* [code, set-base-code]:

fixes $r :: ('a :: \{ceq, ccompare\} \times 'a)\ set$ **shows**

$irrefl\text{-}on\ A\ r \longleftrightarrow$

$(case\ ID\ CEQ('a)\ of\ Some\ eq \Rightarrow (\forall (x,\ y) \in r.\ x \in A \longrightarrow y \in A \longrightarrow \neg eq\ x\ y) \mid$
 $None \Rightarrow$

$case\ ID\ CCOMPARE('a)\ of\ None \Rightarrow Code.abort\ (STR\ "irrefl\text{-}on:\ ceq = None$
 $\&\ ccompare = None'')\ (\lambda\cdot.\ irrefl\text{-}on\ A\ r)$

$\langle proof \rangle \quad | \text{ Some } c \Rightarrow (\forall (x, y) \in r. x \in A \longrightarrow y \in A \longrightarrow c \ x \ y \neq Eq))$

lemma *wf-code* [code, set-base-code]:

fixes *rbt* :: ('a :: ccompare × 'a) set-rbt

and *dxs* :: ('b :: ceq × 'b) set-dlist

shows

wf-code (DList-set *dxs*) =
 (case ID CEQ('b) of None $\Rightarrow$ Code.abort (STR "wf-code DList-set: ceq = None")) (λ-. *wf-code* (DList-set *dxs*))
 | Some - $\Rightarrow$ acyclic (DList-set *dxs*)
wf-code (RBT-set *rbt*) =
 (case ID CCOMPARE('a) of None $\Rightarrow$ Code.abort (STR "wf-code RBT-set: ccompare = None")) (λ-. *wf-code* (RBT-set *rbt*))
 | Some - $\Rightarrow$ acyclic (RBT-set *rbt*)
wf-code (Set-Monad *xs*) = acyclic (Set-Monad *xs*)
 $\langle proof \rangle$

lemma *bacc-code* [code, set-base-code]:

bacc *R* 0 = - snd ' *R*

bacc *R* (Suc *n*) = (let *rec* = *bacc* *R* *n* in *rec* ∪ - snd ' (Set.filter (λ(*y*, *x*). *y* ∉ *rec*) *R*))
 $\langle proof \rangle$

lemma *acc-code* [code, set-base-code]:

fixes *A* :: ('a :: {finite, card-UNIV} × 'a) set **shows**

Wellfounded.acc *A* = *bacc* *A* (of-phantom (card-UNIV :: 'a card-UNIV))

$\langle proof \rangle$

lemma *sorted-list-of-set-code* [code, set-base-code]:

fixes *dxs* :: 'a :: {linorder, ceq} set-dlist

and *rbt* :: 'b :: {linorder, ccompare} set-rbt

shows

sorted-list-of-set (RBT-set *rbt*) =
 (case ID CCOMPARE('b) of None $\Rightarrow$ Code.abort (STR "sorted-list-of-set RBT-set: ccompare = None")) (λ-. *sorted-list-of-set* (RBT-set *rbt*))
 | Some - $\Rightarrow$ sort (RBT-Set2.keys *rbt*)
 — We must sort the keys because *ccompare*'s ordering need not coincide with *linorder*'s.
sorted-list-of-set (DList-set *dxs*) =
 (case ID CEQ('a) of None $\Rightarrow$ Code.abort (STR "sorted-list-of-set DList-set: ceq = None")) (λ-. *sorted-list-of-set* (DList-set *dxs*))
 | Some - $\Rightarrow$ sort (list-of-dlist *dxs*)
sorted-list-of-set (Set-Monad *xs*) = sort (remdups *xs*)
 $\langle proof \rangle$

lemma *image-filter-set-conv-fold*:

```

⟨Option.image-filter f (set xs) =
  fold (λx A. case f x of None ⇒ A | Some y ⇒ insert y A) xs {}⟩
⟨proof⟩

```

lemma *image-filter-code* [code, set-base-code]:

```

fixes dxs :: 'a :: ceq set-dlist
and rbt :: 'b :: ccompare set-rbt
shows
  Option.image-filter h (RBT-set rbt) =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "Option.image-filter
RBT-set: ccompare = None") (λ-. Option.image-filter h (RBT-set rbt))
    | Some - ⇒ RBT-Set2.fold (λx A. case h x of None ⇒ A |
Some y ⇒ insert y A) rbt {})
    (is ?rbt)
  Option.image-filter g (DList-set dxs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Option.image-filter DList-set:
ceq = None") (λ-. Option.image-filter g (DList-set dxs))
    | Some - ⇒ DList-Set.fold (λx A. case g x of None ⇒ A | Some y
⇒ insert y A) dxs {})
    (is ?dlist)
  Option.image-filter f (Set-Monad xs) = Set-Monad (List.map-filter f xs)
    (is ?set-monad)
⟨proof⟩

```

lemma *these-code* [code, set-base-code]:

```

⟨Option.these A = Option.image-filter (λx. x) A⟩
⟨proof⟩

```

lemma *can-select-code*:

```

fixes xs :: 'a :: ceq list
and dxs :: 'a :: ceq set-dlist
and rbt :: 'b :: ccompare set-rbt
shows
  Set.can-select R (RBT-set rbt) =
    (case ID CCOMPARE('b) of None ⇒ Code.abort (STR "Set.can-select
RBT-set: ccompare = None") (λ-. Set.can-select R (RBT-set rbt))
    | Some - ⇒ singleton-list-fusion (filter-generator R
rbt-keys-generator) (RBT-Set2.init rbt))
    (is ?rbt)
  Set.can-select Q (DList-set dxs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Set.can-select DList-set: ceq
= None") (λ-. Set.can-select Q (DList-set dxs))
    | Some - ⇒ DList-Set.length (DList-Set.filter Q dxs) = 1)
    (is ?dlist)
  Set.can-select P (Set-Monad xs) =
    (case ID CEQ('a) of None ⇒ Code.abort (STR "Set.can-select Set-Monad:
ceq = None") (λ-. Set.can-select P (Set-Monad xs))
    | Some eq ⇒ case filter P xs of Nil ⇒ False | x # xs ⇒ list-all
(eq x) xs)

```

(is ?Set-Monad)
 ⟨proof⟩

lemma [code]:
 ⟨Set.can-select P A $\longleftrightarrow$ is-singleton (Set.filter P A)⟩
 ⟨proof⟩

lemma pred-of-set-code [code, set-base-code]:
fixes dxs :: 'a :: ceq set-dlist
and rbt :: 'b :: ccompare set-rbt
shows
 pred-of-set (RBT-set rbt) =
 (case ID CCOMPARE('b) of None $\Rightarrow$ Code.abort (STR "pred-of-set RBT-set:
 ccompare = None") (λ-. pred-of-set (RBT-set rbt))
 | Some - $\Rightarrow$ RBT-Set2.fold (sup ∘ Predicate.single) rbt bot)
 pred-of-set (DList-set dxs) =
 (case ID CEQ('a) of None $\Rightarrow$ Code.abort (STR "pred-of-set DList-set: ceq =
 None") (λ-. pred-of-set (DList-set dxs))
 | Some - $\Rightarrow$ DList-Set.fold (sup ∘ Predicate.single) dxs bot)
 pred-of-set (Set-Monad xs) = fold (sup ∘ Predicate.single) xs bot
 ⟨proof⟩

'a Predicate.pred is implemented as a monad, so we keep the monad when converting to 'a set. For this case, insert-monad and union-monad avoid the unnecessary dictionary construction.

definition insert-monad :: 'a $\Rightarrow$ 'a set $\Rightarrow$ 'a set
where [simp]: insert-monad = insert

definition union-monad :: 'a set $\Rightarrow$ 'a set $\Rightarrow$ 'a set
where [simp]: union-monad = (∪)

lemma insert-monad-code [code, set-base-code]:
 insert-monad x (Set-Monad xs) = Set-Monad (x # xs)
 ⟨proof⟩

lemma union-monad-code [code, set-base-code]:
 union-monad (Set-Monad xs) (Set-Monad ys) = Set-Monad (xs @ ys)
 ⟨proof⟩

lemma set-of-pred-code [code, set-base-code]:
 set-of-pred (Predicate.Seq f) =
 (case f () of seq.Empty $\Rightarrow$ Set-Monad []
 | seq.Insert x P $\Rightarrow$ insert-monad x (set-of-pred P)
 | seq.Join P xq $\Rightarrow$ union-monad (set-of-pred P) (set-of-seq xq))
 ⟨proof⟩

lemma set-of-seq-code [code, set-base-code]:
 set-of-seq seq.Empty = Set-Monad []
 set-of-seq (seq.Insert x P) = insert-monad x (set-of-pred P)

set-of-seq (*seq.Join P xq*) = *union-monad* (*set-of-pred P*) (*set-of-seq xq*)
 <proof>

hide-const (**open**) *insert-monad union-monad*

3.12.5 Type class instantiations

datatype *set-impl* = *Set-IMPL*

declare

set-impl.eq.simps [code del]
set-impl.size [code del]
set-impl.rec [code del]
set-impl.case [code del]

lemma [code, set-base-code]:

fixes *x* :: *set-impl*
shows *size x* = 0
and *size-set-impl x* = 0

<proof>

definition *set-Choose* :: *set-impl* **where** [simp]: *set-Choose* = *Set-IMPL*

definition *set-Collect* :: *set-impl* **where** [simp]: *set-Collect* = *Set-IMPL*

definition *set-DList* :: *set-impl* **where** [simp]: *set-DList* = *Set-IMPL*

definition *set-RBT* :: *set-impl* **where** [simp]: *set-RBT* = *Set-IMPL*

definition *set-Monad* :: *set-impl* **where** [simp]: *set-Monad* = *Set-IMPL*

code-datatype *set-Choose set-Collect set-DList set-RBT set-Monad*

definition *set-empty-choose* :: 'a *set* **where** [simp]: *set-empty-choose* = {}

lemma *set-empty-choose-code* [code, set-base-code]:

(*set-empty-choose* :: 'a :: {ceq, ccompare} *set*) =
 (case *CCOMPARE*('a) of Some - $\Rightarrow$ *RBT-set RBT-Set2.empty*
 | None $\Rightarrow$ case *CEQ*('a) of None $\Rightarrow$ *Set-Monad []* | Some - $\Rightarrow$ *DList-set*
 (*DList-Set.empty*))

<proof>

definition *set-impl-choose2* :: *set-impl* $\Rightarrow$ *set-impl* $\Rightarrow$ *set-impl*

where [simp]: *set-impl-choose2* = (λ - . *Set-IMPL*)

lemma *set-impl-choose2-code* [code, set-base-code]:

set-impl-choose2 set-Monad set-Monad = *set-Monad*
set-impl-choose2 set-RBT set-RBT = *set-RBT*
set-impl-choose2 set-DList set-DList = *set-DList*
set-impl-choose2 set-Collect set-Collect = *set-Collect*
set-impl-choose2 x y = *set-Choose*

<proof>

definition *set-empty* :: *set-impl* $\Rightarrow$ 'a *set*

where $[simp]: set-empty = (\lambda-. \{\})$

lemma *set-empty-code* $[code, set-base-code]:$
 $set-empty\ set-Collect = Collect-set\ (\lambda-. False)$
 $set-empty\ set-DList = DList-set\ DList-Set.empty$
 $set-empty\ set-RBT = RBT-set\ RBT-Set2.empty$
 $set-empty\ set-Monad = Set-Monad\ []$
 $set-empty\ set-Choose = set-empty-choose$
 $\langle proof \rangle$

class *set-impl* =
fixes *set-impl* :: ('a, set-impl) phantom

syntax (*input*)
 $-SET-IMPL :: type \Rightarrow logic\ (\langle (1SET'-IMPL/(1'(-))) \rangle)$

syntax-consts
 $-SET-IMPL == set-impl$

$\langle ML \rangle$

lemma *empty-code* $[code, set-base-code, code-unfold]:$
 $(\{\} :: 'a :: set-impl\ set) = set-empty\ (of-phantom\ SET-IMPL('a))$
 $\langle proof \rangle$

3.12.6 Generator for the *set-impl*-class

This generator registers itself at the derive-manager for the classes *set-impl*. Here, one can choose the desired implementation via the parameter.

- **instantiation** $type :: (type, \dots, type)\ (rbt, dlist, collect, monad, choose,$
or arbitrary constant name) *set-impl*

This generator can be used for arbitrary types, not just datatypes.

$\langle ML \rangle$

derive (*dlist*) *set-impl* unit bool
derive (*rbt*) *set-impl* nat
derive (*set-RBT*) *set-impl* int
derive (*dlist*) *set-impl* Enum.finite-1 Enum.finite-2 Enum.finite-3
derive (*rbt*) *set-impl* integer natural
derive (*rbt*) *set-impl* char

instantiation *sum* :: (*set-impl*, *set-impl*) *set-impl* **begin**

definition $SET-IMPL('a + 'b) = Phantom('a + 'b)$
 $(set-impl-choose2\ (of-phantom\ SET-IMPL('a))\ (of-phantom\ SET-IMPL('b)))$

instance $\langle proof \rangle$

end

```

instantiation prod :: (set-impl, set-impl) set-impl begin
definition SET-IMPL('a * 'b) = Phantom('a * 'b)
  (set-impl-choose2 (of-phantom SET-IMPL('a)) (of-phantom SET-IMPL('b)))
instance <proof>
end

derive (choose) set-impl list
derive (rbt) set-impl String.literal

instantiation option :: (set-impl) set-impl begin
definition SET-IMPL('a option) = Phantom('a option) (of-phantom SET-IMPL('a))
instance <proof>
end

derive (monad) set-impl fun
derive (choose) set-impl set

instantiation phantom :: (type, set-impl) set-impl begin
definition SET-IMPL(('a, 'b) phantom) = Phantom (('a, 'b) phantom) (of-phantom
  SET-IMPL('b))
instance <proof>
end

```

We enable automatic implementation selection for sets constructed by *set*, although they could be directly converted using *Set-Monad* in constant time. However, then it is more likely that the parameters of binary operators have different implementations, which can lead to less efficient execution.

However, we test whether automatic selection picks *Set-Monad* anyway and take a short-cut.

```

definition set-aux :: set-impl ⇒ 'a list ⇒ 'a set
where [simp]: set-aux - = set

lemma set-aux-code [code, set-base-code]:
  defines conv ≡ foldl (λs (x :: 'a). insert x s)
  shows
    set-aux set-Monad = Set-Monad
    set-aux set-Choose =
      (case CCOMPARE('a :: {ccompare, ceq}) of Some - ⇒ conv (RBT-set
        RBT-Set2.empty)
       | None ⇒ case CEQ('a) of None ⇒ Set-Monad
        | Some - ⇒ conv (DList-set DList-Set.empty)) (is ?thesis2)
    set-aux impl = conv (set-empty impl) (is ?thesis1)
  <proof>

lemma set-code [code, set-base-code]:
  fixes xs :: 'a :: set-impl list
  shows set xs = set-aux (of-phantom (ID SET-IMPL('a))) xs

```

<proof>

3.12.7 Pretty printing for sets

code-post marks contexts (as hypothesis) in which we use *code_post* as a decision procedure rather than a pretty-printing engine. The intended use is to enable more rules when proving assumptions of rewrite rules.

definition *code-post* :: bool **where** *code-post* = True

lemma *conj-code-post* [*code-post*]:
assumes *code-post*
shows True & $x \longleftrightarrow x$ False & $x \longleftrightarrow$ False
<proof>

A flag to switch post-processing of sets on and off. Use **declare pretty_sets[*code_post* del]** to disable pretty printing of sets in value.

definition *code-post-set* :: bool
where *pretty-sets* [*code-post*, *simp*]: *code-post-set* = True

definition *collapse-RBT-set* :: 'a set-rbt $\Rightarrow$ 'a :: ccompare set $\Rightarrow$ 'a set
where *collapse-RBT-set* r M = set (RBT-Set2.keys r) $\cup$ M

lemma *RBT-set-collapse-RBT-set* [*code-post*]:
fixes r :: 'a :: ccompare set-rbt
assumes *code-post* $\Rightarrow$ *is-ccompare* TYPE('a) **and** *code-post-set*
shows RBT-set r = *collapse-RBT-set* r {}
<proof>

lemma *collapse-RBT-set-Branch* [*code-post*]:
collapse-RBT-set (Mapping-RBT (Branch c l x v r)) M =
collapse-RBT-set (Mapping-RBT l) (insert x (*collapse-RBT-set* (Mapping-RBT
r) M))
<proof>

lemma *collapse-RBT-set-Empty* [*code-post*]:
collapse-RBT-set (Mapping-RBT rbt.Empty) M = M
<proof>

definition *collapse-DList-set* :: 'a :: ceq set-dlist $\Rightarrow$ 'a set
where *collapse-DList-set* dxs = set (DList-Set.list-of-dlist dxs)

lemma *DList-set-collapse-DList-set* [*code-post*]:
fixes dxs :: 'a :: ceq set-dlist
assumes *code-post* $\Rightarrow$ *is-ceq* TYPE('a) **and** *code-post-set*
shows DList-set dxs = *collapse-DList-set* dxs
<proof>

lemma *collapse-DList-set-empty* [*code-post*]: *collapse-DList-set* (Abs-dlist []) = {}

<proof>

lemma *collapse-DList-set-Cons* [code-post]:
 collapse-DList-set (*Abs-dlist* (*x # xs*)) = *insert* *x* (*collapse-DList-set* (*Abs-dlist* *xs*))
<proof>

lemma *Set-Monad-code-post* [code-post]:
 assumes *code-post-set*
 shows *Set-Monad* [] = {}
 and *Set-Monad* (*x#xs*) = *insert* *x* (*Set-Monad* *xs*)
<proof>

declare [[code drop:
 <Gcd :: - ⇒ nat>
 <Lcm :: - ⇒ nat>
 <Gcd :: - ⇒ int>
 <Lcm :: - ⇒ int>
 Gcd-fin
 Lcm-fin
 <Inf :: - ⇒ 'a Predicate.pred>
 <Sup :: - ⇒ 'a Predicate.pred>
]]

end

theory *Mapping-Impl* **imports**
 RBT-Mapping2
 AssocList
 HOL-Library.Mapping
 Set-Impl
 Containers-Generator
begin

3.13 Different implementations of maps

3.13.1 Map implementations

definition *Assoc-List-Mapping* :: (*'a, 'b*) *alist* ⇒ (*'a, 'b*) *mapping*
where [simp]: *Assoc-List-Mapping* *al* = *Mapping.Mapping* (*DAList.lookup* *al*)

definition *RBT-Mapping* :: (*'a :: ccompare, 'b*) *mapping-rbt* ⇒ (*'a, 'b*) *mapping*
where [simp]: *RBT-Mapping* *t* = *Mapping.Mapping* (*RBT-Mapping2.lookup* *t*)

code-datatype *Assoc-List-Mapping* *RBT-Mapping* *Mapping*

3.13.2 Map operations

lemma *lookup-Mapping-code* [*code*, *no-atp*]:
 $\text{Mapping.lookup (RBT-Mapping } t) = \text{RBT-Mapping2.lookup } t$
 $\text{Mapping.lookup (Assoc-List-Mapping } al) = \text{DAList.lookup } al$
 ⟨*proof*⟩

lemma *is-empty-transfer* [*transfer-rule*]:
includes *lifting-syntax*
shows (*pcr-mapping* (=) (=) ==> (=)) ($\lambda m. m = \text{Map.empty}$) *Mapping.is-empty*
<proof>

lemma *is-empty-Mapping* [code]:
fixes $t :: ('a :: ccompare, 'b) \text{ mapping-rbt}$ **shows**
 $\text{Mapping.is-empty (RBT-Mapping } t) \longleftrightarrow$
 $(\text{case ID CCOMPARE('a) of None} \Rightarrow \text{Code.abort (STR "is-empty RBT-Mapping: ccompare = None") } (\lambda_. \text{Mapping.is-empty (RBT-Mapping } t)))$
 $\quad | \text{Some } _ \Rightarrow \text{RBT-Mapping2.is-empty } t)$
 $\text{Mapping.is-empty (Assoc-List-Mapping } al) \longleftrightarrow al = \text{DAList.empty}$
 $\langle \text{proof} \rangle$

$$\begin{aligned}
& \textbf{lemma } \textit{update-Mapping} [\textit{code}]: \\
& \quad \textbf{fixes } t :: ('a :: \textit{ccompare}, 'b) \textit{mapping-rbt} \textbf{ shows} \\
& \quad \textit{Mapping.update } k \ v \ (\textit{RBT-Mapping } t) = \\
& \quad (\textit{case ID CCOMPARE}('a) \textit{ of None} \Rightarrow \textit{Code.abort } (\textit{STR "update RBT-Mapping:} \\
& \textit{ccompare = None"})) \ (\lambda\cdot. \textit{Mapping.update } k \ v \ (\textit{RBT-Mapping } t)) \\
& \quad \quad \quad | \textit{Some } - \Rightarrow \textit{RBT-Mapping } (\textit{RBT-Mapping2.insert } k \ v \ t)) \ (\textbf{is} \\
& \quad \textit{?RBT}) \\
& \quad \textit{Mapping.update } k \ v \ (\textit{Assoc-List-Mapping } al) = \textit{Assoc-List-Mapping } (\textit{DAList.update} \\
& \quad k \ v \ al) \\
& \quad \textit{Mapping.update } k \ v \ (\textit{Mapping } m) = \textit{Mapping } (m(k \mapsto v)) \\
& \langle \textit{proof} \rangle
\end{aligned}$$
$$\begin{array}{l}
\textbf{lemma } \textit{delete-Mapping} [code]: \\
\textbf{fixes } t :: ('a :: ccompare, 'b) \textit{mapping-rbt} \textbf{ shows} \\
\textit{Mapping.delete } k \textit{ (RBT-Mapping } t) = \\
(\textit{case ID CCOMPARE('a) of None} \Rightarrow \textit{Code.abort (STR "delete RBT-Mapping:} \\
\textit{ccompare = None") } (\lambda\cdot. \textit{Mapping.delete } k \textit{ (RBT-Mapping } t)) \\
\quad | \textit{Some } - \Rightarrow \textit{RBT-Mapping (RBT-Mapping2.delete } k \textit{ } t)) \\
\textit{Mapping.delete } k \textit{ (Assoc-List-Mapping } al) = \textit{Assoc-List-Mapping (AssocList.delete} \\
k \textit{ } al) \\
\textit{Mapping.delete } k \textit{ (Mapping } m) = \textit{Mapping (m(k := None))} \\
\langle \textit{proof} \rangle
\end{array}$$

theorem *rbt-comp-lookup-map-const*: $\text{rbt-comp-lookup } c \ (\text{RBT-Impl.map } (\lambda \cdot. f) \ t)$
 $= \text{map-option } f \circ \text{rbt-comp-lookup } c \ t$
 $\langle \text{proof} \rangle$

lemma *keys-Mapping* [code]:
 fixes $t :: ('a :: ccompare, 'b) \text{mapping-rbt}$ **shows**

```

Mapping.keys (RBT-Mapping t) = RBT-set (RBT-Mapping2.map (λ- -. ()) t)
(is ?RBT)
Mapping.keys (Assoc-List-Mapping al) = AssocList.keys al (is ?Assoc-List)
Mapping.keys (Mapping m) = Collect (λk. m k ≠ None) (is ?Mapping)
⟨proof⟩

```

lemma *Mapping-size-transfer* [transfer-rule]:
includes *lifting-syntax*
shows (pcr-mapping (=) (=) ==> (=)) (card ∘ dom) Mapping.size
⟨proof⟩

lemma *size-Mapping* [code]:
fixes *t* :: ('a :: ccompare, 'b) mapping-rbt **shows**
Mapping.size (RBT-Mapping t) =
(case ID CCOMPARE('a) of None ⇒ Code.abort (STR "size RBT-Mapping:
ccompare = None") (λ-. Mapping.size (RBT-Mapping t))
| Some - ⇒ length (RBT-Mapping2.entries t))
Mapping.size (Assoc-List-Mapping al) = size al
⟨proof⟩

declare *tabulate-fold* [code]

declare *ordered-keys-def* [code]

declare *Mapping.lookup-default-def* [code]

lemma *filter-code* [code]:
fixes *t* :: ('a :: ccompare, 'b) mapping-rbt **shows**
Mapping.filter P (RBT-Mapping t) =
(case ID CCOMPARE('a) of None ⇒ Code.abort (STR "filter RBT-Mapping:
ccompare = None") (λ-. Mapping.filter P (RBT-Mapping t))
| Some - ⇒ RBT-Mapping (RBT-Mapping2.filter (λ(k, v). P
k v) t))
Mapping.filter P (Assoc-List-Mapping al) = Assoc-List-Mapping (DAList.filter
(λ(k, v). P k v) al)
Mapping.filter P (Mapping m) = Mapping (λk. case m k of None ⇒ None | Some
v ⇒ if P k v then Some v else None)
⟨proof⟩

lemma *map-values-code* [code]:
fixes *t* :: ('a :: ccompare, 'b) mapping-rbt **shows**
Mapping.map-values f (RBT-Mapping t) =
(case ID CCOMPARE('a) of None ⇒ Code.abort (STR "map-values RBT-Mapping:
ccompare = None") (λ-. Mapping.map-values f (RBT-Mapping t))
| Some - ⇒ RBT-Mapping (RBT-Mapping2.map f t))
Mapping.map-values f (Assoc-List-Mapping al) = Assoc-List-Mapping (AssocList.map-values
f al)
Mapping.map-values f (Mapping m) = Mapping (λk. map-option (f k) (m k))
⟨proof⟩

lemma *bulkload-code* [code]:
Mapping.bulkload vs = RBT-Mapping (RBT-Mapping2.bulkload (zip-with-index vs))
 ⟨proof⟩

datatype *mapping-impl* = *Mapping-IMPL*

lemma [code]:
 fixes *x* :: *mapping-impl*
 shows *size x = 0*
 and *size-mapping-impl x = 0*
 ⟨proof⟩

definition *mapping-Choose* :: *mapping-impl*
 where [simp]: *mapping-Choose* = *Mapping-IMPL*

definition *mapping-Assoc-List* :: *mapping-impl*
 where [simp]: *mapping-Assoc-List* = *Mapping-IMPL*

definition *mapping-RBT* :: *mapping-impl*
 where [simp]: *mapping-RBT* = *Mapping-IMPL*

definition *mapping-Mapping* :: *mapping-impl*
 where [simp]: *mapping-Mapping* = *Mapping-IMPL*

code-datatype *mapping-Choose mapping-Assoc-List mapping-RBT mapping-Mapping*

definition *mapping-empty-choose* :: ('a, 'b) *mapping*
 where [simp]: *mapping-empty-choose* = *Mapping.empty*

lemma *mapping-empty-choose-code* [code]:
 (*mapping-empty-choose* :: ('a :: ccompare, 'b) *mapping*) =
 (case ID CCOMPARE('a) of Some - $\Rightarrow$ *RBT-Mapping RBT-Mapping2.empty*
 | None $\Rightarrow$ *Assoc-List-Mapping DAList.empty*)
 ⟨proof⟩

definition *mapping-impl-choose2* :: *mapping-impl* $\Rightarrow$ *mapping-impl* $\Rightarrow$ *mapping-impl*
 where [simp]: *mapping-impl-choose2* = (λ -. *Mapping-IMPL*)

lemma *mapping-impl-choose2-code* [code]:
mapping-impl-choose2 mapping-RBT mapping-RBT = mapping-RBT
mapping-impl-choose2 mapping-Assoc-List mapping-Assoc-List = mapping-Assoc-List
mapping-impl-choose2 mapping-Mapping mapping-Mapping = mapping-Mapping
mapping-impl-choose2 x y = mapping-Choose
 ⟨proof⟩

definition *mapping-empty* :: ('a, 'b) *mapping*
 where [simp]: *mapping-empty* = (λ -. *Mapping.empty*)

```

lemma mapping-empty-code [code]:
  mapping-empty mapping-RBT = RBT-Mapping RBT-Mapping2.empty
  mapping-empty mapping-Assoc-List = Assoc-List-Mapping DAList.empty
  mapping-empty mapping-Mapping = Mapping ( $\lambda$ -. None)
  mapping-empty mapping-Choose = mapping-empty-choose
  <proof>

```

3.13.3 Type classes

```

class mapping-impl =
  fixes mapping-impl :: ('a, mapping-impl) phantom

syntax (input)
  -MAPPING-IMPL :: type => logic (⟨(1MAPPING'-IMPL/(1'(-'))⟩)⟩)

syntax-consts
  -MAPPING-IMPL == mapping-impl

  <ML>

```

```

lemma Mapping-empty-code [code, code-unfold]:
  (Mapping.empty :: ('a :: mapping-impl, 'b) mapping) =
    mapping-empty (of-phantom MAPPING-IMPL('a))
  <proof>

```

3.13.4 Generator for the *mapping-impl*-class

This generator registers itself at the derive-manager for the classes *mapping-impl*. Here, one can choose the desired implementation via the parameter.

- `instantiation type :: (type,...,type) (rbt,assoclist,mapping,choose,
or arbitrary constant name) mapping-impl`

This generator can be used for arbitrary types, not just datatypes.

<ML>

```

derive (assoclist) mapping-impl unit bool
derive (rbt) mapping-impl nat
derive (mapping-RBT) mapping-impl int
derive (assoclist) mapping-impl Enum.finite-1 Enum.finite-2 Enum.finite-3
derive (rbt) mapping-impl integer natural
derive (rbt) mapping-impl char

```

```

instantiation sum :: (mapping-impl, mapping-impl) mapping-impl
begin

```

definition $MAPPING-IMPL('a + 'b) = Phantom('a + 'b)$
 $(mapping-impl-choose2 (of-phantom MAPPING-IMPL('a)) (of-phantom MAPPING-IMPL('b)))$

instance $\langle proof \rangle$

end

instantiation $prod :: (mapping-impl, mapping-impl) mapping-impl$ **begin**

definition $MAPPING-IMPL('a * 'b) = Phantom('a * 'b)$
 $(mapping-impl-choose2 (of-phantom MAPPING-IMPL('a)) (of-phantom MAPPING-IMPL('b)))$

instance $\langle proof \rangle$

end

derive $(choose) mapping-impl list$

derive $(rbt) mapping-impl String.literal$

instantiation $option :: (mapping-impl) mapping-impl$
begin

definition $MAPPING-IMPL('a option) = Phantom('a option) (of-phantom MAPPING-IMPL('a))$

instance $\langle proof \rangle$

end

derive $(choose) mapping-impl set$

instantiation $phantom :: (type, mapping-impl) mapping-impl$
begin

definition $MAPPING-IMPL(('a, 'b) phantom) = Phantom (('a, 'b) phantom)$
 $(of-phantom MAPPING-IMPL('b))$

instance $\langle proof \rangle$

end

declare $[[code drop:$
 $rec-mapping-impl$
 $case-mapping-impl$
 $\langle HOL.equal :: mapping-impl \Rightarrow \rightarrow \rangle$
 $Mapping.combine-with-key$
 $Mapping.combine$

]]

code-identifier

code-module *Mapping* $\rightarrow$ (*SML*) *Mapping-Impl*
| **code-module** *Mapping-Impl* $\rightarrow$ (*SML*) *Mapping-Impl*

end

theory *Map-To-Mapping* **imports**

Mapping-Impl

begin

3.14 Infrastructure for operation identification

To convert theorems from $'a \Rightarrow 'b$ *option* to $('a, 'b)$ *mapping* using lifting / transfer, we first introduce constants for the empty map and map lookup, then apply lifting / transfer, and finally eliminate the non-converted constants again.

Dynamic theorem list of rewrite rules that are applied before *Transfer.transfer*

$\langle ML \rangle$

Dynamic theorem list of rewrite rules that are applied after *Transfer.transfer*

$\langle ML \rangle$

context includes *lifting-syntax*

begin

definition *map-empty* :: $'a \Rightarrow 'b$ *option*

where [*code-unfold*]: *map-empty* = *Map.empty*

declare *map-empty-def*[*containers-post*, *symmetric*, *containers-pre*]

declare *Mapping.empty.transfer*[*transfer-rule del*]

lemma *map-empty-transfer* [*transfer-rule*]:

(*pcr-mapping A B*) *map-empty Mapping.empty*

$\langle proof \rangle$

definition *map-apply* :: $('a \Rightarrow 'b$ *option*) $\Rightarrow 'a \Rightarrow 'b$ *option*

where [*code-unfold*]: *map-apply* = $(\lambda m. m)$

lemma *eq-map-apply*: $m\ x \equiv map-apply\ m\ x$

$\langle proof \rangle$

declare *eq-map-apply*[*symmetric*, *abs-def*, *containers-post*]

We cannot use *eq-map-apply* as a fold rule for operator identification, because it would loop. We use a *simproc* instead.

$\langle ML \rangle$

lemma *map-apply-parametric* [*transfer-rule*]:

((*A* $\implies$ *B*) $\implies$ *A* $\implies$ *B*) *map-apply* *map-apply*
 $\langle proof \rangle$

lemma *map-apply-transfer* [*transfer-rule*]:

(*pcr-mapping* *A* *B* $\implies$ *A* $\implies$ *rel-option* *B*) *map-apply* *Mapping.lookup*
 $\langle proof \rangle$

definition *map-update* :: '*a* $\Rightarrow$ '*b* option $\Rightarrow$ ('*a* $\Rightarrow$ '*b* option) $\Rightarrow$ ('*a* $\Rightarrow$ '*b* option)

where *map-update* *x* *f* = *f*(*x* := *y*)

lemma *map-update-parametric* [*transfer-rule*]:

assumes [*transfer-rule*]: *bi-unique* *A*
shows (*A* $\implies$ *rel-option* *B* $\implies$ (*A* $\implies$ *rel-option* *B*) $\implies$ (*A* $\implies$ *rel-option* *B*)) *map-update* *map-update*
 $\langle proof \rangle$

context **begin**

$\langle ML \rangle$

lift-definition *update'* :: '*a* $\Rightarrow$ '*b* option $\Rightarrow$ ('*a*, '*b*) *mapping* $\Rightarrow$ ('*a*, '*b*) *mapping*

is *map-update* **parametric** *map-update-parametric* $\langle proof \rangle$

lemma *update'-code* [*simp*, *code*, *code-unfold*]:

update' *x* *None* = *Mapping.delete* *x*
update' *x* (*Some* *y*) = *Mapping.update* *x* *y*
 $\langle proof \rangle$

end

declare *map-update-def*[*abs-def*, *containers-post*] *map-update-def*[*symmetric*, *containers-pre*]

definition *map-is-empty* :: ('*a* $\Rightarrow$ '*b* option) $\Rightarrow$ *bool*

where *map-is-empty* *m* $\longleftrightarrow$ *m* = *Map.empty*

lemma *map-is-empty-folds*:

m = *map-empty* $\longleftrightarrow$ *map-is-empty* *m*
map-empty = *m* $\longleftrightarrow$ *map-is-empty* *m*
 $\langle proof \rangle$

```

declare map-is-empty-folds[containers-pre]
  map-is-empty-def[abs-def, containers-post]

lemma map-is-empty-transfer [transfer-rule]:
  assumes bi-total A
  shows (pcr-mapping A B ==> (=)) map-is-empty Mapping.is-empty
  <proof>

end

<ML>

hide-const (open) map-apply map-empty map-is-empty map-update
hide-fact (open) map-apply-def map-empty-def eq-map-apply

end

theory Containers imports
  Set-Linorder
  Collection-Order
  Collection-Eq
  Collection-Enum
  Equal
  Mapping-Impl
  Map-To-Mapping
begin

end

```

3.15 Compatibility with Regular-Sets

```

theory Compatibility-Containers-Regular-Sets imports
  Containers
  Regular-Sets.Regexp-Method
begin

```

Adaptation theory to make *regexp* work when *Containers.Containers* are loaded.

Warning: Each invocation of *regexp* takes longer than without *Containers.Containers* because the code generator takes longer to generate the evaluation code for *regexp*.

```

datatype-compat rex
derive ceq rex
derive compare rex
derive (choose) set-impl rex

```

notepad begin

$\langle proof \rangle$

end

end

Chapter 4

User guide

This user guide shows how to use and extend the lightweight containers framework (LC). For a more theoretical discussion, see [5]. This user guide assumes that you are familiar with refinement in the code generator [1, 2]. The theory *Containers-Userguide* generates it; so if you want to experiment with the examples, you can find their source code there. Further examples can be found in the `Examples` folder.

4.1 Characteristics

- **Separate type classes for code generation**

LC follows the ideal that type classes for code generation should be separate from the standard type classes in Isabelle. LC's type classes are designed such that every type can become an instance, so well-sortedness errors during code generation can always be remedied.

- **Multiple implementations**

LC supports multiple simultaneous implementations of the same container type. For example, the following implements at the same time (i) the set of *bool* as a distinct list of the elements, (ii) *int set* as a RBT of the elements or as the RBT of the complement, and (iii) sets of functions as monad-style lists:

```
value ({True}, {1 :: int}, - {2 :: int, 3}, {λx :: int. x * x, λy. y + 1})
```

The LC type classes are the key to simultaneously supporting different implementations.

- **Extensibility**

The LC framework is designed for being extensible. You can add new containers, implementations and element types any time.

4.2 Getting started

Add the entry theory *Containers.Containers* for LC to the end of your imports. This will reconfigure the code generator such that it implements the types *'a set* for sets and *('a, 'b) mapping* for maps with one of the data structures supported. As with all the theories that adapt the code generator setup, it is important that *Containers.Containers* comes at the end of the imports.

Note: LC should not be used together with the theory *HOL-Library.Code-Cardinality*. Run the following command, e.g., to check that LC works correctly and implements sets of *ints* as red-black trees (RBT):

```
value [code] {1 :: int}
```

This should produce $\{1\}$. Without LC, sets are represented as (complements of) a list of elements, i.e., *set [1]* in the example.

If your exported code does not use your own types as elements of sets or maps and you have not declared any code equation for these containers, then your **export-code** command will use LC to implement *'a set* and *('a, 'b) mapping*.

Our running example will be arithmetic expressions. The function *vars e* computes the variables that occur in the expression *e*

```
type-synonym vname = string
datatype expr = Var vname | Lit int | Add expr expr
fun vars :: expr  $\Rightarrow$  vname set where
  vars (Var v) = {v}
| vars (Lit i) = {}
| vars (Add e1 e2) = vars e1  $\cup$  vars e2
```

```
value vars (Var "x")
```

To illustrate how to deal with type variables, we will use the following variant where variable names are polymorphic:

```
datatype 'a expr' = Var' 'a | Lit' int | Add' 'a expr' 'a expr'
fun vars' :: 'a expr'  $\Rightarrow$  'a set where
  vars' (Var' v) = {v}
| vars' (Lit' i) = {}
| vars' (Add' e1 e2) = vars' e1  $\cup$  vars' e2

value vars' (Var' (1 :: int))
```

4.3 New types as elements

This section explains LC's type classes and shows how to instantiate them. If you want to use your own types as the elements of sets or the keys of maps, you must instantiate up to eight type classes: *ceq* (§4.3.1), *ccompare* (§4.3.2), *set-impl* (§4.3.3), *mapping-impl* (§4.3.3), *cenum* (§4.3.4), *finite-UNIV* (§4.3.5), *card-UNIV* (§4.3.5), and *cproper-interval* (§4.3.5). Otherwise, well-sortedness errors like the following will occur:

```
*** Wellsortedness error:
*** Type expr not of sort {ceq,ccompare}
*** No type arity expr :: ceq
*** At command "value"
```

In detail, the sort requirements on the element type *'a* are:

- *ceq* (§4.3.1), *ccompare* (§4.3.2), and *set-impl* (§4.3.3) for *'a* set in general
- *cenum* (§4.3.4) for set comprehensions $\{x. P\ x\}$,
- *card-UNIV*, *cproper-interval* for *'a* set set and any deeper nesting of sets (§4.3.5),¹ and
- *equal*,² *ccompare* (§4.3.2) and *mapping-impl* (§4.3.3) for (*'a*, *'b*) mapping.

4.3.1 Equality testing

The type class *ceq* defines the operation $CEQ('a) :: ('a \Rightarrow 'a \Rightarrow bool) option$ for testing whether two elements are equal.³ The test is embedded in an *option* value to allow for types that do not support executable equality test such as $'a \Rightarrow 'b$. Whenever possible, $CEQ('a)$ should provide an executable equality operator. Otherwise, membership tests on such sets will raise an exception at run-time.

¹These type classes are only required for set complements (see §4.7.2).

²We deviate here from the strict separation of type classes, because it does not make sense to store types in a map on which we do not have equality, because the most basic operation *Mapping.lookup* inherently requires equality.

³Technically, the type class *ceq* defines the operation *ceq*. As usage often does not fully determine *ceq*'s type, we use the notation $CEQ('a)$ that explicitly mentions the type. In detail, $CEQ('a)$ is translated to $CEQ('a) :: ('a \Rightarrow 'a \Rightarrow bool) option$ including the type constraint. We do the same for the other type class operators: *ccompare* constrains the operation *ccompare* (§4.3.2), *SET-IMPL('a)* constrains the operation *set-impl*, (§4.3.3), *MAPPING-IMPL('a)* (constrains the operation *mapping-impl*, (§4.3.3), and *CENUM('a)* constrains the operation *cenum*, §4.3.4.

For data types, the *derive* command can automatically instantiate *ceq*, we only have to tell it whether an equality operation should be provided or not (parameter *no*).

```
derive (eq) ceq expr
```

```
datatype example = Example
```

```
derive (no) ceq example
```

In the remainder of this subsection, we look at how to manually instantiate a type for *ceq*. First, the simple case of a type constructor *simple-tycon* without parameters that already is an instance of *equal*:

```
typedecl simple-tycon
```

```
axiomatization where simple-tycon-equal: OFCLASS(simple-tycon, equal-class)
```

```
instance simple-tycon :: equal <proof>
```

```
instantiation simple-tycon :: ceq begin
```

```
definition CEQ(simple-tycon) = Some (=)
```

```
instance <proof>
```

```
end
```

For polymorphic types, this is a bit more involved, as the next example with *'a expr'* illustrates (note that we could have delegated all this to *derive*). First, we need an operation that implements equality tests with respect to a given equality operation on the polymorphic type. For data types, we can use the relator which the transfer package (method *transfer*) requires and the BNF package generates automatically. As we have used the old datatype package for *'a expr'*, we must define it manually:

```
context fixes R :: 'a  $\Rightarrow$  'b  $\Rightarrow$  bool begin
```

```
fun expr'-rel :: 'a expr'  $\Rightarrow$  'b expr'  $\Rightarrow$  bool
```

```
where
```

```
  expr'-rel (Var' v)      (Var' v')       $\longleftrightarrow$  R v v'
| expr'-rel (Lit' i)      (Lit' i')        $\longleftrightarrow$  i = i'
| expr'-rel (Add' e1 e2) (Add' e1' e2')  $\longleftrightarrow$  expr'-rel e1 e1'  $\wedge$  expr'-rel e2 e2'
| expr'-rel -            -                 $\longleftrightarrow$  False
```

```
end
```

If we give HOL equality as parameter, the relator is equality:

```
lemma expr'-rel-eq: expr'-rel (=) e1 e2  $\longleftrightarrow$  e1 = e2
```

```
<proof>
```

Then, the instantiation is again canonical:

```
instantiation expr' :: (ceq) ceq begin
```

definition

CEQ('a expr') =
 (case ID CEQ('a) of None $\Rightarrow$ None | Some eq $\Rightarrow$ Some (expr'-rel eq))

instance

$\langle proof \rangle$

end

Note the following two points: First, the instantiation should avoid to use (=) on terms of the polymorphic type. This keeps the LC framework separate from the type class *equal*, i.e., every choice of 'a in 'a *expr'* can be of sort *ceq*. The easiest way to achieve this is to obtain the equality test from *CEQ('a)*. Second, we use *ID CEQ('a)* instead of *CEQ('a)*. In proofs, we want that the simplifier uses assumptions like *CEQ('a) = Some ...* for rewriting. However, *CEQ('a)* is a nullary constant, so the simplifier reverses such an equation, i.e., it only rewrites *Some ...* to *CEQ('a)*. Applying the identity function *ID* to *CEQ('a)* avoids this, and the code generator eliminates all occurrences of *ID*. Although *ID = id* by definition, do not use the conventional *id* instead of *ID*, because *id CEQ('a)* immediately simplifies to *CEQ('a)*.

4.3.2 Ordering

LC takes the order for storing elements in search trees from the type class *ccompare* rather than *compare*, because we cannot instantiate *compare* for some types (e.g., 'a *set* as ($\subseteq$) is not linear). Similar to *CEQ('a)* in class *CEQ('b)*, the class *ccompare* specifies an optional comparator *CCOMPARE('a) :: (('a $\Rightarrow$ 'a $\Rightarrow$ order)) option*. If you cannot or do not want to implement a comparator on your type, you can default to *None*. In that case, you will not be able to use your type as elements of sets or as keys in maps implemented by search trees.

If the type is a data type or instantiates *compare* and we wish to use that comparator also for the search tree, instantiation is again canonical: For our data type *expr*, *derive* does everything!

derive ccompare expr

In general, the pattern for type constructors without parameters looks as follows:

axiomatization where simple-tycon-compare: OFCLASS(simple-tycon, compare-class)

instance simple-tycon :: compare $\langle proof \rangle$

derive (compare) ccompare simple-tycon

For polymorphic types like 'a *expr'*, we should not do everything manually: First, we must define a comparator that takes the comparator on the

type variable $'a$ as a parameter. This is necessary to maintain the separation between Isabelle/HOL's type classes (like *compare*) and LC's. Such a comparator is again easily defined by *derive*.

derive ccompare expr'

thm ccompare- expr' -def comparator- expr' -simps

4.3.3 Heuristics for picking an implementation

Now, we have defined the necessary operations on *expr* and $'a \text{ expr}'$ to store them in a set or use them as the keys in a map. But before we can actually do so, we also have to say which data structure to use. The type classes *set-impl* and *mapping-impl* are used for this.

They define the overloaded operations $\text{SET-IMPL}('a) :: ('a, \text{set-impl}) \text{phantom}$ and $\text{MAPPING-IMPL}('a) :: ('a, \text{mapping-impl}) \text{phantom}$, respectively. The phantom type $('a, 'b) \text{phantom}$ from theory *HOL-Library.Pantom-Type* is isomorphic to $'b$, but formally depends on $'a$. This way, the type class operations meet the requirement that their type contains exactly one type variable. The Haskell and ML compiler will get rid of the extra type constructor again.

For sets, you can choose between *set-Collect* (characteristic function P like in $\{x. P x\}$), *set-DList* (distinct list), *set-RBT* (red-black tree), and *set-Monad* (list with duplicates). Additionally, you can define *set-impl* as *set-Choose* which picks the implementation based on the available operations (RBT if *ccompare* provides a linear order, else distinct lists if *CEQ('a)* provides equality testing, and lists with duplicates otherwise). *set-Choose* is the safest choice because it picks only a data structure when the required operations are actually available. If *set-impl* picks a specific implementation, Isabelle does not ensure that all required operations are indeed available.

For maps, the choices are *mapping-Assoc-List* (associative list without duplicates), *mapping-RBT* (red-black tree), and *mapping-Mapping* (closures with function update). Again, there is also the *mapping-Choose* heuristics.

For simple cases, *derive* can be used again (even if the type is not a data type). Consider, e.g., the following instantiations: *expr set* uses RBTs, $(\text{expr}, -)$ *mapping* and $'a \text{ expr}' \text{ set}$ use the heuristics, and $('a \text{ expr}', -)$ *mapping* uses the same implementation as $('a, -)$ *mapping*.

derive (rbt) set-impl expr

derive (choose) mapping-impl expr

derive (choose) set-impl expr'

More complex cases such as taking the implementation preference of a type parameter must be done manually.

```

instantiation  $\text{expr}' :: (\text{mapping-impl}) \text{ mapping-impl}$  begin
definition
  MAPPING-IMPL('a  $\text{expr}'$ ) =
    Phantom('a  $\text{expr}'$ ) (of-phantom MAPPING-IMPL('a))
instance  $\langle \text{proof} \rangle$ 
end

```

To see the effect of the different configurations, consider the following examples where *empty* refers to *Mapping.empty*. For that, we must disable pretty printing for sets as follows:

```
declare pretty-sets[code-post del]
```

value [code]	result
$\{\} :: \text{expr set}$	<i>RBT-set (Mapping-RBT Empty)</i>
$\text{empty} :: (\text{expr}, \text{unit}) \text{ mapping}$	<i>RBT-Mapping (Mapping-RBT Empty)</i>
$\{\} :: \text{string expr}' \text{ set}$	<i>RBT-set (Mapping-RBT Empty)</i>
$\{\} :: (\text{nat} \Rightarrow \text{nat}) \text{ expr}' \text{ set}$	<i>Set-Monad []</i>
$\{\} :: \text{bool expr}' \text{ set}$	<i>RBT-set (Mapping-RBT Empty)</i>
$\text{empty} :: (\text{bool expr}', \text{unit}) \text{ mapping}$	<i>Assoc-List-Mapping (Alist [])</i>

For *expr*, *mapping-Choose* picks RBTs, because *compare* provides a comparison operation for *expr*. For '*a expr*', the effect of *set-Choose* is more pronounced: *compare* is not *None*, so neither is *compare*, and *set-Choose* picks RBTs. As $\text{nat} \Rightarrow \text{nat}$ neither provides equality tests (*ceq*) nor comparisons (*compare*), neither does $(\text{nat} \Rightarrow \text{nat}) \text{ expr}'$, so we use lists with duplicates. The last two examples show the difference between inheriting a choice and choosing freshly: By default, *bool* prefers distinct (associative) lists over RBTs, because there are just two elements. As *bool expr'* inherits the choice for maps from *bool*, an associative list implements $\text{empty} :: (\text{bool expr}', \text{unit}) \text{ mapping}$. For sets, in contrast, *set-impl* discards '*a*'s preferences and picks RBTs, because there is a comparison operation.

Finally, let's enable pretty-printing for sets again:

```
declare pretty-sets [code-post]
```

4.3.4 Set comprehensions

If you use the default code generator setup that comes with Isabelle, set comprehensions $\{x. P x\} :: 'a \text{ set}$ are only executable if the type '*a* has sort *enum*. Internally, Isabelle's code generator transforms set comprehensions into an explicit list of elements which it obtains from the list *enum* of all of '*a*'s elements. Thus, the type must be an instance of *enum*, i.e., finite in

particular. For example, $\{c. \text{CHR } "A" \leq c \wedge c \leq \text{CHR } "D"\}$ evaluates to *set "ABCD"*, the set of the characters A, B, C, and D.

For compatibility, LC also implements such an enumeration strategy, but avoids the finiteness restriction. The type class *cenum* mimicks *enum*, but its single parameter *cEnum* $:: ('a \text{ list} \times (('a \Rightarrow \text{bool}) \Rightarrow \text{bool}) \times (('a \Rightarrow \text{bool}) \Rightarrow \text{bool})) \text{ option}$ combines all of *enum*'s parameters, namely a list of all elements, a universal and an existential quantifier. *option* ensures that every type can be an instance as *CENUM*('a) can always default to *None*.

For types that define *CENUM*('a), set comprehensions evaluate to a list of their elements. Otherwise, set comprehensions are represented as a closure. This means that if the generated code contains at least one set comprehension, all element types of a set must instantiate *cenum*. Infinite types default to *None*, and enumerations for finite types are canonical, see *Containers.Collection-Enum* for examples.

```
instantiation expr :: cenum begin
definition CENUM(expr) = None
instance <proof>
end
```

```
derive (no) cenum expr'
derive compare-order expr
```

For example, **value** ($\{b. b = \text{True}\}, \{x. \text{compare } x (\text{Lit } 0) = \text{Lt}\}$) yields $(\{\text{True}\}, \text{Collect-set } -)$

LC keeps complements of such enumerated set comprehensions, i.e., $-\{b. b = \text{True}\}$ evaluates to *Complement {True}*. If you want that the complement operation actually computes the elements of the complements, you have to replace the code equations for *uminus* as follows:

```
declare Set-uminus-code[code del] Set-uminus-cenum[code]
```

Then, $-\{b. b = \text{True}\}$ becomes $\{\text{False}\}$, but this applies to all complement invocations. For example, *UNIV* $:: \text{bool set}$ becomes $\{\text{False}, \text{True}\}$.

4.3.5 Nested sets

To deal with nested sets such as *expr set set*, the element type must provide three operations from three type classes:

- *finite-UNIV* from theory *HOL-Library.Cardinality* defines the constant *finite-UNIV* $:: ('a, \text{bool}) \text{ phantom}$ which designates whether the type is finite.

- *card-UNIV* from theory *HOL-Library.Cardinality* defines the constant *card-UNIV* :: ('a, nat) phantom which returns *CARD('a)*, i.e., the number of values in 'a. If 'a is infinite, *CARD('a) = 0*.
- *cproper-interval* from theory *Containers.Collection-Order* defines the function *cproper-interval* :: 'a option $\Rightarrow$ 'a option $\Rightarrow$ bool. If the type 'a is finite and *ccompare* yields a linear order on 'a, then *cproper-interval* *x y* returns whether the open interval between *x* and *y* is non-empty. The bound *None* denotes unboundedness.

Note that the type class *finite-UNIV* must not be confused with the type class *finite*. *finite-UNIV* allows the generated code to examine whether a type is finite whereas *finite* requires that the type in fact is finite.

For datatypes, the theory *Containers.Card-Datatype* defines some machinery to assist in proving that the type is (in)finite and has a given number of elements – see *Examples/Card_Datatype_Ex.thy* for examples. With this, it is easy to instantiate *card-UNIV* for our running examples:

```
lemma inj-expr [simp]: inj Lit    inj Var    inj Add    inj (Add e)
<proof>
```

```
lemma infinite-UNIV-expr:  $\neg$  finite (UNIV :: expr set)
including card-datatype
<proof>
```

```
instantiation expr :: card-UNIV begin
definition finite-UNIV = Phantom(expr) False
definition card-UNIV = Phantom(expr) 0
instance
  <proof>
end
```

```
lemma inj-expr' [simp]: inj Lit'    inj Var'    inj Add'    inj (Add' e)
<proof>
```

```
lemma infinite-UNIV-expr':  $\neg$  finite (UNIV :: 'a expr' set)
including card-datatype
<proof>
```

```
instantiation expr' :: (type) card-UNIV begin
definition finite-UNIV = Phantom('a expr') False
definition card-UNIV = Phantom('a expr') 0
instance
  <proof>
```

end

As *expr* and '*a expr*' are infinite, instantiating *cproper-interval* is trivial, because *cproper-interval* only makes assumptions about its parameters for finite types. Nevertheless, it is important to actually define *cproper-interval*, because the code generator requires a code equation.

```
instantiation expr :: cproper-interval begin
definition cproper-interval-expr :: expr proper-interval
  where cproper-interval-expr - - = undefined
instance <proof>
end
```

```
instantiation expr' :: (compare) cproper-interval begin
definition cproper-interval-expr' :: 'a expr' proper-interval
  where cproper-interval-expr' - - = undefined
instance <proof>
end
```

Instantiation of *proper-interval*

To illustrate what to do with finite types, we instantiate *proper-interval* for *expr*. Like *compare* relates to *compare*, the class *cproper-interval* has a counterpart *proper-interval* without the finiteness assumption. Here, we first have to gather the simplification rules of the comparator from the derive invocation, especially, how the strict order of the comparator, *lt-of-comp*, can be defined.

Since the order on lists is not yet shown to be consistent with the comparators that are used for lists, this part of the userguide is currently not available.

4.4 New implementations for containers

This section explains how to add a new implementation for a container type. If you do so, please consider to add your implementation to this AFP entry.

4.4.1 Model and verify the data structure

First, you of course have to define the data structure and verify that it has the required properties. As our running example, we use a trie to implement ('a, 'b) *mapping*. A trie is a binary tree whose the nodes store the values, the keys are the paths from the root to the given node. We use lists of *boolans* for the keys where the *boolean* indicates whether we should go to the left or right child.

For brevity, we skip this step and rather assume that the type $'v$ *trie-raw* of tries has following operations and properties:

```
type-synonym trie-key = bool list
axiomatization
  trie-empty :: 'v trie-raw and
  trie-update :: trie-key  $\Rightarrow$  'v  $\Rightarrow$  'v trie-raw  $\Rightarrow$  'v trie-raw and
  trie-lookup :: 'v trie-raw  $\Rightarrow$  trie-key  $\Rightarrow$  'v option and
  trie-keys :: 'v trie-raw  $\Rightarrow$  trie-key set
where trie-lookup-empty: trie-lookup trie-empty = Map.empty
and trie-lookup-update:
  trie-lookup (trie-update k v t) = (trie-lookup t)(k  $\mapsto$  v)
and trie-keys-dom-lookup: trie-keys t = dom (trie-lookup t)
```

This is only a minimal example. A full-fledged implementation has to provide more operations and – for efficiency – should use more than just *booleans* for the keys.

$\langle proof \rangle \langle proof \rangle$

4.4.2 Generalise the data structure

As $('k, 'v)$ *mapping* store keys of arbitrary type $'k$, not just *trie-key*, we cannot use $'v$ *trie-raw* directly. Instead, we must first convert arbitrary types $'k$ into *trie-key*. Of course, this is not always possible, but we only have to make sure that we pick tries as implementation only if the types do. This is similar to red-black trees which require an order. Hence, we introduce a type class to convert arbitrary keys into trie keys. We make the conversions optional such that every type can instantiate the type class, just as LC does for *ceq* and *compare*.

```
type-synonym 'a cbl = (('a  $\Rightarrow$  bool list)  $\times$  (bool list  $\Rightarrow$  'a)) option
class cbl =
  fixes cbl :: 'a cbl
  assumes inj-to-bl: ID cbl = Some (to-bl, from-bl)  $\implies$  inj to-bl
  and to-bl-inverse: ID cbl = Some (to-bl, from-bl)  $\implies$  from-bl (to-bl a) =
a
begin
abbreviation from-bl where from-bl  $\equiv$  snd (the (ID cbl))
abbreviation to-bl where to-bl  $\equiv$  fst (the (ID cbl))
end
```

It is best to immediately provide the instances for as many types as possible. Here, we only present two examples: *unit* provides conversion functions, $'a \Rightarrow 'b$ does not.

```
instantiation unit :: cbl begin
```

```

definition cbl = Some ( $\lambda$ -. [],  $\lambda$ -. ())
instance  $\langle proof \rangle$ 
end

```

```

instantiation fun :: (type, type) cbl begin
definition cbl = (None :: ('a  $\Rightarrow$  'b) cbl)
instance  $\langle proof \rangle$ 
end

```

4.4.3 Hide the invariants of the data structure

Many data structures have invariants on which the operations rely. You must hide such invariants in a **typedef** before connecting to the container, because the code generator cannot handle explicit invariants. The type must be inhabited even if the types of the elements do not provide the required operations. The easiest way is often to ignore all invariants in that case.

In our example, we require that all keys in the trie represent encoded values.

```

typedef (overloaded) ('k :: cbl, 'v) trie =
  {t :: 'v trie-raw.
   trie-keys t  $\subseteq$  range (to-bl :: 'k  $\Rightarrow$  trie-key)  $\vee$  ID (cbl :: 'k cbl) = None}
 $\langle proof \rangle$ 

```

Next, transfer the operations to the new type. The transfer package does a good job here.

setup-lifting type-definition-trie — also sets up code generation

```

lift-definition empty :: ('k :: cbl, 'v) trie
is trie-empty
 $\langle proof \rangle$ 

```

```

lift-definition lookup :: ('k :: cbl, 'v) trie  $\Rightarrow$  'k  $\Rightarrow$  'v option
is  $\lambda$ t. trie-lookup t  $\circ$  to-bl  $\langle proof \rangle$ 

```

```

lift-definition update :: 'k  $\Rightarrow$  'v  $\Rightarrow$  ('k :: cbl, 'v) trie  $\Rightarrow$  ('k, 'v) trie
is trie-update  $\circ$  to-bl
 $\langle proof \rangle$ 

```

```

lift-definition keys :: ('k :: cbl, 'v) trie  $\Rightarrow$  'k set
is  $\lambda$ t. from-bl ' trie-keys t  $\langle proof \rangle$ 

```

And now we go for the properties. Note that some properties hold only if the type class operations are actually provided, i.e., *cbl* $\neq$ *None* in our example.

lemma lookup-empty: lookup empty = Map.empty

<proof>

context

fixes $t :: ('k :: \text{cbl}, 'v) \text{ trie}$

assumes ID-cbl: ID (cbl :: 'k cbl) $\neq \text{None}$

begin

lemma lookup-update: lookup (update k v t) = (lookup t)(k $\mapsto$ v)

<proof>

lemma keys-conv-dom-lookup: keys t = dom (lookup t)

<proof>

end

4.4.4 Connecting to the container

Connecting to the container (*'a, 'b mapping* in our example) takes three steps:

1. Define a new pseudo-constructor
2. Implement the container operations for the new type
3. Configure the heuristics to automatically pick an implementation
4. Test thoroughly

Thorough testing is particularly important, because Isabelle does not check whether you have implemented all your operations, whether you have configured your heuristics sensibly, nor whether your implementation always terminates.

Define a new pseudo-constructor

Define a function that returns the abstract container view for a data structure value, and declare it as a datatype constructor for code generation with **code-datatype**. Unfortunately, you have to repeat all existing pseudo-constructors, because there is no way to extract the current set of pseudo-constructors from the code generator. We call them pseudo-constructors, because they do not behave like datatype constructors in the logic. For example, ours are neither injective nor disjoint.

definition Trie-Mapping :: ('k :: cbl, 'v) trie $\Rightarrow$ ('k, 'v) mapping

where [simp, code del]: Trie-Mapping t = Mapping.Mapping (lookup t)

code-datatype Assoc-List-Mapping RBT-Mapping Mapping Trie-Mapping

Implement the operations

Next, you have to prove and declare code equations that implement the container operations for the new implementation. Typically, these just dispatch to the operations on the type from §4.4.3. Some operations depend on the type class operations from §4.4.2 being defined; then, the code equation must check that the operations are indeed defined. If not, there is usually no way to implement the operation, so the code should raise an exception. Logically, we use the function *Code.abort* of type *String.literal* $\Rightarrow$ (*unit* $\Rightarrow$ 'a') $\Rightarrow$ 'a' with definition $\lambda\text{-}f. f \text{ ()}$, but the generated code raises an exception **Fail** with the given message (the unit closure avoids non-termination in strict languages). This function gets the exception message and the unit-closure of the equation's left-hand side as argument, because it is then trivial to prove equality.

Again, we only show a small set of operations; a realistic implementation should cover as many as possible.

context **fixes** *t* :: ('k :: cbl, 'v) trie **begin**

lemma lookup-Trie-Mapping [code]:

Mapping.lookup (Trie-Mapping *t*) = lookup *t*

— Lookup does not need the check on *cbl*, because we have defined the pseudo-constructor *Trie-Mapping* in terms of *lookup*

<proof>

lemma update-Trie-Mapping [code]:

Mapping.update *k v* (Trie-Mapping *t*) =

(case ID *cbl* :: 'k cbl of

None $\Rightarrow$ Code.abort (STR "update Trie-Mapping: cbl = None") ($\lambda\text{-}$

Mapping.update *k v* (Trie-Mapping *t*))

| Some - $\Rightarrow$ Trie-Mapping (update *k v t*))

<proof>

lemma keys-Trie-Mapping [code]:

Mapping.keys (Trie-Mapping *t*) =

(case ID *cbl* :: 'k cbl of

None $\Rightarrow$ Code.abort (STR "keys Trie-Mapping: cbl = None") ($\lambda\text{-}$

Mapping.keys (Trie-Mapping *t*))

| Some - $\Rightarrow$ keys *t*)

<proof>

end

These equations do not replace the existing equations for the other constructors, but they do take precedence over them. If there is already a generic

implementation for an operation *foo*, say $foo\ A = gen\text{-}foo\ A$, and you prove a specialised equation $foo\ (Trie\text{-}Mapping\ t) = trie\text{-}foo\ t$, then when you call *foo* on some *Trie-Mapping t*, your equation will kick in. LC exploits this sequentiality especially for binary operators on sets like $(\cap)$, where there are generic implementations and faster specialised ones.

Configure the heuristics

Finally, you should setup the heuristics that automatically picks a container implementation based on the types of the elements (§4.3.3).

The heuristics uses a type with a single value, e.g., *mapping-impl* with value *Mapping-IMPL*, but there is one pseudo-constructor for each container implementation in the generated code. All these pseudo-constructors are the same in the logic, but they are different in the generated code. Hence, the generated code can distinguish them, but we do not have to commit to anything in the logic. This allows to reconfigure and extend the heuristic at any time.

First, define and declare a new pseudo-constructor for the heuristics. Again, be sure to redeclare all previous pseudo-constructors.

definition `mapping-Trie :: mapping-impl`
where `[simp]: mapping-Trie = Mapping-IMPL`

code-datatype

`mapping-Choose mapping-Assoc-List mapping-RBT mapping-Mapping mapping-Trie`

Then, adjust the implementation of the automatic choice. For every initial value of the container (such as the empty map or the empty set), there is one new constant (e.g., *mapping-empty-choose* and *set-empty-choose*) equivalent to it. Its code equation, however, checks the available operations from the type classes and picks an appropriate implementation.

For example, the following prefers red-black trees over tries, but tries over associative lists:

lemma `mapping-empty-choose-code [code]:`
`(mapping-empty-choose :: ('a :: {ccompare, cbl}, 'b) mapping) =`
`(case ID CCOMPARE('a) of Some - => RBT-Mapping RBT-Mapping2.empty`
`| None =>`
`case ID (cbl :: 'a cbl) of Some - => Trie-Mapping empty`
`| None => Assoc-List-Mapping DList.empty)`
`⟨proof⟩`

There is also a second function for every such initial value that dispatches on the pseudo-constructors for *mapping-impl*. This function is used to pick

the right implementation for types that specify a preference.

lemma mapping-empty-code [code]:

```
mapping-empty mapping-Trie = Trie-Mapping empty
⟨proof⟩
```

For (k, v) *mapping*, LC also has a function *mapping-impl-choose2* which is given two preferences and returns one (for *a set*, it is called *set-impl-choose2*). Polymorphic type constructors like $a + b$ use it to pick an implementation based on the preferences of *a* and *b*. By default, it returns *mapping-Choose*, i.e., ignore the preferences. You should add a code equation like the following that overrides this choice if both preferences are your new data structure:

lemma mapping-impl-choose2-Trie [code]:

```
mapping-impl-choose2 mapping-Trie mapping-Trie = mapping-Trie
⟨proof⟩
```

If your new data structure is better than the existing ones for some element type, you should reconfigure the type's preference. As all preferences are logically equal, you can prove (and declare) the appropriate code equation. For example, the following prefers tries for keys of type *unit*:

lemma mapping-impl-unit-Trie [code]:

```
MAPPING-IMPL(unit) = Phantom(unit) mapping-Trie
⟨proof⟩
```

value Mapping.empty :: (unit, int) mapping

You can also use your new pseudo-constructor with *derive* in instantiations, just give its name as option:

derive (mapping-Trie) mapping-impl simple-tycon

4.5 Changing the configuration

As containers are connected to data structures only by refinement in the code generator, this can always be adapted later on. You can add new data structures as explained in §4.4. If you want to drop one, you redeclare the remaining pseudo-constructors with **code-datatype** and delete all code equations that pattern-match on the obsolete pseudo-constructors. The command **code-thms** will tell you which constants have such code equations. You can also freely adapt the heuristics for picking implementations as described in §4.4.4.

One thing, however, you cannot change afterwards, namely the decision whether an element type supports an operation and if so how it does, because this decision is visible in the logic.

4.6 New containers types

We hope that the above explanations and the examples with sets and maps suffice to show what you need to do if you add a new container type, e.g., priority queues. There are three steps:

1. **Introduce a type constructor for the container.**

Your new container type must not be a composite type, like $'a \Rightarrow 'b$ *option* for maps, because refinement for code generation only works with a single type constructor. Neither should you reuse a type constructor that is used already in other contexts, e.g., do not use $'a$ *list* to model queues.

Introduce a new type constructor if necessary (e.g., $('a, 'b)$ *mapping* for maps) – if your container type already has its own type constructor, everything is fine.

2. **Implement the data structures**

and connect them to the container type as described in §4.4.

3. **Define a heuristics for picking an implementation.**

See [5] for an explanation.

4.7 Troubleshooting

This section describes some difficulties in using LC that we have come across, provides some background for them, and discusses how to overcome them. If you experience other difficulties, please contact the author.

4.7.1 Nesting of mappings

Mappings can be arbitrarily nested on the value side, e.g., $('a, ('b, 'c)$ *mapping*) *mapping*. However, $('a, 'b)$ *mapping* cannot currently be the key of a mapping, i.e., code generation fails for $((('a, 'b)$ *mapping*, $'c)$ *mapping*. Similarly, you cannot have a set of mappings like $('a, 'b)$ *mapping set* at the moment. There are no issues to make this work, we have just not seen the need for it. If you need to generate code for such types, please get in touch with the author.

4.7.2 Wellsortedness errors

LC uses its own hierarchy of type classes which is distinct from Isabelle/HOL's. This ensures that every type can be made an instance of LC's type classes.

Consequently, you must instantiate these classes for your own types. The following lists where you can find information about the classes and examples how to instantiate them:

type class	user guide	theory
<i>card-UNIV</i>	§4.3.5	<i>HOL–Library.Cardinality</i>
<i>cenum</i>	§4.3.4	<i>Containers.Collection-Enum</i>
<i>ceq</i>	§4.3.1	<i>Containers.Collection-Eq</i>
<i>compare</i>	§4.3.2	<i>Containers.Collection-Order</i>
<i>cproper-interval</i>	§4.3.5	<i>Containers.Collection-Order</i>
<i>finite-UNIV</i>	§4.3.5	<i>HOL–Library.Cardinality</i>
<i>mapping-impl</i>	§4.3.3	<i>Containers.Mapping-Impl</i>
<i>set-impl</i>	§4.3.3	<i>Containers.Set-Impl</i>

The type classes *card-UNIV* and *cproper-interval* are only required to implement the operations on set complements. If your code does not need complements, you can manually declare the code equations not involving *Complement* using the theorem collection *set-base-code*. It is also recommended that you remove the pseudo-constructor *Complement* from the code generator. Note that some set operations like $A - B$ and *UNIV* have no code equations any more.

```
code-datatype Collect-set DList-set RBT-set Set-Monad
declare set-base-code [code]
```

4.7.3 Exception raised at run-time

Not all combinations of data and container implementation are possible. For example, you cannot implement a set of functions with a RBT, because there is no order on $'a \Rightarrow 'b$. If you try, the code will raise an exception **Fail** (with an exception message) or **Match**. They can occur in three cases:

1. You have misconfigured the heuristics that picks implementations (§4.3.3), or you have manually picked an implementation that requires an operation that the element type does not provide. Printing a stack trace for the exception may help you in locating the error.
2. You are trying to invoke an operation on a set complement which cannot be implemented on a complement representation, e.g., $(\cdot)$. If the element type is enumerable, provide an instance of *cenum* and choose to represent complements of sets of enumerable types by the elements rather than the elements of the complement (see §4.3.4 for how to do this).
3. You use set comprehensions on types which do not provide an enumeration (i.e., they are represented as closures) or you chose to represent a map as a closure.

A lot of operations are not implementable for closures, in particular those that return some element of the container

Inspect the code equations with **code-thms** and look for calls to *Collect-set* and *Mapping* which are LC's constructor for sets and maps as closures.

Note that the code generator preprocesses set comprehensions like $\{i < 4 \mid i. 2 < i\}$ to $(\lambda i. i < 4) \text{ ' } \{i. 2 < i\}$, so this is a set comprehension over *int* rather than *bool*.

$\langle ML \rangle$

4.7.4 LC slows down my code

Normally, this will not happen, because LC's data structures are more efficient than Isabelle's list-based implementations. However, in some rare cases, you can experience a slowdown:

1. **Your containers contain just a few elements.**

In that case, the overhead of the heuristics to pick an implementation outweighs the benefits of efficient implementations. You should identify the tiny containers and disable the heuristics locally. You do so by replacing the initial value like $\{\}$ and *Mapping.empty* with low-overhead constructors like *Set-Monad* and *Mapping*. For example, if *tiny-set-code*: $\text{tiny-set} = \{1, 2\}$ is your code equation with a tiny set, the following changes the code equation to directly use the list-based representation, i.e., disables the heuristics:

lemma empty-Set-Monad: $\{\} = \text{Set-Monad } []$ $\langle \text{proof} \rangle$

declare tiny-set-code[code del, unfolded empty-Set-Monad, code]

If you want to globally disable the heuristics, you can also declare an equation like *empty-Set-Monad* as [code].

2. **The element type contains many type constructors and some type variables.**

LC heavily relies on type classes, and type classes are implemented as dictionaries if the compiler cannot statically resolve them, i.e., if there are type variables. For type constructors with type variables (like $'a \times 'b$), LC's definitions of the type class parameters recursively calls itself on the type variables, i.e., $'a$ and $'b$. If the element type is polymorphic, the compiler cannot precompute these recursive calls and therefore they have to be constructed repeatedly at run time. If you wrap your complicated type in a new type constructor, you can define optimised equations for the type class parameters.

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