Completeness of Decreasing Diagrams for the Least Uncountable Cardinality

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Abstract

In [8] it was formally proved that the decreasing diagrams method [7] is sound for proving confluence: if a binary relation r has LD property defined in [8], then it has CR property defined in [6].

In this formal theory it is proved that if the cardinality of r does not exceed the first uncountable cardinal, then r has CR property if and only if r has LD property. As a consequence, the decreasing diagrams method is complete for proving confluence of relations of the least uncountable cardinality.

A paper that describes details of this proof has been submitted to the FSCD 2025 conference. This formalization extends formalizations [1, 5, 4, 2] and the paper [3].

Contents

1 Preliminaries

1.1 Formal definition of finite levels of the DCR hierarchy

theory *Finite-DCR-Hierarchy*
  imports *Main*
begin

1.1.1 Auxiliary definitions

definition *confl-rel*
  where *confl-rel* $r \equiv (\forall\, a\ b\ c.\ (a,b) \in r\widehat{\ }{*} \wedge (a,c) \in r\widehat{\ }{*} \longrightarrow (\exists\ d.\ (b,d) \in r\widehat{\ }{*} \wedge (c,d) \in r\widehat{\ }{*})\ )$

definition *jn00* $::\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn00\ r0\ b\ c \equiv (\exists\ d.\ (b,d) \in r0\widehat{\ }{=} \wedge (c,d) \in r0\widehat{\ }{=})$

definition *jn01* $::\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn01\ r0\ r1\ b\ c \equiv (\exists\, b'\ d.\ (b,b') \in r1\widehat{\ }{=} \wedge (b',d) \in r0\widehat{\ }{*} \wedge (c,d) \in r0\widehat{\ }{*})$

definition *jn10* $::\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn10\ r0\ r1\ b\ c \equiv (\exists\, c'\ d.\ (b,d) \in r0\widehat{\ }{*} \wedge (c,c') \in r1\widehat{\ }{=} \wedge (c',d) \in r0\widehat{\ }{*})$

definition *jn11* $::\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn11\ r0\ r1\ b\ c \equiv (\exists\, b'\ b''\ c'\ c''\ d.\ \ (b,b') \in r0\widehat{\ }{*} \wedge (b',b'') \in r1\widehat{\ }{=} \wedge (b'',d) \in r0\widehat{\ }{*}$
$\wedge (c,c') \in r0\widehat{\ }{*} \wedge (c',c'') \in r1\widehat{\ }{=} \wedge (c'',d) \in r0\widehat{\ }{*})$

definition *jn02* $::\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn02\ r0\ r1\ r2\ b\ c \equiv (\exists\, b'\ d.\ \ (b,b') \in r2\widehat{\ }{=} \wedge (b',d) \in (r0 \cup r1)\widehat{\ }{*} \wedge (c,d) \in (r0 \cup r1)\widehat{\ }{*}\ )$

definition *jn12* $::\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn12\ r0\ r1\ r2\ b\ c \equiv (\exists\, b'\ b''\ d.\ \ (b,b') \in (r0)\widehat{\ }{*} \wedge (b',b'') \in r2\widehat{\ }{=} \wedge (b'',d) \in (r0 \cup r1)\widehat{\ }{*}$
$\wedge (c,d) \in (r0 \cup r1)\widehat{\ }{*})$

definition *jn22* $::\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn22\ r0\ r1\ r2\ b\ c \equiv (\exists\, b'\ b''\ c'\ c''\ d.\ \ (b,b') \in (r0 \cup r1)\widehat{\ }{*} \wedge (b',b'') \in r2\widehat{\ }{=} \wedge (b'',d) \in (r0 \cup r1)\widehat{\ }{*}$

$\wedge$ $(c,c') \in (r0 \cup r1)\hat{}*$ $\wedge$ $(c',c'') \in r2\hat{}=$ $\wedge$ (c'',d) $\in (r0 \cup r1)\hat{}*)$

definition *LD2* :: *'a rel* ⇒ *'a rel* ⇒ *'a rel* ⇒ *bool*
where
LD2 r r0 r1 ≡ ($r = r0 \cup r1$
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r0 \wedge (a,c) \in r0 \longrightarrow$ *jn00 r0 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r0 \wedge (a,c) \in r1 \longrightarrow$ *jn01 r0 r1 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r1 \wedge (a,c) \in r1 \longrightarrow$ *jn11 r0 r1 b c*))

definition *LD3* :: *'a rel* ⇒ *'a rel* ⇒ *'a rel* ⇒ *'a rel* ⇒ *bool*
where
LD3 r r0 r1 r2 ≡ ($r = r0 \cup r1 \cup r2$
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r0 \wedge (a,c) \in r0 \longrightarrow$ *jn00 r0 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r0 \wedge (a,c) \in r1 \longrightarrow$ *jn01 r0 r1 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r1 \wedge (a,c) \in r1 \longrightarrow$ *jn11 r0 r1 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r0 \wedge (a,c) \in r2 \longrightarrow$ *jn02 r0 r1 r2 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r1 \wedge (a,c) \in r2 \longrightarrow$ *jn12 r0 r1 r2 b c*)
$\wedge$ ($\forall$ *a b c*. $(a,b) \in r2 \wedge (a,c) \in r2 \longrightarrow$ *jn22 r0 r1 r2 b c*))

definition *DCR2* :: *'a rel* ⇒ *bool*
where
DCR2 r ≡ (∃ *r0 r1*. *LD2 r r0 r1*)

definition *DCR3* :: *'a rel* ⇒ *bool*
where
DCR3 r ≡ (∃ *r0 r1 r2*. *LD3 r r0 r1 r2*)

definition $\mathfrak{L}1$ :: (*nat* ⇒ *'U rel*) ⇒ *nat* ⇒ *'U rel*
where
$\mathfrak{L}1\ g\ \alpha \equiv \bigcup \{A.\ \exists\ \alpha'.\ (\alpha' < \alpha) \wedge A = g\ \alpha'\}$

definition $\mathfrak{L}v$ :: (*nat* ⇒ *'U rel*) ⇒ *nat* ⇒ *nat* ⇒ *'U rel*
where
$\mathfrak{L}v\ g\ \alpha\ \beta \equiv \bigcup \{A.\ \exists\ \alpha'.\ (\alpha' < \alpha \vee \alpha' < \beta) \wedge A = g\ \alpha'\}$

definition $\mathfrak{D}$ :: (*nat* ⇒ *'U rel*) ⇒ *nat* ⇒ *nat* ⇒ (*'U* × *'U* × *'U* × *'U*) *set*
where
$\mathfrak{D}\ g\ \alpha\ \beta = \{(b,b',b'',d).\ (b,b') \in (\mathfrak{L}1\ g\ \alpha)\hat{}* \wedge (b',b'') \in (g\ \beta)\hat{}= \wedge (b'',d) \in (\mathfrak{L}v\ g\ \alpha\ \beta)\hat{}*\}$

definition *DCR-generating* :: (*nat* ⇒ *'U rel*) ⇒ *bool*
where
DCR-generating g ≡ ($\forall$ $\alpha\ \beta$ *a b c*. $(a,b) \in (g\ \alpha) \wedge (a,c) \in (g\ \beta)$
$\longrightarrow$ ($\exists$ $b'\ b''\ c'\ c''\ d$. $(b,b',b'',d) \in (\mathfrak{D}\ g\ \alpha\ \beta) \wedge (c,c',c'',d) \in (\mathfrak{D}\ g\ \beta\ \alpha)$))

1.1.2 Result

The next definition formalizes the condition "an ARS with a reduction relation r belongs to the class DCR_n", where n is a natural number.

definition *DCR* :: *nat* ⇒ *'U rel* ⇒ *bool*
where
DCR n r ≡ (∃ *g*::(*nat* ⇒ *'U rel*). *DCR-generating g* ∧ *r* = ⋃ { *r'*. ∃ *α'*. *α'* < *n* ∧ *r'* = *g α'* })

end

1.2 Completeness of the DCR3 method for proving confluence of relations of the least uncountable cardinality

theory *DCR3-Method*
imports
HOL−Cardinals.Cardinals
Abstract−Rewriting.Abstract-Rewriting
Finite-DCR-Hierarchy
begin

1.2.1 Auxiliary definitions

abbreviation *ω-ord* **where** *ω-ord* ≡ *natLeq*

definition *sc-ord*::*'U rel* ⇒ *'U rel* ⇒ *bool*
where *sc-ord α α'* ≡ (*α* <*o α'* ∧ (∀ *β*::*'U rel*. *α* <*o β* ⟶ *α'* ≤*o β*))

definition *lm-ord*::*'U rel* ⇒ *bool*
where *lm-ord α* ≡ *Well-order α* ∧ ¬ (*α* = {} ∨ *isSuccOrd α*)

definition *nord* :: *'U rel* ⇒ *'U rel* **where** *nord α* = (*SOME α'*::*'U rel*. *α'* =*o α*)

definition *𝒪*::*'U rel set* **where** *𝒪* ≡ *nord* ‘ {*α*. *Well-order α*}

definition *oord*::*'U rel rel* **where** *oord* ≡ (*Restr ordLeq 𝒪*)

definition *CCR* :: *'U rel* ⇒ *bool*
where
CCR r = (∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. ∃ *c* ∈ *Field r*. (*a*,*c*) ∈ *r*^* ∧ (*b*,*c*) ∈ *r*^*)

definition *Conelike* :: *'U rel* ⇒ *bool*
where
Conelike r = (*r* = {} ∨ (∃ *m* ∈ *Field r*. ∀ *a* ∈ *Field r*. (*a*,*m*) ∈ *r*^*))

definition *dncl* :: *'U rel* ⇒ *'U set* ⇒ *'U set*
where
dncl r A = ((*r*^*)^−*1*)‘‘*A*

definition $Inv :: 'U\ rel \Rightarrow 'U\ set\ set$
where
$Inv\ r = \{\ A :: 'U\ set\ .\ r\ ``\ A \subseteq A\ \}$

definition $SF :: 'U\ rel \Rightarrow 'U\ set\ set$
where
$SF\ r = \{\ A :: 'U\ set.\ Field\ (Restr\ r\ A) = A\ \}$

definition $SCF{::}'U\ rel \Rightarrow ('U\ set)\ set$ **where**
$SCF\ r \equiv \{\ B{::}('U\ set)\ .\ B \subseteq Field\ r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in B.\ (a,b) \in r\hat{}*)\ \}$

definition $cfseq :: 'U\ rel \Rightarrow (nat \Rightarrow 'U) \Rightarrow bool$
where
$cfseq\ r\ xi \equiv ((\forall\ a \in Field\ r.\ \exists\ i.\ (a,\ xi\ i) \in r\hat{}*) \wedge (\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r))$

definition $rpth :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow nat \Rightarrow (nat \Rightarrow 'U)\ set$
where
$rpth\ r\ a\ b\ n \equiv \{\ f{::}(nat \Rightarrow 'U).\ f\ 0 = a \wedge f\ n = b \wedge (\forall\ i{<}n.\ (f\ i,\ f(Suc\ i)) \in r)$
$\}$

definition $\mathcal{F} :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow 'U\ set\ set$
where
$\mathcal{F}\ r\ a\ b \equiv \{\ F{::}'U\ set.\ \exists\ n{::}nat.\ \exists\ f \in rpth\ r\ a\ b\ n.\ F = f`\{i.\ i{\le}n\}\ \}$

definition $\mathfrak{f} :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow 'U\ set$
where
$\mathfrak{f}\ r\ a\ b \equiv (if\ (\mathcal{F}\ r\ a\ b \neq \{\})\ then\ (SOME\ F.\ F \in \mathcal{F}\ r\ a\ b)\ else\ \{\})$

definition $dnEsc ::\ 'U\ rel \Rightarrow 'U\ set \Rightarrow 'U \Rightarrow 'U\ set\ set$
where
$dnEsc\ r\ A\ a \equiv \{\ F.\ \exists\ b.\ ((b \notin dncl\ r\ A) \wedge (F \in \mathcal{F}\ r\ a\ b) \wedge (F \cap A = \{\}))\ \}$

definition $dnesc ::\ 'U\ rel \Rightarrow 'U\ set \Rightarrow 'U \Rightarrow 'U\ set$
where
$dnesc\ r\ A\ a = (if\ (dnEsc\ r\ A\ a \neq \{\})\ then\ (SOME\ F.\ F \in dnEsc\ r\ A\ a)\ else\ \{$
$a\ \})$

definition $escl ::\ 'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow 'U\ set$
where
$escl\ r\ A\ B = \bigcup\ ((dnesc\ r\ A)\ `\ B)$

definition $clterm$ **where** $clterm\ s'\ r \equiv (Conelike\ s' \longrightarrow Conelike\ r)$

definition $spthlen{::}'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow nat$
where
$spthlen\ r\ a\ b \equiv (LEAST\ n{::}nat.\ (a,b) \in r\hat{}\hat{}n)$

definition $spth :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow (nat \Rightarrow 'U)\ set$
where

$spth\ r\ a\ b = rpth\ r\ a\ b\ (spthlen\ r\ a\ b)$

definition $\mathfrak{U}$::$'U\ rel \Rightarrow ('U\ rel)\ set$ **where**
$\mathfrak{U}\ r \equiv \{\ s::('U\ rel)\ .\ CCR\ s \wedge s \subseteq r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in Field\ s.\ (a,b) \in r\hat{\ }*)\ \}$

definition $RCC\text{-}rel$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow bool$ **where**
$RCC\text{-}rel\ r\ \alpha \equiv (\mathfrak{U}\ r = \{\} \wedge \alpha = \{\}) \vee (\exists\ s \in \mathfrak{U}\ r.\ |s| =o\ \alpha \wedge (\ \forall\ s' \in \mathfrak{U}\ r.\ |s| \leq o\ |s'|\ ))$

definition RCC :: $'U\ rel \Rightarrow 'U\ rel\ (\|\text{-}\|)$
where $\|r\| \equiv (SOME\ \alpha.\ RCC\text{-}rel\ r\ \alpha)$

definition Den::$'U\ rel \Rightarrow ('U\ set)\ set$ **where**
$Den\ r \equiv \{\ B::('U\ set)\ .\ B \subseteq Field\ r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in B.\ (a,b) \in r\hat{\ }=)\ \}$

definition $Span$::$'U\ rel \Rightarrow ('U\ rel)\ set$ **where**
$Span\ r \equiv \{\ s.\ s \subseteq r \wedge Field\ s = Field\ r\ \}$

definition $scf\text{-}rel$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow bool$ **where**
$scf\text{-}rel\ r\ \alpha \equiv (\exists\ B \in SCF\ r.\ |B| =o\ \alpha \wedge (\ \forall\ B' \in SCF\ r.\ |B| \leq o\ |B'|\ ))$

definition scf :: $'U\ rel \Rightarrow 'U\ rel$
where $scf\ r \equiv (SOME\ \alpha.\ scf\text{-}rel\ r\ \alpha)$

definition $w\text{-}dncl$ :: $'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$
where
$w\text{-}dncl\ r\ A = \{\ a \in dncl\ r\ A.\ \forall\ b.\ \forall\ F \in \mathcal{F}\ r\ a\ b.\ (\ b \notin dncl\ r\ A \longrightarrow F \cap A \neq \{\}\ )\ \}$

definition $\mathfrak{L}$ :: $('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$\mathfrak{L}\ f\ \alpha \equiv \bigcup\ \{A.\ \exists\ \alpha'.\ \alpha' <o\ \alpha \wedge A = f\ \alpha'\}$

definition Dbk :: $('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set\ (\ \nabla\ \text{-}\ \text{-}\ )$
where
$\nabla\ f\ \alpha \equiv f\ \alpha - (\mathfrak{L}\ f\ \alpha)$

definition $\mathcal{Q}$ :: $'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$\mathcal{Q}\ r\ f\ \alpha \equiv (f\ \alpha - (dncl\ r\ (\mathfrak{L}\ f\ \alpha)))$

definition $\mathcal{W}$ :: $'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$\mathcal{W}\ r\ f\ \alpha \equiv (f\ \alpha - (w\text{-}dncl\ r\ (\mathfrak{L}\ f\ \alpha)))$

definition $\mathcal{N}1$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}1\ r\ \alpha0 \equiv \{\ f\ .\ \forall \alpha\ \alpha'.\ (\ \alpha \leq o\ \alpha0 \wedge \alpha' \leq o\ \alpha\ ) \longrightarrow (f\ \alpha') \subseteq (f\ \alpha)\ \}$

definition $\mathcal{N}2$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}2\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge \neg\ (\alpha = \{\} \vee isSuccOrd\ \alpha)\ ) \longrightarrow (\nabla\ f\ \alpha) = \{\}\ \}$

definition $\mathcal{N}3$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}3\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge (\alpha = \{\} \vee isSuccOrd\ \alpha)\ ) \longrightarrow$
$(\ \omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha| \longrightarrow ((escl\ r\ (\mathfrak{L}\ f\ \alpha)\ (f\ \alpha) \subseteq (f\ \alpha)) \wedge (clterm\ (Restr\ r\ (f\ \alpha))\ r))\ )\ \}$

definition $\mathcal{N}4$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}4\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge (\alpha = \{\} \vee isSuccOrd\ \alpha)\ ) \longrightarrow$
$(\ \forall\, a \in (\mathfrak{L}\ f\ \alpha).\ (\ r``\{a\} \subseteq w\text{-}dncl\ r\ (\mathfrak{L}\ f\ \alpha)\ ) \vee (\ r``\{a\} \cap (\mathcal{W}\ r\ f\ \alpha) \neq \{\}\ )\ )\ \}$

definition $\mathcal{N}5$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}5\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow (f\ \alpha) \in SF\ r\ \}$

definition $\mathcal{N}6$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}6\ r\ \alpha 0 \equiv \{\ f.\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow CCR\ (Restr\ r\ (f\ \alpha))\ \}$

definition $\mathcal{N}7$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}7\ r\ \alpha 0 \equiv \{\ f.\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow (\ \alpha < o\ \omega\text{-}ord \longrightarrow |f\ \alpha| < o\ \omega\text{-}ord\ ) \wedge (\omega\text{-}ord \leq o\ \alpha \longrightarrow |f\ \alpha| \leq o\ \alpha)\ \}$

definition $\mathcal{N}8$ :: $'U\ rel \Rightarrow 'U\ set\ set \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}8\ r\ Ps\ \alpha 0 \equiv \{\ f.\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \wedge (\alpha = \{\} \vee isSuccOrd\ \alpha) \wedge (\ (\exists\ P.\ Ps = \{P\}) \vee (\neg\ finite\ Ps \wedge |Ps| \leq o\ |f\ \alpha|\ )) \longrightarrow$
$(\forall\ P \in Ps.\ ((f\ \alpha) \cap P) \in SCF\ (Restr\ r\ (f\ \alpha)))\ \}$

definition $\mathcal{N}9$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}9\ r\ \alpha 0 \equiv \{\ f\ .\ \omega\text{-}ord \leq o\ \alpha 0 \longrightarrow Field\ r \subseteq (f\ \alpha 0)\ \}$

definition $\mathcal{N}10$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}10\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow ((\exists\ y{::}'U.\ \mathcal{Q}\ r\ f\ \alpha = \{y\}) \longrightarrow (Field\ r \subseteq dncl\ r\ (f\ \alpha)))\ \}$

definition $\mathcal{N}11$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}11\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge isSuccOrd\ \alpha) \longrightarrow \mathcal{Q}\ r\ f\ \alpha = \{\} \longrightarrow (Field$

$r \subseteq dncl\ r\ (f\ \alpha))\ \}$

definition $\mathcal{N}12$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}12\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ \alpha \le o\ \alpha 0 \longrightarrow \omega\text{-}ord \le o\ \alpha \longrightarrow \omega\text{-}ord \le o\ |\mathfrak{L}\ f\ \alpha|\ \}$

definition $\mathcal{N}$ :: $'U\ rel \Rightarrow 'U\ set\ set \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}\ r\ Ps \equiv \{\ f \in (\mathcal{N}1\ r\ |Field\ r|\ ) \cap (\mathcal{N}2\ r\ |Field\ r|\ ) \cap (\mathcal{N}3\ r\ |Field\ r|\ ) \cap (\mathcal{N}4\ r\ |Field\ r|\ )$
$\cap\ (\mathcal{N}5\ r\ |Field\ r|\ ) \cap (\mathcal{N}6\ r\ |Field\ r|\ ) \cap (\mathcal{N}7\ r\ |Field\ r|\ ) \cap (\mathcal{N}8\ r\ Ps\ |Field\ r|\ )$
$\cap\ (\mathcal{N}9\ r\ |Field\ r| \cap \mathcal{N}10\ r\ |Field\ r| \cap \mathcal{N}11\ r\ |Field\ r| \cap \mathcal{N}12\ r\ |Field\ r|\ ).$
$(\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow f\ \alpha = f\ \beta)\ \}$

definition $\mathcal{T}$ :: $('U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set) \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{T}\ F \equiv \{\ f::'U\ rel \Rightarrow 'U\ set\ .$
$f\ \{\} = \{\}$
$\wedge\ (\forall\ \alpha 0\ \alpha::'U\ rel.\ (sc\text{-}ord\ \alpha 0\ \alpha \longrightarrow f\ \alpha = F\ \alpha 0\ (f\ \alpha 0)))$
$\wedge\ (\forall\ \alpha.\ (lm\text{-}ord\ \alpha \longrightarrow f\ \alpha = \bigcup\ \{\ D.\ \exists\ \beta.\ \beta <o\ \alpha \wedge D = f\ \beta\ \}))$
$\wedge\ (\forall \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow f\ \alpha = f\ \beta)\ \}$

definition $\mathcal{E}p$ **where** $\mathcal{E}p\ r\ Ps\ A\ A' \equiv$
$(((\exists\ P.\ Ps = \{P\}) \vee ((\neg\ finite\ Ps) \wedge |Ps| \le o\ |A|\ ))$
$\longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A')\ ))$

definition $\mathcal{E}$ :: $'U\ rel \Rightarrow 'U \Rightarrow 'U\ set \Rightarrow 'U\ set\ set \Rightarrow 'U\ set\ set$
where
$\mathcal{E}\ r\ a\ A\ Ps \equiv \{\ A'.$
$(a \in Field\ r \longrightarrow a \in A') \wedge A \subseteq A'$
$\wedge\ (\ |A| <o\ \omega\text{-}ord \longrightarrow |A'| <o\ \omega\text{-}ord\ ) \wedge (\ \omega\text{-}ord \le o\ |A| \longrightarrow |A'| \le o\ |A|\ )$
$\wedge\ (A \in SF\ r \longrightarrow ($
$A' \in SF\ r$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ (\ \forall\ a{\in}A.\ (r``\{a\} \subseteq w\text{-}dncl\ r\ A) \vee (r``\{a\} \cap (A' - w\text{-}dncl\ r\ A) \neq \{\})$
$)$
$\wedge\ ((\exists\ y.\ A' - dncl\ r\ A \subseteq \{y\}) \longrightarrow (Field\ r \subseteq (dncl\ r\ A')))$
$\wedge\ \mathcal{E}p\ r\ Ps\ A\ A'$
$\wedge\ (\ \omega\text{-}ord \le o\ |A| \longrightarrow escl\ r\ A\ A' \subseteq A' \wedge clterm\ (Restr\ r\ A')\ r))\ )\ \}$

definition $wbase$::$'U\ rel \Rightarrow 'U\ set \Rightarrow ('U\ set)\ set$ **where**
$wbase\ r\ A \equiv \{\ B::'U\ set.\ A \subseteq w\text{-}dncl\ r\ B\ \}$

definition $wrank\text{-}rel$ :: $'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ rel \Rightarrow bool$ **where**
$wrank\text{-}rel\ r\ A\ \alpha \equiv (\exists\ B \in wbase\ r\ A.\ |B| =o\ \alpha \wedge (\ \forall\ B' \in wbase\ r\ A.\ |B| \le o\ |B'|\ ))$

definition $wrank$ :: $'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ rel$

where *wrank r A* ≡ (*SOME* α. *wrank-rel r A* α)

definition *Mwn* :: $'U$ *rel* ⇒ $'U$ *rel* ⇒ $'U$ *set*
where
Mwn r α = { *a* ∈ *Field r*. α <*o wrank r* (*r* ‘‘{*a*}) }

definition *Mwnm* :: $'U$ *rel* ⇒ $'U$ *set*
where
Mwnm r = { *a* ∈ *Field r*. ‖*r*‖ ≤*o wrank r* (*r* ‘‘{*a*}) }

definition *wesc-rel* :: $'U$ *rel* ⇒ ($'U$ *rel* ⇒ $'U$ *set*) ⇒ $'U$ *rel* ⇒ $'U$ ⇒ $'U$ ⇒ *bool*
where
wesc-rel r f α *a b* ≡ (*b* ∈ $\mathcal{W}$ *r f* α ∧ (*a*,*b*) ∈ (*Restr r* ($\mathcal{W}$ *r f* α))^*
∧ (∀β. α <*o* β ∧ β <*o* |*Field r*| ∧ (β = {} ∨ *isSuccOrd* β) ⟶ (*r*‘‘{*b*} ∩ ($\mathcal{W}$ *r f* β) ≠ {})))

definition *wesc* :: $'U$ *rel* ⇒ ($'U$ *rel* ⇒ $'U$ *set*) ⇒ $'U$ *rel* ⇒ $'U$ ⇒ $'U$
where
wesc r f α *a* ≡ (*SOME b*. *wesc-rel r f* α *a b*)

definition *cardLeN1*::$'a$ *set* ⇒ *bool*
where
cardLeN1 A ≡ (∀ *B* ⊆ *A*.
(∀ *C* ⊆ *B* . ((∃ *D f*. *D* ⊂ *C* ∧ *C* ⊆ *f*‘*D*) ⟶ (∃ *f*. *B* ⊆ *f*‘*C*)))
∨ (∃ *g* . *A* ⊆ *g*‘*B*))

1.2.2 Auxiliary lemmas

lemma *lem-Ldo-ldogen-ord*:
assumes ∀α β *a b c*. α ≤ β ⟶ (*a*, *b*) ∈ *g* α ∧ (*a*, *c*) ∈ *g* β ⟶
(∃ b' b'' c' c'' *d*. (*b*, b', b'', *d*) ∈ $\mathfrak{D}$ *g* α β ∧ (*c*, c', c'', *d*) ∈ $\mathfrak{D}$ *g* β α)
shows *DCR-generating g*
⟨*proof*⟩

lemma *lem-rtr-field*: (*x*,*y*) ∈ *r*^* ⟹ (*x* = *y*) ∨ (*x* ∈ *Field r* ∧ *y* ∈ *Field r*)
⟨*proof*⟩

lemma *lem-fin-fl-rel*: *finite* (*Field r*) = *finite r*
⟨*proof*⟩

lemma *lem-Relprop-fld-sat*:
fixes *r s*::$'U$ *rel*
assumes *a1*: *s* ⊆ *r* **and** *a2*: s' = *Restr r* (*Field s*)
shows *s* ⊆ s' ∧ *Field* s' = *Field s*
⟨*proof*⟩

lemma *lem-Relprop-sat-un*:
fixes *r*::$'U$ *rel* **and** *S*::$'U$ *set set* **and** A'::$'U$ *set*
assumes *a1*: ∀ *A*∈*S*. *Field* (*Restr r A*) = *A* **and** *a2*: A' = ⋃ *S*

shows *Field* (*Restr r A'*) = *A'*
⟨*proof*⟩

lemma *lem-nord-r*: *Well-order* $\alpha \Longrightarrow$ *nord* α =*o* α ⟨*proof*⟩

lemma *lem-nord-l*: *Well-order* $\alpha \Longrightarrow \alpha$ =*o* *nord* α ⟨*proof*⟩

lemma *lem-nord-eq*: α =*o* $\beta \Longrightarrow$ *nord* α = *nord* β ⟨*proof*⟩

lemma *lem-nord-req*: *Well-order* $\alpha \Longrightarrow$ *Well-order* $\beta \Longrightarrow$ *nord* α = *nord* $\beta \Longrightarrow \alpha$ =*o* β
⟨*proof*⟩

lemma *lem-Onord*: $\alpha \in \mathcal{O} \Longrightarrow \alpha$ = *nord* α ⟨*proof*⟩

lemma *lem-Oeq*: $\alpha \in \mathcal{O} \Longrightarrow \beta \in \mathcal{O} \Longrightarrow \alpha$ =*o* $\beta \Longrightarrow \alpha = \beta$ ⟨*proof*⟩

lemma *lem-Owo*: $\alpha \in \mathcal{O} \Longrightarrow$ *Well-order* α ⟨*proof*⟩

lemma *lem-fld-oord*: *Field oord* = $\mathcal{O}$ ⟨*proof*⟩

lemma *lem-nord-less*: α <*o* $\beta \Longrightarrow$ *nord* $\beta \neq$ *nord* $\alpha \land$ (*nord* α, *nord* β) $\in$ *oord*
⟨*proof*⟩

lemma *lem-nord-ls*: α <*o* $\beta \Longrightarrow$ *nord* α <*o* *nord* β
⟨*proof*⟩

lemma *lem-nord-le*: $\alpha \leq$*o* $\beta \Longrightarrow$ *nord* $\alpha \leq$*o* *nord* β
⟨*proof*⟩

lemma *lem-nordO-ls-l*: α <*o* $\beta \Longrightarrow$ *nord* $\alpha \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nordO-ls-r*: α <*o* $\beta \Longrightarrow$ *nord* $\beta \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nordO-le-l*: $\alpha \leq$*o* $\beta \Longrightarrow$ *nord* $\alpha \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nordO-le-r*: $\alpha \leq$*o* $\beta \Longrightarrow$ *nord* $\beta \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nord-ls-r*: α <*o* $\beta \Longrightarrow \alpha$ <*o* *nord* β
⟨*proof*⟩

lemma *lem-nord-ls-l*: α <*o* $\beta \Longrightarrow$ *nord* α <*o* β
⟨*proof*⟩

lemma *lem-nord-le-r*: $\alpha \leq$*o* $\beta \Longrightarrow \alpha \leq$*o* *nord* β
⟨*proof*⟩

lemma *lem-nord-le-l*: $\alpha \leq$*o* $\beta \Longrightarrow$ *nord* $\alpha \leq$*o* β
⟨*proof*⟩

lemma *lem-oord-wo*: *Well-order oord*
⟨*proof*⟩

lemma *lem-lmord-inf*:
fixes α::$'U$ *rel*
assumes *lm-ord* α
shows $\neg$ *finite* (*Field* α)
⟨*proof*⟩

lemma *lem-sucord-ex*:
fixes α β::$'U$ *rel*
assumes α $<o$ β
shows $\exists$ α'::$'U$ *rel*. *sc-ord* α α'
⟨*proof*⟩

lemma *lem-osucc-eq*: *isSuccOrd* $\alpha \Longrightarrow \alpha =o \beta \Longrightarrow$ *isSuccOrd* β
⟨*proof*⟩

lemma *lem-ord-subemp*: (α::$'a$ *rel*) $\leq o$ ({}::$'b$ *rel*) $\Longrightarrow \alpha = \{\}$
⟨*proof*⟩

lemma *lem-ordint-sucord*:
fixes $\alpha 0$::$'a$ *rel* **and** α::$'b$ *rel*
assumes $\alpha 0$ $<o$ α $\wedge$ ($\forall$ γ::$'b$ *rel*. $\alpha 0$ $<o$ $\gamma \longrightarrow \alpha \leq o$ γ)
shows *isSuccOrd* α
⟨*proof*⟩

lemma *lem-sucord-ordint*:
fixes α::$'U$ *rel*
assumes *Well-order* α $\wedge$ *isSuccOrd* α
shows $\exists$ $\alpha 0$::$'U$ *rel*. $\alpha 0$ $<o$ α $\wedge$ ($\forall$ γ::$'U$ *rel*. $\alpha 0$ $<o$ $\gamma \longrightarrow \alpha \leq o$ γ)
⟨*proof*⟩

lemma *lem-sclm-ordind*:
fixes P::$'U$ *rel* $\Rightarrow$ *bool*
assumes *a1*: P {}
  and *a2*: $\forall$ $\alpha 0$ α::$'U$ *rel*. (*sc-ord* $\alpha 0$ α $\wedge$ P $\alpha 0$ $\longrightarrow$ P α)
  and *a3*: $\forall$ α. ((*lm-ord* α $\wedge$ ($\forall$ β. β $<o$ $\alpha \longrightarrow P$ β)) $\longrightarrow P$ α)
shows $\forall$ α. *Well-order* $\alpha \longrightarrow P$ α
⟨*proof*⟩

lemma *lem-ordseq-rec-sets*:
fixes E::$'U$ *set* **and** F::$'U$ *rel* $\Rightarrow$ $'U$ *set* $\Rightarrow$ $'U$ *set*
assumes $\forall$ α β. $\alpha =o \beta \longrightarrow F$ $\alpha = F$ β
shows $\exists$ f::($'U$ *rel* $\Rightarrow$ $'U$ *set*).
    f {} = E
  $\wedge$ ($\forall$ $\alpha 0$ α::$'U$ *rel*. (*sc-ord* $\alpha 0$ $\alpha \longrightarrow f$ $\alpha = F$ $\alpha 0$ (f $\alpha 0$)))
  $\wedge$ ($\forall$ α. *lm-ord* $\alpha \longrightarrow f$ $\alpha = \bigcup$ { D. $\exists$ β. β $<o$ α $\wedge$ $D = f$ β })

$\wedge$ ($\forall$ α β. $\alpha =o$ $\beta \longrightarrow f$ $\alpha = f$ β)
⟨*proof*⟩

lemma *lem-lmord-prec*:
fixes α::'*a rel* **and** α'::'*b rel*
assumes *a1*: $\alpha' <o$ α **and** *a2*: *isLimOrd* α
shows $\exists$ β::('*a rel*). $\alpha' <o$ $\beta \wedge \beta <o$ α
⟨*proof*⟩

lemma *lem-inford-ge-w*:
fixes α::'*U rel*
assumes *Well-order* α **and** $\neg$ *finite* (*Field* α)
shows ω-*ord* $\leq o$ α
⟨*proof*⟩

lemma *lem-ge-w-inford*:
fixes α::'*U rel*
assumes ω-*ord* $\leq o$ α
shows $\neg$ *finite* (*Field* α)
⟨*proof*⟩

lemma *lem-fin-card*: *finite* $|A|$ = *finite* A
⟨*proof*⟩

lemma *lem-cardord-emp*: *Card-order* ({}::'*U rel*)
⟨*proof*⟩

lemma *lem-card-emprel*: |{}::'*U rel*| $=o$ ({}::'*U rel*)
⟨*proof*⟩

lemma *lem-cord-lin*: *Card-order* $\alpha \Longrightarrow$ *Card-order* $\beta \Longrightarrow$ ($\alpha \leq o$ β) = ($\neg$ ($\beta <o$ α)) ⟨*proof*⟩

lemma *lem-co-one-ne-min*:
fixes α::'*U rel* **and** a::'*a*
assumes *Well-order* α **and** $\alpha \neq \{\}$
shows $|\{a\}| \leq o$ α
⟨*proof*⟩

lemma *lem-rel-inf-fld-card*:
fixes r::'*U rel*
assumes $\neg$ *finite* r
shows |*Field* r| $=o$ $|r|$
⟨*proof*⟩

lemma *lem-cardreleq-cardfldeq-inf*:
fixes *r1 r2*:: '*U rel*
assumes *a1*: |*r1*| $=o$ |*r2*| **and** *a2*: $\neg$ *finite r1* $\vee$ $\neg$ *finite r2*
shows |*Field r1*| $=o$ |*Field r2*|

⟨*proof*⟩

lemma *lem-card-un-bnd*:
fixes S::$'a$ *set set* **and** α::$'U$ *rel*
assumes *a3*: $\forall A \in S.\ |A| \leq o\ \alpha$ **and** *a4*: $|S| \leq o\ \alpha$ **and** *a5*: ω-*ord* $\leq o\ \alpha$
shows $| \bigcup\ S\ | \leq o\ \alpha$
⟨*proof*⟩

lemma *lem-ord-suc-ge-w*:
fixes $\alpha 0\ \alpha$::$'U$ *rel*
assumes *a1*: ω-*ord* $\leq o\ \alpha$ **and** *a2*: *sc-ord* $\alpha 0\ \alpha$
shows ω-*ord* $\leq o\ \alpha 0$
⟨*proof*⟩

lemma *lem-restr-ordbnd*:
fixes r::$'U$ *rel* **and** A::$'U$ *set* **and** α::$'U$ *rel*
assumes *a1*: ω-*ord* $\leq o\ \alpha$ **and** *a2*: $|A| \leq o\ \alpha$
shows $|\mathit{Restr}\ r\ A| \leq o\ \alpha$
⟨*proof*⟩

lemma *lem-card-inf-lim*:
fixes r::$'U$ *rel*
assumes *a1*: *Card-order* α **and** *a2*: ω-*ord* $\leq o\ \alpha$
shows $\neg(\ \alpha = \{\} \vee \mathit{isSuccOrd}\ \alpha\ )$
⟨*proof*⟩

lemma *lem-card-nreg-inf-osetlm*:
fixes α::$'U$ *rel*
assumes *a1*: *Card-order* α **and** *a2*: $\neg$ *regularCard* α **and** *a3*: $\neg$ *finite* (*Field* α)
shows $\exists\ S$::$'U$ *rel set*. $|S| < o\ \alpha \wedge (\forall\ \alpha' \in S.\ \alpha' < o\ \alpha) \wedge (\forall\ \alpha'$::$'U$ *rel*. $\alpha' < o\ \alpha$
$\longrightarrow (\exists\ \beta \in S.\ \alpha' \leq o\ \beta))$
⟨*proof*⟩

lemma *lem-card-un-bnd-stab*:
fixes S::$'a$ *set set* **and** α::$'U$ *rel*
assumes *stable* α **and** $\forall A \in S.\ |A| < o\ \alpha$ **and** $|S| < o\ \alpha$
shows $| \bigcup\ S\ | < o\ \alpha$
⟨*proof*⟩

lemma *lem-finwo-cardord*: *finite* $\alpha \Longrightarrow$ *Well-order* $\alpha \Longrightarrow$ *Card-order* α
⟨*proof*⟩

lemma *lem-finwo-le-w*: *finite* $\alpha \Longrightarrow$ *Well-order* $\alpha \Longrightarrow \alpha < o$ *natLeq*
⟨*proof*⟩

lemma *lem-wolew-fin*: $\alpha < o$ *natLeq* $\Longrightarrow$ *finite* α
⟨*proof*⟩

lemma *lem-wolew-nat*:

assumes $a1$: $\alpha <o\ natLeq$ **and** $a2$: $n = card\ (Field\ \alpha)$
shows $\alpha =o\ (natLeq\text{-}on\ n)$
$\langle proof \rangle$

lemma *lem-cntset-enum*: $|A| =o\ natLeq \Longrightarrow (\exists\ f.\ A = f\ `\ (UNIV::nat\ set))$
$\langle proof \rangle$

lemma *lem-oord-int-card-le-inf*:
fixes $\alpha::'U\ rel$
assumes $\omega\text{-}ord \leq o\ \alpha$
shows $|\{\ \gamma \in \mathcal{O}::'U\ rel\ set.\ \gamma <o\ \alpha\ \}| \leq o\ \alpha$
$\langle proof \rangle$

lemma *lem-oord-card-le-int-inf*:
fixes $\alpha::'U\ rel$
assumes $a1$: $Card\text{-}order\ \alpha$ **and** $a2$: $\omega\text{-}ord \leq o\ \alpha$
shows $\alpha \leq o\ |\{\ \gamma \in \mathcal{O}::'U\ rel\ set.\ \gamma <o\ \alpha\ \}|$
$\langle proof \rangle$

lemma *lem-ord-int-card-le-inf*:
fixes $\alpha::'U\ rel$ **and** $f :: 'U\ rel \Rightarrow 'a$
assumes $\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow f\ \alpha = f\ \beta$ **and** $\omega\text{-}ord \leq o\ \alpha$
shows $|f\ `\ \{\ \gamma::'U\ rel.\ \gamma <o\ \alpha\ \}| \leq o\ \alpha$
$\langle proof \rangle$

lemma *lem-card-setcv-inf-stab*:
fixes $\alpha::'U\ rel$ **and** $A::'U\ set$
assumes $a1$: $Card\text{-}order\ \alpha$ **and** $a2$: $\omega\text{-}ord \leq o\ \alpha$ **and** $a3$: $|A| \leq o\ \alpha$
shows $\exists\ f::('U\ rel \Rightarrow 'U).\ A \subseteq f\ `\{\ \gamma::'U\ rel.\ \gamma <o\ \alpha\ \} \wedge (\forall\ \gamma 1\ \gamma 2.\ \gamma 1 =o\ \gamma 2$
$\longrightarrow f\ \gamma 1 = f\ \gamma 2)$
$\langle proof \rangle$

lemma *lem-jnfix-gen*:
fixes $I::'i\ set$ **and** $leI::'i\ rel$ **and** $L::'l\ set$
and $t::'i \times 'l \Rightarrow 'i \Rightarrow 'n$ **and** $jnN::'n \Rightarrow 'n \Rightarrow 'n$
assumes $a1$:$\neg\ finite\ L$
and $a2$: $|L| <o\ |I|$
and $a3$: $\forall \alpha \in I.\ (\alpha,\alpha) \in leI$
and $a4$: $\forall \alpha \in I.\ \forall \beta \in I.\ \forall \gamma \in I.\ (\alpha,\beta) \in leI \wedge (\beta,\gamma) \in leI \longrightarrow (\alpha,\gamma) \in leI$
and $a5$: $\forall \alpha \in I.\ \forall \beta \in I.\ (\alpha,\beta) \in leI \vee (\beta,\alpha) \in leI$
and $a6$: $\forall \beta \in I.\ |\{\alpha \in I.\ (\alpha,\beta) \in leI\}| \leq o\ |L|$
and $a7$: $\forall \alpha \in I.\ \exists \alpha' \in I.\ (\alpha,\alpha') \in leI \wedge (\alpha',\alpha) \notin leI$
shows $\exists\ h.\ \forall \alpha \in I.\ \forall \beta \in I.\ \forall i \in L.\ \forall j \in L.\ \exists\ \gamma \in I.\ (\alpha,\gamma) \in leI \wedge (\beta,\gamma) \in leI \wedge (\gamma,\alpha) \notin leI$
$\wedge\ (\gamma,\beta) \notin leI$
$\wedge\ h\ \gamma = jnN\ (t\ (\alpha,i)\ \gamma)\ (t\ (\beta,j)\ \gamma)$
$\langle proof \rangle$

lemma *lem-jnfix-card*:
fixes $\kappa::'U\ rel$ **and** $L::'l\ set$ **and** $t::('U\ rel) \times 'l \Rightarrow 'U\ rel \Rightarrow 'n$ **and** $jnN::'n \Rightarrow 'n$

$\Rightarrow$ $'n$
  and S::$'U$ *rel set*
assumes *a1*: *Card-order* κ **and** *a2*: $\neg$ *finite* L **and** *a3*: $|L| <o\ \kappa$
  and *a4*: $\forall\ \alpha \in S.\ |Field\ \alpha| \leq o\ |L|$
  and *a5*: $S \subseteq \mathcal{O}$ **and** *a6*: $|\{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}| \leq o\ |S|$
  and *a7*: $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$
shows $\exists\ h.\ \forall\ \alpha \in S.\ \forall\ \beta \in S.\ \forall i{\in}L.\ \forall j{\in}L.$
  $(\exists\ \gamma \in S.\ \alpha <o\ \gamma \wedge \beta <o\ \gamma \wedge h\ \gamma = jnN\ (t\ (\alpha,i)\ \gamma)\ (t\ (\beta,j)\ \gamma)\ )$
⟨*proof*⟩

lemma *lem-cardsuc-ls-fldcard*:
fixes κ::$'a$ *rel* **and** α::$'b$ *rel*
assumes *a1*: *Card-order* κ **and** *a2*: $\alpha <o$ *cardSuc* κ
shows $|Field\ \alpha| \leq o\ \kappa$
⟨*proof*⟩

lemma *lem-jnfix-cardsuc*:
fixes L::$'l$ *set* **and** κ::$'U$ *rel* **and** t::$('U\ rel)\times'l \Rightarrow\ 'U\ rel \Rightarrow\ 'n$ **and** jnN::$'n \Rightarrow\ 'n$ $\Rightarrow$ $'n$
  and S::$'U$ *rel set*
assumes *a1*: $\neg$ *finite* L **and** *a2*: $\kappa =o$ *cardSuc* $|L|$
  and *a3*: $S \subseteq \{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}$ **and** *a4*: $|\{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}| \leq o\ |S|$
  and *a5*: $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$
shows $\exists\ h.\ \forall\ \alpha \in S.\ \forall\ \beta \in S.\ \forall i{\in}L.\ \forall j{\in}L.$
  $(\exists\ \gamma \in S.\ \alpha <o\ \gamma \wedge \beta <o\ \gamma \wedge h\ \gamma = jnN\ (t\ (\alpha,i)\ \gamma)\ (t\ (\beta,j)\ \gamma)\ )$
⟨*proof*⟩

lemma *lem-Relprop-cl-ccr*:
fixes r::$'U$ *rel*
shows *Conelike* $r \Longrightarrow$ *CCR* r
⟨*proof*⟩

lemma *lem-Relprop-ccr-confl*:
fixes r::$'U$ *rel*
shows *CCR* $r \Longrightarrow$ *confl-rel* r
⟨*proof*⟩

lemma *lem-Relprop-fin-ccr*:
fixes r::$'U$ *rel*
shows *finite* $r \Longrightarrow$ *CCR* r = *Conelike* r
⟨*proof*⟩

lemma *lem-Relprop-ccr-ch-un*:
fixes S::$'U$ *rel set*
assumes *a1*: $\forall s{\in}S.$ *CCR* s **and** *a2*: $\forall s1{\in}S.\ \forall s2{\in}S.\ s1 \subseteq s2 \vee s2 \subseteq s1$
shows *CCR* $(\bigcup\ S)$
⟨*proof*⟩

lemma *lem-Relprop-restr-ch-un*:
fixes $C::'U\ set\ set$ **and** $r::'U\ rel$
assumes $\forall A1 \in C.\ \forall A2 \in C.\ A1 \subseteq A2 \vee A2 \subseteq A1$
shows $Restr\ r\ (\bigcup\ C) = \bigcup\ \{\ s.\ \exists\ A \in C.\ s = Restr\ r\ A\ \}$
⟨*proof*⟩

lemma *lem-Inv-restr-rtr*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $A \in Inv\ r$
shows $r\hat{}* \cap (A \times (UNIV::'U\ set)) \subseteq (Restr\ r\ A)\hat{}*$
⟨*proof*⟩

lemma *lem-Inv-restr-rtr2*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $A \in Inv\ r$
shows $r\hat{}* \cap (A \times (UNIV::'U\ set)) \subseteq (Restr\ r\ A)\hat{}* \cap ((UNIV::'U\ set) \times A)$
⟨*proof*⟩

lemma *lem-inv-rtr-mem*:
fixes $r::'U\ rel$ **and** $A::'U\ set$ **and** $a\ b::'U$
assumes $A \in Inv\ r$ **and** $a \in A$ **and** $(a,b) \in r\hat{}*$
shows $b \in A$
⟨*proof*⟩

lemma *lem-Inv-ccr-restr*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $CCR\ r$ **and** $A \in Inv\ r$
shows $CCR\ (Restr\ r\ A)$
⟨*proof*⟩

lemma *lem-Inv-cl-restr*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $Conelike\ r$ **and** $A \in Inv\ r$
shows $Conelike\ (Restr\ r\ A)$
⟨*proof*⟩

lemma *lem-Inv-ccr-restr-invdiff*:
fixes $r::'U\ rel$ **and** $A\ B::'U\ set$
assumes $a1$: $CCR\ (Restr\ r\ A)$ **and** $a2$: $B \in Inv\ (r\hat{}-1)$
shows $CCR\ (Restr\ r\ (A - B))$
⟨*proof*⟩

lemma *lem-Inv-dncl-invbk*: $dncl\ r\ A \in Inv\ (r\hat{}-1)$
⟨*proof*⟩

lemma *lem-inv-sf-ext*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $A \subseteq Field\ r$
shows $\exists\ A' \in SF\ r.\ A \subseteq A' \wedge (finite\ A \longrightarrow finite\ A') \wedge ((\neg\ finite\ A) \longrightarrow |A'| =o$

$|A|$)
⟨*proof*⟩

lemma *lem-inv-sf-un*:
assumes $S \subseteq SF\ r$
shows $(\bigcup\ S) \in SF\ r$
⟨*proof*⟩

lemma *lem-Inv-ccr-sf-inv-diff*:
fixes $r{::}'U\ rel$ **and** $A\ B{::}'U\ set$
assumes *a1*: $A \in SF\ r$ **and** *a2*: $CCR\ (Restr\ r\ A)$ **and** *a3*: $B \in Inv\ (r\hat{}-1)$
shows $(A-B) \in SF\ r \vee (\exists\ y{::}'U.\ (A-B) = \{y\})$
⟨*proof*⟩

lemma *lem-Inv-ccr-sf-dn-diff*:
fixes $r{::}'U\ rel$ **and** $A\ D\ A'{::}'U\ set$
assumes *a1*: $A \in SF\ r$ **and** *a2*: $CCR\ (Restr\ r\ A)$ **and** *a3*: $A' = (A - (dncl\ r\ D))$
shows $((A' \in SF\ r) \wedge CCR\ (Restr\ r\ A')) \vee (\exists\ y{::}'U.\ A' = \{y\})$
⟨*proof*⟩

lemma *lem-rseq-tr*:
fixes $r{::}'U\ rel$ **and** $xi{::}nat \Rightarrow\ 'U$
assumes $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\forall\ i\ j.\ i < j \longrightarrow (xi\ i \in Field\ r \wedge (xi\ i,\ xi\ j) \in r\hat{}+)$
⟨*proof*⟩

lemma *lem-rseq-rtr*:
fixes $r{::}'U\ rel$ **and** $xi{::}nat \Rightarrow\ 'U$
assumes $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\forall\ i\ j.\ i \leq j \longrightarrow (xi\ i \in Field\ r \wedge (xi\ i,\ xi\ j) \in r\hat{}*)$
⟨*proof*⟩

lemma *lem-rseq-svacyc-inv-tr*:
fixes $r{::}'U\ rel$ **and** $xi{::}nat \Rightarrow\ 'U$ **and** $a{::}'U$
assumes *a1*: *single-valued* r **and** *a2*: $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\bigwedge\ i.\ (xi\ i,\ a) \in r\hat{}+ \Longrightarrow (\exists\ j.\ i{<}j \wedge a = xi\ j)$
⟨*proof*⟩

lemma *lem-rseq-svacyc-inv-rtr*:
fixes $r{::}'U\ rel$ **and** $xi{::}nat \Rightarrow\ 'U$ **and** $a{::}'U$
assumes *a1*: *single-valued* r **and** *a2*: $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\bigwedge\ i.\ (xi\ i,\ a) \in r\hat{}* \Longrightarrow (\exists\ j.\ i{\leq}j \wedge a = xi\ j)$
⟨*proof*⟩

lemma *lem-ccrsv-cfseq*:
fixes $r{::}'U\ rel$
assumes *a1*: $r \neq \{\}$ **and** *a2*: $CCR\ r$ **and** *a3*: *single-valued* r **and** *a4*: $\forall\, x{\in}Field\ r.\ r``\{x\} \neq \{\}$
shows $\exists\ xi.\ cfseq\ r\ xi$

⟨*proof*⟩

lemma *lem-cfseq-fld*: *cfseq r xi* $\Longrightarrow$ *xi* ‘ *UNIV* $\subseteq$ *Field r*
⟨*proof*⟩

lemma *lem-cfseq-inv*: *cfseq r xi* $\Longrightarrow$ *single-valued r* $\Longrightarrow$ *xi* ‘ *UNIV* $\in$ *Inv r*
⟨*proof*⟩

lemma *lem-scfinv-scf-int*: $A \in SCF\ r \cap Inv\ r \Longrightarrow B \in SCF\ r \Longrightarrow (A \cap B) \in SCF\ r$
⟨*proof*⟩

lemma *lem-scf-minr*: $a \in Field\ r \Longrightarrow B \in SCF\ r \Longrightarrow \exists\ b \in B.\ (a,b) \in (r \cap ((UNIV{-}B) \times UNIV))\hat{}{*}$
⟨*proof*⟩

lemma *lem-cfseq-ncl*:
fixes *r*::*'U rel* **and** *xi*::*nat* $\Rightarrow$ *'U*
assumes *a1*: *cfseq r xi* **and** *a2*: $\neg$ *Conelike r*
shows $\forall\ n.\ \exists\ k.\ n \le k \wedge (xi\ (Suc\ k),\ xi\ k) \notin r\hat{}{*}$
⟨*proof*⟩

lemma *lem-cfseq-inj*:
fixes *r*::*'U rel* **and** *xi*::*nat* $\Rightarrow$ *'U*
assumes *a1*: *cfseq r xi* **and** *a2*: *acyclic r*
shows *inj xi*
⟨*proof*⟩

lemma *lem-cfseq-rmon*:
fixes *r*::*'U rel* **and** *xi*::*nat* $\Rightarrow$ *'U*
assumes *a1*: *cfseq r xi* **and** *a2*: *single-valued r* **and** *a3*: *acyclic r*
shows $\forall\ i\ j.\ (xi\ i,\ xi\ j) \in r\hat{}{+} \longrightarrow i < j$
⟨*proof*⟩

lemma *lem-rseq-hd*:
assumes $\forall i{<}n.\ (f\ i,\ f\ (Suc\ i)) \in r$
shows $\forall i{\le}n.\ (f\ 0,\ f\ i) \in r\hat{}{*}$
⟨*proof*⟩

lemma *lem-rseq-tl*:
assumes $\forall i{<}n.\ (f\ i,\ f\ (Suc\ i)) \in r$
shows $\forall i{\le}n.\ (f\ i,\ f\ n) \in r\hat{}{*}$
⟨*proof*⟩

lemma *lem-ccext-ntr-rpth*: $(a,b) \in r\hat{}\hat{}n = (rpth\ r\ a\ b\ n \neq \{\})$
⟨*proof*⟩

lemma *lem-ccext-rtr-rpth*: $(a,b) \in r\hat{}{*} \Longrightarrow \exists\ n.\ rpth\ r\ a\ b\ n \neq \{\}$
⟨*proof*⟩

lemma *lem-ccext-rpth-rtr*: *rpth r a b n* $\neq$ {} $\Longrightarrow$ (*a*,*b*) $\in$ *r*^*
⟨*proof*⟩

lemma *lem-ccext-rtr-Fne*:
fixes *r*::′*U rel* **and** *a b*::′*U*
shows (*a*,*b*) $\in$ *r*^* = ($\mathcal{F}$ *r a b* $\neq$ {})
⟨*proof*⟩

lemma *lem-ccext-fprop*: $\mathcal{F}$ *r a b* $\neq$ {} $\Longrightarrow$ $\mathfrak{f}$ *r a b* $\in$ $\mathcal{F}$ *r a b* ⟨*proof*⟩

lemma *lem-ccext-ffin*: *finite* ($\mathfrak{f}$ *r a b*)
⟨*proof*⟩

lemma *lem-ccr-fin-subr-ext*:
fixes *r s*::′*U rel*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: *finite s*
shows $\exists$ *s*′::(′*U rel*). *finite s*′ $\wedge$ *CCR s*′ $\wedge$ *s* $\subseteq$ *s*′ $\wedge$ *s*′ $\subseteq$ *r*
⟨*proof*⟩

lemma *lem-Ccext-fint*:
fixes *r s*::′*U rel* **and** *a b*::′*U*
assumes *a1*: *Restr r* ($\mathfrak{f}$ *r a b*) $\subseteq$ *s* **and** *a2*: (*a*,*b*) $\in$ *r*^*
shows {*a*, *b*} $\subseteq$ $\mathfrak{f}$ *r a b* $\wedge$ ($\forall$ *c* $\in$ $\mathfrak{f}$ *r a b*. (*a*,*c*) $\in$ *s*^* $\wedge$ (*c*,*b*) $\in$ *s*^*)
⟨*proof*⟩

lemma *lem-Ccext-subccr-eqfld*:
fixes *r r*′::′*U rel*
assumes *CCR r* **and** *r* $\subseteq$ *r*′ **and** *Field r*′ = *Field r*
shows *CCR r*′
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-pext*:
fixes *r s*::′*U rel* **and** *x*::′*U*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: *finite s* **and** *a5*: *x* $\in$ *Field r*
shows $\exists$ *s*′::(′*U rel*). *finite s*′ $\wedge$ *CCR s*′ $\wedge$ *s* $\subseteq$ *s*′ $\wedge$ *s*′ $\subseteq$ *r* $\wedge$ *x* $\in$ *Field s*′
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-dext*:
fixes *r*::′*U rel* **and** *A*::′*U set*
assumes *a1*: *CCR r* **and** *a2*: *A* $\subseteq$ *Field r* **and** *a3*: *finite A*
shows $\exists$ *s*::(′*U rel*). *finite s* $\wedge$ *CCR s* $\wedge$ *s* $\subseteq$ *r* $\wedge$ *A* $\subseteq$ *Field s*
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-pext*:
fixes *r s*::′*U rel* **and** *x*::′*U*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: $\neg$ *finite s* **and** *a5*: *x* $\in$ *Field r*
shows $\exists$ *s*′::(′*U rel*). *CCR s*′ $\wedge$ *s* $\subseteq$ *s*′ $\wedge$ *s*′ $\subseteq$ *r* $\wedge$ |*s*′| =*o* |*s*| $\wedge$ *x* $\in$ *Field s*′
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-set-ext*:
fixes $r\ s{::}'U\ rel$ **and** $A{::}'U\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $finite\ s$ **and** *a4*: $A \subseteq Field\ r$ **and** *a5*: $finite\ A$
shows $\exists\ s'{::}('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge finite\ s' \wedge A \subseteq Field\ s'$
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-set-ext*:
fixes $r\ s{::}'U\ rel$ **and** $A{::}'U\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $\neg\ finite\ s$ **and** *a4*: $A \subseteq Field\ r$ **and** *a5*: $|A| \leq o\ |Field\ s|$
shows $\exists\ s'{::}('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge |s'| =o\ |s| \wedge A \subseteq Field\ s'$
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-pext5*:
fixes $r{::}'U\ rel$ **and** $A\ B{::}'U\ set$ **and** $x{::}'U$
assumes *a1*: $CCR\ r$ **and** *a2*: $finite\ A$ **and** *a3*: $A \in SF\ r$
shows $\exists\ A'{::}('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge finite\ A'$
$\wedge\ (\forall a{\in}A.\ r``\{a\}{\subseteq}B \vee r``\{a\}{\cap}(A'{-}B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y{::}'U.\ A'{-}B = \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B))$
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-pext5*:
fixes $r{::}'U\ rel$ **and** $A\ B{::}'U\ set$ **and** $x{::}'U$
assumes *a1*: $CCR\ r$ **and** *a2*: $\neg\ finite\ A$ **and** *a3*: $A \in SF\ r$
shows $\exists\ A'{::}('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge |A'| =o\ |A|$
$\wedge\ (\forall a{\in}A.\ r``\{a\}{\subseteq}B \vee r``\{a\}{\cap}(A'{-}B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y{::}'U.\ A'{-}B = \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B))$
⟨*proof*⟩

lemma *lem-Ccext-subccr-pext5*:
fixes $r{::}'U\ rel$ **and** $A\ B{::}'U\ set$ **and** $x{::}'U$
assumes $CCR\ r$ **and** $A \in SF\ r$
shows $\exists\ A'{::}('U\ set).\ (x \in Field\ r \longrightarrow x \in A')$
$\wedge\ A \subseteq A'$
$\wedge\ A' \in SF\ r$
$\wedge\ (\forall a{\in}A.\ ((r``\{a\}{\subseteq}B) \vee (r``\{a\}{\cap}(A'{-}B) \neq \{\})))$
$\wedge\ ((\exists\ y{::}'U.\ A'{-}B = \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B))$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ ((finite\ A \longrightarrow finite\ A') \wedge (\ (\neg\ finite\ A) \longrightarrow |A'| =o\ |A|\ ))$
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-set-ext-scf*:
fixes $r\ s{::}'U\ rel$ **and** $A\ P{::}'U\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $finite\ s$ **and** *a4*: $A \subseteq Field\ r$ **and** *a5*: $finite\ A$

and $a6$: $P \in SCF\ r$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge finite\ s' \wedge A \subseteq Field\ s'$
$\wedge\ ((Field\ s' \cap P) \in SCF\ s')$
$\langle proof \rangle$

lemma *lem-ccext-scf-sat*:
assumes $s \subseteq r$ **and** $Field\ s = Field\ r$
shows $SCF\ s \subseteq SCF\ r$
$\langle proof \rangle$

lemma *lem-Ccext-infsubccr-set-ext-scf2*:
fixes $r\ s::'U\ rel$ **and** $A::'U\ set$ **and** $Ps::'U\ set\ set$
assumes $a1$: $CCR\ r$ **and** $a2$: $s \subseteq r$ **and** $a3$: $\neg\ finite\ s$ **and** $a4$: $A \subseteq Field\ r$
and $a5$: $|A| \leq o\ |Field\ s|$ **and** $a6$: $Ps \subseteq SCF\ r \wedge |Ps| \leq o\ |Field\ s|$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge |s'| =o\ |s| \wedge A \subseteq Field\ s'$
$\wedge\ (\forall\ P \in Ps.\ (Field\ s' \cap P) \in SCF\ s')$
$\langle proof \rangle$

lemma *lem-Ccext-finsubccr-pext5-scf2*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes $a1$: $CCR\ r$ **and** $a2$: $finite\ A$ **and** $a3$: $A \in SF\ r$ **and** $a4$: $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge$
$finite\ A'$
$\wedge\ (\forall a \in A.\ r``\{a\} \subseteq B \vee r``\{a\} \cap (A' - B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y::'U.\ A' - B' = \{y\}) \longrightarrow Field\ r \subseteq (A' \cup B'))$
$\wedge\ ((\exists\ P.\ Ps = \{P\}) \longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r$
$A')))$
$\langle proof \rangle$

lemma *lem-Ccext-infsubccr-pext5-scf2*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes $a1$: $CCR\ r$ **and** $a2$: $\neg\ finite\ A$ **and** $a3$: $A \in SF\ r$ **and** $a4$: $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge$
$|A'| =o\ |A|$
$\wedge\ (\forall a \in A.\ r``\{a\} \subseteq B \vee r``\{a\} \cap (A' - B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y::'U.\ A' - B' = \{y\}) \longrightarrow Field\ r \subseteq (A' \cup B'))$
$\wedge\ (\ |Ps| \leq o\ |A| \longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A'))\ )$
$\langle proof \rangle$

lemma *lem-Ccext-subccr-pext5-scf2*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes $CCR\ r$ **and** $A \in SF\ r$ **and** $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A')$
$\wedge\ A \subseteq A'$
$\wedge\ A' \in SF\ r$
$\wedge\ (\forall a \in A.\ ((r``\{a\} \subseteq B) \vee (r``\{a\} \cap (A' - B) \neq \{\})))$
$\wedge\ ((\exists\ y::'U.\ A' - B' = \{y\}) \longrightarrow Field\ r \subseteq (A' \cup B'))$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ ((finite\ A \longrightarrow finite\ A') \wedge (\ (\neg\ finite\ A) \longrightarrow |A'| =o\ |A|\ ))$

$\wedge$ ($((\exists\ P.\ Ps = \{P\}) \vee ((\neg\ finite\ Ps) \wedge |Ps| \leq o\ |A|\ )) \longrightarrow$
$(\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A')))$
$\langle proof \rangle$

lemma *lem-dnEsc-el*: $F \in dnEsc\ r\ A\ a \Longrightarrow a \in F \wedge finite\ F$ $\langle proof \rangle$

lemma *lem-dnEsc-emp*: $dnEsc\ r\ A\ a = \{\} \Longrightarrow dnesc\ r\ A\ a = \{\ a\ \}$ $\langle proof \rangle$

lemma *lem-dnEsc-ne*: $dnEsc\ r\ A\ a \neq \{\} \Longrightarrow dnesc\ r\ A\ a \in dnEsc\ r\ A\ a$
$\langle proof \rangle$

lemma *lem-dnesc-in*: $a \in dnesc\ r\ A\ a \wedge finite\ (dnesc\ r\ A\ a)$
$\langle proof \rangle$

lemma *lem-escl-incr*: $B \subseteq escl\ r\ A\ B$ $\langle proof \rangle$

lemma *lem-escl-card*: $(finite\ B \longrightarrow finite\ (escl\ r\ A\ B)) \wedge (\neg\ finite\ B \longrightarrow |escl\ r\ A\ B| \leq o\ |B|\ )$
$\langle proof \rangle$

lemma *lem-Ccext-infsubccr-set-ext-scf3*:
fixes $r\ s::'U\ rel$ **and** $A\ A0::'U\ set$ **and** $Ps::'U\ set\ set$
assumes $a1$: $CCR\ r$ **and** $a2$: $s \subseteq r$ **and** $a3$: $\neg\ finite\ s$ **and** $a4$: $A \subseteq Field\ r$
and $a5$: $|A| \leq o\ |Field\ s|$ **and** $a6$: $Ps \subseteq SCF\ r \wedge |Ps| \leq o\ |Field\ s|$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge |s'| =o\ |s| \wedge A \subseteq Field\ s'$
$\wedge (\forall\ P{\in}Ps.\ (Field\ s' \cap P) \in SCF\ s') \wedge (escl\ r\ A0\ (Field\ s') \subseteq Field\ s')$
$\wedge (\exists\ D.\ s' = Restr\ r\ D) \wedge (Conelike\ s' \longrightarrow Conelike\ r)$
$\langle proof \rangle$

lemma *lem-Ccext-infsubccr-pext5-scf3*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes $a1$: $CCR\ r$ **and** $a2$: $\neg\ finite\ A$ **and** $a3$: $A \in SF\ r$ **and** $a4$: $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge |A'| =o\ |A|$
$\wedge (\forall a{\in}A.\ r``\{a\}{\subseteq}B \vee r``\{a\}{\cap}(A'{-}B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge ((\exists\ y::'U.\ A'{-}B' \subseteq \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B'))$
$\wedge (\ |Ps| \leq o\ |A| \longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A'))\ )$
$\wedge (escl\ r\ A\ A' \subseteq A') \wedge clterm\ (Restr\ r\ A')\ r$
$\langle proof \rangle$

lemma *lem-Ccext-finsubccr-pext5-scf3*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes $a1$: $CCR\ r$ **and** $a2$: $finite\ A$ **and** $a3$: $A \in SF\ r$ **and** $a4$: $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge finite\ A'$
$\wedge (\forall a{\in}A.\ r``\{a\}{\subseteq}B \vee r``\{a\}{\cap}(A'{-}B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge ((\exists\ y::'U.\ A'{-}B' \subseteq \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B'))$
$\wedge ((\exists\ P.\ Ps = \{P\}) \longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A')))$

⟨*proof*⟩

lemma *lem-Ccext-subccr-pext5-scf3*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$ **and** $C::'U\ set \Rightarrow bool$
assumes *a1*: $CCR\ r$ **and** *a2*: $A \in SF\ r$ **and** *a3*: $Ps \subseteq SCF\ r$
and *a4*: $C = (\lambda\ A'::'U\ set.\ (x \in Field\ r \longrightarrow x \in A')$
$\wedge\ A \subseteq A'$
$\wedge\ A' \in SF\ r$
$\wedge\ (\forall\ a{\in}A.\ ((r``\{a\}{\subseteq}B) \vee (r``\{a\}{\cap}(A'{-}B) \neq \{\})))$
$\wedge\ ((\exists\ y::'U.\ A'{-}B' \subseteq \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B'))$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ ((finite\ A \longrightarrow finite\ A') \wedge (\ (\neg\ finite\ A) \longrightarrow |A'| =o\ |A|\ ))$
$\wedge\ (\ ((\exists\ P.\ Ps = \{P\}) \vee ((\neg\ finite\ Ps) \wedge |Ps| \leq o\ |A|\ )) \longrightarrow$
$(\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A')))$
$\wedge\ (\ (\neg\ finite\ A) \longrightarrow ((escl\ r\ A\ A' \subseteq A') \wedge (clterm\ (Restr\ r\ A')$
$r)))\ )$
shows $\exists\ A'::('U\ set).\ C\ A'$
⟨*proof*⟩

lemma *lem-acyc-un-emprd*:
fixes $r\ s::\ 'U\ rel$
assumes *a1*: $acyclic\ r \wedge acyclic\ s$ **and** *a2*: $(Range\ r) \cap (Domain\ s) = \{\}$
shows $acyclic\ (r \cup s)$
⟨*proof*⟩

lemma *lem-spthlen-rtr*: $(a,b) \in r\widehat{\ }{*} \Longrightarrow (a,b) \in r\widehat{\ }\widehat{\ }(spthlen\ r\ a\ b)$
⟨*proof*⟩

lemma *lem-spthlen-tr*: $(a,b) \in r\widehat{\ }{*} \wedge a \neq b \Longrightarrow (a,b) \in r\widehat{\ }\widehat{\ }(spthlen\ r\ a\ b) \wedge spthlen\ r\ a\ b > 0$
⟨*proof*⟩

lemma *lem-spthlen-min*: $(a,b) \in r\widehat{\ }\widehat{\ }n \Longrightarrow spthlen\ r\ a\ b \leq n$
⟨*proof*⟩

lemma *lem-spth-inj*:
fixes $r::'U\ rel$ **and** $a\ b::'U$ **and** $f::nat \Rightarrow 'U$ **and** $n::nat$
assumes *a1*: $f \in spth\ r\ a\ b$ **and** *a2*: $n = spthlen\ r\ a\ b$
shows $inj\text{-}on\ f\ \{i.\ i{\leq}n\}$
⟨*proof*⟩

lemma *lem-rtn-rpth-inj*: $(a,b) \in r\widehat{\ }\widehat{\ }n \Longrightarrow n = spthlen\ r\ a\ b \Longrightarrow \exists\ f\ .\ f \in rpth\ r\ a\ b\ n \wedge inj\text{-}on\ f\ \{i.\ i \leq n\}$
⟨*proof*⟩

lemma *lem-rtr-rpth-inj*: $(a,b) \in r\widehat{\ }{*} \Longrightarrow \exists\ f\ n\ .\ f \in rpth\ r\ a\ b\ n \wedge inj\text{-}on\ f\ \{i.\ i \leq n\}$

⟨*proof*⟩

lemma *lem-sum-ind-ex*:
assumes *a1*: $g = (\lambda n{::}nat.\ \sum i{<}n.\ f\ i)$
  and *a2*: $\forall\, i{::}nat.\ f\ i > 0$
shows $\exists\ n\ k.\ (m{::}nat) = g\ n + k \wedge k < f\ n$
⟨*proof*⟩

lemma *lem-sum-ind-un*:
assumes *a1*: $g = (\lambda n{::}nat.\ \sum i{<}n.\ f\ i)$
  and *a2*: $\forall\, i{::}nat.\ f\ i > 0$
  and *a3*: $(m{::}nat) = g\ n + k \wedge k < f\ n$
  and *a4*: $m = g\ n' + k' \wedge k' < f\ n'$
shows $n = n' \wedge k = k'$
⟨*proof*⟩

lemma *lem-flatseq*:
fixes $r{::}'U\ rel$ **and** $xi{::}nat \Rightarrow {}'U$
assumes $\forall\, n.\ (xi\ n,\ xi\ (Suc\ n)) \in r\hat{}{*} \wedge (xi\ n \neq xi\ (Suc\ n))$
shows $\exists\ g\ yi.\ (\ \forall\, n.\ (yi\ n,\ yi\ (Suc\ n)) \in r\ )$
$\wedge\ (\forall\ i{::}nat.\ \forall\ j{::}nat.\ i < j \longleftrightarrow g\ i < g\ j\ )$
$\wedge\ (\forall\ i{::}nat.\ yi\ (g\ i) = xi\ i)$
$\wedge\ (\forall\ i{::}nat.\ inj\text{-}on\ yi\ \{\ k.\ g\ i \le k \wedge k \le g\ (Suc\ i)\ \}\ )$
$\wedge\ (\forall\ k{::}nat.\ \exists\ i{::}nat.\ g\ i \le k \wedge Suc\ k \le g\ (Suc\ i))$
$\wedge\ (\forall\ k\ i\ i'.\ g\ i \le k \wedge Suc\ k \le g\ (Suc\ i) \wedge g\ i' \le k \wedge Suc\ k \le g\ (Suc\ i') \longrightarrow i = i'\ )$
⟨*proof*⟩

lemma *lem-sv-un3*:
fixes $r1\ r2\ r3{::}'U\ rel$
assumes $single\text{-}valued\ (r1 \cup r3)$ **and** $single\text{-}valued\ (r2 \cup r3)$ **and** $Field\ r1 \cap Field\ r2 = \{\}$
shows $single\text{-}valued\ (r1 \cup r2 \cup r3)$
⟨*proof*⟩

lemma *lem-cfcomp-d2uset*:
fixes $\kappa{::}'U\ rel$ **and** $r{::}'U\ rel$ **and** $W{::}'U\ rel \Rightarrow {}'U\ set$ **and** $R{::}'U\ rel \Rightarrow {}'U\ rel$
  and $S{::}'U\ rel\ set$
assumes *a1*: $\kappa =o\ cardSuc\ |UNIV{::}nat\ set|$
  and *a3*: $T = \{\ t{::}'U\ rel.\ t \neq \{\} \wedge CCR\ t \wedge single\text{-}valued\ t \wedge acyclic\ t \wedge (\forall\, x{\in}Field\ t.\ t``\{x\} \neq \{\})\ \}$
  and *a4*: $Refl\ r$

  and *a5*: $S \subseteq \{\alpha \in \mathcal{O}{::}'U\ rel\ set.\ \alpha <o\ \kappa\}$
  and *a6*: $|\{\alpha \in \mathcal{O}{::}'U\ rel\ set.\ \alpha <o\ \kappa\}| \le o\ |S|$
  and *a7*: $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$

  and *a8*: $Field\ r = (\bigcup \alpha{\in}S.\ W\ \alpha)$ **and** *a9*: $\forall\, \alpha{\in}S.\ \forall\ \beta{\in}S.\ \alpha \neq \beta \longrightarrow W\ \alpha \cap W\ \beta = \{\}$

and *a10*: $\bigwedge \alpha.\ \alpha \in S \Longrightarrow R\ \alpha \in T \wedge R\ \alpha \subseteq r \wedge |W\ \alpha| \leq o\ |UNIV::nat\ set|$
$\wedge\ Field\ (R\ \alpha) = W\ \alpha \wedge \neg\ Conelike\ (Restr\ r\ (W\ \alpha))$
and *a11*: $\bigwedge \alpha\ x.\ \alpha \in S \Longrightarrow x \in W\ \alpha \Longrightarrow \exists\ a.$
$((x,a) \in (Restr\ r\ (W\ \alpha))\hat{}* \wedge (\forall\ \beta \in S.\ \alpha <o\ \beta \longrightarrow (r``\{a\} \cap W\ \beta)$
$\neq \{\}))$
shows $\exists\ r'.\ CCR\ r' \wedge DCR\ 2\ r' \wedge r' \subseteq r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in Field\ r'.\ (a,b)$
$\in r\hat{}*)$
⟨*proof*⟩

lemma *lem-uset-cl-ext*:
fixes $r::'U\ rel$ **and** $s::'U\ rel$
assumes $s \in \mathfrak{U}\ r$ **and** *Conelike s*
shows *Conelike r*
⟨*proof*⟩

lemma *lem-uset-cl-singleton*:
fixes $r::'U\ rel$
assumes *Conelike r* **and** $r \neq \{\}$
shows $\exists\ m::'U.\ \exists\ m'::'U.\ \{(m',m)\} \in \mathfrak{U}\ r$
⟨*proof*⟩

lemma *lem-rcc-emp*: $\|\{\}\| = \{\}$
⟨*proof*⟩

lemma *lem-rcc-rccrel*:
fixes $r::'U\ rel$
shows *RCC-rel* $r\ \|r\|$
⟨*proof*⟩

lemma *lem-rcc-uset-ne*:
assumes $\mathfrak{U}\ r \neq \{\}$
shows $\exists\ s \in \mathfrak{U}\ r.\ |s| =o\ \|r\| \wedge (\ \forall\ s' \in \mathfrak{U}\ r.\ |s| \leq o\ |s'|\ )$
⟨*proof*⟩

lemma *lem-rcc-uset-emp*:
assumes $\mathfrak{U}\ r = \{\}$
shows $\|r\| = \{\}$
⟨*proof*⟩

lemma *lem-rcc-uset-mem-bnd*:
assumes $s \in \mathfrak{U}\ r$
shows $\|r\| \leq o\ |s|$
⟨*proof*⟩

lemma *lem-rcc-cardord*: *Card-order* $\|r\|$
⟨*proof*⟩

lemma *lem-uset-ne-rcc-inf*:
fixes $r::'U\ rel$

assumes $\neg$ ($\|r\|$ $<o$ ω-*ord*)
shows $\mathfrak{U}$ $r \neq \{\}$
⟨*proof*⟩

lemma *lem-rcc-inf*: (ω-*ord* $\leq o$ $\|r\|$) = ($\neg$ ($\|r\|$ $<o$ ω-*ord*))
⟨*proof*⟩

lemma *lem-Rcc-eq1-12*:
fixes r::$'U$ *rel*
shows *CCR* $r \Longrightarrow r \in \mathfrak{U}$ r
⟨*proof*⟩

lemma *lem-Rcc-eq1-23*:
fixes r::$'U$ *rel*
assumes $r \in \mathfrak{U}$ r
shows (r = ({}::$'U$ *rel*)) $\vee$ (({}::$'U$ *rel*) $<o$ $\|r\|$)
⟨*proof*⟩

lemma *lem-Rcc-eq1-31*:
fixes r::$'U$ *rel*
assumes (r = ({}::$'U$ *rel*)) $\vee$ (({}::$'U$ *rel*) $<o$ $\|r\|$)
shows *CCR* r
⟨*proof*⟩

lemma *lem-Rcc-eq2-12*:
fixes r::$'U$ *rel* **and** a::$'a$
assumes *Conelike* r
shows $\|r\|$ $\leq o$ $|\{a\}|$
⟨*proof*⟩

lemma *lem-Rcc-eq2-23*:
fixes r::$'U$ *rel* **and** a::$'a$
assumes $\|r\|$ $\leq o$ $|\{a\}|$
shows $\|r\|$ $<o$ ω-*ord*
⟨*proof*⟩

lemma *lem-Rcc-eq2-31*:
fixes r::$'U$ *rel*
assumes *CCR* r **and** $\|r\|$ $<o$ ω-*ord*
shows *Conelike* r
⟨*proof*⟩

lemma *lem-Rcc-range*:
fixes r::$'U$ *rel*
shows $\|r\|$ $\leq o$ |*UNIV*::($'U$ *set*)|
⟨*proof*⟩

lemma *lem-rcc-nccr*:
fixes r::$'U$ *rel*

assumes $\neg$ (*CCR r*)
shows $\|r\| = \{\}$
$\langle proof \rangle$

lemma *lem-Rcc-relcard-bnd*:
fixes r::'*U rel*
shows $\|r\| \leq o\ |r|$
$\langle proof \rangle$

lemma *lem-Rcc-inf-lim*:
fixes r::'*U rel*
assumes ω-*ord* $\leq o\ \|r\|$
shows $\neg(\ \|r\| = \{\} \vee isSuccOrd\ \|r\|\ )$
$\langle proof \rangle$

lemma *lem-rcc-uset-ne-ccr*:
fixes r::'*U rel*
assumes $\mathfrak{U}\ r \neq \{\}$
shows *CCR r*
$\langle proof \rangle$

lemma *lem-rcc-uset-tr*:
fixes $r\ s\ t$::'*U rel*
assumes *a1*: $s \in \mathfrak{U}\ r$ **and** *a2*: $t \in \mathfrak{U}\ s$
shows $t \in \mathfrak{U}\ r$
$\langle proof \rangle$

lemma *lem-scf-emp*: *scf* $\{\} = \{\}$
$\langle proof \rangle$

lemma *lem-scf-scfrel*:
fixes r::'*U rel*
shows *scf-rel r* (*scf r*)
$\langle proof \rangle$

lemma *lem-scf-uset*:
shows $\exists\ A \in SCF\ r.\ |A| =o\ scf\ r \wedge (\ \forall\ B \in SCF\ r.\ |A| \leq o\ |B|\ )$
$\langle proof \rangle$

lemma *lem-scf-uset-mem-bnd*:
assumes $B \in SCF\ r$
shows *scf r* $\leq o\ |B|$
$\langle proof \rangle$

lemma *lem-scf-cardord*: *Card-order* (*scf r*)
$\langle proof \rangle$

lemma *lem-scf-inf*: (ω-*ord* $\leq o$ (*scf r*)) = ($\neg$ ((*scf r*) $<o$ ω-*ord*))
$\langle proof \rangle$

lemma *lem-scf-eq1-12*:
fixes $r::'U\ rel$
shows *Field r* $\in$ *SCF r*
⟨*proof*⟩

lemma *lem-scf-range*:
fixes $r::'U\ rel$
shows $(scf\ r) \leq o\ |UNIV::('U\ set)|$
⟨*proof*⟩

lemma *lem-scf-relfldcard-bnd*:
fixes $r::'U\ rel$
shows $(scf\ r) \leq o\ |Field\ r|$
⟨*proof*⟩

lemma *lem-scf-ccr-scf-rcc-eq*:
fixes $r::'U\ rel$
assumes *CCR r*
shows $\|r\| =o\ (scf\ r)$
⟨*proof*⟩

lemma *lem-scf-ccr-scf-uset*:
fixes $r::'U\ rel$
assumes *CCR r* **and** $\neg$ *Conelike r*
shows $\exists\ s \in \mathfrak{U}\ r.\ (\neg\ finite\ s) \wedge |Field\ s| =o\ (scf\ r)$
⟨*proof*⟩

lemma *lem-Scf-scfprops*:
fixes $r::'U\ rel$
shows $(\ (scf\ r) \leq o\ |UNIV::('U\ set)|\ ) \wedge (\ (scf\ r) \leq o\ |Field\ r|\ )$
⟨*proof*⟩

lemma *lem-scf-ccr-finscf-cl*:
assumes *CCR r*
shows $finite\ (Field\ (scf\ r)) = Conelike\ r$
⟨*proof*⟩

lemma *lem-sv-uset-sv-span*:
fixes $r\ s::'U\ rel$
assumes $a1$: $s \in \mathfrak{U}\ r$ **and** $a2$: *single-valued s*
shows $\exists\ r1.\ r1 \in Span\ r \wedge CCR\ r1 \wedge single\text{-}valued\ r1 \wedge s \subseteq r1 \wedge (acyclic\ s \longrightarrow acyclic\ r1)$
⟨*proof*⟩

lemma *lem-incrfun-nat*: $\forall\ i::nat.\ f\ i < f\ (Suc\ i) \Longrightarrow \forall\ i\ j.\ i \leq j \longrightarrow f\ i + (j-i) \leq f\ j$
⟨*proof*⟩

lemma *lem-sv-uset-rcceqw*:
fixes $r::'U\ rel$
assumes $a1: \|r\| =o\ \omega\text{-}ord$
shows $\exists\ r1 \in \mathfrak{U}\ r.\ single\text{-}valued\ r1 \land acyclic\ r1 \land (\forall\ x \in Field\ r1.\ r1``\{x\} \neq \{\})$
⟨*proof*⟩

lemma *lem-sv-span-scflew*:
fixes $r::'U\ rel$
assumes $CCR\ r$ **and** $scf\ r \leq o\ \omega\text{-}ord$
shows $\exists\ r1.\ r1 \in Span\ r \land CCR\ r1 \land single\text{-}valued\ r1$
⟨*proof*⟩

lemma *lem-sv-span-scfeqw*:
fixes $r::'U\ rel$
assumes $CCR\ r$ **and** $scf\ r =o\ \omega\text{-}ord$
shows $\exists\ r1.\ r1 \in Span\ r \land r1 \neq \{\} \land CCR\ r1 \land single\text{-}valued\ r1 \land acyclic\ r1 \land (\forall x \in Field\ r1.\ r1``\{x\} \neq \{\})$
⟨*proof*⟩

lemma *lem-Ldo-den-ccr-uset*:
fixes $r\ s::'U\ rel$
assumes $CCR\ s$ **and** $s \subseteq r \land Field\ s \in Den\ r$
shows $s \in \mathfrak{U}\ r$
⟨*proof*⟩

lemma *lem-Ldo-ds-reduc*:
fixes $r\ s::'U\ rel$ **and** $n0::nat$
assumes $a1: CCR\ s \land DCR\ n0\ s$ **and** $a2: s \subseteq r$ **and** $a3: Field\ s \in Den\ r$ **and** $a4: Field\ s \in Inv\ (r - s)$
shows $CCR\ r \land DCR\ (Suc\ n0)\ r$
⟨*proof*⟩

lemma *lem-Ldo-sat-reduc*:
fixes $r\ s::'U\ rel$ **and** $n::nat$
assumes $a1: s \in Span\ r$ **and** $a2: CCR\ s \land DCR\ n\ s$
shows $CCR\ r \land DCR\ (Suc\ n)\ r$
⟨*proof*⟩

lemma *lem-Ldo-uset-reduc*:
fixes $r\ s::'U\ rel$ **and** $n0::nat$
assumes $a1: s \in \mathfrak{U}\ r$ **and** $a2: DCR\ n0\ s$ **and** $a3: n0 \neq 0$
shows $DCR\ (Suc\ n0)\ r$
⟨*proof*⟩

lemma *lem-Ldo-addid*:
fixes $r::'U\ rel$ **and** $r'::'U\ rel$ **and** $n0::nat$ **and** $A::'U\ set$
assumes $a1: DCR\ n0\ r$ **and** $a2: r' = r \cup \{(a,b).\ a = b \land a \in A\}$ **and** $a3: n0 \neq 0$
shows $DCR\ n0\ r'$

⟨*proof*⟩

lemma *lem-Ldo-removeid*:
fixes $r::'U\ rel$ **and** $r'::'U\ rel$ **and** $n0::nat$
assumes *a1*: $DCR\ n0\ r$ **and** *a2*: $r' = r - \{(a,b).\ a = b\}$
shows $DCR\ n0\ r'$
⟨*proof*⟩

lemma *lem-Ldo-eqid*:
fixes $r::'U\ rel$ **and** $r'::'U\ rel$ **and** $n::nat$
assumes *a1*: $DCR\ n\ r$ **and** *a2*: $r' - \{(a,b).\ a = b\} = r - \{(a,b).\ a = b\}$ **and** *a3*: $n \neq 0$
shows $DCR\ n\ r'$
⟨*proof*⟩

lemma *lem-wdn-range-lb*: $A \subseteq w\text{-}dncl\ r\ A$
⟨*proof*⟩

lemma *lem-wdn-range-ub*: $w\text{-}dncl\ r\ A \subseteq dncl\ r\ A$ ⟨*proof*⟩

lemma *lem-wdn-mon*: $A \subseteq A' \Longrightarrow w\text{-}dncl\ r\ A \subseteq w\text{-}dncl\ r\ A'$ ⟨*proof*⟩

lemma *lem-wdn-compl*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
shows $UNIV - w\text{-}dncl\ r\ A = \{a.\ \exists\ b.\ b \notin dncl\ r\ A \wedge (a,b) \in (Restr\ r\ (UNIV-A))\hat{}*\}$
⟨*proof*⟩

lemma *lem-cowdn-uset*:
fixes $r::'U\ rel$ **and** $A\ A'\ W::'U\ set$
assumes *a1*: $CCR\ (Restr\ r\ A')$ **and** *a2*: $escl\ r\ A\ A' \subseteq A'$
and *a3*: $Q = A' - dncl\ r\ A$ **and** *a4*: $W = A' - w\text{-}dncl\ r\ A$ **and** *a5*: $Q \in SF\ r$
shows $Restr\ r\ Q \in \mathfrak{U}\ (Restr\ r\ W)$
⟨*proof*⟩

lemma *lem-shrel-L-eq*:
fixes $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$ **and** $\beta::'U\ rel$
assumes $\alpha =o\ \beta$
shows $\mathfrak{L}\ f\ \alpha = \mathfrak{L}\ f\ \beta$
⟨*proof*⟩

lemma *lem-shrel-dbk-eq*:
fixes $f::'U\ rel \Rightarrow 'U\ set$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$ **and** $\beta::'U\ rel$
assumes $f \in \mathcal{N}\ r\ Ps$ **and** $\alpha =o\ \beta$ **and** $\alpha \leq o\ |Field\ r|$ **and** $\beta \leq o\ |Field\ r|$
shows $(\nabla\ f\ \alpha) = (\nabla\ f\ \beta)$
⟨*proof*⟩

lemma *lem-L-emp*: $\alpha =o\ (\{\}::'U\ rel) \Longrightarrow \mathfrak{L}\ f\ \alpha = \{\}$
⟨*proof*⟩

lemma *lem-der-qinv1*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$ **and** $x\ y::'U$
assumes *a1*: $x \in \mathcal{Q}\ r\ f\ \alpha$ **and** *a2*: $(x,y) \in r\widehat{\ }*$ **and** *a3*: $y \in (f\ \alpha)$
shows $y \in \mathcal{Q}\ r\ f\ \alpha$
$\langle proof \rangle$

lemma *lem-der-qinv2*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$ **and** $x\ y::'U$
assumes *a1*: $x \in \mathcal{Q}\ r\ f\ \alpha$ **and** *a2*: $(x,y) \in (Restr\ r\ (f\ \alpha))\widehat{\ }*$ **and** *a3*: $y \in (f\ \alpha)$
shows $(x,y) \in (Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha))\widehat{\ }*$
$\langle proof \rangle$

lemma *lem-der-qinv3*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$
assumes *a1*: $A \subseteq (f\ \alpha)$ **and** *a2*: $\forall\ x \in (f\ \alpha).\ \exists\ y \in A.\ (x,y) \in (Restr\ r\ (f\ \alpha))\widehat{\ }*$
shows $\forall\ x \in (\mathcal{Q}\ r\ f\ \alpha).\ \exists\ y \in (A \cap (\mathcal{Q}\ r\ f\ \alpha)).\ (x,y) \in (Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha))\widehat{\ }*$
$\langle proof \rangle$

lemma *lem-der-inf-qrestr-ccr1*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes $f \in \mathcal{N}\ r\ Ps$ **and** $\alpha \leq o\ |Field\ r|$
shows $CCR\ (Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha))$
$\langle proof \rangle$

lemma *lem-Nfdn-aemp*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r$ **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ scf\ r$ **and** *a4*: $Field\ r \subseteq dncl\ r\ (f\ \alpha)$
shows $\alpha = \{\}$
$\langle proof \rangle$

lemma *lem-der-qccr-lscf-sf*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r$ **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ scf\ r$
shows $(\mathcal{Q}\ r\ f\ \alpha) \in SF\ r$
$\langle proof \rangle$

lemma *lem-der-q-uset*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r$ **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ scf\ r$ **and** *a4*: $isSuccOrd\ \alpha$
shows $Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha) \in \mathfrak{U}\ (Restr\ r\ (f\ \alpha))$
$\langle proof \rangle$

lemma *lem-qw-range*: $f \in \mathcal{N}\ r\ Ps \Longrightarrow \alpha \leq o\ |Field\ r| \Longrightarrow \mathcal{W}\ r\ f\ \alpha \subseteq Field\ r$
$\langle proof \rangle$

lemma *lem-der-qw-eq*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha\ \beta::'U\ rel$

assumes $f \in \mathcal{N}\ r\ Ps$ **and** $\alpha =o\ \beta$
shows $\mathcal{W}\ r\ f\ \alpha = \mathcal{W}\ r\ f\ \beta$
⟨*proof*⟩

lemma *lem-Der-inf-qw-disj*:
fixes $r::'U\ rel$ **and** $\alpha\ \beta::'U\ rel$
assumes *Well-order* α **and** *Well-order* β
shows $(\neg\ (\alpha =o\ \beta)) \longrightarrow (\mathcal{W}\ r\ f\ \alpha) \cap (\mathcal{W}\ r\ f\ \beta) = \{\}$
⟨*proof*⟩

lemma *lem-der-inf-qw-restr-card*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $\neg$ *finite* r **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ |Field\ r|$
shows $|Restr\ r\ (\mathcal{W}\ r\ f\ \alpha)| <o\ |Field\ r|$
⟨*proof*⟩

lemma *lem-QS-subs-WS*: $\mathcal{Q}\ r\ f\ \alpha \subseteq \mathcal{W}\ r\ f\ \alpha$
⟨*proof*⟩

lemma *lem-WS-limord*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $\neg$ *finite* r **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ |Field\ r|$
and *a4*: $\neg\ (\alpha = \{\} \vee isSuccOrd\ \alpha)$
shows $\mathcal{W}\ r\ f\ \alpha = \{\}$
⟨*proof*⟩

lemma *lem-der-inf-qw-restr-uset*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: *Refl* $r \wedge \neg$ *finite* r **and** *a2*: $f \in \mathcal{N}\ r\ Ps$
and *a3*: $\alpha <o\ |Field\ r|$ **and** *a4*: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$
shows $Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha) \in \mathfrak{U}\ (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha))$
⟨*proof*⟩

lemma *lem-der-inf-qw-restr-ccr*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: *Refl* $r \wedge \neg$ *finite* r **and** *a2*: $f \in \mathcal{N}\ r\ Ps$
and *a3*: $\alpha <o\ |Field\ r|$ **and** *a4*: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$
shows $CCR\ (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha))$
⟨*proof*⟩

lemma *lem-der-qw-uset*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r \wedge$ *Refl* $r \wedge \neg$ *finite* r **and** *a2*: $f \in \mathcal{N}\ r\ Ps$
and *a3*: $\alpha <o\ scf\ r$ **and** *a4*: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$ **and** *a5*: *isSuccOrd* α
shows $Restr\ r\ (\mathcal{W}\ r\ f\ \alpha) \in \mathfrak{U}\ (Restr\ r\ (f\ \alpha))$
⟨*proof*⟩

lemma *lem-Shinf-N1*:
fixes $r::'U\ rel$ **and** $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$

assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
shows $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}1\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N2*:
fixes $r::'U\ rel$ **and** $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
shows $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}2\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N3*:
fixes $r::'U\ rel$ **and** $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a3*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \in \textit{SF}\ r \longrightarrow$
    $(\omega\textit{-ord} \le o\ |A| \longrightarrow \textit{escl}\ r\ A\ (F\ \alpha\ A) \subseteq (F\ \alpha\ A) \wedge \textit{clterm}\ (\textit{Restr}\ r\ (F$
$\alpha\ A))\ r)$
shows $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}3\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N4*:
fixes $r::'U\ rel$ **and** $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a4*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \in \textit{SF}\ r \longrightarrow (\forall\ a{\in}A.\ r``\{a\} \subseteq \textit{w-dncl}\ r\ A$
$\vee\ r``\{a\} \cap (F\ \alpha\ A - \textit{w-dncl}\ r\ A) \neq \{\})$
shows $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}4\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N5*:
fixes $r::'U\ rel$ **and** $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
assumes *a5*: $\forall\ \alpha\ A.\ (\textit{Well-order}\ \alpha \wedge A \in \textit{SF}\ r) \longrightarrow (F\ \alpha\ A) \in \textit{SF}\ r$
shows $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N6*:
fixes $r::'U\ rel$ **and** $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a6*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \in \textit{SF}\ r \longrightarrow \textit{CCR}\ (\textit{Restr}\ r\ (F\ \alpha\ A))$
shows $\forall\alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}6\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N7*:

fixes r::$'U$ *rel* **and** F::$'U$ *rel* $\Rightarrow$ $'U$ *set* $\Rightarrow$ $'U$ *set* **and** f::$'U$ *rel* $\Rightarrow$ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a7*: $\forall\ \alpha\ A.$ ($|A|$ <*o* ω-*ord* $\longrightarrow$ $|F\ \alpha\ A|$ <*o* ω-*ord*)
      $\wedge$ (ω-*ord* $\leq$*o* $|A|$ $\longrightarrow$ $|F\ \alpha\ A|$ $\leq$*o* $|A|$)
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}7\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N8*:
fixes r::$'U$ *rel* **and** F::$'U$ *rel* $\Rightarrow$ $'U$ *set* $\Rightarrow$ $'U$ *set* **and** f::$'U$ *rel* $\Rightarrow$ $'U$ *set* **and** *Ps*::$'U$ *set set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a7*: $\forall\ \alpha\ A.$ ($|A|$ <*o* ω-*ord* $\longrightarrow$ $|F\ \alpha\ A|$ <*o* ω-*ord*)
      $\wedge$ (ω-*ord* $\leq$*o* $|A|$ $\longrightarrow$ $|F\ \alpha\ A|$ $\leq$*o* $|A|$)
  and *a8*: $\forall\alpha\ A.\ A \in$ *SF* $r \longrightarrow \mathcal{E}p\ r$ *Ps* $A\ (F\ \alpha\ A)$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}8\ r$ *Ps* α
⟨*proof*⟩

lemma *lem-Shinf-N9*:
fixes r::$'U$ *rel* **and** g::$'U$ *rel* $\Rightarrow$ $'U$
 and F::$'U$ *rel* $\Rightarrow$ $'U$ *set* $\Rightarrow$ $'U$ *set* **and** f::$'U$ *rel* $\Rightarrow$ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a2*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow g\ \alpha \in$ *Field* $r \longrightarrow g\ \alpha \in F\ \alpha\ A$
  and *a11*: ω-*ord* $\leq$*o* $|$*Field* $r|$ $\longrightarrow$ *Field* $r \subseteq g$ ‘ { γ::$'U$ *rel*. γ <*o* $|$*Field* $r|$ }
shows $f \in \mathcal{N}9\ r\ |$*Field* $r|$
⟨*proof*⟩

lemma *lem-Shinf-N10*:
fixes r::$'U$ *rel* **and** F::$'U$ *rel* $\Rightarrow$ $'U$ *set* $\Rightarrow$ $'U$ *set* **and** f::$'U$ *rel* $\Rightarrow$ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a10*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \in$ *SF* $r \longrightarrow$
    $((\exists\, y.\ \ (F\ \alpha\ A) -$ *dncl* $r\ A \subseteq \{y\}) \longrightarrow$ (*Field* $r \subseteq$ *dncl* $r\ (F\ \alpha\ A)))$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}10\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N11*:
fixes r::$'U$ *rel* **and** F::$'U$ *rel* $\Rightarrow$ $'U$ *set* $\Rightarrow$ $'U$ *set* **and** f::$'U$ *rel* $\Rightarrow$ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a10*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \in$ *SF* $r \longrightarrow$
    $((\exists\, y.\ \ (F\ \alpha\ A) -$ *dncl* $r\ A \subseteq \{y\}) \longrightarrow$ (*Field* $r \subseteq$ *dncl* $r\ (F\ \alpha\ A)))$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}11\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N12*:
fixes $r::'U\ rel$ **and** $g::'U\ rel \Rightarrow 'U$
and $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes $a0$: $f \in \mathcal{T}\ F$
and $a1$: $\forall\alpha.\ Well\text{-}order\ \alpha \longrightarrow f \in \mathcal{N}1\ r\ \alpha$
and $a2$: $\forall\ \alpha\ A.\ Well\text{-}order\ \alpha \longrightarrow g\ \alpha \in Field\ r \longrightarrow g\ \alpha \in F\ \alpha\ A$
and $a11$: $\omega\text{-}ord \le o\ |Field\ r| \longrightarrow Field\ r = g\ `\ \{\ \gamma::'U\ rel.\ \gamma <o\ |Field\ r|\ \}$
and $a2'$: $\forall\alpha::'U\ rel.\ \omega\text{-}ord \le o\ \alpha \wedge \alpha \le o\ |Field\ r| \longrightarrow \omega\text{-}ord \le o\ |g\ `\ \{\gamma.\ \gamma <o$
$\alpha\}|$
shows $f \in \mathcal{N}12\ r\ |Field\ r|$
⟨*proof*⟩

lemma *lem-Shinf-E-ne*:
fixes $r::'U\ rel$ **and** $a0::'U$ **and** $A::'U\ set$ **and** $Ps::'U\ set\ set$
assumes $a2$: $CCR\ r$ **and** $a3$: $Ps \subseteq SCF\ r$
shows $\mathcal{E}\ r\ a0\ A\ Ps \neq \{\}$
⟨*proof*⟩

lemma *lem-oseq-fin-inj*:
fixes $g::'U\ rel \Rightarrow 'a$ **and** $I::'U\ rel \Rightarrow 'U\ rel\ set$ **and** $A::'a\ set$
assumes $a1$: $I = (\lambda\ \alpha'.\ \{\ \alpha::'U\ rel.\ \alpha <o\ \alpha'\ \})$
and $a2$: $\omega\text{-}ord \le o\ |A|$
and $a3$: $\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow g\ \alpha = g\ \beta$
shows $\exists\ h.\ (\forall\ \alpha'.\ g`(I\ \alpha') \subseteq h`(I\ \alpha') \wedge h`(I\ \alpha') \subseteq g`(I\ \alpha') \cup A)$
$\wedge\ (\forall\ \alpha'.\ \omega\text{-}ord \le o\ \alpha' \longrightarrow \omega\text{-}ord \le o\ |h`(I\ \alpha')|\ )$
$\wedge\ (\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow h\ \alpha = h\ \beta)$
⟨*proof*⟩

lemma *lem-Shinf-N-ne*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$
assumes $CCR\ r$ **and** $Ps \subseteq SCF\ r$
shows $\mathcal{N}\ r\ Ps \neq \{\}$
⟨*proof*⟩

lemma *lem-wrankrel-eq*: $wrank\text{-}rel\ r\ A0\ \alpha \Longrightarrow \alpha =o\ \beta \Longrightarrow wrank\text{-}rel\ r\ A0\ \beta$
⟨*proof*⟩

lemma *lem-wrank-wrankrel*:
fixes $r::'U\ rel$ **and** $A0::'U\ set$
shows $wrank\text{-}rel\ r\ A0\ (wrank\ r\ A0)$
⟨*proof*⟩

lemma *lem-wrank-uset*:
fixes $r::'U\ rel$ **and** $A0::'U\ set$
shows $\exists\ A \in wbase\ r\ A0.\ |A| =o\ wrank\ r\ A0 \wedge (\ \forall\ B \in wbase\ r\ A0.\ |A| \le o\ |B|$
$)$
⟨*proof*⟩

lemma *lem-wrank-uset-mem-bnd*:
fixes $r::'U\ rel$ **and** $A0\ B::'U\ set$
assumes $B \in wbase\ r\ A0$
shows $wrank\ r\ A0 \leq o\ |B|$
⟨*proof*⟩

lemma *lem-wrank-cardord*: $Card\text{-}order\ (wrank\ r\ A0)$
⟨*proof*⟩

lemma *lem-wrank-ub*: $wrank\ r\ A0 \leq o\ |A0|$
⟨*proof*⟩

lemma *lem-card-un2-bnd*: $\omega\text{-}ord \leq o\ \alpha \Longrightarrow |A| \leq o\ \alpha \Longrightarrow |B| \leq o\ \alpha \Longrightarrow |A \cup B| \leq o\ \alpha$
⟨*proof*⟩

lemma *lem-card-un2-lsbnd*: $\omega\text{-}ord \leq o\ \alpha \Longrightarrow |A| < o\ \alpha \Longrightarrow |B| < o\ \alpha \Longrightarrow |A \cup B| < o\ \alpha$
⟨*proof*⟩

lemma *lem-wrank-un-bnd*:
fixes $r::'U\ rel$ **and** $S::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes $a1$: $\forall\ A \in S.\ wrank\ r\ A \leq o\ \alpha$ **and** $a2$: $|S| \leq o\ \alpha$ **and** $a3$: $\omega\text{-}ord \leq o\ \alpha$
shows $wrank\ r\ (\bigcup\ S) \leq o\ \alpha$
⟨*proof*⟩

lemma *lem-wrank-un-bnd-stab*:
fixes $r::'U\ rel$ **and** $S::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes $a1$: $\forall\ A \in S.\ wrank\ r\ A < o\ \alpha$ **and** $a2$: $|S| < o\ \alpha$ **and** $a3$: $stable\ \alpha$
shows $wrank\ r\ (\bigcup\ S) < o\ \alpha$
⟨*proof*⟩

lemma *lem-wrank-fw*:
fixes $r::'U\ rel$ **and** $K::'U\ set$ **and** $\alpha::'U\ rel$
assumes $a1$: $\omega\text{-}ord \leq o\ \alpha$ **and** $a2$: $wrank\ r\ K \leq o\ \alpha$ **and** $a3$: $\forall\ b \in K.\ wrank\ r\ (r``\{b\}) \leq o\ \alpha$
shows $wrank\ r\ (\bigcup b \in K.\ (r``\{b\})) \leq o\ \alpha$
⟨*proof*⟩

lemma *lem-wrank-fw-stab*:
fixes $r::'U\ rel$ **and** $K::'U\ set$ **and** $\alpha::'U\ rel$
assumes $a1$: $\omega\text{-}ord \leq o\ \alpha \wedge stable\ \alpha$ **and** $a2$: $wrank\ r\ K < o\ \alpha$ **and** $a3$: $\forall\ b \in K.\ wrank\ r\ (r``\{b\}) < o\ \alpha$
shows $wrank\ r\ (\bigcup b \in K.\ (r``\{b\})) < o\ \alpha$
⟨*proof*⟩

lemma *lem-wnb-neib*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$
assumes $a1$: $\omega\text{-}ord \leq o\ \alpha$ **and** $a2$: $\alpha < o\ \|r\|$

shows $\forall\ a \in Field\ r.\ \exists\ b \in Mwn\ r\ \alpha.\ (a,b) \in r\hat{}*$
⟨*proof*⟩

lemma *lem-wnb-neib3*:
fixes $r::'U\ rel$
assumes $a1$: $\omega\text{-}ord <o \|r\|$ **and** $a2$: $stable\ \|r\|$
shows $\forall\ a \in Field\ r.\ \exists\ b \in Mwnm\ r.\ (a,b) \in r\hat{}*$
⟨*proof*⟩

lemma *lem-scfgew-ncl*: $\omega\text{-}ord \leq o\ scf\ r \Longrightarrow \neg\ Conelike\ r$
⟨*proof*⟩

lemma *lem-wnb-P-ncl-reg-grw*:
fixes $r::'U\ rel$
assumes $a1$: $CCR\ r$ **and** $a2$: $\omega\text{-}ord <o\ scf\ r$ **and** $a3$: $regularCard\ (scf\ r)$
shows $\exists\ P \in SCF\ r.\ (\forall\ \alpha::'U\ rel.\ \alpha <o\ scf\ r \longrightarrow (\forall\ a \in P.\ \alpha <o\ wrank\ r\ (r``\{a\})$
$))$
⟨*proof*⟩

lemma *lem-wnb-P-ncl-nreg*:
fixes $r::'U\ rel$
assumes $a1$: $CCR\ r$ **and** $a2$: $\omega\text{-}ord \leq o\ scf\ r$ **and** $a3$: $\neg\ regularCard\ (scf\ r)$
shows $\exists\ Ps::'U\ set\ set.\ Ps \subseteq SCF\ r \wedge |Ps| <o\ scf\ r$
$\wedge\ (\forall\ \alpha::'U\ rel.\ \alpha <o\ scf\ r \longrightarrow (\exists\ P \in Ps.\ \forall\ a \in P.\ \alpha <o\ wrank$
$r\ (r``\{a\})\ ))$
⟨*proof*⟩

lemma *lem-Wf-ext-arc*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow\ 'U\ set$ **and** $\alpha::'U\ rel$ **and** $a::'U$
assumes $a1$: $scf\ r =o\ |Field\ r|$ **and** $a2$: $f \in \mathcal{N}\ r\ Ps$
and $a3$: $\forall \gamma::'U\ rel.\ \gamma <o\ scf\ r \longrightarrow (\forall a \in P.\ \gamma <o\ wrank\ r\ (r``\{a\}))$
and $a4$: $\omega\text{-}ord \leq o\ \alpha$ **and** $a5$: $a \in f\ \alpha \cap P$
shows $\bigwedge\ \beta.\ \alpha <o\ \beta\ \wedge\ \beta <o\ |Field\ r|\ \wedge\ (\beta = \{\}\ \vee\ isSuccOrd\ \beta) \Longrightarrow (r``\{a\} \cap$
$(\mathcal{W}\ r\ f\ \beta) \neq \{\})$
⟨*proof*⟩

lemma *lem-Wf-esc-pth*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow\ 'U\ set$ **and** $\alpha::'U\ rel$
assumes $a1$: $Refl\ r \wedge \neg\ finite\ r$ **and** $a2$: $f \in \mathcal{N}\ r\ Ps$
and $a3$: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$ **and** $a4$: $\alpha <o\ |Field\ r|$
shows $\bigwedge\ F.\ F \in SCF\ (Restr\ r\ (f\ \alpha)) \Longrightarrow$
$\forall\ a \in \mathcal{W}\ r\ f\ \alpha.\ \exists\ b \in (F \cap (\mathcal{W}\ r\ f\ \alpha)).\ (a,b) \in (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha))\hat{}*$
⟨*proof*⟩

lemma *lem-Nf-lewfbnd*:
assumes $a1$: $f \in \mathcal{N}\ r\ Ps$ **and** $a2$: $\alpha \leq o\ |Field\ r|$ **and** $a3$: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$
shows $\omega\text{-}ord \leq o\ \alpha$
⟨*proof*⟩

lemma *lem-regcard-iso*: $\kappa =o\ \kappa' \Longrightarrow regularCard\ \kappa' \Longrightarrow regularCard\ \kappa$
⟨*proof*⟩

lemma *lem-cardsuc-inf-gwreg*: $\neg$ *finite* $A \Longrightarrow \kappa =o\ cardSuc\ |A| \Longrightarrow \omega\text{-}ord <o\ \kappa$
$\wedge$ *regularCard* κ
⟨*proof*⟩

lemma *lem-ccr-rcscf-struct*:
fixes *r*::'*U rel*
assumes *a1*: *Refl r* **and** *a2*: *CCR r* **and** *a3*: $\omega\text{-}ord <o$ *scf r* **and** *a4*: *regularCard* (*scf r*)
and *a5*: *scf r* $=o$ $|Field\ r|$
shows $\exists$ *Ps*. $\exists\ f \in \mathcal{N}\ r\ Ps$.
$\forall \alpha.\ \omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha| \wedge \alpha <o\ |Field\ r| \wedge isSuccOrd\ \alpha \longrightarrow$
$CCR\ (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha)) \wedge |Restr\ r\ (\mathcal{W}\ r\ f\ \alpha)| <o\ |Field\ r|$
$\wedge\ (\forall a \in \mathcal{W}\ r\ f\ \alpha.\ wesc\text{-}rel\ r\ f\ \alpha\ a\ (wesc\ r\ f\ \alpha\ a))$
⟨*proof*⟩

lemma *lem-oint-infcard-sc-cf*:
fixes $\alpha 0$::'*a rel* **and** κ::'*U rel* **and** *S*::'*U rel set*
assumes *a1*: *Card-order* κ **and** *a2*: $\omega\text{-}ord \leq o\ \kappa$
and *a3*: $S = \{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha 0 \leq o\ \alpha \wedge isSuccOrd\ \alpha \wedge \alpha <o\ \kappa\}$
shows $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$
⟨*proof*⟩

lemma *lem-oint-infcard-gew-sc-cfbnd*:
fixes $\alpha 0$::'*a rel* **and** κ::'*U rel* **and** *S*::'*U rel set*
assumes *a1*: *Card-order* κ **and** *a2*: $\omega\text{-}ord \leq o\ \kappa$ **and** *a3*: $\alpha 0 <o\ \kappa$ **and** *a4*: $\alpha 0$ $=o\ \omega\text{-}ord$
and *a5*: $S = \{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha 0 \leq o\ \alpha \wedge isSuccOrd\ \alpha \wedge \alpha <o\ \kappa\}$
shows $|\{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}| \leq o\ |S|$
$\wedge\ (\exists\ f.\ (\forall\ \alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha 0 \leq o\ \alpha \wedge \alpha <o\ \kappa \longrightarrow \alpha \leq o\ f\ \alpha \wedge f\ \alpha \in S))$
⟨*proof*⟩

lemma *lem-rcc-uset-rcc-bnd*:
assumes $s \in \mathfrak{U}\ r$
shows $\|r\| \leq o\ \|s\|$
⟨*proof*⟩

lemma *lem-dc2-ccr-scf-lew*:
fixes *r*::'*U rel*
assumes *a1*: *CCR r* **and** *a2*: *scf r* $\leq o\ \omega\text{-}ord$
shows *DCR 2 r*
⟨*proof*⟩

lemma *lem-dc3-ccr-refl-scf-wsuc*:
fixes *r*::'*U rel*
assumes *a1*: *Refl r* **and** *a2*: *CCR r*
and *a3*: $|Field\ r| =o\ cardSuc\ |UNIV::nat\ set|$ **and** *a4*: *scf r* $=o$ $|Field\ r|$

shows *DCR 3 r*
⟨*proof*⟩

lemma *lem-dc3-ccr-scf-lewsuc*:
fixes *r*::*′U rel*
assumes *a1*: *CCR r* **and** *a2*: $|Field\ r| \leq o\ cardSuc\ |UNIV::nat\ set|$
shows *DCR 3 r*
⟨*proof*⟩

lemma *lem-Cprf-conf-ccr-decomp*:
fixes *r*::*′U rel*
assumes *confl-rel r*
shows $\exists\ S::('U\ rel\ set).\ (\forall s\in S.\ CCR\ s) \wedge (r = \bigcup\ S) \wedge (\forall\ s1\in S.\ \forall s2\in S.\ s1 \neq s2 \longrightarrow Field\ s1 \cap Field\ s2 = \{\}\ )$
⟨*proof*⟩

lemma *lem-Cprf-dc-disj-fld-un*:
fixes *S*::*′U rel set* **and** *n*::*nat*
assumes *a1*: $\forall\ s1\in S.\ \forall s2\in S.\ s1\neq s2 \longrightarrow Field\ s1 \cap Field\ s2 = \{\}$
and *a2*: $\forall\ s\in S.\ DCR\ n\ s$
shows $DCR\ n\ (\bigcup\ S)$
⟨*proof*⟩

lemma *lem-dc3-to-d3*:
fixes *r*::*′U rel*
assumes *DCR 3 r*
shows *DCR3 r*
⟨*proof*⟩

lemma *lem-dc3-confl-lewsuc*:
fixes *r*::*′U rel*
assumes *a1*: *confl-rel r* **and** *a2*: $|Field\ r| \leq o\ cardSuc\ |UNIV::nat\ set|$
shows *DCR 3 r*
⟨*proof*⟩

lemma *lem-cle-eqdef*: $|A| \leq o\ |B| = (\exists\ g\ .\ A \subseteq g`B)$
⟨*proof*⟩

lemma *lem-cardLeN1-eqdef*:
fixes *A*::*′a set*
shows $cardLeN1\ A = (\ |A| \leq o\ cardSuc\ |\{n::nat\ .\ True\}|\ )$
⟨*proof*⟩

lemma *lem-cleN1-eqdef*:
fixes $r::('U\times'U)\ set$
shows $(\ |r| \leq o\ cardSuc\ |\{n::nat\ .\ True\}|\ )$
$\longleftrightarrow (\ \forall\ s \subseteq r.\ (\ \ (\ \forall\ t \subseteq s\ .\ ((\exists\ t'\ f.\ t' \subset t \wedge t \subseteq f`t') \longrightarrow (\exists\ f.\ s \subseteq f`t\ ))\ )$
$\vee\ (\ \exists\ g\ .\ r \subseteq g`s\ )$
$)\ )$

⟨*proof*⟩

1.2.3 Result

The next theorem has the following meaning: if the cardinality of a confluent binary relation r does not exceed the first uncountable cardinal, then confluence of r can be proved with the help of the decreasing diagrams method using no more than 3 labels (e.g. 0, 1, 2 ordered in the usual way).

theorem *thm-main*:
fixes *r*::('*U*×'*U*) *set*
assumes ∀ *a b c* . (*a*,*b*) ∈ *r*^∗ ∧ (*a*,*c*) ∈ *r*^∗ ⟶ (∃ *d*. (*b*,*d*) ∈ *r*^∗ ∧ (*c*,*d*) ∈ *r*^∗)
  and |*r*| ≤*o cardSuc* |{*n*::*nat* . *True*}|
shows ∃ *r0 r1 r2* . (
    (*r* = (*r0* ∪ *r1* ∪ *r2*))
  ∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r0*
    ⟶ (∃ *d*.
        (*b*,*d*) ∈ *r0*^=
      ∧ (*c*,*d*) ∈ *r0*^=))
  ∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r1*
    ⟶ (∃ *b*′ *d*.
        (*b*,*b*′) ∈ *r1*^= ∧ (*b*′,*d*) ∈ *r0*^∗
      ∧ (*c*,*d*) ∈ *r0*^∗))
  ∧ (∀ *a b c*. (*a*,*b*) ∈ *r1* ∧ (*a*,*c*) ∈ *r1*
    ⟶ (∃ *b*′ *b*′′ *c*′ *c*′′ *d*.
        (*b*,*b*′) ∈ *r0*^∗ ∧ (*b*′,*b*′′) ∈ *r1*^= ∧ (*b*′′,*d*) ∈ *r0*^∗
      ∧ (*c*,*c*′) ∈ *r0*^∗ ∧ (*c*′,*c*′′) ∈ *r1*^= ∧ (*c*′′,*d*) ∈ *r0*^∗))
  ∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r2*
    ⟶ (∃ *b*′ *d*.
        (*b*,*b*′) ∈ *r2*^= ∧ (*b*′,*d*) ∈ (*r0* ∪ *r1*)^∗
      ∧ (*c*,*d*) ∈ (*r0* ∪ *r1*)^∗))
  ∧ (∀ *a b c*. (*a*,*b*) ∈ *r1* ∧ (*a*,*c*) ∈ *r2*
    ⟶ (∃ *b*′ *b*′′ *d*.
        (*b*,*b*′) ∈ *r0*^∗ ∧ (*b*′,*b*′′) ∈ *r2*^= ∧ (*b*′′,*d*) ∈ (*r0* ∪ *r1*)^∗
      ∧ (*c*,*d*) ∈ (*r0* ∪ *r1*)^∗))
  ∧ (∀ *a b c*. (*a*,*b*) ∈ *r2* ∧ (*a*,*c*) ∈ *r2*
    ⟶ (∃ *b*′ *b*′′ *c*′ *c*′′ *d*.
        (*b*,*b*′) ∈ (*r0* ∪ *r1*)^∗ ∧ (*b*′,*b*′′) ∈ *r2*^= ∧ (*b*′′,*d*) ∈ (*r0* ∪ *r1*)^∗
      ∧ (*c*,*c*′) ∈ (*r0* ∪ *r1*)^∗ ∧ (*c*′,*c*′′) ∈ *r2*^= ∧ (*c*′′,*d*) ∈ (*r0* ∪ *r1*)^∗
))
)
⟨*proof*⟩

end

1.3 Optimality of the DCR3 method for proving confluence of relations of the least uncountable cardinality

theory *DCR3-Optimality*
 imports

HOL−Cardinals.Cardinals
Finite-DCR-Hierarchy
begin

1.3.1 Auxiliary definitions

datatype *Lev* = l*0* | l*1* | l*2* | l*3* | l*4* | l*5* | l*6* | l*7* | l*8*

type-synonym *'U rD* = *Lev* × *'U set* × *'U set* × *'U set*

fun *rP* :: *Lev* ⇒ *'U set* ⇒ *'U set* ⇒ *'U set* ⇒ *Lev* ⇒ *'U set* ⇒ *'U set* ⇒ *'U set* ⇒ *bool*
where
rP l*0* *A B C n' A' B' C'* = (*A* = {} ∧ *B* = {} ∧ *C* = {} ∧ *n'* = l*1* ∧ *finite A'* ∧ *B'* = {} ∧ *C'* = {})
| *rP* l*1* *A B C n' A' B' C'* = (*finite A* ∧ *B* = {} ∧ *C* = {} ∧ *n'* = l*2* ∧ *A'* = *A* ∧ *B'* = {} ∧ *C'* = {})
| *rP* l*2* *A B C n' A' B' C'* = (*finite A* ∧ *B* = {} ∧ *C* = {} ∧ *n'* = l*3* ∧ *A'* = *A* ∧ *finite B'* ∧ *C'* = {})
| *rP* l*3* *A B C n' A' B' C'* = (*finite A* ∧ *finite B* ∧ *C* = {} ∧ *n'* = l*4* ∧ *A'* = *A* ∧ *B'* = *B* ∧ *C'* = {})
| *rP* l*4* *A B C n' A' B' C'* = (*finite A* ∧ *finite B* ∧ *C* = {} ∧ *n'* = l*5* ∧ *A'* = *A* ∧ *B'* = *B* ∧ *finite C'*)
| *rP* l*5* *A B C n' A' B' C'* = (*finite A* ∧ *finite B* ∧ *finite C* ∧ *n'* = l*6* ∧ *A'* = *A* ∧ *B'* = *B* ∧ *C'* = *C*)
| *rP* l*6* *A B C n' A' B' C'* = (*finite A* ∧ *finite B* ∧ *finite C* ∧ *n'* = l*7* ∧ *A'* = *A* ∪ *B* ∪ *C* ∧ *B'* = *A'* ∧ *C'* = *A'*)
| *rP* l*7* *A B C n' A' B' C'* = (*finite A* ∧ *B* = *A* ∧ *C* = *A* ∧ *n'* = l*8* ∧ *A'* = *A* ∧ *B'* = *A'* ∧ *C'* = *A'*)
| *rP* l*8* *A B C n' A' B' C'* = (*finite A* ∧ *B* = *A* ∧ *C* = *A* ∧ *n'* = l*7* ∧ *A* ⊂ *A'* ∧ *finite A'* ∧ *B'* = *A'* ∧ *C'* = *A'*)

definition *rC* :: *'U set* ⇒ *'U set* ⇒ *'U set* ⇒ *'U set* ⇒ *bool*
where
rC S A B C = (*A* ⊆ *S* ∧ *B* ⊆ *S* ∧ *C* ⊆ *S*)

definition *rE* :: *'U set* ⇒ (*'U rD*) *rel*
where
rE S = { ((*n1*, *A1*, *B1*, *C1*), (*n2*, *A2*, *B2*, *C2*)). *rP n1 A1 B1 C1 n2 A2 B2 C2* ∧ *rC S A1 B1 C1* ∧ *rC S A2 B2 C2* }

fun *lev-next* :: *Lev* ⇒ *Lev*
where
lev-next l*0* = l*1*
| *lev-next* l*1* = l*2*
| *lev-next* l*2* = l*3*
| *lev-next* l*3* = l*4*
| *lev-next* l*4* = l*5*
| *lev-next* l*5* = l*6*

| *lev-next* l6 = l7
| *lev-next* l7 = l8
| *lev-next* l8 = l7

fun *levrd* :: *'U rD* ⇒ *Lev*
where
levrd (*n*, *A*, *B*, *C*) = *n*

fun *wrd* :: *'U rD* ⇒ *'U set*
where
wrd (*n*, *A*, *B*, *C*) = *A* ∪ *B* ∪ *C*

definition *Wrd* :: *'U rD set* ⇒ *'U set*
where
Wrd S = (⋃ (*wrd* ‘ *S*))

definition *bkset* :: *'a rel* ⇒ *'a set* ⇒ *'a set*
where
bkset r A = ((*r*^*)^−*1*)‘‘*A*

1.3.2 Auxiliary lemmas

lemma *lem-rtr-field*: (*x*,*y*) ∈ *r*^* ⟹ (*x* = *y*) ∨ (*x* ∈ *Field r* ∧ *y* ∈ *Field r*)
⟨*proof*⟩

lemma *lem-fin-fl-rel*: *finite* (*Field r*) = *finite r*
⟨*proof*⟩

lemma *lem-rel-inf-fld-card*:
fixes *r*::*'U rel*
assumes ¬ *finite r*
shows |*Field r*| =*o* |*r*|
⟨*proof*⟩

lemma *lem-confl-field*: *confl-rel r* = (∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. ∀ *c* ∈ *Field r*.
(*a*,*b*) ∈ *r*^* ∧ (*a*,*c*) ∈ *r*^* ⟶
(∃ *d* ∈ *Field r*. (*b*,*d*) ∈ *r*^* ∧ (*c*,*d*) ∈ *r*^*))
⟨*proof*⟩

lemma *lem-d2-to-dc2*:
fixes *r*::*'U rel*
assumes *DCR2 r*
shows *DCR 2 r*
⟨*proof*⟩

lemma *lem-dc2-to-d2*:
fixes *r*::*'U rel*
assumes *DCR 2 r*
shows *DCR2 r*

⟨*proof*⟩

lemma *lem-rP-inv*: $rP\ n\ A\ B\ C\ n'\ A'\ B'\ C' \Longrightarrow (\ A \subseteq A' \wedge B \subseteq B' \wedge C \subseteq C' \wedge \mathit{finite}\ A \wedge \mathit{finite}\ B \wedge \mathit{finite}\ C \wedge \mathit{finite}\ A' \wedge \mathit{finite}\ B' \wedge \mathit{finite}\ C'\ )$
⟨*proof*⟩

lemma *lem-infset-finext*:
fixes $S::'U\ set$ **and** $A::'U\ set$
assumes $\neg\ \mathit{finite}\ S$ **and** $\mathit{finite}\ A$ **and** $A \subseteq S$
shows $\exists\ B.\ B \subseteq S \wedge A \subset B \wedge \mathit{finite}\ B$
⟨*proof*⟩

lemma *lem-rE-df*:
fixes $S::'U\ set$
shows $(u,v) \in rE\ S \Longrightarrow (u,w) \in rE\ S \Longrightarrow (v,t) \in (rE\ S)\hat{}{=} \Longrightarrow (w,t) \in (rE\ S)\hat{}{=} \Longrightarrow v = w$
⟨*proof*⟩

lemma *lem-rE-succ-lev*:
fixes $S::'U\ set$
assumes $(u,v) \in rE\ S$
shows $\mathit{levrd}\ v = (\mathit{lev\text{-}next}\ (\mathit{levrd}\ u))$
⟨*proof*⟩

lemma *lem-rE-levset-inv*:
fixes $S::'U\ set$ **and** $L\ u\ v$
assumes *a1*: $(u,v) \in (rE\ S)\hat{}{*}$ **and** *a2*: $\mathit{levrd}\ u \in L$ **and** *a3*: $\mathit{lev\text{-}next}\ `\ L \subseteq L$
shows $\mathit{levrd}\ v \in L$
⟨*proof*⟩

lemma *lem-rE-levun*:
fixes $S::'U\ set$
shows $u \in \mathit{Domain}\ (rE\ S) \Longrightarrow \mathit{levrd}\ u \in \{\mathit{l1}, \mathit{l3}, \mathit{l5}\} \Longrightarrow \exists\ v.\ (rE\ S)``\{u\} \subseteq \{v\}$
⟨*proof*⟩

lemma *lem-rE-domfield*:
fixes $S::'U\ set$
assumes $\neg\ \mathit{finite}\ S$
shows $\mathit{Domain}\ (rE\ S) = \mathit{Field}\ (rE\ S)$
⟨*proof*⟩

lemma *lem-wrd-fin-field-rE*:
fixes $S::'U\ set$
assumes $\neg\ \mathit{finite}\ S$
shows $u \in \mathit{Field}\ (rE\ S) \Longrightarrow \mathit{finite}\ (\mathit{wrd}\ u)$
⟨*proof*⟩

lemma *lem-rE-rtr-wrd-mon*:
fixes $S::'U\ set$ **and** $u\ v::'U\ rD$

shows $(u,v) \in (rE\ S)\widehat{\ }* \Longrightarrow wrd\ u \subseteq wrd\ v$
$\langle proof \rangle$

lemma *lem-Wrd-bkset-rE*: $Wrd\ (bkset\ (rE\ S)\ U) = Wrd\ U$
$\langle proof \rangle$

lemma *lem-Wrd-rE-field-subs-cnt*:
fixes $S{::}'U\ set$ **and** $U{::}('U\ rD)\ set$
assumes $\neg\ finite\ S$
shows $U \subseteq Field\ (rE\ S) \Longrightarrow |U| \leq o\ |UNIV{::}nat\ set| \Longrightarrow |Wrd\ U| \leq o\ |UNIV{::}nat\ set|$
$\langle proof \rangle$

lemma *lem-rE-dn-cnt*:
fixes $S{::}'U\ set$ **and** $U{::}('U\ rD)\ set$
assumes $\neg\ finite\ S$
shows $U \subseteq Field\ (rE\ S) \Longrightarrow |U| \leq o\ |UNIV{::}nat\ set| \Longrightarrow V \subseteq bkset\ (rE\ S)\ U \Longrightarrow |Wrd\ V| \leq o\ |UNIV{::}nat\ set|$
$\langle proof \rangle$

lemma *lem-rE-succ-Wrd-univ*: $(u,w) \in (rE\ S) \Longrightarrow levrd\ u \in \{l0, l2, l4\} \Longrightarrow S - wrd\ w \subseteq Wrd\ (((rE\ S)``\{u\}) - \{w\})$
$\langle proof \rangle$

lemma *lem-rE-succ-nocntbnd*:
fixes $S{::}'U\ set$ **and** $u0{::}'U\ rD$ **and** $v0{::}'U\ rD$ **and** $U{::}('U\ rD)\ set$
assumes $a0$: $\neg\ |S| \leq o\ |UNIV{::}nat\ set|$ **and** $a1$: $(u0,\ v0) \in (rE\ S)$ **and** $a2$: $levrd\ u0 \in \{l0, l2, l4\}$
  and $a3$: $U \subseteq Field\ (rE\ S)$ **and** $a4$: $((rE\ S)\ ``\ \{u0\}) - \{v0\} \subseteq bkset\ (rE\ S)\ U$
shows $\neg\ |U| \leq o\ |UNIV{::}nat\ set|$
$\langle proof \rangle$

lemma *lem-rE-succ-nocntbnd2*:
fixes $S{::}'U\ set$ **and** $u0{::}'U\ rD$ **and** $v0{::}'U\ rD$
assumes $a0$: $\neg\ |S| \leq o\ |UNIV{::}nat\ set|$
  and $a1$: $(u0,\ v0) \in (rE\ S)$ **and** $a2$: $levrd\ u0 \in \{l0, l2, l4\}$
  and $a3$: $r \subseteq (rE\ S)$ **and** $a4$: $\forall\ u.\ |r``\{u\}| \leq o\ |UNIV{::}nat\ set|$
  and $a5$: $((rE\ S)\ ``\ \{u0\}) - \{v0\} \subseteq bkset\ (rE\ S)\ ((r\widehat{\ }*)``\{u0\})$
shows $False$
$\langle proof \rangle$

lemma *lem-rE-diamsubr-un*:
fixes $S{::}'U\ set$
assumes $a1$: $r0 \subseteq (rE\ S)$ **and** $a2$: $\forall\ a\ b\ c.\ (a,b) \in r0 \wedge (a,c) \in r0 \longrightarrow (\exists\ d.\ (b,d) \in r0\widehat{\ }= \wedge (c,d) \in r0\widehat{\ }=)$
shows $\forall\ u.\ \exists\ v.\ r0``\{u\} \subseteq \{v\}$
$\langle proof \rangle$

lemma *lem-rE-succ-nocntbnd3*:

fixes *S*::′*U set* **and** *u0*::′*U rD* **and** *v0*::′*U rD*
assumes *a0*: ¬ |*S*| ≤*o* |*UNIV*::*nat set*|
and *a1*: *LD2* (*rE S*) *r0 r1*
and *a2*: (*u0*, *v0*) ∈ (*rE S*) **and** *a3*: *levrd u0* ∈ {l*0*, l*2*, l*4*}
and *a4*: *r* = {(*u*,*v*) ∈ *rE S*. *u* = *v0*} ∪ *r0*
and *a5*: ((*rE S*) ‘‘ {*u0*}) − {*v0*} ⊆ *bkset* (*rE S*) ((*r*^∗)‘‘{*u0*})
shows *False*
⟨*proof*⟩

lemma *lem-rE-one*:
fixes *S*::′*U set* **and** *u0*::′*U rD* **and** *v0*::′*U rD*
assumes *a0*: ¬ |*S*| ≤*o* |*UNIV*::*nat set*| **and** *a1*: *LD2* (*rE S*) *r0 r1*
and *a2*: (*u0*, *v0*) ∈ *r0* **and** *a3*: *levrd u0* ∈ {l*0*, l*2*, l*4*}
shows *False*
⟨*proof*⟩

lemma *lem-rE-jn0*:
fixes *S*::′*U set* **and** *u1*::′*U rD* **and** *u2*::′*U rD* **and** *v*::′*U rD*
assumes *a1*: (*u1*,*v*) ∈ (*rE S*) **and** *a2*: (*u2*,*v*) ∈ (*rE S*) **and** *a3*: *u1* ≠ *u2*
shows *levrd v* ∈ {l*7*, l*8*}
⟨*proof*⟩

lemma *lem-rE-jn1*:
fixes *S*::′*U set* **and** *u1*::′*U rD* **and** *u2*::′*U rD* **and** *v*::′*U rD*
assumes *a1*: (*u1*,*v*) ∈ (*rE S*) **and** *a2*: (*u2*,*v*) ∈ (*rE S*)^∗ **and** *a3*: (*u1*,*u2*) ∉ (*rE S*) ∧ (*u2*,*u1*) ∉ (*rE S*)^∗
shows *levrd v* ∈ {l*7*, l*8*}
⟨*proof*⟩

lemma *lem-rE-jn2*:
fixes *S*::′*U set* **and** *u1*::′*U rD* **and** *u2*::′*U rD* **and** *v*::′*U rD*
assumes *a1*: (*u1*,*v*) ∈ (*rE S*)^∗ **and** *a2*: (*u2*,*v*) ∈ (*rE S*)^∗ **and** *a3*: (*u1*,*u2*) ∉ (*rE S*)^∗ ∧ (*u2*,*u1*) ∉ (*rE S*)^∗
shows *levrd v* ∈ {l*7*, l*8*}
⟨*proof*⟩

lemma *lem-rel-pow2fw*: (*u*,*u1*) ∈ *r* ∧ (*u1*,*v*) ∈ *r* ⟶ (*u*,*v*) ∈ *r*^^*2*
⟨*proof*⟩

lemma *lem-rel-pow3fw*: (*u*,*u1*) ∈ *r* ∧ (*u1*,*u2*) ∈ *r* ∧ (*u2*,*v*) ∈ *r* ⟶ (*u*,*v*) ∈ *r*^^*3*
⟨*proof*⟩

lemma *lem-rel-pow3*: (*u*,*v*) ∈ *r*^^*3* ⟹ ∃ *u1 u2*. (*u*,*u1*) ∈ *r* ∧ (*u1*,*u2*) ∈ *r* ∧ (*u2*,*v*) ∈ *r*
⟨*proof*⟩

lemma *lem-rel-pow4*: (*u*,*v*) ∈ *r*^^*4* ⟹ ∃ *u1 u2 u3*. (*u*,*u1*) ∈ *r* ∧ (*u1*,*u2*) ∈ *r* ∧ (*u2*,*u3*) ∈ *r* ∧ (*u3*,*v*) ∈ *r*
⟨*proof*⟩

lemma *lem-rel-pow5*: $(u,v) \in r\widehat{\ }\widehat{\ }5 \Longrightarrow \exists\ u1\ u2\ u3\ u4.\ (u,u1) \in r \wedge (u1,u2) \in r \wedge (u2,u3) \in r \wedge (u3,u4) \in r \wedge (u4,v) \in r$
⟨*proof*⟩

lemma *lem-rE-l1-l78-dist*:
fixes $S::'U\ set$
assumes *a1*: $levrd\ u = \mathrm{l}1$ **and** *a2*: $levrd\ v \in \{\mathrm{l}7, \mathrm{l}8\}$ **and** *a3*: $n \leq 5$
shows $(u,v) \notin (rE\ S)\widehat{\ }\widehat{\ }n$
⟨*proof*⟩

lemma *lem-rE-notLD2*:
fixes $S::'U\ set$ **and** $r0\ r1::('U\ rD)\ rel$
assumes *a0*: $\neg\ |S| \leq o\ |UNIV::nat\ set|$ **and** *a1*: $LD2\ (rE\ S)\ r0\ r1$
shows *False*
⟨*proof*⟩

lemma *lem-rE-dominv*:
fixes $S::'U\ set$
assumes $\neg$ *finite S*
shows $u \in Domain\ (rE\ S) \Longrightarrow (u,v) \in (rE\ S)\widehat{\ }* \Longrightarrow v \in Domain\ (rE\ S)$
⟨*proof*⟩

lemma *lem-rE-next*:
fixes $S::'U\ set$
assumes $\neg$ *finite S* **and** $u \in Domain\ (rE\ S)$
shows $\exists\ v.\ (u,v) \in (rE\ S) \wedge v \in Domain\ (rE\ S) \wedge levrd\ v = (lev\text{-}next\ (levrd\ u))$
⟨*proof*⟩

lemma *lem-rE-reachl8*:
fixes $S::'U\ set$
assumes $\neg$ *finite S* **and** $u \in Domain\ (rE\ S)$
shows $\exists\ v.\ (u,v) \in (rE\ S)\widehat{\ }* \wedge v \in Domain\ (rE\ S) \wedge levrd\ v = \mathrm{l}8$
⟨*proof*⟩

lemma *lem-rE-jn*:
fixes $S::'U\ set$
assumes *a0*: $\neg$ *finite S* **and** *a1*: $u1 \in Domain\ (rE\ S)$ **and** *a2*: $u2 \in Domain\ (rE\ S)$
shows $\exists\ t.\ (u1,t) \in (rE\ S)\widehat{\ }* \wedge (u2,t) \in (rE\ S)\widehat{\ }*$
⟨*proof*⟩

lemma *lem-rE-confl*:
fixes $S::'U\ set$
assumes $\neg$ *finite S*
shows *confl-rel* $(rE\ S)$
⟨*proof*⟩

lemma *lem-rE-dc3dc2*:

fixes *S*::*'U set*
assumes $\neg$ |*S*| $\le o$ |*UNIV*::*nat set*|
shows *confl-rel* (*rE S*) $\wedge$ ($\neg$ *DCR2* (*rE S*))
⟨*proof*⟩

lemma *lem-rE-cardbnd*:
fixes *S*::*'U set*
assumes $\neg$ *finite S*
shows |*rE S*| $\le o$ |*S*|
⟨*proof*⟩

lemma *lem-fmap-rel*:
fixes *f r s a0 b0*
assumes *a1*: (*a0*, *b0*) $\in$ *r*^* **and** *a2*: $\forall$ *a b*. (*a*,*b*) $\in$ *r* $\longrightarrow$ (*f a*, *f b*) $\in$ *s*
shows (*f a0*, *f b0*) $\in$ *s*^*
⟨*proof*⟩

lemma *lem-fmap-confl*:
fixes *r*::*'a rel* **and** *f*::*'a* $\Rightarrow$ *'b*
assumes *a1*: *inj-on f* (*Field r*) **and** *a2*: *confl-rel r*
shows *confl-rel* {(*u*,*v*). $\exists$ *a b*. *u* = *f a* $\wedge$ *v* = *f b* $\wedge$ (*a*,*b*) $\in$ *r*}
⟨*proof*⟩

lemma *lem-fmap-dcn*:
fixes *r*::*'a rel* **and** *f*::*'a* $\Rightarrow$ *'b*
assumes *a1*: *inj-on f* (*Field r*) **and** *a2*: *DCR n r*
shows *DCR n* {(*u*,*v*). $\exists$ *a b*. *u* = *f a* $\wedge$ *v* = *f b* $\wedge$ (*a*,*b*) $\in$ *r*}
⟨*proof*⟩

lemma *lem-not-dcr2*:
assumes *cardSuc* |*UNIV*::*nat set*| $\le o$ |*UNIV*::*'U set*|
shows $\exists$ *r*::*'U rel*. *confl-rel r* $\wedge$ |*r*| $\le o$ *cardSuc* |*UNIV*::*nat set*| $\wedge$ ($\neg$ *DCR2 r*)
⟨*proof*⟩

1.3.3 Result

The next theorem has the following meaning: if the set of elements of type *'U* is uncountable, then there exists a confluent binary relation *r* on *'U* such that the cardinality of *r* does not exceed the first uncountable cardinal and confluence of *r* cannot be proved using the decreasing diagrams method with 2 labels.

theorem *thm-example-not-dcr2*:
assumes *cardSuc* |{*n*::*nat*. *True*}| $\le o$ |{*x*::*'U*. *True*}|
shows $\exists$ *r*::*'U rel*. (
($\forall$ *a b c*. (*a*,*b*) $\in$ *r*^* $\wedge$ (*a*,*c*) $\in$ *r*^* $\longrightarrow$ ($\exists$ *d*. (*b*,*d*) $\in$ *r*^* $\wedge$ (*c*,*d*) $\in$ *r*^*)
)
$\wedge$ |*r*| $\le o$ *cardSuc* |{*n*::*nat*. *True*}|
$\wedge$ ($\neg$ ($\exists$ *r0 r1*. (

($r = (r0 \cup r1)$)
$\wedge$ ($\forall$ a b c. $(a,b) \in r0 \wedge (a,c) \in r0$
$\longrightarrow$ ($\exists$ d.
$(b,d) \in r0\widehat{\ }=$
$\wedge$ $(c,d) \in r0\widehat{\ }=$))
$\wedge$ ($\forall$ a b c. $(a,b) \in r0 \wedge (a,c) \in r1$
$\longrightarrow$ ($\exists$ b' d.
$(b,b') \in r1\widehat{\ }= \wedge (b',d) \in r0\widehat{\ }*$
$\wedge$ $(c,d) \in r0\widehat{\ }*$))
$\wedge$ ($\forall$ a b c. $(a,b) \in r1 \wedge (a,c) \in r1$
$\longrightarrow$ ($\exists$ b' b'' c' c'' d.
$(b,b') \in r0\widehat{\ }* \wedge (b',b'') \in r1\widehat{\ }= \wedge (b'',d) \in r0\widehat{\ }*$
$\wedge$ $(c,c') \in r0\widehat{\ }* \wedge (c',c'') \in r1\widehat{\ }= \wedge (c'',d) \in r0\widehat{\ }*$))))
))
⟨*proof*⟩

corollary *cor-example-not-dcr2*:
shows $\exists$ *r*::(*nat set*) *rel*. (
($\forall$ a b c. $(a,b) \in r\widehat{\ }* \wedge (a,c) \in r\widehat{\ }* \longrightarrow$ ($\exists$ d. $(b,d) \in r\widehat{\ }* \wedge (c,d) \in r\widehat{\ }*$)
)
$\wedge$ $|r| \leq o$ *cardSuc* $|\{n::nat.\ True\}|$
$\wedge$ ($\neg$ ($\exists$ *r0 r1* . (
($r = (r0 \cup r1)$)
$\wedge$ ($\forall$ a b c. $(a,b) \in r0 \wedge (a,c) \in r0$
$\longrightarrow$ ($\exists$ d.
$(b,d) \in r0\widehat{\ }=$
$\wedge$ $(c,d) \in r0\widehat{\ }=$))
$\wedge$ ($\forall$ a b c. $(a,b) \in r0 \wedge (a,c) \in r1$
$\longrightarrow$ ($\exists$ b' d.
$(b,b') \in r1\widehat{\ }= \wedge (b',d) \in r0\widehat{\ }*$
$\wedge$ $(c,d) \in r0\widehat{\ }*$))
$\wedge$ ($\forall$ a b c. $(a,b) \in r1 \wedge (a,c) \in r1$
$\longrightarrow$ ($\exists$ b' b'' c' c'' d.
$(b,b') \in r0\widehat{\ }* \wedge (b',b'') \in r1\widehat{\ }= \wedge (b'',d) \in r0\widehat{\ }*$
$\wedge$ $(c,c') \in r0\widehat{\ }* \wedge (c',c'') \in r1\widehat{\ }= \wedge (c'',d) \in r0\widehat{\ }*$))))
))
⟨*proof*⟩

end

1.4 DCR implies LD Property

theory *Main-Result-DCR-N1*
imports
DCR3-Method
Decreasing−Diagrams.Decreasing-Diagrams
begin

1.4.1 Auxiliary definitions

definition *map-seq-labels* :: $('b \Rightarrow 'c) \Rightarrow ('a,'b)\ seq \Rightarrow ('a,'c)\ seq$
where
map-seq-labels $f\ \sigma = (fst\ \sigma,\ map\ (\lambda(\alpha,a).\ (f\ \alpha,\ a))\ (snd\ \sigma))$

fun *map-diag-labels* :: $('b \Rightarrow 'c) \Rightarrow$
$('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq \Rightarrow$
$('a,'c)\ seq \times ('a,'c)\ seq \times ('a,'c)\ seq \times ('a,'c)\ seq$
where
map-diag-labels $f\ (\tau,\sigma,\sigma',\tau') = ((map\text{-}seq\text{-}labels\ f\ \tau), (map\text{-}seq\text{-}labels\ f\ \sigma), (map\text{-}seq\text{-}labels\ f\ \sigma'), (map\text{-}seq\text{-}labels\ f\ \tau'))$

fun *f-to-ls* :: $(nat \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a\ list$
where
f-to-ls $f\ 0 = []$
| *f-to-ls* $f\ (Suc\ n) = (f\text{-}to\text{-}ls\ f\ n)\ @\ [(f\ n)]$

1.4.2 Auxiliary lemmas

lemma *lem-ftofs-len*: $length\ (f\text{-}to\text{-}ls\ f\ n) = n$ ⟨*proof*⟩

lemma *lem-irr-inj-im-irr*:
fixes $r{::}'a\ rel$ **and** $r'{::}'b\ rel$ **and** $f{::}'a \Rightarrow 'b$
assumes *irrefl* r **and** *inj-on* f $(Field\ r)$
and $r' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \wedge b' = f\ b \wedge (a,b) \in r\}$
shows *irrefl* r'
⟨*proof*⟩

lemma *lem-tr-inj-im-tr*:
fixes $r{::}'a\ rel$ **and** $r'{::}'b\ rel$ **and** $f{::}'a \Rightarrow 'b$
assumes *trans* r **and** *inj-on* f $(Field\ r)$
and $r' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \wedge b' = f\ b \wedge (a,b) \in r\}$
shows *trans* r'
⟨*proof*⟩

lemma *lem-lpeak-expr*: *local-peak* $lrs\ (\tau,\ \sigma) = (\exists\ a\ b\ c\ \alpha\ \beta.\ (a,\alpha,b) \in lrs \wedge (a,\beta,c) \in lrs \wedge \tau = (a,[(\alpha,b)]) \wedge \sigma = (a,[(\beta,c)]))$
⟨*proof*⟩

lemma *lem-map-seq*:
fixes $lrs{::}('a,'b)\ lars$ **and** $f{::}'b \Rightarrow 'c$ **and** $lrs'{::}('a,'c)\ lars$ **and** $\sigma{::}('a,'b)\ seq$
assumes *a1*: $lrs' = \{(a,l',b).\ \exists l.\ l' = f\ l \wedge (a,l,b) \in lrs\ \}$
and *a2*: $\sigma \in$ *Decreasing-Diagrams.seq lrs*
shows $(map\text{-}seq\text{-}labels\ f\ \sigma) \in$ *Decreasing-Diagrams.seq* lrs'
⟨*proof*⟩

lemma *lem-map-diag*:
fixes $lrs{::}('a,'b)\ lars$ **and** $f{::}'b \Rightarrow 'c$ **and** $lrs'{::}('a,'c)\ lars$
and $d{::}('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq$

assumes *a1*: $lrs' = \{(a,l',b).\ \exists l.\ l' = f\ l \wedge (a,l,b) \in lrs\ \}$
  and *a2*: *diagram lrs d*
shows *diagram lrs′* (*map-diag-labels f d*)
⟨*proof*⟩

lemma *lem-map-D-loc*:
fixes *cmp cmp′ s1 s2 s3 s4 f*
assumes *a1*: *Decreasing-Diagrams.D cmp s1 s2 s3 s4*
  and *a2*: *trans cmp* **and** *a3*: *irrefl cmp* **and** *a4*: *inj-on f* (*Field cmp*)
  and *a5*: $cmp' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \wedge b' = f\ b \wedge (a,b) \in cmp\}$
  and *a6*: *length s1* = *1* **and** *a7*: *length s2* = *1*
shows *Decreasing-Diagrams.D cmp′* (*map f s1*) (*map f s2*) (*map f s3*) (*map f s4*)
⟨*proof*⟩

lemma *lem-map-DD-loc*:
fixes *lrs*::(*′a*,*′b*) *lars* **and** *cmp*::*′b rel* **and** *lrs′*::(*′a*,*′c*) *lars* **and** *cmp′*::*′c rel* **and** f::*′b* ⇒ *′c*
assumes *a1*: *trans cmp* **and** *a2*: *irrefl cmp* **and** *a3*: *inj-on f* (*Field cmp*)
  and *a4*: $cmp' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \wedge b' = f\ b \wedge (a,b) \in cmp\}$
  and *a5*: $lrs' = \{(a,l',b).\ \exists l.\ l' = f\ l \wedge (a,l,b) \in lrs\ \}$
  and *a6*: *length* (*snd* (*fst d*)) = *1* **and** *a7*: *length* (*snd* (*fst* (*snd d*))) = *1*
  and *a8*: *DD lrs cmp d*
shows *DD lrs′ cmp′* (*map-diag-labels f d*)
⟨*proof*⟩

lemma *lem-ddseq-mon*: $lrs1 \subseteq lrs2 \Longrightarrow$ *Decreasing-Diagrams.seq lrs1* ⊆ *Decreasing-Diagrams.seq lrs2*
⟨*proof*⟩

lemma *lem-dd-D-mon*:
fixes *cmp1 cmp2* α β *s1 s2*
assumes *a1*: *trans cmp1* ∧ *irrefl cmp1* **and** *a2*: *trans cmp2* ∧ *irrefl cmp2* **and** *a3*: *cmp1* ⊆ *cmp2*
  and *a4*: *Decreasing-Diagrams.D cmp1* [α] [β] *s1 s2*
shows *Decreasing-Diagrams.D cmp2* [α] [β] *s1 s2*
⟨*proof*⟩

1.4.3 Result

The next lemma has the following meaning: every ARS in the finite DCR hierarchy has the LD property.

lemma *lem-dcr-to-ld*:
fixes *n*::*nat* **and** *r*::*′U rel*
assumes *DCR n r*
shows *LD* (*UNIV*::*nat set*) *r*
⟨*proof*⟩

2 Main theorem

The next theorem has the following meaning: if the cardinality of a binary relation r does not exceed the first uncountable cardinal ($cardSuc\ |UNIV::nat\ set|$), then the following two conditions are equivalent:

1. r is confluent ($Abstract\text{-}Rewriting.CR\ r$)

2. r can be proven confluent using the decreasing diagrams method with natural numbers as labels ($Decreasing\text{-}Diagrams.LD\ (UNIV::nat\ set)\ r$).

theorem *N1-completeness*:
fixes $r::'a\ rel$
assumes $|r| \leq o\ cardSuc\ |UNIV::nat\ set|$
shows $Abstract\text{-}Rewriting.CR\ r = Decreasing\text{-}Diagrams.LD\ (UNIV::nat\ set)\ r$
$\langle proof \rangle$

end

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