

# Ceva's Theorem

Mathias Schack Rabing

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## Abstract

This entry contains a definition of the area the triangle constructed by three points. Building on this, some basic geometric properties about the area of a triangle are derived. These properties are used to prove Ceva's theorem.

## Contents

**theory** *Ceva*

**imports**

*Triangle.Triangle*

**begin**

**definition** *Triangle-area* :: 'a::real-inner  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  real

**where** *Triangle-area* x y z = abs(sin (angle x y z)) \* dist x y \* dist y z

**lemma** *Triangle-area-per1* : *Triangle-area* a b c = *Triangle-area* b c a

*<proof>*

**lemma** *Triangle-area-per2* : *Triangle-area* a b c = *Triangle-area* b a c

*<proof>*

**lemma** *collinear-angle*:

**fixes** a b c :: 'a::euclidean-space

**shows** collinear {a, b, c}  $\Longrightarrow$  a  $\neq$  b  $\Longrightarrow$  b  $\neq$  c  $\Longrightarrow$  angle a b c  $\in$  {0, pi}

*<proof>*

**lemma** *Triangle-area-0* :

**fixes** c :: 'a::euclidean-space

**shows** *Triangle-area* a b c = 0  $\longleftrightarrow$  collinear {a,b,c}

*<proof>*

**lemma** *Angle-longer-side* :

**fixes** a :: 'a :: euclidean-space

**assumes**  $Col : \textit{between} (b,d) c$   
**assumes**  $NeqBC : b \neq c$   
**shows**  $\textit{angle} a b c = \textit{angle} a b d$   
 <proof>

**lemma**  $\textit{Triangle-area-comb} :$   
**fixes**  $c :: 'a::\textit{euclidean-space}$   
**assumes**  $Col : \textit{between} (b,c) m$   
**shows**  $\textit{Triangle-area} a b m + \textit{Triangle-area} a c m = \textit{Triangle-area} a b c$   
 <proof>

**lemma**  $\textit{Triangle-area-cal} :$   
**fixes**  $a :: 'a::\textit{euclidean-space}$   
**assumes**  $Col : \textit{collinear} \{a,m,b\}$   
**shows**  $\exists k. \textit{dist} a m * k = \textit{Triangle-area} a c m \wedge \textit{dist} b m * k = \textit{Triangle-area} b c m$   
 <proof>

**lemma**  $\textit{Triangle-area-comb-alt} :$   
**fixes**  $a :: 'a::\textit{euclidean-space}$   
**assumes**  $Col1 : \textit{collinear} \{a,m,b\}$   
**assumes**  $Col2 : \textit{collinear} \{c,k,m\}$   
**shows**  $\textit{Goal} : \exists h. \textit{dist} a m * h = \textit{Triangle-area} a c k \wedge \textit{dist} b m * h = \textit{Triangle-area} b c k$   
 <proof>

**lemma**  $\textit{Cevas} :$   
**fixes**  $a :: 'a::\textit{euclidean-space}$   
**assumes**  $\textit{MidCol} : \textit{collinear} \{a,k,d\} \wedge \textit{collinear} \{b,k,e\} \wedge \textit{collinear} \{c,k,f\}$   
**assumes**  $\textit{TriCol} : \textit{collinear} \{a,f,b\} \wedge \textit{collinear} \{a,e,c\} \wedge \textit{collinear} \{b,d,c\}$   
**assumes**  $\textit{Triangle} : \neg \textit{collinear} \{a,b,c\}$   
**shows**  $\textit{dist} a f * \textit{dist} b d * \textit{dist} c e = \textit{dist} f b * \textit{dist} d c * \textit{dist} e a$   
 <proof>

end