A Verified Code Generator from Isabelle/HOL to CakeML

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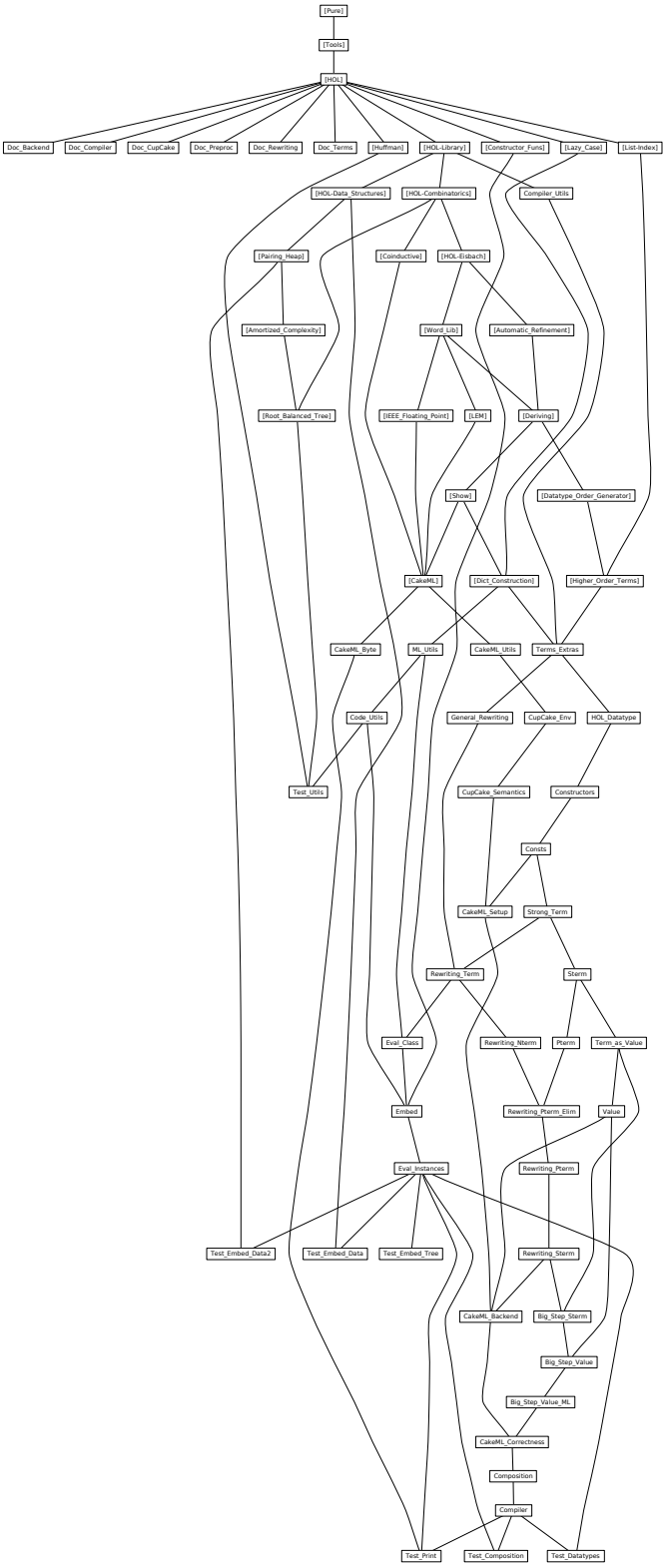
[Pure]
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Composition
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Test_Print
Test_Composition
Test_Datatypes

Chapter 1

Terms

theory *Doc-Terms*
imports *Main*
begin

end

1.1 Additional material over the *Higher-Order-Terms* AFP entry

theory *Terms-Extras*
imports
  ../Utils/Compiler-Utils
  Higher-Order-Terms.Pats
  Dict-Construction.Dict-Construction
begin

no-notation *Mpat-Antiquot.mpaq-App* (**infixl** ‹$$› *900*)

ML-file *hol-term.ML*

primrec *basic-rule* :: - ⇒ *bool* **where**
basic-rule (*lhs*, *rhs*) ⟷
  linear lhs ∧
  is-const (*fst* (*strip-comb lhs*)) ∧
  ¬ *is-const lhs* ∧
  frees rhs |⊆| *frees lhs*

lemma *basic-ruleI*[*intro*]:
  assumes *linear lhs*
  assumes *is-const* (*fst* (*strip-comb lhs*))
  assumes ¬ *is-const lhs*
  assumes *frees rhs* |⊆| *frees lhs*
  shows *basic-rule* (*lhs*, *rhs*)

using *assms* **by** *simp*

primrec *split-rule* :: (*term* × *'a*) ⇒ (*name* × (*term list* × *'a*)) **where**
split-rule (*lhs*, *rhs*) = (*let* (*name*, *args*) = *strip-comb lhs in* (*const-name name*, (*args*, *rhs*)))

fun *unsplit-rule* :: (*name* × (*term list* × *'a*)) ⇒ (*term* × *'a*) **where**
unsplit-rule (*name*, (*args*, *rhs*)) = (*name* $$ *args*, *rhs*)

lemma *split-unsplit*: *split-rule* (*unsplit-rule t*) = *t*
by (*induct t rule*: *unsplit-rule.induct*) (*simp add*: *strip-list-comb const-name-def*)

lemma *unsplit-split*:
  assumes *basic-rule r*
  shows *unsplit-rule* (*split-rule r*) = *r*
using *assms*
by (*cases r*) (*simp add*: *split-beta*)

datatype *pat* = *Patvar name* | *Patconstr name pat list*

fun *mk-pat* :: *term* ⇒ *pat* **where**
mk-pat pat = (*case strip-comb pat of* (*Const s*, *args*) ⇒ *Patconstr s* (*map mk-pat args*) | (*Free s*, []) ⇒ *Patvar s*)

declare *mk-pat.simps*[*simp del*]

lemma *mk-pat-simps*[*simp*]:
  mk-pat (*name* $$ *args*) = *Patconstr name* (*map mk-pat args*)
  mk-pat (*Free name*) = *Patvar name*
apply (*auto simp*: *mk-pat.simps strip-list-comb-const*)
apply (*simp add*: *const-term-def*)
done

primrec *patvars* :: *pat* ⇒ *name fset* **where**
patvars (*Patvar name*) = {| *name* |} |
patvars (*Patconstr - ps*) = *ffUnion* (*fset-of-list* (*map patvars ps*))

lemma *mk-pat-frees*:
  assumes *linear p*
  shows *patvars* (*mk-pat p*) = *frees p*
using *assms* **proof** (*induction p rule*: *linear-pat-induct*)
  case (*comb name args*)

  have *map* (*patvars* ∘ *mk-pat*) *args* = *map frees args*
    using *comb* **by** *force*
  hence *fset-of-list* (*map* (*patvars* ∘ *mk-pat*) *args*) = *fset-of-list* (*map frees args*)
    by *metis*
  thus *?case*
    by (*simp add*: *freess-def*)

qed *simp*

This definition might seem a little counter-intuitive. Assume we have two defining equations of a function, e.g. *map*: *map f* [] = [] *map f* (*x* # *xs*) = *f x* # *map f xs* The pattern "matrix" is compiled right-to-left. Equal patterns are grouped together. This definition is needed to avoid the following situation: *map f* [] = [] *map g* (*x* # *xs*) = *g x* # *map g xs* While this is logically the same as above, the problem is that *f* and *g* are overlapping but distinct patterns. Hence, instead of grouping them together, they stay separate. This leads to overlapping patterns in the target language which will produce wrong results. One way to deal with this is to rename problematic variables before invoking the compiler.

fun *pattern-compatible* :: *term* ⇒ *term* ⇒ *bool* **where**
pattern-compatible (*t*$_1$ $ *t*$_2$) (*u*$_1$ $ *u*$_2$) ⟷ *pattern-compatible t*$_1$ *u*$_1$ ∧ (*t*$_1$ = *u*$_1$ ⟶ *pattern-compatible t*$_2$ *u*$_2$) |
pattern-compatible t u ⟷ *t* = *u* ∨ *non-overlapping t u*

lemmas *pattern-compatible-simps*[*simp*] =
pattern-compatible.simps[*folded app-term-def*]

lemmas *pattern-compatible-induct* = *pattern-compatible.induct*[*case-names app-app*]

lemma *pattern-compatible-refl*[*intro?*]: *pattern-compatible t t*
by (*induct t*) *auto*

corollary *pattern-compatile-reflP*[*intro*!]: *reflp pattern-compatible*
by (*auto intro*: *pattern-compatible-refl reflpI*)

lemma *pattern-compatible-cases*[*consumes 1*]:
assumes *pattern-compatible t u*
obtains (*eq*) *t* = *u*
| (*non-overlapping*) *non-overlapping t u*
using *assms* **proof** (*induction arbitrary*: *thesis rule*: *pattern-compatible-induct*)
case (*app-app t*$_1$ *t*$_2$ *u*$_1$ *u*$_2$)

show *?case*
proof (*cases t*$_1$ = *u*$_1$ ∧ *t*$_2$ = *u*$_2$)
case *True*
with *app-app* **show** *thesis*
by *simp*
next
case *False*
from *app-app* **have** *pattern-compatible t*$_1$ *u*$_1$ *t*$_1$ = *u*$_1$ ⟹ *pattern-compatible t*$_2$ *u*$_2$
by *auto*
with *False* **have** *non-overlapping* (*t*$_1$ $ *t*$_2$) (*u*$_1$ $ *u*$_2$)
using *app-app* **by** (*metis non-overlapping-appI1 non-overlapping-appI2*)
thus *thesis*

by (*rule app-app.prems*(*2*))
qed
qed *auto*

inductive *rev-accum-rel* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool* **for** *R*
where
nil: *rev-accum-rel R* [] [] |
snoc: *rev-accum-rel R xs ys* ⟹ (*xs* = *ys* ⟹ *R x y*) ⟹ *rev-accum-rel R* (*xs* @ [*x*]) (*ys* @ [*y*])

lemma *rev-accum-rel-refl*[*intro*]: *reflp R* ⟹ *rev-accum-rel R xs xs*
unfolding *reflp-def*
by (*induction xs rule*: *rev-induct*) (*auto intro*: *rev-accum-rel.intros*)

lemma *rev-accum-rel-length*:
assumes *rev-accum-rel R xs ys*
shows *length xs* = *length ys*
using *assms*
by *induct auto*

context begin

private inductive-cases *rev-accum-relE*[*consumes 1*, *case-names nil snoc*]: *rev-accum-rel P xs ys*

lemma *rev-accum-rel-butlast*[*intro*]:
assumes *rev-accum-rel P xs ys*
shows *rev-accum-rel P* (*butlast xs*) (*butlast ys*)
using *assms* **by** (*cases rule*: *rev-accum-relE*) (*auto intro*: *rev-accum-rel.intros*)

lemma *rev-accum-rel-snoc-eqE*: *rev-accum-rel P* (*xs* @ [*a*]) (*xs* @ [*b*]) ⟹ *P a b*
by (*auto elim*: *rev-accum-relE*)

end

abbreviation *patterns-compatible* :: *term list* ⇒ *term list* ⇒ *bool* **where**
patterns-compatible ≡ *rev-accum-rel pattern-compatible*

abbreviation *patterns-compatibles* :: (*term list* × '*a*) *fset* ⇒ *bool* **where**
patterns-compatibles ≡ *fpairwise* (λ(*pats*$_1$, -) (*pats*$_2$, -). *patterns-compatible pats*$_1$ *pats*$_2$)

lemma *pattern-compatible-combD*:
assumes *length xs* = *length ys pattern-compatible* (*list-comb f xs*) (*list-comb f ys*)
shows *patterns-compatible xs ys*
using *assms* **by** (*induction xs ys rule*: *rev-induct2*) (*auto intro*: *rev-accum-rel.intros*)

lemma *pattern-compatible-combI*[*intro*]:
assumes *patterns-compatible xs ys pattern-compatible f g*

shows *pattern-compatible* (*list-comb f xs*) (*list-comb g ys*)
using *assms*
proof (*induction rule*: *rev-accum-rel.induct*)
case (*snoc xs ys x y*)

then have *pattern-compatible* (*list-comb f xs*) (*list-comb g ys*)
by *auto*

moreover have *pattern-compatible x y* **if** *list-comb f xs* = *list-comb g ys*
proof (*rule snoc*, *rule list-comb-semi-inj*)
show *length xs* = *length ys*
using *snoc* **by** (*auto dest*: *rev-accum-rel-length*)
qed *fact*

ultimately show *?case*
by *auto*
qed (*auto intro*: *rev-accum-rel.intros*)

experiment begin

— The above example can be made concrete here. In general, the following identity does not hold:

lemma *pattern-compatible t u* ⟷ *t* = *u* ∨ *non-overlapping t u*
apply *rule*
apply (*erule pattern-compatible-cases*; *simp*)
apply (*erule disjE*)
apply (*metis pattern-compatible-refl*)
oops

— The counterexample:

definition *pats1* = [*Free* (*Name* ″*f*″), *Const* (*Name* ″*nil*″)]
definition *pats2* = [*Free* (*Name* ″*g*″), *Const* (*Name* ″*cons*″) $ *Free* (*Name* ″*x*″) $ *Free* (*Name* ″*xs*″)]

proposition *non-overlapping* (*list-comb c pats1*) (*list-comb c pats2*)
unfolding *pats1-def pats2-def*
apply (*simp add*: *app-term-def*)
apply (*rule non-overlapping-appI2*)
apply (*rule non-overlapping-const-appI*)
done

proposition ¬ *patterns-compatible pats1 pats2*
unfolding *pats1-def pats2-def*
apply *rule*
apply (*erule rev-accum-rel.cases*)
apply *simp*
apply *auto*

apply (*erule rev-accum-rel.cases*)
apply *auto*
apply (*erule rev-accum-rel.cases*)
apply *auto*
apply (*metis overlapping-var1I*)
done

end

abbreviation *pattern-compatibles* :: (*term* × *'a*) *fset* ⇒ *bool* **where**
pattern-compatibles ≡ *fpairwise* (λ(*lhs*$_1$, -) (*lhs*$_2$, -). *pattern-compatible lhs*$_1$ *lhs*$_2$)

corollary *match-compatible-pat-eq*:
assumes *pattern-compatible* t_1 t_2 *linear* t_1 *linear* t_2
assumes *match* t_1 *u* = *Some* env_1 *match* t_2 *u* = *Some* env_2
shows $t_1 = t_2$
using *assms* **by** (*metis pattern-compatible-cases match-overlapping*)

corollary *match-compatible-env-eq*:
assumes *pattern-compatible* t_1 t_2 *linear* t_1 *linear* t_2
assumes *match* t_1 *u* = *Some* env_1 *match* t_2 *u* = *Some* env_2
shows $env_1 = env_2$
using *assms* **by** (*metis match-compatible-pat-eq option.inject*)

corollary *matchs-compatible-eq*:
assumes *patterns-compatible* ts_1 ts_2 *linears* ts_1 *linears* ts_2
assumes *matchs* ts_1 *us* = *Some* env_1 *matchs* ts_2 *us* = *Some* env_2
shows $ts_1 = ts_2$ $env_1 = env_2$
proof −
fix *name*
have *match* (*name* $$ ts_1) (*name* $$ *us*) = *Some* env_1 *match* (*name* $$ ts_2) (*name* $$ *us*) = *Some* env_2
using *assms* **by** *auto*
moreover have *length* ts_1 = *length* ts_2
using *assms* **by** (*metis matchs-some-eq-length*)
ultimately have *pattern-compatible* (*name* $$ ts_1) (*name* $$ ts_2)
using *assms* **by** (*auto simp*: *const-term-def*)
moreover have *linear* (*name* $$ ts_1) *linear* (*name* $$ ts_2)
using *assms* **by** (*auto intro*: *linear-list-comb'*)
note ∗ = *calculation*

from ∗ **have** *name* $$ ts_1 = *name* $$ ts_2
by (*rule match-compatible-pat-eq*) *fact*+
thus $ts_1 = ts_2$
by (*meson list-comb-inj-second injD*)

from ∗ **show** $env_1 = env_2$
by (*rule match-compatible-env-eq*) *fact*+
qed

```
lemma compatible-find-match:
   assumes pattern-compatibles (fset-of-list cs) list-all (linear ∘ fst) cs is-fmap
(fset-of-list cs)
  assumes match pat t = Some env (pat, rhs) ∈ set cs
  shows find-match cs t = Some (env, pat, rhs)
using assms proof (induction cs arbitrary: pat rhs)
 case (Cons c cs)
 then obtain pat' rhs' where [simp]: c = (pat', rhs')
   by force
 have find-match ((pat', rhs') # cs) t = Some (env, pat, rhs)
   proof (cases match pat' t)
     case None
     have pattern-compatibles (fset-of-list cs)
       using Cons
       by (force simp: fpairwise-alt-def)
     have list-all (linear ∘ fst) cs
       using Cons
       by (auto simp: list-all-iff)
     have is-fmap (fset-of-list cs)
       using Cons
       by (meson fset-of-list-subset is-fmap-subset set-subset-Cons)
     show ?thesis
       apply (simp add: None)
       apply (rule Cons)
          apply fact+
       using Cons None by force
   next
     case (Some env')
     have linear pat linear pat'
       using Cons apply (metis Ball-set comp-apply fst-conv)
       using Cons by simp
     moreover from Cons have pattern-compatible pat pat'
       apply (cases pat = pat')
        apply (simp add: pattern-compatible-refl)
       unfolding fpairwise-alt-def
       by (force simp: fset-of-list-elem)
     ultimately have env' = env pat' = pat
       using match-compatible-env-eq match-compatible-pat-eq
       using Cons Some
       by blast+
     with Cons have rhs' = rhs
       using is-fmapD
       by (metis ‹c = (pat', rhs')› fset-of-list-elem list.set-intros(1))
     show ?thesis
       apply (simp add: Some)
       apply (intro conjI)
       by fact+
   qed
```

thus *?case*
unfolding ‹c = -› **.**
qed *auto*

context *term* **begin**

definition *arity-compatible* :: $'a \Rightarrow 'a \Rightarrow$ *bool* **where**
arity-compatible t_1 t_2 = (
let
($head_1$, $pats_1$) = *strip-comb* t_1;
($head_2$, $pats_2$) = *strip-comb* t_2
in $head_1 = head_2 \longrightarrow length\ pats_1 = length\ pats_2$
)

abbreviation *arity-compatibles* :: ($'a \times 'b$) *fset* $\Rightarrow$ *bool* **where**
arity-compatibles $\equiv$ *fpairwise* (λ(lhs_1, -) (lhs_2, -). *arity-compatible* lhs_1 lhs_2)

definition *head* :: $'a \Rightarrow$ *name* **where**
head $t \equiv$ *const-name* (*fst* (*strip-comb* t))

abbreviation *heads-of* :: (*term* $\times$ $'a$) *fset* $\Rightarrow$ *name fset* **where**
heads-of $rs \equiv$ (*head* $\circ$ *fst*) |‘| rs

end

definition *arity* :: ($'a$ *list* $\times$ $'b$) *fset* $\Rightarrow$ *nat* **where**
arity rs = *fthe-elem'* ((*length* $\circ$ *fst*) |‘| rs)

lemma *arityI*:
assumes *fBall* rs (λ(*pats*, -). *length pats* = n) $rs \neq \{||\}$
shows *arity* $rs = n$
proof −
have (*length* $\circ$ *fst*) |‘| $rs = \{|\ n\ |\}$
using *assms* **by** *force*
thus *?thesis*
unfolding *arity-def fthe-elem'-eq* **by** *simp*
qed

end

1.2 Reflecting HOL datatype definitions

theory *HOL-Datatype*
imports
Terms-Extras
HOL−Library.Datatype-Records
HOL−Library.Finite-Map
Higher-Order-Terms.Name
begin

datatype *typ* =
  TVar name |
  TApp name typ list

datatype-compat *typ*

context begin

qualified definition *tapp-0* **where**
tapp-0 tc = *TApp tc* []

qualified definition *tapp-1* **where**
tapp-1 tc t1 = *TApp tc* [*t1*]

qualified definition *tapp-2* **where**
tapp-2 tc t1 t2 = *TApp tc* [*t1*, *t2*]

end

quickcheck-generator *typ*
  constructors:
    TVar,
    HOL-Datatype.tapp-0,
    HOL-Datatype.tapp-1,
    HOL-Datatype.tapp-2

datatype-record *dt-def* =
  tparams :: *name list*
  constructors :: (*name*, *typ list*) *fmap*

ML-file *hol-datatype.ML*

end

1.3 Constructor information

theory *Constructors*
imports
  Terms-Extras
  HOL-Datatype
begin

type-synonym *C-info* = (*name*, *dt-def*) *fmap*

locale *constructors* =
  fixes *C-info* :: *C-info*
begin

definition *flat-C-info* :: (*string* × *nat* × *string*) *list* **where**
flat-C-info = *do* {
  (*tname*, *Cs*) ← *sorted-list-of-fmap C-info*;
  (*C*, *params*) ← *sorted-list-of-fmap* (*constructors Cs*);
  [(*as-string C*, (*length params*, *as-string tname*))]
}

definition *all-tdefs* :: *name fset* **where**
all-tdefs = *fmdom C-info*

definition *C* :: *name fset* **where**
C = *ffUnion* (*fmdom* |‘| *constructors* |‘| *fmran C-info*)

definition *all-constructors* :: *name list* **where**
all-constructors =
  concat (*map* (λ(-, *Cs*). *map fst* (*sorted-list-of-fmap* (*constructors Cs*))) (*sorted-list-of-fmap C-info*))

end

declare *constructors.C-def*[*code*]
declare *constructors.flat-C-info-def*[*code*]
declare *constructors.all-constructors-def*[*code*]

export-code
  constructors.C constructors.flat-C-info constructors.all-constructors
  checking *Scala*

end

1.4 Special constants

theory *Consts*
imports
  Constructors
  Higher-Order-Terms.Nterm
begin

locale *special-constants* = *constructors*

locale *pre-constants* = *special-constants* +
  fixes *heads* :: *name fset*
begin

definition *all-consts* :: *name fset* **where**
all-consts = *heads* |∪| *C*

abbreviation *welldefined* :: *'a*::*term* ⇒ *bool* **where**
welldefined t ≡ *consts t* |⊆| *all-consts*

sublocale *welldefined*: *simple-syntactic-and welldefined*
by *standard auto*

end

declare *pre-constants.all-consts-def*[*code*]

locale *constants* = *pre-constants* +
assumes *disjnt*: *fdisjnt heads C*
— Conceptually the following assumptions should belong into *constructors*, but I prefer to keep that one assumption-free.
assumes *distinct-ctr*: *distinct all-constructors*
begin

lemma *distinct-ctr'*: *distinct* (*map as-string all-constructors*)
unfolding *distinct-map*
using *distinct-ctr*
by (*auto intro*: *inj-onI dest*: *name.expand*)

end

end

1.5 Term algebra extended with wellformedness

theory *Strong-Term*
imports *Consts*
begin

class *pre-strong-term* = *term* +
fixes *wellformed* :: *'a* ⇒ *bool*
fixes *all-frees* :: *'a* ⇒ *name fset*
assumes *wellformed-const*[*simp*]: *wellformed* (*const name*)
assumes *wellformed-free*[*simp*]: *wellformed* (*free name*)
assumes *wellformed-app*[*simp*]: *wellformed* (*app* u_1 u_2) ⟷ *wellformed* u_1 ∧ *wellformed* u_2
assumes *all-frees-const*[*simp*]: *all-frees* (*const name*) = *fempty*
assumes *all-frees-free*[*simp*]: *all-frees* (*free name*) = {| *name* |}
assumes *all-frees-app*[*simp*]: *all-frees* (*app* u_1 u_2) = *all-frees* u_1 |∪| *all-frees* u_2
begin

abbreviation *wellformed-env* :: (*name*, *'a*) *fmap* ⇒ *bool* **where**
wellformed-env ≡ *fmpred* (λ-. *wellformed*)

end

context *pre-constants* **begin**

definition *shadows-consts* :: *'a*::*pre-strong-term* ⇒ *bool* **where**
shadows-consts t ⟷ ¬ *fdisjnt all-consts* (*all-frees t*)

sublocale *shadows*: *simple-syntactic-or shadows-consts*
by *standard* (*auto simp*: *shadows-consts-def fdisjnt-alt-def*)

abbreviation *not-shadows-consts-env* :: (*name*, *'a*::*pre-strong-term*) *fmap* ⇒ *bool*
where
not-shadows-consts-env ≡ *fmpred* (λ- *s*. ¬ *shadows-consts s*)

end

declare *pre-constants.shadows-consts-def*[*code*]

class *strong-term* = *pre-strong-term* +
assumes *raw-frees-all-frees*: *abs-pred* (λ*t*. *frees t* |⊆| *all-frees t*) *t*
assumes *raw-subst-wellformed*: *abs-pred* (λ*t*. *wellformed t* ⟶ (∀ *env*. *wellformed-env env* ⟶ *wellformed* (*subst t env*))) *t*
begin

lemma *frees-all-frees*: *frees t* |⊆| *all-frees t*
proof (*induction t rule*: *raw-induct*)
case (*abs t*)
show *?case*
by (*rule raw-frees-all-frees*)
qed *auto*

lemma *subst-wellformed*: *wellformed t* ⟹ *wellformed-env env* ⟹ *wellformed* (*subst t env*)
proof (*induction t arbitrary*: *env rule*: *raw-induct*)
case (*abs t*)
show *?case*
by (*rule raw-subst-wellformed*)
qed (*auto split*: *option.splits*)

end

global-interpretation *wellformed*: *subst-syntactic-and wellformed* :: *'a*::*strong-term* ⇒ *bool*
by *standard* (*auto simp*: *subst-wellformed*)

instantiation *term* :: *strong-term* **begin**

fun *all-frees-term* :: *term* ⇒ *name fset* **where**
all-frees-term (*Free x*) = {| *x* |} |
all-frees-term (*t*$_1$ $ *t*$_2$) = *all-frees-term t*$_1$ |∪| *all-frees-term t*$_2$ |

```
all-frees-term (Λ t) = all-frees-term t |
all-frees-term - = {||}

lemma frees-all-frees-term[simp]: all-frees t = frees (t::term)
by (induction t) auto

definition wellformed-term :: term ⇒ bool where
[simp]: wellformed-term t ⟷ Term.wellformed t

instance proof (standard, goal-cases)
  case 8


  have ∗: abs-pred P t if P t for P and t :: term
    unfolding abs-pred-term-def using that
    by auto

  show ?case
    apply (rule ∗)
    unfolding wellformed-term-def
    by (auto simp: Term.subst-wellformed)
qed (auto simp: const-term-def free-term-def app-term-def abs-pred-term-def)

end

instantiation nterm :: strong-term begin

definition wellformed-nterm :: nterm ⇒ bool where
[simp]: wellformed-nterm t ⟷ True

fun all-frees-nterm :: nterm ⇒ name fset where
all-frees-nterm (Nvar x) = {| x |} |
all-frees-nterm (t1 $n t2) = all-frees-nterm t1 |∪| all-frees-nterm t2 |
all-frees-nterm (Λn x. t) = finsert x (all-frees-nterm t) |
all-frees-nterm (Nconst -) = {||}

instance proof (standard, goal-cases)
  case (7 t)
  show ?case
    unfolding abs-pred-nterm-def
    by auto
qed (auto simp: const-nterm-def free-nterm-def app-nterm-def abs-pred-nterm-def)

end

lemma (in pre-constants) shadows-consts-frees:
  fixes t :: 'a::strong-term
  shows ¬ shadows-consts t ⟹ fdisjnt all-consts (frees t)
unfolding fdisjnt-alt-def shadows-consts-def
```

using *frees-all-frees*
by *auto*

abbreviation *wellformed-clauses* :: - ⇒ *bool* **where**
wellformed-clauses cs ≡ *list-all* (λ(*pat*, *t*). *linear pat* ∧ *wellformed t*) *cs* ∧ *distinct* (*map fst cs*) ∧ *cs* ≠ []

end

1.6 Terms with sequential pattern matching

theory *Sterm*
imports *Strong-Term*
begin

datatype *sterm* =
Sconst name |
Svar name |
Sabs (*clauses*: (*term* × *sterm*) *list*) |
Sapp sterm sterm (**infixl** ‹$_s› 70)

datatype-compat *sterm*

derive *linorder sterm*

abbreviation *Sabs-single* (‹Λ_s -. ->› [0, 50] 50) **where**
Sabs-single x rhs ≡ *Sabs* [(*Free x*, *rhs*)]

type-synonym *sclauses* = (*term* × *sterm*) *list*

lemma *sterm-induct*[*case-names Sconst Svar Sabs Sapp*]:
assumes ⋀*x*. *P* (*Sconst x*)
assumes ⋀*x*. *P* (*Svar x*)
assumes ⋀*cs*. (⋀*pat t*. (*pat*, *t*) ∈ *set cs* ⟹ *P t*) ⟹ *P* (*Sabs cs*)
assumes ⋀*t u*. *P t* ⟹ *P u* ⟹ *P* (*t* $_s *u*)
shows *P t*
using *assms*
apply *induction-schema*
apply *pat-completeness*
apply *lexicographic-order*
done

instantiation *sterm* :: *pre-term* **begin**

definition *app-sterm* **where**
app-sterm t u = *t* $_s *u*

fun *unapp-sterm* **where**
unapp-sterm (*t* $_s *u*) = *Some* (*t*, *u*) |

unapp-sterm - = *None*

definition *const-sterm* **where**
const-sterm = *Sconst*

fun *unconst-sterm* **where**
unconst-sterm (*Sconst name*) = *Some name* |
unconst-sterm - = *None*

fun *unfree-sterm* **where**
unfree-sterm (*Svar name*) = *Some name* |
unfree-sterm - = *None*

definition *free-sterm* **where**
free-sterm = *Svar*

fun *frees-sterm* **where**
frees-sterm (*Svar name*) = {|*name*|} |
frees-sterm (*Sconst* -) = {||} |
frees-sterm (*Sabs cs*) = *ffUnion* (*fset-of-list* (*map* (λ(*pat*, *rhs*). *frees-sterm rhs* − *frees pat*) *cs*)) |
frees-sterm (*t* $_s$ *u*) = *frees-sterm t* |∪| *frees-sterm u*

fun *subst-sterm* **where**
subst-sterm (*Svar s*) *env* = (*case fmlookup env s of Some t* ⇒ *t* | *None* ⇒ *Svar s*)
|
subst-sterm (*t*$_1$ $_s$ *t*$_2$) *env* = *subst-sterm t*$_1$ *env* $_s$ *subst-sterm t*$_2$ *env* |
subst-sterm (*Sabs cs*) *env* = *Sabs* (*map* (λ(*pat*, *rhs*). (*pat*, *subst-sterm rhs* (*fmdrop-fset* (*frees pat*) *env*))) *cs*) |
subst-sterm t env = *t*

fun *consts-sterm* :: *sterm* ⇒ *name fset* **where**
consts-sterm (*Svar* -) = {||} |
consts-sterm (*Sconst name*) = {|*name*|} |
consts-sterm (*Sabs cs*) = *ffUnion* (*fset-of-list* (*map* (λ(-, *rhs*). *consts-sterm rhs*) *cs*)) |
consts-sterm (*t* $_s$ *u*) = *consts-sterm t* |∪| *consts-sterm u*

instance
by *standard*
(*auto*
simp: *app-sterm-def const-sterm-def free-sterm-def*
elim: *unapp-sterm.elims unconst-sterm.elims unfree-sterm.elims*
split: *option.splits*)

end

instantiation *sterm* :: *term* **begin**

```
definition abs-pred-sterm :: (sterm ⇒ bool) ⇒ sterm ⇒ bool where
  abs-pred P t ⟷ (∀ cs. t = Sabs cs ⟶ (∀ pat t. (pat, t) ∈ set cs ⟶ P t) ⟶
P t)

lemma abs-pred-sterm-code [code]:
  ‹abs-pred P t = (case t of Sabs cs ⇒ (∀ (-, t) ∈ set cs. P t) ⟶ P t | - ⇒ True)›
  by (cases t) (auto simp add: abs-pred-sterm-def)

lemma abs-pred-stermI [intro]:
  assumes ⋀cs. (⋀pat t. (pat, t) ∈ set cs ⟹ P t) ⟹ P (Sabs cs)
  shows abs-pred P t
using assms unfolding abs-pred-sterm-def by auto

instance proof (standard, goal-cases)
  case (1 P t)
  then show ?case
   by (induction t) (auto simp: const-sterm-def free-sterm-def app-sterm-def abs-pred-sterm-def)
next
  case (2 t)
  show ?case
    apply rule
    apply auto
    apply (subst (3) list.map-id[symmetric])
    apply (rule list.map-cong0)
    apply auto
    by blast
next
  case (3 x t)
  show ?case
    apply rule
    apply clarsimp
    subgoal for cs env pat rhs
      apply (cases x |∈| frees pat)
      subgoal
        apply (rule arg-cong[where f = subst rhs])
        by (auto intro: fmap-ext)
      subgoal premises prems[rule-format]
        apply (subst (2) prems(1)[symmetric, where pat = pat])
        subgoal by fact
        subgoal
          using prems
          unfolding ffUnion-alt-def
          by (auto simp add: fset-of-list.rep-eq elim!: fBallE)
        subgoal
          apply (rule arg-cong[where f = subst rhs])
          by (auto intro: fmap-ext)
        done
      done
    done
```

```
next
  case (4 t)
  show ?case
    apply rule
    apply clarsimp
    subgoal premises prems[rule-format]
      apply (rule prems(1)[OF prems(4)])
      subgoal using prems by auto
      subgoal using prems unfolding fdisjnt-alt-def by auto
      done
    done
next
  case 5
  show ?case
    proof (intro abs-pred-stermI allI impI, goal-cases)
      case (1 cs env)
      show ?case
        proof safe
          fix name
          assume name |∈| frees (subst (Sabs cs) env)
          then obtain pat rhs
            where (pat, rhs) ∈ set cs
              and name |∈| frees (subst rhs (fmdrop-fset (frees pat) env))
              and name |∉| frees pat
           by (auto simp: fset-of-list-elem case-prod-twice comp-def ffUnion-alt-def)

          hence name |∈| frees rhs |−| fmdom (fmdrop-fset (frees pat) env)
            using 1 by (simp add: fmpred-drop-fset)

          hence name |∈| frees rhs |−| frees pat
            using ‹name |∉| frees pat› by blast

          show name |∈| frees (Sabs cs)
            apply (simp add: ffUnion-alt-def)
            apply (rule fBexI[where x = (pat, rhs)])
            unfolding prod.case
             apply (fact ‹name |∈| frees rhs |−| frees pat›)
            unfolding fset-of-list-elem
            by fact

          assume name |∈| fmdom env
          thus False
           using ‹name |∈| frees rhs |−| fmdom (fmdrop-fset (frees pat) env)› ‹name
|∉| frees pat›
            by fastforce
        next
          fix name
          assume name |∈| frees (Sabs cs) name |∉| fmdom env
```

```
        then obtain pat rhs
          where (pat, rhs) ∈ set cs name |∈| frees rhs name |∉| frees pat
          by (auto simp: fset-of-list-elem ffUnion-alt-def)

         moreover hence name |∈| frees rhs |−| fmdom (fmdrop-fset (frees pat)
env) |−| frees pat
          using ‹name |∉| fmdom env› by fastforce

        ultimately have name |∈| frees (subst rhs (fmdrop-fset (frees pat) env))
|−| frees pat
          using 1 by (simp add: fmpred-drop-fset)

        show name |∈| frees (subst (Sabs cs) env)
          apply (simp add: case-prod-twice comp-def)
          unfolding ffUnion-alt-def
          apply (rule fBexI)
           apply (fact ‹name |∈| frees (subst rhs (fmdrop-fset (frees pat) env)) |−|
frees pat›)
          apply (subst fimage-iff)
          apply (rule fBexI[where x = (pat, rhs)])
           apply simp
          using ‹(pat, rhs) ∈ set cs›
          by (auto simp: fset-of-list-elem)
       qed
    qed
next
  case 6
  show ?case
    proof (intro abs-pred-stermI allI impI, goal-cases)
      case (1 cs env)

      — some property on various operations that is only useful in here
       have ∗: fbind (fmimage m (fbind A g)) f = fbind A (λx. fbind (fmimage m
(g x)) f)
        for m A f g
        including fset.lifting and fmap.lifting
        by transfer′ force

      have consts (subst (Sabs cs) env) = fbind (fset-of-list cs) (λ(pat, rhs). consts
rhs |∪| ffUnion (consts |‘| fmimage (fmdrop-fset (frees pat) env) (frees rhs)))
        apply (simp add: funion-image-bind-eq)
        apply (rule fbind-cong[OF refl])
        apply (clarsimp split: prod.splits)
        apply (subst 1)
         apply (subst (asm) fset-of-list-elem, assumption)
        apply simp
        by (simp add: funion-image-bind-eq)
      also have ... = fbind (fset-of-list cs) (consts ∘ snd) |∪| fbind (fset-of-list cs)
(λ(pat, rhs). ffUnion (consts |‘| fmimage (fmdrop-fset (frees pat) env) (frees rhs)))
```

apply (*subst fbind-fun-funion*[*symmetric*])
apply (*rule fbind-cong*[*OF refl*])
by *auto*
also have ... = *consts* (*Sabs cs*) |∪| *fbind* (*fset-of-list cs*) (λ(*pat*, *rhs*). *ffUnion* (*consts* |‘| *fmimage* (*fmdrop-fset* (*frees pat*) *env*) (*frees rhs*)))
apply (*rule cong*[*OF cong*, *OF refl - refl*, **where** *f1* = *funion*])
apply (*subst funion-image-bind-eq*[*symmetric*])
unfolding *consts-sterm.simps*
apply (*rule arg-cong*[**where** *f* = *ffUnion*])
apply (*subst fset-of-list-map*)
apply (*rule fset.map-cong*[*OF refl*])
by *auto*
also have ... = *consts* (*Sabs cs*) |∪| *fbind* (*fmimage env* (*fbind* (*fset-of-list cs*) (λ(*pat*, *rhs*). *frees rhs* |−| *frees pat*))) *consts*
apply (*subst funion-image-bind-eq*)
apply (*subst fmimage-drop-fset*)
apply (*rule cong*[*OF cong*, *OF refl refl*, **where** *f1* = *funion*])
apply (*subst* ∗)
apply (*rule fbind-cong*[*OF refl*])
by *auto*
also have ... = *consts* (*Sabs cs*) |∪| *ffUnion* (*consts* |‘| *fmimage env* (*frees* (*Sabs cs*)))
by (*simp only*: *frees-sterm.simps fset-of-list-map fmimage-Union funion-image-bind-eq*)

finally show *?case* **.**
qed
qed (*auto simp*: *abs-pred-sterm-def*)

end

lemma *no-abs-abs*[*simp*]: ¬ *no-abs* (*Sabs cs*)
by (*subst no-abs.simps*) (*auto simp*: *term-cases-def*)

lemma *closed-except-simps*:
closed-except (*Svar x*) *S* ⟷ *x* |∈| *S*
closed-except (*t*$_1$ $_s *t*$_2$) *S* ⟷ *closed-except t*$_1$ *S* ∧ *closed-except t*$_2$ *S*
closed-except (*Sabs cs*) *S* ⟷ *list-all* (λ(*pat*, *t*). *closed-except t* (*S* |∪| *frees pat*)) *cs*
closed-except (*Sconst name*) *S* ⟷ *True*
proof *goal-cases*
case *3*
show *?case*
proof (*standard*, *goal-cases*)
case *1*
then show *?case*
apply (*auto simp*: *list-all-iff ffUnion-alt-def fset-of-list-elem closed-except-def*)
apply (*drule ffUnion-least-rev*)
apply *auto*

```
    by (smt (z3) case-prod-conv fbspec fimageI fminusI fset-of-list-elem fset-rev-mp)
  next
    case 2
    then show ?case
    by (fastforce simp: list-all-iff ffUnion-alt-def fset-of-list-elem closed-except-def)
  qed
qed (auto simp: ffUnion-alt-def closed-except-def)

lemma closed-except-sabs:
  assumes closed (Sabs cs) (pat, rhs) ∈ set cs
  shows closed-except rhs (frees pat)
using assms unfolding closed-except-def
apply auto
by (metis bot.extremum-uniqueI fempty-iff ffUnion-subset-elem fimageI fminusI
fset-of-list-elem old.prod.case)

instantiation sterm :: strong-term  begin

fun wellformed-sterm :: sterm ⇒ bool where
wellformed-sterm (t1 $s t2) ⟷ wellformed-sterm t1 ∧ wellformed-sterm t2 |
wellformed-sterm (Sabs cs) ⟷ list-all (λ(pat, t). linear pat ∧ wellformed-sterm
t) cs ∧ distinct (map fst cs) ∧ cs ≠ [] |
wellformed-sterm - ⟷ True

primrec all-frees-sterm :: sterm ⇒ name fset where
all-frees-sterm (Svar x) = {|x|} |
all-frees-sterm (t1 $s t2) = all-frees-sterm t1 |∪| all-frees-sterm t2 |
all-frees-sterm (Sabs cs) = ffUnion (fset-of-list (map (λ(P, T). P |∪| T) (map
(map-prod frees all-frees-sterm) cs))) |
all-frees-sterm (Sconst -) = {||}

instance proof (standard, goal-cases)
  case (7 t)
  show ?case
    apply (intro abs-pred-stermI allI impI)
    apply simp
    apply (rule ffUnion-least)
    apply (rule fBallI)
    apply auto
    apply (subst ffUnion-alt-def)
    apply simp
    apply (rule-tac x = (a, b) in fBexI)
    by (auto simp: fset-of-list-elem)
next
  case (8 t)
  show ?case
    apply (intro abs-pred-stermI allI impI)
    apply (simp add: list.pred-map comp-def case-prod-twice, safe)
    subgoal
```

apply (*subst list-all-iff*)
apply (*rule ballI*)
apply *safe*[*1*]
apply (*fastforce simp*: *list-all-iff*)
subgoal premises *prems*[*rule-format*]
apply (*rule prems*)
apply (*fact prems*)
using *prems* **apply** (*fastforce simp*: *list-all-iff*)
using *prems* **by** *force*
done
subgoal
apply (*subst map-cong*[*OF refl*])
by *auto*
done
qed (*auto simp*: *const-sterm-def free-sterm-def app-sterm-def*)

end

lemma *match-sabs*[*simp*]: ¬ *is-free t* ⟹ *match t* (*Sabs cs*) = *None*
by (*cases t*) *auto*

context *pre-constants* **begin**

lemma *welldefined-sabs*: *welldefined* (*Sabs cs*) ⟷ *list-all* (λ(-, *t*). *welldefined t*) *cs*
apply (*auto simp*: *list-all-iff ffUnion-alt-def dest*!: *ffUnion-least-rev*)
apply (*subst* (*asm*) *list-all-iff-fset*[*symmetric*])
apply (*auto simp*: *list-all-iff fset-of-list-elem*)
done

lemma *shadows-consts-sterm-simps*[*simp*]:
shadows-consts (*t*$_1$ $_s *t*$_2$) ⟷ *shadows-consts t*$_1$ ∨ *shadows-consts t*$_2$
shadows-consts (*Svar name*) ⟷ *name* |∈| *all-consts*
shadows-consts (*Sabs cs*) ⟷ *list-ex* (λ(*pat*, *t*). ¬ *fdisjnt all-consts* (*frees pat*) ∨ *shadows-consts t*) *cs*
shadows-consts (*Sconst name*) ⟷ *False*
proof (*goal-cases*)
case *3*
show *?case*
unfolding *shadows-consts-def list-ex-iff*
apply *rule*
subgoal
by (*force simp*: *ffUnion-alt-def fset-of-list-elem fdisjnt-alt-def elim*!: *ballE*)
subgoal
apply (*auto simp*: *fset-of-list-elem fdisjnt-alt-def*)
by (*auto simp*: *fset-eq-empty-iff ffUnion-alt-def fset-of-list-elem elim*!: *allE fBallE*)
done
qed (*auto simp*: *shadows-consts-def fdisjnt-alt-def*)

lemma *subst-shadows*:
assumes ¬ *shadows-consts* (*t*::*sterm*) *not-shadows-consts-env* Γ
shows ¬ *shadows-consts* (*subst t* Γ)
using *assms* **proof** (*induction t arbitrary*: Γ *rule*: *sterm-induct*)
case (*Sabs cs*)
show *?case*
apply (*simp add*: *list-ex-iff case-prod-twice*)
apply (*rule ballI*)
subgoal for *c*
apply (*cases c*, *hypsubst-thin*, *simp*)
apply (*rule conjI*)
subgoal using *Sabs*(*2*) **by** (*fastforce simp*: *list-ex-iff*)
apply (*rule Sabs*(*1*))
apply *assumption*
subgoal using *Sabs*(*2*) **by** (*fastforce simp*: *list-ex-iff*)
subgoal using *Sabs*(*3*) **by** *force*
done
done
qed (*auto split*: *option.splits*)

end

end

1.7 Terms with explicit pattern matching

theory *Pterm*
imports
../Utils/Compiler-Utils
Consts
Sterm — Inclusion of this theory might seem a bit strange. Indeed, it is only for technical reasons: to allow for a **quickcheck** setup.
begin

datatype *pterm* =
Pconst name |
Pvar name |
Pabs (*term* × *pterm*) *fset* |
Papp pterm pterm (**infixl** ‹$$_p$$› *70*)

primrec *sterm-to-pterm* :: *sterm* ⇒ *pterm* **where**
sterm-to-pterm (*Sconst name*) = *Pconst name* |
sterm-to-pterm (*Svar name*) = *Pvar name* |
sterm-to-pterm (*t* $$_s$$ *u*) = *sterm-to-pterm t* $$_p$$ *sterm-to-pterm u* |
sterm-to-pterm (*Sabs cs*) = *Pabs* (*fset-of-list* (*map* (*map-prod id sterm-to-pterm*) *cs*))

```
quickcheck-generator pterm
  — will print some fishy "constructor" names, but at least it works
  constructors: sterm-to-pterm

lemma sterm-to-pterm-total:
  obtains t' where t = sterm-to-pterm t'
proof (induction t arbitrary: thesis)
  case (Pconst x)
  then show ?case
    by (metis sterm-to-pterm.simps)
next
  case (Pvar x)
  then show ?case
    by (metis sterm-to-pterm.simps)
next
  case (Pabs cs)
 from Pabs.IH obtain cs' where cs = fset-of-list (map (map-prod id sterm-to-pterm)
cs')
    apply atomize-elim
    proof (induction cs)
      case empty
      show ?case
        apply (rule exI[where x = []])
        by simp
    next
      case (insert c cs)
      obtain pat rhs where c = (pat, rhs) by (cases c) auto

      have ∃ cs'. cs = fset-of-list (map (map-prod id sterm-to-pterm) cs')
        apply (rule insert)
        using insert.prems unfolding finsert.rep-eq
        by blast
      then obtain cs' where cs = fset-of-list (map (map-prod id sterm-to-pterm)
cs')
        by blast

      obtain rhs' where rhs = sterm-to-pterm rhs'
        apply (rule insert.prems[of (pat, rhs) rhs])
        unfolding ‹c = -› by simp+

      show ?case
        apply (rule exI[where x = (pat, rhs') # cs'])
        unfolding ‹c = -› ‹cs = -› ‹rhs = -›
        by (simp add: id-def)
    qed
  hence Pabs cs = sterm-to-pterm (Sabs cs')
    by simp
  then show ?case
    using Pabs by metis
```

next
  case (*Papp t1 t2*)
  then obtain *t1' t2'* **where** *t1 = sterm-to-pterm t1' t2 = sterm-to-pterm t2'*
    by *metis*
  then have *t1 $_p t2 = sterm-to-pterm (t1' $_s t2')*
    by *simp*
  with *Papp* **show** *?case*
    by *metis*
qed

lemma *pterm-induct*[*case-names Pconst Pvar Pabs Papp*]:
  assumes $\bigwedge x.\ P\ (Pconst\ x)$
  assumes $\bigwedge x.\ P\ (Pvar\ x)$
  assumes $\bigwedge cs.\ (\bigwedge pat\ t.\ (pat,\ t) \mid\in\mid cs \Longrightarrow P\ t) \Longrightarrow P\ (Pabs\ cs)$
  assumes $\bigwedge t\ u.\ P\ t \Longrightarrow P\ u \Longrightarrow P\ (t\ \$_p\ u)$
  shows *P t*
proof (*rule pterm.induct*, *goal-cases*)
  case (*3 cs*)
  show *?case*
    apply (*rule assms*)
    using *3*
    by *auto*
qed *fact+*

instantiation *pterm* :: *pre-term* **begin**

definition *app-pterm* **where**
app-pterm t u = t $_p u

fun *unapp-pterm* **where**
unapp-pterm (t $_p u) = Some (t, u) |
unapp-pterm - = None

definition *const-pterm* **where**
const-pterm = Pconst

fun *unconst-pterm* **where**
unconst-pterm (Pconst name) = Some name |
unconst-pterm - = None

definition *free-pterm* **where**
free-pterm = Pvar

fun *unfree-pterm* **where**
unfree-pterm (Pvar name) = Some name |
unfree-pterm - = None

function (*sequential*) *subst-pterm* **where**
subst-pterm (Pvar s) env = (case fmlookup env s of Some t ⇒ t | None ⇒ Pvar

```
s) |
subst-pterm (t1 $p t2) env = subst-pterm t1 env $p subst-pterm t2 env |
subst-pterm (Pabs cs) env = Pabs ((λ(pat, rhs). (pat, subst-pterm rhs (fmdrop-fset
(frees pat) env))) |‘| cs) |
subst-pterm t - = t
by pat-completeness auto

termination
proof (relation measure (size ∘ fst), goal-cases)
  case 4
  then show ?case
    apply auto
    including fset.lifting apply transfer
    apply (rule le-imp-less-Suc)
    apply (rule sum-nat-le-single[where y = (a, (b, size b)) for a b])
    by auto
qed auto

primrec consts-pterm :: pterm ⇒ name fset where
consts-pterm (Pconst x) = {|x|} |
consts-pterm (t1 $p t2) = consts-pterm t1 |∪| consts-pterm t2 |
consts-pterm (Pabs cs) = ffUnion (snd |‘| map-prod id consts-pterm |‘| cs) |
consts-pterm (Pvar -) = {||}

primrec frees-pterm :: pterm ⇒ name fset where
frees-pterm (Pvar x) = {|x|} |
frees-pterm (t1 $p t2) = frees-pterm t1 |∪| frees-pterm t2 |
frees-pterm (Pabs cs) = ffUnion ((λ(pv, tv). tv − frees pv) |‘| map-prod id frees-pterm
|‘| cs) |
frees-pterm (Pconst -) = {||}

instance
by standard
   (auto
      simp: app-pterm-def const-pterm-def free-pterm-def
      elim: unapp-pterm.elims unconst-pterm.elims unfree-pterm.elims
      split: option.splits)

end

corollary subst-pabs-id:
  assumes ⋀pat rhs. (pat, rhs) |∈| cs ⟹ subst rhs (fmdrop-fset (frees pat) env)
= rhs
  shows subst (Pabs cs) env = Pabs cs
apply (subst subst-pterm.simps)
apply (rule arg-cong[where f = Pabs])
apply (rule fset-map-snd-id)
apply (rule assms)
apply assumption
```

done

corollary *frees-pabs-alt-def*:
frees (*Pabs cs*) = *ffUnion* ((λ(*pat*, *rhs*). *frees rhs* − *frees pat*) |‘| *cs*)
apply *simp*
apply (*rule arg-cong*[**where** *f* = *ffUnion*])
apply (*rule fset.map-cong*[*OF refl*])
by *auto*

lemma *sterm-to-pterm-frees*[*simp*]: *frees* (*sterm-to-pterm t*) = *frees t*
proof (*induction t*)
case (*Sabs cs*)
show *?case*
apply *simp*
apply (*rule arg-cong*[**where** *f* = *ffUnion*])
apply (*rule fimage-cong*[*OF refl*])
apply *clarsimp*
apply (*subst Sabs*)
by (*auto simp*: *fset-of-list-elem snds.simps*)
qed *auto*

lemma *sterm-to-pterm-consts*[*simp*]: *consts* (*sterm-to-pterm t*) = *consts t*
proof (*induction t*)
case (*Sabs cs*)
show *?case*
apply *simp*
apply (*rule arg-cong*[**where** *f* = *ffUnion*])
apply (*rule fimage-cong*[*OF refl*])
apply *clarsimp*
apply (*subst Sabs*)
by (*auto simp*: *fset-of-list-elem snds.simps*)
qed *auto*

lemma *subst-sterm-to-pterm*:
subst (*sterm-to-pterm t*) (*fmmap sterm-to-pterm env*) = *sterm-to-pterm* (*subst t env*)
proof (*induction t arbitrary*: *env rule*: *sterm-induct*)
case (*Sabs cs*)
show *?case*
apply *simp*
apply (*rule fset.map-cong0*)
apply (*auto split*: *prod.splits*)
apply (*rule Sabs*)
by (*auto simp*: *fset-of-list.rep-eq*)
qed (*auto split*: *option.splits*)

instantiation *pterm* :: *term* **begin**

definition *abs-pred-pterm* :: (*pterm* ⇒ *bool*) ⇒ *pterm* ⇒ *bool* **where**

abs-pred P t ⟷ (∀ *cs*. *t* = *Pabs cs* ⟶ (∀ *pat t*. (*pat*, *t*) |∈| *cs* ⟶ *P t*) ⟶ *P t*)

lemma *abs-pred-pterm-code* [*code*]:
‹*abs-pred P t* ⟷ (*case t of Pabs cs* ⇒ (∀ (-, *t*)|∈|*cs*. *P t*) ⟶ *P t* | - ⇒ *True*)›
by (*cases t*) (*auto simp add*: *abs-pred-pterm-def*)

context begin

private lemma *abs-pred-trivI0*: *P t* ⟹ *abs-pred P* (*t*::*pterm*)
unfolding *abs-pred-pterm-def* **by** *auto*

instance proof (*standard*, *goal-cases*)
case (*1 P t*)
then show *?case*
by (*induction t rule*: *pterm-induct*)
(*auto simp*: *const-pterm-def free-pterm-def app-pterm-def abs-pred-pterm-def*)
next

case (*2 t*)
show *?case*
unfolding *abs-pred-pterm-def*
apply *clarify*
apply (*rule subst-pabs-id*)
by *blast*
next
case (*3 x t*)
show *?case*
unfolding *abs-pred-pterm-def*
apply *clarsimp*
apply (*rule fset.map-cong0*)
apply (*rename-tac c*)
apply (*case-tac c*; *hypsubst-thin*)
apply *simp*
subgoal for *cs env pat rhs*
apply (*cases x* |∈| *frees pat*)
subgoal
apply (*rule arg-cong*[**where** *f* = *subst rhs*])
by (*auto intro*: *fmap-ext*)
subgoal premises *prems*[*rule-format*]
apply (*subst* (*2*) *prems*(*1*)[*symmetric*, **where** *pat* = *pat*])
subgoal
by *fact*
subgoal
using *prems* **unfolding** *ffUnion-alt-def*

```
        by (auto simp: fset-of-list.rep-eq elim!: fBallE)
      subgoal
        apply (rule arg-cong[where f = subst rhs])
        by (auto intro: fmap-ext)
      done
    done
  done
next
  case (4 t)
  show ?case
    unfolding abs-pred-pterm-def
    apply clarsimp
    apply (rule fset.map-cong0)
    apply clarsimp
    subgoal premises prems[rule-format] for cs env1 env2 a b
      apply (rule prems(2)[OF prems(5)])
      using prems unfolding fdisjnt-alt-def by auto
    done
next
  case 5
  show ?case
    proof (rule abs-pred-trivI0, clarify)
      fix t :: pterm
      fix env :: (name, pterm) fmap

      obtain t' where t = sterm-to-pterm t'
        by (rule sterm-to-pterm-total)
      obtain env' where env = fmmap sterm-to-pterm env'
        by (metis fmmap-total sterm-to-pterm-total)

      show frees (subst t env) = frees t − fmdom env if fmpred (λ-. closed) env
        unfolding ‹t = -› ‹env = -›
        apply simp
        apply (subst subst-sterm-to-pterm)
        apply simp
        apply (rule subst-frees)
        using that unfolding ‹env = -›
        apply simp
        apply (rule fmpred-mono-strong; assumption?)
        unfolding closed-except-def by simp
    qed
next
  case 6
  show ?case
    proof (rule abs-pred-trivI0, clarify)
      fix t :: pterm
      fix env :: (name, pterm) fmap

      obtain t' where t = sterm-to-pterm t'
```

```
      by (rule sterm-to-pterm-total)
    obtain env' where env = fmmap sterm-to-pterm env'
      by (metis fmmap-total sterm-to-pterm-total)

    show consts (subst t env) = consts t |∪| ffUnion (consts |‘| fmimage env (frees
t))
      unfolding ‹t = -› ‹env = -›
      apply simp
      apply (subst comp-def)
      apply simp
      apply (subst subst-sterm-to-pterm)
      apply simp
      apply (rule subst-consts')
      done
  qed
qed (rule abs-pred-trivI0)

end

end

lemma no-abs-abs[simp]: ¬ no-abs (Pabs cs)
by (subst no-abs.simps) (auto simp: term-cases-def)

lemma sterm-to-pterm:
  assumes no-abs t
  shows sterm-to-pterm t = convert-term t
using assms proof induction
  case (free name)
  show ?case
    apply simp
    apply (simp add: free-sterm-def free-pterm-def)
    done
next
  case (const name)
  show ?case
    apply simp
    apply (simp add: const-sterm-def const-pterm-def)
    done
next
  case (app t1 t2)
  then show ?case
    apply simp
    apply (simp add: app-sterm-def app-pterm-def)
    done
qed

abbreviation Pabs-single (‹Λp -. -› [0, 50] 50) where
Pabs-single x rhs ≡ Pabs {| (Free x, rhs) |}
```

lemma *closed-except-simps*:
closed-except (*Pvar x*) *S* ⟷ *x* |∈| *S*
closed-except (*t*$_1$ $_p *t*$_2$) *S* ⟷ *closed-except t*$_1$ *S* ∧ *closed-except t*$_2$ *S*
closed-except (*Pabs cs*) *S* ⟷ *fBall cs* (λ(*pat*, *t*). *closed-except t* (*S* |∪| *frees pat*))
closed-except (*Pconst name*) *S* ⟷ *True*
proof *goal-cases*
case *3*
show *?case*
proof (*standard*, *goal-cases*)
case *1*
then show *?case*
apply (*auto simp*: *ffUnion-alt-def closed-except-def*)
apply (*drule ffUnion-least-rev*)
apply *auto*
by (*smt* (*z3*) *case-prod-conv fBall-alt-def fminus-iff fset-rev-mp id-apply map-prod-simp*)
next
case *2*
then show *?case*
by (*fastforce simp*: *ffUnion-alt-def closed-except-def*)
qed
qed (*auto simp*: *ffUnion-alt-def closed-except-def*)

instantiation *pterm* :: *pre-strong-term* **begin**

function (*sequential*) *wellformed-pterm* :: *pterm* ⇒ *bool* **where**
wellformed-pterm (*t*$_1$ $_p *t*$_2$) ⟷ *wellformed-pterm t*$_1$ ∧ *wellformed-pterm t*$_2$ |
wellformed-pterm (*Pabs cs*) ⟷ *fBall cs* (λ(*pat*, *t*). *linear pat* ∧ *wellformed-pterm t*) ∧ *is-fmap cs* ∧ *pattern-compatibles cs* ∧ *cs* ≠ {|||} |
wellformed-pterm - ⟷ *True*
by *pat-completeness auto*

termination
proof (*relation measure size*, *goal-cases*)
case *4*
then show *?case*
apply *auto*
including *fset.lifting* **apply** *transfer*
apply (*rule le-imp-less-Suc*)
apply (*rule sum-nat-le-single*[**where** *y* = (*a*, (*b*, *size b*)) **for** *a b*])
by *auto*
qed *auto*

primrec *all-frees-pterm* :: *pterm* ⇒ *name fset* **where**
all-frees-pterm (*Pvar x*) = {|*x*|} |
all-frees-pterm (*t*$_1$ $_p *t*$_2$) = *all-frees-pterm t*$_1$ |∪| *all-frees-pterm t*$_2$ |
all-frees-pterm (*Pabs cs*) = *ffUnion* ((λ(*P*, *T*). *P* |∪| *T*) |‘| *map-prod frees all-frees-pterm* |‘| *cs*) |

```
all-frees-pterm (Pconst -) = {||}

instance
by standard (auto simp: const-pterm-def free-pterm-def app-pterm-def)

end

lemma sterm-to-pterm-all-frees[simp]: all-frees (sterm-to-pterm t) = all-frees t
proof (induction t)
  case (Sabs cs)
  show ?case
    apply simp
    apply (rule arg-cong[where f = ffUnion])
    apply (rule fimage-cong[OF refl])
    apply clarsimp
    apply (subst Sabs)
    by (auto simp: fset-of-list-elem snds.simps)
qed auto

instance pterm :: strong-term proof (standard, goal-cases)
  case (1 t)
  obtain t' where t = sterm-to-pterm t'
    by (metis sterm-to-pterm-total)
  show ?case
    apply (rule abs-pred-trivI)
    unfolding ‹t = -› sterm-to-pterm-all-frees sterm-to-pterm-frees
    by (rule frees-all-frees)
next
  case (2 t)
  show ?case
    unfolding abs-pred-pterm-def
    apply (intro allI impI)
    apply (simp add: case-prod-twice, intro conjI)
    subgoal by blast
    subgoal by (auto intro: is-fmap-image)
    subgoal
      unfolding fpairwise-image fpairwise-alt-def
      by (auto elim!: fBallE)
    done
qed

lemma wellformed-PabsI:
  assumes is-fmap cs pattern-compatibles cs cs ≠ {||}
  assumes ⋀pat t. (pat, t) |∈| cs ⟹ linear pat
  assumes ⋀pat t. (pat, t) |∈| cs ⟹ wellformed t
  shows wellformed (Pabs cs)
using assms by auto

corollary subst-closed-pabs:
```

assumes (*pat*, *rhs*) |∈| *cs closed* (*Pabs cs*)
shows *subst rhs* (*fmdrop-fset* (*frees pat*) *env*) = *rhs*
using *assms* **by** (*subst subst-closed-except-id*) (*auto simp*: *fdisjnt-alt-def closed-except-simps*)

lemma (**in** *constants*) *shadows-consts-pterm-simps*[*simp*]:
shadows-consts (*t*$_1$ $_p *t*$_2$) ⟷ *shadows-consts t*$_1$ ∨ *shadows-consts t*$_2$
shadows-consts (*Pvar name*) ⟷ *name* |∈| *all-consts*
shadows-consts (*Pabs cs*) ⟷ *fBex cs* (λ(*pat*, *t*). *shadows-consts pat* ∨ *shadows-consts t*)
shadows-consts (*Pconst name*) ⟷ *False*
proof *goal-cases*
case *3*

show *?case*
unfolding *shadows-consts-def*
apply *rule*
subgoal
by (*force simp*: *ffUnion-alt-def fset-of-list-elem fdisjnt-alt-def elim*!: *ballE*)
subgoal
apply (*auto simp*: *fset-of-list-elem fdisjnt-alt-def*)
by (*auto simp*: *fset-eq-empty-iff ffUnion-alt-def fset-of-list-elem elim*!: *allE fBallE*)
done
qed (*auto simp*: *shadows-consts-def fdisjnt-alt-def*)

end

1.8 Irreducible terms (values)

theory *Term-as-Value*
imports *Sterm*
begin

1.9 Viewing *sterm* as values

declare *list.pred-mono*[*mono*]

context *constructors* **begin**

inductive *is-value* :: *sterm* ⇒ *bool* **where**
abs: *is-value* (*Sabs cs*) |
constr: *list-all is-value vs* ⟹ *name* |∈| *C* ⟹ *is-value* (*name* $$ *vs*)

lemma *value-distinct*:
Sabs cs ≠ *name* $$ *ts* (**is** *?P*)
name $$ *ts* ≠ *Sabs cs* (**is** *?Q*)
proof −

```
  show ?P
    apply (rule list-comb-cases'[where f = const name and xs = ts])
     apply (auto simp: const-sterm-def is-app-def elim: unapp-sterm.elims)
    done
  thus ?Q
    by simp
qed

abbreviation value-env :: (name, sterm) fmap ⇒ bool where
value-env ≡ fmpred (λ-. is-value)

lemma svar-value[simp]: ¬ is-value (Svar name)
proof
  assume is-value (Svar name)
  thus False
    apply (cases rule: is-value.cases)
    apply (fold free-sterm-def)
    by simp
qed

lemma value-cases:
  obtains (comb) name vs where list-all is-value vs t = name $$ vs name |∈| C
        | (abs) cs where t = Sabs cs
        | (nonvalue) ¬ is-value t
proof (cases t)
  case Svar
  thus thesis using nonvalue by simp
next
  case Sabs
  thus thesis using abs by (auto intro: is-value.abs)
next
  case (Sconst name)

  have list-all is-value [] by simp
  have t = name $$ [] unfolding Sconst by (simp add: const-sterm-def)
  show thesis
    using comb is-value.cases abs nonvalue by blast
next
  case Sapp

  show thesis
    proof (cases is-value t)
      case False
      thus thesis using nonvalue by simp
    next
      case True
      then obtain name vs where list-all is-value vs t = name $$ vs name |∈| C
        unfolding Sapp
        by cases auto
```

```
    thus thesis using comb by simp
  qed
qed

end

fun smatch' :: pat ⇒ sterm ⇒ (name, sterm) fmap option where
smatch' (Patvar name) t = Some (fmap-of-list [(name, t)]) |
smatch' (Patconstr name ps) t =
 (case strip-comb t of
   (Sconst name', vs) ⇒
    (if name = name' ∧ length ps = length vs then
      map-option (foldl (++f) fmempty) (those (map2 smatch' ps vs))
    else
      None)
 | - ⇒ None)

lemmas smatch'-induct = smatch'.induct[case-names var constr]

context constructors begin

context begin

private lemma smatch-list-comb-is-value:
 assumes is-value t
 shows match (name $$ ps) t = (case strip-comb t of
   (Sconst name', vs) ⇒
    (if name = name' ∧ length ps = length vs then
      map-option (foldl (++f) fmempty) (those (map2 match ps vs))
    else
      None)
 | - ⇒ None)
using assms
apply cases
apply (auto simp: strip-list-comb split: option.splits)
apply (subst (2) const-sterm-def)
apply (auto simp: matchs-alt-def)
done

lemma smatch-smatch'-eq:
 assumes linear pat is-value t
 shows match pat t = smatch' (mk-pat pat) t
using assms
proof (induction pat arbitrary: t rule: linear-pat-induct)
 case (comb name args)

 show ?case
  using ‹is-value t›
  proof (cases rule: is-value.cases)
```

```
      case (abs cs)
      thus ?thesis
        by (force simp: strip-list-comb-const)
    next
      case (constr args' name')

     have map2 match args args' = map2 smatch' (map mk-pat args) args' if length
args = length args'
        using that constr(2) comb(2)
        by (induct args args' rule: list-induct2) auto

      thus ?thesis
        using constr
        apply (auto
                    simp: smatch-list-comb-is-value strip-list-comb map-option-case
strip-list-comb-const
              intro: is-value.intros)
        apply (subst (2) const-sterm-def)
        apply (auto simp: matchs-alt-def)
        done
    qed
qed simp

end

end

end
```

1.10 A dedicated value type

```
theory Value
imports Term-as-Value
begin

datatype value =
  is-Vconstr: Vconstr name value list |
  Vabs sclauses (name, value) fmap |
  Vrecabs (name, sclauses) fmap name (name, value) fmap

type-synonym vrule = name × value

setup ‹Sign.mandatory-path quickcheck›


datatype value =
  Vconstr name value list |
  Vabs sclauses (name × value) list |
  Vrecabs (name × sclauses) list name (name × value) list
```

primrec *Value* :: *quickcheck.value* ⇒ *value* **where**
Value (*quickcheck.Vconstr s vs*) = *Vconstr s* (*map Value vs*) |
Value (*quickcheck.Vabs cs* Γ) = *Vabs cs* (*fmap-of-list* (*map* (*map-prod id Value*) Γ)) |
Value (*quickcheck.Vrecabs css name* Γ) = *Vrecabs* (*fmap-of-list css*) *name* (*fmap-of-list* (*map* (*map-prod id Value*) Γ))

setup ‹*Sign.parent-path*›

quickcheck-generator *value*
  constructors: *quickcheck.Value*

fun *vmatch* :: *pat* ⇒ *value* ⇒ (*name*, *value*) *fmap option* **where**
vmatch (*Patvar name*) *v* = *Some* (*fmap-of-list* [(*name*, *v*)]) |
vmatch (*Patconstr name ps*) (*Vconstr name*′ *vs*) =
  (*if name* = *name*′ ∧ *length ps* = *length vs then*
    map-option (*foldl* (+$_f$) *fmempty*) (*those* (*map2 vmatch ps vs*))
  else
    None) |
vmatch - - = *None*

lemmas *vmatch-induct* = *vmatch.induct*[*case-names var constr*]

locale *value-pred* =
  fixes *P* :: (*name*, *value*) *fmap* ⇒ *sclauses* ⇒ *bool*
  fixes *Q* :: *name* ⇒ *bool*
  fixes *R* :: *name fset* ⇒ *bool*
begin

primrec *pred* :: *value* ⇒ *bool* **where**
pred (*Vconstr name vs*) ⟷ *Q name* ∧ *list-all id* (*map pred vs*) |
pred (*Vabs cs* Γ) ⟷ *pred-fmap id* (*fmmap pred* Γ) ∧ *P* Γ *cs* |
pred (*Vrecabs css name* Γ) ⟷
    pred-fmap id (*fmmap pred* Γ) ∧
    pred-fmap (*P* Γ) *css* ∧
    name |∈| *fmdom css* ∧
    R (*fmdom css*)

declare *pred.simps* [*simp del*]

lemma *pred-alt-def*[*simp*, *code*]:
  pred (*Vconstr name vs*) ⟷ *Q name* ∧ *list-all pred vs*
  pred (*Vabs cs* Γ) ⟷ *fmpred* (λ-. *pred*) Γ ∧ *P* Γ *cs*
  pred (*Vrecabs css name* Γ) ⟷ *fmpred* (λ-. *pred*) Γ ∧ *pred-fmap* (*P* Γ) *css* ∧ *name* |∈| *fmdom css* ∧ *R* (*fmdom css*)
by (*auto simp*: *list-all-iff pred.simps*)

For technical reasons, we don't introduce an abbreviation for *fmpred* (λ-.

pred) *env* here. This locale is supposed to be interpreted with **global-interpretation** (or **sublocale** and a **defines** clause. However, this does not affect abbreviations: the abbreviation would still refer to the locale constant, not the constant introduced by the interpretation.

lemma *vmatch-env*:
  assumes *vmatch pat v* = *Some env pred v*
  shows *fmpred* (λ-. *pred*) *env*
using *assms* **proof** (*induction pat v arbitrary*: *env rule*: *vmatch-induct*)
  case (*constr name ps name' vs*)
  hence
    map-option (*foldl* (+$+_f$) *fmempty*) (*those* (*map2 vmatch ps vs*)) = *Some env*
    name = *name' length ps* = *length vs*
    by (*auto split*: *if-splits*)
  then obtain *envs* **where** *env* = *foldl* (+$+_f$) *fmempty envs map2 vmatch ps vs* = *map Some envs*
    by (*blast dest*: *those-someD*)

  moreover have *fmpred* (λ-. *pred*) *env* **if** *env* ∈ *set envs* **for** *env*
    proof −
      from *that* **have** *Some env* ∈ *set* (*map2 vmatch ps vs*)
        unfolding ‹*map2* - - - = -› **by** *simp*
      then obtain *p v* **where** *p* ∈ *set ps v* ∈ *set vs vmatch p v* = *Some env*
        apply (*rule map2-elemE*)
        by *auto*
      hence *pred v*
        using *constr* **by** (*simp add*: *list-all-iff*)
      show *?thesis*
        by (*rule constr*; *safe?*) *fact+*
    qed

    ultimately show *?case*
      by *auto*
qed *auto*

end

primrec *value-to-sterm* :: *value* ⇒ *sterm* **where**
value-to-sterm (*Vconstr name vs*) = *name* $$ *map value-to-sterm vs* |
value-to-sterm (*Vabs cs* Γ) = *Sabs* (*map* (λ(*pat*, *t*). (*pat*, *subst t* (*fmdrop-fset* (*frees pat*) (*fmmap value-to-sterm* Γ)))) *cs*) |
value-to-sterm (*Vrecabs css name* Γ) =
  Sabs (*map* (λ(*pat*, *t*). (*pat*, *subst t* (*fmdrop-fset* (*frees pat*) (*fmmap value-to-sterm* Γ)))) (*the* (*fmlookup css name*)))

This locale establishes a connection between a predicate on *value*s with the corresponding predicate on *sterm*s, by means of *value-to-sterm*.

locale *pre-value-sterm-pred* = *value-pred* +
  fixes *S*

```
  assumes value-to-sterm: pred v ⟹ S (value-to-sterm v)
begin

corollary value-to-sterm-env:
  assumes fmpred (λ-. pred) Γ
  shows fmpred (λ-. S) (fmmap value-to-sterm Γ)
unfolding fmpred-map proof
  fix name v
  assume fmlookup Γ name = Some v
  with assms have pred v by (metis fmpredD)
  thus S (value-to-sterm v) by (rule value-to-sterm)
qed

end

locale value-sterm-pred = value-pred + S: simple-syntactic-and S for S +
  assumes const: ⋀name. Q name ⟹ S (const name)
  assumes abs: ⋀Γ cs.
    (⋀n v. fmlookup Γ n = Some v ⟹ pred v ⟹ S (value-to-sterm v)) ⟹
    fmpred (λ-. pred) Γ ⟹
    P Γ cs ⟹
    S (Sabs (map (λ(pat, t). (pat, subst t (fmmap value-to-sterm (fmdrop-fset (frees
pat) Γ)))) cs))
begin

sublocale pre-value-sterm-pred
proof
  fix v
  assume pred v
  then show S (value-to-sterm v)
    proof (induction v)
      case (Vconstr x1 x2)
      show ?case
        apply simp
        unfolding S.list-comb
        apply rule
         apply (rule const)
        using Vconstr by (auto simp: list-all-iff)
    next
      case (Vabs x1 x2)
      show ?case
        apply auto
        apply (rule abs)
        using Vabs
        by (auto intro: fmran'I)
    next
      case (Vrecabs x1 x2 x3)
      show ?case
        apply auto
```

```
      apply (rule abs)
      using Vrecabs
      by (auto simp: fmlookup-dom-iff fmpred-iff intro: fmran'I)
    qed
qed

end

global-interpretation vwellformed:
  value-sterm-pred
    λ-. wellformed-clauses
    λ-. True
    λ-. True
    wellformed
  defines vwellformed = vwellformed.pred
proof (standard, goal-cases)
  case (2 Γ cs)
  hence cs ≠ []
    by simp

  moreover have wellformed (subst rhs (fmdrop-fset (frees pat) (fmmap value-to-sterm
Γ)))
    if (pat, rhs) ∈ set cs for pat rhs
    proof −
      show ?thesis
        apply (rule subst-wellformed)
        subgoal using 2 that by (force simp: list-all-iff)
        apply (rule fmpred-drop-fset)
        using 2 by auto
    qed

  moreover have distinct (map (fst ∘ (λ(pat, t). (pat, subst t (fmmap value-to-sterm
(fmdrop-fset (frees pat) Γ))))) cs)
    apply (subst map-cong[OF refl, where g = fst])
    using 2 by auto

  ultimately show ?case
    using 2 by (auto simp: list-all-iff)
qed (auto simp: const-sterm-def)

abbreviation wellformed-venv ≡ fmpred (λ-. vwellformed)

global-interpretation vclosed:
  value-sterm-pred
    λΓ cs. list-all (λ(pat, t). closed-except t (fmdom Γ |∪| frees pat)) cs
    λ-. True
    λ-. True
    closed
  defines vclosed = vclosed.pred
```

proof (*standard*, *goal-cases*)
case (*2* Γ *cs*)
show *?case*
apply (*simp add*: *list-all-iff case-prod-twice Sterm.closed-except-simps*)
apply *safe*
apply (*subst closed-except-def*)
apply (*subst subst-frees*)
apply *simp*
subgoal
apply (*rule fmpred-drop-fset*)
apply (*rule fmpredI*)
apply (*rule 2*)
apply *assumption*
using *2* **by** *auto*
subgoal
using *2* **by** (*auto simp*: *list-all-iff closed-except-def*)
done
qed *simp*

abbreviation *closed-venv* ≡ *fmpred* (λ-. *vclosed*)

context *pre-constants* **begin**

sublocale *vwelldefined*:
value-sterm-pred
λ- *cs*. *list-all* (λ(-, *t*). *welldefined t*) *cs*
λ*name*. *name* |∈| *C*
λ*dom*. *dom* |⊆| *heads*
welldefined
defines *vwelldefined* = *vwelldefined.pred*
proof (*standard*, *goal-cases*)
case (*2* Γ *cs*)
note *fset-of-list-map*[*simp del*]

show *?case*
apply *simp*
apply (*rule ffUnion-least*)
apply (*rule fBallI*)
apply (*subst* (*asm*) *fset-of-list-elem*)
apply *simp*
apply (*erule imageE*)
apply (*simp add*: *case-prod-twice*)
subgoal for - *x*
apply (*cases x*)
apply *simp*
apply (*rule subst-consts*)
subgoal
using *2* **by** (*fastforce simp*: *list-all-iff*)
subgoal

```
      apply simp
      apply (rule fmpred-drop-fset)
      unfolding fmpred-map
      apply (rule fmpredI)
      using 2 by auto
    done
  done
qed (simp add: all-consts-def)

lemmas vwelldefined-alt-def = vwelldefined.pred-alt-def

end

declare pre-constants.vwelldefined-alt-def [code]

context constructors begin

sublocale vconstructor-value:
  pre-value-sterm-pred
    λ- -. True
    λname. name |∈| C
    λ-. True
    is-value
  defines vconstructor-value = vconstructor-value.pred
proof
  fix v
  assume value-pred.pred (λ- -. True) (λname. name |∈| C) (λ-. True) v
  then show is-value (value-to-sterm v)
    proof (induction v)
      case (Vconstr name vs)
      hence list-all is-value (map value-to-sterm vs)
        by (fastforce simp: list-all-iff value-pred.pred-alt-def)
      show ?case
        unfolding value-to-sterm.simps
        apply (rule is-value.constr)
         apply fact
        using Vconstr by (simp add: value-pred.pred-alt-def)
    qed (auto simp: disjnt-def intro: is-value.intros)
qed

lemmas vconstructor-value-alt-def = vconstructor-value.pred-alt-def

abbreviation vconstructor-value-env ≡ fmpred (λ-. vconstructor-value)

definition vconstructor-value-rs :: vrule list ⇒ bool where
vconstructor-value-rs rs ⟷
  list-all (λ(-, rhs). vconstructor-value rhs) rs ∧
  fdisjnt (fset-of-list (map fst rs)) C
```

end

declare *constructors.vconstructor-value-alt-def* [*code*]
declare *constructors.vconstructor-value-rs-def* [*code*]

context *pre-constants* **begin**

sublocale *not-shadows-vconsts*:
  value-sterm-pred
    λ- *cs*. *list-all* (λ(*pat*, *t*). *fdisjnt all-consts* (*frees pat*) ∧ ¬ *shadows-consts t*) *cs*
    λ-. *True*
    λ-. *True*
    λ*t*. ¬ *shadows-consts t*
  defines *not-shadows-vconsts* = *not-shadows-vconsts.pred*
proof (*standard*, *goal-cases*)
  case (*2* Γ *cs*)
  show *?case*
    apply (*simp add*: *list-all-iff list-ex-iff case-prod-twice*)
    apply (*rule ballI*)
    subgoal for *x*
      apply (*cases x*, *simp*)
      apply (*rule conjI*)
      subgoal
        using *2* **by** (*force simp*: *list-all-iff*)
      apply (*rule subst-shadows*)
      subgoal
        using *2* **by** (*force simp*: *list-all-iff*)
      apply *simp*
      apply (*rule fmpred-drop-fset*)
      apply (*rule fmpredI*)
      using *2* **by** *auto*
    done
qed (*auto simp*: *const-sterm-def app-sterm-def*)

lemmas *not-shadows-vconsts-alt-def* = *not-shadows-vconsts.pred-alt-def*

abbreviation *not-shadows-vconsts-env* ≡ *fmpred* (λ- *s*. *not-shadows-vconsts s*)

end

declare *pre-constants.not-shadows-vconsts-alt-def* [*code*]

fun *term-to-value* :: *sterm* ⇒ *value* **where**
term-to-value t =
  (*case strip-comb t of*
    (*Sconst name*, *args*) ⇒ *Vconstr name* (*map term-to-value args*)
  | (*Sabs cs*, []) ⇒ *Vabs cs fmempty*)

lemma (**in** *constructors*) *term-to-value-to-sterm*:

```
  assumes is-value t
  shows value-to-sterm (term-to-value t) = t
using assms proof induction
  case (constr vs name)

  have map (value-to-sterm ∘ term-to-value) vs = map id vs
    proof (rule list.map-cong0, unfold comp-apply id-apply)
      fix v
      assume v ∈ set vs
      with constr show value-to-sterm (term-to-value v) = v
        by (simp add: list-all-iff)
    qed
  thus ?case
    apply (simp add: strip-list-comb-const)
    apply (subst const-sterm-def)
    by simp
qed simp

lemma vmatch-dom:
  assumes vmatch pat v = Some env
  shows fmdom env = patvars pat
using assms proof (induction pat v arbitrary: env rule: vmatch-induct)
  case (constr name ps name' vs)
  hence
    map-option (foldl (++ₓ) fmempty) (those (map2 vmatch ps vs)) = Some env
    name = name' length ps = length vs
    by (auto split: if-splits)
  then obtain envs where env = foldl (++ₓ) fmempty envs map2 vmatch ps vs
= map Some envs
    by (blast dest: those-someD)

  moreover have fset-of-list (map fmdom envs) = fset-of-list (map patvars ps)
    proof safe
      fix names
      assume names |∈| fset-of-list (map fmdom envs)
      hence names ∈ set (map fmdom envs)
        unfolding fset-of-list-elem .
      then obtain env where env ∈ set envs names = fmdom env
        by auto
      hence Some env ∈ set (map2 vmatch ps vs)
        unfolding ‹map2 - - - = -› by simp
      then obtain p v where p ∈ set ps v ∈ set vs vmatch p v = Some env
        by (auto elim: map2-elemE)
      moreover hence fmdom env = patvars p
        using constr by fastforce
      ultimately have names ∈ set (map patvars ps)
        unfolding ‹names = -› by simp
      thus names |∈| fset-of-list (map patvars ps)
        unfolding fset-of-list-elem .
```

```
    next
      fix names
      assume names |∈| fset-of-list (map patvars ps)
      hence names ∈ set (map patvars ps)
        unfolding fset-of-list-elem .
      then obtain p where p ∈ set ps names = patvars p
        by auto
      then obtain v where v ∈ set vs vmatch p v ∈ set (map2 vmatch ps vs)
        using ‹length ps = length vs› by (auto elim!: map2-elemE1)
      then obtain env where env ∈ set envs vmatch p v = Some env
        unfolding ‹map2 vmatch ps vs = -› by auto
      moreover hence fmdom env = patvars p
        using constr ‹name = name'› ‹length ps = length vs› ‹p ∈ set ps› ‹v ∈ set
vs›
        by fastforce
      ultimately have names ∈ set (map fmdom envs)
        unfolding ‹names = -› by auto
      thus names |∈| fset-of-list (map fmdom envs)
        unfolding fset-of-list-elem .
    qed

  ultimately show ?case
    by (auto simp: fmdom-foldl-add)
qed auto

fun vfind-match :: sclauses ⇒ value ⇒ ((name, value) fmap × term × sterm)
option where
vfind-match [] - = None |
vfind-match ((pat, rhs) # cs) t =
  (case vmatch (mk-pat pat) t of
    Some env ⇒ Some (env, pat, rhs)
  | None ⇒ vfind-match cs t)

lemma vfind-match-elem:
  assumes vfind-match cs t = Some (env, pat, rhs)
  shows (pat, rhs) ∈ set cs vmatch (mk-pat pat) t = Some env
using assms
by (induct cs) (auto split: option.splits)

inductive veq-structure :: value ⇒ value ⇒ bool where
abs-abs: veq-structure (Vabs - -) (Vabs - -) |
recabs-recabs: veq-structure (Vrecabs - - -) (Vrecabs - - -) |
constr-constr: list-all2 veq-structure ts us ⟹ veq-structure (Vconstr name ts) (Vconstr
name us)

lemma veq-structure-simps[code, simp]:
  veq-structure (Vabs cs1 Γ1) (Vabs cs2 Γ2)
  veq-structure (Vrecabs css1 name1 Γ1) (Vrecabs css2 name2 Γ2)
   veq-structure (Vconstr name1 ts) (Vconstr name2 us) ⟷ name1 = name2 ∧
```

```
list-all2 veq-structure ts us
by (auto intro: veq-structure.intros elim: veq-structure.cases)

lemma veq-structure-refl[simp]: veq-structure t t
by (induction t) (auto simp: list.rel-refl-strong)

global-interpretation vno-abs: value-pred λ- -. False λ-. True λ-. False
  defines vno-abs = vno-abs.pred .

lemma veq-structure-eq-left:
  assumes veq-structure t u vno-abs t
  shows t = u
using assms proof (induction rule: veq-structure.induct)
  case (constr-constr ts us name)
  have ts = us if list-all vno-abs ts
    using constr-constr.IH that
    by induction auto
  with constr-constr show ?case
    by auto
qed auto

lemma veq-structure-eq-right:
  assumes veq-structure t u vno-abs u
  shows t = u
using assms proof (induction rule: veq-structure.induct)
  case (constr-constr ts us name)
  have ts = us if list-all vno-abs us
    using constr-constr.IH that
    by induction auto
  with constr-constr show ?case
    by auto
qed auto

fun vmatch' :: pat ⇒ value ⇒ (name, value) fmap option where
vmatch' (Patvar name) v = Some (fmap-of-list [(name, v)]) |
vmatch' (Patconstr name ps) v =
  (case v of
    Vconstr name' vs ⇒
      (if name = name' ∧ length ps = length vs then
        map-option (foldl (++f) fmempty) (those (map2 vmatch' ps vs))
      else
        None)
  | - ⇒ None)

lemma vmatch-vmatch'-eq: vmatch p v = vmatch' p v
proof (induction rule: vmatch.induct)
  case (2 name ps name' vs)
  then show ?case
    apply auto
```

```
    apply (rule map-option-cong[OF - refl])
    apply (rule arg-cong[where f = those])
    apply (rule map2-cong[OF refl refl])
    apply blast
    done
qed auto

locale value-struct-rel =
  fixes Q :: value ⇒ value ⇒ bool
  assumes Q-impl-struct: Q t1 t2 ⟹ veq-structure t1 t2
   assumes Q-def[simp]: Q (Vconstr name ts) (Vconstr name' us) ⟷ name =
name' ∧ list-all2 Q ts us
begin

lemma eq-left: Q t u ⟹ vno-abs t ⟹ t = u
using Q-impl-struct by (metis veq-structure-eq-left)

lemma eq-right: Q t u ⟹ vno-abs u ⟹ t = u
using Q-impl-struct by (metis veq-structure-eq-right)

context begin

private lemma vmatch'-rel:
  assumes Q t1 t2
  shows rel-option (fmrel Q) (vmatch' p t1) (vmatch' p t2)
using assms(1) proof (induction p arbitrary: t1 t2)
  case (Patconstr name ps)
  with Q-impl-struct have veq-structure t1 t2
    by blast

  thus ?case
    proof (cases rule: veq-structure.cases)
      case (constr-constr ts us name')

      {
        assume length ps = length ts

        have list-all2 (rel-option (fmrel Q)) (map2 vmatch' ps ts) (map2 vmatch'
ps us)
          using ‹list-all2 veq-structure ts us› Patconstr ‹length ps = length ts›
          unfolding ‹t1 = -› ‹t2 = -›
          proof (induction arbitrary: ps)
            case (Cons t ts u us ps0)
            then obtain p ps where ps0 = p # ps
              by (cases ps0) auto

            have length ts = length us
              using Cons by (auto dest: list-all2-lengthD)
```

```
          hence Q t u
            using ‹Q (Vconstr name' (t # ts)) (Vconstr name' (u # us))›
            by (simp add: list-all-iff)
          hence rel-option (fmrel Q) (vmatch' p t) (vmatch' p u)
            using Cons unfolding ‹ps0 = -› by simp

           moreover have list-all2 (rel-option (fmrel Q)) (map2 vmatch' ps ts)
(map2 vmatch' ps us)
            apply (rule Cons)
            subgoal
              apply (rule Cons)
              unfolding ‹ps0 = -› apply simp
              by assumption
            subgoal
             using ‹Q (Vconstr name' (t # ts)) (Vconstr name' (u # us))› ‹length
ts = length us›
              by (simp add: list-all-iff)
            subgoal
              using ‹length ps0 = length (t # ts)›
              unfolding ‹ps0 = -› by simp
            done

          ultimately show ?case
            unfolding ‹ps0 = -›
            by auto
        qed auto

       hence rel-option (list-all2 (fmrel Q)) (those (map2 vmatch' ps ts)) (those
(map2 vmatch' ps us))
        by (rule rel-funD[OF those-transfer])

      have
        rel-option (fmrel Q)
          (map-option (foldl (++f) fmempty) (those (map2 vmatch' ps ts)))
          (map-option (foldl (++f) fmempty) (those (map2 vmatch' ps us)))
        apply (rule rel-funD[OF rel-funD[OF option.map-transfer]])
         apply transfer-prover
        by fact
    }
    note ∗ = this

    have length ts = length us
      using constr-constr by (auto dest: list-all2-lengthD)

    thus ?thesis
      unfolding ‹t1 = -› ‹t2 = -›
      apply auto
      apply (rule ∗)
      by simp
```

```
    qed auto
qed auto

lemma vmatch-rel: Q t1 t2 ⟹ rel-option (fmrel Q) (vmatch p t1) (vmatch p t2)
unfolding vmatch-vmatch'-eq by (rule vmatch'-rel)

lemma vfind-match-rel:
  assumes list-all2 (rel-prod (=) R) cs1 cs2
  assumes Q t1 t2
   shows rel-option (rel-prod (fmrel Q) (rel-prod (=) R)) (vfind-match cs1 t1)
(vfind-match cs2 t2)
using assms(1) proof induction
  case (Cons c1 cs1 c2 cs2)
  moreover obtain pat1 rhs1 where c1 = (pat1, rhs1) by fastforce
  moreover obtain pat2 rhs2 where c2 = (pat2, rhs2) by fastforce

  ultimately have pat1 = pat2 R rhs1 rhs2
    by auto

  have rel-option (fmrel Q) (vmatch (mk-pat pat1) t1) (vmatch (mk-pat pat1) t2)
    by (rule vmatch-rel) fact
  thus ?case
    proof cases
      case None
      thus ?thesis
        unfolding ‹c1 = -› ‹c2 = -› ‹pat1 = -›
        using Cons by auto
    next
      case (Some Γ1 Γ2)
      thus ?thesis
        unfolding ‹c1 = -› ‹c2 = -› ‹pat1 = -›
        using ‹R rhs1 rhs2›
        by auto
    qed
qed simp

lemmas vfind-match-rel' =
  vfind-match-rel[
    where R = (=) and cs1 = cs and cs2 = cs for cs,
    unfolded prod.rel-eq,
    OF list.rel-refl, OF refl]

end end

hide-fact vmatch-vmatch'-eq
hide-const vmatch'

global-interpretation veq-structure: value-struct-rel veq-structure
by standard auto
```

abbreviation *env-eq* **where**
env-eq ≡ *fmrel* (λ*v t*. *t* = *value-to-sterm v*)

lemma *env-eq-eq*:
assumes *env-eq venv senv*
shows *senv* = *fmmap value-to-sterm venv*
proof (*rule fmap-ext*, *unfold fmlookup-map*)
fix *name*
from *assms* **have** *rel-option* (λ*v t*. *t* = *value-to-sterm v*) (*fmlookup venv name*) (*fmlookup senv name*)
by *auto*
thus *fmlookup senv name* = *map-option value-to-sterm* (*fmlookup venv name*)
by *cases auto*
qed

context *constructors* **begin**

context begin

private lemma *vmatch-eq0*: *rel-option env-eq* (*vmatch p v*) (*smatch' p* (*value-to-sterm v*))
proof (*induction p v rule*: *vmatch-induct*)
case (*constr name ps name' vs*)

have
rel-option env-eq
(*map-option* (*foldl* (++$_f$) Γ) (*those* (*map2 vmatch ps vs*)))
(*map-option* (*foldl* (++$_f$) Γ′) (*those* (*map2 smatch' ps* (*map value-to-sterm vs*))))
if *length ps* = *length vs* **and** *name* = *name'* **and** *env-eq* Γ Γ′ **for** Γ Γ′
using *that constr*
proof (*induction arbitrary*: Γ Γ′ *rule*: *list-induct2*)
case (*Cons p ps v vs*)
hence *rel-option env-eq* (*vmatch p v*) (*smatch' p* (*value-to-sterm v*))
by *auto*
thus *?case*
proof *cases*
case (*Some* Γ$_1$ Γ$_2$)
thus *?thesis*
apply (*simp add*: *option.map-comp comp-def*)
apply (*rule Cons*)
using *Cons* **by** *auto*
qed *simp*
qed *fastforce*

thus *?case*
apply (*auto simp*: *strip-list-comb-const*)
apply (*subst const-sterm-def*, *simp*)+

done
qed *auto*

corollary *vmatch-eq*:
assumes *linear p vconstructor-value v*
shows *rel-option env-eq* (*vmatch* (*mk-pat p*) *v*) (*match p* (*value-to-sterm v*))
using *assms*
by (*metis smatch-smatch'-eq vmatch-eq0 vconstructor-value.value-to-sterm*)

end

end

abbreviation *match-related* **where**
match-related ≡ (λ(Γ$_1$, *pat*$_1$, *rhs*$_1$) (Γ$_2$, *pat*$_2$, *rhs*$_2$). *rhs*$_1$ = *rhs*$_2$ ∧ *pat*$_1$ = *pat*$_2$ ∧ *env-eq* Γ$_1$ Γ$_2$)

lemma (**in** *constructors*) *find-match-eq*:
assumes *list-all* (*linear* ∘ *fst*) *cs vconstructor-value v*
shows *rel-option match-related* (*vfind-match cs v*) (*find-match cs* (*value-to-sterm v*))
using *assms* **proof** (*induct cs*)
case (*Cons c cs*)
then obtain *p t* **where** *c* = (*p*, *t*) **by** *fastforce*
hence *rel-option env-eq* (*vmatch* (*mk-pat p*) *v*) (*match p* (*value-to-sterm v*))
using *Cons* **by** (*fastforce intro*: *vmatch-eq*)
thus *?case*
using *Cons* **unfolding** ‹*c* = -›
by *cases auto*
qed *auto*

inductive *erelated* :: *value* ⇒ *value* ⇒ *bool* (‹- ≈$_e$ -›) **where**
constr: *list-all2 erelated ts us* ⟹ *Vconstr name ts* ≈$_e$ *Vconstr name us* |
abs: *fmrel-on-fset* (*ids* (*Sabs cs*)) *erelated* Γ$_1$ Γ$_2$ ⟹ *Vabs cs* Γ$_1$ ≈$_e$ *Vabs cs* Γ$_2$ |
rec-abs:
pred-fmap (λ*cs*. *fmrel-on-fset* (*ids* (*Sabs cs*)) *erelated* Γ$_1$ Γ$_2$) *css* ⟹
Vrecabs css name Γ$_1$ ≈$_e$ *Vrecabs css name* Γ$_2$

code-pred *erelated* **.**

global-interpretation *erelated*: *value-struct-rel erelated*
proof
fix *v*$_1$ *v*$_2$
assume *v*$_1$ ≈$_e$ *v*$_2$
thus *veq-structure v*$_1$ *v*$_2$
by *induction* (*auto intro*: *list.rel-mono-strong*)
next
fix *name name'* **and** *ts us* :: *value list*
show *Vconstr name ts* ≈$_e$ *Vconstr name' us* ⟷ (*name* = *name'* ∧ *list-all2*

erelated ts us)
  by (*auto intro*: *erelated.intros elim*: *erelated.cases*)
qed

lemma *erelated-refl*[*intro*]: $t \approx_e t$
proof (*induction t*)
  case *Vrecabs*
  thus *?case*
    apply (*auto intro*!: *erelated.intros fmpredI fmrel-on-fset-refl-strong*)
    apply (*auto intro*: *fmran'I*)
    done
qed (*auto intro*!: *erelated.intros list.rel-refl-strong fmrel-on-fset-refl-strong fmran'I*)

export-code
  value-to-sterm vmatch vwellformed vclosed erelated-i-i pre-constants.vwelldefined
  constructors.vconstructor-value-rs pre-constants.not-shadows-vconsts term-to-value
  vfind-match veq-structure vno-abs
  checking *Scala*

end

Chapter 2

A smaller version of CakeML: *CupCakeML*

theory *Doc-CupCake*
imports *Main*
begin

end

2.1 CupCake environments

theory *CupCake-Env*
imports *../Utils/CakeML-Utils*
begin

fun *cake-no-abs* :: *v* ⇒ *bool* **where**
cake-no-abs (*Conv* - *vs*) ⟷ *list-all cake-no-abs vs* |
cake-no-abs - ⟷ *False*

fun *is-cupcake-pat* :: *Ast.pat* ⇒ *bool* **where**
is-cupcake-pat (*Ast.Pvar* -) ⟷ *True* |
is-cupcake-pat (*Ast.Pcon* (*Some* (*Short* -)) *xs*) ⟷ *list-all is-cupcake-pat xs* |
is-cupcake-pat - ⟷ *False*

fun *is-cupcake-exp* :: *exp* ⇒ *bool* **where**
is-cupcake-exp (*Ast.Var* (*Short* -)) ⟷ *True* |
is-cupcake-exp (*Ast.App oper es*) ⟷ *oper* = *Ast.Opapp* ∧ *list-all is-cupcake-exp es* |
is-cupcake-exp (*Ast.Con* (*Some* (*Short* -)) *es*) ⟷ *list-all is-cupcake-exp es* |
is-cupcake-exp (*Ast.Fun* - *e*) ⟷ *is-cupcake-exp e* |
is-cupcake-exp (*Ast.Mat e cs*) ⟷ *is-cupcake-exp e* ∧ *list-all* (λ(*p*, *e*). *is-cupcake-pat p* ∧ *is-cupcake-exp e*) *cs* ∧ *cake-linear-clauses cs* |
is-cupcake-exp - ⟷ *False*

abbreviation *cupcake-clauses* :: (*Ast.pat* × *exp*) *list* ⇒ *bool* **where**
cupcake-clauses ≡ *list-all* (λ(*p*, *e*). *is-cupcake-pat p* ∧ *is-cupcake-exp e*)

fun *cupcake-c-ns* :: *c-ns* ⇒ *bool* **where**
cupcake-c-ns (*Bind cs mods*) ⟷
  mods = [] ∧ *list-all* (λ(-, -, *tid*). *case tid of TypeId* (*Short* -) ⇒ *True* | - ⇒ *False*)
cs

locale *cakeml-static-env* =
  fixes *static-cenv* :: *c-ns*
  assumes *static-cenv*: *cupcake-c-ns static-cenv*
begin

definition *empty-sem-env* :: *v sem-env* **where**
empty-sem-env = (| *sem-env.v* = *nsEmpty*, *sem-env.c* = *static-cenv* |)

lemma *v-of-empty-sem-env*[*simp*]: *sem-env.v empty-sem-env* = *nsEmpty*
unfolding *empty-sem-env-def* **by** *simp*

lemma *c-of-empty-sem-env*[*simp*]: *c empty-sem-env* = *static-cenv*
unfolding *empty-sem-env-def* **by** *simp*

fun *is-cupcake-value* :: *SemanticPrimitives.v* ⇒ *bool*
and *is-cupcake-all-env* :: *all-env* ⇒ *bool* **where**
is-cupcake-value (*Conv* (*Some* (-, *TypeId* (*Short* -))) *vs*) ⟷ *list-all is-cupcake-value vs* |
is-cupcake-value (*Closure env* - *e*) ⟷ *is-cupcake-exp e* ∧ *is-cupcake-all-env env* |
is-cupcake-value (*Recclosure env es* -) ⟷ *list-all* (λ(-, -, *e*). *is-cupcake-exp e*) *es* ∧ *is-cupcake-all-env env* |
is-cupcake-value - ⟷ *False* |
is-cupcake-all-env (| *sem-env.v* = *Bind v0* [], *sem-env.c* = *c0* |) ⟷ *c0* = *static-cenv* ∧ *list-all* (*is-cupcake-value* ∘ *snd*) *v0* |
is-cupcake-all-env - ⟷ *False*

lemma *is-cupcake-all-envE*:
  assumes *is-cupcake-all-env env*
  obtains *v c* **where** *env* = (| *sem-env.v* = *Bind v* [], *sem-env.c* = *c* |) *c* = *static-cenv list-all* (*is-cupcake-value* ∘ *snd*) *v*
using *assms*
by (*auto elim*!: *is-cupcake-all-env.elims*)

fun *is-cupcake-ns* :: *v-ns* ⇒ *bool* **where**
is-cupcake-ns (*Bind v0* []) ⟷ *list-all* (*is-cupcake-value* ∘ *snd*) *v0* |
is-cupcake-ns - ⟷ *False*

lemma *is-cupcake-nsE*:
  assumes *is-cupcake-ns ns*
  obtains *v* **where** *ns* = *Bind v* [] *list-all* (*is-cupcake-value* ∘ *snd*) *v*
using *assms* **by** (*rule is-cupcake-ns.elims*)

lemma *is-cupcake-all-envD*:
assumes *is-cupcake-all-env env*
shows *is-cupcake-ns* (*sem-env.v env*) *cupcake-c-ns* (*c env*)
using *assms static-cenv* **by** (*auto elim*!: *is-cupcake-all-envE*)

lemma *is-cupcake-all-envI*:
assumes *is-cupcake-ns* (*sem-env.v env*) *sem-env.c env* = *static-cenv*
shows *is-cupcake-all-env env*
using *assms static-cenv*
apply (*cases env*)
apply *simp*
subgoal for *v c*
apply (*cases v*)
apply *simp*
subgoal for *x1 x2*
by (*cases x2*) *auto*
done
done

end

end

2.2 CupCake semantics

theory *CupCake-Semantics*
imports
CupCake-Env
CakeML.Matching
CakeML.Big-Step-Unclocked-Single
begin

fun *cupcake-nsLookup* :: (*'m*,*'n*,*'v*)*namespace* ⇒ *'n* ⇒ *'v option* **where**
cupcake-nsLookup (*Bind v1* -) *n* = *map-of v1 n*

lemma *cupcake-nsLookup-eq*[*simp*]: *nsLookup ns* (*Short n*) = *cupcake-nsLookup ns n*
by (*cases ns*) *auto*

fun *cupcake-pmatch* :: ((*string*),(*string*),(*nat*∗*tid-or-exn*))*namespace* ⇒ *pat* ⇒ *v* ⇒(*string*∗*v*)*list* ⇒((*string*∗*v*)*list*)*match-result* **where**
cupcake-pmatch cenv (*Pvar x*) *v0 env* = *Match* ((*x*, *v0*)# *env*) |
cupcake-pmatch cenv (*Pcon* (*Some* (*Short n*)) *ps*) (*Conv* (*Some* (*n'*, *t'*)) *vs*) *env* =
(*case cupcake-nsLookup cenv n of*
Some (*l*, *t*)=>
if same-tid t t' ∧ (*List.length ps* = *l*) *then*
if same-ctor (*n*, *t*) (*n'*,*t'*) *then*
Matching.fold2 (λ*p v m. case m of*

```
            Match env ⇒ cupcake-pmatch cenv p v env
          | m ⇒ m) Match-type-error ps vs (Match env)
        else
          No-match
      else
        Match-type-error
  | - => Match-type-error) |
cupcake-pmatch cenv - - - = Match-type-error

fun cupcake-match-result :: - ⇒ v ⇒(pat*exp)list ⇒ v ⇒ (exp × pat × (char list
× v) list, v)result where
cupcake-match-result - - [] err-v = Rerr (Rraise err-v) |
cupcake-match-result cenv v0 ((p, e) # pes) err-v =
 (if distinct (pat-bindings p []) then
   (case cupcake-pmatch cenv p v0 [] of
     Match env' ⇒ Rval (e, p, env') |
     No-match ⇒ cupcake-match-result cenv v0 pes err-v |
     Match-type-error ⇒ Rerr (Rabort Rtype-error))
  else
     Rerr (Rabort Rtype-error))

lemma cupcake-match-resultE:
 assumes cupcake-match-result cenv v0 pes err-v = Rval (e, p, env')
 obtains init rest
   where pes = init @ (p, e) # rest
     and distinct (pat-bindings p [])
      and list-all (λ(p, e). cupcake-pmatch cenv p v0 [] = No-match ∧ distinct
(pat-bindings p [])) init
     and cupcake-pmatch cenv p v0 [] = Match env'
using assms
proof (induction pes)
 case (Cons pe pes)
 obtain p0 e0 where pe = (p0, e0)
   by fastforce

 show thesis
   proof (cases distinct (pat-bindings p0 []))
     case True
     thus ?thesis
       proof (cases cupcake-pmatch cenv p0 v0 [])
         case No-match
         show ?thesis
           proof (rule Cons)
             fix init rest
             assume pes = init @ (p, e) # rest
              assume list-all (λ(p, e). cupcake-pmatch cenv p v0 [] = No-match ∧
distinct (pat-bindings p [])) init
             assume distinct (pat-bindings p [])
             assume cupcake-pmatch cenv p v0 [] = Match env'
```

```
          moreover have pe # pes = ((p0, e0) # init) @ (p, e) # rest
            unfolding ‹pes = -› ‹pe = -› by simp

              moreover have list-all (λ(p, e). cupcake-pmatch cenv p v0 [] =
No-match ∧ distinct (pat-bindings p [])) ((p0, e0) # init)
            apply auto
            subgoal by fact
            subgoal using True by simp
            subgoal using ‹list-all - -› by simp
            done

          moreover have distinct (pat-bindings p [])
            by fact

          ultimately show thesis
            using Cons by blast
        next
          show cupcake-match-result cenv v0 pes err-v = Rval (e, p, env')
            using Cons No-match True unfolding ‹pe = -› by auto
        qed
      next
        case Match
        with Cons show ?thesis
          using True unfolding ‹pe = -› by force
      next
        case Match-type-error
        with Cons show ?thesis
          using True unfolding ‹pe = -› by force
      qed
    next
      case False
      hence False
        using Cons unfolding ‹pe = -› by force
      thus ?thesis ..
    qed
qed simp

lemma cupcake-pmatch-eq:
  is-cupcake-pat pat ⟹ pmatch-single envC s pat v0 env = cupcake-pmatch envC
pat v0 env
proof (induct rule: pmatch-single.induct)
  case 4
  from is-cupcake-pat.elims(2)[OF 4(2)] show ?case
    proof cases
      case 2
      then show ?thesis
        using 4(1) apply -
        apply simp
```

```
      apply (auto split: option.splits match-result.splits)
      apply (rule Matching.fold2-cong)
        apply (auto simp: fun-eq-iff split: match-result.splits)
      apply (metis in-set-conv-decomp-last list.pred-inject(2) list-all-append)
      done
  qed simp
qed auto

lemma cupcake-match-result-eq:
  cupcake-clauses pes ⟹
    match-result env s v pes err-v =
      map-result (λ(e, -, env'). (e, env')) id (cupcake-match-result (c env) v pes
err-v)
by (induction pes) (auto split: match-result.splits simp: cupcake-pmatch-eq pmatch-single-equiv)

context cakeml-static-env begin

lemma cupcake-nsBind-preserve:
  is-cupcake-ns ns ⟹ is-cupcake-value v0 ⟹ is-cupcake-ns (nsBind k v0 ns)
by (cases ns) (auto elim: is-cupcake-ns.elims)

lemma cupcake-build-rec-preserve:
 assumes is-cupcake-all-env cl-env is-cupcake-ns env list-all (λ(-, -, e). is-cupcake-exp
e) fs
  shows is-cupcake-ns (build-rec-env fs cl-env env)
proof −
  have is-cupcake-ns (foldr (λ(f, -) env'. nsBind f (Recclosure cl-env fs0 f) env')
fs env)
    if list-all (λ(-, -, e). is-cupcake-exp e) fs0
    for fs0
    using ‹is-cupcake-ns env›
    proof (induction fs arbitrary: env)
      case (Cons f fs)
      show ?case
        apply (cases f, simp)
        apply (rule cupcake-nsBind-preserve)
         apply (rule Cons.IH)
         apply (rule Cons)
        using that assms by auto
    qed auto
  thus ?thesis
    unfolding build-rec-env-def
    using assms
    by (simp add: cond-case-prod-eta)
qed

lemma cupcake-v-update-preserve:
  assumes is-cupcake-all-env env is-cupcake-ns (f (sem-env.v env))
  shows is-cupcake-all-env (sem-env.update-v f env)
```

using *assms*
by (*metis is-cupcake-all-env.simps*(*1*) *is-cupcake-all-envE is-cupcake-nsE sem-env.collapse sem-env.record-simps*(*1*) *sem-env.record-simps*(*2*) *sem-env.sel*(*2*))

lemma *cupcake-nsAppend-preserve*: *is-cupcake-ns ns1* ⟹ *is-cupcake-ns ns2* ⟹ *is-cupcake-ns* (*nsAppend ns1 ns2*)
by (*auto elim*!: *is-cupcake-ns.elims*)

lemma *cupcake-alist-to-ns-preserve*: *list-all* (*is-cupcake-value* ∘ *snd*) *env* ⟹ *is-cupcake-ns* (*alist-to-ns env*)
unfolding *alist-to-ns-def*
by *simp*

lemma *cupcake-opapp-preserve*:
assumes *do-opapp vs* = *Some* (*env*, *e*) *list-all is-cupcake-value vs*
shows *is-cupcake-all-env env is-cupcake-exp e*
proof −
obtain *cl v0* **where** *vs* = [*cl*, *v0*]
using *assms*
by (*cases vs rule*: *do-opapp.cases*) *auto*
with *assms* **have** *is-cupcake-value cl is-cupcake-value v0*
by *auto*

have *is-cupcake-all-env env* ∧ *is-cupcake-exp e*
using ‹*do-opapp vs* = -› **proof** (*cases rule*: *do-opapp-cases*)
case (*closure env*′ *n arg*)
then show *?thesis*
using ‹*is-cupcake-value cl*› ‹*is-cupcake-value v0*› ‹*vs* = [*cl*, *v0*]›
by (*auto intro*: *cupcake-v-update-preserve cupcake-nsBind-preserve dest*:*is-cupcake-all-envD*(*1*))
next
case (*recclosure env*′ *funs name n*)
hence *is-cupcake-all-env env*′
using ‹*is-cupcake-value cl*› ‹*vs* = [*cl*, *v0*]› **by** *simp*
have (*name*, *n*, *e*) ∈ *set funs*
using *recclosure* **by** (*auto dest*: *map-of-SomeD*)
hence *is-cupcake-exp e*
using ‹*is-cupcake-value cl*› ‹*vs* = [*cl*, *v0*]› *recclosure*
by (*auto simp*: *list-all-iff*)
thus *?thesis*
using ‹*is-cupcake-all-env env*′› ‹*is-cupcake-value cl*› ‹*is-cupcake-value v0*›
‹*vs* = [*cl*, *v0*]› *recclosure*
unfolding ‹*env* = -›
using *cupcake-build-rec-preserve cupcake-nsBind-preserve cupcake-v-update-preserve is-cupcake-all-envD*(*1*)
by *auto*
qed

thus *is-cupcake-all-env env is-cupcake-exp e*
by *simp*+

qed

context begin

lemma *cup-pmatch-list-length-neq*:
length vs ≠ *length ps* ⟹ *Matching.fold2*(λ*p v m*. *case m of*
Match env ⇒ *cupcake-pmatch cenv p v env*
| *m* ⇒ *m*) *Match-type-error ps vs m* = *Match-type-error*
by (*induction ps vs arbitrary*:*m rule*:*List.list-induct2'*) *auto*

lemma *cup-pmatch-list-nomatch*:
length vs = *length ps* ⟹ *Matching.fold2*(λ*p v m*. *case m of*
Match env ⇒ *cupcake-pmatch cenv p v env*
| *m* ⇒ *m*) *Match-type-error ps vs No-match* = *No-match*
by (*induction ps vs rule*:*List.list-induct2'*) *auto*

lemma *cup-pmatch-list-typerr*:
length vs = *length ps* ⟹ *Matching.fold2*(λ*p v m*. *case m of*
Match env ⇒ *cupcake-pmatch cenv p v env*
| *m* ⇒ *m*) *Match-type-error ps vs Match-type-error* = *Match-type-error*
by (*induction ps vs rule*:*List.list-induct2'*) *auto*

private lemma *cupcake-pmatch-list-preserve*:
assumes ⋀*p v env*. *p* ∈ *set ps* ∧ *v* ∈ *set vs* ⟶ *list-all* (*is-cupcake-value* ∘ *snd*)
env ⟶ *if-match* (*list-all* (*is-cupcake-value* ∘ *snd*)) (*cupcake-pmatch cenv p v env*)
list-all (*is-cupcake-value* ∘ *snd*) *env*
shows *if-match* (*list-all* (λ*a*. *is-cupcake-value* (*snd a*))) (*Matching.fold2*
(λ*p v m*. *case m of*
Match env ⇒ *cupcake-pmatch cenv p v env*
| *m* ⇒ *m*)
Match-type-error ps vs (*Match env*))
using *assms* **proof** (*induction ps vs arbitrary*: *env rule*:*list-induct2'*)
case (*4 p ps v vs*)
show *?case*
proof (*cases cupcake-pmatch cenv p v env*)
case *No-match*
then show *?thesis*
by (*cases length ps* = *length vs*) (*auto simp*:*cup-pmatch-list-nomatch cup-pmatch-list-length-neq*)
next
case *Match-type-error*
then show *?thesis*
by (*cases length ps* = *length vs*) (*auto simp*:*cup-pmatch-list-typerr cup-pmatch-list-length-neq*)
next
case (*Match env'*)
then have *env'*: *list-all* (*is-cupcake-value* ∘ *snd*) *env'*
using *4* **by** *fastforce*
then show *?thesis*
apply (*cases length ps* = *length vs*)
using *4 Match* **by** *fastforce*+

qed
qed (*auto simp*: *comp-def*)

private lemma *cupcake-pmatch-preserve0*:
is-cupcake-pat pat ⟹
is-cupcake-value v0 ⟹
list-all (*is-cupcake-value* ∘ *snd*) *env* ⟹
cupcake-c-ns envC ⟹
if-match (*list-all* (*is-cupcake-value* ∘ *snd*)) (*cupcake-pmatch envC pat v0 env*)
proof (*induction rule*: *cupcake-pmatch.induct*)
case (*2 cenv n ps n′ t′ vs env*)
have *p*:*p* ∈ *set ps* ⟹ *is-cupcake-pat p* **for** *p*
using *2* **by** (*metis Ball-set is-cupcake-pat.simps*(*2*))
have *v*:*v* ∈ *set vs* ⟹ *is-cupcake-value v* **for** *v*
using *2* **by** (*metis Ball-set is-cupcake-value.elims*(*2*) *v.distinct*(*11*) *v.distinct*(*13*) *v.inject*(*2*))
show *?case*
by (*auto intro*!: *cupcake-pmatch-list-preserve split*:*if-splits option.splits*) (*metis 2 p v*)+
qed (*auto split*: *option.splits if-splits elim*: *is-cupcake-pat.elims is-cupcake-value.elims*)

lemma *cupcake-pmatch-preserve*:
is-cupcake-pat pat ⟹
is-cupcake-value v0 ⟹
list-all (*is-cupcake-value* ∘ *snd*) *env* ⟹
cupcake-c-ns envC ⟹
cupcake-pmatch envC pat v0 env = *Match env′* ⟹
list-all (*is-cupcake-value* ∘ *snd*) *env′*
by (*metis if-match.simps*(*1*) *cupcake-pmatch-preserve0*)+

end

lemma *cupcake-match-result-preserve*:
cupcake-c-ns envC ⟹
cupcake-clauses pes ⟹
is-cupcake-value v ⟹
if-rval (λ(*e*, *p*, *env′*). *is-cupcake-pat p* ∧ *is-cupcake-exp e* ∧ *list-all* (*is-cupcake-value* ∘ *snd*) *env′*)
(*cupcake-match-result envC v pes err-v*)
apply (*induction pes*)
apply (*auto split*: *match-result.splits*)
apply (*rule cupcake-pmatch-preserve*)
apply *auto*
done

lemma *static-cenv-lookup*:
assumes *cupcake-nsLookup static-cenv i* = *Some* (*len*, *b*)
obtains *name* **where** *b* = *TypeId* (*Short name*)
using *assms static-cenv*

apply (*cases static-cenv*; *cases b*)
apply (*auto simp*: *list-all-iff split*: *prod.splits tid-or-exn.splits id0.splits dest*!: *map-of-SomeD elim*!: *ballE allE*)
using *static-cenv*
apply (*auto simp*: *list-all-iff split*: *prod.splits tid-or-exn.splits id0.splits dest*!: *map-of-SomeD elim*!: *ballE allE*)
done

lemma *cupcake-build-conv-preserve*:
fixes *v*
assumes *list-all is-cupcake-value vs build-conv static-cenv* (*Some* (*Short i*)) *vs* = *Some v*
shows *is-cupcake-value v*
using *assms*
by (*auto simp*: *build-conv.simps split*: *option.splits elim*: *static-cenv-lookup*)

lemma *cupcake-nsLookup-preserve*:
assumes *is-cupcake-ns ns nsLookup ns n* = *Some v0*
shows *is-cupcake-value v0*
proof −
obtain *vs* **where** *list-all* (*is-cupcake-value* ∘ *snd*) *vs ns* = *Bind vs* []
using *assms*
by (*auto elim*: *is-cupcake-ns.elims*)
show *?thesis*
proof (*cases n*)
case (*Short id*)
hence (*id*, *v0*) ∈ *set vs*
using *assms* **unfolding** ‹*ns* = -› **by** (*auto dest*: *map-of-SomeD*)
thus *?thesis*
using ‹*list-all* (*is-cupcake-value* ∘ *snd*) *vs*›
by (*auto simp*: *list-all-iff*)
next
case *Long*
hence *nsLookup ns n* = *None*
unfolding ‹*ns* = -› **by** *simp*
thus *?thesis*
using *assms* **by** *auto*
qed
qed

corollary *match-all-preserve*:
assumes *cupcake-match-result cenv v0 pes err-v* = *Rval* (*e*, *p*, *env′*) *cupcake-c-ns cenv*
assumes *is-cupcake-value v0 cupcake-clauses pes*
shows *list-all* (*is-cupcake-value* ∘ *snd*) *env′ is-cupcake-exp e is-cupcake-pat p*
proof −
from *assms* **obtain** *init rest*
where *pes* = *init* @ (*p*, *e*) # *rest* **and** *cupcake-pmatch cenv p v0* [] = *Match env′*

```
    by (elim cupcake-match-resultE)
  hence (p, e) ∈ set pes
    by simp
  thus is-cupcake-exp e is-cupcake-pat p
    using assms by (auto simp: list-all-iff)

  show list-all (is-cupcake-value ∘ snd) env'
    by (rule cupcake-pmatch-preserve[where env = []]) (fact | simp)+
qed

end

fun list-all2-shortcircuit where
list-all2-shortcircuit P (x # xs) (y # ys) ⟷ (case y of Rval - ⇒ P x y ∧
list-all2-shortcircuit P xs ys | Rerr - ⇒ P x y) |
list-all2-shortcircuit P [] [] ⟷ True |
list-all2-shortcircuit P - - ⟷ False

lemma list-all2-shortcircuit-induct[consumes 1, case-names nil cons-val cons-err]:
  assumes list-all2-shortcircuit P xs ys
  assumes R [] []
  assumes ⋀x xs y ys. P x (Rval y) ⟹ list-all2-shortcircuit P xs ys ⟹ R xs ys
⟹ R (x # xs) (Rval y # ys)
  assumes ⋀x xs y ys. P x (Rerr y) ⟹ R (x # xs) (Rerr y # ys)
  shows R xs ys
using assms
proof (induction P xs ys rule: list-all2-shortcircuit.induct)
  case (1 P x xs y ys)
  thus ?case
    by (cases y) auto
qed auto

lemma list-all2-shortcircuit-mono[mono]:
  assumes R ≤ Q
  shows list-all2-shortcircuit R ≤ list-all2-shortcircuit Q
proof
  fix xs ys
  assume list-all2-shortcircuit R xs ys
  thus list-all2-shortcircuit Q xs ys
    using assms by (induction xs ys rule: list-all2-shortcircuit-induct) auto
qed

lemma list-all2-shortcircuit-weaken: list-all2-shortcircuit P xs ys ⟹ (⋀xs ys. P
xs ys ⟹ Q xs ys) ⟹ list-all2-shortcircuit Q xs ys
by (metis list-all2-shortcircuit-mono predicate2I rev-predicate2D)

lemma list-all2-shortcircuit-rval[simp]:
  list-all2-shortcircuit P xs (map Rval ys) ⟷ list-all2 (λx y. P x (Rval y)) xs ys
(is ?lhs ⟷ ?rhs)
```

proof
assume *?lhs* **thus** *?rhs*
by (*induction map Rval ys*::('*b*, '*c*) *result list arbitrary*: *ys rule*: *list-all2-shortcircuit-induct*) *auto*
next
assume *?rhs* **thus** *?lhs*
by (*induction rule*: *list-all2-induct*) *auto*
qed

inductive *cupcake-evaluate-single* :: *all-env* ⇒ *exp* ⇒ (*v*, *v*) *result* ⇒ *bool* **where**
con1:
do-con-check (*c env*) *cn* (*length es*) ⟹
list-all2-shortcircuit (*cupcake-evaluate-single env*) (*rev es*) *rs* ⟹
sequence-result rs = *Rval vs* ⟹
build-conv (*c env*) *cn* (*rev vs*) = *Some v0* ⟹
cupcake-evaluate-single env (*Con cn es*) (*Rval v0*) |
con2:
¬ *do-con-check* (*c env*) *cn* (*List.length es*) ⟹
cupcake-evaluate-single env (*Con cn es*) (*Rerr* (*Rabort Rtype-error*)) |
con3:
do-con-check (*c env*) *cn* (*List.length es*) ⟹
list-all2-shortcircuit (*cupcake-evaluate-single env*) (*rev es*) *rs* ⟹
sequence-result rs = *Rerr err* ⟹
cupcake-evaluate-single env (*Con cn es*) (*Rerr err*) |
var1:
nsLookup (*sem-env.v env*) *n* = *Some v0* ⟹ *cupcake-evaluate-single env* (*Var n*)
(*Rval v0*) |
var2:
nsLookup (*sem-env.v env*) *n* = *None* ⟹ *cupcake-evaluate-single env* (*Var n*)
(*Rerr* (*Rabort Rtype-error*)) |
fn:
cupcake-evaluate-single env (*Fun n e*) (*Rval* (*Closure env n e*)) |
app1:
list-all2-shortcircuit (*cupcake-evaluate-single env*) (*rev es*) *rs* ⟹
sequence-result rs = *Rval vs* ⟹
do-opapp (*rev vs*) = *Some* (*env*′, *e*) ⟹
cupcake-evaluate-single env′ *e bv* ⟹
cupcake-evaluate-single env (*App Opapp es*) *bv* |
app3:
list-all2-shortcircuit (*cupcake-evaluate-single env*) (*rev es*) *rs* ⟹
sequence-result rs = *Rval vs* ⟹
do-opapp (*rev vs*) = *None* ⟹
cupcake-evaluate-single env (*App Opapp es*) (*Rerr* (*Rabort Rtype-error*)) |
app6:
list-all2-shortcircuit (*cupcake-evaluate-single env*) (*rev es*) *rs* ⟹
sequence-result rs = *Rerr err* ⟹
cupcake-evaluate-single env (*App op0 es*) (*Rerr err*) |
mat1:
cupcake-evaluate-single env e (*Rval v0*) ⟹

cupcake-match-result (*c env*) *v0 pes Bindv* = *Rval* (*e′*, -, *env′*) ⟹
cupcake-evaluate-single (*env* (| *sem-env.v* := *nsAppend* (*alist-to-ns env′*) (*sem-env.v env*) |)) *e′ bv* ⟹
cupcake-evaluate-single env (*Mat e pes*) *bv* |
mat1error:
cupcake-evaluate-single env e (*Rval v0*) ⟹
cupcake-match-result (*c env*) *v0 pes Bindv* = *Rerr err* ⟹
cupcake-evaluate-single env (*Mat e pes*) (*Rerr err*) |
mat2:
cupcake-evaluate-single env e (*Rerr err*) ⟹
cupcake-evaluate-single env (*Mat e pes*) (*Rerr err*)

context *cakeml-static-env* **begin**

context begin

private lemma *cupcake-list-preserve0*:
list-all2-shortcircuit
(λ*e r*. *cupcake-evaluate-single env e r* ∧ (*is-cupcake-all-env env* ⟶ *is-cupcake-exp e* ⟶ *if-rval is-cupcake-value r*)) *es rs* ⟹
is-cupcake-all-env env ⟹ *list-all is-cupcake-exp es* ⟹ *sequence-result rs* = *Rval vs* ⟹ *list-all is-cupcake-value vs*
proof (*induction es rs arbitrary*: *vs rule*:*list-all2-shortcircuit-induct*)
case (*cons-val - - - rs*)
thus *?case*
by (*cases sequence-result rs*) *auto*
qed *auto*

private lemma *cupcake-single-preserve0*:
cupcake-evaluate-single env e res ⟹ *is-cupcake-all-env env* ⟹ *is-cupcake-exp e* ⟹ *if-rval is-cupcake-value res*
proof (*induction rule*:*cupcake-evaluate-single.induct*)
case (*con1 env cn es rs vs v0*)
then obtain *tid* **where** *cn*: *cn* = *Some* (*Short tid*) **and** *list-all is-cupcake-exp* (*rev es*)
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
hence *list-all is-cupcake-value* (*rev vs*) **and** *c env* = *static-cenv*
using *cupcake-list-preserve0 con1*
by (*fastforce elim*:*is-cupcake-all-envE*)+

then show *?case*
using *cupcake-build-conv-preserve con1 cn*
by *fastforce*
next
case (*app1 env es rs vs env′ e bv*)
hence *list-all is-cupcake-exp* (*rev es*)
by *fastforce*
hence *list-all is-cupcake-value* (*rev vs*)
using *app1 cupcake-list-preserve0* **by** *force*

hence *is-cupcake-exp e* **and** *is-cupcake-all-env env′*
using *app1 cupcake-opapp-preserve* **by** *blast+*
then show *?case*
using *app1* **by** *blast*
next
case (*mat1 env e v0 pes e′ uu env′ bv*)
hence *cupcake-c-ns* (*c env*) *cupcake-clauses pes is-cupcake-value v0*
by (*auto dest*: *is-cupcake-all-envD*)
hence *list-all* (*is-cupcake-value* ∘ *snd*) *env′* **and** *e′*: *is-cupcake-exp e′*
using *cupcake-match-result-preserve*[**where** *envC* = *c env* **and** *v* = *v0* **and** *pes* = *pes* **and** *err-v* = *Bindv*, *unfolded mat1*, *simplified*]
by *auto*
have *is-cupcake-all-env* (*update-v* (λ-. *nsAppend* (*alist-to-ns env′*) (*sem-env.v env*)) *env*)
apply (*rule cupcake-v-update-preserve*)
apply *fact*
apply (*rule cupcake-nsAppend-preserve*)
apply (*rule cupcake-alist-to-ns-preserve*)
apply *fact*
apply (*rule is-cupcake-all-envD*)
apply *fact*
done
then show *?case*
using *mat1 e′* **by** *blast*
qed (*auto intro*: *cupcake-nsLookup-preserve dest*: *is-cupcake-all-envD*)

lemma *cupcake-single-preserve*:
cupcake-evaluate-single env e (*Rval res*) ⟹ *is-cupcake-all-env env* ⟹ *is-cupcake-exp e* ⟹ *is-cupcake-value res*
by (*fastforce dest*:*cupcake-single-preserve0*)

lemma *cupcake-list-preserve*:
list-all2-shortcircuit (*cupcake-evaluate-single env*) *es rs* ⟹
is-cupcake-all-env env ⟹ *list-all is-cupcake-exp es* ⟹ *sequence-result rs* = *Rval vs* ⟹ *list-all is-cupcake-value vs*
by (*induction es rs arbitrary*:*vs rule*:*list-all2-shortcircuit-induct*) (*fastforce dest*:*cupcake-single-preserve*)+

private lemma *cupcake-list-correct-rval*:
assumes *list-all2-shortcircuit*
(λ*e r*.
cupcake-evaluate-single env e r ∧
(*is-cupcake-all-env env* ⟶ *is-cupcake-exp e* ⟶ (∀ (*s*::′*a state*). ∃ *s′*. *evaluate env s e* (*s′*, *r*))))
es rs is-cupcake-all-env env list-all is-cupcake-exp es sequence-result rs = *Rval vs*
shows ∃ *s′*. *evaluate-list* (*evaluate env*) (*s*::′*a state*) *es* (*s′*,*Rval vs*)
using *assms* **proof** (*induction es rs arbitrary*: *s vs rule*:*list-all2-shortcircuit-induct*)
case (*cons-val e es y ys*)
have *e*: *is-cupcake-exp e list-all is-cupcake-exp es*

```
    using cons-val by fastforce+
  then obtain vs' where ys: sequence-result ys = Rval vs'
    using cons-val by fastforce
  hence vs: Rval vs = Rval (y # vs')
    using cons-val by fastforce

  from e obtain s' where evaluate env s e (s', Rval y)
    using cons-val by fastforce
  from e ys obtain s'' where evaluate-list (evaluate env) s' es (s'', Rval vs')
    using cons-val by fastforce

  show ?case
    unfolding vs
    by (rule; rule evaluate-list.cons1) fact+
qed (auto intro:evaluate-list.intros)

private lemma cupcake-list-correct-rerr:
  assumes list-all2-shortcircuit
     (λe r.
         cupcake-evaluate-single env e r ∧
       (is-cupcake-all-env env ⟶ is-cupcake-exp e ⟶ (∀(s::'a state). ∃s'. evaluate
env s e (s', r))))
     es rs is-cupcake-all-env env list-all is-cupcake-exp es sequence-result rs = Rerr
err
  shows ∃s'. evaluate-list (evaluate env) (s::'a state) es (s',Rerr err)
using assms proof (induction es rs arbitrary: s err rule:list-all2-shortcircuit-induct)
  case (cons-val e es y ys)
  then have is-cupcake-exp e list-all is-cupcake-exp es
    by fastforce+
  moreover have err: sequence-result ys = Rerr err
    using cons-val
    by (cases sequence-result ys) (auto simp: error-result.map-id)

  ultimately show ?case
    using cons3 cons-val
    by fast
qed (auto intro:evaluate-list.intros)

private lemma cupcake-list-correct0:
  assumes list-all2-shortcircuit
     (λe r.
         cupcake-evaluate-single env e r ∧
       (is-cupcake-all-env env ⟶ is-cupcake-exp e ⟶ (∀(s::'a state). ∃s'. evaluate
env s e (s', r))))
     es rs is-cupcake-all-env env list-all is-cupcake-exp es
  shows ∃s'. evaluate-list (evaluate env) (s::'a state) es (s',sequence-result rs)
 using assms by (cases sequence-result rs) (fastforce intro: cupcake-list-correct-rval
cupcake-list-correct-rerr)+
```

lemma *cupcake-single-correct*:
assumes *cupcake-evaluate-single env e res is-cupcake-all-env env is-cupcake-exp e*
shows $\exists s'$. *Big-Step-Unclocked-Single.evaluate env s e* (*s'*,*res*)
using *assms* **proof** (*induction arbitrary*:*s rule*:*cupcake-evaluate-single.induct*)
case (*con1 env cn es rs vs v0*)
then have *list-all is-cupcake-exp* (*rev es*)
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
then show *?case*
using *cupcake-list-correct-rval evaluate.con1 con1*
by *blast*
next
case (*con3 env cn es rs err*)
then have *list-all is-cupcake-exp* (*rev es*)
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
then show *?case*
using *cupcake-list-correct-rerr con3 evaluate.con3*
by *blast*
next
case (*app1 env es rs vs env' e bv*)
hence *es*: *list-all is-cupcake-exp* (*rev es*)
by *fastforce*
hence *list-all is-cupcake-value* (*rev vs*)
using *app1 cupcake-list-preserve list-all2-shortcircuit-weaken*
by (*metis* (*no-types*, *lifting*) *list-all-rev*)
hence *is-cupcake-exp e* **and** *is-cupcake-all-env env'*
using *app1 cupcake-opapp-preserve* **by** *blast+*

then show *?case*
using *cupcake-list-correct-rval es app1 evaluate.app1*
by *blast*
next
case (*app3 env es rs vs*)
hence *list-all is-cupcake-exp* (*rev es*)
by *simp*
then show *?case*
using *cupcake-list-correct-rval evaluate.app3 app3*
by *blast*
next
case (*app6 env es rs err op0*)
hence *list-all is-cupcake-exp* (*rev es*)
by *simp*
then show *?case*
using *cupcake-list-correct-rerr app6 evaluate.app6*
by *blast*
next
case (*mat1 env e v0 pes e' uu env' bv*)
hence *e*: *is-cupcake-exp e* **and** *cupcake-c-ns* (*c env*) **and** *pes*: *cupcake-clauses pes* **and** *is-cupcake-value v0*

```
    by (fastforce dest: is-cupcake-all-envD cupcake-single-preserve)+
  hence list-all (is-cupcake-value ∘ snd) env' and e': is-cupcake-exp e'
    using cupcake-match-result-preserve[where envC = c env and v = v0 and
pes = pes and err-v = Bindv, unfolded mat1, simplified]
    by blast+
  have env': is-cupcake-all-env (update-v (λ-. nsAppend (alist-to-ns env') (sem-env.v
env)) env)
    apply (rule cupcake-v-update-preserve)
     apply fact
    apply (rule cupcake-nsAppend-preserve)
     apply (rule cupcake-alist-to-ns-preserve)
     apply fact
    apply (rule is-cupcake-all-envD)
    apply fact
    done

  from e obtain s' where evaluate env s e (s', Rval v0)
    using mat1 by blast
  have match-result env s' v0 pes Bindv = Rval (e', env')
    using mat1 cupcake-match-result-eq[OF pes, where env = env and v = v0
and err-v = Bindv and s = s']
    by fastforce
  from e' env' obtain s'' where evaluate (update-v (λ-. nsAppend (alist-to-ns
env') (sem-env.v env)) env) s' e' (s'', bv)
    using mat1 by blast

  show ?case
    by rule+ fact+
next
  case (mat1error env e v0 pes err)
  hence is-cupcake-exp e and pes: cupcake-clauses pes
    by (auto dest: is-cupcake-all-envD)

  then obtain s' where Big-Step-Unclocked-Single.evaluate env s e (s', Rval v0)
    using mat1error by blast
  hence match-result env s' v0 pes Bindv = Rerr err
    using cupcake-match-result-eq[OF pes, where env = env and s = s' and v =
v0 and err-v = Bindv] unfolding mat1error
    by (simp add: error-result.map-id)

  show ?case
    by (rule; rule evaluate.mat1b) fact+
next
  case (mat2 - e)
  hence is-cupcake-exp e
    by simp
  then show ?case
    using mat2 evaluate.mat2 by blast
qed (blast intro:evaluate.intros)+
```

lemma *cupcake-list-correct*:
assumes *list-all2-shortcircuit* (*cupcake-evaluate-single env*) *es rs is-cupcake-all-env env list-all is-cupcake-exp es*
shows ∃ *s'*. *evaluate-list* (*evaluate env*) (*s*::'*a state*) *es* (*s'*,*sequence-result rs*)
using *assms* **by** (*fastforce intro*:*cupcake-list-correct0 list-all2-shortcircuit-weaken cupcake-single-correct*)+

private lemma *cupcake-list-complete0*:
evaluate-list
(λ*s e r*. *evaluate env s e r* ∧ (*is-cupcake-all-env env* ⟶ *is-cupcake-exp e* ⟶ *cupcake-evaluate-single env e* (*snd r*))) *s1 es res* ⟹
is-cupcake-all-env env ⟹ *list-all is-cupcake-exp es* ⟹ ∃ *rs*. *list-all2-shortcircuit* (*cupcake-evaluate-single env*) *es rs* ∧ *sequence-result rs* = (*snd res*)
proof (*induction rule*:*evaluate-list.induct*)
case *empty*
have *list-all2-shortcircuit* (*cupcake-evaluate-single env*) [] []
by *fastforce*
then show *?case*
by *fastforce*
next
case (*cons1 s1 e s2 v es s3 vs*)
then obtain *rs* **where** *list-all2-shortcircuit* (*cupcake-evaluate-single env*) *es rs*
and *sequence-result rs* = *Rval vs*
and *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*e* # *es*) (*Rval v* # *rs*)
by *fastforce*+
then show *?case*
by *fastforce*
next
case (*cons2 s1 e s2 err es*)
hence *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*e* # *es*) [*Rerr err*]
by *simp*
then show *?case*
by *fastforce*
next
case (*cons3 s1 e s2 v es s3 err*)
then obtain *rs* **where** *list-all2-shortcircuit* (*cupcake-evaluate-single env*) *es rs*
and *err*:*sequence-result rs* = *Rerr err*
and *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*e* # *es*) (*Rval v* # *rs*)
by *fastforce*
moreover have *sequence-result* (*Rval v* # *rs*) = *Rerr err*
by (*auto simp*: *error-result.map-id err*)
ultimately show *?case*
by *fastforce*
qed

private lemma *cupcake-single-complete0*:
evaluate env s e res ⟹ *is-cupcake-all-env env* ⟹ *is-cupcake-exp e* ⟹ *cupcake-evaluate-single env e* (*snd res*)

proof (*induction rule*:*evaluate.induct*)
case (*con1 env cn es vs v s1 s2*)
hence *list-all is-cupcake-exp* (*rev es*)
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
hence *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*rev es*) (*map Rval vs*)
using *cupcake-list-complete0 con1* **by** *fastforce*
show *?case*
by (*simp*|*rule*|*fact*)+
next
case (*con3 env cn es s1 s2 err*)
hence *list-all is-cupcake-exp* (*rev es*)
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
then obtain *rs* **where** *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*rev es*) *rs sequence-result rs* = *Rerr err*
using *con3* **by** (*fastforce dest*:*cupcake-list-complete0*)
show *?case*
by (*simp*;*rule cupcake-evaluate-single.con3*) *fact*+
next
case (*app1 env s1 es s2 vs env*′ *e bv*)
then obtain *rs* **where** *rs*: *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*rev es*) *rs sequence-result rs* = *Rval vs*
by (*fastforce dest*:*cupcake-list-complete0*)
hence *list-all is-cupcake-exp* (*rev es*)
using *app1* **by** *fastforce*
hence *list-all is-cupcake-value vs list-all is-cupcake-value* (*rev vs*)
using *cupcake-list-preserve app1 rs* **by** *fastforce*+
hence *is-cupcake-exp e is-cupcake-all-env env*′
using *app1 cupcake-opapp-preserve* **by** *fastforce*+
hence *cupcake-evaluate-single env*′ *e* (*snd bv*)
using *app1* **by** *fastforce*
show *?case*
by *rule fact*+
next
case (*app3 env s1 es s2 vs*)
hence *list-all is-cupcake-exp* (*rev es*)
by *simp*
obtain *rs* **where** *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*rev es*) *rs sequence-result rs* = *Rval vs*
using *app3 cupcake-list-complete0* **by** *fastforce*
show *?case*
by (*simp*|*rule*|*fact*)+
next
case (*app6 env s1 es s2 err op0*)
obtain *rs* **where** *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*rev es*) *rs sequence-result rs* = *Rerr err*
using *cupcake-list-complete0 app6* **by** *fastforce*
show *?case*
by (*simp*|*rule*|*fact*)+
next

case (*mat1 env s1 e s2 v1 pes e′ env′ bv*)
hence *is-cupcake-exp e* **and** *cupcake-c-ns* (*c env*) **and** *pes*:*cupcake-clauses pes* **and** *is-cupcake-value v1*
by (*fastforce dest*: *is-cupcake-all-envD cupcake-single-preserve*)+

moreover obtain *uu* **where** *cupcake-match-result* (*c env*) *v1 pes Bindv* = *Rval* (*e′*, *uu*, *env′*)
using *cupcake-match-result-eq*[*OF pes*,**where** *env* = *env* **and** *s*= *s2* **and** *v* = *v1* **and** *err-v* = *Bindv*, *unfolded mat1*]
by (*cases cupcake-match-result* (*c env*) *v1 pes Bindv*) *auto*

ultimately have *list-all* (*is-cupcake-value* ∘ *snd*) *env′ is-cupcake-exp e′*
using *cupcake-match-result-preserve*[**where** *envC* = *c env* **and** *v* = *v1* **and** *pes* = *pes* **and** *err-v* = *Bindv*]
by *fastforce*+
moreover have *is-cupcake-all-env* (*update-v* (*λ-. nsAppend* (*alist-to-ns env′*) (*sem-env.v env*)) *env*)
apply (*rule cupcake-v-update-preserve*)
apply *fact*
apply (*rule cupcake-nsAppend-preserve*)
apply (*rule cupcake-alist-to-ns-preserve*)
apply *fact*
apply (*rule is-cupcake-all-envD*)
apply *fact*
done

ultimately have *cupcake-evaluate-single env e* (*Rval v1*)
and *cupcake-evaluate-single* (*update-v* (*λ-. nsAppend* (*alist-to-ns env′*) (*sem-env.v env*)) *env*) *e′* (*snd bv*)
using *mat1* **by** *fastforce*+

show *?case*
by (*rule cupcake-evaluate-single.mat1*) *fact*+
next
case (*mat1b env s1 e s2 v1 pes err*)
hence *is-cupcake-exp e* **and** *pes*: *cupcake-clauses pes*
by (*auto dest*: *is-cupcake-all-envD*)

have *cupcake-evaluate-single env e* (*Rval v1*)
using *mat1b* **by** *fastforce*
have *cupcake-match-result* (*c env*) *v1 pes Bindv* = *Rerr err*
using *cupcake-match-result-eq*[*OF pes*, **where** *env* = *env* **and** *s* = *s2* **and** *v* = *v1* **and** *err-v* = *Bindv*] **unfolding** *mat1b*
by (*cases* (*cupcake-match-result* (*c env*) *v1 pes Bindv*)) (*auto simp*:*error-result.map-id*)
show *?case*
by (*simp*; *rule cupcake-evaluate-single.mat1error*) *fact*+
qed (*fastforce intro*: *cupcake-evaluate-single.intros*)+

lemma *cupcake-single-complete*:

evaluate env s e (*s'*, *res*) ⟹ *is-cupcake-all-env env* ⟹ *is-cupcake-exp e* ⟹ *cupcake-evaluate-single env e res*
by (*fastforce dest*:*cupcake-single-complete0*)

lemma *cupcake-list-complete*:
evaluate-list (*evaluate env*) *s1 es res* ⟹
is-cupcake-all-env env ⟹ *list-all is-cupcake-exp es* ⟹ ∃ *rs*. *list-all2-shortcircuit* (*cupcake-evaluate-single env*) *es rs* ∧ *sequence-result rs* = (*snd res*)
by (*fastforce intro*:*cupcake-list-complete0 cupcake-single-complete evaluate-list-mono-strong*)

private lemma *cupcake-list-state-preserve0*:
assumes *evaluate-list* (λ*s e res*. *Big-Step-Unclocked-Single.evaluate env s e res* ∧ (*is-cupcake-all-env env* ⟶ *is-cupcake-exp e* ⟶ *s* = *fst res*)) *s es res*
list-all is-cupcake-exp es is-cupcake-all-env env
shows *s* = (*fst res*)
using *assms* **by** (*induction rule*:*evaluate-list.induct*) *auto*

lemma *cupcake-state-preserve*:
assumes *Big-Step-Unclocked-Single.evaluate env s e res is-cupcake-all-env env is-cupcake-exp e*
shows *s* = (*fst res*)
using *assms* **proof** (*induction arbitrary*: *rule*: *evaluate.induct*)
case (*con1 env cn es vs v s1 s2*)
hence *list-all is-cupcake-exp es*
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
then show *?case*
using *con1* **by** (*fastforce dest*:*cupcake-list-state-preserve0*)
next
case (*con3 env cn es s1 s2 err*)
hence *list-all is-cupcake-exp es*
by (*cases rule*: *is-cupcake-exp.cases*[**where** *x* = *Con cn es*]) *auto*
then show *?case*
using *con3* **by** (*fastforce dest*:*cupcake-list-state-preserve0*)
next
case (*app1 env s1 es s2 vs env' e bv*)
have *list-all is-cupcake-exp* (*rev es*)
using *app1* **by** *fastforce*
then obtain *rs* **where** *rs*: *list-all2-shortcircuit* (*cupcake-evaluate-single env*) (*rev es*) *rs sequence-result rs* = *Rval vs*
using *app1* **by** (*fastforce dest*:*evaluate-list-mono-strong*[*THEN cupcake-list-complete*])
hence *list-all is-cupcake-value vs list-all is-cupcake-value* (*rev vs*)
using *cupcake-list-preserve app1 rs* **by** *fastforce*+
hence *is-cupcake-exp e is-cupcake-all-env env'*
using *app1 cupcake-opapp-preserve* **by** *fastforce*+
then show *?case*
using *app1* **by** (*fastforce dest*:*cupcake-list-state-preserve0*)
next
case (*mat1 env s1 e s2 v1 pes e' env' bv*)
hence *is-cupcake-exp e* **and** *cupcake-c-ns* (*c env*) **and** *pes*:*cupcake-clauses pes*

and *is-cupcake-value v1*
by (*fastforce dest*: *is-cupcake-all-envD cupcake-single-complete cupcake-single-preserve*)+

moreover obtain *uu* **where** *cupcake-match-result* (*c env*) *v1 pes Bindv* = *Rval* (*e′*, *uu*, *env′*)
using *cupcake-match-result-eq*[*OF pes*,**where** *env* = *env* **and** *s*= *s2* **and** *v* = *v1* **and** *err-v* = *Bindv*, *unfolded mat1*]
by (*cases cupcake-match-result* (*c env*) *v1 pes Bindv*) *auto*

ultimately have *list-all* (*is-cupcake-value* ∘ *snd*) *env′ is-cupcake-exp e′*
using *cupcake-match-result-preserve*[**where** *envC* = *c env* **and** *v* = *v1* **and** *pes* = *pes* **and** *err-v* = *Bindv*]
by *fastforce*+
moreover have *is-cupcake-all-env* (*update-v* (*λ-. nsAppend* (*alist-to-ns env′*) (*sem-env.v env*)) *env*)
apply (*rule cupcake-v-update-preserve*)
apply *fact*
apply (*rule cupcake-nsAppend-preserve*)
apply (*rule cupcake-alist-to-ns-preserve*)
apply *fact*
apply (*rule is-cupcake-all-envD*)
apply *fact*
done
ultimately show *?case*
using *mat1* **by** *fastforce*
qed (*fastforce dest*:*cupcake-list-state-preserve0*)+

corollary *cupcake-single-correct-strong*:
assumes *cupcake-evaluate-single env e res is-cupcake-all-env env is-cupcake-exp e*
shows *Big-Step-Unclocked-Single.evaluate env s e* (*s*,*res*)
using *assms cupcake-single-correct cupcake-state-preserve* **by** *fastforce*

corollary *cupcake-single-complete-weak*:
evaluate env s e (*s*, *res*) ⟹ *is-cupcake-all-env env* ⟹ *is-cupcake-exp e* ⟹ *cupcake-evaluate-single env e res*
using *cupcake-single-complete* **by** *fastforce*

end end

hide-const (**open**) *c*

end

Chapter 3

Term rewriting

theory *Doc-Rewriting*
imports *Main*
begin

end
theory *General-Rewriting*
imports *Terms-Extras*
begin

locale *rewriting* =
  fixes $R :: 'a::term \Rightarrow 'a \Rightarrow bool$
  assumes *R-fun*: $R\ t\ t' \Longrightarrow R\ (app\ t\ u)\ (app\ t'\ u)$
  assumes *R-arg*: $R\ u\ u' \Longrightarrow R\ (app\ t\ u)\ (app\ t\ u')$
begin

lemma *rt-fun*:
  $R^{**}\ t\ t' \Longrightarrow R^{**}\ (app\ t\ u)\ (app\ t'\ u)$
by (*induct rule*: *rtranclp.induct*) (*auto intro*: *rtranclp.rtrancl-into-rtrancl R-fun*)

lemma *rt-arg*:
  $R^{**}\ u\ u' \Longrightarrow R^{**}\ (app\ t\ u)\ (app\ t\ u')$
by (*induct rule*: *rtranclp.induct*) (*auto intro*: *rtranclp.rtrancl-into-rtrancl R-arg*)

lemma *rt-comb*:
  $R^{**}\ t_1\ u_1 \Longrightarrow R^{**}\ t_2\ u_2 \Longrightarrow R^{**}\ (app\ t_1\ t_2)\ (app\ u_1\ u_2)$
by (*metis rt-fun rt-arg rtranclp-trans*)

lemma *rt-list-comb*:
  assumes *list-all2* R^{**} *ts us* R^{**} *t u*
  shows R^{**} (*list-comb t ts*) (*list-comb u us*)
using *assms*
by (*induction ts us arbitrary*: *t u rule*: *list.rel-induct*) (*auto intro*: *rt-comb*)

end

end

3.1 Higher-order term rewriting using de-Bruijn indices

theory *Rewriting-Term*
imports
../Terms/General-Rewriting
../Terms/Strong-Term
begin

3.1.1 Matching and rewriting

type-synonym *rule* = *term* × *term*

inductive *rewrite* :: *rule fset* ⇒ *term* ⇒ *term* ⇒ *bool* (‹-/ ⊢/ - ⟶/ -› [*50*,*0*,*50*] *50*) **for** *rs* **where**
step: *r* |∈| *rs* ⟹ *rs* ⊢ *t* → *u* ⟹ *rs* ⊢ *t* ⟶ *u* |
beta: *rs* ⊢ (Λ *t* $ *t'*) ⟶ *t* [*t'*]$_\beta$ |
fun: *rs* ⊢ *t* ⟶ *t'* ⟹ *rs* ⊢ *t* $ *u* ⟶ *t'* $ *u* |
arg: *rs* ⊢ *u* ⟶ *u'* ⟹ *rs* ⊢ *t* $ *u* ⟶ *t* $ *u'*

global-interpretation *rewrite*: *rewriting rewrite rs* **for** *rs*
by *standard* (*auto intro*: *rewrite.intros simp*: *app-term-def*)+

abbreviation *rewrite-rt* :: *rule fset* ⇒ *term* ⇒ *term* ⇒ *bool* (‹-/ ⊢/ - ⟶*/ -› [*50*,*0*,*50*] *50*) **where**
rewrite-rt rs ≡ (*rewrite rs*)**

lemma *rewrite-beta-alt*: *t* [*t'*]$_\beta$ = *u* ⟹ *wellformed t'* ⟹ *rs* ⊢ (Λ *t* $ *t'*) ⟶ *u*
by (*metis rewrite.beta*)

3.1.2 Wellformedness

primrec *rule* :: *rule* ⇒ *bool* **where**
rule (*lhs*, *rhs*) ⟷ *basic-rule* (*lhs*, *rhs*) ∧ *Term.wellformed rhs*

lemma *ruleI*[*intro*]:
assumes *basic-rule* (*lhs*, *rhs*)
assumes *Term.wellformed rhs*
shows *rule* (*lhs*, *rhs*)
using *assms* **by** *simp*

lemma *split-rule-fst*: *fst* (*split-rule r*) = *head* (*fst r*)
unfolding *head-def* **by** (*smt* (*z3*) *prod.case-eq-if prod.collapse prod.inject split-rule.simps*)

locale *rules* = *constants C-info heads-of rs* **for** *C-info* **and** *rs* :: *rule fset* +
assumes *all-rules*: *fBall rs rule*

```
  assumes arity: arity-compatibles rs
  assumes fmap: is-fmap rs
  assumes patterns: pattern-compatibles rs
  assumes nonempty: rs ≠ {||}
  assumes not-shadows: fBall rs (λ(lhs, -). ¬ shadows-consts lhs)
  assumes welldefined-rs: fBall rs (λ(-, rhs). welldefined rhs)
begin

lemma rewrite-wellformed:
  assumes rs ⊢ t ⟶ t' wellformed t
  shows wellformed t'
using assms
proof (induction rule: rewrite.induct)
  case (step r t u)
  obtain lhs rhs where r = (lhs, rhs)
    by force
  hence wellformed rhs
    using all-rules step by force
  show ?case
    apply (rule wellformed.rewrite-step)
      apply (rule step(2)[unfolded ‹r = -›])
     apply fact+
    done
next
  case (beta u t)
  show ?case
    unfolding wellformed-term-def
    apply (rule replace-bound-wellformed)
    using beta by auto
qed auto

lemma rewrite-rt-wellformed: rs ⊢ t ⟶* t' ⟹ wellformed t ⟹ wellformed t'
by (induction rule: rtranclp.induct) (auto intro: rewrite-wellformed simp del: well-
formed-term-def)

lemma rewrite-closed: rs ⊢ t ⟶ t' ⟹ closed t ⟹ closed t'
proof (induction t t' rule: rewrite.induct)
  case (step r t u)
  obtain lhs rhs where r = (lhs, rhs)
    by force
  with step have rule (lhs, rhs)
    using all-rules by blast
  hence frees rhs |⊆| frees lhs
    by simp
  moreover have (lhs, rhs) ⊢ t → u
    using step unfolding ‹r = -› by simp
  show ?case
    apply (rule rewrite-step-closed)
    apply fact+
```

done
next
case (*beta t t'*)
have *frees t* $[t']_\beta$ |⊆| *frees t* |∪| *frees t'*
by (*rule replace-bound-frees*)
with *beta* **show** *?case*
unfolding *closed-except-def* **by** *auto*
qed (*auto simp*: *closed-except-def*)

lemma *rewrite-rt-closed*: *rs* ⊢ *t* ⟶* *t'* ⟹ *closed t* ⟹ *closed t'*
by (*induction rule*: *rtranclp.induct*) (*auto intro*: *rewrite-closed*)

end

end

3.2 Higher-order term rewriting using explicit bound variable names

theory *Rewriting-Nterm*
imports
Rewriting-Term
Higher-Order-Terms.Term-to-Nterm
../Terms/Strong-Term
begin

3.2.1 Definitions

type-synonym *nrule* = *term* × *nterm*

abbreviation *nrule* :: *nrule* ⇒ *bool* **where**
nrule ≡ *basic-rule*

fun (**in** *constants*) *not-shadowing* :: *nrule* ⇒ *bool* **where**
not-shadowing (*lhs*, *rhs*) ⟷ ¬ *shadows-consts lhs* ∧ ¬ *shadows-consts rhs*

locale *nrules* = *constants C-info heads-of rs* **for** *C-info* **and** *rs* :: *nrule fset* +
assumes *all-rules*: *fBall rs nrule*
assumes *arity*: *arity-compatibles rs*
assumes *fmap*: *is-fmap rs*
assumes *patterns*: *pattern-compatibles rs*
assumes *nonempty*: *rs* ≠ {||}
assumes *not-shadows*: *fBall rs not-shadowing*
assumes *welldefined-rs*: *fBall rs* (λ(-, *rhs*). *welldefined rhs*)

3.2.2 Matching and rewriting

inductive *nrewrite* :: *nrule fset* ⇒ *nterm* ⇒ *nterm* ⇒ *bool* (‹-/ ⊢$_n$/ - ⟶/ -› [*50*,*0*,*50*] *50*) **for** *rs* **where**

step: $r \mid\in\mid rs \Longrightarrow r \vdash t \to u \Longrightarrow rs \vdash_n t \longrightarrow u \mid$
beta: $rs \vdash_n ((\Lambda_n\ x.\ t)\ \$_n\ t') \longrightarrow subst\ t\ (fmap\text{-}of\text{-}list\ [(x,\ t')]) \mid$
fun: $rs \vdash_n t \longrightarrow t' \Longrightarrow rs \vdash_n t\ \$_n\ u \longrightarrow t'\ \$_n\ u \mid$
arg: $rs \vdash_n u \longrightarrow u' \Longrightarrow rs \vdash_n t\ \$_n\ u \longrightarrow t\ \$_n\ u'$

global-interpretation *nrewrite*: *rewriting nrewrite rs* **for** *rs*
by *standard* (*auto intro*: *nrewrite.intros simp*: *app-nterm-def*)+

abbreviation *nrewrite-rt* :: *nrule fset* ⇒ *nterm* ⇒ *nterm* ⇒ *bool* (‹-/ ⊢$_n$/ - ⟶*/ -› [*50*,*0*,*50*] *50*) **where**
nrewrite-rt rs ≡ (*nrewrite rs*)**

lemma (**in** *nrules*) *nrewrite-closed*:
  assumes $rs \vdash_n t \longrightarrow t'$ *closed t*
  shows *closed t'*
using *assms* **proof** *induction*
  case (*step r t u*)
  obtain *lhs rhs* **where** *r* = (*lhs*, *rhs*)
    by *force*
  with *step* **have** *nrule* (*lhs*, *rhs*)
    using *all-rules* **by** *blast*
  hence *frees rhs* |⊆| *frees lhs*
    by *simp*
  have (*lhs*, *rhs*) ⊢ *t* → *u*
    using *step* **unfolding** ‹*r* = -› **by** *simp*

  show *?case*
    apply (*rule rewrite-step-closed*)
      apply *fact*+
    done
next
  case *beta*
  show *?case*
    apply *simp*
    apply (*subst closed-except-def*)
    apply (*subst subst-frees*)
    using *beta* **unfolding** *closed-except-def* **by** *auto*
qed (*auto simp*: *closed-except-def*)

corollary (**in** *nrules*) *nrewrite-rt-closed*:
  assumes $rs \vdash_n t \longrightarrow* t'$ *closed t*
  shows *closed t'*
using *assms*
by *induction* (*auto intro*: *nrewrite-closed*)

3.2.3 Translation from *Term-Class.term* to *nterm*

context begin

```
private lemma term-to-nterm-all-vars0:
  assumes wellformed' (length Γ) t
  shows ∃ T. all-frees (fst (run-state (term-to-nterm Γ t) x)) |⊆| fset-of-list Γ |∪|
frees t |∪| T ∧ fBall T (λy. y > x)
using assms proof (induction Γ t arbitrary: x rule: term-to-nterm-induct)
  case (bound Γ i)
  hence Γ ! i |∈| fset-of-list Γ
    by (simp add: fset-of-list-elem)
  with bound show ?case
    by (auto simp: State-Monad.return-def)
next
  case (abs Γ t)

  obtain T
    where all-frees (fst (run-state (term-to-nterm (next x # Γ) t) (next x))) |⊆|
fset-of-list (next x # Γ) |∪| frees t |∪| T
      and fBall T ((<) (next x))
    apply atomize-elim
    apply (rule abs(1))
    using abs by auto

  show ?case
    apply (simp add: split-beta create-alt-def)
    apply (rule exI[where x = finsert (next x) T])
    apply (intro conjI)
    subgoal by auto
    subgoal using ‹all-frees (fst (run-state - (next x))) |⊆| -› by simp
    subgoal
      apply simp
      apply (rule conjI)
       apply (rule next-ge)
      using ‹fBall T ((<) (next x))›
      by (metis fBallE fBallI fresh-class.next-ge order.strict-trans)
    done
next
  case (app Γ t1 t2 x1)
  obtain t1' x2 where run-state (term-to-nterm Γ t1) x1 = (t1', x2)
    by fastforce
  moreover obtain T1
    where all-frees (fst (run-state (term-to-nterm Γ t1) x1)) |⊆| fset-of-list Γ |∪|
frees t1 |∪| T1
      and fBall T1 ((<) x1)
    apply atomize-elim
    apply (rule app(1))
    using app by auto
  ultimately have all-frees t1' |⊆| fset-of-list Γ |∪| frees t1 |∪| T1
    by simp

  obtain T2
```

```
    where all-frees (fst (run-state (term-to-nterm Γ t2) x2)) |⊆| fset-of-list Γ |∪|
frees t2 |∪| T2
      and fBall T2 ((<) x2)
    apply atomize-elim
    apply (rule app(2))
    using app by auto
  moreover obtain t2' x' where run-state (term-to-nterm Γ t2) x2 = (t2', x')
    by fastforce
  ultimately have all-frees t2' |⊆| fset-of-list Γ |∪| frees t2 |∪| T2
    by simp

  have x1 ≤ x2
    apply (rule state-io-relD[OF term-to-nterm-mono])
    apply fact
    done

  show ?case
    apply simp
    unfolding ‹run-state (term-to-nterm Γ t1) x1 = -›
    apply simp
    unfolding ‹run-state (term-to-nterm Γ t2) x2 = -›
    apply simp
    apply (rule exI[where x = T1 |∪| T2])
    apply (intro conjI)
    subgoal using ‹all-frees t1' |⊆| -› by auto
    subgoal using ‹all-frees t2' |⊆| -› by auto
    subgoal
      apply auto
      using ‹fBall T1 ((<) x1)› apply auto[]
      using ‹fBall T2 ((<) x2)› ‹x1 ≤ x2›
      by auto
    done
qed auto

lemma term-to-nterm-all-vars:
  assumes wellformed t fdisjnt (frees t) S
  shows fdisjnt (all-frees (fresh-frun (term-to-nterm [] t) (T |∪| S))) S
proof −
  let ?Γ = []
  let ?x = fresh-fNext (T |∪| S)
  from assms have wellformed' (length ?Γ) t
    by simp
  then obtain F
    where all-frees (fst (run-state (term-to-nterm ?Γ t) ?x)) |⊆| fset-of-list ?Γ |∪|
frees t |∪| F
      and fBall F (λy. y > ?x)
    by (metis term-to-nterm-all-vars0)

  have fdisjnt F (T |∪| S) if S ≠ {||}
```

```
    apply (rule fdisjnt-ge-max)
    apply (rule fBall-pred-weaken[OF - ‹fBall F (λy. y > ?x)›])
    apply (rule less-trans)
     apply (rule fNext-ge-max)
    using that by auto
  show ?thesis
    apply (rule fdisjnt-subset-left)
     apply (subst fresh-frun-def)
     apply (subst fresh-fNext-def[symmetric])
     apply fact
    apply simp
    apply (rule fdisjnt-union-left)
     apply fact
    using ‹- ⟹ fdisjnt F (T |∪| S)› by (auto simp: fdisjnt-alt-def)
qed

end

fun translate-rule :: name fset ⇒ rule ⇒ nrule where
translate-rule S (lhs, rhs) = (lhs, fresh-frun (term-to-nterm [] rhs) (frees lhs |∪|
S))

lemma translate-rule-alt-def:
  translate-rule S = (λ(lhs, rhs). (lhs, fresh-frun (term-to-nterm [] rhs) (frees lhs
|∪| S)))
by auto

definition compile' where
compile' C-info rs = translate-rule (pre-constants.all-consts C-info (heads-of rs))
|`| rs

context rules begin

definition compile :: nrule fset where
compile = translate-rule all-consts |`| rs

lemma compile'-compile-eq[simp]: compile' C-info rs = compile
unfolding compile'-def compile-def ..

lemma compile-heads: heads-of compile = heads-of rs
unfolding compile-def translate-rule-alt-def head-def[abs-def]
by force

lemma compile-rules: nrules C-info compile
proof
  have fBall compile (λ(lhs, rhs). nrule (lhs, rhs))
    proof safe
      fix lhs rhs
      assume (lhs, rhs) |∈| compile
```

```
      then obtain rhs'
        where (lhs, rhs') |∈| rs
          and rhs: rhs = fresh-frun (term-to-nterm [] rhs') (frees lhs |∪| all-consts)
        unfolding compile-def by force
      then have rule: rule (lhs, rhs')
        using all-rules by blast

      show nrule (lhs, rhs)
        proof
          from rule show linear lhs is-const (fst (strip-comb lhs)) ¬ is-const lhs by
auto

          have frees rhs |⊆| frees rhs'
            unfolding rhs using rule
            by (metis rule.simps term-to-nterm-vars)
          also have frees rhs' |⊆| frees lhs
            using rule by auto
          finally show frees rhs |⊆| frees lhs .
        qed
    qed

  thus fBall compile nrule
    by fast
next
  show arity-compatibles compile
    proof safe
      fix lhs1 rhs1 lhs2 rhs2
      assume (lhs1, rhs1) |∈| compile (lhs2, rhs2) |∈| compile
      then obtain rhs1' rhs2' where (lhs1, rhs1') |∈| rs (lhs2, rhs2') |∈| rs
        unfolding compile-def by force
      thus arity-compatible lhs1 lhs2
        using arity by (blast dest: fpairwiseD)
    qed
next
  have is-fmap rs
    using fmap by simp
  thus is-fmap compile
    unfolding compile-def translate-rule-alt-def
    by (rule is-fmap-image)
next
  have pattern-compatibles rs
    using patterns by simp
  thus pattern-compatibles compile
    unfolding compile-def translate-rule-alt-def
    by (auto dest: fpairwiseD)
next
  show fdisjnt (heads-of compile) C
    using disjnt by (simp add: compile-heads)
next
```

```
  have fBall compile not-shadowing
    proof safe
      fix lhs rhs
      assume (lhs, rhs) |∈| compile
      then obtain rhs'
        where rhs = fresh-frun (term-to-nterm [] rhs') (frees lhs |∪| all-consts)
          and (lhs, rhs') |∈| rs
        unfolding compile-def translate-rule-alt-def by auto
      hence rule (lhs, rhs') ¬ shadows-consts lhs
        using all-rules not-shadows by blast+
       moreover hence wellformed rhs' frees rhs' |⊆| frees lhs fdisjnt all-consts
(frees lhs)
        unfolding shadows-consts-def by simp+

      moreover have ¬ shadows-consts rhs
        apply (subst shadows-consts-def)
        apply simp
        unfolding ‹rhs = -›
        apply (rule fdisjnt-swap)
        apply (rule term-to-nterm-all-vars)
         apply fact
        apply (rule fdisjnt-subset-left)
         apply fact
        apply (rule fdisjnt-swap)
        apply fact
        done

      ultimately show not-shadowing (lhs, rhs)
        unfolding compile-heads by simp
    qed
  thus fBall compile (constants.not-shadowing C-info (heads-of compile))
    unfolding compile-heads .

  have fBall compile (λ(-, rhs). welldefined rhs)
    unfolding compile-heads
    unfolding compile-def ball-simps
    apply (rule fBall-pred-weaken[OF - welldefined-rs])
    subgoal for x
      apply (cases x)
      apply simp
      apply (subst fresh-frun-def)
      apply (subst term-to-nterm-consts[THEN pred-state-run-state])
      by auto
    done
 thus fBall compile (λ(-, rhs). consts rhs |⊆| pre-constants.all-consts C-info (heads-of
compile))
    unfolding compile-heads .
next
  show compile ≠ {||}
```

using *nonempty* **unfolding** *compile-def* **by** *auto*
next
show *distinct all-constructors*
by (*fact distinct-ctr*)
qed

sublocale *rules-as-nrules*: *nrules C-info compile*
by (*fact compile-rules*)

end

3.2.4 Correctness of translation

theorem (**in** *rules*) *compile-correct*:
assumes *compile* $\vdash_n$ *u* $\longrightarrow$ *u'* *closed u*
shows *rs* $\vdash$ *nterm-to-term' u* $\longrightarrow$ *nterm-to-term' u'*
using *assms* **proof** (*induction u u'*)
case (*step r u u'*)
moreover obtain *pat rhs'* **where** *r* = (*pat*, *rhs'*)
by *force*
ultimately obtain *nenv* **where** *match pat u* = *Some nenv u'* = *subst rhs' nenv*
by *auto*
then obtain *env* **where** *nrelated.P-env* [] *env nenv match pat* (*nterm-to-term* [] *u*) = *Some env*
by (*metis nrelated.match-rel option.exhaust option.rel-distinct*(*1*) *option.rel-inject*(*2*))

have *closed-env nenv*
using *step* ‹*match pat u* = *Some nenv*› **by** (*intro closed.match*)

from *step* **obtain** *rhs*
where *rhs'* = *fresh-frun* (*term-to-nterm* [] *rhs*) (*frees pat* |∪| *all-consts*) (*pat*, *rhs*) |∈| *rs*
unfolding ‹*r* = -› *compile-def* **by** *auto*
with *assms* **have** *rule* (*pat*, *rhs*)
using *all-rules* **by** *blast*
hence *rhs* = *nterm-to-term* [] *rhs'*
unfolding ‹*rhs'* = -›
by (*simp add*: *term-to-nterm-nterm-to-term fresh-frun-def*)

have *compile* $\vdash_n$ *u* $\longrightarrow$ *u'*
using *step* **by** (*auto intro*: *nrewrite.step*)
hence *closed u'*
by (*rule rules-as-nrules.nrewrite-closed*) *fact*

show *?case*
proof (*rule rewrite.step*)
show (*pat*, *rhs*) $\vdash$ *nterm-to-term* [] *u* $\rightarrow$ *nterm-to-term* [] *u'*
apply (*subst nterm-to-term-eq-closed*)
apply *fact*

```
      apply simp
      apply (rule exI[where x = env])
      apply (rule conjI)
       apply fact
      unfolding ‹rhs = -›
      apply (subst nrelated-subst)
        apply fact
       apply fact
      unfolding fdisjnt-alt-def apply simp
      unfolding ‹u' = subst rhs' nenv› by simp
    qed fact
next
  case beta
  show ?case
    apply simp
    apply (subst subst-single-eq[symmetric, simplified])
    apply (subst nterm-to-term-subst-replace-bound[where n = 0])
    subgoal using beta by (simp add: closed-except-def)
    subgoal by simp
    subgoal by simp
    subgoal by simp (rule rewrite.beta)
    done
qed (auto intro: rewrite.intros simp: closed-except-def)
```

3.2.5 Completeness of translation

```
context rules begin

context
  notes [simp] = closed-except-def fdisjnt-alt-def
begin

private lemma compile-complete0:
  assumes rs ⊢ t ⟶ t' closed t wellformed t
  obtains u' where compile ⊢$_n$ fst (run-state (term-to-nterm [] t) s) ⟶ u' u' ≈$_\alpha$
fst (run-state (term-to-nterm [] t') s')
apply atomize-elim
using assms proof (induction t t' arbitrary: s s')
  case (step r t t')
  let ?$t_n$ = fst (run-state (term-to-nterm [] t) s)
  let ?$t_n$' = fst (run-state (term-to-nterm [] t') s')
  from step have closed t closed ?$t_n$
    using term-to-nterm-vars0[where Γ = []]
    by force+
  from step have nterm-to-term' ?$t_n$ = t
    by (auto intro!: term-to-nterm-nterm-to-term0)

  obtain pat rhs' where r = (pat, rhs')
    by fastforce
```

with *step* **obtain** *env'* **where** *match pat t = Some env' t' = subst rhs' env'*
by *fastforce*
with ‹- = *t*› **have** *rel-option* (*nrelated.P-env* []) (*match pat t*) (*match pat* (*?t*$_n$))
by (*metis nrelated.match-rel*)
with *step* ‹- = *Some env'*›
obtain *env* **where** *nrelated.P-env* [] *env' env match pat ?t*$_n$ = *Some env*
by (*metis* (*no-types*, *lifting*) *option-rel-Some1*)
with ‹*closed ?t*$_n$› **have** *closed-env env*
by (*blast intro*: *closed.match*)

from *step* **obtain** *rhs*
where *rhs = fresh-frun* (*term-to-nterm* [] *rhs'*) (*frees pat* |∪| *all-consts*) (*pat*, *rhs*) |∈| *compile*
unfolding ‹*r* = -› *compile-def*
by *force*
with *step* **have** *rule* (*pat*, *rhs'*)
unfolding ‹*r* = -›
using *all-rules* **by** *fast*
hence *nterm-to-term' rhs = rhs'*
unfolding ‹*rhs* = -›
by (*simp add*: *fresh-frun-def term-to-nterm-nterm-to-term*)
obtain *u'* **where** *subst rhs env = u'*
by *simp*
have *t' = nterm-to-term' u'*
unfolding ‹*t'* = -›
unfolding ‹- = *rhs'*›[*symmetric*]
apply (*subst nrelated-subst*)
apply *fact+*
using ‹- = *u'*›
by *simp+*

have *compile* ⊢$_n$ *?t*$_n$ ⟶ *u'*
apply (*rule nrewrite.step*)
apply *fact*
apply *simp*
apply (*intro conjI exI*)
apply *fact+*
done
with ‹*closed ?t*$_n$› **have** *closed u'*
by (*blast intro*:*rules-as-nrules.nrewrite-closed*)
with ‹*t'* = *nterm-to-term'* -› **have** *u'* ≈$_\alpha$ *?t*$_n$*'*
by (*force intro*: *nterm-to-term-term-to-nterm*[**where** Γ = [] **and** Γ' = [],*simplified*])

show *?case*
apply (*intro conjI exI*)
apply (*rule nrewrite.step*)
apply *fact*
apply *simp*
apply (*intro conjI exI*)

```
    apply fact+
   done
next
 case (beta t t')
 let ?name = next s
 let ?t_n = fst (run-state (term-to-nterm [?name] t) (?name))
  let ?t_n' = fst (run-state (term-to-nterm [] t') (snd (run-state (term-to-nterm
[?name] t) (?name))))

 from beta have closed t closed (t [t']_β)  closed (Λ t $ t') closed  t'
   using replace-bound-frees
   by fastforce+
 moreover from beta have wellformed' (Suc 0) t wellformed t'
   by simp+
 ultimately have t = nterm-to-term [?name] ?t_n
          and t' = nterm-to-term' ?t_n'
          and ∗:frees ?t_n = {|?name|} ∨ frees ?t_n = fempty
          and closed ?t_n'
   using term-to-nterm-vars0[where Γ = [?name]]
   using term-to-nterm-vars0[where Γ = []]
   by (force simp: term-to-nterm-nterm-to-term0)+

 hence ∗∗:t [t']_β = nterm-to-term' (subst-single ?t_n ?name ?t_n')
   by (auto simp: nterm-to-term-subst-replace-bound[where n = 0])

 from ‹closed ?t_n'› have closed-env (fmap-of-list [(?name, ?t_n')])
   by auto

 show ?case
   apply (rule exI)
   apply (auto simp: split-beta create-alt-def)
    apply (rule nrewrite.beta)
   apply (subst subst-single-eq[symmetric])
   apply (subst ∗∗)
  apply (rule nterm-to-term-term-to-nterm[where Γ = [] and Γ' = [], simplified])
   apply (subst subst-single-eq)
   apply (subst subst-frees[OF ‹closed-env ›])
   using ∗ by force
next
 case (fun t t' u)
 hence closed  t closed u closed (t $ u)
     and wellform:wellformed t wellformed u
   by fastforce+
 from fun obtain u'
   where compile ⊢_n fst (run-state (term-to-nterm [] t) s) ⟶ u'
       u' ≈_α fst (run-state (term-to-nterm [] t') s')
   by force
 show ?case
   apply (rule exI)
```

apply (*auto simp*: *split-beta create-alt-def*)
apply (*rule nrewrite.fun*)
apply *fact*
apply *rule*
apply *fact*
apply (*subst term-to-nterm-alpha-equiv*[*of* [] [], *simplified*])
using ‹*closed u*› ‹*wellformed u*› **by** *auto*
next
case (*arg u u' t*)
hence *closed t closed u closed* (*t* $ *u*)
and *wellform*:*wellformed t wellformed u*
by *fastforce*+
from *arg* **obtain** *t'*
where *compile* ⊢$_n$ *fst* (*run-state* (*term-to-nterm* [] *u*) (*snd* (*run-state* (*term-to-nterm* [] *t*) *s*))) ⟶ *t'*
t' ≈$_\alpha$ *fst* (*run-state* (*term-to-nterm* [] *u'*) (*snd* (*run-state* (*term-to-nterm* [] *t*) *s'*)))
by *force*
show *?case*
apply (*rule exI*)
apply (*auto simp*: *split-beta create-alt-def*)
apply *rule*
apply *fact*
apply *rule*
apply (*subst term-to-nterm-alpha-equiv*[*of* [] [], *simplified*])
using ‹*closed t*› ‹*wellformed t*› **apply** *force*+
by *fact*
qed

lemma *compile-complete*:
assumes *rs* ⊢ *t* ⟶ *t'* *closed t wellformed t*
obtains *u'* **where** *compile* ⊢$_n$ *term-to-nterm' t* ⟶ *u' u'* ≈$_\alpha$ *term-to-nterm' t'*
unfolding *term-to-nterm'-def*
using *assms* **by** (*metis compile-complete0*)

end

end

3.2.6 Splitting into constants

type-synonym *crules* = (*term list* × *nterm*) *fset*
type-synonym *crule-set* = (*name* × *crules*) *fset*

abbreviation *arity-compatibles* :: (*term list* × *'a*) *fset* ⇒ *bool* **where**
arity-compatibles ≡ *fpairwise* (λ(*pats*$_1$, -) (*pats*$_2$, -). *length pats*$_1$ = *length pats*$_2$)

lemma *arity-compatible-length*:
assumes *arity-compatibles rs* (*pats*, *rhs*) |∈| *rs*

shows *length pats* = *arity rs*
proof −
have *fBall rs* (λ(*pats*$_1$, -). *fBall rs* (λ(*pats*$_2$, -). *length pats*$_1$ = *length pats*$_2$))
using *assms* **unfolding** *fpairwise-alt-def* **by** *blast*
hence *fBall rs* (λ*x*. *fBall rs* (λ*y*. (*length* ∘ *fst*) *x* = (*length* ∘ *fst*) *y*))
by *force*
hence (*length* ∘ *fst*) (*pats*, *rhs*) = *arity rs*
using *assms*(*2*) **unfolding** *arity-def fthe-elem′-eq* **by** (*rule singleton-fset-holds*)
thus *?thesis*
by *simp*
qed

locale *pre-crules* = *constants C-info fst* |‘| *rs* **for** *C-info* **and** *rs* :: *crule-set*

locale *crules* = *pre-crules* +
assumes *fmap*: *is-fmap rs*
assumes *nonempty*: *rs* ≠ {||}
assumes *inner*:
fBall rs (λ(-, *crs*).
arity-compatibles crs ∧
is-fmap crs ∧
patterns-compatibles crs ∧
crs ≠ {||} ∧
fBall crs (λ(*pats*, *rhs*).
linears pats ∧
pats ≠ [] ∧
fdisjnt (*freess pats*) *all-consts* ∧
¬ *shadows-consts rhs* ∧
frees rhs |⊆| *freess pats* ∧
welldefined rhs))

lemma (**in** *pre-crules*) *crulesI*:
assumes ⋀*name crs*. (*name*, *crs*) |∈| *rs* ⟹ *arity-compatibles crs*
assumes ⋀*name crs*. (*name*, *crs*) |∈| *rs* ⟹ *is-fmap crs*
assumes ⋀*name crs*. (*name*, *crs*) |∈| *rs* ⟹ *patterns-compatibles crs*
assumes ⋀*name crs*. (*name*, *crs*) |∈| *rs* ⟹ *crs* ≠ {||}
assumes ⋀*name crs pats rhs*. (*name*, *crs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *crs* ⟹ *linears pats*
assumes ⋀*name crs pats rhs*. (*name*, *crs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *crs* ⟹ *pats* ≠ []
assumes ⋀*name crs pats rhs*. (*name*, *crs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *crs* ⟹ *fdisjnt* (*freess pats*) *all-consts*
assumes ⋀*name crs pats rhs*. (*name*, *crs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *crs* ⟹ ¬ *shadows-consts rhs*
assumes ⋀*name crs pats rhs*. (*name*, *crs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *crs* ⟹ *frees rhs* |⊆| *freess pats*
assumes ⋀*name crs pats rhs*. (*name*, *crs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *crs* ⟹ *welldefined rhs*
assumes *is-fmap rs rs* ≠ {||}

shows *crules C-info rs*
proof *unfold-locales*
show *is-fmap rs*
using *assms*(*11*) **.**
next
show *rs* ≠ {||}
using *assms*(*12*) **.**
next
show *fBall rs* (λ(-, *crs*). *Rewriting-Nterm.arity-compatibles crs* ∧ *is-fmap crs* ∧
patterns-compatibles crs ∧ *crs* ≠ {||} ∧
fBall crs (λ(*pats*, *rhs*). *linears pats* ∧ *pats* ≠ [] ∧ *fdisjnt* (*freess pats*) *all-consts*
∧
¬ *shadows-consts rhs* ∧ *frees rhs* |⊆| *freess pats* ∧ *consts rhs* |⊆| *all-consts*))
using *assms*(*1*−*10*)
by (*intro prod-BallI conjI*) *metis*+
qed

lemmas *crulesI*[*intro*!] = *pre-crules.crulesI*[*unfolded pre-crules-def*]

definition *consts-of* :: *nrule fset* ⇒ *crule-set* **where**
consts-of = *fgroup-by split-rule*

lemma *consts-of-heads*: *fst* |‘| *consts-of rs* = *heads-of rs*
unfolding *consts-of-def*
by (*simp add*: *split-rule-fst comp-def*)

lemma (**in** *nrules*) *consts-rules*: *crules C-info* (*consts-of rs*)
proof
have *is-fmap rs*
using *fmap* **by** *simp*
thus *is-fmap* (*consts-of rs*)
unfolding *consts-of-def* **by** *auto*

show *consts-of rs* ≠ {||}
using *nonempty* **unfolding** *consts-of-def*
by (*meson fgroup-by-nonempty*)

show *constants C-info* (*fst* |‘| *consts-of rs*)
proof
show *fdisjnt* (*fst* |‘| *consts-of rs*) *C*
using *disjnt* **by** (*auto simp*: *consts-of-heads*)
next
show *distinct all-constructors*
by (*fact distinct-ctr*)
qed

fix *name crs*
assume *crs*: (*name*, *crs*) |∈| *consts-of rs*

thus *crs* $\neq$ {||}
unfolding *consts-of-def*
by (*meson femptyE fgroup-by-nonempty-inner*)

show *arity-compatibles crs patterns-compatibles crs*
proof *safe*
fix $pats_1$ rhs_1 $pats_2$ rhs_2
assume ($pats_1$, rhs_1) |∈| *crs* ($pats_2$, rhs_2) |∈| *crs*
with *crs* **obtain** lhs_1 lhs_2
where *rs*: (lhs_1, rhs_1) |∈| *rs* (lhs_2, rhs_2) |∈| *rs* **and**
split: *split-rule* (lhs_1, rhs_1) = (*name*, ($pats_1$, rhs_1))
split-rule (lhs_2, rhs_2) = (*name*, ($pats_2$, rhs_2))
unfolding *consts-of-def* **by** (*force simp*: *split-beta*)

hence *arity*: *arity-compatible* lhs_1 lhs_2
using *arity* **by** (*force dest*: *fpairwiseD*)

from *rs* **have** *const*: *is-const* (*fst* (*strip-comb* lhs_1)) *is-const* (*fst* (*strip-comb* lhs_2))
using *all-rules* **by** *force+*

have *name* = *const-name* (*fst* (*strip-comb* lhs_1)) *name* = *const-name* (*fst* (*strip-comb* lhs_2))
using *split* **by** (*auto simp*: *split-beta*)
with *const* **have** *fst* (*strip-comb* lhs_1) = *Const name fst* (*strip-comb* lhs_2) = *Const name*
apply (*fold const-term-def*)
subgoal by *simp*
subgoal by *fastforce*
done
hence *fst*: *fst* (*strip-comb* lhs_1) = *fst* (*strip-comb* lhs_2)
by *simp*

with *arity* **have** *length* (*snd* (*strip-comb* lhs_1)) = *length* (*snd* (*strip-comb* lhs_2))
unfolding *arity-compatible-def*
by (*simp add*: *split-beta*)

with *split* **show** *length* $pats_1$ = *length* $pats_2$
by (*auto simp*: *split-beta*)

have *pattern-compatible* lhs_1 lhs_2
using *rs patterns* **by** (*auto dest*: *fpairwiseD*)
moreover have lhs_1 = *name* $$ $pats_1$
using *split*(*1*) *const*(*1*) **by** (*auto simp*: *split-beta*)
moreover have lhs_2 = *name* $$ $pats_2$
using *split*(*2*) *const*(*2*) **by** (*auto simp*: *split-beta*)
ultimately have *pattern-compatible* (*name* $$ $pats_1$) (*name* $$ $pats_2$)
by *simp*

```
    thus patterns-compatible pats1 pats2
      using ‹length pats1 = -› by (auto dest: pattern-compatible-combD)
  qed

  show is-fmap crs
  proof
    fix pats rhs1 rhs2
    assume (pats, rhs1) |∈| crs (pats, rhs2) |∈| crs
    with crs obtain lhs1 lhs2
      where rs: (lhs1, rhs1) |∈| rs (lhs2, rhs2) |∈| rs and
          split: split-rule (lhs1, rhs1) = (name, (pats, rhs1))
                 split-rule (lhs2, rhs2) = (name, (pats, rhs2))
      unfolding consts-of-def by (force simp: split-beta)

    have lhs1 = lhs2
    proof (rule ccontr)
      assume lhs1 ≠ lhs2
      then consider (fst) fst (strip-comb lhs1) ≠ fst (strip-comb lhs2)
                  | (snd) snd (strip-comb lhs1) ≠ snd (strip-comb lhs2)
        by (metis list-strip-comb)
      thus False
      proof cases
        case fst
       moreover have is-const (fst (strip-comb lhs1)) is-const (fst (strip-comb
lhs2))
          using rs all-rules by force+
        ultimately show ?thesis
          using split const-name-simps by (fastforce simp: split-beta)
      next
        case snd
        with split show ?thesis
          by (auto simp: split-beta)
      qed
    qed

    with rs show rhs1 = rhs2
      using ‹is-fmap rs› by (auto dest: is-fmapD)
  qed

  fix pats rhs
  assume (pats, rhs) |∈| crs
  then obtain lhs where (lhs, rhs) |∈| rs pats = snd (strip-comb lhs)
    using crs unfolding consts-of-def by (force simp: split-beta)
  hence nrule (lhs, rhs)
    using all-rules by blast
  hence linear lhs frees rhs |⊆| frees lhs
    by auto
  thus linears pats
    unfolding ‹pats = -› by (intro linears-strip-comb)
```

```
  have ¬ is-const lhs is-const (fst (strip-comb lhs))
    using ‹nrule -› by auto
  thus pats ≠ []
    unfolding ‹pats = -› using ‹linear lhs›
    apply (cases lhs)
       apply (fold app-term-def)
    by (auto split: prod.splits)

  from ‹nrule (lhs, rhs)› have frees (fst (strip-comb lhs)) = {||}
    by (cases fst (strip-comb lhs)) (auto simp: is-const-def)
  hence frees lhs = freess (snd (strip-comb lhs))
    by (subst frees-strip-comb) auto
  thus frees rhs |⊆| freess pats
    unfolding ‹pats = -› using ‹frees rhs |⊆| frees lhs› by simp

  have ¬ shadows-consts rhs
    using ‹(lhs, rhs) |∈| rs› not-shadows
    by force
  thus ¬ pre-constants.shadows-consts C-info (fst |‘| consts-of rs) rhs
    by (simp add: consts-of-heads)

  have fdisjnt all-consts (frees lhs)
    using ‹(lhs, rhs) |∈| rs› not-shadows
    by (force simp: shadows-consts-def)
  moreover have freess pats |⊆| frees lhs
    unfolding ‹pats = -› ‹frees lhs = -›
    by simp
  ultimately have fdisjnt (freess pats) all-consts
    by (metis fdisjnt-subset-right fdisjnt-swap)

  thus fdisjnt (freess pats) (pre-constants.all-consts C-info (fst |‘| consts-of rs))
    by (simp add: consts-of-heads)

  show pre-constants.welldefined C-info (fst |‘| consts-of rs) rhs
    using welldefined-rs ‹(lhs, rhs) |∈| rs›
    by (force simp: consts-of-heads)
qed

sublocale nrules ⊆ nrules-as-crules?: crules C-info consts-of rs
by (fact consts-rules)
```

3.2.7 Computability

```
export-code
  translate-rule consts-of arity nterm-to-term
  checking Scala

end
```

3.3 Higher-order term rewriting with explicit pattern matching

theory *Rewriting-Pterm-Elim*
imports
Rewriting-Nterm
../Terms/Pterm
begin

3.3.1 Intermediate rule sets

type-synonym *irules* = (*term list* × *pterm*) *fset*
type-synonym *irule-set* = (*name* × *irules*) *fset*

locale *pre-irules* = *constants C-info fst* |‘| *rs* **for** *C-info* **and** *rs* :: *irule-set*

locale *irules* = *pre-irules* +
assumes *fmap*: *is-fmap rs*
assumes *nonempty*: *rs* ≠ {||}
assumes *inner*:
fBall rs (λ(-, *irs*).
arity-compatibles irs ∧
is-fmap irs ∧
patterns-compatibles irs ∧
irs ≠ {||} ∧
fBall irs (λ(*pats*, *rhs*).
linears pats ∧
abs-ish pats rhs ∧
closed-except rhs (*freess pats*) ∧
fdisjnt (*freess pats*) *all-consts* ∧
wellformed rhs ∧
¬ *shadows-consts rhs* ∧
welldefined rhs))

lemma (**in** *pre-irules*) *irulesI*:
assumes ⋀*name irs*. (*name*, *irs*) |∈| *rs* ⟹ *arity-compatibles irs*
assumes ⋀*name irs*. (*name*, *irs*) |∈| *rs* ⟹ *is-fmap irs*
assumes ⋀*name irs*. (*name*, *irs*) |∈| *rs* ⟹ *patterns-compatibles irs*
assumes ⋀*name irs*. (*name*, *irs*) |∈| *rs* ⟹ *irs* ≠ {||}
assumes ⋀*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ *linears pats*
assumes ⋀*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ *abs-ish pats rhs*
assumes ⋀*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ *fdisjnt* (*freess pats*) *all-consts*
assumes ⋀*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ *closed-except rhs* (*freess pats*)
assumes ⋀*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ *wellformed rhs*

assumes $\bigwedge$*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ ¬ *shadows-consts rhs*
assumes $\bigwedge$*name irs pats rhs*. (*name*, *irs*) |∈| *rs* ⟹ (*pats*, *rhs*) |∈| *irs* ⟹ *welldefined rhs*
assumes *is-fmap rs rs* ≠ {||}
shows *irules C-info rs*
proof *unfold-locales*
show *is-fmap rs*
using *assms*(*12*) **.**
next
show *rs* ≠ {||}
using *assms*(*13*) **.**
next
show *fBall rs* (λ(-, *irs*). *Rewriting-Nterm.arity-compatibles irs* ∧ *is-fmap irs* ∧
patterns-compatibles irs ∧ *irs* ≠ {||} ∧
fBall irs (λ(*pats*, *rhs*). *linears pats* ∧ *abs-ish pats rhs* ∧ *closed-except rhs* (*freess pats*) ∧
fdisjnt (*freess pats*) *all-consts* ∧ *pre-strong-term-class.wellformed rhs* ∧
¬ *shadows-consts rhs* ∧ *consts rhs* |⊆| *all-consts*))
using *assms*(*1*−*11*)
by (*intro prod-BallI conjI*) *metis*+
qed

lemmas *irulesI*[*intro*!] = *pre-irules.irulesI*[*unfolded pre-irules-def*]

Translation from *nterm* to *pterm*

fun *nterm-to-pterm* :: *nterm* ⇒ *pterm* **where**
nterm-to-pterm (*Nvar s*) = *Pvar s* |
nterm-to-pterm (*Nconst s*) = *Pconst s* |
nterm-to-pterm (t_1 $\$_n$ t_2) = *nterm-to-pterm* t_1 $\$_p$ *nterm-to-pterm* t_2 |
nterm-to-pterm (Λ_n *x*. *t*) = (Λ_p *x*. *nterm-to-pterm t*)

lemma *nterm-to-pterm-inj*: *nterm-to-pterm x* = *nterm-to-pterm y* ⟹ *x* = *y*
by (*induction y arbitrary*: *x*) (*auto elim*: *nterm-to-pterm.elims*)

lemma *nterm-to-pterm*:
assumes *no-abs t*
shows *nterm-to-pterm t* = *convert-term t*
using *assms*
apply *induction*
apply *auto*
by (*auto simp*: *free-nterm-def free-pterm-def const-nterm-def const-pterm-def app-nterm-def app-pterm-def*)

lemma *nterm-to-pterm-frees*[*simp*]: *frees* (*nterm-to-pterm t*) = *frees t*
by (*induct t*) *auto*

lemma *closed-nterm-to-pterm*[*intro*]: *closed-except* (*nterm-to-pterm t*) (*frees t*)

unfolding *closed-except-def* **by** *simp*

lemma (**in** *constants*) *shadows-nterm-to-pterm*[*simp*]: *shadows-consts* (*nterm-to-pterm t*) = *shadows-consts t*
by (*induct t*) (*auto simp*: *shadows-consts-def fdisjnt-alt-def*)

lemma *wellformed-nterm-to-pterm*[*intro*]: *wellformed* (*nterm-to-pterm t*)
by (*induct t*) *auto*

lemma *consts-nterm-to-pterm*[*simp*]: *consts* (*nterm-to-pterm t*) = *consts t*
by (*induct t*) *auto*

Translation from *crule-set* to *irule-set*

definition *translate-crules* :: *crules* ⇒ *irules* **where**
translate-crules = *fimage* (*map-prod id nterm-to-pterm*)

definition *compile* :: *crule-set* ⇒ *irule-set* **where**
compile = *fimage* (*map-prod id translate-crules*)

lemma *compile-heads*: *fst* |‘| *compile rs* = *fst* |‘| *rs*
unfolding *compile-def* **by** *simp*

lemma (**in** *crules*) *compile-rules*: *irules C-info* (*compile rs*)
proof
  have *is-fmap rs*
    using *fmap* **by** *simp*
  thus *is-fmap* (*compile rs*)
    unfolding *compile-def map-prod-def id-apply* **by** (*rule is-fmap-image*)

  show *compile rs* ≠ {||}
    using *nonempty* **unfolding** *compile-def* **by** *auto*

  show *constants C-info* (*fst* |‘| *compile rs*)
    proof
      show *fdisjnt* (*fst* |‘| *compile rs*) *C*
        using *disjnt* **unfolding** *compile-def*
        by *force*
    next
      show *distinct all-constructors*
        by (*fact distinct-ctr*)
    qed

  fix *name irs*
  assume *irs*: (*name*, *irs*) |∈| *compile rs*
  then obtain *irs′* **where** (*name*, *irs′*) |∈| *rs irs* = *translate-crules irs′*
    unfolding *compile-def* **by** *force*
  hence *arity-compatibles irs′*
    using *inner*

by (*metis* (*no-types*, *lifting*) *case-prodD*)
thus *arity-compatibles irs*
unfolding ‹*irs* = *translate-crules irs'*› *translate-crules-def*
by (*force dest*: *fpairwiseD*)

have *patterns-compatibles irs'*
using ‹(*name*, *irs'*) |∈| *rs*› *inner*
by (*blast dest*: *fpairwiseD*)
thus *patterns-compatibles irs*
unfolding ‹*irs* = -› *translate-crules-def*
by (*auto dest*: *fpairwiseD*)

have *is-fmap irs'*
using ‹(*name*, *irs'*) |∈| *rs*› *inner* **by** *auto*
thus *is-fmap irs*
unfolding ‹*irs* = *translate-crules irs'*› *translate-crules-def map-prod-def id-apply*
by (*rule is-fmap-image*)

have *irs'* ≠ {||}
using ‹(*name*, *irs'*) |∈| *rs*› *inner* **by** *auto*
thus *irs* ≠ {||}
unfolding ‹*irs* = *translate-crules irs'*› *translate-crules-def* **by** *simp*

fix *pats rhs*
assume (*pats*, *rhs*) |∈| *irs*
then obtain *rhs'* **where** (*pats*, *rhs'*) |∈| *irs'* *rhs* = *nterm-to-pterm rhs'*
unfolding ‹*irs* = *translate-crules irs'*› *translate-crules-def* **by** *force*
hence *linears pats pats* ≠ [] *frees rhs'* |⊆| *freess pats* ¬ *shadows-consts rhs'*
using *fbspec*[*OF inner* ‹(*name*, *irs'*) |∈| *rs*›]
by *blast*+

show *linears pats* **by** *fact*
show *closed-except rhs* (*freess pats*)
unfolding ‹*rhs* = *nterm-to-pterm rhs'*›
using ‹*frees rhs'* |⊆| *freess pats*›
by (*metis dual-order.trans closed-nterm-to-pterm closed-except-def*)

show *wellformed rhs*
unfolding ‹*rhs* = *nterm-to-pterm rhs'*› **by** *auto*

have *fdisjnt* (*freess pats*) *all-consts*
using ‹(*pats*, *rhs'*) |∈| *irs'*› ‹(*name*, *irs'*) |∈| *rs*› *inner*
by *auto*
thus *fdisjnt* (*freess pats*) (*pre-constants.all-consts C-info* (*fst* |‘| *compile rs*))
unfolding *compile-def* **by** *simp*

have ¬ *shadows-consts rhs*
unfolding ‹*rhs* = -› **using** ‹¬ *shadows-consts* -› **by** *simp*
thus ¬ *pre-constants.shadows-consts C-info* (*fst* |‘| *compile rs*) *rhs*

unfolding *compile-heads* **.**

show *abs-ish pats rhs*
using ‹*pats* ≠ []› **unfolding** *abs-ish-def* **by** *simp*

have *welldefined rhs′*
using *fbspec*[*OF inner* ‹(*name*, *irs′*) |∈| *rs*›, *simplified*]
using ‹(*pats*, *rhs′*) |∈| *irs′*›
by *blast*

thus *pre-constants.welldefined C-info* (*fst* |‘| *compile rs*) *rhs*
unfolding *compile-def* ‹*rhs* = -›
by *simp*
qed

sublocale *crules* ⊆ *crules-as-irules*: *irules C-info compile rs*
by (*fact compile-rules*)

Transformation of *irule-set*

definition *transform-irules* :: *irules* ⇒ *irules* **where**
transform-irules rs = (
if arity rs = *0 then rs*
else map-prod id Pabs |‘| *fgroup-by* (λ(*pats*, *rhs*). (*butlast pats*, (*last pats*, *rhs*)))
rs)

lemma *arity-compatibles-transform-irules*:
assumes *arity-compatibles rs*
shows *arity-compatibles* (*transform-irules rs*)
proof (*cases arity rs* = *0*)
case *True*
thus *?thesis*
unfolding *transform-irules-def* **using** *assms* **by** *simp*
next
case *False*
let *?rs′* = *transform-irules rs*
let *?f* = λ(*pats*, *rhs*). (*butlast pats*, (*last pats*, *rhs*))
let *?grp* = *fgroup-by ?f rs*
have *rs′*: *?rs′* = *map-prod id Pabs* |‘| *?grp*
using *False* **unfolding** *transform-irules-def* **by** *simp*
show *?thesis*
proof *safe*
fix *pats*$_1$ *rhs*$_1$ *pats*$_2$ *rhs*$_2$
assume (*pats*$_1$, *rhs*$_1$) |∈| *?rs′* (*pats*$_2$, *rhs*$_2$) |∈| *?rs′*
then obtain *rhs*$_1$′ *rhs*$_2$′ **where** (*pats*$_1$, *rhs*$_1$′) |∈| *?grp* (*pats*$_2$, *rhs*$_2$′) |∈| *?grp*
unfolding *rs′* **by** *auto*
then obtain *pats*$_1$′ *pats*$_2$′ *x y* — dummies
where *fst* (*?f* (*pats*$_1$′, *x*)) = *pats*$_1$ (*pats*$_1$′, *x*) |∈| *rs*
and *fst* (*?f* (*pats*$_2$′, *y*)) = *pats*$_2$ (*pats*$_2$′, *y*) |∈| *rs*

```
      by (fastforce simp: split-beta elim: fgroup-byE2)
    hence pats₁ = butlast pats₁' pats₂ = butlast pats₂' length pats₁' = length pats₂'
      using assms by (force dest: fpairwiseD)+
    thus length pats₁ = length pats₂
      by auto
  qed
qed

lemma arity-transform-irules:
  assumes arity-compatibles rs rs ≠ {||}
  shows arity (transform-irules rs) = (if arity rs = 0 then 0 else arity rs − 1)
proof (cases arity rs = 0)
  case True
  thus ?thesis
    unfolding transform-irules-def by simp
next
  case False
  let ?f = λ(pats, rhs). (butlast pats, (last pats, rhs))
  let ?grp = fgroup-by ?f rs
  let ?rs' = map-prod id Pabs |`| ?grp

  have arity ?rs' = arity rs − 1
    proof (rule arityI)
      show fBall ?rs' (λ(pats, -). length pats = arity rs − 1)
        proof (rule prod-fBallI)
          fix pats rhs
          assume (pats, rhs) |∈| ?rs'
          then obtain cs where (pats, cs) |∈| ?grp rhs = Pabs cs
            by force
          then obtain pats' x — dummy
            where pats = butlast pats' (pats', x) |∈| rs
            by (fastforce simp: split-beta elim: fgroup-byE2)
          hence length pats' = arity rs
            using assms by (metis arity-compatible-length)
          thus length pats = arity rs − 1
            unfolding ‹pats = butlast pats'› using False by simp
        qed
    next
      show ?rs' ≠ {||}
        using assms by (simp add: fgroup-by-nonempty)
    qed

  with False show ?thesis
    unfolding transform-irules-def by simp
qed

definition transform-irule-set :: irule-set ⇒ irule-set where
transform-irule-set = fimage (map-prod id transform-irules)
```

lemma *transform-irule-set-heads*: *fst* |‘| *transform-irule-set rs* = *fst* |‘| *rs*
unfolding *transform-irule-set-def* **by** *simp*

lemma (**in** *irules*) *rules-transform*: *irules C-info* (*transform-irule-set rs*)
proof
  have *is-fmap rs*
    using *fmap* **by** *simp*
  thus *is-fmap* (*transform-irule-set rs*)
   unfolding *transform-irule-set-def map-prod-def id-apply* **by** (*rule is-fmap-image*)

  show *transform-irule-set rs* ≠ {||}
    using *nonempty* **unfolding** *transform-irule-set-def* **by** *auto*

  show *constants C-info* (*fst* |‘| *transform-irule-set rs*)
    proof
      show *fdisjnt* (*fst* |‘| *transform-irule-set rs*) *C*
        using *disjnt* **unfolding** *transform-irule-set-def*
        by *force*
    next
      show *distinct all-constructors*
        by (*fact distinct-ctr*)
    qed

  fix *name irs*
  assume *irs*: (*name*, *irs*) |∈| *transform-irule-set rs*
  then obtain *irs′* **where** (*name*, *irs′*) |∈| *rs irs* = *transform-irules irs′*
    unfolding *transform-irule-set-def* **by** *force*
  hence *arity-compatibles irs′*
    using *inner*
    by (*metis* (*no-types*, *lifting*) *case-prodD*)
  thus *arity-compatibles irs*
   unfolding ‹*irs* = *transform-irules irs′*› **by** (*rule arity-compatibles-transform-irules*)

  have *irs′* ≠ {||}
    using ‹(*name*, *irs′*) |∈| *rs*› *inner* **by** *blast*
  thus *irs* ≠ {||}
    unfolding ‹*irs* = *transform-irules irs′*› *transform-irules-def*
    by (*simp add*: *fgroup-by-nonempty*)

  let *?f* = λ(*pats*, *rhs*). (*butlast pats*, (*last pats*, *rhs*))
  let *?grp* = *fgroup-by ?f irs′*

  have *patterns-compatibles irs′*
    using ‹(*name*, *irs′*) |∈| *rs*› *inner*
    by (*blast dest*: *fpairwiseD*)
  show *patterns-compatibles irs*
    proof (*cases arity irs′* = *0*)
      case *True*
      thus *?thesis*

```
    unfolding ‹irs = transform-irules irs'› transform-irules-def
    using ‹patterns-compatibles irs'› by simp
  next
    case False
    hence irs': irs = map-prod id Pabs |`| ?grp
      unfolding ‹irs = transform-irules irs'› transform-irules-def by simp

    show ?thesis
      proof safe
        fix pats₁ rhs₁ pats₂ rhs₂
        assume (pats₁, rhs₁) |∈| irs (pats₂, rhs₂) |∈| irs
        with irs' obtain cs₁ cs₂ where (pats₁, cs₁) |∈| ?grp (pats₂, cs₂) |∈| ?grp
          by force
        then obtain pats₁' pats₂' and x y — dummies
          where (pats₁', x) |∈| irs' (pats₂', y) |∈| irs'
            and pats₁ = butlast pats₁' pats₂ = butlast pats₂'
          unfolding irs'
          by (fastforce elim: fgroup-byE2)
        hence patterns-compatible pats₁' pats₂'
          using ‹patterns-compatibles irs'› by (auto dest: fpairwiseD)
        thus patterns-compatible pats₁ pats₂
          unfolding ‹pats₁ = -› ‹pats₂ = -›
          by auto
      qed
  qed

have is-fmap irs'
  using ‹(name, irs') |∈| rs› inner by blast
show is-fmap irs
  proof (cases arity irs' = 0)
    case True
    thus ?thesis
      unfolding ‹irs = transform-irules irs'› transform-irules-def
      using ‹is-fmap irs'› by simp
  next
    case False
    hence irs': irs = map-prod id Pabs |`| ?grp
      unfolding ‹irs = transform-irules irs'› transform-irules-def by simp

    show ?thesis
      proof
        fix pats rhs₁ rhs₂
        assume (pats, rhs₁) |∈| irs (pats, rhs₂) |∈| irs
        with irs' obtain cs₁ cs₂
          where (pats, cs₁) |∈| ?grp rhs₁ = Pabs cs₁
            and (pats, cs₂) |∈| ?grp rhs₂ = Pabs cs₂
          by force
        moreover have is-fmap ?grp
          by auto
```

```
      ultimately show rhs₁ = rhs₂
        by (auto dest: is-fmapD)
    qed
  qed

fix pats rhs
assume (pats, rhs) |∈| irs

show linears pats
  proof (cases arity irs' = 0)
    case True
    thus ?thesis
      using ‹(pats, rhs) |∈| irs› ‹(name, irs') |∈| rs› inner
      unfolding ‹irs = transform-irules irs'› transform-irules-def
      by auto
  next
    case False
    hence irs': irs = map-prod id Pabs |`| ?grp
      unfolding ‹irs = transform-irules irs'› transform-irules-def by simp
    then obtain cs where (pats, cs) |∈| ?grp
      using ‹(pats, rhs) |∈| irs› by force
    then obtain pats' x — dummy
      where fst (?f (pats', x)) = pats (pats', x) |∈| irs'
      by (fastforce simp: split-beta elim: fgroup-byE2)
    hence pats = butlast pats'
      by simp
    moreover have linears pats'
      using ‹(pats', x) |∈| irs'› ‹(name, irs') |∈| rs› inner
      by auto
    ultimately show ?thesis
      by auto
  qed

have fdisjnt (freess pats) all-consts
  proof (cases arity irs' = 0)
    case True
    thus ?thesis
      using ‹(pats, rhs) |∈| irs› ‹(name, irs') |∈| rs› inner
      unfolding ‹irs = transform-irules irs'› transform-irules-def
      by auto
  next
    case False
    hence irs': irs = map-prod id Pabs |`| ?grp
      unfolding ‹irs = transform-irules irs'› transform-irules-def by simp
    then obtain cs where (pats, cs) |∈| ?grp
      using ‹(pats, rhs) |∈| irs› by force
    then obtain pats' x — dummy
      where fst (?f (pats', x)) = pats (pats', x) |∈| irs'
      by (fastforce simp: split-beta elim: fgroup-byE2)
```

```
    hence pats = butlast pats'
      by simp
    moreover have fdisjnt (freess pats') all-consts
      using ‹(pats', x) |∈| irs'› ‹(name, irs') |∈| rs› inner by auto
    ultimately show ?thesis
      by (metis subsetI in-set-butlastD freess-subset fdisjnt-subset-left)
  qed

 thus fdisjnt (freess pats) (pre-constants.all-consts C-info (fst |`| transform-irule-set
rs))
   unfolding transform-irule-set-def by simp

 show closed-except rhs (freess pats)
   proof (cases arity irs' = 0)
     case True
     thus ?thesis
       using ‹(pats, rhs) |∈| irs› ‹(name, irs') |∈| rs› inner
       unfolding ‹irs = transform-irules irs'› transform-irules-def
       by auto
   next
     case False
     hence irs': irs = map-prod id Pabs |`| ?grp
       unfolding ‹irs = transform-irules irs'› transform-irules-def by simp
     then obtain cs where (pats, cs) |∈| ?grp rhs = Pabs cs
       using ‹(pats, rhs) |∈| irs› by force

     show ?thesis
       unfolding ‹rhs = Pabs cs› closed-except-simps
       proof safe
         fix pat t
         assume (pat, t) |∈| cs
         then obtain pats' where (pats', t) |∈| irs' ?f (pats', t) = (pats, (pat, t))
           using ‹(pats, cs) |∈| ?grp› by auto
         hence closed-except t (freess pats')
           using ‹(name, irs') |∈| rs› inner by auto

         have pats' ≠ []
           using ‹arity-compatibles irs'› ‹(pats', t) |∈| irs'› False
           by (metis list.size(3) arity-compatible-length)
         hence pats' = pats @ [pat]
           using ‹?f (pats', t) = (pats, (pat, t))›
           by (fastforce simp: split-beta snoc-eq-iff-butlast)
         hence freess pats |∪| frees pat = freess pats'
           unfolding freess-def by auto

         thus closed-except t (freess pats |∪| frees pat)
           using ‹closed-except t (freess pats')› by simp
       qed
   qed
```

```
show wellformed rhs
  proof (cases arity irs' = 0)
    case True
    thus ?thesis
      using ‹(pats, rhs) |∈| irs› ‹(name, irs') |∈| rs› inner
      unfolding ‹irs = transform-irules irs'› transform-irules-def
      by auto
  next
    case False
    hence irs': irs = map-prod id Pabs |`| ?grp
      unfolding ‹irs = transform-irules irs'› transform-irules-def by simp
    then obtain cs where (pats, cs) |∈| ?grp rhs = Pabs cs
      using ‹(pats, rhs) |∈| irs› by force

    show ?thesis
      unfolding ‹rhs = Pabs cs›
      proof (rule wellformed-PabsI)
        show cs ≠ {||}
          using ‹(pats, cs) |∈| ?grp› ‹irs' ≠ {||}›
          by (meson femptyE fgroup-by-nonempty-inner)
      next
        show is-fmap cs
          proof
            fix pat t1 t2
            assume (pat, t1) |∈| cs (pat, t2) |∈| cs
            then obtain pats1' pats2'
              where (pats1', t1) |∈| irs' ?f (pats1', t1) = (pats, (pat, t1))
                and (pats2', t2) |∈| irs' ?f (pats2', t2) = (pats, (pat, t2))
              using ‹(pats, cs) |∈| ?grp› by force
            moreover hence pats1' ≠ [] pats2' ≠ []
              using ‹arity-compatibles irs'› False
              unfolding prod.case
              by (metis list.size(3) arity-compatible-length)+
            ultimately have pats1' = pats @ [pat] pats2' = pats @ [pat]
              unfolding split-beta fst-conv snd-conv
              by (metis prod.inject snoc-eq-iff-butlast)+
            with ‹is-fmap irs'› show t1 = t2
              using ‹(pats1', t1) |∈| irs'› ‹(pats2', t2) |∈| irs'›
              by (blast dest: is-fmapD)
          qed
      next
        show pattern-compatibles cs
          proof safe
            fix pat1 rhs1 pat2 rhs2
            assume (pat1, rhs1) |∈| cs (pat2, rhs2) |∈| cs
            then obtain pats1' pats2'
              where (pats1', rhs1) |∈| irs' ?f (pats1', rhs1) = (pats, (pat1, rhs1))
                and (pats2', rhs2) |∈| irs' ?f (pats2', rhs2) = (pats, (pat2, rhs2))
```

```
            using ‹(pats, cs) |∈| ?grp›
            by force
          moreover hence pats1' ≠ [] pats2' ≠ []
            using ‹arity-compatibles irs'› False
            unfolding prod.case
            by (metis list.size(3) arity-compatible-length)+
          ultimately have pats1' = pats @ [pat1] pats2' = pats @ [pat2]
            unfolding split-beta fst-conv snd-conv
            by (metis prod.inject snoc-eq-iff-butlast)+
          moreover have patterns-compatible pats1' pats2'
        using ‹(pats1', rhs1) |∈| irs'› ‹(pats2', rhs2) |∈| irs'› ‹patterns-compatibles
irs'›
            by (auto dest: fpairwiseD)
          ultimately show pattern-compatible pat1 pat2
            by (auto elim: rev-accum-rel-snoc-eqE)
        qed
      next
        fix pat t
        assume (pat, t) |∈| cs
        then obtain pats' where (pats', t) |∈| irs' pat = last pats'
          using ‹(pats, cs) |∈| ?grp› by auto
        moreover hence pats' ≠ []
          using ‹arity-compatibles irs'› False
          by (metis list.size(3) arity-compatible-length)
        ultimately have pat ∈ set pats'
          by auto

        moreover have linears pats'
          using ‹(pats', t) |∈| irs'› ‹(name, irs') |∈| rs› inner by auto
        ultimately show linear pat
          by (metis linears-linear)

        show wellformed t
          using ‹(pats', t) |∈| irs'› ‹(name, irs') |∈| rs› inner by auto
      qed
    qed

  have ¬ shadows-consts rhs
    proof (cases arity irs' = 0)
      case True
      thus ?thesis
        using ‹(pats, rhs) |∈| irs› ‹(name, irs') |∈| rs› inner
        unfolding ‹irs = transform-irules irs'› transform-irules-def
        by auto
    next
      case False
      hence irs': irs = map-prod id Pabs |`| ?grp
        unfolding ‹irs = transform-irules irs'› transform-irules-def by simp
      then obtain cs where (pats, cs) |∈| ?grp rhs = Pabs cs
```

```
      using ‹(pats, rhs) |∈| irs› by force

    show ?thesis
      unfolding ‹rhs = -›
      proof
        assume shadows-consts (Pabs cs)
       then obtain pat t where (pat, t) |∈| cs shadows-consts t ∨ shadows-consts
pat
          by force
        then obtain pats' where (pats', t) |∈| irs' pat = last pats'
          using ‹(pats, cs) |∈| ?grp› by auto
        moreover hence pats' ≠ []
          using ‹arity-compatibles irs'› False
          by (metis list.size(3) arity-compatible-length)
        ultimately have pat ∈ set pats'
          by auto

        show False
          using ‹shadows-consts t ∨ shadows-consts pat›
          proof
            assume shadows-consts t
            thus False
              using ‹(name, irs') |∈| rs› ‹(pats', t) |∈| irs'› inner by auto
          next
            assume shadows-consts pat

            have fdisjnt (freess pats') all-consts
              using ‹(name, irs') |∈| rs› ‹(pats', t) |∈| irs'› inner by auto
            have fdisjnt (frees pat) all-consts
              apply (rule fdisjnt-subset-left)
               apply (subst freess-single[symmetric])
               apply (rule freess-subset)
               apply simp
               apply fact+
              done

            thus False
              using ‹shadows-consts pat›
              unfolding shadows-consts-def fdisjnt-alt-def by auto
        qed
      qed
  qed
  thus ¬ pre-constants.shadows-consts C-info (fst |‘| transform-irule-set rs) rhs
    by (simp add: transform-irule-set-heads)

  show abs-ish pats rhs
    proof (cases arity irs' = 0)
      case True
      thus ?thesis
```

```
    using ‹(pats, rhs) |∈| irs› ‹(name, irs′) |∈| rs› inner
    unfolding ‹irs = transform-irules irs′› transform-irules-def
    by auto
  next
   case False
   hence irs′: irs = map-prod id Pabs |`| ?grp
    unfolding ‹irs = transform-irules irs′› transform-irules-def by simp
   then obtain cs where (pats, cs) |∈| ?grp rhs = Pabs cs
    using ‹(pats, rhs) |∈| irs› by force
   thus ?thesis
    unfolding abs-ish-def by (simp add: is-abs-def term-cases-def)
  qed

have welldefined rhs
  proof (cases arity irs′ = 0)
   case True
   hence ‹(pats, rhs) |∈| irs′›
    using ‹(pats, rhs) |∈| irs› ‹(name, irs′) |∈| rs› inner
    unfolding ‹irs = transform-irules irs′› transform-irules-def
    by (smt (z3) fBallE split-conv)
   thus ?thesis
    unfolding transform-irule-set-def
    using fbspec[OF inner ‹(name, irs′) |∈| rs›, simplified]
    by force
  next
   case False
   hence irs′: irs = map-prod id Pabs |`| ?grp
    unfolding ‹irs = transform-irules irs′› transform-irules-def by simp
   then obtain cs where (pats, cs) |∈| ?grp rhs = Pabs cs
    using ‹(pats, rhs) |∈| irs› by force
   show ?thesis
    unfolding ‹rhs = -›
    apply simp
    apply (rule ffUnion-least)
    unfolding ball-simps
    apply rule
    apply (rename-tac x, case-tac x, hypsubst-thin)
    apply simp
    subgoal premises prems for pat t
     proof −
      from prems obtain pats′ where (pats′, t) |∈| irs′
       using ‹(pats, cs) |∈| ?grp› by auto
      hence welldefined t
       using fbspec[OF inner ‹(name, irs′) |∈| rs›, simplified]
       by blast
      thus ?thesis
       unfolding transform-irule-set-def
       by simp
     qed
```

done
qed
thus *pre-constants.welldefined C-info* (*fst* |‘| *transform-irule-set rs*) *rhs*
unfolding *transform-irule-set-heads* **.**
qed

Matching and rewriting

definition *irewrite-step* :: *name* ⇒ *term list* ⇒ *pterm* ⇒ *pterm* ⇒ *pterm option*
where
irewrite-step name pats rhs t = *map-option* (*subst rhs*) (*match* (*name* $$ *pats*) *t*)

abbreviation *irewrite-step′* :: *name* ⇒ *term list* ⇒ *pterm* ⇒ *pterm* ⇒ *pterm* ⇒ *bool* (‹-, -, - ⊢$_i$/ - →/ -› [50,0,50] 50) **where**
name, *pats*, *rhs* ⊢$_i$ *t* → *u* ≡ *irewrite-step name pats rhs t* = *Some u*

lemma *irewrite-stepI*:
assumes *match* (*name* $$ *pats*) *t* = *Some env subst rhs env* = *u*
shows *name*, *pats*, *rhs* ⊢$_i$ *t* → *u*
using *assms* **unfolding** *irewrite-step-def* **by** *simp*

inductive *irewrite* :: *irule-set* ⇒ *pterm* ⇒ *pterm* ⇒ *bool* (‹-/ ⊢$_i$/ - ⟶/ -› [50,0,50] 50) **for** *irs* **where**
step: ⟦ (*name*, *rs*) |∈| *irs*; (*pats*, *rhs*) |∈| *rs*; *name*, *pats*, *rhs* ⊢$_i$ *t* → *t′* ⟧ ⟹ *irs* ⊢$_i$ *t* ⟶ *t′* |
beta: ⟦ *c* |∈| *cs*; *c* ⊢ *t* → *t′* ⟧ ⟹ *irs* ⊢$_i$ *Pabs cs* $$_p$ *t* ⟶ *t′* |
fun: *irs* ⊢$_i$ *t* ⟶ *t′* ⟹ *irs* ⊢$_i$ *t* $$_p$ *u* ⟶ *t′* $$_p$ *u* |
arg: *irs* ⊢$_i$ *u* ⟶ *u′* ⟹ *irs* ⊢$_i$ *t* $$_p$ *u* ⟶ *t* $$_p$ *u′*

global-interpretation *irewrite*: *rewriting irewrite rs* **for** *rs*
by *standard* (*auto intro*: *irewrite.intros simp*: *app-pterm-def*)+

abbreviation *irewrite-rt* :: *irule-set* ⇒ *pterm* ⇒ *pterm* ⇒ *bool* (‹-/ ⊢$_i$/ - ⟶*/ -› [50,0,50] 50) **where**
irewrite-rt rs ≡ (*irewrite rs*)**

lemma (**in** *irules*) *irewrite-closed*:
assumes *rs* ⊢$_i$ *t* ⟶ *u closed t*
shows *closed u*
using *assms* **proof** *induction*
case (*step name rs pats rhs t t′*)
then obtain *env* **where** *match* (*name* $$ *pats*) *t* = *Some env t′* = *subst rhs env*
unfolding *irewrite-step-def* **by** *auto*
hence *closed-env env*
using *step* **by** (*auto intro*: *closed.match*)

show *?case*
unfolding ‹*t′* = -›
apply (*subst closed-except-def*)

apply (*subst subst-frees*)
apply *fact*
apply (*subst match-dom*)
apply *fact*
apply (*subst frees-list-comb*)
apply *simp*
apply (*subst closed-except-def*[*symmetric*])
using *inner step* **by** *fastforce*
next
case (*beta c cs t t'*)
then obtain *pat rhs* **where** *c* = (*pat*, *rhs*)
by (*cases c*) *auto*
with *beta* **obtain** *env* **where** *match pat t* = *Some env t'* = *subst rhs env*
by *auto*
moreover have *closed t*
using *beta* **unfolding** *closed-except-def* **by** *simp*
ultimately have *closed-env env*
using *beta* **by** (*auto intro*: *closed.match*)

show *?case*
unfolding ‹*t'* = *subst rhs env*›
apply (*subst closed-except-def*)
apply (*subst subst-frees*)
apply *fact*
apply (*subst match-dom*)
apply *fact*
apply *simp*
apply (*subst closed-except-def*[*symmetric*])
using *inner beta* ‹*c* = -› **by** (*auto simp*: *closed-except-simps*)
qed (*auto simp*: *closed-except-def*)

corollary (**in** *irules*) *irewrite-rt-closed*:
assumes *rs* ⊢$_i$ *t* ⟶* *u closed t*
shows *closed u*
using *assms* **by** *induction* (*auto intro*: *irewrite-closed*)

Correctness of translation

abbreviation *irelated* :: *nterm* ⇒ *pterm* ⇒ *bool* (‹- ≈$_i$ -› [*0*,*50*] *50*) **where**
n ≈$_i$ *p* ≡ *nterm-to-pterm n* = *p*

global-interpretation *irelated*: *term-struct-rel-strong irelated*
by *standard*
(*auto simp*: *app-pterm-def app-nterm-def const-pterm-def const-nterm-def elim*: *nterm-to-pterm.elims*)

lemma *irelated-vars*: *t* ≈$_i$ *u* ⟹ *frees t* = *frees u*
by *auto*

lemma *irelated-no-abs*:
  assumes $t \approx_i u$
  shows *no-abs t* $\longleftrightarrow$ *no-abs u*
using *assms*
apply (*induction arbitrary*: *t*)
 apply (*auto elim*!: *nterm-to-pterm.elims*)
  apply (*fold const-nterm-def const-pterm-def free-nterm-def free-pterm-def app-pterm-def app-nterm-def*)
by *auto*

lemma *irelated-subst*:
  assumes $t \approx_i u$ *irelated.P-env nenv penv*
  shows *subst t nenv* $\approx_i$ *subst u penv*
using *assms* **proof** (*induction arbitrary*: *nenv penv u rule*: *nterm-to-pterm.induct*)
  case (*1 s*)
  then show *?case*
    by (*auto elim*!: *fmrel-cases*[**where** $x = s$])
next
  case *4*
  from *4*(*2*)[*symmetric*] **show** *?case*
    apply *simp*
    apply (*rule 4*)
     apply *simp*
    using *4*(*3*)
    by (*simp add*: *fmrel-drop*)
qed *auto*

lemma *related-irewrite-step*:
  assumes *name*, *pats*, *nterm-to-pterm rhs* $\vdash_i u \to u'$ $t \approx_i u$
  obtains t' **where** *unsplit-rule* (*name*, *pats*, *rhs*) $\vdash t \to t'$ $t' \approx_i u'$
proof −
  let *?rhs'* = *nterm-to-pterm rhs*
  let *?x* = *name* $$ *pats*

  from *assms* **obtain** *env* **where** *match ?x u* = *Some env u'* = *subst ?rhs' env*
    unfolding *irewrite-step-def* **by** *blast*
  then obtain *nenv* **where** *match ?x t* = *Some nenv irelated.P-env nenv env*
    using *assms*
   by (*metis Option.is-none-def not-None-eq option.rel-distinct*(*1*) *option.sel rel-option-unfold irelated.match-rel*)

  show *thesis*
    proof
      show *unsplit-rule* (*name*, *pats*, *rhs*) $\vdash t \to$ *subst rhs nenv*
        using ‹*match ?x t* = -› **by** *auto*
    next
      show *subst rhs nenv* $\approx_i u'$
        unfolding ‹*u'* = -› **using** ‹*irelated.P-env nenv env*›
        by (*auto intro*: *irelated-subst*)

```
    qed
qed

theorem (in nrules) compile-correct:
  assumes compile (consts-of rs) ⊢ᵢ u ⟶ u' t ≈ᵢ u closed t
  obtains t' where rs ⊢ₙ t ⟶ t' t' ≈ᵢ u'
using assms(1−3) proof (induction arbitrary: t thesis rule: irewrite.induct)
  case (step name irs pats rhs u u')
  then obtain crs where irs = translate-crules crs (name, crs) |∈| consts-of rs
    unfolding compile-def by force
  moreover with step obtain rhs' where rhs = nterm-to-pterm rhs' (pats, rhs')
|∈| crs
    unfolding translate-crules-def by force
  ultimately obtain rule where split-rule rule = (name, (pats, rhs')) rule |∈| rs
    unfolding consts-of-def by blast
  hence nrule rule
    using all-rules by blast

  obtain t' where unsplit-rule (name, pats, rhs') ⊢ t → t' t' ≈ᵢ u'
    using ‹name, pats, rhs ⊢ᵢ u → u'› ‹t ≈ᵢ u› unfolding ‹rhs = nterm-to-pterm
rhs'›
    by (elim related-irewrite-step)
  hence rule ⊢ t → t'
    using ‹nrule rule› ‹split-rule rule = (name, (pats, rhs'))›
    by (metis unsplit-split)

  show ?case
    proof (rule step.prems)
      show rs ⊢ₙ t ⟶ t'
        apply (rule nrewrite.step)
         apply fact
        apply fact
        done
    next
      show t' ≈ᵢ u'
        by fact
    qed
next
  case (beta c cs u u')
  then obtain pat rhs where c = (pat, rhs) (pat, rhs) |∈| cs
    by (cases c) auto
  obtain v w where t = v $ₙ w v ≈ᵢ Pabs cs w ≈ᵢ u
    using ‹t ≈ᵢ Pabs cs $ₚ u› by (auto elim: nterm-to-pterm.elims)
 obtain x nrhs irhs where v = (Λₙ x. nrhs) cs = {| (Free x, irhs) |} nrhs ≈ᵢ irhs
    using ‹v ≈ᵢ Pabs cs› by (auto elim: nterm-to-pterm.elims)
  hence t = (Λₙ x. nrhs) $ₙ w Λₙ x. nrhs ≈ᵢ Λₚ x. irhs
    unfolding ‹t = v $ₙ w› using ‹v ≈ᵢ Pabs cs› by auto

  have pat = Free x rhs = irhs
```

```
    using ‹cs = {| (Free x, irhs) |}› ‹(pat, rhs) |∈| cs› by auto
  hence (Free x, irhs) ⊢ u → u'
    using beta ‹c = -› by simp
  hence u' = subst irhs (fmap-of-list [(x, u)])
    by simp

  show ?case
    proof (rule beta.prems)
      show rs ⊢n t ⟶ subst nrhs (fmap-of-list [(x, w)])
        unfolding ‹t = (Λn x. nrhs) $n w›
        by (rule nrewrite.beta)
    next
      show subst nrhs (fmap-of-list [(x, w)]) ≈i u'
        unfolding ‹u' = subst irhs -›
        apply (rule irelated-subst)
         apply fact
        apply simp
        apply rule
         apply rule
        apply fact
        done
    qed
next
  case (fun v v' u)
  obtain w x where t = w $n x w ≈i v x ≈i u
    using ‹t ≈i v $p u› by (auto elim: nterm-to-pterm.elims)
  with fun obtain w' where rs ⊢n w ⟶ w' w' ≈i v'
    unfolding closed-except-def by auto

  show ?case
    proof (rule fun.prems)
      show rs ⊢n t ⟶ w' $n x
        unfolding ‹t = w $n x›
        by (rule nrewrite.fun) fact
    next
      show w' $n x ≈i v' $p u
        by auto fact+
    qed
next
  case (arg u u' v)
  obtain w x where t = w $n x w ≈i v x ≈i u
    using ‹ t ≈i v $p u› by (auto elim: nterm-to-pterm.elims)
  with arg obtain x' where rs ⊢n x ⟶ x' x' ≈i u'
    unfolding closed-except-def by auto

  show ?case
    proof (rule arg.prems)
      show rs ⊢n t ⟶ w $n x'
        unfolding ‹t = w $n x›
```

```
      by (rule nrewrite.arg) fact
    next
      show w $n x' ≈i v $p u'
        by auto fact+
    qed
qed

corollary (in nrules) compile-correct-rt:
  assumes compile (consts-of rs) ⊢i u ⟶* u' t ≈i u closed t
  obtains t' where rs ⊢n t ⟶* t' t' ≈i u'
using assms proof (induction arbitrary: thesis t)
  case (step u' u'')

  obtain t' where rs ⊢n t ⟶* t' t' ≈i u'
    using step by blast

  obtain t'' where rs ⊢n t' ⟶* t'' t'' ≈i u''
    proof (rule compile-correct)
      show compile (consts-of rs) ⊢i u' ⟶ u'' t' ≈i u'
        by fact+
    next
      show closed t'
        using ‹rs ⊢n t ⟶* t'› ‹closed t›
        by (rule nrewrite-rt-closed)
    qed blast

  show ?case
    proof (rule step.prems)
      show rs ⊢n t ⟶* t''
        using ‹rs ⊢n t ⟶* t'› ‹rs ⊢n t' ⟶* t''› by auto
    qed fact
qed blast
```

Completeness of translation

```
lemma (in nrules) compile-complete:
  assumes rs ⊢n t ⟶ t' closed t
  shows compile (consts-of rs) ⊢i nterm-to-pterm t ⟶ nterm-to-pterm t'
  using assms proof induction
  case (step r t t')
  then obtain pat rhs' where r = (pat, rhs')
    by force
  then have (pat, rhs') |∈| rs (pat, rhs') ⊢ t → t'
    using step by blast+
  then have nrule (pat, rhs')
    using all-rules by blast
  then obtain name pats where (name, (pats, rhs')) = split-rule r pat = name
$$ pats
    unfolding split-rule-def ‹r = -›
```

```
    apply atomize-elim
    by (auto simp: split-beta)

  obtain crs where (name, crs) |∈| consts-of rs (pats, rhs') |∈| crs
    using step ‹- = split-rule r› ‹r = -›
    by (metis consts-of-def fgroup-by-complete fst-conv snd-conv)
  then obtain irs where irs = translate-crules crs
    by blast
  then have (name, irs) |∈| compile (consts-of rs)
    unfolding compile-def
    using ‹(name, -) |∈| -›
    by (metis fimageI id-def map-prod-simp)
  obtain rhs where rhs = nterm-to-pterm rhs' (pats, rhs) |∈| irs
    using ‹irs = -› ‹- |∈| crs›
    unfolding translate-crules-def
    by (metis fimageI id-def map-prod-simp)

  from step obtain env' where match pat t = Some env' t' = subst rhs' env'
    unfolding ‹r = -› using rewrite-step.simps
    by force
  then obtain env where match pat (nterm-to-pterm t) = Some env irelated.P-env
env' env
    by (metis irelated.match-rel option-rel-Some1)
  then have subst rhs env = nterm-to-pterm t'
    unfolding ‹t' = -›
    apply −
    apply (rule sym)
    apply (rule irelated-subst)
    unfolding ‹rhs = -›
    by auto

  have name, pats, rhs ⊢ᵢ nterm-to-pterm t → nterm-to-pterm t'
    apply (rule irewrite-stepI)
    using ‹match - - = Some env› unfolding ‹pat = -›
     apply assumption
    by fact

  show ?case
    by rule fact+
next
  case (beta x t t')
  obtain c where c = (Free x, nterm-to-pterm t)
    by blast
  from beta have closed (nterm-to-pterm t')
    using closed-nterm-to-pterm[where t = t']
    unfolding closed-except-def
    by auto
  show ?case
    apply simp
```

apply *rule*
using ‹*c* = -›
by (*fastforce intro*: *irelated-subst*[*THEN sym*])+
next
case (*fun t t′ u*)
show *?case*
apply *simp*
apply *rule*
apply (*rule fun*)
using *fun*
unfolding *closed-except-def*
apply *simp*
done
next
case (*arg u u′ t*)
show *?case*
apply *simp*
apply *rule*
apply (*rule arg*)
using *arg*
unfolding *closed-except-def*
by *simp*
qed

Correctness of transformation

abbreviation *irules-deferred-matches* :: *pterm list* ⇒ *irules* ⇒ (*term* × *pterm*) *fset* **where**
irules-deferred-matches args ≡ *fselect*
(λ(*pats*, *rhs*). *map-option* (λ*env*. (*last pats*, *subst rhs env*)) (*matchs* (*butlast pats*) *args*))

context *irules* **begin**

inductive *prelated* :: *pterm* ⇒ *pterm* ⇒ *bool* (‹- ≈$_p$ -› [*0*,*50*] *50*) **where**
const: *Pconst x* ≈$_p$ *Pconst x* |
var: *Pvar x* ≈$_p$ *Pvar x* |
app: t_1 ≈$_p$ u_1 ⟹ t_2 ≈$_p$ u_2 ⟹ t_1 $_p$ t_2 ≈$_p$ u_1 $_p$ u_2 |
pat: *rel-fset* (*rel-prod* (=) *prelated*) cs_1 cs_2 ⟹ *Pabs* cs_1 ≈$_p$ *Pabs* cs_2 |
defer:
(*name*, *rsi*) |∈| *rs* ⟹ *0* < *arity rsi* ⟹
rel-fset (*rel-prod* (=) *prelated*) (*irules-deferred-matches args rsi*) *cs* ⟹
list-all closed args ⟹
name $$ *args* ≈$_p$ *Pabs cs*

inductive-cases *prelated-absE*[*consumes 1*, *case-names pat defer*]: *t* ≈$_p$ *Pabs cs*

lemma *prelated-refl*[*intro*!]: *t* ≈$_p$ *t*
proof (*induction t*)

```
  case Pabs
  thus ?case
    by (auto simp: snds.simps intro!: prelated.pat rel-fset-refl-strong rel-prod.intros)
qed (auto intro: prelated.intros)

sublocale prelated: term-struct-rel prelated
by standard (auto simp: const-pterm-def app-pterm-def intro: prelated.intros elim:
prelated.cases)

lemma prelated-pvars:
  assumes t ≈_p u
  shows frees t = frees u
using assms proof (induction rule: prelated.induct)
  case (pat cs_1 cs_2)
  show ?case
    apply simp
    apply (rule arg-cong[where f = ffUnion])
    apply (rule rel-fset-image-eq)
     apply fact
    apply auto
    done
next
  case (defer name rsi args cs)

  {
    fix pat t
    assume (pat, t) |∈| cs
    with defer obtain t'
      where (pat, t') |∈| irules-deferred-matches args rsi frees t = frees t'
      by (auto elim: rel-fsetE2)
    then obtain pats rhs env
      where pat = last pats (pats, rhs) |∈| rsi
        and matchs (butlast pats) args = Some env t' = subst rhs env
      by auto

    have closed-except rhs (freess pats) linears pats
      using ‹(pats, rhs) |∈| rsi› ‹(name, rsi) |∈| rs› inner by auto

    have arity-compatibles rsi
      using defer inner
      by (metis (no-types, lifting) case-prodD)
    have length pats > 0
      by (subst arity-compatible-length) fact+
    hence pats = butlast pats @ [last pats]
      by simp

    note ‹frees t = frees t'›
    also have frees t' = frees rhs − fmdom env
      unfolding ‹t' = -›
```

```
      apply (rule subst-frees)
      apply (rule closed.matchs)
       apply fact+
      done
    also have ... = frees rhs − freess (butlast pats)
      using ‹matchs - - = -› by (metis matchs-dom)
    also have ... |⊆| frees pats − freess (butlast pats)
      using ‹closed-except - -›
      by (auto simp: closed-except-def)
    also have ... = frees (last pats) |−| freess (butlast pats)
      by (subst ‹pats = -›) (simp add: funion-fminus)
    also have ... = frees (last pats)
      proof (rule fminus-triv)
        have fdisjnt (freess (butlast pats)) (freess [last pats])
          using ‹linears pats› ‹pats = -›
          by (metis linears-appendD)
        thus frees (last pats) |∩| freess (butlast pats) = {||}
          by (fastforce simp: fdisjnt-alt-def)
      qed
    also have ... = frees pat unfolding ‹pat = -› ..
    finally have frees t |⊆| frees pat .
  }
  hence closed (Pabs cs)
    unfolding closed-except-simps
    by (auto simp: closed-except-def)
  moreover have closed (name $$ args)
    unfolding closed-list-comb by fact
  ultimately show ?case
    unfolding closed-except-def by simp
qed auto

corollary prelated-closed: t ≈p u ⟹ closed t ⟷ closed u
unfolding closed-except-def
by (auto simp: prelated-pvars)

lemma prelated-no-abs-right:
  assumes t ≈p u no-abs u
  shows t = u
using assms
apply (induction rule: prelated.induct)
    apply auto
 apply (fold app-pterm-def)
 apply auto
done

corollary env-prelated-refl[intro!]: prelated.P-env env env
by (auto intro: fmap.rel-refl)
```

The following, more general statement does not hold: $t \approx_p u \Longrightarrow$ *rel-option*

prelated.P-env (*match x t*) (*match x u*) If t and u are related because of the *prelated.defer* rule, they have completely different shapes. Establishing *is-abs t* = *is-abs u* as a precondition would rule out this case, but at the same time be too restrictive.

Instead, we use ⟦*match ?x ?u* = *Some ?env*; *?t* $\approx_p$ *?u*; $\bigwedge$*env′*. ⟦*match ?x ?t* = *Some env′*; *prelated.P-env env′ ?env*⟧ $\Longrightarrow$ *?thesis*⟧ $\Longrightarrow$ *?thesis*.

lemma *prelated-subst*:
  assumes t_1 $\approx_p$ t_2 *prelated.P-env* env_1 env_2
  shows *subst* t_1 env_1 $\approx_p$ *subst* t_2 env_2
using *assms* **proof** (*induction arbitrary*: env_1 env_2 *rule*: *prelated.induct*)
  case (*var x*)
  thus *?case*
    proof (*cases rule*: *fmrel-cases*[**where** *x* = *x*])
      case *none*
      thus *?thesis*
        by (*auto intro*: *prelated.var*)
    next
      case (*some t u*)
      thus *?thesis*
        by *simp*
    qed
next
  case (*pat* cs_1 cs_2)
  let *?drop* = λ*env*. λ(*pat*::*term*). *fmdrop-fset* (*frees pat*) *env*
  from *pat* **have** *prelated.P-env* (*?drop* env_1 *pat*) (*?drop* env_2 *pat*) **for** *pat*
    by *blast*
  with *pat* **show** *?case*
    by (*auto intro*!: *prelated.pat rel-fset-image*)
next
  case (*defer name rsi args cs*)
  have *name* $$ *args* $\approx_p$ *Pabs cs*
    apply (*rule prelated.defer*)
      apply *fact*+
     apply (*rule fset.rel-mono-strong*)
      apply *fact*
     apply *force*
    apply *fact*
    done
  moreover have *closed* (*name* $$ *args*)
    unfolding *closed-list-comb* **by** *fact*
  ultimately have *closed* (*Pabs cs*)
    by (*metis prelated-closed*)

  let *?drop* = λ*env*. λ*pat*. *fmdrop-fset* (*frees pat*) *env*
  let *?f* = λ*env*. (λ(*pat*, *rhs*). (*pat*, *subst rhs* (*?drop env pat*)))

  have *name* $$ *args* $\approx_p$ *Pabs* (*?f* env_2 |‘| *cs*)
    proof (*rule prelated.defer*)

```
      show (name, rsi) |∈| rs 0 < arity rsi list-all closed args
        using defer by auto
    next
      {
        fix pat1 rhs1
        fix pat2 rhs2
        assume (pat2, rhs2) |∈| cs
        assume pat1 = pat2 rhs1 ≈p rhs2
        have rhs1 ≈p subst rhs2 (fmdrop-fset (frees pat2) env2)
          by (subst subst-closed-pabs) fact+
      }
      hence rel-fset (rel-prod (=) prelated) (id |`| irules-deferred-matches args rsi)
(?f env2 |`| cs)
        by (force intro!: rel-fset-image[OF ‹rel-fset - - -›])

     thus rel-fset (rel-prod (=) prelated) (irules-deferred-matches args rsi) (?f env2
|`| cs)
        by simp
    qed

  moreover have map (λt. subst t env1) args = args
    apply (rule map-idI)
    apply (rule subst-closed-id)
    using defer by (simp add: list-all-iff)

  ultimately show ?case
    by (simp add: subst-list-comb)
qed (auto intro: prelated.intros)

lemma prelated-step:
  assumes name, pats, rhs ⊢i u → u' t ≈p u
  obtains t' where name, pats, rhs ⊢i t → t' t' ≈p u'
proof −
  let ?lhs = name $$ pats
  from assms obtain env where match ?lhs u = Some env u' = subst rhs env
    unfolding irewrite-step-def by blast
  then obtain env' where match ?lhs t = Some env' prelated.P-env env' env
    using assms by (auto elim: prelated.related-match)
  hence subst rhs env' ≈p subst rhs env
    using assms by (auto intro: prelated-subst)

  show thesis
    proof
      show name, pats, rhs ⊢i t → subst rhs env'
        unfolding irewrite-step-def using ‹match ?lhs t = Some env'›
        by simp
    next
      show subst rhs env' ≈p u'
        unfolding ‹u' = subst rhs env›
```

```
      by fact
  qed
qed


lemma prelated-beta: — same problem as prelated.related-match
  assumes (pat, rhs2) ⊢ t2 → u2 rhs1 ≈p rhs2 t1 ≈p t2
  obtains u1 where (pat, rhs1) ⊢ t1 → u1 u1 ≈p u2
proof −
  from assms obtain env2 where match pat t2 = Some env2 u2 = subst rhs2 env2
    by auto
  with assms obtain env1 where match pat t1 = Some env1 prelated.P-env env1
env2
    by (auto elim: prelated.related-match)
  with assms have subst rhs1 env1 ≈p subst rhs2 env2
    by (auto intro: prelated-subst)

  show thesis
    proof
      show (pat, rhs1) ⊢ t1 → subst rhs1 env1
        using ‹match pat t1 = -› by simp
    next
      show subst rhs1 env1 ≈p u2
        unfolding ‹u2 = -› by fact
    qed
qed

theorem transform-correct:
  assumes transform-irule-set rs ⊢i u ⟶ u' t ≈p u closed t
  obtains t' where rs ⊢i t ⟶* t' — zero or one step and t' ≈p u'
using assms(1−3) proof (induction arbitrary: t thesis rule: irewrite.induct)
  case (beta c cs2 u2 x2')
  obtain v u1 where t = v $p u1 v ≈p Pabs cs2 u1 ≈p u2
    using ‹t ≈p Pabs cs2 $p u2› by cases
  with beta have closed u1
    by (simp add: closed-except-def)

  obtain pat rhs2 where c = (pat, rhs2) by (cases c) auto

  from ‹v ≈p Pabs cs2› show ?case
    proof (cases rule: prelated-absE)
      case (pat cs1)
      with beta ‹c = -› obtain rhs1 where (pat, rhs1) |∈| cs1 rhs1 ≈p rhs2
        by (auto elim: rel-fsetE2)
      with beta obtain x1' where (pat, rhs1) ⊢ u1 → x1' x1' ≈p x2'
        using ‹u1 ≈p u2› assms ‹c = -›
        by (auto elim: prelated-beta simp del: rewrite-step.simps)

      show ?thesis
```

```
      proof (rule beta.prems)
        show rs ⊢ᵢ t ⟶* x₁'
          unfolding ‹t = -› ‹v = -›
          by (intro r-into-rtranclp irewrite.beta) fact+
      next
        show x₁' ≈ₚ x₂'
          by fact
    qed
  next
    case (defer name rsi args)
    with beta ‹c = -› obtain rhs₁' where (pat, rhs₁') |∈| irules-deferred-matches
args rsi rhs₁' ≈ₚ rhs₂
      by (auto elim: rel-fsetE2)
    then obtain envₐ rhs₁ pats
     where matchs (butlast pats) args = Some envₐ pat = last pats rhs₁' = subst
rhs₁ envₐ
        and (pats, rhs₁) |∈| rsi
      by auto
    hence linears pats
      using ‹(name, rsi) |∈| rs› inner unfolding irules-def by auto

    have arity-compatibles rsi
      using defer inner
      by (metis (no-types, lifting) case-prodD)
    have length pats > 0
      by (subst arity-compatible-length) fact+
    hence pats = butlast pats @ [pat]
      unfolding ‹pat = -› by simp

   from beta ‹c = -› obtain envᵦ where match pat u₂ = Some envᵦ x₂' = subst
rhs₂ envᵦ
      by fastforce
   with ‹u₁ ≈ₚ u₂› obtain envᵦ' where match pat u₁ = Some envᵦ' prelated.P-env
envᵦ' envᵦ
      by (metis prelated.related-match)

    have closed-env envₐ
      by (rule closed.matchs) fact+
    have closed-env envᵦ'
      apply (rule closed.matchs[where pats = [pat] and ts = [u₁]])
       apply simp
       apply fact
      apply simp
      apply fact
      done

    have fmdom envₐ = freess (butlast pats)
      by (rule matchs-dom) fact
    moreover have fmdom envᵦ' = frees pat
```

```
      by (rule match-dom) fact
    moreover have fdisjnt (freess (butlast pats)) (frees pat)
      using ‹pats = -› ‹linears pats›
      by (metis freess-single linears-appendD(3))
    ultimately have fdisjnt (fmdom env_a) (fmdom env_b')
      by simp

    show ?thesis
      proof (rule beta.prems)
        have rs ⊢_i name $$ args $_p u_1 ⟶ subst rhs_1' env_b'
          proof (rule irewrite.step)
            show (name, rsi) |∈| rs (pats, rhs_1) |∈| rsi
              by fact+
          next
            show name, pats, rhs_1 ⊢_i name $$ args $_p u_1 → subst rhs_1' env_b'
              apply (rule irewrite-stepI)
               apply (fold app-pterm-def)
               apply (subst list-comb-snoc)
               apply (subst matchs-match-list-comb)
               apply (subst ‹pats = -›)
               apply (rule matchs-appI)
                apply fact
               apply simp
               apply fact
              unfolding ‹rhs_1' = -›
              apply (rule subst-indep')
               apply fact+
              done
          qed
        thus rs ⊢_i t ⟶* subst rhs_1' env_b'
          unfolding ‹t = -› ‹v = -›
          by (rule r-into-rtranclp)
      next
        show subst rhs_1' env_b' ≈_p x_2'
          unfolding ‹x_2' = -›
          by (rule prelated-subst) fact+
      qed
  qed
next
  case (step name rs_2 pats rhs u u')
  then obtain rs_1 where rs_2 = transform-irules rs_1 (name, rs_1) |∈| rs
    unfolding transform-irule-set-def by force
  hence arity-compatibles rs_1
    using inner
    by (metis (no-types, lifting) case-prodD)

  show ?case
    proof (cases arity rs_1 = 0)
      case True
```

hence $rs_2 = rs_1$
unfolding ‹rs_2 = -› *transform-irules-def* **by** *simp*
with *step* **have** (*pats*, *rhs*) |∈| rs_1
by *simp*
from *step* **obtain** t' **where** *name*, *pats*, *rhs* $\vdash_i t \to t'$ $t' \approx_p u'$
using *assms*
by (*auto elim*: *prelated-step*)

show *?thesis*
proof (*rule step.prems*)
show *rs* $\vdash_i t \longrightarrow* t'$
by (*intro conjI exI r-into-rtranclp irewrite.step*) *fact+*
qed *fact*
next
let *?f* = λ(*pats*, *rhs*). (*butlast pats*, *last pats*, *rhs*)
let *?grp* = *fgroup-by ?f* rs_1

case *False*
hence rs_2 = *map-prod id Pabs* |‘| *?grp*
unfolding ‹rs_2 = -› *transform-irules-def* **by** *simp*
with *step* **obtain** *cs* **where** *rhs* = *Pabs cs* (*pats*, *cs*) |∈| *?grp*
by *force*

from *step* **obtain** env_2 **where** *match* (*name* $$ *pats*) *u* = *Some* env_2 u' = *subst rhs* env_2
unfolding *irewrite-step-def* **by** *auto*
then obtain $args_2$ **where** *u* = *name* $$ $args_2$ *matchs pats* $args_2$ = *Some* env_2
by (*auto elim*: *match-list-combE*)
with *step* **obtain** $args_1$ **where** *t* = *name* $$ $args_1$ *list-all2 prelated* $args_1$ $args_2$
by (*auto elim*: *prelated.list-combE*)

then obtain env_1 **where** *matchs pats* $args_1$ = *Some* env_1 *prelated.P-env* env_1 env_2
using ‹*matchs pats* $args_2$ = -› **by** (*metis prelated.related-matchs*)
hence *fmdom* env_1 = *freess pats*
by (*auto simp*: *matchs-dom*)

obtain cs' **where** u' = *Pabs* cs'
unfolding ‹u' = -› ‹*rhs* = -› **by** *auto*

hence cs' = (λ(*pat*, *rhs*). (*pat*, *subst rhs* (*fmdrop-fset* (*frees pat*) env_2))) |‘| *cs*
using ‹u' = *subst rhs* env_2› **unfolding** ‹*rhs* = -›
by *simp*

show *?thesis*
proof (*rule step.prems*)
show *rs* $\vdash_i t \longrightarrow* t$

```
      by (rule rtranclp.rtrancl-refl)
    next
     show t ≈p u'
      unfolding ‹u' = Pabs cs'› ‹t = -›
      proof (intro prelated.defer rel-fsetI; safe?)
       show (name, rs1) |∈| rs
        by fact
      next
       show 0 < arity rs1
        using False by simp
      next
       show list-all closed args1
        using ‹closed t› unfolding ‹t = -› closed-list-comb .
      next
       fix pat rhs'
       assume (pat, rhs') |∈| irules-deferred-matches args1 rs1
       then obtain pats' rhs env
        where (pats', rhs) |∈| rs1
          and matchs (butlast pats') args1 = Some env pat = last pats' rhs'
= subst rhs env
        by auto
       with False have pats' ≠ []
        using ‹arity-compatibles rs1›
        by (metis list.size(3) arity-compatible-length)
       hence butlast pats' @ [last pats'] = pats'
        by simp

       from ‹(pats, cs) |∈| ?grp› obtain patse rhse
        where (patse, rhse) |∈| rs1 pats = butlast patse
        by (auto elim: fgroup-byE2)

       have patterns-compatible (butlast pats') pats
        unfolding ‹pats = -›
        apply (rule rev-accum-rel-butlast)
       using ‹(pats', rhs) |∈| rs1› ‹(patse, rhse) |∈| rs1› ‹(name, rs1) |∈| rs›
inner
        by (blast dest: fpairwiseD)

           interpret irules': irules C-info transform-irule-set rs by (rule
rules-transform)

       have butlast pats' = pats env = env1
         apply (rule matchs-compatible-eq)
        subgoal by fact
        subgoal
         apply (rule linears-butlastI)
         using ‹(pats', rhs) |∈| rs1› ‹(name, rs1) |∈| rs› inner by auto
        subgoal
         using ‹(pats, -) |∈| rs2› ‹(name, rs2) |∈| transform-irule-set rs›
```

```
    using irules'.inner by auto
    apply fact+
  subgoal
    apply (rule matchs-compatible-eq)
       apply fact
      apply (rule linears-butlastI)
    using ‹(pats', rhs) |∈| rs1› ‹(name, rs1) |∈| rs› inner
    apply auto []
    using ‹(pats, -) |∈| rs2› ‹(name, rs2) |∈| transform-irule-set rs›
    using irules'.inner apply auto[]
    by fact+
  done

let ?rhs-subst = λenv. subst rhs (fmdrop-fset (frees pat) env)

have fmdom env2 = freess pats
  using ‹match (- $$ -) - = Some env2›
  by (simp add: match-dom)

show fBex cs' (rel-prod (=) prelated (pat, rhs'))
  unfolding ‹rhs' = -›
  proof (rule fBexI, rule rel-prod.intros)
    have fdisjnt (freess (butlast pats')) (frees (last pats'))
      apply (subst freess-single[symmetric])
      apply (rule linears-appendD)
      apply (subst ‹butlast pats' @ [last pats'] = pats'›)
      using ‹(pats', rhs) |∈| rs1› ‹(name, rs1) |∈| rs› inner
      by auto

    show subst rhs env ≈p ?rhs-subst env2
      apply (rule prelated-subst)
       apply (rule prelated-refl)
      unfolding fmfilter-alt-defs
      apply (subst fmfilter-true)
      subgoal premises prems for x y
        using fmdomI[OF prems]
        unfolding ‹pat = -› ‹fmdom env2 = -›
        apply (subst (asm) ‹butlast pats' = pats›[symmetric])
        using ‹fdisjnt (freess (butlast pats')) (frees (last pats'))›
        by (auto simp: fdisjnt-alt-def)
      subgoal
        unfolding ‹env = -›
        by fact
      done
  next
    have (pat, rhs) |∈| cs
      unfolding ‹pat = -›
      apply (rule fgroup-byD[where a = (x, y) for x y])
        apply fact
```

```
            apply simp
            apply (intro conjI)
             apply fact
            apply (rule refl)+
           apply fact
           done
         thus (pat, ?rhs-subst env2) |∈| cs'
          unfolding ‹cs' = -› by force
       qed simp
     next
       fix pat rhs'
       assume (pat, rhs') |∈| cs'
       then obtain rhs
         where (pat, rhs) |∈| cs
          and rhs' = subst rhs (fmdrop-fset (frees pat) env2 )
         unfolding ‹cs' = -› by auto
       with ‹(pats, cs) |∈| ?grp› obtain pats'
         where (pats', rhs) |∈| rs1 pats = butlast pats' pat = last pats'
         by auto
       with False have length pats' ≠ 0
         using ‹arity-compatibles -› by (metis arity-compatible-length)
       hence pats' = pats @ [pat]
         unfolding ‹pats = -› ‹pat = -› by auto
       moreover have linears pats'
         using ‹(pats', rhs) |∈| rs1› ‹(name, rs1) |∈| -› inner by auto
       ultimately have fdisjnt (fmdom env1) (frees pat)
         unfolding ‹fmdom env1 = -›
         by (auto dest: linears-appendD)

       let ?rhs-subst = λenv. subst rhs (fmdrop-fset (frees pat) env)

     show fBex (irules-deferred-matches args1 rs1) (λe. rel-prod (=) prelated
e (pat, rhs'))
       unfolding ‹rhs' = -›
       proof (rule fBexI, rule rel-prod.intros)
         show ?rhs-subst env1 ≈p ?rhs-subst env2
           using ‹prelated.P-env env1 env2› inner
           by (auto intro: prelated-subst)
       next
         have matchs (butlast pats') args1 = Some env1
           using ‹matchs pats args1 = -› ‹pats = -› by simp
         moreover have subst rhs env1 = ?rhs-subst env1
           apply (rule arg-cong[where f = subst rhs])
           unfolding fmfilter-alt-defs
           apply (rule fmfilter-true[symmetric])
           using ‹fdisjnt (fmdom env1) -›
           by (auto simp: fdisjnt-alt-def intro: fmdomI)
         ultimately show (pat, ?rhs-subst env1) |∈| irules-deferred-matches
args1 rs1
```

```
            using ‹(pats', rhs) |∈| rs₁› ‹pat = last pats'›
            by auto
        qed simp
      qed
    qed
  qed
next
  case (fun v v' u)
  obtain w x where t = w $p x w ≈p v x ≈p u closed w
    using ‹t ≈p v $p u› ‹closed t› by cases (auto simp: closed-except-def)
  with fun obtain w' where rs ⊢i w ⟶* w' w' ≈p v'
    by blast

  show ?case
    proof (rule fun.prems)
      show rs ⊢i t ⟶* w' $p x
        unfolding ‹t = -›
        by (intro irewrite.rt-comb[unfolded app-pterm-def] rtranclp.rtrancl-refl) fact
    next
      show w' $p x ≈p v' $p u
        by (rule prelated.app) fact+
    qed
next
  case (arg u u' v)
  obtain w x where t = w $p x w ≈p v x ≈p u closed x
    using ‹t ≈p v $p u› ‹closed t› by cases (auto simp: closed-except-def)
  with arg obtain x' where rs ⊢i x ⟶* x' x' ≈p u'
    by blast

  show ?case
    proof (rule arg.prems)
      show rs ⊢i t ⟶* w $p x'
        unfolding ‹t = w $p x›
        by (intro irewrite.rt-comb[unfolded app-pterm-def] rtranclp.rtrancl-refl) fact
    next
      show w $p x' ≈p v $p u'
        by (rule prelated.app) fact+
    qed
qed

end
```

Completeness of transformation

```
lemma (in irules) transform-completeness:
  assumes rs ⊢i t ⟶ t' closed t
  shows transform-irule-set rs ⊢i t ⟶* t'
using assms proof induction
  case (step name irs' pats' rhs' t t')
```

```
 then obtain irs where irs = transform-irules irs' (name, irs) |∈| transform-irule-set
rs
    unfolding transform-irule-set-def
    by (metis fimageI id-apply map-prod-simp)
  show ?case
  proof (cases arity irs' = 0)
    case True
    hence irs = irs'
      unfolding ‹irs = -›
      unfolding transform-irules-def
      by simp
    with step have (pats', rhs') |∈| irs name, pats', rhs' ⊢ᵢ t → t'
      by blast+
    have transform-irule-set rs ⊢ᵢ t ⟶* t'
      apply (rule r-into-rtranclp)
      apply rule
      by fact+

    show ?thesis by fact
  next
    let ?f = λ(pats, rhs). (butlast pats, last pats, rhs)
    let ?grp = fgroup-by ?f irs'
    note closed-except-def [simp add]
    case False
    then have irs = map-prod id Pabs |`| ?grp
      unfolding ‹irs = -›
      unfolding transform-irules-def
      by simp
    with False have irs = transform-irules irs'
      unfolding transform-irules-def
      by simp
    obtain pat pats where pat = last pats' pats = butlast pats'
      by blast
    from step False have length pats' ≠ 0
      using arity-compatible-length inner
      by (smt (verit, ccfv-threshold) case-prodD)
    then have pats' = pats @ [pat]
      unfolding ‹pat = -› ‹pats = -›
      by simp
    from step have linears pats'
      using inner by auto
    then have fdisjnt (freess pats) (frees pat)
      unfolding ‹pats' = -›
      using linears-appendD(3) freess-single
      by force

    from step obtain cs where (pats, cs) |∈| ?grp
      unfolding ‹pats = -›
      by (metis (no-types, lifting) fgroup-by-complete fst-conv prod.simps(2))
```

with *step* **have** (*pat*, *rhs'*) |∈| *cs*
unfolding ‹*pat* = -› ‹*pats* = -›
by (*meson fgroup-byD old.prod.case*)
have (*pats*, *Pabs cs*) |∈| *irs*
using ‹*irs* = *map-prod id Pabs* |`| *?grp*› ‹(*pats*, *cs*) |∈| -›
by (*metis* (*no-types*, *lifting*) *eq-snd-iff fst-conv fst-map-prod id-def rev-fimage-eqI snd-map-prod*)
from *step* **obtain** *env'* **where** *match* (*name* $$ *pats'*) *t* = *Some env' subst rhs' env'* = *t'*
using *irewrite-step-def* **by** *auto*
have *name* $$ *pats'* = (*name* $$ *pats*) $ *pat*
unfolding ‹*pats'* = -›
by (*simp add*: *app-term-def*)
then obtain t_0 t_1 env_0 env_1 **where** *t* = t_0 $\$_p$ t_1 *match* (*name* $$ *pats*) t_0 = *Some* env_0 *match pat* t_1 = *Some* env_1 *env'* = env_0 $+\!\!+_f$ env_1
using *match-appE-split*[*OF* ‹*match* (*name* $$ *pats'*) - = -›[*unfolded* ‹*name* $$ *pats'* = -›], *unfolded app-pterm-def*]
by *blast*
with *step* **have** *closed* t_0 *closed* t_1
by *auto*
then have *closed-env* env_0 *closed-env* env_1
using *match-vars*[*OF* ‹*match* - t_0 = -›] *match-vars*[*OF* ‹*match* - t_1 = -›]
unfolding *closed-except-def*
by *auto*
obtain $t_0{}'$ **where** *subst* (*Pabs cs*) env_0 = $t_0{}'$
by *blast*
then obtain *cs'* **where** $t_0{}'$ = *Pabs cs' cs'* = ((λ(*pat*, *rhs*). (*pat*, *subst rhs* (*fmdrop-fset* (*frees pat*) env_0))) |`| *cs*)
using *subst-pterm.simps*(*3*) **by** *blast*
obtain *rhs* **where** *subst rhs'* (*fmdrop-fset* (*frees pat*) env_0) = *rhs*
by *blast*
then have (*pat*, *rhs*) |∈| *cs'*
unfolding ‹*cs'* = -›
using ‹- |∈| *cs*›
by (*metis* (*mono-tags*, *lifting*) *old.prod.case rev-fimage-eqI*)
have env_0 $+\!\!+_f$ env_1 = (*fmdrop-fset* (*frees pat*) env_0) $+\!\!+_f$ env_1
apply (*subst fmadd-drop-left-dom*[*symmetric*])
using ‹*match pat* - = -› *match-dom*
by *metis*
have *fdisjnt* (*fmdom* env_0) (*fmdom* env_1)
using *match-dom*
using ‹*match pat* - = -› ‹*match* (*name* $$ *pats*) - = -›
using ‹*fdisjnt* - -›
unfolding *fdisjnt-alt-def*
by (*metis matchs-dom match-list-combE*)
have *subst rhs* env_1 = *t'*
unfolding ‹- = *rhs*›[*symmetric*]
unfolding ‹- = *t'*›[*symmetric*]
unfolding ‹*env'* = -›

unfolding ‹env_0 ++$_f$ - = -›
apply (*subst subst-indep'*)
using ‹*closed-env* env_0›
apply *blast*
using ‹*fdisjnt* (*fmdom* -) -›
unfolding *fdisjnt-alt-def*
by *auto*

show *?thesis*
unfolding ‹t = -›
apply *rule*
apply (*rule r-into-rtranclp*)
apply (*rule irewrite.intros*(*3*))
apply *rule*
apply *fact*+
apply (*rule irewrite-stepI*)
apply *fact*+
unfolding ‹t_0' = -›
apply *rule*
apply *fact*
using ‹*match pat* t_1 = -› ‹*subst rhs* - = -›
by *force*
qed
qed (*auto intro*: *irewrite.rt-comb*[*unfolded app-pterm-def*] *intro*!: *irewrite.intros simp*: *closed-except-def*)

Computability

export-code
compile transform-irules
checking *Scala SML*

end

3.3.2 Pure pattern matching rule sets

theory *Rewriting-Pterm*
imports *Rewriting-Pterm-Elim*
begin

type-synonym *prule* = *name* × *pterm*

primrec *prule* :: *prule* ⇒ *bool* **where**
prule (-, *rhs*) ⟷ *wellformed rhs* ∧ *closed rhs* ∧ *is-abs rhs*

lemma *pruleI*[*intro*!]: *wellformed rhs* ⟹ *closed rhs* ⟹ *is-abs rhs* ⟹ *prule* (*name*, *rhs*)
by *simp*

locale *prules* = *constants C-info fst* |‘| *rs* **for** *C-info* **and** *rs* :: *prule fset* +

assumes *all-rules*: *fBall rs prule*
assumes *fmap*: *is-fmap rs*
assumes *not-shadows*: *fBall rs* (λ(-, *rhs*). $\neg$ *shadows-consts rhs*)
assumes *welldefined-rs*: *fBall rs* (λ(-, *rhs*). *welldefined rhs*)

Rewriting

inductive *prewrite* :: *prule fset* $\Rightarrow$ *pterm* $\Rightarrow$ *pterm* $\Rightarrow$ *bool* (‹-/ $\vdash_p$/ - $\longrightarrow$/ -› [*50*,*0*,*50*] *50*) **for** *rs* **where**
step: (*name*, *rhs*) |$\in$| *rs* $\Longrightarrow$ *rs* $\vdash_p$ *Pconst name* $\longrightarrow$ *rhs* |
beta: *c* |$\in$| *cs* $\Longrightarrow$ *c* $\vdash$ *t* $\rightarrow$ *t'* $\Longrightarrow$ *rs* $\vdash_p$ *Pabs cs* $\$_p$ *t* $\longrightarrow$ *t'* |
fun: *rs* $\vdash_p$ *t* $\longrightarrow$ *t'* $\Longrightarrow$ *rs* $\vdash_p$ *t* $\$_p$ *u* $\longrightarrow$ *t'* $\$_p$ *u* |
arg: *rs* $\vdash_p$ *u* $\longrightarrow$ *u'* $\Longrightarrow$ *rs* $\vdash_p$ *t* $\$_p$ *u* $\longrightarrow$ *t* $\$_p$ *u'*

global-interpretation *prewrite*: *rewriting prewrite rs* **for** *rs*
by *standard* (*auto intro*: *prewrite.intros simp*: *app-pterm-def*)+

abbreviation *prewrite-rt* :: *prule fset* $\Rightarrow$ *pterm* $\Rightarrow$ *pterm* $\Rightarrow$ *bool* (‹-/ $\vdash_p$/ - $\longrightarrow$*/ -› [*50*,*0*,*50*] *50*) **where**
prewrite-rt rs $\equiv$ (*prewrite rs*)**

Translation from *irule-set* to *prule fset*

definition *finished* :: *irule-set* $\Rightarrow$ *bool* **where**
finished rs = *fBall rs* (λ(-, *irs*). *arity irs* = *0*)

definition *translate-rhs* :: *irules* $\Rightarrow$ *pterm* **where**
translate-rhs = *snd* $\circ$ *fthe-elem*

definition *compile* :: *irule-set* $\Rightarrow$ *prule fset* **where**
compile = *fimage* (*map-prod id translate-rhs*)

lemma *compile-heads*: *fst* |‘| *compile rs* = *fst* |‘| *rs*
unfolding *compile-def* **by** *simp*

Correctness of translation

lemma *arity-zero-shape*:
assumes *arity-compatibles rs arity rs* = *0 is-fmap rs rs* $\neq$ {||}
obtains *t* **where** *rs* = {| ([], *t*) |}
proof −
from *assms* **obtain** *ppats prhs* **where** (*ppats*, *prhs*) |$\in$| *rs*
by *fast*

moreover {
fix *pats rhs*
assume (*pats*, *rhs*) |$\in$| *rs*
with *assms* **have** *length pats* = *0*
by (*metis arity-compatible-length*)
hence *pats* = []

```
    by simp
  }
  note all = this

  ultimately have proto: ([], prhs) |∈| rs by auto

  have fBall rs (λ(pats, rhs). pats = [] ∧ rhs = prhs)
    proof safe
      fix pats rhs
      assume cur: (pats, rhs) |∈| rs
      with all show pats = [] .
      with cur have ([], rhs) |∈| rs by auto

      with proto show rhs = prhs
        using assms by (auto dest: is-fmapD)
    qed
  hence fBall rs (λr. r = ([], prhs))
    by blast
  with assms have rs = {| ([], prhs) |}
    by (simp add: singleton-fset-is)
  thus thesis
    by (rule that)
qed

lemma (in irules) compile-rules:
  assumes finished rs
  shows prules C-info (compile rs)
proof
  show is-fmap (compile rs)
    using fmap
    unfolding compile-def map-prod-def id-apply
    by (rule is-fmap-image)
next
  show fdisjnt (fst |`| compile rs) C
    unfolding compile-def
    using disjnt by simp
next
  have
    fBall (compile rs) prule
    fBall (compile rs) (λ(-, rhs). ¬ shadows-consts rhs)
    fBall (compile rs) (λ(-, rhs). welldefined rhs)
    proof (safe del: fsubsetI)
      fix name rhs
      assume (name, rhs) |∈| compile rs
      then obtain irs where (name, irs) |∈| rs rhs = translate-rhs irs
        unfolding compile-def by force
      hence is-fmap irs irs ≠ {||} arity irs = 0
        using assms inner unfolding finished-def by blast+
      moreover have arity-compatibles irs
```

```
      using ‹(name, irs) |∈| rs› inner
      by (metis (no-types, lifting) case-prodD)
    ultimately obtain u where irs = {| ([], u) |}
      by (metis arity-zero-shape)
    hence rhs = u and u: ([], u) |∈| irs
      unfolding ‹rhs = -› translate-rhs-def by simp+
    hence abs-ish [] u
      using inner ‹(name, irs) |∈| rs› by fastforce
    thus is-abs rhs
      unfolding abs-ish-def ‹rhs = u› by simp

    show wellformed rhs
      using u ‹(name, irs) |∈| rs› inner unfolding ‹rhs = u›
      by fastforce

    have closed-except u {||}
      using u inner ‹(name, irs) |∈| rs›
      by fastforce
    thus closed rhs
      unfolding ‹rhs = u› .

    {
      assume shadows-consts rhs
      hence shadows-consts u
        unfolding compile-def ‹rhs = u› by simp
      moreover have ¬ shadows-consts u
        using inner ‹([], u) |∈| irs› ‹(name, irs) |∈| rs› by fastforce
      ultimately show False by blast
    }

    have welldefined u
      using fbspec[OF inner ‹(name, irs) |∈| rs›, simplified] ‹([], u) |∈| irs›
      by blast
    thus welldefined rhs
      unfolding ‹rhs = u› compile-def
      by simp
  qed
 thus
  fBall (compile rs) prule
   fBall (compile rs) (λ(-, rhs). ¬ pre-constants.shadows-consts C-info (fst |‘| compile rs) rhs)
  fBall (compile rs) (λ(-, rhs). pre-constants.welldefined C-info (fst |‘| compile rs) rhs)
   unfolding compile-heads by auto
next
 show distinct all-constructors
  by (fact distinct-ctr)
qed
```

```
theorem (in irules) compile-correct:
  assumes compile rs ⊢p t ⟶ t′ finished rs
  shows rs ⊢i t ⟶ t′
using assms(1) proof induction
  case (step name rhs)
  then obtain irs where rhs = translate-rhs irs (name, irs) |∈| rs
    unfolding compile-def by force
  hence arity-compatibles irs
    using inner
    by (metis (no-types, lifting) case-prodD)

  have is-fmap irs irs ≠ {||} arity irs = 0
    using assms inner ‹(name, irs) |∈| rs› unfolding finished-def by blast+
  then obtain u where irs = {| ([], u) |}
    using ‹arity-compatibles irs›
    by (metis arity-zero-shape)

  show ?case
    unfolding ‹rhs = -›
    apply (rule irewrite.step)
      apply fact
    unfolding ‹irs = -› translate-rhs-def irewrite-step-def
    by (auto simp: const-term-def)
qed (auto intro: irewrite.intros)

theorem (in irules) compile-complete:
  assumes rs ⊢i t ⟶ t′ finished rs
  shows compile rs ⊢p t ⟶ t′
using assms(1) proof induction
  case (step name irs params rhs t t′)
  hence arity-compatibles irs
    using inner
    by (metis (no-types, lifting) case-prodD)

  have is-fmap irs irs ≠ {||} arity irs = 0
    using assms inner step unfolding finished-def by blast+
  then obtain u where irs = {| ([], u) |}
    using ‹arity-compatibles irs›
    by (metis arity-zero-shape)
  with step have name, [], u ⊢i t → t′
    by simp
  hence t = Pconst name
    unfolding irewrite-step-def
    by (cases t) (auto split: if-splits simp: const-term-def)
  hence t′ = u
    using ‹name, [], u ⊢i t → t′›
    unfolding irewrite-step-def
    by (cases t) (auto split: if-splits simp: const-term-def)
```

have (*name*, t') |∈| *compile rs*
unfolding *compile-def*
proof
show (*name*, t') = *map-prod id translate-rhs* (*name*, *irs*)
using ‹*irs* = -› ‹t' = *u*›
by (*simp add*: *split-beta translate-rhs-def*)
qed *fact*
thus *?case*
unfolding ‹*t* = -›
by (*rule prewrite.step*)
qed (*auto intro*: *prewrite.intros*)

export-code
compile finished
checking *Scala*

end

3.4 Sequential pattern matching

theory *Rewriting-Sterm*
imports *Rewriting-Pterm*
begin

type-synonym *srule* = *name* × *sterm*

abbreviation *closed-srules* :: *srule list* ⇒ *bool* **where**
closed-srules ≡ *list-all* (*closed* ∘ *snd*)

primrec *srule* :: *srule* ⇒ *bool* **where**
srule (-, *rhs*) ⟷ *wellformed rhs* ∧ *closed rhs* ∧ *is-abs rhs*

lemma *sruleI*[*intro*!]: *wellformed rhs* ⟹ *closed rhs* ⟹ *is-abs rhs* ⟹ *srule* (*name*, *rhs*)
by *simp*

locale *srules* = *constants C-info fst* |‘| *fset-of-list rs* **for** *C-info* **and** *rs* :: *srule list* +
assumes *all-rules*: *list-all srule rs*
assumes *distinct*: *distinct* (*map fst rs*)
assumes *not-shadows*: *list-all* (λ(-, *rhs*). ¬ *shadows-consts rhs*) *rs*
assumes *swelldefined-rs*: *list-all* (λ(-, *rhs*). *welldefined rhs*) *rs*
begin

lemma *map*: *is-map* (*set rs*)
using *distinct* **by** (*rule distinct-is-map*)

lemma *clausesE*:
assumes (*name*, *rhs*) ∈ *set rs*

obtains *cs* **where** *rhs* = *Sabs cs*
proof −
from *assms* **have** *is-abs rhs*
using *all-rules* **unfolding** *list-all-iff* **by** *auto*
then obtain *cs* **where** *rhs* = *Sabs cs*
by (*cases rhs*) (*auto simp*: *is-abs-def term-cases-def*)
with *that* **show** *thesis* **.**
qed

end

Rewriting

inductive *srewrite-step* **where**
cons-match: *srewrite-step* ((*name*, *rhs*) # *rest*) *name rhs* |
cons-nomatch: *name* ≠ *name′* ⟹ *srewrite-step rs name rhs* ⟹ *srewrite-step* ((*name′*, *rhs′*) # *rs*) *name rhs*

lemma *srewrite-stepI0*:
assumes (*name*, *rhs*) ∈ *set rs is-map* (*set rs*)
shows *srewrite-step rs name rhs*
using *assms* **proof** (*induction rs*)
case (*Cons r rs*)
then obtain *name′ rhs′* **where** *r* = (*name′*, *rhs′*) **by** *force*
show *?case*
proof (*cases name* = *name′*)
case *False*
show *?thesis*
unfolding ‹*r* = -›
apply (*rule srewrite-step.cons-nomatch*)
subgoal by *fact*
apply (*rule Cons*)
using *False Cons*(*2*) ‹*r* = -› **apply** *force*
using *Cons*(*3*) **unfolding** *is-map-def* **by** *auto*
next
case *True*
have *rhs* = *rhs′*
apply (*rule is-mapD*)
apply *fact*
unfolding ‹*r* = -›
using *Cons*(*2*) ‹*r* = -› **apply** *simp*
using *True* **apply** *simp*
done
show *?thesis*
unfolding ‹*r* = -› ‹*name* = -› ‹*rhs* = -›
by (*rule srewrite-step.cons-match*)
qed
qed *auto*

lemma (**in** *srules*) *srewrite-stepI*: (*name*, *rhs*) ∈ *set rs* ⟹ *srewrite-step rs name rhs*
using *map*
by (*metis srewrite-stepI0*)

hide-fact *srewrite-stepI0*

inductive *srewrite* :: *srule list* ⇒ *sterm* ⇒ *sterm* ⇒ *bool* (‹-/ ⊢$_s$/ - ⟶/ -› [*50*,*0*,*50*] *50*) **for** *rs* **where**
step: *srewrite-step rs name rhs* ⟹ *rs* ⊢$_s$ *Sconst name* ⟶ *rhs* |
beta: *rewrite-first cs t t'* ⟹ *rs* ⊢$_s$ *Sabs cs* $_s *t* ⟶ *t'* |
fun: *rs* ⊢$_s$ *t* ⟶ *t'* ⟹ *rs* ⊢$_s$ *t* $_s *u* ⟶ *t'* $_s *u* |
arg: *rs* ⊢$_s$ *u* ⟶ *u'* ⟹ *rs* ⊢$_s$ *t* $_s *u* ⟶ *t* $_s *u'*

code-pred *srewrite* **.**

abbreviation *srewrite-rt* :: *srule list* ⇒ *sterm* ⇒ *sterm* ⇒ *bool* (‹-/ ⊢$_s$/ - ⟶*/ -› [*50*,*0*,*50*] *50*) **where**
srewrite-rt rs ≡ (*srewrite rs*)**

global-interpretation *srewrite*: *rewriting srewrite rs* **for** *rs*
by *standard* (*auto intro*: *srewrite.intros simp*: *app-sterm-def*)+

code-pred (*modes*: *i* ⇒ *i* ⇒ *o* ⇒ *bool*) *srewrite-step* **.**
code-pred (*modes*: *i* ⇒ *i* ⇒ *o* ⇒ *bool*) *srewrite* **.**

Translation from *pterm* to *sterm*

In principle, any function of type ('*a* × '*b*) *fset* ⇒ ('*a* × '*b*) *list* that orders by keys would do here. However, For simplicity's sake, we choose a fixed one (*ordered-fmap*) here.

primrec *pterm-to-sterm* :: *pterm* ⇒ *sterm* **where**
pterm-to-sterm (*Pconst name*) = *Sconst name* |
pterm-to-sterm (*Pvar name*) = *Svar name* |
pterm-to-sterm (*t* $_p *u*) = *pterm-to-sterm t* $_s *pterm-to-sterm u* |
pterm-to-sterm (*Pabs cs*) = *Sabs* (*ordered-fmap* (*map-prod id pterm-to-sterm* |‘| *cs*))

lemma *pterm-to-sterm*:
  assumes *no-abs t*
  shows *pterm-to-sterm t* = *convert-term t*
using *assms* **proof** *induction*
  case (*free name*)
  show *?case*
    apply *simp*
    apply (*simp add*: *free-sterm-def free-pterm-def*)
    done
next
  case (*const name*)

show *?case*
apply *simp*
apply (*simp add*: *const-sterm-def const-pterm-def*)
done
next
case (*app* t_1 t_2)
then show *?case*
apply *simp*
apply (*simp add*: *app-sterm-def app-pterm-def*)
done
qed

sterm-to-pterm has to be defined, for technical reasons, in *CakeML-Codegen.Pterm*.

lemma *pterm-to-sterm-wellformed*:
assumes *wellformed t*
shows *wellformed* (*pterm-to-sterm t*)
using *assms* **proof** (*induction t rule*: *pterm-induct*)
case (*Pabs cs*)
show *?case*
apply *simp*
unfolding *map-prod-def id-apply*
apply (*intro conjI*)
subgoal
apply (*subst list-all-iff-fset*)
apply (*subst ordered-fmap-set-eq*)
apply (*rule is-fmap-image*)
using *Pabs* **apply** *simp*
apply (*rule fBallI*)
apply (*erule fimageE*)
apply *auto*[]
using *Pabs*(*2*) **apply** *auto*[]
apply (*rule Pabs*)
using *Pabs*(*2*) **by** *auto*
subgoal
apply (*rule ordered-fmap-distinct*)
apply (*rule is-fmap-image*)
using *Pabs*(*2*) **by** *simp*
subgoal
apply (*subgoal-tac cs* $\neq$ {||})
including *fset.lifting* **apply** *transfer*
unfolding *ordered-map-def*
using *Pabs*(*2*) **by** *auto*
done
qed *auto*

lemma *pterm-to-sterm-sterm-to-pterm*:
assumes *wellformed t*
shows *sterm-to-pterm* (*pterm-to-sterm t*) = *t*
using *assms* **proof** (*induction t*)

case (*Pabs cs*)
note *fset-of-list-map*[*simp del*]
show *?case*
apply *simp*
unfolding *map-prod-def id-apply*
apply (*subst ordered-fmap-image*)
subgoal
apply (*rule is-fmap-image*)
using *Pabs* **by** *simp*
apply (*subst ordered-fmap-set-eq*)
subgoal
apply (*rule is-fmap-image*)
apply (*rule is-fmap-image*)
using *Pabs* **by** *simp*
subgoal
apply (*subst fset.map-comp*)
apply (*subst map-prod-def*[*symmetric*])+
unfolding *o-def*
apply (*subst prod.map-comp*)
apply (*subst id-def*[*symmetric*])+
apply *simp*
apply (*subst map-prod-def*)
unfolding *id-def*
apply (*rule fset-map-snd-id*)
apply *simp*
apply (*rule Pabs*)
using *Pabs*(*2*) **by** (*auto simp*: *snds.simps*)
done
qed *auto*

corollary *pterm-to-sterm-frees*: *wellformed t* $\Longrightarrow$ *frees* (*pterm-to-sterm t*) = *frees t*
by (*metis pterm-to-sterm-sterm-to-pterm sterm-to-pterm-frees*)

corollary *pterm-to-sterm-closed*:
closed-except t S $\Longrightarrow$ *wellformed t* $\Longrightarrow$ *closed-except* (*pterm-to-sterm t*) *S*
unfolding *closed-except-def*
by (*simp add*: *pterm-to-sterm-frees*)

corollary *pterm-to-sterm-consts*: *wellformed t* $\Longrightarrow$ *consts* (*pterm-to-sterm t*) = *consts t*
by (*metis pterm-to-sterm-sterm-to-pterm sterm-to-pterm-consts*)

corollary (**in** *constants*) *pterm-to-sterm-shadows*:
wellformed t $\Longrightarrow$ *shadows-consts t* $\longleftrightarrow$ *shadows-consts* (*pterm-to-sterm t*)
unfolding *shadows-consts-def*
by (*metis pterm-to-sterm-sterm-to-pterm sterm-to-pterm-all-frees*)

definition *compile* :: *prule fset* $\Rightarrow$ *srule list* **where**

compile rs = *ordered-fmap* (*map-prod id pterm-to-sterm* |‘| *rs*)

Correctness of translation

context *prules* **begin**

lemma *compile-heads*: *fst* |‘| *fset-of-list* (*compile rs*) = *fst* |‘| *rs*
 unfolding *compile-def*
 apply (*subst ordered-fmap-set-eq*)
  apply (*subst map-prod-def*, *subst id-apply*)
  apply (*rule is-fmap-image*)
  apply (*rule fmap*)
 apply *simp*
 done

lemma *compile-rules*: *srules C-info* (*compile rs*)
proof
 show *list-all srule* (*compile rs*)
  using *fmap all-rules*
  unfolding *compile-def list-all-iff*
  including *fset.lifting*
  apply *transfer*
  apply (*subst ordered-map-set-eq*)
  subgoal by *simp*
  subgoal
   unfolding *map-prod-def id-def*
   by (*erule is-map-image*)
  subgoal
   apply (*rule ballI*)
   apply *safe*
   subgoal
    apply (*rule pterm-to-sterm-wellformed*)
    apply *fastforce*
    done
   subgoal
    apply (*rule pterm-to-sterm-closed*)
     apply *fastforce*
    apply *fastforce*
    done
   subgoal for - - *a b*
    apply (*erule ballE*[**where** *x* = (*a*, *b*)])
     apply (*cases b*; *auto*)
       apply (*auto simp*: *is-abs-def term-cases-def*)
    done
   done
  done
next
 show *distinct* (*map fst* (*compile rs*))
  unfolding *compile-def*

```
    apply (rule ordered-fmap-distinct)
    unfolding map-prod-def id-def
    apply (rule is-fmap-image)
    apply (rule fmap)
    done
next
  have list-all (λ(-, rhs). welldefined rhs) (compile rs)
    unfolding compile-def
    apply (subst ordered-fmap-list-all)
    subgoal
      apply (subst map-prod-def)
      apply (subst id-apply)
      apply (rule is-fmap-image)
      by (fact fmap)
    apply simp
    apply (rule fBallI)
    subgoal for x
      apply (cases x, simp)
      apply (subst pterm-to-sterm-consts)
      using all-rules apply force
      using welldefined-rs by force
    done
   thus list-all (λ(-, rhs). consts rhs |⊆| pre-constants.all-consts C-info (fst |‘|
fset-of-list (compile rs))) (compile rs)
    by (simp add: compile-heads)
next
  interpret c: constants - fset-of-list (map fst (compile rs))
    by (simp add: constants-axioms compile-heads)
  have all-consts: c.all-consts = all-consts
    by (simp add: compile-heads)

  note fset-of-list-map[simp del]
  have list-all (λ(-, rhs). ¬ shadows-consts rhs) (compile rs)
    unfolding compile-def
    apply (subst list-all-iff-fset)
    apply (subst ordered-fmap-set-eq)
     apply (subst map-prod-def)
    unfolding id-apply
     apply (rule is-fmap-image)
     apply (fact fmap)
    apply simp
    apply (rule fBall-pred-weaken[where P = λ(-, rhs). ¬ shadows-consts rhs])
    subgoal for x
      apply (cases x, simp)
      apply (subst (asm) pterm-to-sterm-shadows)
      using all-rules apply force
      by simp
    subgoal
      using not-shadows by force
```

done
thus *list-all* (λ(-, *rhs*). ¬ *pre-constants.shadows-consts C-info* (*fst* |‘| *fset-of-list* (*compile rs*)) *rhs*) (*compile rs*)
unfolding *compile-heads all-consts* **.**
next
show *fdisjnt* (*fst* |‘| *fset-of-list* (*compile rs*)) *C*
unfolding *compile-def*
apply (*subst fset-of-list-map*[*symmetric*])
apply (*subst ordered-fmap-keys*)
apply (*subst map-prod-def*)
apply (*subst id-apply*)
apply (*rule is-fmap-image*)
using *fmap disjnt* **by** *auto*
next
show *distinct all-constructors*
by (*fact distinct-ctr*)
qed

sublocale *prules-as-srules*: *srules C-info compile rs*
by (*fact compile-rules*)

end

global-interpretation *srelated*: *term-struct-rel-strong* (λ*p s*. *p* = *sterm-to-pterm s*)
proof (*standard*, *goal-cases*)
case (*5 name t*)
then show *?case* **by** (*cases t*) (*auto simp*: *const-sterm-def const-pterm-def split*: *option.splits*)
next
case (*6* u_1 u_2 *t*)
then show *?case* **by** (*cases t*) (*auto simp*: *app-sterm-def app-pterm-def split*: *option.splits*)
qed (*auto simp*: *const-sterm-def const-pterm-def app-sterm-def app-pterm-def*)

lemma *srelated-subst*:
assumes *srelated.P-env penv senv*
shows *subst* (*sterm-to-pterm t*) *penv* = *sterm-to-pterm* (*subst t senv*)
using *assms*
proof (*induction t arbitrary*: *penv senv*)
case (*Svar name*)
thus *?case*
by (*cases rule*: *fmrel-cases*[**where** *x* = *name*]) *auto*
next
case (*Sabs cs*)
show *?case*
apply *simp*
including *fset.lifting*
apply (*transfer′ fixing*: *cs penv senv*)

```
    unfolding set-map image-comp
    apply (rule image-cong[OF refl])
    unfolding comp-apply
    apply (case-tac x)
    apply hypsubst-thin
    apply simp
    apply (rule Sabs)
     apply assumption
    apply (simp add: snds.simps)
    apply rule
    apply (rule Sabs)
    done
qed auto

context begin

private lemma srewrite-step-non-empty: srewrite-step rs' name rhs ⟹ rs' ≠ []
by (induct rule: srewrite-step.induct) auto

private lemma compile-consE:
  assumes (name, rhs') # rest = compile rs is-fmap rs
  obtains rhs where rhs' = pterm-to-sterm rhs (name, rhs) |∈| rs rest = compile
(rs − {| (name, rhs) |})
proof −
  from assms have ordered-fmap (map-prod id pterm-to-sterm |‘| rs) = (name,
rhs') # rest
    unfolding compile-def
    by simp
  hence (name, rhs') ∈ set (ordered-fmap (map-prod id pterm-to-sterm |‘| rs))
    by simp

  have (name, rhs') |∈| map-prod id pterm-to-sterm |‘| rs
    apply (rule ordered-fmap-sound)
    subgoal
      unfolding map-prod-def id-apply
      apply (rule is-fmap-image)
      apply fact
      done
    subgoal by fact
    done
  then obtain rhs where rhs' = pterm-to-sterm rhs (name, rhs) |∈| rs
    by auto

  have rest = compile (rs − {| (name, rhs) |})
    unfolding compile-def
    apply (subst inj-on-fimage-set-diff[where C = rs])
    subgoal
      apply (rule inj-onI)
      apply safe
```

```
      apply auto
     using ‹is-fmap rs› by (blast dest: is-fmapD)
   subgoal by simp
   subgoal using ‹(name, rhs) |∈| rs› by simp
   subgoal
     apply simp
     apply (subst ordered-fmap-remove)
      apply (subst map-prod-def)
     unfolding id-apply
      apply (rule is-fmap-image)
      apply fact
     using ‹(name, rhs) |∈| rs› apply force
     apply (subst ‹rhs' = pterm-to-sterm rhs›[symmetric])
     apply (subst ‹ordered-fmap - = -›[unfolded id-def])
     by simp
   done

  show thesis
   by (rule that) fact+
qed

private lemma compile-correct-step:
  assumes srewrite-step (compile rs) name rhs is-fmap rs fBall rs prule
  shows (name, sterm-to-pterm rhs) |∈| rs
using assms proof (induction compile rs name rhs arbitrary: rs)
  case (cons-match name rhs' rest)
  then obtain rhs where rhs' = pterm-to-sterm rhs (name, rhs) |∈| rs
   by (auto elim: compile-consE)

  show ?case
   unfolding ‹rhs' = -›
   apply (subst pterm-to-sterm-sterm-to-pterm)
   using fbspec[OF ‹fBall rs prule› ‹(name, rhs) |∈| rs›] apply force
   by fact
next
  case (cons-nomatch name name1 rest rhs rhs1')
  then obtain rhs1 where rhs1' = pterm-to-sterm rhs1 (name1, rhs1) |∈| rs rest
= compile (rs − {| (name1, rhs1) |})
   by (auto elim: compile-consE)

  let ?rs' = rs − {| (name1, rhs1) |}
  have (name, sterm-to-pterm rhs) |∈| ?rs'
   proof (intro cons-nomatch)
     show rest = compile ?rs'
      by fact

     show is-fmap (rs |−| {|(name1, rhs1)|})
      using ‹is-fmap rs›
      by (rule is-fmap-subset) auto
```

```
      show fBall ?rs' prule
        using cons-nomatch by blast
    qed
  thus ?case
    by simp
qed

lemma compile-correct0:
  assumes compile rs ⊢s u ⟶ u' prules C rs
  shows rs ⊢p sterm-to-pterm u ⟶ sterm-to-pterm u'
using assms proof induction
  case (beta cs t t')
  then obtain pat rhs env where (pat, rhs) ∈ set cs match pat t = Some env t'
= subst rhs env
    by (auto elim: rewrite-firstE)

 then obtain env' where match pat (sterm-to-pterm t) = Some env' srelated.P-env
env' env
    by (metis option.distinct(1) option.inject option.rel-cases srelated.match-rel)
  hence subst (sterm-to-pterm rhs) env' = sterm-to-pterm (subst rhs env)
    by (simp add: srelated-subst)

  let ?rhs' = sterm-to-pterm rhs

  have (pat, ?rhs') |∈| fset-of-list (map (map-prod id sterm-to-pterm) cs)
    using ‹(pat, rhs) ∈ set cs›
    including fset.lifting
    by transfer' force

  note fset-of-list-map[simp del]
  show ?case
    apply simp
    apply (rule prewrite.intros)
     apply fact
    unfolding rewrite-step.simps
    apply (subst map-option-eq-Some)
    apply (intro exI conjI)
     apply fact
    unfolding ‹t' = -›
    by fact
next
  case (step name rhs)
  hence (name, sterm-to-pterm rhs) |∈| rs
    unfolding prules-def prules-axioms-def
    by (metis compile-correct-step)
  thus ?case
    by (auto intro: prewrite.intros)
qed (auto intro: prewrite.intros)
```

end

lemma (**in** *prules*) *compile-correct*:
  assumes *compile rs* $\vdash_s$ *u* $\longrightarrow$ *u'*
  shows *rs* $\vdash_p$ *sterm-to-pterm u* $\longrightarrow$ *sterm-to-pterm u'*
by (*rule compile-correct0*) (*fact* | *standard*)+

hide-fact *compile-correct0*

Completeness of translation

global-interpretation *srelated'*: *term-struct-rel-strong* (λ*p s*. *pterm-to-sterm p* = *s*)
proof (*standard*, *goal-cases*)
  case (*1 t name*)
  then show *?case* **by** (*cases t*) (*auto simp*: *const-sterm-def const-pterm-def split*: *option.splits*)
next
  case (*3 t* u_1 u_2)
  then show *?case* **by** (*cases t*) (*auto simp*: *app-sterm-def app-pterm-def split*: *option.splits*)
qed (*auto simp*: *const-sterm-def const-pterm-def app-sterm-def app-pterm-def*)

corollary *srelated-env-unique*:
  srelated'.P-env penv senv $\Longrightarrow$ *srelated'.P-env penv senv'* $\Longrightarrow$ *senv* = *senv'*
apply (*subst* (*asm*) *fmrel-iff*)+
apply (*subst* (*asm*) *option.rel-sel*)+
apply (*rule fmap-ext*)
by (*metis option.exhaust-sel*)

lemma *srelated-subst'*:
  assumes *srelated'.P-env penv senv wellformed t*
  shows *pterm-to-sterm* (*subst t penv*) = *subst* (*pterm-to-sterm t*) *senv*
using *assms* **proof** (*induction t arbitrary*: *penv senv*)
  case (*Pvar name*)
  thus *?case*
    by (*cases rule*: *fmrel-cases*[**where** *x* = *name*]) *auto*
next
  case (*Pabs cs*)
  hence *is-fmap cs*
    by *force*

  show *?case*
    apply *simp*
    unfolding *map-prod-def id-apply*
    apply (*subst ordered-fmap-image*[*symmetric*])
     apply *fact*
    apply (*subst fset.map-comp*[*symmetric*])

```
    apply (subst ordered-fmap-image[symmetric])
    subgoal by (rule is-fmap-image) fact
    apply (subst ordered-fmap-image[symmetric])
     apply fact
    apply auto
    apply (drule ordered-fmap-sound[OF ‹is-fmap cs›])
    subgoal for pat rhs
      apply (rule Pabs)
        apply assumption
       apply auto
      using Pabs by force+
    done
qed auto

lemma srelated-find-match:
  assumes find-match cs t = Some (penv, pat, rhs) srelated'.P-env penv senv
  shows find-match (map (map-prod id pterm-to-sterm) cs) (pterm-to-sterm t) =
Some (senv, pat, pterm-to-sterm rhs)
proof −
  let ?cs' = map (map-prod id pterm-to-sterm) cs
  let ?t' = pterm-to-sterm t
  have ∗: list-all2 (rel-prod (=) (λp s. pterm-to-sterm p = s)) cs ?cs'
    unfolding list.rel-map
    by (auto intro: list.rel-refl)

  obtain senv0
      where find-match ?cs' ?t' = Some (senv0, pat, pterm-to-sterm rhs) srelated'.P-env penv senv0
    using srelated'.find-match-rel[OF ∗ refl, where t = t, unfolded assms]
    unfolding option-rel-Some1 rel-prod-conv
    by auto
  with assms have senv = senv0
    by (metis srelated-env-unique)
  show ?thesis
    unfolding ‹senv = -› by fact
qed

lemma (in prules) compile-complete:
  assumes rs ⊢p t ⟶ t' wellformed t
  shows compile rs ⊢s pterm-to-sterm t ⟶ pterm-to-sterm t'
using assms proof induction
  case (step name rhs)
  then show ?case
    apply simp
    apply rule
    apply (rule prules-as-srules.srewrite-stepI)
    unfolding compile-def
    apply (subst fset-of-list-elem[symmetric])
    apply (subst ordered-fmap-set-eq)
```

```
    apply (insert fmap)
    apply (rule is-fmapI)
    apply (force dest: is-fmapD)
   by (metis fimage-eqI id-def map-prod-simp)
next
 case (beta c cs t t')
 from beta obtain pat rhs penv where c = (pat, rhs) match pat t = Some penv
subst rhs penv = t'
   by (metis (no-types, lifting) map-option-eq-Some rewrite-step.simps surj-pair)
 then obtain senv where match pat (pterm-to-sterm t) = Some senv srelated'.P-env
penv senv
   by (metis option-rel-Some1 srelated'.match-rel)
 have wellformed rhs
   using beta ‹c = -› prules.all-rules prule.simps
   by force
 then have subst (pterm-to-sterm rhs) senv = pterm-to-sterm t'
   using srelated-subst' ‹- = t'› ‹srelated'.P-env - -›
   by metis
 have (pat, pterm-to-sterm rhs) |∈| map-prod id pterm-to-sterm |`| cs
   using beta ‹c = -›
   by (metis fimage-eqI id-def map-prod-simp)
 have is-fmap cs
   using beta
   by auto
 have find-match (ordered-fmap cs) t = Some (penv, pat, rhs)
   apply (rule compatible-find-match)
   subgoal
     apply (subst ordered-fmap-set-eq[OF ‹is-fmap cs›])+
     using beta by simp
   subgoal
     unfolding list-all-iff
     apply rule
     apply (rename-tac x, case-tac x)
     apply simp
     apply (drule ordered-fmap-sound[OF ‹is-fmap cs›])
     using beta by auto
   subgoal
     apply (subst ordered-fmap-set-eq)
     by fact
   subgoal
     by fact
   subgoal
     using beta(1) ‹c = -› ‹is-fmap cs›
     using fset-of-list-elem ordered-fmap-set-eq by fast
   done

 show ?case
   apply simp
   apply rule
```

apply (*subst* ‹- = *pterm-to-sterm t'*›[*symmetric*])
apply (*rule find-match-rewrite-first*)
unfolding *map-prod-def id-apply*
apply (*subst ordered-fmap-image*[*symmetric*])
apply *fact*
apply (*subst map-prod-def*[*symmetric*])
apply (*subst id-def*[*symmetric*])
apply (*rule srelated-find-match*)
by *fact*+
qed (*auto intro*: *srewrite.intros*)

Computability

export-code *compile*
checking *Scala*

end

3.5 Big-step semantics

theory *Big-Step-Sterm*
imports
Rewriting-Sterm
../Terms/Term-as-Value
begin

3.5.1 Big-step semantics evaluating to irreducible *sterm*s

inductive (**in** *constructors*) *seval* :: *srule list* ⇒ (*name*, *sterm*) *fmap* ⇒ *sterm* ⇒ *sterm* ⇒ *bool* (‹-, -/ ⊢$_s$/ - ↓/ -› [*50*,*0*,*50*] *50*) **for** *rs* **where**
const: (*name*, *rhs*) ∈ *set rs* ⟹ *rs*, Γ ⊢$_s$ *Sconst name* ↓ *rhs* |
var: *fmlookup* Γ *name* = *Some val* ⟹ *rs*, Γ ⊢$_s$ *Svar name* ↓ *val* |
abs: *rs*, Γ ⊢$_s$ *Sabs cs* ↓ *Sabs* (*map* (λ(*pat*, *t*). (*pat*, *subst t* (*fmdrop-fset* (*frees pat*) Γ))) *cs*) |
comb:
rs, Γ ⊢$_s$ *t* ↓ *Sabs cs* ⟹ *rs*, Γ ⊢$_s$ *u* ↓ *u'* ⟹
find-match cs u' = *Some* (*env*, -, *rhs*) ⟹
rs, Γ ++$_f$ *env* ⊢$_s$ *rhs* ↓ *val* ⟹
rs, Γ ⊢$_s$ *t* $$_s$ *u* ↓ *val* |
constr:
name |∈| *C* ⟹
list-all2 (*seval rs* Γ) *ts us* ⟹
rs, Γ ⊢$_s$ *name* $$ *ts* ↓ *name* $$ *us*

lemma (**in** *constructors*) *seval-closed*:
assumes *rs*, Γ ⊢$_s$ *t* ↓ *u closed-srules rs closed-env* Γ *closed-except t* (*fmdom* Γ)
shows *closed u*
using *assms* **proof** *induction*
case (*const name rhs* Γ)

```
  thus ?case
    by (auto simp: list-all-iff)
next
  case (comb Γ t cs u u' env pat rhs val)
  hence closed (Sabs cs) closed u'
    by (auto simp: closed-except-def)
  moreover have (pat, rhs) ∈ set cs match pat u' = Some env
    using comb by (auto simp: find-match-elem)
  ultimately have closed-except rhs (frees pat)
    by (auto dest: closed-except-sabs)

  show ?case
    proof (rule comb)
      have closed-env env
        by (rule closed.match) fact+
      thus closed-env (Γ ++_f env)
        using ‹closed-env Γ› by auto
    next
      have closed-except rhs (fmdom Γ |∪| frees pat)
        using ‹closed-except rhs -›
        unfolding closed-except-def by auto
      hence closed-except rhs (fmdom Γ |∪| fmdom env)
        using ‹match pat u' = Some env› by (metis match-dom)
      thus closed-except rhs (fmdom (Γ ++_f env))
        using comb by simp
    qed fact
next
  case (abs Γ cs)
  show ?case
    apply (subst subst-sterm.simps[symmetric])
    apply (subst closed-except-def)
    apply (subst subst-frees)
    apply fact+
    apply (subst fminus-fsubset-conv)
    apply (subst closed-except-def[symmetric])
    apply (subst funion-fempty-right)
    apply fact
    done
next
  case (constr name Γ ts us)
  have list-all closed us
    using ‹list-all2 - - -› ‹closed-except (list-comb - -) -›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      thus ?case
        using constr unfolding closed.list-comb
        by auto
    qed simp
  thus ?case
```

```
    unfolding closed.list-comb
    by (simp add: closed-except-def)
qed auto

lemma (in srules) seval-wellformed:
  assumes rs, Γ ⊢s t ↓ u wellformed t wellformed-env Γ
  shows wellformed u
using assms proof induction
  case (const name rhs Γ)
  thus ?case
    using all-rules
    by (auto simp: list-all-iff)
next
  case (comb Γ t cs u u' env pat rhs val)
  hence (pat, rhs) ∈ set cs match pat u' = Some env
    by (auto simp: find-match-elem)

  show ?case
    proof (rule comb)
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› comb
        by (auto simp: list-all-iff)
    next
      have wellformed-env env
        apply (rule wellformed.match)
         apply fact
        apply (rule comb)
        using comb apply simp
        apply fact+
        done
      thus wellformed-env (Γ ++f env)
        using comb by auto
    qed
next
  case (abs Γ cs)
  thus ?case
    by (metis subst-sterm.simps subst-wellformed)
next
  case (constr name Γ ts us)
  have list-all wellformed us
    using ‹list-all2 - - -› ‹wellformed (list-comb - -)›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      thus ?case
        using constr by (auto simp: wellformed.list-comb app-sterm-def)
    qed simp
  thus ?case
    by (simp add: wellformed.list-comb const-sterm-def)
qed auto
```

lemma (**in** *constants*) *seval-shadows*:
assumes *rs*, Γ ⊢$_s$ *t* ↓ *u* ¬ *shadows-consts t*
assumes *list-all* (λ(-, *rhs*). ¬ *shadows-consts rhs*) *rs*
assumes *not-shadows-consts-env* Γ
shows ¬ *shadows-consts u*
using *assms* **proof** *induction*
case (*const name rhs* Γ)
thus *?case*
unfolding *srules-def*
by (*auto simp*: *list-all-iff*)
next
case (*comb* Γ *t cs u u′ env pat rhs val*)
hence ¬ *shadows-consts* (*Sabs cs*) ¬ *shadows-consts u′*
by *auto*
moreover from *comb* **have** (*pat*, *rhs*) ∈ *set cs match pat u′* = *Some env*
by (*auto simp*: *find-match-elem*)
ultimately have ¬ *shadows-consts rhs*
by (*auto simp*: *list-ex-iff*)

moreover have *not-shadows-consts-env env*
using *comb* ‹*match pat u′* = -› **by** (*auto intro*: *shadows.match*)

ultimately show *?case*
using *comb* **by** *blast*
next
case (*abs* Γ *cs*)
show *?case*
apply (*subst subst-sterm.simps*[*symmetric*])
apply (*rule subst-shadows*)
apply *fact*+
done
next
case (*constr name* Γ *ts us*)
have *list-all* (*Not* ∘ *shadows-consts*) *us*
using ‹*list-all2* - - -› ‹¬ *shadows-consts* (*name* $$ *ts*)›
proof (*induction ts us rule*: *list.rel-induct*)
case (*Cons v vs u us*)
thus *?case*
using *constr* **by** (*auto simp*: *shadows.list-comb const-sterm-def app-sterm-def*)
qed *simp*
thus *?case*
by (*auto simp*: *shadows.list-comb list-ex-iff list-all-iff const-sterm-def*)
qed *auto*

lemma (**in** *constructors*) *seval-list-comb-abs*:
assumes *rs*, Γ ⊢$_s$ *name* $$ *args* ↓ *Sabs cs*
shows *name* ∈ *dom* (*map-of rs*)
using *assms*

proof (*induction* Γ *name* $$ *args Sabs cs arbitrary*: *args cs*)
case (*constr name′* - - *us*)
hence *Sabs cs* = *name′* $$ *us* **by** *simp*
hence *False*
by (*cases rule*: *list-comb-cases*) (*auto simp*: *const-sterm-def app-sterm-def*)
thus *?case* **..**
next
case (*comb* Γ *t cs′ u u′ env pat rhs*)

hence *strip-comb* (*t* $$_s$$ *u*) = *strip-comb* (*name* $$ *args*)
by *simp*
hence *strip-comb t* = (*Sconst name*, *butlast args*) *u* = *last args*
apply −
subgoal
apply (*simp add*: *strip-list-comb-const*)
apply (*fold app-sterm-def const-sterm-def*)
by (*auto split*: *prod.splits*)
subgoal
apply (*simp add*: *strip-list-comb-const*)
apply (*fold app-sterm-def const-sterm-def*)
by (*auto split*: *prod.splits*)
done
hence *t* = *name* $$ *butlast args*
apply (*fold const-sterm-def*)
by (*metis list-strip-comb fst-conv snd-conv*)

thus *?case*
using *comb* **by** *auto*
qed (*auto elim*: *list-comb-cases simp*: *const-sterm-def app-sterm-def intro*: *weak-map-of-SomeI*)

lemma (**in** *constructors*) *is-value-eval-id*:
assumes *is-value t closed t*
shows *rs*, Γ ⊢$_s$ *t* ↓ *t*
using *assms* **proof** *induction*
case (*abs cs*)

have *rs*, Γ ⊢$_s$ *Sabs cs* ↓ *Sabs* (*map* (λ(*pat*, *t*). (*pat*, *subst t* (*fmdrop-fset* (*frees pat*) Γ))) *cs*)
by (*rule seval.abs*)
moreover have *subst* (*Sabs cs*) Γ = *Sabs cs*
using *abs* **by** (*metis subst-closed-id*)
ultimately show *?case*
by *simp*
next
case (*constr vs name*)
have *list-all2* (*seval rs* Γ) *vs vs*
proof (*rule list.rel-refl-strong*)
fix *v*
assume *v* ∈ *set vs*

```
    moreover hence closed v
      using constr
      unfolding closed.list-comb
      by (auto simp: list-all-iff)
    ultimately show rs, Γ ⊢s v ↓ v
      using ‹list-all - -›
      by (force simp: list-all-iff)
  qed
  with ‹name |∈| C› show ?case
    by (rule seval.constr)
qed

lemma (in constructors) ssubst-eval:
  assumes rs, Γ ⊢s t ↓ t' Γ' ⊆f Γ closed-env Γ value-env Γ
  shows rs, Γ ⊢s subst t Γ' ↓ t'
using assms proof induction
  case (var Γ name val)
  show ?case
    proof (cases fmlookup Γ' name)
      case None
      thus ?thesis
        using var by (auto intro: seval.intros)
    next
      case (Some val')
      with var have val' = val
        unfolding fmsubset-alt-def by force
      show ?thesis
        apply simp
        apply (subst Some)
        apply (subst ‹val' = -›)
        apply simp
        apply (rule is-value-eval-id)
        using var by auto
    qed
next
  case (abs Γ cs)
  hence subst (subst (Sabs cs) Γ') Γ = subst (Sabs cs) Γ
    by (metis subst-twice fmsubset-pred)
  moreover have rs, Γ ⊢s subst (Sabs cs) Γ' ↓ subst (subst (Sabs cs) Γ') Γ
    apply simp
    apply (subst map-map[symmetric])
    apply (rule seval.abs)
    done
  ultimately have rs, Γ ⊢s subst (Sabs cs) Γ' ↓ subst (Sabs cs) Γ
    by metis
  thus ?case by simp
next
  case (constr name Γ ts us)
  hence list-all2 (λt. seval rs Γ (subst t Γ')) ts us
```

```
    by (blast intro: list.rel-mono-strong)
  with constr show ?case
    by (auto simp: subst-list-comb list-all2-map1 intro: seval.constr)
qed (auto intro: seval.intros)

lemma (in constructors) seval-agree-eq:
  assumes rs, Γ ⊢ₛ t ↓ u fmrestrict-fset S Γ = fmrestrict-fset S Γ′ closed-except t
S
  assumes S |⊆| fmdom Γ closed-srules rs closed-env Γ
  shows rs, Γ′ ⊢ₛ t ↓ u
using assms proof (induction arbitrary: Γ′ S)
  case (var Γ name val)
  hence name |∈| S
    by (simp add: closed-except-def)
  hence fmlookup Γ name = fmlookup Γ′ name
    using ‹fmrestrict-fset S Γ = -›
    unfolding fmfilter-alt-defs
    including fmap.lifting
    by transfer′ (auto simp: map-filter-def fun-eq-iff split: if-splits)
  with var show ?case
    by (auto intro: seval.var)
next
  case (abs Γ cs)

  — Intentionally local: not really a useful lemma outside of its scope
  have ∗: fmdrop-fset S (fmrestrict-fset T m) = fmrestrict-fset (T |∪| S) (fmdrop-fset
S m) for S T m
    unfolding fmfilter-alt-defs fmfilter-comp
    by (rule fmfilter-cong) auto

  {
    fix pat t
    assume (pat, t) ∈ set cs
    with abs have closed-except t (S |∪| frees pat)
      by (auto simp: Sterm.closed-except-simps list-all-iff)

    have
      subst t (fmdrop-fset (frees pat) (fmrestrict-fset S Γ)) = subst t (fmdrop-fset
(frees pat) Γ)
      apply (subst ∗)
      apply (rule subst-restrict-closed)
      apply fact
      done

    moreover have
      subst t (fmdrop-fset (frees pat) (fmrestrict-fset S Γ′)) = subst t (fmdrop-fset
(frees pat) Γ′)
      apply (subst ∗)
      apply (rule subst-restrict-closed)
```

```
    apply fact
    done

  ultimately have subst t (fmdrop-fset (frees pat) Γ) = subst t (fmdrop-fset
(frees pat) Γ′)
    using abs by metis
 }

 hence map (λ(pat, t). (pat, subst t (fmdrop-fset (frees pat) Γ))) cs =
       map (λ(pat, t). (pat, subst t (fmdrop-fset (frees pat) Γ′))) cs
   by auto

 thus ?case
   by (metis seval.abs)
next
 case (comb Γ t cs u u′ env pat rhs val)
 have fmdom env = frees pat
   apply (rule match-dom)
   apply (rule find-match-elem)
   apply fact
   done

 show ?case
   proof (rule seval.comb)
     show rs, Γ′ ⊢s t ↓ Sabs cs rs, Γ′ ⊢s u ↓ u′
       using comb by (auto simp: Sterm.closed-except-simps)
   next
     show rs, Γ′ ++f env ⊢s rhs ↓ val
       proof (rule comb)
         have fmrestrict-fset (S |∪| fmdom env) (Γ ++f env) = fmrestrict-fset (S
|∪| fmdom env) (Γ′ ++f env)
           using comb(8)
           unfolding fmfilter-alt-defs
           including fmap.lifting and fset.lifting
               by transfer′ (auto simp: map-filter-def fun-eq-iff map-add-def split:
option.splits if-splits)

         thus fmrestrict-fset (S |∪| frees pat) (Γ ++f env) = fmrestrict-fset (S |∪|
frees pat) (Γ′ ++f env)
           unfolding ‹fmdom env = -› .
       next
         have closed-except t S
           using comb by (simp add: Sterm.closed-except-simps)

         have closed (Sabs cs)
           apply (rule seval-closed)
           apply fact+
           using ‹closed-except t S› ‹S |⊆| fmdom Γ›
           unfolding closed-except-def apply simp
```

```
          done

        have (pat, rhs) ∈ set cs
          using ‹find-match - - = -› by (rule find-match-elem)
        hence closed-except rhs (frees pat)
          using ‹closed (Sabs cs)› by (auto dest: closed-except-sabs)
        thus closed-except rhs (S |∪| frees pat)
          unfolding closed-except-def by auto
      next
        show S |∪| frees pat |⊆| fmdom (Γ ++_f env)
          apply simp
          apply (intro conjI)
          using comb(10) apply blast
          unfolding ‹fmdom env = -› by blast
      next
        have closed-except u S
          using comb by (auto simp: closed-except-def)

        show closed-env (Γ ++_f env)
          apply rule
           apply fact
          apply (rule closed.match[where t = u' and pat = pat])
          subgoal
            by (rule find-match-elem) fact
          subgoal
            apply (rule seval-closed)
              apply fact+
           using ‹closed-except u S› ‹S |⊆| fmdom Γ› unfolding closed-except-def
by blast
          done
      qed fact
    qed fact
next
  case (constr name Γ ts us)
  show ?case
    apply (rule seval.constr)
     apply fact
    apply (rule list.rel-mono-strong)
     apply fact
    using constr
    unfolding closed.list-comb list-all-iff
    by auto
qed (auto intro: seval.intros)
```

Correctness wrt *srewrite*

```
context srules begin context begin

private lemma seval-correct0:
```

assumes *rs*, $\Gamma \vdash_s t \downarrow u$ *closed-except t* (*fmdom* Γ) *closed-env* Γ
shows *rs* $\vdash_s$ *subst t* $\Gamma \longrightarrow* u$
using *assms* **proof** *induction*
case (*const name rhs* Γ)

have *srewrite-step rs name rhs*
by (*rule srewrite-stepI*) *fact*
thus *?case*
by (*auto intro*: *srewrite.intros*)
next
case (*comb* Γ *t cs u u' env pat rhs val*)
hence *closed-except t* (*fmdom* Γ) *closed-except u* (*fmdom* Γ)
by (*simp add*: *Sterm.closed-except-simps*)+
moreover have *closed-srules rs*
using *all-rules*
unfolding *list-all-iff* **by** *fastforce*
ultimately have *closed* (*Sabs cs*) *closed u'*
using *comb* **by** (*metis seval-closed*)+

from *comb* **have** (*pat*, *rhs*) $\in$ *set cs match pat u' = Some env*
by (*auto simp*: *find-match-elem*)
hence *closed-except rhs* (*frees pat*)
using ‹*closed* (*Sabs cs*)› **by** (*auto dest*: *closed-except-sabs*)
hence *frees rhs* $|\subseteq|$ *frees pat*
by (*simp add*: *closed-except-def*)
moreover have *fmdom env = frees pat*
using ‹*match pat u' = -*› **by** (*auto simp*: *match-dom*)
ultimately have *frees rhs* $|\subseteq|$ *fmdom env*
by *simp*
hence *subst rhs* (Γ $+\!\!+_f$ *env*) = *subst rhs env*
by (*rule subst-add-shadowed-env*)

have *rs* $\vdash_s$ *subst t* Γ $\$_s$ *subst u* $\Gamma \longrightarrow* Sabs\ cs\ \$_s\ u'$
using *comb* **by** (*force intro*: *srewrite.rt-comb*[*unfolded app-sterm-def*] *simp*: *Sterm.closed-except-simps*)
also have *rs* $\vdash_s$ *Sabs cs* $\$_s$ *u'* $\longrightarrow*$ *subst rhs env*
using *comb* ‹*closed u'*› **by** (*force intro*: *srewrite.beta find-match-rewrite-first*)
also have *rs* $\vdash_s$ *subst rhs env* $\longrightarrow*$ *subst rhs* (Γ $+\!\!+_f$ *env*)
unfolding ‹*subst rhs* (Γ $+\!\!+_f$ *env*) = -› **by** *simp*
also have *rs* $\vdash_s$ *subst rhs* (Γ $+\!\!+_f$ *env*) $\longrightarrow*$ *val*
proof (*rule comb*)
show *closed-except rhs* (*fmdom* (Γ $+\!\!+_f$ *env*))
using *comb* ‹*match pat u' = Some env*› ‹*fmdom env = -*› ‹*frees rhs* $|\subseteq|$ *frees pat*›
by (*auto simp*: *closed-except-def*)
next
show *closed-env* (Γ $+\!\!+_f$ *env*)
using *comb* ‹*match pat u' = Some env*› ‹*closed u'*›
by (*blast intro*: *closed.match*)

qed

finally show *?case* **by** *simp*
next
case (*constr name* Γ *ts us*)
show *?case*
apply (*simp add*: *subst-list-comb*)
apply (*rule srewrite.rt-list-comb*)
subgoal
apply (*simp add*: *list.rel-map*)
apply (*rule list.rel-mono-strong*[*OF constr*(*2*)])
apply *clarify*
apply (*elim impE*)
using *constr*(*3*) **apply** (*erule closed.list-combE*)
apply (*rule constr*)+
apply (*auto simp*: *const-sterm-def*)
done
subgoal by *auto*
done
qed *auto*

corollary *seval-correct*:
assumes *rs*, *fmempty* $\vdash_s$ *t* $\downarrow$ *u closed t*
shows *rs* $\vdash_s$ *t* $\longrightarrow*$ *u*
proof −
have *closed-except t* (*fmdom fmempty*)
using *assms* **by** *simp*
with *assms* **have** *rs* $\vdash_s$ *subst t fmempty* $\longrightarrow*$ *u*
by (*fastforce intro*!: *seval-correct0*)
thus *?thesis*
by *simp*
qed

end end

end
theory *Big-Step-Value*
imports
Big-Step-Sterm
../Terms/Value
begin

3.5.2 Big-step semantics evaluating to *value*

primrec *vrule* :: *vrule* ⇒ *bool* **where**
vrule (-, *rhs*) ⟷ *vwellformed rhs* ∧ *vclosed rhs* ∧ ¬ *is-Vconstr rhs*

locale *vrules* = *constants C-info fst* |‘| *fset-of-list rs* **for** *C-info* **and** *rs* :: *vrule list*
+

assumes *all-rules*: *list-all vrule rs*
assumes *distinct*: *distinct* (*map fst rs*)
assumes *not-shadows*: *list-all* (λ(-, *rhs*). *not-shadows-vconsts rhs*) *rs*
assumes *vconstructor-value-rs*: *vconstructor-value-rs rs*
assumes *vwelldefined-rs*: *list-all* (λ(-, *rhs*). *vwelldefined rhs*) *rs*
begin

lemma *map*: *is-map* (*set rs*)
using *distinct* **by** (*rule distinct-is-map*)

end

abbreviation *value-to-sterm-rules* :: *vrule list* ⇒ *srule list* **where**
value-to-sterm-rules ≡ *map* (*map-prod id value-to-sterm*)

inductive (**in** *special-constants*)
veval :: (*name* × *value*) *list* ⇒ (*name*, *value*) *fmap* ⇒ *sterm* ⇒ *value* ⇒ *bool* (‹-, -/ ⊢$_v$/ - ↓/ -› [*50*,*0*,*50*] *50*) **for** *rs* **where**
const: (*name*, *rhs*) ∈ *set rs* ⟹ *rs*, Γ ⊢$_v$ *Sconst name* ↓ *rhs* |
var: *fmlookup* Γ *name* = *Some val* ⟹ *rs*, Γ ⊢$_v$ *Svar name* ↓ *val* |
abs: *rs*, Γ ⊢$_v$ *Sabs cs* ↓ *Vabs cs* Γ |
comb:
rs, Γ ⊢$_v$ *t* ↓ *Vabs cs* Γ′ ⟹ *rs*, Γ ⊢$_v$ *u* ↓ *u*′ ⟹
vfind-match cs u′ = *Some* (*env*, -, *rhs*) ⟹
rs, Γ′ +$+_f$ *env* ⊢$_v$ *rhs* ↓ *val* ⟹
rs, Γ ⊢$_v$ *t* $$_s$ *u* ↓ *val* |
rec-comb:
rs, Γ ⊢$_v$ *t* ↓ *Vrecabs css name* Γ′ ⟹
fmlookup css name = *Some cs* ⟹
rs, Γ ⊢$_v$ *u* ↓ *u*′ ⟹
vfind-match cs u′ = *Some* (*env*, -, *rhs*) ⟹
rs, Γ′ +$+_f$ *env* ⊢$_v$ *rhs* ↓ *val* ⟹
rs, Γ ⊢$_v$ *t* $$_s$ *u* ↓ *val* |
constr:
name |∈| *C* ⟹
list-all2 (*veval rs* Γ) *ts us* ⟹
rs, Γ ⊢$_v$ *name* $$ *ts* ↓ *Vconstr name us*

lemma (**in** *vrules*) *veval-wellformed*:
assumes *rs*, Γ ⊢$_v$ *t* ↓ *v wellformed t wellformed-venv* Γ
shows *vwellformed v*
using *assms* **proof** *induction*
case *const*
thus *?case*
using *all-rules*
by (*auto simp*: *list-all-iff*)
next
case *comb*
show *?case*

```
    apply (rule comb)
     using comb by (auto simp: list-all-iff dest: vfind-match-elem intro: vwell-
formed.vmatch-env)
next
  case (rec-comb Γ t css name Γ' cs u u' env pat rhs val)
  hence (pat, rhs) ∈ set cs vmatch (mk-pat pat) u' = Some env
    by (metis vfind-match-elem)+

  show ?case
    proof (rule rec-comb)
      have wellformed t
        using rec-comb by simp
      have vwellformed (Vrecabs css name Γ')
        by (rule rec-comb) fact+
      thus wellformed rhs
        using rec-comb ‹(pat, rhs) ∈ set cs›
        by (auto simp: list-all-iff)

      have wellformed-venv Γ'
        using ‹vwellformed (Vrecabs css name Γ')› by simp
      moreover have wellformed-venv env
        using rec-comb ‹vmatch (mk-pat pat) u' = Some env›
        by (auto intro: vwellformed.vmatch-env)
      ultimately show wellformed-venv (Γ' ++_f env)
        by blast
    qed
next
  case (constr name Γ ts us)
  have list-all vwellformed us
    using ‹list-all2 - - -› ‹wellformed (list-comb - -)›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      with constr show ?case
        unfolding wellformed.list-comb by auto
    qed simp
  thus ?case
    by (simp add: list-all-iff)
qed auto

lemma (in vrules) veval-closed:
  assumes rs, Γ ⊢_v t ↓ v closed-except t (fmdom Γ) closed-venv Γ
  assumes wellformed t wellformed-venv Γ
  shows vclosed v
using assms proof induction
  case (const name rhs Γ)
  hence (name, rhs) ∈ set rs
    by (auto dest: map-of-SomeD)
  thus ?case
    using const all-rules
```

```
    by (auto simp: list-all-iff)
next
  case (comb Γ t cs Γ' u u' env pat rhs val)
  hence pat: (pat, rhs) ∈ set cs vmatch (mk-pat pat) u' = Some env
    by (metis vfind-match-elem)+

  show ?case
    proof (rule comb)
      have vclosed u'
        using comb by (auto simp: Sterm.closed-except-simps)
      have closed-venv env
        by (rule vclosed.vmatch-env) fact+
      thus closed-venv (Γ' ++_f env)
        using comb by (auto simp: Sterm.closed-except-simps)
    next
      have wellformed t
        using comb by simp
      have vwellformed (Vabs cs Γ')
        by (rule veval-wellformed) fact+
      thus wellformed rhs
        using pat by (auto simp: list-all-iff)

      have wellformed-venv Γ'
        using ‹vwellformed (Vabs cs Γ')› by simp
      moreover have wellformed-venv env
        using comb pat
        by (auto intro: vwellformed.vmatch-env veval-wellformed)
      ultimately show wellformed-venv (Γ' ++_f env)
        by blast

      have vclosed (Vabs cs Γ')
        using comb by (auto simp: Sterm.closed-except-simps)
      hence closed-except rhs (fmdom Γ' |∪| frees pat)
        using pat by (auto simp: list-all-iff)

      moreover have fmdom env = frees pat
        using ‹vwellformed (Vabs cs Γ')› pat
        by (auto simp: vmatch-dom mk-pat-frees list-all-iff)

      ultimately show closed-except rhs (fmdom (Γ' ++_f env))
        using ‹vclosed (Vabs cs Γ')›
        by simp
    qed
next
  case (rec-comb Γ t css name Γ' cs u u' env pat rhs val)
  hence pat: (pat, rhs) ∈ set cs vmatch (mk-pat pat) u' = Some env
    by (metis vfind-match-elem)+

  show ?case
```

```
    proof (rule rec-comb)
      have vclosed u'
        using rec-comb by (auto simp: Sterm.closed-except-simps)
      have closed-venv env
        by (rule vclosed.vmatch-env) fact+
      thus closed-venv (Γ' ++_f env)
        using rec-comb by (auto simp: Sterm.closed-except-simps)
    next
      have wellformed t
        using rec-comb by simp
      have vwellformed (Vrecabs css name Γ')
        by (rule veval-wellformed) fact+
      thus wellformed rhs
        using pat rec-comb by (auto simp: list-all-iff)

      have wellformed-venv Γ'
        using ‹vwellformed (Vrecabs css name Γ')› by simp
      moreover have wellformed-venv env
        using rec-comb pat
        by (auto intro: vwellformed.vmatch-env veval-wellformed)
      ultimately show wellformed-venv (Γ' ++_f env)
        by blast

      have wellformed-clauses cs
       using ‹vwellformed (Vrecabs css name Γ')› ‹fmlookup css name = Some cs›
        by auto

      have vclosed (Vrecabs css name Γ')
        using rec-comb by (auto simp: Sterm.closed-except-simps)
      hence closed-except rhs (fmdom Γ' |∪| frees pat)
        using rec-comb pat by (auto simp: list-all-iff)
      moreover have fmdom env = frees pat
        using ‹wellformed-clauses cs› pat
        by (auto simp: list-all-iff vmatch-dom mk-pat-frees)
      ultimately show closed-except rhs (fmdom (Γ' ++_f env))
        using ‹vclosed (Vrecabs css name Γ')›
        by simp
    qed
  next
    case (constr name Γ ts us)
    have list-all vclosed us
      using ‹list-all2 - - -› ‹closed-except (- $$ -) -› ‹wellformed (name $$ ts)›
      proof (induction ts us rule: list.rel-induct)
        case (Cons v vs u us)
        with constr show ?case
          unfolding closed.list-comb wellformed.list-comb
          by (auto simp: list-all-iff Sterm.closed-except-simps)
      qed simp
    thus ?case
```

by (*simp add*: *list-all-iff*)
qed (*auto simp*: *Sterm.closed-except-simps*)

lemma (**in** *vrules*) *veval-constructor-value*:
assumes *rs*, $\Gamma \vdash_v t \downarrow v$ *vconstructor-value-env* Γ
shows *vconstructor-value v*
using *assms* **proof** *induction*
case (*comb* Γ *t cs* Γ' *u u' env pat rhs val*)
hence (*pat*, *rhs*) ∈ *set cs vmatch* (*mk-pat pat*) $u' = Some\ env$
by (*metis vfind-match-elem*)+
hence *vconstructor-value-env* ($\Gamma' +\!\!+_f$ *env*)
using *comb* **by** (*auto intro*: *vconstructor-value.vmatch-env*)
thus *?case*
using *comb* **by** *auto*
next
case (*constr name* Γ *ts us*)
hence *list-all vconstructor-value us*
by (*auto elim*: *list-all2-rightE*)
with *constr* **show** *?case*
by *simp*
next
case *const*
thus *?case*
using *vconstructor-value-rs*
by (*auto simp*: *list-all-iff vconstructor-value-rs-def*)
next
case (*rec-comb* Γ *t css name* Γ' *cs u u' env pat rhs val*)
hence (*pat*, *rhs*) ∈ *set cs vmatch* (*mk-pat pat*) $u' = Some\ env$
by (*metis vfind-match-elem*)+
hence *vconstructor-value-env* ($\Gamma' +\!\!+_f$ *env*)
using *rec-comb* **by** (*auto intro*: *vconstructor-value.vmatch-env*)
thus *?case*
using *rec-comb* **by** *auto*
qed (*auto simp*: *list-all-iff vconstructor-value-rs-def*)

lemma (**in** *vrules*) *veval-welldefined*:
assumes *rs*, $\Gamma \vdash_v t \downarrow v$ *fmpred* (λ-. *vwelldefined*) Γ *welldefined t*
shows *vwelldefined v*
using *assms* **proof** *induction*
case *const*
thus *?case*
using *vwelldefined-rs assms*
unfolding *list-all-iff*
by (*auto simp*: *list-all-iff*)
next
case (*comb* Γ *t cs* Γ' *u u' env pat rhs val*)
hence *vwelldefined* (*Vabs cs* Γ')
by *auto*

```
  show ?case
    proof (rule comb)
      have fmpred (λ-. vwelldefined) Γ'
        using ‹vwelldefined (Vabs cs Γ')›
        by simp
      moreover have fmpred (λ-. vwelldefined) env
        apply (rule vwelldefined.vmatch-env)
         apply (rule vfind-match-elem)
        using comb by auto
      ultimately show fmpred (λ-. vwelldefined) (Γ' ++_f env)
        by auto
    next
      have (pat, rhs) ∈ set cs
        using comb by (metis vfind-match-elem)
      thus welldefined rhs
        using ‹vwelldefined (Vabs cs Γ')›
        by (auto simp: list-all-iff)
    qed
next
  case (rec-comb Γ t css name Γ' cs u u' env pat rhs val)
  have (pat, rhs) ∈ set cs
    by (rule vfind-match-elem) fact
  show ?case
    proof (rule rec-comb)
      show fmpred (λ-. vwelldefined) (Γ' ++_f env)
        proof
          show fmpred (λ-. vwelldefined) env
         using rec-comb by (auto dest: vfind-match-elem intro: vwelldefined.vmatch-env)
        next
          show fmpred (λ-. vwelldefined) Γ'
            using rec-comb by auto
        qed
    next
      have vwelldefined (Vrecabs css name Γ')
        using rec-comb by auto

      thus welldefined rhs
        apply simp
        apply (elim conjE)
        apply (drule fmpredD[where m = css])
        using ‹(pat, rhs) ∈ set cs› rec-comb by (auto simp: list-all-iff)
    qed
next
  case (constr name Γ ts us)
  have list-all vwelldefined us
    using ‹list-all2 - - -› ‹welldefined (- $$ -)›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      with constr show ?case
```

```
      unfolding welldefined.list-comb
      by auto
  qed simp
 with constr show ?case
  by (simp add: list-all-iff all-consts-def)
next
 case abs
 thus ?case
  unfolding welldefined-sabs by auto
qed auto
```

Correctness wrt *constructors.seval*

```
context vrules begin

definition rs' :: srule list where
rs' = value-to-sterm-rules rs

lemma value-to-sterm-srules: srules C-info rs'
proof
 show distinct (map fst rs')
  unfolding rs'-def
  using distinct by auto
next
 show list-all srule rs'
  unfolding rs'-def list.pred-map
  apply (rule list.pred-mono-strong[OF all-rules])
  apply (auto intro: vclosed.value-to-sterm vwellformed.value-to-sterm)
  subgoal by (auto intro: vwellformed.value-to-sterm)
  subgoal by (auto intro: vclosed.value-to-sterm)
  subgoal for a b by (cases b) (auto simp: is-abs-def term-cases-def)
  done
next
 show fdisjnt (fst |`| fset-of-list rs') C
  using vconstructor-value-rs unfolding rs'-def vconstructor-value-rs-def
  by auto
 interpret c: constants - fst |`| fset-of-list rs'
  by standard (fact | fact distinct-ctr)+
 have all-consts: c.all-consts = all-consts
  unfolding c.all-consts-def all-consts-def
  by (simp add: rs'-def)
 have shadows-consts: c.shadows-consts rhs = shadows-consts rhs for rhs :: sterm
  by (induction rhs; fastforce simp: all-consts list-ex-iff)

 have list-all (λ(-, rhs). ¬ shadows-consts rhs) rs'
  unfolding rs'-def
  unfolding list.pred-map map-prod-def id-def case-prod-twice list-all-iff
  apply auto
  unfolding comp-def all-consts-def
```

```
    using not-shadows
    by (fastforce simp: list-all-iff dest: not-shadows-vconsts.value-to-sterm)
  thus list-all (λ(-, rhs). ¬ c.shadows-consts rhs) rs'
    unfolding shadows-consts .

  have list-all (λ(-, rhs). welldefined rhs) rs'
    unfolding rs'-def list.pred-map
    apply (rule list.pred-mono-strong[OF vwelldefined-rs])
    subgoal for z
      apply (cases z; hypsubst-thin)
      apply simp
      apply (erule vwelldefined.value-to-sterm)
      done
    done
  moreover have map fst rs = map fst rs'
    unfolding rs'-def by simp
  ultimately have list-all (λ(-, rhs). welldefined rhs) rs'
    by simp
  thus list-all (λ(-, rhs). c.welldefined rhs) rs'
    unfolding all-consts .
next
  show distinct all-constructors
    by (fact distinct-ctr)
qed

end
```

When we evaluate *sterm*s using *veval*, the result is a *value* which possibly contains a closure (constructor *Vabs*). Such a closure is essentially a case-lambda (like *Sabs*), but with an additionally captured environment of type *string* ⇀ *value* (which is usually called Γ'). The contained case-lambda might not be closed.

The proof idea is that we can always substitute with Γ' and obtain a regular *sterm* value. The only interesting part of the proof is the case when a case-lambda gets applied to a value, because in that process, a hidden environment is *unveiled*. That environment may not bear any relation to the active environment Γ at all. But pattern matching and substitution proceeds only with that hidden environment.

```
context vrules begin

context begin

private lemma veval-correct0:
  assumes rs, Γ ⊢v t ↓ v wellformed t wellformed-venv Γ
  assumes closed-except t (fmdom Γ) closed-venv Γ
  assumes vconstructor-value-env Γ
  shows rs', fmmap value-to-sterm Γ ⊢s t ↓ value-to-sterm v
using assms proof induction
```

```
  case (constr name Γ ts us)

  have list-all2 (seval rs' (fmmap value-to-sterm Γ)) ts (map value-to-sterm us)
    unfolding list-all2-map2
    proof (rule list.rel-mono-strong[OF ‹list-all2 - - -›], elim conjE impE)
      fix t u
      assume t ∈ set ts u ∈ set us
      assume rs, Γ ⊢ᵥ t ↓ u

      show wellformed t  closed-except t (fmdom Γ)
        using ‹t ∈ set ts› constr
        unfolding wellformed.list-comb closed.list-comb list-all-iff
        by auto
    qed (rule constr | assumption)+

  thus ?case
    using ‹name |∈| C›
    by (auto intro: seval.constr)
next
  case (comb Γ t cs Γ' u u' venv pat rhs val)

  — We first need to establish a ton of boring side-conditions.

  hence vmatch (mk-pat pat) u' = Some venv
    by (auto dest: vfind-match-elem)

  have wellformed t
    using comb by simp
  have vwellformed (Vabs cs Γ')
    by (rule veval-wellformed) fact+

  hence
    list-all (linear ∘ fst) cs
    wellformed-venv Γ'
    by (auto simp: list-all-iff split-beta)

  have rel-option match-related (vfind-match cs u') (find-match cs (value-to-sterm
u'))
    apply (rule find-match-eq)
     apply fact
    apply (rule veval-constructor-value)
     apply fact+
    done

  then obtain senv
    where find-match cs (value-to-sterm u') = Some (senv, pat, rhs)
      and env-eq venv senv
    using ‹vfind-match - - = -›
    by cases auto
```

hence (*pat*, *rhs*) ∈ *set cs match pat* (*value-to-sterm u′*) = *Some senv*
by (*auto dest*: *find-match-elem*)
hence *fmdom senv* = *frees pat*
by (*simp add*: *match-dom*)

moreover have *senv* = *fmmap value-to-sterm venv*
using ‹*env-eq venv senv*›
by (*rule env-eq-eq*)

ultimately have *fmdom venv* = *frees pat*
by *simp*

have *closed-except t* (*fmdom* Γ) *wellformed t*
using *comb* **by** (*simp add*: *closed-except-def*)+
have *vclosed* (*Vabs cs* Γ′)
by (*rule veval-closed*) *fact*+

have *vconstructor-value* (*Vabs cs* Γ′) *vconstructor-value u′*
by (*rule veval-constructor-value*; *fact*)+
hence *vconstructor-value-env* Γ′
by *simp*
have *vconstructor-value-env venv*
by (*rule vconstructor-value.vmatch-env*) *fact*+

have *wellformed u*
using *comb* **by** *simp*
have *vwellformed u′*
by (*rule veval-wellformed*) *fact*+
have *wellformed-venv venv*
by (*rule vwellformed.vmatch-env*) *fact*+

have *closed-except u* (*fmdom* Γ)
using *comb* **by** (*simp add*: *closed-except-def*)
have *vclosed u′*
by (*rule veval-closed*) *fact*+
have *closed-venv venv*
by (*rule vclosed.vmatch-env*) *fact*+

have *closed-venv* Γ′
using ‹*vclosed* (*Vabs cs* Γ′)› **by** *simp*

let *?subst* = λ*pat t*. *subst t* (*fmdrop-fset* (*frees pat*) (*fmmap value-to-sterm* Γ′))

1. We know the following (induction hypothesis): *rs′*, *fmmap value-to-sterm* (Γ′ ++$_f$ *venv*) ⊢$_s$ *rhs* ↓ *value-to-sterm val*
2. ... first, we can deduce using *ssubst-eval* that this is equivalent to substituting *rhs* first: *rs′*, *fmmap value-to-sterm* (Γ′ ++$_f$ *venv*) ⊢$_s$ *subst rhs* (*fmdrop-fset* (*frees pat*) (*fmmap value-to-sterm* Γ′)) ↓ *value-to-sterm val*

3. ... second, we can replace the hidden environment Γ' by the active environment Γ using *seval-agree-eq* because it does not contain useful information at this point: *rs'*, *fmmap value-to-sterm* (Γ $+\!\!+_f$ *venv*) $\vdash_s$ *subst rhs* (*fmdrop-fset* (*frees pat*) (*fmmap value-to-sterm* Γ')) $\downarrow$ *value-to-sterm val*

4. ... finally we can apply a step in the original semantics and arrive at the conclusion: *rs'*, *fmmap value-to-sterm* $\Gamma \vdash_s t$ \$$_s$ $u \downarrow$ *value-to-sterm val*

have *rs'*, *fmmap value-to-sterm* (Γ' $+\!\!+_f$ *venv*) $\vdash_s$ *?subst pat rhs* $\downarrow$ *value-to-sterm val*
proof (*rule ssubst-eval*)
show *rs'*, *fmmap value-to-sterm* (Γ' $+\!\!+_f$ *venv*) $\vdash_s$ *rhs* $\downarrow$ *value-to-sterm val*
proof (*rule comb*)
have *linear pat closed-except rhs* (*fmdom* Γ' |∪| *frees pat*)
using ‹(*pat*, *rhs*) ∈ *set cs*› ‹*vwellformed* (*Vabs cs* Γ')› ‹*vclosed* (*Vabs cs* Γ')›
by (*auto simp*: *list-all-iff*)
hence *closed-except rhs* (*fmdom* Γ' |∪| *fmdom venv*)
using ‹*vmatch* (*mk-pat pat*) u' = *Some venv*›
by (*auto simp*: *mk-pat-frees vmatch-dom*)
thus *closed-except rhs* (*fmdom* (Γ' $+\!\!+_f$ *venv*))
by *simp*
next
show *wellformed-venv* (Γ' $+\!\!+_f$ *venv*)
using ‹*wellformed-venv* Γ'› ‹*wellformed-venv venv*›
by *blast*
next
show *closed-venv* (Γ' $+\!\!+_f$ *venv*)
using ‹*closed-venv* Γ'› ‹*closed-venv venv*›
by *blast*
next
show *vconstructor-value-env* (Γ' $+\!\!+_f$ *venv*)
using ‹*vconstructor-value-env* Γ'› ‹*vconstructor-value-env venv*›
by *blast*
next
show *wellformed rhs*
using ‹(*pat*, *rhs*) ∈ *set cs*› ‹*vwellformed* (*Vabs cs* Γ')›
by (*fastforce simp*: *list-all-iff*)
qed
next
have *fmdrop-fset* (*fmdom venv*) $\Gamma' \subseteq_f \Gamma' +\!\!+_f$ *venv*
including *fmap.lifting* **and** *fset.lifting*
by *transfer'*
(*auto simp*: *map-drop-set-def map-filter-def map-le-def map-add-def split*: *if-splits*)
thus *fmdrop-fset* (*frees pat*) (*fmmap value-to-sterm* Γ') $\subseteq_f$ *fmmap value-to-sterm* (Γ' $+\!\!+_f$ *venv*)
unfolding ‹*fmdom venv* = *frees pat*›
by (*metis fmdrop-fset-fmmap fmmap-subset*)
next

```
    show closed-env (fmmap value-to-sterm (Γ' ++f venv))
      apply auto
      apply rule
       apply (rule vclosed.value-to-sterm-env, fact)+
      done
  next
    show value-env (fmmap value-to-sterm (Γ' ++f venv))
      apply auto
      apply rule
       apply (rule vconstructor-value.value-to-sterm-env, fact)+
      done
  qed

 show ?case
  proof (rule seval.comb)
    have rs', fmmap value-to-sterm Γ ⊢s t ↓ value-to-sterm (Vabs cs Γ')
      using comb by (auto simp: closed-except-def)
     thus rs', fmmap value-to-sterm Γ ⊢s t ↓ Sabs (map (λ(pat, t). (pat, ?subst
pat t)) cs)
      by simp
  next
    show rs', fmmap value-to-sterm Γ ⊢s u ↓ value-to-sterm u'
      using comb by (simp add: closed-except-def)
  next
   show rs', fmmap value-to-sterm Γ ++f senv ⊢s ?subst pat rhs ↓ value-to-sterm
val
      proof (rule seval-agree-eq)
        show rs', fmmap value-to-sterm Γ' ++f fmmap value-to-sterm venv ⊢s
?subst pat rhs ↓ value-to-sterm val
          using ‹rs', fmmap value-to-sterm (Γ' ++f venv) ⊢s ?subst pat rhs ↓
value-to-sterm val› by simp
      next
        show fmrestrict-fset (frees pat) (fmmap value-to-sterm Γ' ++f fmmap
value-to-sterm venv) =
             fmrestrict-fset (frees pat) (fmmap value-to-sterm Γ ++f senv)
          unfolding ‹senv = -›
          apply (subst ‹fmdom venv = frees pat›[symmetric])+
          apply (subst fmdom-map[symmetric])
          apply (subst fmadd-restrict-right-dom)
          apply (subst fmdom-map[symmetric])
          apply (subst fmadd-restrict-right-dom)
          apply simp
          done
      next
        have closed (value-to-sterm (Vabs cs Γ'))
          using ‹vclosed (Vabs cs Γ')›
          by (rule vclosed.value-to-sterm)
       thus closed-except (subst rhs (fmdrop-fset (frees pat) (fmmap value-to-sterm
Γ'))) (frees pat)
```

```
        using ‹(pat, rhs) ∈ set cs›
        by (auto simp: Sterm.closed-except-simps list-all-iff)
      next
        show closed-env (fmmap value-to-sterm Γ′ ++_f fmmap value-to-sterm
venv)
        using ‹closed-venv Γ′› ‹closed-venv venv›
        by (auto intro: vclosed.value-to-sterm-env)
      next
          show frees pat |⊆| fmdom (fmmap value-to-sterm Γ′ ++_f fmmap
value-to-sterm venv)
        using ‹fmdom venv = frees pat›
        by fastforce
      next
       show closed-srules rs′
        using all-rules
        unfolding rs′-def list-all-iff
        by (fastforce intro: vclosed.value-to-sterm)
      qed
    next
     show find-match (map (λ(pat, t). (pat, ?subst pat t)) cs) (value-to-sterm u′)
= Some (senv, pat, ?subst pat rhs)
      using ‹find-match - - = -›
      by (auto simp: find-match-map)
    qed
next
  — Basically a verbatim copy from the comb case

  case (rec-comb Γ t css name Γ′ cs u u′ venv pat rhs val)

  hence vmatch (mk-pat pat) u′ = Some venv
    by (auto dest: vfind-match-elem)

  have cs = the (fmlookup css name)
    using rec-comb by simp

  have wellformed t
    using rec-comb by simp
  have vwellformed (Vrecabs css name Γ′)
    by (rule veval-wellformed) fact+
  hence vwellformed (Vabs cs Γ′) — convenient hack: cs is not really part of a
Vabs
    using rec-comb by auto

  hence
    list-all (linear ∘ fst) cs
    wellformed-venv Γ′
    by (auto simp: list-all-iff split-beta)

  have rel-option match-related (vfind-match cs u′) (find-match cs (value-to-sterm
```

u′))
apply (*rule find-match-eq*)
apply *fact*
apply (*rule veval-constructor-value*)
apply *fact*+
done

then obtain *senv*
where *find-match cs* (*value-to-sterm u′*) = *Some* (*senv*, *pat*, *rhs*)
and *env-eq venv senv*
using ‹*vfind-match* - - = -›
by *cases auto*
hence (*pat*, *rhs*) ∈ *set cs match pat* (*value-to-sterm u′*) = *Some senv*
by (*auto dest*: *find-match-elem*)
hence *fmdom senv* = *frees pat*
by (*simp add*: *match-dom*)

moreover have *senv* = *fmmap value-to-sterm venv*
using ‹*env-eq venv senv*›
by (*rule env-eq-eq*)

ultimately have *fmdom venv* = *frees pat*
by *simp*

have *closed-except t* (*fmdom* Γ) *wellformed t*
using *rec-comb* **by** (*simp add*: *closed-except-def*)+
have *vclosed* (*Vrecabs css name* Γ′)
by (*rule veval-closed*) *fact*+
hence *vclosed* (*Vabs cs* Γ′)
using *rec-comb* **by** *auto*

have *vconstructor-value u′*
by (*rule veval-constructor-value*) *fact*+
have *vconstructor-value* (*Vrecabs css name* Γ′)
by (*rule veval-constructor-value*) *fact*+
hence *vconstructor-value-env* Γ′
by *simp*
have *vconstructor-value-env venv*
by (*rule vconstructor-value.vmatch-env*) *fact*+

have *wellformed u*
using *rec-comb* **by** *simp*
have *vwellformed u′*
by (*rule veval-wellformed*) *fact*+
have *wellformed-venv venv*
by (*rule vwellformed.vmatch-env*) *fact*+

have *closed-except u* (*fmdom* Γ)
using *rec-comb* **by** (*simp add*: *closed-except-def*)

```
  have vclosed u'
    by (rule veval-closed) fact+
  have closed-venv venv
    by (rule vclosed.vmatch-env) fact+

  have closed-venv Γ'
    using ‹vclosed (Vabs cs Γ')› by simp

  let ?subst = λpat t. subst t (fmdrop-fset (frees pat) (fmmap value-to-sterm Γ'))
  have rs', fmmap value-to-sterm (Γ' ++_f venv) ⊢_s ?subst pat rhs ↓ value-to-sterm
val
    proof (rule ssubst-eval)
      show rs', fmmap value-to-sterm (Γ' ++_f venv) ⊢_s rhs ↓ value-to-sterm val
        proof (rule rec-comb)
          have linear pat closed-except rhs (fmdom Γ' |∪| frees pat)
            using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ')› ‹vclosed (Vabs cs
Γ')›
            by (auto simp: list-all-iff)
          hence closed-except rhs (fmdom Γ' |∪| fmdom venv)
            using ‹vmatch (mk-pat pat) u' = Some venv›
            by (auto simp: mk-pat-frees vmatch-dom)
          thus closed-except rhs (fmdom (Γ' ++_f venv))
            by simp
        next
          show wellformed-venv (Γ' ++_f venv)
            using ‹wellformed-venv Γ'› ‹wellformed-venv venv›
            by blast
        next
          show closed-venv (Γ' ++_f venv)
            using ‹closed-venv Γ'› ‹closed-venv venv›
            by blast
        next
          show vconstructor-value-env (Γ' ++_f venv)
            using ‹vconstructor-value-env Γ'› ‹vconstructor-value-env venv›
            by blast
        next
          show wellformed rhs
            using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ')›
            by (fastforce simp: list-all-iff)
        qed
    next
      have fmdrop-fset (fmdom venv) Γ' ⊆_f Γ' ++_f venv
        including fmap.lifting and fset.lifting
        by transfer'
           (auto simp: map-drop-set-def map-filter-def map-le-def map-add-def split:
if-splits)
    thus fmdrop-fset (frees pat) (fmmap value-to-sterm Γ') ⊆_f fmmap value-to-sterm
(Γ' ++_f venv)
        unfolding ‹fmdom venv = frees pat›
```

```
      by (metis fmdrop-fset-fmmap fmmap-subset)
    next
     show closed-env (fmmap value-to-sterm (Γ' ++f venv))
      apply auto
      apply rule
       apply (rule vclosed.value-to-sterm-env, fact)+
      done
    next
     show value-env (fmmap value-to-sterm (Γ' ++f venv))
      apply auto
      apply rule
       apply (rule vconstructor-value.value-to-sterm-env, fact)+
      done
    qed

  show ?case
    proof (rule seval.comb)
      have rs', fmmap value-to-sterm Γ ⊢s t ↓ value-to-sterm (Vrecabs css name
Γ')
       using rec-comb by (auto simp: closed-except-def)
      thus rs', fmmap value-to-sterm Γ ⊢s t ↓ Sabs (map (λ(pat, t). (pat, ?subst
pat t)) cs)
       unfolding ‹cs = -› by simp
    next
     show rs', fmmap value-to-sterm Γ ⊢s u ↓ value-to-sterm u'
      using rec-comb by (simp add: closed-except-def)
    next
     show rs', fmmap value-to-sterm Γ ++f senv ⊢s ?subst pat rhs ↓ value-to-sterm
val
      proof (rule seval-agree-eq)
        show rs', fmmap value-to-sterm Γ' ++f fmmap value-to-sterm venv ⊢s
?subst pat rhs ↓ value-to-sterm val
          using ‹rs', fmmap value-to-sterm (Γ' ++f venv) ⊢s ?subst pat rhs ↓
value-to-sterm val› by simp
      next
        show fmrestrict-fset (frees pat) (fmmap value-to-sterm Γ' ++f fmmap
value-to-sterm venv) =
             fmrestrict-fset (frees pat) (fmmap value-to-sterm Γ ++f senv)
        unfolding ‹senv = -›
        apply (subst ‹fmdom venv = frees pat›[symmetric])+
        apply (subst fmdom-map[symmetric])
        apply (subst fmadd-restrict-right-dom)
        apply (subst fmdom-map[symmetric])
        apply (subst fmadd-restrict-right-dom)
        apply simp
        done
      next
       have closed (value-to-sterm (Vabs cs Γ'))
        using ‹vclosed (Vabs cs Γ')›
```

```
      by (rule vclosed.value-to-sterm)
    thus closed-except (subst rhs (fmdrop-fset (frees pat) (fmmap value-to-sterm
Γ′))) (frees pat)
      using ‹(pat, rhs) ∈ set cs›
      by (auto simp: Sterm.closed-except-simps list-all-iff)
    next
      show closed-env (fmmap value-to-sterm Γ′ ++_f fmmap value-to-sterm
venv)
      using ‹closed-venv Γ′› ‹closed-venv venv›
      by (auto intro: vclosed.value-to-sterm-env)
    next
        show frees pat |⊆| fmdom (fmmap value-to-sterm Γ′ ++_f fmmap
value-to-sterm venv)
      using ‹fmdom venv = frees pat›
      by fastforce
    next
     show closed-srules rs′
      using all-rules
      unfolding rs′-def list-all-iff
      by (fastforce intro: vclosed.value-to-sterm)
    qed
  next
    show find-match (map (λ(pat, t). (pat, ?subst pat t)) cs) (value-to-sterm u′)
= Some (senv, pat, ?subst pat rhs)
    using ‹find-match - - = -›
    by (auto simp: find-match-map)
  qed
next
 case (const name rhs Γ)
 show ?case
  apply (rule seval.const)
  unfolding rs′-def
  using ‹(name, rhs) ∈ -› by force
next
 case abs
 show ?case
  by (auto simp del: fmdrop-fset-fmmap intro: seval.abs)
qed (auto intro: seval.var seval.abs)

lemma veval-correct:
 assumes rs, fmempty ⊢_v t ↓ v wellformed t closed t
 shows rs′, fmempty ⊢_s t ↓ value-to-sterm v
proof −
 have rs′, fmmap value-to-sterm fmempty ⊢_s t ↓ value-to-sterm v
  using assms
  by (auto intro: veval-correct0 simp del: fmmap-empty)
 thus ?thesis
  by simp
qed
```

end end

end

3.5.3 Big-step semantics with conflation of constants and variables

theory *Big-Step-Value-ML*
imports *Big-Step-Value*
begin

definition *mk-rec-env* :: (*name*, *sclauses*) *fmap* ⇒ (*name*, *value*) *fmap* ⇒ (*name*, *value*) *fmap* **where**
mk-rec-env css Γ′ = *fmmap-keys* (λ*name cs*. *Vrecabs css name* Γ′) *css*

context *special-constants* **begin**

inductive *veval′* :: (*name*, *value*) *fmap* ⇒ *sterm* ⇒ *value* ⇒ *bool* (‹-/ ⊢$_v$/ - ↓/ -› [*50*,*0*,*50*] *50*) **where**
const: *name* |∉| *C* ⟹ *fmlookup* Γ *name* = *Some val* ⟹ Γ ⊢$_v$ *Sconst name* ↓ *val* |
var: *fmlookup* Γ *name* = *Some val* ⟹ Γ ⊢$_v$ *Svar name* ↓ *val* |
abs: Γ ⊢$_v$ *Sabs cs* ↓ *Vabs cs* Γ |
comb:
Γ ⊢$_v$ *t* ↓ *Vabs cs* Γ′ ⟹ Γ ⊢$_v$ *u* ↓ *u′* ⟹
vfind-match cs u′ = *Some* (*env*, -, *rhs*) ⟹
Γ′ ++$_f$ *env* ⊢$_v$ *rhs* ↓ *val* ⟹
Γ ⊢$_v$ *t* $$_s$ *u* ↓ *val* |
rec-comb:
Γ ⊢$_v$ *t* ↓ *Vrecabs css name* Γ′ ⟹
fmlookup css name = *Some cs* ⟹
Γ ⊢$_v$ *u* ↓ *u′* ⟹
vfind-match cs u′ = *Some* (*env*, -, *rhs*) ⟹
Γ′ ++$_f$ *mk-rec-env css* Γ′ ++$_f$ *env* ⊢$_v$ *rhs* ↓ *val* ⟹
Γ ⊢$_v$ *t* $$_s$ *u* ↓ *val* |
constr: *name* |∈| *C* ⟹ *list-all2* (*veval′* Γ) *ts us* ⟹ Γ ⊢$_v$ *name* $$ *ts* ↓ *Vconstr name us*

lemma *veval′-sabs-svarE*:
assumes Γ ⊢$_v$ *Sabs cs* $$_s$ *Svar n* ↓ *v*
obtains *u′ env pat rhs*
where *fmlookup* Γ *n* = *Some u′*
vfind-match cs u′ = *Some* (*env*, *pat*, *rhs*)
Γ ++$_f$ *env* ⊢$_v$ *rhs* ↓ *v*
using *assms* **proof** *cases*
case (*constr name ts*)
hence *strip-comb* (*Sabs cs* $$_s$ *Svar n*) = *strip-comb* (*name* $$ *ts*)
by *simp*

```
  hence False
    apply (fold app-sterm-def)
    apply (simp add: strip-list-comb-const)
    apply (simp add: const-sterm-def)
    done
  thus ?thesis by simp
next
  case rec-comb
  hence False by cases
  thus ?thesis by simp
next
  case (comb cs' Γ' u' env pat rhs)
  moreover have fmlookup Γ n = Some u'
    using ‹Γ ⊢v Svar n ↓ u'›
    proof cases
      case (constr name ts)
      hence False
        by (fold free-sterm-def) simp
      thus ?thesis by simp
    qed auto
  moreover have cs = cs' Γ = Γ'
    using ‹Γ ⊢v Sabs cs ↓ Vabs cs' Γ'›
    by (cases; auto)+

  ultimately show ?thesis
    using that by auto
qed

lemma veval'-wellformed:
  assumes Γ ⊢v t ↓ v wellformed t wellformed-venv Γ
  shows vwellformed v
using assms proof induction
  case comb
  show ?case
    apply (rule comb)
      using comb by (auto simp: list-all-iff dest: vfind-match-elem intro: vwell-
formed.vmatch-env)
next
  case (rec-comb Γ t css name Γ' cs u u' env pat rhs val)
  have (pat, rhs) ∈ set cs
    by (rule vfind-match-elem) fact
  show ?case
    proof (rule rec-comb)
      show wellformed-venv (Γ' ++f mk-rec-env css Γ' ++f env)
        proof (intro fmpred-add)
          show wellformed-venv Γ'
            using rec-comb by auto
        next
          show wellformed-venv env
```

```
        using rec-comb by (auto dest: vfind-match-elem intro: vwellformed.vmatch-env)
       next
        show wellformed-venv (mk-rec-env css Γ')
         unfolding mk-rec-env-def
         using rec-comb by (auto intro: fmdomI)
       qed
    next
     have vwellformed (Vrecabs css name Γ')
      unfolding mk-rec-env-def
      using rec-comb by (auto intro: fmdom'I)
     thus wellformed rhs
      using ‹(pat, rhs) ∈ set cs› rec-comb by (auto simp: list-all-iff)
    qed
next
  case (constr name Γ ts us)
  have list-all vwellformed us
    using ‹list-all2 - - -› ‹wellformed (- $$ -)›
    proof (induction ts us rule: list.rel-induct)
     case (Cons v vs u us)
     thus ?case
      using constr by (auto simp: app-sterm-def wellformed.list-comb)
    qed simp
  thus ?case
    by (simp add: list-all-iff)
qed auto

lemma (in constants) veval'-shadows:
  assumes Γ ⊢v t ↓ v not-shadows-vconsts-env Γ ¬ shadows-consts t
  shows not-shadows-vconsts v
using assms proof induction
  case comb
  show ?case
    apply (rule comb)
   using comb by (auto simp: list-all-iff dest: vfind-match-elem intro: not-shadows-vconsts.vmatch-env)
next
  case (rec-comb Γ t css name Γ' cs u u' env pat rhs val)
  have (pat, rhs) ∈ set cs
    by (rule vfind-match-elem) fact
  show ?case
    proof (rule rec-comb)
     show not-shadows-vconsts-env (Γ' ++f mk-rec-env css Γ' ++f env)
      proof (intro fmpred-add)
       show not-shadows-vconsts-env env
       using rec-comb by (auto dest: vfind-match-elem intro: not-shadows-vconsts.vmatch-env)
      next
       show not-shadows-vconsts-env (mk-rec-env css Γ')
        unfolding mk-rec-env-def
        using rec-comb by (auto intro: fmdomI)
      next
```

```
        show not-shadows-vconsts-env Γ'
          using rec-comb by auto
      qed
    next
      have not-shadows-vconsts (Vrecabs css name Γ')
        using rec-comb by auto
      thus ¬ shadows-consts rhs
        using ‹(pat, rhs) ∈ set cs› rec-comb by (auto simp: list-all-iff)
    qed
next
  case (constr name Γ ts us)
  have list-all (not-shadows-vconsts) us
    using ‹list-all2 - - -› ‹¬ shadows-consts (name $$ ts)›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      thus ?case
        using constr by (auto simp: shadows.list-comb app-sterm-def)
    qed simp
  thus ?case
    by (simp add: list-all-iff)
qed (auto simp: list-all-iff list-ex-iff)

lemma veval'-closed:
  assumes Γ ⊢v t ↓ v closed-except t (fmdom Γ) closed-venv Γ
  assumes wellformed t wellformed-venv Γ
  shows vclosed v
using assms proof induction
  case (comb Γ t cs Γ' u u' env pat rhs val)
  hence vclosed (Vabs cs Γ')
    by (auto simp: closed-except-def)

  have (pat, rhs) ∈ set cs vmatch (mk-pat pat) u' = Some env
    by (rule vfind-match-elem; fact)+
  hence fmdom env = patvars (mk-pat pat)
    by (simp add: vmatch-dom)

  have vwellformed (Vabs cs Γ')
    apply (rule veval'-wellformed)
    using comb by auto
  hence linear pat
    using ‹(pat, rhs) ∈ set cs›
    by (auto simp: list-all-iff)
  hence fmdom env = frees pat
    unfolding ‹fmdom env = -›
    by (simp add: mk-pat-frees)

  show ?case
    proof (rule comb)
      show wellformed rhs
```

```
      using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ′)›
      by (auto simp: list-all-iff)
    next
     show closed-venv (Γ′ ++f env)
      apply rule
      using ‹vclosed (Vabs cs Γ′)› apply auto[]
      apply (rule vclosed.vmatch-env)
       apply (rule vfind-match-elem)
      using comb by (auto simp: closed-except-def)
    next
     show closed-except rhs (fmdom (Γ′ ++f env))
      using ‹vclosed (Vabs cs Γ′)› ‹fmdom env = frees pat› ‹(pat, rhs) ∈ set cs›
      by (auto simp: list-all-iff)
    next
     show wellformed-venv (Γ′ ++f env)
      apply rule
      using ‹vwellformed (Vabs cs Γ′)› apply auto[]
      apply (rule vwellformed.vmatch-env)
       apply (rule vfind-match-elem)
       apply fact
      apply (rule veval′-wellformed)
      using comb by auto
    qed
next
  case (rec-comb Γ t css name Γ′ cs u u′ env pat rhs val)
  have (pat, rhs) ∈ set cs vmatch (mk-pat pat) u′ = Some env
    by (rule vfind-match-elem; fact)+
  hence fmdom env = patvars (mk-pat pat)
    by (simp add: vmatch-dom)

  have vwellformed (Vrecabs css name Γ′)
    apply (rule veval′-wellformed)
    using rec-comb by auto
  hence wellformed-clauses cs
    using rec-comb by auto
  hence linear pat
    using ‹(pat, rhs) ∈ set cs›
    by (auto simp: list-all-iff)
  hence fmdom env = frees pat
    unfolding ‹fmdom env = -›
    by (simp add: mk-pat-frees)
  show ?case
    proof (rule rec-comb)
     show closed-venv (Γ′ ++f mk-rec-env css Γ′ ++f env)
      proof (intro fmpred-add)
       show closed-venv Γ′
        using rec-comb by (auto simp: closed-except-def)
      next
       show closed-venv env
```

```
          using rec-comb by (auto simp: closed-except-def dest: vfind-match-elem
intro: vclosed.vmatch-env)
        next
          show closed-venv (mk-rec-env css Γ')
            unfolding mk-rec-env-def
            using rec-comb by (auto simp: closed-except-def intro: fmdomI)
        qed
    next
      have vclosed (Vrecabs css name Γ')
        using mk-rec-env-def
        using rec-comb by (auto simp: closed-except-def intro: fmdom'I)
      hence closed-except rhs (fmdom Γ' |∪| frees pat)
        apply simp
        apply (elim conjE)
        apply (drule fmpredD[where m = css])
         apply (rule rec-comb)
        using ‹(pat, rhs) ∈ set cs›
        unfolding list-all-iff by auto

      thus closed-except rhs (fmdom (Γ' ++f mk-rec-env css Γ' ++f env))
        unfolding closed-except-def
        using ‹fmdom env = frees pat›
        by auto
    next
      show wellformed rhs
        using ‹wellformed-clauses cs› ‹(pat, rhs) ∈ set cs›
        by (auto simp: list-all-iff)
    next
      show wellformed-venv (Γ' ++f mk-rec-env css Γ' ++f env)
        proof (intro fmpred-add)
          show wellformed-venv Γ'
            using ‹vwellformed (Vrecabs css name Γ')› by auto
        next
          show wellformed-venv env
            using rec-comb by (auto dest: vfind-match-elem intro: veval'-wellformed
vwellformed.vmatch-env)
        next
          show wellformed-venv (mk-rec-env css Γ')
            unfolding mk-rec-env-def
            using ‹vwellformed (Vrecabs css name Γ')› by (auto intro: fmdomI)
        qed
    qed
next
  case (constr name Γ ts us)
  have list-all vclosed us
    using ‹list-all2 - - -› ‹closed-except (- $$ -) -› ‹wellformed (- $$ -)›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      with constr show ?case
```

```
      unfolding closed.list-comb wellformed.list-comb
      by (auto simp: Sterm.closed-except-simps)
    qed simp
  thus ?case
    by (simp add: list-all-iff)
qed (auto simp: Sterm.closed-except-simps)

primrec vwelldefined' :: value ⇒ bool where
vwelldefined' (Vconstr name vs) ⟷ list-all vwelldefined' vs |
vwelldefined' (Vabs cs Γ) ⟷
  pred-fmap id (fmmap vwelldefined' Γ) ∧
  list-all (λ(pat, t). consts t |⊆| (fmdom Γ |∪| C)) cs ∧
  fdisjnt C (fmdom Γ) |
vwelldefined' (Vrecabs css name Γ) ⟷
  pred-fmap id (fmmap vwelldefined' Γ) ∧
  pred-fmap (λcs.
    list-all (λ(pat, t). consts t |⊆| fmdom Γ |∪| (C |∪| fmdom css)) cs ∧
    fdisjnt C (fmdom Γ)) css ∧
  name |∈| fmdom css ∧
  fdisjnt C (fmdom css)

lemma vmatch-welldefined':
  assumes vmatch pat v = Some env vwelldefined' v
  shows fmpred (λ-. vwelldefined') env
using assms proof (induction pat v arbitrary: env rule: vmatch-induct)
  case (constr name ps name' vs)
  hence
    map-option (foldl (++f) fmempty) (those (map2 vmatch ps vs)) = Some env
    name = name' length ps = length vs
    by (auto split: if-splits)
  then obtain envs where env = foldl (++f) fmempty envs map2 vmatch ps vs
= map Some envs
    by (blast dest: those-someD)

  moreover have fmpred (λ-. vwelldefined') env if env ∈ set envs for env
    proof −
      from that have Some env ∈ set (map2 vmatch ps vs)
        unfolding ‹map2 - - - = -› by simp
      then obtain p v where p ∈ set ps v ∈ set vs vmatch p v = Some env
        by (auto elim: map2-elemE)
      hence vwelldefined' v
        using constr by (simp add: list-all-iff)
      show ?thesis
        by (rule constr; safe?) fact+
    qed

    ultimately show ?case
      by auto
qed auto
```

lemma *sconsts-list-comb*:
  consts (*list-comb f xs*) |⊆| *S* ⟷ *consts f* |⊆| *S* ∧ *list-all* (λ*x*. *consts x* |⊆| *S*) *xs*
by (*induction xs arbitrary*: *f*) *auto*

lemma *sconsts-sabs*:
 consts (*Sabs cs*) |⊆| *S* ⟷ *list-all* (λ(-, *t*). *consts t* |⊆| *S*) *cs*
 apply (*auto simp*: *list-all-iff ffUnion-alt-def dest*!: *ffUnion-least-rev*)
  apply (*subst* (*asm*) *list-all-iff-fset*[*symmetric*])
  apply (*auto simp*: *list-all-iff fset-of-list-elem*)
 done

lemma (**in** *constants*) *veval′-welldefined′*:
 assumes Γ ⊢$_v$ *t* ↓ *v fdisjnt C* (*fmdom* Γ)
 assumes *consts t* |⊆| *fmdom* Γ |∪| *C fmpred* (λ-. *vwelldefined′*) Γ
 assumes *wellformed t wellformed-venv* Γ
 assumes ¬ *shadows-consts t not-shadows-vconsts-env* Γ
 shows *vwelldefined′ v*
using *assms* **proof** *induction*
 case (*abs* Γ *cs*)
 thus *?case*
  unfolding *sconsts-sabs*
  by (*auto simp*: *list-all-iff list-ex-iff*)
next
 case (*comb* Γ *t cs* Γ′ *u u′ env pat rhs val*)
 hence (*pat*, *rhs*) ∈ *set cs*
  by (*auto dest*: *vfind-match-elem*)
 moreover have *vwelldefined′* (*Vabs cs* Γ′)
  using *comb* **by** *auto*
 ultimately have *consts rhs* |⊆| *fmdom* Γ′ |∪| *C*
  by (*auto simp*: *list-all-iff*)

 have *vwellformed* (*Vabs cs* Γ′)
  apply (*rule veval′-wellformed*)
  using *comb* **by** *auto*
 hence *linear pat*
  using ‹(*pat*, *rhs*) ∈ *set cs*›
  by (*auto simp*: *list-all-iff*)
 hence *frees pat* = *patvars* (*mk-pat pat*)
  by (*simp add*: *mk-pat-frees*)
 hence *fmdom env* = *frees pat*
  apply *simp*
  apply (*rule vmatch-dom*)
  apply (*rule vfind-match-elem*)
  apply (*rule comb*)
  done

 have *not-shadows-vconsts* (*Vabs cs* Γ′)

```
  apply (rule veval'-shadows)
  using comb by auto

have vwelldefined' (Vabs cs Γ')
  using comb by auto

show ?case
  proof (rule comb)
    show consts rhs |⊆| fmdom (Γ' ++_f env) |∪| C
      using ‹consts rhs |⊆| fmdom Γ' |∪| C›
      by auto
  next
    have vwelldefined' u'
      using comb by auto
    hence fmpred (λ-. vwelldefined') env
      using comb
      by (auto intro: vmatch-welldefined' dest: vfind-match-elem)
    thus fmpred (λ-. vwelldefined') (Γ' ++_f env)
      using ‹vwelldefined' (Vabs cs Γ')› by auto
  next
    have fdisjnt C (fmdom Γ')
      using ‹vwelldefined' (Vabs cs Γ')›
      by simp
    moreover have fdisjnt C (fmdom env)
      unfolding ‹fmdom env = frees pat›
      using ‹(pat, rhs) ∈ set cs› ‹not-shadows-vconsts (Vabs cs Γ')›
      by (auto simp: list-all-iff all-consts-def fdisjnt-alt-def)
    ultimately show fdisjnt C (fmdom (Γ' ++_f env))
      unfolding fdisjnt-alt-def by auto
  next
    show wellformed rhs
      using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ')›
      by (auto simp: list-all-iff)
  next
    have wellformed-venv Γ'
      using ‹vwellformed (Vabs cs Γ')› by simp
    moreover have wellformed-venv env
      apply (rule vwellformed.vmatch-env)
       apply (rule vfind-match-elem)
       apply fact
      apply (rule veval'-wellformed)
      using comb by auto
    ultimately show wellformed-venv (Γ' ++_f env)
      by blast
  next
    have not-shadows-vconsts-env Γ'
      using ‹not-shadows-vconsts (Vabs cs Γ')› by simp
    moreover have not-shadows-vconsts-env env
      apply (rule not-shadows-vconsts.vmatch-env)
```

```
        apply (rule vfind-match-elem)
        apply fact
       apply (rule veval'-shadows)
       using comb by auto
     ultimately show not-shadows-vconsts-env (Γ' ++ₓ env)
       by blast
   next
     show ¬ shadows-consts rhs
       using ‹not-shadows-vconsts (Vabs cs Γ')› ‹(pat, rhs) ∈ set cs›
       by (auto simp: list-all-iff)
   qed
next
  case (rec-comb Γ t css name Γ' cs u u' env pat rhs val)
  hence (pat, rhs) ∈ set cs
    by (auto dest: vfind-match-elem)
  moreover have vwelldefined' (Vrecabs css name Γ')
    using rec-comb by auto
  ultimately have consts rhs |⊆| fmdom Γ' |∪| (C |∪| fmdom css)
    using ‹fmlookup css name = Some cs›
    by (auto simp: list-all-iff dest!: fmpredD[where m = css])

  have vwellformed (Vrecabs css name Γ')
    apply (rule veval'-wellformed)
    using rec-comb by auto
  hence wellformed-clauses cs
    using rec-comb by auto
  hence linear pat
    using ‹(pat, rhs) ∈ set cs›
    by (auto simp: list-all-iff)
  hence frees pat = patvars (mk-pat pat)
    by (simp add: mk-pat-frees)
  hence fmdom env = frees pat
    apply simp
    apply (rule vmatch-dom)
    apply (rule vfind-match-elem)
    apply (rule rec-comb)
    done

  have not-shadows-vconsts (Vrecabs css name Γ')
    apply (rule veval'-shadows)
    using rec-comb by auto

  have vwelldefined' (Vrecabs css name Γ')
    using rec-comb by auto

  show ?case
    proof (rule rec-comb)
      show consts rhs |⊆| fmdom (Γ' ++ₓ mk-rec-env css Γ' ++ₓ env) |∪| C
        using ‹consts rhs |⊆| -› unfolding mk-rec-env-def
```

```
      by auto
    next
      have fmpred (λ-. vwelldefined') Γ'
        using ‹vwelldefined' (Vrecabs css name Γ')› by auto
      moreover have fmpred (λ-. vwelldefined') (mk-rec-env css Γ')
        unfolding mk-rec-env-def
        using rec-comb by (auto intro: fmdomI)
      moreover have fmpred (λ-. vwelldefined') env
        using rec-comb by (auto dest: vfind-match-elem intro: vmatch-welldefined')
      ultimately show fmpred (λ-. vwelldefined') (Γ' ++f mk-rec-env css Γ' ++f
env)
        by blast
    next
      have fdisjnt C (fmdom Γ')
        using rec-comb by auto
      moreover have fdisjnt C (fmdom env)
        unfolding ‹fmdom env = frees pat›
      using ‹fmlookup css name = Some cs› ‹(pat, rhs) ∈ set cs› ‹not-shadows-vconsts
(Vrecabs css name Γ')›
        apply auto
        apply (drule fmpredD[where m = css])
        by (auto simp: list-all-iff all-consts-def fdisjnt-alt-def)
      moreover have fdisjnt C (fmdom (mk-rec-env css Γ'))
        unfolding mk-rec-env-def
        using ‹vwelldefined' (Vrecabs css name Γ')›
        by simp
      ultimately show fdisjnt C (fmdom (Γ' ++f mk-rec-env css Γ' ++f env))
        unfolding fdisjnt-alt-def by auto
    next
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› ‹wellformed-clauses cs›
        by (auto simp: list-all-iff)
    next
      have wellformed-venv Γ'
        using ‹vwellformed (Vrecabs css name Γ')› by simp
      moreover have wellformed-venv (mk-rec-env css Γ')
        unfolding mk-rec-env-def
        using ‹vwellformed (Vrecabs css name Γ')›
        by (auto intro: fmdomI)
      moreover have wellformed-venv env
        apply (rule vwellformed.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
        apply (rule veval'-wellformed)
        using rec-comb by auto
      ultimately show wellformed-venv (Γ' ++f mk-rec-env css Γ' ++f env)
        by blast
    next
      have not-shadows-vconsts-env Γ'
```

```
      using ‹not-shadows-vconsts (Vrecabs css name Γ')› by simp
    moreover have not-shadows-vconsts-env (mk-rec-env css Γ')
      unfolding mk-rec-env-def
      using ‹not-shadows-vconsts (Vrecabs css name Γ')›
      by (auto intro: fmdomI)
    moreover have not-shadows-vconsts-env env
      apply (rule not-shadows-vconsts.vmatch-env)
       apply (rule vfind-match-elem)
       apply fact
      apply (rule veval'-shadows)
      using rec-comb by auto
    ultimately show not-shadows-vconsts-env (Γ' ++_f mk-rec-env css Γ' ++_f
env)
      by blast
  next
    show ¬ shadows-consts rhs
      using rec-comb ‹not-shadows-vconsts (Vrecabs css name Γ')› ‹(pat, rhs) ∈
set cs›
      by (auto simp: list-all-iff)
  qed
next
  case (constr name Γ ts us)
  have list-all vwelldefined' us
    using ‹list-all2 - - -› ‹consts (name $$ ts) |⊆| -›
    using ‹wellformed (name $$ ts)› ‹¬ shadows-consts (name $$ ts)›
    proof (induction ts us rule: list.rel-induct)
      case (Cons v vs u us)
      with constr show ?case
        unfolding wellformed.list-comb shadows.list-comb
        by (auto simp: consts-list-comb)
    qed simp
  thus ?case
    by (simp add: list-all-iff)
qed auto

end
```

Correctness wrt *veval*

context *vrules* **begin**

The following relation can be characterized as follows:

- Values have to have the same structure. (We prove an interpretation of *value-struct-rel*.)
- For closures, the captured environments must agree on constants and variables occurring in the body. The first *value* argument is from *veval* (i.e. from *CakeML-Codegen.Big-Step-Value*), the second from *veval'*.

coinductive *vrelated* :: *value* ⇒ *value* ⇒ *bool* (‹⊢$_v$/ - ≈ -› [*0*, *50*] *50*) **where**
constr: *list-all2 vrelated ts us* ⟹ ⊢$_v$ *Vconstr name ts* ≈ *Vconstr name us* |
abs:
fmrel-on-fset (*frees* (*Sabs cs*)) *vrelated* Γ$_1$ Γ$_2$ ⟹
fmrel-on-fset (*consts* (*Sabs cs*)) *vrelated* (*fmap-of-list rs*) Γ$_2$ ⟹
⊢$_v$ *Vabs cs* Γ$_1$ ≈ *Vabs cs* Γ$_2$ |
rec-abs:
pred-fmap (λ*cs*.
fmrel-on-fset (*frees* (*Sabs cs*)) *vrelated* Γ$_1$ Γ$_2$ ∧
fmrel-on-fset (*consts* (*Sabs cs*)) *vrelated* (*fmap-of-list rs*) (Γ$_2$ ++$_f$ *mk-rec-env css* Γ$_2$)) *css* ⟹
name |∈| *fmdom css* ⟹
⊢$_v$ *Vrecabs css name* Γ$_1$ ≈ *Vrecabs css name* Γ$_2$

Perhaps unexpectedly, *vrelated* is not reflexive. The reason is that it does not just check syntactic equality including captured environments, but also adherence to the external rules.

sublocale *vrelated*: *value-struct-rel vrelated*
proof
fix *t*$_1$ *t*$_2$
assume ⊢$_v$ *t*$_1$ ≈ *t*$_2$
thus *veq-structure t*$_1$ *t*$_2$
apply (*induction t*$_1$ *arbitrary*: *t*$_2$)
apply (*erule vrelated.cases*; *auto*)
apply (*erule list.rel-mono-strong*)
apply *simp*
apply (*erule vrelated.cases*; *auto*)
apply (*erule vrelated.cases*; *auto*)
done
next
fix *name name′* **and** *ts us* :: *value list*
show ⊢$_v$ *Vconstr name ts* ≈ *Vconstr name′ us* ⟷ (*name* = *name′* ∧ *list-all2 vrelated ts us*)
proof *safe*
assume ⊢$_v$ *Vconstr name ts* ≈ *Vconstr name′ us*
thus *name* = *name′ list-all2 vrelated ts us*
by (*cases*; *auto*)+
qed (*auto intro*: *vrelated.intros*)
qed

The technically involved relation *vrelated* implies a weaker, but more intuitive property: If ⊢$_v$ *t* ≈ *u* then *t* and *u* are equal after termification (i.e. conversion with *value-to-sterm*). In fact, if both terms are ground terms, it collapses to equality. This follows directly from the interpretation of *value-struct-rel*.

lemma *veval′-correct*:
assumes Γ$_2$ ⊢$_v$ *t* ↓ *v*$_2$ *wellformed t wellformed-venv* Γ$_2$
assumes ¬ *shadows-consts t not-shadows-vconsts-env* Γ$_2$

```
  assumes welldefined t
  assumes fmpred (λ-. vwelldefined) Γ1
  assumes fmrel-on-fset (frees t) vrelated Γ1 Γ2
  assumes fmrel-on-fset (consts t) vrelated (fmap-of-list rs) Γ2
  obtains v1 where rs, Γ1 ⊢v t ↓ v1 ⊢v v1 ≈ v2
apply atomize-elim
using assms proof (induction arbitrary: Γ1)
  case (const name Γ2 val2)
  hence fmrel-on-fset {|name|} vrelated (fmap-of-list rs) Γ2
    by simp
  have rel-option vrelated (fmlookup (fmap-of-list rs) name) (fmlookup Γ2 name)
    apply (rule fmrel-on-fsetD[where S = {|name|}])
     apply simp
    apply fact
    done
  then obtain val1 where fmlookup (fmap-of-list rs) name = Some val1 ⊢v val1
≈ val2
    using const by cases auto
  hence (name, val1) ∈ set rs
    by (auto dest: fmap-of-list-SomeD)

  show ?case
    apply (intro conjI exI)
     apply (rule veval.const)
     apply fact+
    done
next
  case (var Γ2 name val2)
  hence fmrel-on-fset {|name|} vrelated Γ1 Γ2
    by simp
  have rel-option vrelated (fmlookup Γ1 name) (fmlookup Γ2 name)
    apply (rule fmrel-on-fsetD[where S = {|name|}])
     apply simp
    apply fact
    done
  then obtain val1 where fmlookup Γ1 name = Some val1 ⊢v val1 ≈ val2
    using var by cases auto
  show ?case
    apply (intro conjI exI)
     apply (rule veval.var)
     apply fact+
    done
next
  case abs
  thus ?case
    by (auto intro!: veval.abs vrelated.abs)
next
  case (comb Γ2 t cs Γ'2 u u'2 env2 pat rhs val2)
  hence ∃ v. rs, Γ1 ⊢v t ↓ v ∧ ⊢v v ≈ Vabs cs Γ'2
```

by (*auto intro*: *fmrel-on-fsubset*)
then obtain v **where** $\vdash_v v \approx \mathit{Vabs}\ cs\ \Gamma'_2\ rs, \Gamma_1 \vdash_v t \downarrow v$
by *blast*
moreover then obtain Γ'_1
where $v = \mathit{Vabs}\ cs\ \Gamma'_1$
and *fmrel-on-fset* (*frees* (*Sabs cs*)) *vrelated* $\Gamma'_1\ \Gamma'_2$
and *fmrel-on-fset* (*consts* (*Sabs cs*)) *vrelated* (*fmap-of-list rs*) Γ'_2
by *cases auto*
ultimately have $rs, \Gamma_1 \vdash_v t \downarrow \mathit{Vabs}\ cs\ \Gamma'_1$
by *simp*

have $\exists u_1'.\ rs, \Gamma_1 \vdash_v u \downarrow u_1' \wedge \vdash_v u_1' \approx u'_2$
using *comb* **by** (*auto intro*: *fmrel-on-fsubset*)
then obtain u'_1 **where** $\vdash_v u'_1 \approx u'_2\ rs, \Gamma_1 \vdash_v u \downarrow u'_1$
by *blast*

have *rel-option* (*rel-prod* (*fmrel vrelated*) (=)) (*vfind-match cs* u'_1) (*vfind-match cs* u'_2)
by (*rule vrelated.vfind-match-rel'*) *fact*
then obtain env_1 **where** *vfind-match cs* u'_1 = *Some* (env_1, *pat*, *rhs*) *fmrel vrelated* env_1 env_2
using ‹*vfind-match cs* u'_2 = -›
by *cases auto*

have (*pat*, *rhs*) ∈ *set cs*
by (*rule vfind-match-elem*) *fact*

have *vwellformed* (*Vabs cs* Γ'_2)
apply (*rule veval'-wellformed*)
apply *fact*
using ‹*wellformed* (*t* $_s$ *u*)› **apply** *simp*
apply *fact+*
done
hence *wellformed-venv* Γ'_2
by *simp*

have *vwelldefined v*
apply (*rule veval-welldefined*)
apply *fact+*
using *comb* **by** *simp*
hence *vwelldefined* (*Vabs cs* Γ'_1)
unfolding ‹*v* = -› **.**

have *linear pat*
using ‹(*pat*, *rhs*) ∈ *set cs*› ‹*vwellformed* (*Vabs cs* Γ'_2)›
by (*auto simp*: *list-all-iff*)

have *fmdom* env_1 = *patvars* (*mk-pat pat*)
apply (*rule vmatch-dom*)

```
    apply (rule vfind-match-elem)
    apply fact
    done
  with ‹linear pat› have fmdom env₁ = frees pat
    by (simp add: mk-pat-frees)

  have fmdom env₂ = patvars (mk-pat pat)
    apply (rule vmatch-dom)
    apply (rule vfind-match-elem)
    apply fact
    done
  with ‹linear pat› have fmdom env₂ = frees pat
    by (simp add: mk-pat-frees)

  note fset-of-list-map[simp del]
  have ∃ val₁. rs, Γ'₁ ++ₓ env₁ ⊢ᵥ rhs ↓ val₁ ∧ ⊢ᵥ val₁ ≈ val₂
    proof (rule comb)
      show fmrel-on-fset (frees rhs) vrelated (Γ'₁ ++ₓ env₁) (Γ'₂ ++ₓ env₂)
        proof
          fix name
          assume name |∈| frees rhs
          show rel-option vrelated (fmlookup (Γ'₁ ++ₓ env₁) name) (fmlookup (Γ'₂
++ₓ env₂) name)
            proof (cases name |∈| frees pat)
              case True
              hence name |∈| fmdom env₁ name |∈| fmdom env₂
                using ‹fmdom env₁ = frees pat› ‹fmdom env₂ = frees pat›
                by simp+
             hence fmlookup (Γ'₁ ++ₓ env₁) name = fmlookup env₁ name fmlookup
(Γ'₂ ++ₓ env₂) name = fmlookup env₂ name
                by auto
              thus ?thesis
                using ‹fmrel vrelated env₁ env₂›
                by auto
            next
              case False
              hence name |∉| fmdom env₁ name |∉| fmdom env₂
                using ‹fmdom env₁ = frees pat› ‹fmdom env₂ = frees pat›
                by simp+
              hence fmlookup (Γ'₁ ++ₓ env₁) name = fmlookup Γ'₁ name fmlookup
(Γ'₂ ++ₓ env₂) name = fmlookup Γ'₂ name
                by auto

              moreover have name |∈| frees (Sabs cs)
                using False ‹name |∈| frees rhs› ‹(pat, rhs) ∈ set cs›
                apply (auto simp: ffUnion-alt-def)
                apply (rule fBexI[where x = frees rhs |−| frees pat])
                 apply (auto simp: fset-of-list-elem)
                done
```

```
            ultimately show ?thesis
              using ‹fmrel-on-fset (frees (Sabs cs)) vrelated Γ′1 Γ′2›
              by (auto dest: fmrel-on-fsetD)
          qed
        qed
      next
        have not-shadows-vconsts (Vabs cs Γ′2)
          apply (rule veval′-shadows)
           apply fact+
          using comb by auto
        thus ¬ shadows-consts rhs
          using ‹(pat, rhs) ∈ set cs›
          by (auto simp: list-all-iff)

        show not-shadows-vconsts-env (Γ′2 ++f env2)
          apply rule
          using ‹not-shadows-vconsts (Vabs cs Γ′2)› apply simp
          apply (rule not-shadows-vconsts.vmatch-env)
           apply (rule vfind-match-elem)
           apply fact
          apply (rule veval′-shadows)
           apply fact
           apply fact
          using comb by auto

        show fmrel-on-fset (consts rhs) vrelated (fmap-of-list rs) (Γ′2 ++f env2)
          proof
            fix name
            assume name |∈| consts rhs
            hence name |∈| consts (Sabs cs)
              using ‹(pat, rhs) ∈ set cs›
              by (auto intro!: fBexI simp: fset-of-list-elem ffUnion-alt-def)
            hence rel-option vrelated (fmlookup (fmap-of-list rs) name) (fmlookup Γ′2
name)
              using ‹fmrel-on-fset (consts (Sabs cs)) vrelated (fmap-of-list rs) Γ′2›
              by (auto dest: fmrel-on-fsetD)
            moreover have name |∉| fmdom env2
              proof
                assume name |∈| fmdom env2
                hence fmlookup env2 name ≠ None
                  by (meson fmdom-notI)
                then obtain v where fmlookup env2 name = Some v
                  by blast
                hence name |∈| fmdom env2
                  by (auto intro: fmdomI)
                hence name |∈| frees pat
                  using ‹fmdom env2 = frees pat›
                  by simp
```

```
            have welldefined rhs
              using ‹vwelldefined (Vabs cs Γ'1)› ‹(pat, rhs) ∈ set cs›
              by (auto simp: list-all-iff)
            hence name |∈| fst |`| fset-of-list rs |∪| C
              using ‹name |∈| consts rhs›
              by (auto simp: all-consts-def)
            moreover have ¬ shadows-consts pat
              using ‹not-shadows-vconsts (Vabs cs Γ'2)› ‹(pat, rhs) ∈ set cs›
              by (auto simp: list-all-iff shadows-consts-def all-consts-def)
            ultimately show False
              using ‹name |∈| frees pat›
              unfolding shadows-consts-def fdisjnt-alt-def all-consts-def
              by auto
          qed
          ultimately show rel-option vrelated (fmlookup (fmap-of-list rs) name)
(fmlookup (Γ'2 ++f env2) name)
          by simp
      qed
    next
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ'2)›
        by (auto simp: list-all-iff)
    next
      have wellformed-venv Γ'2
        by fact
      moreover have wellformed-venv env2
        apply (rule vwellformed.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
        apply (rule veval'-wellformed)
          apply fact
        using ‹wellformed (t $s u)› apply simp
        apply fact+
        done
      ultimately show wellformed-venv (Γ'2 ++f env2)
        by blast
    next
      show welldefined rhs
        using ‹vwelldefined (Vabs cs Γ'1)› ‹(pat, rhs) ∈ set cs›
        by (auto simp: list-all-iff)
    next
      have fmpred (λ-. vwelldefined) Γ'1
        using ‹vwelldefined (Vabs cs Γ'1)› by simp

      moreover have fmpred (λ-. vwelldefined) env1
        apply (rule vwelldefined.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
```

```
      apply (rule veval-welldefined)
        apply fact
       apply fact
      using comb apply simp
      done

    ultimately show fmpred (λ-. vwelldefined) (Γ'₁ ++_f env₁)
      by blast
  qed

  then obtain val₁ where rs, Γ'₁ ++_f env₁ ⊢_v rhs ↓ val₁ ⊢_v val₁ ≈ val₂
    by blast

  show ?case
    apply (intro conjI exI)
     apply (rule veval.comb)
       apply fact+
    done
next
  — Almost verbatim copy from comb case.
  case (rec-comb Γ₂ t css name Γ'₂ cs u u'₂ env₂ pat rhs val₂)
  hence ∃ v. rs, Γ₁ ⊢_v t ↓ v ∧ ⊢_v v ≈ Vrecabs css name Γ'₂
    by (auto intro: fmrel-on-fsubset)
  then obtain v where ⊢_v v ≈ Vrecabs css name Γ'₂ rs, Γ₁ ⊢_v t ↓ v
    by blast
  moreover then obtain Γ'₁
    where v = Vrecabs css name Γ'₁
      and fmrel-on-fset (frees (Sabs cs)) vrelated Γ'₁ Γ'₂
        and fmrel-on-fset (consts (Sabs cs)) vrelated (fmap-of-list rs) (Γ'₂ ++_f
mk-rec-env css Γ'₂)
    using ‹fmlookup css name = Some cs›
    by cases auto
  ultimately have rs, Γ₁ ⊢_v t ↓ Vrecabs css name Γ'₁
    by simp

  have ∃ u₁'. rs, Γ₁ ⊢_v u ↓ u₁' ∧ ⊢_v u₁' ≈ u'₂
    using rec-comb by (auto intro: fmrel-on-fsubset)
  then obtain u'₁ where ⊢_v u'₁ ≈ u'₂ rs, Γ₁ ⊢_v u ↓ u'₁
    by blast

  have rel-option (rel-prod (fmrel vrelated) (=)) (vfind-match cs u'₁) (vfind-match
cs u'₂)
    by (rule vrelated.vfind-match-rel') fact
   then obtain env₁ where vfind-match cs u'₁ = Some (env₁, pat, rhs) fmrel
vrelated env₁ env₂
    using ‹vfind-match cs u'₂ = -›
    by cases auto

  have (pat, rhs) ∈ set cs
```

```
  by (rule vfind-match-elem) fact

have vwellformed (Vrecabs css name Γ'2)
  apply (rule veval'-wellformed)
    apply fact
  using ‹wellformed (t $s u)› apply simp
  apply fact+
  done
hence wellformed-venv Γ'2 vwellformed (Vabs cs Γ'2)
  using rec-comb by auto

have vwelldefined v
  apply (rule veval-welldefined)
    apply fact+
  using rec-comb by simp
hence vwelldefined (Vrecabs css name Γ'1)
  unfolding ‹v = -› .
hence vwelldefined (Vabs cs Γ'1)
  using rec-comb by auto

have linear pat
  using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ'2)›
  by (auto simp: list-all-iff)

have fmdom env1 = patvars (mk-pat pat)
  apply (rule vmatch-dom)
  apply (rule vfind-match-elem)
  apply fact
  done
with ‹linear pat› have fmdom env1 = frees pat
  by (simp add: mk-pat-frees)

have fmdom env2 = patvars (mk-pat pat)
  apply (rule vmatch-dom)
  apply (rule vfind-match-elem)
  apply fact
  done
with ‹linear pat› have fmdom env2 = frees pat
  by (simp add: mk-pat-frees)

note fset-of-list-map[simp del]
have ∃ val1. rs, Γ'1 ++f env1 ⊢v rhs ↓ val1 ∧ ⊢v val1 ≈ val2
  proof (rule rec-comb)
    have not-shadows-vconsts (Vrecabs css name Γ'2)
      apply (rule veval'-shadows)
        apply fact+
      using rec-comb by auto
    thus ¬ shadows-consts rhs
      using ‹(pat, rhs) ∈ set cs› rec-comb
```

by (*auto simp*: *list-all-iff*)
hence *fdisjnt all-consts* (*frees rhs*)
by (*rule shadows-consts-frees*)

have *not-shadows-vconsts-env* Γ'_2
using ‹*not-shadows-vconsts* (*Vrecabs css name* Γ'_2)›
by *simp*

moreover have *not-shadows-vconsts-env* (*mk-rec-env css* Γ'_2)
unfolding *mk-rec-env-def*
using ‹*not-shadows-vconsts* (*Vrecabs css name* Γ'_2)›
by (*auto intro*: *fmdomI*)

moreover have *not-shadows-vconsts-env* env_2
apply (*rule not-shadows-vconsts.vmatch-env*)
apply (*rule vfind-match-elem*)
apply *fact*
apply (*rule veval'-shadows*)
apply *fact*
apply *fact*
using *rec-comb* **by** *auto*

ultimately show *not-shadows-vconsts-env* (Γ'_2 $++_f$ *mk-rec-env css* Γ'_2 $++_f$ env_2)
by *auto*

have *not-shadows-vconsts* (*Vabs cs* Γ'_2)
using ‹*not-shadows-vconsts* (*Vrecabs - - -*)› *rec-comb*
by *auto*

show *fmrel-on-fset* (*frees rhs*) *vrelated* (Γ'_1 $++_f$ env_1) (Γ'_2 $++_f$ *mk-rec-env css* Γ'_2 $++_f$ env_2)
proof
fix *name*
assume *name* $|\in|$ *frees rhs*
moreover have *fmdom css* $|\subseteq|$ *all-consts*
using ‹*vwelldefined* (*Vrecabs - - -*)› **unfolding** *all-consts-def*
by *auto*
ultimately have *name* $|\notin|$ *fmdom css*
using ‹*fdisjnt* - (*frees rhs*)›
unfolding *fdisjnt-alt-def*
by (*metis* (*full-types*) *fempty-iff finterI fset-rev-mp*)

show *rel-option vrelated* (*fmlookup* (Γ'_1 $++_f$ env_1) *name*) (*fmlookup* (Γ'_2 $++_f$ *mk-rec-env css* Γ'_2 $++_f$ env_2) *name*)
proof (*cases name* $|\in|$ *frees pat*)
case *True*
hence *name* $|\in|$ *fmdom* env_1 *name* $|\in|$ *fmdom* env_2
using ‹*fmdom* env_1 = *frees pat*› ‹*fmdom* env_2 = *frees pat*›

```
            by simp+
          hence
            fmlookup (Γ'₁ ++ₓ env₁) name = fmlookup env₁ name
            fmlookup (Γ'₂ ++ₓ mk-rec-env css Γ'₂ ++ₓ env₂) name = fmlookup
env₂ name
            by auto
          thus ?thesis
            using ‹fmrel vrelated env₁ env₂›
            by auto
        next
          case False
          hence name |∉| fmdom env₁ name |∉| fmdom env₂
            using ‹fmdom env₁ = frees pat› ‹fmdom env₂ = frees pat›
            by simp+
          hence
            fmlookup (Γ'₁ ++ₓ env₁) name = fmlookup Γ'₁ name
            fmlookup (Γ'₂ ++ₓ mk-rec-env css Γ'₂ ++ₓ env₂) name = fmlookup
Γ'₂ name
            unfolding mk-rec-env-def using ‹name |∉| fmdom css›
            by auto

          moreover have name |∈| frees (Sabs cs)
            using False ‹name |∈| frees rhs› ‹(pat, rhs) ∈ set cs›
            apply (auto simp: ffUnion-alt-def)
            apply (rule fBexI[where x = frees rhs |−| frees pat])
             apply (auto simp: fset-of-list-elem)
            done

          ultimately show ?thesis
            using ‹fmrel-on-fset (frees (Sabs cs)) vrelated Γ'₁ Γ'₂›
            by (auto dest: fmrel-on-fsetD)
        qed
      qed

    show fmrel-on-fset (consts rhs) vrelated (fmap-of-list rs) (Γ'₂ ++ₓ mk-rec-env
css Γ'₂ ++ₓ env₂)
      proof
        fix name
        assume name |∈| consts rhs
        hence name |∈| consts (Sabs cs)
          using ‹(pat, rhs) ∈ set cs›
          by (auto intro!: fBexI simp: fset-of-list-elem ffUnion-alt-def)
       hence rel-option vrelated (fmlookup (fmap-of-list rs) name) (fmlookup (Γ'₂
++ₓ mk-rec-env css Γ'₂) name)
          using ‹fmrel-on-fset (consts (Sabs cs)) vrelated (fmap-of-list rs) (Γ'₂
++ₓ mk-rec-env css Γ'₂)›
          by (auto dest: fmrel-on-fsetD)
        moreover have name |∉| fmdom env₂
          proof
```

```
            assume name |∈| fmdom env₂
            hence fmlookup env₂ name ≠ None
              by (meson fmdom-notI)
            then obtain v where fmlookup env₂ name = Some v
              by blast
            hence name |∈| fmdom env₂
              by (auto intro: fmdomI)
            hence name |∈| frees pat
              using ‹fmdom env₂ = frees pat›
              by simp

            have vwelldefined (Vabs cs Γ′₁)
              using ‹vwelldefined (Vrecabs css - Γ′₁)›
              using rec-comb
              by auto

            hence welldefined rhs
              using ‹(pat, rhs) ∈ set cs› rec-comb
              by (auto simp: list-all-iff)
            hence name |∈| fst |`| fset-of-list rs |∪| C
              using ‹name |∈| consts rhs› all-consts-def
              by blast
            moreover have ¬ shadows-consts pat
              using ‹not-shadows-vconsts (Vabs cs Γ′₂)› ‹(pat, rhs) ∈ set cs›
              by (auto simp: list-all-iff shadows-consts-def all-consts-def)
            ultimately show False
              using ‹name |∈| frees pat›
              unfolding shadows-consts-def fdisjnt-alt-def all-consts-def
              by auto
          qed
          ultimately show rel-option vrelated (fmlookup (fmap-of-list rs) name)
(fmlookup (Γ′₂ ++ᶠ mk-rec-env css Γ′₂ ++ᶠ env₂) name)
          by simp
      qed
    next
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ′₂)›
        by (auto simp: list-all-iff)
    next
      have wellformed-venv Γ′₂
        by fact
      moreover have wellformed-venv env₂
        apply (rule vwellformed.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
        apply (rule veval′-wellformed)
          apply fact
        using ‹wellformed (t $ₛ u)› apply simp
        apply fact+
```

```
      done
    moreover have wellformed-venv (mk-rec-env css Γ'₂)
      unfolding mk-rec-env-def
      using ‹vwellformed (Vrecabs - - -)›
      by (auto intro: fmdomI)
    ultimately show wellformed-venv (Γ'₂ ++_f mk-rec-env css Γ'₂ ++_f env₂)
      by blast
  next
    show welldefined rhs
      using ‹vwelldefined (Vabs cs Γ'₁)› ‹(pat, rhs) ∈ set cs›
      by (auto simp: list-all-iff)
  next
    have fmpred (λ-. vwelldefined) Γ'₁
      using ‹vwelldefined (Vabs cs Γ'₁)› by simp

    moreover have fmpred (λ-. vwelldefined) env₁
      apply (rule vwelldefined.vmatch-env)
       apply (rule vfind-match-elem)
       apply fact
      apply (rule veval-welldefined)
        apply fact
       apply fact
      using rec-comb apply simp
      done

    ultimately show fmpred (λ-. vwelldefined) (Γ'₁ ++_f env₁)
      by blast
  qed

 then obtain val₁ where rs, Γ'₁ ++_f env₁ ⊢_v rhs ↓ val₁ ⊢_v val₁ ≈ val₂
  by blast

 show ?case
  apply (intro conjI exI)
   apply (rule veval.rec-comb)
      apply fact+
  done
next
 case (constr name Γ₂ ts us₂)

 have list-all2 (λt u₂. (∃u₁. rs, Γ₁ ⊢_v t ↓ u₁ ∧ ⊢_v u₁ ≈ u₂)) ts us₂
  using ‹list-all2 - ts us₂›
  proof (rule list.rel-mono-strong, elim conjE impE allE exE)
    fix t u₂
    assume t ∈ set ts u₂ ∈ set us₂
    assume Γ₂ ⊢_v t ↓ u₂

    show wellformed t welldefined t ¬ shadows-consts t
      using constr ‹t ∈ set ts›
```

```
      unfolding welldefined.list-comb wellformed.list-comb shadows.list-comb
      by (auto simp: list-all-iff list-ex-iff)

    show
      wellformed-venv Γ₂
      not-shadows-vconsts-env Γ₂
      fmpred (λ-. vwelldefined) Γ₁
      by fact+

    have consts t |∈| fset-of-list (map consts ts)
      using ‹t ∈ set ts› by (simp add: fset-of-list-elem)
    hence consts t |⊆| consts (name $$ ts)
      unfolding consts-list-comb
      by (metis ffUnion-subset-elem le-supI2)
    thus fmrel-on-fset (consts t) vrelated (fmap-of-list rs) Γ₂
      using constr by (blast intro: fmrel-on-fsubset)

    have frees t |∈| fset-of-list (map frees ts)
      using ‹t ∈ set ts› by (simp add: fset-of-list-elem)
    hence frees t |⊆| frees (name $$ ts)
      unfolding frees-list-comb const-sterm-def freess-def
      by (auto intro!: ffUnion-subset-elem)
    thus fmrel-on-fset (frees t) vrelated Γ₁ Γ₂
      using constr by (blast intro: fmrel-on-fsubset)
  qed auto

  then obtain us₁ where list-all2 (veval rs Γ₁) ts us₁ list-all2 vrelated us₁ us₂
    by induction auto

  thus ?case
    using constr
    by (auto intro: veval.constr vrelated.constr)
qed

lemma veval'-correct':
  assumes Γ₂ ⊢ᵥ t ↓ v₂ wellformed t wellformed-venv Γ₂
  assumes ¬ shadows-consts t not-shadows-vconsts-env Γ₂
  assumes welldefined t
  assumes closed t
  assumes fmrel-on-fset (consts t) vrelated (fmap-of-list rs) Γ₂
  obtains v₁ where rs, fmempty ⊢ᵥ t ↓ v₁ ⊢ᵥ v₁ ≈ v₂
proof (rule veval'-correct[where Γ₁ = fmempty])
  show fmpred (λ-. vwelldefined) fmempty by simp
next
  show fmrel-on-fset (frees t) vrelated fmempty Γ₂
    using ‹closed t› unfolding closed-except-def by auto
qed (rule assms)+

end
```

Preservation of extensional equality

lemma (**in** *constants*) *veval′-agree-eq*:
assumes $\Gamma \vdash_v t \downarrow v$ *fmrel-on-fset* (*ids t*) *erelated* $\Gamma'\ \Gamma$
assumes *closed-venv* Γ *closed-except t* (*fmdom* Γ)
assumes *wellformed t wellformed-venv* Γ *fdisjnt C* (*fmdom* Γ)
assumes *consts t* $|\subseteq|$ *fmdom* Γ $|\cup|$ *C fmpred* (λ-. *vwelldefined′*) Γ
assumes $\neg$ *shadows-consts t not-shadows-vconsts-env* Γ
obtains v' **where** $\Gamma' \vdash_v t \downarrow v'\ v' \approx_e v$
using *assms* **proof** (*induction arbitrary*: Γ' *thesis*)
case (*const name* Γ *val*)
hence *name* $|\in|$ *ids* (*Sconst name*)
unfolding *ids-def* **by** *simp*
with *const* **have** *rel-option erelated* (*fmlookup* Γ' *name*) (*fmlookup* Γ *name*)
by (*auto dest*: *fmrel-on-fsetD*)
then obtain *val′* **where** *fmlookup* Γ' *name* = *Some val′ val′* $\approx_e$ *val*
using ‹*fmlookup* Γ *name* = *Some val*›
by *cases auto*
thus *?case*
using *const* **by** (*auto intro*: *veval′.const*)
next
case (*var* Γ *name val*)
hence *name* $|\in|$ *ids* (*Svar name*)
unfolding *ids-def* **by** *simp*
with *var* **have** *rel-option erelated* (*fmlookup* Γ' *name*) (*fmlookup* Γ *name*)
by (*auto dest*: *fmrel-on-fsetD*)
then obtain *val′* **where** *fmlookup* Γ' *name* = *Some val′ val′* $\approx_e$ *val*
using ‹*fmlookup* Γ *name* = *Some val*›
by *cases auto*
thus *?case*
using *var* **by** (*auto intro*: *veval′.var*)
next
case (*abs* Γ *cs*)
hence *Vabs cs* $\Gamma' \approx_e$ *Vabs cs* Γ
by (*auto intro*: *erelated.abs*)
thus *?case*
using *abs* **by** (*auto intro*: *veval′.abs*)
next
case (*comb* Γ *t cs* Γ_Λ *u* v_2 *env pat rhs val*)

have *fmrel-on-fset* (*ids t*) *erelated* $\Gamma'\ \Gamma$
apply (*rule fmrel-on-fsubset*)
apply *fact*
unfolding *ids-def* **by** *auto*
then obtain v_1' **where** $\Gamma' \vdash_v t \downarrow v_1'\ v_1' \approx_e$ *Vabs cs* Γ_Λ
using *comb* **by** (*auto simp*: *closed-except-def*)
then obtain Γ_Λ' **where** $v_1' =$ *Vabs cs* Γ_Λ' *fmrel-on-fset* (*ids* (*Sabs cs*)) *erelated* $\Gamma_\Lambda'\ \Gamma_\Lambda$
by (*auto elim*: *erelated.cases*)

have *fmrel-on-fset* (*ids u*) *erelated* Γ' Γ
apply (*rule fmrel-on-fsubset*)
apply *fact*
unfolding *ids-def* **by** *auto*
then obtain v_2' **where** $\Gamma' \vdash_v u \downarrow v_2'$ $v_2' \approx_e v_2$
apply −
apply (*erule comb.IH*(*2*))
using *comb* **by** (*auto simp*: *closed-except-def*)

have *rel-option* (*rel-prod* (*fmrel erelated*) (=)) (*vfind-match cs* v_2') (*vfind-match cs* v_2)
using ‹$v_2' \approx_e v_2$› **by** (*rule erelated.vfind-match-rel'*)
then obtain *env'* **where** *fmrel erelated env' env vfind-match cs* v_2' = *Some* (*env'*, *pat*, *rhs*)
using *comb* **by** *cases auto*

have *vclosed* (*Vabs cs* Γ_Λ)
apply (*rule veval'-closed*)
using *comb* **by** (*auto simp*: *closed-except-def*)
have *vclosed* v_2
apply (*rule veval'-closed*)
using *comb* **by** (*auto simp*: *closed-except-def*)

have *closed-except* (*Sabs cs*) (*fmdom* Γ_Λ)
using ‹*vclosed* (*Vabs cs* Γ_Λ)› **by** (*auto simp*: *Sterm.closed-except-simps*)
hence *frees* (*Sabs cs*) |⊆| *fmdom* Γ_Λ
unfolding *closed-except-def* **.**

have *vwellformed* (*Vabs cs* Γ_Λ)
apply (*rule veval'-wellformed*)
apply *fact*
using *comb* **by** *auto*

have (*pat*, *rhs*) ∈ *set cs*
by (*rule vfind-match-elem*) *fact*
hence *linear pat*
using ‹*vwellformed* (*Vabs cs* Γ_Λ)›
by (*auto simp*: *list-all-iff*)
hence *frees pat* = *patvars* (*mk-pat pat*)
by (*simp add*: *mk-pat-frees*)
hence *fmdom env* = *frees pat*
apply *simp*
apply (*rule vmatch-dom*)
apply (*rule vfind-match-elem*)
apply (*rule comb*)
done

have *vwelldefined'* (*Vabs cs* Γ_Λ)
apply (*rule veval'-welldefined'*)

```
      apply fact
  using comb by auto
hence consts rhs |⊆| fmdom Γ_Λ |∪| C fdisjnt C (fmdom Γ_Λ)
  using ‹(pat, rhs) ∈ set cs›
  by (auto simp: list-all-iff)

have not-shadows-vconsts (Vabs cs Γ_Λ)
  apply (rule veval'-shadows)
  using comb by auto

obtain val' where Γ_Λ' ++_f env' ⊢_v rhs ↓ val' val' ≈_e val
  proof (erule comb.IH)
    show closed-venv (Γ_Λ ++_f env)
      apply rule
      using ‹vclosed (Vabs cs Γ_Λ)› apply simp
      apply (rule vclosed.vmatch-env)
       apply (rule vfind-match-elem)
       apply fact
      apply fact
      done
  next
    show wellformed rhs
      using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ_Λ)›
      by (auto simp: list-all-iff)
  next
    show wellformed-venv (Γ_Λ ++_f env)
      apply rule
      using ‹vwellformed (Vabs cs Γ_Λ)› apply simp
      apply (rule vwellformed.vmatch-env)
       apply (rule vfind-match-elem)
       apply (rule comb)
      apply (rule veval'-wellformed)
        apply fact
      using comb by auto
  next
    show closed-except rhs (fmdom (Γ_Λ ++_f env))
      using ‹(pat, rhs) ∈ set cs› ‹vclosed (Vabs cs Γ_Λ)› ‹fmdom env = frees pat›
      by (auto simp: list-all-iff)

    have fmdom env = fmdom env'
      using ‹fmrel erelated env' env›
      by (metis fmrel-fmdom-eq)

    show fmrel-on-fset (ids rhs) erelated (Γ_Λ' ++_f env') (Γ_Λ ++_f env)
      proof
        fix id
        assume id |∈| ids rhs

       thus rel-option erelated (fmlookup (Γ_Λ' ++_f env') id) (fmlookup (Γ_Λ ++_f
```

```
env) id)
        unfolding ids-def
        proof (cases rule: funion-strictE)
          case A
          hence id |∈| fmdom env |∪| fmdom Γ_Λ
            using ‹closed-except rhs (fmdom (Γ_Λ ++_f env))›
            unfolding closed-except-def
            by auto

          thus ?thesis
            proof (cases rule: funion-strictE)
              case A
              hence id |∈| fmdom env'
                 using ‹fmdom env = frees pat› ‹fmdom env = fmdom env'› by
simp
              with A show ?thesis
                using ‹fmrel erelated env' env› by auto
            next
              case B
              hence id |∉| frees pat
                using ‹fmdom env = frees pat› by simp
              hence id |∈| frees (Sabs cs)
                apply auto
                unfolding ffUnion-alt-def
                apply simp
                apply (rule fBexI[where x = (pat, rhs)])
                using ‹id |∈| frees rhs› apply simp
                unfolding fset-of-list-elem
                apply (rule ‹(pat, rhs) ∈ set cs›)
                done
              hence id |∈| ids (Sabs cs)
                unfolding ids-def by simp

              have id |∉| fmdom env'
                using B unfolding ‹fmdom env = fmdom env'› by simp
              thus ?thesis
                using ‹id |∉| fmdom env›
                apply simp
                apply (rule fmrel-on-fsetD)
                 apply (rule ‹id |∈| ids (Sabs cs)›)
                apply (rule ‹fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ›)
                done
            qed
        next
          case B
          have id |∈| consts (Sabs cs)
            apply auto
            unfolding ffUnion-alt-def
            apply simp
```

```
          apply (rule fBexI[where x = (pat, rhs)])
           apply simp
           apply fact
          unfolding fset-of-list-elem
          apply (rule ‹(pat, rhs) ∈ set cs›)
          done
        hence id |∈| ids (Sabs cs)
          unfolding ids-def by auto

        show ?thesis
          using ‹fmdom env = fmdom env'›
          apply auto
           apply (rule fmrelD)
           apply (rule ‹fmrel erelated env' env›)
          apply (rule fmrel-on-fsetD)
           apply (rule ‹id |∈| ids (Sabs cs)›)
          apply (rule ‹fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ›)
          done
      qed
    qed
next
  show fmpred (λ-. vwelldefined') (Γ_Λ ++_f env)
    proof
      have vwelldefined' (Vabs cs Γ_Λ)
        apply (rule veval'-welldefined')
              apply fact
        using comb by auto
      thus fmpred (λ-. vwelldefined') Γ_Λ
        by simp
    next
      have vwelldefined' v_2
        apply (rule veval'-welldefined')
              apply fact
        using comb by auto

      show fmpred (λ-. vwelldefined') env
        apply (rule vmatch-welldefined')
         apply (rule vfind-match-elem)
         apply fact+
        done
    qed
next
  have fdisjnt C (fmdom Γ_Λ)
    using ‹vwelldefined' (Vabs cs Γ_Λ)› by simp
  moreover have fdisjnt C (fmdom env)
    unfolding ‹fmdom env = -›
    using ‹(pat, rhs) ∈ set cs› ‹not-shadows-vconsts (Vabs cs Γ_Λ)›
    by (auto simp: list-all-iff all-consts-def fdisjnt-alt-def)
  ultimately show fdisjnt C (fmdom (Γ_Λ ++_f env))
```

```
      unfolding fdisjnt-alt-def by auto
    next
      show ¬ shadows-consts rhs
        using ‹(pat, rhs) ∈ set cs› ‹not-shadows-vconsts (Vabs cs Γ_Λ)›
        by (auto simp: list-all-iff)
    next
      have not-shadows-vconsts-env Γ_Λ
        using ‹not-shadows-vconsts (Vabs cs Γ_Λ)› by auto
      moreover have not-shadows-vconsts-env env
        apply (rule not-shadows-vconsts.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
        apply (rule veval'-shadows)
        using comb by auto
      ultimately show not-shadows-vconsts-env (Γ_Λ ++_f env)
        by blast
    next
      show consts rhs |⊆| fmdom (Γ_Λ ++_f env) |∪| C
        using ‹consts rhs |⊆| -›
        by auto
    qed

  moreover have Γ' ⊢_v t $_s u ↓ val'
    proof (rule veval'.comb)
      show Γ' ⊢_v t ↓ Vabs cs Γ_Λ'
        using ‹Γ' ⊢_v t ↓ v_1'›
        unfolding ‹v_1' = -› .
    qed fact+

  ultimately show ?case
    using comb by metis
next
  case (rec-comb Γ t css name Γ_Λ cs u v_2 env pat rhs val)

  have fmrel-on-fset (ids t) erelated Γ' Γ
    apply (rule fmrel-on-fsubset)
    apply fact
    unfolding ids-def by auto
  then obtain v_1' where Γ' ⊢_v t ↓ v_1' v_1' ≈_e Vrecabs css name Γ_Λ
    using rec-comb by (auto simp: closed-except-def)
  then obtain Γ_Λ'
    where v_1' = Vrecabs css name Γ_Λ'
      and pred-fmap (λcs. fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ) css
    by (auto elim: erelated.cases)

  have fmrel-on-fset (ids u) erelated Γ' Γ
    apply (rule fmrel-on-fsubset)
     apply fact
    unfolding ids-def by auto
```

then obtain v_2' **where** $\Gamma' \vdash_v u \downarrow v_2'$ $v_2' \approx_e v_2$
apply −
apply (*erule rec-comb.IH*(*2*))
using *rec-comb* **by** (*auto simp*: *closed-except-def*)

have *rel-option* (*rel-prod* (*fmrel erelated*) (=)) (*vfind-match cs* v_2') (*vfind-match cs* v_2)
using ‹$v_2' \approx_e v_2$› **by** (*rule erelated.vfind-match-rel'*)
then obtain *env'* **where** *fmrel erelated env' env vfind-match cs* v_2' = *Some* (*env'*, *pat*, *rhs*)
using *rec-comb* **by** *cases auto*

have *vclosed* (*Vrecabs css name* Γ_Λ)
apply (*rule veval'-closed*)
using *rec-comb* **by** (*auto simp*: *closed-except-def*)
hence *vclosed* (*Vabs cs* Γ_Λ)
using *rec-comb* **by** (*auto simp*: *closed-except-def*)
have *vclosed* v_2
apply (*rule veval'-closed*)
using *rec-comb* **by** (*auto simp*: *closed-except-def*)

have *closed-except* (*Sabs cs*) (*fmdom* Γ_Λ)
using ‹*vclosed* (*Vabs cs* Γ_Λ)› **by** (*auto simp*: *Sterm.closed-except-simps*)
hence *frees* (*Sabs cs*) |⊆| *fmdom* Γ_Λ
unfolding *closed-except-def* **.**

have *vwellformed* (*Vrecabs css name* Γ_Λ)
apply (*rule veval'-wellformed*)
apply *fact*
using *rec-comb* **by** *auto*
hence *vwellformed* (*Vabs cs* Γ_Λ)
using *rec-comb* **by** (*auto simp*: *closed-except-def*)

have (*pat*, *rhs*) ∈ *set cs*
by (*rule vfind-match-elem*) *fact*
hence *linear pat*
using ‹*vwellformed* (*Vabs cs* Γ_Λ)›
by (*auto simp*: *list-all-iff*)
hence *frees pat* = *patvars* (*mk-pat pat*)
by (*simp add*: *mk-pat-frees*)
hence *fmdom env* = *frees pat*
apply *simp*
apply (*rule vmatch-dom*)
apply (*rule vfind-match-elem*)
apply (*rule rec-comb*)
done

have *vwelldefined'* (*Vrecabs css name* Γ_Λ)
apply (*rule veval'-welldefined'*)

```
        apply fact
    using rec-comb by auto
  hence consts rhs |⊆| fmdom Γ_Λ |∪| (C |∪| fmdom css) fdisjnt C (fmdom Γ_Λ)
    using ‹(pat, rhs) ∈ set cs› ‹fmlookup css name = Some cs›
    by (auto simp: list-all-iff dest!: fmpredD[where m = css])

  have not-shadows-vconsts (Vrecabs css name Γ_Λ)
    apply (rule veval'-shadows)
    using rec-comb by auto
  hence not-shadows-vconsts (Vabs cs Γ_Λ)
    using rec-comb by auto

  obtain val' where Γ_Λ' ++_f mk-rec-env css Γ_Λ' ++_f env' ⊢_v rhs ↓ val' val' ≈_e
val
    proof (erule rec-comb.IH)
      show closed-venv (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env)
        apply rule
         apply rule
        using ‹vclosed (Vabs cs Γ_Λ)› apply simp
        unfolding mk-rec-env-def
        using ‹vclosed (Vrecabs css name Γ_Λ)› apply (auto intro: fmdomI)[]
        apply (rule vclosed.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
        apply fact
        done
    next
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› ‹vwellformed (Vabs cs Γ_Λ)›
        by (auto simp: list-all-iff)
    next
      show wellformed-venv (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env)
        apply rule
         apply rule
        using ‹vwellformed (Vabs cs Γ_Λ)› apply simp
        unfolding mk-rec-env-def
        using ‹vwellformed (Vrecabs css name Γ_Λ)› apply (auto intro: fmdomI)[]
        apply (rule vwellformed.vmatch-env)
         apply (rule vfind-match-elem)
         apply fact
        apply (rule veval'-wellformed)
          apply fact
        using rec-comb by auto
    next
      have closed-except rhs (fmdom (Γ_Λ ++_f env))
        using ‹(pat, rhs) ∈ set cs› ‹vclosed (Vabs cs Γ_Λ)› ‹fmdom env = frees pat›
        by (auto simp: list-all-iff closed-except-def)
      thus closed-except rhs (fmdom (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env))
        unfolding closed-except-def
```

```
      by auto

    have fmdom env = fmdom env'
      using ‹fmrel erelated env' env›
      by (metis fmrel-fmdom-eq)

    have fmrel-on-fset (ids rhs) erelated (mk-rec-env css Γ_Λ') (mk-rec-env css Γ_Λ)
      unfolding mk-rec-env-def
      apply rule
      apply simp
      unfolding option.rel-map
      apply (rule option.rel-refl)
      apply (rule erelated.intros)
      apply (rule ‹pred-fmap (λcs. fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ)
css›)
      done

    have fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ
      using ‹pred-fmap (λcs. fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ) css›
rec-comb
      by auto

    have fmdom (mk-rec-env css Γ_Λ) = fmdom (mk-rec-env css Γ_Λ')
      unfolding mk-rec-env-def by auto

    show fmrel-on-fset (ids rhs) erelated (Γ_Λ' ++_f mk-rec-env css Γ_Λ' ++_f env')
(Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env)
      proof
        fix id
        assume id |∈| ids rhs

       thus rel-option erelated (fmlookup (Γ_Λ' ++_f mk-rec-env css Γ_Λ' ++_f env')
id) (fmlookup (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env) id)
          unfolding ids-def
          proof (cases rule: funion-strictE)
            case A
            hence id |∈| fmdom env |∪| fmdom Γ_Λ
              using ‹closed-except rhs (fmdom (Γ_Λ ++_f env))›
              unfolding closed-except-def
              by auto

            thus ?thesis
              proof (cases rule: funion-strictE)
                case A
                hence id |∈| fmdom env'
                  using ‹fmdom env = frees pat› ‹fmdom env = fmdom env'› by
simp
                with A show ?thesis
                  using ‹fmrel erelated env' env› by auto
```

```
      next
        case B
        hence id |∉| frees pat
          using ‹fmdom env = frees pat› by simp
        hence id |∈| frees (Sabs cs)
          apply auto
          unfolding ffUnion-alt-def
          apply simp
          apply (rule fBexI[where x = (pat, rhs)])
          using ‹id |∈| frees rhs› apply simp
          unfolding fset-of-list-elem
          apply (rule ‹(pat, rhs) ∈ set cs›)
          done
        hence id |∈| ids (Sabs cs)
          unfolding ids-def by simp

        have id |∉| fmdom env'
          using B unfolding ‹fmdom env = fmdom env'› by simp
        thus ?thesis
          using ‹id |∉| fmdom env› ‹fmdom (mk-rec-env css Γ_Λ) = fmdom
(mk-rec-env css Γ_Λ')›
          apply auto
           apply (rule fmrel-on-fsetD)
            apply (rule ‹id |∈| ids rhs›)
           apply (rule ‹fmrel-on-fset (ids rhs) erelated (mk-rec-env css Γ_Λ')
(mk-rec-env css Γ_Λ)›)
          apply (rule fmrel-on-fsetD)
           apply (rule ‹id |∈| ids (Sabs cs)›)
          apply (rule ‹fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ›)
          done
      qed
    next
      case B
      have id |∈| consts (Sabs cs)
        apply auto
        unfolding ffUnion-alt-def
        apply simp
        apply (rule fBexI[where x = (pat, rhs)])
         apply simp
         apply fact
        unfolding fset-of-list-elem
        apply (rule ‹(pat, rhs) ∈ set cs›)
        done
      hence id |∈| ids (Sabs cs)
        unfolding ids-def by auto

      show ?thesis
        using ‹fmdom env = fmdom env'› ‹fmdom (mk-rec-env css Γ_Λ) =
fmdom (mk-rec-env css Γ_Λ')›
```

```
            apply auto
              apply (rule fmrelD)
              apply (rule ‹fmrel erelated env' env›)
             apply (rule fmrel-on-fsetD)
              apply (rule ‹id |∈| ids rhs›)
               apply (rule ‹fmrel-on-fset (ids rhs) erelated (mk-rec-env css Γ_Λ') (mk-rec-env css Γ_Λ)›)
             apply (rule fmrelD)
             apply (rule ‹fmrel erelated env' env›)
            apply (rule fmrel-on-fsetD)
             apply (rule ‹id |∈| ids (Sabs cs)›)
            apply (rule ‹fmrel-on-fset (ids (Sabs cs)) erelated Γ_Λ' Γ_Λ›)
            done
        qed
      qed
  next
    show fmpred (λ-. vwelldefined') (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env)
      proof (intro fmpred-add)
        have vwelldefined' (Vrecabs css name Γ_Λ)
          apply (rule veval'-welldefined')
                apply fact
          using rec-comb by auto
        thus fmpred (λ-. vwelldefined') Γ_Λ fmpred (λ-. vwelldefined') (mk-rec-env css Γ_Λ)
          unfolding mk-rec-env-def
          by (auto intro: fmdomI)
      next
        have vwelldefined' v_2
          apply (rule veval'-welldefined')
                apply fact
          using rec-comb by auto

        show fmpred (λ-. vwelldefined') env
          apply (rule vmatch-welldefined')
           apply (rule vfind-match-elem)
           apply fact+
          done
      qed
  next
    have fdisjnt C (fmdom env)
      unfolding ‹fmdom env = -›
      using ‹(pat, rhs) ∈ set cs› ‹not-shadows-vconsts (Vabs cs Γ_Λ)›
      by (auto simp: list-all-iff all-consts-def fdisjnt-alt-def)
    moreover have fdisjnt C (fmdom css)
      using ‹vwelldefined' (Vrecabs css name Γ_Λ)› by simp
    ultimately show fdisjnt C (fmdom (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env))
      using ‹fdisjnt C (fmdom Γ_Λ)›
      unfolding fdisjnt-alt-def mk-rec-env-def by auto
  next
```

```
    show ¬ shadows-consts rhs
      using ‹(pat, rhs) ∈ set cs› ‹not-shadows-vconsts (Vabs cs Γ_Λ)›
      by (auto simp: list-all-iff)
  next
    have not-shadows-vconsts-env Γ_Λ
      using ‹not-shadows-vconsts (Vabs cs Γ_Λ)› by auto
    moreover have not-shadows-vconsts-env env
      apply (rule not-shadows-vconsts.vmatch-env)
       apply (rule vfind-match-elem)
       apply fact
      apply (rule veval'-shadows)
      using rec-comb by auto
    moreover have not-shadows-vconsts-env (mk-rec-env css Γ_Λ)
      unfolding mk-rec-env-def
      using ‹not-shadows-vconsts (Vrecabs css name Γ_Λ)›
      by (auto intro: fmdomI)
    ultimately show not-shadows-vconsts-env (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f
env)
      by blast
  next
    show consts rhs |⊆| fmdom (Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env) |∪| C
      using ‹consts rhs |⊆| -› unfolding mk-rec-env-def
      by auto
  qed

 moreover have Γ' ⊢_v t $_s u ↓ val'
  proof (rule veval'.rec-comb)
    show Γ' ⊢_v t ↓ Vrecabs css name Γ_Λ'
      using ‹Γ' ⊢_v t ↓ v_1'›
      unfolding ‹v_1' = -› .
  qed fact+

 ultimately show ?case
  using rec-comb by metis
next
 case (constr name Γ ts us)

 have list-all (λt. fmrel-on-fset (ids t) erelated Γ' Γ ∧ closed-except t (fmdom Γ)
∧ wellformed t ∧ consts t |⊆| fmdom Γ |∪| C ∧ ¬ shadows-consts t) ts
  apply (rule list-allI)
  apply rule
   apply (rule fmrel-on-fsubset)
    apply (rule constr)
  subgoal
    unfolding ids-list-comb
    by (induct ts; auto)
  subgoal
    apply (intro conjI)
    subgoal
```

```
      using ‹closed-except (name $$ ts) (fmdom Γ)›
      unfolding closed.list-comb by (auto simp: list-all-iff)
    subgoal
      using ‹wellformed (name $$ ts)›
      unfolding wellformed.list-comb by (auto simp: list-all-iff)
    subgoal
      using ‹consts (name $$ ts) |⊆| fmdom Γ |∪| C›
      unfolding consts-list-comb
      by (metis Ball-set constr.prems(8) special-constants.sconsts-list-comb)
    subgoal
      using ‹¬ shadows-consts (name $$ ts)›
      unfolding shadows.list-comb by (auto simp: list-ex-iff)
    done
  done

 obtain us' where list-all3 (λt u u'. Γ' ⊢v t ↓ u' ∧ u' ≈e u) ts us us'
  using ‹list-all2 - - -› ‹list-all - ts›
  proof (induction arbitrary: thesis rule: list.rel-induct)
   case (Cons t ts u us)
   then obtain us' where list-all3 (λt u u'. Γ' ⊢v t ↓ u' ∧ u' ≈e u) ts us us'
    by auto
   have
    fmrel-on-fset (ids t) erelated Γ' Γ closed-except t (fmdom Γ)
    wellformed t consts t |⊆| fmdom Γ |∪| C ¬ shadows-consts t
    using Cons by auto

   then obtain u' where Γ' ⊢v t ↓ u' u' ≈e u
    using ‹closed-venv Γ› ‹wellformed-venv Γ› ‹fdisjnt C (fmdom Γ)› ‹fmpred
(λ-. vwelldefined') Γ›
    using ‹not-shadows-vconsts-env Γ› Cons.hyps
    by blast

   show ?case
    apply (rule Cons.prems)
    apply (rule list-all3-cons)
     apply fact
    apply (rule conjI)
     apply fact+
    done
  qed auto

 show ?case
  apply (rule constr.prems)
   apply (rule veval'.constr[where us = us'])
    apply fact
  using ‹list-all3 - ts us us'›
   apply (induct; auto)
  apply (rule erelated.intros)
  using ‹list-all3 - ts us us'›
```

apply (*induct*; *auto*)
done
qed

end

Chapter 4

Preprocessing of code equations

theory *Doc-Preproc*
imports *Main*
begin

end

4.1 A type class for correspondence between HOL expressions and terms

theory *Eval-Class*
imports
  ../Rewriting/Rewriting-Term
  ../Utils/ML-Utils
  Deriving.Derive-Manager
  Dict-Construction.Dict-Construction
begin

no-notation *Mpat-Antiquot.mpaq-App* (**infixl** ‹$$› *900*)
hide-const (**open**) *Strong-Term.wellformed*
declare *Strong-Term.wellformed-term-def*[*simp del*]

class *evaluate* =
  fixes *eval* :: *rule fset* ⇒ *term* ⇒ *'a* ⇒ *bool* (‹-/ ⊢/ (- ≈/ -)› [*50*,*0*,*50*] *50*)
  assumes *eval-wellformed*: *rs* ⊢ *t* ≈ *a* ⟹ *wellformed t*
begin

definition *eval'* :: *rule fset* ⇒ *term* ⇒ *'a* ⇒ *bool* (‹-/ ⊢/ (- ↓/ -)› [*50*,*0*,*50*] *50*)
where
rs ⊢ *t* ↓ *a* ⟷ *wellformed t* ∧ (∃ *t'*. *rs* ⊢ *t* ⟶* *t'* ∧ *rs* ⊢ *t'* ≈ *a*)

lemma *eval'I*[*intro*]:

assumes *wellformed t rs* $\vdash$ *t* $\longrightarrow*$ *t′ rs* $\vdash$ *t′* $\approx$ *a*
shows *rs* $\vdash$ *t* $\downarrow$ *a*
using *assms* **unfolding** *eval′-def* **by** *auto*

lemma *eval′E*[*elim*]:
assumes *rs* $\vdash$ *t* $\downarrow$ *a*
obtains *t′* **where** *wellformed t rs* $\vdash$ *t* $\longrightarrow*$ *t′ rs* $\vdash$ *t′* $\approx$ *a*
using *assms* **unfolding** *eval′-def* **by** *auto*

lemma *eval-trivI*: *rs* $\vdash$ *t* $\approx$ *a* $\Longrightarrow$ *rs* $\vdash$ *t* $\downarrow$ *a*
by (*auto dest*: *eval-wellformed*)

lemma *eval-compose*:
assumes *wellformed t rs* $\vdash$ *t* $\longrightarrow*$ *t′ rs* $\vdash$ *t′* $\downarrow$ *a*
shows *rs* $\vdash$ *t* $\downarrow$ *a*
proof −
from ‹*rs* $\vdash$ *t′* $\downarrow$ *a*› **obtain** *t″* **where** *rs* $\vdash$ *t′* $\longrightarrow*$ *t″ rs* $\vdash$ *t″* $\approx$ *a*
by *blast*
moreover hence *rs* $\vdash$ *t* $\longrightarrow*$ *t″*
using *assms* **by** *auto*
ultimately show *rs* $\vdash$ *t* $\downarrow$ *a*
using *assms* **by** *auto*
qed

end

instantiation *fun* :: (*evaluate*, *evaluate*) *evaluate* **begin**

definition *eval-fun* **where**
eval-fun rs t a $\longleftrightarrow$ *wellformed t* $\wedge$ ($\forall$ *x* t_x. *rs* $\vdash$ t_x $\downarrow$ *x* $\longrightarrow$ *rs* $\vdash$ *t* \$ t_x $\downarrow$ *a x*)

instance
by *standard* (*simp add*: *eval-fun-def*)

end

corollary *eval-funD*:
assumes *rs* $\vdash$ *t* $\approx$ *f rs* $\vdash$ t_x $\downarrow$ *x*
shows *rs* $\vdash$ *t* \$ t_x $\downarrow$ *f x*
using *assms* **unfolding** *eval-fun-def* **by** *blast*

corollary *eval′-funD*:
assumes *rs* $\vdash$ *t* $\downarrow$ *f rs* $\vdash$ t_x $\downarrow$ *x*
shows *rs* $\vdash$ *t* \$ t_x $\downarrow$ *f x*
proof −
from *assms* **obtain** *t′* **where** *rs* $\vdash$ *t* $\longrightarrow*$ *t′ rs* $\vdash$ *t′* $\approx$ *f*
by *auto*
have *wellformed* (*t* \$ t_x)
using *assms* **by** *auto*

moreover have $rs \vdash t \$ t_x \longrightarrow* t' \$ t_x$
using ‹$rs \vdash t \longrightarrow* t'$› **by** (*rule rewrite.rt-fun*[*unfolded app-term-def*])
moreover have $rs \vdash t' \$ t_x \downarrow f\ x$
using ‹$rs \vdash t' \approx f$› *assms*(*2*) **by** (*rule eval-funD*)
ultimately show $rs \vdash t \$ t_x \downarrow f\ x$
by (*rule eval-compose*)
qed

lemma *eval-ext*:
assumes *wellformed f* $\bigwedge x\ t.\ rs \vdash t \downarrow x \Longrightarrow rs \vdash f \$ t \downarrow a\ x$
shows $rs \vdash f \approx a$
using *assms* **unfolding** *eval-fun-def* **by** *blast*

lemma *eval'-ext*:
assumes *wellformed f* $\bigwedge x\ t.\ rs \vdash t \downarrow x \Longrightarrow rs \vdash f \$ t \downarrow a\ x$
shows $rs \vdash f \downarrow a$
apply (*rule eval'I*[*OF* ‹*wellformed f*›])
apply (*rule rtranclp.rtrancl-refl*)
apply (*rule eval-ext*)
using *assms* **by** *auto*

lemma *eval'-ext-alt*:
fixes $f :: 'a{::}evaluate \Rightarrow 'b{::}evaluate$
assumes *wellformed' 1 t* $\bigwedge u\ x.\ rs \vdash u \downarrow x \Longrightarrow rs \vdash t\ [u]_\beta \downarrow f\ x$
shows $rs \vdash \Lambda\ t \downarrow f$
proof (*rule eval'-ext*)
show *wellformed* $(\Lambda\ t)$
using *assms* **by** *simp*
next
fix $x :: 'a$ **and** u
assume $rs \vdash u \downarrow x$
show $rs \vdash \Lambda\ t \$ u \downarrow f\ x$
proof (*rule eval-compose*)
show *wellformed* $(\Lambda\ t \$ u)$
using *assms* ‹$rs \vdash u \downarrow x$› **by** *auto*
next
show $rs \vdash \Lambda\ t \$ u \longrightarrow* t\ [u]_\beta$
using ‹$rs \vdash u \downarrow x$› **by** (*auto intro*: *rewrite.beta*)
next
show $rs \vdash t\ [u]_\beta \downarrow f\ x$
using *assms* ‹$rs \vdash u \downarrow x$› **by** *auto*
qed
qed

lemma *eval-impl-wellformed*[*dest*]: $rs \vdash t \approx a \Longrightarrow$ *wellformed' n t*
by (*auto dest*: *wellformed-inc eval-wellformed*[*unfolded wellformed-term-def*])

lemma *eval'-impl-wellformed*[*dest*]: $rs \vdash t \downarrow a \Longrightarrow$ *wellformed' n t*
unfolding *eval'-def* **by** (*auto dest*: *wellformed-inc*)

lemma *wellformed-unpack*:
  $wellformed'\ n\ (t\ \$\ u) \Longrightarrow wellformed'\ n\ t$
  $wellformed'\ n\ (t\ \$\ u) \Longrightarrow wellformed'\ n\ u$
  $wellformed'\ n\ (\Lambda\ t) \Longrightarrow wellformed'\ (Suc\ n)\ t$
by *auto*

lemma *replace-bound-aux*:
  $n < 0 \longleftrightarrow False$
  $Suc\ n < Suc\ m \longleftrightarrow n < m$
  $0 < Suc\ n \longleftrightarrow True$
  $((0::nat) = 0) \longleftrightarrow True$
  $(0 = Suc\ m) \longleftrightarrow False$
  $(Suc\ m = Suc\ n) \longleftrightarrow n = m$
  $(Suc\ m = 0) \longleftrightarrow False$
  $(if\ True\ then\ P\ else\ Q) = P$
  $(if\ False\ then\ P\ else\ Q) = Q$
  $int\ (0::nat) = 0$
by *auto*

named-theorems *eval-data-intros*
named-theorems *eval-data-elims*

context begin

private definition *rewrite-step-term* :: $term \times term \Rightarrow term \Rightarrow term\ option$
where
$rewrite\text{-}step\text{-}term = rewrite\text{-}step$

private lemmas *rewrite-rt-fun* = *rewrite.rt-fun*[*unfolded app-term-def*]
private lemmas *rewrite-rt-arg* = *rewrite.rt-arg*[*unfolded app-term-def*]

ML-file *tactics.ML*

end

method-setup *wellformed* = ‹*Scan.succeed* (*SIMPLE-METHOD'* o *Tactics.wellformed-tac*)›

end

4.2 Deep embedding of Pure terms into term-rewriting logic

theory *Embed*
imports
  Constructor-Funs.Constructor-Funs
  ../Utils/Code-Utils

Eval-Class
keywords *embed* :: *thy-goal*
begin

fun *non-overlapping′* :: *term* ⇒ *term* ⇒ *bool* **where**
non-overlapping′ (*Const x*) (*Const y*) ⟷ *x* ≠ *y* |
non-overlapping′ (*Const* -) (- $ -) ⟷ *True* |
non-overlapping′ (- $ -) (*Const* -) ⟷ *True* |
non-overlapping′ (*t*$_1$ $ *t*$_2$) (*u*$_1$ $ *u*$_2$) ⟷ *non-overlapping′ t*$_1$ *u*$_1$ ∨ *non-overlapping′* *t*$_2$ *u*$_2$ |
non-overlapping′ - - ⟷ *False*

lemma *non-overlapping-approx*:
assumes *non-overlapping′ t u*
shows *non-overlapping t u*
using *assms*
by (*induct t u rule*: *non-overlapping′.induct*) *fastforce+*

fun *pattern-compatible′* :: *term* ⇒ *term* ⇒ *bool* **where**
pattern-compatible′ (*t*$_1$ $ *t*$_2$) (*u*$_1$ $ *u*$_2$) ⟷ *pattern-compatible′ t*$_1$ *u*$_1$ ∧ (*t*$_1$ = *u*$_1$ ⟶ *pattern-compatible′ t*$_2$ *u*$_2$) |
pattern-compatible′ t u ⟷ *t* = *u* ∨ *non-overlapping′ t u*

lemma *pattern-compatible-approx*:
assumes *pattern-compatible′ t u*
shows *pattern-compatible t u*
using *assms*
proof (*induction t u rule*: *pattern-compatible.induct*)
case *2-1*
thus *?case*
by (*force simp*: *non-overlapping-approx*)
next
case *2-5*
thus *?case*
by (*force simp*: *non-overlapping-approx*)
qed *auto*

abbreviation *pattern-compatibles′* :: (*term* × *′a*) *fset* ⇒ *bool* **where**
pattern-compatibles′ ≡ *fpairwise* (λ(*lhs*$_1$, -) (*lhs*$_2$, -). *pattern-compatible′ lhs*$_1$ *lhs*$_2$)

definition *rules′* :: *C-info* ⇒ *rule fset* ⇒ *bool* **where**
rules′ C-info rs ⟷
fBall rs rule ∧
arity-compatibles rs ∧
is-fmap rs ∧
pattern-compatibles′ rs ∧
rs ≠ {||} ∧
fBall rs (λ(*lhs*, -). ¬ *pre-constants.shadows-consts C-info* (*heads-of rs*) *lhs*) ∧
fdisjnt (*heads-of rs*) (*constructors.C C-info*) ∧

fBall rs (λ(-, *rhs*). *pre-constants.welldefined C-info* (*heads-of rs*) *rhs*) ∧
distinct (*constructors.all-constructors C-info*)

lemma *rules-approx*:
assumes *rules′ C-info rs*
shows *rules C-info rs*
proof
show *fBall rs rule arity-compatibles rs is-fmap rs rs* ≠ {||}
and *fBall rs* (λ(*lhs*, -). ¬ *pre-constants.shadows-consts C-info* (*heads-of rs*) *lhs*)
and *fBall rs* (λ(-, *rhs*). *pre-constants.welldefined C-info* (*heads-of rs*) *rhs*)
and *fdisjnt* (*heads-of rs*) (*constructors.C C-info*)
and *distinct* (*constructors.all-constructors C-info*)
using *assms* **unfolding** *rules′-def* **by** *simp*+
next
have *pattern-compatibles′ rs*
using *assms* **unfolding** *rules′-def* **by** *simp*
thus *pattern-compatibles rs*
by (*rule fpairwise-weaken*) (*blast intro*: *pattern-compatible-approx*)
qed

lemma *embed-ext*: *f* ≡ *g* ⟹ *f x* ≡ *g x*
by *auto*

ML-file *embed.ML*

consts *lift-term* :: *′a* ⇒ *term* (‹⟨-⟩›)

setup‹
let
fun embed ((*Const* (@{*const-name lift-term*}, -)) $ *t*) = *HOL-Term.mk-term false t*
| *embed* (*t* $ *u*) = *embed t* $ *embed u*
| *embed t* = *t*
in Context.theory-map (*Syntax-Phases.term-check 99 lift* (*K* (*map embed*))) *end*
›

end

4.3 Default instances

theory *Eval-Instances*
imports *Embed*
begin

ML-file *eval-instances.ML*

setup ‹*Eval-Instances.setup*›

derive *evaluate nat bool list unit prod sum option char num name term*

end

Chapter 5

Final stage: Translation to CakeML

theory *Doc-Backend*
imports *Main*
begin

end

5.1 Basic CakeML setup

theory *CakeML-Setup*
imports
../CupCakeML/CupCake-Semantics
CakeML.CakeML-Code
../Terms/Consts
begin

global-interpretation *name*: *rekey Name*
rewrites *inv Name = as-string*
proof −
have *bij Name*
by (*metis bijI' name.exhaust name.inject*)
show *rekey Name*
by *standard fact*

show *inv Name = as-string*
by (*metis inv-equality name.exhaust-sel name.sel*)
qed

global-interpretation *name-as-string*: *rekey as-string*
by (*rule name.inv*)

hide-const (**open**) *Lem-string.concat*

hide-const (**open**) *sem-env.c*
hide-const (**open**) *sem-env.v*

definition *empty-locn* :: *locn* **where**
empty-locn = (| *row* = *0*, *col* = *0*, *offset* = *0* |)

definition *empty-locs* :: *locs* **where**
empty-locs = (*empty-locn*, *empty-locn*)

definition *empty-state* :: *unit SemanticPrimitives.state* **where**
empty-state = (| *clock* = *0*, *refs* = [], *ffi* = *empty-ffi-state*, *defined-types* = {}, *defined-mods* = {} |)

fun *fmap-of-ns* :: ('*b*, *string*, '*a*) *namespace* ⇒ (*name*, '*a*) *fmap* **where**
fmap-of-ns (*Bind xs* -) = *fmap-of-list* (*map* (*map-prod Name id*) *xs*)

lemma *fmlookup-ns*[*simp*]: *fmlookup* (*fmap-of-ns ns*) *k* = *cupcake-nsLookup ns* (*as-string k*)
by (*cases ns*) (*simp add*: *fmlookup-of-list map-prod-def name.map-of-rekey option.map-ident*)

lemma *fmap-of-nsBind*[*simp*]: *fmap-of-ns* (*nsBind* (*as-string k*) *v0 ns*) = *fmupd k v0* (*fmap-of-ns ns*)
by (*cases ns*) *auto*

lemma *fmap-of-nsAppend*[*simp*]: *fmap-of-ns* (*nsAppend ns1 ns2*) = *fmap-of-ns ns2* ++$_f$ *fmap-of-ns ns1*
by (*cases ns1*; *cases ns2*) *simp*

lemma *fmap-of-alist-to-ns*[*simp*]: *fmap-of-ns* (*alist-to-ns xs*) = *fmap-of-list* (*map* (*map-prod Name id*) *xs*)
unfolding *alist-to-ns-def* **by** *simp*

lemma *fmap-of-nsEmpty*[*simp*]: *fmap-of-ns nsEmpty* = *fmempty*
unfolding *nsEmpty-def* **by** *simp*

context begin

private lemma *build-rec-env-fmap0*:
fmap-of-ns (*foldr* (λ(*f*, *x*, *e*). *nsBind f* (*Recclosure env*$_\Lambda$ *funs' f*)) *funs env*) =
fmap-of-ns env ++$_f$ *fmap-of-list* (*map* (λ(*f*, -). (*Name f*, *Recclosure env*$_\Lambda$ *funs' f*)) *funs*)
apply (*induction funs arbitrary*: *env*)
apply *auto*
by (*metis* (*no-types*, *lifting*) *fmap-of-nsBind name.sel*)

definition *cake-mk-rec-env* **where**
cake-mk-rec-env funs env = *fmap-of-list* (*map* (λ(*f*, -). (*Name f*, *Recclosure env funs f*)) *funs*)

lemma *build-rec-env-fmap*:
fmap-of-ns (*build-rec-env funs env*$_\Lambda$ *env*) = *fmap-of-ns env* ++$_f$ *cake-mk-rec-env funs env*$_\Lambda$
unfolding *build-rec-env-def cake-mk-rec-env-def*
by (*rule build-rec-env-fmap0*)

end

5.2 Constructors according to CakeML

definition *cake-tctor* :: *string* ⇒ *tctor* **where**
cake-tctor name = (*if name* = ''*fun*'' *then Ast.TC-fn else Ast.TC-name* (*Short name*))

primrec *typ-to-t* :: *typ* ⇒ *Ast.t* **where**
typ-to-t (*TVar name*) = *Ast.Tvar* (*as-string name*) |
typ-to-t (*TApp name args*) = *Ast.Tapp* (*map typ-to-t args*) (*cake-tctor* (*as-string name*))

context *constructors* **begin**

definition *as-static-cenv* :: *c-ns* **where**
as-static-cenv = *Bind* (*rev* (*map* (*map-prod id* (*map-prod id* (*TypeId* ∘ *Short*))) *flat-C-info*)) []

lemma *as-static-cenv-cakeml-static-env*: *cakeml-static-env as-static-cenv*
unfolding *cakeml-static-env-def as-static-cenv-def*
by (*auto simp*: *list.pred-map split*: *prod.splits*)

sublocale *cake-static-env?*: *cakeml-static-env as-static-cenv*
by (*rule as-static-cenv-cakeml-static-env*)

definition *as-cake-type-def* :: *Ast.type-def* **where**
as-cake-type-def =
map (λ(*name*, *dt-def*). (*map as-string* (*tparams dt-def*), *as-string name*,
map (λ(*C*, *params*). (*as-string C*, *map typ-to-t params*))
(*sorted-list-of-fmap* (*constructors dt-def*))))
(*sorted-list-of-fmap C-info*)

definition *cake-dt-prelude* :: *Ast.dec* **where**
cake-dt-prelude = *Ast.Dtype empty-locs as-cake-type-def*

definition *cake-all-types* :: *tid-or-exn set* **where**
cake-all-types = (*TypeId* ∘ *Short* ∘ *as-string*) ‘ *fset all-tdefs*

definition *empty-state-with-types* :: *unit SemanticPrimitives.state* **where**
empty-state-with-types =
(| *clock* = *0*, *refs* = [], *ffi* = *empty-ffi-state*, *defined-types* = *cake-all-types*, *defined-mods* = {} |)

lemma *empty-state-with-types-alt-def*:
empty-state-with-types = *empty-state* (| *defined-types* := *cake-all-types* |)
unfolding *empty-state-with-types-def empty-state-def*

by (*auto simp*: *datatype-record-update*)

end

5.2.1 Running the generated type declarations through the semantics

context *constants* **begin**

context begin

private lemma *state-types-update*:
update-defined-types (λ-. *cake-all-types* ∪ *defined-types empty-state*) *empty-state*
=
empty-state-with-types
unfolding *empty-state-with-types-def empty-state-def*

by (*simp add*: *datatype-record-update*)

private lemma *env-types-update*: *build-tdefs* [] *as-cake-type-def* = *as-static-cenv*
unfolding *as-cake-type-def-def as-static-cenv-def build-tdefs-def alist-to-ns-def flat-C-info-def*
apply (*auto simp*: *List.bind-def map-concat*)
apply (*rule arg-cong*[**where** *f* = *concat*])
by (*auto simp*: *map-concat comp-def split-beta*)

private lemmas *evaluate-type* =
evaluate-dec.dtype1[
where *new-tdecs* = *cake-all-types* **and** *s* = *empty-state* **and** *mn* = [] **and** *tds*
= *as-cake-type-def*,
unfolded state-types-update env-types-update,
folded empty-sem-env-def]

private lemma *type-defs-to-new-tdecs*:
type-defs-to-new-tdecs [] *as-cake-type-def* =
set (*map* (λ*name*. *TypeId* (*Short* (*as-string name*))) (*sorted-list-of-fset* (*fmdom*
C-info)))
unfolding *cake-all-types-def type-defs-to-new-tdecs-def as-cake-type-def-def all-tdefs-def*
by (*simp add*: *case-prod-twice sorted-list-of-fmap-def*)

private lemma *cakeml-convoluted1*: *foldr* (λ(*n*, *ts*). (#) *n*) *ys xs* = *map fst ys* @
xs
by (*induction ys arbitrary*: *xs*) *auto*

private lemma *cakeml-convoluted2*: *foldr* ($\lambda x\ y.\ f\ x\ @\ y$) *xs ys* = *concat* (*map f xs*) @ *ys*
by (*induction xs arbitrary*: *ys*) *auto*

private lemma *check-dup-ctors-alt-def*: *check-dup-ctors tds* $\longleftrightarrow$ *distinct* (*tds* $\gg\!=$ (λ(-, -, *cons*). *map fst cons*))
unfolding *check-dup-ctors-def*
apply *simp*
apply (*rule arg-cong*[**where** *f* = *distinct*])
apply (*subst foldr-cong*[*OF refl refl*, **where** *g* = $\lambda x\ a.$ *map fst* (*snd* (*snd x*)) @ *a*])
subgoal
  apply (*subst split-beta*)
  apply (*subst split-beta*)
  by (*rule cakeml-convoluted1*)
subgoal
  apply (*subst cakeml-convoluted2*)
  apply *auto*
  unfolding *List.bind-def*
  apply (*rule arg-cong*[**where** *f* = *concat*])
  by *auto*
done

lemma *evaluate-dec-prelude*:
  evaluate-dec t [] *env empty-state cake-dt-prelude* (*empty-state-with-types*, *Rval empty-sem-env*)
unfolding *cake-dt-prelude-def*
proof (*rule evaluate-type*, *intro conjI*)
  show *check-dup-ctors as-cake-type-def*
    using *distinct-ctr'*
   unfolding *check-dup-ctors-alt-def List.bind-def as-cake-type-def-def all-constructors-def*
    by (*auto simp*: *comp-def split-beta map-concat*)
next
  show *allDistinct* (*map* ($\lambda x.$ *case x of* (*tvs*, *tn*, *ctors*) $\Rightarrow$ *tn*) *as-cake-type-def*)
    unfolding *all-distinct-alt-def as-cake-type-def-def*
    apply (*auto simp*: *comp-def case-prod-twice*)
    apply (*rule name-as-string.fst-distinct*)
    unfolding *sorted-list-of-fmap-def*
    by (*auto simp*: *comp-def*)
next
  show *cake-all-types* = *type-defs-to-new-tdecs* [] *as-cake-type-def*
    unfolding *cake-all-types-def type-defs-to-new-tdecs all-tdefs-def*
    by *simp*
next
  show *disjnt cake-all-types* (*defined-types empty-state*)
    unfolding *empty-state-def disjnt-def* **by** *simp*
qed

end

end

Computability

declare *constructors.as-static-cenv-def*[*code*]
declare *constructors.as-cake-type-def-def*[*code*]
declare *constructors.cake-dt-prelude-def*[*code*]

export-code *constructors.as-static-cenv constructors.cake-dt-prelude*
  checking *Scala*

end

5.3 CakeML backend

theory *CakeML-Backend*
imports
  CakeML-Setup
  ../Terms/Value
  ../Rewriting/Rewriting-Sterm
begin

5.3.1 Compilation

fun *mk-ml-pat* :: *pat* ⇒ *Ast.pat* **where**
mk-ml-pat (*Patvar s*) = *Ast.Pvar* (*as-string s*) |
mk-ml-pat (*Patconstr s args*) = *Ast.Pcon* (*Some* (*Short* (*as-string s*))) (*map mk-ml-pat args*)

lemma *mk-pat-cupcake*[*intro*]: *is-cupcake-pat* (*mk-ml-pat pat*)
by (*induct pat*) (*auto simp*: *list-all-iff*)

context begin

private fun *frees′* :: *term* ⇒ *name list* **where**
frees′ (*Free x*) = [*x*] |
frees′ (*t*$_1$ $ *t*$_2$) = *frees′ t*$_2$ @ *frees′ t*$_1$ |
frees′ (Λ *t*) = *frees′ t* |
frees′ - = []

private lemma *frees′-eq*[*simp*]: *fset-of-list* (*frees′ t*) = *frees t*
by (*induction t*) *auto*

private lemma *frees′-list-comb*: *frees′* (*list-comb f xs*) = *concat* (*rev* (*map frees′ xs*)) @ *frees′ f*
by (*induction xs arbitrary*: *f*) (*auto simp*: *app-term-def*)

private lemma *frees′-distinct*: *linear pat* ⟹ *distinct* (*frees′ pat*)
proof (*induction pat*)
  case (*App t u*)

```
  hence distinct (frees' u @ frees' t)
    by (fastforce intro: distinct-append-fset fdisjnt-swap)
  thus ?case
    by simp
qed auto

private fun pat-bindings' :: Ast.pat ⇒ name list where
pat-bindings' (Ast.Pvar n) = [Name n] |
pat-bindings' (Ast.Pcon - ps) = concat (rev (map pat-bindings' ps)) |
pat-bindings' (Ast.Pref p) = pat-bindings' p |
pat-bindings' (Ast.Ptannot p -) = pat-bindings' p |
pat-bindings' - = []

private lemma pat-bindings'-eq:
  map Name (pats-bindings ps xs) = concat (rev (map pat-bindings' ps)) @ map
Name xs
  map Name (pat-bindings p xs) = pat-bindings' p @ map Name xs
by (induction ps xs and p xs rule: pats-bindings-pat-bindings.induct) (auto simp:
ac-simps)

private lemma pat-bindings'-empty-eq: map Name (pat-bindings p []) = pat-bindings'
p
by (simp add: pat-bindings'-eq)

private lemma pat-bindings'-eq-frees: linear p ⟹ pat-bindings' (mk-ml-pat (mk-pat
p)) = frees' p
proof (induction rule: mk-pat.induct)
  case (1 t)

  show ?case
    using ‹linear t› proof (cases rule: linear-strip-comb-cases)
      case (comb s args)
      have map (pat-bindings' ∘ mk-ml-pat ∘ mk-pat) args = map frees' args
        proof (rule list.map-cong0, unfold comp-apply)
          fix x
          assume x ∈ set args
          moreover hence linear x
            using 1 comb by (metis linears-linear linears-strip-comb snd-conv)
          ultimately show pat-bindings' (mk-ml-pat (mk-pat x)) = frees' x
            using 1 comb by auto
        qed

      hence concat (rev (map (pat-bindings' ∘ mk-ml-pat ∘ mk-pat) args)) = concat
(rev (map frees' args))
        by metis
      with comb show ?thesis
        apply (fold const-term-def)
        apply (auto simp: strip-list-comb-const frees'-list-comb comp-assoc)
        apply (unfold const-term-def)
```

```
      apply simp
      done
  qed auto
qed

lemma mk-pat-distinct: linear pat ⟹ distinct (pat-bindings (mk-ml-pat (mk-pat
pat)) [])
by (metis pat-bindings'-eq-frees pat-bindings'-empty-eq frees'-distinct distinct-map)

end

locale cakeml = pre-constants
begin

fun
  mk-exp :: name fset ⇒ sterm ⇒ exp and
  mk-clauses :: name fset ⇒ (term × sterm) list ⇒ (Ast.pat × exp) list and
  mk-con :: name fset ⇒ sterm ⇒ exp where
mk-exp - (Svar s) = Ast.Var (Short (as-string s)) |
mk-exp - (Sconst s) = (if s |∈| C then Ast.Con (Some (Short (as-string s))) [] else
Ast.Var (Short (as-string s))) |
mk-exp S (t1 $s t2) = Ast.App Ast.Opapp [mk-con S t1, mk-con S t2] |
mk-exp S (Sabs cs) = (
  let n = fresh-fNext S in
   Ast.Fun (as-string n) (Ast.Mat (Ast.Var (Short (as-string n))) (mk-clauses S
cs))) |
mk-con S t =
  (case strip-comb t of
    (Sconst c, args) ⇒
      if c |∈| C then Ast.Con (Some (Short (as-string c))) (map (mk-con S) args)
else mk-exp S t
   | - ⇒ mk-exp S t) |
mk-clauses S cs = map (λ(pat, t). (mk-ml-pat (mk-pat pat), mk-con (frees pat |∪|
S) t)) cs

context begin

private lemma mk-exp-cupcake0:
  wellformed t ⟹ is-cupcake-exp (mk-exp S t)
  wellformed-clauses cs ⟹ cupcake-clauses (mk-clauses S cs) ∧ cake-linear-clauses
(mk-clauses S cs)
  wellformed t ⟹ is-cupcake-exp (mk-con S t)
proof (induction rule: mk-exp-mk-clauses-mk-con.induct)
  case (5 S t)
  show ?case
    apply (simp split!: prod.splits sterm.splits if-splits)
    subgoal premises prems for args c
      proof −
        from prems have t = c $$ args
```

```
      apply (fold const-sterm-def)
      by (metis fst-conv list-strip-comb snd-conv)
    show ?thesis
      apply (auto simp: list-all-iff simp del: mk-con.simps)
      apply (rule 5(1))
        apply (rule prems(1)[symmetric])
       apply (rule refl)
      apply (rule prems)
     apply assumption
      using ‹wellformed t› ‹t = -›
      apply (auto simp: wellformed.list-comb list-all-iff)
      done
  qed
 using 5 by (auto split: prod.splits sterm.splits)
qed (auto simp: Let-def list-all-iff intro: mk-pat-distinct)

declare mk-con.simps[simp del]

lemma mk-exp-cupcake:
  wellformed t ⟹ is-cupcake-exp (mk-exp S t)
  wellformed t ⟹ is-cupcake-exp (mk-con S t)
by (metis mk-exp-cupcake0)+

end

definition mk-letrec-body where
mk-letrec-body S rs = (
  map (λ(name, rhs).
    (as-string name, (
      let n = fresh-fNext S in
     (as-string n, Ast.Mat (Ast.Var (Short (as-string n))) (mk-clauses S (sterm.clauses
rhs))))))) rs
)

definition compile-group :: name fset ⇒ srule list ⇒ Ast.dec where
compile-group S rs = Ast.Dletrec empty-locs (mk-letrec-body S rs)

definition compile :: srule list ⇒ Ast.prog where
compile rs = [Ast.Tdec (compile-group all-consts rs)]

end

declare cakeml.mk-con.simps[code]
declare cakeml.mk-exp.simps[code]
declare cakeml.mk-clauses.simps[code]
declare cakeml.mk-letrec-body-def[code]
declare cakeml.compile-group-def[code]
declare cakeml.compile-def[code]
```

locale *cakeml′* = *cakeml* + *constants*

context *srules* **begin**

sublocale *srules-as-cake?*: *cakeml′ C-info fst* |‘| *fset-of-list rs* **by** *standard*

lemma *mk-letrec-cupcake*:
list-all (λ(-, -, *exp*). *is-cupcake-exp exp*) (*mk-letrec-body S rs*)
unfolding *mk-letrec-body-def*
using *all-rules*
apply (*auto simp*: *Let-def list-all-iff intro*!: *mk-pat-cupcake mk-exp-cupcake mk-pat-distinct*)
subgoal for *a b*
apply (*erule ballE*[**where** *x* = (*a*, *b*)]; *cases b*)
apply (*auto simp*: *list-all-iff is-abs-def term-cases-def*)
done
subgoal for *a b*
apply (*erule ballE*[**where** *x* = (*a*, *b*)]; *cases b*)
apply (*auto simp*: *list-all-iff is-abs-def term-cases-def*)
done
done

end

definition *compile′* **where**
compile′ C-info rs = *cakeml.compile C-info* (*fst* |‘| *fset-of-list rs*) *rs*

lemma (**in** *srules*) *compile′-compile-eq*: *compile′ C-info rs* = *compile rs*
unfolding *compile′-def* **..**

5.3.2 Computability

export-code *cakeml.compile*
checking *Scala*

5.3.3 Correctness of semantic functions

abbreviation *related-pat* :: *term* ⇒ *Ast.pat* ⇒ *bool* **where**
related-pat t p ≡ (*p* = *mk-ml-pat* (*mk-pat t*))

context *cakeml′* **begin**

inductive *related-exp* :: *sterm* ⇒ *exp* ⇒ *bool* **where**
var: *related-exp* (*Svar name*) (*Ast.Var* (*Short* (*as-string name*))) |
const: *name* |∉| *C* ⟹ *related-exp* (*Sconst name*) (*Ast.Var* (*Short* (*as-string name*)))
|
constr: *name* |∈| *C* ⟹ *list-all2 related-exp ts es* ⟹
related-exp (*name* $$ *ts*) (*Ast.Con* (*Some* (*Short* (*as-string name*))) *es*) |
app: *related-exp* t_1 u_1 ⟹ *related-exp* t_2 u_2 ⟹ *related-exp* (t_1 $\$_s$ t_2) (*Ast.App*
Ast.Opapp [u_1, u_2]) |
fun: *list-all2* (*rel-prod related-pat related-exp*) *cs ml-cs* ⟹

```
      n |∉| ids (Sabs cs) ⟹ n |∉| all-consts ⟹
        related-exp (Sabs cs) (Ast.Fun (as-string n) (Ast.Mat (Ast.Var (Short
(as-string n))) ml-cs)) |
mat: list-all2 (rel-prod related-pat related-exp) cs ml-cs ⟹
     related-exp scr ml-scr ⟹
       related-exp (Sabs cs $_s scr) (Ast.Mat ml-scr ml-cs)

lemma related-exp-is-cupcake:
  assumes related-exp t e wellformed t
  shows is-cupcake-exp e
using assms proof induction
  case (fun cs ml-cs n)
  hence list-all (λ(pat, t). linear pat ∧ wellformed t) cs by simp
  moreover have cupcake-clauses ml-cs ∧ cake-linear-clauses ml-cs
    using ‹list-all2 - cs ml-cs› ‹list-all - cs›
    proof induction
      case (Cons c cs ml-c ml-cs)
      obtain ml-p ml-e where ml-c = (ml-p, ml-e) by fastforce
      obtain p t where c = (p, t) by fastforce

      have ml-p = mk-ml-pat (mk-pat p)
        using Cons unfolding ‹ml-c = -› ‹c = -› by simp
      thus ?case
        using Cons unfolding ‹ml-c = -› ‹c = -› by (auto intro: mk-pat-distinct)
    qed simp

  ultimately show ?case
    by auto
next
  case (mat cs ml-cs scr ml-scr)
  hence list-all (λ(pat, t). linear pat ∧ wellformed t) cs by simp
  moreover have cupcake-clauses ml-cs ∧ cake-linear-clauses ml-cs
    using ‹list-all2 - cs ml-cs› ‹list-all - cs›
    proof induction
      case (Cons c cs ml-c ml-cs)
      obtain ml-p ml-e where ml-c = (ml-p, ml-e) by fastforce
      obtain p t where c = (p, t) by fastforce

      have ml-p = mk-ml-pat (mk-pat p)
        using Cons unfolding ‹ml-c = -› ‹c = -› by simp
      thus ?case
        using Cons unfolding ‹ml-c = -› ‹c = -› by (auto intro: mk-pat-distinct)
    qed simp

  ultimately show ?case
    using mat by auto
next
  case (constr name ts es)
  hence list-all wellformed ts
```

```
    by (simp add: wellformed.list-comb)
  with ‹list-all2 - ts es› have list-all is-cupcake-exp es
    by induction auto
  thus ?case
    by simp
qed auto

definition related-fun :: (term × sterm) list ⇒ name ⇒ exp ⇒ bool where
related-fun cs n e ⟷
  n |∉| ids (Sabs cs) ∧ n |∉| all-consts ∧ (case e of
    (Ast.Mat (Ast.Var (Short n')) ml-cs) ⇒
      n = Name n' ∧ list-all2 (rel-prod related-pat related-exp) cs ml-cs
  | - ⇒ False)

lemma related-fun-alt-def:
  related-fun cs n (Ast.Mat (Ast.Var (Short (as-string n))) ml-cs) ⟷
    list-all2 (rel-prod related-pat related-exp) cs ml-cs ∧
    n |∉| ids (Sabs cs) ∧ n |∉| all-consts
unfolding related-fun-def
by auto

lemma related-funE:
  assumes related-fun cs n e
  obtains ml-cs
    where e = Ast.Mat (Ast.Var (Short (as-string n))) ml-cs n |∉| ids (Sabs cs)
n |∉| all-consts
      and list-all2 (rel-prod related-pat related-exp) cs ml-cs
using assms unfolding related-fun-def
by (simp split: exp0.splits id0.splits)

lemma related-exp-fun:
  related-fun cs n e ⟷ related-exp (Sabs cs) (Ast.Fun (as-string n) e) ∧ n |∉| ids
(Sabs cs) ∧ n |∉| all-consts
  (is ?lhs ⟷ ?rhs)
proof
  assume ?rhs
  hence related-exp (Sabs cs) (Ast.Fun (as-string n) e) by simp
  thus ?lhs
    by cases (auto simp: related-fun-def dest: name.expand)
next
  assume ?lhs
  thus ?rhs
    by (auto intro: related-exp.fun elim: related-funE)
qed

inductive related-v :: value ⇒ v ⇒ bool where
conv:
  list-all2 related-v us vs ⟹
    related-v (Vconstr name us) (Conv (Some (as-string name, -)) vs) |
```

closure:
related-fun cs n e $\Longrightarrow$
fmrel-on-fset (*ids* (*Sabs cs*)) *related-v* Γ (*fmap-of-ns* (*sem-env.v env*)) $\Longrightarrow$
related-v (*Vabs cs* Γ) (*Closure env* (*as-string n*) *e*) |
rec-closure:
fmrel-on-fset (*fbind* (*fmran css*) (*ids* $\circ$ *Sabs*)) *related-v* Γ (*fmap-of-ns* (*sem-env.v env*)) $\Longrightarrow$
fmrel (λ*cs*. λ(*n*, *e*). *related-fun cs n e*) *css* (*fmap-of-list* (*map* (*map-prod Name* (*map-prod Name id*)) *exps*)) $\Longrightarrow$
related-v (*Vrecabs css name* Γ) (*Recclosure env exps* (*as-string name*))

abbreviation *var-env* :: (*name*, *value*) *fmap* $\Rightarrow$ (*string* $\times$ *v*) *list* $\Rightarrow$ *bool* **where**
var-env Γ *ns* $\equiv$ *fmrel related-v* Γ (*fmap-of-list* (*map* (*map-prod Name id*) *ns*))

lemma *related-v-ext*:
assumes *related-v v ml-v*
assumes $v' \approx_e v$
shows *related-v v' ml-v*
using *assms* **proof** (*induction arbitrary*: *v'*)
case (*conv us ml-us name*)
obtain *ts* **where** $v' =$ *Vconstr name ts list-all2 erelated ts us*
using ‹$v' \approx_e$ *Vconstr name us*›
by *cases auto*

have *list-all2 related-v ts ml-us*
by (*rule list-all2-trans*[*OF* - ‹*list-all2 erelated ts us*› *conv*(*1*)]) *auto*

thus *?case*
using *conv* **unfolding** ‹$v' =$ -›
by (*auto intro*: *related-v.conv*)
next
case (*closure cs n e* Γ_2 *env*)
obtain Γ_1 **where** $v' =$ *Vabs cs* Γ_1 *fmrel-on-fset* (*ids* (*Sabs cs*)) *erelated* Γ_1 Γ_2
using ‹$v' \approx_e$ -›
by *cases auto*

have *fmrel-on-fset* (*ids* (*Sabs cs*)) *related-v* Γ_1 (*fmap-of-ns* (*sem-env.v env*))
apply *rule*
subgoal premises *prems* **for** *x*
apply (*insert prems*)
apply (*drule fmrel-on-fsetD*)
apply (*rule closure*)
subgoal
apply (*insert prems*)
apply (*drule fmrel-on-fsetD*)
apply (*rule* ‹*fmrel-on-fset* (*ids* (*Sabs cs*)) *erelated* Γ_1 Γ_2›)
apply (*metis* (*mono-tags*, *lifting*) *option.rel-sel*)
done
done

done

show *?case*
unfolding ‹$v' = $ -›
by (*rule related-v.closure*) *fact+*
next
case (*rec-closure css* Γ_2 *env exps name*)
obtain Γ_1 **where** $v' =$ *Vrecabs css name* Γ_1 *pred-fmap* (λ*cs. fmrel-on-fset* (*ids* (*Sabs cs*)) *erelated* Γ_1 Γ_2) *css*
using ‹$v' \approx_e$ -›
by *cases auto*

have *fmrel-on-fset* (*fbind* (*fmran css*) (*ids* $\circ$ *Sabs*)) *related-v* Γ_1 (*fmap-of-ns* (*sem-env.v env*))
apply (*rule fmrel-on-fsetI*)
subgoal premises *prems* **for** *x*
apply (*insert prems*)
apply (*drule fmrel-on-fsetD*)
apply (*rule rec-closure*)
subgoal
apply (*insert prems*)
apply (*erule fbindE*)
apply (*drule pred-fmapD*[*OF* ‹*pred-fmap - css*›])
unfolding *comp-apply*
apply (*drule fmrel-on-fsetD*)
apply *assumption*
apply (*metis* (*mono-tags*, *lifting*) *option.rel-sel*)
done
done
done

show *?case*
unfolding ‹$v' = $ -›
by (*rule related-v.rec-closure*) *fact+*
qed

context begin

private inductive *match-result-related* :: (*string* $\times$ *v*) *list* $\Rightarrow$ (*string* $\times$ *v*) *list* *match-result* $\Rightarrow$ (*name*, *value*) *fmap option* $\Rightarrow$ *bool* **for** *eenv* **where**
no-match: *match-result-related eenv No-match None* |
error: *match-result-related eenv Match-type-error* - |
match: *var-env* Γ *eenv-m* $\Longrightarrow$ *match-result-related eenv* (*Match* (*eenv-m* @ *eenv*)) (*Some* Γ)

private corollary *match-result-related-empty*: *match-result-related eenv* (*Match eenv*) (*Some fmempty*)
proof −
have *match-result-related eenv* (*Match* ([] @ *eenv*)) (*Some fmempty*)

```
    by (rule match-result-related.match) auto
  thus ?thesis
    by simp
qed

private fun is-Match :: 'a match-result ⇒ bool where
is-Match (Match -) ⟷ True |
is-Match - ⟷ False

lemma cupcake-pmatch-related:
  assumes related-v v ml-v
  shows match-result-related eenv (cupcake-pmatch as-static-cenv (mk-ml-pat pat)
ml-v eenv) (vmatch pat v)
using assms proof (induction pat arbitrary: v ml-v eenv)
  case (Patvar name)
  have var-env (fmap-of-list [(name, v)]) [(as-string name, ml-v)]
    using Patvar by auto
  hence match-result-related eenv (Match ([(as-string name, ml-v)] @ eenv)) (Some
(fmap-of-list [(name, v)]))
    by (rule match-result-related.match)
  thus ?case
    by simp
next
  case (Patconstr name ps v0)

  show ?case
    using Patconstr.prems
    proof (cases rule: related-v.cases)
      case (conv us vs name' -)

      define f where
      f p v m =
        (case m of
          Match env ⇒ cupcake-pmatch as-static-cenv p v env
        | m ⇒ m) for p v m

      {
        assume name = name'

        assume length ps = length us
        hence list-all2 (λ- -. True) us ps
          by (induct rule: list-induct2) auto
        hence list-all3 (λt p v. related-v t v) us ps vs
          using ‹list-all2 related-v us vs›
          by (rule list-all3-from-list-all2s) auto

        hence ∗: match-result-related eenv
                  (Matching.fold2 f Match-type-error (map mk-ml-pat ps) vs (Match
(eenv-m @ eenv)))
```

```
        (map-option (foldl (++f) Γ) (those (map2 vmatch ps us))) (is ?rel)
    if var-env Γ eenv-m
    for eenv-m Γ
    using Patconstr.IH ‹related-v v0 ml-v› that
    proof (induction us ps vs arbitrary: Γ eenv-m rule: list-all3-induct)
      case (Cons t us p ps v vs)

    have match-result-related (eenv-m @ eenv) (cupcake-pmatch as-static-cenv
(mk-ml-pat p) v (eenv-m @ eenv)) (vmatch p t)
        using Cons by simp

      thus ?case
        proof cases
          case no-match
          thus ?thesis
            unfolding f-def
            apply (cases length (map mk-ml-pat ps) = length vs)
       by (fastforce intro: match-result-related.intros simp:cup-pmatch-list-nomatch
cup-pmatch-list-length-neq)+
        next
          case error
          thus ?thesis
            unfolding f-def
            apply (cases length (map mk-ml-pat ps) = length vs)
       by (fastforce intro: match-result-related.intros simp:cup-pmatch-list-typerr
cup-pmatch-list-length-neq)+
        next
          case (match Γ′ eenv-m′)

          have match-result-related eenv
                 (Matching.fold2 f Match-type-error (map mk-ml-pat ps) vs
(Match ((eenv-m′ @ eenv-m) @ eenv)))
                (map-option (foldl (++f) (Γ ++f Γ′)) (those (map2 vmatch ps
us)))
            proof (rule Cons, rule Cons)
              show var-env (Γ ++f Γ′) (eenv-m′ @ eenv-m)
                using ‹var-env Γ eenv-m› match
                by force
            qed (simp | fact)+

          thus ?thesis
            using match
            unfolding f-def
            by (auto simp: map-option.compositionality comp-def)
        qed
    qed (auto intro: match-result-related.match)

  moreover have var-env fmempty []
    by force
```

```
      ultimately have ?rel [] fmempty
        by fastforce

      hence ?thesis
        using conv ‹length ps = length us›
        unfolding f-def ‹name = name'›
       by (auto intro: match-result-related.intros split: option.splits elim: static-cenv-lookup)
     }
     moreover
     {
      assume name ≠ name'
      with conv have ?thesis
      by (auto intro: match-result-related.intros split: option.splits elim: same-ctor.elims
simp: name.expand)
     }
     moreover
     {
      let ?fold = Matching.fold2 f Match-type-error (map mk-ml-pat ps) vs (Match
eenv)

      assume *: length ps ≠ length us
      moreover have length us = length vs
        using ‹list-all2 related-v us vs› by (rule list-all2-lengthD)
      ultimately have length ps ≠ length vs
        by simp

      moreover have ¬ is-Match (Matching.fold2 f err xs ys init)
        if ¬ is-Match err and length xs ≠ length ys for init err xs ys
        using that f-def
       by (induct f err xs ys init rule: fold2.induct) (auto split: match-result.splits)
      ultimately have ¬ is-Match ?fold
        by simp
      hence ?fold = Match-type-error ∨ ?fold = No-match
        by (cases ?fold) auto

      with * have ?thesis
        unfolding ‹ml-v = -› ‹v0 = -› f-def
        by (auto intro: match-result-related.intros split: option.splits)
     }

     ultimately show ?thesis
       by auto
   qed (auto intro: match-result-related.intros)
qed

lemma match-all-related:
  assumes list-all2 (rel-prod related-pat related-exp) cs ml-cs
  assumes list-all (λ(pat, -). linear pat) cs
```

```
  assumes related-v v ml-v
  assumes cupcake-match-result as-static-cenv ml-v ml-cs Bindv = Rval (ml-rhs,
ml-pat, eenv)
  obtains rhs pat Γ where
    ml-pat = mk-ml-pat (mk-pat pat)
    related-exp rhs ml-rhs
    vfind-match cs v = Some (Γ, pat, rhs)
    var-env Γ eenv
using assms
proof (induction cs ml-cs arbitrary: thesis ml-pat ml-rhs rule: list-all2-induct)
  case (Cons c cs ml-c ml-cs)
  moreover obtain pat0 rhs0 where c = (pat0, rhs0) by fastforce
 moreover obtain ml-pat0 ml-rhs0 where ml-c = (ml-pat0, ml-rhs0) by fastforce
  ultimately have ml-pat0 = mk-ml-pat (mk-pat pat0) related-exp rhs0 ml-rhs0
by auto

  have linear pat0
    using Cons(5) unfolding ‹c = -› by simp+

 have rel: match-result-related [] (cupcake-pmatch as-static-cenv (mk-ml-pat (mk-pat
pat0)) ml-v []) (vmatch (mk-pat pat0) v)
    by (rule cupcake-pmatch-related) fact+

  show ?case
    proof (cases cupcake-pmatch as-static-cenv ml-pat0 ml-v [])
      case Match-type-error
      hence False
        using Cons(7) unfolding ‹ml-c = -›
        by (simp split: if-splits)
      thus thesis ..
    next
      case No-match

      show thesis
        proof (rule Cons(3))
         show cupcake-match-result as-static-cenv ml-v ml-cs Bindv = Rval (ml-rhs,
ml-pat, eenv)
            using Cons(7) No-match unfolding ‹ml-c = -›
            by (simp split: if-splits)
        next
          fix pat rhs Γ
          assume ml-pat = mk-ml-pat (mk-pat pat)
          assume related-exp rhs ml-rhs
          assume vfind-match cs v = Some (Γ, pat, rhs)
          assume var-env Γ eenv

          from rel have match-result-related [] No-match (vmatch (mk-pat pat0) v)
            using No-match unfolding ‹ml-pat0 = -›
            by simp
```

```
        hence vmatch (mk-pat pat0) v = None
          by (cases rule: match-result-related.cases)

        show thesis
          proof (rule Cons(4))
            show vfind-match (c # cs) v = Some (Γ, pat, rhs)
              unfolding ‹c = -›
              using ‹vfind-match cs v = -› ‹vmatch (mk-pat pat0) v = -›
              by simp
          qed fact+
      next
        show list-all (λ(pat, -). linear pat) cs
          using Cons(5) by simp
      qed fact+
    next
      case (Match eenv')
      hence ml-rhs = ml-rhs0 ml-pat = ml-pat0 eenv = eenv'
        using Cons(7) unfolding ‹ml-c = -›
        by (simp split: if-splits)+

      from rel have match-result-related [] (Match eenv') (vmatch (mk-pat pat0) v)

        using Match unfolding ‹ml-pat0 = -› by simp
      then obtain Γ where vmatch (mk-pat pat0) v = Some Γ var-env Γ eenv'
        by (cases rule: match-result-related.cases) auto

      show thesis
        proof (rule Cons(4))
          show ml-pat = mk-ml-pat (mk-pat pat0)
            unfolding ‹ml-pat = -› by fact
        next
          show related-exp rhs0 ml-rhs
            unfolding ‹ml-rhs = -› by fact
        next
          show var-env Γ eenv
            unfolding ‹eenv = -› by fact
        next
          show vfind-match (c # cs) v = Some (Γ, pat0, rhs0)
            unfolding ‹c = -›
            using ‹vmatch (mk-pat pat0) v = Some Γ›
            by simp
        qed
    qed
qed simp

end end

end
```

5.3.4 Correctness of compilation

theory *CakeML-Correctness*
imports
CakeML-Backend
../Rewriting/Big-Step-Value-ML
begin

context *cakeml′* **begin**

lemma *mk-rec-env-related*:
assumes *fmrel* (λ*cs* (*n*, *e*). *related-fun cs n e*) *css* (*fmap-of-list* (*map* (*map-prod Name* (*map-prod Name id*)) *funs*))
assumes *fmrel-on-fset* (*fbind* (*fmran css*) (*ids* ∘ *Sabs*)) *related-v* Γ_Λ (*fmap-of-ns* (*sem-env.v* env_Λ))
shows *fmrel related-v* (*mk-rec-env css* Γ_Λ) (*cake-mk-rec-env funs* env_Λ)
proof (*rule fmrelI*)
fix *name*
have *rel-option* (λ*cs* (*n*, *e*). *related-fun cs n e*) (*fmlookup css name*) (*map-of* (*map* (*map-prod Name* (*map-prod Name id*)) *funs*) *name*)
using *assms* **by** (*auto simp*: *fmap-of-list.rep-eq*)

then have *rel-option* (λ*cs* (*n*, *e*). *related-fun cs* (*Name n*) *e*) (*fmlookup css name*) (*map-of funs* (*as-string name*))
unfolding *name.map-of-rekey′*
by *cases auto*

have ∗: *related-v* (*Vrecabs css name* Γ_Λ) (*Recclosure* env_Λ *funs* (*as-string name*))
using *assms* **by** (*auto intro*: *related-v.rec-closure*)

show *rel-option related-v* (*fmlookup* (*mk-rec-env css* Γ_Λ) *name*) (*fmlookup* (*cake-mk-rec-env funs* env_Λ) *name*)
unfolding *mk-rec-env-def cake-mk-rec-env-def fmap-of-list.rep-eq*
apply (*simp add*: *map-of-map-keyed name.map-of-rekey option.rel-map*)
apply (*rule option.rel-mono-strong*)
apply *fact*
apply (*rule* ∗)
done
qed

lemma *mk-exp-correctness*:
ids t |⊆| *S* ⟹ *all-consts* |⊆| *S* ⟹ ¬ *shadows-consts t* ⟹ *related-exp t* (*mk-exp S t*)
ids (*Sabs cs*) |⊆| *S* ⟹ *all-consts* |⊆| *S* ⟹ ¬ *shadows-consts* (*Sabs cs*) ⟹ *list-all2* (*rel-prod related-pat related-exp*) *cs* (*mk-clauses S cs*)
ids t |⊆| *S* ⟹ *all-consts* |⊆| *S* ⟹ ¬ *shadows-consts t* ⟹ *related-exp t* (*mk-con S t*)
proof (*induction rule*: *mk-exp-mk-clauses-mk-con.induct*)
case (*2 S name*)
show *?case*

```
    proof (cases name |∈| C)
      case True
      hence related-exp (name $$ []) (mk-exp S (Sconst name))
        by (auto intro: related-exp.intros simp del: list-comb.simps)
      thus ?thesis
        by (simp add: const-sterm-def)
    qed (auto intro: related-exp.intros)
next
  case (4 S cs)

  have fresh-fNext (S |∪| all-consts) |∉| S |∪| all-consts
    by (rule fNext-not-member)
  hence fresh-fNext S |∉| S |∪| all-consts
    using ‹all-consts |⊆| S› by (simp add: sup-absorb1)
  hence fresh-fNext S |∉| ids (Sabs cs) |∪| all-consts
    using 4 by auto

  show ?case
    apply (simp add: Let-def)
    apply (rule related-exp.fun)
      apply (rule 4.IH[unfolded mk-clauses.simps])
        apply (rule refl)
       apply fact+
    using ‹fresh-fNext S |∉| ids (Sabs cs) |∪| all-consts› by auto
next
  case (5 S t)
  show ?case
    apply (simp add: mk-con.simps split!: prod.splits sterm.splits if-splits)
    subgoal premises prems for args c
      proof −
        from prems have t = c $$ args
          apply (fold const-sterm-def)
          by (metis fst-conv list-strip-comb snd-conv)
        show ?thesis
          unfolding ‹t = -›
          apply (rule related-exp.constr)
           apply fact
          apply (simp add: list.rel-map)
          apply (rule list.rel-refl-strong)
          apply (rule 5(1))
                apply (rule prems(1)[symmetric])
               apply (rule refl)
          subgoal by (rule prems)
          subgoal by assumption
          subgoal
            using ‹ids t |⊆| S› unfolding ‹t = -›
            apply (auto simp: ids-list-comb)
            by (meson ffUnion-subset-elem fimage-eqI fset-of-list-elem fset-rev-mp)
          subgoal by (rule 5)
```

```
        subgoal
          using ‹¬ shadows-consts t› unfolding ‹t = -›
          unfolding shadows.list-comb
          by (auto simp: list-ex-iff)
        done
    qed
  using 5 by (auto split: prod.splits sterm.splits)
next
 case (6 S cs)

  have list-all2 (λx y. rel-prod related-pat related-exp x (case y of (pat, t) ⇒
(mk-ml-pat (mk-pat pat), mk-con (frees pat |∪| S) t))) cs cs
    proof (rule list.rel-refl-strong, safe intro!: rel-prod.intros)
      fix pat rhs
      assume (pat, rhs) ∈ set cs

      hence consts rhs |⊆| S
        using ‹ids (Sabs cs) |⊆| S›
        unfolding ids-def
        apply auto
        apply (drule ffUnion-least-rev)+
        apply (auto simp: fset-of-list-elem elim!: fBallE)
        done

      have frees rhs |⊆| frees pat |∪| S
        using ‹ids (Sabs cs) |⊆| S› ‹(pat, rhs) ∈ set cs›
        unfolding ids-def
        apply auto
        apply (drule ffUnion-least-rev)+
        apply (auto simp: fset-of-list-elem elim!: fBallE)
        done

      have ¬ shadows-consts rhs
        using ‹(pat, rhs) ∈ set cs› 6
        by (auto simp: list-ex-iff)

      show related-exp rhs (mk-con (frees pat |∪| S) rhs)
        apply (rule 6)
          apply fact
        subgoal by simp
        subgoal
          unfolding ids-def
          using ‹consts rhs |⊆| S› ‹frees rhs |⊆| frees pat |∪| S›
          by auto
        subgoal using 6(3) by auto
        subgoal by fact
        done
    qed
```

thus *?case*
by (*simp add*: *list.rel-map*)
qed (*auto intro*: *related-exp.intros simp*: *ids-def fdisjnt-alt-def*)

context begin

private lemma *semantic-correctness0*:
fixes *exp*
assumes *cupcake-evaluate-single env exp r is-cupcake-all-env env*
assumes *fmrel-on-fset* (*ids t*) *related-v* Γ (*fmap-of-ns* (*sem-env.v env*))
assumes *related-exp t exp*
assumes *wellformed t wellformed-venv* Γ
assumes *closed-venv* Γ *closed-except t* (*fmdom* Γ)
assumes *fmpred* (λ-. *vwelldefined'*) Γ *consts t* |⊆| *fmdom* Γ |∪| *C*
assumes *fdisjnt C* (*fmdom* Γ)
assumes ¬ *shadows-consts t not-shadows-vconsts-env* Γ
shows *if-rval* (λ*ml-v*. ∃ *v*. Γ ⊢$_v$ *t* ↓ *v* ∧ *related-v v ml-v*) *r*
using *assms* **proof** (*induction arbitrary*: Γ *t*)
case (*con1 env cn es ress ml-vs ml-v'*)
obtain *name ts* **where** *cn* = *Some* (*Short* (*as-string name*)) *name* |∈| *C t* = *name* $$ *ts list-all2 related-exp ts es*
using ‹*related-exp t* (*Con cn es*)›
by *cases auto*
with *con1* **obtain** *tid* **where** *ml-v'* = *Conv* (*Some* (*id-to-n* (*Short* (*as-string name*)), *tid*)) (*rev ml-vs*)
by (*auto split*: *option.splits*)

have *ress* = *map Rval ml-vs*
using *con1* **by** *auto*

define *ml-vs'* **where** *ml-vs'* = *rev ml-vs*

note *IH* = ‹*list-all2-shortcircuit - - -*›[
unfolded ‹*ress* = -› *list-all2-shortcircuit-rval list-all2-rev1*,
folded ml-vs'-def]
moreover have
list-all wellformed ts list-all (λ*t*. ¬ *shadows-consts t*) *ts*
list-all (λ*t*. *consts t* |⊆| *fmdom* Γ |∪| *C*) *ts list-all* (λ*t*. *closed-except t* (*fmdom* Γ)) *ts*
subgoal
using ‹*wellformed t*› **unfolding** ‹*t* = -› *wellformed.list-comb* **by** *simp*
subgoal
using ‹¬ *shadows-consts t*› **unfolding** ‹*t* = -› *shadows.list-comb*
by (*simp add*: *list-all-iff list-ex-iff*)
subgoal
using ‹*consts t* |⊆| *fmdom* Γ |∪| *C*›
unfolding *list-all-iff*
by (*metis Ball-set* ‹*t* = *name* $$ *ts*› *con1.prems*(*9*) *special-constants.sconsts-list-comb*)
subgoal

```
    using ‹closed-except t (fmdom Γ)› unfolding ‹t = -› closed.list-comb by simp
   done
 moreover have
   list-all (λt'. fmrel-on-fset (ids t') related-v Γ (fmap-of-ns (sem-env.v env))) ts
   proof (standard, rule fmrel-on-fsubset)
     fix t'
     assume t' ∈ set ts
     thus ids t' |⊆| ids t
       unfolding ‹t = -›
       apply (simp add: ids-list-comb)
       apply (subst (2) ids-def)
       apply simp
       apply (rule fsubset-finsertI2)
       apply (auto simp: fset-of-list-elem intro!: ffUnion-subset-elem)
       done
     show fmrel-on-fset (ids t) related-v Γ (fmap-of-ns (sem-env.v env))
       by fact
   qed

 ultimately obtain us where list-all2 (veval' Γ) ts us list-all2 related-v us ml-vs'
   using ‹list-all2 related-exp ts es›
   proof (induction es ml-vs' arbitrary: ts thesis rule: list.rel-induct)
     case (Cons e es ml-v ml-vs ts0)
     then obtain t ts where ts0 = t # ts related-exp t e by (cases ts0) auto
     with Cons have list-all2 related-exp ts es by simp
      with Cons obtain us where list-all2 (veval' Γ) ts us list-all2 related-v us
ml-vs
       unfolding ‹ts0 = -›
       by auto

     from Cons.hyps[simplified, THEN conjunct2, rule-format, of t Γ]
     obtain u where Γ ⊢ᵥ t ↓ u  related-v u ml-v
       proof
         show
           is-cupcake-all-env env related-exp t e wellformed-venv Γ closed-venv Γ
           fmpred (λ-. vwelldefined') Γ fdisjnt C (fmdom Γ)
           not-shadows-vconsts-env Γ
           by fact+
       next
         show
           wellformed t ¬ shadows-consts t closed-except t (fmdom Γ)
           consts t |⊆| fmdom Γ |∪| C fmrel-on-fset (ids t) related-v Γ (fmap-of-ns
(sem-env.v env))
           using Cons unfolding ‹ts0 = -›
           by auto
       qed blast

     show ?case
       apply (rule Cons(3)[of u # us])
```

```
      unfolding ‹ts0 = -›
       apply auto
         apply fact+
       done
   qed auto

   show ?case
     apply simp
     apply (intro exI conjI)
     unfolding ‹t = -›
      apply (rule veval'.constr)
       apply fact+
     unfolding ‹ml-v' = -›
     apply (subst ml-vs'-def[symmetric])
     apply simp
     apply (rule related-v.conv)
     apply fact
     done
next
  case (var1 env id ml-v)

  from ‹related-exp t (Var id)› obtain name where id = Short (as-string name)
    by cases auto
   with var1 have cupcake-nsLookup (sem-env.v env) (as-string name) = Some
ml-v
    by auto

  from ‹related-exp t (Var id)› consider
      (var) t = Svar name
    | (const) t = Sconst name name |∉| C
    unfolding ‹id = -›
    apply (cases t)
    using name.expand by blast+
  thus ?case
    proof cases
      case var
      hence name |∈| ids t
        unfolding ids-def by simp

       have rel-option related-v (fmlookup Γ name) (cupcake-nsLookup (sem-env.v
env) (as-string name))
        using ‹fmrel-on-fset (ids t) - - -›
        apply −
        apply (drule fmrel-on-fsetD[OF ‹name |∈| ids t›])
        apply simp
        done
      then obtain v where related-v v ml-v fmlookup Γ name = Some v
        using ‹cupcake-nsLookup (sem-env.v env) - = -›
        by cases auto
```

```
    show ?thesis
      unfolding ‹t = -›
      apply simp
      apply (rule exI)
      apply (rule conjI)
       apply (rule veval'.var)
       apply fact+
      done
  next
    case const
    hence name |∈| ids t
      unfolding ids-def by simp

     have rel-option related-v (fmlookup Γ name) (cupcake-nsLookup (sem-env.v
env) (as-string name))
      using ‹fmrel-on-fset (ids t) - - -›
      apply −
      apply (drule fmrel-on-fsetD[OF ‹name |∈| ids t›])
      apply simp
      done
    then obtain v where related-v v ml-v fmlookup Γ name = Some v
      using ‹cupcake-nsLookup (sem-env.v env) - = -›
      by cases auto

    show ?thesis
      unfolding ‹t = -›
      apply simp
      apply (rule exI)
      apply (rule conjI)
       apply (rule veval'.const)
        apply fact+
      done
  qed
next
  case (fn env n u)
  obtain n' where n = as-string n'
    by (metis name.sel)
  obtain cs ml-cs
    where t = Sabs cs u = Mat (Var (Short (as-string n'))) ml-cs n' |∉| ids (Sabs
cs) n' |∉| all-consts
     and list-all2 (rel-prod related-pat related-exp) cs ml-cs
    using ‹related-exp t (Fun n u)› unfolding ‹n = -›
    by cases (auto dest: name.expand)

  obtain ns where fmap-of-ns (sem-env.v env) = fmap-of-list ns
    apply (cases env)
    apply simp
    subgoal for v by (cases v) simp
```

```
    done

  show ?case
    apply simp
    apply (rule exI)
    apply (rule conjI)
    unfolding ‹t = -›
     apply (rule veval'.abs)
    unfolding ‹n = -›
    apply (rule related-v.closure)
    unfolding ‹u = -›
     apply (subst related-fun-alt-def; rule conjI)
      apply fact
     apply (rule conjI; fact)
    using ‹fmrel-on-fset (ids t) - - -›
    unfolding ‹t = -› ‹fmap-of-ns (sem-env.v env) = -›
    by simp
next
  case (app1 env exps ress ml-vs env' exp' bv)
  from ‹related-exp t -› obtain exp1 exp2 t1 t2
    where rev exps = [exp2, exp1] exps = [exp1, exp2] t = t1 $s t2
      and related-exp t1 exp1 related-exp t2 exp2
    by cases auto
  moreover from app1 have ress = map Rval ml-vs
    by auto
  ultimately obtain ml-v1 ml-v2 where ml-vs = [ml-v2, ml-v1]
    using app1(1)
    by (smt (z3) list-all2-shortcircuit-rval list-all2-Cons1 list-all2-Nil)

  have is-cupcake-exp exp1 is-cupcake-exp exp2
    using app1 unfolding ‹exps = -› by (auto dest: related-exp-is-cupcake)
  moreover have fmrel-on-fset (ids t1) related-v Γ (fmap-of-ns (sem-env.v env))
    using app1 unfolding ids-def ‹t = -›
    by (auto intro: fmrel-on-fsubset)
  moreover have fmrel-on-fset (ids t2) related-v Γ (fmap-of-ns (sem-env.v env))
    using app1 unfolding ids-def ‹t = -›
    by (auto intro: fmrel-on-fsubset)
  ultimately have
    cupcake-evaluate-single env exp1 (Rval ml-v1) cupcake-evaluate-single env exp2
(Rval ml-v2) and
    ∃ t1'. Γ ⊢v t1 ↓ t1' ∧ related-v t1' ml-v1 ∃ t2'. Γ ⊢v t2 ↓ t2' ∧ related-v t2' ml-v2
    using app1 ‹related-exp t1 exp1› ‹related-exp t2 exp2›
    unfolding ‹ress = -› ‹exps = -› ‹ml-vs = -› ‹t = -›
    by (auto simp: closed-except-def)

  then obtain v1 v2
    where Γ ⊢v t1 ↓ v1 related-v v1 ml-v1
      and Γ ⊢v t2 ↓ v2 related-v v2 ml-v2
    by blast
```

have *is-cupcake-value ml-*v_1
  by (*rule cupcake-single-preserve*) *fact+*
moreover have *is-cupcake-value ml-*v_2
  by (*rule cupcake-single-preserve*) *fact+*
ultimately have *list-all is-cupcake-value* (*rev ml-vs*)
  unfolding ‹*ml-vs* = -› **by** *simp*

hence *is-cupcake-exp exp' is-cupcake-all-env env'*
  using ‹*do-opapp* - = -› **by** (*metis cupcake-opapp-preserve*)+

have *vclosed* v_1
  proof (*rule veval'-closed*)
    show *closed-except* t_1 (*fmdom* Γ)
      using ‹*closed-except* - (*fmdom* Γ)›
      unfolding ‹t = -›
      by (*simp add*: *closed-except-def*)
  next
    show *wellformed* t_1
      using ‹*wellformed t*› **unfolding** ‹t = -›
      by *simp*
  qed *fact+*
have *vclosed* v_2
  apply (*rule veval'-closed*)
     apply *fact*
  using *app1* **unfolding** ‹t = -› **by** (*auto simp*: *closed-except-def*)

have *vwellformed* v_1
  apply (*rule veval'-wellformed*)
    apply *fact*
  using *app1* **unfolding** ‹t = -› **by** *auto*
have *vwellformed* v_2
  apply (*rule veval'-wellformed*)
    apply *fact*
  using *app1* **unfolding** ‹t = -› **by** *auto*

have *vwelldefined'* v_1
  apply (*rule veval'-welldefined'*)
        apply *fact*
  using *app1* **unfolding** ‹t = -› **by** *auto*
have *vwelldefined'* v_2
  apply (*rule veval'-welldefined'*)
        apply *fact*
  using *app1* **unfolding** ‹t = -› **by** *auto*

have *not-shadows-vconsts* v_1
  apply (*rule veval'-shadows*)
    apply *fact*
  using *app1* **unfolding** ‹t = -› **by** *auto*

```
  have not-shadows-vconsts v2
    apply (rule veval'-shadows)
      apply fact
    using app1 unfolding ‹t = -› by auto

  show ?case
    proof (rule if-rvalI)
      fix ml-v
      assume bv = Rval ml-v
      show ∃ v. Γ ⊢v t ↓ v ∧ related-v v ml-v
        using ‹do-opapp - = -›
        proof (cases rule: do-opapp-cases)
          case (closure env_Λ n)
          then have closure':
            ml-v1 = Closure env_Λ (as-string (Name n)) exp'
              env' = update-v (λ-. nsBind (as-string (Name n)) ml-v2 (sem-env.v
env_Λ)) env_Λ
            unfolding ‹ml-vs = -› by auto
          obtain Γ_Λ cs
            where v1 = Vabs cs Γ_Λ related-fun cs (Name n) exp'
              and fmrel-on-fset (ids (Sabs cs)) related-v Γ_Λ (fmap-of-ns (sem-env.v
env_Λ))
            using ‹related-v v1 ml-v1› unfolding ‹ml-v1 = -›
            by cases auto

          then obtain ml-cs
             where exp' = Mat (Var (Short (as-string (Name n)))) ml-cs Name n
|∉| ids (Sabs cs) Name n |∉| all-consts
              and list-all2 (rel-prod related-pat related-exp) cs ml-cs
            by (auto elim: related-funE)

           hence cupcake-evaluate-single env' (Mat (Var (Short (as-string (Name
n)))) ml-cs) (Rval ml-v)
            using ‹cupcake-evaluate-single env' exp' bv›
            unfolding ‹bv = -›
            by simp

          then obtain m-env v' ml-rhs ml-pat
            where cupcake-evaluate-single env' (Var (Short (as-string (Name n))))
(Rval v')
                and cupcake-match-result (sem-env.c env') v' ml-cs Bindv = Rval
(ml-rhs, ml-pat, m-env)
              and cupcake-evaluate-single (env' (| sem-env.v := nsAppend (alist-to-ns
m-env) (sem-env.v env') |)) ml-rhs (Rval ml-v)
            by cases auto

          have
            closed-venv (fmupd (Name n) v2 Γ_Λ) wellformed-venv (fmupd (Name n)
v2 Γ_Λ)
```

not-shadows-vconsts-env (*fmupd* (*Name n*) v_2 Γ_Λ) *fmpred* (λ-. *vwelldefined′*) (*fmupd* (*Name n*) v_2 Γ_Λ)
using ‹*vclosed* v_1› ‹*vclosed* v_2›
using ‹*vwellformed* v_1› ‹*vwellformed* v_2›
using ‹*not-shadows-vconsts* v_1› ‹*not-shadows-vconsts* v_2›
using ‹*vwelldefined′* v_1› ‹*vwelldefined′* v_2›
unfolding ‹v_1 = -›
by *auto*

have *closed-except* (*Sabs cs*) (*fmdom* (*fmupd* (*Name n*) v_2 Γ_Λ))
using ‹*vclosed* v_1› **unfolding** ‹v_1 = -›
apply (*auto simp*: *Sterm.closed-except-simps list-all-iff*)
apply (*auto simp*: *closed-except-def*)
done

have *consts* (*Sabs cs*) |⊆| *fmdom* (*fmupd* (*Name n*) v_2 Γ_Λ) |∪| *C*
using ‹*vwelldefined′* v_1› **unfolding** ‹v_1 = -›
unfolding *sconsts-sabs*
by (*auto simp*: *list-all-iff*)

have ¬ *shadows-consts* (*Sabs cs*)
using ‹*not-shadows-vconsts* v_1› **unfolding** ‹v_1 = -›
by (*auto simp*: *list-all-iff list-ex-iff*)

have *fdisjnt C* (*fmdom* Γ_Λ)
using ‹*vwelldefined′* v_1› **unfolding** ‹v_1 = -›
by *simp*

have *if-rval* (λ*ml-v*. ∃ *v*. *fmupd* (*Name n*) v_2 Γ_Λ ⊢$_v$ *Sabs cs* $_s$ *Svar* (*Name n*) ↓ *v* ∧ *related-v v ml-v*) *bv*
proof (*rule app1*(*2*))
show *fmrel-on-fset* (*ids* (*Sabs cs* $_s$ *Svar* (*Name n*))) *related-v* (*fmupd* (*Name n*) v_2 Γ_Λ) (*fmap-of-ns* (*sem-env.v env′*))
unfolding *closure′*
apply (*simp del*: *frees-sterm.simps*(*3*) *consts-sterm.simps*(*3*) *name.sel add*: *ids-def split*!: *sem-env.splits*)
apply (*rule fmrel-on-fset-updateI*)
apply (*fold ids-def*)
using ‹*fmrel-on-fset* (*ids* (*Sabs cs*)) *related-v* Γ_Λ -› **apply** *simp*
apply (*rule* ‹*related-v* v_2 *ml-*v_2›)
done
next
show *wellformed* (*Sabs cs* $_s$ *Svar* (*Name n*))
using ‹*vwellformed* v_1› **unfolding** ‹v_1 = -›
by *simp*
next
show *related-exp* (*Sabs cs* $_s$ *Svar* (*Name n*)) *exp′*
unfolding ‹*exp′* = -›
using ‹*list-all2* (*rel-prod related-pat related-exp*) *cs ml-cs*›

```
            by (auto intro:related-exp.intros simp del: name.sel)
          next
           show closed-except (Sabs cs $_s Svar (Name n)) (fmdom (fmupd (Name
n) v_2 Γ_Λ))
            using ‹closed-except (Sabs cs) (fmdom (fmupd (Name n) v_2 Γ_Λ))› by
(simp add: closed-except-def)
          next
            show ¬ shadows-consts (Sabs cs $_s Svar (Name n))
              using ‹¬ shadows-consts (Sabs cs)› ‹Name n |∉| all-consts› by simp
          next
           show consts (Sabs cs $_s Svar (Name n)) |⊆| fmdom (fmupd (Name n)
v_2 Γ_Λ) |∪| C
             using ‹consts (Sabs cs) |⊆| fmdom (fmupd (Name n) v_2 Γ_Λ) |∪| C›
by simp
          next
            show fdisjnt C (fmdom (fmupd (Name n) v_2 Γ_Λ))
              using ‹Name n |∉| all-consts› ‹fdisjnt C (fmdom Γ_Λ)›
              unfolding fdisjnt-alt-def all-consts-def by auto
          qed fact+
        then obtain v where fmupd (Name n) v_2 Γ_Λ ⊢_v Sabs cs $_s Svar (Name
n) ↓ v related-v v ml-v
          unfolding ‹bv = -›
          by auto

        then obtain env pat rhs
          where vfind-match cs v_2 = Some (env, pat, rhs)
            and fmupd (Name n) v_2 Γ_Λ ++_f env ⊢_v rhs ↓ v
          by (auto elim: veval'-sabs-svarE)
        hence (pat, rhs) ∈ set cs vmatch (mk-pat pat) v_2 = Some env
          by (metis vfind-match-elem)+
        hence linear pat wellformed rhs
          using ‹vwellformed v_1› unfolding ‹v_1 = -›
          by (auto simp: list-all-iff)
        hence frees pat = patvars (mk-pat pat)
          by (simp add: mk-pat-frees)
        hence fmdom env = frees pat
          apply simp
          apply (rule vmatch-dom)
          apply (rule ‹vmatch (mk-pat pat) v_2 = Some env›)
          done

        obtain v' where Γ_Λ ++_f env ⊢_v rhs ↓ v' v' ≈_e v
          proof (rule veval'-agree-eq)
            show fmupd (Name n) v_2 Γ_Λ ++_f env ⊢_v rhs ↓ v by fact
          next
            have ∗: Name n |∉| ids rhs if Name n |∉| fmdom env
              proof
                assume Name n |∈| ids rhs
                thus False
```

```
            unfolding ids-def
            proof (cases rule: funion-strictE)
              case A
              moreover have Name n |∉| frees pat
                using that unfolding ‹fmdom env = frees pat› .
              ultimately have Name n |∈| frees (Sabs cs)
                apply auto
                unfolding ffUnion-alt-def
                apply simp
                apply (rule fBexI[where x = (pat, rhs)])
                 apply (auto simp: fset-of-list-elem)
                apply (rule ‹(pat, rhs) ∈ set cs›)
                done
              thus ?thesis
                using ‹Name n |∉| ids (Sabs cs)› unfolding ids-def
                by blast
            next
              case B
              hence Name n |∈| consts (Sabs cs)
                apply auto
                unfolding ffUnion-alt-def
                apply simp
                apply (rule fBexI[where x = (pat, rhs)])
                 apply (auto simp: fset-of-list-elem)
                apply (rule ‹(pat, rhs) ∈ set cs›)
                done
              thus ?thesis
                using ‹Name n |∉| ids (Sabs cs)› unfolding ids-def
                by blast
            qed
          qed

        show fmrel-on-fset (ids rhs) erelated (Γ_Λ ++_f env) (fmupd (Name n)
v_2 Γ_Λ ++_f env)
          apply rule
          apply auto
            apply (rule option.rel-refl; rule erelated-refl)
          using ∗ apply auto[]
           apply (rule option.rel-refl; rule erelated-refl)+
          done
      next
        show closed-venv (fmupd (Name n) v_2 Γ_Λ ++_f env)
          apply rule
           apply fact
          apply (rule vclosed.vmatch-env)
           apply fact
          apply fact
          done
      next
```

```
      show wellformed-venv (fmupd (Name n) v2 Γ_Λ ++_f env)
        apply rule
         apply fact
        apply (rule vwellformed.vmatch-env)
         apply fact
        apply fact
        done
      next
        show closed-except rhs (fmdom (fmupd (Name n) v2 Γ_Λ ++_f env))
          using ‹fmdom env = frees pat› ‹(pat, rhs) ∈ set cs›
          using ‹closed-except (Sabs cs) (fmdom (fmupd (Name n) v2 Γ_Λ))›
          by (auto simp: Sterm.closed-except-simps list-all-iff)
      next
        show wellformed rhs by fact
      next
        show consts rhs |⊆| fmdom (fmupd (Name n) v2 Γ_Λ ++_f env) |∪| C
          using ‹consts (Sabs cs) |⊆| fmdom (fmupd (Name n) v2 Γ_Λ) |∪| C›
‹(pat, rhs) ∈ set cs›
          unfolding sconsts-sabs
          by (auto simp: list-all-iff)
      next
        have fdisjnt C (fmdom env)
          using ‹(pat, rhs) ∈ set cs› ‹¬ shadows-consts (Sabs cs)›
          unfolding ‹fmdom env = frees pat›
          by (auto simp: list-ex-iff fdisjnt-alt-def all-consts-def)
        thus fdisjnt C (fmdom (fmupd (Name n) v2 Γ_Λ ++_f env))
          using ‹Name n |∉| all-consts› ‹fdisjnt C (fmdom Γ_Λ)›
          unfolding fdisjnt-alt-def
          by (auto simp: all-consts-def)
      next
        show ¬ shadows-consts rhs
          using ‹(pat, rhs) ∈ set cs› ‹¬ shadows-consts (Sabs cs)›
          by (auto simp: list-ex-iff)
      next
        have not-shadows-vconsts-env env
          by (rule not-shadows-vconsts.vmatch-env) fact+
        thus not-shadows-vconsts-env (fmupd (Name n) v2 Γ_Λ ++_f env)
          using ‹not-shadows-vconsts-env (fmupd (Name n) v2 Γ_Λ)› by blast
      next
        have fmpred (λ-. vwelldefined') env
          by (rule vmatch-welldefined') fact+
        thus fmpred (λ-. vwelldefined') (fmupd (Name n) v2 Γ_Λ ++_f env)
          using ‹fmpred (λ-. vwelldefined') (fmupd (Name n) v2 Γ_Λ)› by blast
      qed blast

    show ?thesis
      apply (intro exI conjI)
      unfolding ‹t = -›
       apply (rule veval'.comb)
```

```
      using ‹Γ ⊢v t1 ↓ v1› unfolding ‹v1 = -›
        apply blast
       apply fact
      apply fact+
     apply (rule related-v-ext)
      apply fact+
     done
  next
    case (recclosure envΛ funs name n)
    with recclosure have recclosure′:
      ml-v1 = Recclosure envΛ funs name
       env′ = update-v (λ-. nsBind (as-string (Name n)) ml-v2 (build-rec-env
funs envΛ (sem-env.v envΛ))) envΛ
      unfolding ‹ml-vs = -› by auto
    obtain ΓΛ css
      where v1 = Vrecabs css (Name name) ΓΛ
     and fmrel-on-fset (fbind (fmran css) (ids ∘ Sabs)) related-v ΓΛ (fmap-of-ns
(sem-env.v envΛ))
           and fmrel (λcs (n, e). related-fun cs n e) css (fmap-of-list (map
(map-prod Name (map-prod Name id)) funs))
      using ‹related-v v1 ml-v1› unfolding ‹ml-v1 = -›
      by cases auto
   from ‹fmrel - - -› have rel-option (λcs (n, e). related-fun cs (Name n) e)
(fmlookup css (Name name)) (find-recfun name funs)
      apply −
      apply (subst option.rel-sel)
      apply auto
        apply (drule fmrel-fmdom-eq)
        apply (drule fmdom-notI)
         using ‹v1 = Vrecabs css (Name name) ΓΛ› ‹vwellformed v1› apply
auto[1]
      using recclosure(3) apply auto[1]
      apply (erule fmrel-cases[where x = Name name])
       apply simp
      apply (subst (asm) fmlookup-of-list)
      apply (simp add: name.map-of-rekey′)
      by blast

   then obtain cs where fmlookup css (Name name) = Some cs related-fun
cs (Name n) exp′
      using ‹find-recfun - - = -›
      by cases auto

   then obtain ml-cs
      where exp′ = Mat (Var (Short (as-string (Name n)))) ml-cs Name n
|∉| ids (Sabs cs) Name n |∉| all-consts
       and list-all2 (rel-prod related-pat related-exp) cs ml-cs
      by (auto elim: related-funE)
```

hence *cupcake-evaluate-single env′* (*Mat* (*Var* (*Short n*)) *ml-cs*) (*Rval ml-v*)
using ‹*cupcake-evaluate-single env′ exp′ bv*›
unfolding ‹*bv* = -›
by *simp*

then obtain *m-env v′ ml-rhs ml-pat*
where *cupcake-evaluate-single env′* (*Var* (*Short n*)) (*Rval v′*)
and *cupcake-match-result* (*sem-env.c env′*) *v′ ml-cs Bindv* = *Rval* (*ml-rhs*, *ml-pat*, *m-env*)
and *cupcake-evaluate-single* (*env′* (| *sem-env.v* := *nsAppend* (*alist-to-ns m-env*) (*sem-env.v env′*) |)) *ml-rhs* (*Rval ml-v*)
by *cases auto*

have *closed-venv* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ))
using ‹*vclosed* v_1› ‹*vclosed* v_2›
using ‹*fmlookup css* (*Name name*) = *Some cs*›
unfolding ‹v_1 = -› *mk-rec-env-def*
apply *auto*
apply *rule*
apply *rule*
apply (*auto intro*: *fmdomI*)
done
have *wellformed-venv* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ))
using ‹*vwellformed* v_1› ‹*vwellformed* v_2›
using ‹*fmlookup css* (*Name name*) = *Some cs*›
unfolding ‹v_1 = -› *mk-rec-env-def*
apply *auto*
apply *rule*
apply *rule*
apply (*auto intro*: *fmdomI*)
done
have *not-shadows-vconsts-env* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ))
using ‹*not-shadows-vconsts* v_1› ‹*not-shadows-vconsts* v_2›
using ‹*fmlookup css* (*Name name*) = *Some cs*›
unfolding ‹v_1 = -› *mk-rec-env-def*
apply *auto*
apply *rule*
apply *rule*
apply (*auto intro*: *fmdomI*)
done
have *fmpred* (λ-. *vwelldefined′*) (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ))
using ‹*vwelldefined′* v_1› ‹*vwelldefined′* v_2›
using ‹*fmlookup css* (*Name name*) = *Some cs*›
unfolding ‹v_1 = -› *mk-rec-env-def*
apply *auto*
apply *rule*

```
 apply rule
  apply (auto intro: fmdomI)
 done

have closed-except (Sabs cs) (fmdom (fmupd (Name n) v2 ΓΛ))
 using ‹vclosed v1› unfolding ‹v1 = -›
 apply (auto simp: Sterm.closed-except-simps list-all-iff)
 using ‹fmlookup css (Name name) = Some cs›
 apply (auto simp: closed-except-def dest!: fmpredD[where m = css])
 done

 have consts (Sabs cs) |⊆| fmdom (fmupd (Name n) v2 ΓΛ) |∪| (C |∪|
fmdom css)
 using ‹vwelldefined' v1› unfolding ‹v1 = -›
 unfolding sconsts-sabs
 using ‹fmlookup css (Name name) = Some cs›
 by (auto simp: list-all-iff dest!: fmpredD[where m = css])

have ¬ shadows-consts (Sabs cs)
 using ‹not-shadows-vconsts v1› unfolding ‹v1 = -›
 using ‹fmlookup css (Name name) = Some cs›
 by (auto simp: list-all-iff list-ex-iff)

have fdisjnt C (fmdom ΓΛ)
 using ‹vwelldefined' v1› unfolding ‹v1 = -›
 using ‹fmlookup css (Name name) = Some cs›
 by auto

 have if-rval (λml-v. ∃ v. fmupd (Name n) v2 (ΓΛ ++f mk-rec-env css ΓΛ)
⊢v Sabs cs $s Svar (Name n) ↓ v ∧ related-v v ml-v) bv
 proof (rule app1(2))
  have fmrel-on-fset (ids (Sabs cs)) related-v ΓΛ (fmap-of-ns (sem-env.v
env_Λ))
   apply (rule fmrel-on-fsubset)
    apply fact
   apply (subst funion-image-bind-eq[symmetric])
   apply (rule ffUnion-subset-elem)
   apply (subst fimage-iff)
   apply (rule fBexI)
    apply simp
   apply (rule fmranI)
   apply fact
   done

    have fmrel-on-fset (ids (Sabs cs)) related-v (mk-rec-env css ΓΛ)
(cake-mk-rec-env funs env_Λ)
   apply rule
   apply (rule mk-rec-env-related[THEN fmrelD])
    apply (rule ‹fmrel - css ->)
```

apply (*rule* ‹*fmrel-on-fset* (*fbind* - -) *related-v* Γ_Λ -›)
done

show *fmrel-on-fset* (*ids* (*Sabs cs* $_s *Svar* (*Name n*))) *related-v* (*fmupd* (*Name n*) v_2 (Γ_Λ $+\!\!+_f$ *mk-rec-env css* Γ_Λ)) (*fmap-of-ns* (*sem-env.v env'*))
unfolding *recclosure'*
apply (*simp del*: *frees-sterm.simps*(*3*) *consts-sterm.simps*(*3*) *name.sel add*: *ids-def split*!: *sem-env.splits*)
apply (*rule fmrel-on-fset-updateI*)
unfolding *build-rec-env-fmap*
apply (*rule fmrel-on-fset-addI*)
apply (*fold ids-def*)
subgoal
using ‹*fmrel-on-fset* (*ids* (*Sabs cs*)) *related-v* Γ_Λ -› **by** *simp*
subgoal
using ‹*fmrel-on-fset* (*ids* (*Sabs cs*)) *related-v* (*mk-rec-env css* Γ_Λ) -› **by** *simp*
apply (*rule* ‹*related-v* v_2 *ml-*v_2›)
done
next
show *wellformed* (*Sabs cs* $_s *Svar* (*Name n*))
using ‹*vwellformed* v_1› **unfolding** ‹v_1 = -›
using ‹*fmlookup css* (*Name name*) = *Some cs*›
by *auto*
next
show *related-exp* (*Sabs cs* $_s *Svar* (*Name n*)) *exp'*
unfolding ‹*exp'* = -›
apply (*rule related-exp.intros*)
apply *fact*
apply (*rule related-exp.intros*)
done
next
show *closed-except* (*Sabs cs* $_s *Svar* (*Name n*)) (*fmdom* (*fmupd* (*Name n*) v_2 (Γ_Λ $+\!\!+_f$ *mk-rec-env css* Γ_Λ)))
using ‹*closed-except* (*Sabs cs*) (*fmdom* (*fmupd* (*Name n*) v_2 Γ_Λ))›
by (*auto simp*: *list-all-iff closed-except-def*)
next
show ¬ *shadows-consts* (*Sabs cs* $_s *Svar* (*Name n*))
using ‹¬ *shadows-consts* (*Sabs cs*)› ‹*Name n* $|\notin|$ *all-consts*› **by** *simp*
next
show *consts* (*Sabs cs* $_s *Svar* (*Name n*)) $|\subseteq|$ *fmdom* (*fmupd* (*Name n*) v_2 (Γ_Λ $+\!\!+_f$ *mk-rec-env css* Γ_Λ)) $|\cup|$ *C*
using ‹*consts* (*Sabs cs*) $|\subseteq|$ -› **unfolding** *mk-rec-env-def*
by *auto*
next
show *fdisjnt C* (*fmdom* (*fmupd* (*Name n*) v_2 (Γ_Λ $+\!\!+_f$ *mk-rec-env css* Γ_Λ)))
using ‹*Name n* $|\notin|$ *all-consts*› ‹*fdisjnt C* (*fmdom* Γ_Λ)› ‹*vwelldefined'* v_1›

```
            unfolding mk-rec-env-def ‹v₁ = -›
            by (auto simp: fdisjnt-alt-def all-consts-def)
          qed fact+
        then obtain v
          where fmupd (Name n) v₂ (Γ_Λ ++_f mk-rec-env css Γ_Λ) ⊢_v Sabs cs $_s
Svar (Name n) ↓ v related-v v ml-v
          unfolding ‹bv = -›
          by auto

        then obtain env pat rhs
          where vfind-match cs v₂ = Some (env, pat, rhs)
           and fmupd (Name n) v₂ (Γ_Λ ++_f mk-rec-env css Γ_Λ) ++_f env ⊢_v rhs
↓ v
          by (auto elim: veval'-sabs-svarE)
        hence (pat, rhs) ∈ set cs vmatch (mk-pat pat) v₂ = Some env
          by (metis vfind-match-elem)+
        hence linear pat wellformed rhs
          using ‹vwellformed v₁› unfolding ‹v₁ = -›
          using ‹fmlookup css (Name name) = Some cs›
          by (auto simp: list-all-iff)
        hence frees pat = patvars (mk-pat pat)
          by (simp add: mk-pat-frees)
        hence fmdom env = frees pat
          apply simp
          apply (rule vmatch-dom)
          apply (rule ‹vmatch (mk-pat pat) v₂ = Some env›)
          done

      obtain v' where Γ_Λ ++_f mk-rec-env css Γ_Λ ++_f env ⊢_v rhs ↓ v' v' ≈_e v
        proof (rule veval'-agree-eq)
          show fmupd (Name n) v₂ (Γ_Λ ++_f mk-rec-env css Γ_Λ) ++_f env ⊢_v
rhs ↓ v by fact
        next
          have ∗: Name n |∉| ids rhs if Name n |∉| fmdom env
            proof
              assume Name n |∈| ids rhs
              thus False
                unfolding ids-def
                proof (cases rule: funion-strictE)
                  case A
                  moreover have Name n |∉| frees pat
                    using that unfolding ‹fmdom env = frees pat› .
                  ultimately have Name n |∈| frees (Sabs cs)
                    apply auto
                    unfolding ffUnion-alt-def
                    apply simp
                    apply (rule fBexI[where x = (pat, rhs)])
                     apply (auto simp: fset-of-list-elem)
                    apply (rule ‹(pat, rhs) ∈ set cs›)
```

```
      done
    thus ?thesis
      using ‹Name n |∉| ids (Sabs cs)› unfolding ids-def
      by blast
  next
    case B
    hence Name n |∈| consts (Sabs cs)
      apply auto
      unfolding ffUnion-alt-def
      apply simp
      apply (rule fBexI[where x = (pat, rhs)])
       apply (auto simp: fset-of-list-elem)
      apply (rule ‹(pat, rhs) ∈ set cs›)
      done
    thus ?thesis
      using ‹Name n |∉| ids (Sabs cs)› unfolding ids-def
      by blast
  qed
qed

  show fmrel-on-fset (ids rhs) erelated (Γ_Λ ++_f mk-rec-env css Γ_Λ
++_f env) (fmupd (Name n) v_2 (Γ_Λ ++_f mk-rec-env css Γ_Λ) ++_f env)
    apply rule
    apply auto
          apply (rule option.rel-refl; rule erelated-refl)
    using ∗ apply auto[]
        apply (rule option.rel-refl; rule erelated-refl)+
    using ∗ apply auto[]
     apply (rule option.rel-refl; rule erelated-refl)+
    done
next
  show closed-venv (fmupd (Name n) v_2 (Γ_Λ ++_f mk-rec-env css Γ_Λ)
++_f env)
    apply rule
     apply fact
    apply (rule vclosed.vmatch-env)
     apply fact
    apply fact
    done
next
  show wellformed-venv (fmupd (Name n) v_2 (Γ_Λ ++_f mk-rec-env css
Γ_Λ) ++_f env)
    apply rule
     apply fact
    apply (rule vwellformed.vmatch-env)
     apply fact
    apply fact
    done
next
```

show *closed-except rhs* (*fmdom* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ) ++$_f$ *env*))
using ‹*fmdom env* = *frees pat*› ‹(*pat*, *rhs*) ∈ *set cs*›
using ‹*closed-except* (*Sabs cs*) (*fmdom* (*fmupd* (*Name n*) v_2 Γ_Λ))›
apply (*auto simp*: *Sterm.closed-except-simps list-all-iff*)
apply (*erule ballE*[**where** *x* = (*pat*, *rhs*)])
apply (*auto simp*: *closed-except-def*)
done
next
show *wellformed rhs* **by** *fact*
next
show *consts rhs* |⊆| *fmdom* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ) ++$_f$ *env*) |∪| *C*
using ‹*consts* (*Sabs cs*) |⊆| {} › ‹(*pat*, *rhs*) ∈ *set cs*›
unfolding *sconsts-sabs mk-rec-env-def*
by (*auto simp*: *list-all-iff*)
next
have *fdisjnt C* (*fmdom env*)
using ‹(*pat*, *rhs*) ∈ *set cs*› ‹¬ *shadows-consts* (*Sabs cs*)›
unfolding ‹*fmdom env* = *frees pat*›
by (*auto simp*: *list-ex-iff all-consts-def fdisjnt-alt-def*)
moreover have *fdisjnt C* (*fmdom css*)
using ‹*vwelldefined′* v_1› **unfolding** ‹v_1 = {}›
by *simp*
ultimately show *fdisjnt C* (*fmdom* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ) ++$_f$ *env*))
using ‹*Name n* |∉| *all-consts*› ‹*fdisjnt C* (*fmdom* Γ_Λ)›
unfolding *fdisjnt-alt-def mk-rec-env-def*
by (*auto simp*: *all-consts-def*)
next
show ¬ *shadows-consts rhs*
using ‹(*pat*, *rhs*) ∈ *set cs*› ‹¬ *shadows-consts* (*Sabs cs*)›
by (*auto simp*: *list-ex-iff*)
next
have *not-shadows-vconsts-env env*
by (*rule not-shadows-vconsts.vmatch-env*) *fact*+
thus *not-shadows-vconsts-env* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ) ++$_f$ *env*)
using ‹*not-shadows-vconsts-env* (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ))› **by** *blast*
next
have *fmpred* (λ-. *vwelldefined′*) *env*
by (*rule vmatch-welldefined′*) *fact*+
thus *fmpred* (λ-. *vwelldefined′*) (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ) ++$_f$ *env*)
using ‹*fmpred* (λ-. *vwelldefined′*) (*fmupd* (*Name n*) v_2 (Γ_Λ ++$_f$ *mk-rec-env css* Γ_Λ))› **by** *blast*
qed *simp*

```
        show ?thesis
          apply (intro exI conjI)
          unfolding ‹t = -›
           apply (rule veval'.rec-comb)
          using ‹Γ ⊢v t1 ↓ v1› unfolding ‹v1 = -› apply blast
              apply fact+
          apply (rule related-v-ext)
           apply fact+
          done
      qed
    qed
next
  case (mat1 env ml-scr ml-scr' ml-cs ml-rhs ml-pat env' ml-res)

  obtain scr cs
    where t = Sabs cs $s scr related-exp scr ml-scr
      and list-all2 (rel-prod related-pat related-exp) cs ml-cs
    using ‹related-exp t (Mat ml-scr ml-cs)›
    by cases

  have sem-env.c env = as-static-cenv
    using ‹is-cupcake-all-env env›
    by (auto elim: is-cupcake-all-envE)

  obtain scr' where Γ ⊢v scr ↓ scr' related-v scr' ml-scr'
    using mat1(4) unfolding if-rval.simps
    proof
      show
        is-cupcake-all-env env related-exp scr ml-scr wellformed-venv Γ
        closed-venv Γ fmpred (λ-. vwelldefined') Γ fdisjnt C (fmdom Γ)
        not-shadows-vconsts-env Γ
        by fact+
    next
      show fmrel-on-fset (ids scr) related-v Γ (fmap-of-ns (sem-env.v env))
        apply (rule fmrel-on-fsubset)
         apply fact
        unfolding ‹t = -› ids-def
        apply auto
        done
    next
      show
        wellformed scr consts scr |⊆| fmdom Γ |∪| C
        closed-except scr (fmdom Γ) ¬ shadows-consts scr
        using mat1 unfolding ‹t = -› by (auto simp: closed-except-def)
    qed blast

  have list-all (λ(pat, -). linear pat) cs
    using mat1 unfolding ‹t = -› by (auto simp: list-all-iff)
```

```
  obtain rhs pat Γ'
    where ml-pat = mk-ml-pat (mk-pat pat) related-exp rhs ml-rhs
      and vfind-match cs scr' = Some (Γ', pat, rhs)
      and var-env Γ' env'
    using ‹list-all2 - cs ml-cs› ‹list-all - cs› ‹related-v scr' ml-scr'› mat1(2)
    unfolding ‹sem-env.c env = as-static-cenv›
    by (auto elim: match-all-related)

  hence vmatch (mk-pat pat) scr' = Some Γ'
    by (auto dest: vfind-match-elem)
  hence patvars (mk-pat pat) = fmdom Γ'
    by (auto simp: vmatch-dom)

  have (pat, rhs) ∈ set cs
    by (rule vfind-match-elem) fact

  have linear pat
    using ‹(pat, rhs) ∈ set cs› ‹wellformed t› unfolding ‹t = -›
    by (auto simp: list-all-iff)
  hence fmdom Γ' = frees pat
    using ‹patvars (mk-pat pat) = fmdom Γ'›
    by (simp add: mk-pat-frees)

  show ?case
    proof (rule if-rvalI)
      fix ml-rhs'
      assume ml-res = Rval ml-rhs'

      obtain rhs' where Γ ++ₑ Γ' ⊢ᵥ rhs ↓ rhs' related-v rhs' ml-rhs'
        using mat1(5) unfolding ‹ml-res = -› if-rval.simps
        proof
          show is-cupcake-all-env (env (| sem-env.v := nsAppend (alist-to-ns env')
(sem-env.v env) |))
            proof (rule cupcake-v-update-preserve)
              have list-all (is-cupcake-value ∘ snd) env'
                proof (rule match-all-preserve)
                  show cupcake-c-ns (sem-env.c env)
                    unfolding ‹sem-env.c env = -› by (rule static-cenv)
                next
                  have is-cupcake-exp (Mat ml-scr ml-cs)
                    apply (rule related-exp-is-cupcake)
                    using mat1 by auto
                  thus cupcake-clauses ml-cs
                    by simp

                  show is-cupcake-value ml-scr'
                    apply (rule cupcake-single-preserve)
                      apply (rule mat1)
                     apply (rule mat1)
```

```
            using ‹is-cupcake-exp (Mat ml-scr ml-cs)› by simp
          qed fact+
        hence is-cupcake-ns (alist-to-ns env')
          by (rule cupcake-alist-to-ns-preserve)
        moreover have is-cupcake-ns (sem-env.v env)
          by (rule is-cupcake-all-envD) fact
       ultimately show is-cupcake-ns (nsAppend (alist-to-ns env') (sem-env.v
env))
          by (rule cupcake-nsAppend-preserve)
      qed fact
    next
      show related-exp rhs ml-rhs
        by fact
    next
       have ∗: fmdom (fmap-of-list (map (map-prod Name id) env')) = fmdom
Γ'
        using ‹var-env Γ' env'›
        by (metis fmrel-fmdom-eq)

      have ∗∗: id |∈| ids t if id |∈| ids rhs id |∉| fmdom Γ' for id
        using ‹id |∈| ids rhs› unfolding ids-def
        proof (cases rule: funion-strictE)
          case A
          from that have id |∉| frees pat
            unfolding ‹fmdom Γ' = frees pat› by simp
          hence id |∈| frees t
            using ‹(pat, rhs) ∈ set cs›
            unfolding ‹t = -›
            apply auto
            apply (subst ffUnion-alt-def)
            apply simp
            apply (rule fBexI[where x = (pat, rhs)])
            using A apply (auto simp: fset-of-list-elem)
            done
          thus id |∈| frees t |∪| consts t by simp
        next
          case B
          hence id |∈| consts t
            using ‹(pat, rhs) ∈ set cs›
            unfolding ‹t = -›
            apply auto
            apply (subst ffUnion-alt-def)
            apply simp
            apply (rule fBexI[where x = (pat, rhs)])
             apply (auto simp: fset-of-list-elem)
            done
          thus id |∈| frees t |∪| consts t by simp
        qed
```

```
        have fmrel-on-fset (ids rhs) related-v (Γ ++f Γ′) (fmap-of-ns (sem-env.v
env) ++f fmap-of-list (map (map-prod Name id) env′))
          apply rule
          apply simp
          apply safe
          subgoal
            apply (rule fmrelD)
            apply (rule ‹var-env Γ′ env′›)
            done
          subgoal
            unfolding ∗[symmetric]
        using fmdom-of-list fset-of-list-map fset.set-map image-image fst-map-prod
            by simp
          subgoal using ∗
          by (metis (no-types, opaque-lifting) comp-def fimageI fmdom-fmap-of-list
fset-of-list-map fst-comp-map-prod)
          subgoal using ∗∗
            by (metis fmlookup-ns fmrel-on-fsetD mat1.prems(2))
          done

        thus fmrel-on-fset (ids rhs) related-v (Γ ++f Γ′) (fmap-of-ns (sem-env.v
(env (| sem-env.v := nsAppend (alist-to-ns env′) (sem-env.v env) |))))
          by (auto split: sem-env.splits)
      next
        show wellformed-venv (Γ ++f Γ′)
          apply rule
           apply fact
          apply (rule vwellformed.vmatch-env)
           apply fact
          apply (rule veval′-wellformed)
            apply fact
          using mat1 unfolding ‹t = -› by auto
      next
        show closed-venv (Γ ++f Γ′)
          apply rule
           apply fact
          apply (rule vclosed.vmatch-env)
           apply fact
          apply (rule veval′-closed)
             apply fact
          using mat1 unfolding ‹t = -› by (auto simp: closed-except-def)
      next
        show fmpred (λ-. vwelldefined′) (Γ ++f Γ′)
          apply rule
           apply fact
          apply (rule vmatch-welldefined′)
           apply fact
          apply (rule veval′-welldefined′)
                apply fact
```

```
      using mat1 unfolding ‹t = -› by auto
    next
      show not-shadows-vconsts-env (Γ ++f Γ′)
        apply rule
         apply fact
        apply (rule not-shadows-vconsts.vmatch-env)
         apply fact
        apply (rule veval′-shadows)
          apply fact
        using mat1 unfolding ‹t = -› by auto
    next
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› ‹wellformed t› unfolding ‹t = -›
        by (auto simp: list-all-iff)
    next
      show closed-except rhs (fmdom (Γ ++f Γ′))
        apply simp
        unfolding ‹fmdom Γ′ = frees pat›
        using ‹(pat, rhs) ∈ set cs› ‹closed-except t (fmdom Γ)› unfolding ‹t =
-›
        by (auto simp: Sterm.closed-except-simps list-all-iff)
    next
      have consts (Sabs cs) |⊆| fmdom Γ |∪| C
        using mat1 unfolding ‹t = -› by auto
      show consts rhs |⊆| fmdom (Γ ++f Γ′) |∪| C
        apply simp
        unfolding ‹fmdom Γ′ = frees pat›
        using ‹(pat, rhs) ∈ set cs› ‹consts (Sabs cs) |⊆| -›
        unfolding sconsts-sabs
        by (auto simp: list-all-iff)
    next
      have fdisjnt C (fmdom Γ′)
        unfolding ‹fmdom Γ′ = frees pat›
        using ‹¬ shadows-consts t› ‹(pat, rhs) ∈ set cs›
        unfolding ‹t = -›
        by (auto simp: list-ex-iff fdisjnt-alt-def all-consts-def)

      thus fdisjnt C (fmdom (Γ ++f Γ′))
        using ‹fdisjnt C (fmdom Γ)›
        unfolding fdisjnt-alt-def by auto
    next
      show ¬ shadows-consts rhs
        using ‹(pat, rhs) ∈ set cs› ‹¬ shadows-consts t› unfolding ‹t = -›
        by (auto simp: list-ex-iff)
    qed blast

  show ∃ t′. Γ ⊢v t ↓ t′ ∧ related-v t′ ml-rhs′
    unfolding ‹t = -›
    apply (intro exI conjI seval.intros)
```

```
        apply (rule veval'.intros)
          apply (rule veval'.intros)
         apply fact+
       done
   qed
qed auto

theorem semantic-correctness:
  fixes exp
  assumes cupcake-evaluate-single env exp (Rval ml-v) is-cupcake-all-env env
  assumes fmrel-on-fset (ids t) related-v Γ (fmap-of-ns (sem-env.v env))
  assumes related-exp t exp
  assumes wellformed t wellformed-venv Γ
  assumes closed-venv Γ closed-except t (fmdom Γ)
  assumes fmpred (λ-. vwelldefined') Γ consts t |⊆| fmdom Γ |∪| C
  assumes fdisjnt C (fmdom Γ)
  assumes ¬ shadows-consts t not-shadows-vconsts-env Γ
  obtains v where Γ ⊢ᵥ t ↓ v related-v v ml-v
using semantic-correctness0[OF assms]
by auto

end end

end
```

5.4 Converting bytes to integers

```
theory CakeML-Byte
imports
  CakeML.Evaluate-Single
  Show.Show-Instances
begin

definition pat :: Ast.pat where
pat = Ast.Pcon (Some (Short ''String-char-Char'')) (map (λn. Ast.Pvar (''b'' @
show n)) [0..<8])

definition cake-plus :: exp ⇒ exp ⇒ exp where
cake-plus t u = Ast.App (Ast.Opn Ast.Plus) [t, u]

lemma cake-plus-correct:
  assumes evaluate env s u = (s', Rval (Litv (IntLit y)))
  assumes evaluate env s' t = (s'', Rval (Litv (IntLit x)))
  shows evaluate env s (cake-plus t u) = (s'', Rval (Litv (IntLit (x + y))))
unfolding cake-plus-def using assms by simp

definition cake-times :: exp ⇒ exp ⇒ exp where
cake-times t u = Ast.App (Ast.Opn Ast.Times) [t, u]
```

lemma *cake-times-correct*:
assumes *evaluate env s u* = (*s'*, *Rval* (*Litv* (*IntLit y*)))
assumes *evaluate env s' t* = (*s''*, *Rval* (*Litv* (*IntLit x*)))
shows *evaluate env s* (*cake-times t u*) = (*s''*, *Rval* (*Litv* (*IntLit* (*x* ∗ *y*))))
unfolding *cake-times-def* **using** *assms* **by** *simp*

definition *cake-int-of-bool* :: *exp* ⇒ *exp* **where**
cake-int-of-bool e = *Ast.Mat e*
[(*Ast.Pcon* (*Some* (*Short* ''*HOL-False*'')) [], *Lit* (*IntLit 0*)),
(*Ast.Pcon* (*Some* (*Short* ''*HOL-True*'')) [], *Lit* (*IntLit 1*))]

definition *summands* :: *exp list* **where**
summands = *map* (λ*n*. *cake-times* (*Lit* (*IntLit* (*2* ^ *n*))) (*cake-int-of-bool* (*Ast.Var* (*Short* (''*b*'' @ *show n*))))) [*0*..<*8*]

definition *cake-int-of-byte* :: *exp* **where**
cake-int-of-byte =
Ast.Fun ''*x*'' (*Ast.Mat* (*Ast.Var* (*Short* ''*x*'')) [(*pat*, *foldl cake-plus* (*Lit* (*IntLit 0*)) *summands*)])

end

Chapter 6

Composition of phases and full compilation pipeline

theory *Doc-Compiler*
imports *Main*
begin

end

6.1 Composition of correctness results

theory *Composition*
imports *../Backend/CakeML-Correctness*
begin

hide-const (**open**) *sem-env.v*

Term-Class.term $\longrightarrow$ *nterm* $\longrightarrow$ *pterm* $\longrightarrow$ *sterm*

6.1.1 Reflexive-transitive closure of *irules.compile-correct*.

lemma (**in** *prules*) *prewrite-closed*:
  assumes $rs \vdash_p t \longrightarrow t'$ *closed t*
  shows *closed t'*
using *assms* **proof** *induction*
  case (*step name rhs*)
  thus *?case*
    using *all-rules* **by** *force*
next
  case (*beta c*)
  obtain *pat rhs* **where** *c* = (*pat*, *rhs*) **by** (*cases c*) *auto*
  with *beta* **have** *closed-except rhs* (*frees pat*)
    by (*auto simp*: *closed-except-simps*)
  show *?case*

apply (*rule rewrite-step-closed*[*OF* - *beta*(*2*)[*unfolded* ‹*c* = -›]])
using ‹*closed-except rhs* (*frees pat*)› *beta* **by** (*auto simp*: *closed-except-def*)
qed (*auto simp*: *closed-except-def*)

corollary (**in** *prules*) *prewrite-rt-closed*:
assumes *rs* $\vdash_p$ *t* $\longrightarrow*$ *t'* *closed t*
shows *closed t'*
using *assms*
by *induction* (*auto intro*: *prewrite-closed*)

corollary (**in** *irules*) *compile-correct-rt*:
assumes *Rewriting-Pterm.compile rs* $\vdash_p$ *t* $\longrightarrow*$ *t' finished rs*
shows *rs* $\vdash_i$ *t* $\longrightarrow*$ *t'*
using *assms* **proof** (*induction rule*: *rtranclp-induct*)
case *step*
thus *?case*
by (*meson compile-correct rtranclp.simps*)
qed *auto*

6.1.2 Reflexive-transitive closure of *prules.compile-correct*.

lemma (**in** *prules*) *compile-correct-rt*:
assumes *Rewriting-Sterm.compile rs* $\vdash_s$ *u* $\longrightarrow*$ *u'*
shows *rs* $\vdash_p$ *sterm-to-pterm u* $\longrightarrow*$ *sterm-to-pterm u'*
using *assms* **proof** *induction*
case *step*
thus *?case*
by (*meson compile-correct rtranclp.simps*)
qed *auto*

lemma *srewrite-stepD*:
assumes *srewrite-step rs name t*
shows (*name*, *t*) ∈ *set rs*
using *assms* **by** *induct auto*

lemma (**in** *srules*) *srewrite-wellformed*:
assumes *rs* $\vdash_s$ *t* $\longrightarrow$ *t' wellformed t*
shows *wellformed t'*
using *assms* **proof** *induction*
case (*step name rhs*)
hence (*name*, *rhs*) ∈ *set rs*
by (*auto dest*: *srewrite-stepD*)
thus *?case*
using *all-rules* **by** (*auto simp*: *list-all-iff*)
next
case (*beta cs t t'*)
then obtain *pat rhs env* **where** (*pat*, *rhs*) ∈ *set cs match pat t* = *Some env t'*
= *subst rhs env*
by (*elim rewrite-firstE*)

```
  show ?case
    unfolding ‹t' = -›
    proof (rule subst-wellformed)
      show wellformed rhs
        using ‹(pat, rhs) ∈ set cs› beta by (auto simp: list-all-iff)
    next
      show wellformed-env env
        using ‹match pat t = Some env› beta
        by (auto intro: wellformed.match)
    qed
qed auto

lemma (in srules) srewrite-wellformed-rt:
  assumes rs ⊢s t ⟶* t' wellformed t
  shows wellformed t'
using assms
by induction (auto intro: srewrite-wellformed)

lemma vno-abs-value-to-sterm: no-abs (value-to-sterm v) ⟷ vno-abs v for v
by (induction v) (auto simp: no-abs.list-comb list-all-iff)
```

6.1.3 Reflexive-transitive closure of *rules.compile-correct*.

```
corollary (in rules) compile-correct-rt:
  assumes compile ⊢n u ⟶* u' closed u
  shows rs ⊢ nterm-to-term' u ⟶* nterm-to-term' u'
using assms
proof induction
  case (step u' u'')
  hence rs ⊢ nterm-to-term' u ⟶* nterm-to-term' u'
    by auto
  also have rs ⊢ nterm-to-term' u' ⟶ nterm-to-term' u''
   using step by (auto dest: rewrite-rt-closed intro!: compile-correct simp: closed-except-def)
  finally show ?case .
qed auto
```

6.1.4 Reflexive-transitive closure of *irules.transform-correct*.

```
corollary (in irules) transform-correct-rt:
  assumes transform-irule-set rs ⊢i u ⟶* u'' t ≈p u closed t
  obtains t'' where rs ⊢i t ⟶* t'' t'' ≈p u''
using assms proof (induction arbitrary: thesis t)
  case (step u' u'')

  obtain t' where rs ⊢i t ⟶* t' t' ≈p u'
    using step by blast

  obtain t'' where rs ⊢i t' ⟶* t'' t'' ≈p u''
    apply (rule transform-correct)
      apply (rule ‹transform-irule-set rs ⊢i u' ⟶ u''›)
```

```
      apply (rule ‹t' ≈p u'›)
     apply (rule irewrite-rt-closed)
      apply (rule ‹rs ⊢i t ⟶* t'›)
     apply (rule ‹closed t›)
    apply blast
    done

  show ?case
    apply (rule step.prems)
     apply (rule rtranclp-trans)
      apply fact+
    done
qed blast

corollary (in irules) transform-correct-rt-no-abs:
  assumes transform-irule-set rs ⊢i t ⟶* u closed t no-abs u
  shows rs ⊢i t ⟶* u
proof −
  have t ≈p t by (rule prelated-refl)
  obtain t' where rs ⊢i t ⟶* t' t' ≈p u
    apply (rule transform-correct-rt)
       apply (rule assms)
      apply (rule ‹t ≈p t›)
     apply (rule assms)
    apply blast
    done
  thus ?thesis
    using assms by (metis prelated-no-abs-right)
qed

corollary transform-correct-rt-n-no-abs0:
  assumes irules C rs (transform-irule-set ^^ n) rs ⊢i t ⟶* u closed t no-abs u
  shows rs ⊢i t ⟶* u
using assms(1,2) proof (induction n arbitrary: rs)
  case (Suc n)
  interpret irules C rs by fact
  show ?case
    apply (rule transform-correct-rt-no-abs)
      apply (rule Suc.IH)
       apply (rule rules-transform)
    using Suc(3) apply (simp add: funpow-swap1)
     apply fact+
    done
qed auto

corollary (in irules) transform-correct-rt-n-no-abs:
  assumes (transform-irule-set ^^ n) rs ⊢i t ⟶* u closed t no-abs u
  shows rs ⊢i t ⟶* u
by (rule transform-correct-rt-n-no-abs0) (rule irules-axioms assms)+
```

hide-fact *transform-correct-rt-n-no-abs0*

6.1.5 Iterated application of *transform-irule-set*.

definition *max-arity* :: *irule-set* ⇒ *nat* **where**
max-arity rs = *fMax* ((*arity* ∘ *snd*) |‘| *rs*)

lemma *rules-transform-iter0*:
assumes *irules C-info rs*
shows *irules C-info* ((*transform-irule-set* ^^ *n*) *rs*)
using *assms*
by (*induction n*) (*auto intro*: *irules.rules-transform del*: *irulesI*)

lemma (**in** *irules*) *rules-transform-iter*: *irules C-info* ((*transform-irule-set* ^^ *n*) *rs*)
by (*rule rules-transform-iter0*) (*rule irules-axioms*)

lemma *transform-irule-set-n-heads*: *fst* |‘| ((*transform-irule-set* ^^ *n*) *rs*) = *fst* |‘| *rs*
by (*induction n*) (*auto simp*: *transform-irule-set-heads*)

hide-fact *rules-transform-iter0*

definition *transform-irule-set-iter* :: *irule-set* ⇒ *irule-set* **where**
transform-irule-set-iter rs = (*transform-irule-set* ^^ *max-arity rs*) *rs*

lemma *transform-irule-set-iter-heads*: *fst* |‘| *transform-irule-set-iter rs* = *fst* |‘| *rs*
unfolding *transform-irule-set-iter-def* **by** (*simp add*: *transform-irule-set-n-heads*)

lemma (**in** *irules*) *finished-alt-def*: *finished rs* ⟷ *max-arity rs* = *0*
proof
assume *max-arity rs* = *0*
hence ¬ *fBex* ((*arity* ∘ *snd*) |‘| *rs*) (λ*x*. *0* < *x*)
using *nonempty*
unfolding *max-arity-def*
by (*metis fBex-fempty fmax-ex-gr not-less0*)
thus *finished rs*
unfolding *finished-def*
by *force*
next
assume *finished rs*
have *fMax* ((*arity* ∘ *snd*) |‘| *rs*) ≤ *0*
proof (*rule fMax-le*)
show *fBall* ((*arity* ∘ *snd*) |‘| *rs*) (λ*x*. *x* ≤ *0*)
using ‹*finished rs*› **unfolding** *finished-def* **by** *force*
next
show (*arity* ∘ *snd*) |‘| *rs* ≠ {||}
using *nonempty* **by** *force*

```
    qed
  thus max-arity rs = 0
    unfolding max-arity-def by simp
qed

lemma (in irules) transform-finished-id: finished rs ⟹ transform-irule-set rs =
rs
unfolding transform-irule-set-def finished-def transform-irules-def map-prod-def id-apply
by (rule fset-map-snd-id) (auto elim!: fBallE)

lemma (in irules) max-arity-decr: max-arity (transform-irule-set rs) = max-arity
rs − 1
proof (cases finished rs)
  case True
  thus ?thesis
    by (auto simp: transform-finished-id finished-alt-def)
next
  case False
  have (arity ∘ snd) |‘| transform-irule-set rs = (λx. x − 1) |‘| (arity ∘ snd) |‘| rs
    unfolding transform-irule-set-def fset.map-comp
   proof (rule fset.map-cong0, safe, unfold o-apply map-prod-simp id-apply snd-conv)
      fix name irs
      assume (name, irs) |∈| rs
      hence arity-compatibles irs irs ≠ {||}
        using nonempty inner
        unfolding atomize-conj
        by (smt (verit, del-insts) ‹(name, irs) |∈| rs› case-prodD fBallE inner)
      thus arity (transform-irules irs) = arity irs − 1
        by (simp add: arity-transform-irules)
    qed
  hence max-arity (transform-irule-set rs) = fMax ((λx. x − 1) |‘| (arity ∘ snd)
|‘| rs)
    unfolding max-arity-def by simp
  also have ... = fMax ((arity ∘ snd) |‘| rs) − 1
    proof (rule fmax-decr)
      show fBex ((arity ∘ snd) |‘| rs) ((≤) 1)
        using False unfolding finished-def by force
    qed
  finally show ?thesis
    unfolding max-arity-def
    by simp
qed

lemma max-arity-decr'0:
  assumes irules C rs
  shows max-arity ((transform-irule-set ^^ n) rs) = max-arity rs − n
proof (induction n)
  case (Suc n)
  show ?case
```

apply *simp*
apply (*subst irules.max-arity-decr*)
using *Suc assms* **by** (*auto intro*: *irules.rules-transform-iter del*: *irulesI*)
qed *auto*

lemma (**in** *irules*) *max-arity-decr'*: *max-arity* ((*transform-irule-set* ^^ *n*) *rs*) = *max-arity rs* − *n*
by (*rule max-arity-decr'0*) (*rule irules-axioms*)

hide-fact *max-arity-decr'0*

lemma (**in** *irules*) *transform-finished*: *finished* (*transform-irule-set-iter rs*)
unfolding *transform-irule-set-iter-def*
by (*subst irules.finished-alt-def*)
(*auto simp*: *max-arity-decr' intro*: *rules-transform-iter del*: *Rewriting-Pterm-Elim.irulesI*)

Trick as described in §*7.1* in the locale manual.

locale *irules'* = *irules*

sublocale *irules'* ⊆ *irules'-as-irules*: *irules C-info transform-irule-set-iter rs*
unfolding *transform-irule-set-iter-def* **by** (*rule rules-transform-iter*)

sublocale *crules* ⊆ *crules-as-irules'*: *irules' C-info Rewriting-Pterm-Elim.compile rs*
unfolding *irules'-def* **by** (*fact compile-rules*)

sublocale *irules'* ⊆ *irules'-as-prules*: *prules C-info Rewriting-Pterm.compile* (*transform-irule-set-iter rs*)
by (*rule irules'-as-irules.compile-rules*) (*rule transform-finished*)

6.1.6 Big-step semantics

context *srules* **begin**

definition *global-css* :: (*name*, *sclauses*) *fmap* **where**
global-css = *fmap-of-list* (*map* (*map-prod id clauses*) *rs*)

lemma *fmdom-global-css*: *fmdom global-css* = *fst* |‘| *fset-of-list rs*
unfolding *global-css-def* **by** *simp*

definition *as-vrules* :: *vrule list* **where**
as-vrules = *map* (λ(*name*, -). (*name*, *Vrecabs global-css name fmempty*)) *rs*

lemma *as-vrules-fst*[*simp*]: *fst* |‘| *fset-of-list as-vrules* = *fst* |‘| *fset-of-list rs*
unfolding *as-vrules-def*
apply *simp*
apply (*rule fset.map-cong*)
apply (*rule refl*)
by *auto*

lemma *as-vrules-fst'*[*simp*]: *map fst as-vrules = map fst rs*
unfolding *as-vrules-def*
by *auto*

lemma *list-all-as-vrulesI*:
assumes *list-all* (λ(-, *t*). *P fmempty* (*clauses t*)) *rs*
assumes *R* (*fst* |‘| *fset-of-list rs*)
shows *list-all* (λ(-, *t*). *value-pred.pred P Q R t*) *as-vrules*
proof (*rule list-allI*, *safe*)
fix *name rhs*
assume (*name*, *rhs*) ∈ *set as-vrules*
hence *rhs = Vrecabs global-css name fmempty*
unfolding *as-vrules-def* **by** *auto*

moreover have *fmpred* (λ-. *P fmempty*) *global-css*
unfolding *global-css-def list.pred-map*
using *assms* **by** (*auto simp*: *list-all-iff intro*!: *fmpred-of-list*)

moreover have *name* |∈| *fmdom global-css*
unfolding *global-css-def*
apply *auto*
using ‹(*name*, *rhs*) ∈ *set as-vrules*› **unfolding** *as-vrules-def*
including *fset.lifting* **apply** *transfer'*
by *force*

moreover have *R* (*fmdom global-css*)
using *assms* **unfolding** *global-css-def*
by *auto*

ultimately show *value-pred.pred P Q R rhs*
by (*simp add*: *value-pred.pred-alt-def*)
qed

lemma *srules-as-vrules*: *vrules C-info as-vrules*
proof (*standard*, *unfold as-vrules-fst*)
have *list-all* (λ(-, *t*). *vwellformed t*) *as-vrules*
unfolding *vwellformed-def*
apply (*rule list-all-as-vrulesI*)
apply (*rule list.pred-mono-strong*)
apply (*rule all-rules*)
apply (*auto elim*: *clausesE*)
done

moreover have *list-all* (λ(-, *t*). *vclosed t*) *as-vrules*
unfolding *vclosed-def*
apply (*rule list-all-as-vrulesI*)
apply *auto*
apply (*rule list.pred-mono-strong*)

```
    apply (rule all-rules)
   apply (auto elim: clausesE simp: Sterm.closed-except-simps)
   done

 moreover have list-all (λ(-, t). ¬ is-Vconstr t) as-vrules
   unfolding as-vrules-def
   by (auto simp: list-all-iff)

 ultimately show list-all vrule as-vrules
   unfolding list-all-iff by fastforce
next
 show distinct (map fst as-vrules)
   using distinct by auto
next
 show fdisjnt (fst |‘| fset-of-list rs) C
   using disjnt by simp
next
 show list-all (λ(-, rhs). not-shadows-vconsts rhs) as-vrules
   unfolding not-shadows-vconsts-def
   apply (rule list-all-as-vrulesI)
    apply auto
   apply (rule list.pred-mono-strong)
    apply (rule not-shadows)
   by (auto simp: list-all-iff list-ex-iff all-consts-def elim!: clausesE)
next
 show vconstructor-value-rs as-vrules
   unfolding vconstructor-value-rs-def
   apply (rule conjI)
   unfolding vconstructor-value-def
    apply (rule list-all-as-vrulesI)
     apply (simp add: list-all-iff)
    apply simp
   apply simp
   using disjnt by simp
next
 show list-all (λ(-, rhs). vwelldefined rhs) as-vrules
   unfolding vwelldefined-def
   apply (rule list-all-as-vrulesI)
    apply auto
   apply (rule list.pred-mono-strong)
    apply (rule swelldefined-rs)
   apply auto
   apply (erule clausesE)
   apply hypsubst-thin
   apply (subst (asm) welldefined-sabs)
   by simp
next
 show distinct all-constructors
   by (fact distinct-ctr)
```

qed

sublocale *srules-as-vrules*: *vrules C-info as-vrules*
by (*fact srules-as-vrules*)

lemma *rs′-rs-eq*: *srules-as-vrules.rs′* = *rs*
unfolding *srules-as-vrules.rs′-def*
unfolding *as-vrules-def*
apply (*subst map-prod-def*)
apply *simp*
unfolding *comp-def*
apply (*subst case-prod-twice*)
apply (*rule list-map-snd-id*)
unfolding *global-css-def*
using *all-rules map*
apply (*auto simp*: *list-all-iff map-of-is-map map-of-map map-prod-def fmap-of-list.rep-eq*)
subgoal for *a b*
by (*erule ballE*[**where** *x* = (*a*, *b*)], *cases b*, *auto simp*: *is-abs-def term-cases-def*)
done

lemma *veval-correct*:
fixes *v*
assumes *as-vrules*, *fmempty* $\vdash_v$ *t* $\downarrow$ *v wellformed t closed t*
shows *rs*, *fmempty* $\vdash_s$ *t* $\downarrow$ *value-to-sterm v*
using *assms*
by (*rule srules-as-vrules.veval-correct*[*unfolded rs′-rs-eq*])

end

6.1.7 ML-style semantics

context *srules* **begin**

lemma *as-vrules-mk-rec-env*: *fmap-of-list as-vrules* = *mk-rec-env global-css fmempty*
apply (*subst global-css-def*)
apply (*subst as-vrules-def*)
apply (*subst mk-rec-env-def*)
apply (*rule fmap-ext*)
apply (*subst fmlookup-fmmap-keys*)
apply (*subst fmap-of-list.rep-eq*)
apply (*subst fmap-of-list.rep-eq*)
apply (*subst map-of-map-keyed*)
apply (*subst* (*2*) *map-prod-def*)
apply (*subst id-apply*)
apply (*subst map-of-map*)
apply *simp*
apply (*subst option.map-comp*)
apply (*rule option.map-cong*)
apply (*rule refl*)

apply *simp*
apply (*subst global-css-def*)
apply (*rule refl*)
done

abbreviation (*input*) *vrelated* ≡ *srules-as-vrules.vrelated*
notation *srules-as-vrules.vrelated* (‹⊢$_v$/ - ≈ -› [*0*, *50*] *50*)

lemma *vrecabs-global-css-refl*:
  assumes *name* |∈| *fmdom global-css*
  shows ⊢$_v$ *Vrecabs global-css name fmempty* ≈ *Vrecabs global-css name fmempty*
using *assms*
proof (*coinduction arbitrary*: *name*)
  case *vrelated*
  have *rel-option* (λv_1 v_2. (∃ *name*. v_1 = *Vrecabs global-css name fmempty* ∧ v_2 = *Vrecabs global-css name fmempty* ∧ *name* |∈| *fmdom global-css*) ∨ ⊢$_v$ v_1 ≈ v_2) (*fmlookup* (*fmap-of-list as-vrules*) *y*) (*fmlookup* (*mk-rec-env global-css fmempty*) *y*)
for *y*
    apply (*subst as-vrules-mk-rec-env*)
    apply (*rule option.rel-refl-strong*)
    apply (*rule disjI1*)
    apply (*simp add*: *mk-rec-env-def*)
    apply (*elim conjE exE*)
    by (*auto intro*: *fmdomI*)
  with *vrelated* **show** *?case*
    by *fastforce*
qed

lemma *as-vrules-refl-rs*: *fmrel-on-fset* (*fst* |‘| *fset-of-list as-vrules*) *vrelated* (*fmap-of-list as-vrules*) (*fmap-of-list as-vrules*)
apply *rule*
apply (*subst* (*2*) *as-vrules-def*)
apply (*subst* (*2*) *as-vrules-def*)
apply (*simp add*: *fmap-of-list.rep-eq*)
apply (*rule rel-option-reflI*)
apply *simp*
apply (*drule map-of-SomeD*)
apply *auto*
apply (*rule vrecabs-global-css-refl*)
unfolding *global-css-def*
by (*auto simp*: *fset-of-list-elem intro*: *rev-image-eqI*)

lemma *as-vrules-refl-C*: *fmrel-on-fset C vrelated* (*fmap-of-list as-vrules*) (*fmap-of-list as-vrules*)
proof
  fix *c*
  assume *c* |∈| *C*
  hence *c* |∉| *fset-of-list* (*map fst as-vrules*)
    using *srules-as-vrules.vconstructor-value-rs*

unfolding *vconstructor-value-rs-def fdisjnt-alt-def*
by *auto*
hence *c* |∉| *fmdom* (*fmap-of-list as-vrules*)
by *simp*
hence *fmlookup* (*fmap-of-list as-vrules*) *c* = *None*
by (*metis fmdom-notD*)
thus *rel-option vrelated* (*fmlookup* (*fmap-of-list as-vrules*) *c*) (*fmlookup* (*fmap-of-list as-vrules*) *c*)
by *simp*
qed

lemma *veval'-correct''*:
fixes *t v*
assumes *fmap-of-list as-vrules* ⊢$_v$ *t* ↓ *v*
assumes *wellformed t*
assumes ¬ *shadows-consts t*
assumes *welldefined t*
assumes *closed t*
assumes *vno-abs v*
shows *as-vrules*, *fmempty* ⊢$_v$ *t* ↓ *v*
proof −
obtain v_1 **where** *as-vrules*, *fmempty* ⊢$_v$ *t* ↓ v_1 ⊢$_v$ v_1 ≈ *v*
using ‹*fmap-of-list as-vrules* ⊢$_v$ *t* ↓ *v*›
proof (*rule srules-as-vrules.veval'-correct'*, *unfold as-vrules-fst*)
show *wellformed t* ¬ *shadows-consts t closed t consts t* |⊆| *all-consts*
by *fact*+
next
show *wellformed-venv* (*fmap-of-list as-vrules*)
apply *rule*
using *srules-as-vrules.all-rules*
apply (*auto simp*: *list-all-iff*)
done
next
show *not-shadows-vconsts-env* (*fmap-of-list as-vrules*)
apply *rule*
using *srules-as-vrules.not-shadows*
apply (*auto simp*: *list-all-iff*)
done
next
have *fmrel-on-fset* (*fst* |‘| *fset-of-list as-vrules* |∪| *C*) *vrelated* (*fmap-of-list as-vrules*) (*fmap-of-list as-vrules*)
apply (*rule fmrel-on-fset-unionI*)
apply (*rule as-vrules-refl-rs*)
apply (*rule as-vrules-refl-C*)
done

show *fmrel-on-fset* (*consts t*) *vrelated* (*fmap-of-list as-vrules*) (*fmap-of-list as-vrules*)
apply (*rule fmrel-on-fsubset*)

apply *fact+*
using *assms* **by** (*auto simp*: *all-consts-def*)
qed

thus *?thesis*
using *assms* **by** (*metis srules-as-vrules.vrelated.eq-right*)
qed

end

6.1.8 CakeML

context *srules* **begin**

definition *as-sem-env* :: *v sem-env* ⇒ *v sem-env* **where**
as-sem-env env = (‖ *sem-env.v* = *build-rec-env* (*mk-letrec-body all-consts rs*) *env nsEmpty*, *sem-env.c* = *nsEmpty* ‖)

lemma *compile-sem-env*:
evaluate-dec ck mn env state (*compile-group all-consts rs*) (*state*, *Rval* (*as-sem-env env*))
unfolding *compile-group-def as-sem-env-def*
apply (*rule evaluate-dec.dletrec1*)
unfolding *mk-letrec-body-def Let-def*
apply (*simp add*:*comp-def case-prod-twice*)
using *name-as-string.fst-distinct*[*OF distinct*]
by *auto*

lemma *compile-sem-env′*:
fun-evaluate-decs mn state env [(*compile-group all-consts rs*)] = (*state*, *Rval* (*as-sem-env env*))
unfolding *compile-group-def as-sem-env-def mk-letrec-body-def Let-def*
apply (*simp add*: *comp-def case-prod-twice*)
using *name-as-string.fst-distinct*[*OF distinct*]
by *auto*

lemma *compile-prog*[*unfolded combine-dec-result.simps*, *simplified*]:
evaluate-prog ck env state (*compile rs*) (*state*, *combine-dec-result* (*as-sem-env env*) (*Rval* (‖ *sem-env.v* = *nsEmpty*, *sem-env.c* = *nsEmpty* ‖)))
unfolding *compile-def*
apply (*rule evaluate-prog.cons1*)
apply *rule*
apply (*rule evaluate-top.tdec1*)
apply (*rule compile-sem-env*)
apply (*rule evaluate-prog.empty*)
done

lemma *compile-prog′*[*unfolded combine-dec-result.simps*, *simplified*]:
fun-evaluate-prog state env (*compile rs*) = (*state*, *combine-dec-result* (*as-sem-env*

env) (*Rval* (| *sem-env.v* = *nsEmpty*, *sem-env.c* = *nsEmpty* |)))
unfolding *compile-def fun-evaluate-prog-def no-dup-mods-def no-dup-top-types-def prog-to-mods-def prog-to-top-types-def decs-to-types-def*
using *compile-sem-env' compile-group-def* **by** *simp*

definition *sem-env* :: *v sem-env* **where**
sem-env ≡ *extend-dec-env* (*as-sem-env empty-sem-env*) *empty-sem-env*

lemma *cupcake-sem-env*: *is-cupcake-all-env sem-env*
unfolding *as-sem-env-def sem-env-def*
apply (*rule is-cupcake-all-envI*)
apply (*simp add*: *extend-dec-env-def empty-sem-env-def nsEmpty-def*)
apply (*rule cupcake-nsAppend-preserve*)
apply (*rule cupcake-build-rec-preserve*)
apply (*simp add*: *empty-sem-env-def*)
apply (*simp add*: *nsEmpty-def*)
apply (*rule mk-letrec-cupcake*)
apply *simp*
apply (*simp add*: *empty-sem-env-def*)
done

lemma *sem-env-refl*: *fmrel related-v* (*fmap-of-list as-vrules*) (*fmap-of-ns* (*sem-env.v sem-env*))
proof
fix *name*
show *rel-option related-v* (*fmlookup* (*fmap-of-list as-vrules*) *name*) (*fmlookup* (*fmap-of-ns* (*sem-env.v sem-env*)) *name*)
apply (*simp add*:
as-sem-env-def build-rec-env-fmap cake-mk-rec-env-def sem-env-def
fmap-of-list.rep-eq map-of-map-keyed option.rel-map
as-vrules-def mk-letrec-body-def comp-def case-prod-twice)
apply (*rule option.rel-refl-strong*)
apply (*rule related-v.rec-closure*)
apply *auto*[]
apply (*simp add*:
fmmap-of-list[*symmetric*, *unfolded apsnd-def map-prod-def id-def*] *fmap.rel-map*
global-css-def Let-def map-prod-def comp-def case-prod-twice)
apply (*thin-tac map-of rs name* = -)
apply (*rule fmap.rel-refl-strong*)
apply *simp*
subgoal premises *prems* **for** *rhs*
proof −
obtain *name* **where** (*name*, *rhs*) ∈ *set rs*
using *prems*
including *fmap.lifting*
by *transfer'* (*auto dest*: *map-of-SomeD*)

```
      hence is-abs rhs closed rhs welldefined rhs
        using all-rules swelldefined-rs by (auto simp add: list-all-iff)
       then obtain cs where clauses rhs = cs rhs = Sabs cs wellformed-clauses
cs
        using ‹(name, rhs) ∈ set rs› all-rules
        by (cases rhs) (auto simp: list-all-iff is-abs-def term-cases-def)
      show ?thesis
        unfolding related-fun-alt-def ‹clauses rhs = cs›
        proof (intro conjI)
            show list-all2 (rel-prod related-pat related-exp) cs (map (λ(pat, t).
(mk-ml-pat (mk-pat pat), mk-con (frees pat |∪| all-consts) t)) cs)
          unfolding list.rel-map
          apply (rule list.rel-refl-strong)
          apply (rename-tac z, case-tac z, hypsubst-thin)
          apply simp
          subgoal premises prems for pat t
            proof (rule mk-exp-correctness)
              have ¬ shadows-consts rhs
                using ‹(name, rhs) ∈ set rs› not-shadows
                by (auto simp: list-all-iff all-consts-def)
              thus ¬ shadows-consts t
                unfolding ‹rhs = Sabs cs› using prems
                by (auto simp: list-all-iff list-ex-iff)
            next
              have frees t |⊆| frees pat
                using ‹closed rhs› prems unfolding ‹rhs = -›
                apply (auto simp: list-all-iff Sterm.closed-except-simps)
                apply (erule ballE[where x = (pat, t)])
                 apply (auto simp: closed-except-def)
                done
              moreover have consts t |⊆| all-consts
              using ‹welldefined rhs› prems unfolding ‹rhs = -› welldefined-sabs
                by (auto simp: list-all-iff all-consts-def)
              ultimately show ids t |⊆| frees pat |∪| all-consts
                unfolding ids-def by auto
            qed (auto simp: all-consts-def)
          done
        next
          have 1: frees (Sabs cs) = {||}
            using ‹closed rhs› unfolding ‹rhs = Sabs cs›
            by (auto simp: closed-except-def)

          have 2: welldefined rhs
            using swelldefined-rs ‹(name, rhs) ∈ set rs›
            by (auto simp: list-all-iff)

          show fresh-fNext all-consts |∉| ids (Sabs cs)
            apply (rule fNext-not-member-subset)
            unfolding ids-def 1
```

```
        using 2 ‹rhs = -› by (simp add: all-consts-def del: consts-sterm.simps)
      next
        show fresh-fNext all-consts |∉| all-consts
          by (rule fNext-not-member)
      qed
    qed
  done
qed

lemma semantic-correctness':
  assumes cupcake-evaluate-single sem-env (mk-con all-consts t) (Rval ml-v)
  assumes welldefined t closed t ¬ shadows-consts t wellformed t
  obtains v where fmap-of-list as-vrules ⊢v t ↓ v related-v v ml-v
using assms(1) proof (rule semantic-correctness)
  show is-cupcake-all-env sem-env
    by (fact cupcake-sem-env)
next
  show related-exp t (mk-con all-consts t)
    apply (rule mk-exp-correctness)
    using assms
    unfolding ids-def closed-except-def by (auto simp: all-consts-def)
next
  show wellformed t ¬ shadows-consts t by fact+
next
  show closed-except t (fmdom (fmap-of-list as-vrules))
    using ‹closed t› by (auto simp: closed-except-def)
next
  show closed-venv (fmap-of-list as-vrules)
    apply (rule fmpred-of-list)
    using srules-as-vrules.all-rules
    by (auto simp: list-all-iff)

  show wellformed-venv (fmap-of-list as-vrules)
    apply (rule fmpred-of-list)
    using srules-as-vrules.all-rules
    by (auto simp: list-all-iff)
next
  have 1: fmpred (λ-. list-all (λ(pat, t). consts t |⊆| C |∪| fmdom global-css))
global-css
    apply (subst (2) global-css-def)
    apply (rule fmpred-of-list)
    apply (auto simp: map-prod-def)
    subgoal premises prems for pat t
      proof −
        from prems obtain cs where t = Sabs cs
          by (elim clausesE)
        have welldefined t
          using swelldefined-rs prems
          by (auto simp: list-all-iff fmdom-global-css)
```

```
        show ?thesis
          using ‹welldefined t›
          unfolding ‹t = -› welldefined-sabs
          by (auto simp: all-consts-def list-all-iff fmdom-global-css)
      qed
    done

  show fmpred (λ-. vwelldefined') (fmap-of-list as-vrules)
    apply (rule fmpred-of-list)
    unfolding as-vrules-def
    apply simp
    apply (erule imageE)
    apply (auto split: prod.splits)
      apply (subst fdisjnt-alt-def)
      apply simp
      apply (rule 1)
     apply (subst global-css-def)
     apply simp
    subgoal for x1 x2
      apply (rule image-eqI[where x = (x1, x2)])
      by (auto simp: fset-of-list-elem)
    subgoal
      using disjnt by (auto simp: fdisjnt-alt-def fmdom-global-css)
    done
next
  show not-shadows-vconsts-env (fmap-of-list as-vrules)
    apply (rule fmpred-of-list)
    using srules-as-vrules.not-shadows
    unfolding list-all-iff
    by auto
next
  show fdisjnt C (fmdom (fmap-of-list as-vrules))
    using disjnt by (auto simp: fdisjnt-alt-def)
next
  show fmrel-on-fset (ids t) related-v (fmap-of-list as-vrules) (fmap-of-ns (sem-env.v
sem-env))
    unfolding fmrel-on-fset-fmrel-restrict
    apply (rule fmrel-restrict-fset)
    apply (rule sem-env-refl)
    done
next
  show consts t |⊆| fmdom (fmap-of-list as-vrules) |∪| C
    apply (subst fmdom-fmap-of-list)
    apply (subst as-vrules-fst')
    apply simp
    using assms by (auto simp: all-consts-def)
qed blast
```

end

fun *cake-to-value* :: $v \Rightarrow$ *value* **where**
cake-to-value (*Conv* (*Some* (*name*, -)) *vs*) = *Vconstr* (*Name name*) (*map cake-to-value vs*)

context *cakeml′* **begin**

lemma *cake-to-value-abs-free*:
  assumes *is-cupcake-value v cake-no-abs v*
  shows *vno-abs* (*cake-to-value v*)
  using *assms* **by** (*induction v*) (*auto elim*: *is-cupcake-value.elims simp*: *list-all-iff*)

lemma *cake-to-value-related*:
  assumes *cake-no-abs v is-cupcake-value v*
  shows *related-v* (*cake-to-value v*) *v*
using *assms* **proof** (*induction v*)
  case (*Conv c vs*)
  then obtain *name tid* **where** *c* = *Some* ((*as-string name*), *TypeId* (*Short tid*))
    apply (*elim is-cupcake-value.elims*)
    subgoal
      by (*metis name.sel v.simps*(*2*))
    by *auto*
  show *?case*
    unfolding ‹*c* = -›
    apply *simp*
    apply (*rule related-v.conv*)
    apply (*simp add*: *list.rel-map*)
    apply (*rule list.rel-refl-strong*)
    apply (*rule Conv*)
    using *Conv* **unfolding** ‹*c* = -›
    by (*auto simp*: *list-all-iff*)
qed *auto*

lemma *related-v-abs-free-uniq*:
  assumes *related-v* v_1 *ml-v related-v* v_2 *ml-v cake-no-abs ml-v*
  shows $v_1 = v_2$
using *assms* **proof** (*induction arbitrary*: v_2)
  case (*conv* vs_1 *ml-vs name*)
  then obtain vs_2 **where** v_2 = *Vconstr name* vs_2 *list-all2 related-v* vs_2 *ml-vs*
    by (*auto elim*: *related-v.cases simp*: *name.expand*)
  moreover have *list-all cake-no-abs ml-vs*
    using *conv* **by** *simp*
  have *list-all2* (=) vs_1 vs_2
    using ‹*list-all2* - vs_1 -› ‹*list-all2* - vs_2 -› ‹*list-all cake-no-abs ml-vs*›
    by (*induction arbitrary*: vs_2 *rule*: *list.rel-induct*) (*auto simp*: *list-all2-Cons2*)
  thus *?case*
    unfolding ‹v_2 = -›
    by (*simp add*: *list.rel-eq*)

qed *auto*

corollary *related-v-abs-free-cake-to-value*:
assumes *related-v v ml-v cake-no-abs ml-v is-cupcake-value ml-v*
shows v = *cake-to-value ml-v*
using *assms* **by** (*metis cake-to-value-related related-v-abs-free-uniq*)

end

context *srules* **begin**

lemma *cupcake-sem-env-preserve*:
assumes *cupcake-evaluate-single sem-env* (*mk-con S t*) (*Rval ml-v*) *wellformed t*
shows *is-cupcake-value ml-v*
apply (*rule cupcake-single-preserve*[*OF assms*(*1*)])
apply (*rule cupcake-sem-env*)
apply (*rule mk-exp-cupcake*)
apply *fact*
done

lemma *semantic-correctness''*:
assumes *cupcake-evaluate-single sem-env* (*mk-con all-consts t*) (*Rval ml-v*)
assumes *welldefined t closed t* ¬ *shadows-consts t wellformed t*
assumes *cake-no-abs ml-v*
shows *fmap-of-list as-vrules* $\vdash_v$ *t* ↓ *cake-to-value ml-v*
using *assms*
by (*metis cupcake-sem-env-preserve semantic-correctness' related-v-abs-free-cake-to-value*)

end

6.1.9 Composition

context *rules* **begin**

abbreviation *term-to-nterm* **where**
term-to-nterm t ≡ *fresh-frun* (*Term-to-Nterm.term-to-nterm* [] *t*) *all-consts*

abbreviation *sterm-to-cake* **where**
sterm-to-cake ≡ *rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.mk-con all-consts*

abbreviation *term-to-cake t* ≡ *sterm-to-cake* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))
abbreviation *cake-to-term t* ≡ (*convert-term* (*value-to-sterm* (*cake-to-value t*)) :: *term*)

abbreviation *cake-sem-env* ≡ *rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.sem-env*

definition *compiled* ≡ *rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.as-vrules*

lemma *fmdom-compiled*: *fmdom* (*fmap-of-list compiled*) = *heads-of rs*
unfolding *compiled-def*
by (*simp add*:
rules-as-nrules.crules-as-irules'.irules'-as-prules.compile-heads
Rewriting-Pterm.compile-heads transform-irule-set-iter-heads
Rewriting-Pterm-Elim.compile-heads
compile-heads consts-of-heads)

lemma *cake-semantic-correctness*:
assumes *cupcake-evaluate-single cake-sem-env* (*sterm-to-cake t*) (*Rval ml-v*)
assumes *welldefined t closed t* ¬ *shadows-consts t wellformed t*
assumes *cake-no-abs ml-v*
shows *fmap-of-list compiled* ⊢$_v$ *t* ↓ *cake-to-value ml-v*
unfolding *compiled-def*
apply (*rule rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.semantic-correctness''*)
using *assms*
by (*simp-all add*:
rules-as-nrules.crules-as-irules'.irules'-as-prules.compile-heads
Rewriting-Pterm.compile-heads transform-irule-set-iter-heads
Rewriting-Pterm-Elim.compile-heads
compile-heads consts-of-heads all-consts-def)

Lo and behold, this is the final correctness theorem!

theorem *compiled-correct*:
— If CakeML evaluation of a term succeeds ...
assumes ∃ *k*. *Evaluate-Single.evaluate cake-sem-env* (*s* (| *clock* := *k* |)) (*term-to-cake t*) = (*s'*, *Rval ml-v*)
— ... producing a constructor term without closures ...
assumes *cake-no-abs ml-v*
— ... and some syntactic properties of the involved terms hold ...
assumes *closed t* ¬ *shadows-consts t welldefined t wellformed t*
— ... then this evaluation can be reproduced in the term-rewriting semantics
shows *rs* ⊢ *t* ⟶∗ *cake-to-term ml-v*
proof −
let *?heads* = *fst* |`| *fset-of-list rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.as-vrules*
have *?heads* = *heads-of rs*
using *fmdom-compiled* **unfolding** *compiled-def* **by** *simp*

have *wellformed* (*nterm-to-pterm* (*term-to-nterm t*))
by *auto*
hence *wellformed* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))
by (*auto intro*: *pterm-to-sterm-wellformed*)

have *is-cupcake-all-env cake-sem-env*
by (*rule rules-as-nrules.nrules-as-crules.crules-as-irules'.irules'-as-prules.prules-as-srules.cupcake-sem-env*)
have *is-cupcake-exp* (*term-to-cake t*)
by (*rule rules-as-nrules.nrules-as-crules.crules-as-irules'.irules'-as-prules.prules-as-srules.srules-as-cake.mk-fact*

obtain k **where** *Evaluate-Single.evaluate cake-sem-env* (s ⦇ *clock* := k ⦈) (*term-to-cake* t) = (s', *Rval ml-v*)
using *assms* **by** *blast*
then have *Big-Step-Unclocked-Single.evaluate cake-sem-env* (s ⦇ *clock* := (*clock* s') ⦈) (*term-to-cake* t) (s', *Rval ml-v*)
using *unclocked-single-fun-eq* **by** *fastforce*
have *cupcake-evaluate-single cake-sem-env* (*sterm-to-cake* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))) (*Rval ml-v*)
apply (*rule cupcake-single-complete*)
apply *fact+*
done
hence *is-cupcake-value ml-v*
apply (*rule rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.cupcake-sem-env-preserve*)
by (*auto intro*: *pterm-to-sterm-wellformed*)
hence *vno-abs* (*cake-to-value ml-v*)
using ‹*cake-no-abs* -›
by (*metis rules-as-nrules.nrules-as-crules.crules-as-irules'.irules'-as-prules.prules-as-srules.srules-as-cake.ca

hence *no-abs* (*value-to-sterm* (*cake-to-value ml-v*))
by (*metis vno-abs-value-to-sterm*)

hence *no-abs* (*sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*)))
by (*metis sterm-to-pterm convert-term-no-abs*)

have *welldefined* (*term-to-nterm t*)
unfolding *term-to-nterm'-def*
apply (*subst fresh-frun-def*)
apply (*subst pred-stateD*[*OF term-to-nterm-consts*])
apply (*subst surjective-pairing*)
apply (*rule refl*)
apply *fact*
done
have *welldefined* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))
apply (*subst pterm-to-sterm-consts*)
apply *fact*
apply (*subst consts-nterm-to-pterm*)
apply *fact+*
done

have ¬ *shadows-consts t*
using *assms* **unfolding** *shadows-consts-def fdisjnt-alt-def*
by *auto*
hence ¬ *shadows-consts* (*term-to-nterm t*)
unfolding *shadows-consts-def shadows-consts-def*
apply *auto*
using *term-to-nterm-all-vars*[*folded wellformed-term-def*]
by (*metis assms*(*6*) *fdisjnt-swap sup-idem*)

have ¬ *shadows-consts* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))

apply (*subst pterm-to-sterm-shadows*[*symmetric*])
apply *fact*
apply (*subst shadows-nterm-to-pterm*)
unfolding *shadows-consts-def*
apply *simp*
apply (*rule term-to-nterm-all-vars*[**where** T = *fempty*, *simplified*, *THEN fdisjnt-swap*])
apply (*fold wellformed-term-def*)
apply *fact*
using ‹*closed t*› **unfolding** *closed-except-def* **by** (*auto simp*: *fdisjnt-alt-def*)

have *closed* (*term-to-nterm t*)
using *assms* **unfolding** *closed-except-def*
using *term-to-nterm-vars* **unfolding** *wellformed-term-def* **by** *blast*
hence *closed* (*nterm-to-pterm* (*term-to-nterm t*))
using *closed-nterm-to-pterm* **unfolding** *closed-except-def*
by *auto*
have *closed* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))
unfolding *closed-except-def*
apply (*subst pterm-to-sterm-frees*)
apply *fact*
using ‹*closed* (*term-to-nterm t*)› *closed-nterm-to-pterm* **unfolding** *closed-except-def*
by *auto*

have *fmap-of-list compiled* $\vdash_v$ *pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)) $\downarrow$ *cake-to-value ml-v*
by (*rule cake-semantic-correctness*) *fact+*
hence *fmap-of-list rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.as-vrules* $\vdash_v$ *pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)) $\downarrow$ *cake-to-value ml-v*
using *assms* **unfolding** *compiled-def* **by** *simp*
hence *rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.as-vrules*, *fmempty* $\vdash_v$ *pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)) $\downarrow$ *cake-to-value ml-v*
proof (*rule rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.veval'-correct''*)
show ¬ *rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.shadows-consts* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)))
using ‹¬ *shadows-consts* (-::*sterm*)› ‹*?heads* = *heads-of rs*› **by** *auto*
next
show *consts* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*))) |⊆| *rules-as-nrules.crules-as-irules'.irule*
using ‹*welldefined* (*pterm-to-sterm* -)› ‹*?heads* = -› **by** *auto*
qed *fact+*
hence *Rewriting-Sterm.compile* (*Rewriting-Pterm.compile* (*transform-irule-set-iter* (*Rewriting-Pterm-Elim.compile* (*consts-of compile*)))), *fmempty* $\vdash_s$ *pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)) $\downarrow$ *value-to-sterm* (*cake-to-value ml-v*)
by (*rule rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.veval-correct*) *fact+*
hence *Rewriting-Sterm.compile* (*Rewriting-Pterm.compile* (*transform-irule-set-iter* (*Rewriting-Pterm-Elim.compile* (*consts-of compile*)))) $\vdash_s$ *pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*)) $\longrightarrow*$ *value-to-sterm* (*cake-to-value ml-v*)

by (*rule rules-as-nrules.crules-as-irules'.irules'-as-prules.prules-as-srules.seval-correct*) *fact*
hence *Rewriting-Pterm.compile* (*transform-irule-set-iter* (*Rewriting-Pterm-Elim.compile* (*consts-of compile*))) $\vdash_p$ *sterm-to-pterm* (*pterm-to-sterm* (*nterm-to-pterm* (*term-to-nterm t*))) $\longrightarrow*$ *sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*))
by (*rule rules-as-nrules.crules-as-irules'.irules'-as-prules.compile-correct-rt*)
hence *Rewriting-Pterm.compile* (*transform-irule-set-iter* (*Rewriting-Pterm-Elim.compile* (*consts-of compile*))) $\vdash_p$ *nterm-to-pterm* (*term-to-nterm t*) $\longrightarrow*$ *sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*))
by (*subst* (*asm*) *pterm-to-sterm-sterm-to-pterm*) *fact*
hence *transform-irule-set-iter* (*Rewriting-Pterm-Elim.compile* (*consts-of compile*)) $\vdash_i$ *nterm-to-pterm* (*term-to-nterm t*) $\longrightarrow*$ *sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*))
by (*rule rules-as-nrules.crules-as-irules'.irules'-as-irules.compile-correct-rt*)
(*rule rules-as-nrules.crules-as-irules.transform-finished*)
have *Rewriting-Pterm-Elim.compile* (*consts-of compile*) $\vdash_i$ *nterm-to-pterm* (*term-to-nterm t*) $\longrightarrow*$ *sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*))
apply (*rule rules-as-nrules.crules-as-irules.transform-correct-rt-n-no-abs*)
using ‹*transform-irule-set-iter* - $\vdash_i$ - $\longrightarrow*$ -› **unfolding** *transform-irule-set-iter-def*
apply *simp*
apply *fact+*
done

then obtain t' **where** *compile* $\vdash_n$ *term-to-nterm t* $\longrightarrow*$ t' $t' \approx_i$ *sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*))
using ‹*closed* (*term-to-nterm t*)›
by (*metis rules-as-nrules.compile-correct-rt*)
hence *no-abs* t'
using ‹*no-abs* (*sterm-to-pterm* -)›
by (*metis irelated-no-abs*)

have *rs* $\vdash$ *nterm-to-term'* (*term-to-nterm t*) $\longrightarrow*$ *nterm-to-term'* t'
by (*rule compile-correct-rt*) *fact+*
hence *rs* $\vdash$ *t* $\longrightarrow*$ *nterm-to-term'* t'
apply (*subst* (*asm*) *fresh-frun-def*)
apply (*subst* (*asm*) *term-to-nterm-nterm-to-term*[**where** S = *fempty* **and** t = *t*, *simplified*])
apply (*fold wellformed-term-def*)
apply *fact*
using *assms* **unfolding** *closed-except-def* **by** *auto*

have *nterm-to-pterm* t' = *sterm-to-pterm* (*value-to-sterm* (*cake-to-value ml-v*))
using ‹$t' \approx_i$ -›
by *auto*
hence (*convert-term* t' :: *pterm*) = *convert-term* (*value-to-sterm* (*cake-to-value ml-v*))
apply (*subst* (*asm*) *nterm-to-pterm*)
apply *fact*
apply (*subst* (*asm*) *sterm-to-pterm*)

```
    apply fact
   apply assumption
   done
 hence nterm-to-term' t' = convert-term (value-to-sterm (cake-to-value ml-v))
   apply (subst nterm-to-term')
    apply (rule ‹no-abs t'›)
   apply (rule convert-term-inj)
   subgoal premises
     apply (rule convert-term-no-abs)
     apply fact
     done
   subgoal premises
     apply (rule convert-term-no-abs)
     apply fact
     done
   apply (subst convert-term-idem)
    apply (rule ‹no-abs t'›)
   apply (subst convert-term-idem)
    apply (rule ‹no-abs (value-to-sterm (cake-to-value ml-v))›)
   apply assumption
   done

 thus ?thesis
   using ‹rs ⊢ t ⟶* nterm-to-term' t'› by simp
qed

end

end
```

6.2 Executable compilation chain

```
theory Compiler
imports Composition
begin

definition term-to-exp :: C-info ⇒ rule fset ⇒ term ⇒ exp where
term-to-exp C-info rs t =
  cakeml.mk-con C-info (heads-of rs |∪| constructors.C C-info)
    (pterm-to-sterm (nterm-to-pterm (fresh-frun (term-to-nterm [] t) (heads-of rs
|∪| constructors.C C-info))))

lemma (in rules) Compiler.term-to-exp C-info rs = term-to-cake
  unfolding term-to-exp-def by (simp add: all-consts-def)

primrec compress-pterm :: pterm ⇒ pterm where
compress-pterm (Pabs cs) = Pabs (fcompress (map-prod id compress-pterm |`| cs))
|
compress-pterm (Pconst name) = Pconst name |
```

compress-pterm (*Pvar name*) = *Pvar name* |
compress-pterm (*t* $_p$ *u*) = *compress-pterm t* $_p$ *compress-pterm u*

lemma *compress-pterm-eq*[*simp*]: *compress-pterm t* = *t*
by (*induction t*) (*auto simp*: *subst-pabs-id fset-map-snd-id map-prod-def*)

definition *compress-crule-set* :: *crule-set* ⇒ *crule-set* **where**
compress-crule-set = *fcompress* ∘ *fimage* (*map-prod id fcompress*)

definition *compress-irule-set* :: *irule-set* ⇒ *irule-set* **where**
compress-irule-set = *fcompress* ∘ *fimage* (*map-prod id* (*fcompress* ∘ *fimage* (*map-prod id compress-pterm*)))

definition *compress-prule-set* :: *prule fset* ⇒ *prule fset* **where**
compress-prule-set = *fcompress* ∘ *fimage* (*map-prod id compress-pterm*)

lemma *compress-crule-set-eq*[*simp*]: *compress-crule-set rs* = *rs*
unfolding *compress-crule-set-def* **by** *force*

lemma *compress-irule-set-eq*[*simp*]: *compress-irule-set rs* = *rs*
unfolding *compress-irule-set-def map-prod-def* **by** *simp*

lemma *compress-prule-set*[*simp*]: *compress-prule-set rs* = *rs*
unfolding *compress-prule-set-def* **by** *force*

definition *transform-irule-set-iter* :: *irule-set* ⇒ *irule-set* **where**
transform-irule-set-iter rs = ((*transform-irule-set* ∘ *compress-irule-set*) ^^ *max-arity rs*) *rs*

definition *as-sem-env* :: *C-info* ⇒ *srule list* ⇒ *v sem-env* ⇒ *v sem-env* **where**
as-sem-env C-info rs env =
(| *sem-env.v* =
build-rec-env (*cakeml.mk-letrec-body C-info* (*fset-of-list* (*map fst rs*) |∪| *constructors.C C-info*) *rs*) *env nsEmpty*,
sem-env.c =
nsEmpty |)

definition *empty-sem-env* :: *C-info* ⇒ *v sem-env* **where**
empty-sem-env C-info = (| *sem-env.v* = *nsEmpty*, *sem-env.c* = *constructors.as-static-cenv C-info* |)

definition *sem-env* :: *C-info* ⇒ *srule list* ⇒ *v sem-env* **where**
sem-env C-info rs = *extend-dec-env* (*as-sem-env C-info rs* (*empty-sem-env C-info*)) (*empty-sem-env C-info*)

definition *compile* :: *C-info* ⇒ *rule fset* ⇒ *Ast.prog* **where**
compile C-info =
CakeML-Backend.compile′ C-info ∘
Rewriting-Sterm.compile ∘

compress-prule-set ∘
Rewriting-Pterm.compile ∘
transform-irule-set-iter ∘
compress-irule-set ∘
Rewriting-Pterm-Elim.compile ∘
compress-crule-set ∘
Rewriting-Nterm.consts-of ∘
fcompress ∘
Rewriting-Nterm.compile′ C-info ∘
fcompress

definition *compile-to-env* :: *C-info* ⇒ *rule fset* ⇒ *v sem-env* **where**
compile-to-env C-info =
sem-env C-info ∘
Rewriting-Sterm.compile ∘
compress-prule-set ∘
Rewriting-Pterm.compile ∘
transform-irule-set-iter ∘
compress-irule-set ∘
Rewriting-Pterm-Elim.compile ∘
compress-crule-set ∘
Rewriting-Nterm.consts-of ∘
fcompress ∘
Rewriting-Nterm.compile′ C-info ∘
fcompress

lemma (**in** *rules*) *Compiler.compile-to-env C-info rs* = *rules.cake-sem-env C-info rs*
unfolding *Compiler.compile-to-env-def Compiler.sem-env-def Compiler.as-sem-env-def Compiler.empty-sem-env-def*
unfolding *rules-as-nrules.crules-as-irules′.irules′-as-prules.prules-as-srules.sem-env-def*
unfolding *rules-as-nrules.crules-as-irules′.irules′-as-prules.prules-as-srules.as-sem-env-def*
unfolding *empty-sem-env-def*
by (*auto simp*:
Compiler.compress-irule-set-eq[*abs-def*]
Composition.transform-irule-set-iter-def[*abs-def*]
Compiler.transform-irule-set-iter-def[*abs-def*] *comp-def pre-constants.all-consts-def*)

export-code
term-to-exp compile compile-to-env
checking *Scala*

end