The Busy Beaver Upper-Bound Principle

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Abstract

This development formalizes the Busy Beaver upper-bound principle and its computability consequences for the Turing machines from AFP's Universal Turing Machine entry. The abstract layer works over machine models with finite size classes and unique exact halting times: it defines the time Busy Beaver function and proves that any total upper bound for this function decides zero-input halting. The development then connects the abstract theory to AFP's Universal Turing Machine formalization by defining exact numeric-result halting times, a finite size measure for Turing programs, a concrete Turing-machine Busy Beaver function, a code-indexed Busy Beaver function whose upper bounds decide AFP's special halting problem *K1*, and a strengthened program/input-pair Busy Beaver function whose upper bounds decide AFP's two-argument halting problem *H1*. Using AFP's Turing-computability interface, it further proves that no upper-bound-induced *H1* decider set has a Turing-computable total characteristic function, with a specialization to the decider induced by the concrete pair-indexed Busy Beaver function itself.

Contents

Main Results

The formalization separates a reusable Busy Beaver upper-bound principle from its concrete Turing-machine applications.

- Any total upper bound for the abstract function *BB-time* decides zero-input halting for the corresponding model.
- The concrete function *Turing-BB-time* bounds exact numeric-result halting times for bounded-size Turing programs.
- In AFP's *hpk* locale, code-indexed upper bounds decide the diagonal halting problem *K1*.
- Pair-indexed upper bounds decide the two-argument halting problem *H1*, and the induced *H1* characteristic functions are not Turing-computable total.

theory *Busy-Beaver-Base*
imports *Main*
begin

1 The Busy Beaver Upper-Bound Principle

Radó's Busy Beaver function measures the largest halting time among machines of bounded size [1]. This theory formalizes the Busy Beaver upper-bound principle over a model-parametric notion of programs with a finite size measure and unique exact halting times.

The base locale defines the Busy Beaver time function for zero-input runs and proves that any total upper bound for it decides the corresponding fixed-input halting predicate. A second locale adds a generic computability interface: noncomputability follows from an assumed absence of computable fixed-input halting deciders, and eventual domination is derived under an explicit witness assumption that packages the usual input-to-program compilation argument. A separate theory instantiates the base locale with AFP's Turing machines and, by treating program/input pairs as Busy Beaver objects, obtains a concrete upper-bound decider for AFP's two-argument halting problem together with an explicit Turing-noncomputability consequence for the induced characteristic functions.

1.1 Base model and Busy Beaver time

locale *busy-beaver-base* =
  fixes *size* :: *'prog* ⇒ *nat*
    and *halts-in* :: *'prog* ⇒ *nat* ⇒ *nat* ⇒ *bool*
  assumes *finite-size-le*: *finite* {*p*. *size p* ≤ *n*}
    and *halting-time-unique*:
      halts-in p x t ⟹ *halts-in p x u* ⟹ *t* = *u*
begin

definition *halts* :: *'prog* ⇒ *nat* ⇒ *bool* **where**
  halts p x ⟷ (∃ *t*. *halts-in p x t*)

definition *halting-time* :: *'prog* ⇒ *nat* ⇒ *nat* **where**
  halting-time p x = (*THE t*. *halts-in p x t*)

definition *time-set* :: *nat* ⇒ *nat set* **where**
  time-set n = {*t*. ∃ *p*. *size p* ≤ *n* ∧ *halts-in p 0 t*}

definition *BB-time* :: *nat* ⇒ *nat* **where**
  BB-time n = *Max* (*insert 0* (*time-set n*))

definition *upper-bound-for-BB* :: (*nat* ⇒ *nat*) ⇒ *bool* **where**
  upper-bound-for-BB b ⟷ (∀ *n*. *BB-time n* ≤ *b n*)

definition *halting-decider-from* :: (*nat* ⇒ *nat*) ⇒ *'prog* ⇒ *nat* ⇒ *bool* **where**
  halting-decider-from b p x ⟷ (∃ *t*≤*b* (*size p*). *halts-in p x t*)

lemma *halting-time-eq*:
  assumes *halts-in p x t*
  shows *halting-time p x* = *t*
  ⟨*proof*⟩

lemma *finite-time-set* [*simp*]: *finite* (*time-set n*)
⟨*proof*⟩

lemma *finite-insert-time-set* [*simp*]: *finite* (*insert 0* (*time-set n*))
  ⟨*proof*⟩

lemma *BB-time-ge-time*:
  assumes *size p* ≤ *n*
    and *halts-in p 0 t*
  shows *t* ≤ *BB-time n*
⟨*proof*⟩

lemma *BB-time-upper-bound*:
  assumes ⋀*p t*. *size p* ≤ *n* ⟹ *halts-in p 0 t* ⟹ *t* ≤ *B*
  shows *BB-time n* ≤ *B*
⟨*proof*⟩

lemma *halting-decider-from-sound*:
assumes *halting-decider-from b p x*
shows *halts p x*
⟨*proof*⟩

lemma *halting-decider-from-complete-0*:
assumes *ub*: *upper-bound-for-BB b*
assumes *halts p 0*
shows *halting-decider-from b p 0*
⟨*proof*⟩

theorem *halting-decider-from-correct-0*:
assumes *upper-bound-for-BB b*
shows *halting-decider-from b p 0* ⟷ *halts p 0*
⟨*proof*⟩

lemma *BB-time-is-upper-bound*: *upper-bound-for-BB BB-time*
⟨*proof*⟩

end

1.2 Computability consequences

The following locale deliberately keeps computability assumptions explicit. In particular, eventual domination is not derived from finite size classes and exact halting times alone; the assumption *computable-has-busy-witness* states the compilation witness needed to turn values of a computed function into sufficiently long zero-input halting computations.

locale *busy-beaver-model* = *busy-beaver-base size halts-in*
for *size* :: *'prog* ⇒ *nat*
and *halts-in* :: *'prog* ⇒ *nat* ⇒ *nat* ⇒ *bool* +
fixes *computes* :: *'prog* ⇒ (*nat* ⇒ *nat*) ⇒ *bool*
assumes *compute-halts*:
computes p f ⟹ ∃ *t*. *halts-in p x t*
and *computable-has-busy-witness*:
computes p f ⟹ ∃ *N*. ∀ *n*≥*N*.
∃ *q t*. *size q* ≤ *n* ∧ *halts-in q 0 t* ∧ *f n* ≤ *t*
begin

definition *computable* :: (*nat* ⇒ *nat*) ⇒ *bool* **where**
computable f ⟷ (∃ *p*. *computes p f*)

definition *eventually-dominates* :: (*nat* ⇒ *nat*) ⇒ (*nat* ⇒ *nat*) ⇒ *bool* **where**
eventually-dominates g f ⟷ (∃ *N*. ∀ *n*≥*N*. *f n* ≤ *g n*)

lemma *no-computable-upper-bound-if-halting-undecidable*:
assumes *no-decider*:
⋀*b*. *computable b* ⟹

¬ (∀ p. halting-decider-from b p 0 ⟷ halts p 0)
shows ¬ (∃ b. computable b ∧ upper-bound-for-BB b)
⟨proof⟩

theorem *BB-time-not-computable-if-halting-undecidable*:
assumes *no-decider*:
⋀b. computable b ⟹
¬ (∀ p. halting-decider-from b p 0 ⟷ halts p 0)
shows ¬ computable BB-time
⟨proof⟩

lemma *BB-time-dominates-computed-function-from-size*:
assumes *comp*: computes p f
shows eventually-dominates BB-time f
⟨proof⟩

theorem *BB-time-eventually-dominates-computable*:
assumes computable f
shows eventually-dominates BB-time f
⟨proof⟩

end

end
theory *Turing-Busy-Beaver*
imports
Busy-Beaver-Base
Universal-Turing-Machine.TuringComputable
begin

2 Busy Beaver Upper Bounds for Universal Turing Machines

This theory connects the abstract Busy Beaver upper-bound principle to the Turing machines from AFP's Universal Turing Machine entry [2]. We use halting with a numeric result, the halting predicate that occurs in AFP's formalization of the special halting problems. The first instantiation is a direct Turing-program Busy Beaver function. Inside AFP's *hpk* locale we also define code-indexed variants for the one-argument problem *K1* and the two-argument problem *H1*. Finally, using AFP's Turing-computability interface, we make the noncomputability consequence explicit for the characteristic functions of the induced *H1* decider sets.

Three related Busy Beaver notions appear below. The theory *Busy-Beaver-Base* supplies the model-parametric function *BB-time*, whose objects are abstract programs run on input *0*. The first concrete interpretation instantiates this as *Turing-BB-time* for AFP Turing programs, still using a direct program-

size measure. The code-indexed interpretations are introduced for the halting problems in the Universal Turing Machine entry: the diagonal code-indexed function decides *K1*, while the pair-indexed function folds a machine code and input into a single Busy Beaver object so that upper bounds decide *H1*.

2.1 Exact halting times for numeric results

definition *has-num-result-at* :: *tprog0* ⇒ *nat list* ⇒ *nat* ⇒ *bool* **where**
has-num-result-at tm ns t ⟷
is-final (*steps0* (*1*, [], <*ns*>) *tm t*) ∧
(∃ *k n l*. *steps0* (*1*, [], <*ns*>) *tm t* = (*0*, *Bk* ↑ *k*, <*n*::*nat*> @ *Bk* ↑ *l*))

definition *tm-num-halts-in* :: *tprog0* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
tm-num-halts-in tm input t ⟷
has-num-result-at tm [*input*] *t* ∧ (∀ *u*<*t*. ¬ *has-num-result-at tm* [*input*] *u*)

lemma *tm-num-halts-in-unique*:
assumes *tm-num-halts-in tm input t*
and *tm-num-halts-in tm input u*
shows *t* = *u*
⟨*proof*⟩

lemma *TMC-has-num-res-iff-tm-num-halts*:
TMC-has-num-res tm [*input*] ⟷ (∃ *t*. *tm-num-halts-in tm input t*)
⟨*proof*⟩

2.2 A finite size measure for Turing programs

fun *action-rank* :: *action* ⇒ *nat* **where**
action-rank WB = *0*
| *action-rank WO* = *1*
| *action-rank L* = *2*
| *action-rank R* = *3*
| *action-rank Nop* = *4*

definition *instr-size* :: *instr* ⇒ *nat* **where**
instr-size i = *Suc* (*action-rank* (*fst i*) + *snd i*)

definition *tm-size* :: *tprog0* ⇒ *nat* **where**
tm-size tm = *length tm* + *sum-list* (*map instr-size tm*)

lemma *finite-instr-size-le*: *finite* {*i*::*instr*. *instr-size i* ≤ *n*}
⟨*proof*⟩

lemma *length-le-tm-size*:
length tm ≤ *tm-size tm*
⟨*proof*⟩

lemma *instr-size-le-tm-size*:
assumes $i \in set\ tm$
shows *instr-size* $i \leq$ *tm-size tm*
⟨*proof*⟩

lemma *finite-tm-size-le*: *finite* $\{tm::tprog0.\ tm\text{-}size\ tm \leq n\}$
⟨*proof*⟩

interpretation *turing-busy-beaver*: *busy-beaver-base tm-size tm-num-halts-in*
⟨*proof*⟩

abbreviation *Turing-BB-time* :: *nat* ⇒ *nat* **where**
Turing-BB-time ≡ *turing-busy-beaver.BB-time*

theorem *Turing-BB-time-ge*:
assumes *tm-size tm* ≤ *n*
and *tm-num-halts-in tm 0 t*
shows *t* ≤ *Turing-BB-time n*
⟨*proof*⟩

2.3 The code-indexed Busy Beaver function and AFP's special halting problem

context *hpk*
begin

The next predicate is deliberately diagonal: the formal locale input is not used because the special halting problem *K1* asks whether the machine coded by *c* halts on input *c*.

definition *coded-diagonal-num-halts-in* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *bool* **where**
coded-diagonal-num-halts-in c input t ⟷ *tm-num-halts-in* (*c2t c*) *c t*

interpretation *coded-busy-beaver*: *busy-beaver-base* λ*c*::*nat*. *c coded-diagonal-num-halts-in*
⟨*proof*⟩

abbreviation *Coded-BB-time* :: *nat* ⇒ *nat* **where**
Coded-BB-time ≡ *coded-busy-beaver.BB-time*

lemma *coded-halts-iff-K1*:
coded-busy-beaver.halts c 0 ⟷ [*c*] ∈ *K1*
⟨*proof*⟩

theorem *coded-BB-upper-bound-decides-K1*:
assumes *coded-busy-beaver.upper-bound-for-BB b*
shows *coded-busy-beaver.halting-decider-from b c 0* ⟷ [*c*] ∈ *K1*
⟨*proof*⟩

corollary *K1-not-turing-decidable-again*: ¬ *turing-decidable K1*
⟨*proof*⟩

2.4 A code-indexed Busy Beaver function for the two-argument halting problem

The previous code-indexed function targets the diagonal/special problem *K1*. To cover AFP's two-argument halting problem *H1*, we fold the input into the Busy Beaver object itself: an object is a pair $(c,\ m)$ consisting of a machine code and an input. The size measure $c + m$ has finite bounded classes, and an upper bound for the resulting Busy Beaver function decides membership in *H1*.

definition *coded-pair-num-halts-in* :: $(nat \times nat) \Rightarrow nat \Rightarrow nat \Rightarrow bool$ **where**
  coded-pair-num-halts-in cm input t $\longleftrightarrow$
    tm-num-halts-in (*c2t* (*fst cm*)) (*snd cm*) *t*

definition *coded-pair-size* :: $nat \times nat \Rightarrow nat$ **where**
  coded-pair-size cm = *fst cm* + *snd cm*

lemma *finite-coded-pair-size-le*:
  finite $\{cm::nat \times nat.$ *coded-pair-size cm* $\leq n\}$
⟨*proof*⟩

interpretation *pair-busy-beaver*: *busy-beaver-base coded-pair-size coded-pair-num-halts-in*
⟨*proof*⟩

abbreviation *Pair-BB-time* :: $nat \Rightarrow nat$ **where**
  Pair-BB-time ≡ *pair-busy-beaver.BB-time*

lemma *coded-pair-halts-iff-H1*:
  pair-busy-beaver.halts $(c,\ m)$ *0* $\longleftrightarrow$ $[c,\ m] \in$ *H1*
⟨*proof*⟩

theorem *pair-BB-upper-bound-decides-H1-pair*:
  assumes *pair-busy-beaver.upper-bound-for-BB b*
  shows *pair-busy-beaver.halting-decider-from b* $(c,\ m)$ *0* $\longleftrightarrow$ $[c,\ m] \in$ *H1*
  ⟨*proof*⟩

definition *H1-decider-from* :: $(nat \Rightarrow nat) \Rightarrow nat\ list \Rightarrow bool$ **where**
  H1-decider-from b nl $\longleftrightarrow$
    (*case nl of*
      $[c,\ m] \Rightarrow$ *pair-busy-beaver.halting-decider-from b* $(c,\ m)$ *0*
    | - ⇒ *False*)

theorem *H1-decider-from-correct*:
  assumes *pair-busy-beaver.upper-bound-for-BB b*
  shows *H1-decider-from b nl* $\longleftrightarrow$ *nl* $\in$ *H1*
⟨*proof*⟩

corollary *H1-not-turing-decidable-again*: ¬ *turing-decidable H1*
  ⟨*proof*⟩

For every upper bound, the Boolean decider above defines the same set as *H1*. Hence AFP's existing undecidability theorem for *H1* immediately yields a concrete noncomputability consequence: the characteristic function of any such upper-bound-induced decider set is not Turing-computable, even in the total sense. In particular, this applies to the decider induced by *Pair-BB-time* itself.

definition *H1-decider-set-from* :: (*nat* ⇒ *nat*) ⇒ *nat list set* **where**
H1-decider-set-from b = {*nl*. *H1-decider-from b nl*}

lemma *H1-decider-set-from-eq-H1*:
assumes *pair-busy-beaver.upper-bound-for-BB b*
shows *H1-decider-set-from b* = *H1*
⟨*proof*⟩

theorem *no-turing-decidable-H1-decider-set-from*:
assumes *pair-busy-beaver.upper-bound-for-BB b*
shows ¬ *turing-decidable* (*H1-decider-set-from b*)
⟨*proof*⟩

theorem *no-turing-computable-partial-H1-decider-from*:
assumes *pair-busy-beaver.upper-bound-for-BB b*
shows ¬ *turing-computable-partial* (*chi-fun* (*H1-decider-set-from b*))
⟨*proof*⟩

theorem *no-turing-computable-total-H1-decider-from*:
assumes *pair-busy-beaver.upper-bound-for-BB b*
shows ¬ *turing-computable-total* (*chi-fun* (*H1-decider-set-from b*))
⟨*proof*⟩

theorem *no-turing-computable-total-H1-deciding-upper-bound*:
¬ (∃ *b*. *pair-busy-beaver.upper-bound-for-BB b* ∧
turing-computable-total (*chi-fun* (*H1-decider-set-from b*)))
⟨*proof*⟩

corollary *no-turing-computable-total-Pair-BB-time-decider*:
¬ *turing-computable-total* (*chi-fun* (*H1-decider-set-from Pair-BB-time*))
⟨*proof*⟩

end

end
theory *Busy-Beaver*
imports
Busy-Beaver-Base
Turing-Busy-Beaver
begin

3 The Busy Beaver Entry

This theory is the AFP entry point. It collects the abstract Busy Beaver upper-bound principle and its instantiation for the Turing machines from the AFP Universal Turing Machine development.

end

AI Assistance

AI assistance was used for proof engineering. The final definitions, statements, and proofs are checked by Isabelle.

References

[1] T. Radó. On non-computable functions. *Bell System Technical Journal*, 41(3):877–884, 1962.

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