

# The Balog–Szemerédi–Gowers Theorem

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## Abstract

We formalise the Balog–Szemerédi–Gowers Theorem, a profound result in additive combinatorics which played a central role in Gowers’s proof deriving the first effective bounds for Szemerédi’s Theorem [2]. The proof is of great mathematical interest given that it involves an interplay between different mathematical areas, namely applications of graph theory and probability theory to additive combinatorics involving algebraic objects. This interplay is what made the process of the formalisation, for which we had to develop formalisations of new background material in the aforementioned areas, more rich and technically challenging. We demonstrate how locales, Isabelle’s module system, can be employed to handle such interplays. To treat the graph-theoretic aspects of the proof, we make use of a new, more general undirected graph theory library developed recently by Chelsea Edmonds, which is both flexible and extensible [1]. For the formalisation we followed a proof presented in the 2022 lecture notes by Timothy Gowers "Introduction to Additive Combinatorics" for Part III of the Mathematical Tripos taught at the University of Cambridge [3]. In addition to the main theorem, which, following our source, is formulated for difference sets, we also give an alternative version for sumsets which required a formalisation of an auxiliary triangle inequality following a proof by Yufei Zhao from his book "Graph Theory and Additive Combinatorics" [4]. We moreover formalise a few additional results in additive combinatorics that are not used in the proof of the main theorem. This is the first formalisation of the Balog–Szemerédi–Gowers Theorem in any proof assistant to our knowledge.

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# 1 Miscellaneous technical lemmas

**theory** *Miscellaneous-Lemmas*

**imports**

*HOL-Library.Indicator-Function*

*HOL-Analysis.Convex*

**begin**

**lemma** *set-pairs-filter-subset*:  $A \subseteq B \implies \{p . p \in A \times A \wedge P p\} \subseteq \{p . p \in B \times B \wedge P p\}$   
*<proof>*

**lemma** *card-set-ss-indicator*:

**assumes**  $A \subseteq B$

**assumes** *finite B*

**shows**  $\text{card } A = (\sum p \in B. \text{indicator } A p)$

*<proof>*

**lemma** *card-cartesian-prod-square*: *finite X*  $\implies \text{card } (X \times X) = (\text{card } X)^2$

*<proof>*

**lemma** (*in ordered-ab-group-add*) *diff-strict1-mono*:

**assumes**  $a > a' \ b \leq b'$

**shows**  $a - b > a' - b'$

*<proof>*

**lemma** *card-cartesian-product-6*:  $\text{card } (A \times A \times A \times A \times A \times A) = (\text{card } A)^6$

*<proof>*

**lemma** *card-cartesian-product3*:  $\text{card } (X \times Y \times Z) = \text{card } X * \text{card } Y * \text{card } Z$

*<proof>*

**lemma** *card-le-image-div*:

**fixes** *A*:: 'a set **and** *B*:: 'b set **and** *f*:: 'a  $\implies$  'b set **and** *r*:: real

**assumes** *finite B* **and** *pairwise* ( $\lambda s t. \text{disjnt } (f s) (f t)$ ) *A* **and**  $\forall d \in A. (\text{card } (f d)) \geq r$

**and**  $\forall d \in A. f d \subseteq B$  **and**  $r > 0$

**shows**  $\text{card } A \leq \text{card } B / r$

*<proof>*

**lemma** *list-middle-eq*:

$\text{length } xs = \text{length } ys \implies \text{hd } xs = \text{hd } ys \implies \text{last } xs = \text{last } ys$

$\implies \text{butlast } (tl xs) = \text{butlast } (tl ys) \implies xs = ys$

*<proof>*

**lemma** *list2-middle-singleton*:

```

assumes length xs = 3
shows butlast (tl xs) = [xs ! 1]
⟨proof⟩

```

```

lemma le-powr-half-mult:
  fixes x y z:: real
  assumes x ^ 2 ≤ y * z and 0 ≤ y and 0 ≤ z
  shows x ≤ y powr(1/2) * z powr (1/2)
  ⟨proof⟩

```

```

lemma Cauchy-Schwarz-ineq-sum2:
  fixes f g:: 'a ⇒ real and A:: 'a set
  shows (∑ d ∈ A. f d * g d) ≤
    (∑ d ∈ A. (f d)^2) powr (1/2) * (∑ d ∈ A. (g d)^2) powr (1/2)
  ⟨proof⟩

```

```

end

```

## 2 Background material for the graph-theoretic aspects of the main proof

This section includes a number of lemmas on project specific definitions for graph theory, building on the general undirected graph theory library [1]

```

theory Graph-Theory-Preliminaries
  imports
    Miscellaneous-Lemmas
    Undirected-Graph-Theory.Bipartite-Graphs
    Undirected-Graph-Theory.Connectivity
    Random-Graph-Subgraph-Threshold.Ugraph-Misc
begin

```

### 2.1 On graphs with loops

```

context ulgraph

```

```

begin

```

```

definition degree-normalized:: 'a ⇒ 'a set ⇒ real where
  degree-normalized v S ≡ card (neighbors-ss v S) / (card S)

```

```

lemma degree-normalized-le-1: degree-normalized x S ≤ 1

```

```

⟨proof⟩

```

```

end

```

## 2.2 On bipartite graphs

**context** *bipartite-graph*  
**begin**

**definition** *codegree*:: 'a  $\Rightarrow$  'a  $\Rightarrow$  nat **where**  
*codegree* v u  $\equiv$  card {x  $\in$  V . vert-adj v x  $\wedge$  vert-adj u x}

**lemma** *codegree-neighbors*: *codegree* v u = card (neighborhood v  $\cap$  neighborhood u)  
*<proof>*

**lemma** *codegree-sym*: *codegree* v u = *codegree* u v  
*<proof>*

**definition** *codegree-normalized*:: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a set  $\Rightarrow$  real **where**  
*codegree-normalized* v u S  $\equiv$  *codegree* v u / card S

**lemma** *codegree-normalized-altX*:  
**assumes** x  $\in$  X **and** x'  $\in$  X  
**shows** *codegree-normalized* x x' Y = card (neighbors-ss x Y  $\cap$  neighbors-ss x' Y)  
/ card Y  
*<proof>*

**lemma** *codegree-normalized-altY*:  
**assumes** y  $\in$  Y **and** y'  $\in$  Y  
**shows** *codegree-normalized* y y' X = card (neighbors-ss y X  $\cap$  neighbors-ss y' X)  
/ card X  
*<proof>*

**lemma** *codegree-normalized-sym*: *codegree-normalized* u v S = *codegree-normalized*  
v u S  
*<proof>*

**definition** *bad-pair*:: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a set  $\Rightarrow$  real  $\Rightarrow$  bool **where**  
*bad-pair* v u S c  $\equiv$  *codegree-normalized* v u S < c

**lemma** *bad-pair-sym*:  
**assumes** *bad-pair* v u S c **shows** *bad-pair* u v S c  
*<proof>*

**definition** *bad-pair-set*:: 'a set  $\Rightarrow$  'a set  $\Rightarrow$  real  $\Rightarrow$  ('a  $\times$  'a) set **where**  
*bad-pair-set* S T c  $\equiv$  {(u, v)  $\in$  S  $\times$  S. *bad-pair* u v T c}

**lemma** *bad-pair-set-ss*: *bad-pair-set* S T c  $\subseteq$  S  $\times$  S  
*<proof>*

**lemma** *bad-pair-set-filter-alt*:

*bad-pair-set*  $S T c = \text{Set.filter } (\lambda p . \text{bad-pair } (\text{fst } p) (\text{snd } p) T c) (S \times S)$   
 ⟨proof⟩

**lemma** *bad-pair-set-finite*:  
**assumes** *finite*  $S$   
**shows** *finite* (*bad-pair-set*  $S T c$ )  
 ⟨proof⟩

**lemma** *codegree-is-path-length-two*:  
 $\text{codegree } x x' = \text{card } \{p . \text{connecting-path } x x' p \wedge \text{walk-length } p = 2\}$   
 ⟨proof⟩

**lemma** *codegree-bipartite-eq*:  
 $\forall x \in X. \forall x' \in X. \text{codegree } x x' = \text{card } \{y \in Y. \text{vert-adj } x y \wedge \text{vert-adj } x' y\}$   
 ⟨proof⟩

**lemma** (**in** *fin-bipartite-graph*) *bipartite-deg-square-eq*:  
 $\forall y \in Y. (\sum x' \in X. \sum x \in X. \text{indicator } \{z. \text{vert-adj } x z \wedge \text{vert-adj } x' z\} y)$   
 $= (\text{degree } y)^2$   
 ⟨proof⟩

**lemma** (**in** *fin-bipartite-graph*) *codegree-degree*:  
 $(\sum x' \in X. \sum x \in X. (\text{codegree } x x')) = (\sum y \in Y. (\text{degree } y)^2)$   
 ⟨proof⟩

**lemma** (**in** *fin-bipartite-graph*) *sum-degree-normalized-X-density*:  
 $(\sum x \in X. \text{degree-normalized } x Y) / \text{card } X = \text{edge-density } X Y$   
 ⟨proof⟩

**lemma** (**in** *fin-bipartite-graph*) *sum-degree-normalized-Y-density*:  
 $(\sum y \in Y. \text{degree-normalized } y X) / \text{card } Y = \text{edge-density } X Y$   
 ⟨proof⟩

**end**  
**end**

### 3 Auxiliary probability space results

**theory** *Prob-Space-Lemmas*  
**imports**  
*Random-Graph-Subgraph-Threshold.Prob-Lemmas*  
**begin**  
  
**context** *prob-space*  
  
**begin**  
  
**lemma** *expectation-uniform-count*:

**assumes**  $M = \text{uniform-count-measure } X$  **and** *finite*  $X$   
**shows**  $\text{expectation } f = (\sum_{x \in X} f x) / \text{card } X$

*<proof>*

A lemma to obtain a value for  $x$  where the inequality is satisfied

**lemma** *expectation-obtains-ge*:  
**fixes**  $f :: 'a \Rightarrow \text{real}$   
**assumes**  $M = \text{uniform-count-measure } X$  **and** *finite*  $X$   
**assumes**  $\text{expectation } f \geq c$   
**obtains**  $x$  **where**  $x \in X$  **and**  $f x \geq c$

*<proof>*

The following is the variation on the Cauchy-Schwarz inequality presented in Gowers's notes before Lemma 2.13 [3].

**lemma** *cauchy-schwarz-ineq-var*:  
**fixes**  $X :: 'a \Rightarrow \text{real}$   
**assumes** *integrable*  $M (\lambda x. (X x)^2)$  **and**  $X \in \text{borel-measurable } M$   
**shows**  $\text{expectation } (\lambda x. (X x)^2) \geq (\text{expectation } (\lambda x. (X x)))^2$

*<proof>*

**lemma** *integrable-uniform-count-measure-finite*:  
**fixes**  $g :: 'a \Rightarrow 'b::\{\text{banach, second-countable-topology}\}$   
**shows** *finite*  $A \implies \text{integrable } (\text{uniform-count-measure } A) g$   
*<proof>*

**lemma** *cauchy-schwarz-ineq-var-uniform*:  
**fixes**  $X :: 'a \Rightarrow \text{real}$   
**assumes**  $M = \text{uniform-count-measure } S$   
**assumes** *finite*  $S$   
**shows**  $\text{expectation } (\lambda x. (X x)^2) \geq (\text{expectation } (\lambda x. (X x)))^2$

*<proof>*

An equation for expectation over a discrete random variables distribution:

**lemma** *expectation-finite-uniform-space*:  
**assumes**  $M = \text{uniform-count-measure } S$  **and** *finite*  $S$   
**fixes**  $X :: 'a \Rightarrow \text{real}$   
**shows**  $\text{expectation } X = (\sum_{y \in X} y \cdot \text{prob } \{x \in S \cdot X x = y\} * y)$

*<proof>*

**lemma** *expectation-finite-uniform-indicator*:  
**assumes**  $M = \text{uniform-count-measure } S$  **and** *finite*  $S$   
**shows**  $\text{expectation } (\lambda x. \text{indicator } (T x) y) = \text{prob } \{x \in S \cdot \text{indicator } (T x) y = 1\}$  **(is**  $\text{expectation } ?X = -)$

*<proof>*

**end**  
**end**

## 4 A triangle inequality for sumsets

**theory** *Sumset-Triangle-Inequality*

**imports**

*Pluennecke-Ruzsa-Inequality.Pluennecke-Ruzsa-Inequality*

**begin**

**context** *additive-abelian-group*

**begin**

We show a useful triangle inequality for sumsets that does \*not\* follow from the Ruzsa triangle inequality. The proof follows the exposition in Zhao's book [4].

The following auxiliary lemma corresponds to Lemma 7.3.4 in Zhao's book [4].

**lemma** *triangle-ineq-sumsets-aux:*

**fixes**  $X B Y :: 'a \text{ set}$

**assumes**  $hX: \text{finite } X$  **and**  $hB: \text{finite } B$  **and**  $hXG: X \subseteq G$  **and**  $hBG: B \subseteq G$

**and**

$hXne: X \neq \{\}$  **and**  $hYX: \bigwedge Y. Y \subseteq X \implies Y \neq \{\} \implies \text{card } (\text{sumset } Y B) / \text{card } Y \geq$

$\text{card } (\text{sumset } X B) / \text{card } X$  **and**  $hC: \text{finite } C$  **and**  $hCne: C \neq \{\}$  **and**  $hCG: C \subseteq G$

**shows**  $\text{card } (\text{sumset } X (\text{sumset } C B)) / \text{card } (\text{sumset } X C) \leq \text{card } (\text{sumset } X B) / \text{card } X$

*<proof>*

The following inequality is the result corresponding to Corollary 7.3.6 in Zhao's book [4].

**lemma** *triangle-ineq-sumsets:*

**assumes**  $hA: \text{finite } A$  **and**  $hB: \text{finite } B$  **and**  $hC: \text{finite } C$  **and**

$hAG: A \subseteq G$  **and**  $hBG: B \subseteq G$  **and**  $hCG: C \subseteq G$

**shows**  $\text{card } A * \text{card } (\text{sumset } B C) \leq \text{card } (\text{sumset } A B) * \text{card } (\text{sumset } A C)$

*<proof>*

**end**  
**end**



## 5 Background material in additive combinatorics

This section outlines some background definitions and basic lemmas in additive combinatorics based on the notes by Gowers [3].

**theory** *Additive-Combinatorics-Preliminaries*

**imports**

*Pluennecke-Ruzsa-Inequality.Pluennecke-Ruzsa-Inequality*

**begin**

### 5.1 Additive quadruples and additive energy

**context** *additive-abelian-group*

**begin**

**definition** *additive-quadruple*:: 'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  'a  $\Rightarrow$  bool **where**

*additive-quadruple* a b c d  $\equiv a \in G \wedge b \in G \wedge c \in G \wedge d \in G \wedge a \oplus b = c \oplus d$

**lemma** *additive-quadruple-aux*:

**assumes** *additive-quadruple* a b c d

**shows**  $d = a \oplus b \ominus c$

*<proof>*

**lemma** *additive-quadruple-diff*:

**assumes** *additive-quadruple* a b c d

**shows**  $a \ominus c = d \ominus b$

*<proof>*

**definition** *additive-quadruple-set*:: 'a set  $\Rightarrow$  ('a  $\times$  'a  $\times$  'a  $\times$  'a) set **where**

*additive-quadruple-set* A  $\equiv \{(a, b, c, d) \mid a b c d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge$

*additive-quadruple* a b c d}

**lemma** *additive-quadruple-set-sub*:

*additive-quadruple-set* A  $\subseteq \{(a, b, c, d) \mid a b c d. d = a \oplus b \ominus c \wedge a \in A \wedge b \in A \wedge$

$c \in A \wedge d \in A\}$  *<proof>*

**definition** *additive-energy*:: 'a set  $\Rightarrow$  real **where**

*additive-energy* A  $\equiv \text{card}(\text{additive-quadruple-set } A) / (\text{card } A)^3$

**lemma** *card-ineq-aux-quadruples*:

**assumes** *finite* A

**shows**  $\text{card}(\text{additive-quadruple-set } A) \leq (\text{card } A)^3$

*<proof>*

**lemma** *additive-energy-upper-bound*: *additive-energy* A  $\leq 1$

*<proof>*

## 5.2 On sums

**definition** *f-sum*:: 'a  $\Rightarrow$  'a set  $\Rightarrow$  nat **where**

$$f\text{-sum } d \ A \equiv \text{card } \{(a, b) \mid a \ b. \ a \in A \wedge b \in A \wedge a \oplus b = d\}$$

**lemma** *pairwise-disjnt-sum-1*:

$$\text{pairwise } (\lambda s \ t. \ \text{disjnt } ((\lambda d. \ \{(a, b) \mid a \ b. \ a \in A \wedge b \in A \wedge (a \oplus b = d)\}) \ s) \\ ((\lambda d. \ \{(a, b) \mid a \ b. \ a \in A \wedge b \in A \wedge (a \oplus b = d)\}) \ t)) \ (\text{sumset } A \ A)$$

*<proof>*

**lemma** *pairwise-disjnt-sum-2*:

$$\text{pairwise } \text{disjnt } ((\lambda d. \ \{(a, b) \mid a \ b. \ a \in A \wedge b \in A \wedge a \oplus b = d\}) \ ' (\text{sumset } A \ A))$$

*<proof>*

**lemma** *sum-Union-span*:

**assumes**  $A \subseteq G$

$$\text{shows } \bigcup ((\lambda d. \ \{(a, b) \mid a \ b. \ a \in A \wedge b \in A \wedge (a \oplus b = d)\}) \ ' (\text{sumset } A \ A)) \\ = A \times A$$

*<proof>*

**lemma** *f-sum-le-card*:

**assumes** *finite*  $A$  **and**  $A \subseteq G$

**shows**  $f\text{-sum } d \ A \leq \text{card } A$

*<proof>*

**lemma** *f-sum-card*:

**assumes**  $A \subseteq G$  **and**  $hA$ : *finite*  $A$

**shows**  $(\sum d \in (\text{sumset } A \ A). \ (f\text{-sum } d \ A)) = (\text{card } A)^{\wedge 2}$

*<proof>*

**lemma** *f-sum-card-eq*:

**assumes**  $A \subseteq G$

$$\text{shows } \forall x \in \text{sumset } A \ A. \ (f\text{-sum } x \ A)^{\wedge 2} = \\ \text{card } \{(a, b, c, d) \mid a \ b \ c \ d. \ a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \\ \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus b = x \wedge c \oplus d = x\}$$

*<proof>*

**lemma** *pairwise-disjoint-sum*:

$$\text{pairwise } (\lambda s \ t. \ \text{disjnt } ((\lambda x. \ \{(a, b, c, d) \mid a \ b \ c \ d. \ a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \\ \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus b = x \wedge c \oplus d = x\}) \ s))$$

$((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge$   
*additive-quadruple*  $a \ b \ c \ d \wedge a \oplus b = x \wedge c \oplus d = x\}) \ t)) \ (\text{sumset } A \ A)$   
 ⟨proof⟩

**lemma** *pairwise-disjnt-quadruple-sum:*

*pairwise disjnt*  $((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge$   
*additive-quadruple*  $a \ b \ c \ d \wedge a \oplus b = x \wedge c \oplus d = x\}) \ ' (\text{sumset } A \ A))$   
 ⟨proof⟩

**lemma** *quadruple-sum-Union-eq:*

$\bigcup ((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge$   
*additive-quadruple*  $a \ b \ c \ d \wedge a \oplus b = x \wedge c \oplus d = x\}) \ ' (\text{sumset } A \ A)) =$   
*additive-quadruple-set*  $A$

⟨proof⟩

**lemma** *f-sum-card-quadruple-set:*

**assumes**  $hAG: A \subseteq G$  **and**  $hA: \text{finite } A$   
**shows**  $(\sum d \in (\text{sumset } A \ A). (f\text{-sum } d \ A) \hat{=} 2) = \text{card } (\text{additive-quadruple-set } A)$

⟨proof⟩

**lemma** *f-sum-card-quadruple-set-additive-energy:* **assumes**  $A \subseteq G$  **and** *finite*  $A$

**shows**  $(\sum d \in \text{sumset } A \ A. (f\text{-sum } d \ A) \hat{=} 2) = \text{additive-energy } A * (\text{card } A) \hat{=} 3$

⟨proof⟩

**definition** *popular-sum:: 'a  $\Rightarrow$  real  $\Rightarrow$  'a set  $\Rightarrow$  bool where*

*popular-sum*  $d \ \vartheta \ A \equiv f\text{-sum } d \ A \geq \vartheta * \text{of-real } (\text{card } A)$

**definition** *popular-sum-set:: real  $\Rightarrow$  'a set  $\Rightarrow$  'a set where*

*popular-sum-set*  $\vartheta \ A \equiv \{d \in \text{sumset } A \ A. \text{popular-sum } d \ \vartheta \ A\}$

### 5.3 On differences

The following material is directly analogous to the material given previously on sums. All definitions and lemmas are the corresponding ones for differences. E.g. *f-diff* corresponds to *f-sum*.

**definition** *f-diff:: 'a  $\Rightarrow$  'a set  $\Rightarrow$  nat where*

*f-diff*  $d \ A \equiv \text{card } \{(a, b) \mid a \ b. a \in A \wedge b \in A \wedge a \ominus b = d\}$

**lemma** *pairwise-disjnt-diff-1:*

*pairwise*  $(\lambda s \ t. \text{disjnt } ((\lambda d. \{(a, b) \mid a \ b. a \in A \wedge b \in A \wedge (a \ominus b = d)\}) \ s)$   
 $((\lambda d. \{(a, b) \mid a \ b. a \in A \wedge b \in A \wedge (a \ominus b = d)\}) \ t)) \ (\text{differencerset } A \ A)$

⟨proof⟩

**lemma** *pairwise-disjnt-diff-2:*

*pairwise disjoint*  $((\lambda d. \{(a, b) \mid a \ b. a \in A \wedge b \in A \wedge a \oplus b = d\}) \text{ ' (differenceset } A \ A))$

$\langle \text{proof} \rangle$

**lemma** *diff-Union-span:*

**assumes**  $A \subseteq G$

**shows**  $\bigcup ((\lambda d. \{(a, b) \mid a \ b. a \in A \wedge b \in A \wedge (a \oplus b = d)\}) \text{ ' (differenceset } A \ A)) = A \times A$

$\langle \text{proof} \rangle$

**lemma** *f-diff-le-card:*

**assumes** *finite*  $A$  **and**  $A \subseteq G$

**shows**  $f\text{-diff } d \ A \leq \text{card } A$

$\langle \text{proof} \rangle$

**lemma** *f-diff-card:*

**assumes**  $A \subseteq G$  **and**  $hA$ : *finite*  $A$

**shows**  $(\sum d \in (\text{differenceset } A \ A). f\text{-diff } d \ A) = (\text{card } A) \hat{\ } 2$

$\langle \text{proof} \rangle$

**lemma** *f-diff-card-eq:*

**assumes**  $A \subseteq G$

**shows**  $\forall x \in \text{differenceset } A \ A. (f\text{-diff } x \ A) \hat{\ } 2 = \text{card } \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus c = x \wedge d \oplus b = x\}$

$\langle \text{proof} \rangle$

**lemma** *pairwise-disjoint-diff:*

*pairwise disjoint*  $(\lambda s \ t. \text{disjnt } ((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus c = x \wedge d \oplus b = x\}) \ s)$

$((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus c = x \wedge d \oplus b = x\}) \ t)) \text{ (differenceset } A \ A)$

$\langle \text{proof} \rangle$

**lemma** *pairwise-disjnt-quadruple-diff:*

*pairwise disjoint*  $((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus c = x \wedge d \oplus b = x\}) \text{ ' (differenceset } A \ A))$

$\langle \text{proof} \rangle$

**lemma** *quadruple-diff-Union-eq:*

$\bigcup ((\lambda x. \{(a, b, c, d) \mid a \ b \ c \ d. a \in A \wedge b \in A \wedge c \in A \wedge d \in A \wedge \text{additive-quadruple } a \ b \ c \ d \wedge a \oplus c = x \wedge d \oplus b = x\}) \text{ ' (differenceset } A \ A)) =$

*additive-quadruple-set*  $A$

*<proof>*

**lemma** *f-diff-card-quadruple-set*:

**assumes** *hAG*:  $A \subseteq G$  **and** *hA*: *finite A*

**shows**  $(\sum d \in (\text{differenceset } A \ A). (f\text{-diff } d \ A)^{\wedge 2}) = \text{card } (\text{additive-quadruple-set } A)$

*<proof>*

**lemma** *f-diff-card-quadruple-set-additive-energy*: **assumes**  $A \subseteq G$  **and** *finite A*

**shows**  $(\sum d \in \text{differenceset } A \ A. (f\text{-diff } d \ A)^{\wedge 2}) = \text{additive-energy } A * (\text{card } A)^{\wedge 3}$

*<proof>*

**definition** *popular-diff*::  $'a \Rightarrow \text{real} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$  **where**

*popular-diff*  $d \ \vartheta \ A \equiv f\text{-diff } d \ A \geq \vartheta * \text{of-real } (\text{card } A)$

**definition** *popular-diff-set*::  $\text{real} \Rightarrow 'a \text{ set} \Rightarrow 'a \text{ set}$  **where**

*popular-diff-set*  $\vartheta \ A \equiv \{d \in \text{differenceset } A \ A. \text{popular-diff } d \ \vartheta \ A\}$

**end**

**end**

## 6 Results on lower bounds on additive energy

**theory** *Additive-Energy-Lower-Bounds*

**imports**

*Additive-Combinatorics-Preliminaries*

*Miscellaneous-Lemmas*

**begin**

**context** *additive-abelian-group*

**begin**

The following corresponds to Proposition 2.11 in Gowers's notes [3].

**proposition** *additive-energy-lower-bound-sumset*: **fixes**  $C::\text{real}$

**assumes** *finite A* **and**  $A \subseteq G$  **and**  $(\text{card } (\text{sumset } A \ A)) \leq C * \text{card } A$  **and**  $\text{card } A \neq 0$

**shows**  $\text{additive-energy } A \geq 1/C$

*<proof>*

An analogous version of Proposition 2.11 where the assumption is on a difference set is given below. The proof is identical to the proof of *additive-energy-lower-bound-sumset* above (with the obvious modifications).

**proposition** *additive-energy-lower-bound-differenceset*: **fixes**  $C::\text{real}$

**assumes** *finite A and  $A \subseteq G$  and  $(\text{card}(\text{differenceset } A)) \leq C * \text{card } A$  and  $\text{card } A \neq 0$*

**shows** *additive-energy  $A \geq 1/C$*

*<proof>*

**end**

**end**

## 7 Towards the proof of the Balog–Szemerédi–Gowers Theorem

**theory** *Balog-Szemerédi-Gowers-Main-Proof*

**imports**

*Prob-Space-Lemmas*

*Graph-Theory-Preliminaries*

*Sumset-Triangle-Inequality*

*Additive-Combinatorics-Preliminaries*

**begin**

**context** *additive-abelian-group*

**begin**

After having introduced all the necessary preliminaries in the imported files, we are now ready to follow the chain of the arguments for the main proof as in Gowers’s notes [3].

The following lemma corresponds to Lemma 2.13 in Gowers’s notes [3].

**lemma** (*in fin-bipartite-graph*) *proportion-bad-pairs-subset-bipartite:*

**fixes** *c::real*

**assumes** *c > 0*

**obtains** *X' where  $X' \subseteq X$  and  $\text{card } X' \geq \text{density} * \text{card } X / \text{sqrt } 2$  and*

*$\text{card}(\text{bad-pair-set } X' Y c) / (\text{card } X')^2 \leq 2 * c / \text{density}^2$*

*<proof>*

The following technical probability lemma corresponds to Lemma 2.14 in Gowers’s notes [3].

**lemma** (*in prob-space*) *expectation-condition-card-1:*

**fixes** *X::'a set and f::'a  $\Rightarrow$  real and  $\delta$ ::real*

**assumes** *finite X and  $\forall x \in X. f x \leq 1$  and  $M = \text{uniform-count-measure } X$  and expectation  $f \geq \delta$*

**shows**  *$\text{card} \{x \in X. (f x \geq \delta / 2)\} \geq \delta * \text{card } X / 2$*

*<proof>*

The following technical probability lemma corresponds to Lemma 2.15 in Gowers’s notes.

**lemma** (*in prob-space*) *expectation-condition-card-2:*

**fixes**  $X::'a$  set **and**  $\beta::real$  **and**  $\alpha::real$  **and**  $f::'a \Rightarrow real$   
**assumes** finite  $X$  **and**  $\bigwedge x. x \in X \implies f x \leq 1$  **and**  $\beta > 0$  **and**  $\alpha > 0$   
**and** expectation  $f \geq 1 - \alpha$  **and**  $M = \text{uniform-count-measure } X$   
**shows**  $\text{card } \{x \in X. f x \geq 1 - \beta\} \geq (1 - \alpha / \beta) * \text{card } X$

*<proof>*

The following lemma corresponds to Lemma 2.16 in Gowers's notes [3]. For the proof, we will apply Lemma 2.13 (*proportion-bad-pairs-subset-bipartite*), the technical probability Lemmas 2.14 (*expectation-condition-card-1*) and 2.15 (*expectation-condition-card-2*) as well as background material on graphs with loops and bipartite graphs that was previously presented.

**lemma** (in *fin-bipartite-graph*) *walks-of-length-3-subsets-bipartite*:  
**obtains**  $X'$  **and**  $Y'$  **where**  $X' \subseteq X$  **and**  $Y' \subseteq Y$  **and**  
 $\text{card } X' \geq (\text{edge-density } X Y)^2 * \text{card } X / 16$  **and**  
 $\text{card } Y' \geq \text{edge-density } X Y * \text{card } Y / 4$  **and**  
 $\forall x \in X'. \forall y \in Y'. \text{card } \{p. \text{connecting-walk } x y p \wedge \text{walk-length } p = 3\} \geq$   
 $(\text{edge-density } X Y)^6 * \text{card } X * \text{card } Y / 2^{13}$

*<proof>*

The following lemma corresponds to Lemma 2.17 in Gowers's notes [3].

Note that here we have set(*additive-energy*  $A = 2 * c$  (instead of (*additive-energy*  $A = c$  as in the notes) and we are accordingly considering  $c$ -popular differences (instead of  $c/2$ -popular differences as in the notes) so that we will still have ( $\vartheta = \text{additive-energy } A / 2$ ).

**lemma** *popular-differences-card*: **fixes**  $A::'a$  set **and**  $c::real$   
**assumes** finite  $A$  **and**  $A \subseteq G$  **and** *additive-energy*  $A = 2 * c$   
**shows**  $\text{card } (\text{popular-diff-set } c A) \geq c * \text{card } A$

*<proof>*

The following lemma corresponds to Lemma 2.18 in Gowers's notes [3]. It includes the key argument of the main proof and its proof applies Lemmas 2.16 (*walks-of-length-3-subsets-bipartite*) and 2.17 (*popular-differences-card*). In the proof we will use an appropriately defined bipartite graph as an intermediate/auxiliary construct so as to apply lemma *walks-of-length-3-subsets-bipartite*. As each vertex set of the bipartite graph is constructed to be a copy of a finite subset of an Abelian group, we need flexibility regarding types, which is what prompted the introduction and use of the new graph theory library [1] (that does not impose any type restrictions e.g. by representing vertices as natural numbers).

**lemma** *obtains-subsets-differenceset-card-bound*:  
**fixes**  $A::'a$  set **and**  $c::real$   
**assumes** finite  $A$  **and**  $c > 0$  **and**  $A \neq \{\}$  **and**  $A \subseteq G$  **and** *additive-energy*  $A = 2 * c$

**obtains**  $B$  **and**  $A'$  **where**  $B \subseteq A$  **and**  $B \neq \{\}$  **and**  $\text{card } B \geq c^4 * \text{card } A / 16$   
**and**  $A' \subseteq A$  **and**  $A' \neq \{\}$  **and**  $\text{card } A' \geq c^2 * \text{card } A / 4$   
**and**  $\text{card } (\text{differenceset } A' B) \leq 2^{13} * \text{card } A / c^{15}$

*<proof>*

We now show the main theorem, which is a direct application of lemma *obtains-subsets-differenceset-card-bound* and the Ruzsa triangle inequality. (The main theorem corresponds to Corollary 2.19 in Gowers's notes [3].)

**theorem** *Balog-Szemerédi-Gowers*: **fixes**  $A::'a \text{ set}$  **and**  $c::\text{real}$   
**assumes**  $\text{afin}: \text{finite } A$  **and**  $A \neq \{\}$  **and**  $c > 0$  **and**  $\text{additive-energy } A = 2 * c$   
**and**  $\text{ass}: A \subseteq G$   
**obtains**  $A'$  **where**  $A' \subseteq A$  **and**  $\text{card } A' \geq c^2 * \text{card } A / 4$  **and**  
 $\text{card } (\text{differenceset } A' A') \leq 2^{30} * \text{card } A / c^{34}$

*<proof>*

The following is an analogous version of the Balog–Szemerédi–Gowers Theorem for a sumset instead of a difference set. The proof is similar to that of the original version, again using *obtains-subsets-differenceset-card-bound*, however, instead of the Ruzsa triangle inequality we will use the alternative triangle inequality for sumsets *triangle-ineq-sumsets*.

**theorem** *Balog-Szemerédi-Gowers-sumset*: **fixes**  $A::'a \text{ set}$  **and**  $c::\text{real}$   
**assumes**  $\text{afin}: \text{finite } A$  **and**  $A \neq \{\}$  **and**  $c > 0$  **and**  $\text{additive-energy } A = 2 * c$   
**and**  $\text{ass}: A \subseteq G$   
**obtains**  $A'$  **where**  $A' \subseteq A$  **and**  $\text{card } A' \geq c^2 * \text{card } A / 4$  **and**  
 $\text{card } (\text{sumset } A' A') \leq 2^{30} * \text{card } A / c^{34}$

*<proof>*

**end**  
**end**

## 8 Supplementary results related to intermediate lemmas used in the proof of the Balog–Szemerédi–Gowers Theorem

**theory** *Balog-Szemerédi-Gowers-Supplementary*  
**imports**  
*Balog-Szemerédi-Gowers-Main-Proof*  
**begin**

**context** *additive-abelian-group*

**begin**

Even though it is not applied anywhere in this development, for the sake of completeness we give the following analogous version of Lemma 2.17 (*pop-*



ular-differences-card) but for popular sums instead of popular differences. The proof is identical to that of Lemma 2.17, with the obvious modifications.

**lemma** *popular-sums-card*:  
**fixes**  $A::'a$  set **and**  $c::real$   
**assumes** *finite*  $A$  **and** *additive-energy*  $A = 2 * c$  **and**  $A \subseteq G$   
**shows**  $card$  (*popular-sum-set*  $c A$ )  $\geq c * card A$

*<proof>*

The following is an analogous version of lemma *obtains-subsets-differenceset-card-bound* (2.18 in Gowers's notes [3]) but for a sumset instead of a difference set. It is not used anywhere in this development but we provide it for the sake of completeness. The proof is identical to that of lemma *obtains-subsets-differenceset-card-bound* with *f-diff* changed to *f-sum*, *popular-diff* changed to *popular-sum*,  $\oplus$  interchanged with  $\ominus$ , and instead of lemma *popular-differences-card* we apply its analogous version for popular sums, that is lemma *popular-sums-card*.

**lemma** *obtains-subsets-sumset-card-bound*: **fixes**  $A::'a$  set **and**  $c::real$   
**assumes** *finite*  $A$  **and**  $c > 0$  **and**  $A \neq \{\}$  **and**  $A \subseteq G$  **and** *additive-energy*  $A = 2 * c$   
**obtains**  $B$  **and**  $A'$  **where**  $B \subseteq A$  **and**  $B \neq \{\}$  **and**  $card B \geq c^4 * card A / 16$   
**and**  $A' \subseteq A$  **and**  $A' \neq \{\}$  **and**  $card A' \geq c^2 * card A / 4$   
**and**  $card$  (*sumset*  $A' B$ )  $\leq 2^{13} * card A / c^{15}$

*<proof>*

**end**  
**end**

## References

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